THE ANALYSIS OF DISCONTINUITIES IN CYLINDRICAL TUBES PROPAGATING SOUND WAVES

Thesis by

John W. Miles

In Partial Fulfillment of the Requirements
for the Degree of Aeronautical Engineer

California Institute of Technology
Pasadena, California
September 25, 1944

Acknowledgments

The author wishes to take this opportunity to thank Dr. H. J. Stewart for his guidance and criticism in the preparation of this paper and Miss Barbara Daggett for her careful preparation of the manuscript.

The variational principle which occurs in the analysis was first applied to the analysis of similar problems in the propagation of electromagnetic waves by Dr. Julian Schwinger of the Radiation Laboratory at the Massachusetts Institute of Technology.

Table of Contents

ABSTRACT

The effect of a plane discontinuity on a plane wave propagated in a cylindrical tube is calculated by variational methods. In carrying out the calculations, a transmission line analogy is used and the effect of the discontinuity at a distance is represented by a capacitance placed at the discontinuity.

In section 1 the equations of motion for the propagation of a small disturbance in a cylindrical tube are established, culminating in the two-dimensional wave equation, while the solutions to the equations are discussed in section 2. These solutions constitute an infinite set of modes, in addition to the plane wave usually treated in the literature. In section 3 the analogy between propagation of sound and an electrical transmission line is established, and it is shown that each mode requires a separate transmission line. In section 4 it is shown that the effect of the higher modes excited by a plane discontinuity may be represented by a lumped capacitance, and this capacitance is given by a variational expression which gives a systematic method of calculation yielding an upper bound to the true answer. For the case of a window, a variational principle is produced which gives a lower bound to the true answer. In section 6 this method is applied to a window in a rectangular tube in some detail, in

section 7 it is applied to a change of cross section in a rectangular tube, and in section 8 it is applied to windows and changes of cross section in circular tubes. In section 9 the reflection and transmission factors are calculated, and in section 10 the resonant frequency of certain types of resonators is calculated.

To the author's knowledge this is the first rigorous treatment of the above problems with the exception of an earlier paper calculating the reflection due to a change of cross section in a circular tube.[21]

1. The Equations of Motion

If we assume a fluid medium to be sufficiently dense so that any differential element of length considered is large relative to the mean free path of the molecules and to be free of viscosity, Newton's law may be written

$$(\rho_0+\rho)\left[\frac{\partial\bar{q}}{\partial t}+\bar{q}\cdot\nabla\bar{q}\right]=-\nabla(p_0+p) \qquad (1)$$

where ρ_0 and p_0 are the mean mass densities and pressure in the regions under consideration, respectively, and p are the deviations from these means, and $\bar{q}$ is the vector velocity at any point in the fluid. The equation of continuity demands

$$\frac{\partial}{\partial t}(\rho_0+\rho)+\nabla\cdot\left[(\rho+\rho_0)\bar{q}\right]=0 \qquad (2)$$

If we restrict our treatment to small disturbances such that second order quantities are neglected, (1) and (2) simplify to

$$\rho_0\frac{\partial\bar{q}}{\partial t}=-\nabla p \qquad (3)$$

$$\frac{\partial\rho}{\partial t}+\rho_0\nabla\cdot\bar{q}=0 \qquad (4)$$

If we make the usual assumption that the fluctuations of pressure and density are so rapid that the compression is essentially adiabatic, we may write

$$(p_0+p)(\rho_0+\rho)^{-\gamma}=\text{CONST} \qquad (5)$$

where γ is the ratio of the specific heats Cp/Cv evaluated at p_o and ρ_o. Differentiating (5) and evaluating the constant at p_o and ρ_o, we obtain

$$p = c^2 \rho \tag{6}$$

$$c^2 = \frac{Dp}{D\rho} = \gamma \frac{p_o}{\rho_o} \tag{7}$$

where second order quantities have been neglected. Finally we take the divergence of both sides of (3), interchange the operations of time and space differentiation, substitute the divergence of $\bar{q}$ from (4), and substitute ρ from (6) to obtain

$$\nabla^2 p = \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} \tag{8}$$

which is the well known scalar wave (Helmholtz) equation. We observe that c is the velocity of propagation for the disturbance; in the standard atmosphere (20°C, 760 mm Hg, $\gamma = 1.4$ $\rho_o = 0.00122$ gm-cc^{-1}) (7) yields c = 344 m-sec^{-1} which is in close agreement with the observed velocity in the normal acoustical range (where p and ρ are actually small compared with p_o and ρ_o) thus justifying the assumption of adiabatic compression.

If we now restrict ourselves to the harmonic time dependence $e^{j\omega t}$* (3) becomes

$$\bar{q} = j(\omega \rho_o)^{-1} \nabla p \tag{9}$$

and (8) becomes

$$\nabla^2 p + K^2 p = 0 \quad , \quad K \equiv \frac{\omega}{c} = \frac{2\pi}{\lambda} \tag{10}$$

*In much of the literature the time variation is taken as $e^{-i\omega t}$ in order to facilitate the discussion of wave propagation, but $e^{j\omega t}$ is used here to facilitate the application of impedance concepts.

where K is the wave number, and λ is the wave length. Hence the velocity is derivable from a scalar potential satisfying (10).

We restrict ourselves further to a cylindrical region described by the orthogonal coordinates (u, v, z). Separating out the z portion of the Laplacian operator and then separating out the z dependence of the pressure in the usual manner, we have

$$p(u,v,z) = \phi(u,v)\, e^{\mp jhz} \tag{11}$$

$$\nabla^2 \phi(u,v) + \varkappa^2 \phi(u,v) = 0 \tag{12}$$

$$\varkappa^2 = k^2 - h^2 \tag{13}$$

where the choice of a negative or positive exponent in (11) corresponds to a disturbance propagating in the directions of increasing and decreasing z, respectively. Our problem is now reduced to determining solutions to the two-dimensional equation (12) over a plane surface bounded by the intersection with the particular cylindrical surface being considered and subject there to the boundary condition

$$\nabla \phi \cdot \bar{n} = \frac{\partial \phi}{\partial n} = 0 \tag{14}$$

where $\bar{n}$ is the unit vector normal to the bounding surface. Physically (14) is imposed by the condition that the velocity have no component normal to a fixed surface at that surface.

2. The Solutions to 1 (12)

We now turn our attention to the solutions of 1 (12). It is well known [1] that there will be a doubly infinite set of solutions ϕ_{mn} ("eigenfunctions") corresponding to the set of "eigenvalues" χ_{mn}, the latter being determined by the boundary condition 1 (14). These solutions are easily shown to be orthogonal to integration over the surface in a plane of constant z; to prove this property, we use Green's second identity [2]

$$\iint_S \left[\phi_i \frac{\partial \phi_j}{\partial n} - \phi_j \frac{\partial \phi_i}{\partial n}\right] dS = \iiint_V \left[\phi_i \nabla^2 \phi_j - \phi_j \nabla^2 \phi_i\right] dv \qquad (1)$$

which, for our purposes, may be written

$$\oint \left[\phi_i \frac{\partial \phi_j}{\partial n} - \phi_j \frac{\partial \phi_i}{\partial n}\right] dl = \iint_S \left[\phi_i \nabla^2 \phi_j - \phi_j \nabla^2 \phi_i\right] dS \qquad (2)$$

To prove (2), we consider the volume enclosed between two planes of constant z which are a distance e apart and the cylindrical surface; then if we let e approach zero dS in (1) may be replaced by edl where dl is a line element in the limiting contour since the surface elements in the planes of constant z mutually cancel, (provided, of course, that the surface between the two elements does not contain a discontinuity in ϕ_i or ϕ_j); similarly dv in (1) may be replaced by edS where dS is a surface element in the plane of constant z. Hence the surface and volume integrals in (1) may be replaced by line and surface integrals, respectively, for a surface formed by the intersection of a plane of constant z with a cylindrical surface. If ϕ_i and ϕ_j are now taken to be solutions to 1 (12), the line integral in (2) vanishes identically because of the boundary condition 1 (14), and substituting the Laplacians from 1 (12) in the surface integral we obtain

$$(\varkappa_i^2 - \varkappa_j^2) \iint_S \phi_i \phi_j \, dS = 0 \tag{3}$$

which proves the orthogonality. If i = j it is clear that the only way in which the surface integral in (3) can vanish is to pick the trivial solution $\phi_i = 0$, inasmuch as the integrand is then positive definite. For convenience we normalize the solutions ϕ_{mn} such that

$$\iint_S \phi_{mn} \phi_{pq} \, dS = \delta_p^m \delta_q^n \tag{4}$$

Hence the set ϕ_{mn} is orthonormal. The proof of uniqueness is complicated by the possibility of free oscillations, but it is physically clear that such oscillations must eventually die out. This matter, together with the explicit solutions, has been discussed extensively in the literature (3, 4, 5 and 6), and no attempt will be made to give a proof of uniqueness herein, as it may be considered physically obvious in the face of small but finite attenuation.

If we make the choice $\varkappa = 0$ in 1 (7), we have Laplace's equation, and the only (non-trivial) solution compatible with 1 (14) and (4) is

$$\phi_0 = S^{-1/2} \tag{5}$$

where S is the cross-sectional area of the cylinder. It is obvious that (5) satisfies (4); to show it is the only solution to Laplace's equation compatible with 1 (14), we write Green's first identity [7] as

$$\oint \phi_i \frac{\partial \phi_j}{\partial n}\, d\ell = \iint_S \left[(\nabla \phi_i)\cdot(\nabla \phi_j) + \phi_i \nabla^2 \phi_j \right] dS \qquad (6)$$

which is obtained from the more usual form exactly as in the case of the second identity of (1) and (2). If we let $\phi_i = \phi_j$ be a solution to Laplace's equation satisfying 1 (14), (6) becomes

$$\iint_S (\nabla \phi_i)^2\, dS = 0 \qquad (7)$$

Since the integrand of (7) is positive definite it must vanish identically, and ϕ is a constant.

The solution ϕ_0 is the most important of all the eigenfunctions and is the principal wave corresponding to the usual plane sound wave consisting of alternate longitudinal condensations and rarefactions of the medium. In order to appreciate the importance of the principal wave or mode we investigate further the physical properties of the other modes. From 1 (13), we observe that the propagation constant h_{mn} of the ϕ_{mn} mode is imaginary if γ_{mn} is greater than K; accordingly, there are only a finite number of modes in any set (corresponding to a given cylinder) which are capable of transporting energy, for an imaginary value of the propagation constant implies attenuation. In this respect we remark that the sign of the exponent must be chosen to ensure attenuation, i.e. waves which vanish at infinity are those corresponding to physical reality, and that the attenuation involves no dissipation of energy since the pressure and velocity are 90^o out of phase (cf. 1 (9)). For any given mode we observe that the propagation constant is imaginary for all frequencies below the "cutoff" frequency

$$\omega_{mn} = c\,\varkappa_{mn} \tag{14}$$

as given by 1 (13). The phase velocity of any mode is given by

$$V_{mn} = \frac{\omega}{h_{mn}} = \frac{Kc}{h_{mn}} = \omega(\omega^2 - \omega_{mn}^2)^{-1/2} c \tag{15}$$

which (when real) is always greater than or equal to the velocity c. The velocity with which the signal propagates ("group" velocity) obviously is a function of frequency and is at best a nebulous concept if the signal is not monochromatic. Suppose, however, that the signal consists of a group of waves with wave numbers h distributed continuously from $h_o - e$ to $h_o - e$, the contribution to the signal from the interval dh at h being A (h). The resultant signal intensity at z is

$$S = \int_{h_o - e}^{h_o + e} A(h)\, e^{j(\omega t - hz)}\, dh \tag{16}$$

If we define the signal amplitude as

$$C = \int_{h_o - e}^{h_o + e} A(h)\, e^{j[(\omega - \omega_o)t - (h - h_o)z]}\, dh \tag{17}$$

we may write

$$S = C\, e^{j(\omega_o t - h_o z)} \tag{18}$$

The signal velocity is the time derivative of z and must make C a constant. If e is small the theorem of the mean gives

$$C = 2e\, A(\bar{h})\, e^{j[(\bar{\omega} - \omega_o)t - (\bar{h} - h_o)z]} \tag{19a}$$

$$(h_o - e) \leq \bar{h} \leq h_o + e \tag{19b}$$

Assuming C to be constant and differentiating, we obtain

$$V_s = \frac{dz}{dt} = \frac{\bar{\omega} - \omega_0}{\bar{h} - h_0} \tag{20}$$

where Vs is the signal velocity. If e is an infinitesimal we have

$$V_s = \left(\frac{\partial \omega}{\partial h}\right)_{h=h_0} \tag{21}$$

Evaluating (21) from (15) we obtain

$$V_s = c^2/V_{mn} \tag{22}$$

Referring to (15) we observe that for high frequencies both Vs and Vmn are asymptotic to c, but as ω approaches the cutoff frequency ω mn the phase velocity Vmn approaches infinity, while the signal velocity Vs approaches zero, and below the cutoff frequency Vmn is imaginary (signifying attenuation), while the signal velocity has, of course, no meaning.

We have seen that when a signal is not monochromatic the various components travel with different velocities and phase distortion takes place, the amount of distortion being proportional to the distance traversed along the z axis. Moreover, even a monochromatic signal undergoes phase distortion if more than one mode is allowed to propagate since the different modes have different phase velocities for any given frequency. In practice the attenuation is also a function of the frequency and mode, so that frequency distortion may also be expected. We conclude that the principal mode is unique in being freely propagated for all frequencies with a velocity which is identically equal to c; i.e., the principal mode is independent of the geometry of the cylinder in which it propagates and behaves as if in an infinite medium. In practice only the principal mode should be allowed to propagate (which may be effected by choosing the dimensions of the tube sufficiently small compared to the wavelength), but the other modes must generally be excited at a discontinuity in order to satisfy the boundary condition 1 (14).

3. The Transmission Line Analogy

It is fruitful to compare the transmission of sound in a tube with the propagation of an electric wave along a transmission line. To this end we consider first the principal mode. Since ϕ_0 is constant the gradient of the pressure has only a z component; hence from 1 (9) we may write

$$-j\omega\rho_0 \, w(z) = \frac{\partial}{\partial z} p(z) \tag{1}$$

where w is the z component of q. If we differentiate both sides of (1) with respect to z and use 1 (10) we may write

$$-j\frac{K^2}{\omega\rho_0} p(z) = \frac{\partial}{\partial z} w(z) \tag{2}$$

If we define the characteristic admittance Y_0 as

$$Y_0 = \rho_0 c = Z_0^{-1} \tag{3}$$

(where Z_0 is the characteristic impedance) (1) and (2) become

$$-jK Y_0 \, w(z) = \frac{\partial}{\partial z} p(z) \tag{4}$$

$$-jK Z_0 \, p(z) = \frac{\partial}{\partial z} w(z) \tag{5}$$

which are the partial differential equations of an ordinary transmission line having a voltage w (z) and a current p (z) at the point z [8]. If the time variation is not harmonic, we may replace jK by $\frac{1}{c}\frac{\partial}{\partial t}$. Our choice of voltage and current (implicit in our choice of Z_0) is inverted from that of the elementary electrical analogy found in much of the literature [9] (where voltage $\sim$ p and current $\sim$ w), but it has the distinct advantage that the boundary condition imposed at the end of a closed tube is the vanishing of the normal velocity corresponding to zero voltage at the end of a short circuited transmission line; similarly the boundary condition at the end of an open ended pipe is p = o, corresponding to zero current at the end of an open circuited transmission line. In addition it very often occurs in practice that the disturbance in the fluid medium is made to drive a moving coil mechanism to convert the hydrodynamical energy to electrical energy; in this case the current in the coil will be proportional to the pressure of the fluid, and the equivalent electrical circuit for the tube may be imagined directly connected to the electrical receiver, whereas the choice of pressure as the analogue to voltage requires the acoustic impedance to become an electrical impedance.

In the case where the propagated mode is not the principal mode the characteristic admittance may be generalized as

$$Y_{mn} = \frac{h_{mn}}{K} Y_o = Z_{mn}^{-1} \tag{6}$$

while the generalizations of (4) & (5) are easily seen to be

$$-j h_{mn} Y_{mn} W_{mn}(\upsilon, \nu, z) = \frac{\partial}{\partial z} p_{mn}(\upsilon, \nu, z) \tag{7}$$

$$-j h_{mn} Z_{mn} p_{mn}(\upsilon, \nu, z) = \frac{\partial}{\partial z} W_{mn}(\upsilon, \nu, z) \tag{8}$$

which satisfy 1 (9) & 1 (10) since differentiation with respect to z is effected by the operator jh_{mn}. For the principal mode $h_{mn} = h_o = K$, and (6-8) obviously reduce to (3-5). We remark, however, that (7) & (8) are not true transmission line equations inasmuch as they involve the transverse coordinates u and v; to remove this difficulty we write

$$p_{mn}(u,v,z) = a\, I_{mn}(z)\, \phi_{mn}(u,v) \qquad (9)$$

$$w_{mn}(u,v,z) = b\, V_{mn}(z)\, \phi_{mn}(u,v) \qquad (10)$$

where a and b are arbitrary constants, and $I_{mn}(z)$ and $V_{mn}(z)$ are the "currents" and "voltage" measuring the pressure and longitudinal velocity, respectively, of the mn 'th mode. If we substitute (9) & (10) in (7) & (8) we obtain

$$-j\, h_{mn}\left(\frac{b}{a} Y_{mn}\right) V_{mn}(z) = \frac{\partial}{\partial z} I_{mn}(z) \qquad (11)$$

$$-j\, h_{mn}\left(\frac{a}{b} Z_{mn}\right) I_{mn}(z) = \frac{\partial}{\partial z} V_{mn}(z) \qquad (12)$$

We observe that if the only restriction to be imposed on $V_{mn}(z)$ is the satisfaction of transmission line equations and the linear measurement of w (z) and p (z) the characteristic impedance Z_{mn} is established only up to an arbitrary constant a/b. To complete our transmission line analogy it is expedient to require the time average of the complex power at any surface of constant z in the cylinder to agree with the complex power at the same point on the equivalent transmission line, viz:

$$\frac{1}{2}\iint_S p^*_{mn}(u,v,z)\, q_{mn}(u,v,z)\, dS = \frac{1}{2} V_{mn}(z)\, I^*_{mn}(z) \qquad (13)$$

where the factor of 1/2 effects the r.m.s. time average. Substituting (11) & (12) in (13) we obtain

$$ab \iint_S [\phi_{mn}(u,v)]^2 dS = 1 \tag{14}$$

so that the product ab is associated with the normalization of the eigenfunctions; in particular if we choose the normalization of 2 (4) the product ab is equal to unity, and although this particular choice of normalization is in no way required for the validity of the transmission line analogy, we shall make it in the interest of convenience. Having fixed the product ab we are still free to choose the ratio a/b which is essentially connected with the choice of the characteristic impedance which enters in the transmission line equations (11) & (12). To agree with the choice of characteristic admittance of (3) both a and b are unity, and this is probably the simplest choice. Another convenient choice is to take b equal to the square root of the surface area S and a as the reciprocal of b, whence, for the principal mode force, rather then pressure, is being taken as the analogue of current. Such a choice makes our transmission line analogy a distributed parameter application of the "mobility" or "electromagnetic" mechanical-electrical analogy for lumped parameters.(10) We shall make the choices

$$a = b^{-1} = S^{1/2} = \phi_0^{-1} \tag{15a}$$

as this makes the voltage of the principal mode equal to the total flow in the tube; hence the voltage will be continuous across a change of cross section in the subsequent analyses. For this particular choice of a and b we have

$$Y_0 = \rho_0 c/S \tag{15b}$$

We emphasize again the above particular choices of a and b are by no means imperative and that the equivalent transmission line is fixed only up to these constants. In practice the physically measurable quantities such as reflection factors and standing wave ratios will depend only on the ratios of impedances so that the choice of a/b is dictated purely by analytical convenience.

We have seen that it is possible to represent the propagation of a single mode by an equivalent transmission line, the characteristic impedance and propagation constant of the line depending on the particular mode. It is important to realize that as many transmission lines are required as there are modes; however, for fixed dimensions and frequency there are only a finite number of modes whose transmission lines possess real characteristic impedances, and only these lines propagate energy, while those lines having imaginary characteristic impedances can only store energy, i.e. the pressure and velocity on the latter lines are 90° out of phase. If one considers the propagation of a disturbance down a cylindrical tube it is apparent that those modes having imaginary propagation constants are exponentially damped out; hence the stored energy is concentrated in the vicinity of the source (either a primary source or the scattering effect of a discontinuity) which radiates the non-propagated modes. Accordingly, if we are not interested in the behaviour of the medium in the immediate vicinity of the source, all transmission lines having imaginary characteristic impedances may be lumped into a single network containing only reactive elements and located at the source.

Inasmuch as the impedance presented at the input terminals of an infinite transmission line (the reactive lines may be regarded as extending to infinity as far as their effect at the source is concerned) is the characteristic impedance, the reactance of the element representing any given transmission line is simply the characteristic impedance of that line. Referring to 1 (13) and (6) and choosing the sign of hmn (when imaginary) to ensure attenuation towards infinity, we observe that the characteristic impedance is positive real or purely resistive for propagated modes, and negative imaginary or purely capacitative for non-propagated modes. We must hasten to add, however, that the frequency dependence of the capacitative reactances is not that of a simple lumped capacitative element, but is that of hmn/K (cf. (6)). However, we note that, asymptotically,

$$K/|h_{mn}| = \frac{\omega}{c}\left[\chi_{mn}^2 - \left(\frac{\omega}{c}\right)^2\right]^{-1/2} \sim \frac{\omega}{\chi_{mn} c} \qquad (16)$$

In order to lend some physical significance to the capacitative nature of the non-propagating lines in the actual hydrodynamical case we must find the hydrodynamical analogue to capacity. Electrically, capacitance is the time integral of the current (charge) divided by the voltage which produces it; hence, remembering that time differentiation is replaced by the operator $j\omega$, we may write

$$[C] = \left[\frac{\int p\,dt}{w S}\right] = \left[\frac{p}{j\omega w S}\right] = \left[\frac{F}{\frac{dw}{dt} S^2}\right] = \left[\frac{m}{S^2}\right] = \frac{\text{mass}}{(\text{area})^2} \qquad (17)$$

The stored electrical energy in a capacitance is $1/2\ C\ V^2$, and hydrodynamically we have $1/2\ C\ V^2 = 1/2\ M\ w^2$ = kinetic energy. In our equivalence we have used only the longitudinal velocity w, so that the stored energy in the reactive transmission lines represents a kinetic energy stored in alongitudinal motion of the medium due to the non-propagated modes. Of course, there is additional kinetic energy stored in the transverse motion of the medium, but this motion has no direct effect either on the magnitude or phase of the propagated energy which is our ultimate interest. In establishing a quantitative measure of the energy stored in the vicinity of the discontinuity under consideration, we must remember that our quantities have been normalized (cf. (13)) in such a way that the energy flow in the equivalent circuit represents the total energy flow and not the energy flow per unit area.

Due to the fact that the effects of all non-propagated modes are capacitative, there is no possibility of resonance due to the combination of inductive and capacitative effects in the same z plane; nevertheless it is possible to cancel the capacitative effect of a discontinuity in one plane by a discontinuity in a second plane which, transferred through the equivalent transmission line between the two planes, appears as an inductive effect in the first plane.

Our cylindrical tube has now been replaced by a finite number of transmission lines corresponding to the finite number of transmitted modes plus lumped networks at points where sources and discontinuities exist. This representation will describe accurately the propagation of the sound except in the immediate neighborhood of the sources and discontinuities. The unit of measurement of this "immediate neighborhood" is the wavelength of the disturbance in the tube which is K/hmn times the wavelength in open space.

It can be shown [11] that the equivalent lumped circuit for a dissipationless transmission line of length l, characteristic impedance $Z = (Y^{-1})$, and propagation constant h is either a T or network as shown below.

(18)

(19)

$$(Y_{11} - Y_{12}) = jY \tan\left(\frac{h\ell}{2}\right) \tag{20}$$

$$Y_{12} = -jY \csc(h\ell) \tag{21}$$

The above schematic representations implicitly assume h and Z to be real, corresponding to a propagated mode, but (18-21) are valid for non-propagated modes if it is remembered that both h and Z are negative imaginary for such modes; hence, beyond cutoff, the inductances pictured above become capacitances, and the transmission line for a given mode acts as a high pass filter [12]. Finally, we remark that the lumped networks have no direct physical (hydrodynamical) interpretation since there is no such thing as a lumped hydrodynamical inductance.

Two important special cases in practice are the open ended and closed tubes (corresponding to open and short circuited lines, respectively). Using the above equivalent circuits we may write for the input impedances and admittances

$$Z_{oc} = -jZ \cot(h\ell) \tag{22}$$

$$Z_{sc} = jZ \tan(h\ell) \tag{23}$$

$$Y_{oc} = jY \tan(h\ell) \tag{24}$$

$$Y_{sc} = -jY \cot(h\ell) \tag{25}$$

Notice that the resonant lengths depend on the mode (i.e. h) and are infinite in number. For the more general case of a terminating impedance Z_t or terminating admittance Y_t , the input impedance is

$$Z_i(l) = Z\left[\frac{j\sin(hl) + (Z_t/Z)\cos(hl)}{\cos(hl) + j(Z_t/Z)\sin(hl)}\right] \tag{26}$$

while the input admittance is

$$Y_i(l) = Y\left[\frac{j\sin(hl) + (Y_t/Y)\cos(hl)}{\cos(hl) + j(Y_t/Y)\sin(hl)}\right] \tag{27}$$

We observe that if $hl = \pi$ (i.e. $l = 1/2$ wavelength in the tube) $Z_i = Z_t$, so that a half wavelength of line acts as an one to one transformer, while if $hl = \frac{\pi}{2}$ $Z_i = Z^2/Z_t$, so that a quarter wavelength of line effects inversion about the circle $R = Z$. These and many other important properties are discussed in more detail in any book dealing with high frequency applications of transmission lines (13).

In practice, the excitation of sound in a tube is often accomplished by a vibrating piston which may be driven with either constant velocity or constant force. In the former case the equivalent source representation is a constant voltage generator, while in the latter case a constant current generator is indicated.

Consider now the general discontinuity formed by a region containing arbitrary obstacles and diaphragms and which acts as a junction between n tubes terminating there. We shall assume that the frequency and dimensions are such that only the principal mode propagates freely in each of the tubes, but due to the discontinuity, higher modes will be excited and will affect the phase and amplitude of the principal mode reflections. If voltages and currents V_o^n and I_o^n are defined as in (9) and

(10) in each of the tubes as measures of the longitudinal velocity and pressure, respectively, then due to the linearity of the original equations of motion, the discontinuity may be described by an impedance matrix (Z_{ij}) (the Z_{ij} are not directly related to the characteristic impedances Z_{mn}) such that

$$V_0^n = \sum_{j=1}^{n} Z_{nj} I_0^j \tag{28}$$

where the voltages and currents are measured in arbitrary (but fixed) reference planes in each of the tubes. The equivalent circuit then consists of n transmission lines of characteristic impedances Z^n_o terminating in a 2 n-terminal network described by the impedance matrix (Z_{ij}). Since the impedance matrix describes only the effects of the higher modes, all of its elements will be reactive, but inasmuch as the reference planes in the tubes are arbitrarily fixed, the elements of (Z_{ij}) are not necessarily capacitative, as a section of line may transform a capacitative reactance to an inductive reactance. However, in the special case where the discontinuity is in a single plane of constant z, and all of the n reference planes are made to coincide with this plane, we may further assert that all of the elements of (Z_{ij}) are capacitative. Due to conservation of energy we may show that reciprocity holds and hence that $Z_{ij} = Z_{ji}$.

In the experimental determination of the matrix (Z_{ij}), all the impedance elements are determined relative to Z_o, since quantities which are physically measurable depend only on the ratios of Z_{ij} to Z_o as indicated in the discussion of (6-14) above.

In the subsequent analysis we shall treat only the cases of plane discontinuities, and the assertions of the last two paragraphs will be demonstrated explicitly.

4. The Analysis of a Plane Discontinuity

In the light of the foregoing discussion we shall now establish a general treatment for a discontinuity in a plane of constant z in the form either of an infinitely thin diaphragm or an abrupt change of cross section, or both. We shall assume that the frequency of the disturbance and the dimensions of the cross section are such that only the principal mode is freely propagated. Our task is then to calculate the equivalent circuit to describe the discontinuity arising when a tube of cross section S_1 occupying the region of negative z is coupled through an aperture (or apertures) R in the plane z = 0 to a tube of cross section S_2 occupying the region of positive z, all surfaces being assumed perfectly reflecting. We shall use the superscripts 1 and 2 on all quantities which are different in the tubes 1 and 2, the superscripts 1 and 2 being associated with the top and bottom sign respectively, where a choice of sign exists.

If we assume principal modes of amplitudes $a^{1,2}$ to be incident on the discontinuity from $z = \mp \infty$, respectively, the longitudinal velocity is

$$w^{1,2}(u,v,z) = (a^{1,2}e^{\mp jkz} + b^{1,2}e^{\pm jkz})\,\phi_0^{1,2} + \sum_{m,n} b_{mn}^{1,2}\,\phi_{mn}^{1,2}(u,v)\,e^{\pm j h_{mn}^{1,2} z} \qquad (1)$$

$$b_{mn}^{1,2} = -\delta_0^{mn}\,a^{1,2} + \int_R w(u',v')\,\phi_{mn}^{1,2}(u',v')\,dS' \qquad (2)$$

where w (u',v') is the longitudinal velocity in the plane z = 0 and the $h^{1,2}_{mn}$ are the amplitudes of the reflected modes. Since

(1) is a sum of solutions of the wave equation satisfying the boundary conditions 1 (14) on the cylindrical walls (the eigenvalues $\chi^{1,2}_{mn}$ being so chosen) and reduces the Fourier expansion of $w^{1,2}(u, v, o)$ correctly, it is the required solution by virtue of the uniqueness theorem. Since the integrals in (2) are taken only over the aperture region R in the plane z = 0 the longitudinal velocity vanishes over the remainder of the plane z = 0 (which we shall designate as T) as is required by 1 (14); moreover (1) is a priori continuous across the plane z = 0 in both R and T.

From (1) and 3 (8), the pressure is given by (using 2 (5) and 3 (15))

$$p^{1,2}(u,v,z) = \pm Y_0^{1,2}(a^{1,2}e^{\mp jkz} - b^{1,2}e^{\pm jkz})\,\phi_0^{1,2} \mp \sum_{m,n} Y^{1,2}_{mn}\, b^{1,2}_{mn}\, \phi^{1,2}_{mn}(u,v)\,e^{\pm j h^{1,2}_{mn} z} \qquad (3)$$

We now define the voltages and currents measuring the longitudinal velocities and pressures of the principal modes, after the fashion of 3 (9 & 10), as

$$V_0^{1,2}(z) = (a^{1,2}e^{\mp jkz} + b^{1,2}e^{\pm jkz})(\phi_0^{1,2})^{-1} \qquad (4)$$

$$I_0^{1,2}(z) = Y_0^{1,2}(a^{1,2}e^{\mp jkz} - b^{1,2}e^{\pm jkz})(\phi_0^{1,2})^{-1} \qquad (5)$$

where $Io^{1,2}(z)$ are both flowing into the junction plane z = 0. In the reference plane z = 0 we have, using (2)

$$V_0^{1,2} \equiv V_0^{1,2}(0) = (a^{1,2}+b^{1,2})(\phi_0^{1,2})^{-1} = \int_R w(u',v')\,dS' \qquad (6)$$

$$I_0^{1,2} \equiv I_0^{1,2}(0) = Y_0^{1,2}(a^{1,2} - b^{1,2})(\phi_0^{1,2})^{-1} \qquad (7)$$

Hence we may write

$$V_0^{1} = V_0^{2} \qquad (8)$$

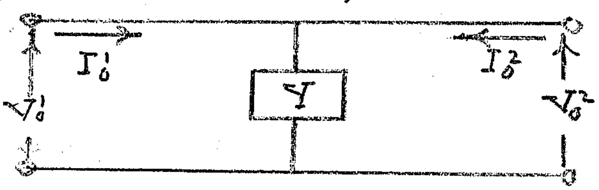
In order to find a relation between the currents, we require that the pressure be continuous across the aperture R (it is obviously discontinuous across T) which, from (3) & (7), yields

$$I_0^1 + I_0^2 = \int_R G(u,v;u',v')\, W(u',v')\, dS' \tag{9}$$

$$G(u,v,u',v') = \sum_{p=1,2} \sum_{m,n} Y_{mn}^{p}\, \phi_{mn}^{p}(u,v)\, \phi_{mn}^{p}(u',v') \tag{10}$$

where G(u,v,u',v') is the Green's function of the problem.

By virtue of (8), the equivalent circuit must consist of a pure shunt element, viz:



for which the circuit equations are (8) and

$$(I_0^1 + I_0^2) = Y V_0^1 \tag{11}$$

where Y is the lumped admittance representing the effect of the higher modes on the dominant modes.

In order to put our equations in a more convenient form for solution we remove the amplitude dependence by writing

$$W(u,v) = (I_0^1 + I_0^2)\, f(u,v) \tag{12}$$

whence (10) becomes

$$\phi_0^1 = \int_R G(u,v,u',v')\, f(u',v')\, dS' \tag{13}$$

which is an integral equation of the first kind (14) for the determination of f(u,v) and having the symmetric kernel

G(u,v,u',v'). (It must be remembered that (13) is valid only in R since the pressure is discontinuous across T.) Substituting (6) and (12) in (11) we have

$$Y = \left[\int_R \phi_0' f(u,v)\, dS\right]^{-1} \tag{14}$$

which determines Y and completes the formal solution of the problem.

To solve the equations (13) and (14) we multiply both sides of (13) by Y f(u,v)dudv, and integrate over the aperture R, and combine with (14) to obtain

$$Y = \frac{\int_R \int_R f(u,v)\, G(u,v,u',v')\, dS'\, dS}{\left[\int_R f(u,v)\, \phi_0'\, dS\right]^2} \tag{15}$$

Substituting (10) in (15) we have

$$Y = \sum_{p=1,2} \sum_{m,n} Y_{mn}^{p} \frac{\left[\int_R f(u,v)\, \phi_{mn}^{p}(u,v)\, dS\right]^2}{\left[\int_R f(u,v)\, \phi_0'\, dS\right]^2} \tag{16}$$

We assert that (16) is a variational expression for Y which is stationary with respect to first order variations about the true value of f (u,v). This can be proved in the usual manner by letting f (u,v) = f^{o}(u,v) d g (u,v) where f^{o} is the true value of f satisfying the integral equation (13), d is an arbitrary parameter, and g is an arbitrary function, substituting in (15) varying (i.e. taking first differences) with respect to d, using (13) and (14) for f^{o}, setting a = 0, and observing that the first variation of Y vanishes. However, we shall use a somewhat different proof which will demonstrate that Y is not only stationary but is an absolute minimum.

Assume that $f^{(o)}$ is the true solution to (15) satisfying the integral equation (13) and giving $Y^{(o)}$ from (14); then consider the function

$$g(u,v) = f(u,v)\left[\int_R f\phi_o' dS\right]^{-1} - f^{(o)}(u,v)\left[\int_R f^{(o)}\phi_o' dS\right]^{-1} \tag{17}$$

where f is a trial function to be inserted in (15). We may certainly assert

$$\int_R\int_R g(u,v)\, G(u,v,u',v')\, g(u',v')\, dS'\, dS \geq 0 \tag{18}$$

due to the symmetry of G (actually we are on the positive imaginary axis); moreover (18) is equal to zero only if g vanishes, i.e. if f and $f^{(o)}$ are identical. If we expand (18) by inserting (17) we obtain

$$\frac{\int_R\int_R fGf\, dS'\, dS}{\left(\int_R f\phi_o' dS\right)^2} - 2\,\frac{\int_R\int_R f^{(o)}Gf\, dS'\, dS}{\int_R f^{(o)}\phi_o' dS \int_R f\phi_o' dS} + \frac{\int_R\int_R f^{(o)}Gf^{(o)}\, dS'\, dS}{\left(\int_R f^{(o)}\phi_o' dS\right)^2} \geq 0 \tag{19}$$

The first and third terms in (19) are simply Y and $Y^{(o)}$, respectively, as is evident from (15). If we multiply both sides of the integral equation for $f^{(o)}$ (13) by fdS and integrate over R the numerator in the second term in (19) cancels the first term in the denominator, while the second term in the denominator is the reciprocal of $Y^{(o)}$ by (14). Hence we may write

$$Y - Y^{(o)} \geq 0 \tag{20}$$

being equal to zero only if f and $f^{(o)}$ are identical. We emphasize again that Y is actually positive imaginary definite rather than positive real definite, but, of course, the factor of j could be cancelled from $Y_{mn}^{1,2}$ in all of the above equations by setting all $Y_i = jB_i$. We have now reduced the situation to one having the aspects of the usual Ritz problem, differing in the fact that the solution is unique and an absolute minimum. Thus if we guess a trial function and insert it in (16) the resulting value of Y will be larger than the true value but the error in Y will generally be less than the error in the trial function since Y is stationary with respect to first order variations about this function. We remark that (15) & (16) are also independent of the amplitude of the function inserted and depend only on its geometrical form.

In order to solve the variational equation (16) we expand f (u,v) in a series of functions and use the variational principle to determine the amplitudes of these functions. The functions obviously must be solutions to the wave equation 1 (12) whose normal derivatives vanish on the boundaries of R since f (u,v) is linearly related to the pressure which serves as a velocity potential; such a set of functions we shall designate as $\phi^3_{pq}(u,v)$. The set ϕ^3_{pq} is a priori orthogonal since it satisfies 1 (12), but there is no particular advantage to making it orthonormal, since (16) is independent of the amplitude of f (u,v); in particular, ϕ^3_0 is simply a constant, and we may give it an amplitude such that the denominator of (16) is unity. Hence we choose the expansion

$$f(u,v) = A_0 + \sum_{p,q} A_{pq}\,\phi^3_{pq}(u,v) \qquad (21)$$

$$\bar{n}\cdot\nabla\phi^3_{pq}(u,v) = 0 \quad \text{on boundaries of R} \qquad (22)$$

$$A_0 \phi_0^1 \int_R dS = 1 \tag{23}$$

If we now substitute (21) in (16) we obtain

$$B = B_0 + 2\sum_{p,q} C_{pq} A_{pq} + \sum_{p,q}\sum_{p',q'} D_{pqp'q'} A_{pq} A_{p'q'} \tag{24}$$

$$B_0 = (A_0)^2 \sum_{s=1,2} \sum_{m,n} B^s_{mn} \left[\int_R \phi^s_{mn}(u,v)\, dS\right]^2 \tag{25}$$

$$C_{pq} = A_0 \sum_{s=1,2} \sum_{m,n} B^s_{mn} \left[\int_R \phi^s_{mn}(u,v)\, dS\right]\left[\int_R \phi^s_{mn}(u,v)\, \phi^s_{pq}(u,v)\, dS\right] \tag{26}$$

$$D_{pqp'q'} = \sum_{s=1,2} \sum_{mn} B^s_{mn} \left[\int_R \phi^s_{mn}(u,v)\, \phi^s_{pq}(u,v)\, dS\right] \cdot \left[\int_R \phi^s_{mn}(u,v)\, \phi^s_{p'q'}(u,v)\, dS\right] \tag{27}$$

where we have written

$$Y = jB, \quad Y^s_{mn} = j\, B^s_{mn} \tag{28}$$

To determine the coefficients Apq we minimize (24) with respect to each of the coefficients Apq in order that any finite number of terms used to approximate (21) will give the best possible approximation to B; the result is

$$\sum_{p',q'} D_{pqp'q'} A_{p'q'} = -C_{pq} \tag{29}$$

Since (29) must be valid for all p and q it constitutes a set of simultaneous equations which is infinite in principle but which, in practice, contains as many equations as the number of terms it is desired to include in the summation of (21). If we substitute (29) in (24) we obtain the simpler expression

$$B = B_0 + \sum_{p,q} C_{pq} A_{pq} \tag{30}$$

Due to the minimumization of B with respect to each of the coefficients Apq the variational principle guarantees that if we solve (29) by an orderly iteration, the value of B given by (30) approaches the true value of B monotonically as additional Apq's are included and always remains above the true value (from which it is implicit that each of the terms in the series of (30) is negative, as is indeed true). By an orderly iteration we mean any process that determines the set bounded by Apq under the approximation that all higher order coefficients are zero (i.e., $A_{rs} = 0$ if $r = p$, $s = q$); in practice there are obviously a variety of procedures which may be adopted, the most expedient depending on the estimated importance of the various coefficients (e.g. in symmetrical cases either or both p and q take only odd values) and whether the calculation is being done manually or with the aid of a machine. Actually, it will usually be found that Bo is sufficiently accurate for most practical purposes (circa 10% or less error).

There are two special categories of the above problem which occur repeatedly in practice, namely the simple change of cross section of a tube where the aperture R coincides with the smaller of the two cross sections (which we shall assume to be S_2) and the simple plane obstacle where the cross sections S_1 and S_2 are identical. In the first case the functions $\overset{3}{\phi}_{mn}$ are identical with the functions $\overset{2}{\phi}_{mn}$, and due to orthogonality, all those terms for which s = 2 in (25 - 27) vanish with the exception of the single term in $D_{pqp'q'}$ for which $m = p = p'$

and $n = q = q'$. In the second case, the terms for $s = 1$ and $s = 2$ are identical, so we simply drop the summation with respect to s and multiply (25-27) by a factor of 2. Hence, the susceptance of a window of cross section S_2 in a tube of cross section S_1 is approximately double the susceptance of the analogous change of cross section.

The above solution gives an answer which is always an upper bound to the true value of B, and no estimate of the error is directly available. In the foregoing treatment, the problem was formulated in terms of the longitudinal velocity in the plane $z = 0$, but an alternative formulation could have been made in terms of the pressure in the plane $z = 0$. We observe, however, that the pressure does not vanish in the region T but is discontinuous there, and in general the Fourier expansion of the pressure in the regions 1 and 2 will involve integrals over different regions, thus complicating the resulting integral equation. In the general problem involving a change of cross section, the formulation in terms of the pressure in the plane $z = 0$ must therefore be regarded as impractical, but in the simple problem involving only a plane obstacle and no change of cross section a formulation in terms of the pressure discontinuity suggests itself.

To effect the formulation just suggested we define the "pressure jump" as

$$p_2(u,v,e) - p_1(u,v,-e) = (I_0^1 + I_0^2)\, g(u,v) \qquad (31)$$

where e is an infinitesimal. In the region R g (u,v), of course, vanishes identically. If we now add the pressure jump of (31) to the left hand side of (9) (which was originally formulated on the basis of continuity of pressure across R) we have an equation which correctly relates the pressures p_1 and p_2 in both the regions R and T of the plane z = 0; hence (13) becomes

$$g(u,v) + \phi_0' = \int_R G(u,v,u',v')\, f(u',v')\, dS' \qquad (32)$$

where, remembering that ϕ_{m^1n} and ϕ_{m^2n} are now identical, G is given by (10) as

$$G(u,v,u',v') = 2 \sum_{m,n} Y_{mn}\, \phi_{mn}(u,v)\, \phi_{mn}(u',v') \qquad (33)$$

If we substitute (33) in (32), remember that f (u,v) vanishes in T, g (u,v) vanishes in R, and that the set ϕ_{mn} is orthonormal we have an inversion (i.e. multiply both sides of (32) by ϕ_{pq} and integrate over R and T

$$2 Y_{mn} \int_R f(u,v)\, \phi_{mn}(u,v)\, dS = \int_T g(u,v)\, \phi_{mn}(u,v)\, dS \qquad (34)$$

$$\int_T g(u,v)\, \phi_0\, dS = -1 \qquad (35)$$

If we now substitute (34) for the integrals in the numerator of (16), (14) for the integral in the denominator, and divide both sides of the result by the square of (35) in order to make Z independent of the amplitude of g (u,v) we obtain

$$\frac{1}{Y} = Z = \frac{1}{2} \sum_{m,n} Z_{mn} \frac{\left[\int_T g(u,v)\, \phi_{mn}(u,v)\, dS\right]^2}{\left[\int_T g(u,v)\, \phi_0\, dS\right]^2} \qquad (36)$$

where the impedances Z and Z mn are the reciprocals of the admittances Y mn.

Inasmuch as (36) has exactly the same form as (16), we may assert immediately that Z as given by (36) is an absolute minimum and hence that Y as given by (36) is an absolute maximum. Thus, for trial functions f(u,v) and g(u,v), (16) and (36) give values of Y which bound the true values from above and below, respectively. In expanding g(u,v) we pick functions ϕ^4_{pq} (u,v) which vanish at the free boundaries (continuity of pressure) and have vanishing normal derivatives at walls (vanishing normal velocities), i.e., functions which, in general, obey mixed boundary conditions. This means that the denominator of (36) will generally be an infinite series, rather than a single term as was the case in the solution of (16). To solve (36) we expand g(u,v) as

$$g(u,v) = \sum_{p,q} B_{pq} \phi^4_{pq} (u,v) \qquad (37)$$

observing that (37) does not include a principal solution ϕ_0 since such a solution cannot vanish at free boundaries. Substituting (37) in (36) we have

$$Z = \left[\sum_{p,q} \sum_{p',q'} D_{pqp'q'} B_{pq} B_{p'q'}\right]\left[\sum_{p,q} C_{pq} B_{pq}\right]^{-2} \qquad (38)$$

$$C_{pq} = \left[\int_T \phi_0 \phi^4_{pq} \, dS\right] \qquad (39)$$

$$D_{pqp'q'} = \frac{1}{2} \sum_{m,n} Z_{mn} \left[\int_T \phi_{mn} \phi^4_{pq} \, dS\right]\left[\int_T \phi_{mn} \phi^4_{p'q'} \, dS\right] \qquad (40)$$

If we now minimize Z in (38) with respect to each of the coefficients Bpq, we obtain the simultaneous equations

$$\sum_{p',q'} D_{pqp'q'} B_{p'q'} = C_{pq} \qquad (41)$$

Multiplying both sides of (41) by Bpq, summing, and substituting in (38) we have

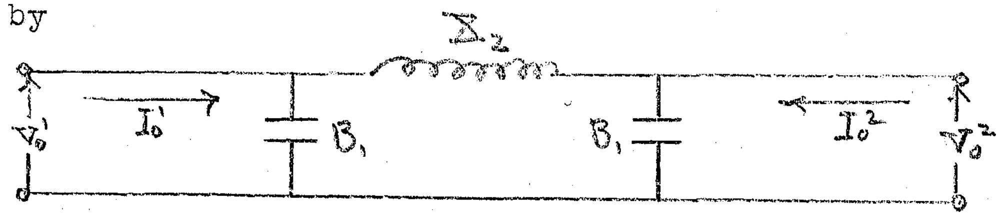
$$Z = \Big[\sum_{p,q} C_{pq} B_{pq} \Big]^{-1} \tag{42}$$

In actual calculations the j will generally be removed from the coefficients Dpqp'q' and the impedances Z_{mn}.

5. Thick Obstacle Corrections

In practice it is often impossible to obtain experimentally an "infinitely thin" obstacle. The effects of thickness will be most important in the case of large apertures (R comparable to S_1 and S_2) where the thickness becomes comparable to the transverse dimensions of the obstacle and, in the case of small apertures, where the thickness becomes comparable to the aperture dimensions.

Rigorously the problem of the thick obstacle should be considered as a double change of cross section, i.e. from S_1 to R and then from R to S_2; unfortunately the thickness is seldom large enough to justify the assumption that there is no interaction of the higher modes excited by the two changes of cross section. Nevertheless, an extremely good approximation to the thickness correction may be obtained by assuming the aperture to be a section of tube propagating only the principal mode. For a plane discontinuity of thickness t having a susceptance of B calculated on the assumption t = 0, the equivalent circuit of the discontinuity is then approximately given by



where, using the results of 3 (20,21):

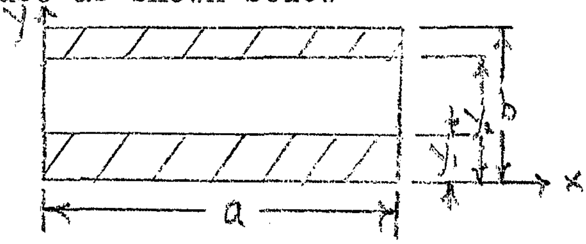
$$B_1 = B/2 + Y_0 \tan\left(\frac{\pi t}{\lambda}\right) \qquad (1)$$

$$X_2 = Z_0 \sin\left(\frac{2\pi t}{\lambda}\right) \qquad (2)$$

We remark that the above representation is obviously wrong in splitting B evenly (i.e. B/2 at each face) when S_1 and S_2 are considerably different, but in such a case the thickness correction will be of relatively small importance compared to the case where S_1 and S_2 are identical, and for small values of t the net behaviour of the network is only slightly affected by the manner of splitting up B.

6. Analysis of a Window in a Rectangular Tube

As a first application of the analysis of the foregoing sections, we shall consider the case of a thin window in a rectangular tube as shown below



While a two dimensional window (i.e. both dimensions less than tube dimensions) is certainly feasible, it has no particular advantage over the slit window shown above (since both will give any capacitance from zero to infinity), and the slit type is more easily constructed. Actually, the most easily constructed type is the asymmetrical window (either $\bar{y}_1$ or $\bar{y}_2$ equal to zero) and would therefore generally be used, although the symmetrical window ($\bar{y}_1 = b - \bar{y}_2$) has the advantage of exciting only even order modes.

The solutions to 1 (12) satisfying 1 (14) in a rectangular tube of width a and heighth b and normalized accord- to 2 (4) are

$$\phi_0 = (ab)^{-1/2} \qquad (1)$$

$$\phi_{mn} = \left[(2-\delta_m^0)(2-\delta_n^0)\right]^{1/2} \phi_0 \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) \qquad (2)$$

$$\mathcal{H}_{mn}^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \tag{3}$$

$$h_{mn}^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - K^2 \tag{4}$$

$$Y_{mn} = \frac{K}{h_{mn}} Y_0 \ , \quad Y_0 = \frac{\rho_0 c}{a b} \tag{5}$$

Since there is no discontinuity in the x coordinate and the incident wave is assumed to consist purely of the principal mode, it is clear that only modes for which m = 0 will be excited, and we shall dispense with the subscript m in the subsequent analysis. The Green's function of 4 (10) then becomes

$$G(y, y') = 4(\phi_0)^2 \sum_1^\infty Y_n \cos\left(\frac{n\pi y}{b}\right)\cos\left(\frac{n\pi y'}{b}\right) \tag{6}$$

We observe that Y_n is asymptotic to $(Kb/n\pi)\,Y_0$ at low frequencies; if we denote the asymptotic limit of the Green's function as G_0, we may write

$$G/Y_0 = G_0/Y_0 + j\frac{4Kb}{\pi}(\phi_0)^2 \sum_1^\infty \Delta_n \cos\left(\frac{n\pi y}{b}\right)\cos\left(\frac{n\pi y'}{b}\right) \tag{7}$$

$$\Delta_n = \left[(n)^2 - (Kb/\pi)^2\right]^{-1/2} - 1/n \tag{8}$$

$$\frac{G_0}{Y_0} = j\frac{4Kb}{\pi}(\phi_0)^2 \sum_1^\infty \frac{1}{n} \cos\left(\frac{n\pi y}{b}\right)\cos\left(\frac{n\pi y'}{b}\right) = -j\frac{2Kb}{\pi}(\phi_0)^2 \log 2\left|\cos\left(\frac{\pi y}{b}\right) - \cos\left(\frac{\pi y'}{b}\right)\right| \tag{9}$$

Since G_0 is essentially the static Green's function, it may be obtained by static methods, such as conformal mapping; however, the series in (9) may be summed directly by expressing the product of the cosines in terms of the cosines of the sum and difference of the arguments, regarding the latter as the real parts of exponentials, recognizing the Taylor series expansion of a logarithm, and taking the real part of the result.

To take advantage of the closed form of G_o we write 4 (15) as

$$Y = jB = \frac{\int_R \int_R f(y)\, G(y,y')\, f(y')\, dy\, dy'}{\left[\int_R f(y)\, \phi_o\, dy\right]^2} + \frac{\int_R \int_R f(y)\left[G(y,y') - G_o(y,y')\right] f(y')\, dy\, dy'}{\left[\int_R f(y)\, \phi_o\, dS\right]^2} \tag{10}$$

where dS = d x d y. Since m is zero, the integrations in x cancel. We observe that the second term in (10) is small compared to the first term since, from (7) and (8), $(G - G_o)$ converges as n^{-3}; thus a perturbation scheme is suggested where we neglect the second part of (10), obtain the resultant function $f_o(y)$ by an exact solution, and then obtain the first perturbation by substituting $f_o(y)$ in (10) where the second part is then included. Hence we obtain, on substituting (7) in (10),

$$\frac{B}{Y_o} = \frac{B^o}{Y_o} + \frac{4kb}{\pi} \sum_1^\infty \Delta_n \frac{\left[\int_R f_o(y) \cos\left(\frac{n\pi y}{b}\right) dy\right]^2}{\left[\int_R f_o(y)\, dy\right]^2} \tag{11}$$

$$\frac{B^o}{Y_o} = \frac{\int_R \int_R f_o(y)\, G(y,y')\, f_o(y')\, dy\, dy'}{\left[\int_R f_o(y)\, \phi_o\, dy\right]^2} \tag{12}$$

Multiplying both sides of (12) by the denominator of the right hand side, differentiating with respect to y, and substituting

(9) we obtain

$$\left(\frac{B^0}{Y_0}\right)\int_R f_0(y')dy' = -\frac{2kb}{\pi}\int_R \log\left[2\left|\cos\left(\frac{\pi y}{b}\right)-\cos\left(\frac{\pi y'}{b}\right)\right|\right] f_0(y')\,dy' \qquad (13)$$

an integral equation which must hold for all y in R. This type of integral equation may be obtained in the general case of section 4 by combining 4 (13) and 4 (14).

As has already been demonstrated, (11) gives an upper bound to B. Since the present problem does not involve a change of cross section, the formulation of 4 (36) will give a lower bound. Separating out the asymptotic limit of 4 (36), writing Z = j X, and following an exactly analogous treatment to that used with (10) above we obtain

$$\left(\frac{X}{Z_0}\right) = \left(\frac{X^0}{Z_0}\right) - \frac{\pi}{kb}\sum_1^\infty \varepsilon_n \frac{\left[\int_T g_0(y)\cos\left(\frac{n\pi y}{b}\right)dy\right]^2}{\left[\int_T g_0(y)\,dy\right]^2} \qquad (14)$$

$$\left(\frac{X^0}{Z_0}\right)\int_T g_0(y')dy' = \frac{1}{2k}\frac{d}{dy}\int_T \frac{g_0(y')\sin\left(\frac{\pi y'}{b}\right)dy'}{\left[\cos\left(\frac{\pi y'}{b}\right)-\cos\left(\frac{\pi y}{b}\right)\right]} \qquad (15)$$

$$\varepsilon_n = \left[n^2 - \left(\frac{kb}{\pi}\right)^2\right]^{1/2} - n \qquad (16)$$

The summation of the series in (15) may be obtained as in the case of (9), except that the series is geometric rather than logarithmic. Since g_0 (y) and f_0(y) both represent exact solutions to the static problem, the static reactance X_0 is given by

$$X^0 = -\frac{1}{B^0} \qquad (17)$$

and it is not necessary to solve (13) and (15) separately.

Moreover, using 4 (14,34 and 35) we may write

$$\int_T g_0(y)\,dy = -b\left(\frac{B^\circ}{Y_0}\right)\int_R f_0(y)\,dy \tag{18}$$

$$\int_T g_0(y)\cos\left(\frac{n\pi y}{b}\right)dy = \frac{2Kb^2}{n\pi}\int_R f_0(y)\cos\left(\frac{n\pi y}{b}\right)dy \tag{19}$$

Substituting (17-19) in (14) we obtain

$$\left(\frac{B}{Y_0}\right)^{-1} = \left(\frac{B^\circ}{Y_0}\right)^{-1} + \frac{4Kb}{\pi}\left(\frac{B^\circ}{Y_0}\right)^{-2}\sum_1^\infty \frac{\varepsilon_n}{n^2}\,\frac{\left[\int_R f_0(y)\cos\left(\frac{n\pi y}{b}\right)dy\right]^2}{\left[\int_R f_0(y)\,dy\right]^2} \tag{20}$$

which gives a lower bound to B in terms of B^o and f_o (y). Of course, the upper and lower bounds to B can be expressed in terms of g_o (y), but the integral equation (13) is more easily solved than (15) since the region R is singly connected, while the region T is doubly connected.

While f_o (y) may be determined by conformal mapping, it is interesting to obtain it by solving the integral equation (13), since the method can be used to determine even better approximations to f (y). In order to solve (13) we make the change of variable

$$\cos\left(\frac{\pi y}{b}\right) = \alpha\cos\theta + \beta \tag{21}$$

$$\begin{matrix}\alpha \\ \beta\end{matrix} = \frac{1}{2}\left[\cos\left(\frac{\pi y_1}{b}\right) \mp \cos\left(\frac{\pi y_2}{b}\right)\right] \tag{22}$$

In the important case of the symmetrical window

$$\alpha = \sin\left[\frac{\pi}{b}(y_2 - y_1)\right], \quad \beta = 0 \tag{23}$$

Substituting (21) in (13) we obtain

$$\left(\frac{B^\circ}{Y_0}\right)\int_0^\pi u(\theta')\,d\theta' = -\frac{2Kb}{\pi}\int_0^\pi \log\left(2\alpha\left|\cos\theta - \cos\theta'\right|\right)u(\theta)\,d\theta' \tag{24}$$

$$u(\theta) = f_0(\theta)\sin\theta\left[1 - (\alpha\cos\theta + \beta)\right]^{-1/2} \tag{25}$$

If we split off the log from (24) and reexpand the log $|\cos\theta - \cos\theta'|$ we obtain

$$\left[\left(\frac{B^{o}}{Y_o}\right) + \frac{2Kb}{\pi}\log\alpha\right]\int_0^{\pi} U(\theta')\,d\theta' = \frac{4Kb}{\pi}\int_0^{\pi} U(\theta')\sum_1^{\infty}\frac{\cos n\theta\cos n\theta'}{n}\,d\theta' \qquad (26)$$

If we expand $U(\theta)$ in the Fourier series

$$U(\theta) = \frac{a_0}{2} + \sum_1^{\infty} a_m\cos m\theta \qquad (27)$$

and substitute

$$\left[\left(\frac{B^{o}}{Y_o}\right) + \frac{2Kb}{\pi}\log\alpha\right]a_0 = \frac{4Kb}{\pi}\sum_1^{\infty} a_m\frac{\cos m\theta}{m} \qquad (28)$$

Hence all the a_m vanish except a_0, while the vanishing of the coefficient of a_0 in (28) yields

$$\left(\frac{B^{o}}{Y_o}\right) = \frac{2Kb}{\pi}\log(1/\alpha) \qquad (29)$$

For a non-trivial solution a_0 does not vanish, and (0) is therefore a constant; hence, from (25)

$$f(y) = C\sin\left(\frac{\pi y}{b}\right)\left\{1 - \left[\frac{\cos\left(\frac{\pi y}{b}\right) - \beta}{\alpha}\right]^2\right\}^{-1/2} \qquad (30)$$

For the symmetrical window of opening $y_2 - y_1 = d$ (29) reduces to

$$\left(\frac{B^{o}}{Y_o}\right) = \frac{4b}{\lambda}\log\left[\csc\left(\frac{\pi d}{2b}\right)\right] \qquad (31)$$

where is the wavelength of the propagated disturbance.

Before investigating the perturbations of (11) and (20) we shall illustrate how higher order corrections may be obtained through exact solutions of integral equations similar to (13).

The approximation made in $f_0(y)$ and B^0 is that K^2 is negligible compared to $\left(\frac{n\pi}{b}\right)^2$ and is obviously poorest for small n ($\frac{n\pi}{b}$ is of course, greater than K for all n since it has been assumed that only the principal mode propagates freely, i.e. that the wave-length is greater than 2b); hence the next approximation is given by including the term n = 1 in the Green's function which is separated out. Denoting this approximation by the superscript 1, (11) and (12) become

$$\left(\frac{B}{\gamma_0}\right) = \left(\frac{B^1}{\gamma_0}\right) + \frac{4Kb}{\pi}\sum_{2}^{\infty}\Delta_n \frac{\left[\int_R f_1(y)\cos\left(\frac{n\pi y}{b}\right)dy\right]^2}{\left[\int_R f_1(y)\,dy\right]^2} \qquad (32)$$

$$\left(\frac{B^1}{\gamma_0}\right)\int_R f(y')\,dy' - \frac{4\Delta_1}{\pi}\int_R f_1(y')\cos\left(\frac{\pi y'}{b}\right)\cos\left(\frac{\pi y}{b}\right)dy' = -\frac{2Kb}{\pi}\int_R \log\left[2\left|\cos\left(\frac{\pi y}{b}\right)-\cos\left(\frac{\pi y'}{b}\right)\right|\right] f_1(y')\,dy' \qquad (33)$$

The solution of (33) may be effected, exactly as in the case of (13), by the change of variable of (21). The results are

$$B^1 = B^0 + \gamma_0\left(\frac{8b}{\lambda}\right)\left[\frac{\beta^2\Delta_1}{1+\alpha^2\Delta_1}\right] \qquad (34)$$

$$\upsilon(\theta) = C\left[1 - \left(\frac{2\alpha\beta\Delta_1}{1+\alpha^2\Delta_1}\right)\cos\theta\right] \qquad (35)$$

where $\upsilon(\theta)$ is defined by (25) if f_0 is replaced by f_1. We remark that for the symmetrical case $\beta = 0$, and the first corrections of (34) and (35) vanish since the mode n = 1 is not excited. If the term n = 2 is also broken off and included in the exact solution of the integral equation we obtain

$$B^2 = B^0 + \gamma_0\left(\frac{8b}{\lambda}\right)\left[\frac{\beta^2\Delta_1}{1+\alpha^2\Delta_1} + \Delta_2\frac{(1-\alpha^2-2\beta^2)^2}{1+2\alpha^4\Delta_2}\right] \qquad (36)$$

which will generally be sufficiently accurate for all practical purposes.

We turn now to the perturbation result of (11). For $n = 1$ and $n = 2$ we use the terms in (36) so that the actual perturbation starts with the term $n = 3$. To evaluate the integrals, we use the change of variable of (21), substitute $\omega_0(0)$ = const., expand $\cos\frac{n\pi y}{b}$ in powers of $\cos\frac{\pi y}{b}$ by 403.3 in Dwight's integral tables, apply the binomial expansion, integrate by 483 in Peirce's integral tables, and simplify the Legendre's duplication formula [15] to obtain

$$(B-B^2) = \gamma_0 \frac{4b}{\pi\lambda}\sum_3^\infty \Delta_n\left[\sum_{r=0}^{\frac{1}{2}(n+1)}(-)^r 2^{(\frac{n}{2}-r-1)}\frac{n\Gamma(n-r)}{\Gamma(r+1)} \cdot \sum_{s=0}^{(n-2r)} 2^{s/2}\alpha^{(n-2r-s)}\beta^s \frac{\Gamma^{-2}\left(\frac{n-2r-s}{2}\right)}{\left(\frac{n-2r-s}{2}\right)}\right]^2 \qquad (37)$$

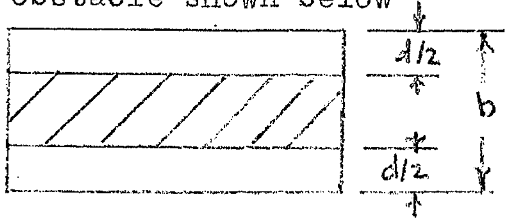
For the symmetrical window where $\beta = 0$ (37) is, of course, considerably simplified. To evaluate the lower bound to B, we observe that the summation in (20) is identical with the summation in (11) if we replace Δn by $\mathcal{E} n/N^2$; hence, from (20), (36) and (37) we obtain

$$\frac{(B^0)^2}{(B^2)} = B^0 + \gamma_0\frac{4b}{\lambda}\left\{\frac{2\beta^2\varepsilon_1}{1+\alpha^2\varepsilon_1} + \frac{\varepsilon_2(1-\alpha^2-2\beta^2)^2}{2+\alpha^4\varepsilon_2} + \frac{1}{\pi}\sum_3^\infty\frac{\varepsilon_n}{n^2}\left[\sum_{r=0}^{\frac{1}{2}(n+1)}(-)^r 2^{(\frac{n}{2}-r-1)}\frac{n\Gamma(n-r)}{\Gamma(r+1)}\sum_{s=0}^{(n-2r)}2^{s/2} \cdot \alpha^{(n-2r-s)}\beta^s\frac{\Gamma^{-2}\left(\frac{n-2r-s}{2}\right)}{\left(\frac{n-2r-s}{2}\right)}\right]^2\right. \qquad (38)$$

As a typical numerical example, we consider the half-open symmetrical window ($y_1 = b/4$, $y_2 = 3b/4$) for $b = \frac{1}{4}\lambda$.

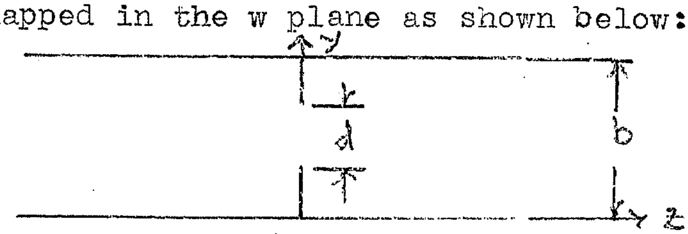
We obtain an upper bound from (37) of $B/Y_o = 0.35512$, while (38) gives a lower bound of $B/Y_o = 0.35510$. Thus we have bounded the exact value of B by quantities differing by less than 0.007%. For this same case, the simple static value (B^o/Y_o) given by (31) is 0.346 and differs from the true value by less than 3%, while (B^2/Y_o) given by (36) is 0.355 to three places. The highest ratio of b/λ encountered in practice will generally be $\frac{1}{2}$, since at this value of b/λ the first higher mode begins to propagate (although not excited by the symmetrical window). For the half-open symmetrical window and $b/\lambda = \frac{1}{2}$, we obtain upper and lower bounds of 0.77034 and 0.77028 for (B/Y_o) (less than 0.01% apart), the static value of (B^o/Y_o) is 0.692 (about 10% from the true value), and (B^2/Y_o) is 0.770 to three places. The 10% error in B^o for $b/\lambda = \frac{1}{2}$, while still not too large for many practical applications, is due to the fact that K is not negligible compared to $\frac{n\pi}{b}$ for all n, being equal to it for n = 1.

From the foregoing it is obvious that the simplest case, for purposes of accurate calculation, is the symmetrical window; however, in practice, the asymmetrical window will be of equal or greater importance. From symmetry, it is clear that the plane $y = \frac{b}{2}$ is a plane of zero normal velocity for the symmetrical window, and hence the asymmetrical window of opening d/2 in a tube of height b/2, is equivalent to a symmetrical window of opening d in a tube of height b. Now the susceptance of both windows is dependent only on the ratios (d/b) and (b/λ), so we may obtain the susceptance of the asymmetrical window by using half the actual wavelength in any of the foregoing results for the susceptance of the symmetrical window. It is of interest to note that the susceptance of the obstacle shown below

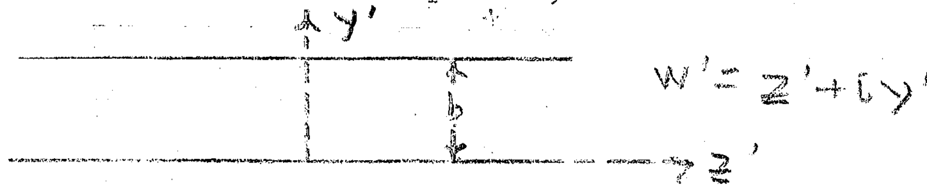


is identical with that of the symmetrical window of opening d as can be seen either by images or by placing a plane of zero normal velocity in the plane $y = b/2$ (by summetry) for the two cases, and observing that they both give the same asymmetrical case.

It was mentioned earlier that the static value B^o of the susceptance may be obtained by conformal mapping. This is true, first because the static approximation $f_o(y)$ is a solution to Laplace's equation (since K is considered negligible compared to $\frac{n\pi}{b}$ the wave equation of 1 (10) reduces to Laplace's equation), and secondly because the $f(y)$ is independent of x, and the problem is therefore two dimensional. The equivalent electrostatic problem is that of finding the excess capacity per unit width due to the window plates in an infinite plane parallel plate condenser. Consider the parallel plates with symmetrical window plates mapped in the w plane as shown below:



(Since the calculation is for purposes of illustration we consider only the symmetrical case.) It is desired to transform the configuration in the w plane to a pair of plane parallel plates in the w' plane without the window plates, viz:



The transformation from the w plane to the w' plane is found to be (16)

$$\cosh\left(\frac{\pi w}{b}\right) = \sin\left(\frac{\pi d}{2b}\right) \cosh\left(\frac{\pi w'}{b}\right) \tag{39}$$

From 3 (16) the hydrodynamical capacity is given by the mass divided by the square of the area, therefore

$$C = \rho_0 \left(\frac{z_1 - z_2}{S}\right) \tag{40}$$

is the capacity per unit width in the w plane, and similarly in the w' plane if we prime C, z_1, and z_2 in (40). In order to obtain the added capacity due to the window plates, the points z_1 and z_2 must be infinitely removed; hence we use the asymptotic form of (39), which between the planes y = y' becomes

$$e^{\pi z/b} = \sin\left(\frac{\pi d}{2b}\right) e^{\pi z'/b} \tag{41}$$

The excess capacity per unit width is then given by

$$C' - C = \frac{\rho_0}{S}\left[(z_1' - z_2') - (z_1 - z_2)\right] = \frac{-2b\rho_0}{\pi S} \log\left[\sin\left(\frac{\pi d}{2b}\right)\right] \tag{42}$$

To obtain the total relative susceptance, we multiply (42) by ω and divide by the characteristic admittance $\rho_0 c/S$ to obtain

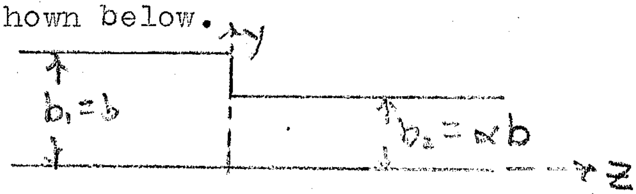
$$\left(\frac{B^\circ}{Y_0}\right) = \frac{4b}{\lambda} \log\left[\csc\left(\frac{\pi d}{2b}\right)\right] \tag{43}$$

(since $K = \frac{\omega}{c} = \frac{2\pi}{\lambda}$) in agreement with (31).

The foregoing example has been treated in some detail in order to illustrate the accuracy that may be expected by virtue of the variational principle. Thus the assumption of the static pressure distribution gives a susceptance (cf. (37)) which is sufficiently accurate for all practical purposes. It is important to realize that the static pressure distribution gives more than the static susceptance, a direct consequence of the variational principle. In the majority of cases, it is not so simple to obtain as much accuracy as in the case of the rectangular window, but the differences (in principle) are only geometrical, and we may expect that very good approximations to the susceptance may be obtained by reasonable approximations to the pressure distribution.

7. Analysis of a Change of Cross Section in a Rectangular Tube

As a second illustration we shall calculate the susceptance due to the change in one dimension of a rectangular tube as shown below.



Although we shall treat only the asymmetrical case, the symmetrical case may be obtained by reflection in the bottom side as in the case of the rectangular window.

The eigenfunctions to the problem are given by 6 (1-5) if we merely use the superscripts 1 and 2 to differentiate between the tubes of heights b_1 and b_2. Since there is no discontinuity in the x direction, and the exciting wave contains only the principal mode, only those modes for which m = 0 will be excited. Substituting 6 (1-5) in 4 (16), dropping the subscript m, carrying out the integrations with respect to x (since the pressure distribution is not a function of x), and writing Y = jB and $\beta = \frac{Kb}{\pi}$ we have

$$\left(\frac{B}{Y_0}\right) = 2\beta \sum_1^{\infty} \left\{ [n^2-\beta^2]^{-1/2} \left[\int_0^{\alpha b} f(y)\cos\left(\frac{n\pi y}{b}\right)dy\right]^2 + \alpha\,[n^2-\alpha^2\beta^2]^{-1/2} \cdot \left[\int_0^{\alpha b} f(y)\cos\left(\frac{n\pi\alpha y}{b}\right)dy\right]^2 \right\} \left\{\int_0^{\alpha b} f(y)\,dy\right\}^{-2} \tag{1}$$

The solution to (1) may be effected as in 4 (21-29) where the functions ϕ^3_{pq} are simply $\phi^2_n(y)$ as given by 4 (1 and 2).

For most practical applications sufficient accuracy could be obtained simply by substituting f(y) equal to a constant in (1). However, more useful approximations may be obtained by static solutions.

Although it is not feasible to solve (in closed form) the integral equation associated with (1), the pressure distribution f(y) may be determined by conformal mapping and then substituted in (1) to give results whose accuracy will be comparable with those of 6 (37) for the analogous window. The equivalent electrostatic problem has been treated in several places (17, 18) (the increase of resistance due to change in conductor cross section is also an equivalent problem.) Although the Schwarz transformation is relatively simple, the resulting function f(y) is too complicated to permit simple evaluation of the integrals in (1); the added capacity due to the discontinuity is found to be (18)

$$C(\alpha) = \frac{K}{\pi}\left[\left(\frac{\alpha^2+1}{\alpha}\right)\cosh^{-1}\left(\frac{1+\alpha^2}{1-\alpha^2}\right) - 2\log\left(\frac{4\alpha}{1-\alpha^2}\right)\right] \tag{2}$$

in M.K.S. units where **K** is the dielectric constant. The formula for electrostatic capacity in M.K.S. units analogous to 6 (40) is

$$C = K\left(\frac{z_1 - z_2}{b}\right) \tag{3}$$

so that **K** must be replaced by $\rho_0 b/S_1$ in (2). (In (3) we use b because the characteristic admittance used is that of tube 1.) Making this substitution in (2), multiplying by ω to obtain the susceptance, dividing by the characteristic admittance of the dominant mode in tube 1 and putting the arc hyperbolic cosine in its logarithmic form we have

$$\frac{B^{o}}{Y_{0}} = 2\beta \log\left[\left(\frac{1-\alpha^{2}}{4\alpha}\right)\left(\frac{1+\alpha}{1-\alpha}\right)^{\frac{1}{2}(\alpha+\alpha^{-1})}\right] \tag{4}$$

where B^{o} is the static approximation to the susceptance.

To obtain the first dynamic corrections to (4) we should split off the static portion of (1), given by setting β = 0 in all the propagation constants, which must be equal to (4), and then substitute the static pressure distribution in the remainder; actually the static pressure distribution is too complex, and we substitute f(y) = const. in the remainder to obtain

$$B = B^{o} + 2\beta Y_{0} \sum_{1}^{\infty} \Delta_{n} \left[\frac{\sin(n\pi\alpha)}{(n\pi\alpha)}\right]^{2} \tag{5}$$

$$\Delta_{n} = \left[n^{2}-\beta^{2}\right]^{-1/2} - n^{-1} \tag{6}$$

A more elementary, but simpler correction can be made by assuming that the dynamic correction factor is something less than $(1-\beta^{2})^{-\frac{1}{2}}$, since this is the correction for the first mode and therefore the maximum possible correction; hence we write

$$B = (1-\beta^{2})^{-1/2} B^{o} \tag{7}$$

We perceive that (7) is an upper bound and may be expected to be good for large α where the series in (1) converges as n^{-3}, but poor for small α, where the early convergence of the series is only as n^{-1}.

Another simple approximation may be obtained by remembering that the susceptance of the analogous window is approximately twice that of the above change of cross section; actually the change of cross section gives somewhat more than half the window susceptance since the plane z = 0 is not a plane of symmetry, and there is some contribution to the susceptance from guide 2. However, the principal discrepancy between the two cases (aside from the factor of 2) is in the nature of the singularity at the edge which has an exterior angle of 270° for the change of cross section and of 360° for the thin window, and inasmuch as the singularity is contained entirely in the static solution one half of the dynamic correction to the window susceptance should be quite accurate. The correction for the symmetrical window is given by the third term in 6 (36) where $\beta = 0$ and $\alpha = \sin(\pi d/2b)$; for the asymmetrical window (analogous to our asymmetrical change of cross section) we must use half the actual wavelength in this correction; hence we obtain

$$B = B^0 + 4\left(\frac{b}{\lambda}\right) Y_0 \left[\frac{\Delta_1 \cos^4\left(\frac{\pi}{2}\alpha\right)}{1 + \Delta_1 \sin^4\left(\frac{\pi}{2}\alpha\right)}\right] \tag{8}$$

where B^0 is given by (4) and Δ_1 is given by (6) above.

For the typical case $\alpha = \beta = \frac{1}{2}$ (6,7 and 8) yield (B/Y_o) = 0.460, 0.452 and 0.452, respectively. It should be expected that (8) will generally be the most accurate of the three approximations; however, (7) is much simpler and almost as good except for small values of α. It may be reasonably expected that (8) should give a lower bound to the true susceptance since half the dynamic correction for the window is presumably too small; hence (7) and (8) appear to give answers accurate to better that 0.2% for the above numerical case.

8. Change of Cross Section and Window in Circular Tube

We shall now consider two very closely related problems which are best treated directly by 4 (24-29): first, the concentric decrease of diameter from $2a_1$ to $2a_2$ in a circular tube, and, second, a thin plate having a concentric hole of diameter $2a_2$ in the plane $z = 0$ in a circular tube of diameter $2a_1$.

The solutions to 1 (12), satisfying 1 (14) in a circular tube of radius a, and normalized according to 2 (4), are

$$\phi_0 = \pi^{-1/2}/a \tag{1}$$

$$\phi_{mn}(r,\theta) = M_{mn} J_m(\chi_{mn} r) \cos(m\theta + \psi_{mn}) \tag{2}$$

$$M_{mn}^{-2} = \frac{2\pi}{(2-\delta_m^0)}\left[\frac{a^2}{2} - \frac{m^2}{2\chi_{mn}^2}\right] J_m^2(\chi_{mn} a) \tag{3}$$

$$J_m'(\chi_{mn} a) = 0 \tag{4}$$

For the particular problems at hand there is radial symmetry, and only those modes for which m = 0 will be excited; hence, dropping the subscript m, (2-4) simplify to

$$\phi_n(r) = \phi_0 \left[J_0(\chi_n r)/J_0(\chi_n a)\right] \tag{5}$$

$$J_1(\chi_n a) = 0 \tag{6}$$

where (6) is the transcendental equation for the determination of the eigenvalues.

For the change of cross section problem the results are given by 4 (24-30) where the functions ϕ^3_{pq} are taken as the set ϕ^2_n; carrying out the integrations by Jahnke and Emde, p. 146, we have

$$B = B_0 + \sum_1^\infty C_p A_p \tag{7}$$

$$B_0 = 4k \sum_1^\infty \left[(\varkappa_n^1)^2 - (k)^2\right]^{-1/2} \left[\frac{J_1(\varkappa_n^1 a_2)}{(\varkappa_n^1 a_2)\, J_0(\varkappa_n^1 a_1)}\right]^2 \tag{8}$$

$$C_p = 4k \sum_1^\infty \left[(\varkappa_n^1)^2 - (k)^2\right]^{-1/2} \left[1 - \left(\frac{\varkappa_p^2}{\varkappa_n^1}\right)^2\right]^{-1} \left[\frac{J_1(\varkappa_n^1 a_2)}{(\varkappa_n^1 a_2)\, J_0(\varkappa_n^1 a_1)}\right]^2 \tag{9}$$

$$D_{pq} = 4k \sum_1^\infty \left[(\varkappa_n^1)^2 - (k)^2\right]^{-1/2} \left[\frac{(\varkappa_n^1)^2}{(\varkappa_n^1)^2 - (\varkappa_p^2)^2}\right] \left[\frac{(\varkappa_n^1)^2}{(\varkappa_n^1)^2 - (\varkappa_q^2)^2}\right] \cdot \left[\frac{J_1(\varkappa_n^1 a_2)}{(\varkappa_n^1 a_2)\, J_0(\varkappa_n^1 a_1)}\right]^2 + \delta_q^p\, k\left[(\varkappa_p^2)^2 - (k)^2\right]^{-1/2} \left(\frac{a_1}{a_2}\right)^2 \tag{10}$$

$$\sum_{q=1}^\infty D_{pq} A_q = -C_p \qquad (p = 1, 2, 3, \cdots \infty) \tag{11}$$

For the case of the window the set ϕ^3_{pq} is again the set ϕ^2_n; hence the results are given by (7-11) above if we merely multiply the right hand sides of (8-10) by a factor of 2 and drop the last term in (10). The results of (7-11) give an upper bound to the true value of B when only a finite number of the coefficients Ap are calculated, approaching the true value uniformly as p is increased. To obtain a lower bound for the case of the window we use 4 (39-43). The functions ϕ^4_{pq} must have radial symmetry, must vanish at r = a_2, and must have vanishing normal derivatives at $r = a_1$; such a set, satisfying 1 (12), is

$$\phi_n'(r) = \phi_0^1 \left[J_0(\chi_n' r) N_0(\chi_n' a_2) - N_0(\chi_n' r) J_0(\chi_n' a_2) \right] \tag{12}$$

$$\left[J_1(\chi_n' a_1) N_0(\chi_n' a_2) - N_1(\chi_n' a_1) J_0(\chi_n' a_2) \right] = 0 \tag{13}$$

where $N_m(x)$ is the Bessel function of the second kind in the notation of Jahnke and Emde. Substituting in 4 (39-43), integrating by Jahnke and Emde, p. 146, and simplifying by the Wronskian relating $N_1(x)$ and $J_0(x)$ [19], we obtain

$$B = \sum_1^\infty C_p' A_p' \tag{14}$$

$$C_p' = 2(\pi \chi_p' a_1)^{-2} \tag{15}$$

$$D'_{pp'} = K(\pi a_1)^{-2} \sum_{n=1}^{\infty} \left[(\chi_n)^2 - (K)^2 \right]^{1/2} \left[\frac{J_0(\chi_n a_2)}{J_0(\chi_n a_1)} \right]^{-2} \cdot \left[(\chi_{p'})^2 - (\chi_n)^2 \right]^{-1} \left[(\chi_p)^2 - (\chi_n)^2 \right]^{-1} \tag{16}$$

$$\sum_{p'=1}^{\infty} D'_{pp'} B_{p'} = -C_p' \quad (p = 1, 2, 3 - - - \infty) \tag{17}$$

The first approximations to the above susceptances are simply B^o and $B^o = (C_1)^2/D'_{11}$, respectively, and in the case of the window the true value lies somewhere between these values, while the accuracy of B^o for the change of cross section will be indicated by the discrepancy between these values for the analogous window. We observe that B^o for the window is exactly twice B^o for the change of cross section. The next approximations are

(18)

(19)

and should be good to within a few percent for all frequencies which satisfy the original requirement that only the principal mode be freely propagated.

For higher order solutions we must resort to the solution of simultaneous equations by iteration if the desired accuracy is high. An interesting method of solution may be effected if we notice that the equations are symmetrical about the diagonal (i.e. Dpq = D qp), whence we may regard the coefficients Ap or Bp as the current flowing in the p'th mesh of an electric circuit having a mesh voltage of Cp, a self-impedance Dpp, and a mutual impedance Dpq with any other mesh q. Since all the coefficients Dpq are real only direct currents would be required, and such "d.c. calculating boards" are owned by almost all electrical utility companies. It would probably be necessary to insert scale factors in Cp, Dpq, and Ap to give convenient electrical units. Although such a scheme makes the solution of a large number of simultaneous equations simple it must be remembered that the computation of Cp and Dpq becomes increasingly difficult as p and q are increased.

9. Calculation of Reflection and Transmission

In many cases it is desired to know the reflection and transmission coefficients due to a discontinuity. For the category of discontinuities considered in section 4 the equivalent circuit is a junction between transmission lines of characteristic admittances Y'_o and Y_o^2, with an admittance jB in shunt at the junction. The voltages and currents on the lines are given by 4 (4) and 4 (5). If we consider the case where the incident wave is in the first tube we may set $a_1 = 1$, $a_2 = 0$ and write

$$R = b^1 \qquad , \quad T = b^2 \left(\frac{S_1}{S_2}\right)^{1/2} \tag{1}$$

for the reflection coefficient R and the transmission coefficient T (note that the transmission coefficient defined is for the longitudinal velocity and may be greater than unity), where 2 (5) and 3 (15) have been used for simplification; from the latter equations we may write

$$\left(\phi_0^2/\phi_0^1\right) = \left(S_1/S_2\right)^{1/2}, \quad \left(Y_0^1/Y_0^2\right) = \left(S_1/S_2\right) \tag{2}$$

From 4 (4) and 4 (5) the voltages and currents in the plane of discontinuity are

$$V_0^1 = (1+R)\,(S_1)^{1/2} \quad , \quad V_0^2 = T\,S_2\,(S_1)^{-1/2} \tag{3}$$

$$I_0^1 = Y_0^1\,(1-R)\,(S_1)^{1/2} \quad , \quad I_0^2 = -T\,S_2\,(S_1)^{-1/2} \tag{4}$$

while from 4 (8) and 4 (12) the circuit equations are

$$V_0^1 = V_0^2. \tag{5}$$

$$(I_0^1 + I_0^2) = jBV_0^1 \tag{6}$$

Solving for R and T we have

$$R = \frac{1-(m+jn)}{1+(m+jn)} \tag{7}$$

$$T = \frac{2m}{1+(m+jn)} \tag{8}$$

$$m = \frac{Y_0^2}{Y_0^1} = \frac{S_1}{S_2} \quad , \quad n = \frac{B}{Y_0^1} \tag{9}$$

If it is desired to define the transmission coefficient for total flow the m should be removed from the numerator of (9). We observe that the effect of the higher modes is to change both the phase and magnitude of the reflected and transmitted waves from those values that would have been obtained if the higher modes had been neglected (i.e. n = 0), the usual assumption in the calculation of reflection at a junction. For the case where there is no change of cross section, but only a window, m is, of course, unity, and (7) and (8) become

$$R = \frac{-j\left(\frac{n}{2}\right)}{1+j\left(\frac{n}{2}\right)} \tag{10}$$

$$T = \frac{1}{1+j\left(\frac{n}{2}\right)} \tag{11}$$

As a numerical example consider the case of an asymmetrical change of height in a rectangular tube where $b_2 = \frac{1}{2}b_1$

so that m = 2, while from 7 (7)

$$n = 0.784\,\beta(1-\beta^2)^{-1/2}, \quad \beta = 2\left(\frac{b}{\lambda}\right) \qquad (12)$$

Then for the case where n is neglected (very low frequencies) we obtain R = -0.33 and T = 1.33. If we next let $\beta = \frac{1}{2}$ and n = 0.452, we obtain R = -0.362 $\angle 15.8^\circ$ and T = 1.316 $\angle -8.55^\circ$, while for β = 0.9 and n = 1.62, we obtain R = -0.56 $\angle 30^\circ$ and T = 1.174 $\angle -28.4^\circ$. It is evident that, even for frequencies which are sufficiently low to prevent the propagation of higher modes, neglecting the higher modes in a discontinuity may lead to serious errors in the calculation of reflection and transmission coefficients, particularly with regard to phase. If we consider the case where the incident wave is in the smaller of the two tubes m = $\frac{1}{2}$ while n is twice the value given in (12). For β = 0 we obtain R = 0.33, T = 0.67, for β = 0.5 R = 0.627 $\angle -42.1^\circ$, T = 0.571 $\angle -31^\circ$, and for β = 0.9 R = 0.919 $\angle -146.4^\circ$, T = 0.28 $\angle -65.2^\circ$. For the incident wave in the smaller tube we observe that the effect of the higher modes is even more pronounced.

Suppose it is desired to cancel the reflection of our discontinuity (i.e. to match impedances) by inserting a window in the tube of the incident wave. It is necessary to determine how far back of the discontinuity to place this window and to evaluate the required susceptance. The equivalent circuit is then as shown below where B_2 is the susceptance due to the discontinuity, B_1 is the window susceptance, and Y_o^2 is the characteristic admittance of the second tube. It is desired that the admittance seen at terminal 1 be Y_o^1, and this admittance is given by 3 (27) as

$$Y_o^1 = jB_1 + Y_o^1 \left[\frac{j\sin(Kl) + \left(\frac{Y_o^2 + jB_2}{Y_o^1}\right)\cos(Kl)}{\cos(Kl) + j\left(\frac{Y_o^2 + jB_2}{Y_o^1}\right)\sin(Kl)} \right] \qquad (13)$$

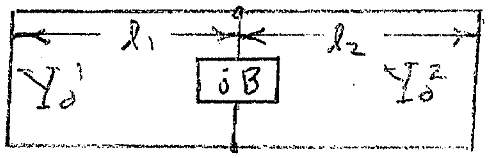
which yields the equations

$$(1-m)\cos(Kl) + (mn_1 - n_2)\sin(Kl) = 0$$
$$(n_1 + n_2)\cos(Kl) + (1 - m - n_1 n_2)\sin(Kl) = 0 \qquad (14)$$

on equating real and imaginary parts and using the notation of (9). The simultaneous solution of (14) will give the desired values of n_1 and l, that value of n_1 being chosen which is positive, and the value of l being taken long enough (i.e. by adding half wavelengths) to ensure that there is no interaction of higher modes. It is probably simpler to use graphical methods of solution via circle diagrams [20]. The problem of impedance matching of transmission lines is discussed extensively in the literature.[20]

10. Application to Cavity Resonators

The impedance concept may be profitably applied to the calculation of a fairly common type of resonators consisting of two tubes joined through the type of discontinuity discussed in section 4 with the remaining ends closed, provided that the closed ends are sufficiently removed from the discontinuity so that no higher modes strike the closed ends. The equivalent circuit is then



and the condition for resonance is, using 3 (25) for the susceptance of a short circuited line,

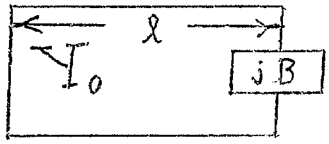
$$\cot\left(\frac{2\pi l_1}{\lambda}\right) + m\cot\left(\frac{2\pi l_2}{\lambda}\right) - n = 0 \tag{1}$$

where m and n are given by 9 (9). If the desired resonant wavelength λ is known then n may be calculated, either l_1 or l_2 may be fixed, and the remaining length may be calculated. If l_1 and l_2 are fixed then the resonant wavelength must be found by successive approximations since n is generally not a simple closed function of this wavelength. Such a resonator furnishes a simple and accurate experimental determination of B.

If the closed ends are replaced by open ends we use 3 (24) to obtain

$$\tan\left(\frac{2\pi l_1}{\lambda}\right) + m\tan\left(\frac{2\pi l_2}{\lambda}\right) + n = 0 \tag{2}$$

A second type of resonator which is susceptible to such treatment is a tube closed at one end with a small hole in the plate which closes the other end, such as the hole through which excitation is received. The susceptance of such a hole may be approximated by half the susceptance such a hole would have as a window in an otherwise continuous tube. The equivalent circuit is then



For resonance we have

$$n = \cot\left(\frac{2\pi l}{\lambda}\right) \qquad (3)$$

If the hole is small n will be large and 1 will be close to half a wavelength, being somewhat greater, since n must be positive, and the extra length is then given by

$$\delta l = \frac{\lambda}{2\pi n} \qquad (4)$$

Hence the resonant wavelength is <u>less</u> than 21 by $100/2\pi n\%$.

It should be remembered that the cross section of the resonators is assumed small compared to the wavelength, so that only the principal mode can resonate.

Bibliography

1. Courant and Hilbert, Methoden der Mathematischen Physik, V II, Ch. 7, Julius Springer, Berlin, 1937;

2. Smythe, W. R., Static and Dynamic Electricity, Section 3.06, McGraw-Hill, New York, 1939;

3. Ballatine, S., J. Franklin Inst., 221 (1936, 469-484;

4. Boussinesq, J., Comptes Renaus, 118 (1894), 271-276, 162-166, 223-226;

5. Liouville, J., J. de Math. (2) 1 (1856), 1-6;

6. Weatherburn, C. E., Quart. J. Math., 46 (1915), 63-82;

7. loc. cit. (2);

8. Slater, J. C., Microwave Transmission, Ch. I, Sec. 6, McGraw-Hill, New York, 1942;

9. Morse, P. M., Vibration and Sound, p. 193, McGraw-Hill, New York, 1936;

10. Miles, J. W., "Applications and Limitations of Electrical Mechanical Analogies, New and Old", J. Acoustical Society, January, 1943;

11. Everitt, Communication Engineering, pp. 172-173, McGraw-Hill, New York, 1937;

12. loc. cit., (2) Section 10.15;

13. loc. cit., (8);

14. Whittaker and Watson, Modern Analysis, Sec. 11.3, Cambridge, 1927;

15. loc. cit.,(14), Sec. 12.15;

16. loc. cit.,(2), Section 4.22;

17. loc. cit., (2), Section 6.13;

18. Walker, Miles, Conjugate Functions for Engineers;

19. Jahnke and Emde, Tables of Functions, p. 144, Dover Publications, New York, 1943;

20. *loc. cit.*, (8), Ch. I;

21. Miles, John, "The Reflection of Sound Due to a Change of Cross Section in a Cylindrical Tube", *J. Acoustical Soc.*, July, 1944.