THE ANALYSIS OF REPEATED GAMES THROUGH EVOLUTION AND LEARNING

Thesis by

Richard Thomas Boylan

In Partial Fulfillment of the Requirements
for the Degree of
Doctor of Philosophy

California Institute of Technology
Pasadena, California
1991
(Submitted September 21, 1990)

Acknowledgements

The author thanks Richard McKelvey for helpful comments, encouragement and many enlightening discussions. The second chapter of the dissertation could not have been written without the help of Roko Aliprantis and especially Kim Border. I would also like to thank for their help: Mahmoud El-Gamal, George Mailath, Tom Palfrey, Guofu Tan, and Thomas Wolff.

Abstract

In this thesis we study the evolution of strategy choices for symmetric, finite, normal games. The second chapter of the dissertation analyzes infinite populations where in each period individuals are randomly and anonymously matched. Individuals are of different types, where a type represents a belief or a strategy choice. After each match individuals are allowed to change types. Thus a stochastic process is defined which describes the evolution of types in the population. The main result in the second chapter is that the evolution of the population can be described through a simpler deterministic system. The third chapter relates the properties of the evolutionary dynamics to standard game theoretic principles. Although individuals act in a purely mechanistic way, in equilibrium, the population as a whole acts like an individual adopting a strategy corresponding to a perfect equilibrium. The fourth chapter analyzes how two learning dynamics for finite normal form games - namely, the Cournot process and fictitious play - can explain experimental data. In doing so the chapter develops econometric techniques that can have a wide application to the analysis of experimental data.

Contents

List of Figures

Chapter 1

INTRODUCTION AND OVERVIEW

1.1 Introduction

The choice of equilibrium concept is crucial in the study of individual decision making. Existing equilibrium concepts rely on assumptions on individuals' beliefs. Unfortunately no equilibrium concept developed so far is satisfactory for all games.[1] An alternative approach is to think of equilibria as stable points of some dynamic model. In order to pursue this approach two questions need to be answered: (i) What dynamics describes the evolution of strategy choices? (ii) What are the properties of such dynamics? In order to answer the first question we need to describe a particular environment, develop a set of axioms that characterize individual behavior, and then compare the theoretical predictions with experimental data.

One class of dynamical models studies the evolution of the distribution of strategies in a large population. At each period individuals are randomly pairwise matched. The distribution of strategies at time $t + 1$ depends on the distribution of strategies at time t and the outcome of the matching

---

[1] The following authors discuss the problem with existing equilibrium concepts: Cho and Kreps (1987), van Damme (1987), Tan and Werlang (1988).

process. Since the population is very large the randomness created by the matching scheme is assumed to be negligible and the process is studied as a deterministic dynamical system. The second chapter shows via an example that such a deterministic process can provide a very poor approximation of the model for finite populations and gives sets of assumptions for which the deterministic approximation is correct.

One model which belongs to the class of models described above is the replicator model. The replicator model is studied in evolutionary biology; it postulates that strategies are genetically determined and that individuals who adopt successful strategies have more offspring than those who adopt unsuccessful strategies. Although in this model individuals do not act as Bayesian maximizers, in many cases the evolutionary forces make individuals look like Bayesian maximizers. The third chapter develops a new equilibrium concept for a generalization of the replicator model and relates properties of equilibria of the generalized replicator model to refinements of Nash equilibria.

In order to analyze dynamic models of strategy choice experimentally, we need to consider models that describe the evolution of strategies in very small populations (possibly two individuals). The two models that have been analyzed the most in this context are the Cournot process and the replicator model. The fourth chapter analyzes some experimental data to determine how well these models describe human behavior. The chapter develops some econometric techniques that could be used in the future to examine the validity of other dynamical processes.

In this section we first discuss why evolutionary forces should be analyzed in conjunction with games. Then we describe the three dynamics analyzed in this dissertation; namely, the Cournot process, fictitious play, and the replicator model. Finally we discuss the reason for perturbing the dynamical processes.

1.1.1 Evolution as a justification of rationality

The assumption that players are rational[2] and that this fact is common knowledge among players is very important in determining the equilibrium strategy of a game: if the assumption does not hold, then the common-knowledge-of-rationality equilibrium strategy may not be played. Suppose that player 1 assigns some positive probability to player 2 being irrational. Then, as illustrated in the example in the next paragraph, player 1 may be irrational (i.e., not maximizing his payoffs) if he plays the equilibrium strategy. Similarly if player 2 assigns some positive probability to player 1 thinking that he is irrational, then he may not want to choose the equilibrium strategy.

Suppose that two individuals play the following game which is called the 'centipede' game.[3] At the first round there are two piles of money: the large pile contains $10 the small pile $0.50. For finitely many rounds two players alternate in deciding whether to stop the game. After each round, if the game is not stopped, the amount in each pile is multiplied by ten. The player who stops the game gets the large pile; the other player gets the small pile. If rationality is common knowledge, then the first player who has the opportunity of ending the game will do so. If rationality is not common

---

[2] One possible definition of rationality is that individuals choose only *rationalizable* strategies (see Bernheim (1984) and Pearce (1984)); i.e., individuals select actions that maximize their expected payoffs and individuals believe that other individuals select only rationalizable strategies.

[3] For the origin of this game see McKelvey and Palfrey (1990).

knowledge then the players may decide against taking the pile.

Economists have argued that evolutionary laws ingrained in the market system lead to the prevalence of rationality. Alchian (1950) argued that firms do not necessarily act as profit maximizers but that the laws of evolution, by selecting for the most successful patterns of behavior, make firms look like profit maximizers. In finance, LeRoy (1989) among others, has given evolutionary arguments for the validity of the efficient market hypothesis.[4] However, if there are enough irrational subjects, rational people may not do better than others. For instance, in a experiment run by McKelvey and Palfrey (1990) where a group of individuals was repeatedly divided into pairs and where each pair played the centipede game, there were two players who clearly did not understand the game they were playing and consistently declined to take the large pile. Those same two players received the two highest payoffs in this experiment.[5] In order to reconcile the economist belief that rationality will prevail and that in the short run rational play may be detrimental, we need to model explicitly the evolutionary forces.

1.1.2 Dynamical processes for the study of games

In order to be able to study the evolution of strategy choices we need to examine the case where the game is played repeatedly. Assume that the same two individuals play a finite normal form game repeatedly. At each round individuals update their beliefs about the opponent's strategy and then select the strategy that maximizes expected payoffs. Such a model was first studied by Cournot. In the Cournot model each person believes that the opponent selects the same strategy the opponent selected in the previous

---

[4] That prices of stock reflect all the information.

[5] This is possible if there are enough people that do not take the pile.

	A	B	C
1	1, 0	0, 0	0, 1
2	0, 1	1, 0	0, 0
3	0, 0	0, 1	1, 0

Figure 1.1: Shapley's example.

period. It is straightforward to construct games for which this process will not converge but instead cycle indefinitely.[6]

Brown (1951) suggested a different learning process which he called 'fictitious play.' In this process individuals assign a probability for each of the opponent's strategies equal to a convex combination of the empirical distribution and his initial beliefs. Robinson (1951) showed that if individuals select strategies according to the fictitious play process, then for all two-person finite zero-sum games individuals will converge to a Nash equilibrium. Shapley (1962) showed that if individuals play the non-zero sum game in Figure 1.1 according to fictitious play then individuals' beliefs and strategies do not converge and in fact cycle indefinitely.

Evolutionary biologists have studied a different dynamic which is called the 'replicator model.' Each member of an infinite population chooses a strategy for the game. At regular periods all members of the population are 'randomly paired' to play a game. Given the distribution of strategies in the population, some strategies will be more successful than others. Strategies that are more successful are adopted more frequently in the successive period. This model can be described by a stochastic difference equation. As the second chapter shows, for infinite populations there exist matching rules for which this stochastic difference equation is equivalent to a determinis-

---

[6] Take for instance a game of matching pennies.

	1A	1B	1C	2A	2B	2C	3A	3B	3C
1A	0	0	-1	1	0	1	1	-1	-1
1B	0	0	-1	0	-1	0	2	0	0
1C	1	1	0	0	-1	0	1	-1	-1
2A	-1	1	-1	0	1	1	0	0	-1
2B	-1	1	-1	-1	0	0	1	1	0
2C	0	2	0	-1	0	0	0	0	-1
3A	-1	0	0	0	0	2	0	-1	0
3B	-1	0	0	-1	-1	1	1	0	1
3C	0	1	1	-1	-1	1	0	-1	0

Figure 1.2: Symmetrized version of Shapley's example.

tic difference equation. When analyzing the process in continuous time, the difference equation is approximated by a differential equation.

Suppose that individuals play the following game: a toss of a coin decides which individual is the row player and which individual is the column player. After observing the outcome of the toss the two individuals play Shapley's example. The normal form of this game is described in Figure 1.2. For this game the replicator dynamics in continuous time will cycle indefinitely.

1.1.3 Perturbations in strategy choices

Many of the equilibrium concepts developed since Selten (1975) have sought to restrict the set of strategies considered by each player.[7] The following example illustrates the reason for this approach. Suppose 51 voters select one of two alternatives, A and B, by secret ballot and majority vote and that all voters prefer A over B. For many choices of alternatives by the voters, none of the voters can unilaterally change the outcome. Therefore, almost any pattern of voting is reasonable. If the voters' strategies are restricted in a way such that all moves are played with positive probability, then each

---

[7] See for instance Myerson (1978), Kohlberg and Mertens (1986).

player is a swing voter with positive probability and will thus want to vote for A.

In the existing literature, game theorists assume that small mistakes insure that players play all possible moves with positive probability. In the previous example, although voting is a simple process, the assumption that players will vote for their favorite alternative by mistake is necessary in order to predict unequivocally that A will be selected.

The existence of small perturbations seems more natural in dynamic processes. Small perturbations of fictitious play may assure a better sampling of the strategies available. In evolutionary biology small mutation rates are believed to lead to the selection of the more favorable traits for a given environment. It thus seems reasonable to study the dynamical processes over arbitrarily small perturbations and determine whether the cycling behavior described in the previous section will persist. Chapter 3 analyzes stationary points that are resistant to small perturbations in the dynamics.

Small perturbations in the dynamics seem also necessary in order to evaluate experimentally the dynamic models of strategy choice. Without small perturbations the likelihood ratios may not be defined because we can observe data that is contrary to all hypothesis. Thus in Chapter 4, in order to examine how well the Cournot process and fictitious play explain experimental data of repeated play of different normal form games, we add a small perturbation to the two models and examine the consequence of changing the level of perturbation.

1.2 Overview

1.2.1 Summary of Chapter 2

Biologists and economists have analyzed populations where each individual interacts with randomly selected individuals. The random matching generates a very complicated stochastic system. Consequently evolutionary biologists have approximated such a system by a deterministic system. The justification of such an approximation is that the population is assumed to be very large and thus some law of large numbers must hold. In this paper we give an example for which this assumption does not hold. We then show that if we assume that the population is infinite then there exist matching rules for which the stochastic and the deterministic systems are the same. Finally, we show that if the process lasts finitely many periods and if the population is large enough, then the deterministic model offers a good approximation of the stochastic model. In doing so we make precise what we mean by population, matching process, and evolution of the population.

1.2.2 Summary of Chapter 3

The paper requires that equilibrium behavior for two person symmetric games be resistant to evolution. In particular, the paper assumes that the evolution of the distribution of strategies in a population can be described according to some generalization of the replicator model. This paper defines an equilibrium concept, 'evolutionary equilibrium,' as the limit of stationary points of an evolutionary process as the proportion of the population that mutates goes to zero. The set of these evolutionary equilibria, is a nonempty subset of the set of perfect equilibria (and thus of the set of Nash equilibria) and a superset of the set of regular equilibria and the set of ESS.

1.2.3 Summary of Chapter 4

In recent years, there has been a surge in the number of experimental studies in economics, and their influence on the field can no longer be denied. On the other hand, one feels that the interaction between the findings in experimental economics and the theoretical and modeling techniques in other subfields of economics has not yet reached its full potential. One of the main reasons that the impact of experimental studies on the rest of the discipline has been limited is our inability to systematically combine what we have learned from all the experiments that have been run to-date. In this paper, we propose a methodology for combining a large number of experiments to make an overall assessment of the empirical justifiability of certain hypotheses regarding human economic behavior. We illustrate our methodology by an application to models of learning in repeated games. Empirical studies of that class of models using experimental or non-experimental data are very sparse in the literature and they are by necessity limited to an experiment-by-experiment analysis. It is clear that no classical statistical procedure will allow us to combine the results of many experiments with vastly different designs to form stylized facts that can benefit the rest of the profession. The application of our proposed (purely Bayesian) procedure allows us to combine two sets of experiments (for a total of nine experiments) to update our beliefs on the relative justifiability of the Cournot and the fictitious play learning hypotheses. Even though some of the experiments individually pointed unequivocally in favor of the Cournot hypothesis, our overall analysis makes us believe that fictitious play is an infinitely more likely hypothesis. The proposed statistical procedure provides a very powerful tool which is quite easy to use. We believe that the extensive use of this and similar tools in literature surveys can lead to a more fruitful interaction between theory and experiments in economics.

Chapter 2

LAWS OF LARGE NUMBERS FOR DYNAMICAL SYSTEMS WITH RANDOMLY MATCHED INDIVIDUALS

2.1 Introduction

There is a large literature (see below) that studies dynamical system with individuals randomly matched in pairs, although the particular way with which people are matched is left unspecified. In this chapter we describe ways of matching individuals and the properties of such matching schemes.

Fudenberg and Levine (1990) examine a model where there are n populations; each population consists of m different types where each type consists of a belief over which strategy the other individuals adopt. The proportion of population i that is of type j is denoted by p_{ij}. Every period each player from a population i is randomly and independently matched with one individual from every other population i' ($i' \neq i$). Fudenberg and Levine assume that the probability with which a player meets a player from population i' and of type j is $p_{i'j}$. The randomly matched individuals play a game selecting strategies according to their beliefs and updating their beliefs according to the observed strategy choices of the other players. Fudenberg and Levine go on to characterize the steady states of the dynamics.

This type of model is very similar to the models that have been studied extensively in population genetics and evolutionary biology.[1] This similarity has led to a large literature on the application of evolutionary biology to game theory.[2] In this paper we assume that there is a large population. Each individual is matched anonymously to exactly one other individual. There are m types of individuals. We keep the terminology vague so that the discussion applies to economics (where types are beliefs), evolutionary biology (where types are strategies), and to population genetics (where types are genotypes or alleles). The set of types is denoted by S, where

$$S = \{s_1, \ldots, s_m\}.$$

The initial proportion of the population of type s_r is denoted by p_r; we assume throughout that, for all s_r in S, p_r is positive. The initial distribution of types in the population is denoted by p where $p = (p_1, \ldots, p_m)$.

Underlying the models by Fudenberg and Leving as well as the biological models is the conjecture that if the population is very large (possibly infinite) there exists a matching scheme such that **for all types s_r and s_v the proportion of individuals of type s_r that are matched with individuals of type s_v is $p_r p_v$ (almost surely).**

In the model given in Fudenberg and Levine (1990) and the biological models, the matching scheme is implemented repeatedly and each individual's type is allowed to change between periods. Then the individuals follow

---

[1] Appendix C contains a brief discussion of these models.

[2] See Chapter 3 and references therein.

a stochastic process governed by the way types are updated and the matching rule. Again, the models are analyzed as deterministic systems. The law of motion is computed by assuming that in every period the set of individuals adopting the same strategy is matched with the population average. Thus there is an implicit conjecture that, if the population is large enough, there exists a matching rule that matches individuals in every period such that the deterministic process provides a good description of the stochastic process.

This paper proves that there is a matching rule for which both conjectures are correct when individuals are represented by the set of natural numbers. Thus this paper answers the following questions that are usually left unanswered in the literature: How is the population characterized? What is the structure of the matching process ? How are types assigned to individuals? What do we mean by each subpopulation facing the distribution of types equal to the population distribution? How do we characterize the evolution of the population from the random matching scheme?

Section 2.2 gives an example of how the dynamics for very large populations differs from the dynamics for infinite populations. Section 2.3 describes the problem of finding a matching technology for infinite populations such that all matches are equally likely. Section 2.4 proves the first conjecture; i.e., that there is a matching scheme such that a law of large numbers holds. Sections 2.5 and 2.6 prove the second conjecture; i.e., that a law of large numbers holds when individuals are randomly matched infinitely many times. Section 2.7 examines a matching scheme for finite populations. Appendix A briefly discusses the literature on the law of large numbers for a continuum of agents. Appendix C gives a brief overview of the biological literature related to matching schemes.

2.2 Matches over very large populations

A natural argument for supporting the conjecture in the introduction is the following: if the population is of size n then the probability that the matching rule does not behave as its expectation is $q(n)$. By the law of large numbers $q(n)$ can be made arbitrarily small by taking n large. In other words for large populations the conjecture is approximately correct. As we will see in Section 2.7 this argument can be formalized if the matching scheme occurs finitely many times. In many cases analyzed in the literature, however, the matching scheme analyzed occurs infinitely many times, and it is thus possible that the small perturbations that occur in each period alter the process significantly in the limit. We construct an example where this problem actually occurs. The example is taken from evolutionary biology because: (i) such models are very important in evolutionary biology, (ii) several economists have applied evolutionary models to game theory, and (iii) the mathematics in this example are very tractable.

Suppose that there is a population consisting of $3M$ individuals, where M is an even number. Individuals have a very simple life: they are born at time t they interact with one randomly selected individual, give birth to new individuals and die at time $t+1$. The matching scheme is left unspecified but we assume that all matches occur with positive probability.[3] Individuals belong to three different types: 1,2,3. If an individual of type r interacts with an individual of type v then this individual has a_{rv} offspring. All offspring are of the same type as the parent. Suppose that the matrix $A = (a_{vw})$ is as

---

[3] One way of describing the random matching scheme is to think of individuals as being drawn successively (without replacement) from an urn. The first and second individuals drawn are matched together, the third and fourth individuals drawn are matched together, and so on.

follows:

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix}.$$

First notice that the population size stays constant: if individuals of type s_v and s_w meet they will have together $2 = a_{vw} + a_{wv}$ offspring and thus keep the population constant. Notice that, because of the 0 entry in the matrix A, at any period t there is a positive probability that one of the types disappears, which we denote by q_e^t. For any population distribution among types (M_1, M_2, M_3) (where $M_1 + M_2 + M_3 = 3M$) there exists a set of matches for which one of the types totally disappears. Since all matches are possible these matches will have positive probability. Denote the probability that one type disappears when the population distribution is (M_1, M_2, M_3) by q_{M_1,M_2,M_2}. Let

$$q_e = \min_{\{M_1+M_2+M_3=3M\}} q_{M_1,M_2,M_3}.$$

Then, for all t, $q_e^t \geq q_e > 0$.

Notice that if a type disappears, it never comes back. Consequently, if the matching scheme is repeated infinitely often, the probability that a type will disappear is greater than

$$1 - \lim_{t \to \infty} (1 - q_e)^t = 1.$$

Let B_t be the set of events for which one of the types disappears by time t. From the previous remarks $\{B_t\}$ is an increasing sequence and $\lim_{t \to \infty} P(B_t) = 1$; therefore one of the types disappears in finite time. It is easy to see that if type s_v disappears then the population will converge to a population composed uniquely of individuals of type $s_{v+1\,(\mathrm{mod}\,3)}$ (almost surely). Therefore the population will converge to one of the vertices of the simplex in finite time (almost surely). Since this result is true irrespective of the population size $3M$, it will also be true as M tends to infinity.

Let p_v be the proportion of the population of type s_v. In evolutionary biology the evolution for the population is analyzed according to the replicator model. The replicator model assumes that a proportion $2p_v p_w$ of the matches are between individuals of type s_v and s_w. Then the proportion of the population of type i at time $t+1$ is related to the population at time t in the following way:

$$p_r^{t+1} = R_r(p^t) \equiv p_r^t \frac{\sum_v p_v^t a_{rv}}{\sum_v \sum_w p_v^t a_{vw} p_w^t} \,. \tag{2.1}$$

First suppose that the initial population is composed of a third of each type. Then the population will remain at the barycenter contrary to the behavior of the population when we examined the random matching rule.

Next suppose that the initial population is not distributed equally among each type. The interior of the simplex is invariant under the map R. Thus if a trajectory starts in the interior of the simplex it cannot reach the boundary in finite time. Also the vertices of the simplex are repellors for the dynamics on the interior of the simplex. Let $W\colon \Delta^3 \to \mathbf{R}$ be such that $W(p) = \frac{1}{p_1 p_2 p_3}$. Weising (1989) shows that W has a unique minimum at the barycenter and that along any trajectory W is strictly increasing. Thus the trajectory approaches the boundary of the simplex asymptotically. By looking at the law of motion we can see that the trajectory oscillates from a neighborhood of $(1,0,0)$, to a neighborhood of $(0,1,0)$, to a neighborhood of $(0,0,1)$ and so on and that the time the trajectory takes to go from one neighborhood to another is increasing. Thus for any period T there is a period $t > T$ such that p^t is far removed from any of the vertices and the ω-limit (the set of accumulation points) of the trajectory is the whole boundary.

2.3 Matches over countably infinite number of agents

This section introduces the notation that will be used in this chapter and discusses the problems of finding a random matching scheme for an infinite population. We assume that the population is countably infinite and is denoted by

$$\mathbf{N} = \{1, 2, \ldots, n, \ldots\}.$$

For convenience we represent the type space by the standard basis for $\mathbf{R}^m$; i.e., s_r is the m dimensional vector with a one on the r^{th} component and zeros on the other components. Let

$$\alpha \colon \mathbf{N} \to S \text{ be such that } \lim_{T \to \infty} \frac{1}{T} \sum_{i=1}^{T} \alpha(i) = p.$$

If $\alpha(i) = s_v$ then individual i is of type s_v. The v^{th} component of the vector p, p_v, represents the proportion of the population of type s_v.

Let Σ be the set of all possible pairwise matchings; i.e.,

$$\Sigma \;=\; \{\sigma \colon \mathbf{N} \to \mathbf{N} \,|\, \sigma \text{ is bijective and for all } i,\ \sigma^2(i) = i \text{ and } \sigma(i) \neq i\}.$$

A few remarks on the conditions that characterize Σ. The first condition says that each individual is matched exactly once. The second condition says that: 'if John is matched with Paul then Paul is matched with John.' The third condition states that an individual cannot be matched to himself.

We first show that there does not exist a probability space $(\Sigma, \mathcal{F}, P)$ such that for all distinct players $i, j, k \in \mathbf{N}$, the event that player i is matched with player j and the event that player i is matched with player k are equally likely. Suppose that $(\Sigma, \mathcal{F}, P)$ is such a probability space. Denote the probability that player i is matched with player j by

$$p(\sigma(i) = j) \equiv P(\{\sigma \in \Sigma | \, \sigma(i) = j\}).$$

In order for p to be well defined we need to assume that for all individuals $i, j \in \mathbf{N}$,

$$\{\sigma \in \Sigma | \, \sigma(i) = j\} \in \mathcal{F}. \tag{2.2}$$

Suppose that $p(\sigma(i) = j) \equiv q > 0$. Notice that since each individual is matched once, the sets $\{\sigma \in \Sigma | \, \sigma(i) = j\}$ and $\{\sigma \in \Sigma | \, \sigma(i) = k\}$ are disjoint. Thus

$$p(\sigma(i) \in \mathbf{N}) = p(\bigcup_{j \in \mathbf{N}} \sigma(i) = j) = \sum_{j \in \mathbf{N}} p(\sigma(i) = j) = \infty.$$

This clearly contradicts the definition of a probability. Alternatively, if $q = 0$ then $p(\sigma(i) \in \mathbf{N}) = 0$ which is not consistent with the fact that individual i is matched once.

Clearly the assumption in equation 2.2 and the assumption that all matches are equally likely lead to this contradiction. In order to weaken the set of measurable sets we need to find another way to express the idea that all matches are equally likely. Alternatively, we could relax the assumption that all matches are equally likely.[4]

2.4 Construction of a probability measure over matches

In this section we will construct a probability space over the set of matches by considering the events: 'the set of matches such that individual i is matched with an individual of type s_v.' We construct the probability space in an

---

[4] Another approach is to assume that there is a continuum of agents. A brief discussion of this approach is contained in Appendix A.

indirect way. First we define a probability over the space $S^{\mathbf{N}}$, 'the set of realizations of matches.' The probability space $(S^{\mathbf{N}}, \mathcal{B}, \mu)$ is defined by the property that the probability with which x_i (where $x \in S^{\mathbf{N}}$, $i \in \mathbf{N}$) equals s_v is the proportion of individuals in the population of type s_v, p_v (for all $s_v \in S$). Then we show that this probability space generates a probability space over matches, $(\Sigma, \mathcal{F}^2_\alpha, P^2_\alpha)$. Thus calling elements in $S^{\mathbf{N}}$ realizations of matches is justified since they can be derived from a probability over matches. This probability space is the (P^2_α) unique probability space for which each individual expects to be matched with the population average. For this probability space the set of events 'individual i is matched with individual j' is not measurable. If we let these events be measurable, as in the probability space $(\Sigma, \mathcal{F}^1_\alpha, P^1_\alpha)$, then these event will not be equally likely, although all individuals have the same probability of being matched with an individual that adopts strategy s_v (for all $s_v \in S$).

We call $S^{\mathbf{N}}$ the set of realizations of the matching process. Let $x \in S^{\mathbf{N}}$. If $x(i) = s_r$ then individual i is matched with an individual of type s_r. The description of the relationship between Σ and $S^{\mathbf{N}}$ and the derivation of the probability space over Σ are the objects of this section.

We consider each element in $S^{\mathbf{N}}$ as the realization of an infinite sequence of *i.i.d.* random variables where the probability that $x(i)$ equals s_v is p_v. If $m = 6$ and $p_v = 1/6$ (for $v = 1, \ldots, 6$) we can think of x as the outcome from rolling a dice infinitely many times. Let $(S^{\mathbf{N}}, \mathcal{B}, \mu)$ be the probability space we just described where $\mathcal{B}$ is the σ-algebra constructed by the finite dimensional rectangles, and μ is the extension of the probability over the finite dimensional rectangles.[5] For all types s_r, let $I_r \subset \mathbf{N}$ be the subset of the population of type s_r; i.e.,

$$I_r(\alpha) = \{i \in \mathbf{N} \mid \alpha(i) = s_r\}.$$

Notice that each set $I_r(\alpha)$ is infinite. Let $X_{rv}(\alpha) \subset S^{\mathbf{N}}$ be the set of realizations of the matching process such that the proportion of individuals that are of type r and are matched with individuals of type v is $p_r p_v$; i.e.,

$$X_{rv}(\alpha) = \{x \in S^{\mathbf{N}} \mid \lim_{T \to \infty} \frac{1}{T} \sum_{i=1}^{T} \alpha_r(i) x_v(i) = p_r p_v\}.$$

Then by the strong law of large numbers, $\mu(X_{rv}(\alpha)) = 1$.[6] Let X_α be the set of realizations of the matching process such that for all $s_r, s_v \in S$, the proportion of individuals that are of type r and are matched with individuals of type v is $p_r p_v$; i.e., $X_\alpha = \bigcap_r \bigcap_v X_{rv}(\alpha)$. Notice that since X_{rv} is the finite intersection of sets of measure one, $\mu(X_\alpha) = 1$.

For all $x \in X_\alpha$ and for all types s_r, s_v, let A_{rv}^x be the set of players of type s_r that are matched with an individual of type s_v; i.e.,

$$A_{rv}^x \equiv \{i \in \mathbf{N} \mid \alpha(i) = s_r, x(i) = s_v\}.$$

---

[5] Formally, a finite dimensional rectangle is a set of the form

$$B = \{x \in X \mid x_i \in B_i \text{ for all } i \text{ in } J\}$$

where J is a finite subset of $\mathbf{N}$ and for all i in J, B_i is a subset of S. Let

$$\mu_J(B) = \prod_{i \in J} (\sum_{s \in B_i} p_s).$$

Let $\mathcal{B}$ be the σ-algebra generated by the finite dimensional rectangles. By Neveu (1965), Proposition V.1.2 there exists a probability measure over $(S^{\mathbf{N}}, \mathcal{B})$ such that for all J-dimensional rectangles, B, $\mu(B) = \mu_J(B)$.

[6] Let $X_i = \alpha_r(i) x_v(i)$. Then, $\{X_i\}$ is a sequence of independent random variables with finite variance, σ_i. Furthermore, $\sum_{i=1}^{\infty} \frac{\sigma_i}{i^2} < \infty$, and thus by the Kolmogorov's law of large numbers (see for instance Rao (1984) Theorem 6, page 60), $\lim_{T \to \infty} \frac{1}{T} \sum_{i=1}^{T} X_i = p_r p_v$ (almost surely).

Notice that each of the sets A^x_{rv} has countably many elements and thus can be enumerated as follows:

$$A^x_{rv} = \{a^x_{rv}(1), a^x_{rv}(2), \ldots\}.$$

For any two different types s_r and s_v let

$$\begin{aligned} \sigma^x_\alpha(a^x_{rr}(i)) &= \begin{cases} a^x_{rr}(i+1) & \text{if } i \text{ is odd,} \\ a^x_{rr}(i-1) & \text{if } i \text{ is even;} \end{cases} \\ \sigma^x_\alpha(a^x_{rv}(i)) &= a^x_{vr}(i). \end{aligned}$$

Clearly $\sigma^x_\alpha \in \Sigma$.

Let $\sigma_\alpha : X_\alpha \rightarrow \Sigma$ be such that $\sigma_\alpha(x) = \sigma^x_\alpha$. The function σ_α is injective since $\alpha \circ \sigma^x_\alpha = x$.

Next we will construct two different probability spaces for matches. In the first probability space the events "individual i is matched with individual j" and "individual i is matched with individual k" are not equally likely. The second probability space is the coarsest measure for which each subpopulation $I_r(\alpha)$ is matched with the population average. Notice that for this probability measure the event "individual i is matched with individual j" are not measurable.[7]

Since σ_α is injective, by identifying x and σ^x_α we can construct a probability over $\Sigma_\alpha = \sigma_\alpha(X_\alpha)$. Formally, let $\mathcal{F}_\alpha$ be the σ-algebra generated by $\sigma_\alpha(\mathcal{B} \bigcap X_\alpha)$ and let $P_\alpha = \mu \circ \sigma_\alpha^{-1}$. Then, $(\Sigma_\alpha, \mathcal{F}_\alpha, P_\alpha)$ is a probability measure.

We can extend this probability over all Σ by letting $\mathcal{F}^1_\alpha$ be the σ-algebra generated by $\mathcal{F}_\alpha$ and $\Sigma \setminus \Sigma_\alpha$ and letting $P^1_\alpha(A) = P_\alpha(A \bigcap \Sigma_\alpha)$.

Theorem 1 $(\Sigma, \mathcal{F}^1_\alpha, P^1_\alpha)$ *is a probability measure for which the event* $\{\sigma \in \Sigma \,|\, \sigma(i) = j\}$ *is measurable.*

---

[7] In Section 3 we showed that for a probability space over Σ either the events "individual i is matched with individual j" are not measurable or they are not equally likely.

Proof: Let B_i^n be the event "i is the n^{th} player selecting strategy $\alpha(i)$ and matched with an individual adopting strategy $\alpha(j)$." Let
$N_i = \{k \leq i | \, \alpha(k) = \alpha(i)\}$. Then

$$B_i^n = \bigcup_{\{N' \subset N_i \,|\, |N'|=n\}} \{x \in S^{\mathbf{N}} | \quad x(i) = \alpha(j), \forall k' \in N', \, x(k') = \alpha(j), \text{ and } \forall k'' \in N_i \setminus N', x(k'') \neq \alpha(j) \quad \}.$$

Thus B_i^n is the finite union of rectangles and is thus measurable. Similarly, let B_j^n be the event "j is the n^{th} player selecting strategy $\alpha(j)$ and matched with an individual adopting strategy $\alpha(i)$." Clearly, B_j^n is measurable. Let $B^n = B_i^n \bigcap B_j^n$ and let $B = \bigcup_{n \leq \min\{i,j\}} B_n$. Clearly, B_n is measurable and $x \in B$ if and only if $\sigma_\alpha(x)(i) = j$. Therefore,

$$\sigma_\alpha(B) = \{\sigma \in \Sigma | \, \sigma(i) = j\} \in \mathcal{F}_\alpha^1,$$

and thus the event that individual i is matched with individual j is measurable. ■

Let $\phi_\alpha \colon \Sigma \rightarrow S^{\mathbf{N}}$ be defined by $\phi_\alpha(\sigma) = \alpha \circ \sigma$ and let $Y_\alpha = \phi_\alpha(\Sigma)$. Let $(S^{\mathbf{N}}, \mathcal{B}', \mu')$ be the completion of the measure space $(S^{\mathbf{N}}, \mathcal{B}, \mu)$.

Lemma 1 $X_\alpha \subset Y_\alpha$. *Consequently,* $\mu'(Y_\alpha) = 1$.

Proof: Let $x \in X(\alpha)$. Then $\sigma_\alpha(x) \in \Sigma$ and $\phi_\alpha(\sigma_\alpha(x)) = x$. Therefore, $x \in Y(\alpha)$. Consequently, $\mu'(Y_\alpha) = 1$ since $X_\alpha \subset Y_\alpha$ and $\mu'(X_\alpha) = 1$. ■

The probability measure μ' is restricted over Y_α by setting:

$$\mathcal{B}_Y = \mathcal{B}' \bigcap Y_\alpha, \; \mu_Y(A) = \mu'(A \bigcap Y_\alpha).$$

Let ${\mathcal{F}_\alpha}^2$ be the σ-algebra on Σ generated by $\phi^{-1}(\mathcal{B}_Y)$ and let $P_\alpha^2 = \mu_Y \circ \phi_\alpha$.

Theorem 2 $(\Sigma, \mathcal{F}^2_\alpha, P^2_\alpha)$ *is a probability measure.*

Proof: Follows from the previous lemma, the fact that $\mu(X_\alpha) = 1$, and Proposition 2.12 (page 21) in Breiman (1968). ∎

For either probability measure we have thus proven the following theorem.

Theorem 3 *Let $I_r(\alpha)$ be the subset of the population of type s_r. Suppose that the proportion of the population of type s_v is p_v. Suppose that people are matched at random according to the matching rule $(\Sigma, \mathcal{F}_\alpha, P_\alpha)$. Then the proportion of the population $I_r(\alpha)$ that is matched with an individual of type s_v is p_v (almost surely).*

2.5 Extension of the probability of the realizations of matches

In the previous section we defined a probability over the set of realizations of the matching process. In order to do that we had to assume that the assignment of types, α, was such that the Cesaro average converges; i.e., $\alpha \in A$ where

$$A = \{\alpha \in S^{\mathbf{N}} | \lim_{T \to \infty} \frac{1}{T} \sum_{i=1}^{T} \alpha(i) \text{ exists and is strictly positive } \}.$$

In this section we dispense with this assumption; i.e., we define a probability over the realization of matches for all $\alpha \in S^{\mathbf{N}}$. This is done by means of a measurable extension of the averaging which we denote by g.

In order to prove the existence of a measurable extension we need both a topological and a measurable structure for $S^{\mathbf{N}}$. The space $S^{\mathbf{N}}$ is endowed with the product topology; this makes $S^{\mathbf{N}}$ a complete, separable, metric space[8] and $(S^{\mathbf{N}}, \mathcal{B})$ a measurable space where $\mathcal{B}$ is the Borel σ-algebra.

Let $G: S^{\mathbf{N}} \twoheadrightarrow \mathbf{R}^m$ be such that for all $\alpha \in S^{\mathbf{N}}$, $G(\alpha)$ is the lim sup of α; i.e.,

$$G(\alpha) = \{a \in \mathbf{R}^m \,|\, \exists \ \{T_n\}_n \text{ such that } \lim_{n\to\infty} \frac{1}{T_n} \sum_{i=1}^{T_n} \alpha(i) = a\}.$$

Notice that since $\{\frac{1}{T} \sum_{i=1}^{T} \alpha(i)\}_{T\in \mathbf{N}}$ is an infinite sequence belonging to the m-dimensional simplex it has a convergent subsequence and thus $G(\alpha)$ is nonempty.

Lemma 2 *The correspondence G is closed-valued and measurable.*

Proof: Fix $\alpha \in S^{\mathbf{N}}$, let $\{a^m\}_m$ be such that $a^m \in G(\alpha)$ and $a^m \to a$. For all $n \in \mathbf{N}$, let $\{T_n^m\}_n$ be such that

$$\lim_{n\to\infty} \frac{1}{T_n^m} \sum_{i=1}^{T_n^m} \alpha(i) = a^m.$$

For each $m \in \mathcal{N}$, let $n(m)$ be such that

$$\Big|\Big|\frac{1}{T_{n(m)}^m} \sum_{i=1}^{T_{n(m)}^m} \alpha(i) - a^m\Big|\Big| < 1/m \quad \text{and} \quad n(m) > n(m-1).$$

Then it is each to check that $\lim_{m\to\infty} \frac{1}{T_{n(m)}^m} \sum_{i=1}^{T_{n(m)}^m} \alpha(i) = a$. Therefore, $a \in G(\alpha)$ and $G(\alpha)$ is closed.

Let $F \subset \mathbf{R}^m$ be closed. Then,

$$\begin{aligned} G^{-1}(F) &= \{\alpha \in S^{\mathbf{N}} \,|\, G(\alpha) \bigcap F \neq \emptyset\} \\ &= \{\alpha \in S^{\mathbf{N}} \,|\, \exists \{T_n\}_n \text{ such that } \lim_{n\to\infty} \frac{1}{T_n} \sum_{i=1}^{T_n} \alpha(i) \in F\} \\ &= \bigcap_{n\in\mathbf{N}} \bigcap_{\tau\in\mathbf{N}} \bigcup_{T>\tau} F_{T,n} \end{aligned}$$

---

[8] One possible metric is d, where for all $\alpha, \beta \in S^{\mathbf{N}}$,

$$d(\alpha, \beta) = \sum_{i\in\mathbf{N}} \frac{|\alpha(i) - \beta(i)|}{2^i(1 + |\alpha(i) - \beta(i)|)}.$$

where

$$\begin{aligned} F_{T,n} &= \{\alpha \in S^{\mathbf{N}} \mid \frac{1}{T}\sum_{i=1}^{T} \alpha(i) \in B(F, \frac{1}{n})\} \text{ and} \\ B(F, \frac{1}{n}) &= \{a \in \mathbf{R}^m \mid \exists b \in F \text{ such that } ||a-b|| < \frac{1}{n}\}. \end{aligned}$$

Since for all T and n, $F_{T,n} \in \mathcal{B}$, $G^{-1}(F) \in \mathcal{B}$ and G is measurable. ∎

Lemma 3 *The correspondence G has a measurable selection g.*

Proof: The Kuratowski–Ryll–Nordziewski Theorem (see for instance Theorem 14.2.1 in Klein and Thompson (1984)) states that any closed-valued $\mathcal{B}$-measurable correspondence into a complete separable metric space has a $\mathcal{B}$-measurable selection. Thus the result follows from Lemma 2. ∎

Notice that for all $\alpha \in A$, $g(\alpha) = \lim_{T\to\infty} \frac{1}{T}\sum_{i=1}^{T} \alpha(i)$ and thus g is the extension we are looking for.

Let B be a finite measurable rectangle in $S^{\mathbf{N}}$; i.e.,

$$B = \{x \in S^{\mathbf{N}} \mid x(i) \in B_i \text{ for all } i \text{ in } J\},$$

where $J \subset \mathbf{N}$ is a finite set and for all i in J, $B_i \subset S$. Then let $\mu(\alpha, B)$ be the probability that if the population selects strategies according to α, then for all $i \in J$, individual i is matched with an individual of type B_i; i.e.,

$$\mu(\alpha, B) = \prod_{i\in J}(\sum_{s\in B_i} g_s(\alpha)).$$

The function $\mu(\alpha, \cdot)$ can clearly be extended so that $(S^{\mathbf{N}}, \mathcal{B}, \mu(\alpha,\cdot))$ is a probability measure.

Lemma 4 *The function μ is a stochastic kernel.*

Proof: In order to check the measurability of $\mu(\cdot, B)$ it is sufficient to consider the case where B is a finite dimensional rectangle (Neveu (1965), page 75). For this case, $\mu(\cdot, B)$ is clearly a continuous function of $g(\alpha)$. Therefore, $\mu(\cdot, B)$ is a measurable function. ∎

2.6 Repeated matching scheme

In this section we define the notation to describe a population which is matched infinitely many times, and in which after each period individuals' type may change. We extend the results of Section 2.4 for this context.

Let $\tau: S \times S \rightarrow S$ be such that if at time t an individual of type s_v is matched with an individual of type s_w then at time $t+1$ the individual is of type $\tau(s_v, s_w)$.[9][10] In order to guarantee that no type disappears immediately we assume that for all types s_r there exist types s_v and s_w (where r, v, w can be equal) such that $t(s_v, s_w) = s_r \in S$. Let $t: S^{\mathbf{N}} \times S^{\mathbf{N}} \rightarrow A$ be such that for all $i \in \mathbf{N}$, $x, \alpha \in S^{\mathbf{N}}$,

$$t(x, \alpha)(i) \equiv \tau(\alpha(i), x(i)).^{11}$$

If at time t individual i is of type $\alpha(i)$ and is matched with an individual of type $x(i)$ then, at time $t+1$, individual i's type is $t(\alpha, x)(i)$. For all $x \in S^{\mathbf{N}}$

---

[9] This law of motion includes the learning models and the evolutionary models with constant populations.

[10] A more general model would allow for a stochastic law of motion; i.e.,

$$\tau: S \times S \times [0,1] \rightarrow S$$

where τ is measurable and $(S, \mathcal{P}(S))$ and $([0,1], \mathcal{B}([0,1]))$ are measurable spaces (where $\mathcal{P}(S)$ is the power set and $\mathcal{B}(S)$ is the Borel σ-algebra). The map becomes stochastic after we define a probability measure over $([0,1], \mathcal{B}([0,1]))$.

[11] If τ is stochastic then $t: S^{\mathbf{N}} \times S^{\mathbf{N}} \times [0,1]^{\mathbf{N}} \rightarrow A$ is such that for all $i \in \mathbf{N}$, $x, \alpha \in S^{\mathbf{N}}$, $\xi \in [0,1]^{\mathbf{N}}$,

$$t(x, \alpha, \xi)(i) \equiv \tau(\alpha(i), x(i), \xi_i).$$

and for all $\alpha \in S^{\mathbf{N}}$, the functions $t\colon S^{\mathbf{N}} \times \{x\} \to S^{\mathbf{N}}$ and $t\colon \{\alpha\} \times S^{\mathbf{N}} \to S^{\mathbf{N}}$ are continuous[12] (and thus measurable) and hence jointly measurable.[13]

Let $Z = S^{\mathbf{N}} \times S^{\mathbf{N}}$ and let $\mathcal{C} = \mathcal{B} \otimes \mathcal{B}$. Let $Q\colon Z \times \mathcal{C} \to [0,1]$ be such that for all $(\alpha, x) \in Z$ and all $(B, B') \in \mathcal{C}$,

$$Q\Big((\alpha,x),(B,B')\Big) = \chi_B\Big(t(\alpha,x)\Big)\,\mu\Big(t(\alpha,x),B'\Big).^{14}$$

$Q\Big((\alpha,x),(\alpha',x')\Big)$ is the probability that if at time t the population has types assigned by α and is matched according to x, then at time $t+1$ the population has types assigned by α' and is matched according to x'.

Lemma 5 *The function Q is a transition probability, i.e., for each $C \in \mathcal{C}$, $Q(\cdot, C)$ is measurable and for each $z \in Z$, $Q(z,\cdot)$ is a probability measure.*

Proof: Fix $z \in Z$. Then $Q(z,\cdot)$ is the product of two probability measures and is thus a probability. Fix $C = (B, B') \in \mathcal{C}$. Notice that $Q(\cdot, C)$ is the product of two measurable functions of t and that t is a measurable function of z. Therefore, $Q(\cdot, C)$ is a measurable function. ∎

The Ionescu Tulcea theorem (see for instance Neveu (1965), Proposition V.1.1) states that if $\{(E_t, \mathcal{F}_t)\}$ is an infinite sequence of measurable spaces

---

[12] Suppose $\alpha^n \to \alpha$ and let $m > 0$. Then there exists an N such that for all $n \geq N$ and for all $i \leq m$, $\alpha^n(i) = \alpha(i)$. Thus for all $n \geq N$ and for all $i \leq m$, $t(\alpha^n, x)(i) = t(\alpha, x)(i)$. Therefore, $t(\alpha^n, x) \to t(\alpha, x)$ and $t(\cdot, x)$ is continuous. The same proof shows that $t(\alpha, \cdot)$ is continuous.

[13] The proof that the continuity of each section implies joint measurability is in Appendix B.

[14] For the case where τ is stochastic let λ be the Lebesgue measure over $[0,1]^{\mathbf{N}}$. For all $\alpha, x \in S^{\mathbf{N}}$, let $\rho_{\alpha,x}$ be the probability measure on $(S^{\mathbf{N}}, \mathcal{B})$ defined by

$$\rho_{\alpha,x}(B) = \lambda\Big(\{\xi \in [0,1]^{\mathbf{N}} | t(\alpha, x, \xi) \in B\}\Big).$$

Then,

$$Q\Big((\alpha,x),(B,B')\Big) = \int_B \mu(t, B')\,d\rho_{\alpha,x}(t).$$

and if $P^{0\cdots t}_{t+1}$ is a transition probability defined with respect to the spaces $(\times^t_{s=0}E_s, \otimes^t_{s=0}\mathcal{F}_s)$ and $(E_{t+1}, \mathcal{F}_{t+1})$, then there exists a unique probability, P_{x_0}, on

$$(\Omega, \mathcal{A}) = \times_t(E_t, \mathcal{F}_t)$$

whose value for every measurable rectangle $\times^T_{t=1}F_t \times^\infty_{s=T+1} E_s$ is given by

$$\begin{aligned} & P_{x_0}[\times^T_{t=1}F_t] \\ = \; & 1_{F_0}(x_0)\int_{F_1} P^0_1(x_0; dx_1)\int_{F_2} P^{01}_2(x_0, x_1; dx_2)\cdots\int_{F_T} P^{0\cdots T-1}_T(x_0\cdots x_{T-1}; dx_T). \end{aligned}$$

Let $\bar{Z} = \times_{n\in\mathbf{N}}Z$ and $\bar{\mathcal{C}} = \otimes_{n\in\mathbf{N}}\mathcal{C}$. Then the Ionescu-Tulcea theorem in conjunction with Lemma 5 gives the following result.

Theorem 4 *There is a unique probability $\bar{Q}_{z_0}$ over $(\bar{Z},\bar{\mathcal{C}})$ such that for every finite dimensional rectangle, $C_1 \times \cdots \times C_J \times^\infty_{n=J+1} S^{\mathbf{N}}$,*

$$\bar{Q}_{z_0}[C_1 \times \cdots \times C_J \times^\infty_{n=J+1} S^{\mathbf{N}}] = \int_{C_1} Q(z_0, z_1; dz_1)\cdots\int_{C_J} Q(z_{J-1}, z_J; dz_J).$$

For each α let X_α be the set of realizations of the matching rule such that each subpopulation is matched with the population average. By the results in Section 4, for all $\alpha \in A$, $\mu(\alpha, X_\alpha) = 1$.

Lemma 6 *The correspondence $X\colon A \twoheadrightarrow S^{\mathbf{N}}$ is measurable and closed valued.*

Proof: The proof is exactly the same as the proof of Lemma 2. ∎

Lemma 7 *The graph of X is measurable.*

Proof: Klein and Thompson (1984) prove (Proposition 13.2.2 and Proposition 13.2.4) that the graph of a closed measurable function is measurable. Thus this result follows directly from Lemma 6. ∎

Theorem 5 *Suppose that the initial population types are described by $\alpha_0 \in A$ and is matched according to $x_0 \in X_{\alpha_0}$. Then at every period each subpopulation is matched with the population average (almost surely).*

Proof: Let $C = \text{graph}\, X$ and let $z \in C$. Clearly, $\chi_{t(z)}(t(z)) = 1$ and by the results in Section 4, $\mu\big(t(z), X_{t(z)}\big) = 1$. Then since $t(z) \in A$ we get that $(t(z), X_{t(z)}) \in \text{graph}\, X$ and thus,

$$Q(z, C) \geq Q\big(z, (t(z), X_{t(z)})\big) = \chi_{t(z)}(t(z))\, \mu\big(t(z), X_{t(z)}\big) = 1.^{15}$$

Since for all $z \in C$, $Q(z, C) = 1$, then for all $J \in \mathbf{N}$,

$$\bar{Q}_{z_0}[C \times \cdots \times C \times_{n=J+1}^{\infty} Z] = \int_C Q(z_0, z_1; dz_1) \cdots \int_C Q(z_{J-1}, z_J; dz_J) = 1.$$

∎

Since in each period each subpopulation is matched with the population average, given an initial population, α, we can compute the distribution of types at any given period t, $g^t(\alpha)$. Specifically, by letting

$$g^1(\alpha) = \sum_v \sum_w g_v(\alpha) g_w(\alpha) t(s_v, s_w),$$

and we define recursively $g^t(\alpha)$ by

$$g^t(\alpha) = g^{t-1}(g^1(\alpha)).$$

Next we want to show that the probability $\bar{Q}_{z_0}$ is generated by some probability measure over the set of matches.

Let $\phi \colon \Sigma^{\mathbf{N}} \times A^{\mathbf{N}} \to \bar{Z}$ be such that

$$\phi(\{\sigma_i\}, \{\alpha_i\}) = \{(\alpha_i \circ \sigma_i, \alpha_i)\}.$$

---

[15] In order to prove that $Q(z, C) = 1$ we had to show that C was measurable. An alternative way to proving the same results is to show that the set C is *thick* (i.e., $A \in \mathcal{C}, A \bigcap C = \emptyset \Rightarrow P(A) = 0$) and thus $\tilde{Q}(x, C) = 1$ where $\tilde{Q}$ is the extension of Q over the trace σ-algebra $\mathcal{C}(C)$ such that $\tilde{Q}(A \bigcap C) = Q(A)$ (see Rao (1981), page 15, Theorem 5).

Let $Y = \phi(\Sigma^{\mathbf{N}} \times A^{\mathbf{N}})$ and let $(\bar{Z}, \bar{\mathcal{C}}', Q')$ be the completion of the measure $(\bar{Z}, \bar{\mathcal{C}}', \bar{Q}_{z_0})$. Using the same arguments as in Lemma 1 we can show that graph $X \subset Y$ and thus $Q'(Y) = 1$. The probability Q' is restricted over Y by setting

$$\mathcal{C}_Y = \mathcal{C} \bigcap Y \quad \text{and} \quad Q_Y(C) = Q'(C \bigcap Y).$$

Let $\mathcal{F}$ be the σ-algebra generated by $\phi^{-1}(\mathcal{C}_Y)$ and let $P = Q_Y \circ \phi$. Using the same argument as in Section 4 we get the following result.

Theorem 6 *$(\Sigma^{\mathbf{N}} \times A^{\mathbf{N}}, \mathcal{F}, P)$ is a probability measure.*

In many applications we will not be interested in the actual matches but only in the evolution of the population. Let $\nu: S^{\mathbf{N}} \times \mathcal{B} \to [0,1]$ be such that for all $\alpha \in S^{\mathbf{N}}$ and $B \in \mathcal{B}$,

$$\nu(\alpha, B) = \mu\Big(\alpha, t_\alpha^{-1}(B)\Big).$$

For all $\alpha, \alpha' \in S^{\mathbf{N}}$, $\nu(\alpha, \alpha')$ is the probability that if at time t the population's types are described by α, then at time $t+1$ the population's types are assigned by α'. In Theorem 6.2, Futia (1982) proves the following result.

Lemma 8 *The function ν is a transition probability; i.e., for each $B \in \mathcal{B}$, $\nu(\cdot, B)$ is measurable and for each $\alpha \in S^{\mathbf{N}}$, $\nu(\alpha, \cdot)$ is a probability.*

Let $\bar{S} = \times_{n\in\mathbf{N}} S^{\mathbf{N}}$ and $\bar{\mathcal{B}} = \bigotimes_{n\in\mathbf{N}} \mathcal{B}$. Then, the Ionescu Tulcea theorem gives the following result.

Theorem 7 *There is a unique probability $\bar{\nu}_{\alpha_0}$ on $(\bar{S}, \bar{\mathcal{B}})$ such that for every finite dimensional rectangle, $B_1 \times \cdots \times B_J \times_{n=J+1}^{\infty} S^{\mathbf{N}}$,*

$$\bar{\nu}_{\alpha_0}[B_1 \times \cdots \times B_J \times_{n=J+1}^{\infty} S^{\mathbf{N}}] = \int_{B_1} \nu(\alpha_0, \alpha_1; d\alpha_1) \cdots \int_{B_J} \nu(\alpha_{J-1}, \alpha_J; d\alpha_J).$$

Proposition 1 *If $\alpha_0 \in A$ then $\nu_{\alpha_0}[A^{\mathbf{N}}] = 1$.*

Proof: Follows since we showed in Section 4 that $\nu(\alpha_0, A) = 1$. ∎

2.7 The law of motion for finite populations and finite number of periods

A justification of the analysis of the deterministic model for finite population is that we are just interested in following the law of motion for a finite number of periods, T, and that for a large enough population the model is approximately correct. This section proves this conjecture.

Let p_v^t be the proportion of individuals of type s_v at time t. The initial proportion p^0 is given while the other proportions are computed by assuming that each type is matched with the population average. For any type s_r let $\{x_r^i\}$ be a sequence of random variables such that (for all i and v) the probability that $x_r^i = s_v$ is p_v and let μ be the probability defined over the whole sequence of $\{x_r^i\}$ (as discussed in section 2.4). If the population size is N, the population is denoted by $P(N)$, where $P(N) = \{1, \ldots, N\}$. For any period $t = 1, \ldots T$, let $\alpha^t \colon P(N) \to S$ denote the assignment of strategies in the population. The initial population α^0 is given while the populations in the other periods are obtained by the law of motion τ (which is described in Section 2.6) and the matching rule. Individual are matched as in section 2.3; i.e., the individuals of type s_r that draw a random variable with value s_v are matched with the individuals of type s_v that draw a random variable with value s_r. Since these groups are finite we will not always be able to match all individuals this way. Thus the remaining individuals are matched in some arbitrary way.

Theorem 8 *For any $T > 0$, $\epsilon > 0$ and for any $\delta > 0$ there exists a positive integer N' such that for all population sizes $N > N'$ and all strategies s_v the following holds: with probability greater than $1 - \delta$ the proportion of the* *population* $P(N)$ *adopting strategy* s_v *at time* T, $\frac{1}{N}\sum_{i=1}^{N}\alpha_v^T(i)$, *is within* ϵ *of* p_v^T.

Proof: The idea behind the proof is the following. We find a uniform upper bound (uniform over time period and strategy) for the proportion of the population adopting a particular strategy in a given time period which is not matched according to the population average (with probability $1-q$). The upper bound and q can be made arbitrarily small by taking the population to be large enough. Then for almost all possible histories we can find a subpopulation which behaves exactly according to the deterministic model. Again the proportion of the population in this subpopulation can be made arbitrarily close to one by taking the population to be large enough.

Fix $T>0$, $\epsilon>0$, and $\delta>0$. Let $\xi>0$ be such that $(1-\frac{\epsilon}{\xi})^{2^T}>1-\epsilon$, let $\underline{p}=\min_{v,t\leq T}p_v^t$, and let N be greater than N' where

$$N'=\frac{T\xi^2}{4\delta\epsilon^2\underline{p}^3(1-\frac{\epsilon}{\xi})^T}\,.$$

Let e^v be the vector with 1 on the v^{th} component and zeros on the other components. Then $\{x_r^i\cdot e^v\}_i$ is a sequence of Bernoulli random variables and thus if n is greater than n', where

$$n'=\frac{T\xi^2}{4\delta\epsilon^2\underline{p}^2}\,,$$

then

$$\mu(|\frac{1}{n}\sum_i x_r^i\cdot e^v-p_v|\geq\frac{\epsilon\underline{p}}{\xi})\leq\frac{\delta}{T}\,.^{16}$$

---

[16] Bernoulli's weak law of large numbers is proved (see Shiryayev (1984), page 47), by showing that if $\{x_i\}$ is a sequence of Bernoulli randoms variables with $E(x_i)=p$ then

$$P\{|\frac{1}{n}\sum_{i=1}^{n}x_i-p|\geq\epsilon\}\leq\frac{1}{4n\epsilon^2}\,.$$

Suppose that the number of individuals adopting any strategy s_r is greater than n and the distribution of strategies in the population is given by p. Then, with probability greater than $1-\frac{\delta}{T}$,

$$\max_{r,v} \frac{|p_r p_v - \frac{1}{|\{i\in P(N)|\alpha(i)=r\}|} \sum_{\{i\in P(N)|\alpha(i)=r\}} x_r^i \cdot e^v|}{p_r p_v} < \frac{\epsilon}{\xi}.$$

Consequently, with probability greater than $1-\delta/T$ the proportion of individuals matched according to the correct proportions is $1-\frac{\epsilon}{\xi}$. But, with probability greater than $1-\frac{\delta}{T}$, at each period there are at least $N\underline{p}(1-\frac{\epsilon}{\xi})^T > n'$ individuals adopting any strategy s_v and thus a proportion $1-\frac{\epsilon}{\xi}$ of the population is matched according to the population proportion. Hence, with probability greater than $1-\delta$ at every period $t = 1, \ldots, T-1$, there is a $(1-\frac{\epsilon}{\xi})$ proportion of the population which is matched according to the population proportions. Define recursively $P_1 = \{$ individuals matched with the population average $\}$, $P_t = \{$ individuals matched with subpopulation P_{t-1} average $\}$. Hence with probability $1-\delta$, $\frac{|P_T|}{N} \geq (1-\frac{\epsilon}{\xi})^{2^T}$. Thus with probability $1-\delta$ the population will be within $1-(1-\frac{\epsilon}{\xi})^{2^T} < \epsilon$ of p^T. ∎

2.8 Conclusion

Biologists and economists have analyzed populations where each individual interacts with randomly selected individuals. The random matching generates a very complicated stochastic system. Consequently biologists have approximated such a system by a deterministic system. The justification of such an approximation is that the population is assumed to be very large and thus some law of large numbers must hold. In the paper we give an example for which this assumption does not hold. This does not mean that this kind of approximation may never hold, but that the correctness of the approximation depends on properties of the law of motion. This paper shows that if we assume that the population is infinite then the stochastic and the deterministic system are the same.

Note that all the proofs hold if the set of type, S, is countably infinite. In this case admissible proportions, p, must be such that: $p > 0$, $\sum_{s \in S} p_s = 1$, and $\lim_{s \to \infty} p_s = 0$. However, we are not aware of any study of such models.

Finally, the matching technologies defined in the paper depend on the particular distribution of strategies in the population. It remains an open question whether matching schemes exist that are independent of the distribution of strategies and for which individuals are matched with individuals that adopt strategies in the same proportion as the population as a whole.

Appendix A: Populations with a continuum of agents

This section follows Feldman and Gilles (1985). Suppose that the population is represented by the unit interval, $I = (0,1]$. Let $\mathcal{I}$ denote the Lebesgue measurable sets of I and let λ be the Lebesgue measure on I. The fact that the population selects strategies according to x is represented by partitioning I as follows:

$$I = (0, x_1] \bigcup (x_1, x_1 + x_2] \bigcup \ldots \bigcup (1 - x_m, 1].$$

Let $\mathcal{B}$ be the Borel sets of I. Let $(\Omega, \mathcal{F}, P)$ be a probability space determined by the randomized matching process. Let $\{X(\cdot, i)\}_{i \in I}$ be a family of random variables, where for all $i \in I$, $X(\cdot, i) \colon \Omega \to \{0,1\}$ is such that

$$p(X(\cdot, i) = 1) \equiv P(\{\omega \colon X(\omega, i) = 1\}) = x_r.$$

The event $X(\omega, i) = 1$ denotes that individual i is matched with an individual who selects strategy r; $X(\omega, i) = 0$ denotes that individual i is matched with an individual who does not select strategy r. Then in order for the conjecture given in the introduction to be correct the following needs to be true: for every $B \in \mathcal{B}$

$$\int_B X(\omega, i) \lambda(di) = x_r \sigma(B). \tag{2.3}$$

For any sample $\omega \in \Omega$, define the set function $v_\omega \colon \mathcal{B} \to \Re_+$ by

$$v_\omega(B) = \int_B X(\omega, i) \lambda(di).$$

Since v_ω is absolutely continuous with respect to λ, $\lambda \gg v_\omega$, by the Radon-Nikodyn theorem there exists a unique measurable function f such that

$$v_\omega(B) = \int_B f(a) \lambda(da). \tag{2.4}$$

But if equation 2.3 holds then by setting $f \equiv x_r$ equation 2.4 is satisfied. Therefore since f is unique $X(\omega, \cdot) = x_r$ (λ) a.e.. But since $x_r \notin \{0,1\}$ this is not an admissible value for X. Thus 2.3 cannot hold for all possible strategy partitions over the player set.

Green (1989) has shown that if the distribution of strategies in the population is described by a different partitioning, then the conjecture may hold.

It still remains to be shown that there exists a probability space $(\Omega, \mathcal{F}, P)$ and a sequence of random variables $\{X(\cdot, i)\}$ which correspond to a random matching process such that the conjecture holds.

Finally, modeling a very large population as a countable set seems a better approximation than modeling it as a continuum.

Appendix B: Joint measurability of a function with continuous sections

Theorem 9 *Let $(X, \mathcal{B})$ be a measure space where X is a separable complete metric and $\mathcal{B}$ is the Borel σ-algebra. Let $f: X \times X \to X$ be such that for all $x \in X$, $f: \{x\} \times X \to X$ and $f: X \times \{x\} \to X$ are continuous. Then f is measurable.*

Proof: Since X is a separable metric space, the Borel σ-algebra $\mathcal{B}$ has a denumerable subfamily, $\mathcal{D}$ generating $\mathcal{B}$ (see for instance Parthasarathy (1967) Theorem 1.8). Let

$$\mathcal{D} = \{D_1, \ldots, D_n, \ldots, \}.$$

Let

$$\begin{aligned} \mathcal{F}^n &= \{F_1 \bigcap F_2 \bigcap \cdots \bigcap F_n | \text{ where } F_i = D_i \text{ or } F_i = X \setminus D_i\} \\ &\equiv \{F_1^n, \ldots, F_{m(n)}^n\}. \end{aligned}$$

Notice that $\mathcal{F}_n$ is a partition of X and that $\mathcal{F}_n \subset \mathcal{B}$. For all i and n choose y_i^n such that $y_i^n \in F_i^n$. Finally, let

$$f_n(x, y) = \sum_{i=1}^{m(n)} f(x, y_i^n) \chi_{X \times F_i^n}.$$

Notice that the continuity of $f: X \times \{c\} \to X$ easily implies the continuity of the function $g: X \times X \to X$ where $g(x, y) = f(x, c)$. Thus $f_n(x, y)$ is a measurable function. Fix $y \in X$ and for all n let $i(n)$ be such that $y \in F_{i(n)}^n$. Notice that $F_{i(n)}^n \subset F_{i(n-1)}^{n-1}$ and that $F_{i(n)}^n \downarrow \{y\}$. Then,

$$\lim_{n \to \infty} f_n(x, y) = \lim_{n \to \infty} f(x, y_{i(n)}^n) = f(x, \lim_{n \to \infty} y_{i(n)}^n) = f(x, y).$$

The function f is hence the pointwise limit of a sequence of measurable functions and is thus measurable. ∎

Appendix C: Brief review of Biology

In this appendix we describe the relationship between the issues raised in this paper and models considered in population genetics and evolutionary biology.

Population Genetics

This section is based on Cavalli-Sforza and Bodmer (1971). In the models we consider, the phenotype (e.g., eye color) is determined by the action of two genes at one locus. Genes are assumed to be of two types (alleles): A_1 and A_2. Individuals are of three types (genotypes): A_1A_1, A_1A_2, and A_2A_2. When two individuals mate they each produce gametes (reproductive cells). Gametes receive one of the parent's genes. An offspring is produced by the union of a gamete from each parent. When two individuals of types aa' and bb' mate they produce offspring of type ab, ab', $a'b$, and $a'b'$ with equal probability. Another assumption describes which individuals mate. "**Matings take place at random with respect to the genetic differences being considered and in a population of infinite size**" (Cavalli-Sforza and Bodmer (1971), page 45). All individuals mate at the same time and then are completely replaced by their offspring. Thus the dynamics of the process depends on the random matching of individuals and the random selection of genotypes for the offspring. Suppose the initial relative frequency of alleles A_1 and A_2 is p_1 and p_2 ($p_1 + p_2 = 1$). Then the Hardy-Weinberg theorem states that in the next period the relative frequency of the genotypes A_1A_1, A_1A_2, A_2A_2 is respectively p_1^2, $2p_1p_2$, and p_2^2. This theorem is 'proved' (just as in the original papers by Hardy (1908) and Weinberg (1908)) by computing the expected proportion with which each of the matches occurs.

No explicit modeling of the matching scheme and no derivation of a law of large numbers is given. Furthermore there seems to be some confusion in the literature about whether the population needs to be infinite or very large. Feller (1967) (page 135) writes: "In a large population the actually observed frequencies of the three genotypes in the filial generation will be close to the theoretical probabilities." And in a footnote at the end of this sentence: "The statements is made precise by the law of large numbers and the central limit theorem, which permit us to estimate the effect of chance fluctuations." Hofbauer and Sigmund (1988) write: "A few more premises were used implicitly in the derivation. For instance we equated 'frequency' with 'probability.' This is admissible in the limiting case of very large populations" (page 9).

The model just described is generalized to the case where genotypes differ in fitness (the expected number of offspring). Denote by w_{ii} the fitness of the genotype A_{ii}. If the initial relative frequency of alleles A_1 and A_2 is p_1 and p_2 then the next period frequency of the genotypes A_1A_1, A_1A_2, A_2A_2 is respectively

$$\frac{w_{11}p_1^2}{d}, \frac{2w_{12}p_1p_2}{d}, \frac{w_{22}p_2^2}{d},$$

where $d = w_{11}p_1^2 + 2w_{12}p_1p_2 + w_{22}p_2^2$.[17] Then the frequency of alleles A_1 in the next period is

$$p_1' = \frac{w_{11}p_1^2 + w_{12}p_1p_2}{d}, \ p_2' = \frac{w_{22}p_2^2 + w_{12}p_1p_2}{d}.$$

This model easily generalizes to the case where there are m alleles, $A_1, \ldots, A_m$. The law of motion for the alleles is the same as equation 2.1 (where $w_{ij} = a_{ij}$). This model has also been analyzed when the matches occur infinitely often; e.g., Fisher (1930) and more recently Losert and Akin (1983) analyze the properties of the limiting distributions of genotypes; i.e., $\lim_{t\to\infty} p^t$.

---

[17] Notice that in genetics it is always assumed that $w_{ij} = w_{ji}$.

The model just described is closely related to the model considered in this chapter. The set of types in this model is the set of genotypes. For instance, in the case where there are only two alleles,

$$S = \{A_1A_1, A_1A_2, A_2A_2\}.$$

The law of motion τ in this model is stochastic; for instance

$$\tau(A_1A_1, A_1A_2) = \tfrac{1}{2}\,\delta_{A_1A_1} + \tfrac{1}{2}\,\delta_{A_1A_2}.$$

Thus in order to prove the Hardy-Weinberg theorem in this context we need to generalize the model for stochastic laws of motions.

The Hardy-Weinberg theorem and the law of motion in equation 2.1 can be derived from a different set of assumptions on the reproductive scheme. Individuals form a large number of gametes. A small proportion of the gametes formed unite with gametes of the opposite sex. Thus, the distribution of genotypes in the next generation is found by a sample of the gametes (and hence of the alleles) in the population. Specifically, if the proportion of alleles A_1 and A_2 is respectively p_1 and p_2, then the next generation is obtained by drawing $2N$ gametes where we assume that the probability of drawing a gamete of type A_1 is p_1 and the probability of drawing a gamete of type A_2 is p_2. Then the expected proportion of alleles of any type is given by equation 2.1 where p_i is the proportion of alleles of type A_i and a_{ij} is the number of offspring of a genotype A_iA_j. Again no law of large numbers has been derived for this system. However, geneticists realize that the deterministic element of the model is due to the assumption of large population.

The stochastic element created by small population is considered in the models of 'random genetic drift.' Suppose that there are N individuals and that the population size stays constant from generation to generation. If the population proportion of allele A_1 is p_1 then the next period population is formed by drawing $2N$ gametes from an urn that contains a proportion p_1 of gametes of type A_1. This model is approximately correct if the number of gametes is large enough (and thus we can consider the sampling procedure as sampling with replacement). The binomial distribution gives the gene frequency distribution after one period. Kimura (1964) derived the gene frequency distribution after n generations. At any generation the mean number of genes of type A_1 is p_1 although eventually one of the alleles is fixed in all populations. The proportion of populations in which A_1 is eventually fixed is p_1. The probability of fixation at generation t, $P(t, p_1)$, depends on the initial gene frequency, p_1, and the population size, N. Kimura (1964) showed that

$$P(t, p_1) \approx 1 - 6p_1(1 - p_1)e^{-\frac{t}{2}N}$$

and Ewens (1969) computed the mean fixation time to be

$$4N[p_1 \log p_1 (1 - p_1) \log(1 - p_1)].$$

Thus this model underscores the difference between the expected gene frequency and the real gene frequency.

Evolutionary Biology

This section follows Maynard Smith (1982), in particular pages 10–23. There is an infinite population of individuals identical in all respects except for how they behave in a two-person contest. In this contest there are only two possible strategies: to act as a hawk (denoted by H) and to act as a dove (denote by D). Each individual is paired off at random with one other individual. Individuals reproduce their kind asexually. The number of offspring for an individual that adopts strategy i and is matched with an individual that adopts strategy j ($i, j \in \{H, D\}$) is $W_0 + E(i, j)$.[18] If p is the frequency of individuals adopting strategy H then it is asserted that the number of offspring of individuals that select strategy i, $E(i)$, is

$$E(i) = W_0 + pE(i, H) + (1 - p)E(i, D)$$

and that the frequency p' of individuals selection H in the next period is

$$p' = p\frac{E(H)}{pE(h) + (1 - p)E(D)}.$$

This model can clearly be generalized to the case where there are m strategies, $s_1, \ldots, s_m$. The law of motion for this system is the one described in equation 2.1 where $a_{rv} \equiv W_0 + E(s_r, s_v)$. Properties of the limit dynamics are studied in Losert and Akin (1983), Akin and Losert (1984), Weising (1989). Again there seems to be some confusion as to whether the population is assumed to be infinite or very large as in Hines (1987).

Suppose that the population stays constant; i.e., $a_{rv} + a_{vr} = 2$. Then if we require the number of individuals of each type to be nonnegative integers, then for all strategies s_r and s_v, $a_{vr} \in \{0, 1, 2\}$. Thus the model described in Section 2.6 describes the replicator model where

$$\tau(s_v, s_w) = \begin{cases} s_v & \text{if } a_{vw} \neq 0 \\ s_w & \text{if } a_{vw} = 0. \end{cases}$$

The results in this chapter show that if the population is countably infinite then equation 2.1 describes the behavior of the process (almost surely).

---

[18] If the parent lives more than one period, the number of offspring includes the parent.

Chapter 3

EVOLUTIONARY EQUILIBRIA RESISTANT TO MUTATION

3.1 Introduction

In this chapter we describe the properties of the replicator model. The replicator model is studied by evolutionary biologists; it postulates that strategies are genetically determined and that individuals who adopt successful strategies have more offspring. Although individuals do not act as Bayesian maximizers, it is possible that the population as a whole acts as a Bayesian maximizer. This chapter characterizes the relationship between properties of the replicator model and refinements of Nash equilibrium. We first discuss how evolution relates to game theory.

Many games cannot be solved in pure strategies. The solution is then for one of the players to play a mixed strategy; in other words to select a strategy according to a specific randomization device. Rubinstein (1988) finds troublesome that optimal behavior should occur by chance. If such criticism is valid it seems useful to give a different interpretation of a game. Instead of imagining two specific players confronting the game, suppose that there is an infinite population of potential players. At each period, players are randomly and anonymously matched. Each player plays a pure strategy. Equilibria in mixed strategies are then interpreted as equilibria where the population is not homogeneous.[1]

One of the justifications for assuming that people use optimal strategies is the belief that such strategies are 'evolutionarily stable.' If (i) particular strategies are transmitted genetically and (ii) evolutionary laws select for optimal strategies, then people will act as optimizers. In most games there are no optimal strategies, since best responses depend on what other players choose. However, there may be modes of behavior that will persist and are immune to genetic drift. This paper examines a specific model for the study of genetic evolution which is called 'the replicator model.'[2] In particular the paper seeks to define and characterize equilibria of the evolutionary process.

There is a very large literature in game theory which discusses different definitions of equilibria. An equilibrium concept which is frequently used by economists is called 'Nash Equilibrium,' although game theorists have given numerous examples where the equilibrium concept is inadequate.[3] A strengthening of the notion of Nash equilibrium has led to the concepts of 'perfect equilibrium' and 'proper equilibrium.'[4] This paper examines the

---

[1] Harsanyi (1973) gives a similar interpretation of mixed strategies. Harsanyi justifies mixed strategies by players' uncertainty on their opponents payoffs (or types). In particular the payoff function for player i of type ξ_i is $u_i + \epsilon\xi_i$. A type knows his own value of ξ_i and knows the distribution function for ξ_j over the set of possible types Ξ_j; in particular, all players know that the expected value of ξ_j is zero. Then in equilibrium each type selects a pure strategy and each player plays the mixed strategy determined by the pure strategy played by each type and the distribution of types. Furthermore, a mixed strategy equilibrium in the complete information game is the limit as ϵ goes to zero of equilibria of incomplete information games.

[2] The following authors discuss the properties of the replicator model: Hines (1987), Hofbauer (1981), Schuster et al. (1981), Taylor and Jonker (1978), Zeeman (1980).

[3] See for instance van Damme (1987).

[4] These equilibrium concepts were introduced in Selten (1975) and Myerson (1978).

relationship between equilibria of the genetic process and existing equilibrium concepts in game theory.[5]

The replicator model describes the distribution of strategies in the population in terms of a differential equation. Thus in order to examine the dynamic equilibria of the replicator model, it is necessary to solve a system of nonlinear differential equations, which is usually done through numerical simulations. Alternatively, there are static equilibrium concepts that can be defined for the replicator model and which can be solved analytically. A particularly well known static equilibrium concept is the evolutionarily stable strategy (denoted in this paper by ESS) which was defined by Maynard Smith and Price (1973). This paper defines a different static equilibrium concept called 'evolutionary equilibrium.' The equilibrium concept is based on an arbitrarily small proportion of genes mutating towards a 'random' strategy.

The second section of the paper describes the replicator model and its relationship with the concepts of Nash and perfect equilibrium. The third section of the paper defines and establishes formal properties of an evolutionary equilibrium. In particular this section proves that an evolutionary equilibrium exists for a large class of payoff matrices. The fourth section analyzes the relationship between evolutionary equilibria and other equilibrium concepts in game theory: the set of evolutionary equilibria is a subset of the set of perfect equilibria and a superset of the set of regular equilibria. Throughout the paper definitions are indicated by italics.

---

[5] The following papers discuss similar issues: Crawford (1988), Friedman (1988), Nachbar (1990), Samuelson (1988). The following papers discuss these issues in some more specialized contexts: Axelrod and Hamilton (1981), Boyd and Lorberbaum (1987), Crawford (1989).

3.2 The replicator model

This section describes the replicator model and reviews results that characterize the relationship between properties of the replicator model and game theoretic equilibria.

Suppose that individuals from a large population are paired randomly.[6] Each individual selects a strategy $i \in \{1, \ldots, n\}$. The scalar x_i is the proportion of individuals who select strategy i and the column vector $x = (x_1, \ldots, x_n)$ describes the proportion of the population that adopts each possible strategy. Thus $x \in \Delta^n$ where

$$\Delta^n \equiv \{x \in \mathbf{R}^n_+ : \sum_{i=1}^{n} x_i = 1\}.$$

If an individual selects strategy i and is matched with an individual that has selected strategy j, a_{ij} ($a_{ij} \geq 0$) individuals will adopt strategy i in the next period. The matrix A, where

$$A \equiv \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix},$$

is called the payoff matrix for the evolutionary game. Then the proportion of the population adopting strategy i at time $t+1$ is

$$x_i^{t+1} = x_i^t \frac{(Ax^t)_i}{x^t \cdot Ax^t}$$

where $(Ax)_i = \sum_{j=1}^{n} a_{ij} x_j$ and $x \cdot Ax = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j$.[7] The next proposition shows that the law of motion is well defined.

Proposition 2 *The strategy simplex is invariant under the replicator difference equation; i.e., if $x^0 \in \Delta^n$ then for $t = 1, 2, \ldots$, $x^t \in \Delta^n$.*

---

[6] Specifically take then individuals to be the natural numbers.

[7] The validity of this statement is proven in Chapter 2.

The last expression can be rewritten as:

$$x_i^{t+1} - x_i^t = x_i^t \frac{(Ax^t)_i - x^t \cdot Ax^t}{x^t \cdot Ax^t}.$$

or by dropping the time subscripts

$$\Delta x_i = x_i \frac{(Ax)_i - x \cdot Ax}{x \cdot Ax} \equiv R_i^D(x). \tag{3.1}$$

The system of difference equations $\Delta x = R^D(x)$ is called the replicator model in discrete time; when we need to specify the payoff function, A, the replicator model is denoted by R_A^D. Let $\Delta t = \frac{\tau}{x \cdot Ax}$ be the time interval between periods (where $\tau \in \mathbf{R}_{++}$); then

$$\Delta x_i = x_i[(Ax)_i - x \cdot Ax]\, \Delta t$$

By letting $\Delta t \to 0$ the last expression can be written as:

$$\dot{x}_i = x_i[(Ax)_i - x \cdot Ax] \equiv R_i^C(x).^{89} \tag{3.2}$$

The system of differential equations $\dot{x} = R^C(x)$ is called the replicator model in continuous time; again when we need to specify the payoff function, A, the replicator model is denoted by R_A^C. Let the function $X : \mathbf{R}^n \times \mathbf{R}_+ \to \mathbf{R}^n$

---

[8] This derivation is entirely heuristic. Let $a = \max_{ij} |a_{ij} - 1|$. Akin and Losert (1984) use Euler's theorem to prove that as a goes to 0 the solution of the replicator dynamics in discrete time converges to the solution of the replicator dynamics in continuous time. However, the convergence is pointwise, not uniform, and thus the limit of the two trajectories can be quite different. For constant sum games, for instance, the replicator dynamics in continuous time is a center while the replicator dynamics in discrete time is an unstable focus (for a definition of these terms see Arnold (1973)).

[9] Hofbauer (1981) noticed that by setting $b_{ij} = a_{ij} - a_{nj}$ and letting $y_i = x_i/x_n$, expression (2) can be rewritten as

$$\dot{y}_i = y_i(b_{in} + \sum_{j=1}^{n-1} b_{ij} y_j) \ \ (i \in \{1, \ldots, n-1\})$$

which is the Volterra-Lotka equation.

be such that for all $t \in \mathbf{R}_+$ and $x^0 \in \Delta^n$,

$$\frac{dX(t,x^0)}{dt} = R^C(X(t,x^0)) \text{ and } X(0,x^0) = x^0.^{10}$$

The next proposition states that this system too is well defined.

Proposition 3 *The strategy simplex is invariant with respect to the replicator differential equation; i.e., if $x^0 \in \Delta^n$ and $t \in \mathbf{R}_+$ then $X(t,x^0) \in \Delta^n$.*

The next theorem describes one of the most important properties of the replicator model.

Theorem 10 *If the vector of strategies (x,x) is a Nash equilibrium of the normal game (A,A^T) then the vector x is a stationary point of the replicator model.*

Unfortunately there are stationary points of the replicator model that are not Nash equilibria of the normal form game.

A partial converse of the theorem was given by Bomze (1986).

Theorem 11 *(Bomze) (i) If the vector x is a stable stationary point*[11] *of the replicator model in continuous time, then the vector of strategies (x,x) is a Nash equilibrium. (ii) If the vector x is an asymptotically stable stationary point*[12] *of the replicator model in continuous time then the vector of strategies (x,x) is an isolated perfect equilibrium.*

---

[10] The existence and uniqueness of the function X follow from the differentiability of the map R^C.

[11] An equilibrium $\tilde{x}$ is *stable* if given any positive scalar ϵ, there is a positive scalar δ such that for all strategies x in the ball centered at $\tilde{x}$ and with radius δ, $x \in B(\tilde{x},\delta) \bigcap \Delta^n$, and for all positive t, $X(x,t) \in B(\tilde{x},\epsilon) \bigcap \Delta^n$.

[12] An equilibrium $\tilde{x}$ is *asymptotically stable* if it is stable and if δ can be chosen such that

$$\forall x \in B(\tilde{x},\delta),\ \lim_{t\to\infty} X(x,t) = \tilde{x}.$$

Unfortunately, asymptotically stable stationary points are difficult to characterize and do not always exist. This paper further characterizes the relationship between the replicator model and game theory. The idea is to look at stationary points of the replicator model that are resistant to mutation.

3.3 Evolutionary equilibrium

This section defines the notions of generalized evolutionary system, evolutionary equilibrium, and nondegenerate payoff matrix. We prove that there exists an evolutionary equilibrium for all nondegenerate payoff matrices.

This chapter examines a class of laws of motion, $\mathcal{H}$, somewhat more general than the replicator model. Let

$$\begin{aligned}\mathcal{H}_A = \; & \{H : \Delta^n \to T\Delta^n | \;\; \exists L : T\Delta^n \to T\Delta^n \text{ such that:} \\ & \qquad\qquad\qquad L \text{ is continuously differentiable, } L_i(y)y_i \geq 0, \\ & \qquad\qquad\qquad (L_i(y)y_i = 0 \Leftrightarrow y_i = 0),\;\; H = L \circ R_A^C\}\end{aligned}$$

where $T\Delta^n$ is the tangent space of the simplex; i.e.,

$$T\Delta^n = \{x \in \mathbf{R}^n \mid \sum_i x_i = 0\}.$$

In order to make the notation less cumbersome, when no confusion can arise, the subscript A in $\mathcal{H}_A$ is dropped.

The important property of this class is that it includes the replicator model in discrete and continuous time: if $L(x) \equiv I_{n\times n}$ then $H = R^C$ and $H \in \mathcal{H}$; if $L(x) \equiv \frac{1}{x\cdot Ax} I_{n\times n}$ then $H = R^D$ and $H \in \mathcal{H}$.[13] Suppose that the

---

[13] The set $\mathcal{H}$ includes other functions such as

$$H(x) = \Big((x_1[(Ax)_1 - x \cdot Ax])^3, -(x_1[(Ax)_1 - x \cdot Ax])^3\Big)$$

where $L : T\Delta^2 \to T\Delta^2$ is defined as $L(y) = (y_1^3, -y_1^3)$.

strategies selected by the offspring are subject to mutation. Assume that the set of possible mutation rates is

$$\begin{aligned} \mathcal{M}^n &= \{m : \Delta^n \to T\Delta^n \,|\, m \text{ is bounded, continuously differentiable} \\ &\quad \text{and } (\forall S \subset \{1,\ldots,n\}) \sum_{i\in S} x_i = 1 \Rightarrow \sum_{i \in S} m_i(x) \leq 0\}. \end{aligned}$$

Notice that we can interpret members of $\mathcal{H}$ and $\mathcal{M}$ as gradient fields. Then the last constraint implies that at the boundary the gradient field points inwards. Again, when it does not lead to confusion, the subscript n is dropped.

For an evolutionary game with payoff matrix A, the generalized replicator model $H \in \mathcal{H}$, the mutation function $m \in \mathcal{M}$, and a scalar $\mu \in (0,1)$, define an *evolutionary system* by the following differential equation:

$$\begin{aligned} \Delta x &= (1-\mu)H(x) + \mu m(x) \quad \text{(discrete version)} \\ \dot{x} &= (1-\mu)H(x) + \mu m(x) \quad \text{(continuous version).} \end{aligned}$$

A vector $\tilde{x}$ is an *evolutionary equilibrium* for the payoff function A and the generalized replicator model $H \in \mathcal{H}$ if for every function m in $\mathcal{M}$, there is a scalar $\mu' \in (0,1)$ and a vector valued function $x : (0, \mu') \to \Delta^n$ such that for all $\mu \in (0, \mu')$,

$$(1-\mu)H(x(\mu)) + \mu m(x(\mu)) = 0$$

and $\lim_{\mu \downarrow 0} x(\mu) = \tilde{x}$.[14]

The following theorem by Jiang (1963) (which generalizes the better known theorem by Fort (1950)) is used in the proof of the existence of evolutionary equilibria. Let X be a compact convex subset of a normed space, let

---

[14] Showing that a vector of strategies is an evolutionary equilibrium by using the definition seems quite hard. In many examples a simpler procedure will be the following: (i) first show that there exists an evolutionary equilibrium; this paper gives a sufficient condition for the existence of equilibria, 'nondegeneracy,' which is straightforward to check. (ii) Serially eliminate all strictly dominated strategies. If there is a unique Nash equilibrium that puts positive weight only on undominated strategies then this will be an evolutionary equilibrium. The validity of this procedure is proven in the rest of the paper.

d be the metric defined on X defined by the norm, and let $C(X, X)$ be the set of continuous function with domain and range in X. Then $(C(X, X), \rho)$ is a metric space where

$$\rho(f, g) = \sup_{x \in X} d(f(x), g(x)).$$

Finally, let $F: C(X, X) \rightarrow X$ be the fixed point correspondence; i.e., for all $f \in C(X, X)$,

$$F(f) = \{x \in X \mid f(x) = x\}.$$

A set D is said to be *totally disconnected* if all the connected subsets of D are singletons.

Theorem 12 *(Jiang Jia-He) Suppose $F(f)$ is a totally disconnected set. Then there is a vector p in $F(f)$ that satisfies the following property. For every neighborhood U of p there is an $\epsilon > 0$ such that:*

$$g \in C(X, X) \textit{ and } \rho(f, g) < \epsilon \Rightarrow F(g) \bigcap U \neq \emptyset.$$

The vector p described in the theorem is called an essential fixed point.

For all subsets of the strategy set $I \subset \{1, \ldots, n\}$, let $A|_I$ be the matrix $(a_{ij})_{i \in I, j \in I}$. A matrix A is *nondegenerate* if for all $I \subset \{1, \ldots, n\}$ such that $\#(I) \geq 2$, the matrix $A|_I$ is nonsingular.[15]

Lemma 9 *If A is nondegenerate then $H \in \mathcal{H}_A$ has finitely many stationary points.*

---

[15] The only property used in the paper is that there are finitely many symmetric equilibria in all the submatrices $A|_S$. I think that a necessary and sufficient condition for the latter property is

$$\forall (i, j) \subset \{1, \ldots, n\},\ a_{ij} = a_{ii} \Rightarrow a_{ji} \neq a_{jj}.$$

Notice that either assumption is much weaker than the Lemke and Howson nondegeneracy condition (for a definition of the Lemke and Howson nondegeneracy condition see van Damme (1987) page 52).

Proof: Let A be a nondegenerate payoff matrix. Notice that by the assumptions on $\mathcal{H}$, in order to show that H has finitely many fixed points, it is sufficient to show that R_A^C has only finitely stationary points. Furthermore since the simplex has only finitely many faces it is sufficient to show that in any face there are finitely many stationary points of R_A^C. Let $S = \{x \in \Delta^n | i \in I \Leftrightarrow x_i > 0\}$. Then $(x_S, x_{-S}) \in S$ is a stationary point of R_A^C if and only if there is a scalar λ such that $A_S x_S = \lambda \mathbf{1}$. Since A_S is invertible each face S has unique stationary point x. ∎

Proposition 4 *For all nondegenerate payoff matrices A there is an evolutionary equilibrium.*

Proof: Fix a payoff matrix A and a generalized replicator function $H \in \mathcal{H}_A$. We first construct a function, H', whose fixed points correspond to the stationary points of H. Then we use Theorem 12 to prove the existence of an essential fixed point. Finally we show that essential fixed points of H' are evolutionary equilibria for A (where the law of motion is H). Let

$$H' : \Delta^n \longrightarrow \mathbf{R}^n \text{ be defined by } H' \equiv I + H.$$

Notice that fixed points of H' are stationary points for H. Unfortunately the function H' does not map its domain, Δ^n, into itself. In order to remedy this problem we extend the function H' to a domain $E\Delta^n$ which is invariant under the extension, $\tilde{H}$. Specifically let

$$\begin{aligned} M &= \max_{x \in \Delta, i \in \{1,\ldots,n\}} |H_i'(x)| + 1; \\ E\Delta^n &= \{x \in \mathbf{R}^n \mid \sum_{i=1}^n x_i = 1, \forall i \in \{1,\ldots,n\}\, x_i \in [-M, M]\}; \end{aligned}$$

$$\alpha : E\Delta^n \rightarrow [0,1)$$

$$\alpha(x) \equiv \min\{\alpha \in [0,1) \,|\, \alpha/n\mathbf{1} + (1-\alpha)x \in \Delta^n\};$$

$$\begin{aligned} \tilde{H} : E\Delta^n &\longrightarrow E\Delta^n \\ \tilde{H}(x) &= H'(\alpha(x)/n\mathbf{1} + (1-\alpha(x))x). \end{aligned}$$

Notice that fixed points of H' are stationary points for H and that all the fixed point of $\tilde{H}$ are in Δ^n[16] (and thus are fixed points of H'). Since by assumption A is nondegenerate, by Lemma 9 there are finitely many stationary points of H and Theorem 12 is applicable. Let $g = (1-\mu)\tilde{H} + \mu(m+I)$. Then for small enough μ, $g : E\Delta^n \to E\Delta^n$ and $\rho(\tilde{H}, g) < \epsilon$. A fixed point of g corresponds to a stationary point of $(1-\mu)H + \mu m$. Thus the set of perturbations allowed in the theorem includes the ones in the definition of evolutionary equilibrium.

Fix a mutation function $m \in \mathcal{M}$. Since all the conditions are satisfied, we use Theorem 12 to prove the existence of an evolutionary equilibrium. Thus there is an $\tilde{x}$ such that for every $\epsilon > 0$ there is a μ_ϵ and a function

$$x_\epsilon(\mu) : (0, \mu_\epsilon) \to \Delta^n$$

such that $\forall \mu \in (0, \mu_\epsilon)$

$$(1-\mu)H(x(\mu)) + \mu m(x_\epsilon(\mu)) = 0$$

and $\sup_{\mu \in (0,\mu_\epsilon)} |x_\epsilon(\mu) - \tilde{x}| < \epsilon$. Then since there are finitely many fixed points of H there is an $\epsilon' > 0$ such that $B(\tilde{x}, \epsilon') \bigcap F(H) = \{\tilde{x}\}$. For $\mu \in (0, \mu_{\epsilon'})$ let $x(\mu) = x_{\epsilon'}(\mu)$. Then $x : (0, \mu_{\epsilon'}) \to \Delta$ is such that

$$(1-\mu)H(x(\mu)) + \mu m(x(\mu)) \equiv 0,$$

and $x(\mu) \to \tilde{x}$. Therefore, $\tilde{x}$ is an evolutionary equilibrium. ∎

---

[16] Proof: Suppose that $\tilde{H}(x) = x$ and $x \notin \Delta^n$. Since $\tilde{H} \cdot \mathbf{1} = 0$, there exists a strategy i such that $x_i < 0$. Let $y \in \partial\Delta_n$ be such that $y = \alpha(x)/n\mathbf{1} + (1-\alpha(x))x$. Then $y_i = 0$. But this implies that $\tilde{H}_i(x) = H_i'(y) \geq 0$. Contradiction.

A few remarks about the condition in the theorem. Clearly, nondegeneracy is not a necessary condition for the existence of an evolutionary equilibrium; the game A, where

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is clearly degenerate and has $(0,0,1)$ as an evolutionary equilibrium. However, some degenerate payoff matrices do not have evolutionary equilibria; let A be

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Let m be such that for all i, $m_i(x) = a_i - x_i$, where $\sum_i a_i = 1$ and $a_i > 0$. Then for all positive μ, the only stationary points of the generalized replicator with mutation are points such that $x_2 = a_2$. Thus there are no evolutionary equilibria.

In this section we showed that if the payoff matrix A is nondegenerate then every law of motion $H \in \mathcal{H}$ has an evolutionary equilibrium. However it is possible that different laws of motion give different set of evolutionary equilibria. The next proposition proves that all laws of motions $H \in \mathcal{H}_R \subset \mathcal{H}$ have the same set of evolutionary equilibria, where

$$\mathcal{H}_R = \{H: \Delta^n \to T\Delta^n \mid \exists f: \Delta \to \mathbf{R}_{++} \text{ such that } f \text{ is continuously differentiable and } H = f \cdot R_C\}.$$

Theorem 13 *Let $H, H' \in \mathcal{H}_R$. Then if $\tilde{x}$ is an evolutionary equilibrium for H then $\tilde{x}$ is an evolutionary equilibrium for H'.*

Proof: Without loss of generality suppose that $H' = f \cdot H$ where f is strictly positive and continuously differentiable. Fix $m' \in \mathcal{M}$ and let $m = \frac{m'}{f} \in \mathcal{M}$.

Since $\tilde{x}$ is an evolutionary equilibrium for $\mathcal{H}$ there is a scalar $\mu' \in (0,1)$ and a vector valued function $x : (0,\mu') \to \Delta^n$ such that for all $\mu \in (0,\mu')$,

$$(1-\mu)H(x(\mu)) + \mu m(x(\mu)) = 0$$

and $\lim_{\mu \downarrow 0} x(\mu) = \tilde{x}$. Consequently, we have that for all $\mu \in (0,\mu')$,

$$(1-\mu)H'(x(\mu)) + \mu m'(x(\mu)) = f(x)[(1-\mu)H(x(\mu)) + \mu m(x(\mu))] = 0$$

and $\lim_{\mu \downarrow 0} x(\mu) = \tilde{x}$. Since m' was arbitrary, $\tilde{x}$ is an evolutionary equilibrium for H'. ∎

3.4 Relationship between evolutionary equilibrium and other equilibrium concepts

The rest of the paper relates evolutionary equilibria to other game theoretic equilibria; i.e., equilibrium concepts that are derived from assumptions on the type of beliefs individuals have and on Bayesian maximization. There are three reasons for being interested in these relationships. (i) Showing that an evolutionary equilibrium corresponds to a game theoretic equilibrium allows us to argue that individuals act 'as if' they are Bayesian maximizers. (ii) There are ways of computing game theoretic equilibria that can be used to compute evolutionary equilibria. (iii) Requiring that a game theoretic equilibrium be evolutionary stable refines the set of equilibria.

The equilibrium concepts that are analyzed are following: (1) Nash equilibrium, (2) undominated Nash equilibrium, (3) perfect equilibrium, (4) strict dominance solvability, (5) regular equilibrium, (6) proper equilibrium, (7) strictly proper equilibrium, (8) ESS, (9) essential equilibrium.

The main results in this section are the following: (a) an evolutionary equilibrium is a symmetric perfect equilibrium; (b) a symmetric regular equilibrium is an evolutionary equilibrium; (c) an ESS is an evolutionary equilibrium; (d) not all symmetric proper equilibria are evolutionary equilibria; (e) not all evolutionary equilibria are strictly perfect equilibria.

3.4.1 Nash equilibrium

Let $\tilde{x} \in \Delta^n$. A vector of strategies $(\tilde{x}, \tilde{x})$ is a *symmetric Nash equilibrium* for the symmetric game (A, A^T) if for all strategies $y \in \Delta^n$, $\tilde{x} \cdot A\tilde{x} \geq y \cdot A\tilde{x}$. The concept of Nash equilibrium is the most widely used equilibrium concept in game theory although it is often considered to be too weak (see however Bernheim (1984) and Pearce (1984)).

Proposition 5 *An evolutionary equilibrium, $\tilde{x}$, is a symmetric Nash equilibrium, $(\tilde{x}, \tilde{x})$.*

Proof: Let $\tilde{x}$ be an evolutionary equilibrium. (i) Suppose that there is a strategy, say 1, such that $\tilde{x}_1 = 0$. In order to prove that $\tilde{x}$ is a Nash equilibrium it suffices to show that $(A\tilde{x})_1 \leq \tilde{x}A\tilde{x}$. Let m be such that $m_1(x) > 0$ for all x in a neighborhood of $\tilde{x}$. Then since $\tilde{x}$ is an evolutionary equilibrium there is a $\mu' > 0$ and a function $x : (0, \mu') \rightarrow \Delta$ such that for every μ in $(0, \mu')$,

$$(1-\mu)H(x(\mu)) + \mu m(x) = 0.$$

This implies that for all μ in $(0, \mu')$,

$$(Ax(\mu))_1 - x(\mu)Ax(\mu) < 0.$$

Thus $(A\tilde{x})_1 \leq \tilde{x}A\tilde{x}$. (ii) Suppose that $\tilde{x}_i > 0$ and $\tilde{x}_j > 0$. Then, $(A\tilde{x})_i - \tilde{x}A\tilde{x} = 0$ and $(A\tilde{x})_j - \tilde{x}A\tilde{x} = 0$. Thus $(A\tilde{x})_i = (A\tilde{x})_j$. Therefore $\tilde{x}$ is a Nash equilibrium.

■

In the next subsection we show that not all perfect equilibria are evolutionary equilibria and thus that not all Nash equilibria are evolutionary equilibria.

3.4.2 Undominated Nash equilibrium

A strategy i is *weakly dominated* if there exists a mixed strategy y such that the payoff for using y is at least as great as the payoff for using i regardless of the other players' strategy and strictly better for some strategy; i.e., for all $x \in \Delta^n$, $y \cdot Ax \geq (Ax)_i$ and there is a $z \in \Delta^n$ such that $y \cdot Az > (Az)_i$.

The principle that strategies that are weakly dominated should not be played is very intuitive, although when used repeatedly it can give results that are surprisingly strong.[17] The next lemma characterizes the relationship between evolutionary equilibria and weakly dominated strategies.

Lemma 10 *Let $\tilde{x}$ be an evolutionary equilibrium. Then $\tilde{x}_i = 0$, if strategy i is weakly dominated.*

Proof: Suppose that mixed strategy y weakly dominates strategy 1 and suppose, without loss of generality, that $1 \notin \text{support}(y)$. Suppose that $\tilde{x}$ is an evolutionary equilibrium and $\tilde{x}_1 > 0$. Let $m \in \mathcal{M}$ be such that in a neighborhood of $\tilde{x}$ (where $x_1 > 0$)

$$m_1(x) \equiv -1, \text{ and for all } i \in \text{support}(y),\ m_i(x) \equiv 1.$$

---

[17] van Damme shows that in a game where player one first decides whether to discard \$1 and then plays a battle of the sexes game, repeated elimination of weakly dominated strategies results to player one getting the highest possible payoff.

Then since $\tilde{x}$ is an evolutionary equilibrium there exists a constant μ' and a function $x : (0, \mu') \rightarrow \Delta$ such that for $\mu \in (0, \mu')$ and for all $i \in \text{support}(y)$,

$$\begin{aligned} (1-\mu)H_i(y) + \mu &= 0, \qquad (3.3) \\ (1-\mu)H_1(y) - \mu &= 0, \qquad (3.4) \end{aligned}$$

and $\lim_{\mu \to 0} x(\mu) = \tilde{x}$. Condition (3.3) implies that

$$x(\mu)Ax(\mu) > (Ax(\mu))_i;$$

condition (3.4) implies that for all $\mu \in (0, \mu')$

$$(Ax(\mu))_1 > x(\mu)Ax(\mu).$$

Thus (3.3) and (3.4) combined give

$$(Ax(\mu))_1 > \sum_i y_i(Ax(\mu))_i$$

which contradicts the assumption of weak domination. ∎

Thus an evolutionary equilibrium is an undominated Nash equilibrium; i.e., a Nash equilibrium where dominated strategies are given zero weight.

3.4.3 Perfect equilibrium

There are several ways in which perturbations have been introduced in solution concepts. Evolutionary equilibria consider perturbations in the law of motion; essential equilibria (which are analyzed later in this section) consider perturbation in the payoff function; finally, perfect equilibria consider equilibria that are 'resistant' to some perturbation of the strategy set.

Let $\mathbf{R}^n_{++} = \{x \in \mathbf{R}^n | (\forall i)\, x_i > 0\}$, $\Delta^n_{++} = \Delta^n \bigcap \mathbf{R}^n_{++}$. Let $\tilde{x} \in \Delta^n$. A vector of strategies $(\tilde{x}, \tilde{x})$ is a *symmetric perfect equilibrium* if there exist sequences $\{\epsilon^t\}$ and $\{x^t\}$, where $\epsilon^t \in \mathbf{R}_{++}$ and $x^t \in \Delta^n_{++}$, such that: (i) for all t, $x^t_i > \epsilon^t$ only if $i \in \text{argmax}_j (Ax^t)_j$; (ii) $\lim_{t\to\infty} \epsilon^t = 0$; (iii) $\lim_{t\to\infty} x^t = \tilde{x}$.

Proposition 6 *An evolutionary equilibrium is a symmetric perfect equilibrium.*

Proof: The result follows from previous lemma and the following result (van Damme (1987), Theorem 3.2.2): for a two person finite normal game an equilibrium is perfect if and only if every weakly dominated strategy is played with probability 0. ∎

The next example shows that not all symmetric perfect equilibria are evolutionary equilibria. Let

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

The vector $(\tilde{x}, \tilde{x})$ where $\tilde{x} = (0, 1, 0)$ is a symmetric perfect equilibrium (since strategy 2 is not weakly dominated) but $\tilde{x}$ is not an evolutionary equilibrium.[18]

3.4.4 Strict dominance solvability

The next proposition says that an evolutionary equilibrium is resistant to the elimination of dominated strategies. Thus restricting the replicator model to rationalizable strategies will not reduce the set of evolutionary equilibria.

Proposition 7 *Suppose that strategy j dominates strategy i and suppose that $\tilde{x}$ is an evolutionary equilibrium for the payoff matrix A. Let A_{-i} be the payoff matrix where the i^{th} row and column have been deleted. Then $\tilde{x}_i$ is an evolutionary equilibrium for the payoff matrix A_{-i}.*

---

[18] Proof: suppose $\tilde{x}$ is an evolutionary equilibrium. Let $m \in \mathcal{M}$ be such that $m(x) = (1, -1, 0)$ for every x in a neighborhood of $\tilde{x}$ and let $x(\mu)$ be the corresponding sequence of stationary points for the generalized replicator model. Since for all x in a neighborhood of $\tilde{x}$, $x \cdot Ax > (Ax)_3$, for small enough μ we must have $x_3(\mu) = 0$. For such μ, $H_1(x(\mu)) > 0$ and thus $(1-\mu)H_1(x(\mu)) + \mu m_1(x(\mu)) > 0$. Thus, $\tilde{x}$ cannot be an evolutionary equilibrium.

Proof: Let $m' \in \mathcal{M}^{n-1}$ be a mutation function. Let $m \in \mathcal{M}^n$ be such that

$$m_j(x) = \begin{cases} m'_j(x) & \text{if } j \neq i \\ 0 & \text{if } j = i. \end{cases}$$

Since $\tilde{x}$ is an evolutionary equilibrium then for small enough $\mu' > 0$ there is a function $x : (0, \mu') \to \Delta^n$ such that $x(\mu) \to \tilde{x}$ and for all $\mu \in (0, \mu')$,

$$(1 - \mu)H_A(x(\mu)) + \mu m(x(\mu)) = 0.$$

Since $\tilde{x}$ is a Nash equilibrium and since strategy i dominates strategy j, then $\tilde{x}A\tilde{x} \geq (A\tilde{x})_j > (A\tilde{x})_i$. Then there is a small enough $\mu'' > 0$ such that for all $\mu \in (0, \mu'')$,

$$x(\mu)Ax(\mu) > (Ax(\mu))_i.$$

Since $m_i(x) = 0$ then $x_i(\mu) = 0$. Therefore, for all $\mu \in (0, \mu'')$,

$$(1 - \mu)H_{A_{-i}}(x_{-i}(\mu)) + \mu m'(x_{-i}(\mu)) = 0,$$

and $\tilde{x}_{-i}$ is an evolutionary equilibrium for A_{-i}. ∎

An equilibrium is *strictly dominance solvable* if it can be obtained by reducing the game to a single cell by iterated deletion of dominated strategies.

Proposition 8 *A strictly dominance solvable equilibrium is an evolutionary equilibrium.*[19]

Proof: Suppose that $\tilde{x}$ is an evolutionary equilibrium, strategy 1 dominates strategy 2 in the normal game $A_{\{1,\dots,n\}-\{4\}}$, and strategy 3 dominates strategy 4 in the game A. By Lemma 10, $\tilde{x}_4 = 0$. Suppose $\tilde{x}_2 > 0$. Choose the function m such that for every x in a neighborhood of $\tilde{x}$, $m_1(x) = 1$, $m_2(x) = -1$. Then for small enough μ, $(Ax(\mu))_2 > (Ax(\mu))_1$. But this is impossible since $x_4(\mu) \to 0$. Thus $\tilde{x}_2 = 0$. ∎

---

[19] Thus an evolutionary equilibrium is ecologically solvable as defined by Nachbar (1990).

3.4.5 Regular equilibrium

The concept of regular equilibrium was introduced by Harsanyi (1973). The following description of the equilibrium is taken from van Damme (1987) although it is simplified by looking at two person symmetric games. Let $z = (x, y)$ be a vector of strategies for the game (A, A^T). Let $k \in \text{supp}(x)$, let $l \in \text{supp}(y)$ and let $m = (k, l)$. Then let $F(x|k)$ be such that:

$$(\forall i \neq k)\ F_i(x|k) = x_i[(Ay)_i - (Ay)_k] \text{ and } F_k(x|k) = \sum_{i=1}^{n} x_i - 1.$$

Similarly let $F(y|l)$ be such that

$$(\forall i \neq l)\ G_i(y|l) = y_i[(Ax)_i - (Ax)_l] \text{ and } G_l(y|l) = \sum_{i=1}^{n} y_i - 1.$$

Finally let

$$H(z|m) = (F(x|k), G(y|l))^T \text{ and } J(\tilde{z}|m) = \frac{\partial H(z|m)}{\partial z}\,|_{z=\tilde{z}}.$$

Then $\tilde{z}$ is a *regular equilibrium* if for some $m \in \text{supp}(x)\times\text{supp}(y)$, $H(z|m) = 0$ and $\det J(z|m) \neq 0$.

Intuitively, a regular equilibrium is one for which the best response mapping is continuously differentiable at a neighborhood of the Nash equilibrium.

Proposition 9 *A symmetric regular equilibrium is an evolutionary equilibrium.*

Proof: Theorem 9.4.3 in van Damme (1987) states that a Nash equilibrium $(\tilde{x}, \tilde{x})$ is regular if and only if $dR^C/dx|_{x=\tilde{x}}$ is nonsingular. Notice that if $dR^C/dx|_{x=\tilde{x}}$ is nonsingular and μ is small enough then

$$\frac{d}{dx}[(1-\mu)L(R^C(x)) + \mu m(x)]|_{\mu=0,x=\tilde{x}}$$

is nonsingular. Therefore if $(\tilde{x}, \tilde{x})$ is a regular equilibrium then by the implicit function theorem $\tilde{x}$ is an evolutionary equilibrium. ∎

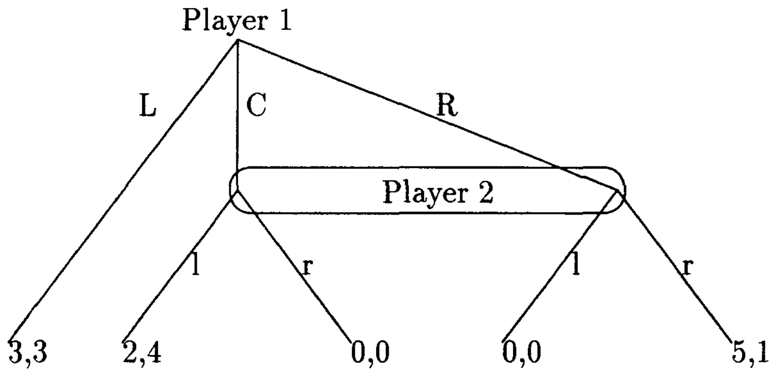

Figure 3.1: Game with a proper equilibrium which is eliminated by forward induction.

The next example shows that not all games with nondegenerate payoff matrices A have a regular equilibrium.

Let

$$(A, A^T) = \begin{pmatrix} 2,2 & 2,2 \\ 2,2 & 1,1 \end{pmatrix}.$$

Clearly the matrix is nondegenerate and the only perfect equilibrium is 'top,' 'left.' The Jacobian of the best response function (as defined by Harsanyi (1973)) at the equilibrium point is

$$\det \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} = 0$$

and thus the game has no regular equilibria.

3.4.6 Proper equilibrium

Let $\tilde{x} \in \Delta^n$. A vector of strategies $(\tilde{x}, \tilde{x})$ is a *symmetric proper equilibrium* if there exist sequences $\{\epsilon^t\}$ and $\{x^t\}$, where $\epsilon^t \in \mathbf{R}_{++}$ and $x^t \in \Delta^n_{++}$, such that: (i) for all t, $x^t_i \leq \epsilon^t x^t_j$ if $(Ax^t)_i < (Ax^t)_j$; (ii) $\lim_{t\to\infty} \epsilon^t = 0$; and (iii)

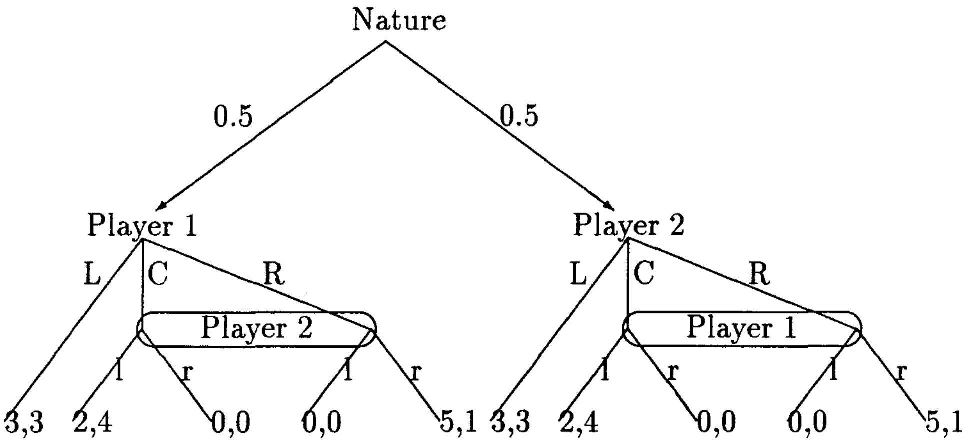

Figure 3.2: Symmetrization of extensive game in Figure 3.1.

$\lim_{t\to\infty} x^t = \tilde{x}$. In this subsection we show that not all proper equilibria are evolutionary equilibria.

The game in Figure 3.1 is used by Tan and Werlang (1988) to show the insufficiency of the concept of proper equilibrium. There are two proper equilibria in the game: Rr and Ll. By a forward induction argument Tan and Werlang argue that since C is dominated by L, C should never be played and therefore l should never be employed.[20]

Therefore properness allows unreasonable equilibria, such as Ll. Are Rr and Ll evolutionary equilibria? We can construct a symmetric game by assuming that two individuals are randomly assigned to the roles of player 1 and player 2. Figure 3.2 shows the extensive form for such a game. An evolutionary game is constructed by normalizing the symmetric extensive game. Figure 3.3 shows the symmetrization of the game in Figure 3.2.

Notice that the matrix is degenerate. The strategy Rr is clearly a strict

---

[20] Ll corresponds to the equilibrium where player 2 warns player 1 that he will play l. Then player 1 has the choice of playing L and receiving 3, playing C and receiving 2, and playing R and receiving 0. If player 2 gets to move he realizes that player 1 did not believe in his bluff. Then player 2 is better off not to follow with his threat and play r.

Rr	6	1	5	0	8	3
Rl	5	0	9	4	8	3
Cr	1	3	0	2	3	5
Cl	0	2	4	6	3	5
Lr	4	3	3	3	6	6
Ll	3	7	7	7	6	6

Figure 3.3: Symmetrization of the game in Figure 3.2.

Nash equilibrium and thus a regular equilibrium (van Damme (1987), Theorem 2.3.3) and thus an evolutionary equilibrium (Proposition 9. Notice that $\tilde{x} = (0,0,0,0,0,1)$ is a symmetric proper equilibrium.[21] Suppose that $\tilde{x}$ is an evolutionary equilibrium. Then for all x in a neighborhood of $\tilde{x}$ let

$$m(x) = (1,0,0,0,1,-2).$$

Choose μ small enough so that $x(\mu)Ax(\mu) > 5.5$ and $x_6 > 0.9$. Then $x_2(\mu) = x_3(\mu) = x_4(\mu) = 0$. The assumptions on m also give that $x_1(\mu) > 0$ and $(Ax)_6 > xAx > (Ax)_5$ which is impossible given the payoff function. Thus $\tilde{x}$ is not an evolutionary equilibrium.

3.4.7 Strictly Perfect Equilibrium

A Nash equilibrium is strictly perfect if it is resistant to all perturbations of the strategy set. Formally, $(\tilde{x}, \tilde{x})$ is a *strictly perfect equilibrium* if there exists a vector $\eta \in \mathbf{R}^n_{++}$ such that for all sequences $\{\eta^t\}$, where $\eta^t_i \in (0, \eta_i)$, there exists a sequence $\{x^t\}$, where $x^t \in \Delta^n$, such that: (i) for all i and t, $x^t_i \geq \eta^t_i$; (ii) $x^t_i > \eta^t_i$ implies that $i \in \text{argmax}_j (Ax^t)_j$; (iii) $\lim_{t \to \infty} \eta^t = 0$; (iv) $\lim_{t \to \infty} x^t = \tilde{x}$.

The concept of strict perfect equilibrium resembles the concept of evolu-

---

[21] Just set $x_i = \frac{\epsilon^2}{1+\epsilon+4\epsilon^2}$ if $i \neq 5, 6$, $x_5 = \frac{\epsilon}{1+\epsilon+4\epsilon^2}$, and $x_6 = \frac{1}{1+\epsilon+4\epsilon^2}$.

tionary equilibrium but as the following examples illustrates not every nondegenerate game has a symmetric strictly perfect equilibrium.

Let

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix}.$$

Then the game (A, A^T) has no symmetric strictly perfect equilibrium but has a unique evolutionary equilibrium $\tilde{x} = (1, 0, 0)$.[22]

3.4.8 ESS

The most widely used equilibrium concept in evolutionary game theory is the concept of evolutionary stable strategy (ESS). While evolutionary equilibria consider dynamic perturbation, ESS considers stable perturbation. A strategy $\tilde{x} \in \Delta^n$ is an *ESS* if for any other strategy $y \in \Delta^n - \{\tilde{x}\}$ there is an ϵ' such that for all $\epsilon \in (0, \epsilon')$

$$\tilde{x}A(\epsilon y + (1-\epsilon)\tilde{x}) > yA(\epsilon y + (1-\epsilon)\tilde{x}).$$

Thus contrary to the notion of evolutionary equilibrium, ESS considers mutation in a static framework. This condition can be rewritten in the following

---

[22] Proof: The vector $(\tilde{x}, \tilde{x})$ is the unique symmetric perfect equilibrium, A is nondegenerate and thus $\tilde{x}$ is an evolutionary equilibrium. Consider the perturbation

$$\frac{1}{2\epsilon+\epsilon^2}(\epsilon, \epsilon^2, \epsilon).$$

Suppose $x(\epsilon) \to (1, 0, 0)$. Then

$$x_1(\epsilon) + x_2(\epsilon) \geq x_1(\epsilon) + x_3(\epsilon);$$

i.e., $x_2(\epsilon) \geq x_3(\epsilon)$. This implies that

$$x_1(\epsilon) + x_3(\epsilon) \geq x_1(\epsilon) + x_2(\epsilon),$$

or $x_2(\epsilon) = x_3(\epsilon)$. But this is possible only if

$$x_1(\epsilon) - x_3(\epsilon) \geq x_1(\epsilon) + x_3(\epsilon).$$

Contradiction.

way. Strategy $\tilde{x}$ is an ESS if for all strategies y different than $\tilde{x}$ one of the two conditions holds:

$$(i) \qquad \tilde{x} \cdot A\tilde{x} > y \cdot A\tilde{x}$$

$$(ii) \quad \tilde{x} \cdot A\tilde{x} = y \cdot A\tilde{x} \text{ and } \tilde{x} \cdot Ay > yAy.$$

The next proposition relates ESS to the replicator model in continuous time.

Proposition 10 *(Zeeman) An ESS is an asymptotically stable stationary point of the replicator model* R^C.[23]

Unfortunately the requirements of ESS and asymptotically stable stationary points seem too strict as the following example shows. Let

$$A = \begin{pmatrix} \epsilon & 1 & -1 \\ -1 & \epsilon & 1 \\ 1 & -1 & \epsilon \end{pmatrix}$$

where $\epsilon \in (0, 1/3)$. The only Nash equilibrium is $\tilde{x} = (1/3, 1/3, 1/3)$. The Hessian of the law of motion is negative definite, therefore: (1) $\tilde{x}$ is not asymptotically stable (2) and thus $\tilde{x}$ is not an ESS (3) and the game has not asymptotically stable stationary points and no ESS; (4) since the determinant of the Hessian is nonzero, $\tilde{x}$ is regular (5) and thus an evolutionary equilibrium.

Suppose that $\epsilon < 0$. Then $\tilde{x}$ is an ESS and is thus an asymptotically stable stationary point for the replicator model. The replicator model in

---

[23] The proposition does not hold for R^D. Let

$$A = \begin{pmatrix} 0 & 1+\epsilon & -1 \\ -1 & 0 & 1+\epsilon \\ 1+\epsilon & -1 & 0 \end{pmatrix}$$

where $\epsilon > -1$. Then $(1/3, 1/3, 1/3)$ is an ESS but not an asymptotically stable equilibrium for R^D.

discrete time in not stable at $\tilde{x}$ since one of the eigenvalues of the linearized system is greater than one. Thus ESS are not necessarily asymptotically stable points of the replicator model in discrete time.

Proposition 11 *A hyperbolic stationary point of the replicator model in continuous time is an evolutionary equilibrium.*

Proof: A stationary point is hyperbolic if and only if all the eigenvalues are negative. Thus the proposition follows from the implicit function theorem.

■

Proposition 12 *An ESS is an evolutionary equilibrium for the law of motion R^C.*

Proof: Let $\tilde{x}$ be an ESS. In the proofs of Theorem 9.2.8, 9.4.8 van Damme (1987) shows that there is an open ball U centered at $\tilde{x}$ such that the function

$$\begin{aligned} V : U &\rightarrow \mathbf{R} \\ V(x) &\equiv \prod_i x_i^{\tilde{x}_i} \end{aligned}$$

is a Lyapunov function and such that $\tilde{x}$ is the only fixed point of R^C in U. Take c to be large enough so that $V^{-1}(c) \subset U$. Let X be the solution of the differential equation $\dot{x} = R^C$. For $x \in U$, let $F(x) \equiv X(1, x)$. Then F is continuous and maps $V^{-1}(c)$ into $V^{-1}(c)$. Then by an argument similar to the one in proposition 3 one can show that there $\tilde{x}$ is an evolutionary equilibrium for R^C. ■

Finally notice that not all ESS are regular equilibria since the game matrix discussed in Section 3.4.5 has no regular equilibria but has $(1, 0)$ as the unique ESS.

3.4.9 Essential equilibrium

A Nash equilibrium (x, y) is *essential*[24] for a game (A, B) if for an arbitrarily small perturbation of the payoff matrix (A', B') there is a Nash equilibrium to (A', B') close to (x, y). This notion predates the concepts of hyperstable equilibrium introduced by Kohlberg and Mertens (1986).[25]

A symmetric Nash equilibrium $(\tilde{x}, \tilde{x})$ for the game (A, A^T) is *symmetric essential* if for any symmetric game with payoffs close enough to A there is a symmetric Nash equilibrium close enough to $(\tilde{x}, \tilde{x})$. The next propositions characterize the set of symmetric essential equilibria.

Proposition 13 *(Bomze) A regular equilibrium is a symmetric essential equilibrium.*

Proposition 14 *(van Damme) An ESS is a symmetric essential equilibrium.*

Proposition 15 *Restrict the set of mutation function of the form*

$$m_i(x) = x_i[(Cx)_i + xCx].$$

Then a symmetric essential equilibrium is an evolutionary equilibrium.

Proof: Suppose $\tilde{x}$ is a symmetric essential equilibrium of the game (A, A^T). Let $m_i(x) = x_i[(Cx)_i + xCx]$ and let $A_\mu = (1 - \mu)A + \mu C$. Then

$$\begin{aligned} \dot{x}(\mu) = \; & (1-\mu)x_i(\mu)[(Ax(\mu))_i - x(\mu)Ax(\mu)] \\ & +\mu x_i(\mu)[(Cx(\mu))_i + x(\mu)Cx(\mu)] \\ = \; & x_i(\mu)[(A_\mu x(\mu))_i - x(\mu)A_\mu x(\mu)] \qquad = 0 \end{aligned}$$

---

[24] The equilibrium concept is defined by Wu and Jia-He (1962).

[25] A subset, H, of the set of Nash equilibria for the game (A, B) is *hyperstable* if it is minimal according to the following condition: given any small perturbation of the payoff matrix, (A', B'), there is a Nash equilibrium to (A', B'), (x', y'), close to the set H.

if $(x(\mu), x(\mu))$ is a Nash equilibrium of the game (A_μ, A_μ^T). But since $\tilde{x}$ is a symmetric essential equilibrium then for any perturbation of the payoff there is a Nash equilibrium arbitrarily close to $\tilde{x}$. Therefore $\tilde{x}$ is an evolutionary equilibrium. ∎

3.5 Conclusion

In this chapter we constructed a model where: (i) the proportion of the population that adopts a strategy is dependent on the distribution of strategies in the population and (ii) mutation affects the dynamics. Models have been studied that included each of these effects separately but to our knowledge nobody has studied them together.

An equilibrium in this paper is a mix of strategies which is a stationary point for the law of motion of the replicator model with arbitrarily small levels of mutation. We show that such an equilibrium exists for all nondegenerate payoff matrices.

An individual facing the population at equilibrium is faced with the same problem as an individual facing a player who selects a particular selection of the set of perfect equilibria: the set of evolutionary equilibria. Several examples in the paper illustrate that evolutionary selection has the property of getting rid of some unintuitive equilibria.

Chapter 4

FICTITIOUS PLAY: A STATISTICAL STUDY OF MULTIPLE ECONOMIC EXPERIMENTS

This chapter is joint work with Mahmoud El-Gamal.

4.1 Introduction

In recent years, there has been a marked growth in the number of experimental studies in Economics, and their influence on the field can no longer be denied.[1] The interaction between experimental studies and other parts of the discipline has also been flourishing, but we believe that it has not yet reached its potential. We believe that the main reason for the slow development of those interactions is our inability to systematically (in some rigorously statistical manner) integrate a large number of experimental studies to develop stylized facts that can inspire and benefit from work in other subfields of economics. It is true that a number of stylized facts have already been established by experimental studies and have been taken into account by

---

[1] A review of experimental findings can be found in Hoffman and Spitzer (1985), Plott (1982), and Smith (1982).

modeling economists, but the progress of the science can no doubt be greatly enhanced by a systematic procedure of generating such stylized facts. This is the issue that we address in this paper. It was clear to us from the start that no classical procedure can be used to combine the results of different experiments in generating stylized facts. That is clearly the case since the different experiments have completely different designs and were performed to investigate different issues. To complicate things further, one cannot ignore the fact that experimenters are aware of the other experiments that their colleagues have run, and hence a classical procedure will need to take into consideration what amounts to pre-testing of the hypotheses in question; a task that defies even the most capable of statisticians.

We propose to analyze the data in a Bayesian fashion, endowing ourselves with a belief on the relative validity of a number of possible theories of human economic behavior and updating our beliefs using the experimental results available to us. Two points make our proposed procedure especially valuable. The first is the fact that the order in which we observe the experimental results (and hence the order of updating of our beliefs) is irrelevant. This is not a surprising result, but we include a rigorous statement and proof in Section 4 for completeness. The second point is specific to the example we use in this paper. We analyzed a total of nine experiments to update our beliefs about two alternative hypotheses, and found that some of the experiments pointed unequivocally in favor of one, and others pointed unequivocally in favor of the other. The overall analysis of all nine experiments, however, made us believe that one hypothesis is infinitely more likely than the other. This suggests that analyzing experiment by experiment and hoping for consensus of the results from all experiments is not a good procedure. It is clear that running a new experiment every time we have a lack of consensus and giving the latest result the full weight is not a good procedure either. This leaves us with the need to combine a large number of available studies in making stylized facts, and leads us back to our Bayesian procedure.

We believe that the best way to illustrate a new procedure is by example, and for this paper, we chose to analyze different learning procedures in repeated games. The motivation for the learning literature is the limitation of the most common game theoretic assumption that players in a repeated game will choose to play a Nash equilibrium. The framework is that of a number of players each of whom chooses to play the action that maximizes her expected payoff subject to her beliefs about her opponent's strategies. The Nash equilibrium is the one where the expectations of all the players are fulfilled. The number of such equilibria that arise in many games can be quite large, and that is a clear restriction on the usefulness of the equilibrium.[2]

A more dynamic justification of theoretical choices of actions for the players is based on different learning algorithms. One assumes that the players start with beliefs on their opponents' possible actions, and update their beliefs as they play each stage of the game and observe their opponents' actual actions. In this paper, we consider two of the most popular learning hypotheses in repeated games. The first hypothesis was proposed by Cournot and assumes that each agent will assume that her opponents will choose the same action they chose in the previous period (see Moulin (1985)). The second hypothesis that we analyze is the so-called fictitious play hypothesis (see Brown (1951), Robinson (1951), Shapley (1962), Brock et al. (1988)). The

---

[2] One way of reducing the number of equilibria is to restrict the player's set of beliefs. This approach is adopted by Banks and Sobel (1987), Cho and Kreps (1987), Grossman and Perry (1986). However such restrictions are somewhat arbitrary.

process assumes that an individual has Dirichlet priors over her opponents' strategies and that at each round the player updates her beliefs according to Bayes' rules. It turns out that these assumptions are equivalent to assuming that each player selects the strategy that maximizes her payoff given beliefs that correspond to a convex combination of the initial beliefs and the empirical distribution. The experimental literature investigating different learning algorithm is very sparse, and limited to an experiment-by-experiment study. We illustrate our proposed procedure by starting with prior probabilities about the relative plausibility of those two learning hypotheses described above, use a simulation technique to compute the likelihood functions (the probability of observing the data in each experiment under each of the two hypotheses), and then use those likelihood functions to do Bayesian of our priors for each individual experiment, and then for all nine experiments combined.

Figure 4.1 is a plot of the economist's posterior as a function of her prior and the smearing parameter ϵ to be explained later. For now, just look at the posterior over the smeared fictitious play hypothesis (the height of the graph) at different values of the prior over that hypothesis at small (say less that .5) values of ϵ. At $\epsilon = 0$, the hypothesis is strictly fictitious play (and we cannot compute the posterior), and for positive values, ϵ is the probability that any particular person in any particular stage of any particular experiment gets to choose her action purely randomly. It is clear that the posterior for most reasonable values of epsilon and for all positive priors is very close to unity, and hence our posterior belief indicates that the smeared fictitious play hypothesis is infinitely more likely than the smeared Cournot hypothesis.

The rest of this paper builds up and justifies the necessary machinery to achieve Figure 4.1. Section 2 discusses one class of games three of which we studied. Section 3 discusses the other class of games, six of which we studied. Section 4 describes and justifies the econometric procedure that we follow. The paper ends with a series of 9 appendices for the nine experiments.

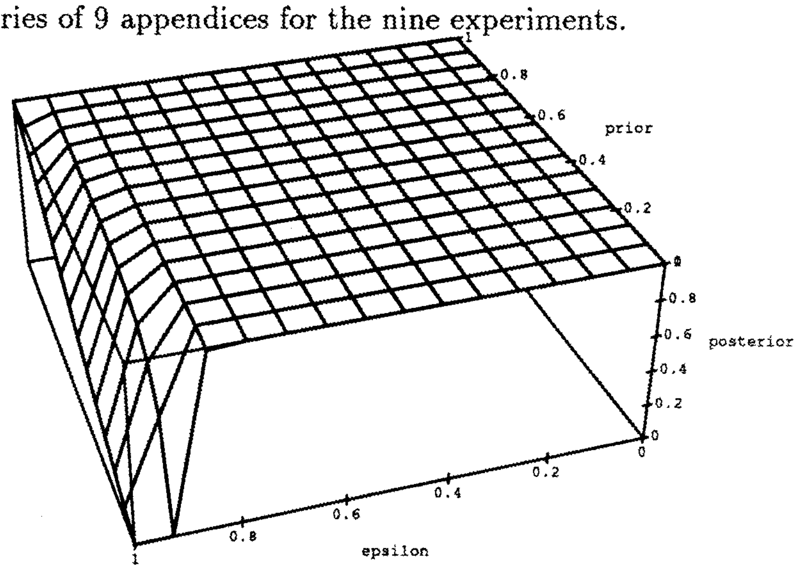

Figure 4.1: Posterior probability of agents playing according to fictitious play using all 9 experiments.

The reader should keep in mind that the procedure we adopt in this paper is not specific to the particular hypothesis we are testing (namely, whether individuals select strategies according to fictitious play) or to the type of data we have (namely, experimental data). Indeed, the procedure we adopt is a general way of aggregating data and evaluating theories.

4.2 Games with one opponent

This section is based on some experiments that were run by Knott and Miller (1987). In this series of three experiments (labeled A, B, and C), each of which has individuals matched in pairs and play the games reproduced in Figures 4.2, 4.3, and 4.4 respectively ten times. Thus, for instance, if in a given repetition of game A, an individual selects strategy $S2$ while her opponent selects strategy $S1$ then the individual receives 200 pennies (the payoffs are the ones for the row choser).

	S1	S2	S3	S4	S5	S6	S7	S8	S9	S10	S11	S12	S13	S14	S15
S1	100	100	75	85	85	90	90	90	90	90	90	90	90	90	90
S2	200	120	100	100	100	100	100	100	100	100	100	100	100	100	100
S3	0	140	125	115	115	110	110	110	110	110	110	110	110	110	110
S4	-100	160	150	130	125	120	120	120	120	120	120	120	120	120	120
S5	-200	180	175	145	135	130	130	130	130	130	130	130	130	130	130
S6	-300	200	200	160	145	140	140	140	140	140	140	140	140	140	140
S7	-400	220	220	175	155	155	150	150	150	150	150	150	150	150	150
S8	-450	0	250	190	165	160	160	160	160	160	160	160	160	160	160
S9	-475	-100	0	205	175	170	170	170	170	170	170	170	170	170	170
S10	-500	-125	-100	0	185	180	175	175	180	180	180	180	180	180	180
S11	-525	-150	-125	-100	0	190	180	180	190	190	190	190	190	190	190
S12	-550	-175	-150	-125	-100	0	195	190	195	195	200	200	200	200	200
S13	-575	-200	-175	-150	-125	-100	0	250	200	200	210	210	210	210	210
S14	-600	-225	-200	-175	-150	-125	-100	0	250	250	250	250	250	220	220
S15	-625	-250	-225	-200	-175	-150	-125	-100	80	100	120	140	160	180	200

Figure 4.2: Payoff matrix for Knott and Miller's experiment A.

	S1	S2	S3	S4	S5	S6	S7	S8	S9	S10	S11	S12	S13	S14	S15
S1	100	100	90	90	90	90	80	-100	-125	-150	-175	-200	-225	-250	-625
S2	800	120	100	100	100	100	100	0	-100	-125	-150	-175	-200	-225	-600
S3	700	400	125	110	110	110	110	110	0	-100	-125	-150	-175	-200	-575
S4	600	450	350	130	120	120	120	120	120	0	-100	-125	-150	-175	-550
S5	500	500	375	150	135	130	130	130	130	130	0	-100	-125	-150	-525
S6	400	600	400	200	200	140	140	140	140	140	50	0	-100	-125	-500
S7	350	800	425	210	210	150	150	150	150	150	100	50	0	-100	-475
S8	200	300	450	250	220	160	160	160	160	160	160	120	50	0	-450
S9	100	200	240	400	230	170	170	170	170	170	170	140	100	20	-400
S10	90	100	200	200	300	200	180	180	180	180	180	160	150	40	-300
S11	65	65	65	100	100	280	200	190	190	190	190	180	175	60	-200
S12	60	60	60	60	80	100	250	200	200	200	200	200	200	80	-100
S13	55	55	55	55	55	55	210	220	210	210	210	210	210	100	0
S14	50	50	50	50	50	50	50	50	220	220	220	220	220	220	220
S15	40	40	40	40	40	40	40	60	80	100	120	140	160	180	200

Figure 4.3: Payoff matrix for Knott and Miller's experiment B.

For each experiment, all the interesting aspects of the theoretical and observed behavior are depicted in a separate appendix. For instance, Appendix A deals with experiment A, Appendix B with experiment B and

	S1	S2	S3	S4	S5	S6	S7	S8	S9	S10	S11	S12	S13	S14	S15
S1	100	75	85	85	90	90	90	90	90	90	90	90	90	90	90
S2	350	120	100	100	100	100	100	100	100	100	100	100	100	100	100
S3	300	140	125	115	115	110	110	110	110	110	110	110	110	110	110
S4	275	130	150	130	125	120	120	120	120	120	120	120	120	120	120
S5	250	120	125	145	135	130	130	130	130	130	130	130	130	130	130
S6	200	110	110	125	145	140	140	140	140	140	140	140	140	140	140
S7	100	100	100	100	100	155	150	300	350	250	150	150	150	150	150
S8	-350	0	50	50	75	100	125	250	400	450	450	160	160	160	160
S9	-400	-100	0	25	50	75	100	200	300	425	400	375	350	325	170
S10	-500	-125	-100	0	25	50	75	175	225	325	375	350	325	300	300
S11	-525	-150	-125	-100	0	25	50	125	200	225	350	325	300	275	250
S12	-550	-175	-150	-125	-100	0	25	100	175	200	200	250	250	250	225
S13	-575	-200	-175	-150	-125	-100	0	75	150	175	175	175	200	200	200
S14	-600	-225	-200	-175	-150	-125	-100	0	125	150	150	150	150	175	175
S15	-625	-250	-225	-200	-175	-150	-125	-100	0	100	120	140	140	150	150

Figure 4.4: Payoff matrix for Knott and Miller's experiment C.

Appendix C with experiment C. In each appendix, the first two figures show a Monte-Carlo distribution of actions in each of the ten stages of the game. Specifically, in Appendix A, Figure 4.11 displays the simulated distribution of those plays under Cournot updating. To obtain this distribution, we simulate 1000 pairs (2000 individuals) and endow them with randomly drawn (uniform over the unit simplex) initial priors over their opponents' possible actions. Figure 4.11 shows the obtained empirical distribution. We then let each pair go through the 10 stages of the game where they are made to update according to the Cournot rule. Similarly, Figure 4.12 shows the distribution obtained by running a similar simulation except that the agents are made to update using the fictitious play rule. The third figure in each appendix shows the actual observed proportions of play of each action in each period. For example Figure 4.13 shows the proportion of times that each of the 15 actions was played in each of the ten periods. The proportion here is taken over the 8 pairs (16 individuals) that played the game in experiment A. The fourth figure in each appendix is the result of our econometric analysis of each individual experiment which will be explained in section 4.

By looking at the best response mapping we see that each game has a unique pure strategy Nash equilibrium: (S14, S14) for experiments A and experiment B and (S7, S7) for experiment C. By inspection of the first two pictures in Appendix A,B, and C we notice also that both the Cournot process and fictitious play converge to the pure strategy Nash equilibrium and that the convergence of fictitious play is much slower. The behavior of the subjects in the three experiments is very similar to fictitious play: subjects seem to converge to the pure strategy Nash equilibrium at a somewhat slower rate than fictitious play.

4.3 Games with multiple opponents

This section is based on some experiments that were run by Cooper et al. (1990). In each experiment, there are 11 players. Each player plays twice against each of the other players where the matchings in each round are determined at random. Agents do not know the identity of the player they are matched with and after each play they find out which strategy the opponent selected. We analyze 6 of the experiments run by the authors. These experiments are labeled experiment 3 through 8, and the payoff matrices are depicted in Figures 4.5 through 4.10. The structure of appendices 3 through 8 is identical to that of appendices A through C, with the first three figures in each appendix depicting the simulated proportion under Cournot, the simulated proportion under fictitious play, and the observed proportions respectively. Notice that in each figure of appendices 3 through 8 that there are only three strategies; the fourth strategy was only mandated by the limitations of our graphics package.

In simulating the 1000 experiments for each of the hypotheses and each of the experiments, we actually drew 1000 ensembles of 11 agents and endowed each of them with an initial belief that is drawn uniformly over the unit simplex. We then replicated the exact matching scheme that occurred in each experiment keeping track of the beliefs of all 11 individuals in our simulated ensemble. For each experiment, then, we followed the full evolution of actions and beliefs for the 11 simulated individuals for a total of 1000 ensembles. The depicted simulated distributions for Cournot and fictitious play in appendices 3 through 8 are the empirical distribution over 1000 ensembles where the individuals were made to update according to the Cournot rules or the fictitious play rules as the figure states.

	S1	S2	S3
S1	350,350	350,250	1000,0
S2	250,350	550,550	0,0
S3	0,1000	0,0	600,600

Figure 4.5: Payoff matrix for Cooper et al.'s experiment 3.

	S1	S2	S2
S1	350,350	350,250	700,0
S2	250,350	550,550	0,0
S3	0,700	0,0	600,600

Figure 4.6: Payoff matrix for Cooper et al.'s experiment 4.

All the games analyzed have two pure strategy Nash equilibria: (S1,S1) and (S2,S2). Suppose that n_i players select strategy Si at a given period and suppose that players select strategies according to the Cournot process. Then all the players that were matched with players that selected strategies

	S1	S2	S3
S1	350,350	350,250	700,0
S2	250,350	550,550	1000,0
S3	0,700	0,1000	600,600

Figure 4.7: Payoff matrix for Cooper et al.'s experiment 5.

	S1	S2	S3
S1	350,350	350,250	700,0
S2	250,350	550,550	650,0
S3	0,700	0,650	600,600

Figure 4.8: Payoff matrix for Cooper et al.'s experiment 6.

	S1	S2	S3
S1	350,350	350,250	700,0
S2	250,350	550,550	0,0
S3	0,700	0,0	500,500

Figure 4.9: Payoff matrix for Cooper et al.'s experiment 7.

	S1	S2	S3
S1	350,350	350,250	1000,0
S2	250,350	550,550	0,0
S3	0,1000	0,0	500,500

Figure 4.10: Payoff matrix for Cooper et al.'s experiment 8.

S1 and S3 will assign probability 1 that their next opponent selects strategy S1 and S3 and thus will select strategy S1. Similarly, all players that are matched with players selecting strategy S2 will select strategy S2. Thus in any period $t \geq 2$, $n_1 + n_3$ individuals select strategy S1 and n_2 individuals select strategy S2. Thus the only difference in the Cournot processes corresponding to different payoff matrices is the proportion of initial beliefs that lead the players to selecting strategy S1 and strategy S2.

In games 3, 4, 7, 8 for almost all initial beliefs strategy S1 is a best response. Suppose an individual acts according to fictitious play. Suppose that an individual has the rare belief for which strategy S2 is a best response and is matched with another individual who adopts strategy S1. Then the second individual increases the probability she assigns to other individuals selecting strategy S2 but not sufficiently to change his strategy choice. On the other hand the first individual increases his belief that individuals select strategy S1 and thus changes her strategy choice to 1. Thus in experiments 3,4,5,7,8, if fictitious play holds, the proportion of individuals that selects strategy S1 converges to 1.

Game 5 is similar to games 3, 4, 7, 8 except that for almost all beliefs an individual selects strategy S2 and thus the proportion of individuals that selects strategy S2 converges to 1.

In game 6 the proportion of initial beliefs for which strategy S1 and S2 are best responses are about equal. However, strategy S1 is dominated by strategy S2 if strategy S3 is not played. Therefore, when individuals behave according to fictitious play the proportion of individuals that selects strategy S2 converges to 1.

The results of experiment 3,4 and 5 are consistent with fictitious play and the Cournot process since in the first two experiments the proportion of the subjects that selects strategy S1 seems to converge to 1 and in the third experiment the proportion of the individuals selecting strategy S2 seems to converge to 1.

In game 6 the two models make very different predictions: the Cournot process predicts that strategy S1 and strategy S2 are equally likely while fictitious play predicts convergence to strategy S2. The experimental data strongly favors fictitious play since the proportion of the subjects selecting strategy S2 seems to converge to 1.

The results of experiment 7,8 are inconsistent with both fictitious play and the Cournot process since both models predict that almost all subjects should select strategy S1 while in fact most of the subjects select strategy S2.

4.4 Econometric Analysis

4.4.1 The simulation procedure

In sections 2 and 3, we described the simulation procedure we used to get approximations of the likelihood function. There are two major questions that we expect the reader to ask:

1. In both experiments, why, the reader may ask, do we draw initial beliefs of all of our robots from the uniform distribution over the unit simplex? Surely, the reader may add, the distribution of beliefs should depend on the actual payoff matrix that the players get to observe. We totally agree with that statement. The problem with incorporating this type of theoretical analysis of the payoff matrix, however, should be obvious. For indeed it is the very game-theoretic contemplation of the payoff

matrices that would convince us that all players should have point mass beliefs that their opponents will play the Nash strategy, and nothing other than Nash strategies should ever be observed. That clearly takes us back full circle and does not add to our knowledge of the justifiability of the notion of Nash equilibrium on the basis of different learning algorithms.

2. One alternative approach that we chose not to follow is to put our method on its head and start with an assumption about one particular learning algorithm being the true one, and then proceeding to get an estimate of the initial distribution of beliefs. This is the procedure followed by McKelvey and Palfrey (1990). It is clear that for our purposes, we are trying to find a reasonable learning algorithm, and assuming that agents actually follow any particular algorithm will be no more justified than assuming that they all play Nash in the first place. Also, once we use our data to estimate the initial distribution of beliefs (under some parametrization of course), we cannot update our beliefs on the truth of our maintained learning hypothesis.

3. A related question to the previous one is why we do not condition our simulations on the actual observed data. In other words, we could have the robots respond by updating to the actual moves that the human subjects used. The answer is quite simple. We want, for the purposes of our Bayesian updating, to compute the likelihood function under the maintained hypothesis that initial beliefs are drawn at random, and under our two alternative models. We then compare our prior beliefs on the two models with the (theoretically) simulated likelihood

function and the observed data, and do our updating. Using the data in computing the likelihood function will interact with our assumption on the distribution of initial priors (and hence with the likelihood function) in ways that we cannot account for and would constitute a form of "data-mining."

4.4.2 The updating procedure

Now we go back to the issue of Bayesian updating of the economist's belief over the two hypotheses and discuss the derivation of the fourth figure of each appendix and the overall result of our analysis depicted in Figure 4.1.[3] Formally, let p_c^t be the experimenter's subjective probability at time t that individuals act according to the Cournot process. Let p_f^t be the experiment subjective probability at time t that individuals act according to fictitious play. Let q_c^t be the probability of observing the observed strategy choices at time t given that the individuals act according to the Cournot process. Let q_f^t be the probability of observing the observed strategy choices at time t given that the individuals act according to fictitious play. The experimenter updates her beliefs according to Bayes' rule. Then the posteriors, p_c^t, p_f^t $(t > 0)$, are determined from the priors, p_c^0, p_f^0 and the observations $\{q_c^t, q_f^t\}$ in the manner described in the following lemma. An obvious corollary to this lemma is that the order with which the experiments are analyzed does not affect the belief of the experimenter.

Lemma 11 *For all* $t \in \mathbf{N}$,

$$p_c^t = \frac{p_c^0 q_c^1 q_c^2 \cdots q_c^t}{p_c^0 q_c^1 q_c^2 \cdots q_c^t + p_f^0 q_f^1 q_f^2 \cdots q_f^t}$$

---

[3] The asymptotic aspects of the evolution of the economist's beliefs following that approach is discussed in more rigor and in more general contexts in El-Gamal and Sundaram (1989), El-Gamal and Sundaram (1990).

$$p_f^t = \frac{p_f^0 q_f^1 q_f^2 \cdots q_f^t}{p_c^0 q_c^1 q_c^2 \cdots q_c^t + p_f^0 q_f^1 q_f^2 \cdots q_f^t}$$

Proof: We will prove the lemma by induction. By Bayes' rule the equalities hold for $t = 1$. Suppose that the equalities hold for $t = k$; i.e.,

$$\begin{aligned} p_c^k &= \frac{p_c^0 q_c^1 q_c^2 \cdots q_c^k}{p_c^0 q_c^1 q_c^2 \cdots q_c^k + p_f^0 q_f^1 q_f^2 \cdots q_f^k}, \\ p_f^k &= \frac{p_f^0 q_f^1 q_f^2 \cdots q_f^k}{p_c^0 q_c^1 q_c^2 \cdots q_c^k + p_f^0 q_f^1 q_f^2 \cdots q_f^k}. \end{aligned}$$

By Bayes' rule,

$$p_c^{k+1} = \frac{p_c^k q_c^{k+1}}{p_c^k q_c^{k+1} + p_f^k q_f^{k+1}}.$$

Substituting the values for p_c^k and p_f^k in the previous expression we get:

$$\begin{aligned} p_c^{k+1} &= \frac{\frac{p_c^0 q_c^1 q_c^2 \cdots q_c^k}{p_c^0 q_c^1 q_c^2 \cdots q_c^k + p_f^0 q_f^1 q_f^2 \cdots q_f^k} q_c^{k+1}}{\frac{p_c^0 q_c^1 q_c^2 \cdots q_c^k}{p_c^0 q_c^1 q_c^2 \cdots q_c^k + p_f^0 q_f^1 q_f^2 \cdots q_f^k} q_c^{k+1} + \frac{p_f^0 q_f^1 q_f^2 \cdots q_f^k}{p_c^0 q_c^1 q_c^2 \cdots q_c^k + p_f^0 q_f^1 q_f^2 \cdots q_f^k} q_f^{k+1}} \\ &= \frac{p_c^0 q_c^1 q_c^2 \cdots q_c^{k+1}}{p_c^0 q_c^1 q_c^2 \cdots q_c^{k+1} + p_f^0 q_f^1 1 q_f^2 \cdots q_f^{k+1}}. \end{aligned}$$

Thus the equalities hold for $t = k + 1$ and thus by induction for all integers t. ∎

4.4.3 Smearing the hypotheses

If we were to put all of the probabilistic mass in our prior on rational agent theories, then a single observation which is inconsistent with all the theories in the support of the prior will make Bayesian updating impossible (it will produce a zero numerator and denominator in the Bayesian updating formula). In the Cooper et al.'s experiments there are observations of agents playing strictly dominated actions which cannot be justified under any beliefs. There are also observations of actions that, although not strictly dominated, cannot be justified under either of our learning algorithms. To make Bayesian updating applicable, we include in our models a small probability of a completely random action taking place. Specifically, we shall let each model predict actions based on the updating rule in question with probability $(1-\epsilon)$ and using a uniform (completely arbitrary) decision rule with probability ϵ. We shall then examine the predictive power of the two models for different values of ϵ.

The argument for such a procedure (other than the fact that it allows us to do Bayesian updating) is that people early on in the experiment do not understand the experiment fully, in particular if they do not wish to spend the effort (the payoffs are not particularly high: a payoff of 600, for instance, gives the subject a lottery with a probability .6 of winning one dollar). Cooper et al. (1990) show that with time there is a statistically significant decrease in the number of times a dominated strategy is selected. Furthermore, individuals have different mental costs in figuring out the experiment. Thus there is a strong argument for a different error rate for each data point. The problem with this approach is that the error component then explains all the experimental data, and thus does not allow the experimenter to differentiate between the different theories. Thus we opted to select a single ϵ for all out agents at all time periods. As the sample size gets large, the average 'correct' ϵ will get small. Thus we will be interested in the validity of the model as ϵ goes to zero. Alternatively, the value for ϵ can be taken to be the average number of mistakes over the experiment.

The two models we compare are then the following:

$$\text{Model 1: } a' = \begin{cases} a_c & \text{with probability } 1-\epsilon \\ a & \text{with probability } \frac{\epsilon}{A}, \end{cases}$$

$$\text{Model 2: } a' = \begin{cases} a_f & \text{with probability } 1-\epsilon \\ a & \text{with probability } \frac{\epsilon}{A}, \end{cases}$$

where a' is the action chosen by the experimental subjects, a_c is the action that maximizes their expected payoff given their beliefs that are updated according to the Cournot procedure, a_f is the same as a_c with fictitious play updating, and $a \in \{1, 2, \ldots, A\}$ is any strategy.

The simulation results depicted in the previous sections give us Monte-Carlo approximations to the probability in each period of the experiment that a particular action will be played. Given those probabilities and the actual observed actions of the individuals in each period, we can compute q_i^t, the probability that model i generates the data observed at time t, and then proceed with Bayesian updating as described above. The computations we use for q_i^t will be reduced form in the following sense. We shall treat the actual action of an agent in a period as a data point which is assumed to be a random draw according the probability distribution obtained from the simulated experiments with probability $(1-\epsilon)$ and as a uniform random draw with probability ϵ. Let the observed data points be indexed by $(n,t) \in \{1,2,...,N\} \times \{1,2,...,T\}$, let the actions available to each individual n in period t be $a \in \{1,2,...,A\}^4$, and let the simulated probability under the pure version of $Model\, i$ of action a in period t be $q_{a,t}^i$. Then

$$q_i^t = \prod_{n=1}^{N} \prod_{t=1}^{T} \left((1-\epsilon) q_{i,a_{n,t}}^t + \epsilon \frac{1}{A} \right)$$

where $a_{n,t}$ is the actual action chosen by agent n in period t.

$$Bad = \{(n,t) \in \{1,...,N\} \times \{1,...,T\} | q_{1,a_{n,t}}^t = 0 \text{ and } q_{2,a_{n,t}}^t = 0\}$$

---

[4] In general, A may depend on n and t, but since it does not in our experiments, we do not need to include this complication.

and let *Good* be the set of points that can be explained by at least one of the theories; i.e.,

$$Good = (\{1, ..., N\} \times \{1, ..., T\}) \setminus Bad.$$

Lemma 12 *For all $\epsilon > 0$ Bayesian updating ignores all points that are incompatible with both models.*

Proof: Notice that

$$q_i^t = (\frac{\epsilon}{A})^{\#(Bad)} \times \prod_{(n,t) \in Good} \left((1-\epsilon) q_{i,a_{n,t}}^t + \frac{\epsilon}{A} \right),$$

where $\#(A)$ is the number of elements in the set A. Then substituting in the Bayesian updating rule above, we can cancel the common factor $(\frac{\epsilon}{A})^{\#(Bad)}$ from the numerator and denominator. Thus Bayesian updating ignores these points; i.e.,

$$p_1^{t+1} = \frac{(\prod_{Good} (1-\epsilon) q_{c,a_{n,t}}^t + \frac{\epsilon}{A}) . p_1^t}{\prod_{Good} (1-\epsilon) q_{c,a_{n,t}}^t + \frac{\epsilon}{A}) . p_1^t + \prod_{Good} (1-\epsilon) q_{f,a_{n,t}}^t + \frac{\epsilon}{A}) . p_2^t}$$

■

The fourth figure in each of the nine appendices shows the posterior beliefs using the likelihood function for that experiment alone as a function of the economist's prior and the level of smearing ϵ. As we discussed in the previous section the behavior of the subjects in the experiments ran by Knott and Miller is very similar to the one predicted by fictitious play. Notice that for reasonable values of ϵ (i.e., less than sixty percent of the actions are caused by random error) and as long as the experimenter's priors put positive weight on fictitious play, then after seeing the outcome of any of the Knott and Miller's experiments, the experimenter believes with probability very close to 1 that subjects act according to fictitious play.

In Cooper et al.'s experiment 3, the proportion of subjects that selects strategy S1 converges to 1 but at a much slower rate than fictitious play. Thus, for reasonable values of ϵ, the experimenter's posteriors give probability very close to 1 to the event that subjects act according to the Cournot process. In experiment 4 subjects converge to strategy S1 at a faster rate; thus for small values of ϵ (such as $\epsilon = 0.1$) the experimenter's posteriors put probability very close to 1 on the Cournot process. For high values of ϵ (such as $\epsilon = 0.4$) the experimenter's posteriors will assign almost all the weight to fictitious play. Finally, intermediate values of ϵ will give posteriors that put positive weight on both theories.

In Cooper et al.'s experiment 5 the proportion of the population that adopts strategy S2 converges very quickly to 1. The convergence is not quite as fast as the convergence of the Cournot process and thus, for any reasonable prior and reasonable value of ϵ, the posteriors put probability very close to 1 on fictitious play.

Cooper et al.'s experiment 6 is by far the most interesting example since for this game the dynamics of fictitious play and the Cournot process are totally different. The behavior of the subjects is very similar to fictitious play and thus not surprisingly the posteriors of the experimenter put probability very close to 1 on fictitious play.

In Cooper et al.'s experiments 7 and 8 the Cournot process and fictitious play indicate that almost all subjects should adopt strategy S1. In fact almost all subjects adopt strategy S2. The posteriors after observing these experiments put probability very close to 1 on the Cournot process; however, both models are clearly inadequate for this experiment. Overall, these results lead an experimenter to believe that fictitious play describes the data better than the Cournot process and this observation is represented in Figure 1.

4.5 Concluding Remarks

In this paper, we have proposed and illustrated a fully Bayesian procedure for updating of our beliefs on a number of economic hypotheses. We showed that in the nine experiments that we analyzed, some pointed unequivocally in favor of the Cournot learning hypothesis and some pointed unequivocally in favor of the fictitious play learning hypothesis. No classical statistical procedure will allow us to combine all nine experiments to decide on what we have learned from all of them combined. Our overall Bayesian analysis of the nine experiments, however, shows that starting from any prior on the relative validity of the two learning hypotheses, we end up believing that the fictitious play learning hypothesis is infinitely more likely. It is possible that a different collection of experiments may give the opposite result, but then we can use the data from those other experiments together with the nine we have already analyzed and obtain an overall measure of the relative validity of the pair of hypotheses in question. There is a danger that published studies may be biased in one direction or another, but that is an even stronger motivation to try to run more experiments with different designs and combine all the observations. The stylized facts derived with this approach will hopefully generate more theories. We hope, for instance, that we will have new learning theories to test against fictitious play.

Appendix A

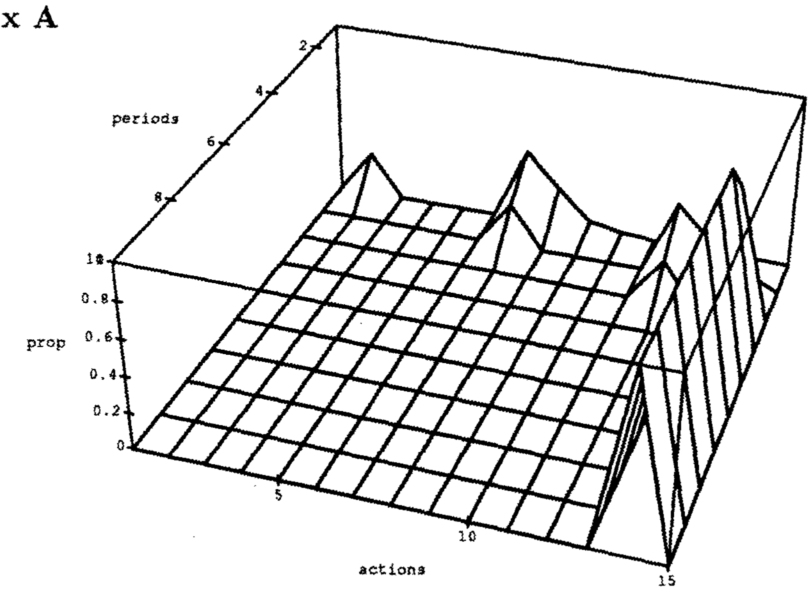

Figure 4.11: Simulation of Knott and Miller's experiment A with the Cournot process.

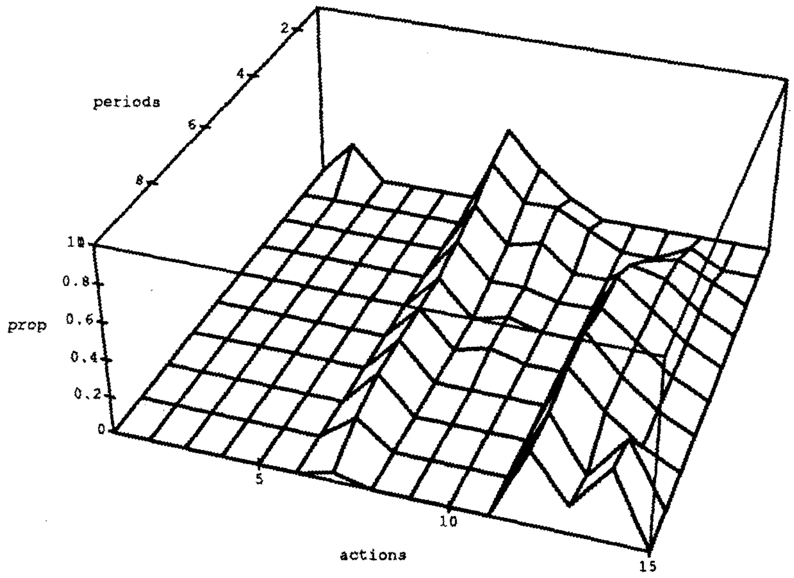

Figure 4.12: Simulation of Knott and Miller's experiment A with fictitious play.

Appendix A

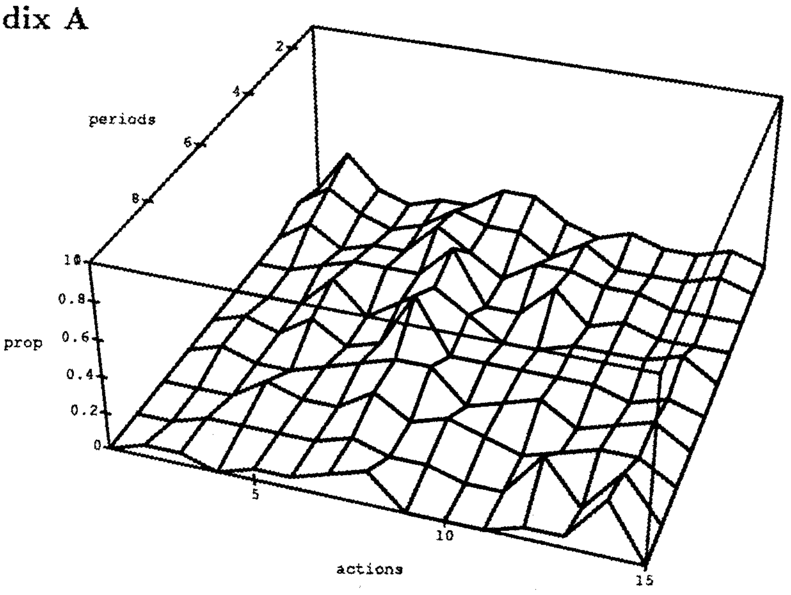

Figure 4.13: Observations from Knott and Miller's experiment A.

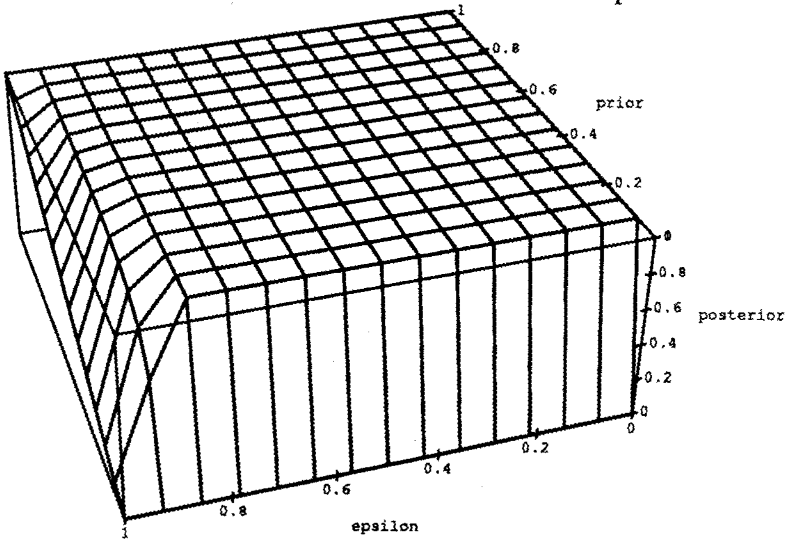

Figure 4.14: Posterior probability of agents playing according to fictitious play in Knott and Miller's experiment A.

Appendix B

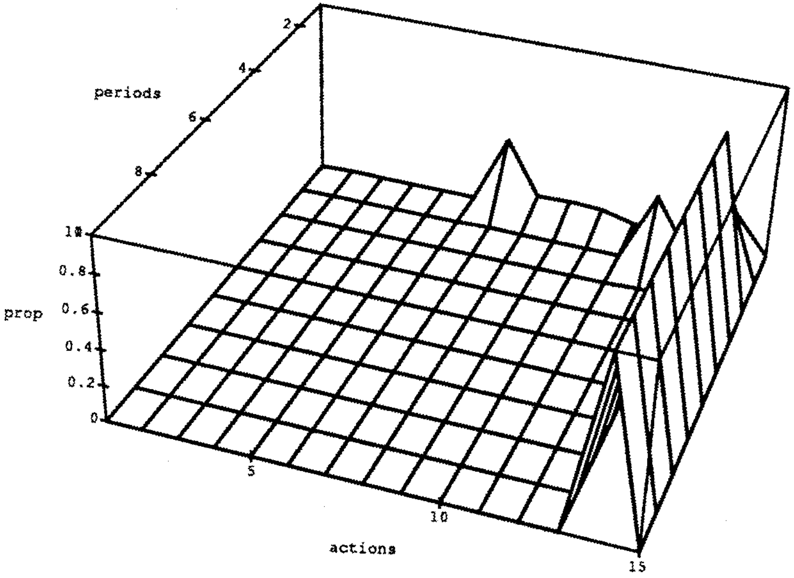

Figure 4.15: Simulation of Knott and Miller's experiment B with the Cournot process.

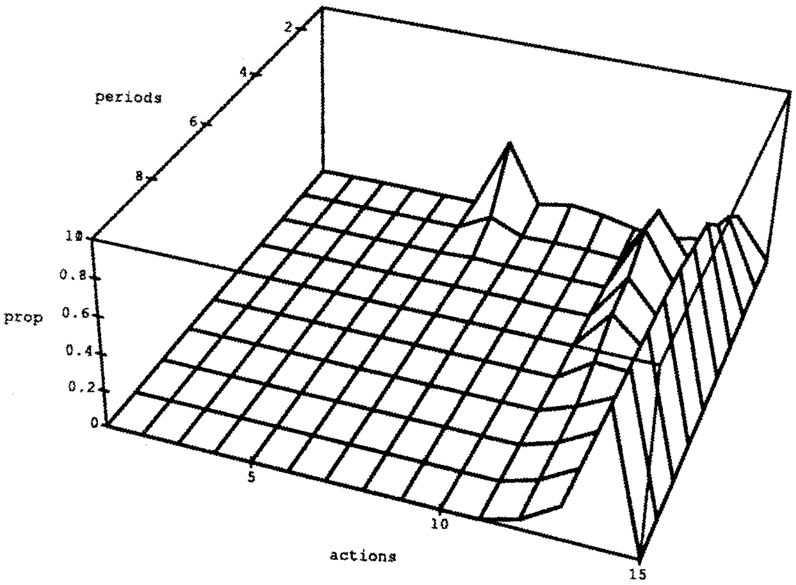

Figure 4.16: Simulation of Knott and Miller's experiment B with fictitious play.

Appendix B

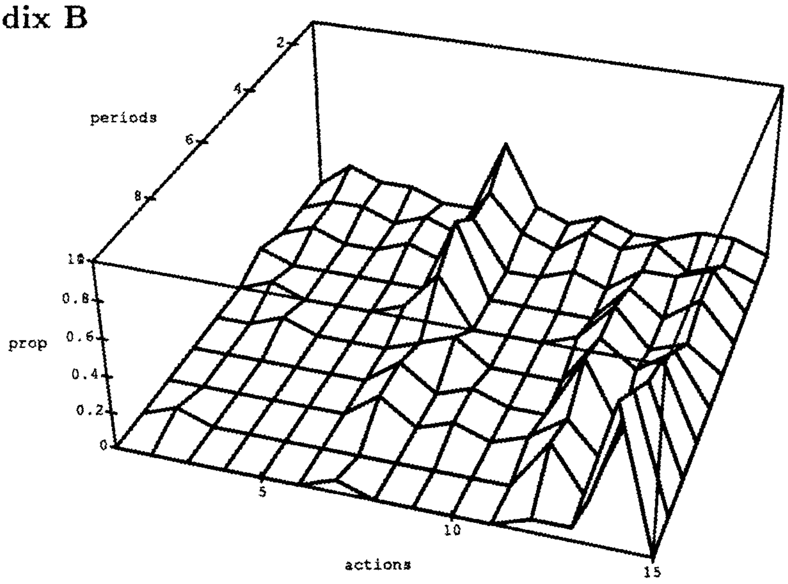

Figure 4.17: Observations from Knott and Miller's experiment B.

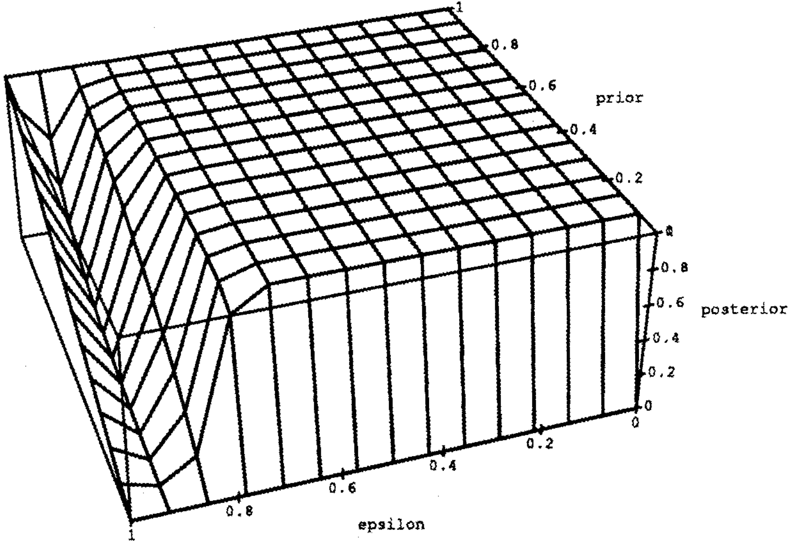

Figure 4.18: Posterior probability of agents playing according to fictitious play in Knott and Miller's experiment B.

Appendix C

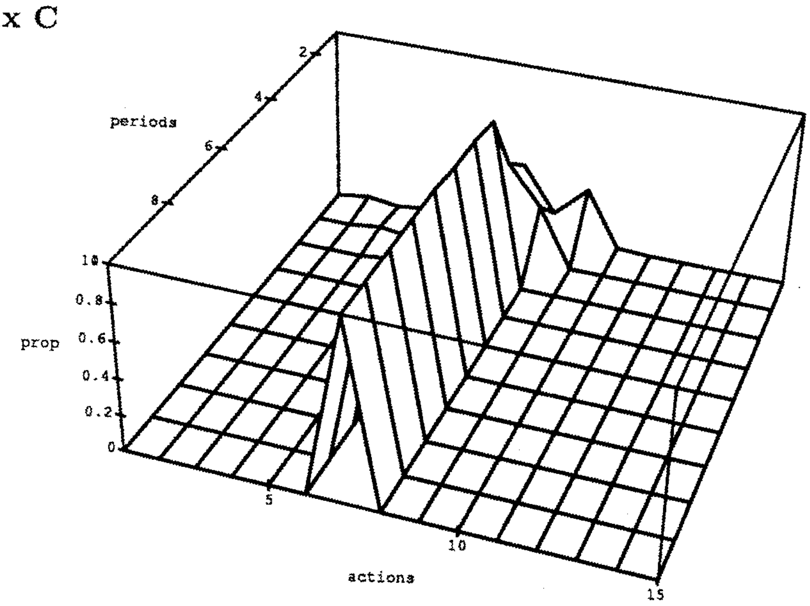

Figure 4.19: Simulation of Knott and Miller's experiment C with the Cournot process.

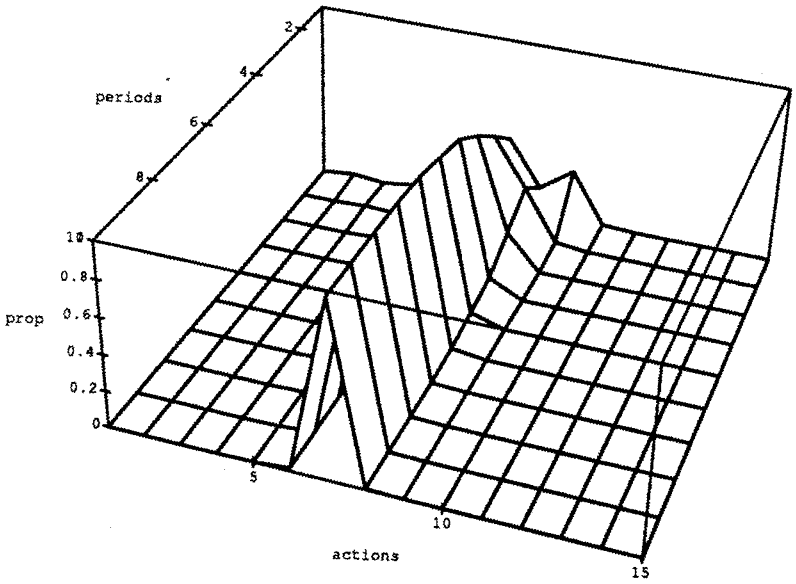

Figure 4.20: Simulation of Knott and Miller's experiment C with fictitious play.

Appendix C

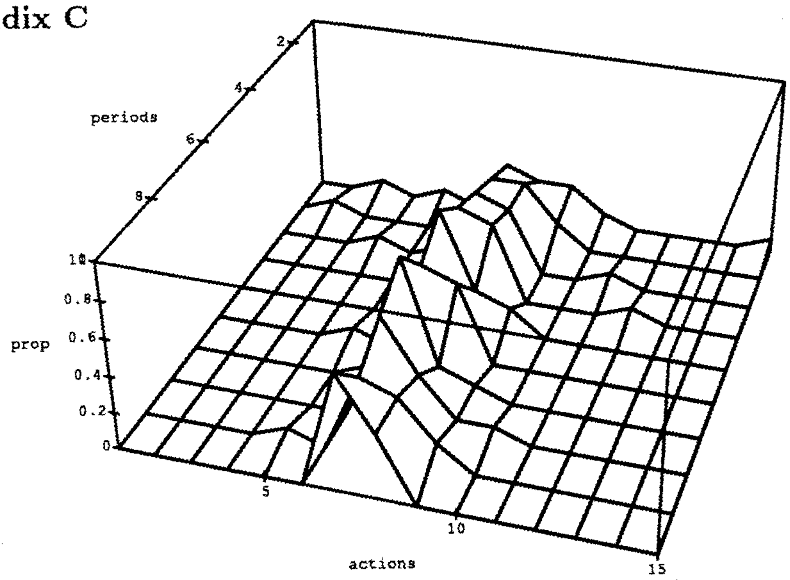

Figure 4.21: Observations from Knott and Miller's experiment C.

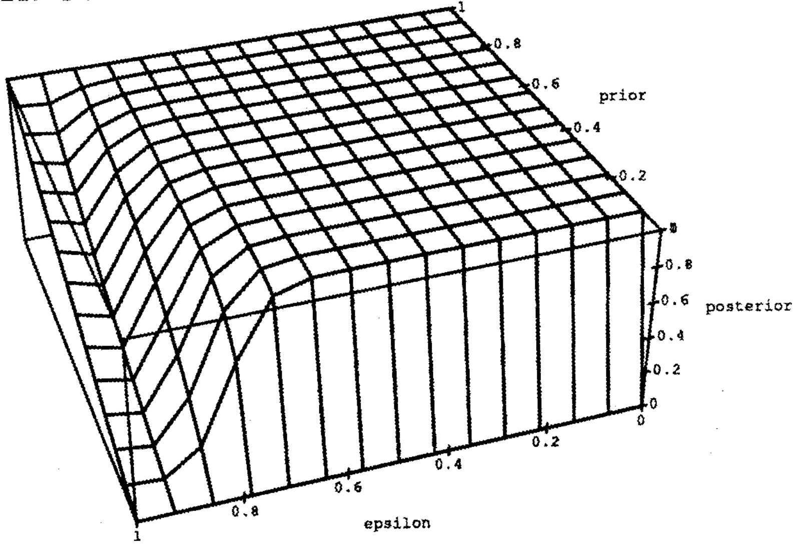

Figure 4.22: Posterior probability of agents playing according to fictitious play in Knott and Miller's experiment C.

Appendix 3

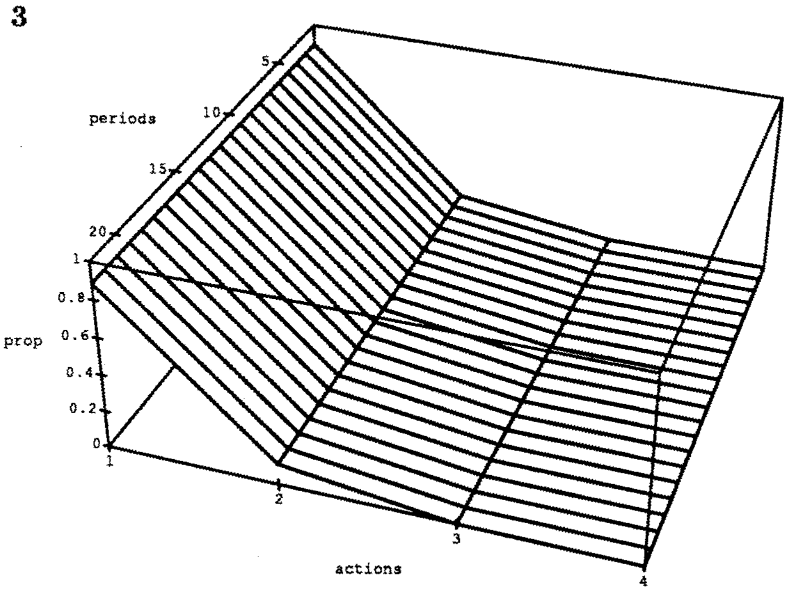

Figure 4.23: Simulation of Cooper et al.'s experiment 3 with the Cournot process.

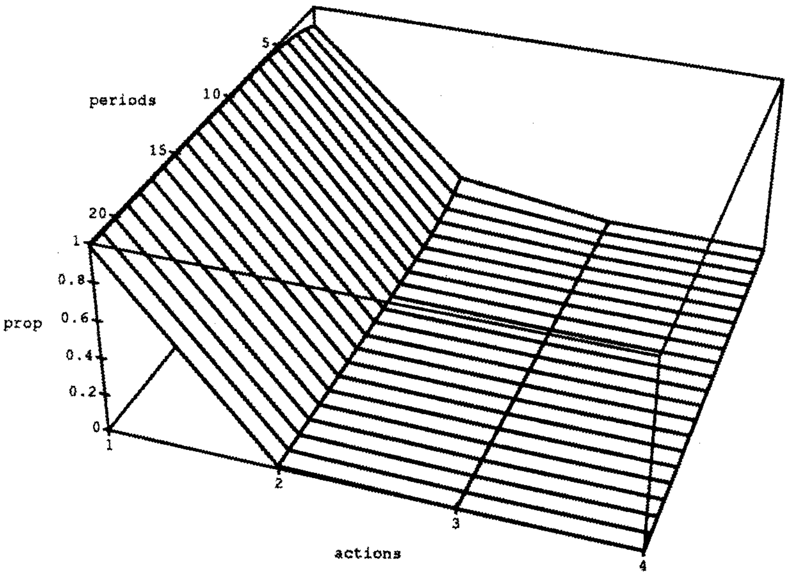

Figure 4.24: Simulation of Cooper et al.'s experiment 3 with fictitious play.

Appendix 3

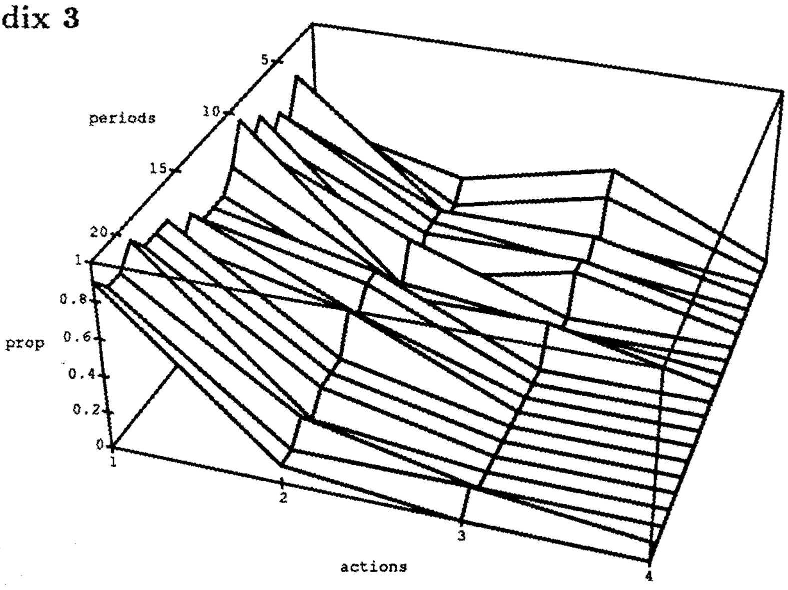

Figure 4.25: Observations in Cooper et al.'s experiment 3.

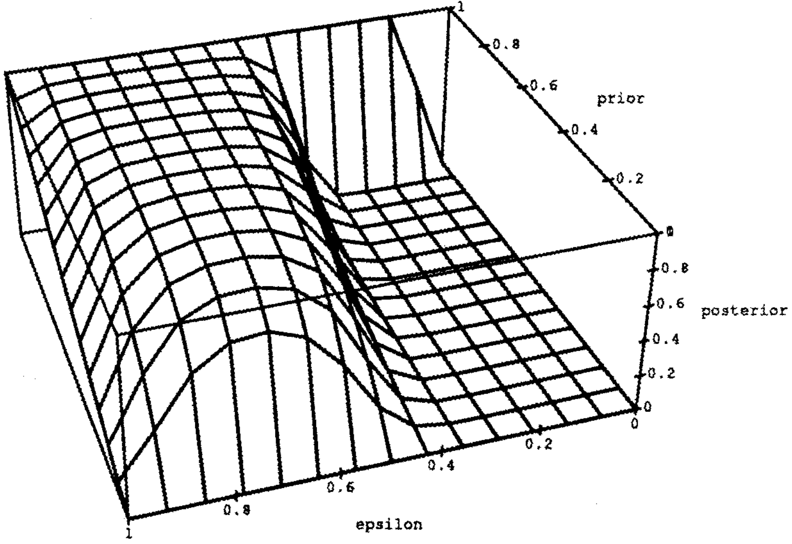

Figure 4.26: Posterior probability of agents playing according to fictitious play in Cooper et al.'s experiment 3.

Appendix 4

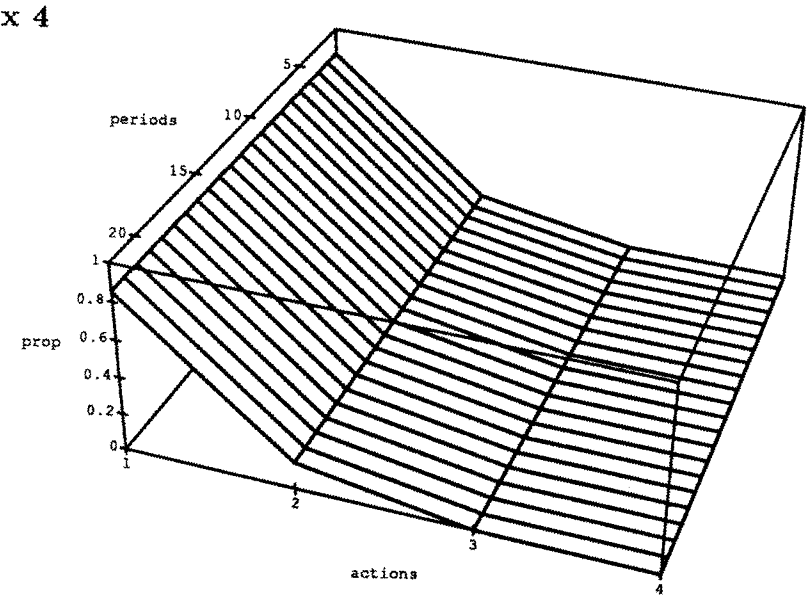

Figure 4.27: Simulation of Cooper et al.'s experiment 4 with the Cournot process.

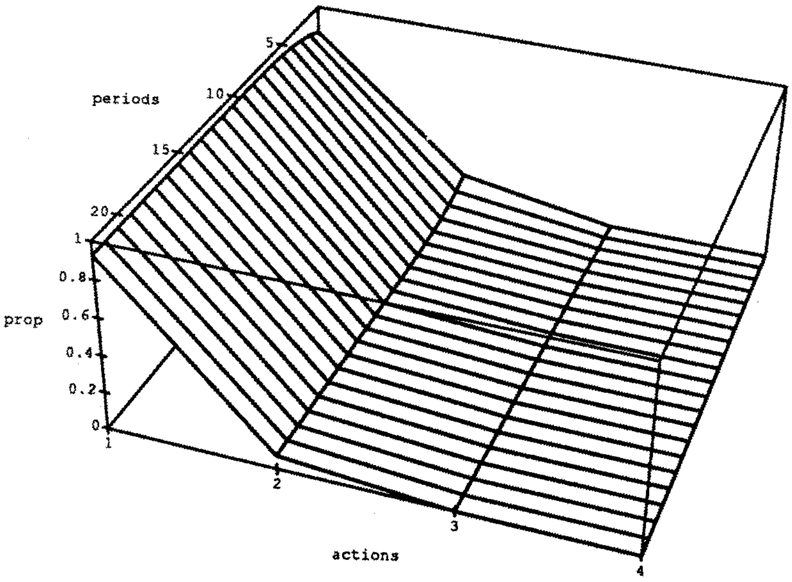

Figure 4.28: Simulation of Cooper et al.'s experiment 4 with fictitious play.

Appendix 4

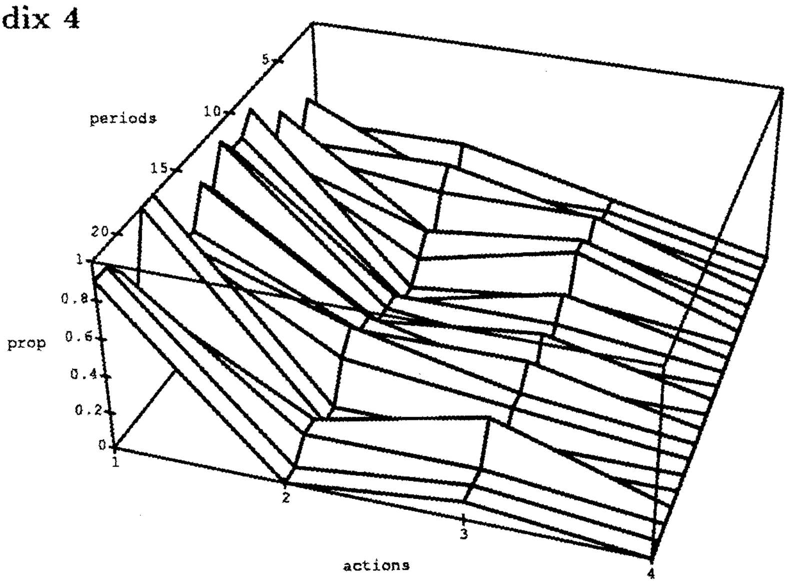

Figure 4.29: Observations in Cooper et al.'s experiment 4.

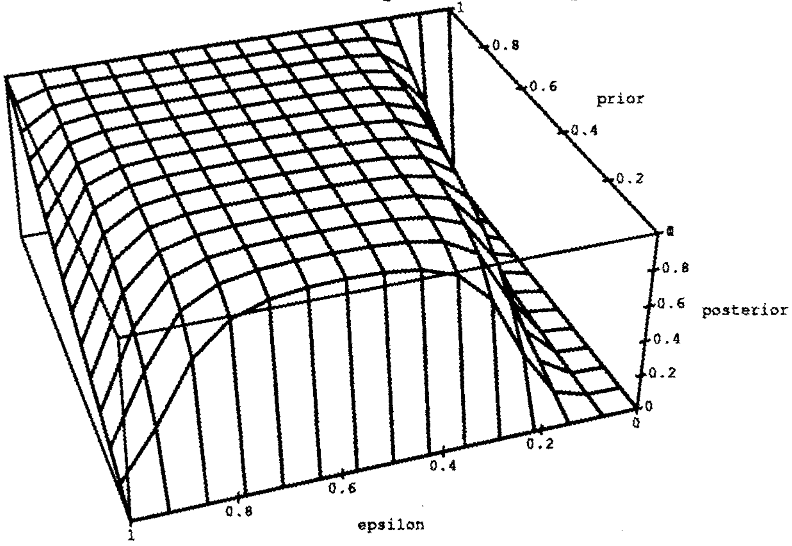

Figure 4.30: Posterior probability of agents playing according to fictitious play in Cooper et al.'s experiment 4.

Appendix 5

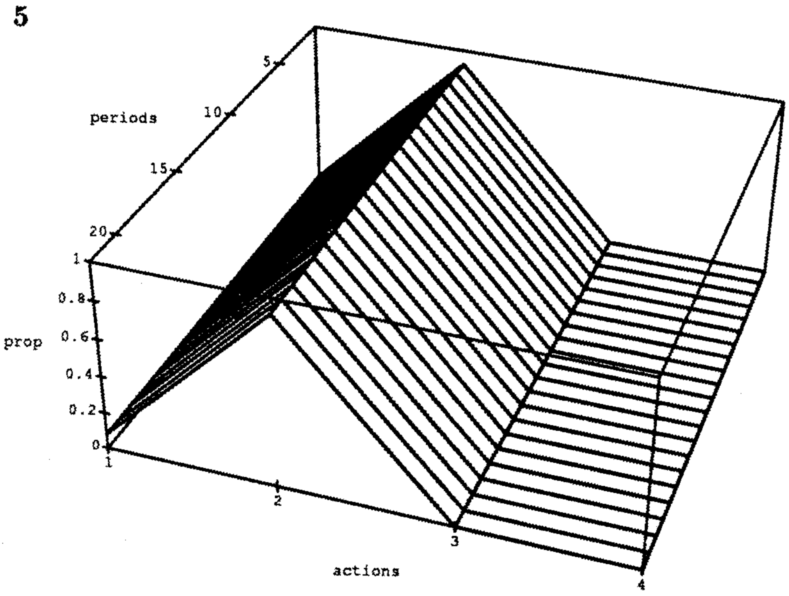

Figure 4.31: Simulation of Cooper et al.'s experiment 5 with the Cournot process.

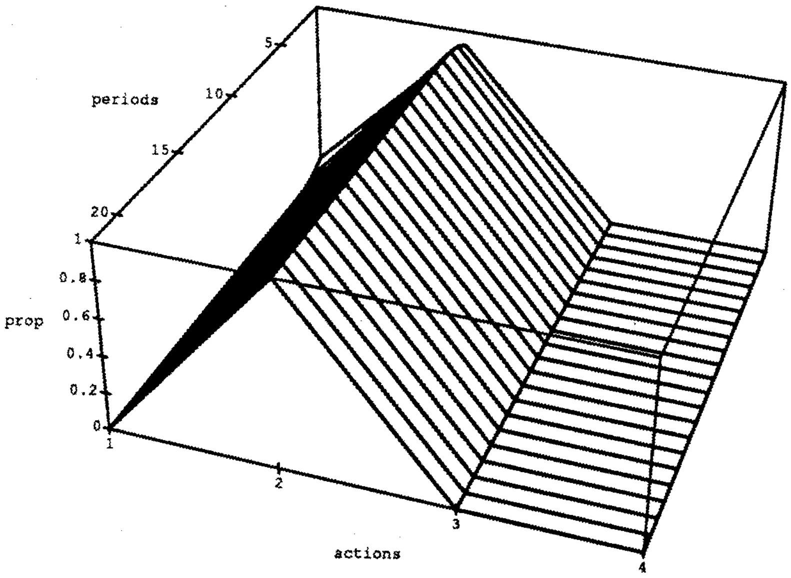

Figure 4.32: Simulation of Cooper et al.'s experiment 5 with fictitious play.

Appendix 5

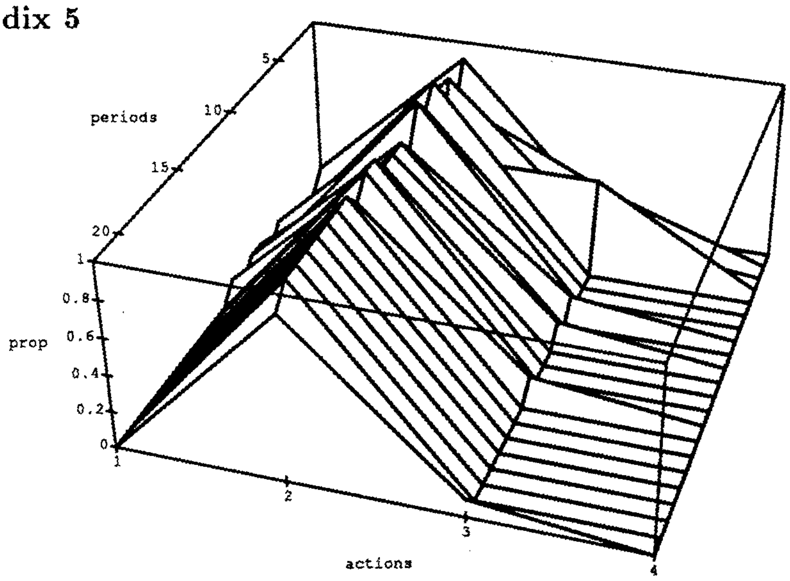

Figure 4.33: Observations in Cooper et al.'s experiment 5.

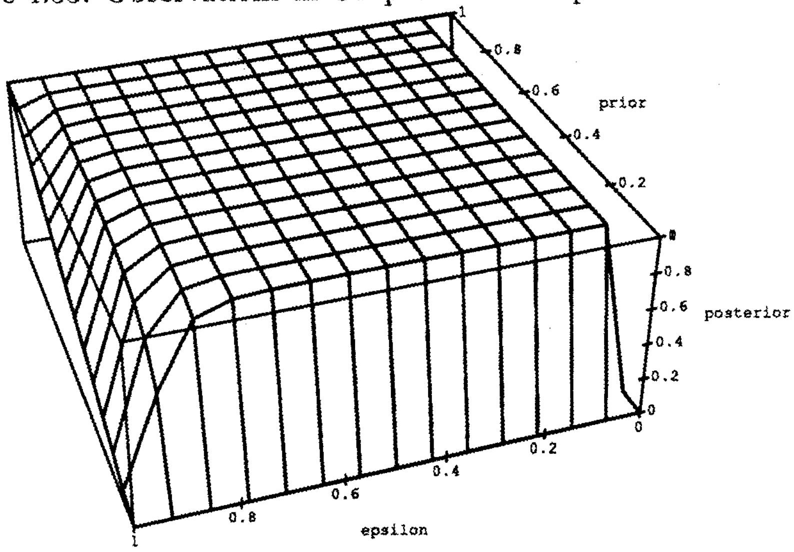

Figure 4.34: Posterior probability of agents playing according to fictitious play in Cooper et al.'s experiment 5.

Appendix 6

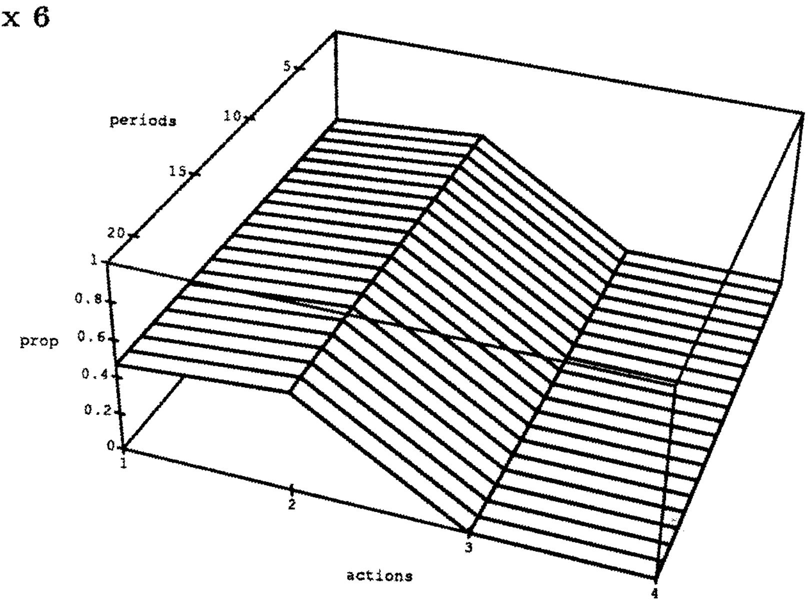

Figure 4.35: Simulation of Cooper et al.'s experiment 6 with the Cournot process.

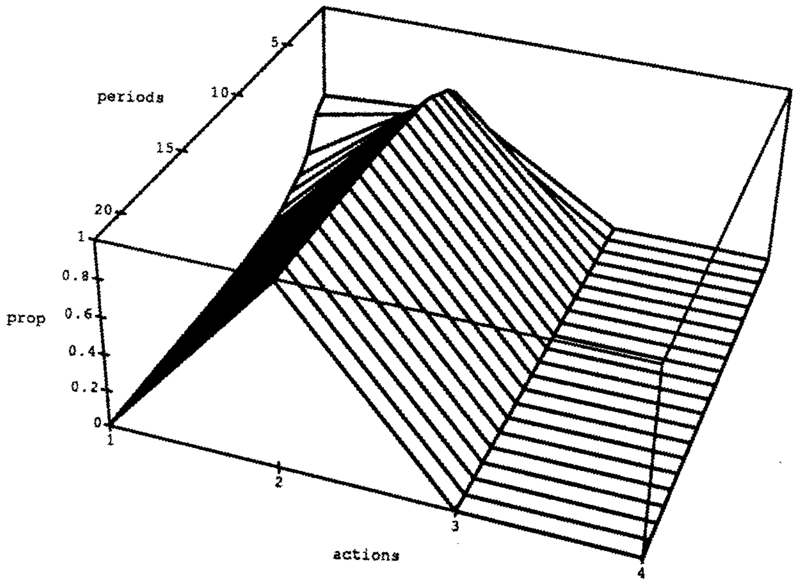

Figure 4.36: Simulation of Cooper et al.'s experiment 6 with fictitious play.

Appendix 6

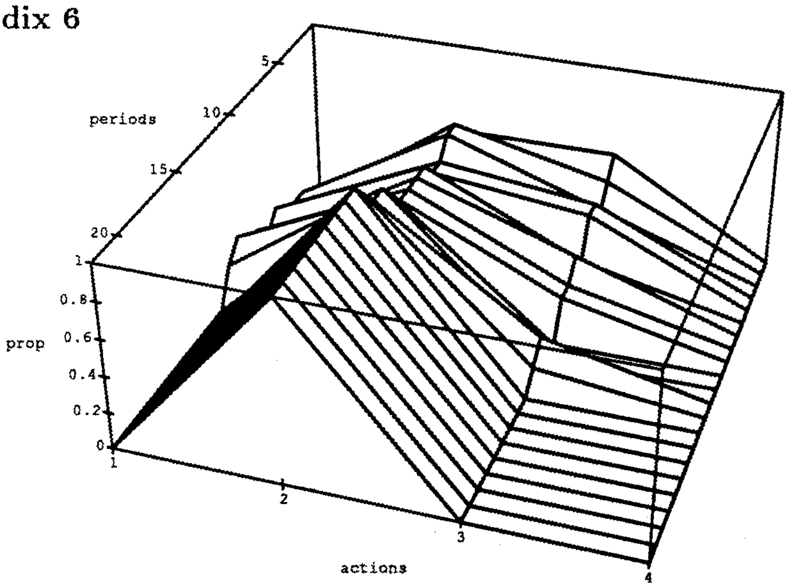

Figure 4.37: Observations in Cooper et al.'s experiment 6 process.

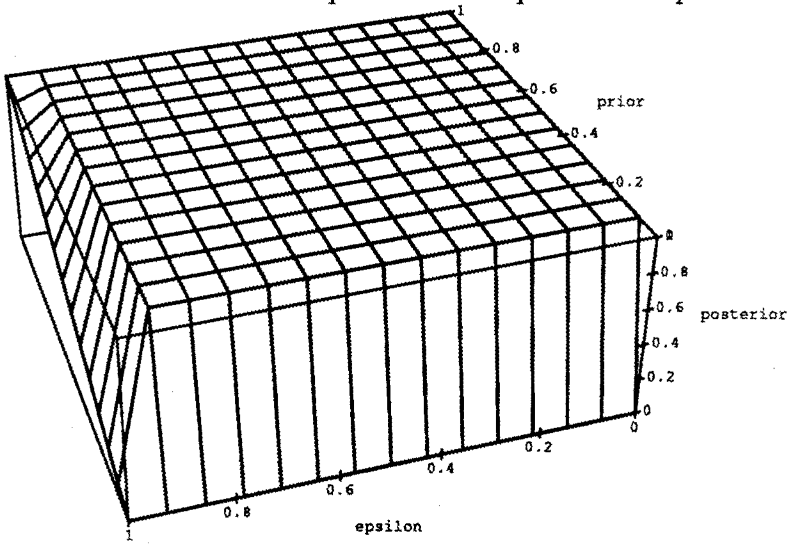

Figure 4.38: Posterior probability of agents playing according to fictitious play in Cooper et al.'s experiment 6.

Appendix 7

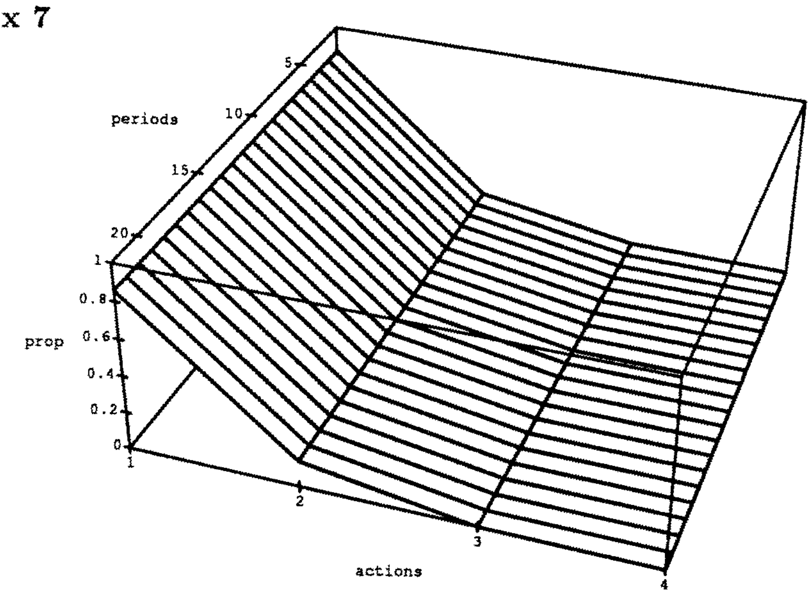

Figure 4.39: Simulation of Cooper et al.'s experiment 7 with the Cournot process.

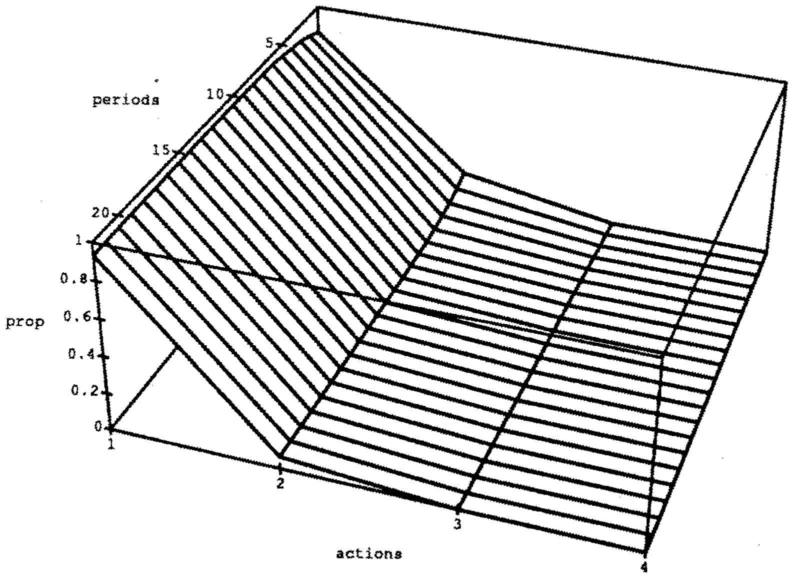

Figure 4.40: Simulation of Cooper et al.'s experiment 7 with fictitious play.

Appendix 7

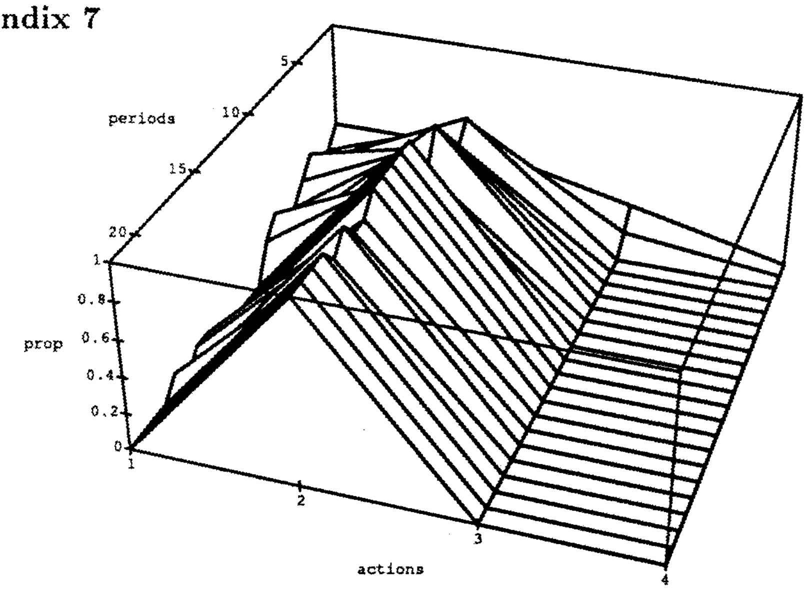

Figure 4.41: Observations in Cooper et al.'s experiment 7 process.

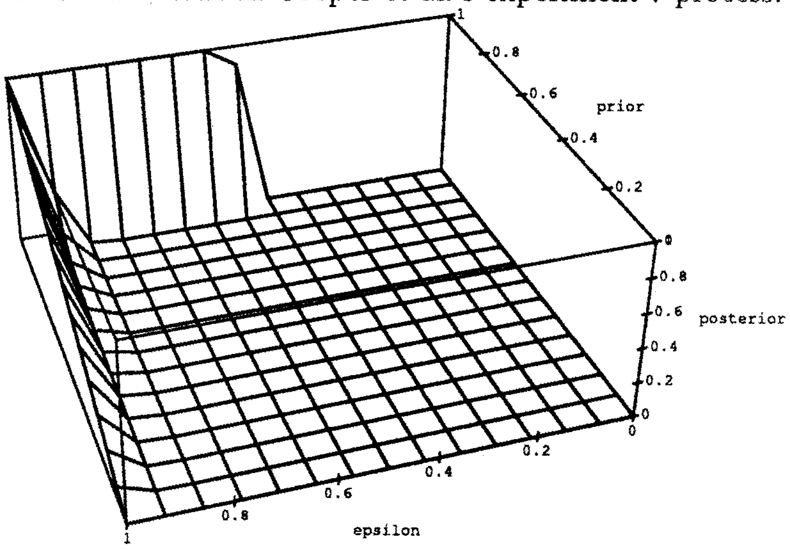

Figure 4.42: Posterior probability of agents playing according to fictitious play in Cooper et al.'s experiment 7.

Appendix 8

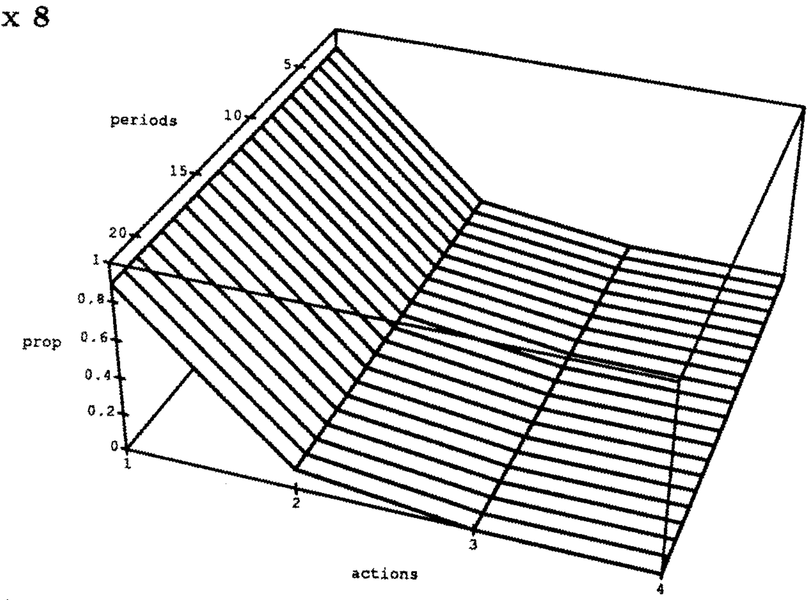

Figure 4.43: Simulation of Cooper et al.'s experiment 8 with the Cournot process.

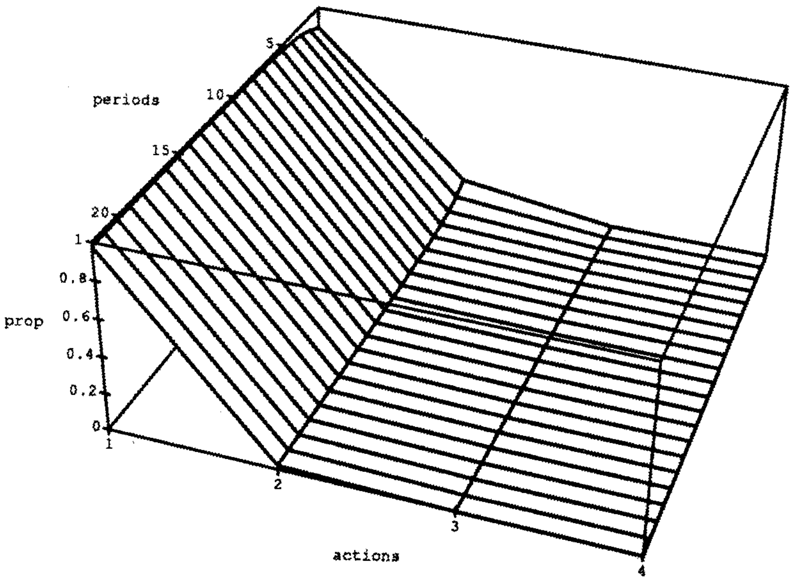

Figure 4.44: Simulation of Cooper et al.'s experiment 8 with fictitious play.

Appendix 8

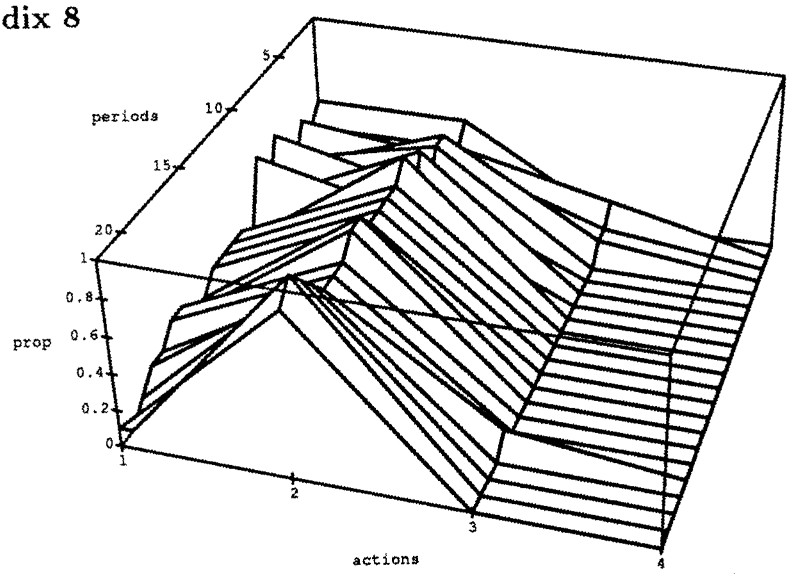

Figure 4.45: Observations in Cooper et al.'s experiment 8 process.

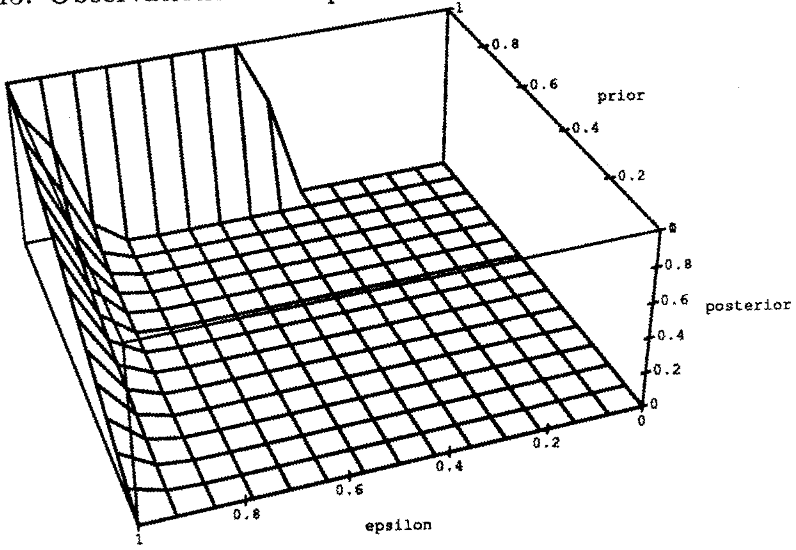

Figure 4.46: Posterior probability of agents playing according to fictitious play in Cooper et al.'s experiment 8.

Bibliography

Akin, E. and V. Losert. 1984. Evolutionary dynamics of zero-sum games. *Journal of Mathematical Biology* 20:231–258.

Alchian, A. 1950. Uncertainty, evolution and economic theory. *Journal of Political Economy* 58:211–221.

Arnold, V. I. 1973. *Ordinary differential equations.* MIT Press.

Axelrod, R. and W. Hamilton. 1981. The evolution of cooperation. *Science* 211:1390–1398.

Banks, J. and J. Sobel. 1987. Equilibrium selection in signaling game. *Econometrica* 55:647–663.

Bernheim, B. D. 1984. Rationalizable strategic behavior. *Econometrica* 52:1007–1028.

Bomze, I. M. 1986. Non-cooperative 2-person games in biology: a classification. *International Journal of Game Theory* 15:31–59.

Boyd, R. and J. Lorberbaum. 1987. No pure strategy is evolutionary stable in the repeated prisoner's dilemma game. *Nature* 237:58–58.

Breiman, L. 1968. *Probability.* Reading, Mass.: Addison-Wesley.

Brock, W., R. Marimon, J. Rust, and T. Sargent. 1988. Informationally decentralized learning algorithms for finite-player, finite-action games of incomplete information. Mimeograph.

Brown, G. 1951. Iterative solutions of games by fictitious play. In Koopmans, T., ed., *Activity Analysis of Production and Allocation*, pages 377–380. John Wiley & Sons, New York.

Cavalli-Sforza, L. L. and W. F. Bodmer. 1971. *The genetics of human populations.* San Francisco: W. H. Freeman and Company.

Cho, I.-K. and D. Kreps. 1987. Signalling games and stable equilibria. *Quarterly Journal of Economics* 102:179–221.

Cooper, R., D. DeJong, R. Forsythe, and T. Ross. 1990. Selection criteria in coordination games: Some experimental results. *American Economic Review* 80:218–234.

Crawford, V. 1988. Learning and mixed-strategy equilibria in evolutionary games. Discussion Paper 88-53.

Crawford, V. 1989. An "evolutionary" explanation of Van Huyck, Battalio, and Beil's experimental results on coordination. Discussion Paper 89.

El-Gamal, M. and R. Sundaram. 1989. Bayesian economists ... bayesian agents i: An alternative approach to optimal learning. Social Science Working Paper # 705.

El-Gamal, M. and R. Sundaram. 1990. $\aleph_0$ experiments in search of a theory. Work in progress.

Ewens, W. J. 1969. *Population genetics.* London: Methuen.

Feldman, M. and C. Gilles. 1985. An expository note on individual risk without aggregate uncertainty. *Journal of Economic Theory* 35.

Feller, W. 1967. *An introduction to probability theory and its applications*, third ed., volume I. New York: John Wiley & Sons, Inc.

Fisher, R. A. 1930. *The genetical theory of natural selection.* Oxford: Clarendon Press.

Fort, M. K. 1950. Essential and non-essential fixed points. *American Journal of Mathematics* 72:315–322.

Friedman, D. 1988. On evolutionary games in economics. Mimeograph.

Fudenberg, D. and D. Levine. 1990. Steady state learning and self-confirming equilibrium. Mimeograph.

Futia, C. A. 1982. Invariant distributions and the limiting behavior of Markovian economic models. *Econometrica* 50:377–408.

Green, E. 1989. Individual-level randomness in a nonatomic population. Working Paper # 227.

Grossman, S. and M. Perry. 1986. Perfect sequential equilibria. *Journal of Economic Theory* 39:97–119.

Hardy, G. H. 1908. Mendelian proportions in mixed population. *Science* 28:49–50. *Reprinted in* J. H. Peters, ed. *Classic Papers in Genetics.* Englewood Cliffs, N. J.: Prentice-Hall, 1959.

Harsanyi, J. C. 1973. Games with randomly disturbed payoffs: a new rationale for mixed-strategy equilibrium points. *International Journal of Game Theory* 2:1–23.

Hines, W. G. S. 1987. Evolutionary stable strategies: a review of basic theory. *Theoretical Population Biology* 31:195–272.

Hofbauer, J. 1981. On the occurrence of limit cycles in the Volterra-Lotka equation. *Nonlinear analysis* 5:1003–1007.

Hofbauer, J. and K. Sigmund. 1988. *The theory of evolution and dynamical systems.* Cambridge: Cambridge University Press.

Hoffman, E. and M. Spitzer. 1985. Experimental law and economics. *Columbia Law Review* 85:991–1036.

Jiang, J.-H. 1963. Essential component of the set of fixed points of the multivalued mappings and its application to the theory of games. *Scientia Sinica* 12:951–964.

Kimura, M. 1964. *Diffusion models in population genetics.* London: Methuen.

Klein, E. and A. Thompson. 1984. *Theory of correspondences.* New York: Wiley.

Knott, J. and G. Miller. 1987. The dynamics of convergence and cooperation: Rational choice and adaptive incrementalism. Mimeographed.

Kohlberg, E. and J.-F. Mertens. 1986. On the strategic stability of equilibria. *Econometrica* 54:1003–1038.

LeRoy, S. F. 1989. Efficient capital markets and martingales. *Journal of Economic Literature* 27:1583–1621.

Losert, V. and E. Akin. 1983. Dynamics of games and genes: discrete versus continuous time. *Journal of Mathematical Biology* 17:241–251.

Maynard Smith, J. 1982. *Evolution and the theory of games.* Cambridge: Cambridge University Press.

Maynard Smith, J. and G. R. Price. 1973. The logic of animal conflicts. *Nature* 246:15–18.

McKelvey, R. and T. Palfrey. 1990. An experimental study of the centipede game. Social Science Working Paper # 737.

Moulin, H. 1985. *Game theory for the social sciences.* New York: New York University Press.

Myerson, R. B. 1978. Refinements of the Nash equilibrium concept. *International Journal of Game Theory* 7:73–80.

Nachbar, J. 1990. "Evolutionary" selection dynamics in games: convergence and limit properties. *International Journal of Game Theory* 19:59–89.

Neveu, J. 1965. *Mathematical foundations of the calculus of probability.* San Francisco: Holden-Day.

Parthasarathy, K. R. 1967. *Probability measures on metric spaces.* New York: Academic Press.

Pearce, D. 1984. Rationalizable strategic behavior and the problem of perfection. *Econometrica* 52:1029–1050.

Plott, C. 1982. Industrial organization theory and experimental economics. *Journal of Economic Literature* 20:1485–1527.

Rao, M. M. 1981. *Foundations of stochastic analysis.* New York: Academic Press.

Rao, M. M. 1984. *Probability theory with applications.* New York: Academic Press.

Robinson, J. 1951. An iterative method for solving a game. *Annals of Mathematics* 54:296–301.

Rubinstein, A. 1988. Comments on the interpretation of game theory. STICERD discussion paper no. 88/181.

Samuelson, L. 1988. Evolutionary foundations of solution concepts for finite, two player, normal-form games. In Vardi (1988), pages 211–225.

Schuster, P., K. Sigmund, J. Hofbauer, and R. Wolff. 1981. Selfregulation of behavior in animal societies. part i: symmetric contests. *Biological Cybernetics* 40:1–8.

Selten, R. 1975. Reexamination of the perfectness concept for equilibrium points in extensive games. *International Journal of Game Theory* 4:25–55.

Shapley, L. S. 1962. On the nonconvergence of fictitious play. Memorandum RM-3026-PR.

Shiryayev, A. N. 1984. *Probability.* New York: Springer-Verlag.

Smith, V. 1982. Microeconomic systems as an experimental science. *American Economics Review* 72:923–955.

Tan, T. C. and S. Werlang. 1988. A guide to knowledge and games. In Vardi (1988), pages 163–177.

Taylor, P. D. and L. B. Jonker. 1978. A guide to knowledge and games. *Mathematical Biosciences* 40:145–156.

van Damme, E. 1987. *Stability and perfection of Nash equilibria.* Berlin: Springer-Verlag.

Vardi, M. Y., ed. 1988. *Theoretical aspects of reasoning about knowledge.* Los Altos: Morgan Kaufmann.

Weinberg, W. 1908. On the demonstration of heredity in man. *Reprinted in* S. H. Boyer, ed. *Papers on Human Genetics.* Englewood Cliffs, N.J.: Prentice-Hall, 1963.

Weising, F. 1989. Evolutionary stability and dynamics stability in generalized 'rock-scissors-paper' games. Working Paper No. 27.

Wu, W.-T. and J. Jia-He. 1962. Essential equilibrium point of n-person non-cooperative games. *Scientia Sinica* 11:1307–1322.

Zeeman, E. C. 1980. Population dynamics from game theory. In Nitecki, Z. and C. Robinson, eds., *Proc. Int. Conf. Global Theory of Dynamical Systems*, pages 471–497. Lecture Notes in Mathematics vol. 819, Springer-Verlag, Berlin.