Fourier Transforms of Certain Classes of Integrable Functions

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Abstract

Let G be a locally compact Abelian group with character group $\hat{G}$. $M(G)$ will denote the class of all bounded Radon measures on G and $P(G)$ will denote the class of all continuous positive definite functions on G. For $\mu \in M(G)$ we write $\hat{\mu}(\hat{x}) = \int_G \overline{(x, \hat{x})} d\mu(x)$ and for $\nu \in M(\hat{G})$ we write $\check{\nu}(x) = \int_{\hat{G}} (x, \hat{x}) d\nu(\hat{x})$. $[L^1(G) \cap P(G)]$ will denote the linear space spanned by $L^1(G) \cap P(G)$. We find necessary and sufficient conditions on $\hat{\mu}$ in order that $\mu \in L^1(G) \cap L^p(G)$ for $1 \leq p \leq \infty$. <u>Theorem</u> 5, <u>Chapter</u> II: $\mu \in L^1(G) \cap L^p(G)$ for $1 < p \leq \infty$ if and only if there exists a constant $K > 0$ such that $\left| \int_{\hat{G}} f(\hat{x}) \hat{\mu}(\hat{x}) d\hat{x} \right| \leq K \| \check{f} \|_q$ for all $f \in [L^1(\hat{G}) \cap P(\hat{G})]$ where $\frac{1}{p} + \frac{1}{q} = 1$. <u>Theorem</u> 6, <u>Chapter</u> II: $\mu \in L^1(G) \cap L^p(G)$ for $1 < p \leq \infty$ if and only if $\hat{\mu} \hat{f} \in (L^1(G) \cap L^p(G))^\wedge$ for all $f \in L^1(G)$. <u>Theorems</u> 3 <u>and</u> 4, <u>Chapter</u> III: $\mu \in L^1(G)$ if and only if there exists some p, $1 \leq p < \infty$, such that for each $\epsilon > 0$ there exists a $\delta > 0$ with the property that $\left| \int_{\hat{G}} f(\hat{x}) \hat{\mu}(\hat{x}) d\hat{x} \right| \leq \epsilon \| \check{f} \|_p$ whenever $f \in [L^1(\hat{G}) \cap P(\hat{G})]$ and $\| \check{f} \|_p \leq \delta \| \check{f} \|_\infty$. By taking G to be the unit circle and $p = 2$ in Theorems 3 and 4, Chapter III, we obtain a generalization of a theorem by R. Salem (Comptes Rendus Vol. 192 (1931)). Taking G to be the additive group of reals and $p = 1$ gives a generalization of a theorem by A. Berry (Annals of Math. (2) Vol. 32 (1931)).

Introduction

Consider an arbitrary locally compact Abelian group G, and let $M(G)$ denote the set of all bounded Radon measures defined on G. With suitable definitions of norm, vector addition and multiplication and scalar multiplication $M(G)$ becomes a commutative Banach algebra with an identity. The character group $\hat{G}$ is homeomorphically embedded as an open subset in the maximal ideal space of $M(G)$, and the restriction of the Gelfand representation of $M(G)$ to $\hat{G}$ is known as the Fourier-Stieltjes transform or simply the Fourier transform. This thesis is devoted to characterizing the Fourier transform of elements of certain subspaces of $M(G)$. In particular we give characterizations of those continuous functions on $\hat{G}$ which are transforms of elements of $L^1(G) \cap L^p(G)$ for $1 \leq p \leq \infty$.

In Chapter I we collect various known definitions and results on topological groups and Fourier transforms which are necessary for an understanding of Chapters II and III. Chapter I also contains a rather complete outline of the theory of a bounded Radon measure as it applies to locally compact groups. Chapter II is devoted to characterizing the Fourier transform of elements from $L^1(G) \cap L^p(G)$ for $1 < p \leq \infty$. The main theorem of Chapter II characterizes $\hat{f}$ for $f \in L^1(G) \cap L^p(G)$ in terms of a continuity property of the linear functional $\int_{\hat{G}} g(\hat{x})\hat{f}(\hat{x})d\hat{x}$, $g \in L^1(\hat{G})$. From this we deduce a multiplier theorem for the Fourier transforms of $L^1(G) \cap L^p(G)$. A closely related theorem concerning certain linear transformations of $L^1(G)$ into $L^1(G) \cap L^p(G)$ is also included in Chapter II. In Chapter III we treat $L^1(G) \cap L^p(G)$ for $p = 1$ i.e. $L^1(G)$. Here we again characterize $\hat{f}$ for $f \in L^1(G)$ in terms of a continuity property of the linear functional $\int_{\hat{G}} g(\hat{x})\hat{f}(\hat{x})\,d\hat{x}$, $g \in L^1(\hat{G})$. As a consequence of this result we obtain two classical theorems: by taking G to be the unit circle we have a theorem by R. Salem and by taking G to be the real line we have a corresponding theorem due to A. C. Berry.

Throughout this thesis numbers in brackets $[\cdot]$ refer to references listed at the end of the thesis. We abbreviate the phrase "if and only if" by "iff", and we indicate the end of a proof with the symbol ‖.

Chapter I

Function Spaces and Topological Groups

This section contains a brief summary of the topological and group theoretic concepts used throughout this thesis. Topological notation and terminology which is not explained here can be found in Kelley [1]. If X is an arbitrary topological space, $C(X)$ will denote the linear space of all continuous, bounded, complex valued functions defined on X. For $f \in C(X)$ we define $\|f\|_\infty = \sup |f(x)|$, $x \in X$. Under pointwise multiplication and with the norm $\|\cdot\|_\infty$, $C(X)$ is a Banach algebra. $K(X)$ will denote the subalgebra of those functions $f \in C(X)$ for which there exists a compact set $A \subset X$ such that $f(x) = 0$ for $x \in A'$ (= the complement of A). In general A depends upon f. The closure of $K(X)$ in the norm of $C(X)$ is denoted by $C_\infty(X)$ and consists of those and only those functions $f \in C(X)$ with the property that for every $\epsilon > 0$ there exists a compact set $A \subset X$ such that $|f(x)| < \epsilon$ for all $x \in A'$. We say for brevity that the elements of $C_\infty(X)$ vanish at infinity. $C_\infty(X)$ is a closed subalgebra of $C(X)$. $C^+(X)$, $K^+(X)$ and $C^+_\infty(X)$ will denote the sets of non-negative, real valued functions f $(f \geq 0)$ in $C(X)$, $K(X)$ and $C_\infty(X)$ respectively. In all of our applications X will be a locally compact Hausdorff space. This condition insures the existence for every compact set $C \subset X$ and every open set U with $C \subset U$ of a function $f \in K^+(X)$ with the properties that $0 \leq f(x) \leq 1$ for all $x \in X$, $f(x) = 1$ for $x \in C$ and $f(x) = 0$ for $x \in U'$.

A topological group G is a group which is also a topological space such that the mapping $(x, y) \rightarrow x - y$ of $G \times G$ onto G is continuous. We will denote the group elements by the letters x, y, z, r, s, t etc., and we will denote the group operation by +. Throughout this thesis G will be a locally compact Abelian group which is specifically assumed to be a Hausdorff space. With these assumptions the space G is normal [2]. It is always possible to choose a complete neighborhood system, $\{V_\alpha \mid \alpha \in \mathcal{A}\}$, of the group identity with the properties that i) V_α is compact for all $\alpha \in \mathcal{A}$, ii) each V_α is symmetric i.e. $x \in V_\alpha$ implies $-x \in V_\alpha$. The index set $\mathcal{A}$ is ordered by the relation $\alpha \geq \beta$ iff $V_\alpha \subset V_\beta$. With this ordering $\mathcal{A}$ is directed upward, for given any α, β then $\gamma \geq \alpha$ and $\gamma \geq \beta$ for $V_\gamma = V_\alpha \cap V_\beta$. If V is any compact symmetric neighborhood of the identity then $V^\infty = \lim_{n\to\infty} nV$ is a σ-compact, closed and open subgroup of G. This means that G is always a union (in general uncountable) of the σ-compact cosets of V^∞. If $f \in K(G)$ then f is uniformly continuous. This means that given $\epsilon > 0$ there exists a symmetric neighborhood, V, of the identity such that $|f(x) - f(y)| < \epsilon$ whenever $x - y \in V$.

A continuous homomorphism of G into the multiplicative group of complex numbers with absolute value one is called a character of G. The symbol $(x, \hat{x})$ will denote the value of the character $\hat{x}$ at the point $x \in G$. If $\hat{x}_1$ and $\hat{x}_2$ are any two characters then $(\cdot, \hat{x}_1)(\cdot, \hat{x}_2)$ is another character which we denote by $(\cdot, \hat{x}_1 + \hat{x}_2)$. Under this operation the set of characters becomes an Abelian group $\hat{G}$ called the character group of G. For each compact set $C \subset G$ and each $\epsilon > 0$ consider the set $U(C, \epsilon) = \{\hat{x} \mid |(x, \hat{x}) - 1| < \epsilon \text{ for all } x \in C\}$. By taking the sets $U(C, \epsilon)$ as a basis for the neighborhoods of the identity in $\hat{G}$, $\hat{G}$ becomes a locally compact Abelian group. Thus $\hat{G}$

will always denote the character group of G with this topology. The famous Pontrjagin Duality Theorem states that $(\hat{G})^{\wedge}$ and G are algebraically and topologically isomorphic. Other properties of topological groups can be found in [3] and [4].

Measure and Integration

Although there exist in the literature several excellent accounts of the theory of Radon measures, no single reference seems to supply a complete background for the measure theoretic aspects of this thesis. The purpose of this section is to supply that background. Most of the definitions and results collected here have been taken directly from the works of Bourbaki [5], Hewitt [6], Hewitt and Zuckermann [7], Edwards [8] and Halmos [9]. For measure theoretic terminology not explained here we refer to Halmos [9].

X will be a locally compact Hausdorff space, $\mathbb{A}$ will denote the smallest σ-ring containing all compact subsets of X and $\mathbb{B}$ will denote the smallest σ-algebra containing the compact subsets of X. We will refer to $\mathbb{B}$ as the collection of Borel subsets of X. $\mathbb{B}^*$ will be the smallest σ-algebra containing all closed subsets of X.

We begin by considering a positive linear functional M defined on the real valued functions of K(X). Such functionals are called Radon measures by Bourbaki, but we will reserve this term for the measure induced by the functional. If M is a positive linear functional on K(X) then M has the following continuity property: if $f_n \in K^+ = K^+(X)$ for $n = 1, 2, 3, \ldots,$ if all f_n vanish outside some fixed compact set and if $f_n(x) \downarrow 0$ pointwise then $M(f_n) \downarrow 0$. Using M we define an outer measure μ^* on certain subsets of X such that $M(f) = \int_X f(x) d\mu^*(x)$

for $f \in K(X)$, f real. The rest of this section is devoted to the theory of this measure.

Definition 1 For each compact set $C \subset X$ define $\mu(C) = \inf M(f)$, $\chi_C \leq f$, $f \in K^+$, where χ_C denotes the characteristic function of C.

Definition 2 For each open set $U \subset X$ define $\mu_*(U) = \sup \mu(C)$, $C \subset U$, C compact.

Definition 3 For an arbitrary set $E \subset X$ define $\mu^*(E) = \inf \mu_*(U)$, $E \subset U$, U open.

Theorem 1 (Hewitt [6] p. 72) μ^* is an outer measure.

Definition 4 A set E is said to be μ^*-measurable if $\mu^*(S) = \mu^*(S \cap E) + \mu^*(S \cap E')$ for all $S \subset X$. The class of μ^*-measurable sets will be denoted by F_μ and the restriction of μ^* to F_μ will be written as μ.

Theorem 2 (Edwards [8] p. 144) F_μ is a σ-algebra which always contains all compact subsets of X and all open subsets of X. Also μ is a measure on F_μ.

Theorem 3 (Halmos [9] p. 237) If C is a compact subset of X then $\mu(C) = \mu^*(C)$. Thus the two definitions of μ^* agree on compact sets.

Theorem 4 (Hewitt [6] p. 74) If $F \in \mathsf{F}_\mu$ and if $\mu(F) < +\infty$ then there exists a set $A \in \mathsf{A}$ such that $F = A \cup P$, $A \cap P = \phi$

and $\mu^*(P) = 0$. A can be taken as a countable union of compact sets.

Theorem 5 If $\bar{\mu}$ is the outer measure generated by (X, F_μ, μ) then $\bar{\mu} = \mu^*$.

Proof Let E be an arbitrary subset of X. We will show that $\bar{\mu}(E) = \mu^*(E)$. Since F_μ is a σ-algebra E is covered by sets of F_μ and we have $\bar{\mu}(E) = \inf \mu(F)$, $E \subset F \in \mathsf{F}_\mu$. On the other hand $\mu^*(E) = \inf \mu(U)$, $E \subset U$, U open. Since F_μ contains all open sets it follows that $\bar{\mu}(E) \leq \mu^*(E)$. If $\bar{\mu}(E) = +\infty$ then $\bar{\mu}(E) = \mu^*(E)$. Assume that $\bar{\mu}(E) < +\infty$ and let $\epsilon > 0$ be arbitrary. Then there exists a set $F \in \mathsf{F}_\mu$, $E \subset F$ such that $\mu(F) - \bar{\mu}(E) \leq \frac{\epsilon}{2}$. Since $\mu(F) = \mu^*(F)$ there also exists an open set U with $F \subset U$ and such that $\mu(U) - \mu(F) \leq \frac{\epsilon}{2}$. Adding the inequalities gives $\mu(U) \leq \bar{\mu}(E) + \epsilon$, $E \subset U$. Since ϵ is arbitrary we have $\mu^*(E) \leq \bar{\mu}(E)$. Thus $\bar{\mu}(E) = \mu^*(E)$ for $E \subset X$ and hence $\bar{\mu} = \mu^*$.‖

Theorem 6 (Edwards [8] p. 174) The restriction of μ to A is a regular measure. This means that for $A \in \mathsf{A}$ we have

$$\mu(A) = \sup \mu(C), \quad C \subset A, \quad C \text{ compact}$$

and

$$\mu(A) = \inf \mu(U), \quad A \subset U \in \mathsf{A}, \quad U \text{ open.}$$

Theorem 7 (Hewitt [7]) If $f \in K(X)$, $f(x)$ real then $M(f) = \int_X f(x)d\mu(x)$ where the integral is the usual Lebesgue-

Stieltjes integral.

F_μ will in general depend upon μ which in turn depends upon the initial functional M. It is desirable however to be able to fix our attention on some one σ-algebra of measurable sets which is suitable in some sense for the measure and integration theory and which is independent of μ. Theorem 8 provides a result in this direction.

Theorem 8 Let ν be the restriction of μ to B^* and let ν^* be the outer measure generated by (X, B^*, ν). Then $\nu^* = \mu^*$. In particular the ν^*-measurable sets are just the elements of F_μ.

Proof It was shown in Theorem 5 that the outer measure generated by (X, F_μ, μ) was μ^*. Thus in order to prove that $\nu^* = \mu^*$ it is sufficient to show that i) $\nu^* = \mu$ on F_μ and ii) $\mu^* = \nu$ on B^* (Zaanen [10] p. 23). Since $\mathsf{B}^* \subset \mathsf{F}_\mu$ (Theorem 2) ii) is trivially true. Now suppose $F \in \mathsf{F}_\mu$. We will show that $\nu^*(F) = \mu(F)$. By definition $\nu^*(F) = \inf \nu(B)$, $F \subset B \in \mathsf{B}^*$, and by Definition 3 $\mu^*(F) = \inf \mu_*(U)$, $F \subset U$, U open. Since F, $U \in \mathsf{F}_\mu$ we can write $\mu^*(F) = \mu(F)$ and $\mu_*(U) = \mu^*(U)$ giving $\mu(F) = \inf \mu^*(U)$, $F \subset U$, U open. Since $U \in \mathsf{B}^*$ $\nu(U) = \mu^*(U)$. Thus $\nu^*(F) \leq \mu(F)$. By writing $\nu^*(F) = \inf \mu(B)$, $F \subset B \in \mathsf{B}^*$ and $\mu(F) = \inf \mu(E)$, $F \subset E \in \mathsf{F}_\mu$ and recalling that $\mathsf{B}^* \subset \mathsf{F}_\mu$ we see that $\mu(F) \leq \nu^*(F)$. This combined with the opposite inequality gives $\nu^* = \mu$ on F_μ. ▮

It is not known in general whether the above theorem is true with $\mathcal{B}^*$ replaced by $\mathcal{B}$. Nevertheless this is true in those situations which are of interest in harmonic analysis. We now turn our attention to these special cases.

The translates f_s of a function f defined on the group G are defined by $f_s(x) = f(x+s)$. If M is a positive linear functional defined on $K(G)$ such that $M(f) = M(f_s)$ for all $f \in K$ and $s \in G$ then M is said to be translation invariant and is called a Haar integral. The measure induced by M is called a Haar measure. A well known theorem (Weil [3]) insures the existence and uniqueness to within a positive multiplicative factor of a non-trivial Haar measure for any arbitrary locally compact topological group. If the group is not Abelian we speak of left and right invariance and left and right Haar measures. The next theorem summarizes the most important properties of Haar measure.

<u>Theorem 9</u> If G is a locally compact Abelian group and m is a Haar measure on G then

i) $m(F) = m(F+s)$ for all $F \in \mathcal{F}_m$ and all $s \in G$

ii) $m(U) > 0$ for U open and non-empty

iii) $m(G) < +\infty$ iff G is compact

iv) $m(\{s\}) \neq 0$, $s \in G$ iff G is discrete.

<u>Definition 5</u> μ is said to be bounded if $\mu(X) < +\infty$.

We can now prove a theorem which links the measure spaces $(G, \mathcal{F}_\mu, \mu)$ and $(G, \mathcal{B}, \mu)$ for certain measures μ. Theorem 10 is a refinement of Theorem 8.

Theorem 10 Let G be an arbitrary locally compact Abelian group and let m denote a Haar measure on G. If μ is bounded or if $\mu = m$ then the outer measure generated by $(G, \mathcal{B}, \nu)$ is μ^* where ν denotes the restriction of μ to $\mathcal{B}$.

Proof Denote the outer measure generated by $(G, \mathcal{B}, \nu)$ by ν^*. The proof of Theorem 8 with $\mathcal{B}^*$ replaced by $\mathcal{B}$ shows that it is sufficient to prove that $\nu^* = \mu$ on $\mathcal{F}_\mu$. This proof also shows that $\mu(F) \leq \nu^*(F)$ for all $F \in \mathcal{F}_\mu$. Hence it is sufficient to prove that $\mu(F) = \nu^*(F)$ in the case that $\mu(F) < +\infty$. If $F \in \mathcal{F}_\mu$ and $\mu(F) < +\infty$ then by Theorem 4 $F = A \cup P$ where $A \cap P = \phi$, $A \in \mathcal{A}$, and $\mu(P) = 0$. Since $A \in \mathcal{A} \subset \mathcal{B}$ $\nu(A) = \nu^*(A) = \mu(A)$. Thus the proof is reduced to showing that $P \in \mathcal{F}_\mu$ and $\mu(P) = 0$ imply $\nu^*(P) = 0$. For both μ bounded and $\mu = m$ the proof centers on the fact that $G = \bigcup_{\gamma \in \Gamma} S_\gamma$ where each S_γ is closed and open and σ-compact, and the S_γ are all disjoint. Assume first that μ is bounded. Then $\mu(S_\gamma) = 0$ except for a countable number of the γ's, $(S_\gamma \in \mathcal{B})$. Let $\Gamma_1 = \{\gamma \mid \gamma \in \Gamma, \mu(S_\gamma) \neq 0\}$. Then $S = \bigcup_{\gamma \in \Gamma_1} S_\gamma$ is σ-compact and hence $S \in \mathcal{A} \subset \mathcal{B}$ and $S' \in \mathcal{B}$. $S' = \bigcup_{\gamma \in \Gamma - \Gamma_1} S_\gamma$, an open set. If C is any compact subset of S' then the fact that $\mu(S_\gamma) = 0$ for $\gamma \in \Gamma - \Gamma_1$ implies that $\mu(C) = 0$. Thus by Definition 3 we have $\nu(S') = \nu^*(S') = \mu(S') = 0$. For P defined as above we have $P = (P \cap S) \cup (P \cap S')$, and hence $\nu^*(P) \leq \nu^*(P \cap S) + \nu^*(P \cap S') = \nu^*(P \cap S)$. $\mu(P \cap S) = 0$ and $\mu(P \cap S) = \inf \mu(U)$, $P \cap S \subset U$, U open. Clearly we can replace U by $U \cap S$. Since $U \cap S$ is open and σ-bounded it follows (Halmos [9] p. 219) that $U \cap S \in \mathbf{A}$ and hence that $\mu(U \cap S) = \nu(U \cap S)$. Thus $P \cap S$ is covered by sets $U \cap S$ of arbitrarily small ν-measure. It follows that $\nu^*(P) \leq \nu^*(P \cap S) = 0$. This proves the case where μ is bounded. Now assume that $\mu = m$. In this case we first prove a lemma of some interest itself.

<u>Lemma 1</u> If U is open and $m(U) < +\infty$ then $U \in \mathbf{A}$.

<u>Proof of Lemma</u> $U \cap S_\gamma$ is open for all γ. By Theorem 9, ii) $m(U \cap S_\gamma) > 0$ for $U \cap S_\gamma \neq \phi$. This and the fact that $m(U) < +\infty$ imply that $U \cap S_\gamma = \phi$ except for a countable number of the γ's. Write $S = \underset{\gamma}{U} S_\gamma$ the union taken over those γ for which $U \cap S_\gamma \neq \phi$. As before S is σ-compact and hence $S \in \mathbf{A}$. Thus $U = U \cap S$ is σ-bounded and again as before $U \in \mathbf{A}$. This proves the lemma.

If $m(P) = \mu(P) = 0$ then for arbitrary $\epsilon > 0$ there exists an open set U with $P \subset U$ and $m(U) < \epsilon$. It follows from Lemma 1 that $U \in \mathbf{A} \subset \mathbf{B}$. Recalling that $\nu^*(P) = \inf \nu(B) = \inf m(B)$, $P \subset B \in \mathbf{B}$ we get $\nu^*(P) = 0$. ▌

If one is concerned with just measure theory the differences which exist between the measure spaces $(X, \mathbf{A}, \mu)$, $(X, \mathbf{B}, \mu)$ and $(X, \mathbf{F}_\mu, \mu)$ are truly significant. Theorem 10 shows that $(X, \mathbf{F}_\mu, \mu)$ can in certain cases be obtained by applying the familiar Carathéodory extension process to $(X, \mathcal{B}, \mu)$. The same extension process applied to $(X, \mathcal{A}, \mu)$ in general does not lead to $(X, \mathcal{F}_\mu, \mu)$. On the other hand starting with $(X, \mathcal{A}, \mu)$ we can always arrive at $(X, \mathcal{F}_\mu, \mu)$ by the extension process indicated in Definitions 2 and 3. These differences which exist in the measure theory completely vanish in the theory of certain L^p spaces. This is explained in the remark following Theorem 11.

Definition 6 Let $\mathcal{S}$ be an arbitrary σ-ring of subsets of X such that $\bigcup_{S \in \mathcal{S}} S = X$ and $\mathcal{S} \subset \mathcal{F}_\mu$. For $1 \leq p < \infty$ define $\mathcal{L}^p(X, \mathcal{S}, \mu)$ to be the set of all $\mathcal{S}$-measurable complex valued functions on X with the property that

$$\| f \|_p = \left(\int_X |f(x)|^p \, d\mu(x) \right)^{\frac{1}{p}} < +\infty.$$

$\mathcal{L}^\infty(X, \mathcal{S}, \mu)$ denotes the space of those $\mathcal{S}$-measurable functions for which

$$\| f \|_\infty = \inf \left\{ \alpha \,\middle|\, \mu\{ x \mid x \in G, \ |f(x)| > \alpha \} = 0 \right\} < +\infty.$$

It should be observed that if $\mathcal{B}^* \subset \mathcal{S}$ and μ is Haar measure then this definition of $\| f \|_\infty$ agrees with that given on page 3 for $f \in C(X)$.

Definition 7 If $\mathcal{N}^p(X, \mathcal{S}, \mu) = \left\{ f \,\middle|\, f \in \mathcal{L}^p(X, \mathcal{S}, \mu), \ \| f \|_p = 0 \right\}$ define

$$L^p(X, \mathcal{S}, \mu) = \mathcal{L}^p(X, \mathcal{S}, \mu) \Big/ \mathcal{N}^p(X, \mathcal{S}, \mu) .$$

Here we repeat the often mentioned fact that the elements of $L^p(X,\mathsf{S},\mu)$ are, strictly speaking, equivalence classes of functions. As usual we will denote these classes by representatives i.e. by the elements of $\mathcal{L}^p(X,\mathsf{S},\mu)$. In the same spirit we write the norm of $L^p(X,\mathsf{S},\mu)$ as $\|\cdot\|_p$. It is well known that with this norm $L^p(X,\mathsf{S},\mu)$ is a Banach space.

<u>Theorem 11</u> (Hewitt [6] p. 74) If $f \in \mathcal{L}^p(X,\mathsf{F}_\mu,\mu)$ for $1 \leq p < \infty$ then there exists a function $\tilde{f} \in \mathcal{L}^p(X,\mathsf{A},\mu)$ such that $\|f - \tilde{f}\|_p = 0$.

This theorem combined with the fact that $\mathsf{A} \subset \mathsf{B} \subset \mathsf{F}_\mu$ shows that the three spaces $L^p(X,\mathsf{A},\mu)$, $L^p(X,\mathsf{B},\mu)$ and $L^p(X,\mathsf{F}_\mu,\mu)$ are isomorphic and isometric Banach spaces. In this thesis we will be mainly concerned with the spaces $L^p(G,\mathsf{B},m)$ where G is a locally compact Abelian group and m is a Haar measure on G. We will write $L^p(G)$ for $L^p(G,\mathsf{B},m)$. In so doing we assume when we write $f \in L^p(G)$ that f is B-measurable.

We end this section with a short discussion of the Radon-Nikodym theorem and absolute continuity of measures. A measure μ is said to be absolutely continuous with respect to a measure ν if $\nu(E) = 0$ implies that $E \in \mathsf{F}_\mu$ and that $\mu(E) = 0$. The classical Radon-Nikodym theorem states that a bounded measure μ which is absolutely continuous with respect to a σ-finite measure ν can be represented in the form $\mu(E) = \int_E f(x)\, d\nu(x)$ for all $E \in \mathsf{F}_\mu$ where f is a representative of a unique class of $L^1(X,\mathsf{F}_\nu,\nu)$. In general a Haar measure is not σ-finite, but it is nevertheless an important fact that the Radon-Nikodym theorem holds for bounded measures which are absolutely continuous with respect to a Haar measure. The reason behind this result is the fact that G is always a disjoint union of σ-compact sets. One of the implications of the Radon-Nikodym theorem for G is that the Banach dual of $L^p(G)$ for $1 \leq p < \infty$ is $L^q(G)$ where $\frac{1}{p} + \frac{1}{q} = 1$. The following theorem on absolute continuity will be used in Chapter III.

Theorem 12 If μ is a bounded measure on a locally compact group G and if m is a Haar measure then μ is absolutely continuous with respect to m iff for every $\epsilon > 0$ there exists a $\delta = \delta(\epsilon, \mu) > 0$ such that $m(E) < \delta$ implies $\mu(E) < \epsilon$ for all sets $E \in \mathbf{A}$ which are open and have compact closure.

Proof If μ is absolutely continuous with respect to m then the condition follows directly from the Radon-Nikodym theorem.

Suppose now that the condition of the theorem holds and that μ is not absolutely continuous with respect to m. Then there exists a set E such that $m(E) = 0$ and such that $\mu^*(E) > 0$. (If $\mu^*(E) = 0$ then $E \in \mathbf{F}_\mu$.) Here μ^* is the μ^* of Definition 4. Since μ is bounded there exists a σ-compact set $S \in \mathbf{A}$ which is both closed and open and is such that $\mu(S') = \mu^*(S') = 0$. (See proof of Theorem 10.) Thus we may as well assume that $E \subset S$. Since S is σ-compact we can cover S with a countable collection of sets $A_n \in \mathbf{A}$ with the property that each A_n is open and $\overline{A}_n$ is compact. Then since $E \subset S$ we have $\mu^*(E) = \mu^*(E \cap S) \leq \sum_{n=1}^{\infty} \mu^*(E \cap A_n)$. Thus it is sufficient to show $\mu^*(E \cap A_n) = 0$. Assume that this is not so and that for $n = N$ $\mu^*(E \cap A_n) = \eta > 0$. Take $\epsilon > 0$ $0 < \epsilon < \eta$ and $\delta = \delta(\epsilon, \mu) > 0$. Since $m(E \cap A_n) = 0$ and since m is outer regular (Definition 3) there exists an open set $B \in \mathcal{A}$ such that $E \subset B \subset A_n$, $\overline{B}$ compact and $m(B) < \delta$. By the hypothesis of the theorem this implies that $\mu(B) < \epsilon$. Since $\mu^*(E \cap A_n) \leq \mu^*(B) = \mu(B)$ we have a contradiction. Hence $\mu^*(E) = 0$ and μ is absolutely continuous with respect to m. ▌

$L^1(G)$, $M(G)$ and Fourier-Stieltjes Transforms

The most widely studied object in abstract harmonic analysis is the space $L^1(G) = L^1(G, \mathcal{B}, m)$. If $f, g \in L^1(G)$ then

$$f * g(x) = \int_G f(x - y)\, g(y)\, dy$$

exists for almost all $x \in G$ (with respect to Haar measure) and $f * g$ is called the convolution of f and g. (Here as in the remainder of the thesis we will denote $dm(x)$ by dx, dy etc.) The Fubini theorem shows that $f * g \in L^1(G)$ and that $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$. This convolution is associative, commutative (for Abelian groups) and satisfies the additional condition that $\alpha(f * g) = \alpha f * g = f * \alpha g$ for all complex numbers α. Thus $L^1(G)$ forms a Banach algebra under the norm $\|\cdot\|_1$ and with convolution for multiplication. It is further true that if $f \in L^1(G)$, $g \in L^p(G)$, $1 \leq p \leq \infty$ then $f * g \in L^p(G)$ and

$\|f * g\|_p \leq \|f\|_1 \|g\|_p$. The fundamental theory of the algebra $L^1(G)$ can be found in Loomis [4].

It is well known that the algebra $L^1(G)$ has an identity iff the group G has the discrete topology. In the case that G is not discrete a so called approximate identity is an indispensible tool for abstract harmonic analysis. Let $\{U_\alpha \mid \alpha \in \mathcal{A}\}$ be a complete system of compact symmetric neighborhoods of the group identity. For each $\alpha \in \mathcal{A}$ let u_α be a positive, bounded and measurable function which vanishes outside U_α and such that $\|u_\alpha\|_1 = 1$. (Since G is a locally compact Hausdorff space we may choose $u_\alpha \in K^+(G)$.) The net $\{u_\alpha \mid \alpha \in \mathcal{A}\}$ is called an approximate identity for $L^1(G)$. If A_α, $\alpha \in \mathcal{A}$ is a set of complex numbers then we will write $\lim_\alpha A_\alpha = A$ if for every $\epsilon > 0$ there exists a $\beta \in \mathcal{A}$ such that $|A - A_\alpha| < \epsilon$ whenever $\alpha \geq \beta$. The following two theorems are fundamental results for approximate identities.

Theorem 13 (Loomis [4] p. 124) Let $\{u_\alpha \mid \alpha \in \mathcal{A}\}$ be an approximate identity for $L^1(G)$. If $f \in L^p(G)$, $1 \leq p < \infty$, then $\lim_\alpha \|u_\alpha * f - f\|_p = 0$.

Theorem 14 If f is uniformly continuous on G then $\lim_\alpha u_\alpha * f(x) = f(x)$ uniformly.

Proof Since $f \in L^\infty(G) = L^1(G)^*$ and since $u_\alpha \in L^1(G)$

$$u_\alpha * f(x) = \int_G f(x - y)\, u_\alpha(y)\, dy$$

for all $x \in G$. Since $\int_G u_\alpha(y)\, dy = 1$ we can write

$$f(x) = \int_G f(x)\, u_\alpha(y)\, dy.$$

Thus

$$u_\alpha * f(x) - f(x) = \int_G [f(x-y) - f(x)]\, u_\alpha(y)\, dy.$$

The uniform continuity of f implies that for every $\epsilon > 0$ there exists a symmetric neighborhood U of the identity such that $|f(x_1) - f(x_2)| \leq \epsilon$ for $x_1 - x_2 \in U$. Choose β such that $U_\beta \subset U$. Then for $\alpha \geq \beta$ $|f(x-y) - f(x)| \leq \epsilon$ when $y \in U_\alpha$. This gives

$$|u_\alpha * f(x) - f(x)| \leq \int_G |f(x-y) - f(x)|\, u_\alpha(y)dy = \int_{U_\alpha} |f(x-y) - f(x)|\, u_\alpha(y)dy \leq \epsilon.$$

Hence $\lim_\alpha u_\alpha * f(x) = f(x)$ uniformly. ∥

Every $f \in L^1(G)$ determines a bounded linear functional F on $K(G)$ where $F(g) = \int_G g(x) f(x)\, dx$, $|F(g)| \leq \| f \|_1 \| g \|_\infty$ for all $g \in K(G)$ and $\| F \| = \| f \|_1$. In the case where G has the discrete topology $L^1(G)$ accounts for all of the Banach dual of $K(G)$ i.e. $K(G)^* = L^1(G)$ (Note $K(G)^* = C_\infty(G)^*$ since $K(G)$ is dense in $C_\infty(G)$). For non-discrete groups this is not the case. The Riesz representation theorem states however that for every $M \in K(G)^*$ there exists a unique complex valued, countably additive set function μ defined on $\mathcal{B}$ of the form $\mu = \mu_1 - \mu_2 + i(\mu_3 - \mu_4)$ where each μ_j is a non-negative, bounded Radon measure and such that $M(g) = \int_G g(x)\, d\mu(x)$, $g \in K(G)$. Conversely each set function of this kind is called a bounded Radon measure. $M(G)$ will denote the set of all bounded Radon measures and $M(G)$ will be identified with $K(G)^* = C_\infty(G)^*$. This identification has been used successfully by Hewitt and Zuckermann [10] for the investigation of $M(G)$. (See Hewitt [2] pp. 132-133 for a survey of this method.) An alternative approach outlined by Rudin [11] pp. 229-230 introduces algebraic and topological structure directly into $M(G)$ considered only as a set of measures. Both of these methods lead to the same results and the particular method one uses is a matter of taste. In the following discussion we use interchangeably whichever approach seems to be most useful for the purpose of this thesis.

For $\lambda, \mu \in M(G)$ and α, β complex we define $(\alpha\lambda + \beta\mu)(E) = \alpha\lambda(E) + \beta\mu(E)$ for all $E \in \mathcal{B}$. With this notion of addition and scalar multiplication $M(G)$ is a vector space over the complex field. A norm is introduced in $M(G)$ by defining $\|\mu\| = \sup \sum_{n=1}^{\infty} |\mu(B_n)|$ where the sup is taken over all countable disjoint coverings of G by sets from $\mathcal{B}$. $M(G)$ endowed with this norm is a Banach space. With this structure $M(G)$ is topologically and algebraically isomorphic to the Banach space $K(G)^*$. In particular we have $\|\mu\| = \sup \left| \int_G g(x) d\mu(x) \right|$, $g \in K(G)$, $\|g\|_\infty \leq 1$. The convolution defined in $L^1(G)$ can be extended in at least two equivalent ways to $M(G)$. The method used by Hewitt and Zuckermann [10] uses the notion of the convolution of functionals on $K(G)$ while the method of Rudin [11] uses the idea of a product measure on $G \times G$. Both of these methods lead to the same result: $(\mu * \lambda)(E) = \int_G \mu(E-x) d\lambda(x)$ for all

$E \in \mathcal{B}$. This convolution is associative, commutative and $\alpha(\mu * \lambda) = \alpha\mu * \lambda = \mu * \alpha\lambda$ for all $\lambda, \mu \in M(G)$ and all complex numbers α. The convolution is also continuous in the norm of $M(G)$ i.e. $\|\mu * \lambda\| \leq \|\mu\| \|\lambda\|$ thus making $M(G)$ a commutative Banach algebra. Unlike $L^1(G)$, $M(G)$ always has an identity, namely that measure which assigns the value one to the group identity and zero to every set which does not contain the identity. Haar measure m is an element of $M(G)$ iff G is compact.

Each $\mu \in M(G)$ can be uniquely decomposed into three measures $\mu = \mu_a + \mu_s + \mu_d$ where μ_a is absolutely continuous with respect to m, μ_s is continuous and singular with respect to m and μ_d is discrete. We denote by $M_a(G)$, $M_s(G)$ and $M_d(G)$ the corresponding subspaces of $M(G)$. The Radon-Nikodym theorem establishes an isometric isomorphism between the Banach algebras $L^1(G)$ and $M_a(G)$. We will often write $\mu \in L^1(G)$ and $\mu \in M_a(G)$ interchangeably. Both $L^1(G)$ and $M_a(G) + M_s(G)$ form closed ideals in $M(G)$. $M_d(G)$ is a closed subalgebra of $M(G)$.

For each $\mu \in M(G)$ there exists a unique non-negative Radon measure $|\mu| \in M(G)$ with the property that $|\mu|(E) \geq |\mu(E)|$ for $E \in \mathcal{B}$ and if $\lambda \in M(G)$, $\lambda(E) \geq |\mu(E)|$ then $\lambda(E) \geq |\mu|(E)$. $|\mu|$ has the additional property that $|\mu| \in M_a(G)$ iff $\mu \in M_a(G)$ and similarly for $M_s(G)$ and $M_d(G)$. For $\mu \in M(G)$ we always have $|\mu|(G) = \|\mu\|$.

A function $f \in C(G)$ is not in general $\mathcal{A}$-measurable or even $\mathcal{B}$-measurable, however for each $\mu \in M(G)$ there always exists a set $A \in \mathcal{A}$ such that $|\mu|(A') = 0$. Since the restriction of f to A is always $\mathcal{A}$-measurable the integral $\int_G f(x)d\mu(x) = \int_A f(x)d\mu(x)$ always exists. With this in mind, we define the Fourier-Stieltjes transform $\hat{\mu}$ of $\mu \in M(G)$ by

$$\hat{\mu}(\hat{x}) = \int_G \overline{(x, \hat{x})}\, d\mu(x).$$

For $\mu \in M(\hat{G})$ we define

$$\check{\mu}(x) = \int_{\hat{G}} (x, \hat{x})\, d\mu(\hat{x}).$$

If $\mu \in M_a(G)$ and $f \in L^1(G)$ is such that $\mu(E) = \int_E f(x)\, dx$ then we write

$$\hat{f}(\hat{x}) = \hat{\mu}(\hat{x}) = \int_G \overline{(x, \hat{x})}\, f(x)\, dx.$$

Some of the main algebraic and topological properties of the Fourier-Stieltjes transform are listed below: $\lambda, \mu \in M(G)$

i) $\hat{\mu}$ is a uniformly continuous function on $\hat{G}$

ii) $(\alpha\lambda + \beta\mu)\hat{} = \alpha\hat{\lambda} + \beta\hat{\mu}$ $\quad \alpha, \beta$ complex

iii) $(\lambda * \mu)\hat{} = \hat{\lambda} \cdot \hat{\mu}$

iv) $\|\hat{\mu}\|_\infty \leq \|\mu\|$

v) $\mu = 0$ iff $\hat{\mu}(\hat{x}) = 0$ for all $\hat{x} \in \hat{G}$.

If we define $\mu^*(E) = \overline{\mu(-E)}$ for $\mu \in M(G)$ then we have

vi) $(\mu^*)\hat{}(\hat{x}) = \overline{\hat{\mu}(\hat{x})}$ for all $\hat{x} \in \hat{G}$.

For $f \in L^1(G)$ we define $f^*(x) = \overline{f(-x)}$.

A function p defined on G is said to be positive definite if $\sum_{j=1}^{k} \sum_{i=1}^{k} p(x_i - x_j)\, c_i \bar{c}_j \geq 0$ for all finite sets of points $x_1 \cdots x_k \in G$ and choices of complex numbers $c_1 \cdots c_k$. $P(G)$ will denote the set of all continuous, positive definite functions. An important result in harmonic analysis states that $p \in P(G)$ iff $p = \check{\mu}$ for some positive measure $\mu \in M(\hat{G})$. For any $f \in L^1(G)$, $f * f^*$ is equal almost every where (with respect to $\digamma_m$) to a function $p \in P(G)$. By $[L^1(G) \cap P(G)]$ we will mean the linear space spanned by $L^1(G) \cap P(G)$. If $f \in [L^1(G) \cap P(G)]$ then $\hat{f} \in L^1(\hat{G})$ and if the Haar measures of G and $\hat{G}$ are suitably normalized $f(x) = (\hat{f})^{\vee}(x)$ almost everywhere (Loomis [4] p. 143). Observe also that if $f \in [L^1(G) \cap P(G)]$ then $f \in L^1(G) \cap L^{\infty}(G)$ and hence $f \in L^p(G)$ for all p, $1 \leq p \leq \infty$.

The characterization of the Fourier-Stieltjes transforms for several classes of measures in $M(G)$ forms the main subject of this thesis. Consequently other known results about Fourier-Stieltjes transforms will be explained and referenced as they are used in Chapters II and III. Most of the important results can be found in the survey articles by Hewitt [2] and by Rudin [11]. We mention here one last topic.

Both $M(G)$ and $L^1(G)$ are commutative Banach algebras and the full force of the theory of Banach algebras can be and has been successfully applied for studying these algebras. For $L^1(G)$ the Gelfand representation is just the Fourier transform and $\hat{G}$ is homeomorphic to the maximal ideal space of $L^1(G)$. In the case that G is discrete $L^1(G)$ accounts for all of $M(G)$ and hence $L^1(G)$ and $M(G)$ have the same maximal ideal space. If G is not discrete this is no longer true. Although each $\hat{x} \in \hat{G}$ corresponds to a maximal ideal of $M(G)$, $\hat{G}$ does not exhaust the maximal ideal space of $M(G)$. At present little is known about the other maximal ideals of $M(G)$. A good account of part of this subject can be found in Rudin [11].

Chapter II

An important class of problems in abstract harmonic analysis can be formulated as follows: Given a subset $N \subset M(G)$ and a function $\varphi \in C(\hat{G})$, what are necessary and sufficient conditions which φ must satisfy in order that $\varphi = \hat{\mu}$ for some $\mu \in N$? This chapter is devoted to proving three closely related theorems which answer the question in the case $N = L^1(G) \cap L^p(G)$, $1 < p \leq \infty$. Chapter III is devoted to the case $N = L^1(G)$. In the statements of our theorems we assume that $\varphi = \hat{\mu}$ for some $\mu \in M(G)$. Thus the results stated here can be considered as refinements and additions to certain previous results which give necessary and sufficient conditions in order that an element $\varphi \in C(\hat{G})$ be the Fourier-Stieltjes transform of a measure $\mu \in M(G)$. We begin with a brief account of these older results.

In 1934 S. Bochner [12] proved the following theorem for the case $G = R^1$, the additive group of reals, and in 1955 W. F. Eberlein [13] generalized the theorem for an arbitrary locally compact Abelian group.

Theorem 1 (S. Bochner-W. F. Eberlein) If $\varphi \in C(\hat{G})$ then a necessary and sufficient condition in order that $\varphi = \hat{\mu}$ for some $\mu \in M(G)$ is that there exists a constant $M > 0$ such that

$$\left|\sum_{i=1}^{n} c_i \varphi(\hat{x}_i)\right| \leq M \sup_{x \in G} \left|\sum_{i=1}^{n} c_i(x, \hat{x}_i)\right|$$

for all finite sets of complex numbers (c_i) and points $(\hat{x}_i)$ in $\hat{G}$. If M_o is the smallest value of M satisfying this inequality then $\|\mu\| = M_o$.

In 1934 I. J. Schoenberg [14] proved a theorem intimately related to the one above. Schoenberg proved the theorem for $G = R^1$, but his proof is true verbatum for the general case.

Theorem 2 (I. J. Schoenberg) If $\varphi \in C(\hat{G})$ then a necessary and sufficient condition in order that $\varphi = \hat{\mu}$ for some $\mu \in M(G)$ is that there exist a constant $M > 0$ such that

$$\left| \int_{\hat{G}} f(\hat{x}) \varphi(\hat{x}) \, d\hat{x} \right| \leq M \| \check{f} \|_{\infty}$$

for all $f \in L^1(\hat{G})$. If M_o is the smallest value of M satisfying the inequality then $\|\mu\| = M_o$.

Both of these theorems characterize the Fourier-Stieltjes transform $\hat{\mu}$ in terms of the continuity properties of a certain linear functional. We will refer to a theorem of this kind as a functional characterization or a functional theorem.

The next theorem was proved in its generalized form by H. Helson [15]. We refer to theorems of this type as multiplier theorems.

Theorem 3 (H. Helson) If $\varphi \in C(\hat{G})$ then a necessary and sufficient conditions in order that $\varphi = \hat{\mu}$ for some $\mu \in M(G)$ is that $\varphi \hat{f} \in (L^1(G))^\wedge$ for all $f \in L^1(G)$.

A transformation theorem of which a special case is equivalent to the multiplier theorem of Helson was proved by R. E. Edwards [16].

Theorem 4 (R. E. Edwards) Suppose that $1 \leq p \leq \infty$ and let T be a continuous transformation from $L^1(G)$ into $L^p(G)$ which commutes with translations i.e. $Tf_s = (Tf)_s$ for all $s \in G$. Then T is of the following form:

$$Tf = \mu * f \qquad\qquad f \in L^1(G)$$

where μ depends only upon T and is such that

(i) $\mu \in M(G)$ if $p = 1$

(ii) $\mu \in L^p(G)$ if $1 < p \leq \infty$.

With these results as background we return to the problem of characterizing the Fourier transforms of functions from the spaces $L^1(G) \cap L^p(G)$ for $1 < p \leq \infty$. The first step is to prove a lemma which forms the basis of our functional theorem. We recall that $P(G)$ denotes the class of continuous, positive definite functions on G and that $[L^1(G) \cap P(G)]$ denotes the linear space spanned by $L^1(G) \cap P(G)$.

Lemma 1 $[L^1(G) \cap P(G)]$ is dense in $L^p(G)$ for $1 \leq p < \infty$ and in $C_\infty(G)$. Also $[L^1(G) \cap P(G)]^\wedge = [L^1(\hat{G}) \cap P(\hat{G})]$.

Proof The proof is essentially given in Loomis [4] pp. 142-143. We repeat it here for completeness. Take $f \in L^p(G)$, $1 \leq p < \infty$, and $\epsilon > 0$. Then there exists a continuous function $g \in L^1(G) \cap L^p(G)$ such that

$$\|f - g\|_p < \frac{\epsilon}{2}.$$

We can choose a continuous function u from an approximate identity such that

$$\|g - u * g\|_p < \frac{\epsilon}{2}$$

and hence

$$\|f - u * g\|_p < \epsilon.$$

$u * g = h_1 * h_1^* - h_2 * h_2^* + i\, h_3 * h_3^* - i\, h_4 * h_4^*$ where

$$h_1 = \frac{1}{2}(u + g^*)$$

$$h_2 = \frac{1}{2}(u - g^*)$$

$$h_3 = \frac{1}{2}(u + i\, g^*)$$

$$h_4 = \frac{1}{2}(u - i\, g^*).$$

Thus $u * g$ is a linear combination of elements $h_j * h_j^* \in L^1(G) \cap P(G)$ and $u * g \in [L^1(G) \cap P(G)]$. If $f \in C_\infty(G)$ then for $\epsilon > 0$ there exists a $g \in K(G)$ such that

$$\|f - g\|_\infty < \frac{\epsilon}{2}.$$

g is uniformly continuous and hence by Theorem 14, Chapter I we can choose a function u from an approximate identity such that

$$\|g - u * g\|_\infty < \frac{\epsilon}{2}.$$

Hence

$$\|f - u * g\|_\infty < \epsilon.$$

Since $K(G) \subset L^1(G)$ we have $u * g \in [L^1(G) \cap P(G)]$. To prove the second part of the lemma we observe that if $f \in [L^1(G) \cap P(G)]$ then $\hat{f} \in L^1(\hat{G})$ (Loomis [4] p. 143). Since f is a linear combination of four positive functions in $L^1(G)$ we also have $\hat{f} \in [L^1(\hat{G}) \cap P(\hat{G})]$. Conversely if

$g \in [L^1(\hat{G}) \cap P(\hat{G})]$, $\check{g} \in [L^1(G) \cap P(G)]$ and $g = (\check{g})^\wedge$.
Thus $[L^1(G) \cap P(G)]^\wedge = [L^1(\hat{G}) \cap P(\hat{G})]$ and
$[L^1(G) \cap P(G)] = [L^1(\hat{G}) \cap P(\hat{G})]^\vee$. ∎

We now come to the functional theorem for $L^1(G) \cap L^p(G)$, $1 < p \leq \infty$.

Theorem 5 For $\varphi = \hat{\mu}$, $\mu \in M(G)$ and $f \in L^1(\hat{G})$ write

$$F(f) = \int_{\hat{G}} f(\hat{x})\,\varphi(\hat{x})\,d\hat{x}.$$

In order that $\mu \in L^1(G) \cap L^p(G)$ for $1 < p \leq \infty$ it is necessary and sufficient that there exist a constant $K > 0$ such that

$$|F(f)| \leq K \|\check{f}\|_q \qquad \text{for all } f \in [L^1(\hat{G}) \cap P(\hat{G})]$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof Assume first that $\mu \in L^1(G) \cap L^p(G)$ and that $d\mu(x) = h(x)\,dx$ for $h \in L^1(G) \cap L^p(G)$. Define

$$L(g) = \int_G g(x)\,h(-x)\,dx \qquad \text{for } g \in L^q(G).$$

Then

$$|L(g)| \leq \|h\|_p \|g\|_q.$$

Take $f \in [L^1(\hat{G}) \cap P(\hat{G})]$. Then $\check{f} \in [L^1(G) \cap P(G)] \subset L^q(G)$ and we have by the Fubini theorem

$$L(\check{f}) = \int_G \Big[\int_{\hat{G}} (x, \hat{x}) f(\hat{x})\,d\hat{x}\Big] h(-x)\,dx = \int_{\hat{G}} f(\hat{x})\hat{h}(\hat{x})\,d\hat{x} = \int_{\hat{G}} f(\hat{x})\,\varphi(\hat{x})\,d\hat{x}.$$

Thus for $f \in [L^1(\hat{G}) \cap P(\hat{G})]$, $L(\check{f}) = F(f)$ and we have $|F(f)| \leq \|h\|_p \|\check{f}\|_q$. This proves the necessity of the condition. Now assume $|F(f)| \leq K \|\check{f}\|_q$ for all $f \in [L^1(\hat{G}) \cap P(\hat{G})]$. Define the linear functional L on $[L^1(\hat{G}) \cap P(\hat{G})]^\vee$ as follows: $L(\check{f}) = F(f)$. Then $|L(\check{f})| \leq K \|\check{f}\|_q$ for all $f \in [L^1(\hat{G}) \cap P(\hat{G})]$. This means that L is a bounded linear functional on a dense subset of $L^q(G)$, $1 \leq q < \infty$, and hence L can be extended uniquely to all of $L^q(G)$ without changing its norm. By the Riesz representation theorem for bounded linear functionals on $L^q(G)$ there exists a unique $h \in L^p(G)$ such that

$$L(g) = \int_G g(x)\, h(-x)\, dx \qquad \text{for all } g \in L^q(G).$$

If $f \in [L^1(\hat{G}) \cap P(\hat{G})]$ then

$$F(f) = \int_{\hat{G}} f(\hat{x})\varphi(\hat{x})d\hat{x} = \int_{\hat{G}} f(\hat{x}) \left[\int_G \overline{(x, \hat{x})} d\mu(x)\right] d\hat{x} = \int_G \check{f}(-x) d\mu(x).$$

Since $F(f) = L(\check{f})$ we get

$$\int_G \check{f}(-x)d\mu(x) = \int_G \check{f}(x)h(-x)dx = \int_G \check{f}(-x)h(x)dx$$

for all $\check{f} \in [L^1(\hat{G}) \cap P(\hat{G})]^\vee = [L^1(G) \cap P(G)]$. The fact that $[L^1(G) \cap P(G)]$ is dense in $L^q(G)$ and $C_\infty(G)$ implies that

$$\int_G f(x)\, d\mu(x) = \int_G f(x)\, h(x)\, dx$$

for all $f \in K(G)$. From this it follows that for any compact set $E \subset G$

$$\int_E d|\mu|(x) = \int_E |h(x)|\, dx.$$

Thus for any compact set E

$$\int_E |h(x)|\, dx \leq \|\mu\|$$

and hence $h \in L^1(G) \cap L^p(G)$. ∎

If G is compact the hypothesis of the theorem can be weakened and we get

Corollary 1 If G is a compact Abelian group and $\varphi \in C(\hat{G})$ then a necessary and sufficient condition in order that $\varphi = \hat{\mu}$ for some $\mu \in L^1(G) \cap L^p(G)$, $1 < p \leq \infty$, is that

$$\left| \int_{\hat{G}} f(\hat{x})\varphi(\hat{x})\, d\hat{x} \right| \leq K \| \check{f} \|_q$$

for some $K > 0$ and all $f \in \left[L^1(\hat{G}) \cap P(\hat{G})\right]$.

Proof Since G is compact $\hat{G}$ is discrete and $L^1(\hat{G}) = \left[L^1(\hat{G}) \cap P(\hat{G})\right]$. Also $\| \check{f} \|_q \leq \| \check{f} \|_\infty$ for $f \in L^1(\hat{G})$. Thus the hypothesis of the theorem implies that

$$\left| \int_{\hat{G}} f(\hat{x})\varphi(\hat{x})\, d\hat{x} \right| \leq K \| \check{f} \|_\infty$$

for all $f \in L^1(\hat{G})$. From the Schoenberg theorem p. 23 we know that $\varphi = \hat{\mu}$ for some $\mu \in M(G)$. It follows from Theorem 5 that $\mu \in L^1(G) \cap L^p(G)$. ∎

The next corollary is an immediate result of the theorem.

Corollary 2 Let $\{\mu_\alpha\}$ be a net of Fourier transforms with $\mu_\alpha \in L^1(G) \cap L^p(G)$, $1 < p \leq \infty$, and $\|\mu_\alpha\|_p \leq K < \infty$. If $\mu \in M(G)$ and if

$$\lim_\alpha \int_{\hat{G}} f(\hat{x})\hat{\mu}_\alpha(\hat{x})d\hat{x} = \int_{\hat{G}} f(\hat{x})\hat{\mu}(\hat{x})d\hat{x}$$

for all $f \in [L^1(\hat{G}) \cap P(\hat{G})]$ then $\mu \in L^1(G) \cap L^p(G)$.

Proof From Theorem 5 and the condition of the corollary we have

$$\left| \int_{\hat{G}} f(\hat{x})\hat{\mu}_\alpha(\hat{x})d\hat{x} \right| \leq \|\mu_\alpha\|_p \|\check{f}\|_q \leq K\|\check{f}\|_q$$

for all $f \in [L^1(\hat{G}) \cap P(\hat{G})]$. Since

$$\lim_\alpha \int_{\hat{G}} f(\hat{x})\hat{\mu}_\alpha(\hat{x})d\hat{x} = \int_{\hat{G}} f(\hat{x})\hat{\mu}(\hat{x})d\hat{x}$$

we have

$$\left| \int_{\hat{G}} f(\hat{x})\hat{\mu}(\hat{x})d\hat{x} \right| \leq K\|\check{f}\|_q.$$

Thus by Theorem 5 $\mu \in L^1(G) \cap L^p(G)$. ‖

We now derive our multiplier theorem from the functional theorem.

Theorem 6 If $\varphi \in C(\hat{G})$ then a necessary and sufficient condition in order that $\varphi = \hat{\mu}$ for some $\mu \in L^1(G) \cap L^p(G)$, $1 < p \leq \infty$, is that $\varphi\hat{f} \in (L^1(G) \cap L^p(G))^\wedge$ for all $f \in L^1(G)$.

<u>Proof</u> Suppose that $\varphi = \hat{\mu}$ for $\mu \in L^1(G) \cap L^p(G)$. Then $\mu * f \in L^1(G) \cap L^p(G)$ for all $f \in L^1(G)$ and $\hat{\mu}\hat{f} = \varphi\hat{f} \in (L^1(G) \cap L^p(G))^\wedge$ for all $f \in L^1(G)$. This proves the necessity of the condition. Now suppose that $\varphi\hat{f} \in (L^1(G) \cap L^p(G))^\wedge$, $1 < p \leq \infty$, for all $f \in L^1(G)$. Then in particular $\varphi\hat{f} \in (L^1(G))^\wedge$ for $f \in L^1(G)$ and by the Helson theorem p. 23 $\varphi = \hat{\mu}$ for some $\mu \in M(G)$. Thus we have $\mu * f \in L^1(G) \cap L^p(G)$ for all $f \in L^1(G)$. This transformation is a bounded linear transformation from $L^1(G)$ into $L^1(G)$ with $\|\mu * f\|_1 \leq \|\mu\| \|f\|_1$. We can also show that the transformation is continuous from $L^1(G)$ into $L^p(G)$. For suppose $f_n \to f$ in $L^1(G)$ and $\mu * f_n \to h$ in $L^p(G)$. Then there is a subsequence $\{f_m\} \subset \{f_n\}$ such that $\mu * f_m(x) \to h(x)$ almost everywhere. By the continuity of the transformation from $L^1(G)$ into $L^1(G)$ we know that for some subsequence $\{f_k\} \subset \{f_m\}$ we have $\mu * f_k(x) \to \mu * f(x)$ almost everywhere. Thus $\mu * f(x) = h(x)$ almost everywhere. Hence by the closed graph theorem the transformations is continuous from $L^1(G)$ into $L^p(G)$. Thus there is a constant K such that

$$\|\mu * f\|_p \leq K \|f\|_1$$

for $f \in L^1(G)$. Now let $\{u_\alpha\}$ be an approximate identity for $L^1(G)$. Then $(\mu * u_\alpha)^\wedge = \hat{\mu}\hat{u}_\alpha = \varphi\hat{u}_\alpha \to \varphi$ uniformly on compact sets in $\hat{G}$. Thus for $f \in [L^1(\hat{G}) \cap P(\hat{G})]$ we have

$$F(f) = \int_{\hat{G}} f(\hat{x})\varphi(\hat{x})d\hat{x} = \lim_{\alpha} \int_{\hat{G}} f(\hat{x})\hat{\mu}(\hat{x})\hat{u}_{\alpha}(\hat{x})d\hat{x} = \lim_{\alpha} \int_{G} \check{f}(-x)\mu * u_{\alpha}(x)dx.$$

Thus

$$|F(f)| \leq \lim_{\alpha} \|\mu * u_{\alpha}\|_p \|\check{f}\|_q \leq \lim_{\alpha} K \|u_{\alpha}\|_1 \|\check{f}\|_q$$

and since $\|u_{\alpha}\|_1 = 1$ we get

$$|F(f)| \leq K\|\check{f}\|_q$$

for all $f \in [L^1(\hat{G}) \cap P(\hat{G})]$. By Theorem 5 we have that $\mu \in L^1(G) \cap L^p(G)$. ▌

Theorem 6 can also be proved by using Edwards' theorem p. 24. The continuity of the transformation $f \longrightarrow \mu * f$ in both $L^1(G)$ and $L^p(G)$ implies by Edwards' theorem that $\mu \in M(G)$ and $\mu \in L^p(G)$. From this it follows as in the proof of Theorem 5 pp. 27-28 that $\mu \in L^1(G) \cap L^p(G)$.

We complete the discussion of $L^1(G) \cap L^p(G)$, $1 < p \leq \infty$, with a transformation theorem. Again this result is a direct consequence of Edwards' theorem.

<u>Theorem 7</u> Let T be a linear transformation from $L^1(G)$ into $L^1(G) \cap L^p(G)$, $1 < p \leq \infty$, which commutes with translations and is such that

$$\|Tf\|_1 \leq K\|f\|_1$$

$$\|Tf\|_p \leq K\|f\|_p$$

Then $Tf = \mu * f$ for some $\mu \in L^1(G) \cap L^p(G)$ where μ depends only upon T.

Proof From Edwards' theorem we know that T is of the form $Tf = \mu * f$ where $\mu \in M(G)$ and $\mu \in L^p(G)$, and again we must have $\mu \in L^1(G) \cap L^p(G)$. ▌

Chapter III

This chapter is devoted to characterizing the Fourier transforms of $L^1(G)$. The main theorem is a functional characterization of $(L^1(G))^\wedge$ which includes as special cases a theorem by R. Salem and a theorem by A. C. Berry.

In 1931 R. Salem [17], [18] gave the following necessary and sufficient conditions which must be satisfied by coefficients (a_n, b_n) in order that $\frac{1}{2} a_o + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ is the Fourier series of an absolutely integrable function on the unit circle.

<u>Theorem 1</u> (R. Salem) Let (Z) be the class of functions $\omega(x) = \sum_{n=1}^{\infty} (\alpha_n \cos nx + \beta_n \sin nx)$ which are continuous and differentiable and such that $|\omega(x)| \leq 1$ and the Fourier series of $\omega'(x)$ is absolutely convergent. In order that (a_n, b_n) be the Fourier coefficients of an integrable function it is necessary and sufficient that

(1) The formally integrated series $\sum_{n=1}^{\infty} \left(\frac{a_n}{n} \sin nx - \frac{b_n}{n} \cos nx \right)$ converges to a continuous function $F(x)$;

(2) The expression $\sum_{n=1}^{\infty} (a_n \alpha_n + b_n \beta_n)$ tends to zero when ω varies in (Z) in such a way that $\sum_{n=1}^{\infty} (\alpha_n^2 + \beta_n^2)$ tends to zero.

A year later, in 1932, A. C. Berry [19] proved the following theorem for R, the additive group of reals. The relation between Theorem 1 and Theorem 2 will be apparent after we have proved the generalization of these theorems.

Theorem 2 (A. C. Berry) A necessary and sufficient condition for $\varphi \in C(R)$ to be of the form $\varphi = \hat{\mu}$ for some $\mu \in L^1(R)$ is that

(1) There exists a constant $K > 0$ such that

$$\left| \int_{-\infty}^{\infty} f(x)\varphi(x)\,dx \right| \leq K \|\hat{f}\|_{\infty}$$

for all $f \in L^1(R)$

(2) To every $\epsilon > 0$ there corresponds a $\delta > 0$ such that

$$\left| \int_{-\infty}^{\infty} f(x)\varphi(x)\,dx \right| \leq \epsilon \|\hat{f}\|_{\infty}$$

for all $f \in L^1(R)$ with $\hat{f} \in L^1(R)$ and

$$\|\hat{f}\|_1 \leq \delta \|\hat{f}\|_{\infty}.$$

In both of these theorems condition (1) insures that the function in question is a Fourier-Stieltjes transform of some bounded Radon measure, and condition (2) implies the absolute continuity of the measure. Since the Schoenberg theorem quoted in Chapter II gives a satisfactory generalization of the conditions (1) we will assume from now on that we are dealing with the Fourier transform of some $\mu \in M(G)$. Our results will therefore be generalizations of the conditions (2). For simplicity of presentation we prove the theorems of necessity and sufficiency separately. We begin with a theorem and corollary concerning the necessity of the generalization of the conditions (2).

Theorem 3 If $\mu \in L^1(G)$ then the linear functional

$$M(f) = \int_G f(x)\, d\mu(x)$$

defined for $f \in L^{\infty}(G)$ satisfies the following condition:

For every $\epsilon > 0$ and every p, $1 \leq p < \infty$, there exists a $\delta > 0$ such that

$$|M(f)| \leq \epsilon \| f \|_{\infty}$$

whenever $f \in L^p(G) \cap L^{\infty}(G)$ and

$$\| f \|_p \leq \delta \| f \|_{\infty}.$$

Furthermore δ depends only upon ϵ, p and μ.

Proof Let $\epsilon > 0$ and p, $1 \leq p < \infty$, be given and fixed. For each $f \in L^p(G) \cap L^{\infty}(G)$ define

$$E = \left\{ x \;\middle|\; |f(x)| > \frac{\epsilon}{2\|\mu\|} \| f \|_{\infty} \right\}$$

Here we assume that $\|\mu\| \neq 0$. The theorem is trivial if $\|\mu\| = 0$.

$$|M(f)| \leq \int_G |f(x)|\, d|\mu|(x) = \int_E |f(x)|\, d|\mu|(x) + \int_{G-E} |f(x)|\, d|\mu|(x)$$

For $x \in G - E$, $|f(x)| \leq \frac{\epsilon}{2\|\mu\|} \| f \|_{\infty}$ and hence we have

$$\int_{G-E} |f(x)|\, d|\mu|(x) \leq \frac{\epsilon}{2\|\mu\|} \| f \|_{\infty} \int_{G-E} d|\mu|(x) \leq \frac{\epsilon}{2} \| f \|_{\infty}.$$

For $x \in E$ we have

$$\int_E |f(x)|\, d|\mu|(x) \leq \|f\|_\infty \int_E d|\mu|(x).$$

Thus

$$|M(f)| \leq \left(\int_E d|\mu|(x) + \frac{\epsilon}{2} \right) \|f\|_\infty.$$

The absolute continuity of μ implies the absolute continuity of $|\mu|$. Thus there is an $\eta > 0$ which depends only upon μ and ϵ and is such that $|\mu|(F) \leq \frac{\epsilon}{2}$ whenever $F \in \mathcal{F}_m$ and $m(F) \leq \eta$, where m denotes a Haar measure on G. Choose δ so that

$$0 < \delta \leq \eta^{\frac{1}{p}} \left(\frac{\epsilon}{2\|\mu\|} \right)$$

and suppose that

$$\|f\|_p \leq \delta \|f\|_\infty.$$

Then

$$\int_G \frac{|f(x)|^p}{\|f\|_\infty^p}\, dx \leq \delta^p.$$

Since $|f(x)| > \frac{\epsilon}{2\|\mu\|} \|f\|_\infty$ for $x \in E$ we have

$$\int_E \left(\frac{\epsilon}{2\|\mu\|}\right)^p dx \leq \int_G \frac{|f(x)|^p}{\|f\|_\infty^p}\, dx \leq \delta^p \leq \eta \left(\frac{\epsilon}{2\|\mu\|}\right)^p$$

and hence

$$\int_E dx = m(E) \leq \eta.$$

Thus for $\| f \|_p \leq \delta \| f \|_\infty$, $m(E) \leq \eta$, implying that $\int_E d|\mu|(x) = |\mu|(E) \leq \frac{\epsilon}{2}$. Combining inequalities we get

$$|M(f)| \leq \left(\int_E d|\mu|(x) + \frac{\epsilon}{2} \right) \| f \|_\infty \leq \epsilon \| f \|_\infty$$

whenever $\| f \|_p \leq \delta \| f \|_\infty$. ▌

<u>Corollary 1</u> If $\mu \in L^1(G)$ then the linear functional

$$M(g) = \int_{\hat{G}} g(\hat{x}) \hat{\mu}(\hat{x})\, d\hat{x}$$

defined for $g \in L^1(\hat{G})$ satisfies the following condition:

For every $\epsilon > 0$ and every p, $1 \leq p < \infty$, there exists a $\delta > 0$ which depends only upon ϵ, p and μ and is such that

$$|M(g)| \leq \epsilon \| \check{g} \|_\infty$$

whenever $g \in \left[L^1(\hat{G}) \cap P(\hat{G}) \right]$ and

$$\| \check{g} \|_p \leq \delta \| \check{g} \|_\infty.$$

<u>Proof</u> If $g \in \left[L^1(\hat{G}) \cap P(\hat{G}) \right]$ then $\check{g} \in \left[L^1(G) \cap P(G) \right]$ and hence $\check{g} \in L^p(G) \cap L^\infty(G)$ for all p, $1 \leq p \leq \infty$. By the Fubini theorem we have

$$M(G) = \int_{\hat{G}} g(\hat{x}) \hat{\mu}(\hat{x})\, d\hat{x} = \int_G \check{g}(-x)\, d\mu(x).$$

The result now follows directly from Theorem 3. ▌

By taking $G = R$ and $p = 1$ in Corollary 1 we obtain a generalization of half of Berry's theorem. If $p = 2$ then by the Plancherel theorem ([4] p. 145) we know that $\| g \|_2 = \| \check{g} \|_2$ for $g \in [L^1(\hat{G}) \cap P(\hat{G})]$. Thus taking G = the unit circle and $p = 2$ and replacing the condition $\| \check{g} \|_2 \leq \delta \| \check{g} \|_\infty$ by $\| g \|_2 \leq \delta \| \check{g} \|_\infty$ gives a generalization of Salem's theorem. The next lemma and theorem generalize the other half of these theorems.

Lemma 1 Let $E \in \mathcal{B}$ be any open Borel set with compact closure and let μ be any bounded Radon measure. Then for every $\epsilon > 0$, $\eta > 0$ and each p, $1 \leq p < \infty$, there exists a function $f \in [L^1(G) \cap P(G)]$ such that

$$\| f \|_\infty = 1,$$

$$\left| \mu(E) - \int_G f(x) \, d\mu(x) \right| \leq \epsilon$$

and

$$\left| m(E) - \| f \|_p^p \right| \leq \eta .$$

Proof If $\| \mu \| = 0$ then $\mu(E) = \int_G f(x) d\mu(x) = 0$ for all $f \in [L^1(G) \cap P(G)]$ and one condition of the lemma is trivial. The other condition to be met is independent of μ. Thus assume that $\| \mu \| \neq 0$.

Let x_o be an arbitrary fixed point in E. Since G is a regular topological space we can find a compact neighborhood C of x_o such that $C \subset E$.

E is an open set and hence by the inner regularity of the measures μ and m (Definition 2, Chapter I) there exists a compact set K having the properties that

$$C \subset K \subset E,$$

$$m(E-K) \leq \frac{\epsilon}{2}$$

and

$$|\mu|(E-K) \leq \frac{\eta}{2} .$$

Here we use the fact that $\overline{E}$ is compact so that $m(E) \leq m(\overline{E}) < +\infty$. Having chosen such a K the complete regularity of the topological space G insures the existence of a function $h \in K(G)$ with the following properties:

$$0 \leq h(x) \leq 1 \qquad \text{for } x \in G$$

$$h(x) = 1 \qquad \text{for } x \in K$$

$$h(x) = 0 \qquad \text{for } x \in E'.$$

It is clear that h^p for $1 \leq p < \infty$ also satisfies these three conditions. Furthermore for each p, $1 \leq p < \infty$,

$$0 \leq \chi_E(x) - h^p(x) \leq 1 \qquad \text{for } x \in G$$

$$\chi_E(x) - h^p(x) = 0 \qquad \text{for } x \in K \quad E'.$$

Thus

$$\left| m(E) - \|h\|_p^p \right| = \left| m(E) - \int_G h^p(x)dx \right| = \left| \int_G (\chi_E(x) - h^p(x))dx \right|$$

$$= \int_{E-K} ((\chi_E(x) - h^p(x))dx \leq m(E-K) \leq \frac{\epsilon}{2} .$$

And similarly

$$\left| \mu(E) - \int_G h(x)\, d\mu(x) \right| = \left| \int_G (\chi_E(x) - h(x))\, d\mu(x) \right|$$

$$\leq \int_{E-K} |\chi_E(x) - h(x)|\, d|\mu| x \leq |\mu|(E-K) \leq \frac{\eta}{2}.$$

Compressing these inequalities gives

$$(*) \qquad \begin{aligned} &\left| m(E) - \| h \|_p^p \right| \leq \frac{\epsilon}{2} \\ &\left| \mu(E) - \int_G h(x)\, d\mu(x) \right| \leq \frac{\eta}{2}. \end{aligned}$$

Since $h \in K(G)$ we know that h is uniformly continuous and that $h \in L^p(G)$ for $1 \leq p \leq \infty$. Thus by Theorems 13 and 14, Chapter I we can choose a function u from an approximate identity such that

$$\left| \| h \|_p^p - \| u * h \|_p^p \right| \leq \frac{\epsilon}{2}$$

and

$$\left| h(x) - u * h(x) \right| \leq \frac{\eta}{2\|\mu\|}.$$

The set $x_o - C$ is a compact neighborhood of the identity and hence the function u can be chosen such that $u(x) = 0$ for $x \in x_o - C$. Now let $f = u * h$. Then the last inequality gives

$$\left| \int_G h(x) d\mu(x) - \int_G f(x) d\mu(x) \right| \leq \int_G |h(x) - f(x)|\, d|\mu|(x) \leq \frac{\eta}{2}.$$

These last three inequalities combined with (*) give

$$\left| m(E) - \| f \|_p^p \right| \leq \epsilon$$

and

$$\left| \mu(E) - \int_G f(x)\, d\mu(x) \right| \leq \eta .$$

We have only to show that $\| f \|_\infty = 1$ in order to complete the proof.

$$f(x_o) = \int_G u(x_o - y) h(y) dy = \int_C u\,(x_o - y) h(y) dy = \int_C u(x_o - y) dy = 1$$

since $h(y) = 1$ for $y \in C \subset K$. This combined with the facts that f is continuous and $\| f \|_\infty \leq \| u \|_1 \| h \|_\infty = \| h \|_\infty = 1$ insures that $\| f \|_\infty = 1$. ▌

We are now ready to state and prove the converse of Corollary 1.

<u>Theorem 4</u> A measure $\mu \in M(G)$ is an element of $L^1(G)$ if the functional

$$M(g) = \int_{\hat{G}} g(\hat{x}) \hat{\mu}(\hat{x})\, d\hat{x}$$

defined on $L^1(\hat{G})$ satisfies the following condition:

For some fixed p, $1 \leq p < \infty$, and for every $\epsilon > 0$ there exists a $\delta > 0$, which depends only upon ϵ, p and μ, and is such that

$$|M(g)| \leq \epsilon \| \check{g} \|_\infty$$

whenever $g \in \left[L^1(\hat{G}) \cap P(\hat{G}) \right]$ and

$$\| \check{g} \|_p \leq \delta \| \check{g} \|_\infty .$$

Proof We will show that the condition of the theorem implies the absolute continuity of μ. In doing this we use the criteria expressed by Theorem 12, Chapter I.* That is we will show that for every $\epsilon > 0$ there is a $\Delta = \Delta(\epsilon, \mu)$ such that $|\mu(E)| \leq \epsilon$ whenever $E \in \mathsf{A}$, E open, $\bar{E}$ compact and $m(E) \leq \Delta$. (* See footnote on page 49.)

Let $\epsilon > 0$ be given and fixed and suppose that the condition of the theorem holds for some p, $1 \leq p < \infty$. Let $E \in \mathsf{A}$ be open and have compact closure. Since $\mathsf{A} \subset \mathsf{B}$ we know that $E \in \mathsf{B}$. Since E is open we know (Theorem 9, Chapter I) that $m(E) > 0$. By taking the ϵ to be $\frac{\epsilon}{2}$ and the η to be $m(E)$ in Lemma 1 we know that there exists a function $f \in [L^1(G) \cap P(G)]$ with the properties that

$$\| f \|_\infty = 1 ,$$

$$\left| \mu(E) - \int_G f(x) \, d\mu(x) \right| \leq \frac{\epsilon}{2}$$

and

$$\left| m(E) - \| f \|_p^p \right| \leq m(E) .$$

Let $g(\hat{x}) = \hat{f}(-\hat{x}) \in [L^1(\hat{G}) \cap P(\hat{G})]$. Then

$$f(x) = \int_{\hat{G}} \overline{(x, \hat{x})} \, g(\hat{x}) \, d\hat{x} = \check{g}(-x) .$$

By Fubini's theorem

$$M(g) = \int_{\hat{G}} g(\hat{x}) \hat{\mu}(\hat{x}) \, d\hat{x} = \int_G f(x) \, d\mu(x)$$

which combined with the previous inequalities gives

$$|\mu(E) - M(g)| \leq \frac{\epsilon}{2}$$

and

$$|m(E) - \|\check{g}\|_p^p| \leq m(E).$$

These imply that

$$|\mu(E)| \leq |M(g)| + \frac{\epsilon}{2}$$

and

$$\|\check{g}\|_p^p \leq 2\, m(E).$$

We note that g depends only upon ϵ and E. The hypothesis of the theorem states that there exists a $\delta = \delta(\frac{\epsilon}{2})$ such that

$$|M(h)| \leq \frac{\epsilon}{2} \|\check{h}\|_\infty$$

for $h \in [L^1(\hat{G}) \cap P(\hat{G})]$ and

$$\|\check{h}\|_p \leq \delta \|\check{h}\|_\infty.$$

Take $\Delta = \frac{\delta^p}{2}$. Then Δ depends only upon ϵ, p and μ and not upon g or E. We will show that $m(E) \leq \Delta$ implies $|\mu(E)| \leq \epsilon$. For $m(E) \leq \Delta = \frac{\delta^p}{2}$ implies that

$$\|\check{g}\|_p^p \leq 2\, m(E) \leq \delta^p,$$

and hence

$$\|\check{g}\|_p \leq \delta.$$

Since $\|\check{g}\|_\infty = 1$ we can write

$$\|\check{g}\|_p \leq \delta \|\check{g}\|_\infty.$$

This by the hypothesis of the theorem implies that

$$|M(g)| \leq \frac{\epsilon}{2} .$$

Combining this with the inequality $|\mu(E)| \leq |M(g)| + \frac{\epsilon}{2}$ gives

$$|\mu(E)| \leq \epsilon .$$

Thus $m(E) \leq \Delta$ implies that $|\mu(E)| \leq \epsilon$ and the theorem is proved. ▌

Theorem 4 is seen to be a generalization of Salem's theorem if again we set $p = 2$ and G = the unit circle. Berry's theorem follows from Theorem 4 by taking $p = 1$ and $G = R$.

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* Theorem 12, Chapter I, is stated for a positive measure. If μ is complex we must show that the condition holds for the positive and negative variations of the real and imaginary parts of μ. If $|\mu(E)| \leq \epsilon$ then clearly the same is true for the real and imaginary parts of μ. Thus assume that μ is real, and let μ^+ denote the positive variation of μ. Then there exists a Borel set E^+ such that $\mu^+(G) = \mu^+(E^+) = \mu(E^+)$. Let U be an open Borel set with $E^+ \subset U$ and let E be an open Borel set with compact closure. Then $m(E \cap U) \leq m(E) \leq \delta$, and since $E \cap U$ is open and has compact closure we have $|\mu(E \cap U)| \leq \epsilon$. But $|\mu(E \cap U)| = |\mu^+(E) + \mu(E \cap (U - E^+))|$. By the regularity of μ, $|\mu(U - E^+)|$ can be made arbitrarily small. Thus $|\mu(E \cap U)| \leq \epsilon$ implies that $\mu^+(E) \leq \epsilon$. The same argument works for the negative variation.

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