I. THE PROPAGATION OF SHOCK WAVES IN NON-UNIFORM GASES.

II. THE STABILITY OF THE SPHERICAL SHAPE OF A VAPOR CAVITY IN A LIQUID.

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I. ABSTRACT

The one-dimensional propagation of shock waves in a perfect gas in which the pressure and the density are not necessarily uniform is investigated by seeking similarity solutions of the equations describing the non-isentropic motion of the gas. It is shown that such solutions can be found and that they can be related to specific types of compressive piston motion. In particular, the propagation of the shock resulting from the uniform compressive motion of a piston in a non-uniform gas is studied. For this case a first order, ordinary, non-linear differential equation which determines the shock strength as a function of distance is derived. An analytic solution of this equation is obtained for a gas in which the pressure is constant but the density varies, and for which the ratio of the specific heats, γ , is 3/2. There is no restriction placed upon the permissible density variations. In situations in which the pressure and the density distributions are variable, and in which general values of γ are allowed, numerical results are presented. It is not possible in such cases to derive analytic solutions of the equation. The discussion of the shock propagated by the non-uniform motion of a piston is more difficult. However, some details are given in the case of strong and weak shocks resulting from a decelerative piston motion.

II. ABSTRACT

The stability of the spherical shape of a gas bubble in a liquid is investigated for the case in which the difference between the pressure in the bubble, P_i, and the pressure in the liquid, P_o, is constant. These conditions apply approximately to a vapor bubble growing, ($P_i > P_o$), or collapsing, ($P_i < P_o$), in a liquid at constant external pressure. The general solution for the behavior of a small deformation in the spherical shape of the cavity is readily determined when surface tension is neglected. For a growing bubble the deformation increases slowly and monotonically; for a collapsing bubble the deformation oscillates with small amplitude until the mean radius of the bubble approaches zero, when the magnitude of the deformation increases rapidly. The consistency and applicability of the small amplitude theory is thus demonstrated. A solution is also obtained which includes the effect of surface tension. In this case the distortion amplitude decreases with increasing radius for the expanding bubble and the singularity in the distortion amplitude for the collapsing bubble at zero radius persists.

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PART ONE

THE PROPAGATION OF SHOCK WAVES IN NON-UNIFORM GASES.

I. INTRODUCTION

The classical theory of one-dimensional gas dynamics concerns itself almost exclusively with gas flows which take place under isentropic conditions. Although the types of flow which can be discussed within the confines of this theory include many interesting cases, the restriction to isentropic flows is fundamentally one of mathematical expediency. If one considers the basic equations for the motion of a perfect gas one readily understands the simplification introduced by the isentropic quality of the motion. These equations are hyperbolic, partial, differential equations and, by using the theory of characteristics, Riemann and others succeeded in developing a complete mathematical method to determine their solutions. It may be remarked that the Riemann method is readily applicable only to one-dimensional flows and cannot be easily applied to two- or three-dimensional configurations. If the condition of isentropy is removed the number of characteristics increases by one, to three, and the problem of solving the equations becomes vastly more difficult. In fact, the use of the equations of non-isentropic motion introduces mathematical complications which apparently make it impossible to develop an explicit theory corresponding to the classical one.

A fundamental problem of gas dynamics is the analysis of the mode of propagation of shock waves in gases. In order to discuss this question by the classical methods the assumption that the motion is isentropic must be made. This means that the analysis is restricted to situations in which the gas on both sides of the shock is in a constant state. Accordingly, only the propagation of constant strength, constant velocity shock waves is included and many problems of interest concerning the motion of non-uniform shocks are automatically excluded. A uniform, as distinct from a non-uniform, shock is of constant strength and possesses a straight line path in the coordinate-time plane. For situations in which a shock is traveling through a gas of variable pressure and density the path in this plane is curved and the shock strength is generally not constant. Such shock waves are designated "non-uniform". Due to the nature of the situation the condition of isentropy must be relaxed and the equations of non-isentropic motion used to analyze the propagation of non-uniform shocks.

The interest in this problem, aside from its basic mathematical value, has increased in recent years for several reasons. Firstly, there exists the obvious need to be able to predict the method of propagation of shock waves in non-uniform gases, e.g., in the earth's atmosphere. Here, due to the presence of pressure and density gradients, the non-uniform conditions referred to previously, exist and the exact discussion demands the use of the concept of anisentropy. Secondly, the recent developments in shock tube research stress the value of the theoretical study of this problem. Finally, the realization that certain aspects of problems of astrophysical interest might be elucidated by a study of shock wave phenomena led to many attempts to solve the pertinent equations.

The majority of the methods of solution adopted are of an ad hoc nature and hence present a peculiar problem of precis to one who attempts to survey the published work. It is helpful, however, in presenting a brief survey of the literature to consider the published discussionsunder the headings general, astrophysical, and pure mathematical. It should be added that all the analyses are restricted to the one-dimensional case.

Chandrasekhar, [1], in the first attempt to analyze the propagation of non-uniform shocks assumed that the shock is of "moderate" strength and that the entropy and the appropriate Riemann invariant are constant through the shock. This approximate method is useful and has been generalized by Friedrichs [2]. It is especially practical if the gas ahead of the shock is at rest. Of general interest also are the discussions presented by Jones, [3,4], who used the method of similarity solutions to determine the laws of propagation of shock waves in regions of constant pressure but variable density. The investigation of Reference 3 is concerned primarily with constant strength shocks while that of Reference 4 attempts to extend the analysis to include variable shock strengths as well as the constant energy shock. Much of Jones's work appears to be unduly complicated relative to the results which he obtained. A further reference to his work on this point is made later. An investigation by an entirely different means has been made by Chisnell [5]; his method is approximate as it assumes that the gas through which the shock is traveling is composed of a sequence of layers of slightly different densities but uniform pressures which are separated by contact discontinuities. It is further assumed that second order changes in the shock strength and the density are negligible. This method is valuable as it enables one to give an approximate discussion of the effects of reflected waves.

The first application of shock wave theory to astrophysical problems was made by Burgers, [6], using a method which was later elaborated upon by Robbertse and Burgers, [7], and by Burgers [8]. The aim was to discuss the motion of a gas cloud in interstellar space. Analytic results were obtained in the case of the motion of very strong shocks through a gas in which the pressure is constant and the density decreases inversely as the $\frac{7}{4}$ th power of the distance. To derive results for other, and perhaps more realistic, density variations would demand a separate analysis in each case if Burgers' method were to be used. The method has the further drawback that it is only after the analysis has been completed that one can know the density distribution for which it is valid. Another application of an astrophysical nature has been carried out by Carrus _et_ _al_., [9,10], in an attempt to discuss shock wave propagation in a generalized Roche model.* In this case the very questionable assumption of constant shock strength is made. The solutions are obtained by numerical methods and the reader is referred to the original papers for details. The

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* A classical generalized Roche model consists of "a massive core of finite dimensions but arbitrary structure surrounded by an envelope of infinitesimal weight in which the density falls off as the inverse square of the distance from the center" [9].

restriction to constant shock strengths was relaxed by Whitham, [11], in his analysis of shock wave propagation in stars. Whitham's method is entirely analytic and is based upon a method of correcting solutions of the linearized equations to give an adequate description of the flow due to the non-linear propagation of shock waves. This approximate method is powerful but is restricted in its applicability to certain pressure and density configurations. It is restricted further to the analysis of weak shocks only. It did exhibit, however, the possibility that as the shock propagated, its strength might increase. This possibility also arises in the analysis to be presented later. Finally, one may mention in this category the discussion of the motion of non-uniform shocks in regions of constant density and variable pressure which was given by Hain and v. Hoerner [12]. Their method of solution was based upon the use of characteristics in conjunction with an elaborate numerical analysis.

Simultaneously with the work mentioned, attempts have been made to derive uniqueness and existence theorems for the solutions of the equations of non-isentropic motion of a perfect gas. These have been reported in a series of papers, written from a purely mathematical viewpoint, by Ludford and Martin, [13], and by Martin [14,15].

A disadvantage of many of the analyses referred to is that the pressure and density of the non-uniform gas may not be arbitrarily prescribed. Further, in many cases the analyses are concerned with the propagation of a rather indefinite shock wave;

e.g., no reference is made to the method of generation of the shock or to the energy input behind it. Clearly, if the shock motion were related to a definite rate of energy input the resulting discussion would be of a more practical value. It is one of the advantages of the method of analysis to be presented here that the shock wave under consideration is always well defined.

The present analysis consists of finding solutions of the Lagrangian form of the equations of non-isentropic motion by the method of separation of the variables. In this way a first order, ordinary, differential equation, which determines the shock strength as a function of distance, is derived. It is shown that the flows to be considered can be related to the compressive motion of a piston and hence, that the shocks can be made definite in the sense mentioned above. A further advantage of this method is that there are no restrictions on the permissible forms of pressure and density distributions. The method, however, is applicable only to certain types of piston motion and cannot be applied in general.

The problem to be discussed is formulated exactly as follows: A semi-infinite tube of ideal gas extends along the $\underline{a}$-axis and is closed at the end $a = a_o$ by a piston. The gas is assumed to be non-uniform, i.e., its pressure and density are considered to be functions of $\underline{a}$. At the instant $t = t_o$ the piston is moved into the gas with a velocity which is a given function of time. A shock wave is generated by the sudden motion of the piston. It is required to determine the dependence of the strength of this shock on the distance traveled.

Two types of piston motion will be treated. Firstly, a uniform, and secondly, a decelerative, motion. In order to make the problem more amenable to mathematical analysis, the following, standard approximations are made.

a) The effect of reflected waves, as well as of possible expansion waves due to the retardation of the piston, will be neglected. It is understood that a full discussion of the flow would necessitate an analysis of the interaction of such waves with the shock. This analysis is not attempted here.

b) The gas flow is assumed to be adiabatic. Each volume element of the gas is considered to change its state without heat loss through conduction or radiation. The effect of viscosity is neglected. It is accordingly assumed that each gas particle receives an increase in its entropy when the shock wave passes through it and that it retains this new entropy for all times afterwards. Thus the entropy in the flow behind the shock is a function of position and not of time.

II. DERIVATION OF THE BASIC EQUATIONS.

In the method of analysis to be presented the shock wave velocity is of major importance. This velocity can be written in terms of the strength of the shock and the pressure and density distributions ahead of it. The strength of the shock is defined to be the ratio of the pressure jump across it to the pressure immediately in front of it. In a situation in which the pressure and the density ahead of the shock are variable, the strength will accordingly vary with the distance over which the shock has traveled. The aim of the following investigation is to derive an equation which will determine the dependence of the shock strength on distance. It will be shown that once this dependence is known solutions to the equations of non-isentropic motion can be found. An explicit expression for the shock velocity will now be derived.

The flow parameters immediately behind the shock are denoted by the subscript 1 while the undisturbed pattern ahead is indicated by o. The position of the shock at any instant t is represented by

$$a = \alpha(t) \tag{1}$$

or, inversely, by

$$t = \tau(a) \ . \tag{2}$$

The shock velocity is therefore

$$v = \dot{\alpha}(t) \tag{3}$$

where, as in all the following work, a dot denotes differentiation with respect to the time t. In accordance with the definition already stated the shock strength, $s(a)$, is given by the relation

$$p_1 = p_o(a)\,[1 + s(a)] \quad . \tag{4}$$

The conservation of the mass and of the momentum across the shock are expressed respectively by the relations

$$\rho_1(u_1 - \dot{a}) = \rho_o(u_o - \dot{a}) \tag{5}$$

and

$$\rho_1\, u_1(u_1 - \dot{a}) + p_1 = \rho_o\, u_o(u_o - \dot{a}) + p_o \quad . \tag{6}$$

In addition, the standard Rankine-Hugoniot formula is valid, i.e.,

$$\frac{\rho_1}{\rho_o} = \frac{p_1 + \mu^2\, p_o}{p_o + \mu^2\, p_1} \tag{7}$$

where for conciseness $\frac{\gamma - 1}{\gamma + 1}$ is denoted by μ^2.

The required expression for the shock velocity is found by combining equations 5 and 6 and modifying the result by the use of equations 4 and 7. Thus the shock velocity is

$$\dot{a}(t) = \pm \left[\frac{p_o(a)}{\rho_o(a)}\right]^{1/2} \left[\frac{1 + s(a) + \mu^2}{1 - \mu^2}\right]^{1/2} + u_o \,. \tag{8}$$

The ambiguity of sign in equation 8 is removed when attention is confined to shock waves moving in the positive <u>a</u> direction. For simplicity, the gas is assumed to be quiescent initially and hence u_o is zero in equation 8. Accordingly, the final expression for the velocity is

$$\dot{a}(t) = \left[\frac{p_o(a)}{\rho_o(a)}\right]^{1/2} \left[\frac{1 + s(a) + \mu^2}{1 - \mu^2}\right]^{1/2} \,. \tag{9}$$

The position of the shock, at any time t, is then determined by

$$t = \tau(a) = \int \left[\frac{p_o(a)}{\rho_o(a)}\right]^{-1/2} \left[\frac{1 + s(a) + \mu^2}{1 - \mu^2}\right]^{-1/2} da \quad (10)$$

The density of the gas directly behind the shock is obtained from equations 4 and 7 in the form

$$\rho_1 = \rho_o(a) \frac{1 + s(a) + \mu^2}{1 + \mu^2[1 + s(a)]} \quad . \qquad (11)$$

It may be remarked that certain results pertaining to the propagation of shocks which are assumed to be of constant strength are immediate corollaries of equation 9. For example, if in equation 9 one assumes that the strength and the pressure are constant then

$$\dot{a}(t) \propto [\ \rho_o(a)]^{-1/2} \quad . \qquad (12)$$

Consequently, if the density distribution has a power dependence on the distance a, say

$$\rho_o(a) \propto a^{2m} \ , \qquad (13)$$

the position of the shock is given by an expression of the form

$$t \propto a^{-m+1} \quad . \qquad (14)$$

The converse of this statement is also true, namely, that equation 14 implies equation 13. Further, it follows from equation 12 that if

$$\rho_o(a) \propto a^{-\varepsilon} \ ,$$

where $\varepsilon > 0$, then the shock velocity increases without limit as the shock progresses. Since the strength of the shock is assumed to be constant it is clear that an infinite amount of energy would be required to propagate the shock. These results have been demonstrated in a slightly restricted form and in a different manner by Jones [3,4].

In order to discuss the gas flow in the region traversed by the shock discontinuity, the Lagrangian form of the equations of motion will be utilized. Each particle of the gas will be identified by its abscissa a, i.e., its position before the shock passed over it. The parameters of this particle after the shock has passed will be functions of a and t. Accordingly,

$$\begin{aligned} x &= x(a,t) \\ p &= p(a,t) \\ \rho &= \rho(a,t) \end{aligned} \qquad (15)$$

denote respectively the position, pressure and density of the gas particle a, assuming that the shock has passed through it. In this notation the equations of motion are

$$x_{tt} + \frac{p_a}{x_a} = 0, \qquad (16)$$

$$(\rho x_a)_t = 0, \qquad (17)$$

and

$$(p \rho^{-\gamma})_t = 0, \qquad (18)$$

where subscripts denote partial derivatives. Equation 18 is the mathematical statement of the adiabatic nature of the gas flow. It is to be noted that no external forces have been represented in the equations. While it is appreciated that stability considerations would demand the existence of such forces when density and pressure gradients are admitted, they are neglected on the grounds that in comparison with the forces due to the shock motion they are negligible. Solutions of the equations of motion 16, 17, and 18, will be sought by the method of separation of the variables. For this purpose it is assumed that

$$x(a,t) = f_1(a)\ g_1(t)\ ,$$

$$p(a,t) = f_2(a)\ g_2(t)\ , \qquad (19)$$

and

$$\rho(a,t) = f_3(a)\ g_3(t)\ .$$

By direct substitution of these relations into the equations of motion the following system of ordinary differential equations is obtained;

$$\frac{g_1\ \ddot{g}_1\ g_3}{g_2} = K_1\ , \qquad (20)$$

$$\frac{f_2'}{f_1'\ f_1\ f_3} = -K_1\ , \qquad (21)$$

$$g_1\ g_3 = K_2\ ; \qquad f_1'\ f_3 \neq 0 \qquad (22)$$

and

$$g_2\ g_3^{-\gamma} = K_3\ ; \qquad f_2\ f_3^{-\gamma} \neq 0\ . \qquad (23)$$

In equations 20,...,23 the K_r represent the constants of separation. Derivatives with respect to the time, t, are denoted by dots and with respect to the distance, a, by primes.

The equations 20, 22 and 23 serve to determine a differential equation for any one of the g-functions; e.g., one finds that

$$\ddot{g}_1\ g_1^{\gamma} = \text{constant}\ , \qquad (24)$$

where the constant is a function of the K_r which need not be written explicitly here. Equation 24 can be solved when γ is a rational fraction by the conventional methods. The solution of equation 24 determines $g_2(t)$ and $g_3(t)$ through equations 22 and 23. Hence, this method of solution of the equations of motion imposes, as one would naturally expect, certain restrictions upon the types of flow which can be discussed. There exists a compensatory fact however, in that the f-functions are restricted only in the sense that equation 21 must be satisfied. Accordingly, solutions of the equations will have been found if three functions of a are determined such that at any instant t_1 at which the shock is at position a_1,

$$x(a_1,t_1) = a_1 ,$$

$$p(a_1,t_1) = p_o(a_1)\,[1 + s(a_1)] ,$$

and

$$\rho(a_1,t_1) = \rho_o(a_1)\left[\frac{1 + s(a_1) + \mu^2}{1 + \mu^2[1 + s(a_1)]}\right].$$

It is necessary in addition that the three functions should satisfy equation 21. These requirements are fulfilled by the functions

$$f_1(a) = \frac{a}{g_1[\tau(a)]} , \qquad (25)$$

$$f_2(a) = \frac{p_o(a)\,[1 + s(a)]}{g_2[\tau(a)]} , \qquad (26)$$

and

$$f_3(a) = \frac{\rho_o(a)[1 + s(a) + \mu^2]}{g_3[\tau(a)]\,\{1 + \mu^2[1 + s(a)]\}} , \qquad (27)$$

where

$$\frac{f_2'}{f_1' \, f_1 \, f_3} = -K_1 \, , \tag{21}$$

and where $\tau(a)$ is defined by equation 10. It is to be noted that equation 21 is the equation which essentially determines the shock strength $s(a)$. Before considering this equation in detail it is convenient to introduce the concept of the piston motion.

III. THE PISTON MOTION

A more concrete picture of the physical situation can be given by relating the gas flow to the motion of a piston. In fact, it will now be shown that the g-functions determine the types of compressive piston motion which may be used. It is clear that due to the method of solution employed one may not arbitrarily prescribe the piston motion. For example, it is not possible to discuss the flow resulting from the accelerative motion of a piston within the purview of this analysis.

Since the position of the gas particle $\underline{a}$ is denoted after the shock has passed it by

$$x(a,t) = f_1(a)\ g_1(t)\ ,$$

it follows, that if $a = a_o$ is the starting point of the piston, then for all t

$$f_1(a_o)\ \dot{g}_1(t) = v_p(t)\ . \qquad (28)$$

In equation 28 $v_p(t)$ represents the piston velocity which is in general a function of time. However,

$$f_1(a) = \frac{a}{g_1[\tau(a)]} \qquad (25)$$

and hence

$$v_p(t) = \frac{a_o}{g_1(t_o)}\ \dot{g}_1(t)\ . \qquad (29)$$

Equation 29 presents the relationship between the piston velocity and the g-functions. It will be remembered that $g_1(t)$ must satisfy equation 24, i.e.,

$$\ddot{g}_1\ g_1^{\gamma} = \text{constant}\ . \qquad (24)$$

In order to make the analysis more definite only two solutions of this equation, and accordingly only two types of piston motion, will be considered. Firstly, the solution

$$g_1(t) = \alpha t + \beta \tag{30}$$

which, by equation 29, represents a uniform piston motion. Secondly, the solution

$$g_1(t) = \lambda t^{2/\gamma + 1} \tag{31}$$

which represents a decelerated motion of the piston.* In equations 30 and 31, α , β, and λ are constants of integration. Only the ratio β/α is important; from equation 29 it is found that

$$\frac{\beta}{\alpha} = \frac{a_o}{v_p} - t_o \quad . \tag{32}$$

(1) THE UNIFORM PISTON MOTION.

In discussing the flow due to a uniform compressive piston motion equation 30 is applicable and hence equations 25, 26, and 27 determine the flow properties in the form

$$x(a,t) = \frac{a(t + \beta/\alpha)}{\tau(a) + \beta/\alpha} \quad , \tag{33}$$

$$p(a,t) = \frac{p_o(a)\ [1 + s(a)][t + \beta/\alpha]^{-\gamma}}{[\tau(a) + \beta/\alpha]^{-\gamma}} \quad , \tag{34}$$

and

$$\rho(a,t) = \frac{\rho_o(a)[1 + s(a) + \mu^2][t + \beta/\alpha]^{-1}}{[\tau(a) + \beta/\alpha]^{-1}} \quad . \tag{35}$$

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* This is true for $\gamma > 1$, which is necessarily so on physical grounds.

The solution given by equation 30 implies that

$$f_2'(a) = 0 \tag{36}$$

which can be stated as

$$f_2(a) = \text{constant} = C. \tag{37}$$

The constant, C, can be evaluated and is found to be

$$C = p_o(a_o)\ [1 + s(a_o)]\ \left(\frac{a_o}{v_p}\right)^{\gamma} \tag{38}$$

Equation 36, or 37, serves to determine the shock strength $s(a)$. Before discussing it, however, it will be useful to determine the energy input of the piston.

The work done by the piston, up to time t, which is equivalent to the energy input, $E(t)$, is given by

$$E(t) = \int_{t_o}^{t} p(a_o, t)\ \frac{da}{dt}\ dt$$

which, by equation 34, is expressible in the form

$$E(t) = \frac{v_p}{(\gamma - 1)}\ \frac{p_o(a_o)[1 + s(a_o)]}{[\tau(a_o) + \beta/\alpha]^{-\gamma}} \left[(t_o + \beta/\alpha)^{-\gamma+1} - (t + \beta/\alpha)^{-\gamma+1} \right]$$

Thus, if one allows the time t to become infinite

$$E(t) \rightarrow \frac{a_o}{\gamma - 1}\ p_o(a_o)\ [1 + s(a_o)]\ ; \qquad t \rightarrow \infty \tag{39}$$

where equations 37 and 38 have been utilized. Hence, the energy input of the uniformly moving piston tends to a finite limit as the time interval $t - t_o$ becomes infinite. It will be shown later that this result does not imply that the shock strength $s(a)$ must be a monotonically decreasing function of the abscissa $\underline{a}$.

The shock strength is determined from equation 36, i.e.,

$$\frac{d}{da}\;\frac{p_o(a)\;[1 + s(a)]}{g_2[\,\tau(a)\,]} = 0 \tag{40}$$

which can be written in the form

$$\frac{d}{da}\,p_o(a)\;[1 + s(a)] = -C(1 - \mu^2)^{\frac{1}{2}}\;[p_o(a)]^{\frac{1}{2} + \frac{1}{\gamma}}\;[\rho_o(a)]^{\frac{1}{2}}\;[1 + s(a)]^{1 + \frac{1}{\gamma}}\;[1 + s(a) + \mu^2]^{-\frac{1}{2}} \tag{41}$$

where C is the constant defined by equation 38. If the pressure and density distributions ahead of the shock are specified, equation 41 determines $s(a)$. The equation is non-linear and is not generally integrable in an analytic form. However, a numerical analysis of it is not difficult since it is a first order equation. Analytic solutions can be found in some special cases which are of considerable interest. One of these is now considered.

<u>The Solution for Constant Pressure Distributions</u>.

In this case equation 41 admits of an analytic solution. It is convenient to set

$$\sigma(a) = 1 + s(a)$$

and

$$p_o(a) = p, \qquad \text{a constant,}$$

in equation 41. One finds then

$$\frac{d\sigma(a)}{da} = -\,C\;p^{\frac{1}{\gamma} - \frac{1}{2}}\;[\,\rho_o(a)]^{\frac{1}{2}}\;[\,\sigma(a)]^{1 + \frac{1}{\gamma}}\;[\,\sigma(a) + \mu^2]^{-\frac{1}{2}} \tag{42}$$

This equation shows that the shock strength is a decreasing function of the abscissa á for any density distribution; C is, by equation 38 a positive constant. To solve equation 42 analytically the value $\gamma = \frac{3}{2}$ is taken. While this value does not correspond to any gas found in nature, it is sufficiently close to actual values to justify its use here. The solution of the equation, under this condition, is

$$\frac{3}{2}(1+s)^{-2/3}\left(\frac{6}{5}+s\right)^{1/2} + \frac{9}{4}\left[\frac{\sqrt[6]{5}}{\sqrt[4]{3}}F(\phi,k) + C_1\right] = C\,p^{\frac{2}{3}-\frac{1}{2}}\int [\,\rho_o(a)]^{\frac{1}{2}}\,da \qquad (43)$$

where C_1 is a constant of integration and σ has been replaced by $1 + s$. In equation 43, $F(\phi,k)$ represents the Legendre Standard Form of the Elliptic Integral of the First Kind. The parameters ϕ and k are defined by the equations

$$\cos\phi = \frac{\sqrt[3]{5}\,(1+s)^{1/3} + 1 - \sqrt{3}}{\sqrt[3]{5}\,(1+s)^{1/3} + 1 + \sqrt{3}}\ , \qquad (44)$$

and

$$\sin^{-1} k = \frac{5\pi}{12}\ . \qquad (45)$$

For the purposes of illustration a simple power law density distribution is assumed,* i.e.,

$$\rho_o(a) = \left(\frac{a}{a_o}\right)^m \rho\ . \qquad (46)$$

It is not possible to solve equation 43 explicitly for $s(a)$; the function is exhibited graphically in Figure 1 for $m = -1, 1$ and 2 where the constant $\rho C\,p^{2/3-1/2}$ is taken to be unity and the

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* ρ is a constant with the dimensions of density.

initial shock strength, i.e., $s(a_o)$ is assumed to be 5. It is seen that the shock strength decays more rapidly for increasing values of the exponent m, and that it becomes zero at a finite distance from the point of inception.

The paths of the shock waves are shown in Figure 2. These paths are determined as follows; the position of the shock front at any instant is, from equation 10,

$$t = \tau(a) = \int \left[\frac{p_o(a)}{\rho_o(a)}\right]^{-1/2} \left[\frac{1 + s(a) + \mu^2}{1 - \mu^2}\right]^{-1/2} da \qquad (10)$$

which is expressible as

$$\frac{t - t_o}{T} = \int_1^x \left[\frac{p_o(x)}{\rho_o(x)}\right]^{-1/2} \left[\frac{1 + s(x) + \mu^2}{1 - \mu^2}\right]^{-1/2} dx \qquad (47)$$

where

$$x = \frac{a}{a_o}, \qquad (48)$$

$$T = a_o\left(\frac{p}{\rho}\right)^{-1/2}, \qquad (49)$$

and the shock is assumed to start into motion at time $t = t_o$ at the position $a = a_o$.

In equation 47, $p_o(x)$ is constant, $\mu^2 = \frac{1}{5}$, $\rho_o(x)$ is assumed to be given and $s(x)$ is determined from equation 43. The integral is evaluated by using Simpson's Rule. A study of Figure 2 shows the relative effects of density variations on the shock velocity. The points in Figure 1 at which the shock strength becomes zero correspond in Figure 2 to the points at which the shock velocity becomes sonic.

The flow properties in the region between the piston and the shock are calculable since the values of s(a) to be substituted in equations 33, 34, and 35, are now known.

The Strong Shock Solution.

The law of variation of the shock strength for strong shocks, (s(a) >> 1), when the pressure ahead is constant, is found from equation 41. For, if s(a) >> 1, that equation may be approximated to in the form

$$\frac{ds}{da} = - C\, p^{\frac{1}{\gamma} - \frac{1}{2}} (1 - \mu^2)^{\frac{1}{2}}\; s^{\frac{1}{2} + \frac{1}{\gamma}} [\rho_o(a)]^{\frac{1}{2}} \quad . \qquad (50)$$

This equation shows that

$$s(a) \propto \left[\int [\rho_o(a)]^{\frac{1}{2}} da \right]^{\frac{2\gamma}{\gamma - 2}} \qquad (51)$$

Thus the dependence of the shock strength on the distance traveled is comparatively simple in this case. The path of this shock is

$$t = \tau(a) \propto \int [\rho_o(a)]^{1/2} [s(a)]^{-1/2} da \qquad (52)$$

which is the appropriate modification of equation 10. The flow properties can be determined in the manner indicated previously.

It is possible to derive analytic results also in the weak shock approximation (s(a) << 1). Since, however, this can be done even in the case of variable pressures it will be discussed in the following section.

The Solution for Variable Pressure Distributions.

It has been remarked previously that equation 41 is not generally integrable in analytic form. If the pressure distribution $p_o(a)$ is not constant then the equation can be written in the form

$$\frac{ds}{da} + (1 + s)\frac{1}{p_o}\frac{dp_o}{da} = - C(1 - \mu^2)^{\frac{1}{2}} [p_o(a)]^{\frac{1}{\gamma} - \frac{1}{2}} [1 + s(a)]^{1 + \frac{1}{\gamma}}$$
$$[\rho_o(a)]^{\frac{1}{2}} [1 + s(a) + \mu^2]^{-\frac{1}{2}} . \qquad (53)$$

This equation immediately discloses the possibility that the shock strength, $s(a)$, could be an increasing function of the abscissa a --this despite the fact that the energy input of the piston tends to a finite value. In order to have $s(a)$ increase, however, the pressure distribution $p_o(a)$ must be a decreasing function of a. It is also seen from equation 53 that it would be impossible for the shock strength to increase in a medium in which the pressure is either a constant or an increasing function of distance.

Equation 53 has been integrated numerically by Runge's method for a series of variable pressures and densities. To carry out this integration it is assumed that the density and pressure distributions are given in the form

$$\rho_o(a) = \left(\frac{a}{a_o}\right)^m \rho \qquad (54)$$

and

$$p_o(a) = \left(\frac{a}{a_o}\right)^n p \quad , \qquad (55)$$

Figure 3 shows the variation of the shock strength as a function of a/a_o for combinations of the values $m = 1,2$, $n = -1, +1$. The initial shock strength is chosen to be 5 and the ratio of the specific heats of the gas, γ, to be $\frac{3}{2}$. This choice of γ is, of course, not necessary but is made in order to facilitate a comparison of Figures 3 and 4 with Figures 1 and 2, so that the effects of variable pressures can be more readily appreciated. Calculations have been made for $\gamma = \frac{7}{5}$, the more conventional value. The results of these calculations showed that the effect of such a variation in the value of γ is negligible.

The paths of the shock waves shown in Figure 3 are exhibited in Figure 4. These paths are calculated in the manner described previously with, of course, the modifications necessary to include the pressure variation.

One interesting case exists in which equation 53, or more properly an approximation to it, yields an analytic solution. This situation arises when one considers the propagation of shocks whose strength is such that $[s(a)]^2$ may be neglected. Then equation 41 reduces to

$$\frac{d}{da} p_o(1 + s) = - C \left(\frac{1 - \mu^2}{1 + \mu^2}\right)^{1/2} [p_o(a)]^{\frac{1}{2} + \frac{1}{\gamma}} [\rho_o(a)]^{\frac{1}{2}} \left[1 + \frac{3}{4} \frac{\gamma + 1}{\gamma} s\right] \qquad (56)$$

which can be put in the standard form

$$\frac{ds}{da} + s \left[\frac{1}{p_o} \frac{dp_o}{da} + \frac{3}{4} C \gamma^{-\frac{3}{2}} (\gamma + 1) [p_o(a)]^{\frac{1}{\gamma} - \frac{1}{2}} [\rho_o(a)]^{\frac{1}{2}}\right] = - C \gamma^{-\frac{1}{2}} [p_o(a)]^{\frac{1}{\gamma} - \frac{1}{2}} [\rho_o(a)]^{\frac{1}{2}} - \frac{1}{p_o(a)} \frac{dp_o(a)}{da}, \qquad (57)$$

and is accordingly immediately soluble. When $p_o(a)$ and $\rho_o(a)$

are specified, s(a) is uniquely determined by equation 57; the method of obtaining the solution is straightforward and since it is not intended to pursue this topic farther it is not given here. As in the previous discussions the paths of the shocks can be determined and the resulting gas flow properties calculated.

Constant Strength Shock Waves.

The analysis of the shock waves resulting from the uniform compressive motion of a piston will be concluded with a short reference to shocks of constant strength. The possibility of the propagation of such shocks is of interest since it is claimed that many astrophysical situations can be adequately described by even this restrictive case. When the shock strength is taken to be constant in equation 41 there results an equation of the form

$$\frac{dp_o}{da} = B[p_o(a)]^{\frac{1}{2}+\frac{1}{\gamma}}\,[\rho_o(a)]^{\frac{1}{2}} \qquad (58)$$

where B is a constant. This equation implies that

$$p_o(a) \propto \left[\int [\rho_o(a)]^{\frac{1}{2}}\,da\right]^{\frac{2\gamma}{2-\gamma}} . \qquad (59)$$

Hence, it is only possible to describe the propagation of constant strength shocks by this method of analysis in the presence of some very specialized pressure-density relationships. Equation 41, or 58, also shows that it is not possible to discuss the propagation of such a shock in regions of constant pressure. This latter result was to be expected from the energy considerations already presented.

THE DECELERATIVE PISTON MOTION.

The discussion of the propagation of shock waves which are generated by the non-uniform motion of a piston, specifically the decelerative motion, is mathematically more complicated than that of the uniform motion. In the case of the decelerated piston equation 31 is applicable and leads through equation 29 to the piston velocity

$$v_p(t) = \frac{2}{\gamma + 1} \left(\frac{a_o}{t_o}\right) \left(\frac{t_o}{t}\right)^{\frac{\gamma-1}{\gamma+1}} . \qquad (60)$$

The energy input of this piston motion is determined by the method indicated earlier and is found to be

$$E(t) = \frac{a_o\, p_o(a_o)\, [1 + s(a_o)]}{\gamma - 1} \left[1 - \left(\frac{t_o}{t}\right)^{\frac{2(\gamma-1)}{\gamma+1}} \right] . \qquad (61)$$

The limiting value of this expression as t/t_o tends to infinity is given by

$$E(t) \rightarrow \frac{a_o\, p_o(a_o)\, [1 + s(a_o)]}{\gamma - 1} \; ; \qquad \frac{t}{t_o} \rightarrow \infty \qquad (62)$$

which is identical with equation 39. The flow properties between the piston and the shock are determined by equations 25, 26 and 27 in the form

$$x(a,t) = \frac{a}{[\tau(a)]^{2/\gamma+1}} \; t^{2/\gamma+1} , \qquad (63)$$

$$p(a,t) = \frac{p_o(a)[1 + s(a)]}{[\tau(a)]^{-2\gamma/\gamma+1}} \; t^{-2\gamma/\gamma+1} , \qquad (64)$$

and

$$\rho(a,t) = \frac{\rho_o(a)[1 + s(a) + \mu^2]}{1 + \mu^2[1 + s(a)]} \frac{1}{[\tau(a)]^{-2/\gamma+1}} t^{-2/\gamma+1} \tag{65}$$

where $\tau(a)$ is as usual, given by equation 10.

The equation which determines the shock strength is 21, i.e.,

$$\frac{d}{da} f_2(a) = - K_1 f_1(a) f_3(a) \frac{d}{da} f_1(a) \tag{21}$$

This equation can be written in the form

$$\frac{d}{da}\left[p_o(a)[1 + s(a)][\tau(a)]^{2\gamma/\gamma+1}\right]$$
$$= \frac{2(\gamma-1)}{(\gamma+1)^2} \frac{a\ \rho_o(a)[1 + s(a) + \mu^2]}{1 + \mu^2[1 + s(a)]} \frac{d}{da}\left[a[\tau(a)]^{-2/\gamma+1}\right] \tag{66}$$

which immediately exhibits the mathematical difficulties referred to. Unless one chooses a very artificial value of γ, equation 66 is difficult to handle either analytically or numerically. It will be discussed therefore in two approximate cases; firstly the strong shock, and secondly the weak shock, approximation. The strong shock discussion will be restricted to the situation in which the density distribution decreases inversely with the abscissa a.

The Strong Shock Approximation.

In this approximation it is assumed that $s(a) >> 1$. Hence, equation 66 reduces to

$$\frac{d}{da}\left[p_o(a)\ s(a)[\tau(a)]^{2\gamma/\gamma+1}\right]$$
$$= \frac{2a\ \rho_o(a)}{\gamma + 1} \frac{d}{da}\left[a[\tau(a)]^{-2/\gamma+1}\right] \tag{67}$$

where

$$\tau(a) \approx (1 - \mu^2)^{1/2} \int \left[\frac{p_o(a)}{\rho_o(a)} \right]^{-1/2} [s(a)]^{-1/2} \, da \quad . \qquad (68)$$

If one assumes that the density distribution is given by

$$\rho_o(a) = \frac{a_o}{a} \rho \qquad (69)$$

equation 67 can be integrated, the constant of integration being taken, without lack of generality, to be zero, to give

$$\tau(a) = \left[\frac{2\rho a_o}{\gamma + 1}\right]^{1/2} a^{1/2} [p_o(a) \, s(a)]^{1/2} \, . \qquad (70)$$

Hence

$$s(a) = a_o \, s(a_o) \, p_o(a_o) \, [p_o(a) \, a]^{-1} \qquad (71)$$

for

$$s(a) \gg 1.$$

Equation 71 shows that the shock strength cannot increase if the pressure distribution is either constant or an increasing function of the abscissa a. It increases, however, if $p_o(a)$ decreases faster than the inverse first power of a.

It may be remarked that the method of solution presented is restricted to the density distribution given by the expression 69. Particular solutions of the shock strength equation for other distributions can be found as follows. One assumes that s(a), $p_o(a)$, and $\rho_o(a)$, follow simple power law variations in the abscissa a. When this assumption is applied to the approximate equation 67 a relationship between the three exponents and γ is determined. Hence two of the power law distributions may be arbitrarily prescribed. It should be noted, however, that a solution found by this method is a particular solution of an equation which is basically non-linear.

The Weak Shock Approximation.

It is not necessary in this case to impose any restrictions upon either the pressure or the density distributions. Equation 66 reduces to

$$\frac{ds}{da} = \frac{2(\gamma - 1)}{(\gamma + 1)^2} \frac{a\, \rho_o(a)}{p_o(a)[\tau(a)]^{2\gamma/\gamma+1}} \frac{d}{da}\left[a[\tau(a)]^{-2/\gamma+1} \right]$$

$$- \frac{d}{da}\left[\ln p_o(a)\ [\tau(a)]^{2\gamma/\gamma+1} \right] \qquad (72)$$

where

$$\tau(a) \approx (1 - \mu^2)^{1/2} \int \left[\frac{p_o(a)}{\rho_o(a)} \right]^{-1/2} da \qquad (73)$$

and

$$s(a) << 1.$$

This equation for the shock strength is of the first order and is linear. The solution is readily obtainable when $p_o(a)$ and $\rho_o(a)$ are specified. It is to be observed that, in situations for which either the strong or the weak shock analysis is valid, the parameters of the gas flow are immediately calculable from equations 63, 64, and 65. The paths of the shock waves can also be determined by the methods outlined in the discussion of the uniform piston motion.

IV. CONCLUSION.

The laws of the one-dimensional propagation of shock waves in non-uniform gases are determinable if the gas flow can be related to certain types of compressive piston motion. These types include a uniform motion and a decelerative, but not an accelerative, motion. The solutions of the equations of non-isentropic gas flow show that the resultant shock may be referred to as a constant energy shock since the energy input of the piston in both cases tends to a finite limit as the duration of the piston motion tends to infinity.

In the case of the uniform motion the gas flow generated is one in which the pressure in the gas between the piston and the shock is a function of time only. The shock strength, s(a), is determined as a function of distance by a non-linear, first order, differential equation. This equation is valid for any initial pressure and density distributions. When the pressure distribution is taken to be constant the shock strength is analytically determinable and it is found to be a decreasing function of the distance. This result holds independently of the density distribution. The rate of decrease however is faster for increasing than for decreasing densities. The equation shows further that if the pressure distribution is an increasing function of the distance then the shock strength must be a decreasing function. The effect of an increasing pressure is to accelerate the decrease. The shock strength, however, may increase if the pressure decreases. On this point it should be remarked that each decreasing pressure distribution must be examined on its own merits since no exact statement as regards the necessary rate of decrease has been derived.

The propagation of strong, $s(a) \gg 1$, and weak, $s(a) \ll 1$, shocks is readily determined by using the appropriate approximate forms of the basic differential equation. The paths of all the shock waves mentioned can also be calculated once the shock strength has been determined.

The analysis of the non-uniform motion of the piston leads to a differential equation for the shock strength which is mathematically difficult to work with. However, the propagation of strong shocks in a region in which the density falls off inversely as the first power of the abscissa $\underline{a}$ can be discussed analytically. In this case there is no restriction on the permissible pressure variations. As in the uniform piston motion the shock strength can only increase if the pressure distribution is a decreasing function of the distance. The laws governing the propagation of weak shock waves are determinable for any pressure and density distributions. The remark made above concerning the calculation of the paths of the shock waves is valid here also.

A disadvantage of the method of analysis is that it introduces a fundamental dichotomy of isentropic and non-isentropic flows. It is not possible to determine, by, say, a limiting procedure, the types of isentropic flow which could be described by the methods adopted in this presentation. This restriction arises from the method of analysis which has been employed. To express, mathematically, the condition that the gas flow is isentropic, would demand the inclusion of an extra equation in the group which constitutes the basis of the analysis. Thus it is clear that a limiting procedure is not applicable. This additional equation is essentially a further restriction on the space dependence of the parameters of the flow. Hence, it would reduce the number of types of flow which can be described by the methods developed in the text.

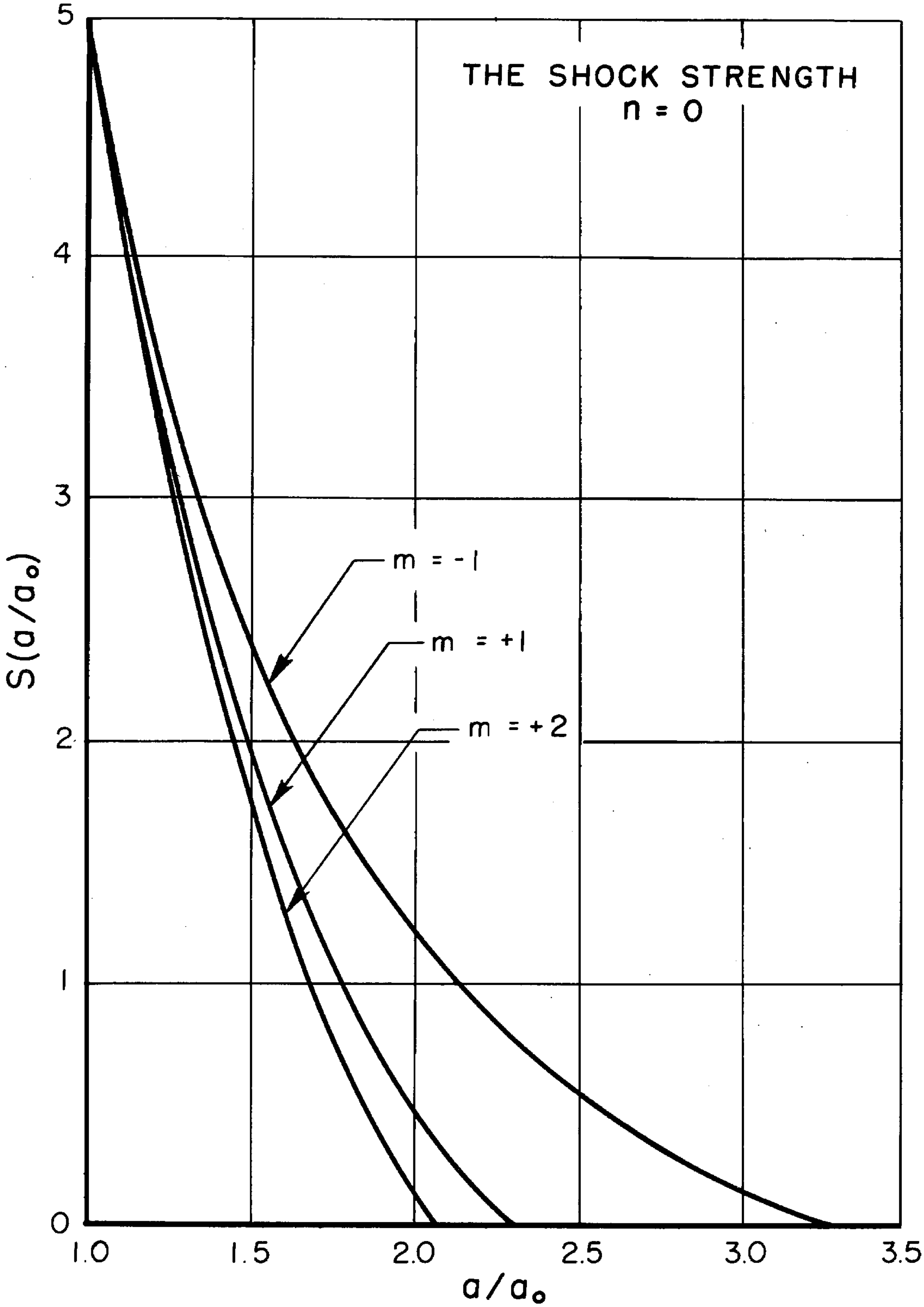

Fig. 1 - The shock strength, s, is shown as a function of a/a_o for three power-law density distributions. The pressure distribution is constant.

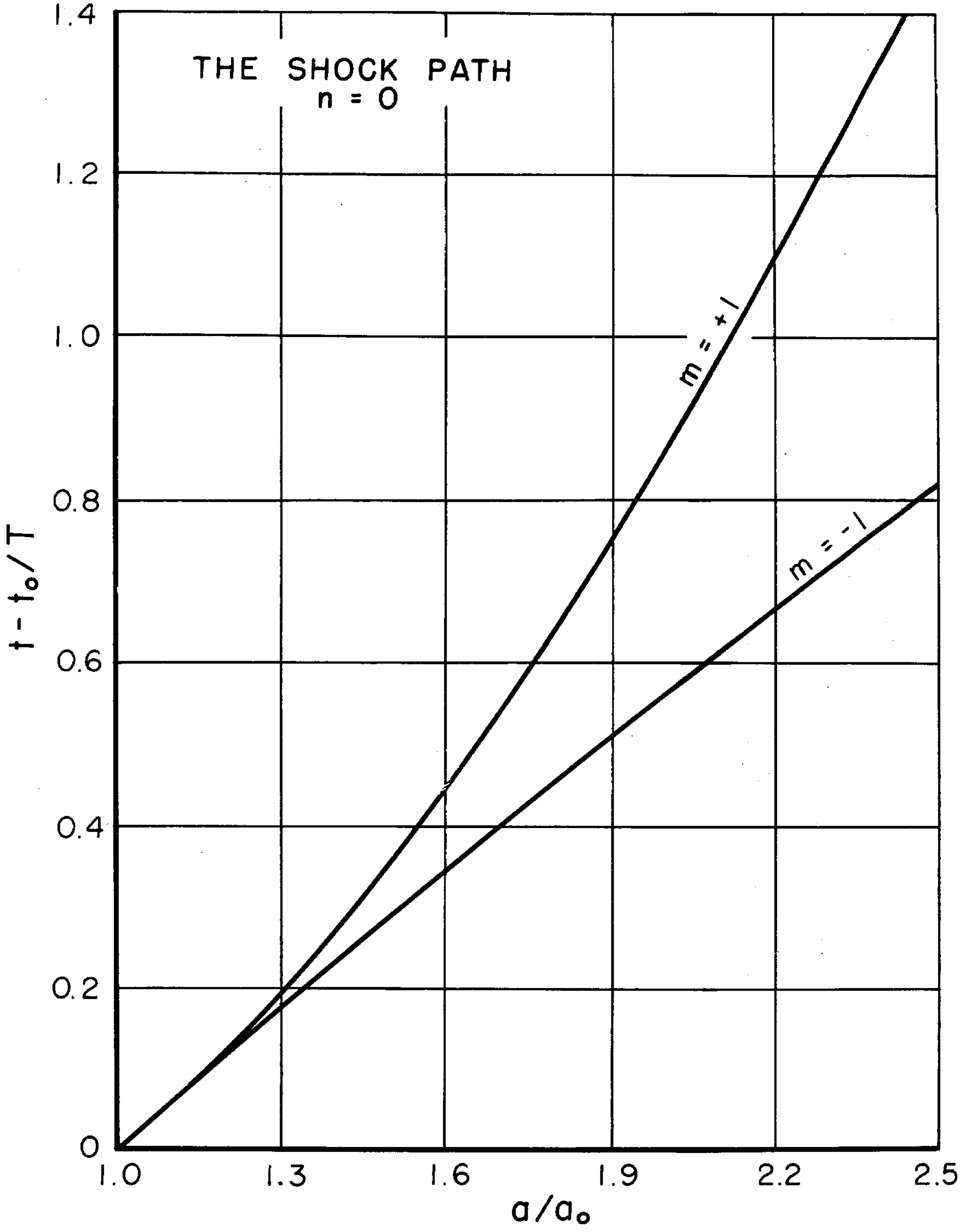

Fig. 2 - The paths of the shock waves for two power-law density distributions are shown. The pressure distribution is constant. The initial shock strength, $s(a_o)$, is 5.

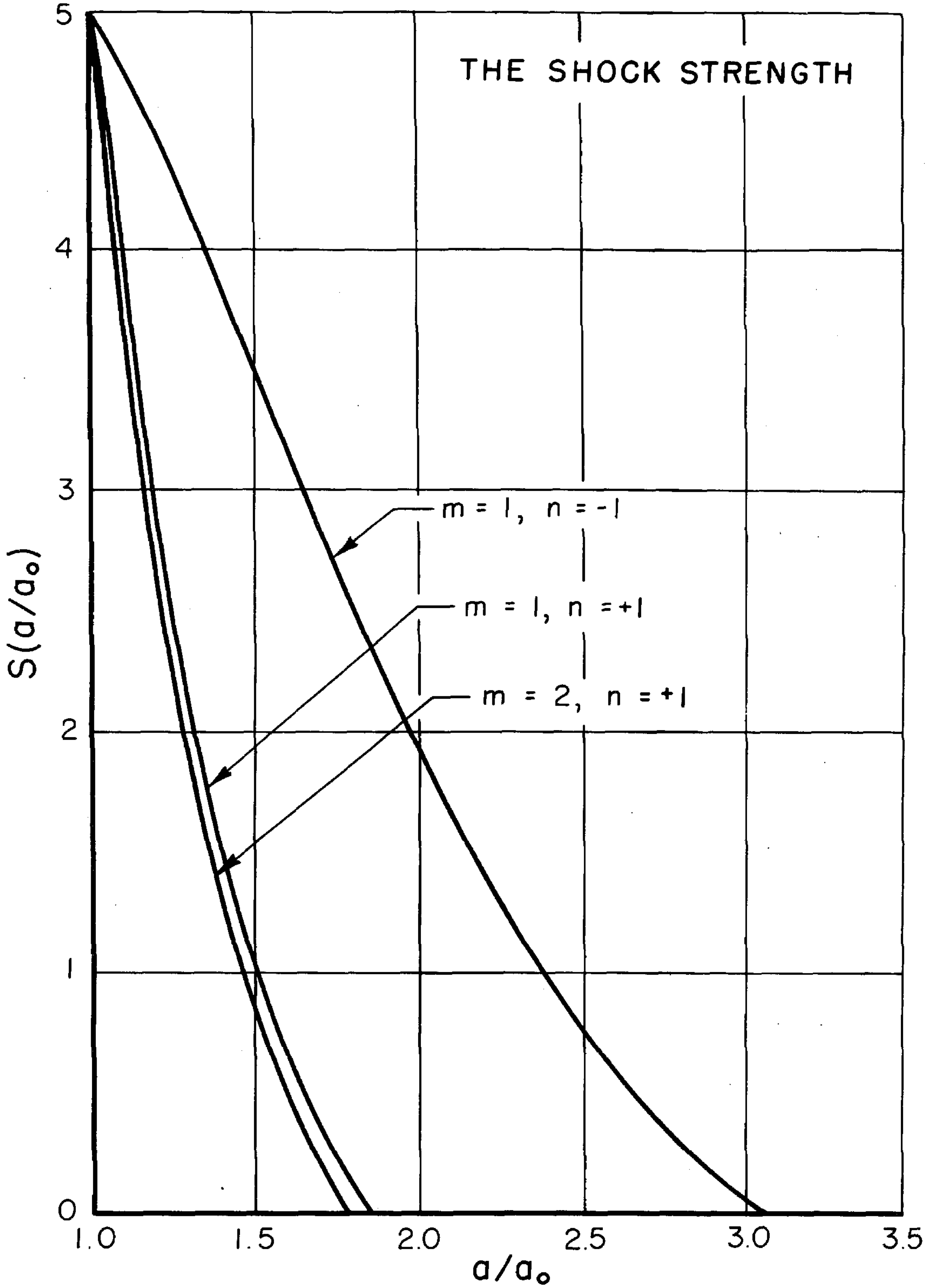

Fig. 3 - The shock strength, s, is shown as a function of a/a_o for three combinations of power-law pressure and density distributions.

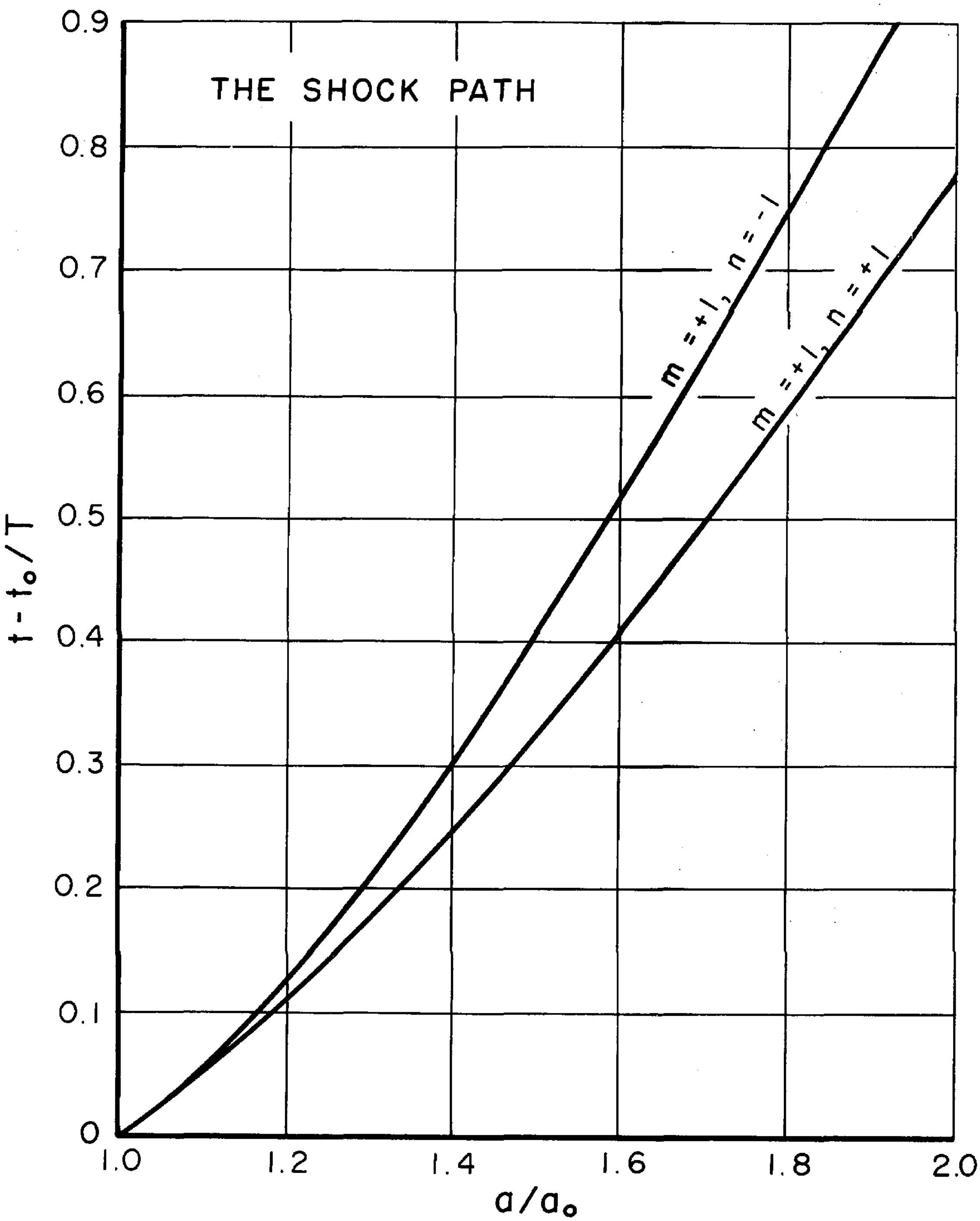

Fig. 4 - The paths of the shock waves for two combinations of power-law pressure and density distributions are shown. The initial shock strength, $s(a_o)$, is 5.

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PART TWO

THE STABILITY OF THE SPHERICAL SHAPE OF A VAPOR CAVITY IN A LIQUID.

I. INTRODUCTION.

The analysis of the effect of tangential forces on the stability of the interface of two fluids constitutes a classical problem in the hydrodynamics of incompressible fluids, namely, the problem of Helmholtz Instability [1]. The effect of normal forces, or accelerations, on the stability of such an interface, however, has been studied only quite recently and is referred to as the problem of Taylor Instability. Whereas the phenomenon of Helmholtz Instability is frequently noticed in nature, that of Taylor Instability is not so readily observed--a fact which may explain the lapse in time between the detection of these two basic types of interface perturbations.

It has been shown by G.I. Taylor, [2], that a plane interface of two fluids of different densities in accelerated motion is stable or unstable according as the acceleration is directed from the heavier to the lighter fluid or conversely. This stability analysis is limited to small amplitude perturbations of the interface and it is found that a small disturbance of the interface begins to grow exponentially with time in the unstable, and to decrease exponentially with time in the stable, situation. While experimental observations agree well with the theory in the small amplitude limit for which the theory is valid, it is known that there are significant deviations in the rate of growth of distortions in the unstable case when their amplitude is large [3].

The influence of viscosity and surface tension on Taylor's results has been analyzed by Bellman and Pennington [4]. Viscosity, as one would expect, retards the growth of the perturbation amplitude but does not remove the instability. The retardation is most marked when the wavelength of the disturbance is small. Surface tension is shown to annihilate the instability for sufficiently small wavelengths.

The analysis of the analogous stability problem in the case of a spherical, rather than a plane, free surface is of considerable interest since one would hope to obtain, through its results, some understanding of the behavior of cavitation bubbles, underwater explosions, and other occurrences of this type. The first attempt to consider the stability of a spherical interface was made by Binnie, [5], in a somewhat superficial study of the problem. Plesset, [6], investigated in a more rigorous manner the stability of the spherical interface of two incompressible, immisicible, non-viscous fluids possessing different densities and he stated the modifications required in extending Taylor's results to this new configuration. Later, Pennington, [7], published a comprehensive analysis of the instability of the surface of a pulsating gas bubble in a liquid, including in his discussion the effects of viscosity and rotational flow. These quantities were not considered by Plesset. The presence of the viscous terms considerably complicates the basic equations and renders it difficult to obtain a clear understanding of the effect of viscosity on the perturbation behavior. Since it is not possible then to present here a reasonably succinct statement of the viscous effects the reader is referred to Pennington's report for the details.

There remain two aspects of the stability question which have not yet been studied. Firstly, since it is known, [8], that the bubble wall, during collapse, may attain velocities comparable to the local velocity of sound it is not inconceivable that the compressibility of the surrounding fluid should be of importance in determining stability criteria. This facet of the problem has never been considered. Secondly, although the general stability criteria have been derived and the gross features of the perturbation behavior recognized it would be of interest to know for how large a range of bubble motion the small amplitude perturbation theory is valid and consistent.

The aim of the present analysis is to attempt to discuss in detail this latter question. For this purpose a particular type of bubble motion is considered. The stability of the spherical shape of a gas bubble in an infinite liquid is investigated for the case in which the difference between the pressure in the bubble, P_i, and the pressure in the liquid, P_o, is constant. These conditions apply approximately to a vapor bubble growing, ($P_i > P_o$), or collapsing, ($P_i < P_o$), in a liquid at constant external pressure.

II. THE DERIVATION OF THE BASIC EQUATIONS

Although the equations which form the basis of this study are available in the literature, it will prove to be instructive to rederive them by a direct method which has not been used in any of the published discussions. To do so the problem is formulated as follows.

An incompressible, non-viscous fluid of density ρ is bounded on the outside by a spherical surface whose radius, $R_1(t)$, is a function of time, and on the inside by an empty* concentric spherical bubble of radius $R_2(t) \ll R_1(t)$. This spherical shell of fluid collapses, or expands, under a constant pressure difference. It is required to discuss the stability of the bubble wall for small perturbations from its spherical shape. In the sequel $R_1(t)$ is to become infinite.

If the interface $R_2(t)$ is distorted from the surface of a sphere to a surface with a radius vector r_s one may write

$$r_s = R_2(t) + \sum_{0}^{\infty} a_n(t)\, Y_n(\theta,\phi) \tag{1}$$

where $Y_n(\theta,\phi)$ is a surface harmonic of degree n and the a_n's, which represent the perturbation amplitudes, are functions of time which are to be determined. The stability of the shape of the cavity can be established by considering whether the interface distortions of small amplitude grow or diminish. More precisely, it is assumed that

---

* The density of the gas in the bubble is assumed to be negligible in comparison with the density of the surrounding fluid.

$$|a_n(t)| \ll R_2(t)$$

and that terms of higher order than the first in a_n and $\dot{a}_n$* are ultimately negligible. Since the fluid surrounding the bubble is incompressible, it follows that the volume contained within the two surfaces, R_1 and r_s, must be constant. It is assumed that the shape of the outer surface is unaffected by the perturbation on the bubble wall. Hence

$$\int r^2 \sin \Theta \, d\Theta \, d\phi \, dr = \frac{4}{3}\pi(R_1^3 - R_2^3) = \text{constant}, \qquad (2)$$

the integral being evaluated over the volume of the fluid. Equation 2 can be written in the form

$$\frac{1}{3}\int (R_1^3 - r_s^3)\, d\omega = \frac{4\pi}{3}(R_1^3 - R_2^3) \qquad (3)$$

where

$$d\omega = \sin \Theta \, d\Theta \, d\phi \qquad (4)$$

is the element of surface area of the unit sphere and the integration takes place over this sphere. On substituting for r_s from equation 1 one finds that equation 3 determines a_o in the form

$$a_o = -\frac{1}{4R_2\sqrt{\pi}} \sum_{1}^{\infty} (n+1)\, a_n^2 \qquad (5)$$

where the following properties of the surface harmonics Y_n have been utilized:

$$\int Y_n Y_m \, d\omega = 0 \qquad n \neq m$$

$$\int Y_n^2 \, d\omega = n + 1 \qquad (6)$$

$$\int Y_o \, d\omega = 4\sqrt{\pi} \quad .$$

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* A dot denotes differentiation with respect to the time.

These relationships depend, of course, upon the normalization of the functions Y_n. This topic is treated in the appendix which will be referred to henceforth as [A].

The velocity potential which describes the unperturbed motion of the two surfaces is

$$\Phi' = \frac{R_1^2 \dot{R}_1}{r}$$

or (7)

$$\Phi' = \frac{R_2^2 \dot{R}_2}{r} ,$$

since

$$R_1^2 \dot{R}_1 = R_2^2 \dot{R}_2 . \tag{8}$$

The latter equality is a direct consequence of the incompressible nature of the fluid. The perturbation of the inner surface modifies the expression for the velocity potential and the resulting flow may be described by the potential function

$$\Phi = \frac{R_2^2 \dot{R}_2}{r} + \sum_0^\infty A_n(t)\, Y_n\, r^{-n-1} \tag{9}$$

in which the A_n's are functions of time. It is assumed that the perturbation term in the potential is a monotonically decreasing function of the radius vector r and that it affects neither the shape nor the velocity of the outer surface $R_1(t)$. This approximation is tantamount to assuming that

$$|\dot{R}_1| \gg \sum_0^\infty (n+1)\, |A_n Y_n|\, R_1^{-n-2} . \tag{10}$$

The functions $A_n(t)$ which appear in the potential function, are determined by the kinematical condition that the cavity surface, as given by the expression 1, should move with the fluid. This condition is

$$\frac{DF}{Dt} = \frac{\partial F}{\partial t} - (\nabla \Phi \cdot \nabla) F = 0 \qquad (11)$$

where

$$F = r_s - R_2 - \sum_{0}^{\infty} a_n Y_n \qquad (12)$$

and $\nabla\Phi$ is to be evaluated from equation 9. On writing out equation 11 explicitly, one finds that

$$\sum \dot{a}_n Y_n + 2\dot{R}_2 R_2^{-1} \sum a_n Y_n + 3\dot{R}_2 R_2^{-2} [\sum (a_n Y_n)]^2 + \sum \frac{(n+1) A_n}{R_2^{n+2}} Y_n$$

$$- \sum \frac{(n+1)(n+2)}{R_2^{n+3}} A_n Y_n \sum a_m Y_m + \sum \frac{A_n}{R_2^{n+3}} \frac{\partial Y_n}{\partial \theta} \sum a_m \frac{\partial Y_m}{\partial \theta}$$

$$+ \sum \frac{A_n}{R_2^{n+3}} \frac{\partial Y_n}{\sin \theta \, d\phi} \sum a_m \frac{\partial \phi_m}{\sin \theta \, d\phi} = 0 \quad . \qquad (13)$$

A more useful relationship can be obtained by integrating this equation over the surface of the unit sphere. The result can be stated, [A], in the form

$$4\sqrt{\pi}\, A_o = 2 \sum \frac{(n+1)^2 a_n A_n}{R_2^{n+1}} - 2R_2 \sum (n+1) \dot{a}_n a_n - 4\dot{R}_2 \sum (n+1) a_n^2 \quad . \qquad (14)$$

The equations of motion of the unperturbed surfaces as well as the equation which determines the perturbation amplitude, as a function of time, will be derived by evaluating the energy of the system. In particular, the Lagrangian of the system will now be calculated. It consists of two terms; firstly, the Surface Energy, E_s, and secondly, the Kinetic Energy, E_K.

The Surface Energy is equal to the product of the surface tension constant σ and the sum of the areas of the two surfaces of the fluid. Since its radius vector, r_s, is a function of Θ and $\emptyset$, the inner surface contributes to the surface energy an amount given by the expression

$$E_{s_i} = \sigma \int r_s^2 \left[1 + \frac{1}{r_s^2} \left(\frac{\partial r_s}{\partial \Theta} \right)^2 + \frac{1}{r_s^2 \sin^2 \Theta} \left(\frac{\partial r_s}{\partial \emptyset} \right)^2 \right]^{1/2} d\omega \quad (15)$$

which may be approximated to by the form

$$E_{s_i} \approx \sigma \int r_s^2 \left[1 + \frac{1}{2r_s^2} \left(\frac{\partial r_s}{\partial \Theta} \right)^2 + \frac{1}{2r_s^2 \sin^2 \Theta} \left(\frac{\partial r_s}{\partial \Theta} \right)^2 \right] d\omega \ . \quad (16)$$

The details of the evaluation of this integral, in common with those of other integrals of the same type which will arise in the course of the analysis, are relegated to the appendix [A]. It follows that

$$E_{s_i} = 4\pi\sigma \ R_2^2 + \frac{\sigma}{2} \sum_1^{\infty} (n-1)(n+1)(n+2) \ a_n^2 \ . \quad (17)$$

The surface energy of the outer surface is clearly $4\pi\sigma R_1^2$ and hence the total surface energy is

$$E_s = 4\pi\sigma (R_1^2 + R_2^2) + \frac{\sigma}{2} \sum_1^{\infty} (n-1)(n+1)(n+2) \ a_n^2 \ . \quad (18)$$

The Kinetic Energy of the fluid is obtained by evaluating

$$E_K = \frac{1}{2} \rho \int (\nabla \Phi)^2 \ r^2 \sin \Theta \ d\Theta \ d\emptyset \ dr \quad (19)$$

over the volume of the fluid. This integral is expressible in the form

$$E_K = \frac{1}{2}\rho\int\left[\left\{\frac{R_2^2 \dot{R}_2}{r^2} + \sum\frac{(n+1)\ A_n Y_n}{r^{n+2}}\right\}^2 + \left\{\sum\frac{A_n}{r^{n+2}}\ \frac{\partial Y_n}{\partial \theta}\right\}^2 + \left\{\sum\frac{A_n}{r^{n+2}\ \sin\theta}\ \frac{\partial Y_n}{\partial \phi}\right\}^2\right] r^2 d\omega \qquad (20)$$

in which $\nabla\Phi$ has been determined from equation 9. On using equations 14 and 13 correct to the first order in a_n, it follows, [A], that

$$E_K = \frac{\rho}{2}\left[4\pi(R_2^3\ \dot{R}_2^2 - R_1^3\ \dot{R}_1^2) - 2\dot{R}_2^2\ R_2\ \sum a_n^2\ (n+1) - 2R_2^2\ \dot{R}_2\ \sum a_n\ \dot{a}_n(n+1) + \sum R_2(\dot{a}_n\ R_2 + 2\dot{R}\ a_n)^2\right], \qquad (21)$$

In evaluating this expression for the kinetic energy the approximation given by the inequality 10 has been adopted. The Lagrangian, L, of the system is the difference of the kinetic energy and the surface energy. Hence,

$$L = \frac{\rho}{2}\left[4\pi(R_2^3\ \dot{R}_2^2 - R_1^3\ \dot{R}_1^2)\right] + 4\pi\sigma\,(R_1^2 + R_2^2) + \sum_1^\infty \rho\, a_n^2\ (n-1)\ [R_2\ \dot{R}_2^2 + \frac{\sigma}{\rho}\ (n+1)(n+2)] + \rho\, R_2^2\ \dot{R}_2^2 \sum a_n\ \dot{a}_n\ (n-1) - \frac{R_2^3}{2}\rho\ \sum \dot{a}_n^2\ . \qquad (22)$$

Accordingly, the energy integral of the unperturbed motion is given in the form

$$\frac{\rho}{2}\left[4\pi(R_2^3\ \dot{R}_2^2 - R_{20}^3\ \dot{R}_{20}^2) - 4\pi(R_1^3\ \dot{R}_1^2 - R_{10}^3\ \dot{R}_{10}^2)\right] + 4\pi\sigma\,(R_2^2 - R_{20}^2) + 4\pi\sigma(R_1^2 - R_{10}^2) = \frac{4\pi}{3}\,(P_o - P_i)(R_{20}^3 - R_2^3) \qquad (23)$$

where the additional subscript zero to a function represents the value of that function at a particular instant $t = t_o$. This may be conveniently taken to be the time at which the motion commences. Equation 23 will be recognized as the generalization of the Rayleigh Bubble Collapse Formula, [9], to include the effects of surface tension and also the effect of considering the surrounding fluid to be of finite, rather than of infinite, extent. In order to recover Rayleigh's expression one modifies equation 23 by assuming that

$$R_1 \to \infty \quad , \qquad \dot{R}_1 \to 0$$

in such a way that

$$R_1^2 \dot{R}_1 \to \text{finite.}$$

This approximation results in the equation

$$2\pi\rho(R_2^3 \dot{R}_2^2 - R_{20}^3 \dot{R}_{20}^2) = \frac{4\pi}{3}(P_o - P_i)(R_{20}^3 - R_2^3) - 4\pi\sigma(R_{20}^2 - R_2^2), \tag{24}$$

which is Rayleigh's expression for the velocity of the bubble wall, with the surface tension term included. The equation of motion of the undisturbed surface is easily found from equation 24 to be

$$R_2 \ddot{R}_2 + \frac{3}{2}\dot{R}_2^2 = \frac{P_i - P_o - 2\sigma/R_2}{\rho} \quad . \tag{25}$$

The Lagrangian of the perturbed motion is obtained from equation 22 in the form

$$\sum a_n^2(n-1)[R_2^2 \dot{R}_2 + \frac{\sigma}{\rho}(n+1)(n+2)] + R_2^2 \dot{R}_2 \sum a_n \dot{a}_n(n-1)$$
$$- \frac{R_2^3}{2} \sum \dot{a}_n^2 = 0 \quad . \tag{26}$$

It is seen then that to the present approximation the perturbation modes are independent and the Lagrangian is that of a damped harmonic oscillator. On writing Lagrange's Equations the following second order differential equation for the a_n's is determined.

$$\ddot{a}_n + \frac{3\dot{R}_2}{R_2}\dot{a}_n - a_n\left[\frac{(n-1)\ddot{R}_2}{R_2} - \frac{\sigma}{\rho R_2^3}(n-1)(n+1)(n+2)\right] = 0. \quad (27)$$

This equation has been obtained previously in the literature by many different methods. Penney and Price, [10], have carried out a numerical solution of equation 27 for $n = 2$ for the case of a pulsating gas bubble in water with an internal pressure in the bubble given by

$$P_i \, R^{3\gamma} = \text{constant}$$

and with a constant pressure P_o in the water at a distance from the bubble. In their computations surface tension is neglected. The numerical solution showed that the distortion amplitude a_2 is much larger when the bubble is near its minimum radius then elsewhere. The following discussion is, however, of a completely analytic nature and is based upon equations 25 and 27.

III. THE SOLUTION OF THE STABILITY PROBLEM.

As a preliminary to the detailed discussion of the stability problem the general features of the asymptotic behavior of the perturbation amplitude a_n are determined.

For the case of an expanding bubble, equation 24 gives*

$$\dot{R}^2 \sim \frac{2P}{3\rho}, \qquad R \to \infty \tag{28}$$

with $P = P_i - P_o > 0$. It then follows from equations 25 and 27 that

$$a_n \to \text{constant}, \qquad R \to \infty . \tag{29}$$

For the case of a collapsing bubble, it is convenient to transform equation 27 by the substitution

$$a_n = [R_o/R]^{3/2}\, b_n \tag{30}$$

into

$$\ddot{b}_n - G(t)\, b_n = 0 \tag{31}$$

with

$$G(t) = \frac{3}{4}\frac{\dot{R}^2}{R^2} + \frac{(n+1/2)}{R}\ddot{R} - (n-1)(n+1)(n+2)\frac{\sigma}{\rho R^3} . \tag{32}$$

From equation 24 one has

$$\dot{R}^2 = \left(\frac{R_o}{R}\right)^3 \left[\dot{R}_o^2 + \frac{2P}{3\rho} + \frac{2\sigma}{\rho R_o}\right] + O\left[\frac{1}{R}\right] \tag{33}$$

where $P = P_o - P_i > 0$ for this case. The radial acceleration, $\ddot{R}$, is determined by equation 25, and the function $G(t)$ is found to be

$$G(t) \sim -\frac{3n}{2}\frac{R_o^3}{R^5}\left[\dot{R}_o^2 + \frac{2P}{3\rho} + \frac{2\sigma}{\rho R_o}\right] \qquad R \to 0 \tag{34}$$

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* The subscript 2 on the bubble radius will be omitted in the following.

except for smaller terms. It is evident that

$$G(t) \sim -\frac{nc^2}{R^5} \tag{35}$$

where c is a real constant. One may now write a W.K.B. approximation to the solution of equation 31, for small R, in the form

$$b_n \sim \frac{e^{\pm \int^t [G(t')]^{1/2} dt'}}{[G(t)]^{1/4}} = \frac{e^{\pm icn^{1/2} \int^t R^{-5/2} dt'}}{R^{-5/4}} \tag{36}$$

The distortion amplitude a_n is then given by

$$a_n \sim R^{-1/4} e^{\pm icn^{1/2} \int^t R^{-5/2} dt'}, \quad R \to 0 \tag{37}$$

so that a_n increases like $R^{-1/4}$ in amplitude and oscillates with increasing frequency as $R \to 0$. This behavior has been found by Birkhoff, [11], by a different procedure. It is of interest that the instability found by Birkhoff near $R = 0$ is qualitatively unaffected by surface tension. The question remains over what range of R is the linearized perturbation theory for the distortion amplitude, a*, valid or consistent. The following problem will therefore be solved. A spherical cavity with radius R_o at time $t = 0$ expands or collapses from rest, $\dot{R}_o = 0$, under a constant pressure difference: at $t = 0$ the cavity is supposed to have a distortion of small amplitude a_o. The subsequent behavior of a for any R is to be determined. Complete solutions for this problem are readily found when surface tension is neglected and these solutions are given first. The effects of surface tension will then be illustrated by some special solutions.

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* The subscript n will be omitted in the following.

Solution for the Expanding Cavity: No Surface Tension.

In the absence of surface tension, the stability equation to be solved simplifies to

$$\ddot{a} + \frac{3\dot{R}}{R}\dot{a} - \frac{(n-1)}{R}\ddot{R}\,a = 0 \quad . \tag{38}$$

One finds from equation 24 that

$$\dot{R}^2 = \frac{2}{3}\frac{P}{\rho}\left[1 - \frac{R_o^3}{R^3}\right] \tag{39}$$

where $P = P_i - P_o > 0$; and from equation 25 that

$$\ddot{R} = \frac{P}{\rho}\,\frac{R_o^3}{R^4} \quad . \tag{40}$$

If the independent variable in equation 38 is changed from t to the volume ratio

$$x = \frac{R_o^3}{R^3} \ , \qquad 0 \leq x \leq 1; \tag{41}$$

there results

$$x(1-x)\frac{d^2a}{dx^2} + \left[\frac{1}{3} - \frac{5}{6}x\right]\frac{da}{dx} - \frac{(n-1)}{6}\,a = 0. \tag{42}$$

Equation 62 will be recognized as the differential equation for the hypergeometric function $F(\alpha, \beta, \gamma, x)$ where the parameters have the values

$$\alpha = \frac{-1 + i(24n - 25)^{1/2}}{12} = \frac{-1}{12} + i\delta$$

$$\beta = \frac{-1 - i(24n - 25)^{1/2}}{12} = \bar{\alpha} \tag{43}$$

$$\gamma = \frac{1}{3} \ .$$

It is convenient to take the general solution of equation 42 in the form, [12],

$$a = AF(\alpha, \ \beta, \tfrac{1}{2} \ ; \ 1-x) + B(1-x)^{1/2} \ F(-\alpha + \tfrac{1}{3}, \ -\beta + \tfrac{1}{3}, \tfrac{3}{2}; \ 1-x) \tag{44}$$

where A and B are constants which are to be determined by the initial conditions. If $a = a_o$ at $t = 0$, or $R = R_o$, then one has

$$A = a_o \tag{45}$$

from the expression 44. Similarly if the initial velocity amplitude for the perturbation is v_o, then

$$v_o = [\tfrac{da}{dt}]_{t=0} = \lim_{x \to 1} [\tfrac{da}{dx} \tfrac{dx}{dt}] \ ,$$

$$= \frac{3B}{2R_o} \ [\frac{2P}{3\rho}]^{1/2} \tag{46}$$

which determines the constant B. The quantity $(\frac{2P}{3\rho})^{1/2}$ is a characteristic velocity and R_o a characteristic length for the system. It is convenient to describe the initial velocity amplitude in terms of the length

$$\ell_o = \frac{v_o \ R_o}{(2P/3\rho)^{1/2}} \tag{47}$$

in which case equation 46 takes the form

$$B = \frac{2 \ \ell_o}{3} \ . \tag{48}$$

The limiting value of a as R tends to infinity, or equivalently as x tends to zero, is determined from equation 44. One has

$$a(\infty) = a_\infty = \sqrt{\pi} \, \Gamma(\tfrac{2}{3}) \left[\frac{a_o}{|\Gamma(\frac{1}{2} - \alpha)|^2} + \frac{\ell_o}{3|\Gamma(\frac{7}{6} + \alpha)|^2} \right]$$

so that for large n

$$a_\infty \sim \frac{e^{\pi\delta}\,\Gamma(2/3)}{2\sqrt{\pi}}\,[a_o\,\delta^{-1/6} + \frac{\ell_o}{3}\,\delta^{-7/6}]\,, \qquad (49)$$

where, from equation 43

$$\delta = \frac{(24n - 25)^{1/2}}{12} \;.$$

Figure 1 shows the variation of a/a_o with R_o/R for various values of n for the case in which the initial velocity amplitude is zero, $\ell_o = 0$. Figure 2 exhibits the variation of a/a_o for a non-zero value of the initial velocity. Of greater significance is the ratio of the distortion amplitude a to the mean radius R; the behavior of $(a/a_o)(R_o/R)$ is shown as a function of R_o/R in Figure 3 for the case in which the initial velocity amplitude is zero, and in Figure 4, for the case in which the initial velocity amplitude is different from zero.

Solution for the Collapsing Cavity: No Surface Tension.

This case can be discussed in exactly the same manner as the previous one. In fact, equation 42 is directly applicable although for convenience it is better to write the solution in the form, [12],

$$a = Ax^{-\alpha}\,F(\alpha,\ \alpha + \tfrac{2}{3},\ \tfrac{1}{2};\ 1 - \tfrac{1}{x})$$

$$+ Bx^{-\alpha}\,(1 - \tfrac{1}{x})^{1/2}\,F(-\beta + \tfrac{1}{2},\ -\beta + 1,\ \tfrac{3}{2};\ 1 - \tfrac{1}{x})$$

$$1 \leq x \leq \infty \qquad (50)$$

Equation 39 is written as

$$\dot{R}^2 = \frac{2}{3}\frac{P}{\rho}\left[\frac{R_o^3}{R^3} - 1\right]$$

where now $P = P_o - P_i > 0$. The constants A and B are found as in the previous case in terms of the initial perturbation amplitude a_o and the initial velocity amplitude v_o;

$$A = a_o$$

$$B = \frac{2}{3} v_o \frac{R_o}{(2P/3\rho)^{1/2}} = \frac{2}{3} \ell_o .$$

The solution can, of course, be written in a variety of forms. In place of equation 50 one can write, for example,

$$a = A'y^{\alpha} F(\alpha, \alpha + \frac{2}{3}, 2\alpha + \frac{7}{6}, y)$$
$$+ B'y^{-\alpha-1/6} F(-\alpha + \frac{1}{2}, -\alpha - \frac{1}{6}, -2\alpha + \frac{5}{6}; y) \qquad (52)$$

where

$$y = \frac{1}{x} = \frac{R^3}{R_o^2}, \qquad 0 \leq y \leq 1 . \qquad (53)$$

A' and B' are linear combinations of a_o and ℓ_o which need not be written explicitly here. From equation 52 one finds that, in the neighborhood of $y = 0$,

$$a \simeq A'y^{-1/12+i\delta} + B'y^{-1/12-i\delta}$$

or

$$a \approx \text{const. } R^{-1/4} \qquad (54)$$

which is the singularity noted by Birkhoff. The variation of the distortion amplitude with mean bubble radius is shown in Figure 5 for $n = 3$ and in Figure 6 for $n = 6$. The quantity of significance is the ratio of a to R; therefore, the variation of $(a/a_o)(R_o/R)$ with R/R_o is shown in Figure 7 for $n = 3$ and in Figure 8 for $n = 6$.

Expanding Cavity With Surface Tension.

For a bubble expanding from rest, $\dot{R}_o = 0$, one has from equation 24

$$\dot{R}^2 = \frac{2P}{3\rho}\left[1 - \frac{R_o^3}{R^3}\right] - \frac{2\sigma}{\rho R}\left[1 - \frac{R_o^2}{R^2}\right] \qquad (55)$$

where $P = P_i - P_o > 0$. The radial acceleration is determined by the relation

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 = \frac{P - 2\sigma/R}{\rho} .$$

If the stability equation 27 is written in terms of the independent variable

$$z = \frac{R_o}{R} \qquad (56)$$

it takes the form

$$z\left[\frac{2}{3} - kz + (k - \frac{2}{3})\, z^3\right]\frac{d^2a}{dz^2} - \left[\frac{2}{3} - kz - \frac{1}{3}(\frac{3k}{2} - 1)\, z^3\right]\frac{da}{dz}$$

$$- (n-1)\left[\frac{k}{2}\,[1 - (n+1)(n+2)] + (1 - \frac{3k}{2})\, z^2\right] a = 0 \qquad (57)$$

where

$$k = \frac{2\sigma}{R_o P} . \qquad (58)$$

Hence k is the ratio of the initial value of the surface tension to the static pressure difference between the inside of the bubble and the liquid. Equation 57 has a neat solution for the value $k = \frac{2}{3}$ in which case it reduces to the hypergeometric differential equation. This value of k is reasonable for vapor bubbles growing in superheated water where it is effectively slightly less than unity, [13]. A convenient form for this solution is

$$a = AF(\alpha, \beta, \tfrac{1}{2};\ 1-z) + B(1-z)^{1/2}\, F(\alpha + \tfrac{1}{2},\ \beta + \tfrac{1}{2}, \tfrac{3}{2};\ 1-z) \quad (59)$$

where

$$\alpha = -\frac{3}{4} + \frac{(9 + 16N)^{1/2}}{4}, \quad (60A)$$

$$\beta = -\frac{3}{4} - \frac{(9 + 16N)^{1/2}}{4}, \quad (60B)$$

and

$$N = \frac{(n-1)}{2}(n^2 + 3n + 1) \quad . \quad (61)$$

If a_o is the initial distortion amplitude and v_o the initial velocity amplitude, one finds

$$A = a_o$$

and

$$B = 2\ \ell_o$$

where

$$\ell_o = \frac{R_o}{(\frac{2P}{3\rho})^{1/2}}\ v_o \ .$$

The variation of a/a_o with R_o/R when $k = 2/3$ is shown in Figure 9 for $n = 2$ and 3. In Figure 10 the variation of $(a/a_o)(R_o/R)$ with R_o/R is shown for these same values of n. When these curves are compared with Figures 1 and 2 or Figures 3 and 4 the stabilizing effect of surface tension is evident. It is also of interest to observe that when surface tension is included a/a_o changes sign as R increases.

Cavity Collapse Under Surface Tension Alone.

An additional case of interest and which is also amenable to analytic investigation occurs for $P_i = P_o$ so that the cavity collapses under the influence of surface tension alone. In order to obtain analytic results for this case the stability equation 27 is modified by the substitution

$$u = \frac{R^2}{R_o^2} \tag{62}$$

so that the resulting equation has u as the independent variable rather than t. The solution of this equation can be written as

$$a = u^m [AF(\alpha, \beta, \tfrac{1}{2}; 1-u) + B(1-u)^{\frac{1}{2}} F(\alpha + \tfrac{1}{2}, \beta + \tfrac{1}{2}, \tfrac{3}{2}; 1-u) \tag{63}$$

where m have the value

$$m = \frac{-1 + i(24n - 25)^{1/2}}{8} \tag{64}$$

and F is the hypergeometric function with parameters α and β determined by the relations

$$\alpha = \frac{m}{2} - \frac{(n-1)}{8}(n^2 + 3n + 4)$$

$$\alpha + \beta = 2m + \frac{3}{4} .$$

The constants A and B are determined in terms of the initial distortion and velocity amplitude a_o and v_o in the form

$$A = a_o$$

$$B = -L_o$$

where

$$L_o = v_o \frac{R_o}{\left(\frac{2\sigma}{R_o \rho}\right)^{1/2}} .$$

It is evident from equation 63 that

$$a \to \text{const. } R^{-1/4} \quad \text{as} \quad R \to 0 .$$

IV. THE CONCLUSION.

For an expanding vapor cavity an initially spherical shape is stable in the sense that the deformation amplitude a remains small compared to R if its initial value a_o is small compared to the initial cavity radius R_o.

The consistency and applicability of the linearized perturbation theory for the distortion amplitude is thus demonstrated. These conclusions drawn from the linearized theory must be qualified for the case in which the surface tension is negligible. As is shown graphically in Figures 3 and 4, a/R as a function of R_o/R has a maximum which increases slowly with n, the order of the surface harmonic. It follows therefore when surface tension is unimportant that needle-like irregularities in the spherical interface may grow to significant amplitudes. The present linearized theory is inadequate to follow the development of these high order distortions of the interface. This instability for large n disappears when surface tension is of significance so that no such restriction need be imposed upon the applicability of the linearized theory in this case.

For a collapsing vapor cavity, on the other hand, the perturbation theory is valid provided that the distortion amplitude is not followed to small cavity radii. If R_o is the initial radius of the spherical cavity then the distortion amplitudes remain small so long as $1 \gtrsim R/R_o \gtrsim 0.2$ where the lower limit is, of course, approximate. The linearized theory is valid then over an interesting and important range of cavity radius. As the radius R tends to zero the distortion amplitudes oscillate in sign with increasing frequency and increase in magnitude in proportion to $R^{-1/4}$. This increase in distortion amplitude as $R^{-1/4}$ is found with and without surface tension. It may be remarked that the linearized theory for the distortion amplitudes breaks down in a range of radii near that for which the present model of the vapor cavity becomes invalid. It is known, [14], that the vapor pressure within a collapsing vapor cavity, such as is encountered in cavitating flow, begins to rise very rapidly as R/R_o becomes smaller than approximately 0.1. Further, it is known from the studies made by Gilmore, [8], that the effects of liquid compressibility become significant when R/R_o becomes less than 0.1.

V. THE APPENDIX

This appendix is devoted to the evaluation of an integral which appears in the derivation of certain results stated in the text of the foregoing analysis. The integral involves products of derivatives of surface harmonics. The surface harmonic Y_n of order n is defined to be

$$Y_n = \sum_{o}^{n} (A_r \cos r\phi + B_r \sin r\phi)\, P_n^r (\cos \theta) \qquad (65)$$

$$= \sum_{o}^{n} U_n^r \qquad (66)$$

where θ and ϕ are the usual angles of colatitude and azimuth in spherical polar coordinates. The $P_n^r(\cos \theta)$ is the associated Legendre function. It is clear that

$$\int U_n^r \, U_m^s \, d\omega = 0 \qquad \begin{matrix} r \neq s \\ m \neq n \end{matrix} \qquad (67)$$

where $d\omega$ is the element of surface of the unit sphere. One is free to choose

$$\int U_n^r \, U_n^r \, d\omega = 1 \qquad (68)$$

which determines the normalization of the surface harmonics. From equations 67 and 68 it follows that

$$\int Y_m \, Y_n \, d\omega = 0 \qquad m \neq n \qquad (69)$$

and

$$\int Y_n^2 \, d\omega = n + 1 \quad . \qquad (70)$$

The integral to be evaluated is of the form

$$\int \left[\frac{\partial Y_m}{\partial \theta} \frac{\partial Y_n}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial Y_m}{\partial \phi} \frac{\partial Y_n}{\partial \phi} \right] d\omega \quad . \qquad (71)$$

If $\mu = \cos\theta$ then this integral becomes

$$\int\left[(1-\mu^2)\frac{\partial Y_n}{\partial\mu}\frac{\partial Y_m}{\partial\mu}+(1-\mu^2)^{-1}\frac{\partial Y_n}{\partial\phi}\frac{\partial Y_m}{\partial\phi}\right]d\omega \quad . (72)$$

This expression can be written in the form

$$\int\left[(1-\mu^2)\sum^{n}\frac{\partial U_n^r}{\partial\mu}\sum^{m}\frac{\partial U_m^s}{\partial\mu}+(1-\mu^2)^{-1}\sum^{n}\frac{\partial U_n^r}{\partial\phi}\sum^{m}\frac{\partial U_m^s}{\partial\phi}\right]d\omega \qquad (73)$$

It is known that U_m^s satisfies the equation

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{\partial U_m^s}{\partial\mu}\right]+\left[m(m+1)-\frac{s^2}{1-\mu^2}\right]U_m^s=0 \quad . (74)$$

On integrating the first term in equation 73 by parts and on using equations 67 and 74, one finds that the integral reduces to

$$\sum_{o}^{n}\int n(n+1)\,{U_n^r}^2\,d\omega \qquad (75)$$

which is

$$= n(n+1)^2 \qquad (76)$$

This value has been used in the derivation of equations 14, 17 and 21 which, of course, have been derived correct to second order terms in a_n.

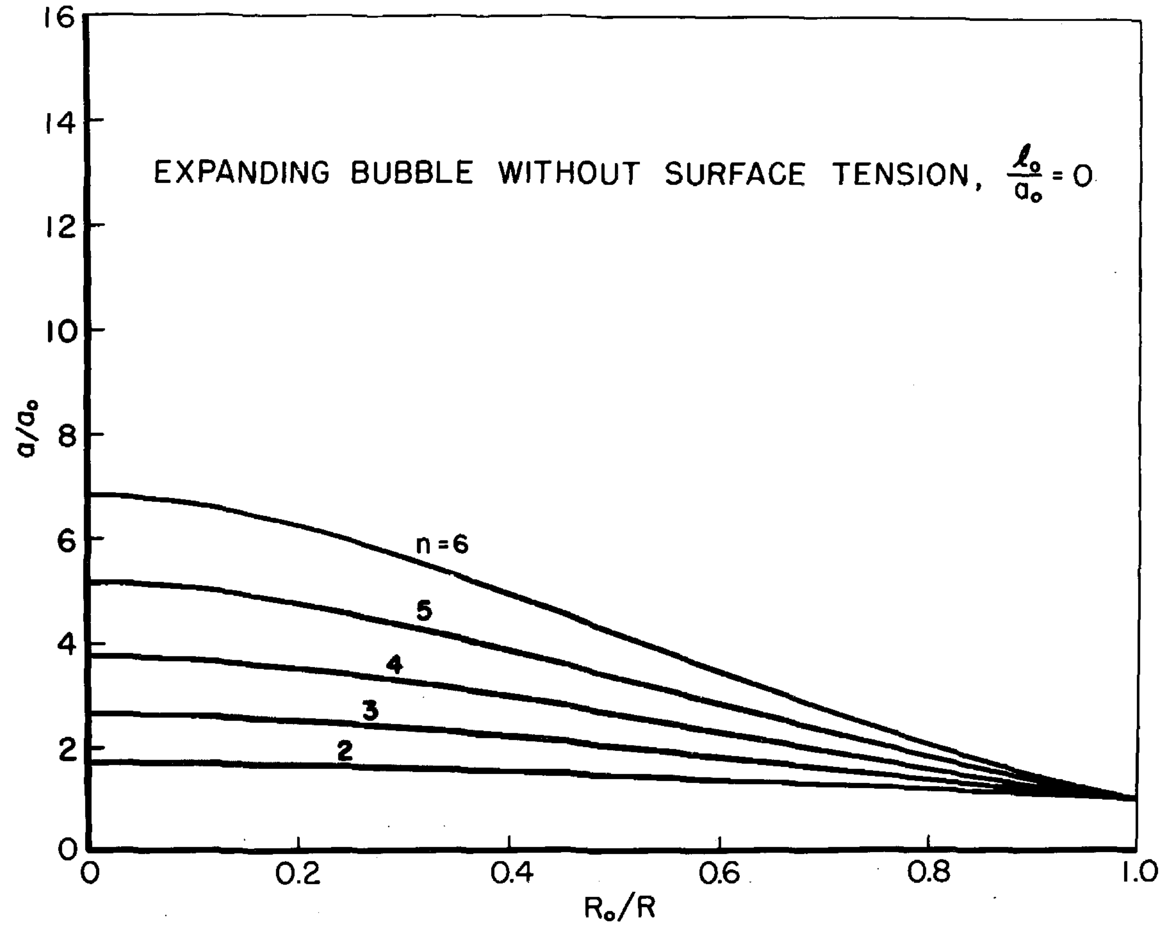

Fig. 1 - The ratio of the distortion amplitude a to its initial value a_o is shown for an expanding vapor cavity as a function of R_o/R where R_o is the initial cavity radius and R its radius at later times. The distortion of the spherical interface is $a_n Y_n$ where Y_n is a spherical harmonic of order n. The initial velocity of the distortion is zero.

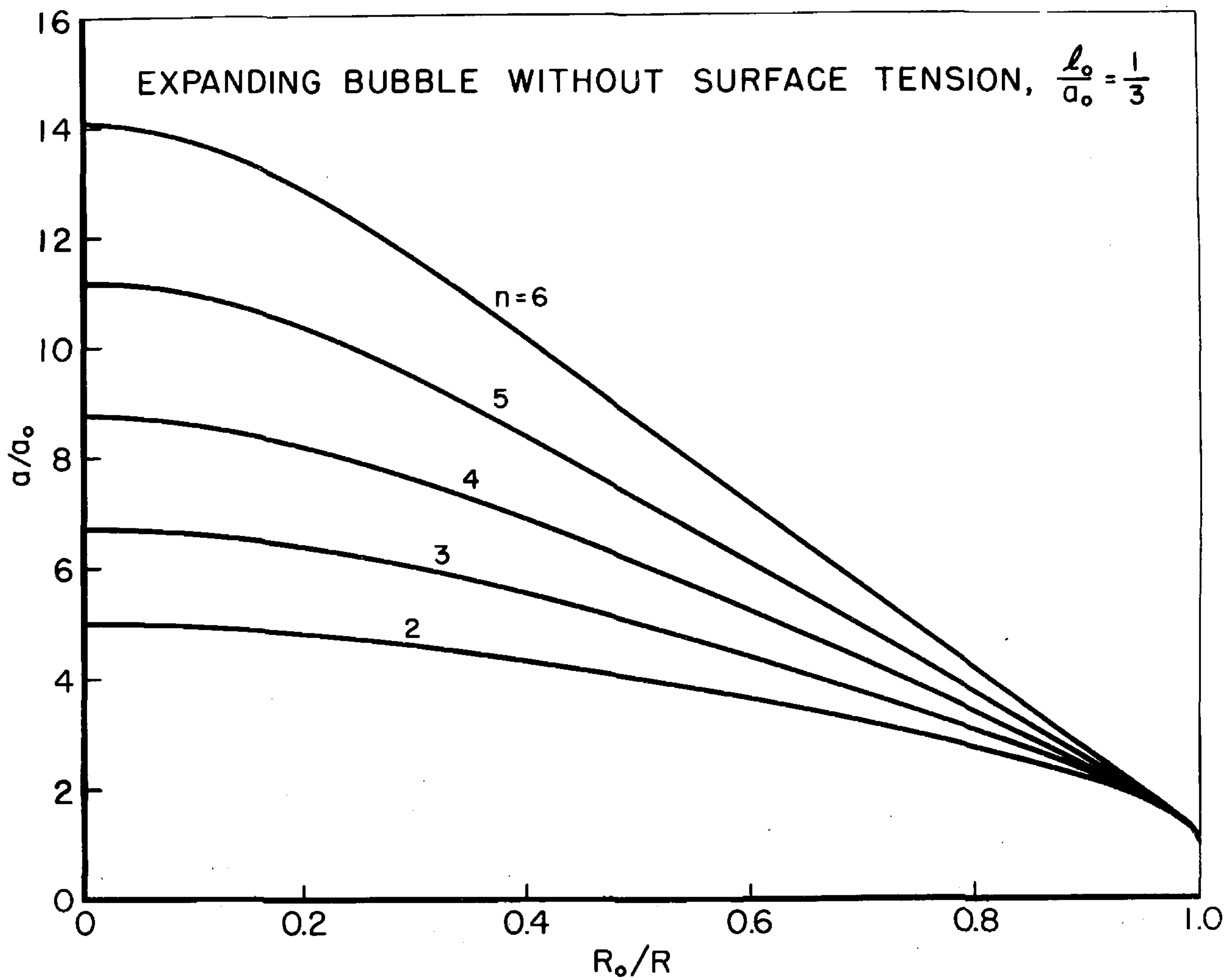

Fig. 2 - The distortion amplitude is shown as a function of cavity radius for a non-zero initial distortion velocity.

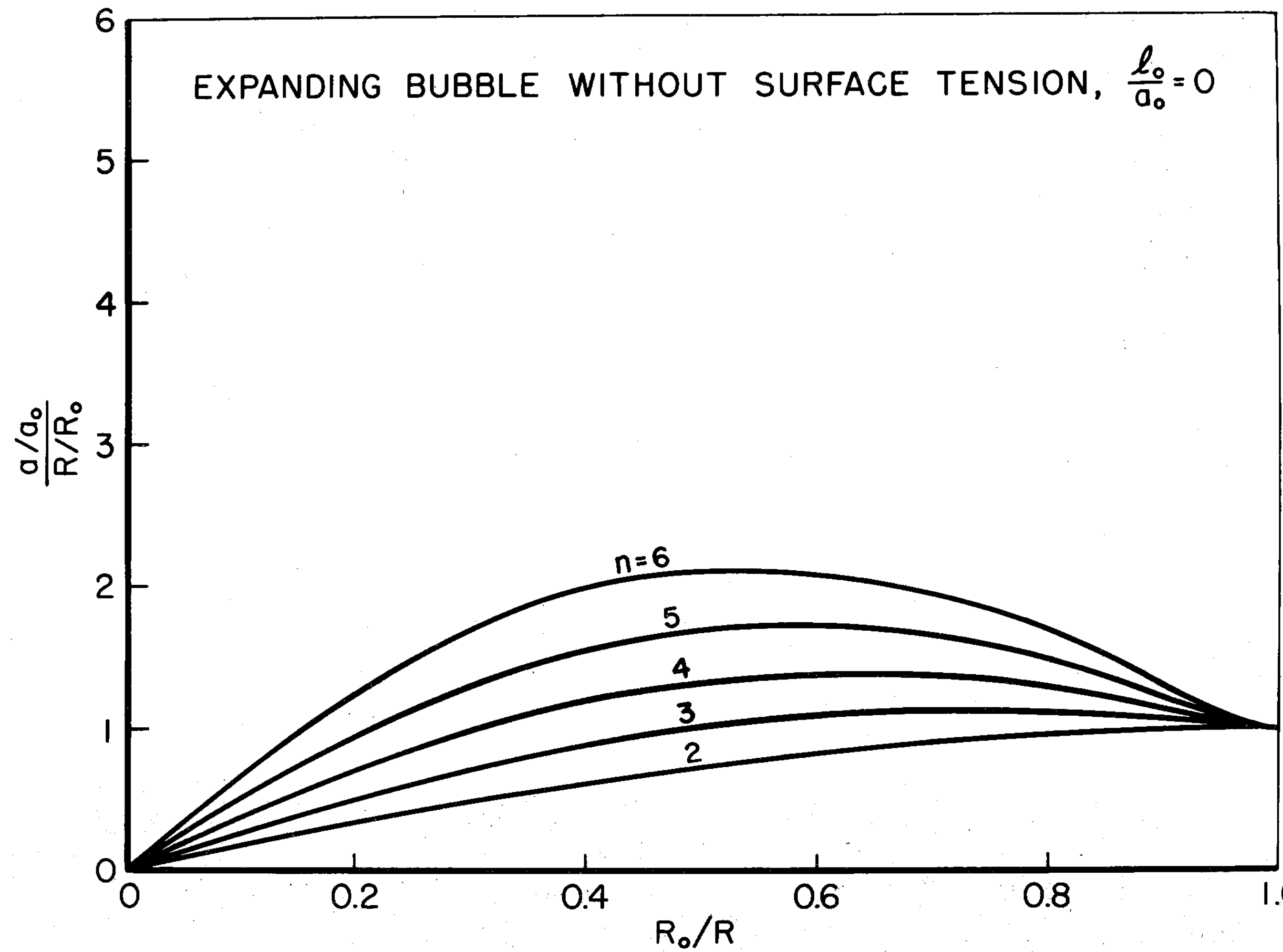

Fig. 3 - The ratio of the distortion amplitude a to the mean cavity radius R (in units of a_o/R_o) is shown as a function of R_o/R for the case shown in Fig. 1.

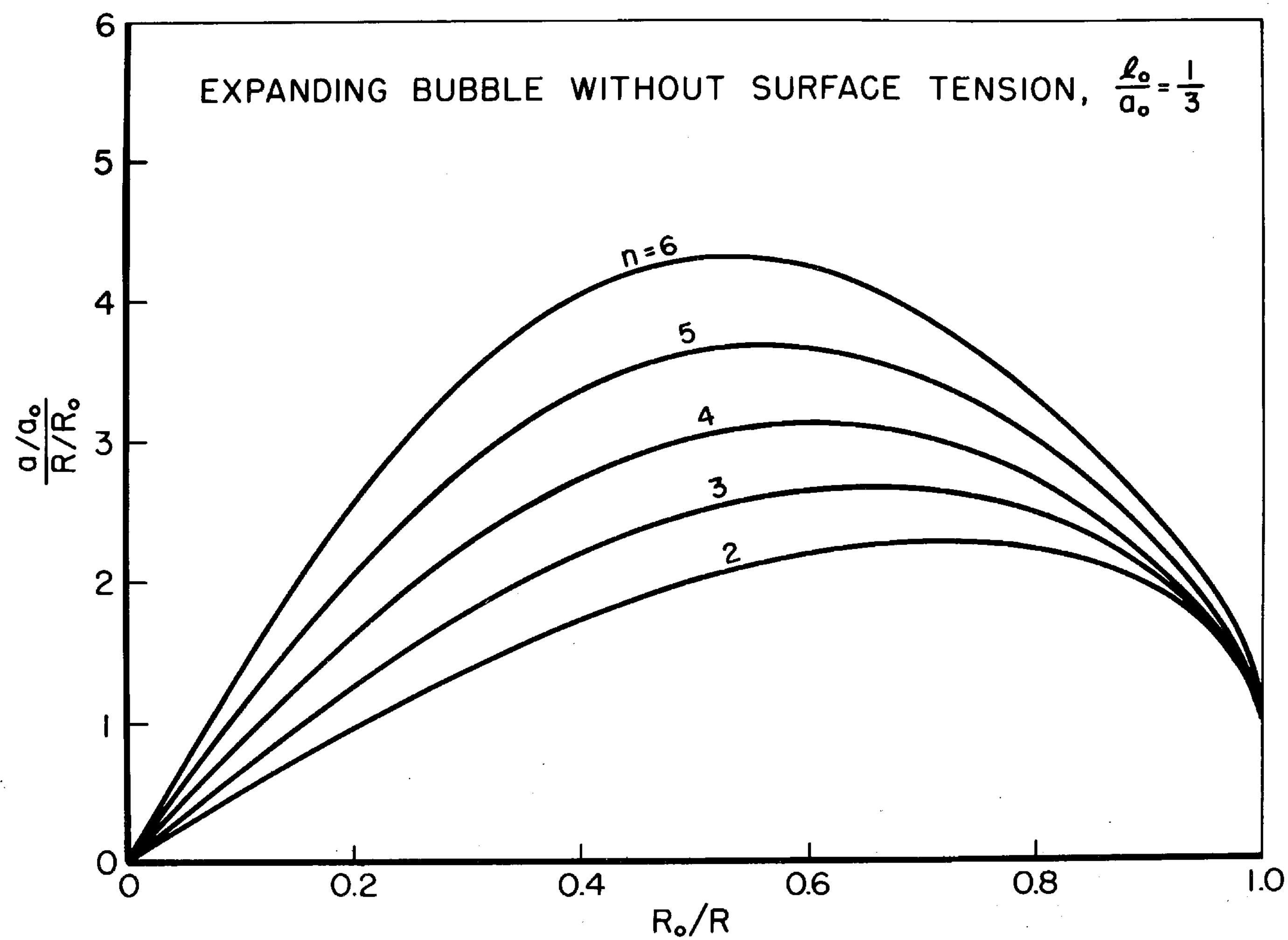

Fig. 4 - The ratio of the distortion amplitude a to the mean cavity radius R (in units of a_o/R_o) is shown as a function of R_o/R when the initial distortion amplitude velocity is non-zero.

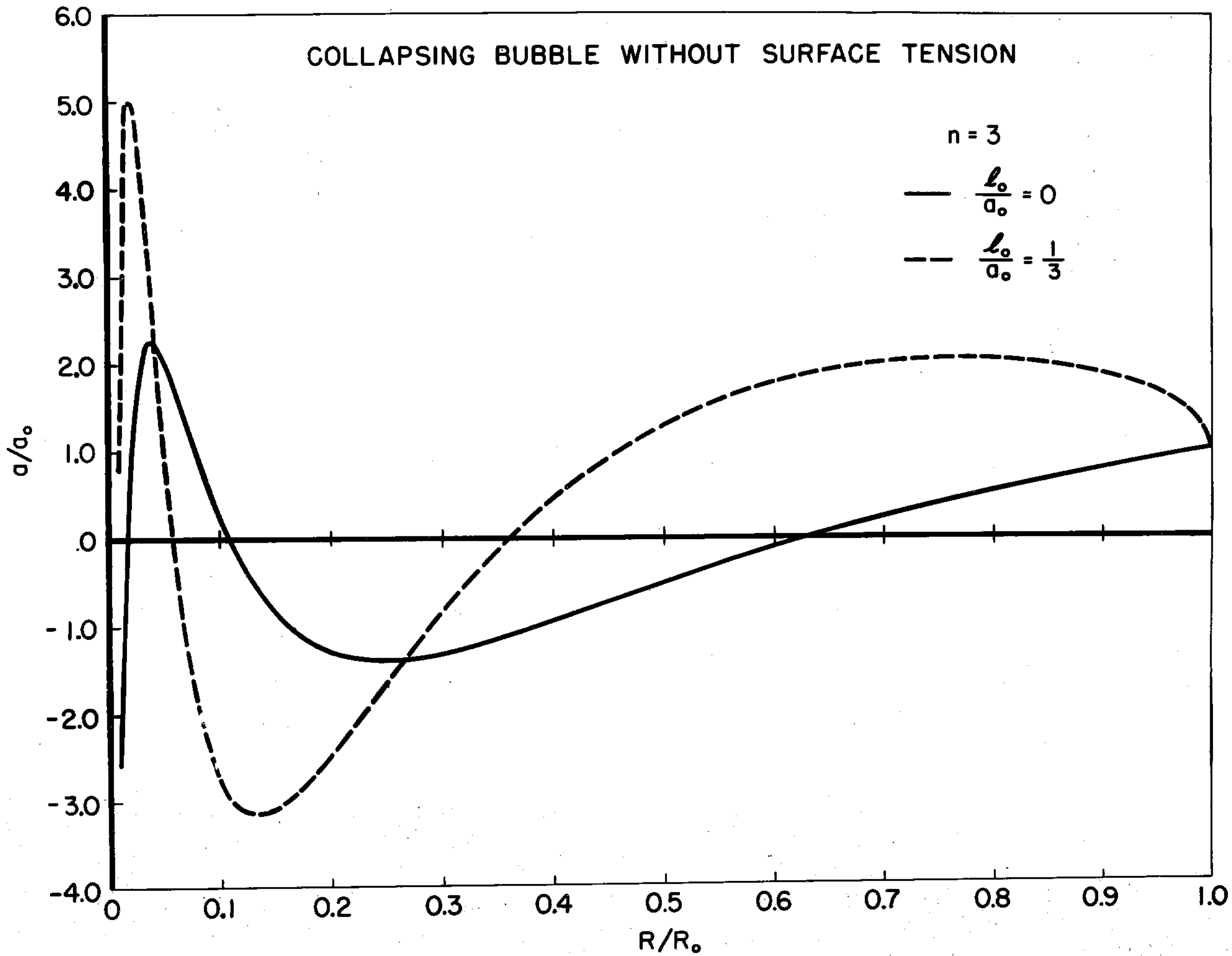

Fig. 5 - The ratio of the distortion amplitude a to its initial value a_o is shown for a collapsing vapor cavity as a function of R/R_o. The case shown is for a distortion described by a spherical harmonic of order $n=3$.

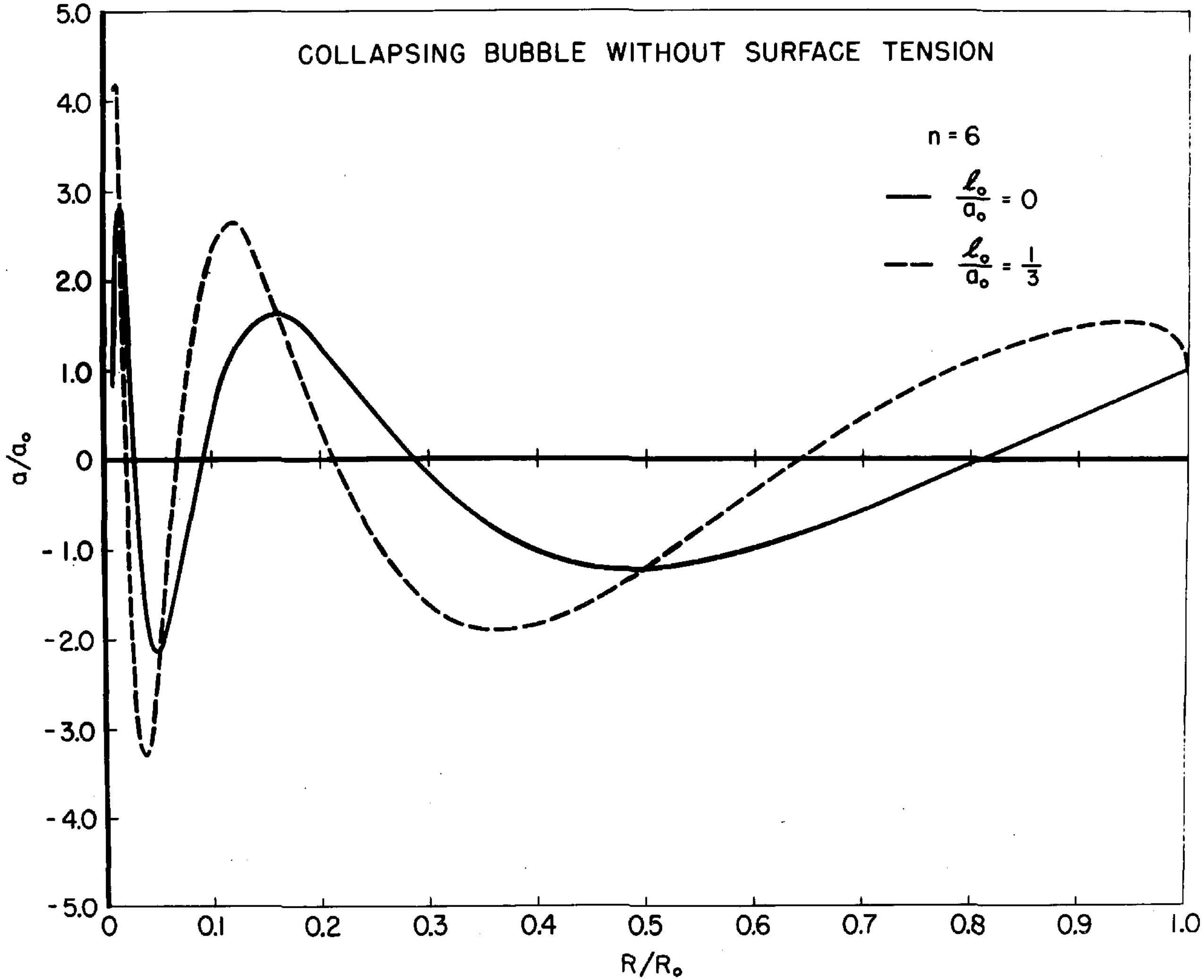

Fig. 6 - The distortion amplitude is shown as a function of cavity radius for the case n = 6.

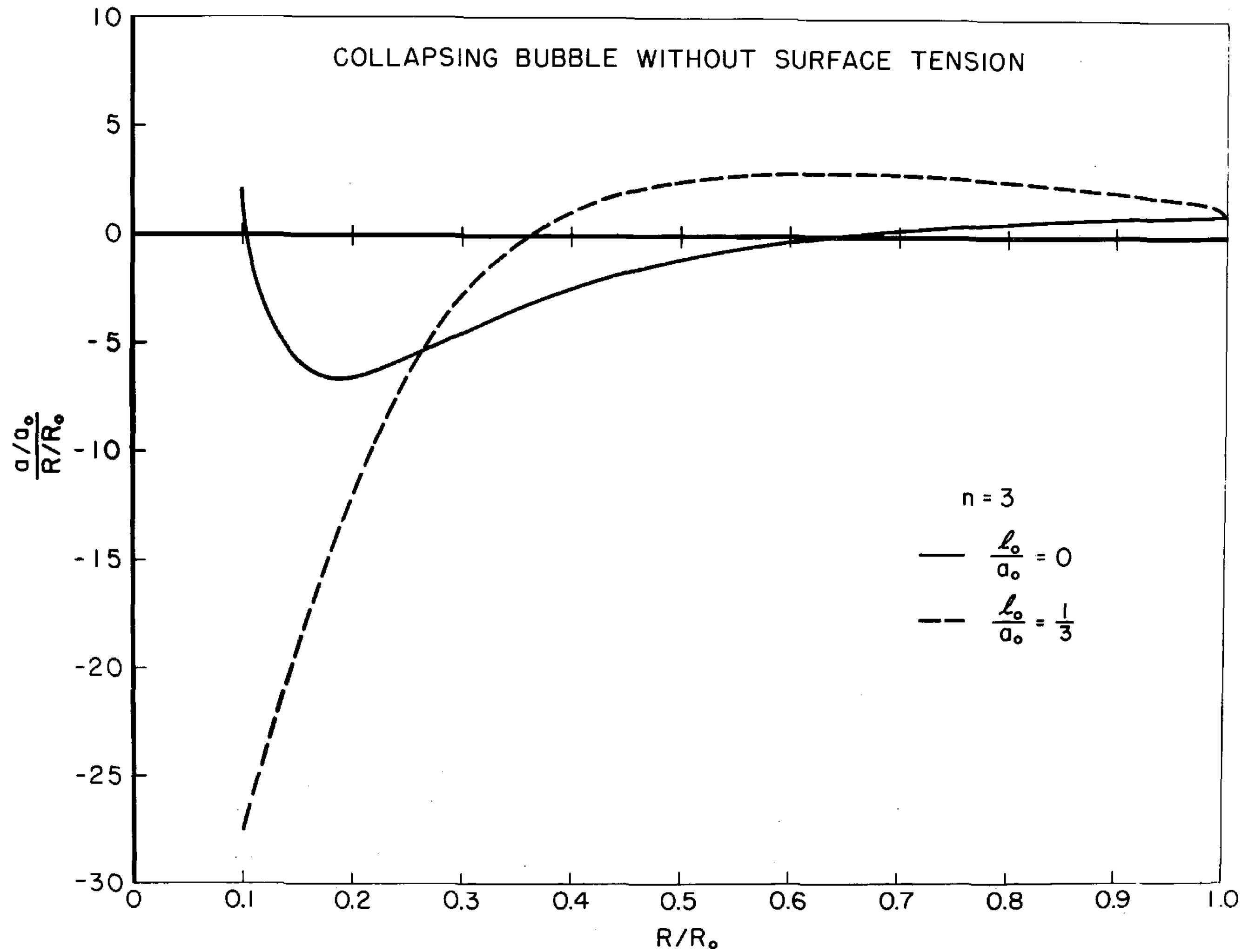

Fig. 7 - The ratio of the distortion amplitude a to the mean cavity radius R (in units of a_o/R_o) is shown as a function of R/R_o for n = 3.

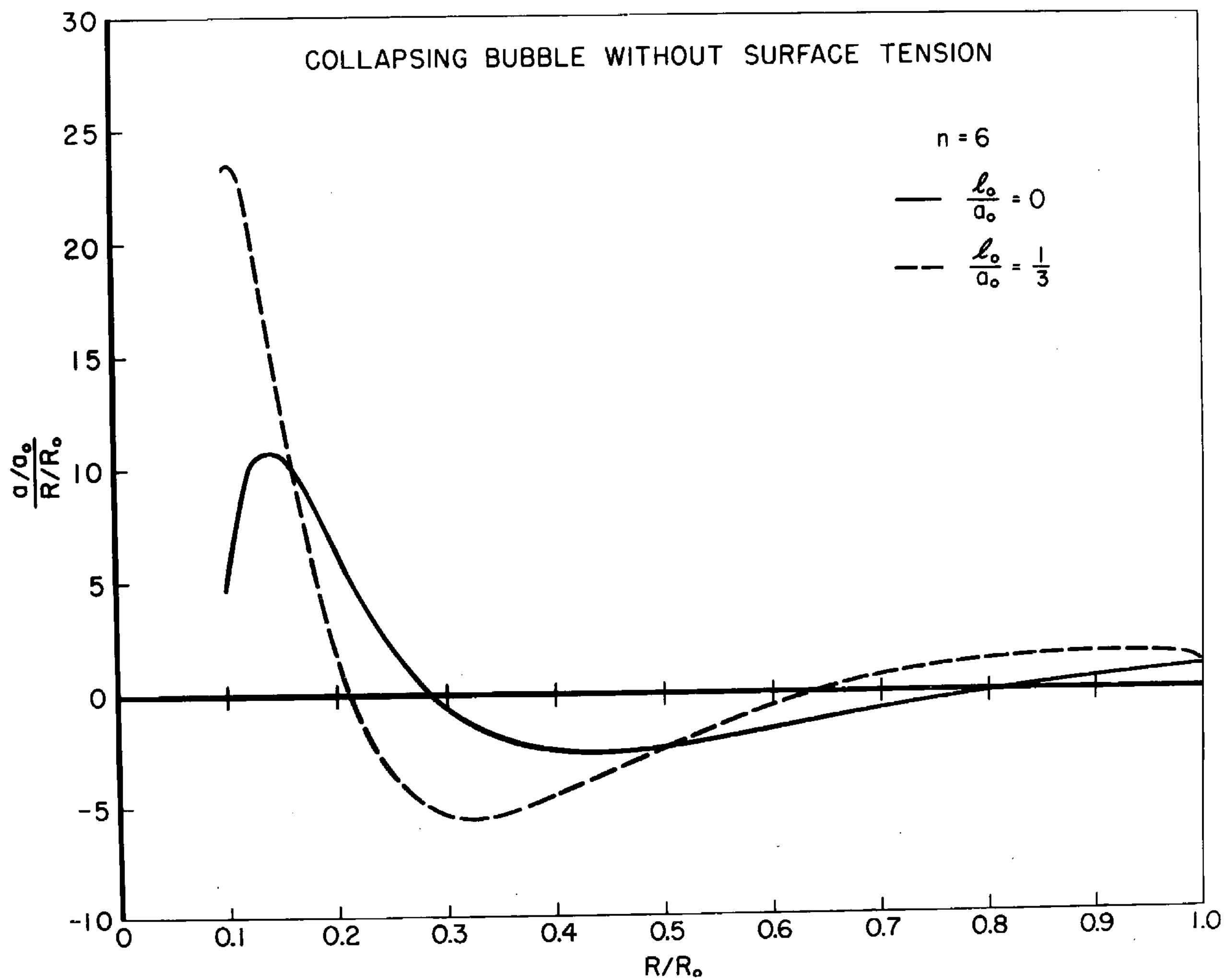

Fig. 8 - The ratio of the distortion amplitude a to the mean cavity radius R (in units of a_o/R_o) is shown as a function of R/R_o for n = 6.

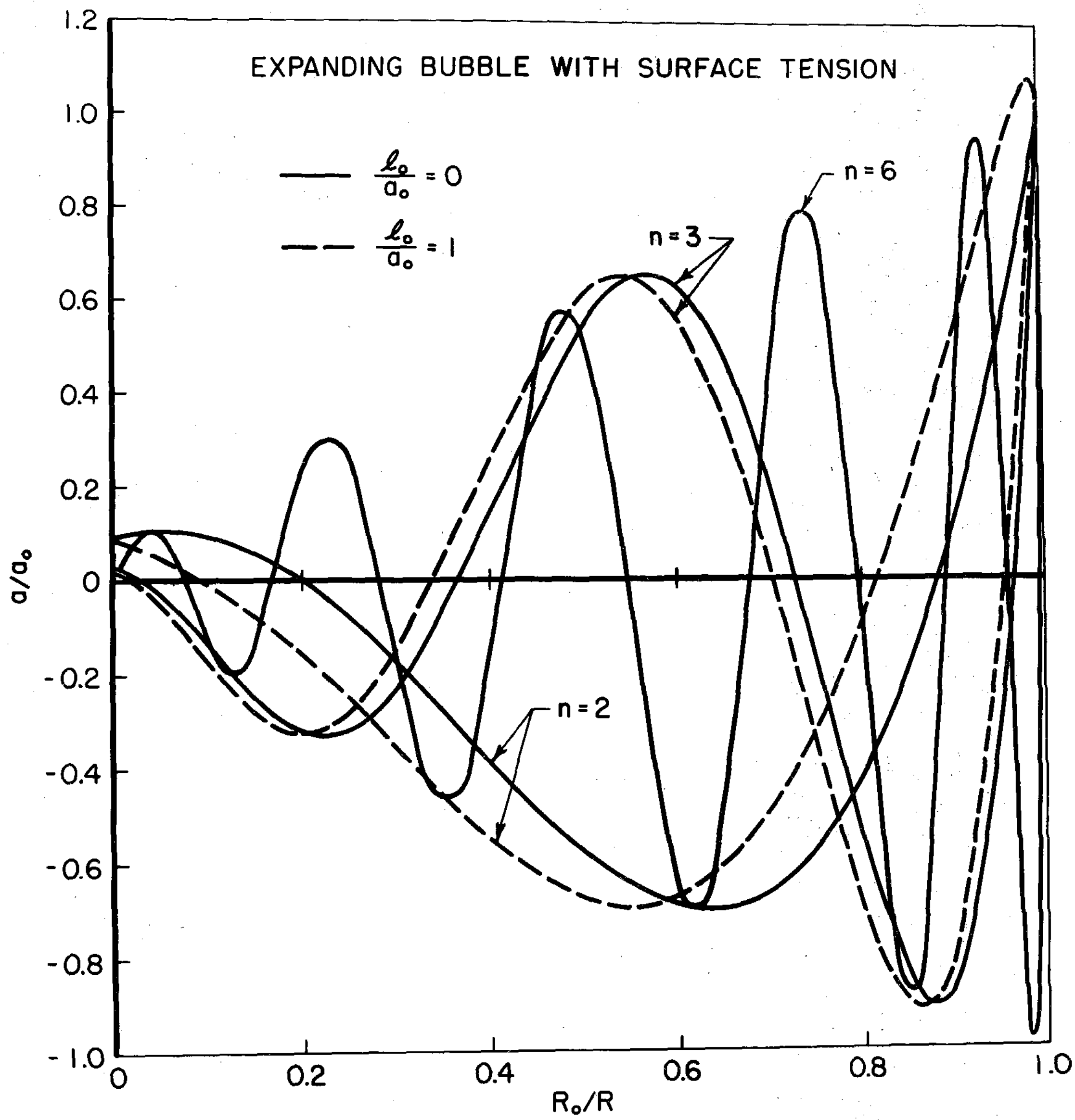

Fig. 9 - The distortion amplitude a relative to its initial value a_o is shown for an expanding cavity as a function of R_o/R for the case in which the effect of surface tension is included. For n=6 the curve with $\ell_o/a_o = 1$ is not shown since it lies quite close to the curve $\ell_o/a_o = 0$.

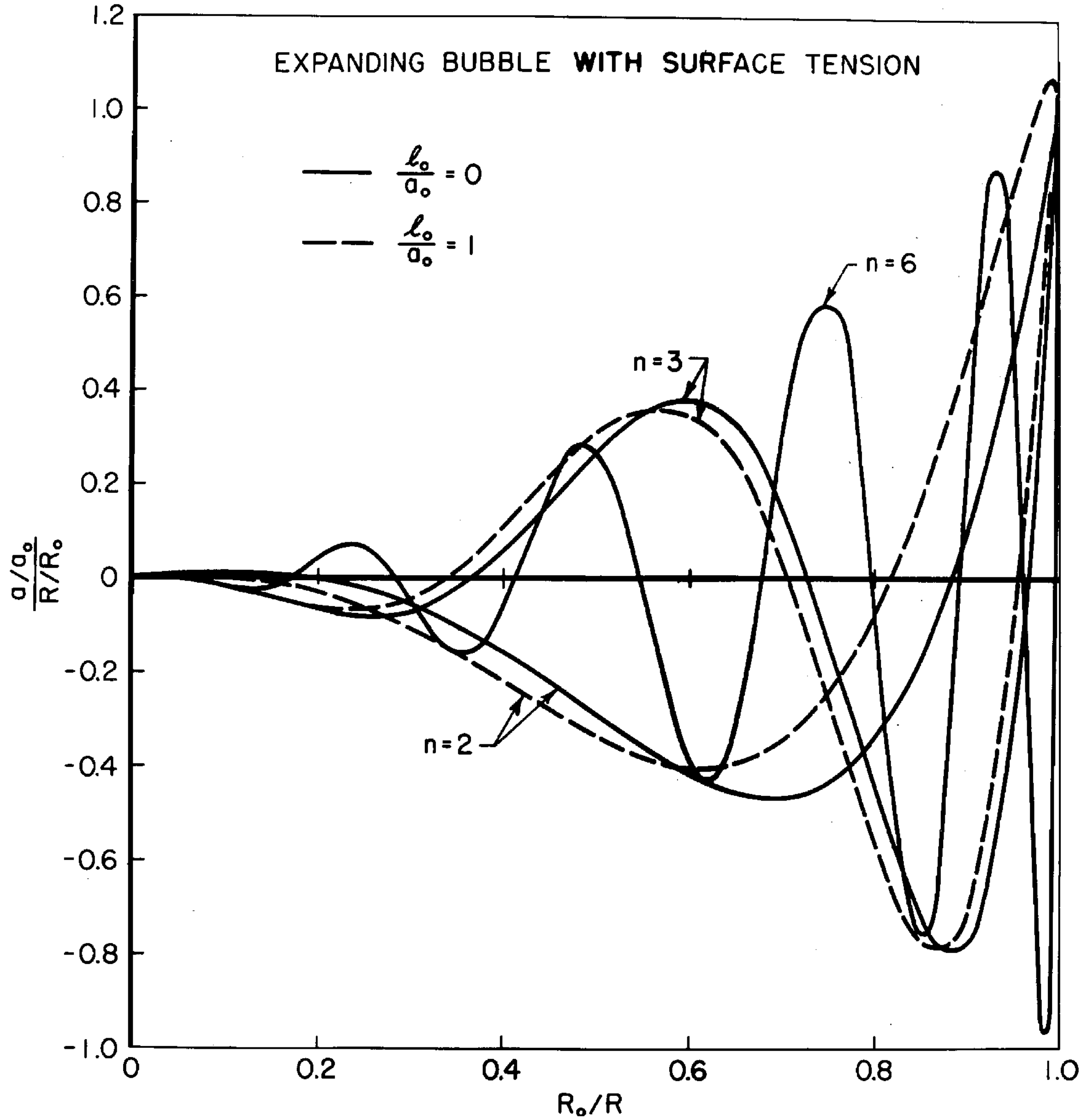

Fig. 10 - The ratio of the distortion amplitude a to the mean cavity radius (in units of a_o/R_o) is shown as a function of R_o/R for the case in which the effect of surface tension is included. For n = 6 the curves $\ell_o/a_o = 1$ and $\ell_o/a_o = 0$ are very near each other.

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