# PART I: MULTIPLE BIFURCATIONS
# PART II: PARALLEL HOMOTOPY METHOD FOR THE REAL NONSYMMETRIC EIGENVALUE PROBLEM

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# Abstract


PART I

Consider an analytic operator equation $G(u, \lambda) = 0$ where $\lambda$ is a real parameter. Suppose 0 is a "simple" eigenvalue of the Fréchet derivative $G_u$ at $(u_0, \lambda_0)$. We give a hierarchy of conditions which completely determine the solution structure of the operator equation. It will be shown that multiple bifurcation as well as simple bifurcation can occur. This extends the standard bifurcation theory from a "simple" eigenvalue in which only one branch bifurcates. When 0 is a multiple eigenvalue, we give some sufficient conditions for multiple bifurcations with a lower bound on the multiplicity of the bifurcation. This theory is applied to some semilinear elliptic partial differential equations on a cylinder with a constant cross–section.

# PART II

We present a homotopy method to compute the eigenvectors and eigenvalues, i.e., the eigenpairs of a given real matrix $A_1$. From the eigenpairs of some real matrix $A_0$, we follow the eigenpairs of

$$A(t) \equiv (1-t)A_0 + tA_1$$

at successive times from $t = 0$ to $t = 1$ using continuation. At $t = 1$, we have the eigenpairs of the desired matrix $A_1$. The following phenomena are present for a general nonsymmetric matrix:

- complex eigenpairs
- ill-conditioned problems due to non-orthogonal eigenvectors
- bifurcation (i.e., crossing of eigenpaths)

These can present computational difficulties if not handled properly. Since each eigenpair can be followed independently, this algorithm is ideal for concurrent computers. We will see that the homotopy method is extremely slow for full matrices but has the potential to compete with other algorithms for sparse matrices as well as matrices with defective eigenvalues.

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# Part I

# MULTIPLE BIFURCATIONS

# Chapter 1

# Introduction

Consider a nonlinear operator equation $G(u, \lambda) = 0$, parametrized by a real number $\lambda$. Suppose a (continuous) branch of solutions $(u(\epsilon), \lambda(\epsilon))$ is known for all $\epsilon$ in some interval about 0. We call this the basic solution branch. When the linearized equation has a one–dimensional null space at $(u_0, \lambda_0) \equiv (u(0), \lambda(0))$, we give some sufficient conditions for bifurcation from the basic solution branch. This is sometimes referred to as bifurcation from a "simple" eigenvalue. We will show that both simple and multiple bifurcations can occur. We indicate how a hierarchy of conditions can be constructed to determine the solution structure of the operator equation. This complements the work of many authors in bifurcation theory. For example, Crandall and Rabinowitz [7] found sufficient conditions for the generic bifurcation from a "simple" eigenvalue, in which case only ONE branch bifurcates.

When the dimension of the null space is greater than one, say $M$, the task of formulating general sufficient conditions is much more difficult. The case $M = 2$ is treated in McLeod and Sattinger [27]. For a general $M$, see Berger [4], Sather [31], Keller and Langford [20], Dancer [9], Sattinger [32] and Rabinowitz [29]. In our work, we impose a strong set of conditions on the operator equation such that there are at least $M$ branches of solutions bifurcating from the basic solution branch. The conditions are chosen so that the roots of the algebraic bifurcation equations, a system of polynomial equations obtained from the original operator equation, can be found and shown to be isolated. This theory can be applied to systems of nonlinear equations in $\mathbb{R}^m$ and semilinear elliptic equations:

$$\Delta u + \lambda f(u) = 0$$

in a 3-dimensional cylinder with a constant cross–section for two classes of functions $f$ (e.g., $f(u) = e^u$ and $f(u) = u(u^2+1)$). These are nontrivial examples where there exist bifurcation points of arbitrarily large multiplicities. In addition, all bifurcation points from the basic solution branch can be found.

In the remainder of this introduction, we will define some concepts and give an outline of the rest of the thesis.

Let $\mathbb{B}_1$ and $\mathbb{B}_2$ be real Banach spaces and $G$ be a smooth operator from $\mathbb{B}_1 \times \mathbb{R}$ into $\mathbb{B}_2$. Let $\mathcal{I}$ be some real interval containing 0. We call the curve $\Gamma \equiv \{(u_0(\epsilon), \lambda_0(\epsilon)) : \forall \epsilon \in \mathcal{I}\}$ a solution branch if $G(u_0(\epsilon), \lambda_0(\epsilon)) = 0$ for every $\epsilon \in \mathcal{I}$ and both $u_0(\epsilon)$ and $\lambda_0(\epsilon)$ are continuous. The point $P \equiv (u_0(0), \lambda_0(0))$ is called a bifurcation point if every neighborhood of $P$ contains a solution not on $\Gamma \backslash P$. We say $\Gamma$ is a basic solution branch from which other branches of solutions bifurcate. If $u_0(\epsilon) \equiv 0$, we also call $\Gamma$ the trivial solution branch. The multiplicity of the bifurcation is defined as the number of mutually distinct solution branches that are different from the basic solution branch and all meeting at $P$. When the multiplicity is one, we call it a simple bifurcation. Otherwise, it is called multiple bifurcation.

We conclude this introduction by giving a synopsis of the rest of the thesis. We give a complete treatment of multiple bifurcation from a "simple" eigenvalue in Chapter 2. Some results for bifurcation from a multiple eigenvalue are stated in Chapter 3, where we give conditions which guarantee a bifurcation point with a lower bound on the multiplicity. By making additional assumptions, we will improve this lower bound. A weak upper bound will also be given. In Chapter 4, we apply this theory to some semilinear elliptic partial differential equations. We show that under some conditions, there exists a bifurcation point of multiplicity at least $n$ for any given positive integer $n$. We also give a procedure to locate all the bifurcation points along the basic solution branch. In the last chapter, we recapitulate and suggest directions of further research.

Some good references for bifurcation theory include Chow and Hale [6] and Sattinger [32].

# Chapter 2

# Multiple Bifurcation from "Simple" Eigenvalues

In this chapter, we extend the standard bifrucation theory from "simple" eigenvalues (see Crandall and Rabinowitz [7]) to show that multiple bifurcations can occur if the basic transversality or range condition is not satisfied. Briefly, this theory concerns solutions of a general equation $G(u, \lambda) = 0$ where $\lambda$ is a real parameter. Suppose $u = 0$ is a solution for all $\lambda$ near 0 and $G_u^0$, the Fréchet derivative of $G$ with respect to $u$ evaluated at $(0, 0)$, has 0 as a "simple" eigenvalue. Then, subject to a transversality condition, the equation has exactly one nontrivial solution branch near $(u, \lambda) = (0, 0)$. We examine what happens when the transversality condition is not satisfied. We will derive a hierarchy of conditions which categorizes the solution set. In particular, we will observe multiple bifurcations.

Note that it is not always possible to transform a given problem to one where the trivial solution is a solution for all $\lambda$ near 0. A simple example is $u^2 - \lambda = 0$. We will also derive conditions for categorizing the solution set near these so called limit points.

The method of proof of all our results will be a Lyapunov–Schmidt reduction followed by two applications of the Implicit Function Theorem. We can also give the exact form of the solutions. Many authors have looked for multiple bifurcations from equations where the dimension of the null space of the linearized operator is greater than one. There does not appear to have been any attempt to seek multiple bifurcations from problems with a one-dimensional null space.

We now establish some notation. The adjoint of $G_u^0$, denoted by $G_u^{0*}$, is the linear mapping from $\mathbb{B}_2^*$ into $\mathbb{B}_1^*$, where $\mathbb{B}_i^*$ is the dual space of $\mathbb{B}_i$. The null and range spaces are abbreviated as $\mathcal{N}()$ and $\mathcal{R}()$ respectively. We recall that a bounded linear operator $A : \mathbb{B}_1 \to \mathbb{B}_2$ is Fredholm of index 0 if $dim\mathcal{N}(A) = codim\mathcal{R}(A) < \infty$. Two important properties that $A$ possesses are:

- $\mathcal{R}(A)$ is closed.
- $\mathcal{R}(A) = \{y \in \mathbb{B}_2 : \ \psi^*(y) = 0, \ \ \forall\psi^* \in \mathcal{N}(A^*)\}$.

In this chapter, we will restrict to the case of a bifurcation from a "simple" eigenvalue. That is, $dim\,\mathcal{N}(G_u^0) = codim\,\mathcal{R}(G_u^0) = 1$.

## 2.1 Bifurcation from the Trivial Solution

We will use the superscript $^0$ notation to mean evaluation of the derivatives of $G$ at the point $(u, \lambda) = (0, 0)$. We begin by stating the following well-known result:

**Theorem 2.1 (Crandall and Rabinowitz [7])** *Let $\mathbb{B}_1$ and $\mathbb{B}_2$ be Banach spaces and let $G$ be a mapping from $\mathbb{B}_1 \times \mathbb{R}$ into $\mathbb{B}_2$. Suppose $G(0,\lambda) = 0$, for all $\lambda$ in some interval containing $0$ and $G_u$, $G_\lambda$ and $G_{u\lambda}$ are continuous in a neighborhood of $(u,\lambda) = (0,0)$. Assume dim $\mathcal{N}(G_u^0) =$ codim $\mathcal{R}(G_u^0) = 1$. Let $\phi$ and $\psi^*$ be nonzero elements of $\mathcal{N}(G_u^0)$ and $\mathcal{N}(G_u^{0*})$ respectively. Assume:*

$$\psi^* G_{u\lambda}^0 \phi \neq 0. \tag{2.2}$$

*Then, there is exactly one nontrivial solution branch near $(0,0)$. It has the form, for $\epsilon$ sufficiently small,*

$$u = \epsilon\phi + \epsilon\Phi(\epsilon), \quad \lambda = \lambda(\epsilon),$$

*where $\lambda(\epsilon)$ and $\Phi(\epsilon)$ are continuous functions such that $\lambda(0) = 0$ and $\Phi(0) = 0$.*

We now seek solutions when the transversality condition (2.2) is violated. For convenience, we state some hypotheses which are assumed for the remainder of this section.

**Hypothesis 2.3**

- *$G$ is a mapping from $\mathbb{B}_1 \times \mathbb{R}$ into $\mathbb{B}_2$, where $\mathbb{B}_1$ and $\mathbb{B}_2$ are real Banach spaces.*
- *$G(0,\lambda) = 0$, for all $\lambda$ in some interval containing $0$.*
- *$G$ has continuous first Fréchet derivatives at $(u,\lambda) = (0,0)$.*

- *$\dim \mathcal{N}(G_u^0) = \operatorname{codim} \mathcal{R}(G_u^0) = 1$ (that is, 0 is a "simple" eigenvalue of $G_u^0$.) We will denote some specific nonzero element of $\mathcal{N}(G_u^0)$ and $\mathcal{N}(G_u^{0*})$ by $\phi$ and $\psi^*$, respectively.*

- $\psi^* G_{u\lambda}^0 \phi = 0$.

We can decompose the Banach spaces into direct sums:

$$\mathbb{B}_1 = \mathcal{N}(G_u^0) \oplus \mathbb{B}_{11}, \qquad \mathbb{B}_2 = \mathcal{R}(G_u^0) \oplus \mathbb{B}_{22}.$$

From (2.3), $\mathbb{B}_{22}$ is a one-dimensional space. Let $\psi$ be the unique element in $\mathbb{B}_{22}$ such that $\psi^*(\psi) = 1$ and let $\mathcal{P}$ be the projection operator from $\mathbb{B}_2$ onto $\mathbb{B}_{22}$. We can represent $\mathcal{P}$ as:

$$\mathcal{P} = \psi\psi^*. \tag{2.4}$$

The projection operator from $\mathbb{B}_2$ onto $\mathcal{R}(G_u^0)$ is $I - \mathcal{P}$, where $I$ is the identity operator.

Assuming (2.3), we can write any element $u$ in $\mathbb{B}_1$ as $\xi\phi + \Phi$, where $\xi \in \mathbb{R}$ and $\Phi \in \mathbb{B}_{11}$. The solution set of $G(u, \lambda) = 0$ is precisely the solution set of the equations $\mathcal{P}G(u, \lambda) = 0$ and $(I - \mathcal{P})G(u, \lambda) = 0$. We first focus on the equation

$$F(\Phi, \xi, \lambda) \equiv (I - \mathcal{P})G(\xi\phi + \Phi, \lambda) = 0.$$

Now $F(0, 0, 0) = 0$ and, as an operator from $\mathbb{B}_{11}$ onto $\mathcal{R}(G_u^0)$, $F_\Phi(0, 0, 0) = (I - \mathcal{P})G_u^0 = G_u^0$ is clearly a continuous isomorphism. Hence by the Implicit Function Theorem, there is a unique continuously differentiable function $\Phi(\xi, \lambda)$ such that

$$F(\Phi(\xi, \lambda), \xi, \lambda) = 0 \tag{2.5}$$

for all $\xi$ and $\lambda$ sufficiently small and

$$\Phi(0,0) = 0. \tag{2.6}$$

The degree of smoothness of $\Phi$ is the same as that of $G$. By taking successive derivatives of (2.5) with respect to $\xi$ and/or $\lambda$ (as many times as $G$ allows) and evaluating at $\xi = \lambda = 0$, we obtain equations for the determination of the derivatives. In particular, all derivatives up to order four are:

$$\Phi^0_\xi = \Phi^0_\lambda = 0, \tag{2.7}$$

$$\Phi^0_{\xi\xi} = X_1,\ \Phi^0_{\xi\lambda} = X_2,\ \Phi^0_{\lambda\lambda} = 0, \tag{2.8}$$

$$\Phi^0_{\xi\xi\xi} = X_3,\ \Phi^0_{\xi\xi\lambda} = X_4,\ \Phi^0_{\xi\lambda\lambda} = X_5,\ \Phi^0_{\lambda\lambda\lambda} = 0, \tag{2.9}$$

and

$$\Phi^0_{\xi\xi\xi\xi} = X_6,\ \Phi^0_{\xi\xi\xi\lambda} = X_7,\ \Phi^0_{\xi\xi\lambda\lambda} = X_8,\ \Phi^0_{\xi\lambda\lambda\lambda} = X_9,\ \Phi^0_{\lambda\lambda\lambda\lambda} = 0. \tag{2.10}$$

The superscript $^0$ on $\Phi$ refers to evaluation at $(\xi, \lambda) = (0,0)$. The elements $X_i$, residing in $\mathbb{B}_{11}$, are defined as the solutions of the following equations:

$$\begin{aligned}
G^0_u X_1 + (I - \mathcal{P})G^0_{uu}\phi^2 &= 0,\\
G^0_u X_2 + (I - \mathcal{P})G^0_{u\lambda}\phi &= 0,\\
G^0_u X_3 + (I - \mathcal{P})(G^0_{uuu}\phi^3 + 3G^0_{uu}\phi X_1) &= 0,\\
G^0_u X_4 + (I - \mathcal{P})(G^0_{uu\lambda}\phi^2 + 2G^0_{uu}\phi X_2 + G^0_{u\lambda}X_1) &= 0,\\
G^0_u X_5 + (I - \mathcal{P})(G^0_{u\lambda\lambda}\phi + 2G^0_{u\lambda}X_2) &= 0,\\
G^0_u X_6 + (I - \mathcal{P})(G^0_{uuuu}\phi^4 + 6G^0_{uuu}\phi^2 X_1 + 3G^0_{uu}X_1^2 + &\\
4G^0_{uu}\phi X_3) &= 0,
\end{aligned}$$

$$\begin{aligned}
G_u^0 X_7 + (I - \mathcal{P})(G^0_{uuu\lambda}\phi^3 + 3G^0_{uuu}\phi^2 X_2 + 3G^0_{uu\lambda}\phi X_1 + &\\
G^0_{u\lambda}X_3 + 3G^0_{uu}\phi X_4 + 3G^0_{uu}X_1X_2) &= 0,\\
G_u^0 X_8 + (I - \mathcal{P})(G^0_{uu\lambda\lambda}\phi^2 + 4G^0_{uu\lambda}\phi X_2 + G^0_{u\lambda\lambda}X_1 + &\\
2G^0_{u\lambda}X_4 + 2G^0_{uu}\phi X_5 + 2G^0_{uu}X_2^2) &= 0,\\
G_u^0 X_9 + (I - \mathcal{P})(G^0_{u\lambda\lambda\lambda}\phi + 3G^0_{u\lambda\lambda}X_2 + 3G^0_{u\lambda}X_5) &= 0.
\end{aligned}$$

With additional smoothness assumption on $G$, the elements $X_i$ exist and are unique. Note that $G^0_{uu}\phi^2$ is shorthand for $G^0_{uu}\phi\phi$.

The solution set of $G(u, \lambda) = 0$ is the same as that of the bifurcation equation:

$$\mathcal{P}G(\xi\phi + \Phi(\xi, \lambda), \lambda) = 0. \tag{2.11}$$

The procedure which we described above is the Lyapunov–Schmidt reduction. Note that (2.11) is just a scalar equation in variables $(\xi, \lambda)$. With further assumptions on $G$, we may deduce its solution structure.

Before stating our first result, we collect together the definition of some constants for future reference:

$$\begin{aligned}
a &= 6\psi^* G^0_{u\lambda}X_2 + 3\psi^* G^0_{u\lambda\lambda}\phi,\\
b &= 6\psi^* G^0_{uu}\phi X_2 + 3\psi^* G^0_{u\lambda}X_1 + 3\psi^* G^0_{uu\lambda}\phi^2,\\
c &= 3\psi^* G^0_{uu}\phi X_1 + \psi^* G^0_{uuu}\phi^3,\\
D &= b^2 - 4ac,\\
d &= 10\psi^* G^0_{u\lambda\lambda\lambda}\phi + 30\psi^* G^0_{u\lambda\lambda}X_2 + 30\psi^* G^0_{u\lambda}X_5,\\
q &= 6\psi^* G^0_{uu\lambda\lambda}\phi^2 + 24\psi^* G^0_{uu\lambda}\phi X_2 + 6\psi^* G^0_{u\lambda\lambda}X_1 + 12\psi^* G^0_{u\lambda}X_4 +\\
&\quad 12\psi^* G^0_{uu}\phi X_5 + 12\psi^* G^0_{uu}X_2^2,
\end{aligned}$$

$$
\begin{aligned}
r &= 4\psi^* G^0_{uuu\lambda}\phi^3 + 12\psi^* G^0_{uuu}\phi^2 X_2 + 12\psi^* G^0_{uu\lambda}\phi X_1 + 4\psi^* G^0_{u\lambda} X_3 + \\
&\quad 12\psi^* G^0_{uu}\phi X_4 + 12\psi^* G^0_{uu} X_1 X_2, \\
s &= \psi^* G^0_{uuuu}\phi^4 + 6\psi^* G^0_{uuu}\phi^2 X_1 + 4\psi^* G^0_{uu}\phi X_3 + 3\psi^* G^0_{uu} X_1^2.
\end{aligned}
$$

**Theorem 2.12** *In addition to (2.3), we assume $G$ is three times continuously Fréchet differentiable in a neighborhood of $(u,\lambda) = (0,0)$,*

$$\psi^* G^0_{uu}\phi^2 = 0, \tag{2.13}$$

*and*

$$a \neq 0.$$

*If $D > 0$, then the equation $G(u,\lambda) = 0$ has exactly two distinct nontrivial real solution branches (given by (2.17) below) near $(0,0)$. If $D < 0$, then the trivial solution is the only real solution near $(0,0)$.*

**Proof:** We introduce a real variable $\epsilon$ and make the following rescaling:

$$\xi = \epsilon\zeta \quad \text{and} \quad \lambda = \epsilon\Lambda, \tag{2.14}$$

where $\zeta^2 + \Lambda^2 = 1$. Recalling the representation (2.4) of $\mathcal{P}$, it is clear that when $\epsilon \neq 0$, the solutions of the bifurcation equation (2.11) correspond to the zeroes of:

$$\frac{6}{\epsilon^3}\psi^* G(\epsilon\zeta\phi + \Phi(\epsilon\zeta, \epsilon\Lambda), \epsilon\Lambda).$$

We can construct a continuous function $g$ from the above by defining it properly at $\epsilon = 0$:

$$g(\zeta,\Lambda,\epsilon) \equiv \begin{cases} \frac{6}{\epsilon^3}\psi^* G(\epsilon\zeta\phi + \Phi(\epsilon\zeta,\epsilon\Lambda),\epsilon\Lambda), & \epsilon \neq 0; \\ \zeta(a\Lambda^2 + b\zeta\Lambda + c\zeta^2), & \epsilon = 0. \end{cases} \tag{2.15}$$

The continuity of $g$ is shown using $\psi^* G^0_{uu}\phi^2 = 0 = \psi^* G^0_{u\lambda}\phi$. Now define the function:

$$h(\zeta, \Lambda, \epsilon) = \left[ \begin{array}{c} g \\ \zeta^2 + \Lambda^2 - 1 \end{array} \right].$$

If $D > 0$, then $h(\zeta, \Lambda, 0) = 0$ has six distinct roots $(\zeta, \Lambda)$:

$$(0,1), \left(k_+, k_+\frac{-b+\sqrt{D}}{2a}\right), \left(k_-, k_-\frac{-b-\sqrt{D}}{2a}\right) \qquad (2.16)$$

and

$$(0,-1), \left(-k_+, -k_+\frac{-b+\sqrt{D}}{2a}\right), \left(-k_-, -k_-\frac{-b-\sqrt{D}}{2a}\right),$$

where $k_\pm = \left(1 + ((-b \pm \sqrt{D})/2a)^2\right)^{-1/2}$. It can be shown that the Jacobian $h_{\zeta,\Lambda}$ evaluated at any of the six roots and $\epsilon = 0$ is nonsingular. (We will call such roots isolated.) Hence by the Implicit Function Theorem, there are continuous functions $\zeta_i(\epsilon)$ and $\Lambda_i(\epsilon)$, $i = 1, \ldots, 6$ such that $h(\zeta_i(\epsilon), \Lambda_i(\epsilon), \epsilon) = 0$ for all $\epsilon$ sufficiently small and $(\zeta_i(0), \Lambda_i(0))$ is one of the six roots listed above. By examining the scaling (2.14), it is apparent that the branch of solution arising from the root of $h(\zeta_i(\epsilon), \Lambda_i(\epsilon), \epsilon) = 0$ is the same as the branch of solution arising from the root of $h(-\zeta_i(\epsilon), -\Lambda_i(\epsilon), -\epsilon) = 0$. Hence locally about $(u, \lambda) = (0,0)$, there are three branches of solutions. Notice that the roots $(\zeta, \Lambda) = (0, \pm 1)$ yield the trivial solution branch. In summary, there are exactly two nontrivial solution branches near the origin $(u, \lambda) = (0,0)$. They are given by:

$$u_i(\epsilon) = \epsilon\zeta_i(\epsilon)\phi + \Phi(\epsilon\zeta_i(\epsilon), \lambda_i(\epsilon)), \quad \lambda_i(\epsilon) = \epsilon\Lambda_i(\epsilon), \quad i = 1, 2 \qquad (2.17)$$

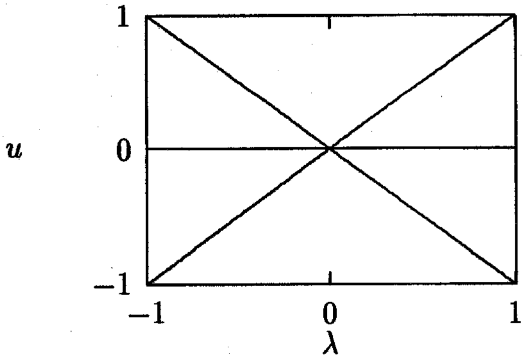


Figure 2.1: Solutions of $u(u-\lambda)(u+\lambda)=0$.

where $\zeta_i(\epsilon)$ and $\Lambda_i(\epsilon)$ are continuous functions and the pairs $(\zeta_i(0), \Lambda_i(0))$ are given by the second and third roots in (2.16). The function $\Phi$ is three times continuously differentiable and it satisfies (2.6) through (2.9).

If $D < 0$, then at $\epsilon = 0$, $h$ has exactly two real roots $(\zeta, \Lambda) = (0, \pm 1)$. They correspond to the trivial solution branch. Hence in a small neighborhood of the origin $(u, \lambda) = (0, 0)$, there are no nontrivial solutions. □

We now give some simple examples. For the equation $u(u-\lambda)(u+\lambda) = 0$, the values of the constants $a$ and $D$ are $-6$ and 144 respectively. Hence there are two nontrivial solution branches. See Figure 2.1. For the equation $u(u^2+\lambda^2) = 0$, $a = 6, D = -144$ and hence the trivial branch is the only real solution branch. For a less trivial example, consider the ordinary differential equation

$$u''(x) + (\pi^2 + \lambda^2)u(x) - u^3(x) = 0$$

with boundary conditions $u(0) = 0 = u(1)$. Here, $a = 3$ and $D = 27$ and hence there are two nontrivial solution branches. This is confirmed by a phase–plane analysis of the equation.

The next theorem concerns the next member of the hierarchy.

**Theorem 2.18** *In addition to (2.3), assume $G$ is five times continuously Fréchet differentiable in a neighborhood of $(u, \lambda) = (0, 0)$ and*

$$\psi^* G^0_{uu}\phi^2 = a = 0. \tag{2.19}$$

*If*

$$bd \neq 0,$$

*then the equation $G(u, \lambda) = 0$ has exactly two distinct nontrivial solution branches near $(0, 0)$. They are given by (2.20) below.*

**Proof:** The proof proceeds exactly as in the last theorem. The function $g$ defined in (2.15) at $\epsilon = 0$ is now

$$\zeta^2(b\Lambda + c\zeta) = 0.$$

Since $b \neq 0$, the pair of roots $(\pm 1, \mp c/b)$ are isolated. (We use the normalization $\zeta^2 + \Lambda^2 = k \equiv 1 + c^2/b^2$.) They yield one branch of solutions. The other two root pairs $(0, \pm k)$ are not isolated. To resolve these double roots, we employ a new scaling:

$$\xi = \frac{\epsilon^2}{2}\zeta \quad \text{and} \quad \lambda = \epsilon\Lambda.$$

Now introduce the function $g_2$:

$$g_2(\zeta, \Lambda, \epsilon) \equiv \begin{cases} \frac{120}{\epsilon^5}\psi^* G(\frac{\epsilon^2}{2}\zeta\phi + \Phi(\frac{\epsilon^2}{2}\zeta, \epsilon\Lambda), \epsilon\Lambda), & \epsilon \neq 0; \\ \zeta\Lambda(5b\zeta + d\Lambda^2), & \epsilon = 0. \end{cases}$$

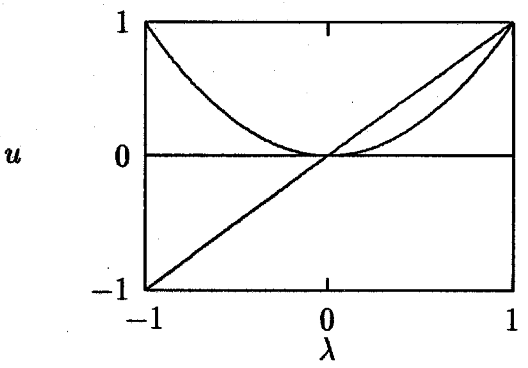


Figure 2.2: Solutions of $u(u - \lambda^2)(u - \lambda) = 0$.

Note that $g_2$ is continuous at $\epsilon = 0$. The system $g_2(\zeta, \Lambda, 0) = 0$ together with $\zeta^2 + \Lambda^2 = l \equiv 1 + d^2/25b^2$ has six roots $(\zeta, \Lambda) = (\pm l, 0), (0, \pm l)$ and $(-d/5b, \pm 1)$. The second and third pairs of these roots correspond to the double roots mentioned previously. Since $bd \neq 0$, the second and third root pairs are isolated. These give rise to two branches of solutions through the origin, one of which is the trivial solution branch. To summarize, the nontrivial solutions are given by:

$$\begin{aligned} u_1(\epsilon) &= \epsilon\zeta_1(\epsilon)\phi + \Phi\left(\epsilon\zeta_1(\epsilon), \lambda_1(\epsilon)\right), \\ \lambda_i(\epsilon) &= \epsilon\Lambda_i(\epsilon), \quad i = 1, 2, \\ u_2(\epsilon) &= \frac{\epsilon^2}{2}\zeta_2(\epsilon)\phi + \Phi\left(\frac{\epsilon^2}{2}\zeta_2(\epsilon), \lambda_2(\epsilon)\right), \end{aligned} \tag{2.20}$$

where $\zeta_i(\epsilon)$ and $\Lambda_i(\epsilon)$ are continuous functions such that $\zeta_1(0) = 1, \Lambda_1(0) = -c/b$ and $\zeta_2(0) = -d/5b$, $\Lambda_2(0) = 1$. The function $\Phi$ is five times continuously differentiable and satisfies (2.6) through (2.10). □

An example belonging to this class is the equation $u(u-\lambda^2)(u-\lambda)=0$. The constants $b, d$ take on the values $-6$ and $60$ respectively. See Figure 2.2.

If the hypotheses are the same as those in Theorem 2.12 except $D=0$, then we have the following result.

**Theorem 2.21** *In addition to (2.3), assume $G$ is five times continuously Fréchet differentiable in a neighborhood of $(u,\lambda)=(0,0)$. Define the following constants:*

$$
\begin{aligned}
y_1 &= \psi^* G^0_{uuuuu}\phi^5 + 10\psi^* G^0_{uuuu}\phi^3 X_1 + 15\psi^* G^0_{uuu}\phi X_1^2 + 10\psi^* G^0_{uuu}\phi^2 X_3 + \\
&\quad 10\psi^* G^0_{uu} X_1 X_3 + 5\psi^* G^0_{uu}\phi X_6 + \\
&\quad \eta(5\psi^* G^0_{uuuu\lambda}\phi^4 + 20\psi^* G^0_{uuuu}\phi^3 X_2 + 30\psi^* G^0_{uuu\lambda}\phi^2 X_1 + \\
&\quad 60\psi^* G^0_{uuu}\phi X_1 X_2 + 30\psi^* G^0_{uuu}\phi^2 X_4 + 15\psi^* G^0_{uu\lambda} X_1^2 + 20\psi^* G^0_{uu\lambda}\phi X_3 + \\
&\quad 30\psi^* G^0_{uu} X_1 X_4 + 20\psi^* G^0_{uu} X_2 X_3 + 20\psi^* G^0_{uu}\phi X_7 + 5\psi^* G^0_{u\lambda} X_6) + \\
&\quad 10\eta^2(\psi^* G^0_{uuu\lambda\lambda}\phi^3 + 6\psi^* G^0_{uuu\lambda}\phi^2 X_2 + 6\psi^* G^0_{uuu}\phi X_2^2 + 3\psi^* G^0_{uuu}\phi^2 X_5 + \\
&\quad 3\psi^* G^0_{uu\lambda\lambda}\phi X_1 + 6\psi^* G^0_{uu\lambda} X_1 X_2 + 6\psi^* G^0_{uu\lambda}\phi X_4 + 3\psi^* G^0_{uu} X_1 X_5 + \\
&\quad 6\psi^* G^0_{uu} X_2 X_4 + 3\psi^* G^0_{uu}\phi X_8 + \psi^* G^0_{u\lambda\lambda} X_3 + 2\psi^* G^0_{u\lambda} X_7) + \\
&\quad 10\eta^3(\psi^* G^0_{uu\lambda\lambda\lambda}\phi^2 + 6\psi^* G^0_{uu\lambda\lambda}\phi X_2 + 6\psi^* G^0_{uu\lambda} X_2^2 + 6\psi^* G^0_{uu\lambda}\phi X_5 + \\
&\quad 6\psi^* G^0_{uu} X_2 X_5 + 2\psi^* G^0_{uu}\phi X_9 + 3\psi^* G^0_{u\lambda\lambda} X_4 + 3\psi^* G^0_{u\lambda} X_8 + \psi^* G^0_{u\lambda\lambda\lambda} X_1) + \\
&\quad \eta^4(30\psi^* G^0_{u\lambda\lambda} X_5 + 20\psi^* G^0_{u\lambda} X_9 + 5\psi^* G^0_{u\lambda\lambda\lambda\lambda}\phi + 20\psi^* G^0_{u\lambda\lambda\lambda} X_2), \\
y_3 &= \frac{5}{2} r + 5\eta q + 3\eta^2 d, \\
y_6 &= 5a, \\
D_1 &= y_3^2 - 4 y_1 y_6, \\
\eta &= -\frac{b}{2a},
\end{aligned}
$$

$$z = \frac{2}{5}d\eta^3 + q\eta^2 + r\eta + s.$$

*Assume*

$$a \neq 0$$

*and*

$$\psi^* G^0_{uu}\phi^2 = z = D = 0. \tag{2.22}$$

*If* $D_1 > 0$, *then the equation* $G(u,\lambda) = 0$ *has exactly two distinct nontrivial real solution branches near* $(0,0)$. *They are given by (2.24) below. If* $D_1 < 0$, *then the trivial solution is the only real solution near* $(0,0)$.

**Proof:** One of the roots $(\zeta,\Lambda) = (0,1)$ in (2.16) is isolated. This corresponds to the trivial branch. Because $D = 0$, the other two roots are the same with $(\zeta,\Lambda) = (k, k\eta)$, where $k = (1 + b^2/4a^2)^{-1/2}$ and hence they are not isolated. To remedy the situation, we use the transformation

$$\xi = \epsilon + \frac{\epsilon^2}{2}\zeta \quad \text{and} \quad \lambda = \epsilon\eta + \frac{\epsilon^2}{2}\Lambda.$$

The function corresponding to (2.15) is defined as:

$$g(\zeta,\Lambda,\epsilon) \equiv \begin{cases} \frac{120}{\epsilon^5}\psi^* G((\epsilon + \frac{\epsilon^2}{2}\zeta)\phi + \Phi(\epsilon + \frac{\epsilon^2}{2}\zeta, \eta\epsilon + \frac{\epsilon^2}{2}\Lambda),\ \eta\epsilon + \frac{\epsilon^2}{2}\Lambda), & \epsilon \neq 0; \\ y_1 + y_2\zeta + y_3\Lambda + y_4\zeta^2 + y_5\zeta\Lambda + y_6\Lambda^2, & \epsilon = 0, \end{cases}$$

where

$$\begin{aligned} y_2 &= 10s + \frac{15}{2}\eta r + 5\eta^2 q + \eta^3 d, \\ y_4 &= 5c, \\ y_5 &= 5b. \end{aligned}$$

The continuity of $g$ at $\epsilon = 0$ can be shown using the assumptions (2.22).

For the normalization condition, we choose $\zeta = 0$ so that $\xi = \epsilon$. Now $g(0, \Lambda, 0)$ has roots:

$$\Lambda = \frac{-y_3 \pm \sqrt{D_1}}{2y_6} \tag{2.23}$$

and they are real and isolated when $D_1 > 0$. (Recall $y_6 = 5a \neq 0$.) For $i = 1, 2$ and $\epsilon$ sufficiently small, the nontrivial solution branches are given by:

$$\begin{aligned} u_i(\epsilon) &= \epsilon\phi + \Phi\left(\epsilon, \lambda_i(\epsilon)\right), \\ \lambda_i(\epsilon) &= \epsilon\eta + \frac{\epsilon^2}{2}\Lambda_i(\epsilon), \end{aligned} \tag{2.24}$$

where $\Lambda_i(\epsilon)$ are continuous functions such that $\Lambda_i(0)$ are the roots (2.23). The function $\Phi$ is five times continuously differentiable and satisfies (2.6) through (2.10). □

We now remark on the choice of normalization in the above proof. When $b \neq 0$, we could also choose the normalization $\Lambda = 0$. Then, the condition of $g$ having two real isolated roots is $y_2^2 - 4y_1y_4 > 0$. Using the assumption $z = D = 0$, it can be shown that this condition is equivalent to $D_1 > 0$.

An example is furnished by $u((u+\lambda)^2 - u^4) = 0$. The value of $D_1$ is 14400 and hence there are two nontrivial solution branches near $(0,0)$. See Figure 2.3. For the example $u((u+\lambda)^2 + u^4) = 0$, the value of $D_1$ is $-14400$ and hence the trivial solution is the only solution near $(0,0)$.

The next result gives conditions for three nontrivial bifurcating solutions.

**Theorem 2.25** *In addition to (2.3), assume $G$ is four times continuously Fréchet differentiable in a neighborhood of $(u, \lambda) = (0, 0)$. Define the follow-*

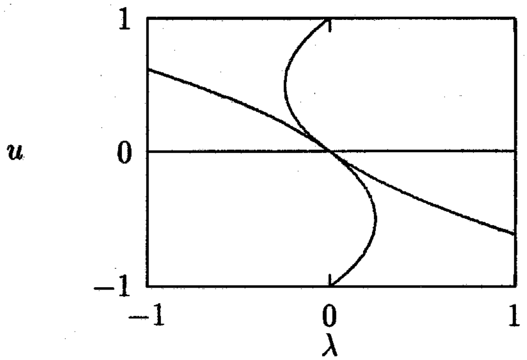


Figure 2.3: Solutions of $u((u+\lambda)^2 - u^4) = 0$.

*ing constants:*

$$p = \frac{2}{5}d,$$
$$D_2 = \left(\frac{3rp - q^2}{9p^2}\right)^3 + \left(\frac{9pqr - 27p^2s - 2q^3}{54p^3}\right)^2.$$

*Suppose*

$$d \neq 0$$

*and*

$$\psi^* G^0_{uu}\phi^2 = a = b = c = 0.$$

*If $D_2 < 0$, then the equation $G(u, \lambda) = 0$ has exactly three distinct nontrivial real solution branches (given by (2.27) below) near $(0, 0)$. If $D_2 > 0$, then there is exactly one nontrivial real solution branch near $(0, 0)$.*

**Proof:** We employ the scaling:

$$\xi = \epsilon\zeta \quad \text{and} \quad \lambda = \epsilon\Lambda$$

and define

$$g(\zeta,\Lambda,\epsilon) \equiv \begin{cases} \frac{24}{\epsilon^4}\psi^* G(\epsilon\zeta\phi + \Phi(\epsilon\zeta,\epsilon\Lambda),\epsilon\Lambda), & \epsilon \neq 0; \\ \zeta(p\Lambda^3 + q\Lambda^2\zeta + r\Lambda\zeta^2 + s\zeta^3), & \epsilon = 0. \end{cases}$$

It is straightforward to show that $g$ is continuous at $\epsilon = 0$.

At $\epsilon = 0$, $(\zeta,\Lambda) = (0,1)$ is one root of $g = 0$ with normalization condition $\zeta^2 + \Lambda^2 = 1$. Factoring out this root, we look at the remaining polynomial $p\Lambda^3 + q\Lambda^2\zeta + r\Lambda\zeta^2 + s\zeta^3 = 0$. Since $p \neq 0$, $(0,\Lambda)$ is not a root of this cubic for any nonzero $\Lambda$. So the roots of this cubic correspond to the roots of

$$p\gamma^3 + q\gamma^2 + r\gamma + s = 0, \tag{2.26}$$

where $\gamma = \frac{\Lambda}{\zeta}$. From the theory of roots of cubic polynomials, (2.26) has three real distinct roots when $D_2 < 0$ and one real root when $D_2 > 0$. We can now summarize the result. When $D_2 < 0$, the system has four real isolated solution branches (one of which is the trivial branch). The nontrivial solutions have the form:

$$u_i(\epsilon) = \epsilon\zeta_i(\epsilon)\phi + \Phi(\epsilon\zeta_i(\epsilon),\lambda_i(\epsilon)), \quad \lambda_i(\epsilon) = \epsilon\Lambda_i(\epsilon), \quad i = 1,2,3 \tag{2.27}$$

where $\zeta_i(\epsilon)$ and $\Lambda_i(\epsilon)$ are continuous functions and $(\zeta_i(0),\Lambda_i(0)) = (1,\gamma_i)$ where $\gamma_i$ are the three roots of (2.26). The function $\Phi$ is four times continuously differentiable and it satisfies (2.6) through (2.10).

When $D_2 > 0$, there is one nontrivial real solution branch. □.

The equation $u\lambda(u-\lambda)(u+\lambda) = 0$ is an example with three nontrivial solution branches. The values of the constants $p, q, r, s, D_2$ are $-24, 0, 24, 0, -1/27$ respectively. See Figure 2.4. An example with one nontrivial real solution

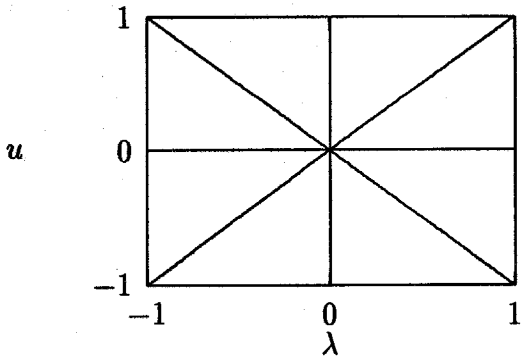


Figure 2.4: Solutions of $u\lambda(u-\lambda)(u+\lambda)=0$.

branch is $u\lambda(u^2+\lambda^2) = 0$. The values of the constants $p, q, r, s, D_2$ are $24, 0, 24, 0, 1/27$ respectively. For a less trivial example, consider the ordinary differential equation

$$u''(x) + (\pi^2 + \lambda^3)u(x) - \lambda u^3(x) = 0$$

with boundary conditions $u(0) = 0 = u(1)$. Here, $p = 12$ and $D_2 = -1/64$ and hence there are three nontrivial solution branches. Again, we can carry out a phase–plane analysis to verify this.

It is clear how to continue to derive conditions for other members of this hierarchy. All the theorems we have discussed so far require the condition $\psi^* G^0_{uu}\phi^2 = 0$. We now consider the cases when this quantity is nonzero.

**Theorem 2.28** *In addition to (2.3), assume $G$ is four times continuously Fréchet differentiable in a neighborhood of $(u,\lambda) = (0,0)$ and*

$$a \cdot \psi^* G^0_{uu}\phi^2 \neq 0.$$

*Then the equation $G(u,\lambda) = 0$ has exactly one nontrivial real solution branch near $(0,0)$. It is given by (2.29) below.*

**Proof:** The proper scaling is:

$$\xi = \frac{\epsilon^2}{2}\zeta \quad \text{and} \quad \lambda = \epsilon\Lambda.$$

The function $g$ is defined as:

$$g(\zeta,\Lambda,\epsilon) \equiv \begin{cases} \frac{24}{\epsilon^4}\psi^* G(\frac{\epsilon^2}{2}\zeta\phi + \Phi(\frac{\epsilon^2}{2}\zeta, \epsilon\Lambda), \epsilon\Lambda), & \epsilon \neq 0; \\ \zeta(3\psi^* G^0_{uu}\phi^2\zeta + 2a\Lambda^2), & \epsilon = 0. \end{cases}$$

It is continuous at $\epsilon = 0$. We take the normalization condition to be $\zeta^2 + \Lambda^2 = 1 + (2a/3\psi^* G^0_{uu}\phi^2)^2$. The nontrivial solution branch is given by (for $\epsilon$ sufficiently small):

$$u(\epsilon) = \frac{\epsilon^2}{2}\zeta(\epsilon)\phi + \Phi\left(\frac{\epsilon^2}{2}\zeta(\epsilon), \lambda(\epsilon)\right), \quad \lambda(\epsilon) = \epsilon\Lambda(\epsilon), \tag{2.29}$$

where $\zeta(\epsilon)$ and $\Lambda(\epsilon)$ are continuous functions such that $\zeta(0) = -\frac{2}{3}a/\psi^* G^0_{uu}\phi^2$ and $\Lambda(0) = 1$. The function $\Phi$ is four times continuously differentiable and satisfies (2.6) through (2.10). □

An example is given by the equation $u(u - \lambda^2) = 0$. Here, $\psi^* G^0_{uu}\phi^2 = 2$ and $a = -6$. See Figure 2.5.

For the next member of this hierarchy, we must come up with some conditions which allow the characterization of the solution set for the case $\psi^* G^0_{uu}\phi^2 \neq 0$ and $a = 0$.

It is straightforward to modify the results for complex solutions ($\lambda$ still real). It is no longer necessary to distinguish between the positive and negative discriminants $D, D_1$ and $D_2$ (see Theorems 2.12, 2.21 and 2.25). As long

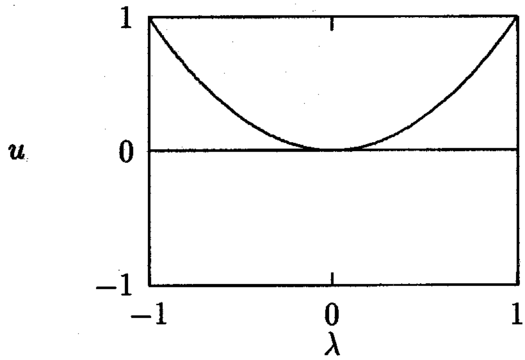


Figure 2.5: Solutions of $u(u-\lambda^2)=0$.

as these quantities are nonzero, there are, respectively, two, two and three nontrivial bifurcating complex solutions.

For the general member of the hierarchy, we can use Newton's diagram (see, for example, Sattinger [32]) to deduce the correct $\epsilon$–scaling(s) of the variables $\xi$ and $\lambda$. We can then find the algebraic bifurcation equation, a polynomial consisting of the lowest order $\epsilon$ terms of the bifurcation equation. Finally, we can determine the conditions for the isolation of the roots of the polynomial.

Our method of proof works for isolated solutions and do not apply to problems with a double zero such as $u(u-\lambda)^2=0$. For this particular equation, $a=c=6$, $b=-12$ and $D=0$. The trivial solution is isolated but the other solution $u=\lambda$ is a double zero. In the situation of Theorem 2.12, a necessary condition for the occurrence of a double zero is $D=0$ and hence two of the roots in (2.16) are the same. To verify that a solution $u_1$ is a double (or higher order) zero, we need to show that the quantity

$$\lim_{u \to u_1} \frac{\| G(u,\lambda) \|}{\| u - u_1 \|^2}$$

exists, where $\| \cdot \|$ is the norm in the appropriate space. For our example, it clearly does.

Our hierarchy also fails for pathological equations such as $G(u,\lambda) = u(u - e^{-1/\lambda^2}) = 0$. All the Taylor coefficients of this equation (expanded about $(0,0)$) are zero except for $G^0_{uu}$. We do not know how to deal with this situation. In Figure 2.6, we summarize all the theorems in this chapter. By calculating the constants in the rectangles, we can proceed down the flowchart to determine the solution structure.

Finally, we remark that if the given equation has a higher degree of smoothness than is specified in the previous theorems, the solution branches will have a higher degree of smoothness as guaranteed by the Implicit Function Theorem.

## 2.2 Limit Point Bifurcations

In this section, we still work with the equation $G(u,\lambda) = 0$. Suppose $G(0,0) = 0$ and $G^0_u$ is a Fredholm operator of index 0 with a nontrivial null space. Let $(u(\epsilon), \lambda(\epsilon))$ be a continuous branch of solutions (for all $\epsilon$ in a neighborhood of 0) such that $(u(0), \lambda(0)) = (0,0)$. We call $(0,0)$ a limit point (or fold point) if

$$\lim_{\epsilon \to 0} \frac{\lambda(\epsilon)}{\epsilon} = 0 \qquad (2.30)$$

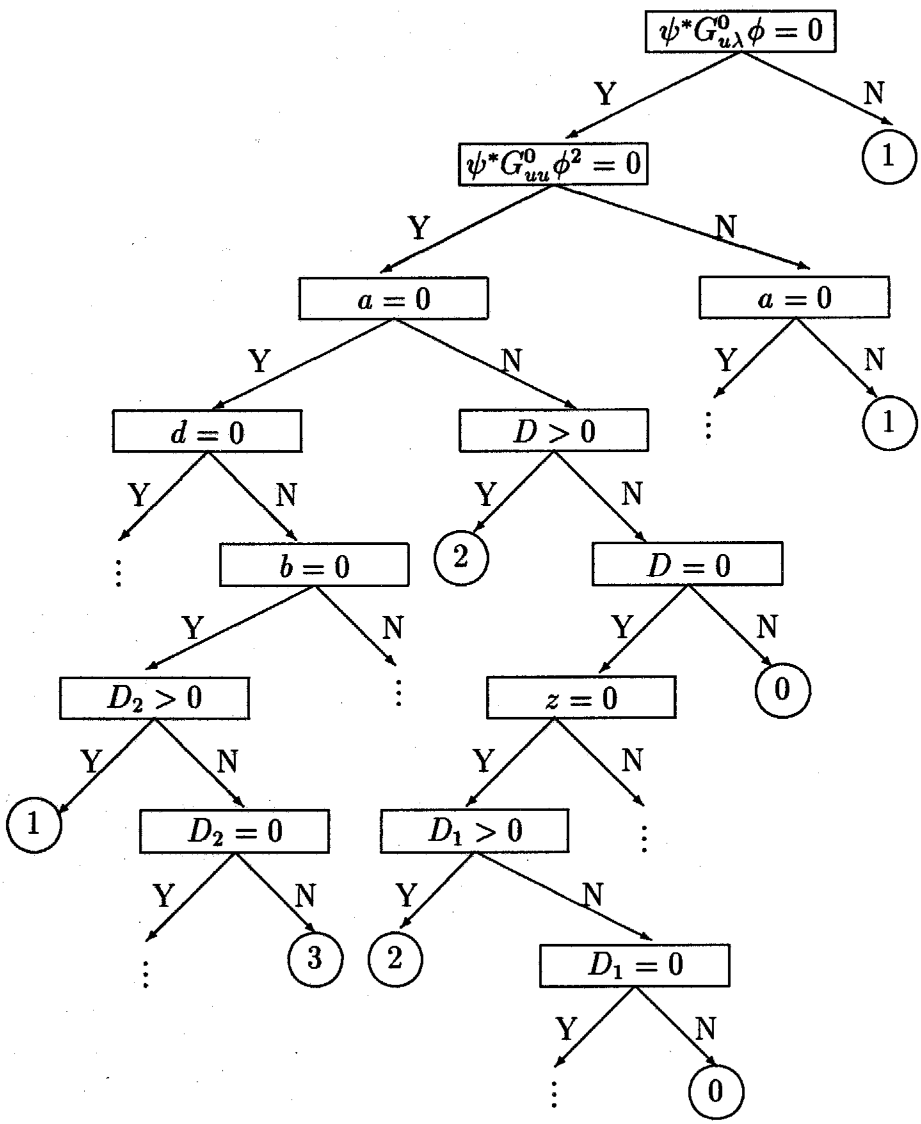


Figure 2.6: Flowchart summarizing the results of this chapter. The number inside each circle is the number of bifurcating solutions.

for every continuous branch of solutions $(u(\epsilon), \lambda(\epsilon))$ passing through $(0,0)$. This is a more general definition than the one given in Decker and Keller [10] where they require that every solution branch $(u(\epsilon), \lambda(\epsilon))$ have continuous first derivatives and

$$G_\lambda^0 \notin \mathcal{R}(G_u^0). \tag{2.31}$$

Indeed, assuming smoothness of the solution branches, we can differentiate $G(u(\epsilon), \lambda(\epsilon)) = 0$ with respect to $\epsilon$ to obtain

$$G_u^0 \dot{u}(0) + G_\lambda^0 \dot{\lambda}(0) = 0,$$

where dot denotes $\epsilon$ derivative. Applying any nonzero vector $\psi^* \in \mathcal{N}(G_u^{0^*})$ to the above, we get

$$\psi^* G_\lambda^0 \cdot \dot{\lambda}(0) = 0.$$

From (2.31), it is easy to see that $\dot{\lambda}(0) = 0$ which clearly implies (2.30). Later on (following Theorem 2.35), we will give an example where the more general definition is needed.

In this section, the Banach spaces are either real or complex. As before, we state some of the hypotheses here for easy reference. They will be assumed for the remainder of this section.

**Hypothesis 2.32**

- *$G$ is a mapping from $\mathbb{B}_1 \times \mathbb{R}$ into $\mathbb{B}_2$.*
- $G(0,0) = 0$.
- *$G$ has continuous first Fréchet derivatives at $(u, \lambda) = (0,0)$.*

- *dim* $\mathcal{N}(G_u^0) =$ *codim* $\mathcal{R}(G_u^0) = 1$. *We will denote some specific nonzero element of* $\mathcal{N}(G_u^0)$ *and* $\mathcal{N}(G_u^{0^*})$ *by* $\phi$ *and* $\psi^*$, *respectively.*

In addition to the elements $X_i$ which were introduced in the last section, we define the elements $X_{10}$ and $X_{11}$ (in $\mathbb{B}_{11}$, a space complementary to $\mathcal{N}(G_u^0)$) by:

$$\begin{aligned} G_u^0 X_{10} + (I - \mathcal{P}) G_\lambda^0 &= 0, \\ G_u^0 X_{11} + (I - \mathcal{P})(G_{uu}^0 \phi X_{10} + G_{u\lambda}^0 \phi) &= 0. \end{aligned}$$

Recall that $\mathcal{P}$ is the projection onto $\mathbb{B}_{22}$, a complement of $\mathcal{R}(G_u^0)$. We also define some constants:

$$\begin{aligned} c &= 3\psi^* G_{uu}^0 \phi X_1 + \psi^* G_{uuu}^0 \phi^3, \\ f &= \psi^* G_{uu}^0 \phi X_{10} + \psi^* G_{u\lambda}^0 \phi. \end{aligned}$$

We begin by stating a well-known result on the generic limit point: a simple quadratic fold. See Keller [17], Henderson and Keller [16].

**Theorem 2.33** *In addition to (2.32), assume G is twice continuously Fréchet differentiable in a neighborhood of* $(u, \lambda) = (0, 0)$ *and*

$$\psi^* G_{uu}^0 \phi^2 \cdot \psi^* G_\lambda^0 \neq 0.$$

*Then, the equation* $G(u, \lambda) = 0$ *has exactly two complex solution branches near* $(0, 0)$. *They are, for j=1,2 and* $\epsilon$ *sufficiently small:*

$$\begin{aligned} u_j(\epsilon) &= \epsilon \zeta_j(\epsilon) \phi + \Phi(\epsilon \zeta_j(\epsilon), \lambda_j(\epsilon)), \\ \lambda_j(\epsilon) &= \frac{\epsilon^2}{2} \Lambda_j(\epsilon), \end{aligned}$$

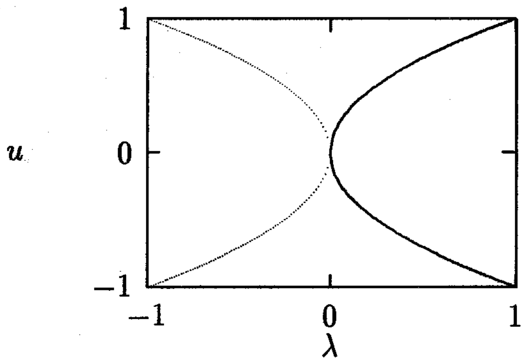


Figure 2.7: Solutions of $u^2 - \lambda = 0$. Dotted line denotes complex solution.

*where the functions $\zeta_j$ and $\Lambda_j$ are continuous and satisfy:*

$$\begin{aligned} \zeta_1(0) &= e^{-i\alpha/2}, & \Lambda_1(0) &= -\rho, \\ \zeta_2(0) &= ie^{-i\alpha/2}, & \Lambda_2(0) &= \rho. \end{aligned}$$

*The real constants $\rho$ and $\alpha$ $(0 \leq \alpha < 2\pi)$ are given by*

$$\rho e^{i\alpha} = \frac{\psi^* G^0_{uu}\phi^2}{\psi^* G^0_\lambda}.$$

*The function $\Phi$ is two times continuously differentiable and satisfies $\Phi(0,0) = 0$, $\Phi^0_\xi = 0$ and $\Phi^0_\lambda = X_{10}$. Furthermore, when $\mathbb{B}_1$ and $\mathbb{B}_2$ are real Banach spaces, then there is a unique real solution branch $(u_1, \lambda_1)$ given above (with $\alpha = 0$) near $(0,0)$.*

The canonical example is $u^2 - \lambda = 0$. See Figure 2.7.

We now state the result for cubic limit points. For related works, see Yang and Keller [36] and references therein.

**Theorem 2.34** *In addition to (2.32), assume $G$ is three times continuously*

*Fréchet differentiable in a neighborhood of* $(u,\lambda)=(0,0)$,

$$\psi^* G^0_{uu}\phi^2 = 0$$

*and*

$$c \cdot \psi^* G^0_\lambda \neq 0.$$

*Then, the equation* $G(u,\lambda)=0$ *has exactly three complex solution branches near* $(0,0)$. *They are, for j=1,2,3 and* $\epsilon$ *sufficiently small:*

$$\begin{aligned} u_j(\epsilon) &= \epsilon\zeta_j(\epsilon)\phi + \Phi(\epsilon\zeta_j(\epsilon),\lambda_j(\epsilon)),\\ \lambda_j(\epsilon) &= \frac{\epsilon^3}{6}\Lambda_j(\epsilon), \end{aligned}$$

*where the functions* $\zeta_j$ *and* $\Lambda_j$ *are continuous and satisfy:*

$$\begin{aligned} \zeta_1(0) &= e^{-i\alpha/3}, & \Lambda_1(0) &= \rho,\\ \zeta_2(0) &= e^{-i(\alpha+2\pi)/3}, & \Lambda_2(0) &= \rho,\\ \zeta_3(0) &= e^{-i(\alpha+4\pi)/3}, & \Lambda_3(0) &= \rho. \end{aligned}$$

*The real constants* $\rho$ *and* $\alpha$ $(0\le\alpha<2\pi)$ *are given by*

$$\rho e^{i\alpha} = -\frac{c}{\psi^* G^0_\lambda}.$$

*The function* $\Phi$ *is three times continuously differentiable and satisfies* $\Phi(0,0)=0$, $\Phi^0_\xi = 0$, $\Phi^0_\lambda = X_{10}$ *and* $\Phi^0_{\xi\xi} = X_1$. *Furthermore, when* $\mathbb{B}_1$ *and* $\mathbb{B}_2$ *are real Banach spaces, then there is a unique real solution branch* $(u_1,\lambda_1)$ *given above (with* $\alpha=0$*) near* $(0,0)$.

**Proof:** We use the scaling:

$$\xi = \epsilon\zeta \;\text{ and }\; \lambda = \frac{\epsilon^3}{6}\Lambda.$$

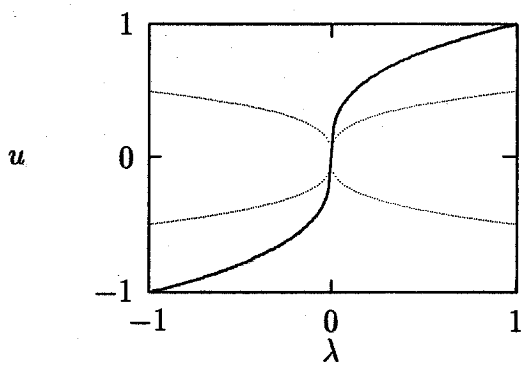


Figure 2.8: Solutions of $u^3 - \lambda = 0$. Dotted lines denote complex solutions.

The function $g$ is defined as:

$$g(\zeta, \Lambda, \epsilon) \equiv \begin{cases} \frac{6}{\epsilon^3}\psi^* G(\epsilon\zeta\phi + \Phi(\epsilon\zeta, \frac{\epsilon^3}{6}\Lambda), \frac{\epsilon^3}{6}\Lambda), & \epsilon \neq 0; \\ c\zeta^3 + \psi^* G^0_\lambda \Lambda, & \epsilon = 0. \end{cases}$$

Note that $g$ is continuous at $\epsilon = 0$. The system $g(\zeta, \Lambda, 0) = 0$ together with the normalization $\Lambda = \rho$ has three complex solution branches. □

The equation $u^3 - \lambda = 0$ is the canonical example exhibiting a cubic limit point. Here $c = 6$ and $\psi^* G^0_\lambda = -1$. See Figure 2.8.

The final theorem gives conditions for a bifurcation with four complex solution branches.

**Theorem 2.35** *In addition to (2.32), assume $G$ is four times continuously Fréchet differentiable in a neighborhood of $(u, \lambda) = (0, 0)$. Define the following constants:*

$$t = 3\psi^* G^0_{uu} X^2_{10} + 6\psi^* G^0_{u\lambda} X_{10} + 3\psi^* G^0_{\lambda\lambda},$$

$$
\begin{aligned}
v &= 6\psi^* G^0_{uuu}\phi^2 X_{10} + 6\psi^* G^0_{uu\lambda}\phi^2 + 6\psi^* G^0_{uu} X_1 X_{10} + 12\psi^* G^0_{uu}\phi X_{11} + \\
&\quad 6\psi^* G^0_{u\lambda} X_1, \\
w &= \psi^* G^0_{uuuu}\phi^4 + 6\psi^* G^0_{uuu}\phi^2 X_1 + 3\psi^* G^0_{uu} X_1^2 + 4\psi^* G^0_{uu}\phi X_3, \\
D_3 &= v^2 - 4tw.
\end{aligned}
$$

*Assume*

$$\psi^* G^0_{uu}\phi^2 = \psi^* G^0_\lambda = c = f = 0$$

*and*

$$t \cdot \psi^* G^0_{\lambda\lambda} \neq 0.$$

*Then,*

- *if $w \neq 0$ and $D_3 \neq 0$, then the equation $G(u,\lambda) = 0$ has exactly four complex solution branches near $(0,0)$. They are, for j=1,2,3,4 and $\epsilon$ sufficiently small:*

$$
\begin{aligned}
u_j(\epsilon) &= \epsilon\zeta_j(\epsilon)\phi + \Phi(\epsilon\zeta_j(\epsilon), \lambda_j(\epsilon)), \\
\lambda_j(\epsilon) &= \frac{\epsilon^2}{2}\Lambda_j(\epsilon),
\end{aligned}
\tag{2.36}
$$

*where the functions $\zeta_j$ and $\Lambda_j$ are continuous and satisfy:*

$$
\begin{aligned}
\zeta_{1,2}(0) &= e^{-i\alpha_\pm/2}, & \Lambda_{1,2}(0) &= \rho_\pm, \\
\zeta_{3,4}(0) &= ie^{-i\alpha_\pm/2}, & \Lambda_{3,4}(0) &= -\rho_\pm.
\end{aligned}
$$

*The real constants $\rho_\pm$ and $\alpha_\pm$ $(0 \leq \alpha_\pm < 2\pi)$ are given by*

$$\rho_\pm e^{i\alpha_\pm} = \frac{-v \pm \sqrt{D_3}}{2t}.$$

- *if $v \neq 0$ and $w = 0$, then there are exactly three complex solution branches near* $(0,0)$. *They have the same form as (2.36) with*

$$\begin{array}{rclrcl} \zeta_1(0) & = & 1, & \Lambda_1(0) & = & 0, \\ \zeta_2(0) & = & e^{-i\beta/2}, & \Lambda_2(0) & = & \delta, \\ \zeta_3(0) & = & ie^{-i\beta/2}, & \Lambda_3(0) & = & -\delta. \end{array}$$

*The real constants $\delta$ and $\beta$* $(0 \leq \beta < 2\pi)$ *are given by*

$$\delta e^{i\beta} = -\frac{v}{t}.$$

- *if $\mathbb{B}_1$ and $\mathbb{B}_2$ are real Banach spaces, then $D_3 >$* $(<$ *resp.*$)$ $0$ *imply there are exactly two (no) real solution branches near* $(0,0)$. *The real solutions are given by* $(u_1, \lambda_1)$ *and* $(u_2, \lambda_2)$ *in (2.36) with* $\alpha_\pm = 0$.

*The function $\Phi$ is four times continuously differentiable and satisfies* $\Phi(0,0) = 0$, $\Phi^0_\xi = 0$, $\Phi^0_\lambda = X_{10}$, $\Phi^0_{\xi\lambda} = X_{11}$ *and* $\Phi^0_{\xi\xi} = X_1$.

**Proof:** We use the scaling:

$$\xi = \epsilon\zeta \quad \text{and} \quad \lambda = \frac{\epsilon^2}{2}\Lambda.$$

The function $g$ is defined as:

$$g(\zeta, \Lambda, \epsilon) \equiv \begin{cases} \frac{24}{\epsilon^4}\psi^* G(\epsilon\zeta\phi + \Phi(\epsilon\zeta, \frac{\epsilon^2}{2}\Lambda), \frac{\epsilon^2}{2}\Lambda), & \epsilon \neq 0; \\ t\Lambda^2 + v\zeta^2\Lambda + w\zeta^4, & \epsilon = 0. \end{cases}$$

□

An example is the equation $(u^2 - \lambda)(u^2 + \lambda) = 0$. The constants $t, v, w, D_3$ have the values $-6, 0, 24$ and $576$. Hence there are two real and two complex solution branches in a neighborhood of $(0,0)$. See the top graph in Figure 2.9.

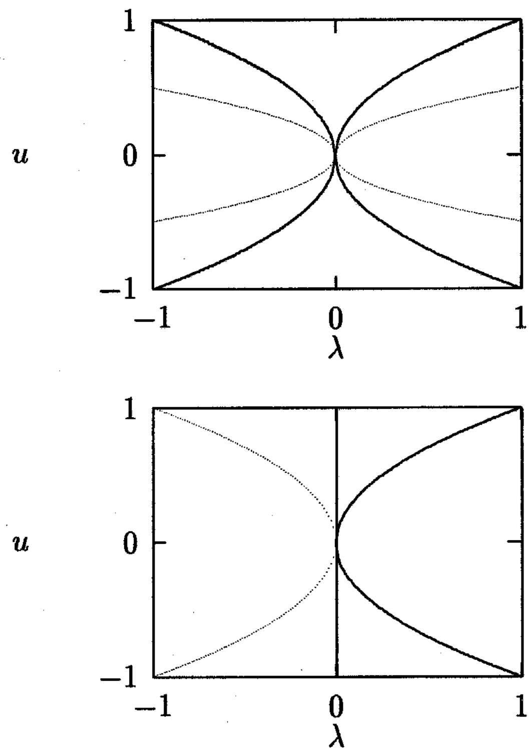


Figure 2.9: Solutions of $(u^2 - \lambda)(u^2 + \lambda) = 0$ are shown in the top graph while solutions of $\lambda(u^2 - \lambda) = 0$ are shown in the bottom graph. Dotted lines denote complex solutions.

Note that in this example, $\psi^* G^0_\lambda = 0$ and hence $(0,0)$ would not be classified as a limit point under the definition of Decker and Keller. The example $u^4 + \lambda^2 = 0$ has no real solution branches near $(0,0)$. Here $t, v, w, D_3$ have the values $6, 0, 24$ and $-576$. The equation $\lambda(u^2 - \lambda) = 0$ has three solution branches, two of which are real. The values of the constants $t, v, w$ and $D_3$ are $-6, 12, 0$ and $144$ respectively. See the bottom graph in Figure 2.9. Finally, consider the ordinary differential equation:

$$u''(x) + \pi^2 u(x) + u^4(x) - \lambda^2 = 0$$

with boundary conditions $u(0) = 0 = u(1)$. Here, $t, v, w, D_3$ have the values $-12/\pi, 0, 128/5\pi, 6144/5\pi^2$. Hence there are two real solution branches.

# Chapter 3

# Multiple Bifurcation from Multiple Eigenvalues

In this chapter, we consider bifurcation from a multiple eigenvalue of $G(u, \lambda)$. We reduce the problem $G(u, \lambda) = 0$ to a finite-dimensional set of equations, called bifurcation equations, using the method of Lyapunov–Schmidt and prove the existence of multiple bifurcating solutions using the Implicit Function Theorem (IFT). Note that in general the bifurcation equations consist of more than one equation. This makes the analysis much harder than in the previous chapter. Consequently, our results are much weaker. Very strong assumptions on $G$ will be made to allow us to find some isolated solutions of the bifurcation equations.

For convenience, we state some hypotheses which are assumed for the remainder of this chapter.

**Hypothesis 3.1**

- *$G$ is a mapping from $\mathbb{B}_1 \times \mathbb{R}$ into $\mathbb{B}_2$, where $\mathbb{B}_1$ and $\mathbb{B}_2$ are real Banach spaces.*

- *$G(0,\lambda) = 0$, for all $\lambda$ in some interval containing* 0.
- *$G$ is twice continuously Fréchet differentiable near $(u,\lambda) = (0,0)$.*
- *$\dim \mathcal{N}(G_u^0) = \operatorname{codim} \mathcal{R}(G_u^0) = M > 0$. Let $\{\phi_1, \ldots, \phi_M\}$ be a basis for $\mathcal{N}(G_u^0)$.*

We can decompose the Banach spaces into direct sums:

$$\mathbb{B}_1 = \mathcal{N}(G_u^0) \oplus \mathbb{B}_{11}, \qquad \mathbb{B}_2 = \mathcal{R}(G_u^0) \oplus \mathbb{B}_{22}.$$

Let $\{\psi_1, \ldots, \psi_M\}$ be a basis for $\mathbb{B}_{22}$. Let $\psi_i^*$ be the unique element in $\mathcal{N}(G_u^{0^*})$ such that $\psi_i^*(\psi_j) = \delta_{ij}$. From (3.1), $G_u^0$ is a Fredholm operator of index 0 and thus the dimension of $\mathcal{N}(G_u^{0^*})$ is $M$. Since $\{\psi_1^*, \ldots, \psi_M^*\}$ is a linearly independent set of elements in $\mathcal{N}(G_u^{0^*})$, they form a basis in that space. Let $\mathcal{P}$ be a projection operator from $\mathbb{B}_2$ onto $\mathbb{B}_{22}$. We can represent $\mathcal{P}$ as:

$$\mathcal{P} = \sum_{i=1}^{M} \psi_i \psi_i^*. \tag{3.2}$$

The projection operator from $\mathbb{B}_2$ onto $\mathcal{R}(G_u^0)$ is $I - \mathcal{P}$, where $I$ is the identity operator.

Any element $u$ in $\mathbb{B}_1$ can be written as $\sum_{i=1}^{M} \xi_i \phi_i + \Phi$, where $\xi_i \in \mathbb{R}$ and $\Phi \in \mathbb{B}_{11}$. The solution set of $G(u,\lambda) = 0$ is precisely the solution set of the equations $\mathcal{P}G(u,\lambda) = 0$ and $(I - \mathcal{P})G(u,\lambda) = 0$. We first focus on the equation

$$F(\Phi, \boldsymbol{\xi}, \lambda) \equiv (I - \mathcal{P})G\left(\sum_{i=1}^{M} \xi_i \phi_i + \Phi, \lambda\right) = 0,$$

where $\boldsymbol{\xi}$ is an $M$–vector whose $i^{th}$ component is $\xi_i$. Now $F(0,0,0) = 0$ and $F_\Phi(0,0,0) = G_u^0$ is a continuous isomorphism from $\mathbb{B}_{11}$ onto $\mathcal{R}(G_u^0)$. Hence by the IFT, there is a twice continuously differentiable function $\Phi(\boldsymbol{\xi}, \lambda)$ such that

$$F(\Phi(\boldsymbol{\xi}, \lambda), \boldsymbol{\xi}, \lambda) = 0 \tag{3.3}$$

for all $(\boldsymbol{\xi}, \lambda)$ near $(0, 0)$ and

$$\Phi(0, 0) = 0. \tag{3.4}$$

By taking successive derivatives of (3.3) with respect to $\xi$ and/or $\lambda$ and evaluating at $\boldsymbol{\xi} = 0$ and $\lambda = 0$, we obtain equations for the determination of the derivatives. In particular, for $i, j = 1, \ldots, M$, some of these derivatives are:

$$\Phi^0_{\xi_i} = \Phi^0_\lambda = \Phi^0_{\lambda\lambda} = 0, \tag{3.5}$$

$$\Phi^0_{\xi_i\xi_j} = \Phi_{ij}, \tag{3.6}$$

where $\Phi_{ij} \in \mathbb{B}_{11}$ is the unique solution to

$$G^0_u \Phi_{ij} + (I - \mathcal{P}) G^0_{uu} \phi_i \phi_j = 0. \tag{3.7}$$

The superscript $^0$ on the derivatives of $\Phi$ refers to evaluation at $(\boldsymbol{\xi}, \lambda) = (0, 0)$. Note that $\Phi_{ij} = \Phi_{ji}$. Now the solution set of $G(u, \lambda) = 0$ is the same as that of the bifurcation equations:

$$\mathcal{P}G\left(\sum_{i=1}^{M} \xi_i \phi_i + \Phi(\boldsymbol{\xi}, \lambda), \lambda\right) = 0, \tag{3.8}$$

where $\Phi$ satisfies (3.4), (3.5) and (3.6). The procedure described above is known as the Lyapunov–Schmidt reduction. With further assumptions on $G$, we may deduce the solution structure of the bifurcation equations.

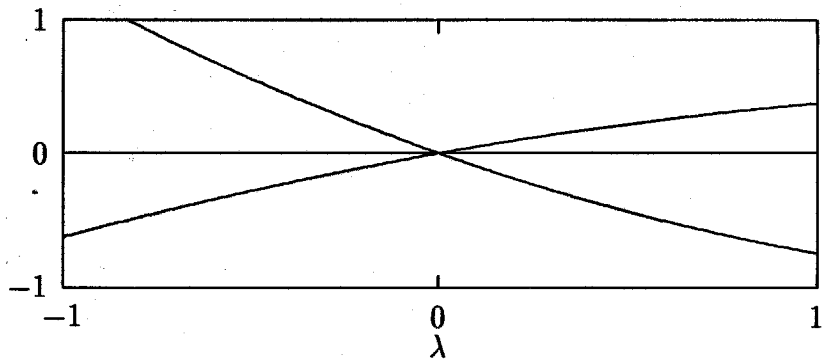


Figure 3.1: Multiple transcritical bifurcation at $(0,0)$ from the trivial solution branch (multiplicity 2).

## 3.1 Multiple Transcritical Bifurcation

We begin by proving a simple result for multiple transcritical bifurcation. This is a type of bifurcation where every branch of bifurcating solution contains solutions over a $\lambda$–interval with $\lambda = 0$ as an interior point. A typical case is illustrated in Figure 3.1.

**Theorem 3.9** *In addition to (3.1), suppose for all $i, j, k \in \{1, \ldots, M\}$:*

$$\psi_j^*(G_{u\lambda}^0 \phi_i) = \delta_{ij}, \tag{3.10}$$

$$\psi_k^*(G_{uu}^0 \phi_i \phi_j) = a_i \delta_{ij} \delta_{ik}, \tag{3.11}$$

*where the real constants $a_i$ are all different from zero, then $G(u,\lambda) = 0$ has exactly $2^M - 1$ distinct nontrivial bifurcating solution branches at $(u,\lambda) = (0,0)$.*

**Proof:** We carry out the Lyapunov–Schmidt reduction as described in the beginning of this chapter. We introduce a real variable $\epsilon$ and make the following rescaling:

$$\boldsymbol{\xi} = \epsilon\boldsymbol{\zeta} \quad \text{and} \quad \lambda = \epsilon\Lambda, \tag{3.12}$$

where $\sum_{i=1}^{M} \zeta_i^2 = 1$. Recalling the representation (3.2) of $\mathcal{P}$, it is clear that when $\epsilon \neq 0$, the solutions of the bifurcation equations (3.8) correspond to the zeroes of:

$$\frac{2}{\epsilon^2}\sum_{i=1}^{M} \psi_i\psi_i^* G\left(\epsilon\sum_{j=1}^{M} \zeta_j\phi_j + \Phi(\epsilon\boldsymbol{\zeta}, \epsilon\Lambda), \epsilon\Lambda\right).$$

We can construct a continuous function $g$ from the above by defining it properly at $\epsilon = 0$:

$$g(\boldsymbol{\zeta}, \Lambda, \epsilon) \equiv \begin{cases} \frac{2}{\epsilon^2}\sum_i \psi_i\psi_i^* G(\epsilon\sum_j \zeta_j\phi_j + \Phi(\epsilon\boldsymbol{\zeta}, \epsilon\Lambda), \epsilon\Lambda), & \epsilon \neq 0; \\ \sum_i \psi_i \left(a_i\zeta_i^2 + 2\zeta_i\Lambda\right), & \epsilon = 0. \end{cases} \tag{3.13}$$

The continuity of $g$ is shown using the L'Hopitals rule and hypotheses (3.10) and (3.11). Now $g(\boldsymbol{\zeta}, \Lambda, 0) = 0$ is a system of $M$ quadratic equations where the variables $\boldsymbol{\zeta}$ are decoupled. This permits us to find its solutions analytically.

Define the function:

$$h(\boldsymbol{\zeta}, \Lambda, \epsilon) = \begin{bmatrix} g \\ \sum_i \zeta_i^2 - 1 \end{bmatrix}.$$

We now show that $h(\boldsymbol{\zeta}, \Lambda, 0) = 0$, sometimes called the algebraic bifurcation equations (ABEs), have exactly $2(2^M - 1)$ roots $(\pm\boldsymbol{\zeta}^{(i)}, \mp\Lambda^{(i)})$, $i = 1, \ldots, 2^M - 1$. From the equation $g(\boldsymbol{\zeta}, \Lambda, 0) = 0$, either $\zeta_i = 0$ or $a_i\zeta_i + 2\Lambda = 0$ for each $i \in \{1, \ldots, M\}$. Suppose the first $k$ components of $\boldsymbol{\zeta}$ are nonzero and the rest are zero. It is easy to show that there are exactly two such solutions to the ABEs. For instance, when $k = 3$, a solution is $\zeta_1 = \frac{\kappa}{a_1}$, $\zeta_2 = \frac{\kappa}{a_2}$, $\zeta_3 = \frac{\kappa}{a_3}$, $\zeta_i = 0$, $i = 3, \ldots, M$ and $\Lambda = -\kappa/2$, where $\kappa = (\frac{1}{a_1^2} + \frac{1}{a_2^2} + \frac{1}{a_3^2})^{-1/2}$. The second solution is the negative of the above solution. To determine the total number of solutions to the ABEs, we sum the number of subsets with $k$ nonzero components with $k$ running from 1 to $M$. By a simple combinatorial argument, this number is $2(2^M - 1)$. To check that these roots are isolated, we need to examine the appropriate Jacobian. At a root $(\boldsymbol{\zeta}, \Lambda)$ and $\epsilon = 0$, the Fréchet derivative of $h$ with respect to $(\boldsymbol{\zeta}, \Lambda)$ is

$$h' \equiv h_{\boldsymbol{\zeta},\Lambda} = \begin{bmatrix} 2(\zeta_1 a_1 + \Lambda)\psi_1 & \cdots & 2(\zeta_M a_M + \Lambda)\psi_M & 2\sum_i \zeta_i \psi_i \\ 2\zeta_1 & \cdots & 2\zeta_M & 0 \end{bmatrix}.$$

It is straightforward to show that the above is an isomorphism from $\mathbb{R}^M \times \mathbb{R}$ onto $\mathbb{B}_{22} \times \mathbb{R}$. We will show it for the root $(\boldsymbol{\zeta}, \Lambda)$ where $\zeta_1 = 1, \zeta_i = 0$ for $i > 1$ and $\Lambda = -a_1/2$. The isolation of the other roots can be shown similarly. Suppose for some $(\boldsymbol{\eta}, \mu) \in \mathbb{R}^M \times \mathbb{R}$,

$$h' \begin{bmatrix} \boldsymbol{\eta} \\ \mu \end{bmatrix} = 0.$$

Writing out the equations, we have

$$\begin{aligned} (a_1\eta_1 + 2\mu)\psi_1 - a_1 \sum_{i\geq 2} \eta_i \psi_i &= 0, \\ 2\eta_1 &= 0. \end{aligned}$$

Since the set $\{\psi_1, \ldots, \psi_M\}$ is linearly independent, the only solution to the above equations is $(\boldsymbol{\eta}, \mu) = (0, 0)$. Thus the matrix $h'$ is invertible. By the IFT there exist unique continuous functions $\boldsymbol{\zeta}(\epsilon)$ and $\Lambda(\epsilon)$ such that

$h(\boldsymbol{\zeta}(\epsilon), \Lambda(\epsilon), \epsilon) = 0$ and $(\boldsymbol{\zeta}(0), \Lambda(0))$ is the given root. Hence one solution of $G(u, \lambda) = 0$, parametrized by $\epsilon$, is

$$u(\epsilon) = \epsilon \sum_i \zeta_i(\epsilon)\phi_i + \Phi(\epsilon\boldsymbol{\zeta}(\epsilon), \lambda(\epsilon)), \qquad \lambda(\epsilon) = \epsilon\Lambda(\epsilon).$$

From the scaling (3.12), it is apparent that the root $(\boldsymbol{\zeta}^{(i)}, \Lambda^{(i)})$ yields the same branch of solutions as the root $(-\boldsymbol{\zeta}^{(i)}, -\Lambda^{(i)})$. Since there is a one–to–one correspondence between each isolated root and each solution of $G(u, \lambda) = 0$, there are exactly $2^M - 1$ nontrivial bifurcating solution branches at $(0, 0)$. □

Note that the $O(\epsilon)$ term in $\lambda(\epsilon)$ is nonzero and hence we have a transcritical bifurcation. When $M = 1$, then assumptions (3.10) and (3.11) are equivalent to

$$\psi_1^*(G_{u\lambda}^0\phi_1) \cdot \psi_1^*(G_{uu}^0\phi_1^2) \neq 0.$$

These are the simplest criteria for a transcritical bifurcation. The following example in $\mathbb{R}^M$ furnishes an example where there are $2^M - 1$ bifurcating solution branches at $(0, 0)$.

$$u_i(u_i - \lambda) + O(u^3 + \lambda u^2 + h.o.t.) = 0, \quad i = 1, \ldots, M.$$

## 3.2 Multiple Pitchfork Bifurcation

If a solution branch lies entirely on one side of the bifurcation point, and similarly for all other solution branches (though not necessarily all on the same side of the bifurcation point), we call this a multiple pitchfork bifurcation. This is because the schematic diagram of each branch suggests a pitchfork (see Figure 3.2). If a branch opens up to the right of the bifurcation point,

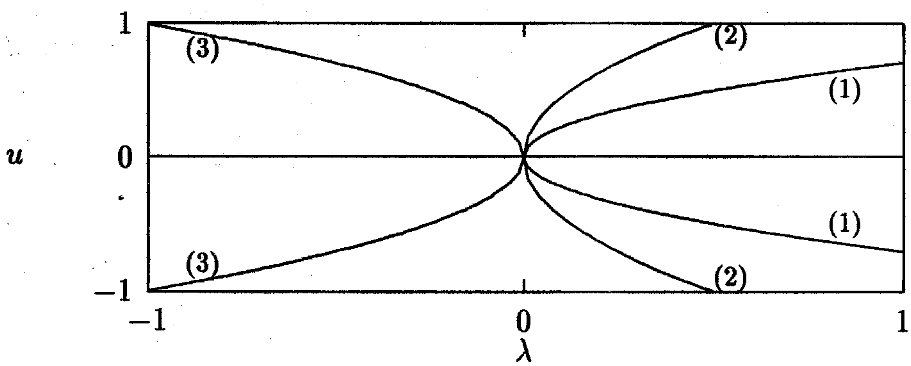


Figure 3.2: Multiple pitchfork bifurcation at $(0,0)$ from the trivial solution (multiplicity 3).

we describe it as being supercritical (branches 1 and 2 in the diagram). If it opens up to the left, it is called a subcritical branch (branch 3 in the diagram).

**Theorem 3.14** *In addition to (3.1), assume $G$ is three times continuously Fréchet differentiable in a neighborhood of $(u,\lambda) = (0,0)$. Assume for all $i,j,k,l \in \{1,\ldots,M\}$:*

*a)* $\psi_j^*(G_{u\lambda}^0\phi_i) = \delta_{ij}$,

*b)* $\psi_k^*(G_{uu}^0\phi_i\phi_j) = 0$,

*c)* $\psi_l^*(G_{uu}^0\phi_i\Phi_{jk}) = 0$, *except possibly when* $i = j = k = l$ *or indices occur in two pairs (e.g.* $i = j$ *and* $k = l$*),*

*d)* $\psi_l^*(G_{uuu}^0\phi_i\phi_j\phi_k) = 0$, *except possibly when* $i = j = k = l$ *or indices occur in two pairs,* *e) Generic Assumption 1:* $a_{ij} \neq 0$ *for* $i \neq j$*, where*

$$a_{ij} = \frac{1}{2}\psi_i^*(G_{uu}^0\phi_i\Phi_{jj}) + \psi_i^*(G_{uu}^0\phi_j\Phi_{ij}) - \frac{1}{2}\psi_j^*(G_{uu}^0\phi_j\Phi_{jj}) + \frac{1}{2}\psi_i^*(G_{uuu}^0\phi_i\phi_j^2) - \frac{1}{6}\psi_j^*(G_{uuu}^0\phi_j^3).$$

*Then* $G(u,\lambda) = 0$ *has at least* $M$ *distinct solution branches bifurcating from the trivial solution branch at* $(0,0)$.

**Proof:** Again, we start with the Lyapunov–Schmidt reduction and introduce a scaling for the bifurcation equations:

$$\boldsymbol{\xi} = \epsilon\zeta \quad \text{and} \quad \lambda = \epsilon^2\Lambda,$$

where $\sum_{i=1}^{M} \zeta_i^2 = 1$. When $\epsilon \neq 0$, the solutions of the bifurcation equations (3.8) correspond to the zeroes of the function $g(\boldsymbol{\zeta}, \Lambda, \epsilon)$ defined as:

$$\begin{cases} \frac{1}{\epsilon^3}\sum_i \psi_i\psi_i^* G(\epsilon\sum_j \zeta_j\phi_j + \Phi(\epsilon\boldsymbol{\zeta}, \epsilon^2\Lambda), \epsilon^2\Lambda), & \epsilon \neq 0; \\ \sum_l \Big[\Lambda\zeta_l + \frac{1}{2}\zeta_l^3\psi_l^*(G_{uu}^0\phi_l\Phi_{ll}) + \\ \quad \frac{1}{2}\zeta_l\sum_{j\neq l}\zeta_j^2\left(\psi_l^*(G_{uu}^0\phi_l\Phi_{jj}) + 2\psi_l^*(G_{uu}^0\phi_j\Phi_{lj})\right) + \\ \quad \frac{1}{6}\zeta_l^3\psi_l^*(G_{uuu}^0\phi_l^3) + \frac{1}{2}\zeta_l\sum_{j\neq l}\zeta_j^2\psi_l^*(G_{uuu}^0\phi_l\phi_j^2)\Big]\psi_l, & \epsilon = 0. \end{cases} \tag{3.15}$$

Using the assumptions (3.14a) through (3.14d), it can be shown that $g$ is continuous at $\epsilon = 0$.

As before, we define the function:

$$h(\boldsymbol{\zeta}, \Lambda, \epsilon) = \begin{bmatrix} g \\ \sum_i \zeta_i^2 - 1 \end{bmatrix}.$$

Let $\boldsymbol{e}_i$ denote the $i^{th}$ column of the $M \times M$ identity matrix. Using the special choice of $\boldsymbol{\zeta}^{(1)} = \boldsymbol{e}_1$, we reduce $g(\boldsymbol{\zeta}, \Lambda, 0) = 0$ to the following linear equation in $\Lambda$,

$$\left[\Lambda + \frac{1}{2}\psi_1^*(G_{uu}^0\phi_1\Phi_{11}) + \frac{1}{6}\psi_1^*(G_{uuu}^0\phi_1^3)\right]\psi_1 = 0.$$

Defining

$$\Lambda^{(1)} = -\frac{1}{2}\psi_1^*(G^0_{uu}\phi_1\Phi_{11}) - \frac{1}{6}\psi_1^*(G^0_{uuu}\phi_1^3),$$

we have $h(\boldsymbol{\zeta}^{(1)}, \Lambda^{(1)}, 0) = 0$. At the point $(\boldsymbol{\zeta}^{(1)}, \Lambda^{(1)})$ and $\epsilon = 0$, we have

$$h' \equiv h_{\boldsymbol{\zeta},\Lambda} = \begin{bmatrix} a_{11}\psi_1 & a_{21}\psi_2 & \cdots & a_{m1}\psi_M & \psi_1 \\ 2 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

We now proceed to show that $h'$ is an isomorphism from $\mathbb{R}^M \times \mathbb{R}$ onto $\mathbb{B}_{22} \times \mathbb{R}$. Suppose for some $(\boldsymbol{\eta}, \mu) \in \mathbb{R}^M \times \mathbb{R}$.

$$h' \begin{bmatrix} \boldsymbol{\eta} \\ \mu \end{bmatrix} = 0.$$

Writing out the equations, we have

$$\begin{aligned} (a_{11}\eta_1 + \mu)\psi_1 + \sum_{i \geq 2} a_{i1}\eta_i\psi_i &= 0, \\ 2\eta_1 &= 0. \end{aligned}$$

Using the facts that the $\{\psi_i\}$ are linearly independent and $a_{i1}$ is nonzero for $i \geq 2$, we easily derive that $\mu = 0$ and $\boldsymbol{\eta} = 0$. Hence $h'$ has a trivial null space. Since it is finite-dimensional, it is also a surjection and thus an isomorphism. From the IFT, we conclude that for sufficiently small $\epsilon$, there exists unique continuous functions $\boldsymbol{\zeta}(\epsilon)$ and $\Lambda(\epsilon)$ such that $h(\boldsymbol{\zeta}(\epsilon), \Lambda(\epsilon), \epsilon) = 0$ and $(\boldsymbol{\zeta}(0), \Lambda(0)) = (\boldsymbol{\zeta}^{(1)}, \Lambda^{(1)})$. This implies a solution of the original equation $G(u, \lambda) = 0$ of the form:

$$u(\epsilon) = \epsilon \sum_i \zeta_i(\epsilon)\phi_i + \Phi(\boldsymbol{\zeta}(\epsilon), \lambda(\epsilon)), \qquad \lambda(\epsilon) = \epsilon^2\Lambda(\epsilon).$$

The same proof works for the other roots of $h(\boldsymbol{\zeta}, \Lambda, 0) = 0$ of the form $\boldsymbol{\zeta}^{(i)} = \boldsymbol{e}_i$, $i = 2, \ldots, M$. Hence, we have found $M$ distinct nontrivial solution branches bifurcating from the trivial solution branch. □

If every root of the ABEs is nonzero, then $(u, \lambda) = (0, 0)$ is a pitchfork bifurcation point because the $O(\epsilon)$ term of $\lambda(\epsilon)$ is zero while the $O(\epsilon^2)$ term, which is equal to a root of the ABEs, is nonzero. We remark that the Generic Assumption 1 (3.14e) is a sufficient condition for $h'$ to be nonsingular. Other authors (e.g., McLeod and Sattinger [27] and Decker and Keller [10]) assume its invertibility in the statement of their theorems.

Note that when $M = 1$, all the assumptions of Theorem 3.14 collapse to two conditions

$$\begin{aligned} \psi_1^*(G_{u\lambda}^0 \phi_1) &\neq 0 \text{ and} \\ \psi_1^*(G_{uu}^0 \phi_1^2) &= 0. \end{aligned}$$

If we make the additional assumption $\psi_1^*(G_{uuu}^0 \phi_1^3) \neq 0$, then we have the usual conditions for the most elementary pitchfork bifurcation.

It is clear that if the assumptions (3.14a) through (3.14e) hold for all $i, j, k, l \in \{1, \ldots, M_1\}$, for some integer $M_1 \leq M$, then there exist at least $M_1$ bifurcating branches.

With further assumptions, we can show the occurrence of more bifurcating solutions. First we make the following definition, using the notation of the above theorem.

**Definition 3.16** *We call the integers $i$ and $j$ with $i \neq j$ and $i, j \in \{1, \ldots, M\}$, an* associated pair $(i, j)_A$ *if the following conditions hold:*

$$\begin{aligned} \psi_i^*(G_{uu}^0 \phi_i \Phi_{ii}) &= \psi_j^*(G_{uu}^0 \phi_j \Phi_{jj}), \\ \psi_i^*(G_{uu}^0 \phi_i \Phi_{jj}) &= \psi_j^*(G_{uu}^0 \phi_j \Phi_{ii}), \end{aligned}$$

$$\begin{aligned}
\psi_i^*(G_{uu}^0\phi_j\Phi_{ij}) &= \psi_j^*(G_{uu}^0\phi_i\Phi_{ij}),\\
\psi_i^*(G_{uuu}^0\phi_i\phi_j^2) &= \psi_j^*(G_{uuu}^0\phi_i^2\phi_j),\\
\psi_i^*(G_{uuu}^0\phi_i^3) &= \psi_j^*(G_{uuu}^0\phi_j^3).
\end{aligned}$$

In words, $i$ and $j$ are symmetric in the above equations.

**Theorem 3.17** *In addition to the hypotheses of Theorem 3.14, we assume there are $K$ associated pairs $(0 \le K \le M(M-1)/2)$. Define*

$$\begin{aligned}
a_{ijk} = \ & -\tfrac{1}{4}\psi_i^*(G_{uu}^0\phi_i\Phi_{ii}) - \tfrac{1}{4}\psi_i^*(G_{uu}^0\phi_i\Phi_{jj}) - \tfrac{1}{2}\psi_i^*(G_{uu}^0\phi_j\Phi_{ij}) - \tfrac{1}{12}\psi_i^*(G_{uuu}^0\phi_i^3) - \\
& \tfrac{1}{4}\psi_i^*(G_{uuu}^0\phi_i\phi_j^2) + \tfrac{1}{4}\psi_k^*(G_{uu}^0\phi_k\Phi_{ii}) + \tfrac{1}{2}\psi_k^*(G_{uu}^0\phi_i\Phi_{ik}) + \tfrac{1}{4}\psi_k^*(G_{uu}^0\phi_k\Phi_{jj}) + \\
& \tfrac{1}{2}\psi_k^*(G_{uu}^0\phi_j\Phi_{jk}) + \tfrac{1}{4}\psi_k^*(G_{uuu}^0\phi_k\phi_i^2) + \tfrac{1}{4}\psi_k^*(G_{uuu}^0\phi_k\phi_j^2).
\end{aligned}$$

*If $a_{ijk} \neq 0$ for every associated pair $(i,j)_A$ and integer $k$ such that $k \neq i, k \neq j$ (Generic Assumption 2), then $G(u,\lambda) = 0$ has at least $M + 2K$ bifurcating solution branches at $(0,0)$.*

**Proof**: The idea of the proof is the same as that of the proof of Theorem 3.14 and we will use the same notation. The extra assumptions here allow us to find $2K$ extra isolated solutions of $h(\boldsymbol{\zeta}, \Lambda, \epsilon) = 0$. Suppose $(1,2)_A$ is an associated pair. Let $\zeta_1^{(1)} = \frac{1}{\sqrt{2}}$, $\zeta_2^{(1)} = s\frac{1}{\sqrt{2}}$, $\zeta_i^{(1)} = 0$, for $i = 3, \ldots, M$ and $s$ is either 1 or $-1$. Then $g(\boldsymbol{\zeta}^{(1)}, \Lambda, 0) = 0$ (see (3.15)) becomes

$$\begin{aligned}
&\frac{1}{\sqrt{2}}\left[\Lambda + \frac{1}{4}\psi_1^*(G_{uu}^0\phi_1\Phi_{11}) + \frac{1}{4}\psi_1^*(G_{uu}^0\phi_1\Phi_{22}) + \frac{1}{2}\psi_1^*(G_{uu}^0\phi_2\Phi_{12}) + \right.\\
&\qquad\qquad \left.\frac{1}{12}\psi_1^*(G_{uuu}^0\phi_1^3) + \frac{1}{4}\psi_1^*(G_{uuu}^0\phi_1\phi_2^2)\right]\psi_1 + \\
&\frac{s}{\sqrt{2}}\left[\Lambda + \frac{1}{4}\psi_2^*(G_{uu}^0\phi_2\Phi_{22}) + \frac{1}{4}\psi_2^*(G_{uu}^0\phi_2\Phi_{11}) + \frac{1}{2}\psi_2^*(G_{uu}^0\phi_1\Phi_{12}) + \right.\\
&\qquad\qquad \left.\frac{1}{12}\psi_2^*(G_{uuu}^0\phi_2^3) + \frac{1}{4}\psi_2^*(G_{uuu}^0\phi_2\phi_1^2)\right]\psi_2 = 0.
\end{aligned}$$

Using the properties of an associated pair, the coefficients of $\psi_1$ and $\psi_2$ are the same and we could uniquely solve for $\Lambda$ as $\Lambda^{(1)} \equiv -\frac{1}{4}\psi_1^*(G^0_{uu}\phi_1\Phi_{11}) - \frac{1}{4}\psi_1^*(G^0_{uu}\phi_1\Phi_{22}) - \frac{1}{2}\psi_1^*(G^0_{uu}\phi_2\Phi_{12}) - \frac{1}{12}\psi_1^*(G^0_{uuu}\phi_1^3) - \frac{1}{4}\psi_1^*(G^0_{uuu}\phi_1\phi_2^2)$. Hence, $h(\zeta^{(1)}, \Lambda^{(1)}, 0) = 0$. Recall that $h'$ is $h_{\zeta,\Lambda}$ evaluated at $(\zeta^{(1)}, \Lambda^{(1)})$ and $\epsilon = 0$. We have

$$h' = \begin{bmatrix} a\psi_1 + sb\psi_2 & sb\psi_1 + a\psi_2 & a_{123}\psi_3 & \cdots & a_{12M}\psi_M & \frac{1}{\sqrt{2}}(\psi_1 + s\psi_2) \\ \frac{2}{\sqrt{2}} & s\frac{2}{\sqrt{2}} & 0 & \cdots & 0 & 0 \end{bmatrix},$$

where

$$\begin{aligned} a &= \frac{1}{2}\psi_1^*(G^0_{uu}\phi_1\Phi_{11}) + \frac{1}{6}\psi_1^*(G^0_{uuu}\phi_1^3) \\ b &= \frac{1}{2}\psi_2^*(G^0_{uu}\phi_2\Phi_{11}) + \psi_2^*(G^0_{uu}\phi_1\Phi_{12}) + \frac{1}{2}\psi_2^*(G^0_{uuu}\phi_2\phi_1^2). \end{aligned}$$

We now show that $h'$ is invertible as an operator from $\mathbb{R}^M \times \mathbb{R}$ onto $\mathbb{B}_{22} \times \mathbb{R}$. Suppose

$$h'\begin{bmatrix} \eta \\ \mu \end{bmatrix} = 0.$$

Writing out the equations, we have, $\eta_i = 0$ for $i = 3, \ldots, M$, $\eta_2 = -s\eta_1$ and

$$\begin{aligned} (a-b)\eta_1 + \frac{1}{\sqrt{2}}\mu &= 0, \\ -s(a-b)\eta_1 + \frac{s}{\sqrt{2}}\mu &= 0. \end{aligned}$$

(We have used Generic Assumption 2 and the linear independence of the set $\{\psi_1, \ldots, \psi_M\}$.)

The above $2 \times 2$ system has only the trivial solution $\eta_1 = \mu = 0$ because its determinant is $\sqrt{2}s(a-b) = -\sqrt{2}sa_{21}$ and is therefore nonzero by Generic Assumption 1 (3.14e). Thus, $h'$ is an isomorphism and the IFT can be used to show the existence of two distinct bifurcating solutions, one for each of $s = \pm 1$. The same proof works for the other associated pairs and hence we have proven the existence of at least $M + 2K$ bifurcating solutions. □

In the above theorem, there are at least two types of solutions. Those where the $O(\epsilon)$ term has one non-zero component from $\mathcal{N}(G_u^0)$, say $\phi_1$, and those with two non-zero components, say $\phi_1$ and $\phi_2$. As we will see later, in examples where there is some form of regularity, solutions with two nonzero components occur naturally.

Decker and Keller [10] obtained an upper bound for their quadratic ABEs. Similarly, we can derive an upper bound for our cubic ABEs. The ABEs without the normalization form a system of $M$ cubic polynomials in $M + 1$ unknowns $(\boldsymbol{\zeta}, \Lambda)$. We can conclude from Bézout's theorem that they have at most $3^M$ isolated roots. Notice that $(\pm\boldsymbol{\zeta}, \Lambda)$ are distinct solutions to the polynomial equations but they are just two different halves of the same branch of our bifurcating solution. Hence an upper bound on the multiplicity is $\frac{1}{2}(3^M - 1)$. Here, the solution $\boldsymbol{\zeta} = 0$ and $\Lambda = 1$ is subtracted out since it corresponds to the basic solution branch. For $M = 1, 2$, this upper bound is equal to our lower bound. We can now infer that there are no bifurcation points of multiplicity 2 or 3 for problems satisfying the hypotheses of Theorem 3.17.

The strong assumptions on $G$ allow us to reduce the bifurcation equations to a tractable form where we could find a solution and show that it is isolated. For a general system of polynomial equations, there are very few tools available to us. One such result is the Birkoff-Kellogg invariant direction theorem which only predicts the existence of a root when $M$ is odd (see Keller and Langford [20]) but doesn't tell us more information about the multiplicity. Krasnoselski [21] and Westreich [34] have actually shown the existence of bifurcation under mild assumptions and when $M$ is odd.

We have obtained local qualitative information about the solution of $G$ using only linear information. However it is often impractical to verify some of the conditions analytically, especially the Generic Assumptions. Since bifurcating solutions of high multiplicities are not very common, we expect any sufficient conditions to be rather strong. The next result is directly analogous to Theorem 3.9.

**Theorem 3.18** *In addition to (3.1), assume $G$ is three times continuously Fréchet differentiable in a neighborhood of $(u,\lambda) = (0,0)$. Suppose for all $i,j,k,l \in \{1,\ldots,M\}$ and some real constant $b$:*

- $\psi_j^*(G^0_{u\lambda}\phi_i) = \delta_{ij}$.
- $\psi_k^*(G^0_{uu}\phi_i\phi_j) = 0$.
- $\psi_l^*(G^0_{uu}\phi_i\Phi_{jk}) = 0$.
- $\psi_i^*(G^0_{uuu}\phi_i\phi_j^2) = 2b$ *for* $i \neq j$.
- $\psi_l^*(G^0_{uuu}\phi_i\phi_j\phi_k) = 0$ *except possibly when indices are all equal or they occur in two pairs.*

*Let $a_i = \frac{1}{6}\psi_i^*(G^0_{uuu}\phi_i^3)$. If for every $i \in \{1,\ldots,M\}$, $a_i - b$ has the same sign (i.e., $a_i - b > 0$ for every $i$ or $a_i - b < 0$ for every $i$), then $G(u,\lambda) = 0$ has exactly $(3^M - 1)/2$ bifurcating solution branches near $(u,\lambda) = (0,0)$.*

**Proof:** The proof is almost identical to that of Theorem 3.9. We just give a brief sketch. We use the scaling

$$\boldsymbol{\xi} = \epsilon\boldsymbol{\zeta} \quad \text{and} \quad \lambda = \epsilon^2\Lambda$$

with $\sum_i \zeta_i^2 = 1$. The function $g(\boldsymbol{\zeta}, \Lambda, \epsilon)$, whose zeroes correspond to those of the bifurcation equations, is defined as:

$$g(\boldsymbol{\zeta}, \Lambda, \epsilon) \;\equiv\; \begin{cases} \frac{1}{\epsilon^3} \sum_{i=1}^{M} \psi_i \psi_i^* G(\epsilon \sum_j \zeta_j \phi_j + \Phi(\epsilon\boldsymbol{\zeta}, \epsilon^2\Lambda), \epsilon^2\Lambda), & \epsilon \neq 0; \\ \sum_{i=1}^{M} \zeta_i \left[\zeta_i^2(a_i - b) + \Lambda + b\right] \psi_i, & \epsilon = 0. \end{cases}$$

It can be shown that $g$ is continuous at $\epsilon = 0$, where the equations are again decoupled enabling us to solve them analytically. The number of roots of the above equation restricting $\boldsymbol{\zeta}$ to have norm one is $3^M - 1$. If the first $k$ components of $\boldsymbol{\zeta}$ are nonzero, each nonzero component $\zeta_i$ must have the value $\pm\sqrt{\kappa/(b - a_i)}$ where $\kappa = (\sum_{i=1}^{k} 1/(b - a_i))^{-1}$ and $\Lambda = \kappa - b$. There are $2^k$ such solutions as a result of the choice of sign for each nonzero component. Hence the total number of roots is

$$\sum_{k=1}^{M} \frac{M!}{(M-k)!\,k!} 2^k = 3^M - 1.$$

In the above sum, the coefficient of $2^k$ is the number of roots with $k$ nonzero components of $\boldsymbol{\zeta}$. It can be shown that each of these roots is isolated. Since a root $(\boldsymbol{\zeta}, \Lambda)$ yields the same branch of solutions as $(-\boldsymbol{\zeta}, \Lambda)$, the exact number of bifurcating branches is $(3^M - 1)/2$. □

An elementary application of the above theorem to the following system of $M$ equations in $M$ variables in $\mathbb{R}^M$

$$u_i(u_i^2 - \lambda) + O(u^4 + \lambda u^2 + h.o.t.) = 0, \quad i = 1, \ldots, M.$$

yields $(3^M - 1)/2$ bifurcating solution branches.

Before stating our final theorem, we define and give some results on finite dimensional gradient systems.

**Definition 3.19** *Let $f$ be a continuous function from $\mathbb{R}^n$ into $\mathbb{R}^n$. Then $f$ is called a gradient system if there exists a continuously differentiable function $F$ from $\mathbb{R}^n$ into $\mathbb{R}$ such that*

$$F_{x_i}(x) = f_i(x),$$

*for every $x \in \mathbb{R}^n$.*

(Note that $f_i$ denotes the $i^{th}$ component of $f$.)

**Lemma 3.20** *Let $f$ be a continuously differentiable function from $\mathbb{R}^n$ into $\mathbb{R}^n$. Then, $f$ is a gradient system if and only if*

$$\frac{\partial f_i}{\partial x_j}(x) = \frac{\partial f_j}{\partial x_i}(x)$$

*for every $i, j \in \{1, \ldots, n\}$ and $x \in \mathbb{R}^n$.*

**Lemma 3.21** *Let $S^n$ denote the unit sphere in $\mathbb{R}^{n+1}$. Suppose $f : S^n \to \mathbb{R}^{n+1}$ is a gradient system such that $f(x) = -f(-x)$ for every $x \in S^n$, then the equation $f(x) = ax$ has at least $n$ distinct pairs of solutions $(\pm x^{(i)}, a^{(i)}), i = 1, \ldots, n$ in $S^n \times \mathbb{R}$.*

For further details and proofs, see, for example, Berger and Berger [3] and Rabinowitz [29]. The next result is actually a special case of a theorem in Keller and Langford [20].

**Theorem 3.22** *In addition to (3.1), assume $G$ is three times continuously Fréchet differentiable in a neighborhood of $(u, \lambda) = (0,0)$. Assume for all $i, j, k, l \in \{1, \ldots, M\}$:*

*a)* $\psi_j^*(G_{u\lambda}^0 \phi_i) = \delta_{ij}$,

*b)* $\psi_k^*(G_{uu}^0 \phi_i \phi_j) = 0$,

*c)* $2\psi_l^*(G_{uu}^0 \phi_i \Phi_{jk}) + \psi_l^*(G_{uu}^0 \phi_k \Phi_{ij}) = 2\psi_k^*(G_{uu}^0 \phi_i \Phi_{jl}) + \psi_k^*(G_{uu}^0 \phi_l \Phi_{ij})$,

*d)* $\psi_i^*(G_{uuu}^0 \phi_j \phi_k \phi_l) = \psi_j^*(G_{uuu}^0 \phi_i \phi_k \phi_l)$.

*Then the bifurcation equations (3.8) have at least $M$ pairs of roots $(\pm\boldsymbol{\xi}^{(i)}, \lambda^{(i)})$, $i = 1, \ldots, M$, near $(\boldsymbol{\xi}, \lambda) = (0,0)$.*

**Proof**: As before, we apply the Lyapunov–Schmidt method to reduce the operator equation $G(u, \lambda) = 0$ to the bifurcation equations. Introduce the scaling:

$$\boldsymbol{\xi} = \epsilon \boldsymbol{\zeta} \quad \text{and} \quad \lambda = \epsilon^2 \Lambda$$

with $\sum_i \zeta_i^2 = 1$. We define the following function whose solutions are precisely those of the bifurcation equations.

$$\tilde{g}(\boldsymbol{\zeta}, \Lambda, \epsilon) \equiv \begin{cases} \frac{6}{\epsilon^3} \sum_i \psi_i \psi_i^* G(\epsilon \sum_j \zeta_j \phi_j + \Phi(\epsilon\boldsymbol{\zeta}, \epsilon^2\Lambda), \epsilon^2\Lambda), & \epsilon \neq 0; \\ \sum_l \Big[ 6\Lambda\zeta_l + 3\sum_{i,j,k} \zeta_i\zeta_j\zeta_k \psi_l^*(G_{uu}^0 \phi_i \Phi_{jk}) + \\ \quad \sum_{i,j,k} \zeta_i\zeta_j\zeta_k \psi_l^*(G_{uuu}^0 \phi_i\phi_j\phi_k) \Big] \psi_l, & \epsilon = 0. \end{cases}$$

Using (3.22a) and (3.22b), we can show that $\tilde{g}$ is continuous at $\epsilon = 0$.

Now define $g(\boldsymbol{\zeta}, \Lambda)$ as a system of $M$ equations equivalent to $\tilde{g}(\boldsymbol{\zeta}, \Lambda, 0)$. (I.e., $g_i$ is the coefficient of $\psi_i$ in $\tilde{g}(\boldsymbol{\zeta}, \Lambda, 0)$.) Let $h(\boldsymbol{\zeta})$ be the vector function whose $l^{th}$ component is

$$h_l \equiv 3 \sum_{i,j,k} \zeta_i\zeta_j\zeta_k\psi_l^*(G_{uu}^0\phi_i\Phi_{jk}) + \sum_{i,j,k} \zeta_i\zeta_j\zeta_k\psi_l^*(G_{uuu}^0\phi_i\phi_j\phi_k).$$

We now show that $h$ is a gradient system.

$$\begin{aligned}\frac{\partial h_l}{\partial \zeta_k} &= 3\sum_{i,j=1}^{M} [2\psi_l^*(G_{uu}^0\phi_i\Phi_{jk}) + \psi_l^*(G_{uu}^0\phi_k\Phi_{ij})]\zeta_i\zeta_j + \\ & \quad 3\sum_{i,j=1}^{M} \psi_l^*(G_{uuu}^0\phi_k\phi_i\phi_j)\zeta_i\zeta_j.\end{aligned}$$

Using the hypotheses (C3) and (C4), $\frac{\partial h_l}{\partial \zeta_k} = \frac{\partial h_k}{\partial \zeta_l}$ and thus $h$ is a gradient system by Lemma 3.20.

By inspection, $h(-\boldsymbol{\zeta}) = -h(\boldsymbol{\zeta})$. Hence from Lemma 3.21, there are at least $M$ distinct elements $\boldsymbol{\xi}^{(i)} \in S^{M-1}$, $i = 1, \ldots, M$, such that $h(\pm\boldsymbol{\zeta}^{(i)}) = \pm a^{(i)}\boldsymbol{\zeta}^{(i)}$, for some real $a^{(i)}$. Define $\Lambda^{(i)} = -a^{(i)}/6$, we have

$$g(\pm\boldsymbol{\zeta}^{(i)}, \Lambda^{(i)}) = \pm 6\Lambda^{(i)}\boldsymbol{\zeta}^{(i)} + h(\pm\boldsymbol{\zeta}^{(i)}) = 0.$$

Thus $g$ has $M$ pairs of distinct roots. □

If the roots are isolated, then there are at least $M$ branches of bifurcating solutions. Note that this theorem requires fewer hypotheses than Theorem 3.14. However it is non-constructive and hence it may be impossible to check the isolation of the roots of the ABEs.

# Chapter 4

# Application to Semilinear Elliptic Equations

In this chapter, we apply our multiple bifurcation theory to a class of non-linear elliptic partial differential equations (PDEs). Let $\Omega$ be a cylinder in $\mathbb{R}^3$ with an arbitrary cross–section, i.e., $(0,1)\times\Omega_2$, where $\Omega_2$ is a bounded, simply–connected domain in $\mathbb{R}^2$ with a piecewise-smooth boundary. We consider the following problem:

$$\left[\begin{array}{cc} \Delta u+\lambda f(u)=0, & (x,y,z)\in\Omega \\ u(0,y,z)=u(1,y,z)=0, & (y,z)\in\Omega_2 \\ \frac{\partial u}{\partial\nu}=0, & (x,y,z)\in(0,1)\times\partial\Omega_2 \end{array}\right], \tag{4.1}$$

The symbol $\Delta$ denotes the Laplacian operator, $\lambda$ is a real parameter, $f$ is a continuous real–valued function and $\frac{\partial u}{\partial\nu}$ denotes the outer normal derivative. We abbreviate the boundary conditions as $\mathcal{B}u=0$. This is called a semilinear equation because the nonlinearity $f$ is only a function of $u$ and not derivatives of $u$. The literature on semilinear elliptic equations is vast. See, for example, Lions [26] and Ni, Peletier and Serrin [28].

In the first section, we deal with the simplest situation. We will require $f(0) = f''(0) = 0$ and demonstrate bifurcation from the trivial solution branch. When $f(0) \neq 0$, the trivial solution is no longer a solution. We will use a solution which is independent of $y$ and $z$ as the basic solution and seek bifurcation from this nontrivial branch. The principles involved are the same as the the former case but a nontrivial basic solution drastically increases the algebra involved.

## 4.1 Bifurcation from the Trivial Solution Branch

In this section, we assume $f$ is smooth and

$$f(0) = f''(0) = 0, \qquad f'(0) \cdot f'''(0) \neq 0.$$

Note that $u = 0$ is a solution of (4.1) for all $\lambda$. We will apply the general theory to show multiple bifurcations along this trivial branch.

It is well–known that a necessary condition for bifurcation of (4.1) at $(u, \lambda) = (0, \lambda_0)$ is that the linearized equation has a nontrivial null space. That is, the problem

$$\begin{aligned} \Delta\phi + \lambda_0 f'(0)\phi &= 0, \ (x, y, z) \in \Omega \\ \mathcal{B}\phi &= 0, \ (x, y, z) \in \partial\Omega \end{aligned} \tag{4.2}$$

has a nontrivial solution $\phi$. We use separation of variables to solve this problem. Because the domain has the product form $(0, 1) \times \Omega_2$ and the Laplacian is linear, it can be shown (Duff and Naylor [11]) that all null functions are separable and have the form $\phi = X(x)T(y, z)$. Substituting this into (4.2), we obtain separate problems for $X$ and $T$:

$$X''(x) + (\lambda_0 f'(0) - \kappa^2)X(x) = 0; \quad X(0) = X(1) = 0, \tag{4.3}$$
$$T_{yy} + T_{zz} + \kappa^2 T = 0; \quad \left.\frac{\partial T}{\partial \nu}\right|_{\partial\Omega_2} = 0. \tag{4.4}$$

In $\mathcal{L}_2(\overline{\Omega}_2)$, the Lebesgue square–integrable functions on $\overline{\Omega}_2$, the eigenvalue problem (4.4) has a countable, unbounded set of eigenvalues $\kappa^2$ with a complete orthonormal set of eigenfunctions $\{T_i\}$. That is, $\int_{\Omega_2} T_i T_j dydz = \delta_{ij}$ and any function $g \in \mathcal{L}_2(\overline{\Omega}_2)$ whose normal derivative vanish on $\partial\Omega_2$ can be represented as $\sum_{i=1}^{\infty} a_i T_i$ for some real constants $a_i$. Suppose $\kappa^2$ is an eigenvalue. Then the solution of (4.3), if it exists, is $X = \sin(p\pi x)$ where $p$ is the positive integer satisfying: $\lambda_0 f'(0) - \kappa^2 = p^2\pi^2$. Rearranging, we obtain:

$$\lambda_0 = \frac{\kappa^2 + \pi^2 p^2}{f'(0)}. \tag{4.5}$$

Hence for a fixed $\lambda_0$, if there are $s$ distinct pairs of numbers $(\kappa_i^2, p_i)$ satisfying (4.5), then the number of linearly independent solutions of (4.2) is $M \equiv m_1 + \ldots + m_s$, where $m_i$ is the multiplicity of the eigenvalue $\kappa_i^2$. (This multiplicity is defined as the number of linearly independent eigenfunctions of (4.4) associated with $\kappa_i^2$.) We label these null functions of (4.2) as $\{\phi_1, \ldots, \phi_M\}$, with

$$\phi_i = X_i(x) T_i(y, z), \tag{4.6}$$

where $X_i$ and $T_i$ satisfy (4.3) and (4.4). We assume that they are normalized, i.e.,

$$\int_\Omega \phi_i^2 = 1.$$

Because the set of eigenfunctions $\{T_i\}$ is orthonormal, we have

$$\int_\Omega \phi_i\phi_j = \int_0^1 X_iX_jdx \int_{\Omega_2} T_iT_jdydz = \delta_{ij}.$$

Next we define the following linear functional:

$$\psi_i^*(g) \equiv \frac{1}{f'(0)} \int_\Omega \phi_i g$$

for any $g \in \mathcal{L}_2(\overline{\Omega})$. Now we are ready verify some of the hypotheses of Theorem 3.14.

Let $G(u,\lambda) = 0$ denote the problem (4.1). Suppose at $\lambda = \lambda_0$, (4.2) has $M$ null functions (4.6). We consider the candidate bifurcation point $(u,\lambda) = (0,\lambda_0)$. We have $G^0_{uu} = \lambda_0 f''(0) = 0$, $G^0_{u\lambda} = f'(0)$ and $G^0_{uuu} = \lambda_0 f'''(0)$. The superscript $^0$ means evaluation of the operator at $(0,\lambda_0)$. Now

$$\psi_j^*(G^0_{u\lambda}\phi_i) = \int_\Omega \phi_i\phi_j = \delta_{ij}.$$

Hence if the conditions (3.14d) and (3.14e) are valid, we have at least $M$ bifurcating solutions. We have shown:

**Theorem 4.7** *Consider the problem (4.1) denoted by $G(u,\lambda) = 0$. Assume $f$ is three times continuously differentiable near $0$ with the following properties:*

$$f(0) = f''(0) = 0, \quad f'(0) \cdot f'''(0) \neq 0.$$

*Suppose $dim\mathcal{N}(G_u(0,\lambda_0)) = M$ with $M$ null functions $\{\phi_1,\ldots,\phi_M\}$. Suppose for every $i,j,k \in \{1,\ldots,M\}$,*

- $\int_\Omega \phi_l\phi_i\phi_j\phi_k = 0$ *for* $l \notin \{i,j,k\}$.

- $\int_\Omega \phi_i^2 \phi_j^2 - \frac{1}{3}\int_\Omega \phi_j^4 \neq 0, \quad i \neq j.$

*Then $G(u,\lambda) = 0$ has at least $M$ nontrivial solution branches bifurcating from the point $(u,\lambda) = (0,\lambda_0)$.*

If the cross–section $\Omega_2$ of the cylinder is a square, we obtain more elegant results.

**Theorem 4.8** *Consider the problem (4.1) denoted by $G(u,\lambda) = 0$ with $\Omega_2 = (0,L)^2$, where $L$ is a positive real number. Assume $f$ is three times continuously differentiable near 0 with the following properties:*

$$f(0) = f''(0) = 0, \quad f'(0) \cdot f'''(0) \neq 0.$$

*Suppose $\dim \mathcal{N}(G_u(0,\lambda_0)) = M$ with $M$ null functions $\{\phi_1,\ldots,\phi_M\}$. Then $G(u,\lambda) = 0$ has exactly $(3^M - 1)/2$ nontrivial solution branches bifurcating from the point $(u,\lambda) = (0,\lambda_0)$.*

**Proof:** For this geometry, the null functions are

$$\phi_i = c_i \sin(p_i \pi x) \cos\left(\frac{m_i \pi y}{L}\right) \cos\left(\frac{n_i \pi z}{L}\right)$$

where

$$\begin{aligned} \lambda_0 &= \frac{\pi^2}{f'(0)}\left(p_i^2 + \frac{m_i^2 + n_i^2}{L^2}\right), \quad i \in \{1,\ldots,M\} \\ c_i &= \begin{cases} 2\sqrt{2}/L, & \text{if } m_i n_i \neq 0; \\ 2/L, & \text{if exactly one of } \{m_i, n_i\} \text{ is zero;} \\ \sqrt{2}/L & \text{otherwise.} \end{cases} \end{aligned}$$

The integer $p_i$ is positive and $m_j$ and $n_k$ are nonnegative integers. By a direct computation, for distinct $i, j \in \{1, \ldots, M\}$,

$$\psi_i^*(G_{uuu}^0 \phi_i^3) = \begin{cases} \frac{27}{8}c, & \text{if } m_i n_i \neq 0; \\ \frac{9}{4}c, & \text{if exactly one of } \{m_i, n_i\} \text{ is zero;} \\ \frac{3}{2}c, & \text{otherwise.} \end{cases}$$

$$\psi_i^*(G_{uuu}^0 \phi_i \phi_j^2) = \frac{3}{2}c,$$

where $c \equiv \frac{\lambda_0 f'''(0)}{L^2 f'(0)}$. Note $\frac{1}{3}\psi_i^*(G_{uuu}^0 \phi_i^3) - \psi_i^*(G_{uuu}^0 \phi_i \phi_j^2)$ has the value $-\frac{3}{8}c$, $-\frac{3}{4}c$ or $-c$ and hence has the same sign for every $i, j \in \{1, \ldots, M\}$. Also, $\psi_l^*(G_{uuu}^0 \phi_i \phi_j \phi_k)$ is proportional to $\int_{\Omega_2} T_i T_j T_k T_l$ which is zero for $l \notin \{i, j, k\}$ by a result from the Basic Lemma (see the last section in this chapter). Thus, there are exactly $(3^M - 1)/2$ nontrivial bifurcating solutions by Theorem 3.18. □

We now make some remarks regarding the nature of these bifurcations. It is immediate from (4.5) that the sign of $\lambda_0$ is equal to the sign of $f'(0)$. It can also be shown that the pitchfork is supercritical if $f'''(0) < 0$ and subcritical if $f'''(0) > 0$. See Figure 4.1. Allgower, Bohmer and Mei [1] have shown the same result holds for the same PDE on a square with Neumann boundary conditions. Note that they require an additional hypothesis that $f$ is odd.

## 4.2 Bifurcation from the Nontrivial Solution Branch

In this section, we study the semilinear equation (4.1) with the nonlinearity $f$ having the property $f(0) \neq 0$. Since $u = 0$ is no longer a solution, we use a solution of (4.1) which is independent of $y$ and $z$ as the basic solution and seek

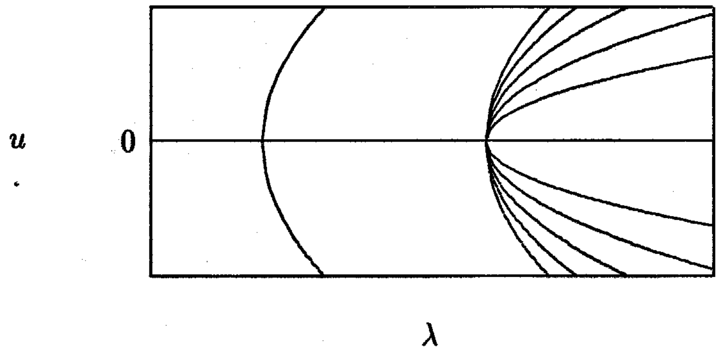


Figure 4.1: Typical bifurcations from the trivial solution branch for the case $f'(0) > 0$ and $f'''(0) > 0$.

bifurcation from this basic solution. First, we investigate some properties of the basic solution mentioned above and its linear eigenvalue problem. Define a function $g$ to be positive* if

$$g(t) > 0, \ \forall t \geq 0.$$

### 4.2.1 A Two-Point Boundary Value Problem

The following two-point boundary value problem:

$$\begin{aligned} u_0''(x) + \lambda f(u_0(x)) &= 0, \quad x \in (0,1); \\ u_0(0) = u_0(1) &= 0, \end{aligned} \tag{4.9}$$

for $f$ continuous has been studied extensively. Some of the earliest work was done by Gelfand [12], Keller and Cohen [19] and Laetsch [22]. We restrict the discussion to the case of a non-negative $u_0(x)$ and $f$ positive*. The following is a summary of the properties of solution(s) $u_0$ as given in Laetsch [22]. For any positive $A > 0$,

- there exist a unique $\lambda$ and exactly one non-negative function $u_0$ which satisfies (4.9) and

$$\max_{x\in[0,1]} u_0(x) = A. \tag{4.10}$$

  Furthermore,

$$\lambda = 2\left[\int_0^A \frac{du}{\sqrt{\int_u^A f(t)dt}}\right]^2, \tag{4.11}$$

  and $\lambda$ is a continuous function of $A$.

- Any solution $u_0$ is symmetric about $x = \frac{1}{2}$ and $u_0(\frac{1}{2}) = A$.

-

$$\lim_{t\to\infty} \frac{f(t)}{t} = \infty \implies \lim_{A\to\infty} \lambda(A) = 0. \tag{4.12}$$

$$\lim_{t\to 0} \frac{f(t)}{t} = \infty \implies \lim_{A\to 0} \lambda(A) = 0. \tag{4.13}$$

The graph of a typical $u_0$ is shown in Figure 4.2.

It is well-known that (4.9) may have more than one solutions for some $\lambda$. In view of the first property (4.10) listed above, it is more convenient to think of $A$, rather than $\lambda$, as the parameter. Henceforth, we will sometimes write quantities depending on $\lambda$ as quantities depending on $A$ so that there will be no ambiguity. Also, we sometimes write $u_0(x, A)$ to emphasize the dependence of $u_0$ on $A$.

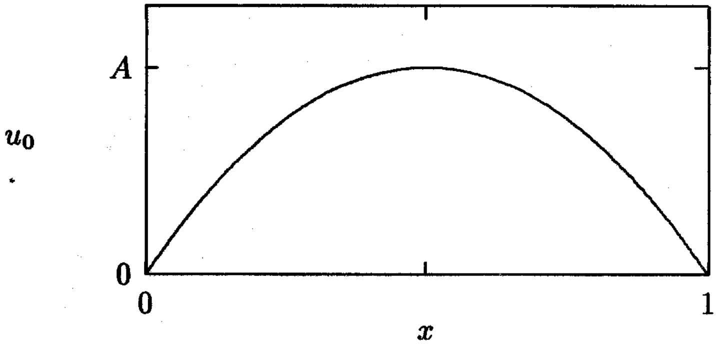


Figure 4.2: A typical graph of $u_0(x)$.

For the remainder of this section, we additionally require $f$ to be a member of $C^1$. The eigenvalue problem associated with (4.9) is

$$w''(x) + (\lambda f'(u_0(x)) + \alpha)\, w(x) = 0, \tag{4.14}$$
$$w(0) = w(1) = 0, \quad \int_0^1 w^2(x)\, dx = 1.$$

This is a regular Sturm-Liouville eigenvalue problem and thus it has a unique, smallest, finite eigenvalue $\alpha$. As we will see later, negative eigenvalues play a significant role in bifurcation. The following two results are due to Beyn [5]:

- Equation (4.14) has at most one non-positive eigenvalue for $A > 0$.
- $\frac{d\lambda(A)}{dA}$ is continuous and

$$\alpha(A) \left\{ \begin{array}{c} > \\ = \\ < \end{array} \right\} 0 \iff \frac{d\lambda(A)}{dA} \left\{ \begin{array}{c} > \\ = \\ < \end{array} \right\} 0. \tag{4.15}$$

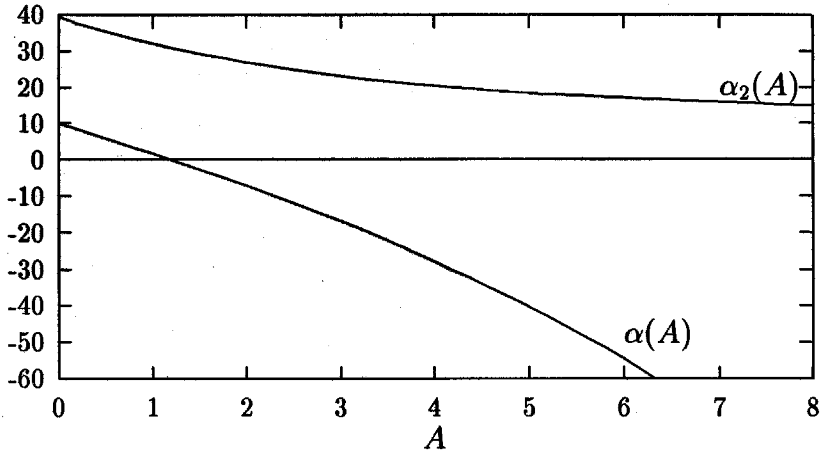


Figure 4.3: The graphs of the smallest ($\alpha$) and second smallest ($\alpha_2$) eigenvalues as a function of $A$ for $f(u) = e^u$.

In Figure 4.3, we plot the behaviour of the two smallest eigenvalues of (4.14) as a function $A$ for the function $f(u) = e^u$. The following lemma gives simple sufficient conditions on $f$ such that $\lim_{A\to\infty} \alpha(A) = -\infty$.

**Lemma 4.16** *Suppose for some real constants $a, c$ with $a > 0$, $c > 1$,*

$$a(c^t - 1) \le f'(t), \quad t \ge 0 \text{ and} \tag{4.17}$$

$$\lim_{A\to\infty} c^{A/2} A^{-1/2} \int_0^A \frac{du}{\sqrt{\int_u^A f(t)dt}} = \infty. \tag{4.18}$$

*Then $\alpha(A)$ is a continuous function of $A$ and takes on every value in $(-\infty, \pi^2)$ as $A$ takes on values in the positive reals.*

**Proof:** The variational principle for the smallest eigenvalue $\alpha(A)$ yields

$$\alpha(A) = \min \int_0^1 (w'(x))^2\, dx - \lambda \int_0^1 f'(u_0(x, A))\, w^2(x) dx. \tag{4.19}$$

The minimization is over all $w \in C[0,1] \cap C^1(0,1)$ such that $w(0) = w(1) = 0$ and $\int_0^1 w^2 = 1$. To obtain an upper bound on $\alpha$, we pick $w(x) = \sqrt{2}\sin(\pi x)$. It is easy to show $\int_0^1 (w'(x))^2\, dx = \pi^2$. Since $u_0$ is concave, we have $u_0(x) \geq 2Ax$ for $x \in [0, \frac{1}{2}]$. Recalling that $u_0$ is symmetric about $\frac{1}{2}$, and using (4.17), (4.19) implies,

$$\begin{aligned} \alpha &\leq \pi^2 - \lambda a \left( \frac{2Ac^A \ln c + \frac{\pi^2}{A \ln c}(c^A - 1)}{\pi^2 + A^2 \ln^2 c} - 1 \right) \\ &\sim -\frac{2a\lambda}{\ln c}\frac{c^A}{A} \quad \text{for large } A. \end{aligned}$$

The assumption (4.18) together with (4.11) imply that $\alpha \to -\infty$ when $A \to \infty$.

From (4.13), $\lambda(0) = 0$ (since $f(0) \neq 0$) and thus (4.14) at $A = 0$ becomes,

$$\begin{gathered} w''(x) + \alpha\, w(x) = 0, \\ w(0) = w(1) = 0, \quad \int_0^1 w^2(x)dx = 1. \end{gathered}$$

Hence the smallest eigenvalue $\alpha$ at $A = 0$ is $\pi^2$.

Finally the continuity of $\alpha(A)$ follows from its variational characterization and the continuity of $u_0(x, A)$ and $f'$. □

We now show that if $f$ satisfies (4.17), then $\lim_{A\to\infty} \lambda(A) = 0$. Integrating (4.17), we obtain

$$f(t) \geq f(0) + a\left(\frac{c^t - 1}{\ln c} - t\right).$$

Thus $f(t)/t$ grows unboundedly for large $t$. Our claim now follows from (4.12). An obvious consequence of (4.17) is that $f'$ is positive*.

A typical plot of the smallest eigenvalue $\alpha$ of (4.14) as a function of $A$ is shown in Figure 4.4. In the bottom plot of the same figure, we show the behaviour of $A$ as $\lambda$ varies.

An example of a function satisfying (4.17) and (4.18) is $f(t) = e^t$. The first relation is clearly true with $a = 1$ and $c = e$. The second relation is demonstrated below:

$$\begin{aligned}
\int_0^A \frac{du}{\sqrt{\int_u^A e^u du}} &= \int_0^A \frac{du}{\sqrt{e^A - e^u}} \\
&= 2\int_0^{\sqrt{e^A-1}} \frac{dt}{e^A - t^2}, \quad \left(t = \sqrt{e^A - e^u}\right) \\
&= e^{-A/2} \ln\left(\frac{e^{A/2} + \sqrt{e^A - 1}}{e^{A/2} - \sqrt{e^A - 1}}\right) \\
&\sim e^{-A/2} A \quad \text{for large } A.
\end{aligned}$$

The relation (4.18) now follows since its left-hand-side is asymptotic to $\sqrt{A}$ for large $A$.

### 4.2.2 Main Results

We restate here the semilinear elliptic problem:

$$\begin{bmatrix} \Delta v + \lambda f(v) = 0, & (x, y, z) \in \Omega \\ \mathcal{B} v = 0, & (x, y, z) \in \partial\Omega \end{bmatrix}. \tag{4.20}$$

We will assume $f$ is smooth and positive*. We first transform (4.20) to an equivalent problem with a trivial branch of solutions. A solution of the two-point boundary value problem (4.9) will be used as the basic solution to (4.1). Recall that for any positive $A$, there corresponds a unique non-negative solution $u_0(x, A)$ of (4.9) such that A is the maximum value of $u_0(x, A)$. Let $\mathcal{I}_A$ be some nonempty open interval such that for all $A \in \mathcal{I}_A$, $\alpha(A) < 0$. (Recall $\alpha$ is the principle eigenvalue in (4.14).) From (4.15), $\frac{d\lambda(A)}{dA} < 0$ and

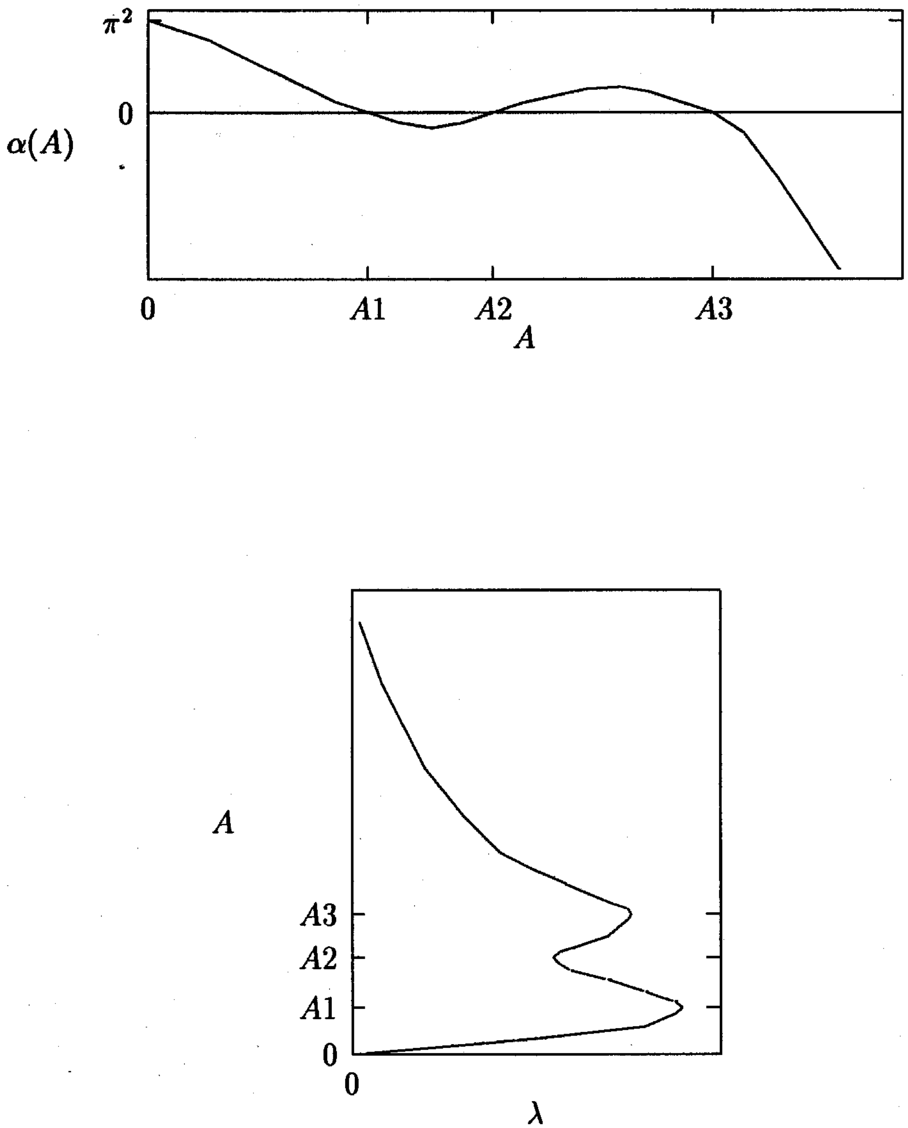


Figure 4.4: Plots of the behaviour of the principle eigenvalue $\alpha$ as a function of $A$ and the corresponding $A$ versus $\lambda$ graph. Note $\alpha$ is negative in $(A_1, A_2) \cup (A_3, \infty)$.

thus we may use $\lambda$ as a parameter for this branch of solutions. We denote this by $(u_0, \lambda)$ for $\lambda$ in the open interval $\mathcal{I}_\lambda$ corresponding to $\mathcal{I}_A$ where $u_0 = u_0(x, \lambda)$. That is,

$$\frac{d^2}{dx^2}u_0 + \lambda f(u_0) = 0, \ \forall \lambda \in \mathcal{I}_\lambda.$$

We make a change of variable $u = v - u_0$ in (4.20), arriving at the equivalent problem:

$$G(u,\lambda) \equiv \begin{bmatrix} \Delta u + \lambda(f(u_0 + u) - f(u_0)) \\ \\ \mathcal{B}u \end{bmatrix} = 0. \tag{4.21}$$

Note that $G(0, \lambda) = 0$ for all real $\lambda$.

Fix some $\lambda_0 \in \mathcal{I}_\lambda$. A necessary condition for bifurcation at $(u, \lambda) = (0, \lambda_0)$ is that $\mathcal{N}(G_u^0)$ is nontrivial. (Superscript $^0$ denotes evaluation at $(u, \lambda) = (0, \lambda_0)$.) Null functions (if they exist) must have the form

$$\phi = X(x)T(y,z). \tag{4.22}$$

As before, we obtain separate equations for $X$ and $T$:

$$X''(x) + (\lambda_0 f'(u_0) - \kappa^2)X(x) = 0; \quad X(0) = X(1) = 0, \tag{4.23}$$

$$T_{yy} + T_{zz} + \kappa^2 T \ = \ 0; \quad \left.\frac{\partial T}{\partial \nu}\right|_{\partial\Omega_2} = 0. \tag{4.24}$$

Comparing (4.14) and (4.23), we see that there exists a nontrivial null space when $-\kappa^2$ is an eigenvalue of (4.14), i.e., when:

$$\alpha = -\kappa^2. \tag{4.25}$$

We call (4.25) the bifurcation condition (BIC). Now we see the significance of a non-positive eigenvalue in (4.14). It is a necessary condition for bifurcation.

Later we will show that only negative eigenvalues need to be considered. We now demonstrate that BIC is also a sufficient condition for bifurcation from negative eigenvalues.

Suppose dim $\mathcal{N}(G_u^0) = M$. Since (4.14) has at most one non–positive eigenvalue, (4.24) must have $M$ equal eigenvalues: $\kappa_1^2 = \cdots = \kappa_M^2 = \kappa^2$ satisfying (BIC). We label the null functions as $\{\phi_1, \ldots, \phi_M\}$, with $\phi_i = X(x)T_i(y,z)$.

For all $g \in \mathcal{L}_2(\overline{\Omega})$, we define the linear functionals

$$\psi_i^*(g) = s\int_\Omega g\phi_i \qquad i = 1, \ldots, M,$$

where $s$ is a nonzero constant that will be defined later. We are ready to check that some of the hypotheses of Theorem 3.14 are satisfied. Now

$$\psi_i^*(G_{u\lambda}^0\phi_j) = s\int_0^1 X^2\left[f'(u_0) + \lambda_0 f''(u_0)\frac{du_0}{d\lambda}\right]_{\lambda=\lambda_0} dx \int_{\Omega_2} T_iT_j\, dydz.$$

Since $T_i$ and $T_j$ are orthonormal, the above quantity is zero when $i \neq j$. If the integral in $x$ is nonzero, we may choose $s$ so that $\psi_i^*(G_{u\lambda}^0\phi_i) = 1$. Also,

$$\psi_l^*(G_{uuu}^0\phi_i\phi_j\phi_k) = s\lambda_0\int_0^1 X^4 f'''(u_0)dx\int_{\Omega_2} T_iT_jT_kT_l\, dydz.$$

We have shown:

**Theorem 4.26** *Consider the problem (4.20) denoted by $\tilde{G}(u,\lambda) = 0$. Suppose $f \in C^3$ and $f(t) > 0$ for all $t \geq 0$. Suppose $u_0 = u_0(x,\lambda)$ is a solution of (4.9) for all $\lambda$ near some $\lambda_0 > 0$. Assume dim $\mathcal{N}(\tilde{G}_u(u_0,\lambda_0)) = M$ with $M$ null functions $\{\phi_1, \ldots, \phi_M\}$ of form (4.22). Suppose for every $i, j, k \in \{1, \ldots, M\}$,*

$$\int_{\Omega_2} T_iT_jT_k = 0, \tag{4.27}$$

$$\int_{\Omega_2} T_i T_j T_k T_l = 0, \quad l \notin \{i,j,k\}. \tag{4.28}$$

*and*

$$\int_0^1 X^2 \left[ f'(u_0) + \lambda_0 f''(u_0) \frac{du_0}{d\lambda} \right]_{\lambda=\lambda_0} dx \neq 0, \quad i = 1, \ldots, M.$$

*If (3.14c) and (3.14e) hold, then (4.20) has at least M solution branches bifurcating from the basic solution branch at* $(u_0, \lambda_0)$.

We remark that if the hypotheses (4.28), (3.14c) and (3.14e) are replaced by (3.22c) and the isolation of those roots of the ABEs, then that same theorem guarantees the existence of at least $M$ branches of bifurcating solutions. There are less conditions involved in the second set of hypotheses but it may be impossible to determine the isolation of the roots because only their existence is known.

We will see in the next section that the additional assumptions (4.17) and (4.18) on $f$ guarantee a null space $G_u$ of arbitrarily large dimension and hence bifurcation of arbitrarily large multiplicity. For the existence of a nontrivial null space, all that is required is that $\alpha$, the smallest eigenvalue of (4.14), is negative and whose magnitude is greater than or equal to $\kappa^2$, the smallest eigenvalue of (4.24). In other words, the BIC holds for at least one value of $\alpha$ and $\kappa^2$. This would guarantee the existence of one bifurcation point.

It is possible to enumerate every bifurcation point along the basic solution branch. In the beginning of this section, we indicated the procedure along one branch. We repeat this for all other branches $(u_0(x, A), \lambda(A))$ where $\alpha(A) < 0$. On the branches where $\alpha(A) > 0$, there can be no bifurcation because the BIC is not satisfied. Finally we show that at points $A_0$ where

$\alpha(A_0) = 0$, there is a locally unique solution. From the BIC and (4.24), it is clear that $M = 1$ and $T_1 = 1$ is the only eigenfunction. In other words, (4.21) has a one-dimensional null space spanned by $\phi_1 = X$. Abbreviating $(u_0(x, A_0), \lambda(A_0))$ by $(u_0, \lambda_0)$, we have

$$\psi_1^* \tilde{G}_\lambda(u_0, \lambda_0) = s \int_0^1 X^2 f(u_0) \neq 0.$$

(Note we use the original formulation of (4.20) $\tilde{G} = 0$ because $G$, as defined in (4.21) has an unbounded $\lambda$ derivative at $A_0$ where $\alpha(A_0) = 0$.) We have shown that $(u_0, \lambda_0)$ satisfies the criteria for a simple limit point. Keller [17] has shown that there is a unique solution in a neighborhood of $(u_0, \lambda_0)$. In summary, we have enumerated all the bifurcation points along the basic solution and they reside on the part of the basic solution branch where $\alpha(A) < 0$ or equivalently $d\lambda/dA < 0$. The BIC is a necessary and sufficient condition for bifurcation for negative $\alpha$. In Figure 4.4, this corresponds to the part of the curve where $A$ lies in the union of $(A_1, A_2)$ and $(A_3, \infty)$.

Assume the hypotheses of Theorem 4.26 hold. We can find an explicit expression for $\Phi_{ij}$, (see (3.7)) for $i, j \in \{1, \ldots, M\}$. In the current context, $\Phi_{ij}$ is the unique solution of

$$\begin{aligned} (\Delta + \lambda_0 f'(u_0))\Phi_{ij} + \lambda_0 f''(u_0)\phi_i\phi_j &= 0, \\ \mathcal{B}\Phi_{ij} &= 0, \end{aligned} \tag{4.29}$$

orthogonal (in $\mathcal{L}_2(\overline{\Omega})$ sense) to the span of $\{\phi_1, \ldots, \phi_M\}$. Recall $\{T_1, T_2, \ldots\}$ is a complete orthonormal set. Thus there exists real constants $d_p$ such that

$$T_i T_j = \sum_{p=1}^{\infty} d_p T_p. \tag{4.30}$$

Since these eigenfunctions satisfy

$$\int_{\Omega_2} T_i T_j T_k = 0, \quad k \in \{1, \ldots, M\},$$

$d_1 = \ldots = d_M = 0$. We look for solution of (4.29) of the form $\Phi_{ij} = \sum_{p=1}^{\infty} R_p(x) T_p(y,z)$. (We are abbreviating $R_{pij}$ by $R_p$.) Substituting into (4.29), we obtain

$$\sum_{p=1}^{\infty} T_p \left[ R_p'' + (\lambda_0 f'(u_0) - \kappa_p^2) R_p + d_p \lambda_0 f''(u_0) X^2 \right] = 0.$$

We now show that for every $p$, there exists a function $R_p(x)$ satisfying

$$R_p'' + (\lambda_0 f'(u_0) - \kappa_p^2) R_p = -d_p \lambda_0 f''(u_0) X^2, \quad R_p(0) = R_p(1) = 0. \qquad (4.31)$$

If $d_p = 0$, then clearly $R_p = 0$. In particular, $R_1 = \cdots = R_M = 0$. We now restrict to the case $p > M$. If $d_p \neq 0$, (4.31) is the inhomogeneous form of the eigenvalue problem (4.14) at $\lambda = \lambda_0$. Denote by $\alpha, \alpha_2, \alpha_3, \ldots$ the eigenvalues of (4.14) with $\alpha < 0 < \alpha_2 \leq \alpha_3 \ldots$. By the Fredholm Alternative, (4.31) has a unique solution if $-\kappa_p^2$ is not equal to $\alpha$ or any $\alpha_i$. This is indeed the case because by the definition of $M$, only elements of $\{\kappa_1, \ldots, \kappa_M\}$ satisfy the (BIC). Furthermore, $\alpha_i > 0$ when $i \geq 2$ and hence cannot be equal to any $-\kappa_p^2$. Hence, we have shown that

$$\Phi_{ij} = \sum_{p=M+1}^{\infty} R_{pij} T_p, \qquad (4.32)$$

where $R_{pij}$ is the unique solution of (4.31). Note that $R_{pij} = R_{pji}$.

For the case of a square cross–section, we have,

**Theorem 4.33** *Consider the problem (4.20) with $\Omega_2 = (0, L)^2$, where $L$ is a positive real number. Denote it by $\tilde{G}(u, \lambda) = 0$. Assume $f \in C^3$ and*

*$f(t) > 0$ for all $t \geq 0$. Suppose $u_0 = u_0(x, \lambda)$ is a solution of (4.9) for all $\lambda$ near some $\lambda_0 > 0$. Assume dim $\mathcal{N}(\tilde{G}_u(u_0, \lambda_0)) = M$ with $M$ null functions $\{\phi_1, \ldots, \phi_M\}$ of the form (4.22). If*

$$\int_0^1 X^2 \left[ f'(u_0) + \lambda_0 f''(u_0) \frac{du_0}{d\lambda} \right]_{\lambda=\lambda_0} dx \neq 0, \quad i = 1, \ldots, M,$$

*and Generic Assumptions 1 and 2 (see Theorems 3.17 and 3.14) hold, then (4.20) has at least $2M$ solution branches bifurcating from the basic solution branch at $(u_0, \lambda_0)$ when $M$ is even and at least $2M - 1$ bifurcating branches when $M$ is odd.*

**Proof:** For a square cross–section, the eigenfunctions of (4.24) have the form

$$\cos\left(\frac{m_i \pi y}{L}\right) \cos\left(\frac{n_i \pi z}{L}\right),$$

with eigenvalue

$$\kappa_i^2 \equiv \frac{\pi^2}{L^2}(m_i^2 + n_i^2), \qquad i = 1, \ldots, M, \tag{4.34}$$

where $m_i, n_i$ are non-negative integers and $-\kappa_1^2 = \cdots = -\kappa_M^2 = -\alpha$, the principle eigenvalue of (4.14) at $\lambda_0$. Note that for fixed $i, j \in \{1, \ldots, M\}$, at most four terms in $\{R_{pij}\}_{p=M+1}^{\infty}$ are nonzero (see (4.32)). This is because $T_i T_j$ has four nonzero components in (4.30). They are:

$$\cos\left(\frac{(m_i \pm m_j)\pi y}{L}\right) \cos\left(\frac{(n_i \pm n_j)\pi z}{L}\right).$$

By the Basic Lemma, we see that (4.27) and (4.28) and (3.14c) hold. Thus by Theorem 4.26, there are at least $M$ bifurcating branches. Now we will show that there are at least $M/2$ or $(M-1)/2$ associated pairs (see Definition 3.16), depending on whether $M$ is even or odd.

First, suppose $M$ is even. Note that the null functions $\{\phi_i\}$ come in pairs. For example, corresponding to $\phi_1$ is the pair $(m_1, n_1)$ of solution to the BIC ($m_1 \neq n_1$ by Corollary 4.37. See next section.) Let $\phi_2$ correspond to the solution $(n_1, m_1)$. It is now straightforward to show that $(1,2)_A$ is an associated pair. Hence we have at least $\frac{M}{2}$ associated pairs. When $M$ is odd, there is exactly one $i$ such that $m_i = n_i$ (see Corollary 4.37). Ignoring this pair, we get at least $\frac{M-1}{2}$ associated pairs. Applying the result of Theorem 3.17, we establish the claim of this theorem. □

In Figure 4.5, we draw some bifurcating branches for the function $f(u) = e^u$.

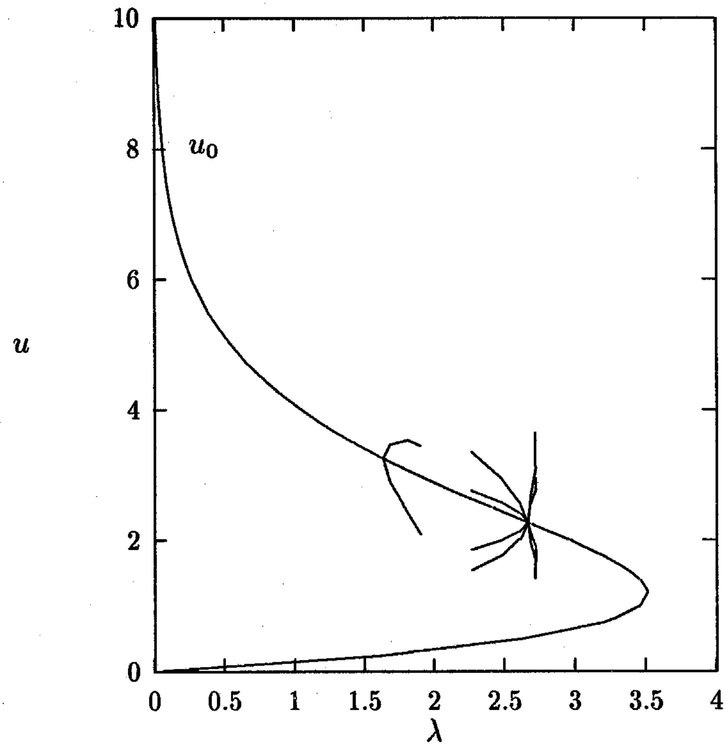


Figure 4.5: Two bifurcation points of the semilinear PDE with $L = 1$, $f(u) = e^u$.

### 4.2.3 An Excursion into Number Theory

In the previous sections, we always assumed that the dimension of a certain null space is some positive number $M$. With a square cross–section and two additional assumptions, we can show that such null spaces exist for *all positive integers* $M$.

Let $N_a$ denote the number of integer solutions $(m, n)$ of

$$N = m^2 + n^2, \tag{4.35}$$

for a given positive integer $N$. Determining $N_a$ is a well-known problem in number theory. From Grosswald [14], we have:

**Theorem 4.36** *Let*

$$N = 2^k \prod_i p_i^{r_i} \prod_j q_j^{s_j}$$

*be the unique representation of a given positive integer $N$ as the product of powers of primes $p_i$ and $q_j$, where $p_i \equiv 1 \pmod 4$, $q_j \equiv 3 \pmod 4$, $r_i$ and $s_j$ are positive integers and $k$ a non-negative integer. Then*

$$N_a = \begin{cases} 4\prod_i (r_i + 1), & \textit{if } \; s_j \textit{ is even } \forall j; \\ 0, & \textit{otherwise.} \end{cases}$$

In particular, we are interested in $N_p$, the number of non-negative integer solutions to (4.35). Using the above result, we show:

**Corollary 4.37** *Let $N$ be represented as in the above theorem with $s_j$ even for all $j$. Then,*

$$N_p = \begin{cases} 1 + \prod_i (r_i + 1), & \textit{if } N \textit{ is a perfect square;} \\ \prod_i (r_i + 1), & \textit{otherwise.} \end{cases}$$

*Furthermore, $N_p$ is odd iff $N$ is of the form $2j^2$.*

**Proof:** We consider three cases. First, if $N$ is neither a perfect square nor of the form $2j^2$, for any integer $j$, then (4.35) has no solutions of the form $(0,j)$ or $(j,j)$. Thus every non-negative solution $(m,n)$ generates eight distinct integral solutions, namely $(\pm m, \pm n)$ and $(\pm n, \pm m)$. Among them only $(m,n)$ and $(n,m)$ are non-negative solutions. Thus, $N_p$ is even and

$$N_p = \frac{N_a}{4} = \prod_i (r_i + 1).$$

Secondly, if $N = 2j^2$ for some positive integer $j$. Then $(j,j)$ is one solution which generates four integral solutions, namely $(\pm j, \pm j)$, but only one non-negative solution $(j,j)$. All the other solutions are of the form $(k,l)$, $k \neq l$ and we can apply the result of the previous paragraph. Thus

$$N_p = \frac{N_a - 4}{4} + 1 = \prod_i (r_i + 1).$$

Since there is only one solution of the form $(j,j)$ and all other solutions come in pairs, $N_p$ must be odd.

Finally, consider $N = j^2$ for some positive $j$. Now $(0,j)$ generates four integral solutions, namely $(0, \pm j), (\pm j, 0)$ and two non-negative solutions $(0,j)$ and $(j,0)$. Hence

$$N_p = \frac{N_a - 4}{4} + 2 = 1 + \prod_i (r_i + 1).$$

Because $N$ is a perfect square, all the $r_i$'s are even and so $N_p$ must be even.

□

Geometrically, $N_p$ corresponds to the number of lattice points on the circle of radius $N$ in the first quadrant of the $m,n$ plane. Table 1 gives a few

| N | solutions (m,n) | number of solutions |
|---|---|---|
| 0 | (0,0) | 1 |
| 1 | (0,1); (1,0) | 2 |
| 2 | (1,1) | 1 |
| 3 | — | 0 |
| 4 | (0,2); (2,0) | 2 |
| 5 | (1,2); (2,1) | 2 |
| ⋮ | ⋮ | ⋮ |
| 25 | (0,5); (3,4); (4,3); (5,0) | 4 |
| ⋮ | ⋮ | ⋮ |
| 50 | (1,7); (5,5); (7,1) | 3 |
| ⋮ | ⋮ | ⋮ |
| 325 | (1,18); (6,17); (10,15); (15,10); (17,6); (18,1) | 6 |
| ⋮ | ⋮ | ⋮ |
| 1250 | (5,35); (17,31); (25,25); (31,17); (35,5) | 5 |
| ⋮ | ⋮ | ⋮ |

Table 4.1: Non-negative integer solutions to $N = m^2 + n^2$.

examples. It is clear that there exist infinitely many values of $N$ for which there are $M$ solutions to (4.35), where $M$ is any given non-negative integer.

Define the problem (4.1) by $G(u,\lambda) = 0$. Assume $f$ is $C^3$, $f(t) > 0\ \forall t \geq 0$, and $f$ satisfies (4.17) and (4.18). Let $u_0 = u_0(x, A)$ be the unique branch of non–negative solutions of (4.9) with the property that $\alpha(A)$ takes on every negative real number as $A$ varies in a semi-infinite interval $(A_1, \infty)$ for some $A_1$. Such an interval exists by Lemma 4.16. We now show the existence of $\lambda_0$ such that at $(u_0, \lambda_0)$, dim $\mathcal{N}(G_u) = M$, for a given integer $M$. We infer from Corollary 4.37 that there is an integer $N$ (in fact infinitely many of them) such that there are $M = N_p$ non-negative integer solutions to $N = m^2 + n^2$. By Lemma 4.16, there exists an $A$ such that

$$-\alpha(A) = \frac{N\pi^2}{L^2}.$$

(See (4.34) and the BIC (4.25).) Thus at $\lambda_0 \equiv \lambda(A)$ and $u_0(x, \lambda_0)$, dim $\mathcal{N}(G_u) = M$. This establishes our claim.

### 4.2.4 The Basic Lemma

**Lemma 4.38 (Basic Lemma)** *Consider the square domain $\Omega_2 \equiv (0, L)^2$. Let $T_1, \ldots, T_M$ be eigenfunctions of (4.24) with a nonzero eigenvalue $\kappa^2$ of multiplicity $M$:*

$$T_i = \cos\left(\frac{m_i \pi y}{L}\right) \cos\left(\frac{n_i \pi z}{L}\right),$$

*with*

$$N \equiv \frac{\kappa^2 L^2}{\pi^2} = m_i^2 + n_i^2, \quad i = 1, \ldots, M. \tag{4.39}$$

*Then for all $i, j, k, l \in \{1, \ldots, M\}$,* *1.* $\int_{\Omega_2} T_i T_j T_k = 0.$

*2.* $\int_{\Omega_2} T_i T_j T_k T_l = 0$ *except possibly when* $i = j = k = l$ *or indices occur in two pairs.*

*3.* $\int_{\Omega_2} T_i T_j T_p = 0$ *except possibly when* $i = j = k = l$ *or indices occur in two pairs, where* $T_p$ *is any one of the following four possible functions:*

$$\cos\left(\frac{(m_k \pm m_l)\pi y}{L}\right)\cos\left(\frac{(n_k \pm n_l)\pi z}{L}\right).$$

**Proof:** We use repeatedly the following facts:

$$\begin{aligned} \int_0^L \cos\left(\frac{m\pi y}{L}\right) dy &= L\delta_{m0}. \\ \int_0^L \cos\left(\frac{m\pi y}{L}\right)\cos\left(\frac{n\pi y}{L}\right) dy &= \frac{L}{2}\delta_{mn}, \quad mn \neq 0. \end{aligned} \tag{4.40}$$

Assertion 1. Without loss of generality, we assume $m_i \leq m_j \leq m_k$. We first treat the case $m_i > 0$. As a consequence of (4.39) $n_i \geq n_j \geq n_k$. Thus,

$$\begin{aligned} \int_{\Omega_2} T_i T_j T_k &= \int_0^L \cos\left(\frac{m_i\pi y}{L}\right)\cos\left(\frac{m_j\pi y}{L}\right)\cos\left(\frac{m_k\pi y}{L}\right) dy \\ &\quad \int_0^L \cos\left(\frac{n_i\pi z}{L}\right)\cos\left(\frac{n_j\pi z}{L}\right)\cos\left(\frac{n_k\pi z}{L}\right) dz \\ &\propto \int_0^L \cos\left(\frac{(m_i+m_j-m_k)\pi y}{L}\right) dy \int_0^L \cos\left(\frac{(n_i-n_j-n_k)\pi z}{L}\right) dz. \end{aligned}$$

To obtain the last line, we make use of the ordering of the constants $m_i$ and $n_j$ and (4.40). From (4.40), the last integral is nonzero iff

$$m_k = m_i + m_j \quad \text{and} \quad n_k = n_i - n_j. \tag{4.41}$$

Adding the squares of the above two equations and using (4.39) we obtain

$$N = 2(n_i n_j - m_i m_j) \tag{4.42}$$

If $n_i n_j = 0$, then above is clearly a contradiction. So we assume $n_i n_j \neq 0$. Write $N = 2^p N_1$, $m_l = 2^{r_l} a_l$, $n_l = 2^{s_l} b_l$, where $p, r_l$ and $s_l$ are non-negative integers and $2 \not| N_1 a_l b_l$. We must consider different cases:

- $r_i < s_i$ and $r_j < s_j$. From (4.39), we have

$$2^p N_1 = 2^{2r_i}(a_i^2 + 2^{2(s_i - r_i)} b_i^2) = 2^{2r_j}(a_j^2 + 2^{2(s_j - r_j)} b_j^2).$$

  From the above, we deduce that $p = 2r_i = 2r_j$. Now (4.42) can be written as

$$2^p N_1 = 2^{r_i + r_j + 1}(2^{s_i + s_j - r_i - r_j} b_i b_j - a_i a_j).$$

  Consequently we have $p = r_i + r_j + 1$, contradicting the previous result.

- In a similar manner, we derive a contradiction for each of the cases $\{ r_i > s_i$ and $r_j > s_j \}$; $\{ r_i > s_i$ and $r_j < s_j\}$; and $\{r_i < s_i$ and $r_j > s_j\}$.

- $r_i = s_i$. We further suppose $r_j \leq s_j$ without loss of generality. Then,

$$2^p N_1 = 2^{2r_i}(a_i^2 + b_i^2) = 2^{2r_j}(a_j^2 + 2^{2(s_j - r_j)} b_j^2).$$

  It is easy to see that $a_i^2 + b_i^2$ has the form $4q + 2$ and we may conclude that $p = 2r_i + 1$ and $r_j = s_j = r_i$. From (4.42), we get,

$$2^p N_1 = 2^{2r_i + 1}(b_i b_j - a_i a_j).$$

  Now the term in the parenthesis on the RHS is even and we must have $p \geq 2r_i + 2$, which is a contradiction.

We have shown that when $m_i > 0$, $\int_{\Omega_2} T_iT_jT_k = 0$. Next we consider the case when $m_i = 0,\ m_j > 0$ and so $N = n_i^2$. Now

$$\int_{\Omega_2} T_iT_jT_k \propto \int_0^L \cos\left(\frac{(m_j - m_k)\pi y}{L}\right) dy \int_0^L \cos\left(\frac{(n_i - n_j - n_k)\pi z}{L}\right) dz. \tag{4.43}$$

The integral is nonzero iff $m_j = m_k$ and $n_i = 2n_j$. Using this in (4.39), we obtain $m_j^2 = 3n_j^2$, which is a contradiction. Finally a similar procedure leads to an inconsistency when $m_i = 0,\ m_j = 0$.

Assertion 2. We must separate into four cases.

- $m_i < m_j < m_k < m_l$. Using (4.40) and the fact $n_i > n_j > n_k > n_l$, we have:

$$\begin{aligned}&\int_{\Omega_2} T_iT_jT_kT_l \propto \\ &\quad \textstyle\int_0^L \left[\cos\left(\frac{(m_i+m_j+m_k-m_l)\pi y}{L}\right) + \cos\left(\frac{(m_i-m_j-m_k+m_l)\pi y}{L}\right)\right] dy \\ &\quad \textstyle\int_0^L \left[\cos\left(\frac{(n_i-n_j-n_k-n_l)\pi z}{L}\right) + \cos\left(\frac{(n_i-n_j-n_k+n_l)\pi z}{L}\right)\right] dz.\end{aligned}$$

  This integral is nonzero iff anyone of the following two cases hold:

  - $m_i + m_j + m_k = m_l$ and $n_i - n_j - n_k = \pm n_l$. Using (4.39) on the sum of the squares of these two equations, we get,

$$\begin{aligned} 0 &= N + m_im_j + m_im_k + m_jm_k + n_jn_k - n_in_j - n_in_k \\ &= (m_i + m_j)(m_i + m_k) + (n_i - n_j)(n_i - n_k) > 0. \end{aligned}$$

    In the last step, we replaced $N$ by $m_i^2 + n_i^2$ so the expression factors and allows us to reach a contradiction.

- o $m_j + m_k - m_i = m_l$ and $n_i - n_j - n_k = \pm n_l$. Using the same technique, we arrive at the following contradiction:

$$0 = (m_j - m_i)(m_k - m_i) + (n_i - n_j)(n_i - n_k) > 0.$$

- $m_i = m_k = m_l \neq m_j$.

$$\int_{\Omega_2} T_i^3 T_j \propto \int_0^L \cos\left(\frac{(3m_i - m_j)\pi y}{L}\right) dy \int_0^L \cos\left(\frac{(3n_i - n_j)\pi z}{L}\right) dz.$$

which is nonzero iff $3m_i - m_j = 0 = 3n_i - n_j$ which is clearly impossible from (4.39).

- $m_i = m_l$ and $m_i, m_j$ and $m_k$ are distinct. Without loss of generality, we assume $m_j > m_k$.

$$\int_{\Omega_2} T_i^2 T_j T_k \propto \int_0^L \left[\cos\left(\frac{(2m_i - m_j - m_k)\pi y}{L}\right) + \cos\left(\frac{(2m_i - m_j + m_k)\pi y}{L}\right)\right]$$
$$\int_0^L \left[\cos\left(\frac{(2n_i + n_j - n_k)\pi z}{L}\right) + \cos\left(\frac{(2n_i - n_j - n_k)\pi z}{L}\right)\right]$$

Again we must consider four different cases.

- o $2m_i = m_j - m_k$ and $2n_i = n_k - n_j$. Squaring and adding these equations and then use (4.39), we get $N = -(m_j m_k + n_j n_k)$ which is clearly false.
- o $2m_i = m_j - m_k$ and $2n_i = n_k + n_j$. The same procedure yields the following inconsistency: $N = n_j n_k - m_j m_k \leq n_j n_k < n_k^2 \leq N$.
- o $2m_i = m_j + m_k$ and $2n_i = n_k - n_j$. Similar.
- o $2m_i = m_j + m_k$ and $2n_i = n_k + n_j$. After some algebra, we obtain the false statement $(m_j - m_k)^2 + (n_j - n_k)^2 = 0$.

Assertion 3. Proof is similar to the above. □

# Chapter 5

# Conclusions

In this thesis, we study some sufficient conditions for multiple bifurcation. In the case of a "simple" eigenvalue, it is possible, at least in theory, to derive a list of sufficient conditions which characterize every possible type of bifurcation for analytic operator equations. For a multiple eigenvalue, the theory is incomplete. We are only able to give some conditions which give a lower bound on the multiplicity of the bifurcation. The main application we give is to semilinear elliptic PDEs in a cylinder with a constant cross–section. If the PDE has a trivial solution, then we can locate all the bifurcation points along the trivial branch with a lower bound on the multiplicity. When the cross–section is a square, we can, in one instance, determine the exact multiplicity of the bifurcation. If there is no trivial solution, then we locate all the bifurcation branches from a basic solution which is the nontrivial, non–negative solution to a two–point boundary value problem associated with the PDE.

Our theory has many shortcomings. The assumptions are rather strong and many of them (especially the Generic Assumptions) are usually impractical to verify analytically. We would like to obtain a tight upper bound on the multiplicity. The theory deals with rather special cases of multiple bifurcations. We only dealt with the cases where the variables in the ABEs are largely decoupled, allowing us to find the roots analytically. An ambitious goal is to come up with a set of sufficient conditions for general bifurcations, similar to the ones for the "simple" eigenvalue case. Another avenue of research is to extend the results to a more general class of problems, say

$$\Delta u + f(x, u, \nabla u, \lambda) = 0.$$

# Part II

# PARALLEL HOMOTOPY METHOD FOR THE REAL NONSYMMETRIC EIGENVALUE PROBLEM

# Chapter 6

# Introduction

Given a real $n \times n$ matrix $A$, we wish to find all its eigenvalues and eigenvectors. That is, we seek $\lambda \in \mathbb{C}$ such that

$$Ax = \lambda x$$

holds for nontrivial $x \in \mathbb{C}^n$. We call $(x, \lambda)$ an eigenpair.

The QR algorithm (see Golub and van Loan [13]) is generally regarded as the best sequential method for computing the eigenpairs. Briefly, the QR algorithm uses a sequence of similarity transformations to reduce a matrix to upper Hessenberg form. It then applies a sequence of Givens rotations from the left and right to reduce the size of the sub-diagonal elements. When these elements are sufficiently small, the diagonal elements are taken to be approximations to the eigenvalues of the matrix. If the matrix is large and sparse, the QR algorithm suffers two serious drawbacks. In the reduction to Hessenberg form, the matrix usually loses its sparsity. Hence the algorithm requires the explicit storage of the entire matrix. This may pose a problem if the matrix is so large that not all its entries can be accommodated within the main memory of the computer. A second drawback is that it is inherently a sequential algorithm due to fact that the Givens rotations must be applied sequentially. Bai and Demmel [2] have circumvented somewhat the second problem by performing a "block" version of the QR algorithm. This improved version seems to work well on vector machines.

## 6.1 Homotopy Method

We present a homotopy method to compute the eigenpairs of a given matrix $A_1$. From the eigenpairs of some real matrix $A_0$, we follow the eigenpairs of

$$A(t) \equiv (1-t)A_0 + tA_1$$

at successive times from $t = 0$ to $t = 1$ using continuation. At $t = 1$, we have the eigenpairs of the desired matrix $A_1$. We call the evolution of an eigenpair as a function of time an eigenpath.

When $A_1$ is a real symmetric tridiagonal matrix with nonzero off-diagonal elements, a very successful homotopy method is known (see Li and Rhee [24]). The following phenomena, while absent in the symmetric tridiagonal case, are present for the general case:

- complex eigenpairs
- ill-conditioned problems due to non-orthogonal eigenvectors
- bifurcation (i.e., crossing of eigenpaths)

These can present computational difficulties if not handled properly. Since each eigenpair can be followed independently, this algorithm is ideal for concurrent computers.

As a simple illustration, we consider $2 \times 2$ matrices where the matrix $A_0$ is diagonal whose elements are the diagonal elements of $A_1$:

$$A_0 = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \quad A_1 = \begin{bmatrix} a & d \\ c & b \end{bmatrix}.$$

The eigenvalues of $A(t)$ are

$$\frac{a + b \pm \sqrt{(a-b)^2 + 4t^2cd}}{2}.$$

Three different situations arise (see Figure 6.1). In the first case, the two eigenpaths never meet for all $t$ in $[0,1]$. In the second case, the eigenpaths meet at one point (called a bifurcation point) with the eigenpaths remaining real throughout. In the third case, there is a bifurcation point with the eigenpaths becoming a complex conjugate pair to the right of the bifurcation point. Typically this is how complex eigenpaths arise from real ones. (Whenever a quantity is said to be complex, we mean it has a nontrivial imaginary component.) The situation for higher dimensional matrices is similar except that an eigenpath can have more than one bifurcation point and that the reverse of case three described above can occur (i.e., complex conjugate pair of eigenpaths occur to the left of the bifurcation point and two real eigenpaths to the right.) See Figure 6.2 for the eigenpaths of a random 10 by 10 matrix.

We conclude this introduction by giving a synopsis of the rest of the thesis. In Chapter 2, the homotopy method along with complex bifurcations will be presented. We will discuss some different types of bifurcations that may arise

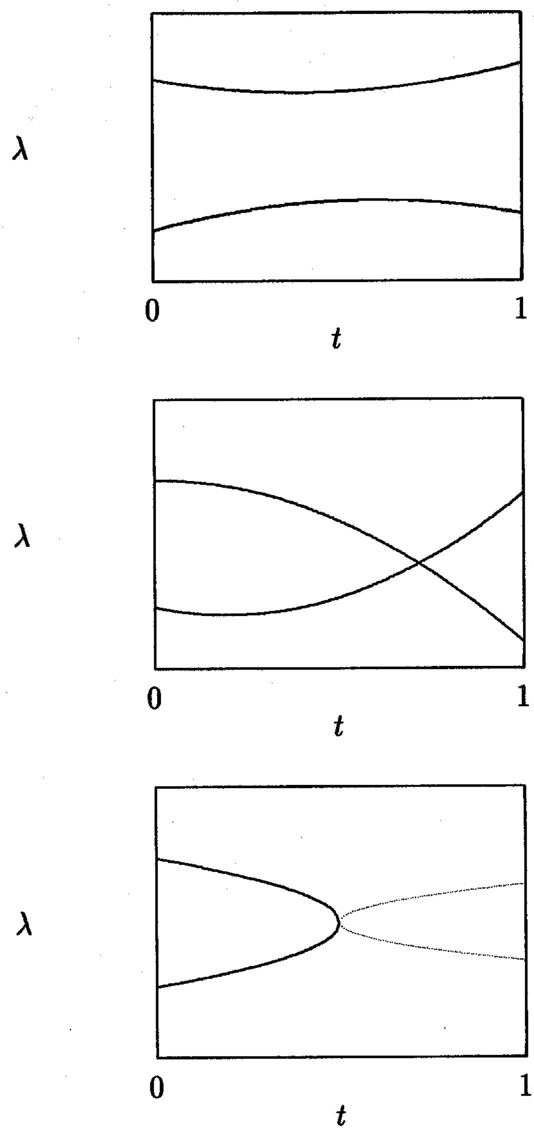


Figure 6.1: Eigenpaths of a 2 by 2 matrix. The dotted lines denote complex eigenpaths.

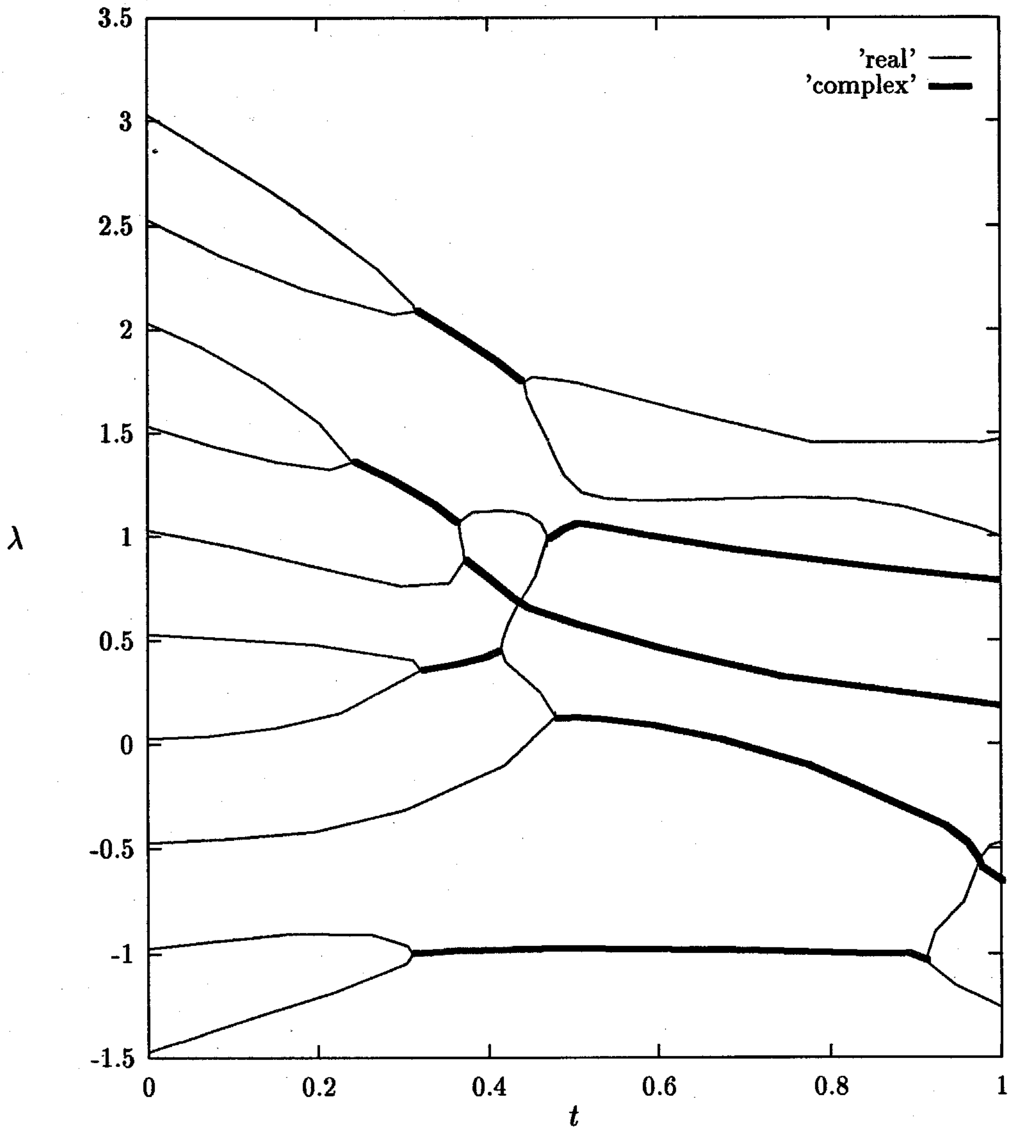


Figure 6.2: Eigenpaths of a random 10 by 10 matrix. Only one path of a complex conjugate pair of eigenpaths is shown.

and identify the generic kind. We will derive an upper bound on the number of bifurcation points of all the eigenpaths. The numerical algorithm will be discussed in Chapter 3. We will describe how to deal with bifurcations, how to solve the linear systems arising from the continuation algorithm, the selection of stepsizes etc.. This will be followed by some numerical results. We will see that the homotopy method is extremely slow for full matrices but has the potential to compete with other algorithms for sparse matrices as well as matrices with defective eigenvalues. In the final chapter, we recapitulate and suggest directions of further research.

For other approaches to the nonsymmetric eigenvalue problem, see for example, Cullum and Willoughby [8], Saad [30], and Shroff [33] . The classic reference for the eigenvalue problem is the treatise by Wilkinson [35].

# Chapter 7

# Homotopy Method and Complex Bifurcation

In this chapter, we discuss some of the various phenomena that may arise on an eigenpath. Usually an eigenpath will be locally unique. That is, there are no other eigenpaths nearby. This can be characterized by a certain Jacobian being nonsingular. When this Jacobian is singular, bifurcation may occur. In other words, two or more eigenpaths may intersect at a point $(u_0, t_0)$. We list some of the possible cases: simple quadratic fold, simple bifurcation point, simple cubic fold and simple pitchfork bifurcation. We will show that the generic kind of bifurcation is the simple quadratic fold. In fact, the transition between real and complex eigenpaths (and vice versa) are via simple quadratic folds.

We first establish some notation. We use the superscripts $^T$ and $^*$ to denote the transpose and the complex conjugate transpose respectively. The null and range spaces of a matrix are written as $\mathcal{N}()$ and $\mathcal{R}()$ respectively. The $i^{th}$ column of the identity matrix $I$ is denoted by $e_i$.

Given a real $n \times n$ matrix $A_1$, we form the homotopy

$$A(t) = (1-t)A_0 + tA_1, \quad 0 \le t \le 1, \tag{7.1}$$

where $A_0$ is a real matrix with real eigenvalues. We write the eigenvalue problem of $A(t)$ as:

$$G(u,t) \equiv \begin{bmatrix} A(t)x - \lambda x \\ n(x) \end{bmatrix} = 0, \tag{7.2}$$

where $u$ is the eigenpair $(x, \lambda)$ of $A(t)$ and $n(x)$ is a normalization equation. For the purposes of analysis, we take

$$n(x) = (x^T x - 1)/2.$$

This may not be valid in some cases. Take for example, any matrix with an eigenvector $x = [1, i]^T$. Here, $x^T x = 0$. The advantage of using this normalization is that the resultant $G$ is analytic and the real and complex cases can be treated identically. The usual normalization $n(x) \equiv x^* x - 1$ is not differentiable except at $x = 0$. In this chapter, we will always assume that every eigenvector $x$ satisfies $x^T x \neq 0$.

Suppose an eigenpair $u_0$ is known at time $t_0$, i.e., $G(u_0, t_0) = 0$. We now describe how to obtain an eigenpair at a later time $t_1$. We must separate the discussion into different cases, depending on whether the Jacobian $G_u^0 \equiv G_u(u_0, t_0)$ is singular or not and on the nature of the singularity.

## 7.1 Nonsingular Jacobian

When $G_u^0$ is nonsingular, then the Implicit Function Theorem tells us that locally about $t_0$, there is a unique solution $u(t)$ with $u(t_0) = u_0$. Differentiating

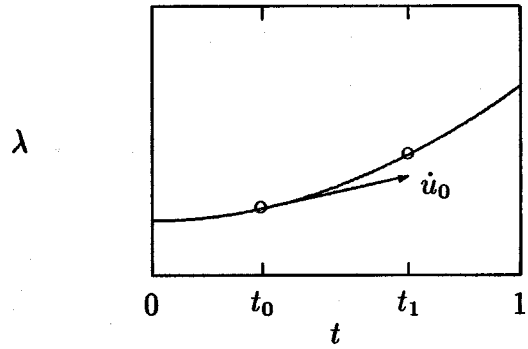


Figure 7.1: Euler-Newton continuation

(7.2) with respective to $t$ and evaluating at $t_0$, we obtain

$$G_u^0 \dot{u}_0 + G_t^0 = 0,$$

where dot denotes t derivative and $G_t^0 \equiv G_t(u_0, t_0)$. Since $G_u^0$ is nonsingular, the above equation has a unique solution $\dot{u}_0$. To obtain the eigenpair at a later time $t_1$, we apply Newton's Method to the equation $G(u, t_1) = 0$ with initial guess $u_0 + (t_1 - t_0)\dot{u}_0$. This is the Euler-Newton continuation method. The Euler step $(t_1 - t_0)\dot{u}_0$ is used to obtain the first Newton iterate (see Figure 7.1). Provided $t_1 - t_0$ is sufficiently small, the Newton iterates will converge quadratically to the eigenpair at $t_1$.

## 7.2 Singular Jacobian: Simple Quadratic Fold

Here we assume the eigenpair $u_0$ is real and

- $G_u^0$ has a one-dimensional null space spanned by say $\phi$, and let $\psi$ span the null space of $G_u^{0^T}$,
- $G_t^0 \notin \mathcal{R}(G_u^0)$,
- $\psi^T(G_{uu}^0\phi^2) \neq 0$.

Note that $G_{uu}^0\phi^2$ is a shorthand for $G_{uu}^0\phi\phi$. The point $(u_0, t_0)$ having the above properties is said to be a simple (real) quadratic fold point of the Equation 7.2. Pictorially, the eigenpath is represented as the solid curve in Figure 7.2.

Since we can no longer use $t$ to parametrize the solution, we employ the following pseudo-arclength method due to Keller [17]. Augment (7.2) with the scalar equation:

$$g(u,t,s) \equiv \phi^T \cdot (u - u_0) - (s - s_0) = 0.$$

This is the equation of a hyperplane whose normal is $\phi$ and is at a distance $s - s_0$ from $u_0$. Now define

$$F(u,t,s) \equiv \begin{bmatrix} G \\ g \end{bmatrix} = 0. \tag{7.3}$$

We have immediately $F(u_0, t_0, s_0) = 0$. It can be shown that the derivative of $F$ with respect to $(u, t)$ and evaluated at $(u_0, t_0, s_0)$

$$F_{(u,t)}^0 \equiv \begin{bmatrix} G_u^0 & G_t^0 \\ \phi^T & 0 \end{bmatrix} \tag{7.4}$$

is nonsingular. Hence again by the Implicit Function Theorem, $F$ has a locally unique solution $(u(s), t(s), s)$ with $u(s_0) = u_0$ and $t(s_0) = t_0$. In fact, the solution has the form:

$$u(s) = u_0 + \phi(s - s_0) + O(s - s_0)^2,$$
$$t(s) = t_0 + \tau(s - s_0)^2 + O(s - s_0)^3, \tag{7.5}$$

where

$$\tau = -\frac{1}{2}\frac{\psi^T(G^0_{uu}\phi^2)}{\psi^* G^0_t}.$$

From the definition of a simple quadratic fold, $\tau$ is well-defined and nonzero. Note that $dt(s_0)/ds = 0$. We can apply the Euler-Newton continuation to the system $F = 0$ and follow the eigenpath around the fold point. Geometrically, the solution of $F = 0$ is the point at which the eigenpath punctures the hyperplane $g = 0$. However, once around the fold point, $t$ will begin to decrease. This is undesirable since our goal is to compute the eigenpair at $t = 1$. It turns out that a complex conjugate pair of eigenpaths will emerge to the right of the fold point. We now elaborate on this point.

Recall that a point $P_0 \equiv (u_0, t_0)$ is called a bifurcation point of the equation $G(u,t) = 0$ if in a neighborhood of $P_0$, there are at least two distinct branches of solutions $(u_1(s), t_1(s))$ and $(u_2(s), t_2(s))$ such that $u_i(s_0) = u_0$ and $t_i(s_0) = t_0$ for $i = 1, 2$. If at least one of these branches is complex, we will call $P_0$ a complex bifurcation point. When $u_0$ is real, (7.2) is a system of real equations. From the last paragraph, we know that locally about the point $P_0$, there is a unique path of *real* solutions. However, when considered as a system of equations over the complex numbers, Henderson and Keller [16] have shown that $P_0$ is a complex bifurcation point with a complex conjugate pair of solutions on the opposite side of the real quadratic fold (see

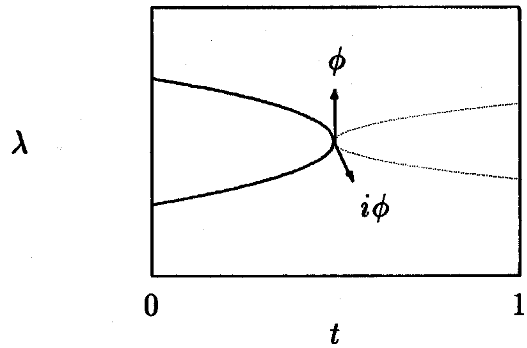


Figure 7.2: Complex conjugate pair of solutions on the opposite side of a simple real quadratic fold point. Dotted lines denote complex solutions.

Figure 7.2). Furthermore the complex solutions have local expansions:

$$u(s) = u_0 + i\phi(s - s_0) + O(s - s_0)^2,$$

$$t(s) = t_0 - \tau(s - s_0)^2 + O(s - s_0)^3.$$

They are very similar to the form of the real solution (7.5). Note that the tangent vector of the complex solution is a rotation of the tangent ($\phi$) of the real solution. We can now use the Euler-Newton continuation with initial step in the direction $i\phi$ to find the complex eigenpairs at a later time.

The result of Henderson and Keller can be generalized to a complex quadratic fold point: i.e., $u_0 \in \mathbb{C}^{n+1}$ and satisfying the three properties outlined at the beginning of this section.

**Theorem 7.6 (Henderson [15])** *Let $G(u,t)$ be an analytic operator from $\mathbb{C}^{n+1} \times \mathbb{R}$ to $\mathbb{C}^{n+1}$. Let $(u_0, t_0)$ be a simple quadratic fold point of $G(u,t) = 0$.*

*Then in a small neighborhood of* $(u_0, t_0)$*, there exist exactly two solution branches. They have the expansions for small* $|\epsilon|$*:*

$$\begin{aligned} u_1(\epsilon) &= u_0 + \epsilon e^{-i\alpha/2}\phi + O(\epsilon^2), \\ t_1(\epsilon) &= t_0 - r\epsilon^2 + O(\epsilon^3), \\ u_2(\epsilon) &= u_0 + i\epsilon e^{-i\alpha/2}\phi + O(\epsilon^2), \\ t_2(\epsilon) &= t_0 + r\epsilon^2 + O(\epsilon^3), \end{aligned}$$

*where*

$$re^{i\alpha} = \frac{\psi^*(G^0_{uu}\phi^2)}{2\psi^* G^0_t}.$$

## 7.3 Singular Jacobian: Simple Quadratic Bifurcation

Here, we assume the eigenpair $u_0$ is real and,

- $G^0_u$ has a one-dimensional null space spanned by say $\phi$, and let $\psi$ span the null space of $G^{0^T}_u$,
- $G^0_t \in \mathcal{R}(G^0_u)$,
- $a \neq 0$ and $b^2 - ac \neq 0$, where

$$\begin{aligned} a &= \psi^T(G^0_{uu}\phi^2), \\ b &= \psi^T(G^0_{uu}\phi\phi_0 + G^0_{ut}\phi), \\ c &= \psi^T(G^0_{uu}\phi_0^2 + 2G^0_{ut}\phi_0), \end{aligned}$$

  and $\phi_0$ is the unique solution of

$$G^0_u\phi_0 = -G^0_t \tag{7.7}$$

orthogonal to $\mathcal{N}(G_u^0)$.

The point $(u_0, t_0)$ having the above properties is called a simple quadratic bifurcation point. In any small neighborhood of $(u_0, t_0)$, there are exactly two distinct branches of solutions passing through the point $(u_0, t_0)$ transcritically. This case is depicted in the middle diagram of Figure 6.1. If $b^2 - ac > 0$, then both branches are real. If $b^2 - ac < 0$, both branches are complex except at the point $(u_0, t_0)$. See Henderson [15] for a more detailed discussion.

The tangent vectors of the two bifurcating branches can be computed and the Euler-Newton continuation can proceed as usual with these new directions. However, in practice, a continuation method usually jumps over a simple quadratic bifurcation point. This is because it is highly unlikely for a numerical step to land exactly at the point.

## 7.4 Singular Jacobian: Cubic Fold Point

Here, we assume the eigenpair $u_0$ is real and,

- $G_u^0$ has a one-dimensional null space spanned by say $\phi$, and let $\psi$ span the null space of $G_u^{0^T}$,
- $G_t^0 \notin \mathcal{R}(G_u^0)$,
- $\psi^T(G_{uu}^0\phi^2) = 0$,
- $\psi^T(G_{uu}^0\phi\phi_1) \neq 0$, where $\phi_1$ is the unique solution of

$$G_u^0\phi_1 = -G_{uu}^0\phi^2 \tag{7.8}$$

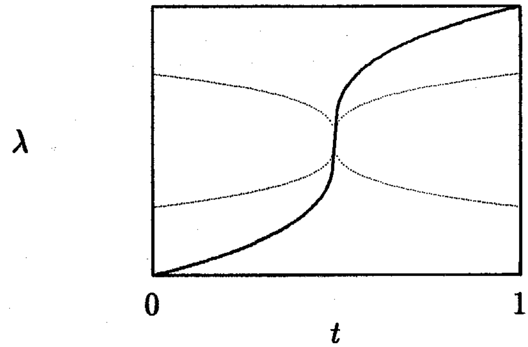


Figure 7.3: Cubic fold point.

orthogonal to $\mathcal{N}(G_u^0)$.

The point $(u_0, t_0)$ having the above properties is called a cubic fold point. There is a unique branch of real solutions near $(u_0, t_0)$ as well as a complex conjugate pair of solutions. See Figure 7.3. Cubic fold points were discussed in Part I of this thesis.

## 7.5 Singular Jacobian: Simple Pitchfork Bifurcation

Here, we assume the eigenpair $u_0$ is real and,

- $G_u^0$ has a one-dimensional null space spanned by say $\phi$, and let $\psi$ span the null space of $G_u^{0^T}$,
- $G_t^0 \in \mathcal{R}(G_u^0)$,
- $\psi^T(G_{uu}^0\phi^2) = 0$,

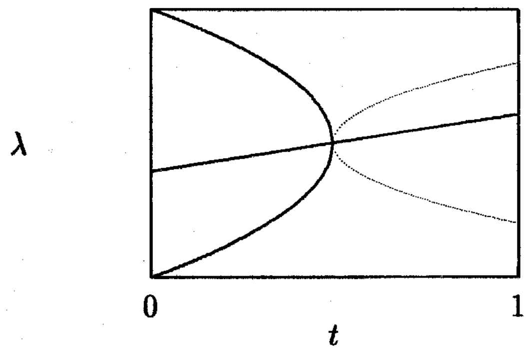


Figure 7.4: Simple Pitchfork Bifurcation.

- $\psi^T(G^0_{uu}\phi\phi_1)\cdot\psi^T(G^0_{uu}\phi_0\phi + G^0_{ut}\phi) \neq 0$, where $\phi_0$ and $\phi_1$ were defined in (7.7) and (7.8).

The point $(u_0, t_0)$ having the above properties is called a simple pitchfork bifurcation point. On one side of the point, there are three real solutions. On the other side, there is one real solution and a complex conjugate eigenpair. The situation is depicted in Figure 7.4. See Henderson [15] for a more detailed discussion.

## 7.6 Generic Singular Jacobians

In the previous sections, we discussed four cases where the Jacobian $G^0_u$ has a one-dimensional null space. It is clear that of all the singular $n \times n$ matrices, those with a one-dimensional null space are generic. Note that for such matrices, the generic case occurs when $u_0$ is real. This is because for a complex $u_0$, its complex conjugate must also be a singular point. Another view is that both the real and imaginary parts of the determinant of $G_u^0$ are zero. For a real $u_0$, there is only one such equation. Of the four cases considered, all but the first are nongeneric because they have nongeneric conditions $\psi^T(G_{uu}^0\phi^2) = 0$ and/or $G_t^0 \in \mathcal{R}(G_u^0)$. The next result characterizes the generic singular Jacobian $G_u^0$.

**Theorem 7.9** *Let $G$ be defined as in Equation 7.2. Suppose for $(u_0, t_0) \in \mathbb{R}^{n+2}$, $G(u_0, t_0) = 0$ and $G_u^0$ is singular with a one-dimensional null space. Let $\phi$ and $\psi$ be spanning vectors for $\mathcal{N}(G_u^0)$ and $\mathcal{N}(G_u^{0^T})$ respectively. Then $\psi^T(G_{uu}^0\phi^2) \neq 0$ iff $\lambda_0$ is an eigenvalue of $A^0 \equiv A(t_0)$ of algebraic multiplicity two and geometric multiplicity one.*

**Proof:** From (7.2), we obtain

$$G_u^0 = \begin{bmatrix} A^0 - \lambda_0 I & -x_0 \\ x_0^T & 0 \end{bmatrix} \in \mathbb{R}^{n+1 \times n+1}.$$

Partition the null vectors as:

$$\phi = \begin{bmatrix} h \\ \nu \end{bmatrix}, \quad \psi^T = [p^T, \mu],$$

where $h, p \in \mathbb{R}^n$ and $\nu, \mu \in \mathbb{R}$. By a direct calculation, we get

$$\psi^T(G_{uu}^0\phi^2) = -2\nu p^T h + \mu \sum_{i=1}^{n} h_i^2. \tag{7.10}$$

($h_i$ is the $i^{th}$ component of $h$.) We rewrite the equation $\psi^T G_u^0 = 0$, using the definitions of $\psi^T$ and $G_u^0$, as

$$[p^T(A^0 - \lambda_0 I) + \mu x_0^T, \ -p^T x_0] = 0. \tag{7.11}$$

Taking the dot product of the first $n$ components of the above vector with $x_0$, we obtain

$$p^T(A^0 - \lambda_0 I)x_0 + \mu x_0^T x_0 = 0.$$

Therefore,

$$\mu = 0. \tag{7.12}$$

The following two cases are the only possible cases in which dim $\mathcal{N}(G_u^0) = 1$.

- CASE 1: $\lambda_0$ is an eigenvalue of $A^0$ with algebraic multiplicity $m \geq 2$ and geometric multiplicity 1. Let

$$J \equiv Q^{-1}(A^0 - \lambda_0 I)Q = \left[\begin{array}{cccc|c} 0 & 1 & & & \\ & 0 & \ddots & & \\ & & \ddots & 1 & \\ & & & 0 & \\ \hline & & & & J_2 \end{array}\right] \tag{7.13}$$

be a Jordan form of $A^0 - \lambda_0 I$ where $J_2$ is nonsingular of dimension $n-m$ and the $x_0$ is the first column of the matrix $Q$ of principal eigenvectors. Note that $G_u^0$ is similar to:

$$\left[\begin{array}{cc} J & -e_1 \\ x^T Q & 0 \end{array}\right].$$

Now from (7.11) and (7.12), we have

$$\begin{aligned} 0 &= p^T(A^0 - \lambda_0 I) \\ &= p^T QJQ^{-1}. \end{aligned}$$

Let $y^T = p^T Q$. Then

$$y^T J = 0.$$

Thus from (7.13), we can take $y^T$ to be $e_m^T$.

From

$$G_u^0 \begin{bmatrix} h \\ \nu \end{bmatrix} = 0,$$

we get,

$$(A^0 - \lambda_0 I)h = \nu x_0. \tag{7.14}$$

Using (7.13) in above, we obtain

$$QJQ^{-1}h = \nu x_0$$

which implies

$$Jw = \nu Q^{-1}x_0 = \nu e_1,$$

where $w = Q^{-1}h$. From (7.13), we obtain the unique solution $w = \nu e_2$. Hence $y^T w = \nu\delta_{m2}$. Finally, from (7.10),

$$\begin{aligned} \psi^T(G_{uu}^0\phi^2) &= -2\nu(p^T Q)(Q^{-1}h) \\ &= -2\nu y^T w \\ &= -2\nu^2\delta_{m2}. \end{aligned}$$

Note that $\nu \neq 0$ since otherwise $w = 0$ which imples $h = 0$. We have reached a contradiction that $\phi$ is the zero vector. Hence $\psi^T(G_{uu}^0\phi^2)$ is nonzero iff $m = 2$.

- CASE 2: $\lambda_0$ is an eigenvalue of $A^0$ with algebraic multiplicity two and geometric multiplicity two. Let

$$J \equiv Q^{-1}(A^0 - \lambda_0 I)Q = \left[\begin{array}{cc|c} 0 & 0 & \\ 0 & 0 & \\ \hline & & J_2 \end{array}\right] \tag{7.15}$$

be a Jordan form of $A^0 - \lambda_0 I$ where $J_2$ is nonsingular and of dimension $n-2$ and $x_0$ is the first column of the matrix $Q$ of principal eigenvectors. As before, we have from (7.14),

$$Jw = \nu e_1,$$

where $w = Q^{-1}h$. From the form of $J$, it is clear that $\nu = 0$. Hence

$$\psi^T(G^0_{uu}\phi^2) = -2\nu p^T h = 0.$$

We have established the claim of the theorem. □

This theorem is a minor variation of a result originally due to H. B. Keller. A similar result also appears in Li, Zong and Cong [25].

The fact that the generic case of of a singular $G^0_u$ occurs when $\lambda_0$ is an eigenvalue of $A^0$ of algebraic multiplicity two and geometric multiplicity one may seem surprising. We now attempt to give an intuitive explanation. Let $X$ be the set of $n \times n$ matrices which have $\lambda_0$ as an eigenvalue of algebraic multiplicity two. Suppose $A$ is a member of $X$. Now $A - \lambda_0 I$ can be similarly transformed to one of:

$$\left[\begin{array}{cc|c} 0 & 1 & \\ 0 & 0 & \\ \hline & & J_1 \end{array}\right] \quad \text{and} \quad \left[\begin{array}{cc|c} 0 & 0 & \\ 0 & 0 & \\ \hline & & J_2 \end{array}\right],$$

where $J_1$ and $J_2$ are some nonsingular matricies. The rank of the left and right matrices are $n-1$ and $n-2$, respectively. Hence in the space $X$, the matrix $A - \lambda_0 I$ with geometric multiplicity one (i.e., similar to the left matrix) is generic.

We can infer from (7.13) (with $m = 2$) that at simple quadratic folds and simple quadratic bifurcation points, the eigenvalue has algebraic multiplicity

| | $\psi^* G_t^0 \neq 0$ | $\psi^* G_t^0 = 0$ |
|---|---|---|
| $\psi^* G_{uu}^0 \phi^2 \neq 0$ | simple quadratic fold | simple quadratic bifurcation |
| $\psi^* G_{uu}^0 \phi^2 = 0$ | simple cubic fold | simple pitchfork bifurcation |

Table 7.1: Summary of some of the different types of points at a singular Jacobian $G_u^0$.

two and geometric multiplicity one. At both cubic fold and simple pitchfork bifurcation point, the algebraic and geometric multiplicities are three and one, respectively. See Table 7.1. The Jacobian $G_u^0$ of course may have other types of nongeneric singularities. For example, the eigenvalue may have multiplicities three and two respectively. Note that in our enumeration of the different types of singular Jacobians with a one-dimensional null space, the case when both the algebraic and geometric multiplicities are two was excluded. Here, there is no bifurcation because there are two linearly independent eigenvectors having the same eigenvalue at the singular point.

The significance of the above theorem is that in practice, we only encounter simple quadratic folds and this is the route by which real eigenpaths become complex.

## 7.7 A Bound on the Number of Bifurcation Points

It is not difficult to show that at a real or complex bifurcation point of (7.2), the algebraic multiplicity of the eigenvalue of $A(t)$ is at least two. Let

$$p(t,\lambda) \equiv \det(A(t) - \lambda I).$$

Since $A(t)$ is linear in $t$, the above is a polynomial in $(t,\lambda)$ of degree $n$. It can be written in the form

$$p(t,\lambda) = a_0(t) + a_1(t)\lambda + \cdots + a_n(t)\lambda^n, \tag{7.16}$$

where $a_i(t)$ is a polynomial in $t$ of degree at most $n-i$ for $i = 0,\ldots,n$ and $a_n(t) = (-1)^n$. Define

$$q(t,\lambda) = \frac{\partial p(t,\lambda)}{\partial \lambda}.$$

From (7.16), it is easy to show that $q$ is a polynomial of degree $n-1$. At a bifurcation point $(t,\lambda)$, we must have

$$p(t,\lambda) = q(t,\lambda) = 0.$$

This is a system of two polynomial equations of degrees $n$ and $n-1$ in two variables. By Bézout's theorem, it has either at most $n(n-1)$ roots or it has a continuum of roots. The latter case can be shown to be impossible by examining the discriminant of $p$. See Li, Zong and Cong [25]. Hence the eigenpaths collectively can have at most $n(n-1)$ bifurcation points.

We remark that some of these roots may have a complex time $t$ and that some roots may lie outside the region of interest (i.e., $t \in [0,1]$). In practice we see on the order of $n$ bifurcation points.

# Chapter 8

# Numerical Algorithm

In this chapter, we describe the numerical implementation of the homotopy algorithm including choice of the initial matrix $A_0$, stepsize selection and the method of solution of the linear systems which arise in the Newton iterations. We will also discuss the transition from real to complex eigenpairs and vice versa. For a more thorough treatment of some of these topics, see Keller [18].

Let the symbols $\Re$ and $\Im$ represent the real and imaginary parts of a quantity. In our implementation of the homotopy algorithm, instead of using the normalization $(x^T x - 1)/2 = 0$, we use:

$$n(x) \equiv \begin{bmatrix} (x^* x - 1)/2 \\ \Im x_J \end{bmatrix} = 0, \tag{8.1}$$

where $x_J$ is the $J^{th}$ component of $x$ and is required to be nonzero. Now writing the eigenvalue equation $A(t)x - \lambda x = 0$ as an equivalent system of $2n$ real equations plus the two real equations from (8.1), there are $2n+2$ real equations in $n+1$ complex variables $(x, \lambda)$ or $2n+2$ real variables. (Note that $t$ is a real parameter and is not counted as a variable.) Another view of why we need two equations in (8.1) is that any complex constant of magnitude one times a normalized eigenvector remains a normalized eigenvector. The normalization (8.1) is not analytic as complex equations but their real forms are infinitely differentiable. In practice we choose $J$ so that $|\Re x_J| \geq |\Re x_i|$ for all $i = 1, \ldots, n$. Li, Zong and Cong [25] use an alternate normalization $c^*x - 1 = 0$ where $c$ is some random vector.

## 8.1 Choice of Initial Matrix $A_0$

We first rescale the given $n \times n$ matrix $A_1$ so that its largest element (in absolute value) has magnitude one. Using Gerschgorin's theorem, we obtain a number $r$ such that all the eigenvalues of $A_1$ must lie within a circle of radius $r$ in the complex plane. We choose $A_0$ to be a diagonal matrix whose elements are the diagonal elements of $A_1$ possibly perturbed so that

$$|A_0(i,i) - A_0(j,j)| > \frac{2r}{n^{3/2}}, \quad i \neq j, \tag{8.2}$$

where $A_0(i,j)$ denotes the $(i,j)$ entry of $A_0$. More precisely, we define $A_0(1,1)$ to be $A_1(1,1)$ and for $i > 1$, we take $A_0(i,i)$ to be $A_1(i,i) + \delta$ where $\delta$ is the smallest number in magnitude which makes $A_0(i,i)$ satisfy (8.2) for all $j < i$. This $A_0$ has unique eigenvalues and the eigenvectors are just the standard unit vectors. With this choice of $A_0$, the initial normalized tangents are unique and easily computed. There is no theoretical justification for the bound in ( 8.2). From numerical experiments on random matrices, the spectrum of $A_0$ with this particular bound seems to be reasonably distributed.

Ideally, $A_0$ should be chosen so that the number of real and complex bifurcation points be minimized. This is because there is extra work involved in locating real fold points. In the example shown in Figure 6.2, by simply reordering the diagonal elements of $A_0$, it is possible for the eigenpaths to have just 3 real fold points. This is the minimum possible because this $A_1$ has six complex eigenvalues. Another desirable property of $A_0$ is that the eigenpaths be well-separated. This decreases the chance of the path-jumping phenomenon. However, it seems extremely difficult to choose a priori an initial matrix which has both the above properties.

## 8.2 Transition at Real Fold Points

The homotopy starts off from $A_0$ and advances using the Euler-Newton continuation, solving the problem in real space. When it detects that it is going backwards in time, then a real fold has been passed. By the theory of the last chapter, there must be a complex conjugate pair of solutions on the opposite of the real fold. We first get a more accurate location of the fold point by using the regular-falsi method to approximate the point at which $dt/ds = 0$. (Recall that this is a necessary condition at a fold point.) With the augmented system, the Jacobian (7.4) is nonsingular so there is no numerical difficulty in the task. We store the location of this fold point in a table for later reference. Using the tangent vector $\phi$ at the fold point, we solve the problem (7.2) in complex space at a later time. This is done by carrying out the Euler-Newton continuation with the initial tangent $i\phi$, in accordance with the theory of Henderson and Keller.

When the partner of the above path comes from the other arm of the same fold, it checks that the fold point has been visited before and it stops further computation. This way, only one path of a complex conjugate pair of eigenpaths is computed.

The reverse of the above situation also arises, although less frequently. That is, time is decreasing when we are advancing along a complex path. Generically, there must be a real fold on the opposite side of this complex path. It can be shown that the Jacobian of the system (7.2), with normalization (8.1), is singular at the (real) fold point. Hence it is not wise to approximate it in the same manner as described in the first paragraph of this section. Our procedure is to choose from the last two computed points, the one that is closer to the fold point (i.e., the one whose eigenvalue has the smallest imaginary part in magnitude). From this point, we take a small step and solve the problem (7.2) in real space. Suppose the new real eigenpair is $u_1$. We then apply the Euler-Newton continuation in both the directions $\dot{u}_1$ and $-\dot{u}_1$. See Figure 8.1. Because the problem is being solved in real space, there is no chance of converging back to the complex solution. On a parallel computer, the node which became idle at a fold point can be invoked to carry out the computation along one of these directions.

When a fold point is approached from the complex side, it is also possible that the continuation step converges to the real solution on the other side of the fold point (instead of turning around the fold and remaining complex). In this case, we carry on as explained in the previous paragraph except that we no longer need to take the first step since the eigenpair is already real.

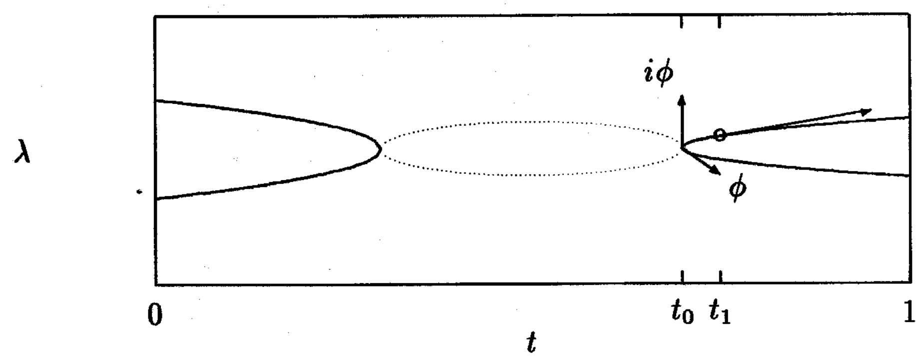


Figure 8.1: Transition from a complex solution to a real solution at a fold point. Dotted lines denote complex solutions.

## 8.3 Computing the Tangent

Suppose two eigenpairs $u_0$ and $u_1$ have been found. We wish to compute the tangent vector at $t_1$. In the formulation (7.3), we have:

$$F_u^1 \dot{u}_1 + F_t^1 \dot{t}_1 = 0,$$

where the superscript [1] denotes evaluation of the Jacobian at $(u_1, t_1)$ and dot denotes $s$ derivative. For a unit tangent, we require in addition:

$$\dot{u}_1^* \dot{u}_1 + \dot{t}_1^2 = 1. \tag{8.3}$$

Note that the above two equations define the tangent up to a sign. To ensure that we are always computing in the same direction, we further impose the condition,

$$\Re(\dot{u}_0^* \dot{u}_1) + \dot{t}_0 \dot{t}_1 > 0.$$

Because (8.3) is nonlinear, we solve instead the system (when $u_0$ is real):

$$\begin{bmatrix} F_u^1 & F_t^1 \\ \dot{u}_0^T & \dot{t}_0 \end{bmatrix} \begin{bmatrix} \phi \\ \alpha \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The tangent $(\dot{u}_1, \dot{t}_1)$ is obtained by normalizing the solution of the above system.

## 8.4 Solution of Nonsymmetric Linear Systems

To compute the tangent vector and in each iteration of the Newton method to find the eigenpairs, a nonsymmetric linear system must be solved. A direct solver is out of the question because the system is large. Unfortunately, iterative solution of nonsymmetric systems is still in the research stage and there are no fast, robust iterative solvers (except for special linear systems). In our implementation, the conjugate gradient method is used to solve the normal equations. That is, to solve $Ax = b$, we apply the conjugate gradient method to the symmetric system $A^T Ax = A^T b$. This has the disadvantages of requiring $A^T$, which may not be readily available and the condition number of the new system is the square of the original one.

## 8.5 Selection of Stepsize

Suppose we have the two eigenpairs $u_0$ and $u_1$. We obtain stepsize $\delta s_2$ for $u_2$ as follows:

$$\delta s_2 = \delta s_1 \; (\Re(\dot{u}_0^* \dot{u}_1) + \dot{t}_0 \dot{t}_1 + .5),$$

where $\delta s_1$ is the stepsize used to obtain $u_1$. The idea is that when the two previous tangents are parallel, then we increase the stepsize by 50%. If the tangents are perpendicular, we decrease the stepsize by a half. We use the above scheme until the time is close to one, when we solve the system $G(u, 1) = 0$.

Whenever a Newton iteration fails to converge after say 8 iterations, we restart it with a stepsize that is one-half of the original one.

## 8.6 Path-jumping

Path-jumping is a serious problem for the homotopy method. This is the phenomenon where the Newton iteration converges to another eigenpath. This occurs when the stepsize is overly ambitious or the linear system involved in the solution of a Newton iterate has a large condition number.

An elegant method of detecting path-jumping is available when the matrix is symmetric tridiagonal with nonzero off-diagonal elements (Li and Rhee [24]). They employ the Sturm sequence property of symmetric matrices. However no satisfactory procedure is known for general matrices. One inefficient way is to use the property that the sum of the eigenvalues of a matrix is equal to the trace of the matrix. Noting that

$$Tr(A(t)) = Tr(A_0) + t(Tr(A_1 - A_0)),$$

almost all path-jumps can be detected by summing the computed eigenvalues and comparing to the above expression for the trace of $A(t)$. However this does not tell us which path has jumped and hence it is necessary to recompute the last step for all eigenpaths. Another drawback is that the computation of the eigenpaths must be synchronized.

We resort to some safeguarding mechanisms to avoid path-jumping. If the change in time and the magnitude of the change in eigenvalue of the newly converged eigenpair is greater than twice the corresponding average changes of the past eigenpairs then we discard the new eigenpair and restart the Newton iteration with one-half the original stepsize.

## 8.7 Parallel Aspects

The homotopy method is fully parallel because each eigenpath can be computed independently of the others. If the sparse matrix $A(t)$ can be stored in each node, then there is no communication overhead at all other than the trivial broadcast of the location of a fold point.

## 8.8 Homotopy Algorithm of Li et al.

Li, Zong and Cong [25] use a different strategy in their homotopy algorithm. They first use Householder transformations to reduce the given matrix to a similar matrix $A_1$ in upper Hessenberg form. Their initial matrix $A_0$ is the same as $A_1$ except one subdiagonal entry is set to zero. They use a divide-and-conquer strategy to obtain the eigenpairs of $A_0$. Because $A_0$ is very close to $A_1$, the eigenpaths will be nearly straight and path–jumping is much less of a problem here. The performance of this method is very encouraging. However, it requires storage of the entire matrix in addition to work storage to find the eigenvalues of $A_0$. For another approach to finding the eigenvalues using homotopy, see Lenard [23].

## 8.9 Numerical Results

We have implemented the homotopy on a SUN Sparc 2 workstation. It runs extremely slowly for large matrices. For a 50 by 50 matrices, the execution times are about five hours. (If we replace the conjugate gradient solver by gaussian elimination with partial pivoting, it takes about 30 minutes.) This is not too surprising if we examine the time complexity of this algorithm. Each of the $n$ eigenpaths requires $O(n^3)$ operations for one Newton iteration. If we assume that the number of Newton iterations and the number of continuation (time) steps are constant, then this is a $O(n^4)$ algorithm. In contrast, the QR algorithm is $O(n^3)$. In practice, the Newton iterations converge after two to four iterations. To minimize the chance of path-jumping, we take rather small stepsizes. The number of continuation steps is a growing function of the size of the matrix. Typical numbers are 10, 15 and 25 for 4 by 4, 10 by 10 and 50 by 50 matrices respectively. Path-jumping occurs up to five times for the random 50 by 50 matrices which we have tested. This results in missing two or three eigenvalues in the computation. We have also tested some 40 by 40 matrices arising from the discretization of reaction-diffusion equations with varying degrees of success. In Figure 8.2, the eigenvalues are well-separated and the homotopy method encountered no difficulty in computing all the eigenpairs.

One nice feature of the homotopy method is that it is able to handle defective (i.e., nondiagonalizable) matrices. This is because even though the

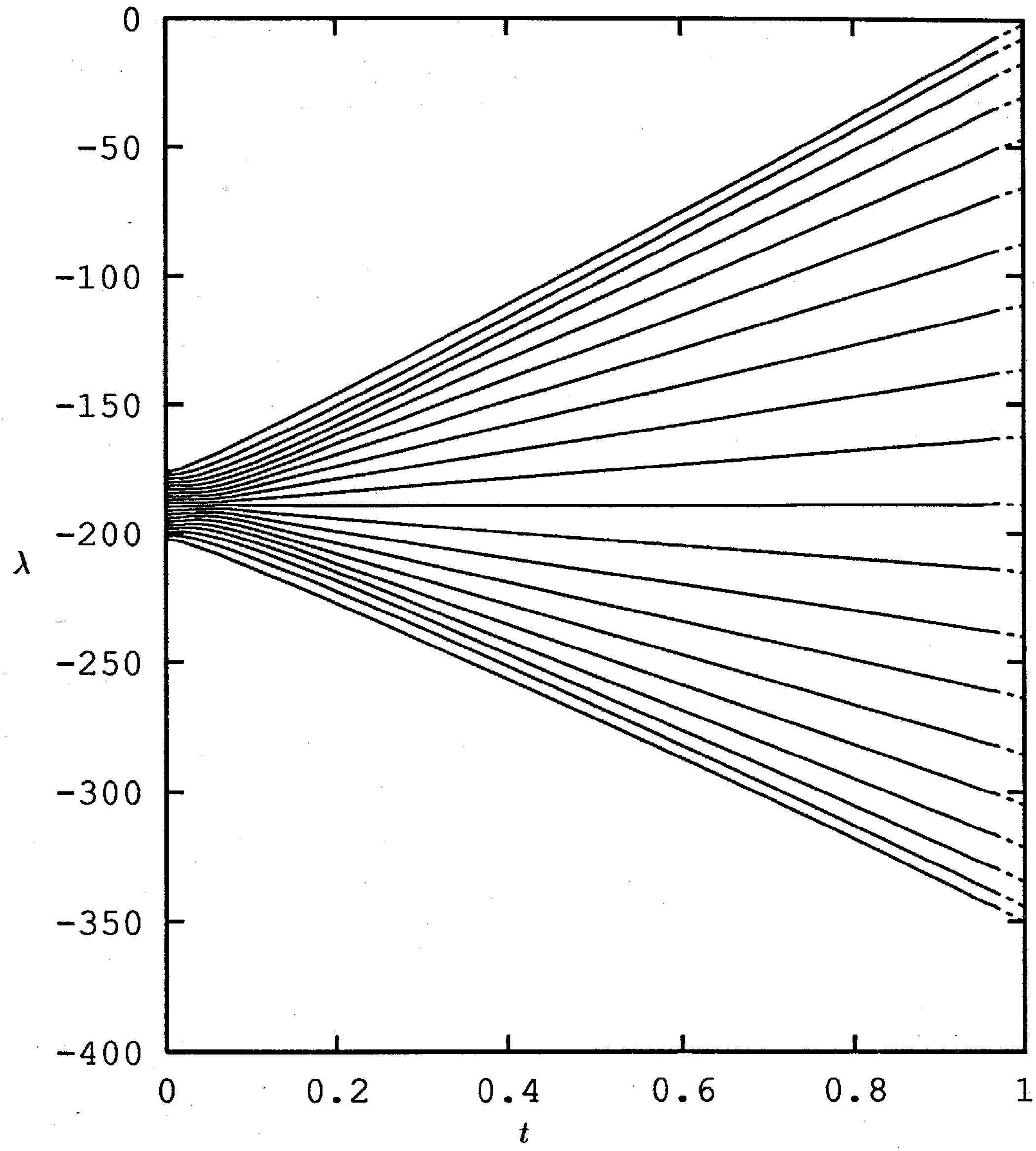


Figure 8.2: Eigenpaths of a 40 by 40 matrix arising from the discretization of a reaction–diffusion equation. Only one path of a complex conjugate pair of eigenpaths is shown.

given matrix is defective, the matrix $A(t)$ is almost always diagonalizable right up to $t = 1$.

Empirically, we notice that the eigenpaths move in a relatively simple fashion as $t$ progresses. That is, there are no wild oscillations and the paths in general maintain their order where they are ordered according to $\Re\lambda$. Thus homotopy method has the potential to efficiently find an eigenvalue with the largest real part. Such eigenvalues are of interest in linear stability theory for partial differential equations. For a sample of twenty 50 by 50 matrices, we follow the two eigenpaths whose initial eigenvalues are largest. In 16 of those cases, the correct eigenvalue was found. We have not been able to devise a mechanism to guarantee that an eigenpath will end up (at $t = 1$) having an eigenvalue with the largest real part.

We have also tried the other normalization $(x^T x - 1)/2 = 0$ and the results are similar.

# Chapter 9

# Conclusion

We have presented a homotopy method to compute the eigenpairs of a real matrix. Starting with a matrix that has real and distinct eigenvalues, Euler-Newton continuation is used to advance the eigenpaths. An eigenpath will remain real unless it encounters a real fold point. On the opposite side of this fold point, two complex conjugate eigenpairs emerge. The reverse situation where two complex conjugate eigenpairs meeting at a real fold point with two real paths bifurcating to the right also occurs. By restricting the solutions in the real space we have shown how to deal with these transitions without numerical difficulties.

For a large, dense $n$ by $n$ matrix, this algorithm has time complexity of $O(n^4)$ and hence it is prohibitively slow. The storage requirement is proportional to the number of nonzero elements of the matrix and thus it is attractive for a sparse matrix. The algorithm seems to work well even if the original matrix has multiple eigenvalues and even defective eigenvectors. This together with the fully parallel nature of the algorithm may make it a competitive method for the large sparse nonsymmetric eigenvalue problem.

However several obstacles must be overcome first. We have already mentioned the path-jumping problem and the inadequacies of our iterative linear solver. Another open problem is to come up with a good initial matrix for the homotopy which would minimize the number of bifurcation points and keep the eigenpaths well-separated. Finally, we would like to determine selected eigenvalues (for example those with the largest real part) by just following one or two eigenpaths.

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