THE NEUTRON-PROTON MASS DIFFERENCE

Thesis by

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Abstract

It is proposed that the neutron-proton and $\pi^{\pm}-\pi^{0}$ mass differences are produced by the electromagnetic field. In order to make the quantum electrodynamic mass corrections convergent, it is necessary to cut off the photon propagation function and the anomalous moment interaction of the neutron and proton for high virtual photon masses. The cutoff masses required to give the observed mass differences are of the order of the proton mass. These cutoffs are in agreement with the relatively small changes nucleon moments appear to undergo when nucleons form nuclei, and with experiments on electron-neutron and electron-proton scattering. A classical interpretation of the neutron-proton mass difference is discussed. A justification of the quantum electrodynamic calculation based on quantum field theory is also given.

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Introduction

In contemporary theories of nuclear interactions, it is assumed that the neutron and proton are two states of a single particle, the "nucleon," and that the $\pi^{\pm}$, π^{0} are three states of a single particle, the "pion." We assign to the nucleon and pion "isotopic spins" of $\frac{1}{2}$ and 1, respectively, in analogy to ordinary spin, but, unlike ordinary spin, isotopic spin states are not transformed under a spatial rotation. We assume that the interactions among nucleons and pions possess isotopic spin symmetry, by which is meant an invariance under transformations analogous to those induced by spatial rotations in the case of ordinary spin. This hypothesis of isotopic spin symmetry has successfully explained many nuclear phenomena. Since isotopic spin symmetry implies charge symmetry, in order to maintain such a hypothesis exactly it is necessary to explain all deviations from charge symmetry in terms of other interactions. In the case of nuclei, such deviations are successfully attributed to the electromagnetic interactions of protons and the neutron-proton mass difference. There are, however, deviations from charge symmetry of a more fundamental nature, since they involve properties of the individual nucleons and pions themselves. These anomalies are:

1) The charged π mesons are about 9 electron masses

heavier than the neutral π meson.

2) The neutron is about 2.5 electron masses heavier than the proton.

We know that the charged mesons and the proton interact with the electromagnetic field in a manner which differs from the neutral meson and neutron, respectively. A natural question to ask is whether the required mass differences can be produced by the electromagnetic field alone. According to quantum electrodynamics, there is a mass correction to any charged particle, of order $e^2/\hbar c$, due to the emission and reabsorption of a virtual photon. Here we encounter our first difficulty with present-day theory, for the resulting integral over all possible virtual photon 4-momenta diverges. To remedy this situation, we must use a Lorentz invariant cutoff which is strong enough to make this integral converge, yet negligible in its effects at low momenta and energies, so that the successful results of quantum electrodynamics are not affected, within experimental accuracy. Such a program is possible, because quantum electrodynamics is a "renormalizable" field theory. By this we mean that if any observable process is computed in terms of the observed mass and charge of the particle, using any cutoff method sufficient to remove the divergences of the theory and to maintain Lorentz and gauge invariance, then the result will be insensitive to the cutoff method, in that a unique limit will exist as the cutoff is weakened to no cutoff.

We can justify such a cutoff either by saying that quantum electrodynamics is wrong for high photon energies, with the exact theory providing a natural cutoff, or, that quantum electrodynamics is correct, but the complex of virtual nucleons, pions, and perhaps other particles resulting from the strong nucleon-pion interactions interacts with the electromagnetic field in such a manner as to cut off the effects of high energy photons. In either case we obtain a cutoff whose form is highly arbitrary in the light of our present knowledge of physics. We can hardly claim that any arbitrary cutoff will yield a given observed anomaly, or even yield the correct sign, but merely that a cutoff of simple analytic form gives the observed result.

In treating the electromagnetic mass corrections of the proton and neutron, another cutoff must be introduced in connection with the anomalous magnetic moments of these particles. These moments cannot be neglected for two reasons. In the first place, these moments, after all, do imply a coupling to the electromagnetic field in addition to the charge of the proton. In the second place, if we neglect them we obtain a positive mass correction for the proton, and no mass correction for the neutron, making the proton heavier than the neutron, contrary to what is observed. For low energies, the coupling of the anomalous moments to the electromagnetic field can be represented by a Pauli term in the Dirac equation. For higher energies, the coupling is not known, but it is reasonable to expect the coupling to cut off at sufficiently high photon energies, due to the fact that the currents producing the anomalous moments have a finite extension in space. Although the precise form of the cutoff is arbitrary in the light of our present knowledge of physics, its existence is physically understandable.

The method used to compute the pion and nucleon mass differences is a perturbation expansion of the quantum electrodynamic mass correction, computed to order $e^2/\hbar c$, using the following cutoffs: For the spin zero pion, we multiply the photon propagation function,* $1/\underline{k}^2$, by the factor $C(\underline{k}^2) = (\Lambda^2/\Lambda^2-\underline{k}^2)^2$. The observed mass difference is obtained for Λ about .86 proton masses. For the spin $\frac{1}{2}$ nucleon, we multiply the photon propagation function, $1/\underline{k}^2$, by the factor $C(\underline{k}^2) = -\Lambda^2/(\underline{k}^2-\Lambda^2)$. We could have used $(\Lambda^2/\Lambda^2-\underline{k}^2)^2$, but this would have required more computation without producing any effects significantly different from $-\Lambda^2/(\underline{k}^2-\Lambda^2)$, and we do not know the correct $C(\underline{k}^2)$ anyway. In the case of the pion, $-\Lambda^2/(\underline{k}^2-\Lambda^2)$ is not a strong enough cutoff. The moment coupling, $(\not{k}\gamma_\mu-\gamma_\mu\not{k})$, is multiplied by the factor $G(\underline{k}^2) = -\lambda^2/(\underline{k}^2-\lambda^2)$. For λ less than $\mu_p^{-\frac{1}{2}}m = .75m$, μ_p being the anomalous magnetic moment of the proton, in

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*R. P. Feynman, Phys. Rev. 76, 769-789 (1949). We use the notation of this article, except that $d^4k = dk_t dk_x dk_y dk_z$, 4-vectors are underlined, and with any 4-vector $\underline{A}$ we can associate the spinor $\not{A} = A_t\gamma_t - A_x\gamma_x - A_y\gamma_y - A_z\gamma_z$.

units of $e\hbar/2mc$, and m its mass, no value of Λ gives the observed mass difference. For $\lambda > \mu_p^{-\frac{1}{2}} m$, however, we can always find a $\Lambda(\lambda)$ which will give the observed mass difference. For example, if $\lambda = 1.5m$, then $\Lambda = 1.5m$; if $\lambda = 1.0m$, then $\Lambda = 5.0m$, approximately. The details of these computations follow this introduction.

The high cutoff for the anomalous moments implies that the currents producing these moments are spread over only a small distance, of the order of the proton Compton wave length, $\hbar/mc$. This is in agreement with the relatively small changes nucleon moments appear to undergo when nucleons form nuclei, and also with experiments on the scattering of electrons by neutrons and protons. These matters will be discussed in more detail after the computation of the neutron-proton mass difference.

The $\pi^{\pm}-\pi^{0}$ Mass Difference

For the spin zero charged π mesons, the electromagnetic mass correction is given, to order e^2, by (1)

$$\Delta m = -e^2 i(2\pi m)^{-1} \int [(2p-k)_\mu((\underline{p}-\underline{k})^2-m^2)^{-1}(2p-k)_\mu - \delta_{\mu\mu}]\cdot \underline{k}^{-2}C(\underline{k}^2)\cdot(2\pi)^{-2}d^4k, \qquad (1)$$

where p_μ is the 4-momentum of the meson and m is its mass. Using the fact that $\underline{p}^2 = m^2$ we can expand (1) as

$$\Delta m = -e^2 i(2\pi)^{-3}m^{-1}\int [4m^2\underline{k}^{-2}(\underline{k}^2-2\underline{p}\cdot\underline{k})^{-1} - (\underline{k}^2-2\underline{p}\cdot\underline{k})^{-1} - 2\underline{k}^{-2}]\cdot C(\underline{k}^2)\cdot d^4k. \qquad (2)$$

We take

$$C(\underline{k}^2) = (\Lambda^2/\Lambda^2-\underline{k}^2)^2. \qquad (3)$$

We can then express Δm as

$$\Delta m = 4e^2m\cdot I_1(\underline{p}) + e^2m^{-1}[I_2(\underline{p})+2\cdot I_2(0)],$$

$$I_1(\underline{p}) = -i(2\pi)^{-3}\Lambda^4\cdot d/d(\Lambda^2)\int \underline{k}^{-2}(\underline{k}^2-\Lambda^2)^{-1}(\underline{k}^2-2\underline{p}\cdot\underline{k})^{-1}d^4k,$$

$$I_2(\underline{p}) = i(2\pi)^{-3}\Lambda^4\int(\underline{k}^2-\Lambda^2)^{-2}(\underline{k}^2-2\underline{p}\cdot\underline{k})^{-1}d^4k. \qquad (4)$$

We can express $\underline{k}^{-2}(\underline{k}^2-\Lambda^2)^{-1}$ as

$$\underline{k}^{-2}(\underline{k}^2-\Lambda^2)^{-1} = \Lambda^{-2}\int_0^{\Lambda^2} (\underline{k}^2-s)^{-2}ds. \qquad (5)$$

Substituting (5) in (4) we get

$$I_1(\underline{p}) = i(2\pi)^{-3}\int_0^{\Lambda^2}\int (\underline{k}^2-s)^{-2}(\underline{k}^2-2\underline{p}\cdot\underline{k})^{-1}d^4kds - \Lambda^{-2}\cdot I_2(\underline{p}),$$

$$I_2(\underline{p}) = i(2\pi)^{-3}\Lambda^4\cdot d/d(\Lambda^2)\int_0^{\Lambda^2}\int (\underline{k}^2-s)^{-2}(\underline{k}^2-2\underline{p}\cdot\underline{k})^{-1}d^4kds. \qquad (6)$$

Substituting (6) in (4) we get

$$\Delta m = e^2m\Big[4\cdot I_3(\underline{p}) + (m^{-2}\Lambda^2 - 4)\Lambda^2\cdot dI_3(\underline{p})/d(\Lambda^2) + 2m^{-2}\Lambda^4\cdot$$

$$\left\{dI_3(\underline{p})/d(\Lambda^2)\right\}_{\underline{p}^2 = 0} + \Big],$$

$$I_3(\underline{p}) = i(2\pi)^{-3}\int_0^{\Lambda^2}\int (\underline{k}^2-s)^{-2}(\underline{k}^2-2\underline{p}\cdot\underline{k})^{-1}d^4kds. \qquad (7)$$

To evaluate $I_3(\underline{p})$, we combine denominators, using the formula

$$(\underline{k}^2-s)^{-2}(\underline{k}^2-2\underline{p}\cdot\underline{k})^{-1} = 2\int_0^{\infty}(\underline{k}^2-2\underline{p}\cdot\underline{k} + (\underline{k}^2-s)x)^{-3}xdx. \qquad (8)$$

We substitute (8) in $I_3(\underline{p})$ and integrate over $\underline{k}$-space, using the formula

$$\int(\alpha\underline{k}^2-2\underline{p}\cdot\underline{k}-\Delta)^{-3}d^4k = -\pi^2 i(2\alpha)^{-1}(\underline{p}^2 + \Delta\alpha)^{-1} \qquad (9)$$

with $\alpha = x + 1$, $\Delta = sx$, $\underline{p}^2 = m^2$, to get

$$I_3(\underline{p}) = i(2\pi)^{-3}\cdot 2 \int_0^\infty\int_0^{\Lambda^2}\int (\alpha\underline{k}^2 - 2\underline{p}\cdot\underline{k} - \Delta)^{-3} d^4k\,ds\,x\,dx = (8\pi)^{-1}\cdot$$

$$\int_0^\infty\int_0^{\Lambda^2}(x+1)^{-1}(m^2+sx(x+1))^{-1}ds\,x\,dx = (8\pi)^{-1}[\log(\Lambda^2/m^2)$$

$$+ \int_0^\infty \log(x(x+1)+\Lambda^{-2}m^2)\cdot(x+1)^{-2}dx]. \qquad (10)$$

Using the fact that $\log(x(x+1)+\Lambda^{-2}m^2) = \log(x+\frac{1}{2}+(\frac{1}{4}-\Lambda^{-2}m^2)^{\frac{1}{2}}) + \log(x+\frac{1}{2}-(\frac{1}{4}-\Lambda^{-2}m^2)^{\frac{1}{2}})$, and the formula

$$\int_0^\infty (x+1)^{-2}\log(x+a)dx = (1-(1-a)^{-1})\log(a), \qquad (11)$$

we get

$$I_3(\underline{p}) = (16\pi)^{-1}m^{-2}\Lambda^2[\log(\Lambda^2/m^2) - (1-4\Lambda^{-2}m^2)^{\frac{1}{2}}\cdot$$

$$\log\left\{(1-(1-4\Lambda^{-2}m^2)^{\frac{1}{2}})^{-1}(1+(1-4\Lambda^{-2}m^2)^{\frac{1}{2}})\right\}]. \qquad (12)$$

Since we anticipate that $4m^2/\Lambda^2 \approx .1$, and we must find $\left\{dI_3(\underline{p})/d(\Lambda^2)\right\}_{\underline{p}^2 = 0+}$ in any case, we expand $I_3(\underline{p})$ in powers of m^2/Λ^2. The result is

$$I_3(\underline{p}) = (16\pi)^{-1}[2(1 + m^2/\Lambda^2 + 2m^4/\Lambda^4 + 0(m^6/\Lambda^6))\log(\Lambda^2/m^2)$$

$$+2 - m^2/\Lambda^2 - (10/3)m^4/\Lambda^4 + 0(m^6/\Lambda^6)],$$

$$\Lambda^2 dI_3(\underline{p})/d(\Lambda^2) = (16\pi)^{-1}\,[-2(m^2/\Lambda^2 + 4m^4/\Lambda^4 + 0(m^6/\Lambda^6))\cdot$$

$$\log(\Lambda^2/m^2) + 2 + 3m^2/\Lambda^2 + (32/3)m^4/\Lambda^4 + 0(m^6/\Lambda^6)]. \qquad (13)$$

Substituting (13) in (7) we get

$$\Delta m = (3/8\pi)e^2 m[\Lambda^2/m^2 + \log(\Lambda^2/m^2) + \tfrac{1}{2} + ((4/3)\log(\Lambda^2/m^2) - (8/9))\cdot$$

$$m^2/\Lambda^2 + 0((m^4/\Lambda^4)\log(\Lambda^2/m^2))]. \qquad (14)$$

If we take m to be 273 electron masses, and Δm to be 9 electron masses,* then $(8\pi/3e^2)\Delta m/m = 37.8$. The value of Λ required to obtain this result is 5.8m = .86 proton masses.

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*Chinowski and Steinberger, Phys. Rev. 93, 586 (1954), give $\Delta m = 8.8 \pm 0.6$ electron masses.

The Neutron-Proton Mass Difference

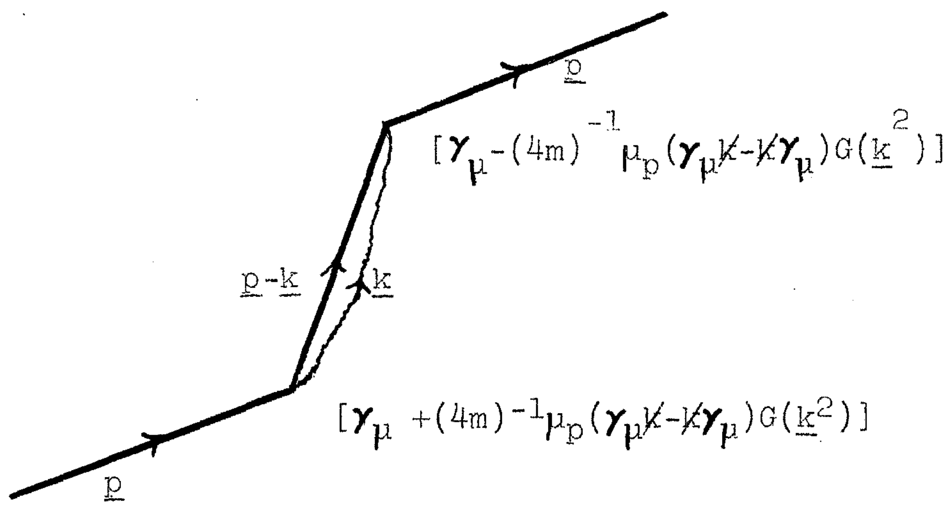

Fig. 1. Feynman diagram for the proton mass correction of order e^2.

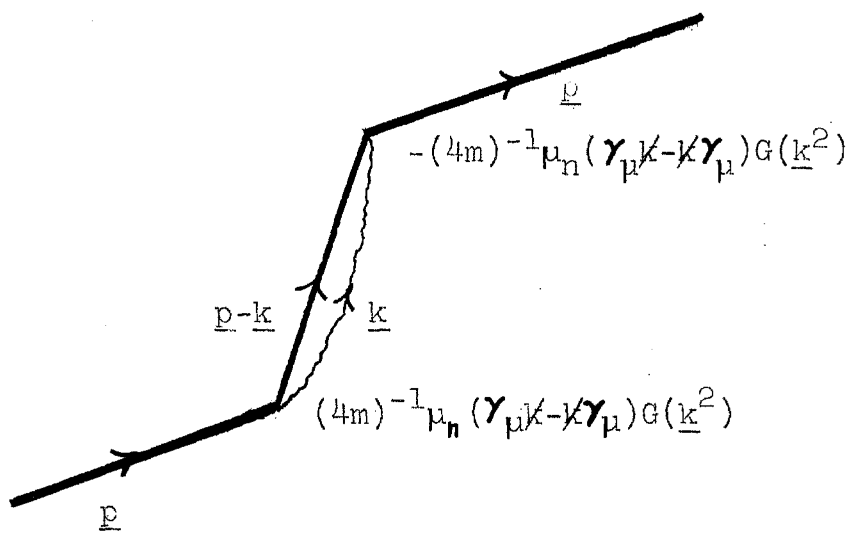

Fig. 2. Feynman diagram for the neutron mass correction of order e^2.

For the spin $\frac{1}{2}$ proton, the electromagnetic mass correction of order e^2 arises from the Feynman diagram given in fig. 1. p is the 4-momentum of the proton, m is its mass,

$\underline{k}$ is the 4-momentum of the emitted virtual photon, and μ_p is the anomalous magnetic moment of the proton in units of $e\hbar/2mc$. It is to be noted that if the amplitude for emission of a photon of 4-momentum $\underline{k}$ is $a(\underline{k})$, then the amplitude for absorbtion of such a photon is $a(-\underline{k})$. In fig. 1, $a(\underline{k}) = [\gamma_\mu + (4m)^{-1}\mu_p(\gamma_\mu k\!\!\!/ - k\!\!\!/\gamma_\mu)G(\underline{k}^2)]$.

The corresponding diagram for the neutron is given in fig. 2. The mass of the neutron will be taken as equal to that of the proton, since the error thus introduced in the mass correction of the neutron is of the same order of magnitude as the higher order mass corrections we are neglecting. $\underline{p}$ is the 4-momentum of the neutron, $\underline{k}$ is the 4-momentum of the emitted virtual photon, and μ_n is the anomalous magnetic moment of the neutron in units of $e\hbar/2mc$.

In the case of both the proton and the neutron, we obtain the mass correction Δm from the relation

$$\Delta m = (\tilde{u}\, u)^{-1}(\tilde{u}\, Mu), \qquad (15)$$

where u is a spinor wave function satisfying the relation $p\!\!\!/ u = mu$, and M is the spinor matrix arising from the sum of all relevant Feynman diagrams. In the cases we are considering, we integrate over all virtual photon 4-momenta.

For the proton, M is given by the following:

$$M = -4\pi e^2 i \int [\gamma_\mu - (4m)^{-1}\mu_p(\gamma_\mu k\!\!\!/ - k\!\!\!/\gamma_\mu)G(\underline{k}^2)](p\!\!\!/ - k\!\!\!/ - m)^{-1}[\gamma_\mu$$

$$+(4m)^{-1}\mu_p(\gamma_\mu \not{k}-\not{k}\gamma_\mu)G(\underline{k}^2)]\cdot \underline{k}^{-2}C(\underline{k}^2)\cdot(2\pi)^{-4}d^4k, \qquad (16)$$

or

$$M = e^2[M_1 + \mu_p M_2 + \mu_p^2 M_3],$$

$$M_1 = -(i/4\pi^3)\int \gamma_\mu(\not{p}-\not{k}-m)^{-1}\gamma_\mu\cdot \underline{k}^{-2}C(\underline{k}^2)d^4k,$$

$$M_2 = (i/16\pi^3 m)\int[\gamma_\mu(\not{p}-\not{k}-m)^{-1}(\not{k}\gamma_\mu-\gamma_\mu\not{k}) - (\not{k}\gamma_\mu-\gamma_\mu\not{k})\cdot$$

$$(\not{p}-\not{k}-m)^{-1}\gamma_\mu]\underline{k}^{-2}G(\underline{k}^2)\cdot C(\underline{k}^2)d^4k,$$

$$M_3 = (i/64\pi^3 m^2)\int(\not{k}\gamma_\mu-\gamma_\mu\not{k})(\not{p}-\not{k}-m)^{-1}(\not{k}\gamma_\mu-\gamma_\mu\not{k})\cdot$$

$$\underline{k}^{-2}G^2(\underline{k}^2)\cdot C(\underline{k}^2)d^4k. \qquad (17)$$

For the neutron, M is given by the following:

$$M = (ie^2\mu_n^2/64\pi^3 m^2)\int(\not{k}\gamma_\mu-\gamma_\mu\not{k})(\not{p}-\not{k}-m)^{-1}(\not{k}\gamma_\mu-\gamma_\mu\not{k})\cdot$$

$$\underline{k}^{-2}G^2(\underline{k}^2)\cdot C(\underline{k}^2)d^4k. \qquad (18)$$

In the computations that follow, we take

$$C(\underline{k}^2) = -\Lambda^2/(\underline{k}^2-\Lambda^2), \qquad G(\underline{k}^2) = -\lambda^2/(\underline{k}^2-\lambda^2). \qquad (19)$$

We first consider the electromagnetic mass correction for the proton. This correction, Δm_p, is given by the following:

$$\Delta m_p = e^2[\Delta m_1 + \mu_p \Delta m_2 + \mu_p^2 \Delta m_3],$$

$$\Delta m_1 = (\tilde{u}\ u)^{-1}(\tilde{u}\ M_1 u), \qquad \Delta m_2 = (\tilde{u}\ u)^{-1}(\tilde{u}\ M_2 u),$$

$$\Delta m_3 = (\tilde{u}\ u)^{-1}(\tilde{u}\ M_3 u). \tag{20}$$

We simplify the expression $\gamma_\mu(\not{p}-\not{k}-m)^{-1}\gamma_\mu$ by using the following relations (2):

$$\not{A}\not{A} = \underline{A}^2, \qquad \gamma_\mu\gamma_\mu = 4, \qquad \gamma_\mu\not{A}\gamma_\mu = -2\not{A}.$$

Thus

$$\gamma_\mu(\not{p}-\not{k}-m)^{-1}\gamma_\mu = \gamma_\mu(\not{p}-\not{k}+m)\gamma_\mu(\underline{p}^2+\underline{k}^2-2\underline{p}\cdot\underline{k}-m^2)^{-1} = (4m-2\not{p}+2\not{k})\cdot$$

$$(\underline{k}^2-2\underline{p}\cdot\underline{k})^{-1} = 2(2m-\not{p}+\not{k})(\underline{k}^2-2\underline{p}\cdot\underline{k})^{-1}.$$

Substituting this in (17) we get

$$M_1 = -(i/2\pi^3)\int(2m-\not{p}+\not{k})(\underline{k}^2-2\underline{p}\cdot\underline{k})^{-1}\cdot\underline{k}^{-2}C(\underline{k}^2)d^4k. \tag{21}$$

Using the fact that $(\tilde{u}\ \not{A}u) = (2m)^{-1}(\tilde{u}\ (\not{p}\not{A}+\not{A}\not{p})u) = m^{-1}\underline{A}\cdot\underline{p}(\tilde{u}\ u)$, and (19), we get

$$\Delta m_1 = -(i/2\pi^3) \int (m+m^{-1}\underline{p}\cdot\underline{k})(\underline{k}^2-2\underline{p}\cdot\underline{k})^{-1}\cdot(-\Lambda^2)\underline{k}^{-2}(\underline{k}^2-\Lambda^2)^{-1}d^4k. \quad (22)$$

Substituting (5) in (22) we get

$$\Delta m_1 = (i/2\pi^3) \int_0^{\Lambda^2}\int (m+m^{-1}\underline{p}\cdot\underline{k})(\underline{k}^2-2\underline{p}\cdot\underline{k})^{-1}(\underline{k}^2-s)^{-2}d^4k\,ds. \quad (23)$$

Substituting (8) in (23) we get

$$\Delta m_1 = (i/\pi^3) \int_0^{\infty}\int_0^{\Lambda^2}\int (m+m^{-1}\underline{p}\cdot\underline{k})(\underline{k}^2-2\underline{p}\cdot\underline{k}+(\underline{k}^2-s)x)^{-3}d^4k\,ds\,x\,dx. \quad (24)$$

We integrate (24) over $\underline{k}$-space, using the formulae

$$\int (\alpha\underline{k}^2-2\underline{p}\cdot\underline{k}-\Delta)^{-3}d^4k = -\pi^2 i(2\alpha)^{-1}(\underline{p}^2+\Delta\alpha)^{-1},$$

$$\int k_\sigma(\alpha\underline{k}^2-2\underline{p}\cdot\underline{k}-\Delta)^{-3}d^4k = -\pi^2 i p_\sigma(2\alpha^2)^{-1}(\underline{p}^2+\Delta\alpha)^{-1}, \quad (25)$$

with $\alpha = x+1$, $\Delta = sx$, $\underline{p}^2 = m^2$, to get

$$\Delta m_1 = (m/2\pi) \int_0^{\infty}\int_0^{\Lambda^2} [(x+1)^{-1}+(x+1)^{-2}](m^2+sx(x+1))^{-1}ds\,x\,dx. \quad (26)$$

It is clear from (26) that $\Delta m_1 > 0$. Integrating (26) over s we get

$$\Delta m_1 = (m/2\pi) \int_0^{\infty} [(x+1)^{-2}+(x+1)^{-3}]\log(1+m^{-2}\Lambda^2 x(x+1))dx,$$

or

$$\Delta m_1 = (m/4\pi)[2J_1(\Lambda^2/m^2) + J_2(\Lambda^2/m^2)],$$

$$J_1(c) = \int_0^\infty (x+1)^{-2}\log(1+cx(x+1))dx,$$

$$J_2(c) = 2\int_0^\infty (x+1)^{-3}\log(1+cx(x+1))dx. \qquad (27)$$

$J_1(c)$ and $J_2(c)$ can be expressed in terms of elementary functions as follows:

$$J_1(c) = \log(c) + \int_0^\infty (x+1)^{-2}\log(x+\tfrac{1}{2}+(\tfrac{1}{4}-c^{-1})^{\frac{1}{2}})dx + \int_0^\infty (x+1)^{-2}\cdot$$

$$\log(x+\tfrac{1}{2}-(\tfrac{1}{4}-c^{-1})^{\frac{1}{2}})dx.$$

Using (11) and symplifying, we get

$$J_1(c) = \tfrac{1}{2}c\cdot\log(c) + \tfrac{1}{2}c(1-4c^{-1})^{\frac{1}{2}}\cdot\log\Big\{(1+(1-4c^{-1})^{\frac{1}{2}})^{-1}\cdot$$

$$(1-(1-4c^{-1})^{\frac{1}{2}})\Big\}. \qquad (28)$$

$$J_2(c) = \log(c) + 2\int_0^\infty (x+1)^{-3}\log(x+\tfrac{1}{2}+(\tfrac{1}{4}-c^{-1})^{\frac{1}{2}})dx + 2\int_0^\infty (x+1)^{-3}\cdot$$

$$\log(x+\tfrac{1}{2}-(\tfrac{1}{4}-c^{-1})^{\frac{1}{2}})dx.$$

Using the formula

$$\int_0^\infty (x+1)^{-3}\log(x+a)dx = \tfrac{1}{2}[(1-(1-a)^{-2})\log(a) - (1-a)^{-1}] \qquad (29)$$

and simplifying, we get

$$J_2(c) = (\tfrac{1}{2}c^2-c)\log(c) + \tfrac{1}{2}c^2(1-4c^{-1})^{\frac{1}{2}}\log\left\{(1+(1-4c^{-1})^{\frac{1}{2}})^{-1}\cdot\right.$$

$$\left.(1-(1-4c^{-1})^{\frac{1}{2}})\right\} - c,$$

$$J_2(c) = c[J_1(c) - \log(c) - 1]. \qquad (30)$$

We simplify the expression $\gamma_\mu(\not{p}-\not{k}-m)^{-1}(\not{k}\gamma_\mu-\gamma_\mu\not{k}) - (\not{k}\gamma_\mu-\gamma_\mu\not{k})(\not{p}-\not{k}-m)^{-1}\gamma_\mu$ by using the following relations (2):

$$\not{A}\not{A} = \underline{A}^2, \qquad \gamma_\mu\gamma_\mu = 4, \qquad \gamma_\mu\not{A}\gamma_\mu = -2\not{A}, \qquad \gamma_\mu\not{A}\not{B}\gamma_\mu = 4\underline{A}\cdot\underline{B}.$$

Thus

$$\gamma_\mu(\not{p}-\not{k}-m)^{-1}(\not{k}\gamma_\mu-\gamma_\mu\not{k}) - (\not{k}\gamma_\mu-\gamma_\mu\not{k})(\not{p}-\not{k}-m)^{-1}\gamma_\mu =$$

$$(\underline{p}^2+\underline{k}^2-2\underline{p}\cdot\underline{k}-m^2)^{-1}\,[\gamma_\mu(\not{p}-\not{k}+m)(\not{k}\gamma_\mu-\gamma_\mu\not{k}) - (\not{k}\gamma_\mu-\gamma_\mu\not{k})\cdot$$

$$(\not{p}-\not{k}+m)\gamma_\mu] = (\underline{k}^2-2\underline{p}\cdot\underline{k})^{-1}\,[\gamma_\mu\not{p}\not{k}\gamma_\mu - \underline{k}^2\gamma_\mu\gamma_\mu + m\gamma_\mu\not{k}\gamma_\mu - \gamma_\mu\not{p}\gamma_\mu\not{k}$$

$$+\gamma_\mu\not{k}\gamma_\mu\not{k} - m\gamma_\mu\gamma_\mu\not{k} + \gamma_\mu\not{k}\not{p}\gamma_\mu - \underline{k}^2\gamma_\mu\gamma_\mu + m\gamma_\mu\not{k}\gamma_\mu - \not{k}\gamma_\mu\not{p}\gamma_\mu + \not{k}\gamma_\mu\not{k}\gamma_\mu$$

$$-m\gamma_\mu\gamma_\mu\not{k}] = (\underline{k}^2-2\underline{p}\cdot\underline{k})^{-1}(8\underline{p}\cdot\underline{k} - 8\underline{k}^2 - 4m\not{k} + 4\underline{p}\cdot\underline{k} - 4\underline{k}^2 - 8m\not{k}) =$$

$$-12(\underline{k}^2-2\underline{p}\cdot\underline{k})^{-1}(\underline{k}^2-\underline{p}\cdot\underline{k}+m\not{k}).$$

Substituting this in (17) we get

$$M_2 = -(3i/4\pi^3 m) \int (\underline{k}^2 - \underline{p}\cdot\underline{k} + m\not{k})(\underline{k}^2 - 2\underline{p}\cdot\underline{k})^{-1}\cdot\underline{k}^{-2}G(\underline{k}^2)\cdot C(\underline{k}^2)d^4k. \tag{31}$$

Using the fact that $(\tilde{u}\ \not{A}u) = (2m)^{-1}(\tilde{u}\ (\not{p}\not{A}+\not{A}\not{p})u) = m^{-1}\underline{A}\cdot\underline{p}(\tilde{u}\ u)$, we get

$$\Delta m_2 = -(3i/4\pi^3 m) \int (\underline{k}^2 - 2\underline{p}\cdot\underline{k})^{-1}\cdot G(\underline{k}^2)\cdot C(\underline{k}^2)d^4k. \tag{32}$$

Substituting (19) in (32), and using the fact that

$$(\underline{k}^2 - \Lambda^2)^{-1}(\underline{k}^2 - \lambda^2)^{-1} = (\Lambda^2 - \lambda^2)^{-1}\int_{\lambda^2}^{\Lambda^2} (\underline{k}^2 - s)^{-2}ds, \tag{33}$$

we get

$$\Delta m_2 = -(3i/4\pi^3 m)\cdot\Lambda^2\lambda^2(\Lambda^2 - \lambda^2)^{-1}\int_{\lambda^2}^{\Lambda^2}\int (\underline{k}^2 - 2\underline{p}\cdot\underline{k})^{-1}(\underline{k}^2 - s)^{-2}d^4k\,ds. \tag{34}$$

Substituting (8) in (34), we get

$$\Delta m_2 = -(3i/2\pi^3 m)\cdot\Lambda^2\lambda^2(\Lambda^2 - \lambda^2)^{-1}\int_0^\infty\int_{\lambda^2}^{\Lambda^2}\int (\underline{k}^2 - 2\underline{p}\cdot\underline{k} + (\underline{k}^2 - s)x)^{-3}\cdot d^4k\,ds\,x\,dx. \tag{35}$$

We integrate (35) over $\underline{k}$-space, using (9) or (25), to get

$$\Delta m_2 = -(3/4\pi m)\cdot\Lambda^2\lambda^2(\Lambda^2 - \lambda^2)^{-1}\int_0^\infty\int_{\lambda^2}^{\Lambda^2} (x+1)^{-1}(m^2 + sx(x+1))^{-1}ds\,x\,dx. \tag{36}$$

It is clear from (36) that $\Delta m_2 < 0$. Integrating (36) over s, we get

$$\Delta m_2 = -(3/4\pi m)\cdot\Lambda^2\lambda^2(\Lambda^2-\lambda^2)^{-1}\int_0^\infty[\log(1+m^{-2}\Lambda^2x(x+1))$$

$$-\log(1+m^{-2}\lambda^2x(x+1))]\ (x+1)^{-2}dx,$$

or, since $\int_0^\infty (x+1)^{-2}\log(1+cx(x+1))dx = J_1(c)$ from (27),

$$\Delta m_2 = -(3/4\pi m)\cdot\Lambda^2\lambda^2(\Lambda^2-\lambda^2)^{-1}[J_1(\Lambda^2/m^2) - J_1(\lambda^2/m^2)]. \qquad (37)$$

We simplify the expression $(\not k\gamma_\mu - \gamma_\mu\not k)(\not p-\not k-m)^{-1}(\not k\gamma_\mu - \gamma_\mu\not k)$ by using the following relations (2):

$$\not A\not A = \underline{A}^2, \quad \gamma_\mu\gamma_\mu = 4, \quad \gamma_\mu\not A\gamma_\mu = -2\not A, \quad \gamma_\mu\not A\not B\gamma_\mu = 4\underline{A}\cdot\underline{B},$$
$$\gamma_u\not A\not B\not C\gamma_u = -2\not C\not B\not A.$$

Thus

$$(\not k\gamma_\mu - \gamma_\mu\not k)(\not p-\not k-m)^{-1}(\not k\gamma_\mu - \gamma_\mu\not k) = (\underline{p}^2+\underline{k}^2-2\underline{p}\cdot\underline{k}-m^2)^{-1}(\not k\gamma_\mu - \gamma_\mu\not k)\cdot$$

$$(\not p-\not k+m)(\not k\gamma_\mu - \gamma_\mu\not k) = (\underline{k}^2-2\underline{p}\cdot\underline{k})^{-1}[\not k\gamma_\mu\not p\not k\gamma_\mu - \not k\gamma_\mu\underline{k}^2\gamma_\mu + m\not k\gamma_\mu\not k\gamma_\mu$$

$$-\not k\gamma_\mu\not p\gamma_\mu\not k + \not k\gamma_\mu\not k\gamma_\mu\not k - m\not k\gamma_\mu\gamma_\mu\not k - \gamma_\mu\not k\not p\not k\gamma_\mu + \gamma_\mu\underline{k}^2\not k\gamma_\mu - m\gamma_\mu\underline{k}^2\gamma_\mu$$

$$+\gamma_\mu\not k\not p\gamma_\mu\not k - \gamma_\mu\underline{k}^2\gamma_\mu\not k + m\gamma_\mu\not k\gamma_\mu\not k] = (\underline{k}^2-2\underline{p}\cdot\underline{k})^{-1}[4\underline{p}\cdot\underline{k}\not k - 4\underline{k}^2\not k$$

$-2m\underline{k}^2 + 2\not{k}\not{p}\not{k} - 2\underline{k}^2\not{k} - 4m\underline{k}^2 + 2\not{k}\not{p}\not{k} - 2\underline{k}^2\not{k} - 4m\underline{k}^2 + 4\underline{k}\cdot\underline{p}\not{k}$

$-4\underline{k}^2\not{k} - 2m\underline{k}^2].$

Using the relation $\not{k}\not{p}\not{k} = -\underline{k}^2\not{p} + 2\underline{p}\cdot\underline{k}\not{k}$ we get

$$(\not{k}\gamma_\mu - \gamma_\mu\not{k})(\not{p}-\not{k}-m)^{-1}(\not{k}\gamma_\mu - \gamma_\mu\not{k}) = (\underline{k}^2-2\underline{p}\cdot\underline{k})^{-1}[8\underline{p}\cdot\underline{k}\not{k} - 12\underline{k}^2\not{k}$$

$$-12m\underline{k}^2 - 4\underline{k}^2\not{p} + 8\underline{p}\cdot\underline{k}\not{k}] = -4(\underline{k}^2-2\underline{p}\cdot\underline{k})^{-1}[(3\underline{k}^2-4\underline{p}\cdot\underline{k})\not{k}$$

$$+\underline{k}^2(\not{p}+3m)] = -4[2\not{k} + \underline{k}^2(\not{p}+3m+\not{k})(\underline{k}^2-2\underline{p}\cdot\underline{k})^{-1}].$$

Substituting this in (17) we get

$$M_3 = -(i/16\pi^3m^2)\int[2\not{k}\underline{k}^{-2} + (\not{p}+3m+\not{k})(\underline{k}^2-2\underline{p}\cdot\underline{k})^{-1}]\,G^2(\underline{k}^2)\cdot C(\underline{k}^2)d^4k. \tag{38}$$

Taking account of the fact that $\int\not{k}\underline{k}^{-2}G^2(\underline{k}^2)\cdot C(\underline{k}^2)d^4k = 0$, and $(\tilde{u}\,\not{A}u) = (2m)^{-1}(\tilde{u}\,(\not{p}\not{A}+\not{A}\not{p})u) = m^{-1}\underline{A}\cdot\underline{p}(\tilde{u}\,u)$ we get

$$\Delta m_3 = -(i/16\pi^3m^2)\int(4m+m^{-1}\underline{p}\cdot\underline{k})(\underline{k}^2-2\underline{p}\cdot\underline{k})^{-1}\cdot G^2(\underline{k}^2)\cdot C(\underline{k}^2)d^4k. \tag{39}$$

We can express $G^2(\underline{k}^2)\cdot C(\underline{k}^2)$ as

$$G^2(\underline{k}^2)\cdot C(\underline{k}^2) = -\lambda^4\Lambda^2\cdot\partial/\partial(\lambda^2)\cdot(\Lambda^2-\lambda^2)^{-1}\int_{\lambda^2}^{\Lambda^2}(\underline{k}^2-s)^{-2}ds. \tag{40}$$

Substituting (40) in (39) we get

$$\Delta m_3 = (i/16\pi^3 m^2)\cdot\lambda^4\Lambda^2\cdot \partial/\partial(\lambda^2)\cdot(\Lambda^2-\lambda^2)^{-1}\int_{\lambda^2}^{\Lambda^2}\int(4m+m^{-1}\underline{p}\cdot\underline{k})\cdot$$

$$(\underline{k}^2-2\underline{p}\cdot\underline{k})^{-1}(\underline{k}^2-s)^{-2}d^4k\,ds. \qquad (41)$$

Substituting (8) in (41) we get

$$\Delta m_3 = (i/8\pi^3 m^2)\cdot\lambda^4\Lambda^2\cdot \partial/\partial(\lambda^2)\cdot(\Lambda^2-\lambda^2)^{-1}\int_0^\infty\int_{\lambda^2}^{\Lambda^2}\int(4m+m^{-1}\underline{p}\cdot\underline{k})\cdot$$

$$(\underline{k}^2-2\underline{p}\cdot\underline{k}+(\underline{k}^2-s)x)^{-3}d^4k\,ds\,x\,dx. \qquad (42)$$

We integrate (42) over $\underline{k}$-space, using (25), to get

$$\Delta m_3 = (16\pi m)^{-1}\cdot\lambda^4\Lambda^2\cdot \partial/\partial(\lambda^2)\cdot(\Lambda^2-\lambda^2)^{-1}\int_0^\infty\int_{\lambda^2}^{\Lambda^2}[4(x+1)^{-1}+(x+1)^{-2}]\cdot$$

$$(m^2+sx(x+1))^{-1}ds\,x\,dx. \qquad (43)$$

We digress for a moment to prove that $\Delta m_3 < 0$*, by proving that $\partial/\partial(\lambda^2)\cdot(\Lambda^2-\lambda^2)^{-1}\int_{\lambda^2}^{\Lambda^2}(m^2+sx(x+1))^{-1}ds < 0.$ Thus, for $\Lambda>\lambda$,

$$\partial/\partial(\lambda^2)\cdot(\Lambda^2-\lambda^2)^{-1}\int_{\lambda^2}^{\Lambda^2}(m^2+sx(x+1))^{-1}ds = -(\Lambda^2-\lambda^2)^{-1}\cdot$$

$$(m^2+\lambda^2x(x+1))^{-1} + (\Lambda^2-\lambda^2)^{-2}\int_{\lambda^2}^{\Lambda^2}(m^2+sx(x+1))^{-1}ds=$$

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*Note that the sign given here is opposite to that published by R. P. Feynman and G. Speisman, Phys. Rev. 94, 500 (1954).

$$-(\Lambda^2-\lambda^2)^{-1}[(m^2+\lambda^2x(x+1))^{-1} -(\Lambda^2-\lambda^2)^{-1}\int_{\lambda^2}^{\Lambda^2}(m^2+sx(x+1))^{-1}\cdot$$

$$ds] < 0,$$

since $(\Lambda^2-\lambda^2)^{-1}\int_{\lambda^2}^{\Lambda^2}(m^2+sx(x+1))^{-1}ds < (m^2+\lambda^2x(x+1))^{-1}$. For $\Lambda<\lambda$ the reasoning is similar, while for $\Lambda=\lambda$,

$$\partial/\partial(\lambda^2)\cdot(\Lambda^2-\lambda^2)^{-1}\int_{\lambda^2}^{\Lambda^2}(m^2+sx(x+1))^{-1}ds= -x(x+1)(m^2+\lambda^2x(x+1))^{-2}$$

and is thus less than zero, completing the proof.

Returning to the evaluation of Δm_3, we integrate (43) over s to get

$$\Delta m_3= (16\pi m)^{-1}\cdot\lambda^4\Lambda^2\cdot\partial/\partial(\lambda^2)\cdot(\Lambda^2-\lambda^2)^{-1}\int_0^\infty[4(x+1)^{-2}+(x+1)^{-3}]\cdot$$

$$[\log(1+m^{-2}\Lambda^2x(x+1)) -\log(1+m^{-2}\lambda^2x(x+1))]dx. \qquad (44)$$

Since $\int_0^\infty(x+1)^{-2}\log(1+cx(x+1))dx= J_1(c)$, and $\int_0^\infty(x+1)^{-3}\cdot\log(1+cx(x+1))dx= \frac{1}{2}J_2(c)$ from (27), we have

$$\Delta m_3= -(32\pi m)^{-1}\cdot\lambda^4\Lambda^2\cdot\partial/\partial(\lambda^2)\cdot(\Lambda^2-\lambda^2)^{-1}[8J_1(\lambda^2/m^2) +J_2(\lambda^2/m^2)$$

$$-8J_1(\Lambda^2/m^2) -J_2(\Lambda^2/m^2)]. \qquad (45)$$

The electromagnetic mass correction for the neutron, Δm_n, is $e^2\mu_n^2\Delta m_3$. Therefore,

$$\Delta m_n = -(e^2\mu_n^2/32\pi m)\cdot\lambda^4\Lambda^2 \cdot \partial/\partial(\lambda^2)\cdot(\Lambda^2-\lambda^2)^{-1}[8J_1(\lambda^2/m^2)$$

$$+J_2(\lambda^2/m^2) - 8J_1(\Lambda^2/m^2) - J_2(\Lambda^2/m^2)]. \qquad (46)$$

We now compute Δm_1, Δm_2, Δm_3, Δm_p, Δm_n, and $\Delta m_n - \Delta m_p$ for various values of Λ and λ, using the following experimental values of $\hbar c/e^2$, μ_p, μ_n, and $(m_n-m_p)/m_p$ (3,4,5):

$\hbar c/e^2 = (137.0377 \pm .0016)$,

$\mu_p = (1.79277 \pm .00006)\cdot e\hbar/2m_p c$,

$\mu_n = (-1.91305 \pm .00009)\cdot e\hbar/2m_p c$,

$(m_n-m_p)/m_p = (.0013784 \pm .000001)$,

where μ_p and μ_n are the anomalous magnetic moments of the proton and neutron, respectively, and m_p and m_n are the masses of the proton and neutron, respectively. The results of these computations are given in fig. 3-5 and tables 1-5 on pages 23-30.

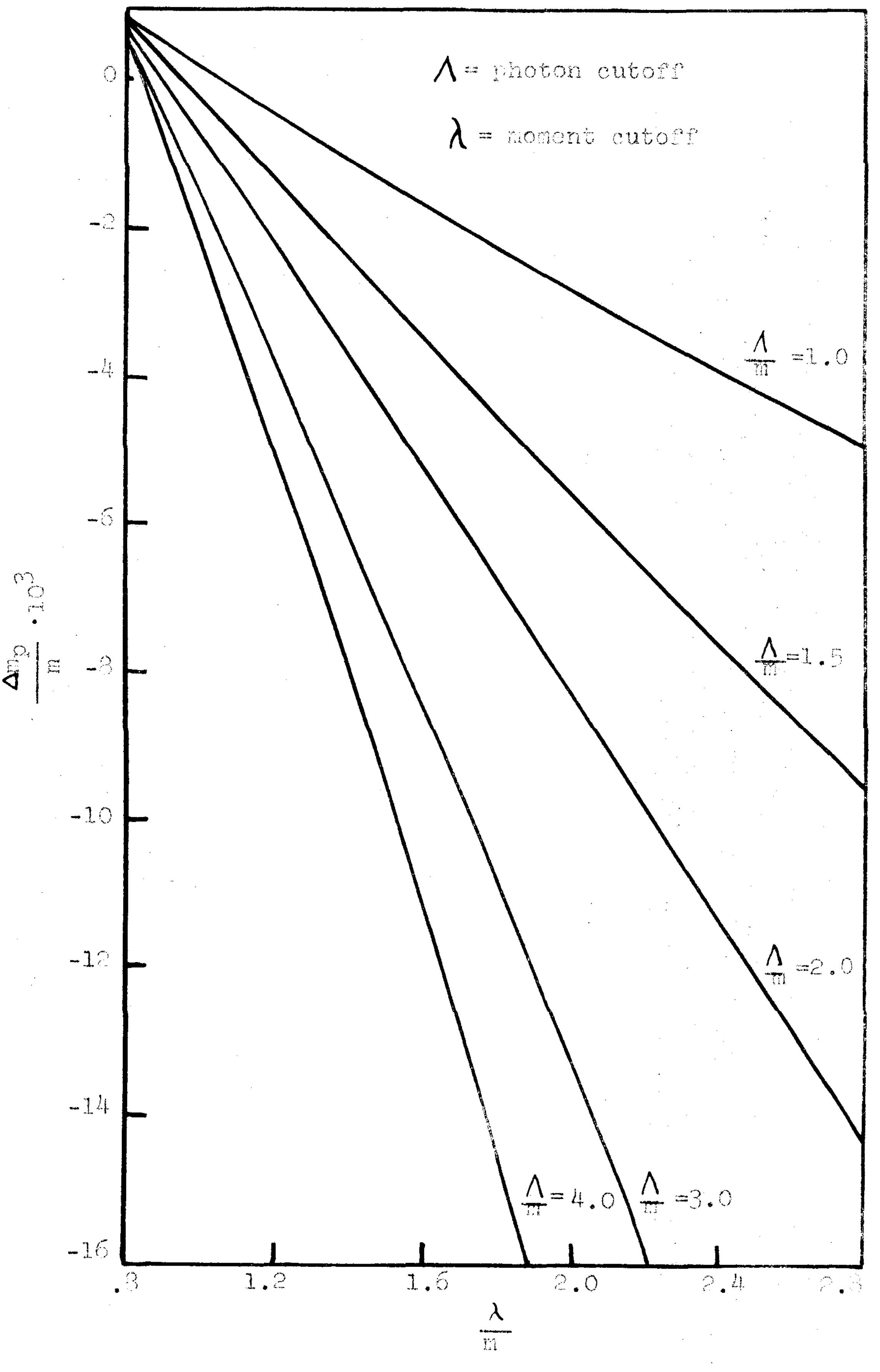

Fig. 3. Proton mass correction.

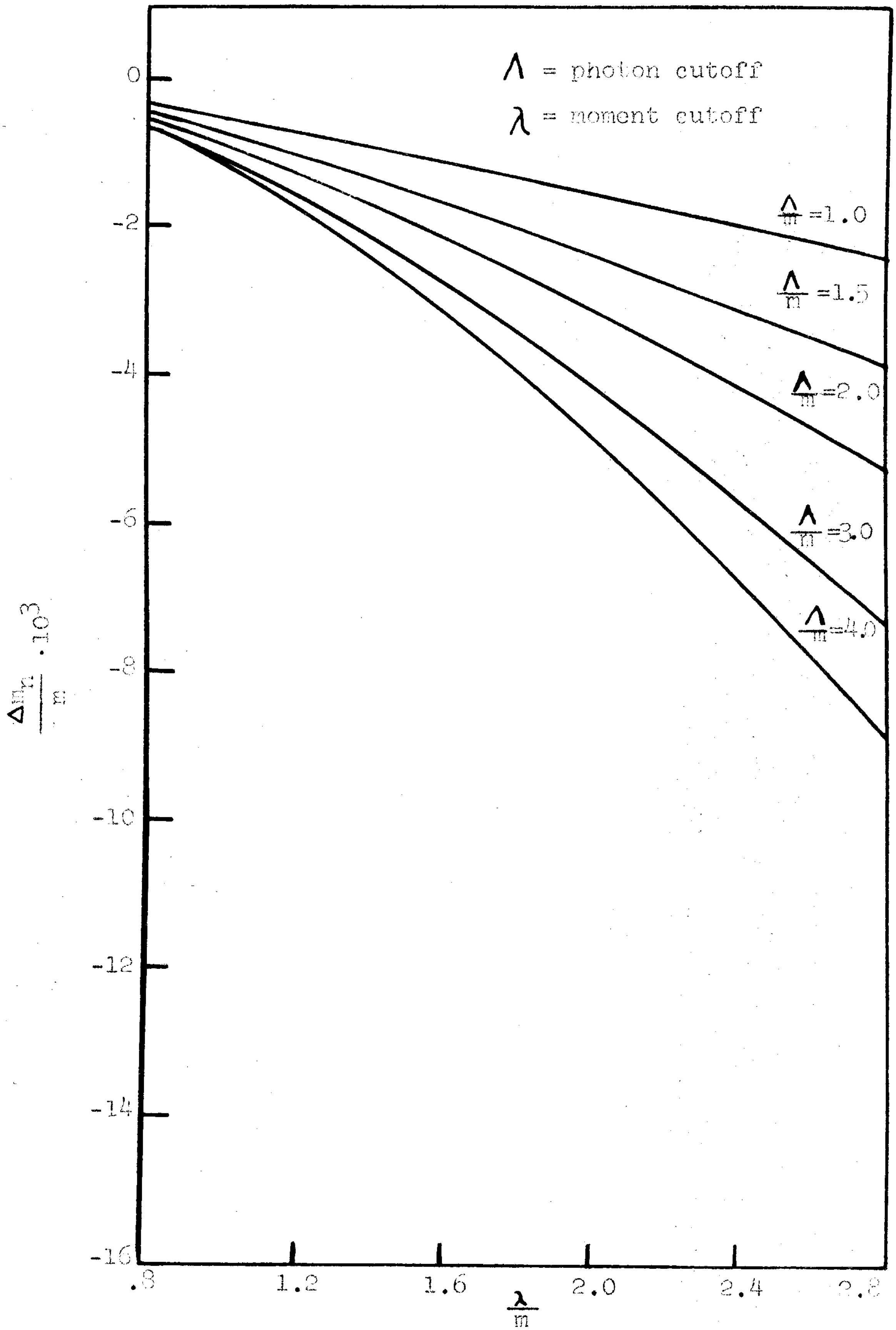

Fig. 4. Neutron mass correction

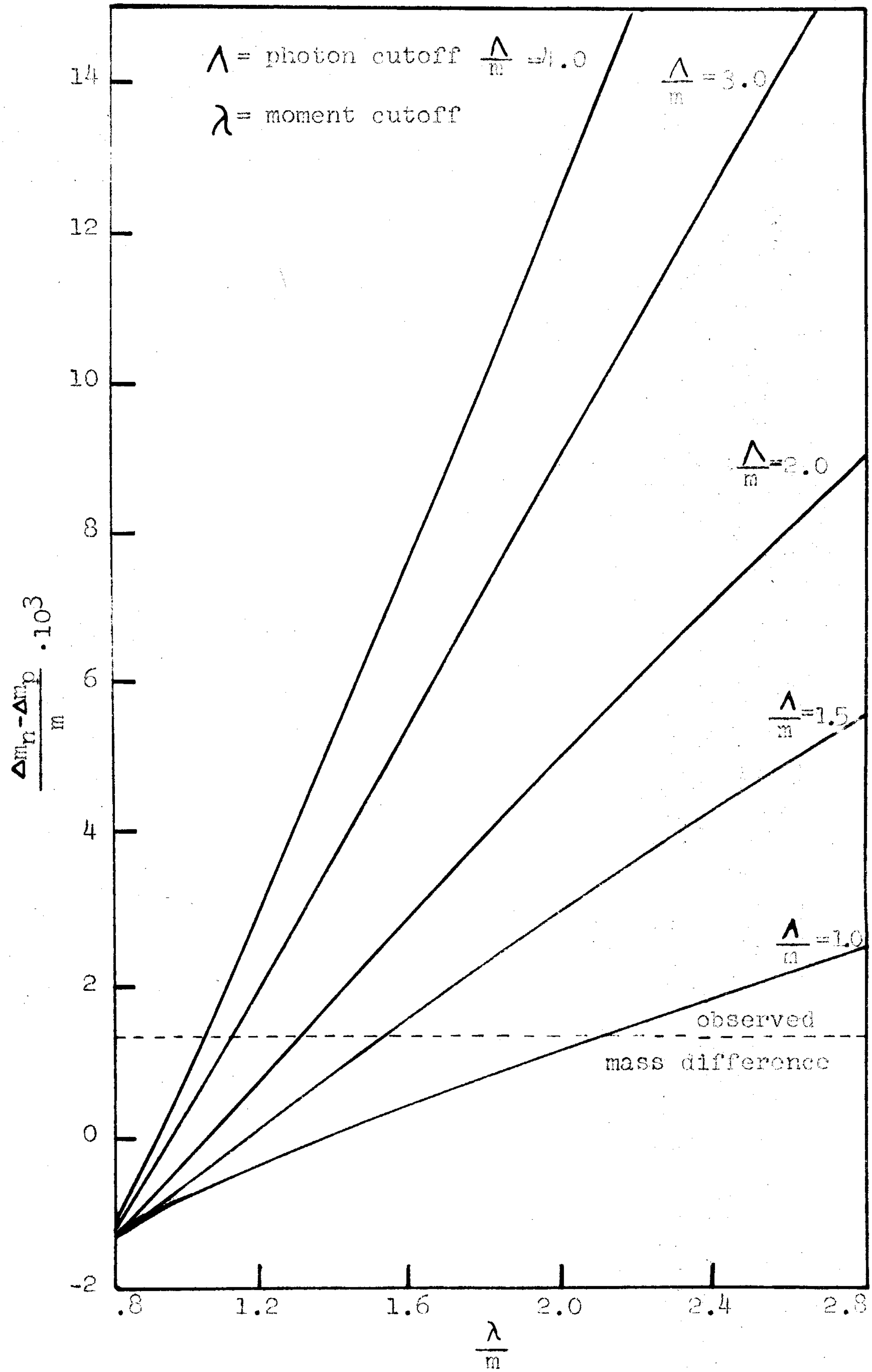

Fig. 5. Neutron-proton mass difference

Table 1. The proton and neutron mass corrections as a function of the moment cutoff, λ , for the photon cutoff, Λ , equal to 1.0m.

$$(e^2\Delta m_1/m)\cdot 10^3 = 2.5791$$

λ/m	$-e^2\mu_p\Delta m_2/m \cdot 10^3$	$-e^2\mu_p^2\Delta m_3/m \cdot 10^3$	$-\Delta m_p/m \cdot 10^3$	$-\Delta m_n/m \cdot 10^3$	$(\Delta m_n-\Delta m_p)/m \cdot 10^3$
0.8	1.4248	.3147	-.8396	.3583	-1.1979
0.9	1.6572	.3912	-.5307	.4454	-.9761
1.1	2.1168	.5557	.0934	.6328	-.5394
1.2	2.3422	.6422	.4053	.7313	-.3260
1.3	2.5638	.7305	.7152	.8318	-.1166
1.4	2.7812	.8200	1.0221	.9337	.0884
1.5	2.9943	.9103	1.3255	1.0366	.2889
1.6	3.2029	1.0012	1.6250	1.1400	.4850
1.8	3.6066	1.1833	2.2108	1.3474	.8634
2.0	3.9926	1.3643	2.7778	1.5535	1.2243
2.2	4.3614	1.5432	3.3255	1.7572	1.5683
2.4	4.7140	1.7190	3.8539	1.9574	1.8965
2.6	5.0512	1.8913	4.3634	2.1536	2.2098
2.8	5.3741	2.0597	4.8547	2.3454	2.5093

Table 2. The proton and neutron mass corrections as a function of the moment cutoff, λ , for the photon cutoff, Λ, equal to 1.5m

$$(e^2 \Delta m_1/m)\cdot 10^3 = 3.4248$$

λ/m	$-e^2\mu_p \Delta m_2/m \cdot 10^3$	$-e^2\mu_p^2 \Delta m_3/m \cdot 10^3$	$-\Delta m_p/m \cdot 10^3$	$-\Delta m_n/m \cdot 10^3$	$(\Delta m_n - \Delta m_p)/m \cdot 10^3$
0.8	2.2047	.4131	-.8070	.4704	-1.2774
0.9	2.5973	.5241	-.3034	.5968	-.9002
1.0	2.9943	.6440	.2135	.7333	-.5198
1.1	3.3929	.7714	.7395	.8784	-.1389
1.2	3.7913	.9053	1.2718	1.0308	.2410
1.3	4.188	1.045	1.808	1.189	.619
1.4	4.581	1.188	2.344	1.353	.991
1.6	5.357	1.486	3.418	1.692	1.726
1.8	6.112	1.793	4.480	2.041	2.439
2.0	6.845	2.105	5.525	2.397	3.128
2.2	7.555	2.420	6.550	2.755	3.795
2.4	8.242	2.734	7.551	3.114	4.437
2.6	8.905	3.048	8.528	3.470	5.058
2.8	9.546	3.358	9.479	3.824	5.655

Table 3. The proton and neutron mass corrections as a function of the moment cutoff, λ, for the photon cutoff, Λ, equal to 2.0m.

$$(e^2\Delta m_1/m)\cdot 10^3 = 4.1173$$

λ/m	$-e^2\mu_p\Delta m_2/m \cdot 10^3$	$-e^2\mu_p^2\Delta m_3/m \cdot 10^3$	$-\Delta m_p/m \cdot 10^3$	$-\Delta m_n/m \cdot 10^3$	$(\Delta m_n-\Delta m_p)/m \cdot 10^3$
0.8	2.8921	.4780	-.7472	.5443	-1.2915
0.9	3.4362	.6142	-.0669	.6993	-.7662
1.0	3.9926	.7635	.6388	.8694	-.2306
1.1	4.5574	.9247	1.3648	1.0530	.3118
1.2	5.1273	1.0964	2.1064	1.2484	.8580
1.3	5.6993	1.2771	2.8596	1.4543	1.4053
1.4	6.2729	1.4660	3.6216	1.6693	1.9523
1.5	6.845	1.662	4.390	1.892	2.498
1.6	7.415	1.864	5.162	2.122	3.040
1.8	8.543	2.282	6.708	2.599	4.109
2.2	10.737	3.159	9.779	3.597	6.182
2.4	11.797	3.609	11.289	4.110	7.179
2.6	12.830	4.062	12.775	4.626	8.149
2.8	13.835	4.517	14.235	5.143	9.092

Table 4. The proton and neutron mass corrections as a function of the moment cutoff, λ, for the photon cutoff, Λ, equal to 3.0m.

$$(e^2\Delta m_1/m)\cdot 10^3 = 5.2089$$

λ/m	$-e^2\mu_p\Delta m_2/m$ $\cdot 10^3$	$-e^2\mu_p^2\Delta m_3/m$ $\cdot 10^3$	$-\Delta m_p/m$ $\cdot 10^3$	$-\Delta m_n/m$ $\cdot 10^3$	$(\Delta m_n-\Delta m_p)/m$ $\cdot 10^3$
0.8	4.0330	.5551	-.6208	.6321	-1.2529
0.9	4.8428	.7237	.3576	.8241	-.4665
1.0	5.6835	.9123	1.3869	1.0388	.3481
1.1	6.5488	1.1195	2.4594	1.2748	1.1846
1.2	7.4336	1.3441	3.5688	1.5305	2.0383
1.3	8.3337	1.5845	4.7093	1.8042	2.9051
1.4	9.2453	1.8396	5.8760	2.0947	3.7813
1.5	10.1654	2.1080	7.0654	2.4003	4.6642
1.6	11.0914	2.3886	8.2711	2.7199	5.5512
1.8	12.9523	2.9821	10.7255	3.3956	7.3299
2.0	14.814	3.612	13.217	4.113	9.104
2.2	16.666	4.271	15.728	4.863	10.865
2.4	18.502	4.954	18.247	5.641	12.606
2.6	20.315	5.655	20.761	6.439	14.322
2.8	22.103	6.370	23.264	7.254	16.010

Table 5. The proton and neutron mass corrections as a function of the moment cutoff, λ, for the photon cutoff, Λ, equal to 4.0m.

$(e^2\Delta m_1/m)\cdot 10^3 = 6.0519$

λ/m	$-e^2\mu_p\Delta m_2/m$ $\cdot 10^3$	$-e^2\mu_p^2\Delta m_3/m$ $\cdot 10^3$	$-\Delta m_p/m$ $\cdot 10^3$	$-\Delta m_n/m$ $\cdot 10^3$	$(\Delta m_n-\Delta m_p)/m$ $\cdot 10^3$
0.8	4.9455	.5975	-.5089	.6804	-1.1893
0.9	5.9770	.7853	.7104	.8942	-.1838
1.0	7.0579	.9977	2.0037	1.1360	.8677
1.1	8.1804	1.2335	3.3620	1.4046	1.9574
1.2	9.3380	1.4916	4.7777	1.6985	3.0792
1.3	10.5250	1.7708	6.2439	2.0164	4.2275
1.4	11.7367	2.0698	7.7546	2.3568	5.3978
1.5	12.9686	2.3873	9.3040	2.7184	6.5856
1.6	14.2171	2.7223	10.8875	3.0998	7.7877
1.8	16.7517	3.4396	14.1394	3.9167	10.2227
2.0	19.319	4.213	17.480	4.797	12.683
2.2	21.903	5.035	20.886	5.733	15.153
2.4	24.490	5.898	24.336	6.716	17.620
2.6	27.072	6.796	27.816	7.738	20.078
2.8	29.641	7.723	31.312	8.794	22.518

From fig. 3-5 or tables 1-5 it is seen that, as the photon cutoff, Λ , increases, the value of the moment cut-off, λ , that is required to give the observed mass difference between the neutron and the proton decreases. As Λ approaches infinity, however, λ does not approach zero, but instead it approaches a finite positive value. This can be seen by considering the dominant asymptotic behavior of Δm_1, Δm_2, Δm_3, and Δm_n-Δm_p.

The dominant asymptotic behavior of Δm_1 is $(3m/4\pi)\cdot\log(\Lambda^2/m^2)$, while that of Δm_2 is $-(3\lambda^2/4\pi m)\log(\Lambda^2/m^2)$. Δm_3 approaches a limit as Λ approaches infinity, so that Δm_3 and Δm_n can be neglected in considering the dominant asymptotic behavior of Δm_n-Δm_p. The dominant asymptotic behavior of Δm_n-Δm_p is thus $(3e^2m/4\pi)(1 -m^{-2}\lambda^2\mu_p)\log(\Lambda^2/m^2)$. Therefore, as Λ approaches infinity, (λ/m) approaches $\mu_p^{-\frac{1}{2}}$= .7469. For (λ/m) less than this value, we cannot obtain the observed mass difference between the neutron and the proton.

Discussion

Classical Interpretation. It is of some interest to look for a classical interpretation of the neutron-proton mass difference, and to compare it to the quantum mechanical result. The first interpretation attempted was proposed by K. Huang and V. F. Weisskopf* in an unpublished communication to R. P. Feynman and myself. This communication was also received by M. Gell-Mann and M. L. Goldberger.**

K. Huang and V. F. Weisskopf considered the proton as formed from charge and current elements assembled from infinity. They insisted that the currents involved be kept constant during the assembly process by some external means, such as batteries. This is necessary, since, when two parallel current loops approach each other, the flux of one loop through the other increases and induces an e. m. f. in the direction opposite to the current. The current loops attract each other, the work done being equal to $-(8\pi)^{-1} \int H^2 d^3x$, while the energy stored in the magnetic field is given by $(8\pi)^{-1} \int H^2 d^3x$. The work done by the batteries is $(4\pi)^{-1} \int H^2 d^3x$, as required by conservation of energy. The assembled proton is required to have the observed magnetic moment and charge. The reasoning for the neutron is similar, except that there are no

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*K. Huang and V. F. Weisskopf are at the Massachusetts Institute of Technology.

**M. Gell-Mann and M. L. Goldberger are at the University of Chicago.

charges involved. K. Huang and V. F. Weisskopf concluded that the proton mass correction has a positive contribution from assembling the charges and a positive contribution from assembling the currents. The reason the quantum mechanical calculation gives a negative magnetic contribution is that the batteries have not been included, and the calculation is therefore incomplete. They believed the bare masses of the neutron and proton were indeed equal, in accordance with the isotopic spin hypothesis, but that the masses appearing in the quantum mechanical equations already included a correction produced by the batteries, and were not necessarily equal.

The trouble with the preceding model arises when we try to make a correspondence to quantum mechanics. The transition to quantum mechanics is made by converting classical canonical variables into operators satisfying certain commutation or anticommutation relations among themselves. What keeping the currents constant corresponds to in quantum mechanics is not known. What we do know, however, is the quantum mechanical analogue of the total angular momentum of a system, and we also know the angular momentum operator of a particle. This suggests that we assemble our current loops in such a manner as to apply no external torques to the current loops, thus keeping the angular momentum of the system constant. For example, two parallel current loops can be brought together in such a manner that the force on one due to the other is colinear with the force charges involved. K. Huang and V. F. Weisskopf concluded that the proton mass correction has a positive contribution from assembling the charges and a positive contribution from assembling the currents. The reason the quantum mechanical calculation gives a negative magnetic contribution is that the batteries have not been included, and the calculation is therefore incomplete. They believed the bare masses of the neutron and proton were indeed equal, in accordance with the isotopic spin hypothesis, but that the masses appearing in the quantum mechanical equations already included a correction produced by the batteries, and were not necessarily equal.

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M. Gell-Mann and M. L. Goldberger, in their reply to K. Huang and V. F. Weisskopf, stressed the difficulty in obtaining a quantum mechanical analogue of the latters' classical interpretation, and proposed the alternative described in the preceding paragraph. They presented the following simplified example to illustrate the two interpretations: The neutron and proton are represented by flywheels spinning with constant angular velocity. The flywheels are assumed to be of identical construction, except that the one representing the proton is charged. The Lagrangian of the flywheel representing the proton may be written as $L=\frac{1}{2}I\dot{\theta}^2 + e\dot{\theta}\int \bar{c}\cdot\bar{A}d^3x - V_{Coulomb} - (8\pi)^{-1}\cdot\int H^2 d^3x$, where I is the moment of inertia of the flywheel, θ is its angular position, and the current density, $\bar{j}$, equals $e\dot{\theta}\bar{c}$. In this example the neutron has no electromagnetic properties, but it is still instructive to compare the two interpretations. The Lagrangian of the flywheel representing the neutron is $\frac{1}{2}I\dot{\theta}^2$.

According to the first interpretation, the proton and neutron have equal angular velocities, $\dot{\theta}$. The energies of the proton and neutron are then $\frac{1}{2}I\dot{\theta}^2 + V_{Coulomb} + (8\pi)^{-1} \cdot \int H^2 d^3x$ and $\frac{1}{2}I\dot{\theta}^2$, respectively. The proton-neutron energy difference is thus $V_{Coulomb} + (8\pi)^{-1}\int H^2 d^3x$.

According to the second interpretation, we first transform to canonical variables. For the proton, the momentum conjugate to θ is $p_\theta = I\dot{\theta} + e\int \bar{c}\cdot\bar{A}d^3x$, while the Hamiltonian is $H = (2I)^{-1}(p_\theta - e\int \bar{c}\cdot\bar{A}d^3x)^2 + V_{Coulomb} + (8\pi)^{-1} \cdot \int H^2 d^3x$. For the neutron, the corresponding quantities are $p_\theta = I\dot{\theta}$ and $H = (2I)^{-1}p_\theta^2$. The proton and neutron are assumed to have equal angular momenta, p_θ. The proton-neutron energy difference is then $V_{Coulomb} + (8\pi)^{-1}\int H^2 d^3x - e\dot{\theta}\int \bar{c}\cdot\bar{A}d^3x - (e^2/2I) \cdot (\int \bar{c}\cdot\bar{A}d^3x)^2$. This can be simplified by using Maxwell's equations with $\bar{J}$ equal to $e\dot{\theta}\bar{c}$. The result is $V_{Coulomb} - (8\pi)^{-1}\int H^2 d^3x - (e^2/2I)(\int \bar{c}\cdot\bar{A}d^3x)^2$, and has a negative magnetic contribution to the energy.

Although we have found a classical interpretation of the neutron-proton mass difference that agrees in sign with the quantum mechanical result, the agreement is only qualitative. Classically, the proton mass correction is of the form $Ae^2 - B(1+\mu_p)^2$, where $1+\mu_p$ is the total magnetic moment of the proton in units of $e\hbar/2m_pc$. This implies that the ratio of the coefficient of μ_p in the mass correction to the coefficient of μ_p^2 should be two. In the quantum mechanical computation, this ratio ranged from five to fifteen.

<u>Justification Based on Quantum Field Theory</u>. Some objections might be raised as to the validity of our results

for the following reason: In our quantum electrodynamic calculation of the neutron-proton mass difference, we have tacitly assumed the electromagnetic interaction with the neutron and proton is linear in the electromagnetic fields. The second order mass corrections then result from treating this interaction in second order perturbation. If there were present an interaction quadratic in the electromagnetic fields, however, we would obtain additional second order mass corrections from treating this interaction in first order perturbation. Such an interaction could furnish the batteries of the Huang-Weisskopf classical interpretation. We will now show, however, that no such term exists. We will show, in fact, that the electromagnetic interaction with the neutron and proton is linear in the electromagnetic fields, assuming that the non-electromagnetic interactions among the elementary particles possess isotopic spin symmetry and can be represented by a Hamiltonian. We will consider interactions, such as the neutrino-electron-nucleon interaction, which are not electromagnetic and do not possess isotopic spin symmetry as negligible in comparison to the electromagnetic interaction.

The proof proceeds as follows: In the Schroedinger representation, the Hamiltonian operator of a relativistic charged field of spin 0, $\frac{1}{2}$, or 1 in an external electromagnetic field can be written as $H + \int j_\mu(\overline{x}) A_\mu(\overline{x},t) d^3x$, where H is the Hamiltonian operator of the free field, $A_\mu(\overline{x},t)$ is the external electromagnetic field, and $j_\mu(\overline{x})$

are the 4-current density operators for the charged field and are independent of $A_\mu(\overline{x},t)$. Dirac (spin $\frac{1}{2}$) fields have a Hamiltonian of this form in the usual formulation, while fields of spin 0 or 1 have such a Hamiltonian when formulated in terms of Kemmer-Duffin matrices (6). We will not concern ourselves with charged fields of spin greater than 1, since no elementary charged particle whose spin is known has a spin greater than 1.

Neglecting all non-electromagnetic interactions which do not possess isotopic spin symmetry, we can write the Hamiltonian operator for the combined elementary fields as $H_0 + H_1 + H_2 + \int J_\mu(\overline{x}) \cdot A_\mu(\overline{x}) d^3x$, where H_0 is the Hamiltonian operator of the free photon field, H_1 is the sum of the free Hamiltonian operators of all the other elementary particle fields, H_2 is the Hamiltonian operator which represents the non-electromagnetic interactions, $J_\mu(\overline{x})$ are the total 4-current density operators, and $A_\mu(\overline{x})$ are the electromagnetic field operators. H_0 and $A_\mu(\overline{x})$ commute with H_1, H_2, and $J_\mu(\overline{x}')$, and H_0, H_1, and H_2 possess isotopic spin symmetry.

Since H_0 commutes with $H_1 + H_2$, they can be simultaneously diagonalized. The resulting eigenstates and eigenvalues possess isotopic spin symmetry. In particular, the masses of the neutron and proton are equal. The Hamiltonian operator in the interaction representation* of these states

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*What we are doing is letting $\psi = e^{-i(H_0+H_1+H_2)t} \cdot \phi$ in the Schroedinger equation, $i\,\partial/\partial t \cdot \psi = (H_0+H_1+H_2+\int J_\mu(\overline{x}) A_\mu(\overline{x}) d^3x)\,\psi$, to get $i\,\partial/\partial t \cdot \phi = \int J_\mu(\overline{x},t) \cdot A_\mu(\overline{x},t) d^3x\,\phi$.

is given by

$$e^{i(H_0+H_1+H_2)t}\cdot\int J_\mu(\bar{x})\cdot A_\mu(\bar{x})d^3x\cdot e^{-i(H_0+H_1+H_2)t} =$$

$$\int e^{i(H_1+H_2)t}\cdot J_\mu(\bar{x})\cdot e^{-i(H_1+H_2)t}\cdot e^{iH_0t}\cdot A_\mu(\bar{x})\cdot e^{-iH_0t}\cdot d^3x=$$

$$\int J_\mu(\bar{x},t)\cdot A_\mu(\bar{x},t)d^3x,$$

where $J_\mu(\bar{x},t)= e^{i(H_1+H_2)t}\cdot J_\mu(\bar{x})\cdot e^{-i(H_1+H_2)t}$, $A_\mu(\bar{x},t) =$

$e^{iH_0t}\cdot A_\mu(\bar{x})\cdot e^{-iH_0t}$.

It is to be noted that $A_\mu(\bar{x},t)$ are the free photon field operators in the Heisenberg representation, and commute with $J_\mu(\bar{x}',t')$. What we have done in our calculation is to approximate the matrix elements of $J_\mu(\bar{x},t)$ between single particle proton and neutron states with nuclear (non-electromagnetic) interactions, by the matrix elements of charge and Pauli operators between free single particle states of the same 4-momentum, and then introduced cutoffs. With or without this approximation, the only process that produces electromagnetic mass corrections to the proton or neutron of order $e^2/\hbar c$ is the emission, propagation, and reabsorption of one virtual photon.

Cutoffs. Finally, we consider the cutoffs we have obtained in computing the neutron-proton mass difference. That these cutoffs are of the order of the proton mass is, at first sight, rather puzzling from a theoretical standpoint. It is more reasonable to expect cutoffs of the order of the pion mass. On the other hand, the relatively small changes nucleon moments appear to undergo when nucleons form nuclei is consistent with high cutoffs. The experiments on electron-neutron and electron-proton scattering are also consistent with cutoffs of the order of the proton mass, and will now be considered in some detail.

Recent experiments on low energy electron-neutron scattering (7) indicate the results are almost entirely accounted for by the magnetic moment of the neutron. We use the convention of expressing the scattering in terms of the depth, V_0, of the square well potential of range $e^2/m_e c^2$ (the classical electron radius) which gives the same scattering matrix at low energies as does the actual interaction. The experimental value of V_0 is (3860 ± 370) ev. The contribution of the magnetic moment to the scattering is 4080 ev (8). This leaves a residual well depth of (-220 ± 370) ev. Such an interaction can arise from the fact that the neutron has a charge distribution. For example, let us take as a model of the neutron a point charge e surrounded by a concentric shell of charge -e, uniformly distributed, and of radius $ae^2/m_e c^2$. This model is a crude analogue of a proton surrounded by a meson cloud of charge -e. The value of V_0 we then obtain is $\frac{1}{2}mc^2 a^2$. If the radius of the shell is b times the proton Compton wave length, $\hbar/mc$, then V_0 is $1400 \cdot b^2$ ev. The range of the neutron charge distribution is thus exceedingly small, being even less than the proton Compton wave length in our crude model. If the range of the current distribution is of the same order as that of the charge distribution, then a moment cutoff of order $\hbar/mc$ is indicated.

Experiments on high energy electron-proton scattering* yield cross sections which are less than those predicted for a proton consisting of a point charge and a point anomalous magnetic moment. If this latter cross section is denoted by σ_0, then the observed cross section is $(1 - 6^{-1}r^2q^2)^2\sigma_0$, where $r = (0.7 \pm 0.25)\cdot 10^{-13}\text{cm} = (3.3 \pm 1.2)\cdot m^{-1}$, and q is the momentum transfer in the center of mass system ($\hbar=c=1$). We can relate the experimental result to cutoffs on the photon propagation function and the anomalous moment interaction. For simplicity, we neglect the cutoff on the anomalous moment. Since its inclusion would tend to lower the cross section, the resulting photon cutoff will be somewhat low. The argument now proceeds as follows: In computing σ_0, the lowest order Feynman diagram involves the emission and reabsorption of one virtual photon for which $\underline{k}^2 = -q^2$, while higher order diagrams give small corrections which we neglect. The point charge, point moment scattering amplitude is thus multiplied by the photon cutoff, $-\Lambda^2/(\underline{k}^2-\Lambda^2)$. The scattering cross section is thus

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*Reported at the 1955 Rochester Conference on High Energy Physics. The experiments were performed at Stanford University.

$(\Lambda^2/\Lambda^2+q^2)^2\sigma_0$. We now compare $(\Lambda^2/\Lambda^2+q^2)^2$ with $(1-6^{-1}r^2q^2)^2$ for small q and obtain $\Lambda^{-1}=6^{-\frac{1}{2}}r=(1.3\pm0.5)\cdot m^{-1}$. This value of Λ is of order m and is thus consistent with the cutoff values needed to give the neutron-proton mass difference.

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