The Maximal Subgroups of the Chevalley Groups $F_4(F)$ Where F is a Finite or Algebraically Closed Field of Characteristic $\neq 2,3$

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Kay Magaard

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Abstract

We find the conjugacy classes of maximal subgroups of the almost simple groups of type $F_4(F)$, where F is a finite or algebraically closed field of characteristic $\neq 2,3$. To do this we study $F_4(F)$ via its representation as the automorphism group of the 27-dimensional exceptional central simple ***Jordan Algebra*** **J** defined over F. A Jordan Algebra over a field of characteristic $\neq 2$ is a nonassociative algebra over a field F satisfying $xy = yx$ and $(x^2y)x = x^2(yx)$ for all its elements x and y . We can represent $\mathrm{Aut}(F_4(F))$ on **J** as the group of semilinear invertible maps preserving the multiplication. Let $G = F_4(F)$ and $G \leq \Gamma \leq \mathrm{Aut}(G)$. We have defined a certain subset of proper nontrivial subalgebras as ***good*** . The principal results are as follows:

SUBALGEBRA THEOREM: Let F be a finite or algebraically closed field of characteristic $\neq 2,3$. Let H be a subgroup of Γ and suppose that H stabilizes a subalgebra. Then H stabilizes a good subalgebra. The conjugacy classes and normalizers of good subalgebras are also given.

STRUCTURE THEOREM: Let H be a subgroup of Γ such that $H \cap G$ is closed but not almost simple. Then H stabilizes a proper nontrivial subalgebra or H is contained in a conjugate of $N_\Gamma(3^3\colon SL_3(3))$. The action of $3^3\colon SL_3(3)$ on **J** is described and it is shown that $3^3\colon SL_3(3)$ is unique up to conjugacy in G .

THEOREM : If L is a closed simple nonabelian subgroup of G, then $N_\Gamma(L)$ is maximal in Γ only if L is one of the following :

$\mathcal{S} = \{F_4(F_0), PSL_2(F), G_2(F), {}^3D_4(2), PSL_3(3), PSU_3(3), PSL_2(q)\colon q \in \{8,9,13,17,25,27\}\}$

For each member $L \in \mathcal{S}$ we identify those representations π_L which could give rise to a maximal subgroup of G and show the existence of $\pi_L(L)$ in G . Up to few exceptions we also determine the number of G conjugacy classes for each equivalence class π_L.

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Introduction

The study of a finite group almost always necessitates some knowledge of its subgroups. In many instances some knowledge of its maximal subgroups is desirable, if not crucial. This is, in part, because the maximal subgroups of a group determine its primitive permutation representations and vice versa. Also, in certain applications of group theory to other fields of mathematics, one needs information about maximal subgroups. One such application is the inverse problem of Galois theory, i.e., when can a finite group be realized as a Galois group of a Galois extension of the rational numbers. See for example Matzat [M]. It is for these and other reasons that the maximal subgroup problem has existed for as long as the subject of group theory itself. However, up to the late 1970's the maximal subgroup problem was only solved for special classes of groups. In 1985 Aschbacher and Scott [As10] showed amongst other things that the maximal subgroup problem can be reduced to two well defined problems:

Let G be a finite group with $L \leq G \leq \mathrm{Aut}(L)$ for some simple nonabelian simple group L, and let V be a faithful irreducible G-module over some field of prime order (dividing $|G|$).

(1) Determine $H^1(G,V)$.

(2) Determine the maximal subgroups of G not containing L.

Any group H satisfying the assumptions of G is called an *almost simple group*.

Over the course of the past ten years, significant progress has been made on the second problem. For example, O'Nan-Scott and Aschbacher proved powerful structure theorems for the Alternating and Symmetric groups, the classical groups, respectively, which in some sense make up the bulk of the finite simple groups. The structure theorems go as follows:

Let G be an almost simple group. Suppose that G is represented as a group of automorphisms of some mathematical object X(G). Then any proper subgroup H of G either stabilizes some member of the set C(G,X(G)) of "natural structures" on X(G) or H is an almost simple group acting irreducibly on X(G).

The natural structures C(G,X(G)) are usually substructures, coproduct or product structures on X(G). For example X(G) is the set $\{1,\ldots,n\}$ when G is the Alternating group of degree n, and when G is a classical group over a field F, then X(G) is a pair (V,f) where V is a vectorspace over F and f is trivial or a nondegenerate symplectic, hermitian or quadratic form on V. Once a structure theorem is proved for a particular group G, enumeration of its maximal subgroups involves listing the almost simple groups that act irreducibly on X(G) and then deciding which of these occur as maximal subgroups of G. At this stage the classification of the finite simple groups and their representation theories are often used to decide which groups possesses suitable irreducible representations on X(G).

For most almost simple groups we have either structure theorems or know explicitly the conjugacy classes of maximal subgroups. Presently, for the families of exceptional groups of Lie type $F_4(F)$, $E_7(F)$, $E_8(F)$ as well as the sporadic groups Baby Monster and the Monster, we have neither. The goal of this thesis is to find the conjugacy classes of maximal subgroups of the almost simple groups of type $F_4(F)$, where F is a finite or algebraically closed field of characteristic $\neq 2,3$.

I believe it's best to study $F_4(F)$ via its representation as the automorphism group of the 27- dimensional exceptional central simple *Jordan Algebra* defined over F. Recall that a Jordan Algebra over a field of characteristic $\neq 2$ is a nonassociative algebra over a field F satisfying $xy = yx$ and $(x^2y)x = x^2(yx)$ for all its elements x and y. A *homomorphism* of a Jordan Algebras is a linear map preserving the multiplication. An algebra is called *simple* when it has no homomorphic images other than 0 and itself. The algebra is called *central simple* when it's simple modulo the center. Recall here that an element z is in the center of a Jordan algebra iff $zx = \lambda x$ for all x in the algebra and $\lambda \in F$ is some fixed element. The results of Albert [A1], Jacobson and Jacobson [Ja1], Jacobson [Ja6], and Schafer [Scha1] classify the central simple Jordan algebras over F. It turns out that all but one family of examples is constructed from some associative algebra A over F by redefining the product of A as $ab = 1/2\ (a*b + b*a)$ for all a,b $\in$ A where $*$ denotes the associative product. The examples that can't be constructed from an associative algebra in this way are called *exceptional.* When F is finite or algebraically closed there is, up to isomorphism, a unique example $\mathbf{J}$ of exceptional type. Its dimension over F is 27. It's automorphism group is $F_4(F)$. We can represent $\mathrm{Aut}(F_4(F))$ on $\mathbf{J}$ as the group of semilinear invertible maps preserving the multiplication. From now on we will let $G = F_4(F)$ and $G \leq \Gamma \leq \mathrm{Aut}(G)$. By a *subalgebra* of $\mathbf{J}$ we mean a subspace containing id (the identity of $\mathbf{J}$) which is closed under multiplication. By a *proper nontrivial subalgebra* we mean a subalgebra $\mathbf{A}$ such that $\langle id \rangle < \mathbf{A} < \mathbf{J}$.

We now give a rough definition of the class of good subalgebras. By a *composition algebra* we mean a pair (V,N) where V an F-algebra with a multiplicative identity and N is a nondegenerate quadratic form on V such that $N(xy) = N(x)N(y)$ for all $x,y \in V$.

Albert and Jacobson [A2] have characterized $\mathbf{J}$ as

$$\left\{ \begin{bmatrix} \gamma_1 & c_1 & \bar{c}_2 \\ \bar{c}_1 & \gamma_2 & c_3 \\ c_2 & \bar{c}_3 & \gamma_3 \end{bmatrix} : \text{ where } \gamma_i \in F \text{ and } c_i \in \mathbf{O} \right\}$$

where $\mathbf{O}$ denotes the unique eight dimensional composition algebra over F, and $^{-}$ denotes the unique automorphism of order 2 of $\mathbf{O}$ such that $c\bar{c} = N(c)\,1$ for all $c \in \mathbf{O}$. We define multiplication on $\mathbf{J}$ by $xy := 1/2(x*y + y*x)$, were $*$ denotes ordinary matrix multiplication. Note that this multiplication is commutative. We define Q to be the quadratic form $Q(x) = 1/2\ \mathrm{tr}(x^2)$ on $\mathbf{J}$.

An *idempotent* is an element $x \in \mathbf{J}$ satisfying $x^2 = x$. An idempotent x is *primitive* if $Q(x) = 1/2$. Let $\mathcal{A}_1 = \langle id,x \rangle$ where x is a primitive idempotent. Let $\mathcal{A}_2 = \langle x,y,z \rangle$ where x,y,z are pairwise orthogonal (wrt. Q) primitive idempotents. Let $\mathcal{A}_3$ be a three dimensional extension field of F contained in $\mathbf{J}$.

Let $\mathbf{J}_{\mathbf{D}} := \left\{ \begin{bmatrix} \gamma_1 & d_1 & \bar{d}_2 \\ \bar{d}_1 & \gamma_2 & d_3 \\ d_2 & \bar{d}_3 & \gamma_3 \end{bmatrix} : d_i \in \mathbf{D} \right\}$

where $\mathbf{D}$ is a proper nondegenerate (wrt. N) subalgebra of $\mathbf{O}$. We remark that up to conjugacy in Aut($\mathbf{O}$) there are four (three) choices for $\mathbf{D}$ when F is finite (algebraically closed) respectively.

Finally let $\mathcal{A}_P$ be the set of subalgebras $\mathbf{B}$ such that $xy = 0$ for all $x,y \in \mathbf{B} \cap id^{\perp}$ plus some technical conditions described in section 6. We remark that the stabilizers of members of $\mathcal{A}_P$ are the maximal parabolic of Γ.

Now we define a subalgebra to be *good* if a is conjugate to a member of

$$\mathcal{G} := \{ \mathcal{A}_i , \mathcal{J}_D, \mathcal{A}_P : 1 \leq i \leq 3 \}.$$

We prove the following:

SUBALGEBRA THEOREM: Let F be a finite or algebraically closed field of characteristic $\neq 2,3$. Let H be a subgroup of Γ and suppose that H stabilizes a subalgebra. Then H stabilizes a good subalgebra.

The structure of the stabilizers of good subalgebras can be found in propositions 2.7 and 2.10 in lemma 5.2 and 5.13 and proposition 6.4.

STRUCTURE THEOREM: Let H be a subgroup of Γ such that $H \cap G$ is closed but not almost simple. Then H stabilizes a proper nontrivial subalgebra or H is contained in a conjugate of $N_\Gamma(3^3: SL_3(3))$.

In lemmas 7.6 , 7.7 and 7.9 we describe the action of $3^3: SL_3(3)$ on J and prove that this $3^3: SL_3(3)$ is unique up to conjugacy in G.

SIMPLE SUBGROUP THEOREM: If L is a closed simple nonabelian subgroup of G then $N_\Gamma(L)$ is maximal in Γ only if L is on the following list and J affords the representation ρ of L:

L	ρ	# of G conj. cl.	nec. cond. for ρ
$F_4(F_0)$	$M(0)\oplus M(\lambda_4)$	1	$[F:F_0]$ is a prime
$PSL_2(F)$	$M(0)\oplus M(8\lambda_1)\oplus M(16\lambda_1)$	1	$p\geq 17$ or $p = 0$
$PSL_2(F)$	$M(0)\oplus M(8\lambda_1)/M(\lambda_1)^\delta\otimes M(3\lambda_1)/M(8\lambda_1)$	1	$p=13$, $\lvert F\rvert \neq 13$

L	ρ	# of G conj. cl.	nec.cond. for ex. of ρ
$PSL_2(13)$	$M(0)\oplus M(8\lambda_1)\backslash M(2\lambda_1)\oplus M(4\lambda_1)\backslash M(8\lambda_1)$	≥ 1	p=13
$G_2(F)$	$M(0)\oplus M(2\lambda_1)$	1	p=7
$PSL_2(7)$	$M(0)\oplus M(6\lambda_1)\oplus M(4\lambda_1)$		
	$\oplus M(4\lambda_1)\backslash M(0)\oplus M(2\lambda_1)\backslash M(4\lambda_1)$	1	p=7
${}^3D_4(2)$	1+26	1	$p\neq 7$
${}^3D_4(2)$	1+26	≥ 1	p=7
$PSL_3(3)$	1+26	≥ 1	all p
$PSU_3(3)$	1+26	≥ 1	p=7
$PSL_2(27)$	1+26	≥ 1	all p
$PSL_2(25)$	1+26	≥ 1	$p\neq 5$, $-1^{1/2}\in F$
$PSL_2(17)$	1+9+16	≥ 1	$p\neq 17$
$PSL_2(13)$	1+12+14	≥ 1	p=7
$PSL_2(9)$	1+8+9+9	≥ 1	$p\neq 5$
$PSL_2(8)$	1+8+9+9	≥ 1	$p\neq 7$

The symbol ? means that it is not known wether an appropriate embedding exists. However, if it did one would get an example of a maximal subgroup. The symbol ≥ 1 means that at least one embedding of the specified type into G exists. However, in case $L \in \{PSL_3(3), PSU_3(3), PSL_2(13)\}$ the only known embedding does not lead to a maximal subgroup. More precise information about the groups on the list can be found in Sections 14, 16 and 17 .

Now some words about the strategy and organization. Section 1 is a collection of facts about composition algebras that are needed in the subsequent discussion of the exceptional 27-dimensional Jordan algebra of section 2 . Section 3 discusses Aschbacher's representation of E_6. Also a proof of the fact that Aschbacher's E_6-

form is isometric to the trilinear form arising from J is given in section 3. In sections 4, 5 and 6 the subalgebra stabilizers of G are analyzed leading to a proof of the Subalgebra Theorem for G in section 6. In section 7 we analyze the normalizers of semisimple abelian subgroups of G. In section 8 we analyze nonlocal subgroups whose generalized Fitting subgroup is not a finite simple group and prove the Structure Theorem for G. In section 9 we generalize the Subalgebra Theorem and the Structure Theorems from G to Γ. Starting from section 10 we deal exclusively with almost simple subgroups of G. In section 10 we show that when a simple subgroup L of G stabilizes a proper nontrivial subalgebra then $N_\Gamma(L)$ is not maximal in Γ. The sections 11 - 14 deal with the simple groups of Lie type defined over the same characteristic as G. The remaining sections deal with the cross- characteristically embedded simple groups of Lie type, the Alternating, and the Sporadic simple groups.

CHAPTER I: A STRUCTURE THEOREM FOR Γ

SECTION 1: FACTS ABOUT COMPOSITION ALGEBRAS

Definition: A *composition algebra* C over F is an (not necessarily associative) F-algebra defined by the following properties:

1. C has a multiplicative identity $\underline{1}$
2. C admits a nondegenerate quadratic form N such that $N(xy) = N(x)N(y)$ for all $x,y \in C$ (N will be called the *associated quadratic form* of C.)

By a homomorphism of composition algebras we mean a linear map which preserves multiplication.

The following is well known and can be found for instance in [Sp1] or [Ja2].

COMPOSITION ALGEBRA THEOREM:

1. **Every composition algebra over F is 1, 2, 4 or 8 dimensional over F.**
2. **Two composition algebras are isomorphic iff their associated quadratic forms are equivalent.**
3. **If a composition algebra C contains an *isotropic vector* (i.e., an $x \in C$ such that $N(x) = 0$) then the associated quadratic form has maximal Witt-index. In particular if F is algebraically closed, then two composition algebras over F are isomorphic iff their dimensions are equal.**
4. **If F is finite or algebraically closed, there is one isomorphism class of m dimensional composition algebras over F, where $m = 1,4,8$. There are two isomorphism classes of two dimensional composition algebras over F as there are two nonequivalent**

choices for N . When F is algebraically closed there is a unique isomorphism class of two dimensional composition algebras over F .

5. There exists an involutory automorphism of C called *conjugation* and denoted by $-$, such that $x\,\overline{x} = \overline{x}\,x = N(x)\,\underline{1}$

Proof: Part 1 can be found in [Ja2] pg. 425. Part 2 is Ex 2, pg. 428 [Ja2]. Part 3 is ex. 3, pg. 428 [Ja2]. Part 5 is on pg. 422 [Ja2]. Part 4 is a consequence of part 3 and the classification of quadratic forms over finite and algebraically closed fields.

By part 5 above we know that $\underline{1}$ is a nondegenerate vector. Thus $C = \langle\underline{1}\rangle \perp \langle\underline{1}\rangle^{\perp}$. We denote by $\pi(x)$ the projection of $x \in C$ onto $\langle\underline{1}\rangle$ with respect to that decomposition.

Definition: Let F be finite or algebraically closed. We will denote by **O** the eight dimensional composition algebra over F. **O** will also be called an *octave*.

The following is also well known .

Aut(**O**) Theorem:

1. Aut(**O**) is the Chevalley group $G_2(F)$.
2. If A,B $\subset$ **O** are isomorphic N-nondegenerate composition subalgebras, then there exists a $g \in$ Aut(**O**) such that $Ag = B$.

Proof: Part 1 is proved in [Sp1] Pg 15 - 20. Part 2 is exercise 2, pg. 428 in [Ja2].

For future reference we give a multiplication table (table 1.1) for **O**, found in [Ja2]ch8 .

Table 1.1

	$\underline{1}$	i_1	i_2	i_3	i_4	i_5	i_6	i_7
$\underline{1}$	$\underline{1}$	i_1	i_2	i_3	i_4	i_5	i_6	i_7
i_1	i_1	$c_1\underline{1}$	i_3	c_1i_2	i_5	c_1i_4	$-i_7$	$-c_1i_6$
i_2	i_2	$-i_3$	$c_2\underline{1}$	$-c_2i_1$	i_6	i_7	c_2i_4	c_2i_5
i_3	i_3	$-c_1i_2$	c_2i_1	$-c_1c_2\underline{1}$	i_7	c_1i_6	$-c_2i_5$	$-c_1c_2i_4$
i_4	i_4	$-i_5$	$-i_6$	$-i_7$	$c_3\underline{1}$	$-c_3i_1$	$-c_3i_2$	$-c_3i_3$
i_5	i_5	$-c_1i_4$	$-i_7$	$-c_1i_6$	c_3i_1	$-c_1c_3\underline{1}$	c_3i_3	$c_1c_3i_2$
i_6	i_6	i_7	$-c_2i_4$	c_2i_5	c_3i_2	$-c_3i_3$	$-c_2c_3\underline{1}$	$-c_2c_3i_1$
i_7	i_7	c_1i_6	$-c_2i_5$	$c_1c_2i_4$	c_3i_3	$-c_1c_3i_2$	$c_2c_3i_1$	$c_1c_2c_3\underline{1}$

The associated quadratic form N of **O** is given by:

$$N(w) = x_0^2 - c_1x_1^2 - c_2x_2^2 + c_1c_2x_3^2 - c_3x_4^2 + c_1c_3x_5^2 + c_2c_3x_6^2 - c_1c_2c_3x_7^2 \quad ,$$

where $w \in \mathbf{O}$ is $w = x_0\underline{1} + \Sigma x_j i_j$. We will choose $c_1 = 1$ and $c_2 = -1 = c_3$ to assure that N will have maximal Witt- index. Note also that $<i_{2j}, i_{2j+1}>$ $j \in \{0,1,2,3\}$ is a hyperbolic pair with respect to N.

Fact 1.2 : With the choice of c_i as above the set $\{\underline{1}, i_k : 1 \leq k \leq 7\}$ becomes an orthogonal (i.e., all basis elements are pairwise orthogonal) basis of **O** with respect to the quadratic form N. Moreover N is nondegenerate.

Fact 1.3 : There are three (four) Aut(**O**) conjugacy classes of nondegenerate composition subalgebras when F is algebraically closed (finite). The following four subalgebras are representatives: $<\underline{1}>$, $<\underline{1},i_1>$, $<\underline{1},i_1,i_2,i_3>$, ($<\underline{1},w>$, where $-N(w) \notin F^2$ and $w \in \underline{1}^{\perp}$). We will denote these by $\underline{1}$, F^2, F^4, K, respectively.

Fact 1.4 : The following are true:

1 . $\overline{x} = (x,1)\,1 - (\,x - (x,1)1)$

2. $(xy,z) = (x,z\overline{y}) = (y,\overline{x}z)$

Proof: This is immediate from the equation $N(xy) = N(x)N(y)$. A proof can be found in [Sp1] pg. 2.

Note that **O** is not an associative algebra, however, the following is true.

Proposition 1.2 : Any proper composition subalgebra of **O** generated by two vectors is associative, and every pair of vectors generates a proper composition subalgebra. In particular the subalgebra generated by two basis vectors of table 1.1 is associative.

Proof: This is well known and a proof can be found in [Sp1] pg. 7 .

Corollary 1.3 : Recall the projection map π from above. Then for all triples $c_i \in \mathbf{O}$ $\pi(c_1(c_2c_3)) = \pi((c_1c_2)c_3))$.

Proof: The composition algebra multiplication is bilinear and π is a linear map. So it is enough to show that the claim holds for any triple of basis vectors $\{\underline{1},i_1,\ldots,i_7\}$. Now observe, using table 1.1, that $\pi(i_r(i_si_t)) \neq 0$ iff $i_si_t \in \langle i_r \rangle$. So the only triples that contribute something nonzero lie in a subalgebra generated by two vectors, i.e., an associative subalgebra. The claim follows.

Proposition 1.4 : Let V be a nontrivial eight dimensional Aut(O) module and N an Aut(O) invariant quadratic form on V. Then (V,N) admits a unique Aut(O) invariant composition algebra structure with identity element 1 and with respect to the quadratic form N.

Proof: Recall that $\mathrm{Aut}(\mathbf{O}) \simeq G_2(F)$. We observe also that $V = M(0) \oplus M(\lambda_1)$ as an

Aut(**O**)-module. Suppose $*$ is another composition algebra multiplication. Then $(x,y,z) := (xy,z)$ is a nontrivial alternating Aut(**O**) invariant trilinear form on $M(\lambda_1) \simeq 1^{\perp}$, where $\perp$ is taken with respect to N. The same is true for $(-,-,-)_*$ defined analogous to $(-,-,-)$ but using $*$. So by [As7] 10.9 we have $(x,y,z)_* = \alpha\,(x,y,z)$ for all $x,y,z \in 1^{\perp}$ and $0 \neq \alpha \in F$. Therefore the following holds:

$$(x*y,z) = \alpha(xy,z) \text{ for all } x,y,z \in 1^{\perp}.$$

From Fact 1.4.2 we get $(xy,1) = (x,\overline{y}) = (x*y,1)$. So for $x,y \in 1^{\perp}$ we can write the following:

$$(+) \qquad x*y - (x*y,1)1 = \sum_{j=1}^{7} (x*y,i_j)i_j = \Sigma\, \alpha(xy,i_j)i_j = \alpha\,(xy - (xy,1)1).$$

If we apply N to (+) after we get the following:

$$(++) \quad N(x*y) = (1-\alpha)^2(xy,1)^2 + \alpha^2\, N(x)N(y).$$

Now we choose $x,y \in 1^{\perp}$ such $x \perp \overline{y}$ and $N(x) \neq 0 \neq N(y)$ and substitute into (++) we get: $N(x)N(y) = N(x*y) = \alpha^2\, N(x)N(y)$. So $\alpha = \pm 1$.

Now we choose x,y so that $(xy,1) = (x,\overline{y}) \neq 0$ and that $N(x) \neq 0 \neq N(y)$. Also set $\alpha = -1$. Substituting this into (++) gives the following:

$N(x)N(y) = N(x*y) = 4(xy,1)^2 + N(x)N(y)$, contradicting $(xy,1) \neq 0$. The only way out is that $\alpha = 1$. Then the claim follows from (+).

SECTION 2: DEFINITION OF J ; FACTS ABOUT J AND Aut(J)

Let F be a finite or algebraically closed field of characteristic $\neq 2,3$ and $G = F_4(F)$. Let **O** be the octave algebra over F. Let **J** be the exceptional 27 dimensional central simple Jordan Algebra over F (c.f. introduction for definitions).

Proposition 2.1: J can be characterized as

$$\left\{ \begin{bmatrix} \gamma_1 & c_1 & \overline{c}_2 \\ \overline{c}_1 & \gamma_2 & c_3 \\ c_2 & \overline{c}_3 & \gamma_3 \end{bmatrix} : \text{ where } \gamma_i \in F \text{ and } c_i \in \mathbf{O} \right\}.$$

We define multiplication on J by $xy := 1/2(x*y + y*x)$, were $*$ denotes ordinary matrix multiplication.

Proof: See [A2].

Note that J is a commutative nonassociative F-algebra. We denote by Aut(J) the set of invertible linear maps $g:J \mapsto J$ satisfying $(xy)g = (xg)(yg)$ for all $x,y \in J$.

Proposition 2.2: The following are true:

1. $G = \mathrm{Aut}(J)$.
2. $J = M(0) \oplus M(\lambda_4)$ as a G module, where $M(\lambda)$ means the irreducible G module of high weight λ wrt. some fixed Cartan subgroup H of G.

Proof : When F is algebraically closed, this is shown in [Sp1] pg. 100 Satz 20. For F finite this is shown in [Ja4].

Definitions: $Q(x) := (1/2)\mathrm{tr}(x^2)$, $(x,y) := Q(x+y) - Q(x) - Q(y)$.

Let $x \in J$ and regard J as an algebra of matrices as in proposition 2.1 and recall π from before the definition of octave in section 1. Also recall N the quadratic form associated with the octaves. Then define:

$\mathrm{Det}(x) := \gamma_1\gamma_2\gamma_3 - \gamma_3 N(c_1) - \gamma_2 N(c_2) - \gamma_3 N(c_3) + \pi(c_1(c_2c_3)) + \pi(\overline{c}_1(\overline{c}_2\overline{c}_3))$.

Let (x,y,z) be the symmetric trilinear form such that $\mathrm{Det}(x) = (1/6)\,(x,x,x)$.

Jacobson in [Ja3] shows that Det(x) is a cubic form over F. As we are working over fields of characteristic $\neq 2,3$ note that (x,y,z) is uniquely given by:

$(x,y,z) = \mathrm{Det}(x+y+z) - \mathrm{Det}(x+y) - \mathrm{Det}(x+z) - \mathrm{Det}(y+z) + \mathrm{Det}(x) + \mathrm{Det}(y) + \mathrm{Det}(z).$

Proposition 2.3:

1. Q is a nondegenerate quadratic form on **J**
2. $Q(id) = 3/2$
3. $Q(x^2) = Q(x)^2$ for all $x \in id^{\perp}$
4. $(xy,z) = (x,yz)$ for all $x,y,z \in \mathbf{J}$
5. $(xy)x^2 = x(yx^2)$
6. $x^3 = (x,id)\, x^2 + (Q(x) - (1/2)\, (x,id)^2)\, x + \mathrm{Det}(x)\, id$ for all $x \in \mathbf{J}$

Proof: [Sp1] pages 64, 65 and 68.

Remark: Proposition 2.3.5 and the commutativity of the product of **J** show that **J** is indeed a Jordan Algebra. Furthermore it follows from proposition 2.3.5 that **J** is power associative, i.e., x^i is a well defined element of **J** for all $x \in \mathbf{J}$ and $i \in \mathbb{Z}^+$; see for example [Ja2] ch. 7.4 .

From now on we will use the symbol $\perp$ to mean orthogonal with respect to Q.

Definition: By a ***subalgebra*** of **J** we mean a subspace of **J** containing id which is closed under the **J** multiplication.

Note that the equation in Proposition 2.3.6 is referred to as *Hamilton's equation*. One easy consequence of Hamilton's equation is:

Lemma 2.4: Let $\mathbf{A}$ be the subspace generated by $\{id, x^i$: i an integer greater than $0\}$. Then $\mathbf{A}$ is a subalgebra of dimension at most 3.

Proof: That $\mathbf{A}$ is a subalgebra is a consequence of proposition 2.3.5. The statement about the dimension follows from Hamilton's equation.

Theorem 2.5:

1. $g \in GL(\mathbf{J})$ is in G iff $(xg,yg) = (x,y)$ and $(xg,yg,zg) = (x,y,z)$ for all $x,y,z \in \mathbf{J}$.
2. $g \in GL(\mathbf{J})$ is in G iff id g $=$ id and $(xg,yg,zg) = (x,y,z)$ for all $x,y,z \in \mathbf{J}$.
3. $\{ g \in GL(\mathbf{J}) : (xg,yg,zg) = (x,y,z)$ for all $x,y,z \in \mathbf{J} \}$ is the universal Chevalley group of type $E_6(F)$.

Proof: Parts 1 and 2 are the contents of Lemma 1 and Theorem on pg 186 of [Ja3]. Part 3 can be found in [Ja5].

Definition: By $x\#y$ we denote the unique element of $\mathbf{J}$ satisfying

$$(x,y,z) = 2(x\#y,z) \text{ for all } z \in \mathbf{J}.$$

Lemma 2.6: $x\#y = xy - (1/2)(x,id)\,y - (1/2)(y,id)\,x - (1/2)(x,y)\,id + (1/2)(x,id)(y,id)\,id$.

Proof: See [Sp1] pg 64.

Definition: An element $x \in \mathbf{J}$ is a ***primitive idempotent*** if $x^2 = x$ and $Q(x) = 1/2$.

Proposition 2.7:

1. G is transitive on primitive idempotents of $\mathbf{J}$.
2. Let $x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Then x is a primitive idempotent.

3. $\mathbf{J} = \langle id,x\rangle \perp E_0(x) \perp E_1(x)$ where $E_i(x) = \{\, y \in \mathbf{J} : xy = i(1/2)\, y\}$.

4. $E_1(x) = \left\{ \begin{bmatrix} 0 & c_1 & \bar{c}_2 \\ \bar{c}_1 & 0 & 0 \\ c_2 & 0 & 0 \end{bmatrix} : c_i \in \mathbf{O} \right\}$ and $E_0(x) = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\gamma & c_3 \\ 0 & \bar{c}_3 & \gamma \end{bmatrix} : \gamma \in F,\ c_3 \in \mathbf{O} \right\}$.

5. The stabilizer of x in G is $\mathrm{Spin}_9^+(F)$ with $E_0(x)$ a nondegenerate 9-dimensional space with respect to Q and $G_x / C_{G_x}(E_0(x)) = \Omega_9(E_0(x),Q)$. G_x acts faithfully on $E_1(x)$.

6. If $\mathbf{A}$ is a Q-nondegenerate subalgebra containing x, then

$$\mathbf{A} = \langle id,x\rangle \perp E_{0,\mathbf{A}}(x) \perp E_{1,\mathbf{A}}(x) \quad \text{where} \quad E_{i,\mathbf{A}}(x) = E_i(x) \cap \mathbf{A}.$$

Proof: See [Sp1] for parts 1, 3, 5 and 6. Parts 2 and 4 are easy computations.

Lemma 2.8: Let $y \in \mathbf{J}$. Then $y\#y = 0$ iff either

1. y is a scalar multiple of a primitive idempotent.
2. $(y,id) = 0$ and $y^2 = 0$.

Proof: This is the content of Satz 10 pg 77 in [Sp1].

Lemma 2.9: If $y \in \mathbf{J}$ and $y^2 = 0$, then $y \in id^{\perp}$ and $y\#y = 0$.

Proof: We use Hamilton's equation (prop. 2.3.6) to deduce that

$$\mathrm{Det}(y)\, id = -(Q(y) - (1/2)(id,y)^2)\, y.$$

Now $Q(y) = (1/2)(y,y) = (1/2)(y^2,id) = 0$. Therefore, since $\{y,id\}$ is a linearly independent set, $\mathrm{Det}(y) = Q(y) = (y,id) = 0$. So now, using lemma 2.6

$$y\#y = y^2 - (1/2)(y,id)\, y - (1/2)(y,id)\, y - (1/2)(y,y)\, id + (1/2)(y,id)^2\, id = 0.$$

Proposition 2.10:

Let $x_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $x_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $x_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and

$W_i := E_1(x_{i+1}) \cap E_1(x_{i+2})$. Then

1. $\mathrm{id} = x_1 + x_2 + x_3$, $(x_i,x_j) = \delta_{ij}$ where δ_{ij} is the Kronecker delta.
2. $\mathbf{J} = \langle x_1,x_2,x_3\rangle \perp W_1 \perp W_2 \perp W_3$.
3. $C_G(\langle x_1,x_2,x_3\rangle) = \mathrm{Spin}_8^+(F)$ and $C_G(\langle x_1,x_2,x_3\rangle)$ induces $\Omega(W_i,Q)$ on W_i.
4. $N_G(\langle x_1,x_2,x_3\rangle)$ = Sym_3 semidirect product $C_G(\langle x_1,x_2,x_3\rangle)$. Sym_3 permutes the set $\{x_i: i = 1,2,3\}$.
5. $\langle x_1,x_2,x_3\rangle$ is a nondegenerate subalgebra of $\mathbf{J}$.

Proof: Parts 1 and 2 are easy calculations using proposition 2.7. Parts 3 and 4 can be found in [Ja4] section 6 or alternatively in [Sp1]. Part 5 is an easy calculation using proposition 2.1.

Let $e_{ij}(c)$ denote the matrix of $\mathbf{J}$ whose ij entry is c, whose ji entry is $\overline{c}$, and all of whose other entries are zero. Then we can identify $\mathbf{O}$ with W_1 via $c \mapsto e_{23}(c)$ $\forall$ c $\in \mathbf{O}$. We will now describe briefly how the (noncommutative) octave multiplication of W_1 (see section 1 for definition of octave) can be recovered from the (commutative) Jordan multiplication of $\mathbf{J}$ (see also [Sp1] pg. 75). Let $v = 2e_{12}(\underline{1})$ and $w = e_{13}(\underline{1})$ and $e_{23}(c),e_{23}(d) \in W_1$. Define $y_+(e_{23}(c)) = 2e_{12}(\overline{c})$ and $y_-(e_{23}(c)) = e_{13}(c)$ and observe that $vy_-(e_{23}(c)) = wy_+(e_{23}(c)) = c$. Similarly define $y_+(e_{23}(d))$ = $2e_{12}(\overline{d})$ and $y_-(e_{23}(d)) = e_{13}(d)$. Now the octave multiplication $\wedge$ of W_1 can be gotten as follows: $e_{23}(c)\wedge e_{23}(d) := y_+(e_{23}(c))\ y_-(e_{23}(d)) = 2e_{12}(\overline{c})\ e_{13}(d) = e_{23}(cd)$.

Convention: When we speak of a *composition subalgebra of* W_1, we mean the image of a composition subalgebra of $\mathbf{O}$ under the above identification of $\mathbf{O}$ with W_1.

Observation 2.11: Under the identification of W_1 with $\mathbf{O}$ above we have $Q|_{W_1} = N$, where N is the quadratic form associated to $\mathbf{O}$ from section 1.

Proof: Let $e_{23}(c) \in W_1$. Then

$$Q(e_{23}(c)) = (1/2)\ \mathrm{tr}(e_{23}(c)^2) = (1/2)\ \mathrm{tr}\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & c\bar{c} & 0 \\ 0 & 0 & c\bar{c} \end{bmatrix}\right)$$

$= (1/2)\ (2N(c))$ (as $c\bar{c} = N(c)\underline{1}$ by part 5 of the Comp. Alg. Thm. of Section 1)

$= N(c)$.

Remark: We can identify W_3 with $\mathbf{O}$ in the same way as we identify W_1 with $\mathbf{O}$. Also we can identify W_2 with $\mathbf{O}$ via $e_{13}(\bar{c}) \mapsto c \ \forall\, c \in \mathbf{O}$.

SECTION 3: 3-FORMS, THE DICKSON 3-FORM AND SOME OF ITS PROPERTIES

All the notation and results of this section up to the definition of Φ before proposition 3.2 are due to Aschbacher and can be found in [As1] - [As5].

Throughout this section let V be a vectorspace over F.

Definition: A triple $\mathcal{T}(T,P,f)$ is a 3-form iff

(F1) f is a trilinear form on V.

(F2) P: V x V $\mapsto$ F is linear in the first variable and satisfies:

$$P(x,ay) = a^2\ P(x,y)$$

$$P(x,y+z) = P(x,y) + P(x,z) + f(x,y,z) \text{ for all } x,y,z \in V \text{ and } a \in F.$$

(F3) $T: V \mapsto F$ satisfies:

$$T(ax) = a^3 T(x)$$

$$T(x+y) = T(x) + T(y) + P(x,y) + P(y,x) \quad \text{for all } x,y \in V \text{ and } a \in F.$$

Remark: As $\mathrm{char}(F) \neq 2,3$, each of the three forms $\{f,P,T\}$ in the 3-form determine the remaining members of the 3-form. For example $T(x) = (1/6)\, f(x,x,x)$ and $P(x,y) = (1/2) f(x,y,y)$.

Definition: P_x is the quadratic form whose associated bilinear form is $f_x := f(x,-,-)$, where $P_x(y) = P(x,y)$.

Definition: By $x\Delta$ we denote the radical of f_x, and $U\Delta := \bigcap_{x \in U} x\Delta$.

Definition: $U\Theta := \{ v \in V : P_v(u) = 0 \text{ for all } u \in U \}$.

Note that $U\Delta$ and $U\Theta$ are linear subspaces of V. Moreover if an isometry of the 3-form fixes the subspace U it must also fix $U\Delta$ and $U\Theta$.

Definitions: If $U \subset U\Delta$ we call U ***singular***. If $U \subset U\Theta$ we call U ***brilliant***.

Note that singular implies brilliant as we assume $\mathrm{char}(F) \neq 2,3$.

Let $S = \{x_1,\ldots,x_r\}$ be a set of singular points.

Definition: S is called a ***special r-tuple*** if $x_i + x_j$ is brilliant nonsingular and $x_i \notin (x_j + x_k)\Theta$ for all pairwise distinct i,j,k. A subspace is ***special*** if it is generated by a special r-tuple.

Now we come to the definition of the Dickson 3-form.

Let V be a 27-dimensional vector space with basis $X = \{ x_i, x_i', x_{ij} : 1 \leq i,j \leq 6 , i<j \}$, subject to the convention $x_{ij} = -x_{ji}$. Let f be the symmetric trilinear form which has monomials $x_i x_j' x_{ij}$ $1 \leq i \neq j \leq 6$ and $x_{1d\,2d}\, x_{3d\,4d}\, x_{5d\,6d}$, where $d \in$ Coset. Coset is some set of coset representatives for Alt_P in Alt, where Alt is the alternating group on $\{1,\dots,6\}$ and P is the partition 12|34|56. The table below lists the monomials with sign +.

Table 3.1

12 34 56	14 23 56	16 25 34
12 35 64	14 25 63	16 23 45
12 36 45	14 26 35	16 24 53
13 24 65	15 26 43	
13 26 54	15 24 36	
13 25 46	15 23 64	

The *Dickson 3-form* is the 3-form (T,P,f) where f is the trilinear form above and P and T- are defined as follows:

$$P(x,y) := 1/2\ f(x,y,y)$$

$$T(x) := 1/6\ f(x,x,x)$$

In [As1] we find two proofs that the isometry group of the 3-form is the universal Chevalley group of type E_6. Moreover the set X is a set of weight vectors for some

Cartan-subgroup H of E_6.

Definition: We define H as the subgroup of E_6 generated by the isometries $h(t)$, $l(a)$, $a \in (F^*)^6$ such that the product of a's coordinates is 1, $t \in F$, where $h(t)$ maps x_6 to $t^2 x_6$, x_6' to $t^3 x_6'$, x_j to $t^{-1} x_j$, x_{j6} to $t^{-2} x_{j6}$ and fixing x_j' for $i,j \leq 5$, and $l(a)$ maps x_i to $a_i x_i$, x_i' to $a_i x_i'$, and x_{ij} to $(a_i a_j)^{-1} x_{ij}$ where a_i denotes the i-th coordinate of a.

Note that H is a Cartan subgroup of $E_6(F)$ and that X is a basis of weight vectors for X.

Definitions: $V_i := < x_j : j \leq i >$ for $1 \leq i \leq 6$ but $i \neq 5$

$V_{15} := < x_{ij} : i,j >$

$U_5 := < V_4, x_5 >$

$V_5 := < V_4, x_{56} >$

$V_{10} := < U_5, x_{6i} : 1 \leq i \leq 5 >$

$V_{12} := < V_6 , x_i' : 1 \leq i \leq 6 >$

$U_3 := < x_{16}, x_{25}, x_{34} >$

$U_6 := < x_{16}, x_{25}, x_{34}, x_{12} - x_{56}, x_2' + x_5, x_1 - x_6' >$

$V_9 := < x_{16}, x_{25}, x_{34}, x_{12}, x_{56}, x_2', x_5, x_1, x_6' >$

$V_{15} := < x_{ij} : 1 \leq i < j \leq 6 >$

Let $\mathcal{U}_i$, $\mathcal{V}_i$ denote the E_6-conjugacy class of U_i resp. V_i. Observe that V_i $1 \leq i \leq 6$ and U_5 are singular, that V_{10} and V_{12} are brilliant, and that $V_{15}\Theta = V_{12}$.

Definition: A ***brilliant subgroup*** of E_6 is a subgroup which is contained in the stabilizer of a member of $\mathcal{V}_i$ where $i \in \{ 1, 2, 3, 5, 6, 9, 10, 12 \}$.

Definition: A subspace U will be called ***totally dark*** if T has no nontrivial zeros in U.

Definition: A subspace U is a member of $\mathcal{U}_9$ if there exists a quadratic field extension K of F such that U^K is a member of $\mathcal{V}_9^K$ and $U \notin \mathcal{V}_9$.

Definition: A *3-decomposition* of V is a decomposition of V into the direct sum of three subspaces $A_1 \oplus A_2 \oplus A_3$ such that each summand is a member of $\mathcal{V}_9$ and $A_i \oplus A_j = A_k\Theta$ for all $\{i,j,k\} = \{1,2,3\}$.

Suppose that F^* contains an element of order 3. Aschbacher shows that there exists a unique class E of elementary abelian groups of order 81 in $\Delta = E_6(F)$: Diagonal which is not contained in a maximal torus of the algebraic group over $E_6(F)$. Under the action of a member of E the space V decomposes into a direct sum of 27 one dimensional Eigenspaces each of which is dark. We will call such a decomposition of V a *27-decomposition* of V.

Let $d = x_{16} + x_{25} + x_{34}$. Aschbacher shows that V is the P_d orthogonal direct sum of d and $d\Theta$ and that the centralizer of d in E_6 is F_4.

Let Φ: $J \mapsto V$ be defined as follows:

$\Phi(e_{11}(1)) = x_{16}$ $\Phi(e_{22}(1)) = x_{25}$ $\Phi(e_{33}(1)) = x_{34}$

$\Phi(e_{12}(\underline{1})) = x_{12} - x_{56}$ $\Phi(e_{13}(\underline{1})) = x_2' + x_5$ $\Phi(e_{23}(\underline{1})) = x_1 - x_6'$

$\Phi(e_{12}(i_1)) = x_{12} + x_{56}$ $\Phi(e_{13}(\bar{i}_1)) = x_2' - x_5$ $\Phi(e_{23}(i_1)) = x_1 + x_6'$

$\Phi(e_{12}(i_2)) = x_{15} + x_{26}$ $\Phi(e_{13}(\bar{i}_2)) = x_2 - x_5'$ $\Phi(e_{23}(i_2)) = x_1' + x_6$

$\Phi(e_{12}(i_3)) = x_{15} - x_{26}$ $\Phi(e_{13}(\bar{i}_3)) = x_2 + x_5'$ $\Phi(e_{23}(i_3)) = x_1' - x_6$

$\Phi(e_{12}(i_4)) = x_3 - x_4'$ $\Phi(e_{13}(\bar{i}_4)) = x_{14} + x_{36}$ $\Phi(e_{23}(i_4)) = x_{45} - x_{23}$

$\Phi(e_{12}(i_5)) = x_3+x'_4$ $\quad\Phi(e_{13}(\bar{i}_5)) = x_{14}- x_{36}$ $\quad\Phi(e_{23}(i_5)) = x_{45}+x_{23}$

$\Phi(e_{12}(i_6)) = x'_3+x_4$ $\quad\Phi(e_{13}(\bar{i}_6)) = x_{13}- x_{46}$ $\quad\Phi(e_{23}(i_6)) = x_{24}+x_{35}$

$\Phi(e_{12}(i_7)) = x'_3 -x_4$ $\quad\Phi(e_{13}(\bar{i}_7)) = x_{13}+x_{46}$ $\quad\Phi(e_{23}(i_7)) = x_{24} -x_{35}$

Proposition 3.2: Φ is an isometry of the trilinear forms (, ,) and f(, ,).

Proof: First we observe that there exists an isometry of the trilinear forms as both of them are E_6- forms and the E_6 forms of V are unique up to conjugacy in GL(V) .

Recall first the definition of W_i from proposition 2.10. We show first that $\Phi|_W$: $W \mapsto \Phi(W)$ is an isometry, where $W = < e_{ii}(1)\ ,\ W_1,\ e_{13}(\underline{1}) >$. Then we observe that $(W_i,\ N)$ (see observation 2.11) is isometric to ($\Phi(W_i)$, $-P_{x_{ij}}$). Finally we will show that Φ is uniquely determined by $\Phi|_W$, i.e., given $\Phi|_W$ there is exactly one map Ψ: $W^{\perp} \mapsto \Phi(W)^{\perp}$ such that $\Phi|_W + \Psi$ is an isometry of E_6-forms. The proof of this last claim is constructive, and in fact we constructed Φ using this construction. Hence we omit further details.

Since we are working over fields of characteristic $\neq 2,3$ it is enough to show that for every $x \in W$ we have $\mathrm{Det}(x) = T(\Phi(x))$ to prove that $\Phi|_W$ is an isometry of trilinear forms.

Let $x\in J$. Now we observe first that x can be written as $x = e_{11}(\alpha_1) + e_{22}(\alpha_2) + e_{33}(\alpha_3) + e_{12}(c_1) + e_{13}(\bar{c}_2) + e_{23}(c_3)$ where $\alpha_i \in F$ and $c_i\in O$.

So then $\mathrm{Det}(x) = \alpha_1\alpha_2\alpha_3 - \Sigma\, \alpha_i\, N(c_{4-i}) + \pi(\, c_1c_2c_3 + \bar{c}_1\bar{c}_2\bar{c}_3)$.

On the other hand

$$T(\Phi(x)) = \alpha_1\alpha_2\alpha_3 + \sum_{i=1}^{3}\alpha_i P(x_{ij}\ ,\ \Phi(e_{rs}(c_{4-i})) + f(\Phi(e_{12}(c_1)),\Phi(e_{13}(\bar{c_2})),\Phi(e_{23}(c_3)))$$

where $ij \in \mathrm{Par}(d)$ and $\{i,r,s\} = \{1,2,3\}$.

A straightforward calculation using the orthogonal basis of table 1.1 and using table 3.1 shows that $N(w) = -\ P_{x_{ij}}(\Phi(w))$ for all $w \in W_i$ subject to our identification of W_i with O. And so $(W_i,\ N)$ is isometric to ($\Phi(W_i)$, $-P_{x_{ij}}$).

If $x \in W$ then in particular $c_1 = 0$, and hence $\Phi(e_{12}(c_1)) = 0$.

Thus $\pi(c_1c_2c_3 + \bar{c}_1\bar{c}_2\bar{c}_3) = 0 = f(\Phi(e_{12}(c_1)), \Phi(e_{13}(\bar{c_2})), \Phi(e_{23}(c_3)))$. So clearly $Det(x) = T(\Phi(w))$ and $\Phi|_W$ is an isometry.

Now it remains to show that Φ is uniquely determined (in the sense above) by its restriction to W.

To do this we will first prove the following:

$e_{12}(i_j)$ is the unique element of V with the property that

1. $< e_{12}(i_j)> = A^{\perp_N} \cap W_3$.
2. $(e_{12}(i_j), e_{13}(\underline{1}), e_{23}(i_j)) = 2\ N(i_j)$.

Where $A = \{ w \in W_3 : (e_{23}(i_j), e_{13}(\underline{1}), w) = 0 \}$ and $\perp_N$ means the perp with respect to N.

If we show that $e_{12}(i_j)$ is unique with respect to (1) and (2) and if ψ is an isometry of E_6- forms, then $\psi(e_{12}(i_j))$ must be the unique element with respect to 1. and 2. as ψ preserves properties 1. and 2. The following holds:

$$(e_{23}(i_j), e_{13}(\underline{1}), e_{12}(w)) = \pi(i_j w + \bar{w}\,\bar{i}_j) \qquad (*)$$

Now observe using table 1.1 that $(*) = 0$ iff $w \in i_j^{\perp}$ and that $(*) = 2\ N(i_j)$ when $w = i_j$.

So (1) and (2) hold for $e_{12}(i_j)$. Uniqueness follows from the linearity of the map $x \mapsto (e_{12}(i_j), e_{13}(i_j), x)$.

Now let Ψ be an isometry of E_6 forms such that $\Psi|_W = \Phi|_W$. Assume for now that Ψ exists. Then $\Psi(e_{12}(i_j))$ is the unique element of V with the property that

1. $<\Psi(e_{12}(i_j))> = A^{\perp_{34}} \cap \Phi(W_3)$.
2. $f(\Psi(e_{12}(i_j)), \Psi(e_{13}(\underline{1})), \Psi(e_{23}(i_j))) = 2\ N(i_j)$.

Where $A = \{ w \in \Phi(W_3) : f(\Psi(e_{23}(i_j)), \Psi(e_{13}(\underline{1})), w) = 0 \}$ and $\perp_{34}$ means the perp with respect to $P_{x_{34}}$.

Similarly $\Psi(e_{13}(i_j))$ is the unique element of V with the property that

1. $<\Psi(e_{13}(i_j))> \ = A^{\perp_{25}} \cap \Phi(W_2)$.

2. $f(\Psi(e_{13}(i_j)), \Psi(e_{12}(\underline{1})), \Psi(e_{23}(i_j))) = 2\ N(i_j)$.

Where $A = \{\ w \in \Phi(W_2) : f(\ \Psi(e_{23}(i_j)), \Psi(e_{12}(\underline{1})), w) = 0\ \}$ and $\perp_{25}$ means the perp with respect to $P_{x_{25}}$.

So we can compute $\Psi(e_{13}(i_j))$ and $\Psi(e_{12}(i_j))$ for all $j \in \{\ 0, \dots ,7\}$.

Now a straightforward but lengthy computation will show that $\Psi = \Phi$, and the claim follows once we show the existence of Ψ.

Now we turn to the existence of Ψ. We note that W is generated by the special triple $\{e_{ii}(1)\colon 1 \leq i \leq 3\}$, a vector $e_{13}(\underline{1}) \in W_2$ such that $P_{e_{22}(1)}(e_{13}(\underline{1})) = -1$ and a basis $\mathfrak{B}_1 = \{e_{23}(\underline{1}), e_{23}(i_1), e_{23}((1/2)(i_{2j} \pm i_{2j+1})) : 1 \leq j \leq 3\}$ of W_1. Moreover it is shown in [Ja4] theorem 9 that $C_{E_6}(\{e_{ii}(1)\}, e_{22}(\underline{1}), e_{23}(\underline{1}))^{\infty} \simeq G_2(F)$. Also it can be shown using the identification of W_1 with $\mathbf{O}$ that $\{e_{23}(i_1), e_{23}((1/2)(i_{2j} \pm i_{2j+1})) : 1 \leq j \leq 3\}$ is a basis of standard vectors (in the sense [As7]) for the 7-dimensional $G_2(F)$ module $<\{e_{23}(i_j) : 1 \leq j \leq 7\}>$ with respect to the $\mathrm{Aut}(\mathbf{O})$ ($\simeq G_2(F)$) invariant trilinear form of proposition 1.4. We indicate here a correspondence. First let us recall the setup of [As7]. Let $Y = \{y_0, y_i, y_i' : 1 \leq i \leq 3\}$ be a basis for a 7-dimensional vectorspace over F. We define an alternating trilinear form h by $y_0y_1y_1' + y_0y_2y_2' + y_0y_3y_3' + y_1y_2y_3 + y_1'y_2'y_3'$, and a bilinear form B by $-2y_0^2 + y_1y_1' + y_2y_2' + y_3y_3'$. The stabilizer of the pair of forms (h,B) in $GL(<Y>)$ is $G_2(F)$. The set Y is called a ***standard set***. Now we give a correspondence of Y to a basis of $\underline{1}^{\perp} < \mathbf{O}$. Let $y_0 \mapsto i_1$, $y_1 \mapsto (1/2)(i_2 + i_3)$, $y_1' \mapsto (1/2)(i_2 - i_3)$, $y_2 \mapsto (1/2)(i_4 + i_5)$, $y_2' \mapsto (1/2)(i_4 - i_5)$, $y_3 \mapsto (1/2)(i_6 - i_7)$, $y_3' \mapsto (1/2)(i_6 + i_7)$.

Now $\Phi(W)$ is generated by the special triple $\{x_{16}, x_{25}, x_{34}\}$, by a basis $\Phi(\mathfrak{B}_1)$ of $x_{25}\Delta \cap x_{34}\Delta$, and a vector $x_2' + x_5 \in x_{16}\Delta \cap x_{34}\Delta$ such that $P_{x_{25}}(x_2' + x_5) = -1$. Moreover we observe that Φ maps $\{e_{ii}(1)\colon 1 \leq i \leq 3\}$ into $\{x_{16}, x_{25}, x_{34}\}$, $\Phi(e_{13}(\underline{1})) = x_2' + x_5$, and $\Phi(e_{23}(\underline{1})) = x_1 - x_6'$. Furthermore by [As3] 5.5 we have

$G_2(F) \simeq C_{E_6}(x_{16}, x_{25}, x_{34}, x'_2+x_5, x_1-x'_6)$. Finally a long but straightforward calculation shows that $\Phi(\mathcal{B}_1(S) \setminus e_{23}(\underline{1}))$ is a basis of standard vectors for the $C_{E_6}(x_{16}, x_{25}, x_{34}, x'_2+x_5, x_1-x'_6)$ - module $<\Phi(\mathcal{B}_1(S) \setminus e_{23}(\underline{1}))>$.

If we can show that E_6 is transitive on triples $\{S, \mathcal{B}_1(S), u(S)\}$, where S is a special triple $\{s_1, s_2, s_3\}$, $\mathcal{B}_1(S)$ is a basis of $s_2\Delta \cap s_3\Delta$, $u(S) \in s_1\Delta \cap s_3\Delta$, and $\mathcal{B}_1(S)$ contains a vector b_1 such that $P_{s_1}(b_1) = -1$, $C_{E_6}(S, b_1, u(S)) \simeq G_2(F)$, and $\mathcal{B}_1(S) \setminus b_1$ is a basis of standard vectors for the $C_{E_6}(S, b_1, u(S))$- module $<\mathcal{B}_1(S) \setminus b_1>$, then we will have the existence of a Ψ such that $\Psi|_W = \Phi|_W$.

Now we show that E_6 is transitive on the set of triples $\{S, \mathcal{B}_1(S), u(S)\}$. Aschbacher showed in [As1] 6.9 that E_6 is transitive on special planes. The content of [As1] 3.15.3 and 4 is that $N_{E_6}(S)^\infty \simeq Spin_8^+(F)$, (W_i, P_{s_i}) is a hyperbolic orthogonal space, $N_{E_6}(S)^\infty$ acts as $\Omega(W_i)$ on (W_i, P_{s_i}), and $N_{E_6}(S)$ induces a triality outer automorphism on $N_{E_6}(S)^\infty$. Now it is well known that $N_{E_6}(S)$ is transitive on the points of W_2 of P_{s_2}- value -1. Also well known is the fact that $M := N_{E_6}(S, u(S))^\infty \simeq Spin_7(F)$ and that M acts irreducibly in a spin representation on W_1. It is known that M is transitive on the set of vectors $w \in W_1$ such that $P_{s_1}(w) = -1$ and that $N_M(w) \simeq G_2(F)$. To complete the proof we observe that the standard bases of the 7-dimensional $G_2(F)$ module correspond to the apartments in the building of $G_2(F)$, and that $G_2(F)$ is transitive on the set of apartments in its building.

We define the multiplication on V in such a way that Φ is an isomorphism of algebras. Define $\bar{Q}(x) := Q(\Phi^{-1}(x))$ for all $x \in V$ and $\bar{P}(ad+w) = a^2$ for all $w \in d\Theta$ and $a \in F$.

Lemma 3.3:

1. $\bar{Q}(x) = -P_d(x) + (9/2)\bar{P}(x)$.

2. $U \subset d\Theta$ is a singular subspace with respect to $\overline{Q}$ iff U is a singular subspace with respect to P_d.

3. $\Phi(id) = d$ and $\Phi(id^{\perp}) = d\Theta$.

Proof: We know that G stabilizes a two dimensional space of quadratic forms of J as J is the direct sum of two absolutely irreducible G modules, both of which are nontrivial under Q. (see prop. 2.2.2)

The space must therefore be spanned by P_d and $\overline{P}$; i.e.,

$$(*)\ \overline{Q}(x) = \alpha\, P_d(x) + \beta\overline{P}(x) \text{ for all } x \in V .$$

We solve for α and β by substituting d and x_{16} into the equation $(*)$. Note that $x_{16} = (1/3)\, d + w$ where $w \in d\Theta$. Then using proposition 2.3 we get:

$$(3/2) = \overline{Q}(d) = \alpha\, P_d(d) + \beta\overline{P}(d) = 3\alpha + \beta \qquad\qquad \text{and}$$

$$(1/2) = \overline{Q}(x_{16}) = \alpha P_d(x_{16}) + \beta\overline{P}(\ (1/3)d + w) = \ (\beta/\ 9).$$

Solving this yields $\alpha = -1$ and $\beta = (9/2)$. This proves part 1.

Now part 2 is an easy consequence of part 1. For part 3 we note first that $\dim(id^{\perp}) = \dim(d\Theta) = 26$. Let $(\ ,\ )$, $f(d,\ ,\ )$ and $\overline{(\ ,\ )}$ denote the bilinear forms associated to Q, P_d and $\overline{P}$ respectively. Now let $x \in d\Theta$. From part 1 we get:

$(id,\Phi^{-1}(x)) = -f(d,d,x) + (9/2)\,\overline{(d,x)} = 0 + 0$, as $\overline{(d,x)} = \overline{P}(d+x) - \overline{P}(d) - \overline{P}(x)$.

This gives part 3.

In view of Proposition 3.2 and Lemma 3.3 we will identify V and J via Φ, Det with T, Q with $\overline{Q}$, id with d, and $id^{\perp}$ with $d\Theta$.

Conventions: We will call $x \in J$ or a subspace U of J *singular (brilliant or dark)* if x resp. U is singular (brilliant, dark) with respect to the 3-form T. We call x resp. U Q-*(non)singular* resp. P_d- *(non)singular* if x resp. U is (non)singular with respect to the quadratic form Q resp. P_d.

Lemma 3.4:

1. Let $x \in \mathbf{J}$. Then x is singular iff $x\#x = 0$.
2. $U \subset \mathbf{J}$ is singular iff $x\#y = 0$ for all $x,y \in U$.

Proof: Let $z \in \mathbf{J}$ and $x,y \in U$. Then $(x,y,z) = 2\,(x\#y,z)$ by definition of #. If $x\#y = 0$ then $(x,y,z) = 0$ for all $z \in \mathbf{J}$ and hence by definition U is singular. If U is singular then $(x,y,z) = 0$ for all $z \in \mathbf{J}$ and $x,y \in U$. Therefore $(x\#y,z) = 0$ for all $z \in \mathbf{J}$ and hence $x\#y = 0$ as Q is a nondegenerate quadratic form and $\mathrm{char}(F) \neq 2$.

Corollary 3.5: Primitive idempotents are singular.

Proof: By proposition 2.7 G is transitive on primitive idempotents. So every primitive idempotent is conjugate to x listed in Proposition 2.7. It is routine to check that $x\#x = 0$. Now Lemma 3.4 applies.

Lemma 3.6: Let S be a brilliant subspace of $\mathbf{J}$ such that $S\Theta$ is a hyperplane, and $s,v \in \mathbf{J} \setminus S\Theta$.

1. Then $P_s|_S$ is similar to $P_v|_S$.
2. A subspace W of S is singular iff W is singular with respect to P_s.

Proof: This is the content of [As1] sec. 2.13.

Definition and Lemma 3.7: If S is a nonsingular subspace satisfying the conditions of lemma 3.6, then S is contained in a unique member of Υ_{10}, denoted by $\Psi_{10}(S)$. S is called a ***hyperbolic subspace.***

Proof: This is the content of [As1] sec. 7.2.

SECTION 4: GENERALITIES ON SUBALGEBRAS OF J

Lemma 4.1: G has two orbits on singular points of **J**. These are the scalar multiples of the primitive idempotents and the vectors satisfying $x^2 = 0$ and $x \neq 0$. (Call the latter *Class two nilpotents.*)

Proof: Note first that by Lemma 3.4, x is singular iff $x \# x = 0$.

If $x \notin d\Theta$ then by lemma 2.8 x is singular iff x is a multiple of a primitive idempotent, and by 2.7.1 G is transitive on primitive idempotents. If $x \in d\Theta$ then by lemma 2.8 x is singular iff $x^2 = 0$. Moreover Aschbacher showed in [As1] 8.6 that G is transitive on singular points of $d\Theta$.

Lemma 4.2: The G orbits on brilliant nonsingular vectors of $d\Theta$ are parameterized by the Q-values. Representatives for the orbits are $x_1 + x_{12}$, $x_1 + \alpha\, x'_6$ where $\alpha \neq 0$.

Proof: If every brilliant $x \in d\Theta$ is of the form s+t where s, t are class two nilpotents, then the claim follows from [As1] 9.2. Now we claim that every x is of the right form. Define $A := \Psi_{10}(x) \cap d\Theta$, where $\Psi_{10}(v)$ is the unique member of Υ_{10} containing v. Now let $s \in \mathbf{J} \setminus x\Theta$ be singular. Then P_s induces a nondegenerate quadratic form on $\Psi_{10}(x)$. Now by lemma 3.6 $a \in A$ is singular iff a is singular with respect to P_s. The claim follows as each nonsingular point of A is contained in a P_s - hyperbolic line contained in A.

Lemma 4.3: ${x_1}^2 = 0$, $(x_1 + x_{12})^2 = x_5$, $(x_1 + \alpha\, x'_6) \# (x_1 + \alpha\, x'_6) = \alpha x_{16}$

Proof: A straightforward computation.

Lemma 4.4: The # square of a brilliant vector is singular.

Proof: By [Sp1] pg. 77 $(x\#x)\#(x\#x) = \mathrm{Det}(x)\,\mathrm{id}$. When x is brilliant then Det(x)

$= 0$, by definition and lemma 3.3.4. So then either $x\#x = 0$, or $x\#x$ is singular by lemma 3.4. The claim follows.

Definition: Let U be a subspace of J. Then $S^2(U) := < \{ xy : x,y \in U \} >$ and $S^{\#2}(U) := < \{ x\#y : x,y \in U \} >$.

Lemma 4.5: Let U be a subspace of J, then $<U,id>$ is a subalgebra iff $S^2(U) \subset <U,id>$ iff $S^{\#2}(U) \subset <U,id>$.

Proof: The first equivalence is obvious. The second one follows from lemma 2.6.

Lemma 4.6: Let U be a subspace of J, then $S^{\#2}(U) = U\Theta^{\perp}$.

Proof: Let $v \in U\Theta$. By definition of #, $0 = (s,t,v) = 2(s\#t,v)$ for all $s,t \in U$. So $U\Theta \subset S^{\#2}(U)^{\perp}$.

Now let $v \in S^{\#2}(U)^{\perp}$. Then $0 = (s\#t,v) = (1/2)(s,t,v)$ for all $s,t \in U$, so $v \in U\Theta$. Thus $S\Theta = S^{\#2}(U)^{\perp}$, so the lemma holds.

Lemma 4.7: If U is a Q-singular subspace of $d\Theta$, then $xy = x\#y$ for all $x,y \in U$. In particular $S^2(U) = S^{\#2}(U)$.

Proof: Let $x,y \in U$. Then by lemma 2.6

$xy = x\#y + (1/2)(x,id)\, y + (1/2)\, (y\, , id)x + (1/2)(x,y)id - (1/2)(x,id)(y,id)\, id.$

Now $(x,y) = 0$ by assumption, and $(x,id) = (y,id) = 0$ in view of lemma 3.3.3. Thus $xy = x\#y$ for all $x,y \in U$. Now the claim follows easily.

Lemma 4.8: Let U be a subspace of J. If for every $s \in U$ we have $s^2 = 0$ (i.e., $S^2(U) = 0$) then U is singular.

Proof: From lemma 2.9 and lemma 3.4 we know that every $s \in U$ is singular and that U

$\subset id^{\perp}$. Now let $s,v \in S$. To complete the proof we need to show that $v \in s\Delta$.

So let $z \in J$ be any vector. From the singularity of $s,v,s+v$ it follows that

$$0 = (s+v,\ s+v\ ,\ z\ ) = (s,s,z) + 2(s,v,z) + (v,v,z)$$

$$= \ 0 \quad + \ 2(s,v,z) + 0.$$

So it follows that $(s,v,z) = 0$ for all $z \in J$. Thus $v \in s\Delta$.

Definition: Let $\mathbb{A}$ be a subalgebra of J. We will denote the Q-radical of $\mathbb{A}$ by $R(\mathbb{A})$. Note that $R(\mathbb{A}) \subset id^{\perp}$ as $id \in \mathbb{A}$.

Lemma 4.9: Let $\mathbb{A}$ be a subalgebra of J, then

1. $R(\mathbb{A})$ is brilliant.
2. Either $R(\mathbb{A})$ or $S^2(R(\mathbb{A}))$ is generated by singular points.
3. Both $R(\mathbb{A})$ and $S^2(R(\mathbb{A}))$ are invariant subspaces of $N_G(\mathbb{A})$.
4. If $R(\mathbb{A}) \neq 0$ then $N(\mathbb{A})$ is contained in a brilliant subgroup of E_6.
5. $S^2(R(\mathbb{A})) \subset R(\mathbb{A})$.

Proof: Let $r \in R(\mathbb{A})$. Then by proposition 2.3.4, $0 = (ab,r) = (a,br)$ for all $a,b \in \mathbb{A}$. Thus $br \in R(\mathbb{A})$ for all $b \in \mathbb{A}$. Thus $S^2(R(\mathbb{A})) \subset R(\mathbb{A})$, proving 5.

Since $R(\mathbb{A})$ is Q-singular $S^{\#2}(\ R(\mathbb{A})) = S^2(R(\mathbb{A}))$ by lemma 4.7.

So $R(\mathbb{A}) \subset R(\mathbb{A})^{\perp} \subset S^2(R(\mathbb{A}))^{\perp} \subset R(\mathbb{A})\Theta$ by lemma 4.6. This shows part 1.

For part 2 note that every vector $x \in R(\mathbb{A})$ is brilliant and Q-singular. So by lemmas 4.4 and 4.7, $S^2(R(\mathbb{A}))$ is either generated by singular points or trivial. In the latter case $R(\mathbb{A})$ is singular by lemma 4.8 , so 2 holds. Part 3 is self evident and part 4 follows from [As2] Theorem 2 which states that the normalizer in E_6 of a brilliant subspace which is generated by singular points is brilliant.

Because of lemma 4.9.4 we will now call subalgebras with nontrivial Q-radical *brilliant*.

Also the following is self evident:

Remark: If $H < G$ and H stabilizes a subalgebra $\mathbf{A}$. Then either:

1. $\mathbf{A}$ is nondegenerate with respect to Q, or

2. H is a brilliant subgroup of G.

A subalgebra $\mathbf{A}$ will be called *nondegenerate* if $R(\mathbf{A}) = 0$.

SECTION 5: NONDEGENERATE SUBALGEBRAS

Springer's Lemma: Let $\mathbf{A}$ be a nondegenerate subalgebra of $\mathbf{J}$ then:

1. $\mathbf{A}$ contains a primitive idempotent iff there exists $0 \neq x \in \mathbf{A}$ such that $\mathrm{Det}(x) = 0$. In particular if $\dim(\mathbf{A}) > 3$ then $\mathbf{A}$ contains a primitive idempotent.

2. If $\dim(\mathbf{A}) \geq 3$ and $\mathbf{A}$ contains a primitive idempotent x then $E_{0,\mathbf{A}}(x) \neq 0$. (See prop. 2.7.6 for the definition of $E_{0,\mathbf{A}}(x)$.)

3. If $\dim(\mathbf{A}) > 3$ and $E_{1,\mathbf{A}}(x) \neq 0$ then $\mathbf{A}$ is isomorphic to

$$\mathbf{J}_{\mathbf{D}} := \left\{ \begin{bmatrix} \gamma_1 & d_1 & \bar{d}_2 \\ \bar{d}_1 & \gamma_2 & d_3 \\ d_2 & \bar{d}_3 & \gamma_3 \end{bmatrix} : d_i \in \mathbf{D} \text{ , where } \mathbf{D} \text{ is a nondegenerate subalgebra of } \mathbf{O} \right\}.$$

4. $\dim(\mathbf{J}_{\mathbf{D}}) = 3 + 3\dim(\mathbf{D})$

Proof: See [Sp1] pg. 76.

Lemma 5.1: Let $\mathbf{A}$ be a two dimensional nondegenerate subalgebra of $\mathbf{J}$, then $\mathbf{A} = \langle id,x \rangle$ where x is a primitive idempotent. G is transitive on two dimensional nondegenerate subalgebras of $\mathbf{J}$.

Proof: Assume that $\mathbf{A}$ does not contain a primitive idempotent. Then in view of 1 of

Springer's Lemma, $\mathbb{A}$ must be totally dark. By [As3] sec 7 there exists a cubic field extension K of F such that $\mathbb{A}^K$ contains exactly three brilliant points, none of which are singular. Moreover it is clear that $\mathbb{A}^K$ is a nondegenerate subalgebra of $\mathbb{J}^K$. Therefore, by 1 of Springer's Lemma, $\mathbb{A}^K$ must contain singular points (recall from corollary 3.5 that primitive idempotents are singular): a *contradiction*. Thus $\mathbb{A}$ contains a primitive idempotent and proposition 2.7.1 completes the proof.

Lemma 5.2: If F is finite, G is transitive on nondegenerate three dimensional subalgebras of $\mathbb{J}$ containing no primitive idempotents. If $\mathbb{A}$ is such a subalgebra then $C_G(\mathbb{A}) = {}^3D_4(F)$ and $N_G(\mathbb{A})/\ C_G(\mathbb{A}) = \mathbb{Z}/3\mathbb{Z}$. Moreover $C_G(\mathbb{A})$ acts irreducibly on $\mathbb{A}^{\perp}$. When F is algebraically closed, $\mathbb{J}$ does not contain three dimensional nondegenerate subalgebras containing no primitive idempotents.

Proof: Let $\mathbb{A}$ satisfy the hypothesis ; then by 1 of Springer's lemma, $\mathbb{A}$ is a totally dark plane containing id. In [As3] sec 7 it is shown that E_6 is transitive on totally dark planes, and that the normalizer in E_6 of a totally dark plane is transitive on the points contained in that plane. This shows that G is transitive on totally dark planes containing id. [As3] sec 7 shows that the centralizer of a totally dark plane is ${}^3D_4(F)$ and that the normalizer in E_6 modulo the centralizer is $\mathbb{Z}_{q^2+q+1}$ extended by $\mathbb{Z}_3$. The stabilizer of a point in a totally dark plane must therefore be $\mathbb{Z}_3$. This is the first claim. Now the minimal dimensional faithful $FC_G(\mathbb{A})$ module is 24 dimensional, proving the second claim.

Now Det( ) is a cubic polynomial and hence has a nontrivial zero on any subspace of dimension ≥ 2 if F is algebraically closed. So the third claim follows from part 1 of Springer's lemma.

Remark: $\mathbb{A}$ is a totally dark subalgebra iff $\mathbb{A}$ is a cubic field extension of F. To see

this observe that the only nontrivial totally dark subalgebras are the ones in lemma 5.2. So any such algebra must be cyclic, i.e., generated by a single element. Now every cyclic subalgebra is associative as a consequence of proposition 2.3.5. Let $\mathbb{A}$ be as in lemma 5.2 and $\lambda \mathrm{id} \neq x \in \mathbb{A}$. Then by Hamilton's equation:
$x^3 = (x,\mathrm{id})\, x^2 + (\, Q(x) - (x,\mathrm{id})^2)\, x + \mathrm{Det}(x)\, \mathrm{id}$. Now as $\mathrm{Det}(x) \neq 0$ we have:
$x(\, x^2 - (x,\mathrm{id})\, x - (Q(x) - (x,\mathrm{id})^2)\mathrm{id}) \,/\, \mathrm{Det}(x) = \mathrm{id}$. So x is invertible and hence $\mathbb{A}$ is a field.

For the converse let $\mathbb{B}$ be a subalgebra which is a cubic extension field of F. We need to show that $\mathrm{Det}(x) \neq 0$ for every $0 \neq x \in \mathbb{B}$. Suppose $x \in \mathbb{B}$ and $\mathrm{Det}(x) = 0$. Then using Hamilton's equation we get $x(x^2-(x,\mathrm{id})x - (Q(x) - (x,\mathrm{id})^2)\mathrm{id}) = \mathrm{Det}(x)\, \mathrm{id} = 0$. So then $x^2 - (x,\mathrm{id})x - (Q(x) - (x,\mathrm{id})^2)\, \mathrm{id} = 0$ as $\mathbb{B}$ is a field. Hence x generates a proper subfield of $\mathbb{B}$. But $\mathbb{B}$ contains only $\langle \mathrm{id} \rangle$ as a proper subfield, as $[\mathbb{B} : F] = 3$. So $x \in \langle \mathrm{id} \rangle$ and hence $x = 0$. This shows the converse.

Lemma 5.3: Let x be a primitive idempotent and $y,z \in E_0(x)$. Then $yz \in \langle \mathrm{id},x \rangle$ and $y^2 = Q(y)\, (\mathrm{id} - x)$.

Proof: See [Sp1] pg. 69.

Lemma 5.4: If F is finite (algebraically closed) G has two (one) orbits of nondegenerate three dimensional subalgebras containing primitive idempotents, respectively. Let x be a primitive idempotent and $y_i \in E_0(x)$ be a vector such that $Q(y_i) = i$, $i \in \{1,\mathrm{non}\}$, where non is a nonsquare of F. Representatives for the orbits can be chosen as follows:

1. $\mathbb{A}_1 = \langle \mathrm{id}\, ,\, x\, ,\, y_1 \rangle$, $C_G(\mathbb{A}_1) = \mathrm{Spin}_8^+(F)$, $N_G(\mathbb{A}_1)/C_G(\mathbb{A}_1) = \mathrm{Sym}(3)$.
2. $\mathbb{A}_{\mathrm{non}} = \langle \mathrm{id}\, ,\, x\, ,\, y_{\mathrm{non}} \rangle$, $C_G(\mathbb{A}_{\mathrm{non}}) = \mathrm{Spin}_8^-(F)$, $N_G(\mathbb{A}_{\mathrm{non}})/C_G(\mathbb{A}_{\mathrm{non}}) = \mathbb{Z}_2$.

Moreover $\mathbb{A}_1$ is a special plane containing id, and x is the unique primitive idempotent contained in A_{non}. Hence the normalizer of A_{non} is not maximal in G. A_{non} does not exist when F is algebraically closed.

Proof: We note first that in view of lemmas 5.1 and 5.3 each A_i is indeed a subalgebra. Moreover an easy calculation shows that each A_i is nondegenerate. Because of Springer's Lemma we know that a three dimensional nondegenerate subalgebra A containing a primitive idempotent x decomposes as $\langle id\rangle \oplus \langle x\rangle \oplus E_{0,A}(x)$. Moreover from proposition 2.7.5 we know that $N_G(x)$ acts as $\Omega_9^+(F)$ on $E_0(x)$. Thus, depending on F, $\Omega_9^+(F)$ has two (resp one) orbits on the points of $E_0(F)$. The y_i listed in the statement of the lemma can be chosen as orbit representatives. This shows that there are at most two (one) conjugacy classes of nondegenerate three dimensional subalgebras containing primitive idempotents and that $\{A_i\}$ contains a set of orbit representatives. We now investigate the centralizers of the A_i's. Now clearly $C_G(A_i) = C_{C(x)}(y_i)$, where $C(x) = C_G(x)$, and therefore the centralizer structure follows from well known results about orthogonal groups. This also shows that the A_i's are not conjugate in G as their centralizers in G are nonisomorphic.
Next we want to survey the primitive idempotents of A_i. Let us first note that by lemma 5.3 $y_i^2 = Q(y_i)$ (id - x). So now let $v = \gamma id + \beta x + \alpha\ y_i \in A_i$ be an idempotent. We assume $\alpha,\beta,\gamma \neq 0$ to avoid trivial solutions. Then solving $v^2 = v$ by comparing coefficients yields the equations $2\gamma\alpha = \alpha$, $\beta^2 + 2\gamma\beta - \alpha^2 Q(y_i) = \beta$, $\gamma^2 + \alpha^2 Q(y_i) = \gamma$. Solving them leads to $\gamma = 1/2$, $Q(y_i) = (1/4)\ \alpha^{-2}$, $\beta^2 = \alpha^2 Q(y_i)$. We now observe that the only time we have a nontrivial solution is when $Q(y_i)$ is a nonzero square in F. Thus we conclude that when i=non, x is the unique primitive idempotent in A_i. Thus the normalizers of A_{non} lies in the normalizer of x. The nontrivial idempotents in A_1 are $\{(1/2)id \pm (1/2)x \pm (1/2)y_1\}$. An idempotent is primitive if its Q-value is 1/2. Now $Q(id \pm x \pm y_1) = Q(id \pm x) + Q(y_1) = 4$ resp 2.

So the primitive idempotents in $\mathbf{A}_1$ are $\{x,(1/2)(id - x \pm y_1)\}$. A routine calculation shows that the idempotents are pairwise orthogonal. Now proposition 2.10.4 yields that the normalizer of $\mathbf{A}_1$ is as claimed. The fact that $\mathbf{A}_1$ is a special plane follows from [As1] sec 2. and corollary 3.5.

Lemma 5.5: Let $\mathbf{A}$ be a nondegenerate subalgebra of dimension ≥ 4. Suppose that $E_{1,\mathbf{A}}(x) = 0$, where $x \in \mathbf{A}$ is a primitive idempotent. Then:

1. $E_{0,\mathbf{A}}(x)$ is nondegenerate with respect to Q.
2. $\langle id\text{-}x\rangle \leq S^2(\mathbf{A}\cap id\Theta) \leq \langle id,x\rangle$.
3. $\langle id,x\rangle$ is an $N_G(\mathbf{A})$ - invariant subalgebra.
4. $N_G(\mathbf{A}) = N_{C_G(x)}(E_{0,\mathbf{A}}(x))$.

Proof: Part 1 is clear as $\mathbf{J} = \langle x,id\rangle \perp E_0(x) \perp E_1(x)$ by proposition 2.7.3. Part 2 is a consequence of lemma 5.3, and the fact that $\mathbf{A}$ is nondegenerate. Parts 3 and 4 follow from part 2 as $\mathbf{A}\cap id\Theta$ and id are $N_G(L)$ invariant.

Recall the setup of proposition 2.10 and that $W_i = E_1(x_{i+1}) \cap E_1(x_{i+2}), i \in \{1,2,3\}$. Also recall the definition of $e_{ij}(c)$ and the identification of W_3 with $\mathbf{O}$ after proposition 2.10.

Lemma 5.6: Let $\mathbf{A}$ be a subalgebra of $\mathbf{J}$ which is isomorphic to $\mathbf{J_D}$ for some $\mathbf{D}$. Then $\mathbf{A}$ is generated as an algebra by a set $\{x_1,x_2,x_3\}$ of three pairwise orthogonal idempotents which generate a three dimensional subalgebra, together with a vector $w \in W_3$ such that $Q(w) = 1$, a nonsingular vector $v \in W_2$ such that $Q(v) = 1$, and a composition subalgebra $\bar{\mathbf{D}} \subset W_1$ whose identity element is $2vw$.

Proof: Let τ: $\mathbf{J_D} \mapsto \mathbf{A}$ be an algebra isomorphism. Let $x_1 = \tau\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right)$,

$$x_2 = \tau(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}) \ , \ x_3 = \tau(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}) \ , \ v = \tau(\begin{bmatrix} 0 & \underline{1} & 0 \\ \underline{1} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}),$$

$$w = \tau(\begin{bmatrix} 0 & 0 & \underline{1} \\ 0 & 0 & 0 \\ \underline{1} & 0 & 0 \end{bmatrix}) \text{ and } \bar{\mathbf{D}} = \{ \tau(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & * \\ 0 & * & 0 \end{bmatrix}) : * \in \mathbf{D} \}.$$ Now the claim follows as any isomorphism of Jordan algebras is an isometry and preserves multiplication (see [Sp1] pg. 76 and Satz 17).

Lemma 5.7: Let v,w and $\bar{\mathbf{D}}$ be as in the last lemma. Then v and w determine a unique multiplication that makes $(W_1, -P_{x_1})$ into an octave with identity element 2vw.. $\bar{\mathbf{D}}$ is a composition subalgebra of $\mathbf{O}$.

Proof: It is shown in [Sp1] pg. 75 (and briefly outlined after proposition 2.10) that a $C_G(x_1,x_2,x_3,v,w)$ - invariant multiplication on W_3 can be defined to make W_3 into an octave with identity element 2vw. Aschbacher has shown in [As3] sec. 5 that $G_2(F) = C_G(x_1,x_2,x_3,v,w)$. Uniqueness follows from proposition 1.4.

Lemma 5.8: If $\mathbf{A}$ is a nondegenerate subalgebra of $\mathbf{J}$ of dimension greater than three, $x \in \mathbf{A}$ is a primitive idempotent and $E_{1,\mathbf{A}}(x) \neq 0$ then $\mathbf{A}$ is conjugate to $\mathbf{J_D}$ for some nondegenerate subalgebra $\mathbf{D}$ of $\mathbf{O}$.

Proof: Let $\mathbf{A}$ be a subalgebra of $\mathbf{J}$ satisfying the hypothesis. Then $\mathbf{A}$ is isomorphic to $\mathbf{J_D}$ by Springer's lemma. In view of the previous two lemmas the claim will follow from the following four claims:

1. **G is transitive on triples of pairwise orthogonal idempotents, which generate a three dimensional nondegenerate subalgebra.**
2. **$C_G(\{x_1,x_2,x_3\})$ is transitive on vectors v of W_2 whose Q-value is 1.**

3. $C_G(\{x_1,x_2,x_3\},v)$ is transitive on vectors w of W_3 whose Q-value is 1.

4. $C_G(\{x_1,x_2,x_3\},v,w)$ is transitive on isomorphic composition subalgebras of $\mathbf{O} = W_1$.

Claim 1 is the content of proposition 2.10.5 together with lemma 5.4. Claim 2 follows from proposition 2.10, lemma 5.4 and Witts theorem. To prove claim 3 we observe that $C_G(\{x_1,x_2,x_3\},v) = Spin_7(F)$, $C_G(\{x_1,x_2,x_3\},v)$ acts irreducibly on W_3, and that $C_G(\{x_1,x_2,x_3\},v)$ acts on W_3 as the image of the stabilizer of a nonsingular point under a triality outer automorphism of $C_G(\{x_1,x_2,x_3\})= Spin_8^+(F)$ (these facts are consequences of [As1] 3.15.4). The claim follows $Spin_7(F)$ is transitive on vetors of W_3 of Q-value 1. To prove claim 4 we observe that $C_G(\{x_1,x_2,x_3\},v,w) \simeq G_2(F) \subset Aut(\mathbf{O})$ by [As1] section 5 and lemma 5.7 respectively. So $C_G(\{x_1,x_2,x_3\},v,w) = Aut(\mathbf{O})$. Now the $Aut(\mathbf{O})$ theorem of section 1 proves the claim.

Now we can list all possible nondegenerate subalgebra types.

Proposition 5.9: Let $\mathbf{A}$ be a nondegenerate subalgebra of $\mathbf{J}$. Then $\mathbf{A}$ is G-conjugate to one of the following:

1. $<id, x>$ where x is a primitive idempotent.
2. A totally dark plane containing id.
3. $<id, x, y_i>$ where $y_i \in E_0(x)$ satisfies $Q(y_i) = i$, $i \in \{ 1, \text{nonsquare} \in F\}$.
4. $<id, x, E>$ where $E \subset E_0(x)$ is a Q-nondegenerate subspace of dimension 2 or more.
5. $\mathbf{J}_{\mathbf{D}}$ for some nondegenerate composition subalgebra $\mathbf{D}$ of $\mathbf{O}$.

Proof: If $\mathbf{A}$ is two dimensional then 1 holds by lemma 5.1. If $\mathbf{A}$ contains no primitive idempotents then lemma 5.2 and Springer's lemma apply and 2 holds. So assume $\dim(\mathbf{A}) \geq 3$ and that $\mathbf{A}$ contains a primitive idempotent x. Then by Springer's lemma, part 2, $E_{0,\mathbf{A}}(x)$ is not 0. Moreover $\mathbf{A} = <id,x> \oplus E_{0,\mathbf{A}}(x) \oplus E_{1,\mathbf{A}}(x)$. If $E_{1,\mathbf{A}}(x)$

$= 0$ then lemmas 5.4 and 5.5 apply and hence 3 or 4 must hold. If $E_{1,\mathbf{A}}(x) \neq 0$ then by Springer's lemma part 3 and lemma 5.8 5 must hold.

The following will be useful.

Lemma 5.10: Let U be a subspace of **J** containing id. Assume that $\mathbf{J} = U \oplus U\Theta$, then U is a nondegenerate subalgebra of **J**.

Proof: By assumption $id \in U$. Let $v \in U\Theta$, then $(id,u,v) = 0$ for all $u \in U$. First we compute using lemma 2.6 and proposition 2.3 that

1 . $id\#u = u - (1/2)(id,id)u - (1/2)(u,id)id - (1/2)(id,u)id + (1/2)(id,id)(u,id)id$

$$= u - (3/2)u \;-\; (u,id)id + (3/2)(u,id)id$$

$$= (1/2)\,(\,-u + (u,id)id\,) \quad \text{for all } u \in \mathbf{J}.$$

2. $id\#id = id$

Thus

$$0=(id,u,v) = 2(id \# u\ ,\ v) \ = (\ -u +(u,id)id\ ,\ v) \ = -(u,v) +(u,id)(v,id) \ \ (*).$$

Also $0 = (id,id,v) = 2(id\#id,v) = 2(id,v)$ yielding $(v,id) = 0$. Substituting this into $(*)$ gives: $(u,v) = 0$ for all $u \in U$. This shows that $U\Theta \subset U^{\perp}$. By assumption $\dim(U\Theta) = \dim(U^{\perp})$ and therefore we actually have $U\Theta = U^{\perp}$. Thus $S^{\#2}(U) = U$ by lemma 4.6, and hence U is a subalgebra by lemma 4.5. As $U \cap U^{\perp} = 0$ U is nondegenerate.

Corollary 5.11: Let U be a subspace of type U_3, totally dark plane, U_6, U_9, V_9 or V_{15} containing id. Then U is a nondegenerate subalgebra of J.

Proof: By lemma 5.10 this is a matter of checking that for every choice of U, $\mathbf{J} = U \oplus U\Theta$. For the respective cases this is shown in [As1] sec 3.15.2, [As3] sec. 7, [As3] sec. 4.5.5, [As4] sec. 3.3.2, [As2] the remark at the end of sec. 3, [As1] sec. 3.8.1.

Definition: We call a subalgebra $\mathbf{A}$ of type $\mathfrak{U}_i$ or $\mathfrak{V}_i$ if either $\mathbf{A}$, or $R(\mathbf{A})$ is of type $\mathfrak{U}_i$ or $\mathfrak{V}_i$.

Lemma 5.12: We assume in the following that each U_i resp. V_i contains id. The following subalgebra isomorphisms hold:

1. $U_3 \simeq \langle id, x, y_1 \rangle$ where x and y_1 are defined as in 5.4.
2. $U_6 \simeq J_{\mathbf{D}}$ where $\mathbf{D} = \langle \underline{1} \rangle$.
3. $U_9 \simeq J_{\mathbf{D}}$ where $\mathbf{D}$ is a quadratic field extension of F.
4. $V_9 \simeq J_{\mathbf{D}}$ where $\mathbf{D}$ is a two dimensional composition subalgebra of $\mathbf{O}$ with Witt index 1 .
5. $V_{15} \simeq J_{\mathbf{D}}$ where $\mathbf{D}$ is a four dimensional composition subalgebra of $\mathbf{O}$.

Proof: For part 1 observe that U_3 contains the three primitive idempotent idempotents x_{16}, x_{25}, x_{34}. So part 1 follows from lemmas 5.11 and 5.4.

Parts 2 and 5 follow from proposition 5.9, corollary 5.11 and a comparison of dimensions (Springer's lemma part 4) after we observe that $E_{1,U_6}(x_{16})$ and $E_{1,V_{15}}(x_{16})$ are not trivial.

For part 3 recall from the definition of $\mathfrak{U}_9$ that U_9^K is of type V_9 for some quadratic field extension K of F. So part 3 will follow once we establish part 4.

Recall the isometry Φ of trilinear forms from section 3. Observe that $\Phi^{-1}(V_9)$ is $\langle e_{ii}(1), e_{ij}(\underline{1}), e_{ij}(i_1) \rangle$. Now observe from section 1 table 1.1 that $\mathbf{D} := \langle \underline{1}, i_1 \rangle$ is a composition subalgebra of $\mathbf{O}$ and that the Witt index of $N|_{\mathbf{D}}$ is one. The claim follows.

Lemma 5.13: The normalizers in G of the $J_{\mathbf{D}}$ are as follows:

1. If $\mathbf{D} = F$, then $N(J_{\mathbf{D}}) = PSO_3(F) \times G_2(F)$, $C_G(J_{\mathbf{D}}) = G_2(F)$.
2. If $\mathbf{D} = F^2$, then $N(J_{\mathbf{D}}) = SL_3(F) o SL_3(F) H_D . \langle \tau \rangle$, $C_G(J_{\mathbf{D}}) = SL_3(F) \langle \tau \rangle$.

3. If $\mathbf{D} = K$ (a quadratic field extension of F), then $N(J_{\mathbf{D}}) = SU_3(F)oSU_3(F).\sigma$.

$C_G(J_{\mathbf{D}}) = SU_3(F)$.

4. If $\mathbf{D} = F^4$, then $N(J_{\mathbf{D}}) = Sp_6(F)oSL_2(F)\ H_D$, $C_G(J_D)=SL_2(F)$.

Proof: The centralizers are given in [Ja4] theorem 9 and are as claimed.

Let $\mathbf{D} = F^4$, then by lemma 5.8 and lemma 5.12.5 $J_{\mathbf{D}} = V_{15}$. Aschbacher has shown in [As1] that V_{15} is the alternating square of V_6 that $N_{E_6}(V_{15}) = SL_6(F)HSL_2(F)$, where H is the Cartan subgroup of $E_6(F)$ defined in section 3, $C_{E_6}(V_{15}) = SL_2$ and that $J = V_{15} \oplus V_6 \oplus V'_6$ as an $SL_6(F)$ -module, where $V'_6 = <x'_1,...,x'_6>$. Moreover Aschbacher has shown that d induces a nondegenerate symplectic form on V_6 and on V'_6 and hence $N_G(V_{15})$ is as claimed as $N_G(H) = H_D$.

Now let $\mathbf{D} = F$. In view of lemma 5.8 and lemma 5.12.2 $J_{\mathbf{D}} = U_6$. Aschbacher shows in [As3] section 5 that $N_{E_6}(U_6) = SL_3(F) \times G_2(F)$, $J = U_6 \bigoplus_{i=1}^{7} W_i$ as an $SL_3(F)$-module, where each W_i is a three dimensional irreducible $SL_3(F)$ module. Moreover because J is a self dual module there is an i such that $Q|_{W_i}$ is nondegenerate. Now recall from lemma 3.3.2 that $Q|_{W_i}$ is nondegenerate iff $P_d|_{W_i}$ is nondegenerate. This shows that $N_G(U_6)/C_G(U_6)$ is contained in $SO(W_i,Q|_{W_i})$ $\simeq PSO_3(F)$. The opposite inclusion follows from lemma 3.3.2 and theorem 2.5.1.

Now let $\mathbf{D} = F^2$. In view of lemma 5.8 and lemma 5.12.4 we need to compute $N_G(V_9)$. Aschbacher shows in [As2] section 3 that $N_{E_6}(V_9) = SL_3(F)oSL_3(F)oSL_3(F)H<\tau>$ where τ acts as a graph automorphism on $C_G(V_9) = SL_3(F)$. Moreover V_9 is a tensor product of two natural $SL_3(F)$- modules and id is the sum of three fundamental tensors. Also $N_{E_6}(V_9)^{\infty}$ acts as $SL_3(F) \times SL_3(F)$ on V_9.

Now let $(m,n) \in SL_3(F) \times SL_3(F)$ and $d = \sum_{i=1}^{3} v_i \otimes w_i \in V \otimes W$, where V,W are natural $SL_3(F)$ modules and $\{v_i\}$ $\{w_i\}$ are bases for V resp. W. Let $(a_{i,j})$ and

$(b_{i,j})$ denote the matrices of m resp. n with respect to the bases $\{v_i\}$ resp. $\{w_i\}$. Then $d = d\,(m,n) = \Sigma\, v_i m \otimes w_i n = \sum_i\sum_j\sum_k a_{i,j}\, b_{i,k}\, v_j \otimes w_k$ iff $(b_{i,j}) = ((a_{i,j})^{-1})^t$ where t denotes the transpose map. So we conclude that $N_{E_6}(V_9)^\infty \cap G$ acts as $SL_3(F)$ on V_9. Also from the definition of τ in [As2] section 3 it is evident that τ centralizes id. So $N_G(V_9)$ is as claimed.

Now let $\mathbf{D} = K$. Then by lemma 5.8 and lemma 5.12.3 we need to compute $N_G(U_9)$. Aschbacher showed in [As3] section 3 that $N_{E_6}(U_9) = SL_3(K)oSU_3(K/F).\sigma$ where σ induces a graph field automorphism on $SL_3(K)$. Moreover $U_9 = V \otimes V^\sigma$ as an $SL_3(K)$ K - module and that id $= \Sigma\, v_i \otimes v_i^\sigma$ where $\{v_i\colon 1 \leq i \leq 3\}$ is a K-basis for V and $N_{E_6}(U_9)$ acts as $SL_3(K)$ on U_9. So now let $m \in SL_3(K)$ and let $(a_{i,j})$ denote the matrix of m with respect to the basis $\{v_i\}$. Then id $=$ id $m = \Sigma\, v_i\, m \otimes v_i^\sigma\, m^\sigma = \sum_{i,j,k} a_{i,j}\, a_{i,k}^\sigma\, v_j \otimes v_k^\sigma$ iff $(a_{i,j}) = (((a_{i,j})^{-1})^t)^\sigma$ where t is the transpose map. So m centralizes id iff $m \in SU_3(K/F)$ and the claim follows as σ centralizes id.

Remark: With a little more work using [Ja1] we could establish the Witt property for nondegenerate subalgebras of $\mathbf{J}$ namely: Let $i\colon \mathbf{A} \mapsto \mathbf{B}$ be an isomorphism of nondegenerate subalgebras of $\mathbf{J}$. Then i extends to an automorphism of $\mathbf{J}$.

We conclude this section with the following:

Lemma 5.14: Let $\mathbf{A}$ be a nondegenerate subalgebra of $\mathbf{J}$. Assume that $N_G(\mathbf{A})$ is maximal in G with respect to stabilizing a nondegenerate subalgebra. Then $\mathbf{A}$ is of type U_3, totally dark plane, V_9, V_{15}, U_6, U_9 or <id,x>, where x is a primitive idempotent. Moreover in each case $\mathbf{A} = C_{\mathbf{J}}(C_G(\mathbf{A}))$.

Proof: By proposition 5.9, $\mathbf{A}$ is in the list of 5.9. The normalizers of algebras of type

$\langle id,x,y_i\rangle$, i=non, and type $\langle id,x\rangle \oplus E_{0,\mathfrak{A}}(x)$ were shown to stabilize the algebra $\langle id,x\rangle$ (see lemmas 5.4 and 5.5) and hence are not maximal in G. In view of lemma 5.12 this leaves only the subalgebras listed in this lemma. To prove the second assertion we check for each case that the complement $\mathfrak{A}\Theta$ of $\mathfrak{A}$ is a direct sum of nontrivial irreducible $C_G(\mathfrak{A})$ modules. For the case U_3 this follows from 2.10.3 or alternatively from [As1] 3.15. For the case totally dark plane this follows from [As3] 7.3. For the cases V_9, U_9, U_6, V_{15} it follows from [As2] 3.5, [As3] 3.3.3, [As3] 5.7, [As1] 3.2 respectively.

SECTION 6: BRILLIANT SUBGROUPS

Recall from section 4 that a brilliant subalgebra is a subalgebra whose Q-radical is nontrivial. Recall from section 3 that a subgroup of G is brilliant if it stabilizes a member of $\{\Upsilon_i : i \in \{1,2,3,5,6,10,9,12\}\}$.

In lemma 4.9.4 we proved that the normalizer of a brilliant subalgebra is a brilliant subgroup of G. In this section we will show that every brilliant subgroup of G is contained in the stabilizer of a nondegenerate subalgebra or a maximal parabolic subgroup of G. The maximal parabolics stabilize certain singular subspaces of idΘ. The next two lemmas show that we may regard the maximal parabolics as subalgebra stabilizers.

Lemma 6.1: Let U be a singular subspace of idΘ. Then $\langle id,U\rangle$ is a subalgebra of J; its radical is U.

Proof: In view of Lemma 4.1 every element of U is class 2 nilpotent and hence also Q-singular. Next we observe that $(u,v) = Q(u+v) - Q(u) - Q(v) = 0 - 0 - 0 = 0$ for all $u,v \in U$. So U is the radical of $\langle id,U\rangle$ as $Q(id)\neq 0$. Now we observe that by lemma 3.4.2 $0 = u\#v$ for all $u,v \in U$. So by lemma 4.5 $\langle id,U \rangle$ is a subalgebra.

Lemma 6.2: Let L be a subgroup of G and suppose that L stabilizes $U \in \Upsilon_i$ $i \in \{1,...,6\}$. Then L stabilizes the singular subspace $U \cap id\Theta$, and hence the subalgebra $\langle id,(U \cap id\Theta) \rangle$. Moreover $U \cap id\Theta \neq 0$ unless $U \in \Upsilon_1$ and U contains a primitive idempotent.

Proof: The first part is evident in view of lemma 6.1 since the Υ_i, $i \in \{1,...,6\}$ are singular. If $U \cap id\Theta = 0$ then $\dim(U) = 1$ and U is spanned by a singular point not in $id\Theta$, which by lemma 4.1 must lie in the F-span of a primitive idempotent.

Definition: Let x be a class two nilpotent. $\mu(x) := P_d$ - radical of $x\Delta$. Let U be a singular subspace of $id^{\perp}$. We define $\mu(U) := \bigcap_{u \in U} \mu(u)$.

Lemma 6.3 : $\mu(x) =$ Q - radical of $x\Delta$.

Proof: G is transitive on class two nilpotents so it suffices to check the claim for $\mu(x_1)$. That's a simple calculation.

Definitions: Let U be a singular subspace of $d\Theta$. U is ***amber*** if $U \subset \mu(U)$. A singular line l is ***scarlet*** if l is not contained in $\mu(l)$. A singular subspace is ***scarlet*** if all of its lines are scarlet. A singular subspace U is ***tangerine*** if for some scarlet line l we have $U = l \oplus (\mu(l) \cap U)$.

Proposition 6.4: Let $\mathbf{A}$ be a subalgebra of $\mathbf{J}$ such that $\mathbf{A} = \langle id \rangle \oplus R(\mathbf{A})$, $R(\mathbf{A})$ is a singular subspace, and $r = \dim(R(\mathbf{A}))$ then:

1. If $R(\mathbf{A})$ is amber then $r = 1$, 2 or 3. G has three orbits on amber subspaces and each orbit is characterized by r with representatives V_r. The stabilizer of an amber

subspace is a maximal parabolic subgroup of G.

2. If $R(\mathbf{A})$ is tangerine then it is contained in a unique maximal tangerine subspace. Each maximal tangerine subspace is conjugate to V_6. The stabilizer of a maximal tangerine subspace is a maximal parabolic subgroup P of G. Moreover P stabilizes a symplectic from on V_6 which is induced by id.

3. If $R(\mathbf{A}) = l$ is a scarlet line then l is contained in the $N_G(\mathbf{A})$ invariant maximal tangerine subspace $l \oplus \mu(l)$.

4. If $r \geq 3$ and $R(\mathbf{A})$ is scarlet then $\mu(R(\mathbf{A})) \neq 0$ is amber.

5. If $R(\mathbf{A})$ is neither amber, scarlet or tangerine then $S \subset R(\mathbf{A}) \subset S + \mu(S)$ where S is scarlet of dimension 3 or 4. In this case $0 \neq \mu(R(\mathbf{A})) \subset \mu(S)$ and $\mu(R(\mathbf{A}))$ is amber.

6. $r \leq 6$.

Proof: Part 1 is the content of [As1] 9.11 and 9.12. Part 6 follows from the fact that V_5 and V_6 are the representatives for the $E_6(F)$ orbits of maximal singular subspaces of $\mathbf{J}$, see [As1] 6.5. Part 2 is the content of [As1] 9.6, 9.12 and 8.4. For part 3 we note that by [As1] 9.2.2 every scarlet line is G-conjugate to $<x_1,x_6>$. Moreover in the proof of [As1] 9.6 it is observed that $\mu(<x_1,x_6>) = <x_2,x_3,x_4,x_5>$. So $\mu(<x_1,x_6>) \oplus <x_1,x_6> = V_6$, which is maximal tangerine by part 2, and part 3 follows as $\mu(U)$ is $N_G(U)$ invariant for any singular subspace U of $\mathbf{J}$. Part 4 is the content of [As1] 9.8 combined with the content of [As1] 9.10.2 and 3. The first half of part 5 is the content of [As1] 9.10.1. Now the representatives for S can be chosen as $<x_3,x_4,x_{56}>$ and $<x_3,x_4,x_{56},x_{15}>$. Then $\mu(S)$ is V_2 respectively $<x_2>$. Now an easy calculation shows $V_2 < \mu(< x_3,x_4,x_{56}> + V_2) < \mu(S) = V_2$ and $<x_2> < \mu(<x_3,x_4,x_{56},x_{15}> + <x_2>) < \mu(S) = <x_2>$. So part 5 follows.

Corollary 6.5: If $\mathbb{A}$ satisfies the hypothesis of proposition 6.4 then $N_G(\mathbb{A})$ is contained in a maximal parabolic subgroup of G. Moreover the following are true:

1. If $R(\mathbb{A}) = V_1 = <x_1>$, then $N_G(\mathbb{A}) \simeq [F]^{15}: Spin_7(F).F^*$, $C_G(\mathbb{A}) \simeq F^{15}: Spin_7(F)$.

2. If $R(\mathbb{A}) = V_2 = <x_1,x_2>$, then $N_G(\mathbb{A}) \simeq [F]^{20}: SL_2(F)\times SL_3(F).F^*$, $C_G(\mathbb{A}) \simeq [F]^{20}:SL_3(F)$.

3. If $R(\mathbb{A}) = V_3 = <x_1,x_2,x_3>$, then $N_G(\mathbb{A}) \simeq [F]^{20}: SL_3(F)\times SL_2(F).F^*$, $C_G(\mathbb{A})\simeq [F]^{20}: SL_2(F)$. Moreover $< id, V_3\Delta>$ and $< id, V_3\Delta^{\perp}>$ are $N_G(\mathbb{A})$ invariant subalgebras of $\mathbf{J}$ and $S^{\#2}(V_3\Delta) = V_3$ and $S^{\#}(V_3,V_3{}^{\perp}) = V_3\Delta$.

4. If $R(\mathbb{A}) = V_6 = <x_1, \dots , x_6>$, then $N_G(\mathbb{A}) \simeq [F]^{15}: Sp_6(F).F^*$, $C_G(\mathbb{A}) \simeq F^{15}$. Moreover $V_6{}^{\perp} \simeq V_6 \oplus V_{15}$ is an $N_G(\mathbb{A})$ invariant subalgebra and $S^{\#}(V_6,V_{15}) = V_6$.

Where $[F]^n$ is some group with n composition factors all of which are isomorphic to the additive group of F.

Proof: First we note that $R(\mathbb{A})$ and $\mu(R(\mathbb{A}))$ are $N_G(\mathbb{A})$ invariant. Unless $R(\mathbb{A})$ is a scarlet line either $R(\mathbb{A})$ or $\mu(R(\mathbb{A}))$ is amber or contained in a unique maximal tangerine subspace, by proposition 6.4. In case $R(\mathbb{A})$ is a scarlet line 6.4.3 applies and $N_G(\mathbb{A})$ stabilizes a member of $\mathcal{V}_6$. Now the first claim follows from 1 and 2 of proposition 6.4.

Part 1. is proved for example in [Co2] or in [Asl] 8.9.2 and 7.8.4. The statements about $N_G(V_6)$ and $C_G(V_6)$ in part 4. can be found for example in [Asl] 8.4 and 8.5. The rest of part 4. is a straightforward calculation. For parts 2. and 3. observe that $N_{N(V_6)}(V_i)$ $i\in\{2,3\}$ is isomorphic to $R:SL_i(F).F^*$, where R denotes the unipotent radical of $N_{N(V_6)}(V_i)$. Also observe that $C_{N(V_6)}(V_i) = R$. Now Aschbacher shows in [Asl] 8.8 that $C_G(V_2)/R(C_G(V_2))^{\infty} \simeq SL_3(F)$, where $R(C_G(V_2))$ is the unipotent radical of $C_G(V_2)$, and so part 2. follows as $R < C_G(V_2)$. Also from [Asl] 8.7.2 it follows that $C_G(V_3)/R(C_G(V_3))^{\infty} \simeq SL_2(F)$ so the first part of 3. follows. The second part of 3. is straightforward calculation.

Orbits on Points Lemma: Let $v \in J$. Then $N_G(v)$ is contained in a Parabolic subgroup or in $N(\mathbf{A})$ where $\mathbf{A}$ is a nondegenerate subalgebra of dimension less than or equal to three.

Proof: Let $v \in J$. Then by lemma 2.4 the subalgebra $\mathbf{A}$ generated by v is at most three dimensional. If this subalgebra is nondegenerate then the claim holds. If the radical of this algebra is singular, then corollary 6.5 applies. The remaining case is the brilliant nonsingular radical. In this case $R(\mathbf{A})$ contains a point conjugate to $<x_1+x_{12}>$. So then by lemma 4.3 and the fact that $S^2(R(\mathbf{A})) \subset R(\mathbf{A})$ (see lemma 4.9.5) $R(\mathbf{A})$ contains a conjugate of $<x_1+x_{12},x_{56}>$. So as $\dim(R(\mathbf{A})) \leq 2$, $R(\mathbf{A})$ must be conjugate to $<x_1+x_{12},x_{56}>$. Now $<x_{56}> = S^2(R(\mathbf{A}))$ is a class two nilpotent, i.e., singular, and $<id,x_{56}>$ is an $N_G(\mathbf{A})$ invariant subalgebra with radical $<x_{56}>$. Hence $<x_{56}>$ is amber by proposition 6.4.1 and hence $N(v)$ is contained is a maximal parabolic.

Lemma 6.6: G has two orbits on subspaces of type Υ_{10}. If $U \in \Upsilon_{10}$ then $N_G(U)$ is contained in the stabilizer of a class 2 nilpotent or a primitive idempotent.

Proof: Let $b \in U \cap id^{\perp}$ be a brilliant nonsingular vector. Then b is conjugate to $x_1 + x_{12}$ or $x_1 + \alpha x_6'$, by lemma 4.2. Now since $U = \Psi_{10}(b)$, U is conjugate to either

$\Psi_{10}(x_1 + x_{12}) = < x_i, x_{i2} : 1 \leq i \neq 2 \leq 6 >$ or

$\Psi_{10}(x_1 + x_6') = < x_1, x_6', x_1', x_6, x_{ij}, x_{kl} : \{1,6,i,j,k,l\} = \{1,2,3,4,5,6\}>$.

So G has at most two orbits on Υ_{10}. Now a simple computation shows that $S^{\#2}(U) = <x_5>$ resp $S^{\#2}(U) = < x_{16}>$. Moreover x_5 is a class 2 nilpotent and x_{16} is a primitive idempotent. Using lemma 4.1 this shows that G has two orbits on Υ_{10} and that $N_G(U)$ is contained in the stabilizer of a class 2 nilpotent or a primitive idempotent.

Lemma 6.7: Let U be a subspace of type $\mathscr{V}_{12}$. Then $N_G(U)$ stabilizes either a $\mathscr{V}_{15}$ type subalgebra or a point of idΘ .

Proof: Let $U \in \mathscr{V}_{12}$, then by [As1] 3.8, $J = U \oplus T$ and $T \in \mathscr{V}_{15}$ and T is $N_G(U)$ invariant. Since U is brilliant id $\notin U$. If id $\in T$ then T is a subalgebra by corollary 5.12. If id $\notin T$ then the two dimensional subspace S generated by the projections of id onto U resp T is an $N_G(U)$ invariant subspace. Moreover $S \cap \langle id \rangle^{\perp}$ is an $N_G(U)$ invariant point. This is the claim.

Lemma 6.8: Let $U \in \mathscr{V}_9$ then $N_G(U)$ either stabilizes a subalgebra of type $\mathscr{V}_9$ or a point of idΘ.

Proof: In [As2] section 3 Aschbacher shows that $N_G(U)$ stabilizes a unique 3-decomposition of J. Let's say $J = U \oplus A \oplus B$ is that decomposition. Let S be the subspace generated by the projections of id onto the subspaces U, A and B. Then S is an $N_G(U)$ invariant subspace containing id. If id projects nontrivially onto U, then the claim holds as the projection of id onto U is an $N_G(U)$ invariant point. If id projects trivially onto U, then $\dim(S) \leq 2$. If $\dim(S) = 1$ then id $\in$ A or B and A or B is a subalgebra by corollary 5.12. If $\dim(S) = 2$ then $id^{\perp} \cap S$ is an $N_G(U)$ invariant point.

At last we have:

Proposition 6.9: Let L be a brilliant subgroup of G. Then L is either contained in a maximal parabolic subgroup of G or L is contained in the stabilizer of some nondegenerate subalgebra. In particular, if L stabilizes a brilliant subalgebra, the claim holds.

Proof: If L is brilliant then L stabilizes a member of $\mathcal{V}_i$, i ∈ { 1,2,3,5,6,9,10,12}. If i ≤ 6 corollary 6.5 applies as these $\mathcal{V}_i$ are singular subspaces. In the other cases lemmas 6.6, 6.7 and 6.8 and the orbit on points lemma apply. Now if L stabilizes a brilliant subalgebra then by lemma 4.9.4 L is a brilliant subgroup and the claim follows as above.

Finally we can combine proposition 6.9, 5.14, and 6.4 to obtain the

SUBALGEBRA THEOREM FOR G: If L be a subgroup of G stabilizing a proper nontrivial subalgebra, then L stabilizes one of the following:

a. A two dimensional subalgebra containing a primitive idempotent.

b. A special plane containing id.

c. A totally dark plane containing id.

d. A member of $\mathcal{U}_6$ containing id.

e. A member of $\mathcal{V}_9$ containing id.

f. A member of $\mathcal{U}_9$ containing id.

g. A member of $\mathcal{V}_{15}$ containing id.

h. An amber or maximal tangerine subspace of idΘ.

Definition: A subalgebra of J is *good* if it is G-conjugate to one of the members listed in the subalgebra theorem.

Note that the subalgebra theorem states that any subgroup stabilizing a proper nontrivial also stabilizes a good subalgebra.

SECTION 7: LOCAL SUBGROUPS OF G

Lemma 7.1: Let N be a normal subgroup of $M < G$. Then $C_J(N)$ is an M invariant subalgebra of J .

Proof: Let $x,y \in C_J(N)$, and $g \in N$. Then $(xy)g = xg\ yg = xy$, so $xy \in C_J(N)$.

From now on throughout this section we will denote by N an abelian group of semisimple elements of G. We will also assume that all the Eigenvalues of N lie in F. Recall the definition of the subgroup H of section 3. Recall also that H is a Cartan subgroup of $E_6(F)$.

We denote the set of weight spaces of a Cartan subgroup K of $E_6(F)$ by $\hat{X}(K)$. Recall from section 3 that $\hat{X}(H)$ is a set of 27 singular points.

Lemma 7.2: If $N < H^g$, $g \in E_6(F)$. Then N centralizes each member of Xg onto which id projects nontrivially. In particular each such member is a primitive idempotent.

Proof: We observe first that $[N,J] \leq \mathrm{id}^{\perp} = \mathrm{id}\Theta$. But N must fix every point $x \in \hat{X}(H^g) \setminus \mathrm{id}\Theta$. But then $[N,x] \in \mathrm{id}\Theta \cap \langle x \rangle$. So $[N,x] = 0$ and the claim follows as x is a multiple of some primitive idempotent by lemma 4.1.

Lemma 7.3: Every Cartan subgroup of G is of the form $C_K(\mathrm{id})$ where K is a Cartan subgroup of $E_6(F)$ and id is in the linear span of three members of $\hat{X}(K)$. In particular if $N < K$ then $C_J(N)$ is a subalgebra containing a subalgebra of type $\mathfrak{U}_3$.

Proof: Aschbacher proved in [As1] 8.15 that $C_H(\mathrm{id})$ is a Cartan subgroup of G, $\{x_{16}, x_{25}, x_{34}\} \subset \hat{X}(H)$. Also $\mathrm{id} \in \langle x_{16}, x_{25}, x_{34} \rangle$, which is a three dimensional nondegenerate subalgebra of type $\mathfrak{U}_3$ (see lemma 5.4 and 5.12).

Lemma 7.4 : If N centralizes a primitive idempotent then N^2 is contained in a Cartan subgroup of G.

Proof: As in the previous lemma $N < C_G(x)$ so N acts on $E_0(x)$, and it suffices to show that N^2 is contained in a Cartan subgroup of $\Omega(E_0(x),Q)$, as every Cartan subgroup of G stabilizing x is a Cartan subgroup of $N_G(x)$ and vice versa. So it suffices to show that there is a subset of weight vectors of N^2 which is a Q-hyperbolic basis of $E_0(x)$ (by a Q- hyperbolic basis we mean a basis consisting of a maximal number of pairwise orthogonal hyperbolic pairs). First we observe that $Q(vn) = Q(v\lambda(n)) = \lambda(n)^2Q(v) = \lambda(n^2)Q(v)$ for every weight vector v of weight λ and every $n \in N$. If λ is a nontrivial weight of N^2, then $Q(v) = 0$ and v is Q-singular.

Now let v be any weight vector corresponding to a nontrivial weight λ of N^2 then, as $E_0(x)$ is Q-nondegenerate, there exists another weight vector $w \in E_0(x) \backslash v^{\perp}$ corresponding to a weight μ of N^2 with the property $0 \neq (v,w)$. Then $\mu = \lambda^{-1}$ and $\{v,w\}$ is a Q- hyperbolic pair of weight vectors of N^2. Moreover we can choose all other weight vectors to lie in $<v,w>^{\perp} \cap E_0(x)$. We write $E_0(x) = H \perp W_0$ where H is the sum of the Q-hyperbolic pairs H_i and W_0 is the zero weight space of N^2. Now W_0 is a nondegenerate subspace of $E_0(x)$ and hence we can extend a Q-hyperbolic basis of H to a Q-hyperbolic basis of $E_0(x)$. As N^2 acts trivially on W_0 such a Q-hyperbolic basis is an N^2 - invariant basis of weight vectors. So we are done.

Lemma 7.5 : If $N^3 \neq 1$, then N^2 is contained in a Cartan subgroup of G.

Proof: In view of lemma 7.4 it is enough to show that N centralizes a primitive idempotent.

By assumption $N^3 \neq 1$ so there exists a weight vector v of N such that N^3 acts nontrivially on $<v>$. So there exists an $n \in N$ and a $v \in J$ such that $vn = \alpha v$ and $\alpha^3 \neq 1$. So then $T(v) = T(vn) = \alpha^3 T(v)$. So v must be brilliant. So by [As3] 6.1 there must exist an N-invariant pair $\{s,U\}$ where $U \in \Upsilon_{10}$ and s is a singular point not in $U\Theta$.

Now by lemma 6.6 G has two orbits on Υ_{10}. Representatives for the orbits are given in the proof of lemma 6.6. In case U is G-conjugate to $\Psi_{10}(x_1+x_6')$ we find that $S^{\#2}(U)$ is an N-invariant primitive idempotent which must be centralized by N (see prop.2.7.5). In case U is G-conjugate to $\Psi_{10}(x_1+x_{12})$ we conjugate so that $\Psi_{10}(x_1+x_{12})$ is N^g invariant for some $g \in G$. Now

(*) $\Psi_{10}(x_1+x_{12}) = <x_{25}> \oplus <x_5> \oplus (E_1(x_{25}) \cap \Psi_{10}(x_1+x_{12}))$.

To see this observe that x_{25} is a primitive idempotent and that $<x_5> = E_0(x_{25}) \cap \Psi_{10}(x_1+x_{12})$. Also observe that $S^2(\Psi(x_1+x_{12}) \cap \mathrm{id}\Theta) = <x_5>$. Now let x be any weight vector of N contained in $\Psi_{10}(x_1+x_{12}) \setminus \mathrm{id}\Theta$. As $[N,J] \leq \mathrm{id}\Theta$ we conclude that $x \in C_J(N)$. So $x^2 \in C_J(N)$ as well. Now we can normalize x so that $x = x_{25} + \beta x_5 + u$ with respect to the decomposition (*), with $\beta \in F$. Now

$$x^2 = (x_{25})^2 + \beta^2(x_5)^2 + u^2 + 2\beta x_{25}x_5 + 2x_{25}u + 2\beta x_5 u$$

$$= x_{25} + 0 + u^2 + 0 + u + 0 \quad (x_5 \text{ is a class 2 nilpotent, } x_5 \in E_0(x_{25}),\ u \in x_5\Delta)$$

$$= x_{25} + \lambda x_5 + u \quad (x_5 \text{ generates } S^2(\Psi_{10}(x_1+x_{12}) \cap \mathrm{id}\Theta))$$

Similarly we can show $(x^2)^2 = x_{25} + u + u^2$. So either x or x^2 is an idempotent of J. An easy computation using lemma 3.3.1 shows that $Q(x) = Q(x^2) = 1/2$. So by definition either x or x^2 is a primitive idempotent.

Note that the proof of 7.5 actually shows that any abelian group L such that $L^3 \neq 1$ centralizes a primitive idempotent. In particular any elementary abelian 2-subgroup of G is contained in the centralizer of a primitive idempotent.

Lemma 7.6: Assume that F contains a primitive third root of unity. If N is not contained in a Cartan subgroup of G and $N^3 = 1$, then $|N| \in \{9,27\}$ and one of the following holds:

1. $C_J(N)$ is a nontrivial totally dark subalgebra , and F is finite of order congruent to 4,7 mod(9).

2. $N_G(N) = 3^3{:}L_3(3)$ and $N_G(N)$ stabilizes a 27-decomposition containing id.

Proof: If N is not contained in a Cartan subgroup of G, then in view of lemmas 7.2 and 7.4 it is not contained in an $E_6(F)$ conjugate of H. Define $M = \langle N, Z(E_6(F)\rangle$, where $Z(E_6(F))$ is the center of $E_6(F)$, which by our choice of F has order 3. Then M is certainly not contained in an $E_6(F)$ conjugate of H, hence M satisfies the hypothesis of [As3] 8.3 and hence $|M| \in \{27,81\}$; the first part of the lemma follows. Moreover, by [As3] 8.3 the Eigenspaces of every element of N make up a 3-decomposition of J. Also the weight spaces of N are dark, else by [As3] 6.1 and 6.3.2 N is contained in an $E_6(F)$ conjugate of H. So no weight space of N can have dimension more than 3, by [As3] 7.3. Also observe that the nontrivial weight spaces of N are Q singular. Lastly, if $|M|=27$ then F is finite of order congruent to 4 or 7 mod(9) [As3] 8.3.1.

If $|M| = 27$ then N has, at most 9 weight spaces, so at least one of them, say W, has dimension ≥ 3. Now if $W = C_J(N)$ then 1 holds. So assume otherwise and let $w \in W$, then by our earlier observation w is dark. So the map $l_w\colon J \mapsto J$ defined by $l_w(v) = w\#v$ is injective. (To see this observe that $v \in \mathrm{Ker}(l_w)$ iff $v \in w\Delta$ and that $w\Delta = 0$ when w is dark.) Moreover l_w permutes the weight spaces of N nontrivially. Also $l_w o l_w o l_w\colon W \mapsto W$ is an isomorphism and $l_w o l_w(W) \subset C_J(N)$. So $\dim(C_J(N)) \geq 3$ and $C_J(N)$ is a totally dark subspace, so 1 holds.

If $|M| = 81$ and every proper subgroup of M containing $Z(E_6(F))$ is contained in an $E_6(F)$- conjugate of H, then the proof of [As3] 8.3 shows that M is an exotic 81 subgroup of $E_6(F)$ whose Eigenspaces form a 27- decomposition.

So now assume that some proper subgroup N_1 of M containing $Z(E_6(F))$ is not contained in an $E_6(F)$ conjugate of H. In particular we may assume in this case that F is finite of order 4 or 7 mod(9) and that $N_1 \cap G$ centralizes a totally dark subalgebra U of **J**. Now we may assume that N contains $N_1 \cap G$, so let $n \in N \setminus (N_1 \cap G)$. If n centralizes U then N centralizes U and case 1 holds. So assume that n acts nontrivially on U and let $\{id, y, y^2\}$ be the Eigenspaces of n on U with respect to the Eigenvalues $1, \omega, \omega^2$ where ω is a primitive cube root of unity. First we observe that

$(*)$ $(id,y) = (id,y^2) = Q(y) = Q(y^2) = 0$ because n preserves the quadratic form Q.

Also since $< id,y,y^2>$ is totally dark we may assume that T(y) is not a cube in F (else $<id,y>$ contains brilliant points) so we may normalize y such that $T(y) = \omega$ and hence

$T(y^2) = (1/6)\, f(y^2, y^2, y^2)$	by definition of T
$= (1/3)(y^2 \# y^2, y^2)$	by definition of #
$= (1/3)(y^4, y^2)$	from lemma 2.6 and the facts $(*)$
$= (1/3)(y^6, id)$	by proposition 2.3.4
$= (1/3)\,((y^3)^2, id)$	by power ass. of **J** (see remark after 2.3.5)
$= (1/3)(\,(T(y)id)^2, id)$	by proposition 2.3.6 and the facts $(*)$
$= (1/3)(id,id)(T(y)^2)$	
$= T(y)^2$	by proposition 2.3.2 and definition of $(\_,\_)$
$= \omega^2.$	

Now an easy calculation shows that $T(id + y + y^2) = T(id) + T(y) + T(y^2) = 0$; a contradiction to the total darkness of $<id,y,y^2>$. Therefore N must centralize a totally dark subalgebra.

Now it remains to show that if M is an exotic 81-group, then $N_G(N) = N{:}L_3(3)$.

Aschbacher showed that $N_{E_6}(M) = M{:}3^3{:}SL_3(3)$, $C_{E_6}(N) = M$ and that $N_{E_6}(M) / M = C_{SL(M)}(Z(E_6)) = 3^3{:}SL_3(3)$. Observe also that N is a complement of $Z(E_6)$ in M. So if $g \in E_6$ and $N^g = N$, then as $Z(E_6)^g = Z(E_6)$, we have $N_{E_6}(N) < N_{E_6}(M)$. And thus $N_{E_6}(N) / N$ is the stabilizer of the pair of subspaces $(Z(E_6),N)$ in $SL(M) \simeq SL_4(3)$. So $N_{E_6}(N) = M : SL_3(3)$, and N is the natural $SL_3(3)$-module.

Now we need to observe that every weight space of N is one dimensional. To see this let $0 \neq w \in W_\lambda$, $\lambda \in Hom(N,F)$. Then w must be a dark vector by our initial observations. So the linear map l_w defined as before by $l_w(x) = w \# x$ is injective and moreover $l_w^2(W_\lambda) < W_0 = C_J(N)$. So if for any weight λ the weight space has dimension greater than one, then $C_J(N)$ has dimension greater than one and hence M is not an exotic 81- group, a contradiction. We conclude that the weight spaces of N are indexed by (correspond bijectively to) $Hom(N,F)$. Now $SL_3(3)$ acts transitively on the nonzero elements of $Hom(N,F)$ and fixes zero. But the zero weight space of N is the subspace of J spanned by id, so $SL_3(3) < G$. As $Z(E_6)$ is not contained in G, we conclude that $N_G(N) = N : SL_3(3)$.

Lemma 7.7: G is transitive on 27-decompositions containing $<id>$.

Proof: By [As3] 8.3 E_6 has one or three orbits on 27-decompositions, parameterized by the equivalence class $mod(F^3)$ of the T -values of the points making up the decomposition. Thus E_6 is transitive on the 27-decompositions containing id. Since the normalizer in E_6 of a 27-decomposition is transitive on the weight spaces of the decomposition (see [As3] 8.3), the claim follows.

Definition: $d_p(q)$ is defined to be the smallest d such that $q^d-1 = 0 \bmod(p)$.

Note that $d_p(q)$ is the dimension of the smallest GF_q - vectorspace V such that p divides $|GL(V)|$.

Definition: Let K be an extension field of F. By J^K we denote the exceptional central simple Jordan algebra over K and by G^K its automorphism group. When $K \simeq \overline{F}$, the algebraic closure of F, we write $\overline{G}$ for $G^{\overline{F}}$ and $\overline{J}$ for $J^{\overline{F}}$.

From now on we will regard G as a subgroup of fixed points of some Frobenius automorphism of G^K resp $\overline{G}$.

Definition: Assume that F^* contains no element of order 3. A *twisted 27-decomposition* is a decomposition $\mathfrak{D}$ of $id^{\perp}$ into 13 pairwise orthogonal two dimensional subspaces such that there exists a quadratic field extension K of F so that $\mathfrak{D}^K$ a 27-decomposition of J^K containing id. Moreover we require that the normalizer in G of this configuration is isomorphic to 3^3: $SL_3(3)$.

Definition: $A_G(U)$ is defined to be $N_G(U)/C_G(U)$.

Definition: Let r be a prime. The *r-rank* of the group L is the maximum of the ranks of the elementary abelian r-subgroups of L and is denoted by $m_r(L)$.

Lemma 7.8: Let r be a prime not equal to 2,3,p and R an r-subgroup of G. Then R is contained in a Cartan subgroup $\overline{T}$ of $\overline{G}$ and is hence abelian. Moreover $A_G(R)$ is isomorphic to a section of the Weyl Group of G. Also $m_r(G) \leq 4$.

Proof: The first part of this statement is in [Sp3] which shows that $R < \overline{T} < \overline{G}$, where $\overline{T}$ is a Cartan subgroup of $\overline{G}$. Hence R is abelian and the r-rank of G is as claimed. We consider $G = Fix_{\overline{G}}(\sigma)$ where σ is some field automorphism with fixed field F. Now in [Sp3] it is shown that $C_{\overline{G}}(R)$ is transitive on the Cartan subgroups of $\overline{G}$ contained in $C_{\overline{G}}(R)$. So by a Frattini argument $N_{\overline{G}}(R) = C_{\overline{G}}(R)\, S$ where $S = N_{N_{\overline{G}}(R)}(\overline{T})$. But $\overline{T}$ is self centralizing and $N_{\overline{G}}(\overline{T})/\overline{T}$ is the Weyl group. So as $A_{\overline{G}} \simeq S/\, S \cap C_{\overline{G}}(R)$ and $S \cap C_{\overline{G}}(R) \geq \overline{T}$ we have $A_{\overline{G}}(R)$ is a section of the Weyl group. As $G \leq \overline{G}$ we have $A_G(R) \leq A_{\overline{G}}(R)$ and the claim follows.

Lemma 7.9: Assume that F^* contains no element of order 3. Twisted 27-decompositions exist in **J**, and G is transitive on twisted 27-decompositions contained in **J**.

Proof: By assumption, F^* does not contain elements of order 3, F is finite and there exists a quadratic field extension K of F such that K^* contains an element of order 3. Now let $N < G^K$ be an elementary abelian 3 group of order 27 whose weight vectors form a 27- decomposition of $\mathbf{J}^K$. Since $|G^K|$ and $|G|$ have the same 3-share and $G < G^K$ we can chose $N < G$. Let σ denote the field automorphism of G^K whose set of fixed points is G. Let M denote the normalizer of N in G^K extended by σ. Now as σ centralizes N we conclude using Lemma 7.6. that $M \simeq (N \oplus \langle\sigma\rangle){:}SL_3(3)$. Now M centralizes $O_2(M) = \langle\sigma\rangle$. Thus $N : SL_3(3) < G$ is the normalizer of N in G.

Next we must show that $N : SL_3(3)$ stabilizes a twisted 27-decomposition. We already know that $N : SL_3(3)$ stabilizes a 27-decomposition of $\mathbf{J}^K$ containing id. Now if $y \in \mathbf{J}^K$ is a weight vector of N, then so is y^2, moreover $Q(y^2) = Q(y) = 0$ when $y \neq$ id and $\langle y, y^2\rangle$ is a Q^K- hyperbolic pair. To see this observe that $(y,y^2) = (y^3,\mathrm{id})$ by 2.3.4 and that $y^3 = Q(y)y + T(y)\mathrm{id} = T(y)\mathrm{id}$ by Hamilton's equation and the fact that $Q(y) = 0$. Combining this gives: $(y,y^2) = T(y)(\mathrm{id},\mathrm{id}) = 3T(y) \neq 0$ by

2.3.2 and the fact that y is dark .

So J^K is the sum of 13 hyperbolic pairs and <id>. Moreover this decomposition is also $SL_3(3)$ invariant. Now we claim that the $N{:}SL_3(3)$-invariant decomposition of J^K induces a decomposition on J. This is the twisted 27-decomposition. To see this we need to observe that σ acts on $<y,y^2>$ and that $C_{<y,y^2>}(\sigma)$ is a line of J. In section 9 there is a description of how σ acts on J^K . Now for all $n \in N$ we have $y\sigma n = yn\sigma = y\lambda_y(n)\sigma = y\sigma\lambda_y(n)^2$ where λ_y is the weight N corresponding to the vector y, showing σ- invariance of $<y,y^2>$. Now let's consider J^K as an $N \oplus <\sigma>$ module. Then we just saw that $<y,y^2>$ is an $K(N \oplus <\sigma>)$ submodule. Now apply 25.7.2 of [As9], which states that in this situation there exists an FN- submodule U of J such that $U^K = <y,y^2>$. So $U = C_{<y,y^2>}(\sigma)$ and $\dim_F(U) = \dim_K(U^K) = 2$. So U is a line in J.

Now we need to show that G is transitive on twisted 27-decompositions. Let D_1 and D_2 be twisted 27-decompositions whose stabilizers are $N_i : SL_3(3)$ i=1,2 respectively. Their stabilizers in G^K are conjugate, say $N_1{}^h = N_2$. Since σ centralizes G we conclude that σ centralizes N_1 and N_2. So $\sigma h\sigma h^{-1} \in C_{G^K}(N_1) = N_1$. So $[\sigma,h]^3 = 1 = [\sigma,h]^{h\sigma h^{-1}}$ forcing $[\sigma,h] = 1$, i.e., $h \in G$. This shows conjugacy.

Lemma 7.10: Let $R < G$ be a 3-group . Then

1. Z(R) is contained in a Cartan subgroup $\overline{T}$ of $\overline{G}$ unless $R = Z(R)$ is elementary abelian of order 27 and R stabilizes a 27 or twisted 27- decomposition of J.

2. If Z(R) is contained in a Cartan subgroup of $\overline{G}$, then $A_G(Z(R))$ is a section of the Weyl group of G.

3. There exists a normal abelian subgroup $A < R$ such that R/A is a section of the Weyl group of G. Hence R/A is a subgroup of an elementary abelian group of order 9.

4. $\dim(C_J(Z(R))) \geq 3$ unless $Z(R) = R$, $|R| = 27$ and R stabilizes a 27 or twisted 27-decomposition.

5. $N_G(R)$ stabilizes one of the following:

   a. A proper nontrivial subalgebra.

   b. A 27 - decomposition.

   c. A twisted 27-decomposition.

Proof: The proof of 2. is the same as in lemma 7.8. For part 3. we use [Sp3] to embed R into $N_{\overline{G}}(\overline{T})$ where $\overline{T}$ is a σ-invariant Cartan subgroup of $\overline{G}$. Then let $A = \overline{T} \cap R$. We observe that A is the kernel of $\pi|_R$ where the canonical projection π: $N_{\overline{G}}(\overline{T}) \mapsto W(\overline{G})$. The claim follows since the Sylow 3-group of $W(\overline{G})$ is elementary abelian of order 9.

Now we will prove part 1. If $Z(R)$ is not elementary abelian $Z(R)$ is contained in a Cartan subgroup of G by lemma 7.5. If $Z(R)$ is elementary abelian and not contained in a Cartan subgroup of G, then by lemma 7.6 $C_J(Z(R))$ is a nontrivial totally dark subalgebra of J or $Z(R)$ stabilizes a 27-decomposition of $\overline{J}$. If $C_J(Z(R))$ is nontrivial then $C_J(Z(R))\otimes\overline{F}$ is a proper nontrivial $Z(R)$ invariant subalgebra of $\overline{J}$ containing brilliant points. So by Springer's lemma part 1 $C_J(Z(R))\otimes\overline{F}$ contains a primitive idempotent and hence by lemma 7.4 $Z(R)$ is contained in a Cartan subgroup of $\overline{G}$. So now assume that $Z(R)$ has order 27, is not contained in a Cartan subgroup of $\overline{G}$, does not stabilize a 27-decomposition of J but stabilizes a 27-decomposition of $\overline{J}$, and in particular F^* does not contain an element of order 3. Let σ be the Frobenius automorphism of $\overline{F}$ such that $G = \mathrm{Fix}(\sigma)$. Then there exists a quadratic field extension K of F contained in $\overline{F}$ such that $\mathrm{Fix}(\sigma^2) = G^K$. By lemma 7.6 $Z(R)$ must stabilize a 27-decomposition of J^K. Now the first part of the proof of lemma 7.9 shows that $Z(R)$ stabilizes a twisted 27-decomposition of J.

It remains to assert that when $Z(R)$ is not contained in a Cartan subgroup of $\overline{G}$ then $R=Z(R)$. Now R is contained in $N_G(Z(R)) \simeq Z(R):SL_3(3)$, by the above and lemmas 7.6 and 7.9. So if $R \neq Z(R)$ then every $g \in R\backslash Z(R)$ induces a nontrivial automorphism of $Z(R)$: a contradiction.

Now part 4 is a consequence of part 1 and lemma 7.2 once we observe that $\dim_{\overline{F}}(C_{\overline{J}}(Z(R)) = \dim_F(C_J(Z(R))$. Part 5 follows from parts 1 and 4 and lemmas 7.6 and 7.9.

Lemma 7.11: Let R be an elementary abelian 2 group. Then:

1. R centralizes a primitive idempotent.
2. $\dim(C_J(R)) \geq 3$.
3. If $|R| = 2$ then $C_J(R) \in \Upsilon_{15}$ or $C_J(R) = \langle id,x,E_0(x)\rangle$ for some primitive idempotent x, and $R = Z(C_G(C_J(R)))$. G has exactly two conjugacy classes of involutions.
4. $A_G(R)$ is a 2,3,7 group.
5. $m_2(G) = 5$.

Proof: Part 1 follows from the proof of lemma 7.5 (see remark after that lemma). So part 5 follows from proposition 2.7.5 and the fact that $m_2(Spin_9(F)) = 5$. For part 3 we observe first that $E_6(F)$ has two conjugacy classes of involutions I_{12} and I_{16}, see [As3] 6.4.1. Moreover $g \in I_{12}$ iff $C_J(g) \in \Upsilon_{15}$ and $g \in I_{16}$ iff $C_J(g) = \langle p\rangle + U$ where $U \in \Upsilon_{10}$ and p is a singular point in $J\backslash U\Theta$. Now an involution $g \in E_6(F)$ lies in G iff $id \in C_J(g)$. So if $g \in I_{12} \cap G$ then g centralizes a subalgebra of type Υ_{15}. Now $C_G(C_J(g)) \simeq SL_2(F)$, by lemmas 5.12.5 and 5.13.4, and $[C_G(C_J(g), J] = [g,J]$. Moreover because g is an involution, g acts as $-Id$ on $[g,J]$. So $g \in Z(C_G(C_J(g)))$. Now $Z(SL_2(F))$ has order 2 and it follows that G is transitive on

$I_{12} \cap G$ as G is transitive on subalgebras of type Υ_{15}.

Now let $g \in I_{16} \cap G$. Then by conjugating g appropriately we may assume using lemma 6.6 that $C_J(g) = \langle x_{16} \rangle + \Psi_{10}(x_1 + x_6')$ or $\langle x_2' + w \rangle + \Psi_{10}(x_1 + x_{12})$ where $w \in U\Theta$. Now since $id = x_{16} + x_{25} + x_{34} \in C_J(g)$ we see that the second possibility is out. Now the remaining parts of 3 follow from proposition 2.7.

For parts 2 and 4 we will assume first that R is not contained in a Cartan subgroup of $E_6(F)$. Then part 2 is the content of [As3] 6.3.4. For part 4 we use [As3] 6.7 to see that $R = R_0 \oplus J$ where $R_0^* = I_{16} \cap R$, that $|R| \leq 4$ $|J| \leq 8$ and J centralizes a subalgebra of type $\mathfrak{U}_6$. So part 4 follows in this case and moreover $A_G(R)$ is a 2,3 group unless $|J|=8$.

Now assume that R is contained in a Cartan subgroup of $E_6(F)$. WLOG we may take $R<H$ and even in Ω_2, the subgroup of elements in H of order 1 or 2, which we may identify with the six dimensional orthogonal space over GF_2 of Witt-index 2. Then by [As3] 6.4 $A_{E_6(F)}(R)$ is a section of $O_6^-(2)$ and is hence a 2,3,5 group, so the same must be true for $A_G(R)$. So now let $g \in A_{E_6(F)}(R)$ be of order 5, then $|R| \geq 16$, and we may also assume WLOG that $[g,R] = R$ and $C_{\Omega_2}(g) = R^{\perp}$, where here the $\perp$ is taken in Ω_2 with respect to the unique quadratic form stabilized by the Weyl group of $E_6(F)$. Assuming $[g,R] = R$ forces $|R| = 16$ and $R^{\perp}$ is a two dimensional nonsingular nondegenerate subspace of Ω_2. Thus there are two possible isometry types for R and hence $A_{O_6^-(2)}(R) \simeq O_4^{\pm}(2)$. Only the order of $O_4^-(2)$ is divisible by 5 and hence R is uniquely determined up to conjugacy in $E_6(F)$. Now the Sylow 5 group of $O_6^-(2)$ has order 5. So up to conjugation in $E_6(F)$ we may assume that g induces the permutation $(x_2,x_3,x_4,x_5,x_6)(x_2',x_3',x_4',x_5',x_6')(x_{i,j} \mapsto x_{i\rho,j\rho})$, where $\rho = (23456) \in Sym_6$, on the weight vectors of H, see section 3, and that $R = \langle l(1,-1,-1,1,1,1)\, g^i : 1 \leq i \leq 5 \rangle$ (see section 3 for the definition of $l(\_,\_,\_,\_,\_,\_)$). So we can see that $C_J(R) = \langle x_1, x_1' \rangle$, and is hence singular. Thus no $E_6(F)$-

conjugate of R will centralize id (recall id is nonsingular) and part 4 follows. For part 2 observe that id is the sum of at least three weight vectors of every Cartan subgroup of $E_6(F)$ containing R. If $R < G$ then each summand of id must be centralized by R and part 2 follows.

Recall that N is a semisimple subgroup of G and all the Eigenvalues of N lie in F. So either N contains a characteristic elementary abelian r-subgroup or N is infinite and N^2 is a characteristic subgroup of N which is contained in a Cartan subgroup of G. Thus lemmas 7.3 , 7.8 , 7.10 and 7.11 amount to the following:

Lemma 7.12: One of the following holds:

1. $C_J(N_0)$ is a proper nontrivial subalgebra, and $0 \neq N_0$ is characteristic in N.
2. $<N, Z(E_6(F))>$ is an exotic 81-group and $N_G(N)$ stabilizes a 27-decomposition.

Lemma 7.13: If $d_r(q) \neq 1$, $r \neq 3$ and R an r-subgroup of G , then $\dim(C_J(R)) \geq 2$.

Proof: Extend the field F to the smallest field K containing F such that $r \mid |K|-1$. Then Lemma 7.12 holds for $R < G^K$. Now $\dim_K(C_{J^K}(R)) = \dim_F(C_J(R))$. So if $\dim_K(C_{J^K}(R)) \geq 2$ our claim holds. If $\dim_K(C_{J^K}(R)) = 1$ then, by Lemma 7.12, $N_{G^K}(R)$ must stabilize a 27-decomposition. So, by lemma 7.9, $N_G(R) = N_{G^K}(R)$ must stabilize a twisted 27-decomposition.

We conclude this section with the following:

Theorem 7.14: Let M be a closed subgroup of G that contains a normal solvable subgroup. Then one of the following holds:

i. M stabilizes a proper nontrivial subalgebra.

ii. M stabilizes a 27-decomposition containing id.

iii. M stabilizes a twisted 27-decomposition.

Proof: By hypothesis M contains a normal abelian subgroup N. If N contains unipotent elements, then M has a unipotent radical. In this case the Borel-Tits theorem applies and M is contained in a maximal parabolic subgroup of G. By proposition 6.4 the maximal parabolic of G stabilize proper nontrivial subalgebras.

So WLOG we can assume that N is a semisimple normal abelian subgroup. So lemmas 7.12 and 7.13 apply and the claim follows.

Note that if case i holds, then M also stabilizes a good subalgebra by the subalgebra theorem.

SECTION 8: NON LOCAL SUBGROUPS WITH NON SIMPLE SOCLE

The following result of [As4] will be our major tool and starting point for this section.

Theorem 8.1: If M is a nonlocal subgroup of G (hence also of $E_6(F)$) with non simple socle, and M is a closed subgroup of G when F is algebraically closed, then one of the following holds:

i. $N_G(M)$ is brilliant.

ii . $N_G(M)$ stabilizes a member of $\mathfrak{U}_6$ or $\mathfrak{U}_9$.

iii. $N_G(M)$ stabilizes a 3-decomposition.

We already saw in the section on subalgebras (proposition 6.8) that brilliant subgroups are subalgebra stabilizers.

Lemma 8.2: Let M be as in the theorem and assume that $N_G(M)$ stabilizes a member U of $\mathfrak{U}_6$ or $\mathfrak{U}_9$. Then $N_G(M)$ stabilizes a subalgebra.

Proof: By hypothesis $J = U \oplus U\Theta$. If id projects nontrivially onto U with respect to this decomposition, then either U is an $N_G(M)$ invariant subalgebra or $N_G(M)$ stabilizes a point. So suppose that id $\in U\Theta$. We claim now that $C_M(U,id) \neq 1$. Suppose that this is true. Then the subalgebra generated by $<U,id> \neq J$ and is $N_G(M)$ invariant; hence the claim.

Suppose that $C_M(U,id) = 1$, then a faithful homomorphic image of M is contained in $A_{E_6}(U)$. But $A_{E_6}(U)$ is an SL_3 and hence M can not have more than one factor in $F^*(M)$. This contradicts our hypothesis on M. So the claim must hold.

Lemma 8.3: Let S be a two dimensional subspace of $id^{\perp}$. Then $N_G(S)$ stabilizes a proper nontrivial subalgebra or $N_G(S)$ is a local subgroup of G.

Proof: We consider first the case $C_G(S) \neq 1$. In this case $C_J(C_G(S))$ is a proper nontrivial $N_G(S)$- invariant subalgebra.

If $C_G(S) = 1$ then $A_G(S) = N_G(S) \leq SL(S) \simeq SL_2(F)$. So as $char(F) \neq 2$, $N_G(S)$ does not contain simple subgroups. So $N_G(S)$ contains a normal solvable subgroup and the claim follows.

Corollary 8.4: Let M be as in theorem 8.1 and assume that $N_G(M)$ stabilizes a 3-decomposition. Then $N_G(M)$ stabilizes a proper nontrivial subalgebra or $N_G(M)$ is a local subgroup.

Proof: The space P generated by the projections of id onto the factors of the 3-decomposition is at most three dimensional and contains id. So $P \cap id^{\perp}$ is at most two dimensional; hence the claim when $P \cap id^{\perp} \neq 0$. If $P \cap id^{\perp} = 0$ then id is contained in a member U of the 3-decomposition. Now U is a (proper nontrivial) subalgebra by corollary 5.11.

The collection of these last lemmas amounts to the following:

Proposition 8.5: Let M be as in theorem 8.1, then $N_G(M)$ stabilizes a proper nontrivial subalgebra of J , or $N_G(M)$ is a local subgroup.

Our efforts finally pay off:

Structure Theorem: Let M be a closed subgroup of G. If $F^*(M)$ is not a simple group, then one of the following holds:

i. M stabilizes a proper nontrivial subalgebra.

ii. M stabilizes a 27- decomposition (containing id).

iii. M stabilizes a twisted 27-decomposition.

Proof: Recall first that char(F) = p (possibly 0). Recall also that $F^*(M) = E(M)F(M)$ and that F(M) is the Fitting subgroup of M. If $F(M) \neq 1$ then Theorem 7.14 applies and the claim follows. So now assume that $F(M) = 1$. Then E(M) is a direct product of more than one simple group. In this case M satisfies the hypothesis of proposition 8.5 and $N_G(F^*(M))$ must therefore stabilize a proper nontrivial subalgebra or $N_G(F^*(M))$ is local and hence satisfies the hypothesis of theorem 7.14.

SECTION 9: OUTER AUTOMORPHISMS , FIELD EXTENSIONS AND A STRUCTURE THEOREM FOR Γ

In this section we will prove the structure theorem for Γ.

Proposition 9.1: $\mathrm{Out}(G) \simeq \mathrm{Gal}(F:F_0)$ where $F_0 \simeq \mathbb{Z}/p\mathbb{Z}$ or $\mathbb{Q}$ and G has no diagonal or graph automorphisms.

Proof: This is proved in [Ca1] when F is finite, where it is also claimed that the result holds when F is a perfect field.

Definition: Recall the basis X of weight vectors for H from section 3. Let $\sigma \in \mathrm{Gal}(F/F_0)$. Let γ_σ be the semilinear map defined by: $(\sum_{x\in X} \lambda_x x)\gamma_\sigma = \sum_{x\in X} \lambda_x^\sigma x$.

Notation: If F_* is a subfield of F, then we denote by J_{F_*} the set $XF_* = \{ \Sigma\, x_i\alpha_i : \alpha_i \in F_*\}$ and by $G_{F_*} = \{ M_X(g) : g \in G$ and $m_{i,j} \in F_*$ for all i,j $\}$.

Lemma 9.2: Let F_σ denote the fixed field of σ.

1. γ_σ preserves the multiplication of J.
2. $C_{\gamma_\sigma}(J) = J_{F_\sigma}$. Moreover J_{F_σ} is the exceptional Jordan algebra over F_0.
3. $\mathrm{Aut}(G) = \langle G, \gamma_\sigma : \sigma \in \mathrm{Gal}(F:F_0) \rangle$.
4. $C_G(\gamma_\sigma) = G_{F_\sigma} = \mathrm{Aut}(J_{F_\sigma})$.

Proof: Let $x,y \in X$. Then we observe by using proposition 3.2 that xy is a linear combination of elements of X with coefficients in the prime field F_0. So clearly $xy\gamma_\sigma = x\gamma_\sigma y\gamma_\sigma$, showing 1. Part 2 is now immediate from part 1. For part 3 we observe that $\bar{g} := (\gamma_\sigma)^{-1}\, g\, \gamma_\sigma \in GL_{27}(F)$ and that $\bar{g}$ preserves the multiplication of J. So

$\bar{g} \in G$ and hence $<G,\gamma_\sigma:\sigma\in Gal(F:F_0)> \subset Aut(G)$. The reverse inclusion follows from proposition 9.1 , proving 3. Part 4 is clear given parts 1,2,3.

Lemma 9.3: If N is normal in a subgroup L of Γ, then $C_{\mathbf{J}}(N)$ is an L invariant subalgebra.

Proof: As in the proof of lemma 7.1 we see that $C_{\mathbf{J}}(N)$ is a subalgebra. Now let $l \in L$, $n \in N$ and $x \in C_{\mathbf{J}}(N)$. Then $xln = x\bar{n}l = xl$, showing that $xl \in C_{\mathbf{J}}(N)$.

Proposition 9.4: Aut(G) does not fuse G-orbits of nondegenerate, amber or maximal tangerine subalgebras.

Proof: The idea of the proof is to observe that the invariants defining the various subalgebras are in the prime field and hence invariant under any field automorphism.

Let $\gamma \in Aut(G) \setminus G$ and $\mathbf{A}$ be a nondegenerate subalgebra. Then $\mathbf{A}$ is G-conjugate to one of the members of the list of proposition 5.9 and $\gamma = g\gamma_\sigma$ where $g\in G$ and γ_σ is as in 9.2. Now it is well known and easy to check that $Q(x\gamma)= Q(x)^\sigma$ and $f(x\gamma,y\gamma,z\gamma) = f(x,y,z)^\sigma$, for all $x,y,z\in\mathbf{J}$. So if z is a primitive idempotent of $\mathbf{J}$ then :

$(z\gamma)^2 = z^2\gamma$ (by 9.2.1)

$= z\gamma$ (because z is an idempotent)

Moreover $Q(z\gamma) = Q(z)^\sigma = 1/2\ \sigma = 1/2$. So $z\gamma$ satisfies the definition of primitive idempotent.

Now we observe that $E_i(z)\ \gamma = E_i(z\gamma)$ as 0 and $1/2 \in F_0$. If $W \leq E_i(z)$ is nondegenerate and has Witt index b, then so does $W\gamma$, as $Q(x)^\sigma = 0$ iff $Q(x)=0$. We recall from section 5 that the G orbit of a nondegenerate subalgebra $\mathbf{A}$ containing a primitive idempotent is determined by the isometry type of $E_{0,\mathbf{A}}(z)$ and $E_{1,\mathbf{A}}(z)$. Since isometry type is not altered by γ the claim holds for all nondegenerate subalgebras containing a primitive idempotent. There is only one orbit of nondegenerate subalgebras containing no primitive idempotents. These are described in lemma 5.2 and are the totally dark three dimensional subalgebras. Now as $f(x,x,x)^{\sigma} = 0$ iff $f(x,x,x) = 0$ we see that γ maps totally dark subalgebras into totally dark subalgebras , completing the argument for the nondegenerate orbits.

Recall from section 6 that the amber and tangerine subalgebras are defined in terms of the invariant $\mu(R(\mathbf{A}))$. Recall if x is a singular point of idΘ, then $\mu(x)$ is the Q-radical of $x\Delta$. Clearly we have $\mu(x\gamma) = \mu(x)\gamma$ and the claim follows for amber, tangerine and scarlet subalgebras.

Corollary 9.5: If $\mathbf{A}$ is a good subalgebra then $N_{\Gamma}(\mathbf{A})/N_{G}(\mathbf{A}) \simeq \Gamma/G$.

Proof: By proposition 9.4 the Γ and the G orbit of $\mathbf{A}$ coincide. So the claim follows from a Frattini argument.

Lemma 9.6: If $\mathfrak{D}$ is a 27- or twisted 27-decomposition, then so is $\mathfrak{D}\gamma$, where $\gamma \in \Gamma \backslash G$.

Proof: First we assume that $\mathfrak{D}$ is a 27-decomposition. Then $\mathfrak{D}\gamma$ is a decomposition of $\mathbf{J}$ into 27 distinct dark weight spaces of N^{γ}, where N is the 27 group giving rise to $\mathfrak{D}$. So $C_{\mathbf{J}}(N^{\gamma}) = \langle id \rangle$ and hence by lemma 7.10.4 the weight spaces of N^{γ} form a 27-decomposition of $\mathbf{J}$.

Now let $\mathfrak{D}$ be a twisted 27-decomposition. Then we extend the field so that $\mathfrak{D}^K$ is a 27-decomposition of $\mathbf{J}^K$. Now by the above $\mathfrak{D}^K\gamma$ is also a 27-decomposition of $\mathbf{J}^K$ and the claim follows.

Definition: A subgroup L of Γ is ***brilliant*** if L stabilizes a member of Υ_i where $i \in \{1,2,3,5,6,9,10,12\}$.

Lemma 9.7: Let L be a brilliant subgroup of Γ. Then L stabilizes a good subalgebra.

Proof: When L stabilizes a singular subspace S, then L stabilizes $\mu(S)$ and the claim follows from Proposition 6.4. When L stabilizes a point or L stabilizes a member of $\mathcal{V}_{10}$, $\mathcal{V}_{12}$ or $\mathcal{V}_9$, then the proofs of section 6 carry over verbatim.

SUBALGEBRA THEOREM FOR Γ: If L is a subgroup of Γ and L stabilizes a proper nontrivial subalgebra, then L stabilizes a good subalgebra.

Proof: If L stabilizes a nondegenerate subalgebra, the proof is identical to that of 5.14. If L stabilizes a brilliant subalgebra then by 4.9.4 and [As2] Thm1 L is a brilliant subgroup of $E_6(F)$ extended by diagonal and field automorphisms. So then L is a brilliant subgroup of Γ and the claim follows from lemma 9.7.

STRUCTURE THEOREM FOR Γ: Let L be a subgroup of Γ and assume that $F^*(L\cap G)$ is not a simple group. Assume also that $L\cap G$ is a closed subgroup of G. Then one of the following holds:

1. L stabilizes a proper nontrivial subalgebra.
2. L stabilizes a 27-decomposition.
3. L stabilizes a twisted 27-decomposition.

Proof: $F^*(L \cap G)$ is normal in L. If the unipotent radical of $F^*(L\cap G) \neq 0$, then by Borel Tits L is contained in a maximal parabolic subgroup of Γ and hence L stabilizes an amber or maximal tangerine subalgebra. If the unipotent radical of $F^*(L\cap G) = 0$ and $F^*(L\cap G)$ contains a characteristic abelian semisimple subgroup N, then either L stabilizes $C_J(N^2)$ which is a proper nontrivial L - invariant subalgebra of J or $|N| = 3^3$ and N is an exotic 27-subgroup. In this case L stabilizes a 27- or twisted 27-decomposition of J by lemma 9.6.

Now if $F^*(L\cap G)$ is a direct product of nonabelian simple groups, theorem 8.1 still holds when G is replaced by Γ (see [As4]), i.e., we know that L is either brilliant, stabilizes a member of $\mathfrak{U}_6$ or $\mathfrak{U}_9$, or stabilizes a 3-decomposition. Now lemma 9.7 handles the first case. The proof of proposition 8.2 goes through verbatim forcing an L invariant subalgebra, handling the second case. If L stabilizes a 3-decomposition then L stabilizes the space P generated by the projections of id onto the summands of the 3-decomposition. Then by lemma 8.3, either $C_{L\cap G}(P) \neq 0$ and hence $C_{\mathbf{J}}(C_{L\cap G}(P))$ is a proper L -invariant subalgebra or $L\cap G$ contains nontrivial normal solvable subgroups. In the latter case the claim follows from the arguments given in the first paragraph of this proof.

SECTION 10: MORE ABOUT SUBALGEBRAS

In this section we assume that L is a simple group. Let $\overline{F}$ denote the algebraic closure of F.

Definitions: Let $A,B < \mathbf{J}$. Then $S(A,B) := < ab : a \in A \text{ and } b \in B >$ and

$$S^{\#}(A,B) := < a\#b : a \in A \text{ and } b \in B >.$$

Lemma 10.1 : Let A,B,C be L-submodules of $\mathbf{J}$. The following are true.

1. **$S(A,B)$ and $S^{\#}(A,B)$ are L- submodules of $\mathbf{J}$.**
2. **If $A = B$ then $S(A,A) = S^2(A)$ and $S^{\#}(A,A) = S^{\#2}(A)$.**
3. **If A , B are subalgebras then $S(A,B) \leq A+B$ iff $A+B$ is a subalgebra of $\mathbf{J}$ iff $S^{\#}(A,B) \leq A+B$.**
4. **$S^{\#}(A,B)$ and $S(A,B)$ are homomorphic images of $A\otimes B$. Moreover $S^2(A)$ and $S^{\#2}(A)$ are homomorphic images of the symmetric square of the module A.**

5. If $A \otimes B$ is a semisimple L-modules then so are $S^2(A,B)$ and $S^{\#2}(A,B)$.
6. If A and B are subalgebras and either $S^{\#}(A,B)$ or $S(A,B)$ has a one dimensional composition factor which is not <id> and $A \otimes B$ is a semisimple L-module, then $\dim(C_{\mathbf{J}}(L)) \geq 2$, and hence $N_{\Gamma}(L)$ is contained in a subalgebra stabilizer.
7. $S(A+B,C) = S(A,C) + S(B,C)$ and $S^{\#}(A+B,C) = S^{\#}(A,C) + S^{\#}(B,C)$.
8. If $S^{\#2}(A)$, $S^{\#2}(B)$, $S^{\#}(A,B) <$ id+A+B, then id+A+B is an L-invariant subalgebra.
9. If $S^2(A)$, $S^2(B)$ and $S(A,B) <$ id+A+B, then id+A+B is an L-invariant subalgebra.

Proof: Part 1 is clear because both # and . are Γ invariant multiplications. Part 2 is obvious. Parts 3, 8 and 9 are also clear given lemma 2.6. For Part 4 notice that the map Φ: $A \times B \mapsto S^{\#}(A,B)$ resp. $S(A,B)$ defined by $\Phi(a,b) = a\#b$ resp ab is bilinear and L-equivariant. Hence, by the universality of the tensor product, Φ must factor though $A \otimes B$ as an L-map. Using the commutativity of $\mathbf{J}$ we see that Φ: $A \times A \mapsto S^{\#2}(A)$ resp. $S^2(A)$ factors through $A \otimes A/ I$ where I is the ideal generated by the commutators. This shows part 4. Parts 5 and 6 follow form Part 4. Part 7 is clear because the multiplications (. and #) on $\mathbf{J}$ are bilinear.

Definition: We call an L-module *small* if its F-dimension is less than or equal to 26.

Lemma 10.2: Let $L \leq G$. If $\{A_i : i \in \{1,...,m\}\}$ are the small L-modules and for all $i,j \in \{1,...,m\}$, $A_i \otimes A_j$ is a semisimple L-module then the socle of $\mathbf{J}$ is an L-invariant subalgebra. Moreover the socle of $\mathbf{J}$ is a semisimple L - module.

Proof: By assumption L is a subgroup of G. Let $U := \bigoplus_{i=1,..,r} U_i$ be the socle of J; notice that the set of U_i's is a subset of the A_i's. Now $S^2(U) = \sum_{i,j} S(U_i,U_j)$ by part 7 of lemma 10.1.

By part 5 of lemma 10.1 and our assumptions on the U_i's we conclude that $S(U_i,U_j)$ is a semisimple L-module contained in J. So $S(U_i,U_j)$ is contained in U for all $i,j \in \{1,...,r\}$. Hence $S^2(U) \subset U$ and from Lemma 4.5 we can conclude that U is a subalgebra of J. Now the claim follows because U is semisimple.

Lemma 10.3: If F is algebraically closed and $L \leq G$, then $id\Theta$ does not have a composition factor of dimension 24 or 25.

Proof: Assume otherwise. Then as J is a selfdual L-module, $id\Theta$ must contain a one or two dimensional L-module U. Because L is assumed to be a simple group, U must be a trivial L-module and hence L centralizes a point of J. But $id\Theta$ regarded as a $C_G(x)$-module, $x \in id\Theta$, does not not have composition factors of dimension greater than 16 (see orbit on points lemma and subalgebra theorem). So as $L < C_G(x)$ for some $x \in id\Theta$, $id\Theta$ regarded as an L-module cannot have a composition factor of dimension greater than 16.

Lemma 10.4: Let U be an L-submodule of $id\Theta$. If $S^{\#2}(U) \leq \langle id \rangle$, then U is singular.

Proof: By lemma 3.4 it suffices to show that every $u \in U$ is singular. Now recall from lemma 4.6 that $U\Theta^{\perp} = S^{\#2}(U) \leq \langle id \rangle$. So $U\Theta \geq id^{\perp} = id\Theta > U$ and hence, by definition, U is brilliant.

If $u \in U$ is nonsingular then by lemma 4.2 u is G- conjugate to x_1+x_2' or $x_1+\alpha x_6'$. In the first case lemmas 4.3 and 4.7 show that $u\#u \in id\Theta$ contradicting our assumption.

In the second case $u\#u$ is singular. This can't happen because $<id>$ does not contain singular points. So every point in U is singular, hence the claim.

Lemma 10.5: Let U be an L-submodule of $id\Theta$. Then $S^{\#2}(U) < id\Theta$ iff U is Q-singular.

Proof: Assume $S^{\#2}(U) \leq id\Theta$. We must show that U is Q-singular. By Lemma 3.3.1 and 2 it suffices to show that for all $u \in U$ we have $P_{id}(u) = 0$. So it suffices to show that $id \in U\Theta$. Now by lemma 4.6 we have $id\Theta > S^{\#2}(U) = U\Theta^{\perp}$. So $U\Theta > id\Theta^{\perp} = id$ and the first half of the claim follows.

Now if U is Q-singular, then for all $u,v \in U$ we have $0 = Q(u+v) = Q(u)+Q(v) + 2(u,v) = -2f(id,u,v) = -(u\#v,id)$; the last two equal signs follow from lemma 3.3 and the definition of $\#$ respectively. So the second half of the claim holds as u,v were arbitrarily chosen.

Lemma 10.6: If U is the unique L-submodule of $id\Theta$ in its quasiequivalence class, then either U has an L invariant complement in $id\Theta$ or U is Q-singular and $id\Theta/U^{\perp} \simeq U^*$ (the dual module of U). In particular if $\dim(U) \geq 14$ or U is the unique $\dim(U)$ dimensional composition factor of the L-module $id\Theta$, then U has an L invariant complement. In any event $U^{\perp}$ is always L invariant.

Proof: These are well known facts about self dual modules. Recall that $id\Theta$ is self dual as Q is a nondegenerate G-invariant quadratic form on $id\Theta$.

Lemma 10.7: Let U be a three dimensional irreducible L submodule of J. Suppose that $\dim(S^{\#2}(U)) > 3$, U is brilliant and contains no singular points and that $S^{\#2}(U)$ is not brilliant. Then $S^{\#2}(U)$ is a member of $\mathfrak{U}_6$.

Proof: Let $\{u_i: 1 \leq i \leq 3\}$ be a basis for U. First we observe that $S^{\#2}(U)$ is generated by $\mathfrak{B} := \{ (u_i+u_j)^{\#2} : 1\leq i\leq j\leq 3 \}$ and that by lemma 4.4 and our hypothesis on U we have that $u^{\#2}$ must be singular for all $0 \neq u \in U$. Next we observe that $\mathfrak{B}$ contains a special triple, otherwise $<\mathfrak{B}> = S^{\#2}(U)$ is brilliant.

Next we find a basis $\mathcal{S} = \{s_i : 1\leq i\leq 3\}$ of U such that $\{ s_i^{\#2} \}$ is a special triple. This can be achieved as follows. Pick three points in $\mathfrak{B}$ which form a special triple, say $\{x_i^{\#2}\}$. Now if $\{x_i\}$ is a basis of U we are done. So suppose otherwise. Pick y $\in U \setminus <x_i : 1\leq i\leq 3>$ then for all $\alpha \in F^*$ $\{ x_1, x_2, x_3+\alpha y\}$ is a basis for U. Now $f(x_1^{\#2}, x_2^{\#2}, (x_3+\alpha y)^{\#2})$ is a polynomial in α of degree 2 with nonzero constant term. So as $|F| > 3$ there exists an $\alpha \neq 0$ such that $f(x_1^{\#2}, x_2^{\#2}, (x_3+\alpha y)^{\#2}) \neq 0$. So we have the desired basis for a proper choice of α.

Next we want to show that $s_i \# s_j \in s_i^{\#2}\Delta \cap s_j^{\#2}\Delta$ for all $i \neq j$. For this it suffices to prove that:

$$(*)\ (s_i\#s_j)\#s_i^{\#2} = 0 \text{ for } i\neq j.$$

Now because the # - square of a singular point is zero and the #-square of an element of U is singular the following holds:

$$0 = ((s_i+\alpha\, s_j)^{\#2})^{\#2} = (s_i^{\#2} + 2\alpha\,(s_i\#s_j) + \alpha^2\, s_j^{\#2})^{\#2}$$
$$= 2\alpha[\, 2(s_i\#s_j)\#s_i^{\#2}) + \alpha(s_i^{\#2}\#\, s_j^{\#2} + 2\,(s_i\#s_j)^{\#2}) + 2\,\alpha^2(s_i\#s_j)\#s_j^{\#2}\,] \text{ for all}$$
$\alpha \in F$.

Now $(*)$ follows from the following easy fact: If $\{v_i : 1\leq i\leq 3\}$ are elements of a vectorspace over a field of characteristic $\neq 2$ and for all $0\neq\alpha \in F$ we have $0 = v_1 + \alpha\, v_2 + \alpha^2\, v_3$ and F contains more than 4 nonzero elements, then for all i $v_i = 0$. As U is brilliant and contains no singular points $0 \neq (\alpha s_i+\beta s_j)^{\#2} = \alpha^2 s_i^{\#2} + \beta^2 s_j^{\#2} + \alpha\beta\, s_i\#s_j$ and singular. Now [As3] 4.3.1 states that if $\{v_1,v_2,v_3\}$ is a special triple, and $x = v_1+ v_2+ w$ is singular, where $w\in v_1\Delta\cap v_2\Delta$. Then $P_{v_3}(w) = -1$, and in particular $w \neq 0$. So if $\alpha,\beta \neq 0$, then we can apply [As3] 4.3.1 we to get $0\neq s_i\#s_j$. So $(s_1+s_2+s_3)^{\#2}$ is singular and not contained in $<s_1^{\#2}, s_2^{\#2}, s_3^{\#2}>$. Then by

[As3] 4.4.1 and 4.5.2 we have that $\{ s_1^{\#2}, s_2^{\#2}, s_3^{\#2}, (s_1+s_2+s_3)^{\#2}\}$ is a special 4 tuple. So $S^{\#2}(U)$ satisfies Aschbacher's definition of $\mathfrak{U}_6$ in Section 1 of [As3]. Hence the claim.

Lemma 10.8: Let U be an irreducible L-submodule of $\mathbf{J}$. If U^* is not a nontrivial homomorphic image of $\mathrm{Sym}^2(U)$, then U is brilliant. Furthermore, if U is self dual then U is brilliant if U is not a submodule or homomorphic image of $\mathrm{Sym}^2(U)$.

Proof: First recall from [As5] 2.1.1 that the space of symmetric bilinear forms on U is isomorphic to the dual of $\mathrm{Sym}^2(U)$. The map Ψ: U $\mapsto$ Symmetric Bilinear forms of U given by $u\Psi = f(u,\text{-},\text{-})$ is an L equivariant map. Now Ψ is either injective or trivial as Ψ is L equivariant and U is irreducible. So $\Psi^*: \mathrm{Sym}^2(U) \mapsto U^*$ is either surjective or trivial. By our assumption we force Ψ^* to be trivial and hence Ψ is trivial. Thus f is trivial on U and hence U is brilliant.

In the next series of lemmas we will investigate the normalizers of simple subgroups that stabilize subalgebras.

Lemma 10.9: Let L be a simple subgroup of $N_G(\mathbf{A})$ where $\mathbf{A}$ is a good subalgebra and suppose that $C_{\mathbf{J}}(L) = \langle id \rangle$. Then $\mathbf{A} \in \mathfrak{U}_6, \mathfrak{V}_9, \mathfrak{U}_9, \mathfrak{V}_{15}, \mathfrak{V}_3, \mathfrak{V}_6$ and L is isomorphic to a subgroup of $PSL_2(F)$, $SL_3(F)$, $SU_3(F)$, $Sp_6(F)$, $SL_3(F)$, $Sp_6(F)$ respectively.

Proof: By assumption on L, $C_G(\mathbf{A}) \cap L = \{e\}$. So an isomorphic image of L is contained in $N_G(\mathbf{A})/C_G(\mathbf{A})$. The claim follows from the subalgebra theorem in section 6 by inspection.

Lemma 10.10: If L stabilizes a subalgebra of type $\mathfrak{U}_6$, then $N_\Gamma(L)$ stabilizes a subalgebra.

Proof: Recall from lemma 5.13 that $N_G(\mathbf{A}) \simeq PSO_3(F) \times G_2(F)$, and that $G_2(F)$ centralizes $\mathbf{A}$. Let L_i be the projection of L into the i'th factor of $N_G(\mathbf{A})$.

First we investigate how $\mathbf{J}$ can break up as an $L_1 \oplus L_2$ - module.

Now $\mathbf{A}^{\perp}$ is the tensor product of the seven dimensional irreducible $G_2(F)$-module with the three dimensional irreducible $PSL_2(F)$ module $M(2\lambda_1)$ (see [As3] 5.7). Moreover as a $G_2(F)$ module $\mathbf{A}^{\perp}$ can be decomposed into $U \perp U\alpha \perp U\alpha^2$, where $\alpha \in N_G(G_2(F))$, such that $S^{\#2}(U\alpha^i)$ is a primitive idempotent contained in $\mathbf{A}$. Now by [As7] L_2 can act either irreducibly on U, in which case $L \simeq PSL(F_0)$, $F_0 \leq F$, and $U \simeq M(6\lambda_1)$, or L_2 fixes a Q-nondegenerate point x of U and $U \simeq \langle x\rangle \oplus W\backslash V$ as an L_2-module where W,V are irreducible three dimensional L_2-modules. We denote the two L_2 actions by L_2^i $i \in \{1,2\}$ respectively .

Now if $L \leq L_1 \oplus L_2^2$ then as an L-module $\mathbf{J} \simeq \mathbf{A} \oplus \langle xL_1\rangle \oplus V \otimes M(3)\backslash W \otimes M(3)$, where $M(3)$ denotes some three dimensional irreducible L-module. As L is isomorphic to a simple subgroup of L_1, L is isomorphic to Alt_5 or $PSL_2(F_0)$, where F_0 is a subfield of F. Now when $L \simeq PSL_2(F_0)$ then $V \otimes M(3)$ is either irreducible or isomorphic to $M(4\lambda_1) \oplus M(2\lambda_2) \oplus M(0)$. When $L \simeq Alt_5$ then $V \otimes M(3)$ isomorphic to $M(5) \oplus M(3) \oplus M(1)$ or $M(5) \oplus M(4)$, where M(i) denotes an irreducible Alt_5 module of dimension i .

Now either $C_{\mathbf{J}}(L) \neq \langle id\rangle$ or $\langle xL_1\rangle$ is the unique three dimensional irreducible L submodule and is hence an $N_\Gamma(L)$ invariant submodule of $\mathbf{J}$. Now $S^{\#2}(\langle xL_1\rangle) = \mathbf{A}$ as $x\#x \in \mathbf{A}$ is a primitive idempotent and as an L-module $\mathbf{A} \simeq M(0) \oplus W$, W a five dimensional irreducible L-module. So then $\mathbf{A}$ is also $N_\Gamma(L)$ invariant.

Now suppose $L \leq L_1 \oplus L_2^1$. Then $L \simeq PSL_2(F_0)$ and as an L - module $\mathbf{J} \simeq \mathbf{A} \oplus M(2\lambda_1) \otimes M(6\lambda_1)^{\delta}$ or $\mathbf{A} \oplus M(4\lambda_1) \oplus M(6\lambda_1) \oplus M(8\lambda_1)$ unless p=7, a case we treat separately. In case $\mathbf{A}^{\perp}$ is irreducible $\mathbf{A}$ is the unique L-invariant module isomorphic to $\mathbf{A}$. So then $\mathbf{A}$ is $N_\Gamma(L)$ -invariant. In case $\mathbf{A}^{\perp}$ is not irreducible, $M(6\lambda_1)$ is $N_\Gamma(L)$-

invariant. Now by lemmas 10.8 and 13.1.3 $M(6\lambda_1)$ is a brilliant subspace of J. If $M(6\lambda_1)$ contains singular points, then by [As2] Thm1 $N_\Gamma(M(6\lambda_1))$ is a brilliant subgroup of Γ, and hence the result follows from 9.7. In case $M(6\lambda_1)$ does not contain singular points, the high weight vector x wrt some Cartan subgroup of L of $M(6\lambda_1)$ must be brilliant and nonsingular. Hence, by Lemma 4.3, its # square is nonzero. So $0 \neq x\#x$ is a high weight vector in J of high weight $12\lambda_1$. So then $\mathrm{id}\Theta$ contains a nontrivial homomorphic image of $M(12\lambda_1)$ or $M(\lambda_1)^\delta \otimes M(\lambda_1)/M(8\lambda_1)$ (= $W(12\lambda_1)$ the Weyl module) or $M(2\lambda_1) \oplus M(0)/M(8\lambda_1)$ depending on whether $p \neq 11$, $p=11$ and $|F| \neq 11$, $|F|=11$ respectively, or $M(\lambda_1)^\delta$; a contradiction to the decomposition of J that we assumed at the beginning of this paragraph.

Now assume $p=7$, $L \leq L_1 \oplus L_2^1$ and $\mathfrak{A}^\perp$ is not irreducible. Then $\mathfrak{A}^\perp \simeq M(2\lambda_1) \otimes M(6\lambda_1)$ as an L-module and hence $J \simeq \mathfrak{A} \oplus M(4\lambda_1)/M(\lambda_1{}^\delta) \otimes M(\lambda_1)/M(4\lambda_1) \oplus M(6\lambda_1)$ (i.e., J contains a nonsplit indecomposable submodule) or $\mathfrak{A} \oplus M(4\lambda_1)/M(2\lambda_1) \oplus M(0)/M(4\lambda_1) \oplus M(6\lambda_1)$ when $|F| \neq 7$, $|F|=7$ respectively. So in the first case $\mathfrak{A}$ is $N_\Gamma(L)$ invariant. In the second case we don't know whether J is completely reducible or contains the projective indecomposable $M(4\lambda_1)/M(2\lambda_1) \oplus M(0)/M(4\lambda_1)$. In case of complete reducibility, $C_J(L)$ is a proper nontrivial $N_\Gamma(L)$ invariant subalgebra see lemma 9.3. In the other case $\mathfrak{A}$ is $N_\Gamma(L)$ invariant.

Lemma 10.11: If L stabilizes a subalgebra of type $\mathfrak{T}_9, \mathfrak{U}_9$ or $\mathfrak{T}_{15}$, then $N_\Gamma(L)$ stabilizes a proper nontrivial subalgebra.

Proof: Extend the field F to $\overline{F}$ if necessary to contain a cube root of unity. Then the subalgebra stabilizer has a nontrivial center. Now let σ be an automorphism of $\overline{\Gamma}$ whose fixed points are Γ. Now let $\tilde{\Gamma} = <\overline{\Gamma},\sigma>$. Observe that $0 \neq C_{\overline{G}}(L)$, and hence $F^*(N_{\overline{G}}(L))$ is not a simple group. So by the structure theorem $N_{\tilde{\Gamma}}(L)$ fixes a subalgebra $\bar{\mathfrak{A}}$ of $\bar{\mathfrak{J}}$. Next we observe that $\sigma \in C_{\tilde{\Gamma}}(N_{\Gamma}(L))$. Thus $\bar{\mathfrak{A}}$ is $\sigma \oplus N_{\Gamma}(L)$ invariant so by 25.7.2 of [As9] there exists an $FN_{\Gamma}(L)$ module $\mathfrak{A}$ such that $\dim_F(\mathfrak{A}) = \dim_{\bar{F}}(\bar{\mathfrak{A}})$. Moreover $\mathfrak{A}$ is the set of fixed points in $\bar{\mathfrak{A}}$ of σ, and hence $\mathfrak{A}$ is a subalgebra by lemma 9.2.1. This is the claim.

Lemma 10.12: If L stabilizes a subalgebra of type $\mathcal{T}_3$ then $N_{\Gamma}(L)$ stabilizes a proper nontrivial subalgebra.

Proof: WLOG let $U = \langle x_1,x_2,x_3 \rangle$ be an L invariant member of $\mathcal{T}_3$. Then a straightforward computation shows: id $<$ $\langle id,U \rangle$ $<$ $\langle id,U\Delta \rangle$ $<$ $U\Delta^{\perp}$ $<$ $U^{\perp}$ $<$ $\mathfrak{J}$ is an L-invariant flag of subspaces of dimensions 1,4,10,18,24 resp. Moreover, it's easy to check that the first four spaces in the flag are indeed subalgebras. If all the $N_{\Gamma}(L)$ conjugates of U are contained in $U\Delta^{\perp}$, then $N_{\Gamma}(L)$ stabilizes the subalgebra of $U\Delta^{\perp}$ which is generated by the $N_{\Gamma}(L)$ conjugates of U.

We assume for the moment that L stabilizes a G-conjugate $\bar{U}$ of U such that $\bar{U}$ is a complement to $U^{\perp}$ in $\mathfrak{J}$. Then $\{ x'_4+a_4, x'_5+a_5, x'_6+a_6\colon a_i \in U^{\perp}\}$ is a basis for $\bar{U}$, for some proper choice of a_i's. By corollary 6.5.3 we have $S^{\#}(U,U^{\perp}) = U\Delta$. Let $W = \langle x'_4,x'_5,x'_6 \rangle$, then we compute that $S^{\#}(U,W) = \langle x_{34},x_{25},x_{16},x_{24},x_{35},x_{14},x_{36},x_{15},x_{26} \rangle$ is a subalgebra of type $\mathcal{T}_9$. Another calculation using 6.5.3 shows that $S^{\#}(U,\bar{U}) \leq U\Delta^{\perp}$, and that $S^{\#}(U,\bar{U})$ has trivial Q-radical. Also, we can compute that $U\Delta$ is the Q-radical of $U\Delta^{\perp}$. So $U\Delta^{\perp} = U\Delta \oplus S^{\#}(U,\bar{U})$ as an L-module. Similarly, we obtain that $\bar{U}\Delta^{\perp} = \bar{U}\Delta \oplus S^{\#}(U,\bar{U})$ as an L-module. Another calculation shows that $U\Delta \cap \bar{U}\Delta = 0$. So $S^{\#}(U,\bar{U}) = U\Delta^{\perp} \cap \bar{U}\Delta^{\perp}$, and is hence an L-invariant nondegenerate subalgebra. So L stabilizes a member of $\mathcal{T}_9$ or $\mathcal{U}_9$. So now lemma 10.11 applies and hence $N_{\Gamma}(L)$ stabilizes some proper nontrivial subalgebra .

Now let V denote the subspace of $\mathcal{J}$ which is generated by the $N_\Gamma(L)$ conjugates of U. By the previous paragraph we may assume that every $N_\Gamma(L)$ conjugate of U lies in $U^\perp$. Thus as V is a direct sum of conjugates of U, we conclude that V is totally Q-singular as $U \le Rad(V)$. So as V is a proper $N_\Gamma(L)$-invariant subspace of $\mathcal{J}$, we will show that either V is a subalgebra or else that there exists an L-submodule $\overline{U}$ as in the previous paragraph.

If V is brilliant, then $N_\Gamma(L)$ is a brilliant subgroup of E_6, by [As2] Thm1, as V is generated by singular points. If V is a direct sum of 2 members of $\mathcal{V}_3$ then V is brilliant and contains singular points. So V is a direct sum of 3 or 4 $N_\Gamma(L)$ conjugates. The latter case is impossible as then $id\Theta \simeq V\backslash M(0) \oplus M(0) \backslash V^*$ has eight composition factors Galois conjugate or dual to U and two composition factors of dimension 1. Then by lemma 12.2 an element of order three in L has trace 2: a contradiction to lemma 12.1.

So $V = U_0 \oplus U_1 \oplus U_2$ with $U = U_0$ and U_i $N_\Gamma(L)$ conjugate to U. Now as $U_i \le U_j^\perp$ for all i,j we have, by corollary 6.5.3, that $S^\#(U_i,U_j) \le U_i\Delta \cap U_j\Delta$ for all i,j. Now as $S^{\#2}(U_i\Delta) = U_i$ and $U_i \cap U_j = 0$ we get that $S^{\#2}(S^\#(U_i,U_j)) = 0$. So by lemma 3.4 $S^\#(U_i,U_j)$ is a singular subspace of $U_i\Delta \cap U_j\Delta$. As V is not brilliant, we may assume that U_i is not contained in $U_j\Delta$ and hence $S^\#(U_i,U_j) \neq 0$, and $U_i \cap S^\#(U_i,U_j) = 0$. As $S^\#(U_i,U_j)$ is L-invariant and we assumed that $C_{id\Theta}(L) = 0$ it follows that $\dim(S^\#(U_i,U_j)) \ge 3$. So now $U_i \oplus S^\#(U_i,U_j)$ is at least six dimensional and singular, as $S^\#(U_i,U_j)$ is a singular subspace of $U_i\Delta$, and hence a member of $\mathcal{V}_6$. So $S^\#(U_i,U_j)$ is the intersection of two members of $\mathcal{V}_6$ and is hence a three dimensional amber subspace by [As1] 9.9, i.e., a member of $\mathcal{V}_3$.

Now V is not brilliant so there exists a vector $v \in V$ such $v = u_0+u_1+u_2$ with $u_i \in U_i$ and $T(v) \neq 0$. Thus $0 \neq T(v) = 6f(u_0,u_1,u_2) = 3\,(u_0, u_1\#u_2)$. So as $S^\#(U_1,U_2)$ is a three dimensional irreducible L-module, it is a complement to $U_0^\perp$ in

J. Moreover, $S^{\#}(U_1,U_2)$ is G conjugate to U and L-invariant. So the argument in the second paragraph of this proof applies and we are done.

Lemma 10.13: If L stabilizes a subalgebra of type $\mathcal{V}_6$, then $N_\Gamma(L)$ stabilizes a subalgebra.

Proof: Let U be a maximal tangerine subspace of idΘ. Then by corollary 6.5.4 $U^\perp$ is a subalgebra of J and $S^{\#}(U,U^\perp) = U$. Now if every $N_\Gamma(L)$ conjugate of U is contained in $U^\perp$, then the subalgebra generated by the $N_\Gamma(L)$ conjugates of U is proper and $N_\Gamma(L)$ invariant.

So assume that $\overline{U}$ is an $N_\Gamma(L)$ conjugate of U not contained in $U^\perp$. Then by [As1] 9.9 1-3 one of the following holds:

i. $U \cap \overline{U}$ is one dimensional.

ii. $U^\perp \cap \overline{U}$ is an amber plane.

iii. $U^\perp \cap \overline{U} = 0$.

If i. holds $C_{id\Theta}(L) \neq 0$ and the claim holds. If ii. holds the claim follows from lemma 10.12. If iii. holds $J = U \oplus (U^\perp \cap \overline{U}^\perp) \oplus \overline{U}$ and $(U^\perp \cap \overline{U}^\perp)$ is an L invariant nondegenerate subalgebra of dimension 15. By inspecting the conjugacy classes of nondegenerate subalgebras, we see that $(U^\perp \cap \overline{U}^\perp)$ is a $\mathcal{V}_{15}$ type subalgebra. Now lemma 10.11 proves the claim.

We summarize lemmas 10.10 to 10.13 in:

Proposition 10.14: If L stabilizes a subalgebra, then $N_\Gamma(L)$ stabilizes a subalgebra.

Proof: If $C_J(L) \neq \langle id \rangle$ then lemma 9.3 gives the claim. If L stabilizes a subalgebra, then L also stabilizes a good subalgebra. In case $C_J(L) = \langle id \rangle$ Lemma 10.10 gives the possible choices of good subalgebras that L can stabilize. Lemmas 10.11 to 10.16

deal with each of the possibilities listed in lemma 10.10.

Corollary 10.15: If L stabilizes a brilliant subspace which is generated by singular points, then $N_{\Gamma}(L)$ stabilizes a subalgebra.

Proof: By [As2] Thm1 L is a brilliant subgroup of $E_6(F)$. Now by proposition 6.9 L stabilizes a subalgebra. So the claim follows from proposition 10.14.

Corollary 10.16 : If K is an extension field of F and L stabilizes a subalgebra of J^K, then $N_{\Gamma}(L)$ stabilizes a subalgebra of J.

Proof: Let σ be the automorphism of Γ^K whose fixed points are Γ. Let $\tilde{\Gamma} = <\Gamma^K,\sigma>$. Then $\sigma \in C_{\tilde{\Gamma}}(N_{\Gamma}(L))$. Now by proposition 10.14 $N_{\tilde{\Gamma}}(L)$ stabilizes a subalgebra A^K. So $\sigma \times N_{\Gamma}(L)$ stabilizes A^K. So there exists an $FN_{\Gamma}(L)$ invariant submodule A of J by [As9] 25.7.2. $A = C_{A^K}(\sigma)$ so A is a subalgebra by lemma 9.2.1.

Lemma 10.17: If the minimal faithful representation of L over $\overline{F}$ has dimension $\geq$ 14, then L is not a subgroup of G unless L has an irreducible $\overline{F}$-representation of dimension 26.

Proof: Let ρ be any 26 dimensional F-representation of L into G. Therefore $L\rho$ centralizes a nondegenerate quadratic form, as G does. We observe first that L can only have one nontrivial composition factor. If this composition factor does not have dimension 26, L centralizes a nontrivial subspace of $id\Theta$ by lemma 10.6. In this case L centralizes a point $\neq$ id. The point centralizers are universal groups of Lie type that have nontrivial representations of degree less than 9. So because L is simple, L must also have a nontrivial representation of degree less or equal to 9: a contradiction to our hypothesis. The claim follows.

At this point we know that any closed simple subgroup $L < G < \overline{G}$ such that $N_\Gamma(L)$ is maximal in Γ does not stabilize a proper nontrivial subalgebra of $\overline{J}$. We will use this fact frequently in showing that certain subgroups of G do not give rise to maximal subgroups.

CHAPTER II : ALMOST SIMPLE SUBGROUPS OF LIE-TYPE DEFINED OVER FIELDS WHOSE CHARACTERISTIC EQUALS char(F)

Throughout this chapter L will denote a simple adjoint group of Lie type of char(F) = p (p=0 allowed). We regard J as an L-module and idΘ and <id> as L-submodules of J. We assume throughout this chapter that F is a splitting field for L and for us an L- module is always an FL-module unless otherwise stated. The arguments in this chapter will be of a more module and representation theoretic nature.

SECTION 11: INITIAL REDUCTIONS

Lemma 11.1 : If L is a simple adjoint group of Lie type in characteristic p , then L embeds into G only if one of the following holds:

1. L is of type ${}^3D_4$.
2. L is of type G_2.
3. L is of type A_i or ${}^2A_i$, $i \in \{1,2\}$.
4. L is of type F_4.

Proof: If L is an orthogonal group $P\Omega_n$ (for us it is irrelevant what the type of L is), then L contains a subgroup 2^{n-1}: Alt_n, when n is odd, and a subgroup of type 2^{n-2}:Alt_{n-1} when n is even. To see this let $\{b_i : 1 \leq i \leq n\}$ be an orthonormal basis of an odd dimensional orthogonal F-space, V. The group generated by the reflections r_{b_i} across the hyperplanes orthogonal to b_i is an elementary abelian 2-group of 2-rank n. Its normalizer in O(V) contains the permutation matrices. Hence the claim for n odd. When n is even any orthogonal F-space will contain a nondegenerate hyperplane and the same construction works. So when $n \geq 5$ L contains an elementary abelian 2 group E and 5 divides $|A_L(E)|$. So in this case L is not a subgroup of G by lemma 7.11.4.

So now we have eliminated all groups containing $P\Omega_n$ $n \geq 5$. This leaves us with the groups listed in the statement of the lemma and the groups of type C_i $i \in \{3,4,5\}$ and ${}^2E_6$. The minimal dimensional faithful representations of the groups not listed in the statement of the lemma have dimension $\geq$ 14 and no 26 dimensional irreducible representations (see [As6]), so these are out by lemma 10.17.

Lemma 11.2: If L is of type G_2, then L centralizes a proper nontrivial subalgebra unless char(F) = 7 and L acts irreducibly on $\mathrm{id}\Theta$.

Proof: In characteristic 7 L possesses an irreducible 26 dimensional representation. For now we will exclude this case from our considerations and treat it separately in Section 14. The other small L-modules have dimensions 1,7, and 14 and have corresponding high weights $0,\lambda_1,\lambda_2$ respectively, see [As6]. Moreover, $\dim(M(2\lambda_1)) =$ 27 if $p \neq 7$ and 26 if $p = 7$. So either $\mathrm{id}\Theta$ contains seven dimensional submodules or else $C_{\mathrm{id}\Theta}(L) \neq 0$ and L centralizes a proper nontrivial subalgebra. Now using [G1] we see that $\mathrm{Sym}^2(M(\lambda_1)) = M(0) \oplus M(2\lambda_1)$ if $p \neq 7$ and is $M(0)/M(2\lambda_1)/M(0)$ when $p = 7$. So if U is a seven dimensional irreducible L submodule of $\mathrm{id}\Theta$ then $\dim(S^{\#2}(U)) \leq 1$. So either $C_{\mathrm{id}\Theta}(L) \neq 0$ or lemma 10.4 applies and U is singular. The latter is impossible as $\mathrm{id}\Theta$ contains no seven dimensional singular subspaces. This is the claim.

Lemma 11.3: If L is of type ${}^3D_4$, then L stabilizes a subalgebra.

Proof: As F is a splitting field for L, we see from [As6] that the only small L modules have dimension 8. From [Jo1] we know that $H^1(L,V) = 0$ for every 8 dimensional L module . So $\dim(C_{id\Theta}(L)) \geq 2$ and the claim follows from lemma 7.1 .

SECTION 12: GROUPS OF LIE TYPE A_2 AND ${}^2A_2$

In this Section we assume that L is of type A_2 or ${}^2A_2$. Here we will start to use our knowledge about the possible character values of involutions and elements of order three in G acting on $id\Theta$. We record this in a lemma.

Lemma 12.1: Let $g \in G$ be an involution, then $tr|_{d\Theta}(g) \in \{-6,2\}$. If $h \in G$ is an element of order 3, then $tr|_{d\Theta}(h) \in \{-1,8\}$.
Proof: The first part follows from [As3] 6.5 and an easy calculation. The second part follows from [As3] 8.2 and an easy calculation.

In this section we will prove that L stabilizes a subalgebra. Combined with corollary 10.16 this will show that an almost simple group of type A_2 or ${}^2A_2$ is never maximal in Γ. From [As6] we get the set of small irreducible modules. These will be listed in the table below. Let g be the involution that acts as the diagonal matrix (-1,-1,1) on the natural L-module. Let h be the element of order three that acts on the natural L-module as the diagonal matrix $(\omega,\omega^{-1},1)$.

Lemma 12.2: The following table lists all small nontrivial irreducible FL-modules up to conjugacy and duality:

	$M(\lambda_1)$	$M(2\lambda_1)$	$M(\lambda_1+\lambda_2)$	$M(\lambda_1)\otimes M(\lambda_1)^\delta$	$M(3\lambda_1)$	$M(2\lambda_1+\lambda_2)$	$M(4\lambda_1)$
Dim(M)	3	6	8	9	10	15	15
tr(g)	-1	2	0	1	-2		
tr(h)	0	0	-1	0	1		

	$M(5\lambda_1)$	$M(2\lambda_1+2\lambda_2)$	$M(3\lambda_1+\lambda_2)$	$M(\lambda_1)^\delta \otimes M(\lambda_1+\lambda_2)$	$M(\lambda_1)\otimes M(2\lambda_1)^\delta$
dim(M)	21	27 (19)	24 (18)	24	18

The character values of g and h are the same on the dual modules and Galois conjugates. The numbers in parentheses denote the value in case $p=5$, and δ denotes a nontrivial field automorphism.

Proof: That the modules listed in the table together with their duals are the complete set of submodules is a consequence of [As6]. The character values are easily computed as follows:

The module $M(\lambda_1)$ is the natural module so the traces of g,h are easily computed. Now $tr(x)|_{A\otimes B} = tr(x)|_A \; tr(x)|_B$, allowing us to compute the traces on $M(\lambda_1)\otimes M(\lambda_1)^\delta$. It is well known that $M(\lambda_1)\otimes M(\lambda_2) = M(\lambda_1+\lambda_2) \oplus M(0)$ so the traces on $M(\lambda_1+\lambda_2)$ are easily obtained. Now in [Sel] it is shown that $M(c\lambda_1)$ is the space of homogeneous polynomials in 3 variables of degree c. From this one can easily get the traces for $c = 2$ or 3.

Lemma 12.3: If idΘ contains a submodule isomorphic to $M(2\lambda_1)$, then $N_\Gamma(L)$ is not maximal in Γ.

Proof: We first recall from [As6] that $Sym^2(M(2\lambda_1)) \simeq M(4\lambda_1) \oplus M(2\lambda_1)$ and that $M(2\lambda_1)\otimes M(2\lambda_1) \simeq M(4\lambda_1) \oplus M(2\lambda_1) \oplus M(2\lambda_1+\lambda_2)$. Further observe that $M(4\lambda_1)$ and $M(2\lambda_1+\lambda_2)$ are not self dual modules. Thus neither of these modules can be submodules of idΘ as idΘ is self dual and hence can not contain nonself dual modules of dimension greater than 13.

Now let $U \simeq M(2\lambda_1)$ be a submodule of idΘ and let $W= S^{\#2}(U)$. By the above $W \simeq U$. If $W < U$ then id+U is a subalgebra by lemma 10.1 and hence $N_\Gamma(L)$ is not maximal by corollary 10.16. By lemma 10.1 the only alternative is $W\cap U$

$= 0$ and $W \simeq U$. Now either $S^{\#}(U,W)$ or $S^{\#2}(W)$ is not contained in $id \oplus U \oplus W$ else $id \oplus U \oplus W$ is an L invariant subalgebra and $N_{\Gamma}(L)$ is not maximal. So there exists a submodule V of $S^{\#}(U,W)$ or $S^{\#2}(W)$ and $V \simeq U$ which is not contained in $id \oplus U \oplus W$. Similarly if $id \oplus U \oplus W \oplus V$ is not a subalgebra, there exists a submodule $T \simeq U$ which is not contained in $id \oplus U \oplus W \oplus V$. Now $\dim((id \oplus U \oplus W \oplus V \oplus T)^{\perp}) = 2$ and hence $\dim(C_{id\Theta}(L)) = 2$. Thus in this last case $N_{\Gamma}(L)$ stabilizes a subalgebra , completing the proof.

Lemma 12.4: If $id\Theta$ contains a submodule isomorphic to $M(\lambda_1)$, then $N_{\Gamma}(L)$ is not maximal in Γ.

Proof: Let $U \simeq M(\lambda_1)$ be a submodule of $id\Theta$. As $Sym^{\#2}(M(\lambda_1)) \simeq M(2\lambda_1)$ we have $S^{\#2}(U)$ is either trivial or isomorphic to $M(2\lambda_1)$. In the first case U is singular and we are done by corollary 10.16. The second case is handled by lemma 12.3.

Lemma 12.5: If $id\Theta$ contains a submodule isomorphic to $M(3\lambda_1)$, then $N_{\Gamma}(L)$ is not maximal in Γ.

Proof: Let $U \simeq M(3\lambda_1)$ be a submodule of $id\Theta$. First we observe that U is Q singular as U is not self dual. So $id\Theta/ U^{\perp} \simeq M(3\lambda_2)$ and $U^{\perp}/ U$ is six dimensional. Now checking traces of g and h shows that the only composition series of $U^{\perp}/ U$ that does not violate lemma 12.1 is the series consisting of trivial composition factors. Now in [As6] it is proved that $Sym^2(U)$ has neither ten nor one dimensional composition factors. From this we conclude that $S^{\#2}(U) = 0$ and hence U is singular. This contradicts that the maximal dimension of a singular subspace is 6. So in fact U can never be a submodule of $id\Theta$, establishing the claim.

Lemma 12.6: The following are true:

1. $\mathrm{Sym}^2(M(\lambda_1+\lambda_2)) \simeq M(2\lambda_1+2\lambda_2) \oplus M(\lambda_1+\lambda_2) \oplus M(0)$ when $p \neq 5$.
2. $\mathrm{Sym}^2(M(\lambda_1+\lambda_2))$ has composition factors $M(2\lambda_1+2\lambda_2)$, $M(0)$, and $M(\lambda_1+\lambda_2)$ occurring with multiplicity 2 when $p=5$.
3. $\mathrm{Sym}(M(\lambda_1)\otimes M(\lambda_1)^\delta) \simeq M(2\lambda_1)\otimes M(2\lambda_1)^\delta \oplus M(\lambda_2)\otimes M(\lambda_2)^\delta$.

Proof: Part 3 follows readily from the well known formula $\mathrm{Sym}^2(A\otimes B) \simeq \mathrm{Sym}^2(A)\otimes \mathrm{Sym}^2(B) \oplus \Lambda^2(A) \otimes\Lambda^2(B)$, see [As5] section 2. Part 1 and 2 can be deduced from [As6] once we observe that $M(0)$ and $M(2(\lambda_1+\lambda_2))$ are submodules of $\mathrm{Sym}^2(M(\lambda_1+\lambda_2))$ and that $\mathrm{Sym}^2(M(\lambda_1+\lambda_2))$ is a self dual module.

Lemma 12.7: If $\mathrm{id}\Theta$ contains a submodule isomorphic to $M(\lambda_1)\otimes M(\lambda_1)^\delta$, then $N_\Gamma(L)$ is not maximal in Γ.

Proof: Let $U\simeq M(\lambda_1)\otimes M(\lambda_1)^\delta$ be a submodule of $\mathrm{id}\Theta$ and let $W= S^{\#2}(U)$. If $\mathrm{id}\oplus U$ is a subalgebra the claim follows. So assume otherwise. Then $\dim(W) = 9$ and $U\cap W = 0$. So $\dim((U\oplus W)^\perp) = 8$. Now only the choice $(U\oplus W)^\perp \simeq M(\lambda_1+\lambda_2)$ does not lead to a contradiction of lemma 12.1. Now we see using lemma 12.6.1 and 2 that $S^{\#2}((U\oplus W)^\perp) < \mathrm{id}\oplus (U\oplus W)^\perp$ and hence L stabilizes a subalgebra. We are done by corollary 10.16.

Lemma 12.8: If $\mathrm{id}\Theta$ contains a submodule isomorphic to $M(\lambda_1+\lambda_2)$, then $N_\Gamma(L)$ is not maximal.

Proof: Let $U \simeq M(\lambda_1+\lambda_2)$ be a submodule of $\mathrm{id}\Theta$ and let $W = S^{\#2}(U)$. If $W < U \oplus \mathrm{id}$ then $U\oplus \mathrm{id}$ is an L invariant subalgebra and we are done by corollary 10.16. If $M(0) < W$ and $M(0) \neq \mathrm{id}$, then $C_{\mathrm{id}\Theta}(L) \neq 0$ and again we are done.

So we may assume that $W\simeq U$ and $W\cap U = 0$. Then $(W\oplus U)^\perp$ is a ten dimensional submodule of idΘ. Now by the previous lemmas $(W\cap U)^{\perp}$ can only contain a submodule isomorphic to $M(\lambda_1+\lambda_2)$. This forces three composition factors isomorphic to U and two trivial composition factors. Now [Jo1] shows that $H^1(L,U) = 0$. Thus $C_{id\Theta}(L) \neq 0$. This gives the claim.

Proposition 12.9: If L is of type A_2 or ${}^2A_2$, then $N_{\Gamma}(L)$ is not maximal in Γ.

Proof: L has no irreducible 26 dimensional representations so L stabilizes a submodule U of idΘ. So either U or $U^{\perp}$ has dimension less than or equal to 13. So WLOG U is isomorphic to one of the modules of dimension less than 13 listed in table 13.1. Now the lemmas 12.3, 4, 5, 7 and 8 handle each of the cases.

SECTION 13: GROUPS OF LIE TYPE A_1

Let $L = PSL_2(F)$. Let $g,h \in SL_2(F)$ be the diagonal matrices $(i,-i)$ and (ω,ω^{-1}) where i, ω are primitive fourth and third roots of unity of F, respectively.

Lemma 13.1 :

1. The small irreducible L-modules are:

   $M(2a\lambda_1)$, $2a < p$ and $2a \leq 26$

   $M(a\lambda_1)\otimes M(b\lambda_1)^{\delta}$, $\delta\neq 1$, $2 \mid (a+b)$ and $(a+1)(b+1) \leq 26$

   $M(a\lambda_1)\otimes M(b\lambda_1)^{\delta}\otimes M(c\lambda_1)^{\alpha}$, $\alpha \neq 1,\delta$, $2 \mid (a+b+c)$ and $(a+1)(b+1)(c+1) \leq 26$

   $M(\lambda_1)\otimes M(\lambda_1)^{\delta}\otimes M(\lambda_1)^{\alpha}\otimes M(\lambda_1)^{\beta}$, $\alpha \neq 1,\delta$ and $\beta \neq \delta,\alpha,1$.

2. $Tr(g)|_{M(2a\lambda_1)} = (-1)^a$, $Tr(g)|_{M(b\lambda_1)} = 0$

   $Tr(h)|_{M(a\lambda_1)} =$ a+1 mod 3 and $Tr(h)|_{M(a\lambda_1)} \in \{-1,0,1\}$

3. $Sym^2(M(a\lambda_1)) \simeq \bigoplus_i M((2a-4i)\lambda_1)$ $i \in \{0,\ldots,[a/2]\}$ if $2a < p$.

   If $a < p < 2a$ define $U_k \simeq M(2k\lambda_1)$ for $0 \leq k < p-a-1$ or $k = (p-1)/2$ and define $U_k \simeq M(\,2\,(\,p-k-1)\,\lambda_1)\,/\,A_k\,/\,M(\,2\,(p-k-1)\,\lambda_1)$ for $(p-1)/2 < k \leq a$. If $|F| \neq p$ then define $A_k \simeq M((2k-p)\lambda_1) \otimes M(\lambda_1)^{\delta}$ and if $|F|=p$ then define $A_k \simeq M(\,(2k-p+1)\lambda_1) \oplus M(\,(2k-p-1)\lambda_1)$. Then $Sym^2(M(a\lambda_1)) \simeq \bigoplus_k U_{a-2k}$ $k \in \{0,...,[a/2]\}$ if $a < p < 2a$. Moreover U_k is indecomposable if $|F| \neq p$.

4. Let U be an L submodule of the L module V. Then:

   $Sym^2(V) \simeq Sym^2(V/U)\,/\,(\,(V/U)\otimes U)\,/\,Sym^2(U)$.

5. Let A,B be L modules. Then:

   $Sym^2(A\otimes B) \simeq Sym^2(A)\otimes Sym^2(B) \oplus \Lambda^2(A) \otimes \Lambda^2(B)$.

Proof: Parts 1 and 3 can be found in [As6]. Part 4 is the content of [As5] 2.2.1. Part 2 is a simple calculation using the fact that $M(a\lambda_1)$ can be identified with space of homogeneous polynomials in two variables of degree a. Moreover we use that $Tr(x)|_{V\otimes W} = Tr(x)|_V\, Tr(x)|_W$.

Lemma 13.2: If $N_\Gamma(L)$ is maximal then **J** contains no submodule isomorphic to a Galois conjugate of $M(2\lambda_1)$.

Proof: Let $U \simeq M(2\lambda_1)$ be an L-submodule of **J**. Then $S^{\#2}(U)$ is a homomorphic image of $M(4\lambda_1) \oplus M(0)$. So U is brilliant by lemma 10.8. So U does not contain singular points else L is brilliant and hence $N_\Gamma(L)$ nonmaximal. If $\dim(S^{\#2}(U))$ is 0 or 1 then either U is singular and hence $N_\Gamma(L)$ is nonmaximal or $C_{id\Theta}(L) \neq 0$ and again $N_\Gamma(L)$ is nonmaximal. If $S^{\#2}(U)$ is brilliant then L is brilliant as $S^{\#2}(U)$ contains singular points. So we can assume that $\dim(S^{\#2}(U)) \geq 5$, $S^{\#2}(U)$ is not brilliant; i.e., the hypotheses of lemma 10.7 are satisfied. So $S^{\#2}(U)$ is a member of $\mathfrak{U}_6$ and as an L-module is isomorphic to $M(0) \oplus M(4\lambda_1)$. Now if $id \in S^{\#2}(U)$ then by lemma 5.10 $S^{\#2}(U)$ is an L invariant subalgebra so we are done by corollary 10.16. If $id \notin S^{\#2}(U)$ then $C_{id\Theta}(L) \neq 0$ and hence $N_\Gamma(L)$ is not maximal.

Lemma 13.3: If $N_\Gamma(L)$ is maximal then no Galois conjugate of $M(\lambda_1)\otimes M(\lambda_1)^\delta$ is isomorphic to a submodule of J.

Proof: Let U be an L - submodule of J isomorphic to $M(\lambda_1)\otimes M(\lambda_1)^\delta$. Recall from 13.1 that $Sym^2(U) \simeq M(2\lambda_1)\otimes M(2\lambda_1)^\delta \oplus M(0)$ and that $Sym^2(M(2\lambda_1)\otimes M(2\lambda_1)^\delta) \simeq M(4\lambda_1)\otimes M(4\lambda_1)^\delta \oplus M(4\lambda_1) \oplus M(4\lambda_1)^\delta \oplus M(0) \oplus M(2\lambda_1)\otimes M(2\lambda_1)^\delta$. Let $S = S^{\#2}(U) \cap id\Theta$ and $T = S^{\#2}(S)$.

Now if $\dim(S) \leq 1$ then either U is singular or $C_{id\Theta}(L) \neq 0$. So then $N_\Gamma(L)$ is not maximal. So we may assume that $S \simeq M(2\lambda_1)\otimes M(2\lambda_1)^\delta$. Now observe that T is not contained in id+S ; else id+S is a ten dimensional subalgebra of J. So T contains a submodule W such that $W\cap U = W\cap S = 0$ and W is isomorphic to S or $M(4\lambda_1)$.

If $W\simeq S$ then $\dim((U\oplus S\oplus W)^\perp) = 4$ and so in view of lemma 13.2 either $C_{id\Theta}(L) \neq 0$ or $(U\oplus S\oplus W)^\perp$ is isomorphic to a Galois conjugate of U . The first case leads to a nonmaximal $N_\Gamma(L)$. The second case leads to a contradiction to lemma 12.1 as the trace of a three element is 2.

If $W\simeq M(4\lambda_1)$ then $S^{\#2}(W)$ is not contained in W+id and $\dim(S^{\#2}(W)) \geq 5$. If $p\neq 7$ $S^{\#2}(W)$ contains a submodule $V \simeq W$ and $V\cap W = V\cap U = V\cap S = 0$. Thus $\dim((U\oplus S\oplus W\oplus V)^\perp) = 3$ and we are done by lemma 13.2. If p=7 and $S^{\#2}(W)$ does not contain a submodule $V\simeq W$ and $W\cap V = 0$, then $S^{\#2}(W)$ contains a submodule $R \simeq M(4\lambda_1)/M(\lambda_1)^\delta\otimes M(\lambda_1)/W$. Now $\dim(U\oplus S\oplus R) = 27$: a contradiction as $\dim(id\Theta) = 26$.

Lemma 13.4: If $N_\Gamma(L)$ is maximal then J does not contain an L submodule that is Galois conjugate to $M(6\lambda_1)$, $M(10\lambda_1)$, $M(\lambda_1)\otimes M(3\lambda_1)^\delta$ or $M(\lambda_1)^\delta\otimes M(5\lambda_1)$, unless $L \simeq PSL_2(7)$.

Proof: Let $U \leq J$ be a Galois conjugate of one of the irreducible L- modules in the statement of the lemma. We observe first that U is brilliant as U is not a homomorphic image of $Sym^2(U)$. So in view of corollary 10.15 we assume that U does not contain singular points. So if $x \in U$ is a highest weight vector of U wrt some Cartan subgroup of L of weight λ, then $0 \neq x\#x$ is a highest weight vector of weight 2λ of $< x\#x\ L > \leq S^{\#2}(U)$. If $U \neq M(6\lambda_1)$ then $\dim(S^{\#2}(U)) \geq 21$ and $U \cap <x\#xL> = 0$, a contradiction as $\dim(id\Theta) = 26$.

So for the rest of the proof $U \simeq M(6\lambda_1)$. Then let $S = <x\#x\ L>$. Now $\dim(S) = 13$ and $S \leq id\Theta$ unless $char(F) = 7$ and $S \geq <id>$. Now $(U \oplus S)^{\perp} \cap id\Theta$ is a six respectively seven dimensional L submodule of $id\Theta$. So in view of the two previous lemmas $id\Theta$ contains a five dim. irreducible submodule T or in case $char(F) = 7$ one of dimension 7. In the latter case the trace of an involution of L is -2 contradicting lemma 12.1. In case $\dim(T) = 5$ J contains a 25 dimensional submodule, namely $U \oplus S \oplus T$. Now the orthogonal complement of this space is a two dimensional L module. Thus $\dim(C_J(L)) \geq 2$ and $N_\Gamma(L)$ is not maximal.

Lemma 13.5: If U is an L-submodule Galois conjugate to $M(4\lambda_1)$ then $N_\Gamma(L)$ is not maximal, unless $L \simeq PSL_2(7)$.

Proof: We consider first the case $p \neq 7$ and $|F| \neq 5$. Let $T = S^{\#2}(U) \cap id\Theta$. T contains either an $S \simeq M(4\lambda_1) \neq U$ or an $M(8\lambda_1)$, else $id + U$ is a subalgebra or $C_{id\Theta}(L) \neq 0$ and $N_\Gamma(L)$ is not maximal. So then at least one of $S^{\#2}(S)$ or $S^{\#}(S,U)$ contains a submodule $W \neq S,U$ and $W \simeq M(a\lambda_1)$ $a \in \{4,8,\}$. Now $0 \neq \dim((U \oplus S \oplus W)^{\perp} \cap id\Theta)) \leq 6$, unless $U \simeq S \simeq W$. In the first case $id\Theta$ has either a three dimensional L-submodule contradicting lemma 13.2 or the involution $g \in L$ has $tr(g) = 1$ on every composition factor of $id\Theta$ and $id\Theta$ has at least five compfactors so that $tr_{id\Theta}(g) \geq 5$ contradicting lemma 12.1. In the second case $S^{\#}(W,X\ )$, $X \in \{U,S,W\}$ must contain another submodule R distinct from U,S,W and isomorphic to

$M(a\lambda_1)$ $a \in \{4,8\}$, else $id+U+S+W$ is a subalgebra. Now in both cases $(U \oplus S \oplus W \oplus R)^{\perp}$ is of dimension ≤ 6. So as before either $id\Theta$ contains a three dimensional irreducible or the trace of an involution of L is ≥ 5 contradicting lemma 12.1.

Now assume $|F| = 5$. Then $L = Alt_5$. It is well known that the irreducible FL modules fall into 2 blocks, $\mathcal{B}_1 = \{M(0), M(2\lambda_1)\}$ and $\mathcal{B}_2 = \{M(4\lambda_1)\}$. So if $id\Theta$ has composition factors isomorphic to $M(0)$ or $M(2\lambda_1)$, then $id\Theta$ contains a submodule isomorphic to $M(0)$ or $M(2\lambda_1)$. Observe that $\dim(M(4\lambda_1))$ does not divide $\dim(id\Theta)$ and hence $id\Theta$ contains a submodule isomorphic to $M(0)$ or $M(2\lambda_1)$. Now either lemma 13.2 or corollary 10.16 applies.

Now let $char(F) = 7$ and $|F| \neq 7$. As before let $T = S^{\#2}(U) \cap id\Theta$. Suppose for the moment that $T = M(4\lambda_1)/\ M(\lambda_1) \otimes M(\lambda_1)^{\delta}/\ U$. We compute that $Sym^2(T) = (T \oplus M(0))/M(4\lambda_1) \otimes A/(T \oplus M(0))$ where $A = (M(\lambda_1) \otimes M(\lambda_1)^{\delta})/M(4\lambda_1)$, by using lemma 13.1.4. Also by definition of T we have $T < S^{\#2}(T)$. Now let $Y :=$ $M(4\lambda_1) \otimes A$ then:

$$Y \simeq (M(5\lambda_1) \otimes M(\lambda_1)^{\delta} \oplus M(3\lambda_1) \otimes M(\lambda_1)^{\delta})/\ (T \oplus M(0) \oplus M(6\lambda_1) \oplus M(2\lambda_1))$$

So as $S^{\#2}(T)$ is a homomorphic image of T, we violate either lemma 13.2 or 13.4, or $id \oplus T$ is a subalgebra in which case we violate proposition 10.14.

So from now on we may assume that whenever $W, V \simeq M(4\lambda_1)$ are submodules of $id\Theta$, then $S^{\#}(W\ ,\ V)$ does not contain a module isomorphic to T. So then we can proceed as in the case $char(F) \neq 5,7$.

So now we assume that $id\Theta$ contains no submodules isomorphic to T. Thus we can now proceed as in the case $p \neq 5,7$. The proof is now complete.

At this point we observe that $N_{\Gamma}(L)$ maximal in Γ forces that every L submodule of $id\Theta$ is at least nine dimensional, unless $L \simeq PSL_2(7)$.

Lemma 13.6: If $S \neq T$ are irreducible L submodules of $\mathrm{id}\Theta$ of dimension ≥ 9, then $(S \oplus T)^{\perp} \cap \mathrm{id}\Theta$ is a submodule of dimension at most 8. If $N_{\Gamma}(L)$ is maximal then $\mathrm{id}\Theta = S \oplus T$.

Proof: Clear in view of lemmas 13.2 , 3 , 4 and 5.

Proposition 13.7: If $N_{\Gamma}(L)$ is maximal , then either $\mathrm{char}(F) \geq 17$ and $\mathrm{id}\Theta = M(8\lambda_1) \oplus M(16\lambda_1)$, or $\mathrm{char}(F) = 13$ $|F| \neq 13$ and $\mathrm{id}\Theta = M(8\lambda_1) \,/\, M(\lambda_1)^{\delta} \otimes M(3\lambda_1) \,/\, M(8\lambda_1)$, or $\mathrm{char}(F) = |F| = 13$ and $\mathrm{id}\Theta = M(8\lambda_1) \backslash M(2\lambda_1) \oplus M(4\lambda_1) \backslash M(8\lambda_1)$, or $L = PSL_2(7)$ and $\mathrm{id}\Theta = M(6\lambda_1) \oplus M(4\lambda_1) \oplus (M(4\lambda_1) \backslash M(0) \oplus M(2\lambda_1) \backslash M(4\lambda_1))$.

Proof : We treat the case $PSL_2(7)$ last. So for now we assume $L \neq PSL_2(7)$.

We observe first that L can not act irreducibly on $\mathrm{id}\Theta$ as then by 13.1 $\mathrm{id}\Theta \simeq M(25\lambda_1)$ and so the trace of an involution is $+1$ hence contradicting 12.1.

So assume for the moment that $\mathrm{id}\Theta \simeq S \oplus T$ with $\dim(S), \dim(T) \geq 9$.

If $S \simeq M(12\lambda_1) \simeq T$ then an element of order 3 in L has trace $+2$ by 13.1 which contradicts 12.1.

If $S \simeq M(2\lambda_1) \otimes M(2\lambda_1)^{\delta}$ then by 14.1 $\mathrm{Sym}^2(S)$ has no 17 dimensional homomorphic image. So this case is out by the above lemma or because $S + \mathrm{id}$ is a subalgebra.

Also S does not have dimension 12 because by 13.1 L has no irreducible 14 dimensional representations.

If $\mathrm{id}\Theta \simeq M(8\lambda_1) \oplus M(16\lambda_1)$ and $p \geq 17$, then no contradiction to earlier lemmas arises. Moreover, in this case one may assume that $S^{\#2}(S) \geq T$ and $S^{\#2}(T) \geq S$.

Now we assume that $S = \mathrm{Soc}(\mathrm{id}\Theta)$ is irreducible of dimension ≥ 9. In case $\dim(S) = 13$ $S^* \simeq \mathrm{id}\Theta \,/\, S \simeq M(12\lambda_1)$ and the element of order 3 in L trace argument eliminates this case.

If $\dim(S) = 12$ then $\mathrm{id}\Theta \,/\, S^{\perp} \simeq S^*$ and $\dim(S^{\perp}/S) = 2$. Now the trace of an element of order 3 in L is, by 13.1.2, zero on any 12 dimensional irreducible L module. So on $\mathrm{id}\Theta$ the element has trace $+2$, contradicting 12.1.

The case $S \simeq M(2\lambda_1)\otimes M(2\lambda_1)^\delta$ is out because $\mathrm{Sym}^2(S)$ is semisimple so either S is a subalgebra or $\mathrm{id}\Theta$ contains another irreducible L submodule distinct from S. The first possibility is out by corollary 10.16 and the second contradicts our assumption $S = \mathrm{Soc}(\mathrm{id}\Theta)$.

So now assume that $S \simeq M(8\lambda_1)$ then $\mathrm{id}\Theta = M(8\lambda_1)/\ U\ /\ M(8\lambda_1)$ indecomposable and $S^{\#2}(\ M(8\lambda_1))$ contains $T/\ M(8\lambda_1)$ indecomposable with $0 \neq T \leq U$. Then by 13.1 $\mathrm{char}(F) = 13$ and $\mathrm{id}\Theta$ is as claimed.

Now we consider the case $L = PSL_2(7)$. In this case lemma 12.1 and the usual considerations eliminate all possibilities other than the the one listed in the statement of the lemma.

We have now exhausted all possibilities, and the claim follows.

SECTION 14: THE EXISTENCE AND UNIQUENESS OF CERTAIN A_1's AND G_2's

Theorem 14.1: If $\mathrm{char}(F) = 7$ then G contains a unique conjugacy class of subgroups isomorphic to $G_2(F)$ and acting irreducibly on $\mathrm{id}\Theta$. Moreover $\mathrm{id}\Theta \simeq M(2\lambda_1)$ as a $G_2(F)$ module.

Proof: This is the content of theorems 1.c and 2.c of Testerman [T2].

Theorem 14.2: If $\mathrm{char}(F) \geq 17$ then G contains a conjugacy class of $PSL_2(F)$'s such that $\mathrm{id}\Theta \simeq M(8\lambda_1) \oplus M(16\lambda_1)$ as a $PSL_2(F)$ module. If $\mathrm{char}(F) = 13$ and $|F| \neq 13$, then G contains a class of $PSL_2(F)$'s such that $\mathrm{id}\Theta \simeq M(8\lambda_1)/\ M(\lambda_1)^\delta$

$\otimes$ $M(3\lambda_1)$ / $M(8\lambda_1)$ as a $PSL_2(F)$ module.

Proof: Testerman constructed this $PSL_2(F)$ in [T3].

Now we want to establish the uniqueness up to conjugacy in G of the $PSL_2(F)$'s when $p \geq 17$. Notice that we built into our assumption that $N_\Gamma(PSL_2(F))$ is maximal. Otherwise $PSL_2(F)$ stabilizes a subalgebra and hence can not act as we assumed.

Lemma 14.3: Let $f \in SY^3(V)$ (the set of set of symmetric trilinear forms on V), $L \leq O(V,f)$ with L perfect, C the conjugacy class of L under $GL(V)$, and B the set of $\langle b\rangle \in SY^3(V)$ with b similar to f. The number of orbits of $\Delta(V,f)$ on $O(V,f) \cap C$ is equal to the number of orbits of $N_{GL(V)}(L)$ on B. Here $\Delta(V,f)$ denotes the subgroup of $GL(V)$ preserving f up to a scalar.

Proof: This is 2.9 of [As5].

Lemma 14.4: We keep the notation of 14.3. If f is the trilinear form of section 3, $N_{GL(V)}(L)$ has one orbit on B, and $C_V(L) = \mathrm{id}$, then G has one orbit on $G \cap C$.

Proof: Let $L \simeq M,N \in G \cap C$. Then by lemma 14.3 there exists $h \in \Delta(V,f)$ such that $M^h = N$. Now as $C_V(M) = C_V(N) = \mathrm{id}$, it follows that $h \in R := N_{\Delta(V,f)}(\langle \mathrm{id}\rangle)$. Now $R \simeq C_{\Delta(V,f)}(G) \oplus G$ as $A_{GL(V)}(G) = G$. So as $C_{\Delta(V,f)}$ does not fuse G conjugacy classes, we can adjust h so that it lies in G. The claim follows.

For the remainder of this section let $L \simeq PSL_2(F)$. Until further notice we assume that $id\Theta \simeq M(8\lambda_1) \oplus M(16\lambda_1)$ as an L- module.

Lemma 14.5: $\dim(\mathrm{Hom}_{FL}(M(a\lambda_1) , \mathrm{Sym}^2(M(b\lambda_1)))) = 1$ if $a \in \{0,8,16\}$ and $b\in \{8,16\}$ or if $a = b = 0$.

Proof: A simple computation using 13.1.

Definitions: Let $f_{i,j}$ denote the nontrivial L-invariant trilinear map $\Psi : M(i\lambda_1) \times M(j\lambda_1) \times M(j\lambda_1) \mapsto F$ defined by $\Psi(x , y , y') = (y,y') (x(\alpha\tau))$ where α is a generator of $\mathrm{Hom}_{FL}(M(i\lambda_1),\mathrm{Sym}^2(M(j\lambda_1)))$ and τ: $\mathrm{Sym}^2(M(j\lambda_1)) \mapsto$ {symmetric bilinear forms of $M(j\lambda_1)$} is an FL - isomorphism. When i=j let f_i denote the nontrivial L- invariant trilinear form defined on $M(i\lambda_1)$, defined as above.

Definition: Let U, W be L-submodules of $id\Theta$ isomorphic to $M(8\lambda_1)$ respectively $M(16\lambda_1)$.

Lemma 14.6: Let f be the trilinear form of section 3. Then

$$f = a_1f_1 + a_8f_8 +a_{16}f_{16} +a_{8,16}f_{8,16} + a_{16,8}f_{16,8}+a_{1,8}f_{1,8} +a_{1,16}f_{1,16},$$

where the a's lie in F. Moreover $a_{16}, a_{8,16}, a_{1,16} \neq 0$.

Proof: We observe that $f(id,x,y) = 0$ for all $x \in U$ and $y\in W$ using the fact that $U = W^{\perp} \cap id\Theta$ and that $- f(id,-,-)|_{id\Theta}$ is the associated bilinear form of $Q|_{id\Theta}$ (see lemma 3.4). Now the first part is self evident from lemma 14.5.

We see that $a_{1,16} \neq 0$ as $W^{\perp} \cap W = 0$. As noted in the proof of proposition 13.7 $W \leq S^{\#2}(U) = U\Theta^{\perp}$. So $id \oplus U = W^{\perp} \geq U\Theta$, and similarly $W\Theta \leq U^{\perp} = W \oplus id$. Now let $x \in U$ be a high weight vector of U of weight $8\lambda_1$. Then $x\#x$ is a high weight vector of W of weight $16\lambda_1$. Now x is brilliant as $8\lambda_1(h) \in F^*$ has order $\neq 1$ or 3 for some h in the Cartan subgroup of L with respect to which the weights are defined. So by lemma 4.4 $x\#x$ is either singular or zero. In the latter case we contradict $W \leq S^{\#2}(U)$ so W contains singular points. So W is not brilliant otherwise L is brilliant by [As2] Thm1 and hence L stabilizes a subalgebra. This shows $a_{16} \neq 0$. If $a_{8,16} = 0$ then $0 = f(y,y,x) = (y\#y , x)$ for all $x \in U$. Thus $y\#y \in U^{\perp} = id \oplus W$ for all $y \in W$. But then W is an L-invariant subalgebra a contradiction.

Let $\overline{F}$ denote the algebraic closure of F, $\overline{J} = J^{\overline{F}}$, $\overline{L} = L^{\overline{F}}$, and $\overline{f} = f^{\overline{F}}$. Then if $\overline{f}$ is unique so is f. So replacing F by $\overline{F}$, we may assume F is algebraically closed. Also $\overline{f} = a_1\overline{f}_1 + a_8\overline{f}_8 + a_{16}\overline{f}_{16} + a_{8,16}\overline{f}_{8,16} + a_{16,8}\overline{f}_{16,8} + a_{1,8}\overline{f}_{1,8} + a_{1,16}\overline{f}_{1,16}$ and $\overline{f}_i$, $\overline{f}_{i,j}$ are preserved by $\overline{L}$, so $\overline{L}$ preserves $\overline{f}$. Thus replacing L by $\overline{L}$ we may assume that $F = \overline{F}$ is algebraically closed. Also we observe that $C_{GL(J)}(L) \simeq (F^*)^3$ is generated by the scalar actions on id, U and W respectively.

Notation: We fix a Cartan subgroup T of $L < G$ and we denote the weight vectors of U and W with respect to T by $\{x_{2i} : -4 \leq i \leq 4\}$ respectively $\{y_{2i} : -8 \leq i \leq 8\}$. Let z be the generator of <id>. Let S_i denote the weight space corresponding to the weight $i\lambda_1$.

Note that we may assume that $T < H_D < H$ where is the Cartan subgroup of $E_6(F)$ defined in section 3.

Lemma 14.7: Let $\mathfrak{A}$ be an F- algebra , $D \leq Aut(\mathfrak{A})$. Assume that D contains only semisimple elements and let S_i denote the weight space of D of weight λ_i. Let S_iS_j denote the space generated by the products. Then $S_iS_j \leq S_{i+j}$, where $\lambda_{i+j}(g)$ is defined as $\lambda_i(g)\lambda_j(g)$ for all $g \in L$.

Proof: Let $s_i \in S_i$, $s_j \in S_j$ and $g \in L$. Then $s_i s_j\, g = \lambda_i(g)\lambda_j(g)\, s_i s_j = \lambda_{i+j}(g)\, s_i s_j$ and the claim follows.

One consequence of lemma 14.7 is that S_0 is a subalgebra. Moreover, as Q is nondegenerate on S_0 and F is algebraically closed, S_0 is a member of $\mathcal{U}_3$ by lemma 5.12.1. Observe also that J is the orthogonal direct sum of the spaces $S_i \oplus S_{-i}$. So as Q is nondegenerate on J the space $S_i \oplus S_{-i}$ is Q-nondegenerate.

Lemma 14.8: The following are true:

1. $\bigoplus_i S_i$ $i = 0 \bmod 8$ is a subalgebra of type Υ_9.
2. $\bigoplus_i S_i$ $i = 0 \bmod 6$ is a subalgebra of type Υ_9.
3. $\bigoplus_i S_i$ $i = 0 \bmod 4$ is a subalgebra of type Υ_{15}.

Proof: We note first that in all three cases $\oplus\, S_i$ is a Q-nondegenerate subspace containing id and S_0. The fact that $\oplus S_i$ is a subalgebra is a consequence of lemma 14.7. Let $x \in S_0$ be a primitive idempotent. Then as S_0 is a member of $\mathcal{U}_3$ we have $E_{1,\oplus S_i}(x) \neq 0$. So, by lemma 5.8, the subalgebras have the claimed types.

Lemma 14.9: Let $\{v_1,v_2,v_3\}$ be the three pairwise orthogonal primitive idempotents spanning S_0. Then we can find $s_i\,, \overline{s}_i \in S_i$ such that

$$W_1 := v_3\Delta \cap v_2\Delta = \langle y_{16},y_{-16},s_4,s_{-4},s_6,s_{-6},s_2,s_{-2}\rangle$$
$$W_2 := v_1\Delta \cap v_3\Delta = \langle y_{12},y_{-12},s_8,s_{-8},y_{10},y_{-10},\overline{s}_2,\overline{s}_{-2}\rangle$$
$$W_3 := v_1\Delta \cap v_3\Delta = \langle \overline{s}_4,\overline{s}_{-4},\overline{s}_8,\overline{s}_{-8},\overline{s}_6,\overline{s}_{-6},y_{14},y_{-14}\rangle\,.$$

Note that each $s_i\,, \overline{s}_i$ is Q-singular and hence singular by lemma 3.6.2.

Proof: We first observe that by lemma 14.7 $y_i \# y_i = 0$ when $|i| > 8$ and hence these y_i are singular by lemma 3.4.1. Now we also observe that $(y_i, y_{-i}) = -(1/2)f(id, y_i, y_{-i}) \neq 0$ by lemmas 14.6 and 14.5. So if $i > 8$ then $y_i + y_{-i}$ brilliant and Q-nonsingular hence squares to $2y_i \# y_{-i}$ which by lemmas 4.2 and 4.3 is a multiple of a primitive idempotent. Moreover, observe that $y_i \# y_{-i} \in S_0$ by lemma 14.7, so we may assume that $y_i \# y_{-i}$ is a multiple of one of the v_j. So without loss of generality we may assume that $y_{16}, y_{-16} \in W_1$.

So now as $R := \oplus S_{8i}$ is a member of $\mathcal{V}_9$ we have $\dim(R \cap W_i) = 2$ (see lemma 5.6). As $R \cap W_1 = \langle y_{16}, y_{-16} \rangle$ it follows that $(W_2 \oplus W_3) \cap R = S_8 \oplus S_{-8}$. So let $\langle s_{\pm 8} \rangle = W_2 \cap R$ and $\langle \bar{s}_{\pm 8} \rangle = W_3 \cap R$.

As $T < H$ every weight vector of H is a weight vector of T. As v_j and y_i $|i| > 8$ are singular; they are also weight vectors of H. So let $Y = \{y_i, v_j, s_k, \bar{s}_k : |i| > 8\}$ be a basis of weight vectors of H. Define a *line* to be a special triple contained in Y. Let Y be the set of points and define incidence by inclusion. Then it was observed in [As1] that the geometry defined in this way can be identified with the building of $\Omega_6^-(2)$ and is hence a generalized quadrangle. Each point is contained in exactly 5 lines and 2 points can lie on, at most, one line.

Now we observe that for $w_i \in S_i$ $f(w_i, w_j, w_k) \neq 0$ only if $i+j+k = 0$. So the lines containing y_{16} must be of the form $\{y_{16}, y_{-16+2i}, \bar{s}_{-2i} : 1 \leq i \leq 3\}$ or $\{y_{16}, y_{-16}, v_1\}$ or $\{y_{16}, s_{-8}, \bar{s}_{-8}\}$. Now we claim that y_{-14}, y_{-12} and y_{-10} are not contained in W_1. Suppose otherwise; then $[v_1, y_{16}, y_{-i}]$ is a triangle in the building of $\Omega_6^-(2)$, a contradiction as, by definition, generalized quadrangles do not contain contain triangles. Similarly we argue that y_{14}, y_{12} and y_{10} are not contained in W_1. So WLOG let $y_{\pm 14} \in W_3$. Then again we can argue as above that $y_{\pm 12}, y_{\pm 10}$ are not contained in W_3. Leaving as the only alternative $y_{\pm 12}$, $y_{\pm 10} \in W_2$.

Now as $R := \oplus S_{6i}$ is a member of Υ_9 we argue as in the second paragraph of this proof that $<s_{\pm 6}> = W_1 \cap R$ and that $<\bar{s}_{\pm 6}> = W_3 \cap R$.

Now if $R := \oplus S_{4i}$ then R is a member of Υ_{15} by lemma 14.8. Also we have shown that $R \cap W_2 = < y_{\pm 12}, s_{\pm 8} >$ and that $\dim(R \cap W_i) = 4$ (see lemma 5.6). So $S_4 \oplus S_{-4} < (W_1 \oplus W_3) \cap R$ and it follows that $<s_{\pm 4}> < R \cap W_1$ and $<\bar{s}_{\pm 4}> < R \cap W_3$.

Finally observe that we have now shown that W_3 is as claimed. Thus $S_2 \oplus S_{-2}$ is contained in $W_1 \oplus W_2$ and so W_1, W_2 are also as claimed. The lemma follows.

Lemma 14.10: $f_{16}(y_{-16}, y_{10}, y_6)$, $f_{8,16}(y_{-16}, y_{10}, x_6)$, $f_{16,8}(x_6, x_6, y_{-12})$, $f_8(x_{-4}, x_{-4}, x_8)$, $f_{1,8}(z, x_4, x_{-4})$, $f_{16}(y_{16}, y_{-12}, y_{-4})$, $f_{8,16}(y_{16}, y_{-12}, x_{-4})$, $f_{1,16}(z, y_i, y_{-i})$ with $i \neq 0$ are all not equal to zero.

Proof: Let $h(\lambda) = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}$ and $g(t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$ then $r_i h(\lambda) = \lambda^i r_i$ for all i and $r_i \in S_i$.

Let $z_m = y_{16-2m}$ and $w_m = x_{8-2m}$.

Then $z_m\, g(t) = \sum_{j=o}^{m} b_{m,j}\, t^{m-j} z_j$ where $b_{m,j}$ denotes the binomial coefficient m over j, see [As 6]. Moreover the same formula holds for w_m in place of z_m.

Let $\tau = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ then by [As6] $z_m \tau = (-1)^m z_m$ and the same formula holds for w_m. Using the invariance of the forms involved we see that $f(s_i, s_j, s_k) = f(s_i h(\lambda), s_j h(\lambda), s_k h(\lambda)) = \lambda^{i+j+k} f(s_i, s_j, s_k)$ for all $\lambda \in F$ and $s_i \in S_i$ $s_j \in S_j$ $s_k \in S_k$. So $f(s_i, s_j, s_k) \neq 0$ only if $i+j+k = 0$.

Now suppose that for a fixed i $f_{16}(y_i, y_j, y_k) = 0$ for all j,k such that $i+j+k = 0$. Then $y_i \in y_j\Delta$ for all j. Thus f_{16} a nontrivial radical. Since L acts irreducibly on U, this implies that $f_{16} = 0$; a contradiction to lemma 14.5.

Now assume that $f_{16}(y_{16}, y_{-16}, y_0) = 0$ then:

$0 = f_{16}(z_0,z_{16},z_9) = f_{16}(z_0g(t),z_{16}g(t),z_9g(t))$

$= \sum_{j,k} b_{16,j}b_{9,k}\ t^{25\ -j-i}f_{16}(z_0,z_j,z_k)$

$= \quad 16t\ f_{16}(z_0,z_{15},z_9) + 9t\ f_{16}(z_0,z_{16},z_8).$

So $f_{16}(y_{16},y_{-14},y_{-2}) = (-9/16)\ f_{16}(y_{16}, y_{-16},y_0) = 0.$

So we conclude that $f_{16}(y_{16},y_{-14},y_{-2}) = 0$ as $f_{16}(y_{16}, y_{-16},y_0) = 0.$

Now using that $f_{16}(y_{16},y_{-14},y_{-4}) = 0$ we get that

$f_{16}(y_{16},y_{-12},y_{-4}) = (-\ 10/15)(-9/16)\ f_{16}(y_{16}, y_{-16},y_0) = 0.$

Continuing this way we get that $f_{16}(y_{16},y_{-16+i},y_{-i}) = \alpha\ f_{16}(y_{16}, y_{-16},y_0) = 0$ for all $i \geq 0$.

Thus y_{16} is contained in the radical of f_{16} a contradiction. The only way out is that $f_{16}(y_{16}, y_{-16},y_0) \neq 0$ and hence $f_{16}(y_{16},y_{-16+2i},y_{-2i}) \neq 0$ for all $0 \leq i \leq 8$. Now we apply τ to get $0 \neq f_{16}(\ y_{16}\tau, y_{-10}\tau,y_{-6}\tau) = f_{16}(y_{-16},y_{10},y_6)$.

All the other claims are proved in the same fashion.

Proposition 14.11: There is a unique class of subgroups of G which are isomorphic to $PSL_2(F)$ such that $J \simeq M(0) \oplus M(8\lambda_1) \oplus M(16\lambda_1)$ as a $PSL_2(F)$ module.

Proof: In view of lemma 14.6 and the fact that $C_{GL(J)}(L) = (F^*)^3$ we may choose $a_{16}= a_{8,16} = a_{1,16} = 1$. If we can now show that the remaining four a's of lemma 14.6 are uniquely determined, we will have shown that L stabilizes a unique E_6 - form and hence by lemma 14.4 we will have proved the claim.

We observe from lemma 14.9 that for $-8 < i < 0$ $\langle s_i \rangle = \mathrm{Ker}(\ f(y_{16},y_{-16-i},\ -\ )|_{S_i})$ and similarly for $0 > i > 8$ $\langle s_i \rangle = \mathrm{Ker}(\ f(y_{-16}, y_{16-i},-\ )|_{S_i})$.

As $f_{16}(y_{-16}, y_{10},y_6) \neq 0 \neq f_{8,16}(y_{-16},y_{10},x_6)$ we compute that $s_6 = \lambda(x_6+\alpha y_6)$ where $\alpha = f(y_{-16},y_{10},x_6)\ /\ f(y_{-16},y_{10},y_6) \neq 0.$

Now recall from lemma 14.9 that s_6 is singular. So we have:

$0 = f(s_6, s_6, y_{-12})$

$= a_{16,8}\ \lambda^2 f_{16,8}\ (x_6, x_6, y_{-12}) + 2\ \lambda^2\alpha\ f(x_6, y_6, y_{-12}) - \lambda^2\ \alpha^2\ f(y_6, y_6, y_{-12}).$

So $a_{16,8}$ is determined as $\lambda \neq 0$ and by lemma 14.10 $f_{16,8}(x_6, x_6, y_{-12}) \neq 0$.

Similarly we have: $s_{-4} = \lambda(x_{-4} + \alpha\ y_{-4})$ where $0 \neq \alpha = f_{8,16}(y_{16}, y_{-12}, x_{-4})/f_{16}(y_{16}, y_{-12}, y_{-4})$ and

$0 = f(s_{-4}, s_{-4}, x_8)$

$= a_8\lambda^2 f_8(x_{-4}, x_{-4}, x_8) + 2\alpha\lambda^2\ f(x_{-4}, y_{-4}, x_8) + \lambda^2\alpha^2\ f(y_{-4}, y_{-4}, x_8).$

So a_8 is determined as $f_8(x_{-4}, x_{-4}, x_8) \neq 0 \neq \lambda$.

Now we also have:

$0 = f(s_4, s_{-4}, v_2) = f(\lambda(x_4 + \beta\ y_4)\ ,\ \mu(x_{-4} + \alpha y_{-4}),\ \gamma_1 z + \gamma_2\ x_0 + \gamma_3 y_0)$

$= a_{1,8}\gamma_1\lambda\mu\ f_{1,8}(x_4, x_{-4}, z) +$ determined quantities.

Now $\lambda, \mu \neq 0$ as $s_4 \neq 0 \neq s_{-4}$. Moreover $0 \neq \gamma_1$ as $0 = f(y_{16}, y_{-16}, v_2) = \gamma_3 f_{16}(y_{16}, y_{-16}, y_0) + \gamma_1 f_{1,16}(y_{16}, y_{-16}, z)$ and $f_{1,16}(y_{16}, y_{-16}, z) \neq 0 \neq f_{16}(y_{16}, y_{-16}, y_0)$. So $a_{1,8}$ is determined as $f_{1,8}(x_4, x_{-4}, z) \neq 0$.

Finally we use that $0 = f(v_2, v_2, z) = a_1\gamma_1^2\ f_1(z,z,z) +$ determined quantities to determine a_1.

So a_1 is determined as $f_1(z,z,z) \neq 0$ and the proof is complete.

Proposition 14.12: Let F_0 be a subfield of F. Then G contains a unique conjugacy class of subgroups isomorphic to $F_4(F_0)$.

Proof: Let $L \simeq F_4(F_0)$. Then it is shown in [As5] that L stabilizes a unique E_6-form so the claim follows from lemma 14.4.

We now want to establish the uniqueness of the $PSL_2(F)$, $char(F) = 13$, described in theorem 14.2. Our methods will be similar to those used in the proof of proposition 14.11 which were previously used in [As5] in a similar situation. For the remainder of this section we will assume that char(F) = 13, $|F| \neq 13$, $L \simeq PSL_2(F)$, and $J \simeq M(0) \oplus M(8\lambda_1)\backslash M(3\lambda_1)\otimes M(\lambda_1)^{\delta}\backslash M(8\lambda_1) \simeq M(0) \oplus W(16\lambda_1)\backslash M(8\lambda_1)$ where $W(16\lambda_1)$ denotes the Weyl module of highest weight $16\lambda_1$. This last isomorphism is in [As6].

Now we establish some notation. Let $Y = \mathrm{Soc}(\mathrm{id}\Theta)$, $W = W(16\lambda_1)/Y$ and $Z = \mathrm{id}\Theta/W(16\lambda_1)$. We fix a Cartan subgroup H_L of L. Then let $\{ y_i \}$ be a basis of weight vectors of Y, $\{w_i$: i= 16,14,12,10,-10,-12,-14,-16$\}$ be a basis of weight vectors spanning W and $\{z_i\}$ be a basis of weight vectors spanning Z. By S_i we will denote the weight space of weight i. For $a \in F$ and $b \in F^*$, let $\alpha_{a,b} \in \mathrm{End}_{FL}(Y\backslash W\backslash Z)$ be the element defined by $y_i\alpha_{a,b} = by_i$, $w_i\alpha_{a,b} = bw_i$, $z_i\alpha_{a,b} = bz_i + ay_i$. Let $f_1 \in \mathrm{Sym}^3(Y\backslash W\backslash Z)$ be such that $f_1 \neq 0$ on Y. We define recursively $f_{i+1} := f_i\alpha_{1,1} - f_i$. Similarly, let $g_1 \in L(M(0),Y\backslash W\backslash Z,Y\backslash W\backslash Z)$ such that $g_1(m,w,v) \neq 0$ for some $m \in M(0)$, w, v $\in$ W, and let $g_{i+1} := g_i\alpha_{1,1} - g_i$.

Lemma 14.13:

1. $\mathrm{Sym}^3(Y\backslash W\backslash Z)$ is spanned by $\{f_1,f_2,f_3,f_4\}$.
2. $\{\alpha_{a,b}\}$ is transitive on $\langle f_1,f_2\rangle \setminus \langle f_2\rangle$ modulo $\langle f_3,f_4\rangle$.
3. $L_{FL}(M(0), Y\backslash W\backslash Z,Y\backslash W\backslash Z)$ is spanned by $\{g_1,g_2\}$.
4. $f = f_1 + a_3f_3 + a_4f_4 + g_1 + b_2g_2 + c\,h$, where h spans $\mathrm{Sym}^3(M(0))$, where $a_i,b_i,c \in F$.

Proof: We prove part 3 first. Observe that g_1 and g_2 are linearly independent. Next we observe that $L_{FL}(M(0), Y\backslash W\backslash Z,Y\backslash W\backslash Z) \simeq \mathrm{Hom}_{FL}(Y\backslash W\backslash Z,Y\backslash W\backslash Z)$ which is clearly two dimensional, and part 3 follows. Part 2 is easy to see. Given parts 1,2,3 we can see part 4 as follows. First, $\mathrm{Sym}^3(M(0)\oplus Y\backslash W\backslash Z)$ is spanned by $\{f_i,g_i,h\}$. If f is an E_6 form, then $f|_Y \neq 0$ otherwise $f|_{Y\backslash W} = 0$ and $(Y\backslash W)\Theta \geq$

$Y\backslash W$ and so by lemma 4.5 $S^{\#2}(Y\backslash W) \leq Y$ and $Y\backslash W \oplus M(0)$ is a subalgebra. But then by the subalgebra theorem $Y\backslash W\backslash Z$ is not indecomposable contradicting our choice of L. So we may assume $f = f_1$ + other terms. By part 2 we may also set $a_2 = 0$. In view of the fact that $f_{id}(-,-)$ is nondegenerate and that $rad(g_{2,id}) \neq 0$, we may further assume that $f = f_1 + g_1$ + other terms. So part 4 follows.

Now we prove part 1. First we want to see that $\{f_1,f_2,f_3,f_4\}$ are linearly independent. To see this we make the following easy to check observations:

a. $f_2(w,v,u) = 0$ for all $w,v,u \in W(16\lambda_1)$, but $f_2(z,w,w)\neq 0$ for some $z\in Z$ and $w\in W(16\lambda_1)$.

b. $f_3(x,w,w) = 0$ for all $w\in W(16\lambda_1)$ and $x \in Y\backslash W\backslash Z$, but $Rad(f_3) \neq 0$.

c. $Rad(f_4) = W(16\lambda_1)$.

To complete the proof of part 1 we need to check that $\dim(Sym^3(Y\backslash W\backslash Z)) \leq 4$. First we want to observe that $\dim(Sym^3(Y\backslash W)) = \dim(Hom_{FL}(Y\backslash W,Sym^2(Y\backslash W))) = 1$. We can see this using 13.1.4 to compute $Sym^2(Y\backslash W) \simeq Sym^2(Y)\backslash Y\otimes W\backslash Sym^2(W)$. We can then compute $Sym^2(W)$ using formula 13.1.5 and then observe that $Hom(Y\backslash W,Sym^2(W)) = 0$. Furthermore we can compute that $Hom(Y\backslash W,Y\otimes W) = 0$ and that $\dim(Hom(Y\backslash W,Sym^2(Y))) = 1$ establish our observation.

So now it suffices to show that the space of trilinear forms $\mathfrak{T}$ which vanish on $Y\backslash W$ is three dimensional. We consider the FL-map $D: \mathfrak{T} \mapsto Hom_{FL}(Z, (Y\backslash W)\otimes(Y\backslash W))$ defined by $D(g)(z) = g(z,-,-)$. Let $\mathfrak{K} = Ker(D)$, then $Dim(\mathfrak{T}/\mathfrak{K}) \leq 1 = \dim(Hom_{FL}(Z, (Y\backslash W)\otimes(Y\backslash W)))$. Now define $E: \mathfrak{K} \mapsto Hom_{FL}(Z, (Y\backslash W)\otimes Z)$ by $E(g)(z) = g(z,\_,\_)$. As $\dim(Hom_{FL}(Z, (Y\backslash W)\otimes Z)) = 1$ $\dim(\mathfrak{K}/Ker(E)) \leq 1$. Finally we observe that $Ker(E) \simeq Hom_{FL}(Z,Z\otimes Z)$, which is one dimensional, as can be computed using lemma 13.1. The claim follows.

As in the case char(F) ≥ 17, we can work over an algebraically closed field. For let $\overline{F}$ denote the algebraic closure of F. Then the f_i, g_i and h are $PSL_2(\overline{F})$ invariant maps.

Lemma 14.14: The following are true:

1. $\bigoplus_i S_i \quad i = 0 \bmod 8$ is a subalgebra of type $\mathcal{V}_9$.
2. $\bigoplus_i S_i \quad i = 0 \bmod 6$ is a subalgebra of type $\mathcal{V}_9$.
3. $\bigoplus_i S_i \quad i = 0 \bmod 4$ is a subalgebra of type $\mathcal{V}_{15}$.

Proof: The proof of lemma 14.8 carries over verbatim.

Lemma 14.15: Let $\{v_1, v_2, v_3\}$ be three pairwise orthogonal primitive idempotents spanning S_0. Then we can find $s_i, \overline{s}_i \in S_i$ such that

$W_1 := v_3\Delta \cap v_2\Delta = \langle w_{16}, w_{-16}, s_4, s_{-4}, s_6, s_{-6}, s_2, s_{-2} \rangle$

$W_2 := v_1\Delta \cap v_3\Delta = \langle w_{12}, w_{-12}, s_8, s_{-8}, w_{10}, w_{-10}, \overline{s}_2, \overline{s}_{-2} \rangle$

$W_3 := v_1\Delta \cap v_3\Delta = \langle \overline{s}_4, \overline{s}_{-4}, \overline{s}_8, \overline{s}_{-8}, \overline{s}_6, \overline{s}_{-6}, w_{14}, w_{-14} \rangle$.

Moreover the $s_i, \overline{s}_i$ are singular.

Proof: The proof of lemma 14.9 carries over verbatim.

Lemma 14.16: The following quantities are not equal to zero: $f_1(w_{16}, w_{-12}, y_{-4})$, $f_1(w_{16}, w_{-10}, y_{-6})$, $f_1(w_{16}, w_{-12}, z_{-4})$, $f_1(w_{16}, w_{-10}, z_{-6})$, $f_3(z_4, z_4, y_{-8})$, $f_4(z_4, z_4, z_{-8})$, $g_2(id, z_0, z_0)$, $h(id, id, id)$.

Proof: The proof is similar to that of lemma 14.10. The idea is to show that if one of the quantities is zero, then the form f_i resp. (g_i, h) is trivial, contradicting the definition of f_i resp. (g_i, h).

Proposition 14.17: Let $\mathrm{char}(F) = 13$, $|F| \neq 13$. Then G contains a unique conjugacy class of subgroups isomorphic to $PSL_2(F)$ and $\mathrm{id}\Theta \simeq M(8\lambda_1)\backslash M(3\lambda_1)\otimes M(\lambda_1)^{\delta}\backslash M(8\lambda_1)$ as an $FPSL_2(F)$- module.

Proof: The proof is analogous to that of proposition 14.11. First we observe that $<s_{-6}> = \mathrm{Ker}(f_1(w_{16},w_{-10},\_)|_{S_{-6}})$. So as $f_1(w_{16},w_{-10},y_{-6}) \neq 0 \neq f_1(w_{16},w_{-10},z_{-6})$ we can write $s_{-6} = y_{-6} + \alpha\, z_{-6}$, where $\alpha = -\, f_1(w_{16},w_{-10},y_{-6}) \,/\, f_1(w_{16},w_{-10},z_{-6})$. For the same reasons we can write $s_4 = y_4 + \beta z_4$.

As s_{-6} is singular by lemma 14.15 we have:

$0 = f(s_4,s_4,y_{-8}) = \beta^2 a_3 f_3(z_4,z_4,y_{-8})$ + determined quantities.

So as $f_3(z_4,z_4,y_{-8}) \neq 0$ we have determined a_3.

Also we have:

$0 = f(s_4,s_4,z_{-8}) = \beta^2 a_4 f_4(z_4,z_4,z_{-8})$ + determined quantities.

So as $f_4(z_4,z_4,z_{-8}) \neq 0$ we have determined a_4.

Now $v_i = \alpha \mathrm{id} + \beta y_0 + \gamma z_0$ where α,β,γ are expressible in determined quantities. We choose i such that $\gamma \neq 0 \neq \alpha$. Then as v_i is singular we have:

$0 = f(v_i,v_i,z_0) = 2\,\alpha\gamma b_2 g_2(\mathrm{id},z_0,z_0)$ + determined quantities. Also as $g_2(\mathrm{id},z_0,z_0) \neq 0$ we have determined b_2.

Finally, we use $0 = f(v_1,v_1,\mathrm{id}) = c\alpha^2 h(\mathrm{id},\mathrm{id},\mathrm{id})$ + determined quantities to determine c. This completes our proof.

CHAPTER III: CROSS CHARACTERISTICALLY EMBEDDED SIMPLE SUBGROUPS, ALTERNATING AND SPORADIC SIMPLE SUBGROUPS

In this chapter let L denote a finite simple group. We will assume that the defining characteristic of the group L is not equal to char(F) = p. For the rest of this chapter $\overline{F}$ will denote the algebraic closure of F.

SECTION 15: INITIAL REDUCTIONS

Proposition 15.1: The structure of the Weyl group of $F_4(F)$ is $2^{\cdot}$ ($Alt_4 \oplus Alt_4$) .2 : 2.
Proof: This is well known. One way of seeing this is as follows. The Weyl group of $E_6(F)$ is $O_6^-(2)$. The Weyl group of $F_4(F)$ is the stabilizer of a singular line in the building of the Weyl group of $E_6(F)$, the structure of which is well known.

Lemma 15.2: The orders of elements of the Weyl group of $F_4(F)$ are : 1,2,3,4,6,8,12.
Proof: Well known.

Lemma 15.3: If $L < G$ and L is of Lie type of characteristic $r \neq 2,3,p$, then $L \simeq PSL_2(q)$ and $q \in \{5,7,13,17,25\}$.

Proof: From lemma 7.8 we know that the Sylow - r- subgroup of G is abelian. So L $\simeq PSL_2(q)$ as all other simple groups of Lie type of char = r have nonabelian Sylow-r-subgroups.

Now the Borel group of $PSL_2(q)$ is a Frobenius group of order $q(q-1)/2$. Again by lemma 7.8 the Frobenius complement has to be a section of the Weyl group of G. So $(q-1)/2$ has to be the order of an element of the Weyl group of G. The claim now follows from lemma 15.2.

Lemma 15.4: If $L < G$ is of Lie type of characteristic 3, then $L \simeq PSL_2(9)$, $PSL_2(27)$, $PSL_3(3)$, $PSU_3(3)$, or ${}^2G_2(3)$.

Proof: Recall from lemma 7.10.3 that a Sylow-3-group R of G contains an abelian normal subgroup A, such that R/A is elementary abelian of order 9, and that $m_3(G) \leq 4$. This restricts the type of L to one of the following: $PSL_2(q)$ and $q \leq 81$, $PSU_3(q)$ and $q \in \{3,9\}$, $PSL_3(q)$ $q \in \{3,9\}$, $P\Omega_5(3) \simeq PSp_4(3)$, ${}^2G_2(3)$, $G_2(3)$ and $PSU_4(3) \simeq P\Omega_6^-(3)$.

Now the argument in lemma 11.1 shows that $P\Omega_5(3)$ and $P\Omega_6^-(3)$ contain a subgroup isomorphic to $2^4{:}\ Alt_5$; so lemma 7.11.4 eliminates these two possibilities. The Borel subgroup of $PSL_2(81)$ is Frobenius of order 81:40. By 7.10.1 the group 3^4 is contained in a Cartan subgroup of $G^{\overline{F}}$. So by 7.10.2 $A_G(3^4)$ is a section of the Weyl group of G. Now 15.2 eliminates $PSL_2(81)$ as a possibility as 40 does not divide the order of the Weyl group of G. The group $PSL_2(9)$ contains a subgroup $3^4{:}PSL_2(9)$. But by lemma 7.10 3^4 is contained in a Cartan subgroup of G and $A_G(3^4)$ is a section of the Weyl group of G. As $PSL_2(9)$ contains elements of order 5 lemma 15.3 eliminates $PSL_3(9)$ as a possiblity.

The group $U_3(9)$ has no faithful $\overline{F}$-representation of degree less than 72 (see [La1]) so it can't embed into G. The only small faithful irreducible $\overline{F}$-representations of $G_2(3)$ have degree 14 (see Atlas and Parkers Character Tables), so $G_2(3)$ is out by lemma 10.17.

Lemma 15.5: If $L < G$ is of Lie type of characteristic 2, then L is isomorphic to one of the following: $PSL_2(q)$ $q \leq 8$, $PSp_4(2)$, $PSL_3(2)$, $G_2(2)'$, ${}^3D_4(2)$ or $PSL_4(2)$.

Proof: Recall from lemma 7.10 that $m_2(G) = 5$, so in general $q \leq 32$. Now recall from lemma 7.10 that $A_G(E)$ is a 2,3,7 - group when E is an elementary abelian 2 group. So if L has Lie rank 1, then $q \leq 8$. So $L \simeq PSL_2(4)$, $PSL_2(8)$, $PSU_3(4)$, $PSU_3(8)$, or $Suz(8)$. Now the small nontrivial irreducible $\overline{F}$-representations of $Suz(8)$ have degree ≥ 14 and hence 10.17 disposes of this case. Now by [La1] we see that the smallest nontrivial irreducible $\overline{F}$-representation of $PSU_3(8)$ has degree at least 56; so this case is out. The case $PSU_3(4)$ is out because the trace of an element of order 3 in $PSU_3(4)$ on any 26 dimensional $\overline{F}$-representation is either 2 or 14; a contradiction to lemma 12.1 (see Atlas and Parkers character tables).

Now suppose that the Lie rank of L is 2. Then L contains subgroups of Lie rank 1 so in general $q \leq 8$. As $m_2(G) = 5$ this leaves only the possibilities listed in the statement of the lemma and the groups $PSL_3(4)$, $PSU_4(2)$, $PSU_5(2)$, and $^2F_4(2)'$. The last four possibilities are out by lemma 7.11.4 as the groups contain subgroups isomorphic to $2^4{:}PSL_2(4)$, $2^4{:}PSL_2(4)$, $3 \times PSU_4(2)$ and $2^4{:}\ 5$ respectively.

Now if L has Lie rank greater than or equal to 3, then as L contains a subgroup of Lie rank 2, we see q=2. Now as $m_2(G) = 5$ the only possibility is $L \simeq PSL_4(2)$. Now the claim is established.

The following result can also be found in [Kl2].

Proposition 15.6: Let L be a sporadic finite simple group. Then L is a subgroup of G iff p=11 and $L \simeq J_1$ or M_{11}.

Proof: It is well known that $PSL_3(4) < M_{22}, M_{23}, M_{24}$, McL, HS, Co.3, Co.2, Co.1. So these groups are out by lemma 15.5. It is also known (see Atlas) that the 3-ranks of the following groups, Suz, Ly, Th are at least 5. It is also known (see Atlas) that the 2-ranks of the groups J_4, He, Fi_{22}, Fi_{23}, Fi_{24}, Ru, HN, B, M are at least 6. So these groups are out by lemmas 7.10 and 7.11. The group J_3 contains a subgroup $PSL_2(16)$, so it is out by proposition 15.5. The group O'N has subgroups $PSL_2(11)$ and $PSL_2(31)$. So O'N is out by corollary 15.3. The small nontrivial irreducible $\overline{F}J_2$ -modules have dimensions 14 and 21 (see Atlas and Parkers tables). So J_2 is out by lemma 10.17. The groups J_1, M_{11} and M_{12} contain $PSL_2(11)$, so by lemma 16.3 these groups can embed into G only if p = 11. Now J_1 was originally constructed as a subgroup of $G_2(11)$ (see [Ja1]). The centralizer in G of a subalgebra of type $\mathfrak{U}_6$ is $G_2(F)$, by subalgebra theorem, which certainly contains $G_2(p)$. So J_1 is a subgroup of G iff p=11.

Now consider M_{11}. From the modular character table we see that M_{11} is a subgroup of $\Omega_9(11)$. Moreover because M_{11} has trivial Schur multiplier, we see that M_{11} is also a subgroup of $Spin_9(11)$. Thus M_{11} is a subgroup of G, because it is a subgroup of the stabilizer of a primitive idempotent. Recall from proposition 2.7.5 that the stabilizer of a primitive idempotent is $Spin_9(F)$. So M_{11} is a subgroup of G iff p=11.

Now it remains to show that M_{12} is not a subgroup of G when p=11. To see this we observe that the small nontrivial 11-modular irreducibles of M_{12} have dimensions 11 and 16, and that M_{12} contains an involution whose trace is 3 resp 0 on these irreducibles. So we see that this involution has trace 10 on any nontrivial 26-dimensional $\overline{F}L$-representation. So M_{12} is out by lemma 12.1.

Proposition 15.7: Let $L \simeq Alt_n$, $n \geq 7$. Then L is not a subgroup of G unless n=7 and p=5. Moreover if p=5 $\dim(C_J(Alt_7)) = 3$ and hence $N_\Gamma(Alt_7)$ is never maximal in Γ.

Proof: We will show first that when $p \neq 5$, then Alt_7 is not a subgroup of G. Then we will show that when $p = 5$ Alt_7 is a subgroup of G only if Alt_7

centralizes a three dimensional subalgebra. Finally, we will show that when p=5 Alt_8 is not a subgroup of G , completing the proof.

The following part of the Alt_7 ordinary character table is an excerpt from the Atlas:

	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8
1A	1	6	10	10	14	14	15	21
2A	1	2	-2	-2	2	2	-1	1
3A	1	3	1	1	2	-1	3	-3
3B	1	0	1	1	-1	2	0	0

For now we will assume that $p \neq 5,7$. The character χ of any 26 dimensional $Alt_7\overline{F}$ - module is a sum of irreducible characters. Now we will survey the possible χ and observe that either $\chi(2A) \notin \{-6,2\}$ or $\chi(3A), \chi(3B) \notin \{-1,8\}$, showing by lemma 12.1 that the module affording χ is not a G module.

If $\chi = \chi_8 + 5\chi_1$, $\chi_7 + \chi_i + \chi_1$ $i = 3,4$, $\chi_7 + \chi_2 + 5\chi_1$, $\chi_i + 2\chi_2$ $i=5,6$,

$\chi_i + \chi_2 + 6\chi_1$ $i=5,6$, $\chi_j + 2\chi_2 + 4\chi_1$ $j=3,4$, $\chi_j + \chi_2 + 10\chi_1$ $j=3,4$,

$\chi_j + 16\chi_1$ $j = 3,4$, or $a\chi_2 + b\chi_1 | \; 6a+b = 26$, then $\chi(2A) \notin \{-6,2\}$

If $\chi = \chi_i + \chi_j + 2\chi_1$ $i=5,6$ $j=3,4$, $\chi_j + \chi_k + \chi_2$ $j,k = 3,4$, then $\chi(3A)$ or $\chi(3B) \notin \{-1,8\}$.

If $\chi_j + \chi_k + 6\chi_1$ and $j,k \in \{3,4\}$, then Alt_7 must stabilize a point. This is impossible because point centralizers do not allow the proposed Alt_7 series as a refinement.

Now assume p=7. The following table is an excerpt from the 7-modular character table (see Parker):

	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6
1A	1	5	10	14	14	21
2A	1	1	-2	2	2	1
3A	1	2	1	2	-1	-3
3B	1	-1	1	-1	2	0

The only 26 dimensional characters that are feasible by Lemma 12.1 are: $\chi_6+\chi_2$ and $2\chi_3+6\chi_1$.

In the first case observe that idΘ is the direct sum of the irreducibles affording $\chi_6$and χ_2 and that $\dim(\mathrm{Sym}^2(\chi_2)) = 15$. So $S^{\#2}(\chi_2) \leq \langle \mathrm{id}, \chi_2\rangle$ and hence $\langle \mathrm{id},\chi_2\rangle$ is a proper nontrivial subalgebra of **J**. Now $\langle \mathrm{id},\chi_2\rangle$ is nondegenerate and hence it contains a primitive idempotent x by Springer's lemma. If $E_1(x) \cap \langle \mathrm{id},\chi_2\rangle = 0$ then lemma 5.5 applies forcing an Alt_7 invariant subalgebra $\langle \mathrm{id},x\rangle < \langle \mathrm{id},\chi_2\rangle$: a contradiction. If $E_1(x)\cap\langle \mathrm{id},\chi_2\rangle \neq 0$ then $\langle \mathrm{id},\chi_2\rangle$ must be a member of $\mathfrak{U}_6$ by lemmas 5.9.5 and 5.12.2. In this case we see from lemma 5.13 $\mathrm{Alt}_7 < N_G(\langle \mathrm{id},\chi_2\rangle)/C_G(\langle \mathrm{id},\chi_2\rangle) \simeq SO_3(F)$: a contradiction as Alt_7 has no nontrivial three dimensional 7-modular representation.

In the other case regard **J** as an Alt_6 module. As 7 does not divide the order of Alt_6, **J** must be semisimple when regarded as an Alt_6 module and hence $\dim(C_{\mathbf{J}}(\mathrm{Alt}_6)$ ≥ 7. The trivial Alt_6 module induced up to Alt_7 is of shape $\chi_1/\chi_2/\chi_1$. So when $x \in C_{\mathrm{id}\Theta}(\mathrm{Alt}_6)$, then $\langle x\ \mathrm{Alt}_7\rangle$ is a homomorphic image of $\chi_1/\chi_2/\chi_1$. So $\langle x\ \mathrm{Alt}_7\rangle = \chi_1$ as we assumed that id$\Theta = 2\chi_3+6\chi_1$ as an Alt_7 module. So we see that $C_{\mathbf{J}}(\mathrm{Alt}_7)$ is a seven dimensional nondegenerate subalgebra of **J**. So by Springer's Lemma $\mathrm{Alt}_7 < C_G(x)$ for some primitive idempotent $x \in C_{\mathbf{J}}(\mathrm{Alt}_7)$. Now $C_G(x)$ has only one composition factor of dimension greater than 10. We assumed that Alt_7 has two ten dimensional composition factors: a contradiction.

Now assume that $p=5$. We will show first that Alt_8 is not a subgroup of G. To see this we observe that Alt_8 contains an involution whose trace on every small module is at least 1 (see the 5-modular character table). Now in all but one case a 26 dimensional Alt_8 representation has more than 2 composition factors. In this case the trace of this involution is more than 2, and hence the claim follows from lemma 12.1. The remaining case is the case 13/13, where the trace on the other involution is 10; again lemma 12.1 yields the claim.

It remains to show that in case $p=5$, Alt_7 must centralize a subalgebra of dimension 3.

Here is an excerpt from the 5-modular character table of Alt_7:

	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7
1A	1	6	8	10	10	13	15
2A	1	2	0	-2	-2	1	-1
3A	1	3	-1	1	1	-2	3
3B	1	0	-1	1	1	1	0

As above, the only 26 dimensional representation not contradicting lemma 12.1 has character $2\chi_1 + 3\chi_3$. Now we restrict this character to $PSL_2(7)$. We observe that $C_{id\Theta}(PSL_2(7))$ is at least two dimensional as 5 does not divide the order of $PSL_2(7)$. An easy calculation using the Atlas and Parkers character tables shows that the trivial $PSL_2(7)$ module induced up to Alt_7 affords the character $2\chi_1 + \chi_6$. Thus if $id\Theta$ affords the character $2\chi_1+3\chi_3$ as an Alt_7 module, then $\dim(C_{id\Theta}(Alt_7)) = 2$ as for any $x \in C_{id\Theta}(PSL_2(7))$ $<xAlt_7>$ is a homomorphic image of $2\chi_1+\chi_6$ and hence a trivial Alt_7 module.

The proposition is finally proved.

SECTION 16: ESTABLISHING NONMAXIMALITY OF CERTAIN SIMPLE SUBGROUPS OF G

The results of the previous section leave the following groups for our consideration:

List 16.1

$Alt_5 \simeq PSL_2(5) \simeq PSL_2(4)$, $Alt_6 \simeq PSp_4(2) \simeq PSL_2(9)$, $PSL_2(7) \simeq PSL_3(2) \simeq PSU_3(2)'$, $PSL_2(8) \simeq {}^2G_2(3)'$, $PSL_2(q)$ $q \in \{13,17,25,27\}$

$PSL_3(3)$, $PSU_3(3) \simeq G_2(2)'$, ${}^3D_4(2)$, J_1 when $p = 11$, M_{11} when $p = 11$

Remark: From now on we will denote the ordinary irreducible characters of our simple groups as in the Atlas [Con1].

Lemma 16.2: Let p=11 . Then $N_\Gamma(J_1)$ is never maximal in Γ.

Proof: From the 11-modular character table and lemma 12.1 we can see that the J_1-module idΘ has three composition factors of dimension 7 and five composition factors of dimension 1. By lemma 7.1 it is enough to show that J_1 centralizes a point of idΘ. Assume for now that this is not the case. Then let A be a seven dimensional J_1-submodule of idΘ. Using the fact that J_1 contains $PSL_2(11)$, we will show in a moment that the symmetric square of A is $<1> \oplus B$ where B is the unique 27 dimensional 11-modular irreducible of J_1. So from lemma 10.1.4 we observe that $\dim(S^{\#2}(A)) \leq 1$.

If $S^{\#2}(A) = 0$, then by lemma 3.4.2 A is singular. This is impossible because the maximal singular subspaces of J are five and six dimensional.

If $S^{\#2}(A) < <id>$, then lemma 10.4 applies and hence A is singular: a contradiction.

So we conclude that $S^{\#2}(A)$ is a point of J distinct from <id>. Now the claim follows as we have shown that $C_{id\Theta}(J_1) \neq 0$.

Now we will show that the symmetric square of the 11-modular irreducible is as claimed. We establish first the 11-modular irreducible is $M(6\lambda_1)$ as a $PSL_2(11)$ module. To see this we note that the character value an element of order 6 on the 7 dimensional J_1 module is -1. The character value of the element of order 6 on the $PSL_2(11)$ modules $M(2\lambda_1)$ and $M(4\lambda_1)$ is 2 and 1 respectively, leaving $M(6\lambda_1)$ as the only possible choice. Now recall from chapter II that $Sym^2(M(6\lambda_1)) = M(8\lambda_1)/M(2\lambda_1) \oplus M(0)/M(8\lambda_1) \oplus M(4\lambda_1) \oplus M(0)$ as a $PSL_2(11)$ module. Now $Sym^2(M(6\lambda_1))$ as a J_1 module can have only 1,7,14 or 27 dimensional composition factors. The trace of an involution respectively an element of order 3 on $Sym^2(M(6\lambda_1))$ is 4 respectively 1. Now it is easy to check using the excerpt of the 11 modular character table of J_1 given below, that the traces can only add up right when the composition factors are as claimed. To see that the module splits we recall that J_1 is a subgroup of $G_2(11)$ and that the symmetric square of the 7 dimensional $G_2(11)$ module is $M(2\lambda_1) \oplus M(0)$.

Here is the excerpt of the 11-modular character table:

1A	2A	3A
1	1	1
7	-1	1
14	-2	-1
27	3	0

Lemma 16.3: Let p=11 . Then $N_\Gamma(M_{11})$ is contained in the stabilizer in Γ of a primitive idempotent.

Proof: By lemma 12.1 the only feasible 26 dimensional characters for M_{11} are the ones whose composition series have composition factors of degrees 1,16,9 or $1^6,10^2$, not necessarily in that order.

The first case leads to semisimple module as idΘ is a self dual module. So in this case M_{11} centralizes a nondegenerate two dimensional subalgebra. So in this case, by lemma 5.1, M_{11} embeds into the stabilizer of a primitive idempotent.

In the second case we regard idΘ as an M_{10} module to see that $\dim(C_{id\Theta}(M_{10})) \geq 6$ as idΘ is a semisimple M_{10} module. Now the trivial M_{10} module induces up to $1\backslash 9\backslash 1$ as an M_{11} module mod 11. So as in the Alt_7 case we see that $\dim(C_{id\Theta}(M_{11})) = 6$ and $C_J(M_{11})$ is a nondegenerate subalgebra containing primitive idempotents. Now $C_J(M_{11})$ is not isomorphic to some $J_{\mathbb{D}}$ for reasons of dimension. So by proposition 5.9 and lemma 5.5 M_{11} must centralize a primitive idempotent. But then J can't have two ten dimensional composition factors as an M_{11} module see prop. 2.7.3. This completes the proof.

Lemma 16.4: $N_\Gamma(Alt_5)$ is never maximal in Γ.

Proof: We observe that the case p=5 was handeled in section 13 as $Alt_5 \simeq PSL_2(5)$. So $p \neq 5$ and idΘ is a semisimple Alt_5 module. So if $N_\Gamma(Alt_5)$ is maximal then idΘ has no trivial composition factors. Now if idΘ has an irreducible three dimensional Alt_5 submodule U then, by lemma 10.8, U is brilliant. If U contains singular points then lemma 10.15 applies. If U does not contain singular points but $S^{\#2}(U)$ is brilliant, then as $S^{\#2}(U)$ contains singular points, lemma 10.15 applies. So U satisfies the hypothesis of lemma 10.7 and hence $S^{\#2}(U)$ is a member of $\mathfrak{U}_6$ and Alt_5 centralizes a point of $S^{\#2}(U)$. If this point is not id, we are done because then Alt_5 centralizes a point of $id\Theta$. If the point centralized is id, then by lemma 5.10, $S^{\#2}(U)$ is a subalgebra and proposition 10.16 applies.

So now we assume that $id\Theta$ is the direct sum of four and five dimensional Alt_5 modules. Thus $id\Theta$ has to be the sum of four four dimensional and two five dimensional Alt_5 modules. This contradicts lemma 12.1 as the trace of the element of order 3 is $4x1 + 2x(-1) = 2 \notin \{-1,8\}$. This completes the proof.

Lemma 16.5: When $p \neq 7$, then $N_\Gamma(PSL_2(7))$ is never maximal in Γ.

Proof: As we assume $p \neq 7$ $id\Theta$ is a semisimple $PSL_2(7)$ module, so if $N_\Gamma(PSL_2(7))$ is to be maximal, then $C_{id\Theta}(PSL_2(7)) = 0$. Checking the character table of $PSL_2(7)$ we see that the only 26 dimensional representations which do not violate lemma 12.1 and have no trivial submodules are afforded by characters of the form $a\chi_2 + b\chi_3 + \chi_6$ where $a+b = 6$, $\deg(\chi_2) = \deg(\chi_3) = 3$ and $\deg(\chi_6) = 8$. Now a simple calculation using the $PSL_2(7)$ character table shows that $Sym^2(\chi_2) = Sym^2(\chi_3) = \chi_4$, the unique irreducible character of degree 6. This shows that when $id\Theta$ affords a character of the form $a\chi_2 + b\chi_3 + \chi_6$, then $S^{\#2}(U) = 0$ for every three dimensional submodule U. So U is singular by lemma 4.8 and hence $PSL_2(7)$ stabilizes a subalgebra. So then $N_\Gamma(PSL_2(7))$ stabilizes a subalgebra by corollary 10.16.

Lemma 16.6: $N_\Gamma(PSU_3(3))$ is not maximal in Γ when $p \neq 7$. When $p=7$ L has an irreducible 26 dimensional representation, and this representation of L is realized in G via the $G_2(7)$ described in section 14.

Proof: Let $L \simeq PSU_3(3)$. First we treat the case when $p \neq 7$. In this case every L - module is semisimple. If L centralizes a point, the statement clearly holds. So then the only character which does not violate lemma 12.1 and contains no trivial characters is $\chi_2 + \chi_2 + \chi_6$ (in Atlas notation) where $\deg(\chi_2) = 6$ and $\deg(\chi_6) = 14$. We compute easily using the character table of $PSU_3(3)$ that $Sym^2(\chi_2) = \chi_7$ (in Atlas notation) and $\deg(\chi_7) = 21$. So, by lemma 10.1.4, $S^{\#2}(A) = 0$, where A is a submodule affording χ_2. So A is singular by lemma 3.4.2. So <id,A> is a $PSU_3(3)$ invariant subalgebra and we are done by corollary 10.16.

Now we consider the case p=7: The Borel group B of $PSU_3(3)$ has order coprime to 7. The trivial B module induced up to $PSU_3(3)$ mod 7 is of the form $1\backslash 26\backslash 1$. Therefore if $id\Theta$ affords a $PSU_3(3)$ character with trivial constituents, then $C_{id\Theta}(B) \neq 0$, and hence $C_{id\Theta}(PSU_3(3)) \neq 0$. So then the arguments above go through in case $PSU_3(3)$ does not act irreducibly on $id\Theta$, as the character χ_7 stays irreducible when reduced mod 7.

We recall from [As7] that $Aut(PSU_3(3))$ is an irreducible subgroup of $G_2(7)$ acting on its seven dimensional irreducible module. Now $PSU_3(3)$ contains a $PSL_2(7)$ such that the 7 modular seven dimensional irreducible is $M(6\lambda_1)$ as a $PSL_2(7)$ module (this can be seen from the ordinary and modular character tables of $PSL_2(7)$ and $PSU_3(3)$). Now from lemma 13.1.3 we know that:

$Sym^2(M(6\lambda_1)) = M(6\lambda_1) \oplus M(0)/M(4\lambda_1)/M(0) \oplus M(4\lambda_1)/(M(2\lambda_1)\oplus M(0))/M(4\lambda_1)$

(is the sum of two indecomposables and an irreducible). So now we can see with the aid of the 7-modular character table that $Sym^2(M(6\lambda_1))$ as a $PSU_3(3)$ module is of the form 1/26/1. Now as a $G_2(7)$ module $Sym^2(M(6\lambda_1))$ is also of the form $1\backslash 26\backslash 1$. So we conclude that the 26 dimensional 7 modular irreducible of $G_2(7)$ stays irreducible upon restriction to $Aut(PSU_3(3))$. So $Aut(PSU_3(3))$ embeds irreducibly into $F_4(7)$ via the $G_2(7)$ of section 14.

Lemma 16.7: Let $L \simeq PSL_3(3)$. Then L is an irreducible subgroup of G. All embeddings of L are equivalent, in $GL_{27}(F)$.

Proof: The irreducible embedding of L can be seen in the stabilizer of a 27- or twisted

27-decomposition of J.

The usual argument using the character tables and lemma 12.1 shows the second part.

Lemma 16.8: Let $L \simeq {}^3D_4(2)$. If $L < G$ then L acts irreducibly on $id\Theta$.

Proof: By [La1] L has no representations of dimension ≤ 13. So the claim follows from 10.17.

Lemma 16.9: Let $L \simeq PSL_2(25)$. Then $N_\Gamma(L)$ is maximal only if $id\Theta$ affords χ_{11}. Moreover G contains such a subgroup when F is a splitting field for $x^2 + 1 = 0$.

Proof: The usual argument using lemma 12.1 shows that $N_\Gamma(L)$ is maximal only if $id\Theta$ affords χ_{11} or χ_{13}. Now the Borel subgroup of L is of the form 5^2: 12. By lemma 7.8 the element of order 12 must be conjugate to an element in the Weyl group of F_4 and hence also to an element in the Weyl group of E_6. Now the Weyl group of E_6 acts as $1 + 6^+ + 20^-$ on J. So from the atlas [Con1] we see that the element of order 12 of the Borel subgroup of L has to have trace -1 or 2 on $id\Theta$. But $\chi_{13}(12a) = \chi_{13}(12b) = 1$ eliminating χ_{13} and leaving only χ_{11} as a possibility.

Now ${}^2F_4(2)'$ is an irreducible subgroup of $E_6(F)$ iff F is a splitting field for $x^2 + 1$. Now it is known that L is a subgroup of ${}^2F_4(2)'$. Now the claim follows from character restriction.

Lemma 16.10: Let $L \simeq PSL_2(27)$. Then $N_\Gamma(L)$ is maximal only if $id\Theta$ affords χ_i $i \in \{6,7,8\}$.

Proof: Clear from 12.1 and [Con1].

Lemma 16.11: Let $L \simeq PSL_2(8)$. Then $N_\Gamma(L)$ is maximal in Γ only if $p \neq 7$ and $id\Theta$ affords the L-module character $\chi_6 + \chi_i + \chi_j$ where $i,j \in \{7,8,9\}$.

Proof: The usual argument when $p \neq 7$. Now let $p=7$ and let $B \simeq 2^3{:}7$ be the Borel subgroup of L. Then recall form the proof of lemma 7.11.4 that the 2^3 is unique up to conjugacy in G and that $C_J(2^3)$ is a member of $\mathfrak{U}_6$. Now as observed in [As5] 19.6 the trivial 2^3 module induced up to L is of the form $1\backslash 8\backslash 1$. So in this case either $C_{id\Theta}(L) \neq 0$ or $id \simeq 8 \oplus 8\backslash 1 \oplus 8\backslash 1$. In the last case let $U = 8 \oplus 8 \oplus 8$. Then $U^\perp$ is a three dimensional submodule of J which must be trivial as an L module as L contains no nontrivial representations over F of dimension < 7.

Lemma 16.12: Let $L \simeq PSL_2(13)$. Then $N_\Gamma(L)$ is maximal only if $p \neq 7$ and $id\Theta$ affords the character $\chi_8 + \chi_i$ $i \in \{4,5,6\}$ or $p = 7$ and $id\Theta$ affords $12 + 14_a$. When $p = 7$ and 13 is a square in F, then the embedding of L via the irreducible $G_2(F)$ affords $12 + 14_a$.

Proof: The usual argument.

Lemma 16.13: Let $L \simeq PSL_2(17)$. Then $N_\Gamma(L)$ is maximal only if $id\Theta$ affords the character $\chi_7 + \chi_i$ $i \in \{2,3\}$.

Proof: The usual argument.

Lemma 16.14 : Let $L \simeq Alt_6$. Then $N_\Gamma(L)$ is a maximal subgroup of Γ only if $p > 5$ and $id\Theta$ affords the character $2\chi_6 + \chi_i$ $i \in \{4,5\}$.

Proof: When $p > 5$ the usual argument using 12.1, $C_{id\Theta}(L) = 0$ and [Con1] gives the claim. So let $p = 5$ then using lemma 12.1 we find that the degrees of the composition factors of $id\Theta$ must be $10^2, 1^6$ or $8^3, 1^2$. In the first case we let M be the normalizer of a Sylow 3 subgroup of L; in this case 5 does not divide $|M|$ and so $\dim(C_{id\Theta}(M)) \geq 6$. Then the character of the trivial representation of M induced up to L in characteristic 5 is $1\backslash 8\backslash 1$. So if L has composition factors $10^2,1^6$, then $\dim(C_{id\Theta}(L)) = 6$, and L stabilizes a subalgebra.

Now we assume the other case. First we observe that $id\Theta$ must be uniserial $8\backslash 1\backslash 8\backslash 1\backslash 8$ otherwise $id\Theta$ has a submodule of dimension 10. As $\dim(H^1(L,8)) = 1$ any such ten dimensional submodule contains a trivial submodule and hence so does $id\Theta$. On the other hand as $\dim(C_{id\Theta}(M)) \geq 2$ we have that either $C_{id\Theta}(L) \neq 0$ or $id\Theta$ contains two distinct submodules $U, W \simeq 8\backslash 1$. This contradicts uniseriality and the claim follows.

SECTION 17: PROOF OF THE SIMPLE SUBGROUP THEOREM

In the last section we restricted the isomorphism type of a simple subgroup giving rise to a maximal subgroup of Γ to the following:

$$PSL_2(q) \quad q \in \{8,9,13,17,25,27\}, \quad PSU_3(3), PSL_3(3), {}^3D_4(2)$$

Moreover, with the exception ${}^3D_4(2)$, we have identified the cross-characteristic embeddings of L into G which could make $N_\Gamma(L)$ into a maximal subgroup of Γ; recall also that we dealt with the non cross-characteristic case in chapter II. However, in the cases ${}^3D_4(2)$, $PSL_2(q)$ $q \in \{8,9,13,17,27\}$ we did not decide if such embeddings of L into G existed. The next few lemmas deal with the existence question.

Lemma 17.1: If $p \neq 7, 13$, then $PSL_2(13)$ does not embed into G via a character specified in lemma 16.12.

Proof: See [Co4].

Recall that $\overline{G} = F_4(\overline{F})$ where $\overline{F}$ is an algebraically closed field.

Lemma 17.2: If $p \neq 17$, then $PSL_2(17)$ embeds into $\overline{G}$ via the character $\chi_2+\chi_8$.
Proof: See [Co4].

Lemma 17.3: If $p \neq 7$, then $PSL_2(8)$ embeds into $\overline{G}$ via the character $\chi_6+\chi_7+\chi_9$.
Proof: See [Co4].

Lemma 17.4: Let $p \neq 7$, then $PSL_2(27)$ embeds into $\overline{G}$ via the character χ_7.
Proof: See [Co4].

Lemma 17.5: ${}^3D_4(2)$ embeds into $\overline{G}$ via the character χ_2. The embedding is unique up to G conjugacy when $p \neq 7$.
Proof: See [Co4] and [Col].

Now we sumarize the results.

Let M be a maximal subgroup of Γ such that $M \cap G$ is a closed proper subgroup of G. If $F^*(M \cap G)$ is not a simple nonabelian group, then the Structure theorem of Section 9 asserts that M either stabilizes a good subalgebra, a 27-decomposition, or a twisted 27-decomposition. Moreover, we know the structure of the stabilizer of each.

Now suppose that $F^*(M \cap G)$ is a simple nonabelian group. Then the results of section 10 show that $F^*(M \cap G)$ does not stabilize a subalgebra. If $F^*(M \cap G)$ is of Lie type of characteristic p, then the results of chapter II restrict $F^*(M \cap G)$ to one of $\{ G_2(F)\ \ p = 7,\ PSL_2(F)\ \ p \geq 13 \text{ or } |F| = 7,\ F_4(F_0)\ \ F_0 < F \}$ and moreover none of these groups stabilizes a good subalgebra. If $F^*(M\cap G)$ is not of Lie type of characteristic p, then the results of Sections 15 and 16 restrict the type one of

$$PSL_2(q)\ \ q\in\{8,9,13,17,25,27\},\ \ PSU_3(3),\ PSL_3(3),\ {}^3D_4(2).$$

With the exception of $PSL_2(9)$ we have exhibited suitable embeddings into $\overline{G}$ in section 16 and in this section. However, the only known embeddings for $PSL_3(3)$, $PSU_3(3)$ $p=7$, $PSL_2(7)$ $p=7$, and $PSL_2(13)$ $p=7$ are contained in proper subgroups of $\overline{G}$.

So we have proved the necessary conditions for maximality as stated in the simple subgroup theorem.

When do we have sufficient conditions for maximality? We mention here, without proof, the following facts:

The stabilizers of good subalgebras are maximal.

If F is finite the stabilizers of 27- or twisted 27-decompositions are maximal iff F is a prime field. If F is not a prime field then the stabilizers are contained in a conjugate of $F_4(F_0)$, where F_0 is the prime field of F.

If F is finite $F_4(F_*)$ is maximal iff $[F:F_*]$ is a prime.

If F is finite and $p \neq 7$, then ${}^3D_4(2)$ is maximal iff F is a prime field.

Similar conditions will hold for $PSL_2(q)$ $q \in \{8,9,17,25,27\}$.

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