Homogeneous Sequences of Cardinals for Ordinal Definable Partition Relations

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Contents

Abstract

In this dissertation we study the consistency strength of the theory **ZFC** & $(\exists\kappa \text{ strong limit})(\forall\mu<\kappa)(\kappa \underset{\mathbf{OD}}{\rightarrow} (\omega)^{\omega}_{\mathbf{V}_\mu})$ $(*)$, and we prove the consistency of this theory relative to the consistency of the existence of a supercompact cardinal and an inaccessible above it. If U is a normal measure on $\mathcal{P}_\kappa(\lambda)$, then $\mathbf{P}_{\mathrm{U}}$ denotes the Supercompact Prikry forcing induced by U. $\kappa \underset{\mathbf{OD}}{\rightarrow} (\omega)^{\omega}_{\mathbf{V}_\mu}$ is the partition relation $\kappa \rightarrow (\omega)^{\omega}_{\mathbf{V}_\mu}$ except that we consider only **OD** colorings of $[\kappa]^\omega$. **Theorems 1,2** are the main results of our thesis.

Theorem 1. If there exists a model of **ZFC** in which κ is a supercompact cardinal and λ is an innaccessible above κ, then we can construct a model **V** of the same properties with the additional property that if U is a normal $\mathcal{P}_\kappa(\lambda)$-measure and G is $\mathbf{P}_{\mathrm{U}}$ – generic over **V**, then **V**[G] does not satisfy the $(\forall\mu<\kappa)\left(\kappa \underset{\mathbf{OD}}{\rightarrow} (\omega)^{\omega}_{\mathbf{V}_\mu}\right)$ partition property. □

If G is a $\mathbf{P}_{\mathrm{U}}$-generic over **V** filter, then we define H to be the set $\mathrm{H} :\overset{\mathrm{def}}{=} \bigcup\left\{\mathcal{P}(\kappa)\cap \mathbf{V}[\mathrm{G}\restriction\alpha] \mid \alpha<\lambda\right\}$, and we consider the inner model **V**(H), which is the smallest inner model of **ZF** that contains H as an element. We prove that **V**(H) satisfies the above partition property $(*)$.

Moreover, **V**(H) satisfies $<\lambda-\mathrm{DC}$ and using this fact we define a forcing $\mathcal{P}$, which is almost-homogeneous, $<\lambda-$closed forcing that forces the AC over **V**(H) and does not add any new sets of rank $<\kappa$.

Theorem 2. If $\mathcal{G}$ is $\mathcal{P}$ -generic over **V**(H) and $\mathbf{V}_1 :\overset{\mathrm{def}}{=} \mathbf{V}(\mathrm{H})[\mathcal{G}]$, then $\mathbf{V}_1 \models [\mathbf{ZFC} + \kappa \text{ strong limit } + \forall\mu<\kappa(\kappa \underset{\mathbf{OD}}{\rightarrow} (\omega)^{\omega}_{\mathbf{V}_\mu})]$. Therefore

$$\mathrm{Con}(\mathbf{ZFC} + (\exists\kappa, \lambda)[\kappa \text{ supercompact } \& \ \lambda \text{ inaccessible } \& \ \kappa < \lambda]) \Rightarrow$$
$$\mathrm{Con}(\mathbf{ZFC} + (\exists\kappa \text{ strong limit})(\forall\mu < \kappa)\Big(\kappa \underset{\mathbf{OD}}{\rightarrow} (\omega)^{\omega}_{\mathbf{V}_\mu}\Big)). \qquad \square$$

Notation

Our Set Theoretic notation follows that of [Jech 78] and [Kunen 80]. **ZFC** denotes the theory of Zermelo Fraenkel Set Theory with the Axiom of Choice. Ordinals and cardinals are denoted by the small letters of the Greek alphabet. **OD** denotes the class of all ordinal definable sets. If X is a set, then $\mathcal{P}(X)$ denotes the power set of X. If κ, λ are cardinals, then $\mathcal{P}_\kappa(\lambda)$ is the set of all subsets of λ of cardinality $< \kappa$. If α is an ordinal, then V_α denotes the set of all sets of rank $< \alpha$ and for any class M, V_α^M is the relativisation of V_α to M. $[\kappa]^\alpha$ denotes the set of all subsets of κ of order-type α which is identified with the set of all (strictly) increasing $\alpha -$ sequences from κ. For the forcing terminology we also follow [Jech 78] and [Kunen 80]. Given a forcing P, Γ denotes the canonical $P -$ name for the $P -$ generic filter. The notation $\check{x}$ of the canonical $P -$ name for an element x of the ground model is supressed. We talk of generic filters as if they exist since all those forcing arguments can be easily translated to rigorous arguments about countable transitive models of finite fragments of **ZFC**.

Introduction and Statement of Results

In this dissertation we prove that an upper bound of the consistency strength of the theory

$$(\P) \qquad \mathbf{ZFC} \quad \& \quad (\exists\kappa \text{ strong limit})(\forall\mu < \kappa)\Big(\kappa \underset{\mathbf{OD}}{\rightarrow} (\omega)^{\omega}_{\mathbf{V}_\mu}\Big)$$

is the existence of a supercompact cardinal and an inaccessible above it. $\kappa \underset{\mathbf{OD}}{\rightarrow} (\omega)^{\omega}_{\mathbf{V}_\mu}$ is the partition relation $\kappa \rightarrow (\omega)^{\omega}_{\mathbf{V}_\mu}$ except that we consider only **OD** colorings of $[\kappa]^\omega$.

This answers an open question proposed by Professor W. Mitchell.

The theorem is proven after the series of the following results:

Let $\mathcal{U}$ be a $\sigma-$complete ultrafilter on a cardinal δ and $\mathbf{P}^*_{\mathcal{U}}$ be the Prikry-tree forcing induced by $\mathcal{U}$ on δ. A condition of this forcing consists of a $\mathcal{U}-$splitting tree on $\mathcal{P}_\kappa(\lambda)$ and a stem which is a node of this tree. The partial order is as in the Prikry forcing.

First, we prove that if σ and τ are two $\mathbf{P}^*_{\mathcal{U}}$ – generic over $\mathbf{V}$ sequences with the property $\mathbf{V}[\sigma] = \mathbf{V}[\tau]$, then they are tail – equivalent. (**1.B.9**)

Using the above fact about the Prikry-tree forcing (i.e., **1.B.9**), we prove the following:

If U is a normal measure on $\mathcal{P}_\kappa(\lambda)$ and if σ and τ are two $\mathbf{P}_{\mathrm{U}}$ – generic over $\mathbf{V}$ sequences with the property $\mathbf{V}[\sigma] = \mathbf{V}[\tau]$, then they are tail – equivalent. (**1.B.10**)

1.B.11 Theorem. We assume that **V** is a transitive model of **ZFC** in which κ is a supercompact cardinal and λ is an inaccessible above κ. Also, U is a normal $\mathcal{P}_\kappa(\lambda)$ – measure and $\mathbf{P}_{\mathrm{U}}$ is the Supercompact Prikry forcing associated with U. If G is $\mathbf{P}_{\mathrm{U}}$ – generic over $\mathbf{V}$, then there exists in $\mathbf{V}[\mathrm{G}]$ a coloring F of $[\kappa]^\omega$ into $\mathbf{V}_\mu$ colors for some $\mu < \kappa$, with the property that F is ordinal definable over $\mathbf{V}[\mathrm{G}]$ with parameters from $\mathbf{V}$ and has no homogeneous set, i.e., $\mathbf{V}[\mathrm{G}] \models \neg(\forall\mu < \kappa)\Big(\kappa \underset{\mathbf{VOD}}{\rightarrow} (\omega)^\omega_{\mathbf{V}_\mu}\Big)$. □

1.B.13 Theorem. If $\mathbf{V}'$ is a transitive model of **ZFC** in which κ is a supercompact cardinal and λ is an inaccessible above κ, then there exists a generic extension $\mathbf{V}$ of $\mathbf{V}'$ in which κ remains supercompact, λ remains inaccessible and the following holds:

If U is a normal $\mathcal{P}_\kappa(\lambda)$ – measure, $\mathbf{P}_{\mathrm{U}}$ is the Supercompact Prikry forcing associated with U and G is $\mathbf{P}_{\mathrm{U}}$ – generic over $\mathbf{V}$, then a sufficient large rank initial segment of the ground model $\mathbf{V}$ is **OD** over the Prikry generic extension $\mathbf{V}[\mathrm{G}]$ and $\mathbf{V}[\mathrm{G}] \models \neg(\forall\mu < \kappa)\Big(\kappa \underset{\mathbf{OD}}{\rightarrow} (\omega)^\omega_{\mathbf{V}_\mu}\Big)$. □

The Theorems **1.B.11** *and* **1.B.13** *suggest that a Supercompact Prikry generic extension of the Universe is not the right model for the theory* (¶). *Instead, we consider an inner model of such generic extension.*

If G is a $\mathbf{P}_{\mathrm{U}}$ – generic over $\mathbf{V}$ filter, then we define H^{G} to be the following set: $\mathrm{H}^{\mathrm{G}} :\stackrel{\mathrm{def}}{=} \bigcup\Big\{\mathcal{P}(\kappa) \cap \mathbf{V}[\mathrm{G}\restriction\alpha] \mid \alpha < \lambda\Big\}$, and we consider the inner model $\mathbf{V}(\mathrm{H})$, for $\mathrm{H} :\stackrel{\mathrm{def}}{=} \mathrm{H}^{\mathrm{G}}$, which is the smallest inner model of **ZF** that contains H as an element. (**2.A.1**)

Then we prove that $\mathrm{H} = \mathcal{P}(\kappa) \cap \mathbf{V}(\mathrm{H})$. Moreover $\mathbf{V}(\mathrm{H}) \subsetneq \mathbf{V}[\mathrm{G}]$ and

$\mathbf{V}(\mathrm{H}) \models \lambda = \kappa^+$. (**2.A.16**)

The next two results show that there exists a plethora of Prikry generic sequences of height $< \lambda$ *in* $\mathbf{V}(\mathrm{H})$.

We let $\mathbf{P}_{\mathcal{U}}$ be the $\mathcal{P}_\kappa(\beta)$ – Supercompact Prikry forcing with respect to $\mathcal{U}$ and $\kappa \leq \beta < 1$. We take $\mathcal{Q} \subset_c \mathbf{r.o.}(\mathbf{P}_{\mathcal{U}})$ and $\mathrm{g} \in \mathbf{V}(\mathrm{H})$ be a $\mathcal{Q}$ – generic over $\mathbf{V}$ filter. We prove that there exists in $\mathbf{V}(\mathrm{H})$ a K which is $\mathbf{P}_{\mathcal{U}}$ – generic over $\mathbf{V}$ such that $\mathrm{K} \cap \mathcal{Q} = \mathrm{g}$. (**2.B.7**)

We suppose that $\langle \bar{z};A \rangle \in \mathbf{P}_{\mathrm{U}}$ and that $\mathrm{x} \in \mathbf{V}(\mathrm{H})$ is a $\mathbf{P}_{\mathcal{U}_\alpha}$–generic sequence, with the property that $\alpha < \beta$ and $\bar{z}\restriction \alpha \subset \mathrm{x} \subset \bar{z}\restriction \alpha \cup A\restriction \alpha$. Then, there exists a $\mathrm{y} \in \mathbf{V}(\mathrm{H})$ such that y is $\mathbf{P}_{\mathcal{U}}$ – generic over $\mathbf{V}$, $\bar{z} \subset \mathrm{y} \subset \bar{z} \cup A$ and $\mathrm{y}\restriction \alpha = \mathrm{x}$. (**2.B.8**)

We define, in $\mathbf{V}(\mathrm{H})$, a forcing $\mathcal{Q}$ with the following *property*:
If $\mathcal{G}^*$ is a $\mathcal{Q}$ -generic filter over $\mathbf{V}(\mathrm{H})$ and G^* is the $\mathbf{P}_{\mathrm{U}}$ – generic over $\mathbf{V}$ filter induced by $\mathcal{G}^*$, then $\mathrm{H} = \mathrm{H}^{\mathrm{G}^*}$. (**3.A.1, 3.A.8**)

The abundance of generic sequences in $\mathbf{V}(\mathrm{H})$ *is the key factor in the proof of the latest property of* $\mathcal{Q}$. *This result is used, in the proof of the partition relation* (¶) *inside the model* $\mathbf{V}(\mathrm{H})$. *(***3.C.1***)*

3.C.1 Theorem. We assume that ϕ is a formula and $\mu < \kappa$ so that $\mathbf{V}(\mathrm{H}) \models \forall \mathrm{s} \in [\kappa]^\omega \exists! \mathrm{x} \in \mathbf{V}_\mu \phi(\mathrm{s}, \mathrm{x}, \vec{\delta}, \mathrm{H})$. Then, there exists an $\mathrm{s} \in [\kappa]^\omega \cap \mathbf{V}(\mathrm{H})$ and an $\mathrm{x}_0 \in \mathbf{V}_\mu$ such that $\mathbf{V}(\mathrm{H}) \models \forall \mathrm{t} \in [\mathrm{s}]^\omega \phi(\mathrm{t}, \mathrm{x}_0, \vec{\delta}, \mathrm{H})$. In particular, $\mathbf{V}(\mathrm{H}) \models (\kappa \text{ is a strong limit}) \;\; \& \;\; \forall \mu < \kappa(\kappa \underset{\mathbf{OD}}{\rightarrow} (\omega)^\omega_{\mathbf{V}_\mu})$. $\square$

Assume that $\mathrm{b} \in \mathrm{H}$. For every formula ϕ and parameters $\mathrm{a} \in \mathbf{V}$, there exists a formula ϕ^* such that

$$\mathbf{V}(\mathrm{H}) \models \phi(\mathrm{a}, \mathrm{b}) \iff \mathbf{V}[\mathrm{b}] \models \phi^*(\mathrm{a}, \mathrm{U}, \kappa, \lambda, \mathrm{b}).$$

The latest (**3.C.3**) is used in the proof of $< \lambda -$ DC inside the inner model $\mathbf{V}(\mathrm{H})$. (**4.A.6**)

The model $\mathbf{V}(\mathrm{H})$ *satisfies the partition relation* (¶) *but unfortunately may not satisfy the Axiom of Choice. Thus we want to force Choice over* $\mathbf{V}(\mathrm{H})$ *and still preserve the partition relation.*

We force AC over $\mathbf{V}(\mathrm{H})$ with $< \lambda -$ sequences of H as conditions and we show that this forcing, say it $\mathcal{P}$, is homogeneous and $< \lambda -$ closed. $< \lambda -$ DC helps in the proof that $\mathcal{P}$ does not add generically any new $< \kappa -$ sequences of κ. (**4.A.11**)

4.B.2 Theorem. Let $\mathbf{V}$ be a transitive model of **ZFC**, in which κ is a supercompact cardinal and λ is an inaccessible cardinal above κ. Let $\mathcal{G}$ be a fixed $\mathcal{P}$ – generic filter over $\mathbf{V}(\mathrm{H})$. If $\mathbf{V}_1 : \stackrel{\mathrm{def}}{=} \mathbf{V}(\mathrm{H})[\mathcal{G}]$, then

$$\mathbf{V}_1 \models [\mathrm{ZFC} + \kappa \text{ is a strong limit } + \forall \mu < \kappa (\kappa \underset{\mathbf{OD}}{\rightarrow} (\omega)^{\omega}_{\mathrm{V}_\mu})]. \qquad \square$$

Chapter 1

OD partitions in a Prikry model

1.A Prikry forcings: Basic facts

We begin with some basic facts about the Supercompact Prikry forcing and the Prikry-tree forcing.

If U is a normal measure on $\mathcal{P}_\kappa(\lambda)$, then $\mathbf{P}_\mathrm{U}$ denotes the Supercompact Prikry forcing induced by U. The version of this forcing which we use is as follows:

1.A.1 Definition *If U is a normal measure over $\mathcal{P}_\kappa(\lambda)$, then the supercompact Prikry forcing induced by U is denoted by $\mathbf{P}_\mathrm{U}$ and*

$$\langle \mathrm{P}_1, ..., \mathrm{P}_\mathrm{n}; \mathrm{A}\rangle \in \mathbf{P}_\mathrm{U}$$

$$\Updownarrow$$

$$\mathrm{P}_1 \underset{\sim}{\subset}, ..., \underset{\sim}{\subset} \mathrm{P}_\mathrm{n} \in \mathcal{P}_\kappa(\lambda) \quad \& \quad \mathrm{A} \in \mathrm{U} \quad \&$$
$$\& \quad (\forall \mathrm{P} \in \mathrm{A})(\mathrm{P}_\mathrm{n} \underset{\sim}{\subset} \mathrm{P}),$$

where $\mathrm{Q} \underset{\sim}{\subset} \mathrm{P}$ *means that* $\mathrm{Q} \subsetneq \mathrm{P} \quad \& \quad |\mathrm{Q}| < |\mathrm{P} \cap \kappa|$. *The sequence* $\langle \mathrm{P}_1, ..., \mathrm{P}_\mathrm{n}\rangle$ *is called the stem of the condition* $\langle \mathrm{P}_1, ..., \mathrm{P}_\mathrm{n}; \mathrm{A}\rangle$.

1.A.1 Definition □

As in the Prikry forcing on a measurable cardinal, it has been proven that $\mathbf{P}_{\mathrm{U}}$ satisfies the so called **Prikry property**, i.e., for any condition and any formula of the forcing language we can shrink the measure one set, leave the stem unchanged and with the new stronger condition we can decide the formula. In addition, as in the Prikry forcing, $\mathbf{P}_{\mathrm{U}}$ has the so called **geometric property**, i.e., a sequence of $\mathcal{P}_\kappa(\lambda)$ – sets is $\mathbf{P}_{\mathrm{U}}$ – generic iff eventually belongs to every U-measure one set. The nontrivial aspect of this result and its corollary–that every subsequence of a $\mathbf{P}_{\mathrm{U}}$ – generic sequence is $\mathbf{P}_{\mathrm{U}}$ – generic–has been proven in [Mathias 73], for the Prikry forcing. However, the same arguments apply for the Supercompact Prikry forcing $\mathbf{P}_{\mathrm{U}}$.

1.A.2 Definition *If $\mathcal{U}$ is a σ –* complete *ultrafilter on a cardinal δ, then the conditions of $\mathbf{P}^*_{\mathcal{U}}$ (the Prikry-tree forcing induced by $\mathcal{U}$ on δ) are of the form* $\langle \mathrm{s;T}\rangle$ *for some $\mathcal{U}$-splitting tree on δ and* $\mathrm{s} \in \mathrm{T}$, *and the order is defined as follows:*

$$\langle \mathrm{s;T}\rangle \leq \langle \mathrm{s';T'}\rangle : \overset{\mathrm{def}}{\Longleftrightarrow} \mathrm{s'} \subseteq \mathrm{s} \quad \& \quad \mathrm{T} \subseteq \mathrm{T'}.$$

1.A.2 Definition □

The Prikry-tree forcing $\mathbf{P}^*_{\mathcal{U}}$ satisfies the **Prikry property** and a variation of the **geometric property**, which is the following: for every $\mathbf{P}^*_{\mathcal{U}}$ – generic sequence g ($g \in {}^{\omega}\delta$), and for every $F : {}^{<\omega}\delta \to \mathcal{U}$, in $\mathbf{V}$, there exists a natural number $k \in \omega$ such that $(\forall N \geq k)(\forall M \geq N) g(M) \in F(g \restriction N)$. These results belong to set-theoretic folklore.

1.B Prikry generic sequences and their models

In this chapter, we are going to show it is consistent that for any G $\mathbf{P}_{\mathrm{U}}$ −generic filter over **V**, the model **V**[G] does not satisfy the partition relation.

Towards this result, first we prove that any two $\mathbf{P}_{\mathrm{U}}$ − generic sequences σ and τ with the property $\mathbf{V}[\sigma] = \mathbf{V}[\tau]$ are tail − equivalent. All the arguments that we are going to use do not depend on the fact that U is a normal measure on $\mathcal{P}_\kappa(\lambda)$.

Hence we are going to show the above result for the Prikry tree-forcing induced by a σ − complete ultrafilter $\mathcal{U}$ on a cardinal δ.

We start with a σ − complete ultrafilter $\mathcal{U}$ on a cardinal δ. The Prikry-tree forcing induced by $\mathcal{U}$ on δ is denoted by $\mathbf{P}^*_{\mathcal{U}}$ and is defined as follows:

1.B.1 Definition *A set T is called a* $\mathcal{U}$ − splitting *tree on* δ *iff* $\mathrm{T} \subseteq \delta^{<\omega}$ *and* $(\forall \mathrm{s} \in \mathrm{T})(\forall \mathrm{t})(\mathrm{t} \subset \mathrm{s} \Rightarrow \mathrm{t} \in \mathrm{T}) \quad \& \quad (\forall \mathrm{s} \in \mathrm{T})(\left\{\alpha \in \delta \mid \mathrm{s}^\frown\alpha \in \mathrm{T}\right\} \in \mathcal{U})$. *The conditions of* $\mathbf{P}^*_{\mathcal{U}}$ *are of the form* $\langle \mathrm{s};\mathrm{T}\rangle$ *for some* $\mathcal{U}$ − splitting *tree T on* δ *and* $\mathrm{s} \in \mathrm{T}$. *In addition if* $\leq$ *denotes the* $\mathbf{P}^*_{\mathcal{U}}$ − order, *then* $\langle \mathrm{s};\mathrm{T}\rangle \leq \langle \mathrm{s}';\mathrm{T}'\rangle :\stackrel{\mathrm{def}}{\Longleftrightarrow} \mathrm{s}' \subseteq \mathrm{s} \quad \& \quad \mathrm{T} \subseteq \mathrm{T}'$. *As usual, for* $\mathrm{s}, \mathrm{s}' \in \delta^{<\omega}$, $\mathrm{s}' \subseteq \mathrm{s}$ *means that there exists some* $\mathrm{k} \leq \mathrm{lh}(\mathrm{s})$ *such that* $\mathrm{s}' = \mathrm{s}\restriction \mathrm{k}$.

1.B.1 Definition □

1.B.2 Definition *Two sequences* $\mathrm{x}, \mathrm{y} \in [\kappa]^\omega$ *are said to be* tail − equivalent *iff for some* $\mathrm{m}', \mathrm{n}' \in \omega$ *and for all* $\mathrm{k} \in \omega$, $\mathrm{x}(\mathrm{n}' + \mathrm{k}) = \mathrm{y}(\mathrm{m}' + \mathrm{k})$.

1.B.2 Definition □

The first result of this thesis is the following theorem:

Theorem A: *If* σ *and* τ *are two* $\mathbf{P}^*_{\mathcal{U}}$ – generic *sequences with the property that* $\mathbf{V}[\sigma] = \mathbf{V}[\tau]$, *then they are* tail – equivalent. $\square$

This theorem is proven in **1.B.9**. For the following results and throughout this part we fix two $\mathbf{P}^*_{\mathcal{U}}$ – generic over $\mathbf{V}$ sequences σ and τ which satisfy the property $\mathbf{V}[\sigma] = \mathbf{V}[\tau]$.

1.B.3 Lemma (H. Woodin) *There exist two reals* α *and* β *(in the ground model* $\mathbf{V}$*) and two sequences* $\langle f_n \mid n \in \omega\rangle$ *and* $\langle g_n \mid n \in \omega\rangle$ *of functions in* $\mathbf{V}$ *such that* $f_n : [\delta]^{\alpha(n)} \to [\delta]^n$ & $g_n : [\delta]^{\beta(n)} \to [\delta]^n$ *with the property that* $f_n(\sigma\restriction \alpha(n)) = \tau\restriction n$ *and* $g_n(\tau\restriction \beta(n)) = \sigma\restriction n$.

Proof: We pick a $\mathbf{P}^*_{\mathcal{U}}$ – name $\dot\tau$ so that for some $k_0 \in \omega$ and some T^* which is a $\mathcal{U}$-splitting tree on δ, $\langle \sigma\restriction k_0; T^*\rangle$ belongs to the $\mathbf{P}^*_{\mathcal{U}}$ – generic filter G induced by σ and

$$\tau = (\dot\tau)^{\mathbf{V}[\sigma]} \quad \& \quad \langle \sigma\restriction k_0; T^*\rangle \Vdash_{\mathbf{P}^*_{\mathcal{U}}} [\dot\tau \text{ is a } \mathbf{P}^*_{\mathcal{U}} - \text{generic sequence}].$$

Let us define in $\mathbf{V}[G]$ a real $\alpha \in [\omega]^\omega$ as follows:

$$\alpha(n) = m \qquad :\overset{\text{def}}{\Longleftrightarrow} \text{m is the least integer} \geq k_0 \text{ such that}$$
$$\exists T[\langle \sigma\restriction m; T\rangle \Vdash_{\mathbf{P}^*_{\mathcal{U}}} (\forall k < n)(\dot\tau(k) = (\tau(k))\check{\ })].$$

Notice that α is well defined, since

$$(\dot\tau)^{\mathbf{V}[\sigma]} = \tau \quad \& \quad \langle \sigma\restriction k_0; T^*\rangle \Vdash_{\mathbf{P}^*_{\mathcal{U}}} [\dot\tau : \check\omega \to \delta \text{ is a } \check{\mathbf{P}}^*_{\mathcal{U}} - \text{generic over } \dot{\mathbf{V}}]$$

hold. Moreover, as $\mathbf{V}$ and $\mathbf{V}[G]$ have the same reals, then $\alpha \in \mathbf{V}$. Next for every $n \in \omega$ we define functions $f_n : [\delta]^{\alpha(n)} \to [\delta]^n$ as follows: Let $\langle \sigma \restriction l^*; S^* \rangle \in G$ such that $\langle \sigma \restriction l^*; S^* \rangle \leq \langle \sigma \restriction k_0; T^* \rangle$ and

$$\langle \sigma \restriction l^*; S^* \rangle \Vdash_{\mathbf{P}^*_{\mathcal{U}}} \Big[\forall n \exists T [\langle \dot{\Gamma} \restriction \check{\alpha}(n); T \rangle \Vdash_{\mathbf{P}^*_{\mathcal{U}}} (\forall k \leq \check{n}) \Big((\dot{\tau})\check{}(k) = ((\dot{\tau}(k))\check{}\,)\check{} \Big) \Big] \Big].$$

Then, for every $\bar{x} \in [\delta]^{\alpha(n)}$, we define

$$f_n(\bar{x}) = \begin{cases} y, & \text{if } \langle \sigma \restriction l^*; S^* \rangle \Vdash_{\mathbf{P}^*_{\mathcal{U}}} [\exists T [\langle \bar{x}; T \rangle \Vdash_{\mathbf{P}^*_{\mathcal{U}}} \forall k < n(\dot{\tau}(k) = (\check{y}(k))\check{}\,)]] \\ \emptyset, & \text{if otherwise.} \end{cases}$$

Each f_n is well defined and by the definability of the forcing relation $\mathbf{P}^*_{\mathcal{U}}$ in $\mathbf{V}$ we have that each f_n is in $\mathbf{V}$. Moreover, $\langle f_n \mid n \in \omega \rangle \in \mathbf{V}$. By the choice of α we conclude that

$$(\forall n < \omega)[f_n(\sigma \restriction \alpha(n)) = \tau \restriction n] \quad \& \quad (\forall x)[f_{n+1}(x) \restriction n = f_n(x \restriction \alpha(n))].$$

The finding of the real α and of the sequence $\langle f_n \mid n \in \omega \rangle$ in $\mathbf{V}$ with the above properties consists of the analysis of the term $\dot{\tau}$ in $\mathbf{V}[\sigma]$.

Similarly, if we find a $\mathbf{P}^*_{\mathcal{U}}$ – name $\dot{\sigma}$ so that $\sigma = (\dot{\sigma})^{\mathbf{V}[\tau]}$ and we interchange the roles of σ and τ in the previous argument, then we will be able to find another real β (in particular $\beta \in [\omega]^\omega$) and another sequence $\langle g_n \mid n \in \omega \rangle \in \mathbf{V}$ of functions such that $g_n : [\delta]^{\beta(n)} \to [\delta]^n$ with the property that $g_n(\tau \restriction \beta(n)) = \sigma \restriction n \quad \& \quad (\forall x)[g_{n+1}(x) \restriction n = g_n(x \restriction \beta(n))]$.

1.B.3 Lemma □

1.B.4 Proposition *Suppose that σ and τ are not* tail – equivalent. *There exist an $n^* \in \omega$, a tree T^*, a $\mathbf{P}^*_{\mathcal{U}}$* – name $\dot{\rho}$, *two reals $\bar{\alpha}, \bar{\beta}$ and two sequences*

$\langle F_n \mid n \in \omega\rangle, \langle G_n \mid n \in \omega\rangle$ *with the following properties: For all* $n \in \omega$, $F_n : [\delta]^{\bar{\alpha}(n)} \to [\delta]^n$ & $G_n : [\delta]^{\bar{\beta}(n)} \to [\delta]^n$ *and*

$$\begin{aligned} \langle\emptyset;T^*\rangle \Vdash_{\mathbf{P}^*_{\mathcal{U}}} \quad & [\dot{\sigma}_{\dot{\Gamma}}, \dot{\rho} \text{ are not tail-equivalent}] \quad \& \\ & \& \quad (\forall n < \omega)[F_n(G_{\bar{\alpha}(n)}(\dot{\rho}\restriction \bar{\beta}(\bar{\alpha}(n))))) = \dot{\rho}\restriction n \quad \& \qquad (\P) \\ & \& \quad G_n(F_{\bar{\beta}(n)}(\dot{\sigma}_{\dot{\Gamma}}\restriction \bar{\alpha}(\bar{\beta}(n))))) = \dot{\sigma}_{\dot{\Gamma}}\restriction n \quad \& \\ & \& \quad F_n(\dot{\sigma}_{\dot{\Gamma}}\restriction \bar{\alpha}(n)) = \dot{\rho}\restriction n \;\&\; G_n(\dot{\rho}\restriction \bar{\beta}(n)) = \dot{\sigma}_{\dot{\Gamma}}\restriction n]. \end{aligned}$$

In addition, for every $\mathbf{P}^*_{\mathcal{U}}$ – generic K *which contains* $\langle\emptyset;T^*\rangle$ *and for* $s :\overset{\text{def}}{=} \sigma\restriction n^*$ *we have that* $(\dot{\rho})^{\mathbf{V}[\sigma_K]} = (\dot{\tau})^{\mathbf{V}[s^\frown\sigma_K]}$.

Proof: We start with the sequences $\langle f_n, g_n \mid n \in \omega\rangle$ found in **1.B.3**. Then we find a condition $\langle s;T\rangle \in G$, where as before G is the $\mathbf{P}^*_{\mathcal{U}}$ – generic filter induced by σ, with the property

$$\begin{aligned} \langle s;T\rangle \Vdash_{\mathbf{P}^*_{\mathcal{U}}} \quad & [\dot{\sigma}_{\dot{\Gamma}}, \dot{\tau} \text{ are not tail-equivalent}] \quad \& \\ & \& \quad (\forall n < \omega)[f_n(\dot{\sigma}_{\dot{\Gamma}}\restriction \alpha(n)) = \dot{\tau}\restriction n \quad \& \quad g_n(\dot{\tau}\restriction \beta(n)) = \dot{\sigma}_{\dot{\Gamma}}\restriction n]. \end{aligned}$$

Let $n^* :\overset{\text{def}}{=} lh(s)$ and $(T)_s$ be the part of the tree T below the node s, i.e., $(T)_s :\overset{\text{def}}{=} \left\{t \in \delta^{<\omega} \mid s^\frown t \in T\right\}$. For all $n < \omega$ we define $\bar{\alpha}(n) :\overset{\text{def}}{=} \alpha(n^* + n) - n^*$, $\bar{\beta}(n) :\overset{\text{def}}{=} \beta(n^* + n)$ and for all $x \in [\delta]^{\bar{\alpha}(n)} \cap (T)_s$, $y \in [\delta]^{\bar{\beta}(n)} \cap (T)_s$ $F_n(x) :\overset{\text{def}}{=} f_{n^*+n}(s^\frown x)\restriction n$, and $G_n(y) :\overset{\text{def}}{=} g_{n^*+n}(y)\restriction [n^*, n^* + n)$. Let K be any $\mathbf{P}^*_{\mathcal{U}}$ – generic filter such that $\langle\emptyset;(T)_s\rangle \in K$. Then $s^\frown\sigma_K$ is a $\mathbf{P}^*_{\mathcal{U}}$ – generic sequence, and if K^* denotes the corresponding $\mathbf{P}^*_{\mathcal{U}}$ – generic filter, then we have that $\langle s;T\rangle \in K^*$.

Next we observe that $\langle\emptyset;(T)_s\rangle \Vdash_{\mathbf{P}^*_{\mathcal{U}}} [(\exists x)(x = (\dot\tau)^{\mathbf{V}[s^\frown\dot\sigma_{\dot\Gamma}]})]$, where $\dot\Gamma$ is the canonical name for the $\mathbf{P}^*_{\mathcal{U}}$ – generic filter and $\dot\sigma_{\dot\Gamma}$ is the canonical name for the generic sequence induced by $\dot\Gamma$. Then the set $\mathcal{A} = \{p \in \mathbf{P}^*_{\mathcal{U}} \mid (\exists\dot\tau')(p \Vdash_{\mathbf{P}^*_{\mathcal{U}}} \left[\dot\tau' = (\dot\tau)^{\mathbf{V}[s^\frown\dot\sigma_{\dot\Gamma}]})\right])\}$ is $\mathbf{P}^*_{\mathcal{U}}$ – dense below $\langle\emptyset;(T)_s\rangle$. Let $\mathcal{I}$ be a maximal antichain below $\langle\emptyset;(T)_s\rangle$ included in $\mathcal{A}$. For every $p \in \mathcal{I}$ choose a $\dot\tau_p$ such that $p \Vdash_{\mathbf{P}^*_{\mathcal{U}}} [\dot\tau_p = (\dot\tau)^{\mathbf{V}[s^\frown\dot\sigma_{\dot\Gamma}]})]$. Let $\dot\rho$ be the $\mathbf{P}^*_{\mathcal{U}}$ – name which is the mixture of $\{\dot\tau_p \mid p \in \mathcal{I}\}$. Since K is a $\mathbf{P}^*_{\mathcal{U}}$ – generic filter which contains the condition $\langle\emptyset;(T)_s\rangle$, then we have that $(\dot\rho)^{\mathbf{V}[K]} = (\dot\tau_p)^{\mathbf{V}[K]}$, whenever $p \in \mathcal{I} \cap K$. Moreover, $s^\frown\sigma_K$ is a $\mathbf{P}^*_{\mathcal{U}}$ – generic sequence, and if K^* denotes the corresponding $\mathbf{P}^*_{\mathcal{U}}$ – generic filter, then we have that $\langle s;T\rangle \in K^*$. By the definition of $\dot\rho$, we conclude that for the unique $p \in \mathcal{I} \cap K$, $(\dot\rho)^{\mathbf{V}[K]} = (\dot\tau_p)^{\mathbf{V}[K]} = (\dot\tau)^{\mathbf{V}[s^\frown\sigma_K]} = (\dot\tau)^{\mathbf{V}[\sigma]} = \tau$.

Since $\langle s;T\rangle \Vdash_{\mathbf{P}^*_{\mathcal{U}}} [(\forall n < \omega)(f_n(\dot\sigma_{\dot\Gamma}\restriction\alpha(n)) = \dot\tau\restriction n \quad \& \quad g_n(\dot\tau\restriction\beta(n)) = \dot\sigma_{\dot\Gamma}\restriction n)]$ holds, then for every $n < \omega$

$$\begin{aligned} F_n((\sigma_{\dot\Gamma})^{\mathbf{V}[K]}\restriction\bar\alpha(n)) &= f_{n^*+n}(s^\frown(\dot\sigma_{\dot\Gamma})^{\mathbf{V}[\sigma_K]}\restriction\alpha(n^*+n) - n^*)\restriction n \\ &= f_{n^*+n}((s^\frown\sigma_K)\restriction\alpha(n^*+n))\restriction n \\ &= (\dot\tau)^{\mathbf{V}[s^\frown\sigma_K]}\restriction n^* + n\restriction n \\ &= ((\dot\tau)^{\mathbf{V}[s^\frown\sigma_K]})\restriction n \\ &= (\dot\rho)^{\mathbf{V}[K]}\restriction n. \end{aligned}$$

Thus $\langle\emptyset;T^*\rangle \Vdash_{\mathbf{P}^*_{\mathcal{U}}} (\forall n < \omega)[F_n(\dot\sigma_{\dot\Gamma}\restriction\bar\alpha(n)) = \dot\rho\restriction n]$. Similarly, for all $n < \omega$ we have that

$$G_n((\dot\rho)^{\mathbf{V}[K]}\restriction\bar\beta(n)) = g_{n^*+n}((\dot\tau)^{\mathbf{V}[K]}\restriction\beta(n^*+n))\restriction[n^*, n^*+n)$$

$$= ((s^\frown\sigma_K)\restriction n^* + n)\restriction [n^*, n^* + n)$$

$$= (\dot{\sigma}_{\dot{\Gamma}})^{V[K]}\restriction n.$$

Thus

$$\langle\emptyset;T^*\rangle \Vdash_{P^*_{\mathcal{U}}} \quad [\dot{\sigma}_{\dot{\Gamma}}, \dot{\rho} \text{ are not tail}-\text{equivalent}] \quad \&$$
$$[F_n(\dot{\sigma}_{\dot{\Gamma}}\restriction\bar{\alpha}(n)) = \dot{\rho}\restriction n \quad \& \quad G_n(\dot{\rho}\restriction\eta(n)) = \dot{\sigma}_{\dot{\Gamma}}\restriction n].$$

This implies (¶). Now the proposition follows if we set $T^* = (T)_s$.

1.B.4 Proposition □

The notation and the assertions of **1.B.4** are used in the next proposition.

1.B.5 Proposition *If $\mathcal{U}^k$ denotes the (finite) k-th iterate of the ultrafilter $\mathcal{U}$, then*

$$(\exists n_1 \in \omega)(\forall n > n_1) \quad (\forall k \in \omega)(\forall A \in \mathcal{U}^k)(\exists B \in \mathcal{U}^{\bar{\alpha}(n+k)})$$
$$(\forall t \in B)[F_{n+k}(t)\restriction [n, n+k) \in A] \quad \&$$
$$(\exists n_2 \in \omega)(\forall n > n_2) \quad (\forall k \in \omega)(\forall A \in \mathcal{U}^k)(\exists B \in \mathcal{U}^{\bar{\beta}(n+k)})$$
$$(\forall t \in B)[G_{n+k}(t)\restriction [n, n+k) \in A].$$

Proof: We assume towards a contradiction that such n_1 does not exist. Then we choose an infinite sequence $\langle n_i, k_i, A_i, B_i \mid i \in \omega\rangle$ such that for all $i \in \omega$

1. $n_i + k_i < n_{i+1}$,

2. $A_i \in \mathcal{U}^{k_i}$,

3. $B_i \in \mathcal{U}^{\bar{\alpha}(n_i+k_i)}$,

4. $(\forall t \in B_i)[F_{n_i+k_i}(t)\restriction[n_i, n_i + k_i) \notin A_i]$.

Using the $\langle B_i \mid i \in \omega\rangle$ we construct a $\mathcal{U}$ – splitting tree over δ, $T^\star$ such that $T^\star \subset T^*$ and $(\forall l \in \omega)(\forall t \in T^\star)(lh(t) = \bar{\alpha}(n_l+k_l) \Rightarrow t \in B_l)$. Next, we consider the set

$$\mathcal{S} \stackrel{\text{def}}{=} \left\{p \in \mathbf{P}^*_{\mathcal{U}} \mid (\exists i)(p \Vdash_{\mathbf{P}^*_{\mathcal{U}}} [\dot{\sigma}_{\dot{\Gamma}}\restriction[n_i, n_i + k_i) \in A_i])\right\}$$

and we show that $\mathcal{S}$ is $\mathbf{P}^*_{\mathcal{U}}$ – dense. Towards this end, we take $\langle u;T\rangle \in \mathbf{P}^*_{\mathcal{U}}$ and we assume that $lh(u) = m_0$. Then, we pick $i \in \omega$ such that $n_i > m_0$ and we shrink appropriately the tree T to another tree T' with the properties that $(\forall t \in T')(lh(t) \geq n_i + k_i \Rightarrow t\restriction[n_i, n_i + k_i) \in A_i)$ and $\langle u;T'\rangle \leq \langle u;T\rangle$. Obviously $\langle u;T'\rangle \in \mathcal{S}$, since for every generic K containing $\langle u;T'\rangle$, the induced generic sequence σ_K is a branch through T'. Hence, $\mathcal{S}$ is indeed $\mathbf{P}^*_{\mathcal{U}}$ – dense. If K is any $\mathbf{P}^*_{\mathcal{U}}$ – generic filter that contains $\langle\emptyset;T^\star\rangle$, then $(\dot{\rho})^{\mathbf{V}[K]}$ is a $\mathbf{P}^*_{\mathcal{U}}$ – generic sequence and so the filter induced by $(\dot{\rho})^{\mathbf{V}[K]}$ meets the set $\mathcal{S}$. Thus there exists an $i \in \omega$ such that $(\dot{\rho})^{\mathbf{V}[K]}\restriction[n_i, n_i + k_i) \in A_i$. In addition the choice of K implies that $F_n(\sigma_K\restriction\bar{\alpha}(n_i + k_i))\restriction[n_i, n_i + k_i) \notin A_i$. Since $\langle\emptyset;T'\rangle \leq \langle\emptyset;T^*\rangle$, then $F_n(\sigma_K\restriction\bar{\alpha}(n_i + k_i)) = (\dot{\rho})^{\mathbf{V}[K]}\restriction n_i + k_i$ and consequently $(\dot{\rho})^{\mathbf{V}[K]}\restriction[n_i, n_i + k_i) \notin A_i$. But the latest is a contradiction.

Similarly, we argue and prove the existence of n_2.

1.B.5 Proposition □

Let us fix the two natural numbers n_1, n_2 given by **1.B.5** and let us take $m_0 :\overset{\text{def}}{=} 1+\max\{n_1, n_2\}$. Then, we consider $\bar{\sigma} :\overset{\text{def}}{=} \sigma^* \restriction [m_0, \omega)$ & $\bar{\tau} :\overset{\text{def}}{=} \tau \restriction [m_0, \omega)$. Since $\mathbf{V}[\sigma^*] = \mathbf{V}[\tau]$, then $\mathbf{V}[\bar{\sigma}] = \mathbf{V}[\bar{\tau}]$.

1.B.6 Lemma *Suppose that the two generic sequences* σ, τ *are not* tail – equivalent. *Then there exist a* $n' \in \omega$, *a* $\mathcal{U}$ – splitting *tree* T'', *two sequences of functions* $\langle \mathbf{f}_n \mid n \in \omega \rangle$, $\langle \mathbf{g}_n \mid n \in [n', \omega) \rangle$, *two reals* γ, η *and a* $\mathbf{P}^*_{\mathcal{U}}$ – name $\dot{\rho}_0$ *which have the following properties:*

$$\langle \emptyset; T'' \rangle \quad \underset{\mathbf{P}^*_{\mathcal{U}}}{\Vdash} [\dot{\sigma}_{\dot{\Gamma}}, \dot{\rho}_0 \text{ are not tail – equivalent}] \quad \&$$
$$\& \quad (\forall n < \omega)\Big[\mathbf{f}_n(\mathbf{g}_{\gamma(n)}(\dot{\rho}_0 \restriction \eta(\gamma(n))) = \dot{\rho}_0 \restriction n \quad \& \qquad (\clubsuit)$$
$$\& \quad \mathbf{g}_n(\mathbf{f}_{\eta(n)}(\dot{\sigma}_{\dot{\Gamma}} \restriction \gamma(\eta(n))))) = \dot{\sigma}_{\dot{\Gamma}} \restriction n \quad \&$$
$$\& \quad \mathbf{f}_n(\dot{\sigma}_{\dot{\Gamma}} \restriction \gamma(n)) = \dot{\rho}_0 \restriction n \quad \& \quad \mathbf{g}_n(\dot{\rho}_0 \restriction \eta(n)) = \dot{\sigma}_{\dot{\Gamma}} \restriction n\Big].$$

In addition, for every $\mathbf{P}^*_{\mathcal{U}}$ – generic K *which contains* $\langle \emptyset; T'' \rangle$ *we have that*

$$\tau \restriction m_0 {}^\frown (\dot{\rho}_0)^{\mathbf{V}[\sigma_K]} = (\dot{\rho})^{\mathbf{V}[\sigma^* \restriction m_0 {}^\frown \sigma_K]},$$

$$\mathbf{f}_n(\bar{\sigma} \restriction \gamma(n)) = \bar{\tau} \restriction n \quad \& \quad \mathbf{g}_n(\bar{\tau} \restriction \eta(n)) = \bar{\sigma} \restriction n,$$

and for every $k \in \omega$ *the following hold:*

$$\Big(A \in \mathcal{U}^k \Rightarrow \{y \in \delta^{\gamma(k)} \mid \mathbf{f}_k(y) \in A\} \in \mathcal{U}^{\gamma(k)}\Big), \qquad (\dagger)$$
$$\Big(A \in \mathcal{U}^k \Rightarrow \{y \in \delta^{\eta(k)} \mid \mathbf{g}_k(y) \in A\} \in \mathcal{U}^{\eta(k)}\Big). \qquad (\dagger\dagger)$$

Proof: We are using the techniques of **1.B.4** for $\langle\emptyset;T^*\rangle$, $\sigma^*, \dot\rho, \bar\alpha, \bar\beta, m_0$ and $\langle F_n, G_n \mid n < \omega\rangle$ as follows: let $n' :\stackrel{\text{def}}{=} m_0$ and $T'' :\stackrel{\text{def}}{=} (T^*)_{\sigma^*\restriction m_0}$ be the subtree of T^* below $\sigma^*\restriction m_0$; for $n \in \omega$, $\gamma(n) :\stackrel{\text{def}}{=} \bar\alpha(n+m_0)-m_0$ and $\eta(n) :\stackrel{\text{def}}{=} \bar\beta(n+m_0)-m_0$; we also define for $x \in [\delta]^{\gamma(n)} \cap T''$, $y \in [\delta]^{\eta(n)} \cap T''$ that

$$f_n(x) :\stackrel{\text{def}}{=} \quad F_{m_0+n}(\sigma^*\restriction m_0{}^\frown x)\restriction [m_0, m_0+n)$$

$$g_n(y) :\stackrel{\text{def}}{=} \quad G_{m_0+n}(\tau\restriction m_0{}^\frown y)\restriction [m_0, m_0+n).$$

In addition, we construct a name $\dot\rho_0$ as an appropriate mixture in a way analogous to the construction of $\dot\rho$ in **1.B.4**, so that for every $\mathbf{P}^*_{\mathcal{U}}$ – generic K which contains $\langle\emptyset;T''\rangle$ we have that $\tau\restriction m_0{}^\frown(\dot\rho_0)^{\mathbf{V}[\sigma_K]} = (\dot\rho)^{\mathbf{V}[\sigma^*\restriction m_0{}^\frown\sigma_K]}$. If K is a $\mathbf{P}^*_{\mathcal{U}}$ – generic filter which contains $\langle\emptyset;T''\rangle$, then K^* denotes the $\mathbf{P}^*_{\mathcal{U}}$ – generic induced by the generic sequence $\sigma^*\restriction m_0{}^\frown\sigma_K$. Thus

$$\begin{aligned}
f_n((\dot\sigma_{\dot\Gamma})^{\mathbf{V}[K]}\restriction\gamma(n)) &= F_{m_0+n}(\sigma^*\restriction m_0{}^\frown(\sigma_K\restriction\bar\alpha(m_0+n)-m_0))\restriction[m_0,m_0+n) \\
&= F_{m_0+n}(\sigma_{K^*}\restriction\bar\alpha(m_0+n))\restriction[m_0,m_0+n) \\
&= ((\dot\rho)^{\mathbf{V}[\sigma^*\restriction m_0{}^\frown\sigma_K]}\restriction m_0+n)\restriction[m_0,m_0+n) \\
&= (\dot\rho)^{\mathbf{V}[\sigma^*\restriction m_0{}^\frown\sigma_K]}\restriction[m_0,m_0+n) \\
&= (\dot\rho_0)^{\mathbf{V}[K]}\restriction n.
\end{aligned}$$

Similarly, for all $n < \omega$

$$\begin{aligned}
g_n(\quad &(\dot\rho_0)^{\mathbf{V}[K]}\restriction\eta(n)) = \\
&= G_{m_0+n}(\tau\restriction m_0{}^\frown(\dot\rho_0)^{\mathbf{V}[K]}\restriction\bar\beta(m_0+n)-m_0)\restriction[m_0,m_0+n) \\
&= G_{m_0+n}((\dot\rho)^{\mathbf{V}[\sigma^*\restriction m_0{}^\frown\sigma_K]}\restriction\bar\beta(m_0+n))\restriction[m_0,m_0+n)
\end{aligned}$$

$$= ((\sigma^* \restriction m_0 {}^\frown \sigma_K) \restriction m_0 + n) \restriction [m_0, m_0 + n)$$

$$= (\dot{\sigma}_{\dot{\Gamma}})^{\mathbf{V}[K]} \restriction n.$$

Hence, $\mathbf{g}_n((\dot{\rho}_0)^{\mathbf{V}[K]} \restriction \eta(n)) = (\dot{\sigma}_{\dot{\Gamma}})^{\mathbf{V}[K]} \restriction n$. Now it is straightforward to show that $\langle \mathbf{f}_n, \mathbf{g}_n \mid n \in \omega \rangle$ satisfy (♣).

Using the trees T^* (as in **1.B.4**) and T'' we construct another $\mathcal{U}$ – splitting tree over δ, which we denote by T^{**}, such that

1. $T^{**} \subset T^*$ and
2. $(\forall t \in T^{**})\Big(\mathrm{lh}(t) \geq m_0 \Rightarrow t \restriction [m_0, \mathrm{lh}(t)) \in T''\Big)$.

Then, it is easy to show that

$$\langle \emptyset; T^{**} \rangle \Vdash_{\mathbf{P}^*_{\mathcal{U}}} \quad (\forall n \in \omega)[F_{m_0+n}(\dot{\sigma}_{\dot{\Gamma}} \restriction \bar{\alpha}(m_0 + n)) = \dot{\rho} \restriction m_0 + n \quad \&$$

$$\& \quad G_{m_0+n}(\dot{\rho} \restriction \bar{\beta}(m_0 + n)) = \dot{\sigma}_{\dot{\Gamma}} \restriction m_0 + n \quad \&$$

$$\& \quad \mathbf{f}_n(\dot{\sigma}_{\dot{\Gamma}} \restriction [m_0, m_0 + \gamma(n))) = \dot{\rho} \restriction [m_0, m_0 + n) \quad \&$$

$$\& \quad \mathbf{g}_n(\dot{\rho} \restriction [m_0, m_0 + \eta(n))) = \dot{\sigma}_{\dot{\Gamma}} \restriction [m_0, m_0 + n)].$$

Therefore,

$$\langle \emptyset; T^{**} \rangle \Vdash_{\mathbf{P}^*_{\mathcal{U}}} \quad (\forall n \in \omega)[F_{m_0+n}(\dot{\sigma}_{\dot{\Gamma}} \restriction \bar{\alpha}(n + m_0)) \restriction [m_0, n + m_0) =$$

$$= \mathbf{f}_n(\dot{\sigma}_{\dot{\Gamma}} \restriction [m_0, m_0 + \gamma(n))) \quad \&$$

$$\& \quad G_{m_0+n}(\dot{\rho} \restriction \bar{\beta}(m_0 + n)) \restriction [m_0, n + m_0) =$$

$$= \mathbf{g}_n(\dot{\rho} \restriction [m_0, m_0 + \eta(n)))].$$

Next, we observe that $(\forall n \in \omega)(\forall k < l \leq n)(\forall A)(A \in \mathcal{U}^n \Rightarrow \{x\restriction [k,l) \in \delta^{l-k} \mid x \in A\} \in \mathcal{U}^{l-k})$. Using the above fact we conclude that for all $k < \omega$,

$$
\begin{aligned}
A \quad & \in \mathcal{U}^k \Rightarrow \\
& \Rightarrow \left\{x \in \delta^{\bar{\alpha}(m_0+k)} \mid F_{m_0+k}(x)\restriction [m_0, m_0+k) \in A\right\} \in \mathcal{U}^{\bar{\alpha}(m_0+k)} \\
& \Rightarrow \left\{x \in \delta^{\bar{\alpha}(m_0+k)} \mid \mathbf{f}_k(x\restriction [m_0, m_0+\gamma(k))) \in A\right\} \in \mathcal{U}^{\bar{\alpha}(m_0+k)} \\
& \Rightarrow \left\{x\restriction [m_0, m_0+\gamma(k)) \in \delta^{\gamma(k)} \mid \mathbf{f}_k(x\restriction [m_0, m_0+\gamma(k))) \in A\right\} \in \mathcal{U}^{\gamma(k)} \\
& \Rightarrow \left\{y \in \delta^{\gamma(k)} \mid \mathbf{f}_k(y) \in A\right\} \in \mathcal{U}^{\gamma(k)}.
\end{aligned}
$$

Thus $(\forall k \in \omega)(A \in \mathcal{U}^k \Rightarrow \left\{y \in \delta^{\gamma(k)} \mid \mathbf{f}_k(y) \in A\right\} \in \mathcal{U}^{\gamma(k)})$. Similarly, we prove that $(\forall k \in \omega)(A \in \mathcal{U}^k \Rightarrow \left\{y \in \delta^{\eta(k)} \mid \mathbf{g}_k(y) \in A\right\} \in \mathcal{U}^{\eta(k)})$.

Moreover, we observe that on T″ the sequence $\langle \mathbf{f}_n \mid n < \omega\rangle$ is coherent in the following sence: For all $x \in T''$, $\mathbf{f}_{n+1}(x)\restriction n = \mathbf{f}_n(x\restriction \gamma(n))$ and in a similar fashion the sequence $\langle \mathbf{g}_n \mid n < \omega\rangle$ is coherent.

1.B.6 Lemma □

1.B.7 •

In the following, we repeat some well-known facts about iterated ultrapowers. We consider the iteration of ultrapowers of **V** using the measure $\mathcal{U}$ and we denote the n − th iterate of **V** by $\mathbf{M}_n$ and the corresponding embeddings by $\mathbf{j}_{0n}$, i.e.,

$\mathbf{M}_0 = \mathbf{V}$

$\mathbf{M}_n$ = Mostowski collapse of $\mathbf{V}^{\delta^k}/\mathcal{U}^k$ for $n < \omega$ and

$\mathbf{M}_\omega = \text{dirlimit}\{\mathbf{M}_n \mid n < \omega\}$.

$\mathbf{j}_{\alpha\beta} : \mathbf{M}_\alpha \to \mathbf{M}_\beta$ is elementary embedding, where $\alpha \leq \beta \leq \omega$ and $\mathbf{j}_{\alpha\alpha} = \text{id}\restriction \mathbf{M}_\alpha$.

We define a sequence $\langle \delta_n \mid 1 \leq n < \omega \rangle$ as follows: For every $1 \leq n < \omega$, $\delta_n :\overset{\text{def}}{=} \mathbf{j}_{n\omega}(\gamma_n)$ where $\gamma_n :\overset{\text{def}}{=} [\text{id}\restriction \mathbf{j}_{0n-1}(\delta)]^{\mathbf{M}_{n-1}}_{\mathbf{j}_{0n-1}(\mathcal{U})}$ and $[f]^{\mathbf{M}_{n-1}}_{\mathbf{j}_{0n-1}(\mathcal{U})}$ denotes the element of $\mathbf{M}_n$ represented in the corresponding ultrapower of $\mathbf{M}_{n-1}$ by the function f. Using the elementarity of $\mathbf{j}$'s we can prove that

$$(\forall A)\Big(A \in \mathcal{U}^k \iff \langle \delta_1, \dots, \delta_k \rangle \in \mathbf{j}_{0\omega}(A)\Big). \tag{$*$}$$

In addition, we can show that

$$\mathbf{M}_\omega = \{\mathbf{j}_{0\omega}(F)(\langle \delta_1, \dots, \delta_n \rangle) \mid 1 \leq n < \omega \quad \& \quad F \in \mathbf{V}\}.$$

To prove the above characterization of $\mathbf{M}_\omega$, we show by induction on $n < \omega$, that

$$\mathbf{M}_{n+1} = \Big\{\mathbf{j}_{0n+1}(f)(\langle \mathbf{j}_{1n+1}(\gamma_1), \dots \mathbf{j}_{n+1n+1}(\gamma_{n+1}) \rangle) \mid f \in \mathbf{V}\Big\}.$$

Since $(*)$ holds, then using the "geometric property" of the Prikry-tree forcing, we show that $\langle \delta_n \mid 1 \leq n < \omega \rangle$ is a $\mathbf{P}^*_{\mathbf{j}_{0\omega}(\mathcal{U})}$ – generic sequence over $\mathbf{M}_\omega$ below $\mathbf{j}_{0\omega}(\delta)$.

Our next step is to use all the properties of the sequences $\langle \mathbf{f}_n, \mathbf{g}_n \mid n < \omega \rangle$ to define another sequence of ordinals $\langle \varepsilon_n \mid 1 \leq n < \omega \rangle$ below $\mathbf{j}_{0\omega}(\delta)$. For $n < \omega$ let us define $s_n :\overset{\text{def}}{=} \mathbf{j}_{0\omega}(\mathbf{f}_n)(\langle \delta_1, \dots, \delta_{\gamma(n)} \rangle)$, and by the coherence of the functions $\mathbf{f}_n$'s we conclude that $\bigcup_{1 \leq \omega} s_n$ is a well-defined sequence. Hence, we let $\langle \varepsilon_n \mid n < \omega \rangle :\overset{\text{def}}{=} \bigcup_{1 \leq n \leq \omega} s_n$, i.e.,

$$(\forall 1 \leq n < \omega)\Big[\langle \varepsilon_1, \dots, \varepsilon_n \rangle = \mathbf{j}_{0\omega}(\mathbf{f}_n)(\langle \delta_1, \dots, \delta_{\gamma(n)} \rangle)\Big]. \tag{¶}$$

1.B.7 □

Under the above notation we prove the following lemma.

1.B.8 Lemma *For every natural number n, we have that* $\varepsilon_n = \delta_n$.

Proof: The basic idea of the proof appears already in the case n = 1. The same argument works for the general case, except that the notation becomes quite involved.

Assume, towards a contradiction, that $\varepsilon_1 < \delta_1$. Then by (¶) of **1.B.7**, we have that

$$\mathbf{j}_{0\omega}(\mathbf{f}_n)(\langle\delta_1,\ldots,\delta_{\gamma(1)}\rangle) < \delta_1. \qquad (¶¶)$$

Before we proceed, we establish the following notation:
For a sequence of functions $\langle g_i \mid 1 \leq i \leq \omega\rangle$ we define $\prod_{1\leq i\leq 1} g_i :\overset{\text{def}}{=} \mathrm{id}$ and for $m > 1$ $\prod_{1\leq i\leq m} g_i :\overset{\text{def}}{=} \underbrace{g_1 \circ \ldots \circ g_m}_{m-\text{times}}$. Also, if g is a function, we define $g^{(0)} :\overset{\text{def}}{=} \mathrm{id}$ and $g^{(1)} :\overset{\text{def}}{=} g$ and for $m > 0$ $g^{(m+1)} :\overset{\text{def}}{=} g^{(m)} \circ g$ and if $x = \langle x_i \mid 1 \leq i \leq n\rangle$ then $(x)_i = x_i$. Using (†) of **1.B.6**, (∗) of **1.B.7** and (¶¶) of **1.B.7**, we define the following sequence of sets:

$$A_1 :\overset{\text{def}}{=} \{x \in \delta^{\gamma(1)} \mid \mathbf{f}_1(x) < (x)_1\}.$$

By (∗) of **1.B.7** we conclude that $A_1 \in \mathcal{U}^{\gamma(1)}$. Then, by induction on $1 \leq n \in \omega$ and by using (†), we define $A_n \in \mathcal{U}^{\gamma^{(n)}(1)}$

$$A_n :\overset{\text{def}}{=} \{x \in \delta^{\gamma^{(n)}(1)} \mid (\prod_{0\leq k\leq n-1} \mathbf{f}_{\gamma^{(k)}(1)})(x) < ((\prod_{1\leq k\leq n-1} \mathbf{f}_{\gamma^{(k)}(1)})(x))_1\}.$$

Since the sequence $\langle A_n \mid n < \omega\rangle$ belongs to **V**, then by the σ – completeness of the measure $\mathcal{U}$ we can find a sequence $z \in \delta^\omega$ such that

$$(\forall n < \omega)[z_n :\stackrel{\text{def}}{=} z \upharpoonright \gamma^{(n)}(1) \in A_n].$$

Let us define $\theta_n :\stackrel{\text{def}}{=} (\prod_{0\leq k\leq n-1} \mathbf{f}_{\gamma^{(k)}(1)})(z_n)$. Then, due to the coherence of the functions $\mathbf{f}_n$'s, we observe that for all $n \in \omega$

$$\theta_n = ((\prod_{1\leq k\leq n} \mathbf{f}_{\gamma^{(k)}(1)})(z_{n+1}))_1.$$

Hence, $(n \in \omega)(\theta_n > \theta_{n+1})$ which contradicts well – foundedness. Therefore, $\varepsilon_1 \not< \delta_1$.

On the other hand if $\delta_1 < \varepsilon_1$, then

$$\delta_1 < \mathbf{j}_{0\omega}(\mathbf{f}_1)(\langle\delta_1, \dots, \delta_{\gamma(1)}\rangle). \tag{¶¶¶}$$

Thus by $(*)$ of **1.B.7** we conclude that $\{x \in \delta^{\gamma(1)} \mid (x)_1 < \mathbf{f}_1(x)\} \in \mathcal{U}^{\gamma(1)}$. Next by applying $(\dagger\dagger)$ of **1.B.6**, we get that

$$B_1 :\stackrel{\text{def}}{=} \{x \in \delta^{\eta(\gamma(1))} \mid x \in T'' \quad \& \quad (\mathbf{g}_{\gamma(1)}(x))_1 < \mathbf{f}_1(\mathbf{g}_{\gamma(1)}(x))\} \in \mathcal{U}^{\eta(\gamma(1))}.$$

Since for all $x \in T''$ we have that $\mathbf{f}_1(\mathbf{g}_{\gamma(1)}(x)) = (x)_1$, then

$$B_1 = \{x \in \delta^{\eta(\gamma(1))} \mid (\mathbf{g}_{\gamma(1)}(x))_1 < (x)_1\} \in \mathcal{U}^{\eta(\gamma(1))}.$$

Then, by induction on $n > 1$, we define

$$B_n :\stackrel{\text{def}}{=} \{x \in \delta^{\eta^{(n)}(\gamma(1))} \mid ((\prod_{0\leq k\leq n-1} \mathbf{g}_{\eta^{(k)}(\gamma(1))})(x))_1 < ((\prod_{1\leq k\leq n-1} \mathbf{g}_{\eta^{(k)}(\gamma(1))})(x))_1\}.$$

Using (∗) of **1.B.7**, we get that $B_n \in \mathcal{U}^{\eta^{(n)}(\gamma(1))}$. Again, by the $\sigma-$completeness of the measure $\mathcal{U}$ we can find a $w \in \delta^\omega$ such that

$$(\forall n < \omega)[w_n :\stackrel{\text{def}}{=} w \upharpoonright \gamma^{(n)}(\gamma(1)) \in B_n].$$

As before we set $\zeta_n :\stackrel{\text{def}}{=} ((\prod_{0 \leq k \leq n-1} g_{\eta^{(k)}(\gamma(1))})(w_n))_1$. But the coherence of the functions g_n's, implies that $\zeta_n = ((\prod_{1 \leq k \leq n} g_{\eta^{(k)}(\gamma(1))})(w_{n+1}))_1$. Hence, $(\forall n \in \omega)(\zeta_n > \zeta_{n+1})$ which contradicts wellfoundness. Thus $\varepsilon_1 \not> \delta_1$. Therefore, $\varepsilon_1 = \delta_1$.

Using exactly the same arguments, we show that $(\forall n < \omega)(\varepsilon_n = \delta_n)$.

1.B.8 Lemma □

1.B.9 Theorem *Let* $\mathcal{U}$ *be a* $\sigma-$complete *ultrafilter on a cardinal* δ *and* $\mathbf{P}^*_{\mathcal{U}}$ *be the Prikry-tree forcing induced by* $\mathcal{U}$ *on* δ*. If* σ *and* τ *are two* $\mathbf{P}^*_{\mathcal{U}}-$generic *over* **V** *sequences with the property* $\mathbf{V}[\sigma] = \mathbf{V}[\tau]$*, then they are* tail – equivalent.

Proof: Suppose that the generic sequences σ, τ are not tail – equivalent. Then the generic sequences $\bar{\sigma}, \bar{\tau}$ are not tail – equivalent (in the notation of **1.B.6**). Then, by (¶) of **1.B.7**, we conclude that

$(\forall 1 \leq n < \omega)[\langle \delta_1, \ldots, \delta_n \rangle = \mathbf{j}_{0\omega}(\mathbf{f}_n)(\langle \delta_1, \ldots, \delta_{\gamma(n)} \rangle)]$. Hence for every n such that $1 \leq n < \omega$, there exists a set $D_n \in \mathcal{U}^{\gamma(n)}$ such that $D_n :\stackrel{\text{def}}{=} \{x \in \delta^{\gamma(n)} \mid \mathbf{f}_n(x) = x \upharpoonright n\}$. From the sequence $\langle D_n \mid 1 \leq n < \omega \rangle$ and using the σ – completeness of the measure $\mathcal{U}$ we construct a $\mathcal{U}$ – splitting tree $\mathcal{T}$ over δ such that

$$(\forall n < \omega)[x \in \mathcal{T} \quad \& \quad lh(x) \geq \gamma(n) \Rightarrow x \upharpoonright \gamma(n) \in D_n].$$

Therefore, for all $n < \omega$ and all x, if $x \in \mathcal{T}$ and $lh(x) = \gamma(n)$, then $\mathbf{f}(x) = x \restriction n$. Let $\mathcal{T}^* :\stackrel{\text{def}}{=} T'' \cap \mathcal{T}$. Then, for all $n < \omega$

$$\langle \emptyset; \mathcal{T}^* \rangle \underset{\mathbf{P}^*_{\mathcal{U}}}{\Vdash} \left[f_n(\dot{\sigma}_{\dot{\Gamma}} \restriction \gamma(n)) = \dot{\rho}_0 \restriction n \ \& \ g_{n'+n}(\dot{\rho}_0 \restriction \eta(n'+n)) = \dot{\sigma}_{\dot{\Gamma}} \restriction n \right].$$

But then the choice of the tree $\mathcal{T}^*$ obligates that for all $n < \omega$

$$\langle \emptyset; \mathcal{T}^* \rangle \underset{\mathbf{P}^*_{\mathcal{U}}}{\Vdash} \dot{\sigma}_{\dot{\Gamma}} \restriction n = \dot{\rho}_0 \restriction n \qquad (\star)$$

and since $\langle \emptyset; \mathcal{T}^* \rangle \leq \langle \emptyset; T'' \rangle$, then

$$\langle \emptyset; \mathcal{T}^* \rangle \underset{\mathbf{P}^*_{\mathcal{U}}}{\Vdash} [\dot{\sigma}_{\dot{\Gamma}}, \dot{\rho}_0 \text{ are not tail} - \text{equivalent}]. \qquad (\star\star)$$

Finally, it is obvious that $(\star)$ and $(\star\star)$ lead to a contradiction. Therefore, the sequences $\bar{\sigma}, \bar{\tau}$ are tail – equivalent and the same is true for the original sequences σ and τ.

1.B.9 Theorem □

1.B.10 Corollary *Let U be a normal measure on* $\mathcal{P}_\kappa(\lambda)$ *and* $\mathbf{P}_U$ *be the Supercompact Prikry forcing induced by U. If* σ *and* τ *are two* $\mathbf{P}_U$ – generic *over* **V** *sequences with the property* $\mathbf{V}[\sigma] = \mathbf{V}[\tau]$, *then they are* tail – equivalent.

Proof: The normality of the measure U implies that the forcing $\mathbf{P}_U$ is equivalent to the forcing $\mathbf{P}^*_U$. To show this equivalence we use the following fact: As it was proven in [Mathias 73], for every sequence $\langle A_x \mid x \in [\mathcal{P}_\kappa(\lambda)]^{<\omega} \rangle$ of sets in U, there exists a set $B \in U$ such that

$$(\forall x \in [\mathcal{P}_\kappa(\lambda)]^{<\omega})\left[\left\{P \in B \mid x \underset{\sim}{\subset} P\right\} \subset A_x\right].$$

Then, we project the measure U on λ and we use **1.B.9** to complete the proof.

1.B.10 Corollary □

1.B.11 Theorem *We assume that* **V** *is a transitive model of* **ZFC**, *in which* κ *is a supercompact cardinal and* λ *is an inaccessible above* κ. *Also,* U *is a normal* $\mathcal{P}_\kappa(\lambda)$ – measure *and* $\mathbf{P}_{\mathrm{U}}$ *is the Supercompact Prikry forcing associated with* U. *If* G *is* $\mathbf{P}_{\mathrm{U}}$ – generic *over* **V**, *then there exists in* $\mathbf{V}[\mathrm{G}]$ *a coloring* F *of* $[\kappa]^\omega$ *into* $\mathbf{V}_\mu$ *colors for some* $\mu < \kappa$, *with the property that* F *is ordinal definable over* $\mathbf{V}[\mathrm{G}]$ *with parameters from* **V** *and has no homogeneous set, i.e.,* $\mathbf{V}[\mathrm{G}] \models \neg(\forall \mu < \kappa)(\kappa \underset{\mathbf{VOD}}{\rightarrow} (\omega)^\omega_{\mathbf{V}_\mu})$.

Proof: First, we choose in **V** a sequence $\langle \mathrm{F}_\alpha \mid \alpha < \kappa \rangle$, with the property that for every $\alpha < \kappa$, $\mathrm{F}_\alpha : [\alpha]^\omega \to 2$ and each F_α has no homogeneous set in **V**. Since for every $\alpha < \kappa$ there are no new ω-sequences of α in the Prikry extension $\mathbf{V}[\mathrm{G}]$, then we conclude that in $\mathbf{V}[\mathrm{G}]$ there are no F_α-homogeneous sequences. Let us fix $\mathrm{x} \in [\kappa]^\omega \cap \mathbf{V}[\mathrm{G}]$. If x is cofinal in κ, then we define

$$\begin{aligned}\mathrm{F(x)} \overset{\text{def}}{=} \quad & \Big\{\alpha \in (\omega^{<\omega})^\omega) \mid \exists \mathrm{G}' \Big[\mathrm{G}' \text{ is } \mathbf{P}_{\mathrm{U}} - \text{generic over } \mathbf{V} \quad \& \\ & \& \quad \mathbf{V}[\mathrm{G}'] = \mathbf{V}[\mathrm{G}] \quad \& \\ & \& \quad \forall \mathrm{n} < \omega \Big(\alpha(\mathrm{n}) = \langle \mathrm{k} \in \omega \mid \sigma^\kappa_{\mathrm{G}'}(\mathrm{n}) \leq \mathrm{x(k)} < \sigma^\kappa_{\mathrm{G}'}(\mathrm{n}+1) \rangle \Big) \Big] \Big\}.\end{aligned}$$

If x is bounded below κ, then we define $\mathrm{F(x)} \overset{\text{def}}{=} \mathrm{F}_{\sup(\mathrm{x})}(\mathrm{x})$.

It follows that there exists a $\mu < \kappa$ such that $\mathrm{F} : [\kappa]^\omega \to \mathbf{V}_\mu$ and that F is **VOD** over $\mathbf{V}[\mathrm{G}]$. In fact, the above coloring F is **OD** with parameters from V_ξ of the ground model, for some large enough ξ. This is true because the relation $\mathbf{V}[G'] = \mathbf{V}[G]$ is equivalent to the fact that the $\mathcal{P}(\xi) \subset \mathbf{V}[G']$ for some ξ large enough, so that we can code in the usual absolute way the $\mathbf{P}_U$ – generic filter G as subset of ξ. In order to express the fact that $\mathcal{P}(\xi) \subset \mathbf{V}[G']$, we need to say that every subset of ξ is the realization through the generic G' of a nice $\mathbf{P}_U$ – name of a subset of ξ, which is obviously an Ordinal Definable property with parameters from V_ξ.

We claim that F has no homogeneous set.

Let us suppose towards a contradiction that F has an homogeneous set namely $h \in [\kappa]^\omega$ such that $|F''[h]^\omega| = 1$. By the choice of F_α's we conclude that such an h must be cofinal in κ. First, we observe that if α, β belong in F(x) with witnesses (two $\mathbf{P}_U$ – generic filters) G_α, G_β then by virtue of **1.B.10** and since $\mathbf{V}[G_\alpha] = \mathbf{V}[G] = \mathbf{V}[G_\beta]$,

$$(\exists n_0, n_1 < \omega)(\forall i < \omega)[\sigma^\kappa_{G_\alpha}(n_0 + i) = \sigma^\kappa_{G_\beta}(n_1 + i)].$$

Thus α, β are tail – equivalent. Let us define $\mathcal{S} \stackrel{\text{def}}{=} \Big\{n \in \omega \mid |\text{ran}(h) \cap [\sigma^\kappa_G(n), \sigma^\kappa_G(n+1))| = 1\Big\}$. Then there are two cases:

Case 1. $\mathcal{S}$ is infinite. We set $h' \in [h]^\omega$ such that for all $n \in \omega$ $|\text{ran}(h') \cap [\sigma^\kappa_G(n), \sigma^\kappa_G(n+1))| \leq 1$. Then we consider the even subsequence h'_e of h', i.e., $h'_e(n) = h'(2n)$, and two reals α, α' with the property that $\alpha \in F(h')$, $\alpha' \in F(h'_e)$ with witness G. The latest implies that for all $n < \omega$, we have that $\alpha(n) = \langle k \in \omega \mid \sigma^\kappa_G(n) \leq h'(k) < \sigma^\kappa_G(n+1)\rangle$ and $\alpha(n) = \langle k \in \omega \mid \sigma^\kappa_G(n) \leq h'_e(k) < \sigma^\kappa_G(n+1)\rangle$. Since $F(h') = F(h'_e)$ then α, α' are tail – equivalent and so

$|\mathrm{ran}(\alpha)\triangle\mathrm{ran}(\alpha')| < \aleph_0$. However, this is a contradiction since α, α' imply that

$$|\mathrm{ran}(\alpha)\triangle\mathrm{ran}(\alpha')| = \aleph_0.$$

Case 2. $\mathcal{S}$ is finite. We set $h^*(n) :\stackrel{\mathrm{def}}{=} h(n+1)$ and we consider $\alpha \in F(h)$ and $\alpha' \in F(h^*)$ with witness G, as in the previous case. Then, as $\alpha, \alpha' \in F(h)$, we can find $n_0, n_1 < \omega$ such that $(\forall i < \omega)[\alpha(n_0 + i) = \alpha'(n_1 + i)]$. Since $\mathcal{S}$ is finite, then we can find a large enough $i < \omega$, such that $|\alpha(n_0 + i)| \geq 2$. Let $k_0 :\stackrel{\mathrm{def}}{=} \min\alpha(n_0 + i)$ and $k_1 :\stackrel{\mathrm{def}}{=} \max\alpha(n_0 + i)$. Since $\alpha(n_0 + i) = \alpha'(n_1 + i)$ and $k_0, k_0 + 1 \in \alpha(n_0 + i)$, then $\sigma^\kappa_G(n_0 + i) \leq h(k_0) < \sigma^\kappa_G(n_0 + i + 1)$ and $\sigma^\kappa_G(n_1 + i) \leq h(k_0 + 1) < \sigma^\kappa_G(n_1 + i + 1)$. But the latest forces that $n_0 = n_1$. Moreover, as $k_1 \in \alpha(n_0 + 1)$, we have that $\sigma^\kappa_G(n_0 + i) \leq h(k_1) < \sigma^\kappa_G(n_0 + i + 1)$ and $\sigma^\kappa_G(n_0 + i) \leq h(k_1 + 1) < \sigma^\kappa_G(n_0 + i + 1)$, which implies that $k_1 + 1 \in \alpha(n_0 + i)$, an obvious contradiction to the maximality of k_1.

Thus h is not F − homogeneous.

Therefore, the **VOD** over **V**[G] coloring F has no homogeneous set in **V**[G]. This completes the proof of the theorem.

1.B.11 Theorem □

1.B.12 •

Before we exhibit an **OD** over **V**[G] partition with no homogeneous sequence, we are going to show that under some special circumstances a rank initial segment of the ground model is **OD** over the Prikry generic extension.

To accomplish the above result, we consider the cardinal ξ which is mentioned in the proof of **1.B.11** (without loss of generality we can assume that ξ is strong limit and thus $|\mathbf{V}_\xi| = \xi$) and had the property that the coloring F was $\mathbf{OD}[V_\xi]$.

Before we force over $\mathbf{V}$ that V_ξ of the ground model $\mathbf{V}$ is **OD** over any Supercompact Prikry generic extension, we have to make sure that κ remains a supercompact cardinal in the generic extension. In order to accomplish that goal, we use R. Laver's forcing Q of cardinality κ. As in [Laver 78], Q is $\kappa - cc$ and in $\mathbf{V}^{Q}$, κ remains supercompact upon forcing with any $< \kappa -$ directed closed forcing. Since Q is $\kappa - cc$, then forcing with Q does not change the behavior of the power-set function above κ. Hence, starting with G.C.H in our ground model, we still have G.C.H above κ in any of the Laver models which we get via Q and moreover λ remains an inaccessible cardinal above κ. We pick a Q-generic extension of the ground model of the supercompact and the inaccessible above it and we consider this generic extension as our new ground model which is denoted by $\mathbf{V}$.

The following arguments are simple variations of those in [Menas 75]. We code $\mathbf{V}_\xi$ by a subset A of ξ using a bijection f of ξ onto $\mathbf{V}_\xi$ in the following standard way: For $\alpha, \beta \in \xi$ we set $\alpha E \beta :\overset{\text{def}}{\Longleftrightarrow} f(\alpha) \in f(\beta)$. Thus $E \subset \xi \times \xi$ is a well – founded binary relation on ξ. Using the canonical absolute bijection of $\xi \times \xi$ onto ξ we code E by $E^* \subset \xi$. Then, it is easy to show that $\langle \mathbf{V}_\xi, \epsilon \rangle$ is the Mostowski transitive collapse of $\langle \xi, E \rangle$ and f is the collapsing map. We call E^*

to be the code of $\mathbf{V}_\xi$ and we observe that $\mathbf{V}_\xi$ can be decoded from E^* in an absolute and ordinal definable way. At this point we recall that our ground model $\mathbf{V}$ satisfies GCH above κ. Let $\nu = 2^{2^\lambda}$.

By using Easton forcing, we force the continuum function to behave in the following fashion:

If $\alpha \in \mathrm{E}^*$, then $2^{\aleph_{\nu+\alpha+1}} = \aleph_{\nu+\alpha+2}$ and

if $\alpha \notin \mathrm{E}^*$, then $2^{\aleph_{\nu+\alpha+1}} = \aleph_{\nu+\alpha+3}$.

Let $\mathbf{V}^*$ denote the generic extension of $\mathbf{V}$ via the Easton forcing. Then for every $\alpha \in \mathrm{E}^*$

$$\alpha \in \mathrm{E}^* \iff \mathbf{V}^* \models 2^{\aleph_{\nu+\alpha+1}} = \aleph_{\nu+\alpha+2}.$$

Since the Easton forcing we use is $< \nu$-directed closed, then U remains a normal measure over $\mathcal{P}_\kappa(\lambda)$ in $\mathbf{V}^*$. Thus $\mathbf{V}^*$ thinks that κ is a λ-supercompact cardinal and that λ is an inaccessible cardinal above κ.

Let us take G to be a $\mathbf{P}_\mathrm{U}$ – generic filter over $\mathbf{V}^*$. Our goal is to show that E^* remains **OD** over $\mathbf{V}^*[\mathrm{G}]$.

Since the forcing $\mathbf{P}_\mathrm{U}$ satisfies the ν–cc, then $\mathbf{V}^*$ and $\mathbf{V}^*[\mathrm{G}]$ agree on cofinalities above ν, and so they have the same cardinals above ν.

Hence, there exists a β such that

$$\aleph_\nu^{\mathbf{V}^*} = \aleph_\beta^{\mathbf{V}^*[\mathrm{G}]}.$$

In addition, the fact that $\mathbf{V}^*$ and $\mathbf{V}^*[\mathrm{G}]$ have the same cardinals implies the following:

$$(\forall\alpha)\Big[(\aleph^{\mathbf{V}^*}_{\nu+\alpha+1}) = (\aleph^{\mathbf{V}^*[\mathrm{G}]}_{\beta+\alpha+1})\Big].$$

Then

$$\begin{aligned}(2^{\aleph_{\beta+\alpha+1}})^{\mathbf{V}^*[\mathrm{G}]} &= (2^{\aleph^{\mathbf{V}^*[\mathrm{G}]}_{\beta+\alpha+1}})^{\mathbf{V}^*[\mathrm{G}]} \\ &= (2^{\aleph^{\mathbf{V}^*}_{\nu+\alpha+1}})^{\mathbf{V}^*[\mathrm{G}]}.\end{aligned}$$

At this point, we use a standard fact about $\nu-$cc forcings. Since $\mathbf{P}_{\mathrm{U}}$ has size $\leq \nu$ in $\mathbf{V}^*$, then

$$(2^{\aleph^{\mathbf{V}^*}_{\nu+\alpha+1}})^{\mathbf{V}^*[\mathrm{G}]} \leq |\nu^{\aleph^{\mathbf{V}^*}_{\nu+\alpha+1}}|^{\mathbf{V}^*} \leq (2^{\aleph^{\mathbf{V}^*}_{\nu+\alpha+1}})^{\mathbf{V}^*}.$$

Thus

$$(2^{\aleph_{\beta+\alpha+1}})^{\mathbf{V}^*[\mathrm{G}]} = (2^{\aleph^{\mathbf{V}^*}_{\nu+\alpha+1}})^{\mathbf{V}^*}.$$

Consequently,

$$(2^{\aleph_{\beta+\alpha+1}})^{\mathbf{V}^*[\mathrm{G}]} = \aleph^{\mathbf{V}^*[\mathrm{G}]}_{\beta+\alpha+2} \iff (2^{\aleph^{\mathbf{V}^*}_{\nu+\alpha+1}})^{\mathbf{V}^*} = \aleph^{\mathbf{V}^*}_{\nu+\alpha+2}.$$

This implies that for all $\alpha < \xi$

$$\alpha \in \mathrm{E}^* \iff \mathbf{V}^*[\mathrm{G}] \models (2^{\aleph_{\beta+\alpha+1}}) = \aleph_{\beta+\alpha+2}.$$

Therefore E^* is **OD** over $\mathbf{V}^*[\mathrm{G}]$. Since E^* is ordinal definable over $\mathbf{V}^*[\mathrm{G}]$, then the collapsing map of $\langle\xi,\mathrm{E}\rangle$ is ordinal definable and consequently every element of $\mathbf{V}^{\mathbf{V}}_{\xi}$ is **OD** over $\mathbf{V}^*[\mathrm{G}]$.

In particular, the set of all $\mathbf{P}_{\mathrm{U}}-$dense sets is **OD** over $\mathbf{V}[\mathrm{G}]$, since it is an element of $\mathbf{V}_\xi^{\mathbf{V}}$. The latest is essential for the next theorem.

As a result of the above considerations, we adopt as our ground model the model $\mathbf{V}^*$ and we denote it again by $\mathbf{V}$.

1.B.12 □

Hence, all the above arguments prove the following theorem.

1.B.13 Theorem *If* $\mathbf{V}'$ *is a transitive model of* **ZFC** *in which* κ *is a supercompact cardinal and* λ *is an inaccessible above* κ*, then there exists a generic extension* $\mathbf{V}$ *of* $\mathbf{V}'$ *in which* κ *remains supercompact,* λ *remains inaccessible and the following holds:*

If U is a normal $\mathcal{P}_\kappa(\lambda)-$measure, $\mathbf{P}_{\mathrm{U}}$ *is the Supercompact Prikry forcing associated with U and G is* $\mathbf{P}_{\mathrm{U}}-$generic over $\mathbf{V}$, *then a sufficient large rank initial segment of the ground model* $\mathbf{V}$ *is* **OD** *over the Prikry generic extension* $\mathbf{V}[\mathrm{G}]$ *and* $\mathbf{V}[\mathrm{G}] \models \neg(\forall \mu < \kappa)\Big(\kappa \underset{\mathbf{OD}}{\rightarrow} (\omega)^{\omega}_{\mathbf{V}_\mu}\Big)$.

1.B.13 Theorem □

The following result is not really crucial for our thesis, even though it uses extensively the methods of [Menas 75] to produce a stronger form of the previous theorem. However, the technology which is used is appealing to the author and thus we make a sketch of the proof.

1.B.14 •

As we proved in **1.B.11**, there exists in **V**[G] a **VOD** partition with no homogeneous sequence. Our goal is to show that it is consistent for the ground model to be **OD** over the Prikry generic extension, thus making the partition that appears in **1.B.11 OD** over the Prikry extension **V**[G], which proves that the partition relation $(\forall\mu < \kappa)\Big(\kappa \underset{\mathbf{OD}}{\rightarrow} (\omega)^{\omega}_{\mathbf{V}_\mu}\Big)$ fails in **V**[G].

Towards this goal, we follow almost identically the proof of a theorem of T. K. Menas [Menas 75] (20. THEOREM p.89) in which he proves that starting with a model of a supercompact cardinal κ and an inaccessible cardinal above κ, there exists a model M and a generic extension N of M with the property that κ is supercompact in both M and N and moreover N $\models$ ZFC$+$**V** $=$ HOD.

We fix λ to be a regular cardinal above κ. The proof, in [Menas 75], involves a reverse Easton forcing construction in which he codes each e(β) rank initial segment of the Universe into the power-set function, where e is an enumeration of all the Beth fixed points. The only modification that we are making in that proof is that we take the enumeration e to start from the first Beth fixed point above λ, say it ν. Hence we construct the model N in which κ remains supercompact. Moreover, as in [Menas 75], if δ is an ordinal, then we code in the usual absolute way $V^N_{e(\delta)}$ by a subset A of $e(\delta)$ and the forcing construction guarantees that

$N \models (\forall\gamma < \epsilon(\delta))[\gamma \in A \iff 2^{\aleph_{e(\delta)+\gamma+1}} = \aleph_{e(\delta)+\gamma+3}]$. Since the decoding of $V^N_{e(\delta)}$ from A is **OD** and absolute, then this shows that every element of $V^N_{e(\delta)}$ is HOD over N. Thus $N \models \mathbf{V} = \mathrm{HOD}$. Let us consider $\mathbf{V} \stackrel{\text{def}}{=} N$ to be our ground model before we force with the Supercompact Prikry forcing.

We recall that ν denotes the first Beth fixed point above λ and $e(0) = \nu$. Let U be a normal measure over $\mathcal{P}_\kappa(\lambda)$ in **V**. We take G to be a $\mathbf{P}_U$ – generic filter over **V**. Our goal is to show that **V** remains **OD** over **V**[G]. Let δ be an ordinal. As before, we code in the usual absolute way $V^{\mathbf{V}}_{e(\delta)}$ by a subset A of $e(\delta)$. Since the forcing $\mathbf{P}_U$ satisfies the ν – cc, then **V** and **V**[G] agree on cofinalities above ν, and so they have the same cardinals above ν. Hence there exists a β such that

$$\aleph^{\mathbf{V}}_{\epsilon(\delta)} = \aleph^{\mathbf{V}[G]}_{\beta}.$$

In addition, the fact that **V** and **V**[G] have the same cardinals implies the following:

$$(\forall\alpha)\Big[(\aleph^{\mathbf{V}}_{e(\delta)+\alpha+1}) = (\aleph^{\mathbf{V}[G]}_{\beta+\alpha+1})\Big].$$

Then

$$\begin{aligned}(2^{\aleph_{\beta+\alpha+1}})^{\mathbf{V}[G]} &= (2^{\aleph^{\mathbf{V}[G]}_{\beta+\alpha+1}})^{\mathbf{V}[G]} \\ &= (2^{\aleph^{\mathbf{V}}_{e(\delta)+\alpha+1}})^{\mathbf{V}[G]}.\end{aligned}$$

At this point, we use a standard fact about ν-cc forcings. Since $\mathbf{P}_{\mathrm{U}}$ has size $\leq \nu$ in $\mathbf{V}$ then

$$(2^{\aleph^{\mathbf{V}}_{e(\delta)+\alpha+1}})^{\mathbf{V}[G]} \leq |\nu^{\aleph^{\mathbf{V}}_{e(\delta)+\alpha+1}}|^{\mathbf{V}} \leq (2^{\aleph^{\mathbf{V}}_{e(\delta)+\alpha+1}})^{\mathbf{V}}.$$

Thus,

$$(2^{\aleph_{\beta+\alpha+1}})^{\mathbf{V}[G]} = (2^{\aleph^{\mathbf{V}}_{e(\delta)+\alpha+1}})^{\mathbf{V}}.$$

Consequently,

$(2^{\aleph_{\beta+\alpha+1}})^{\mathbf{V}[G]} = \aleph^{\mathbf{V}^*[G]}_{\beta+\alpha+3} \iff (2^{\aleph^{\mathbf{V}}_{e(\delta)+\alpha+1}})^{\mathbf{V}} = \aleph^{\mathbf{V}}_{\nu+\alpha+3}$. This implies that for all $\gamma < \mathrm{e}(\delta)$

$$\gamma \in \mathrm{A} \iff \mathbf{V}[\mathrm{G}] \models (2^{\aleph_{\beta+\alpha+1}}) = \aleph_{\beta+\alpha+3}.$$

Therefore, every element of $\mathrm{V}^{\mathbf{V}}_{e(\delta)}$ is HOD over $\mathbf{V}$[G]. This shows that $\mathbf{V}$ is HOD over $\mathbf{V}$[G]. Thus the **VOD** over $\mathbf{V}$[G] coloring F, which appears in **1.B.11** is **OD** over $\mathbf{V}$[G] and has no homogeneous set.

1.B.14 □

Therefore, all the above arguments prove the following theorem.

1.B.15 Theorem *If there exists a transitive model of* **ZFC** *in which κ is a supercompact cardinal and there exists an inaccessible above κ, then for λ any regular cardinal above κ, there exists another model of* **ZFC V** *in which κ is supercompact and the following holds: If U is a normal $\mathcal{P}_\kappa(\lambda)$* – measure, $\mathbf{P}_{\mathrm{U}}$ *is the Supercompact Prikry forcing associated with U and G is* $\mathbf{P}_{\mathrm{U}}$ – generic *over* $\mathbf{V}$, *then the ground model* $\mathbf{V}$ *is* **OD** *over the Prikry generic extension* $\mathbf{V}[G]$ *and* $\mathbf{V}[G] \models \neg(\forall \mu < \kappa)\left(\kappa \underset{\mathbf{OD}}{\rightarrow} (\omega)^{\omega}_{\mathbf{V}_\mu}\right)$.

1.B.15 Theorem □

It is an **open question** whether the above partition relation fails in any Prikry generic extension of any model of **ZFC** with a supercompact cardinal.

Chapter 2

The inner Model V(H)

2.A The power set of κ in V(H)

In the first chapter we proved that if the ground model is prepared appropriately, then the partition relation fails in its Supercompact Prikry generic extension. In order to find a model of the partition relation, we consider a specific inner model of the Prikry extension and we are going to show that it is a symmetric model with a lot of Prikry generic sequences. First, we define the inner model.

2.A.1 Definition *Let U be a fixed normal measure on* $\mathcal{P}_\kappa(\lambda)$ *and* $\mathbf{P}_{\mathrm{U}}$ *be the corresponding Supercompact Prikry forcing. If K is a* $\mathbf{P}_{\mathrm{U}}$ – generic *over* $\mathbf{V}$ *filter, the we define* H^{K} *to be the following set:*

$$\mathrm{H}^{\mathrm{K}} :\stackrel{\mathrm{def}}{=} \bigcup\Big\{\mathcal{P}(\kappa) \cap \mathbf{V}[\mathrm{K}\restriction \alpha] \mid \alpha < \lambda\Big\}.$$

Whenever the generic is denoted by G, then H^{G} *is denoted by H. For a fixed* $\mathbf{P}_{\mathrm{U}}$ – generic *over* $\mathbf{V}$ *filter G we consider the inner model* $\mathbf{V}(\mathrm{H})$ *which is the smallest inner model of ZF that contains H as an element.*

2.A.1 Definition □

Our aim in this section is to show that the power set of κ in the model $\mathbf{V}(\mathrm{H})$ is H. We start with a definition that appears in [Magidor 77].

2.A.2 Definition *Let h be a permutation of* λ *that is the identity on some ordinal* $\alpha \in [\kappa, \lambda)$. *We define an automorphism* $\hat{\mathrm{h}}$ *of* $\mathbf{P}_{\mathrm{U}}$ *as follows:*
If $\langle \mathrm{P}_1, \ldots, \mathrm{P}_n; \mathrm{A} \rangle \in \mathbf{P}_{\mathrm{U}}$ *let* $\hat{\mathrm{h}}(\langle \mathrm{P}_1, \ldots, \mathrm{P}_n; \mathrm{A} \rangle) = \langle \mathrm{h}''\mathrm{P}_1, \ldots, \mathrm{h}''\mathrm{P}_n; \mathrm{h}''\mathrm{A} \rangle$ *where* $\mathrm{h}''\mathrm{A} = \{\mathrm{h}''\mathrm{Q} \mid \mathrm{Q} \in \mathrm{A}\}$.

2.A.2 Definition □

2.A.3 Lemma *(M. Magidor [Magidor 77]) If h is a permutation of* λ *such that for some* $\alpha \in [\kappa, \lambda)$ $\mathrm{h}\restriction \alpha = \mathrm{id}\restriction \alpha$ *then* $\hat{\mathrm{h}}$ *extends to an automorphism of* $\mathrm{r.o.}(\mathbf{P}_{\mathrm{U}})$.

2.A.3 Lemma □

2.A.4 Definition *If h is a permutation of* λ *such that* $\mathrm{h}\restriction \alpha = \mathrm{id}\restriction \alpha$ *for some* $\alpha \in [\kappa, \lambda)$ *then we call the support of* $\hat{\mathrm{h}}$ *to be the largest* β *such that* $\mathrm{h}\restriction \beta = \mathrm{id}\restriction \beta$ *and we denote it by suppt(h).*
We set $\mathcal{A} = \{\hat{\mathrm{h}} \in \mathrm{Aut}(\mathbf{P}_{\mathrm{U}}) \mid \mathrm{suppt(h)} \in [\kappa, \lambda)\}$.

2.A.4 Definition □

2.A.5 Definition *We define* $\dot{g}_\beta$ *to be the following* $\mathbf{V}^{\mathbf{P}_{U_\beta}}$ – name:

$$\begin{aligned}\dot{g}_\beta \quad &= \{\langle \check{y}, p\rangle \mid \exists P_1,\dots,P_n \in \mathcal{P}_\kappa(\lambda)\exists A \in U\exists x \\ &\quad (p = \langle P_1,\dots,P_n;A\rangle \quad \& \\ &\quad \& \quad x \in \{P_1,\dots,P_n\} \quad \& \\ &\quad \& \quad y = x \cap \beta)\}.\end{aligned}$$

If $x = P_i$ *for some* $i \in \{1,\dots,n\}$ *such that* $p = \langle P_1,\dots,P_n;A\rangle$, *then we say that x appears in p.*

2.A.5 Definition □

Obviously $\dot{g}_\beta$ is the canonical $\mathbf{P}_U$ – term for the restriction of any $\mathbf{P}_U$ – generic sequence on β, where β is an ordinal above κ and below λ.
Next we show that all these names are invariant under the automorphisms in $\mathcal{A}$.

2.A.6 Lemma *Let* $\hat{h} \in \mathcal{A}$ *&* $\beta \in [\kappa,\lambda)$. *Assume that* $\mathrm{suppt}(h) = \gamma \geq \beta$. *Then,* $\hat{h}(\dot{g}_\beta) = \dot{g}_\beta$.

Proof: Since $\hat{h}$ is an automorphism of $\mathbf{P}_U$, then $\hat{h}$ is extented on $\mathbf{V}^{\mathbf{P}_U}$ by recursion on the rank of the boolean-valued terms as follows: (for simplicity we use the same symbol for $\hat{h}$ and its extension on $\mathbf{V}^{\mathbf{P}_U}$)

$$\hat{h}(\dot{\tau}) :\stackrel{\mathrm{def}}{=} \left\{\langle \hat{h}(\dot{\sigma}), \hat{h}(p)\rangle \mid \langle \dot{\sigma}, p\rangle \in \dot{\tau}\right\},$$

for any $\dot{\tau} \in \mathbf{V}^{\mathbf{P}_{\mathrm{U}}}$.

In order to show that $\hat{h}(\dot{g}_\beta) \subseteq \dot{g}_\beta$ we take a $\langle \check{y}, p\rangle \in \dot{g}_\beta$ where $y = x \cap \beta$ for some x that appears in p and we recall that

$$\hat{h}(\langle \check{y}, p\rangle) = \langle \hat{h}(\check{y}), \hat{h}(p)\rangle.$$

Assume that $p = \langle P_1, \ldots, P_n; A\rangle$ and $x = P_k$ where $1 \leq k \leq n$.
Then, $\hat{h}(\langle \check{y}, p\rangle) = \langle \check{y}, \hat{h}(p)\rangle$.
Since $\hat{h}(p) = \langle h''P_1, \ldots, h''P_n; h''A\rangle$ and $\mathrm{suppt}(h) = \gamma \geq \beta$ then

$$\begin{aligned} h''P_k \cap \beta &= P_k \cap \beta \\ &= x \cap \beta. \end{aligned}$$

Therefore,

$$\langle \check{y}, \hat{h}(p)\rangle = \langle (h''P_k \cap \beta)\check{}, \hat{h}(p)\rangle$$

and we observe that $h''P_k$ appears in $\hat{h}(p)$. Hence,

$$\hat{h}(\langle \check{y}, p\rangle) = \langle (h''P_k \cap \beta)\check{}, \hat{h}(p)\rangle \in \dot{g}_\beta.$$

So, $\hat{h}(\dot{g}_\beta) \subseteq \dot{g}_\beta$.

Conversely, we consider $\langle \check{y}, p\rangle \in \dot{g}_\beta$, where $y = x \cap \beta$ for some x that appears in p and $x = P_k$ for some k, $1 \leq k \leq n$, and $p = \langle P_1, \ldots, P_n; A\rangle$. Then we let $q :\stackrel{\mathrm{def}}{=} \langle h^{-1}P_1, \ldots, h^{-1}P_n; h^{-1}A\rangle$. Hence,

$$\begin{aligned} x \cap \beta &= P_k \cap \beta \\ &= h^{-1}P_k \cap \beta \end{aligned}$$

since suppt(h) $\geq \beta$. Also $\hat{h}(q) = p$ and

$$\begin{aligned}\langle \check{y}, p\rangle &= \langle \check{y}, \hat{h}(q)\rangle \\ &= \langle (h^{-1}P_k \cap \beta)\check{~}, \hat{h}(q)\rangle \\ &= \langle \hat{h}((h^{-1}P_k \cap \beta)\check{~}), \hat{h}(q)\rangle \\ &= \hat{h}(\langle (h^{-1}P_k \cap \beta)\check{~}, q\rangle)\end{aligned}$$

and $h^{-1}P_k$ appears in q. Hence, $\langle (h^{-1}P_k \cap \beta)\check{~}, q\rangle \in \dot{g}_\beta$ and so $\langle \check{y}, p\rangle \in \hat{h}(\dot{g}_\beta)$. Therefore, $\dot{g}_\beta \subseteq \hat{h}(\dot{g}_\beta)$. This shows that $\hat{h}(\dot{g}_\beta) = \dot{g}_\beta$.

2.A.6 Lemma □

However, the previous result can be improved in terms of the following lemma.

2.A.7 Lemma *Assume that* $\hat{h}_0 \in \mathcal{A}$ *and* suppt$(h_0) = \gamma$.
If $\delta = \sup\{h_0''\beta \cup h_0^{-1}\beta \cup \{\beta, \gamma\}\} + 1$, *then for any* $\hat{h}_1 \in \mathcal{A}$ *with* suppt$(\hat{h}_1) = \delta$ *the following holds:* $\hat{h}_1(\hat{h}_0\dot{g}_\beta) = \hat{h}_0\dot{g}_\beta$.

Proof: We take a $\langle (x\cap\beta)\check{~}, \hat{h}_1\hat{h}_0(p)\rangle \in \hat{h}_1\hat{h}_0(\dot{g}_\beta)$, where x appears in p. Assume that $p = \langle P_1, \ldots, P_n; A\rangle$ and $x = P_k$ where $1 \leq k \leq n$. We show that there exists a q such that

$$\langle (x \cap \beta)\check{~}, \hat{h}_1\hat{h}_0(p)\rangle = \langle (y \cap \beta)\check{~}, \hat{h}_0(q)\rangle$$

for some y that appears in q. By the definition of $\hat{h}_0, \hat{h}_1$ we conclude that

$$\hat{h}_1\hat{h}_0 p = \langle h_1''h_0''P_1, \ldots, h_1''h_0''P_n; h_1''h_0''A\rangle.$$

(Without loss of generality we assume that for all $P \in A$ $P=h_1''P = h_0''P$.)

Let us fix $i \in \{1,\dots,n\}$. Then

$$\begin{aligned} h_1''h_0''P_i &= h_1''[((h_0''P_i)\cap\delta)\cup((h_0''P_i)\setminus\delta)] \\ &= h_1''((h_0''P_i)\cap\delta)\cup h_1''((h_0''P_i)\setminus\delta) \\ &= ((h_0''P_i)\cap\delta)\cup h_1''((h_0P_i)\setminus\delta). \end{aligned}$$

Notice that the second equality follows from the first, since $\mathrm{suppt}(h_1) = \delta$. Then, we find $P_i' \subseteq P_i$ such that $(h_0''P_i)\cap\delta = h_0''P_i'$. If $\zeta \in P_i \cap \beta$, then $h\zeta < \delta$ and so $\zeta \in P_i' \cap \beta$. Hence, $P_i' \cap \beta = P_i \cap \beta$.

Since $\mathrm{suppt}(h_1) = \delta$ and $(h_0''P_i)\setminus\delta \subseteq \lambda\setminus\delta$, then $h_1''(h_0''p_i\setminus\delta) \subseteq \lambda\setminus\delta$.

Assume that $h_1''(h_0''P_i\setminus\delta) = h_0''W_i$. If $\zeta \in W_i \cap P_i'$, then $h_0\zeta \geq \delta$ and $h_0\zeta < \delta$ which is a contradiction. Thus $W_i \cap P_i' = 0$.

Moreover, if $\zeta < \beta$, then $h\zeta < \delta$ so $\zeta \notin W_i$, i.e., $W_i \cap \beta = 0$. We set $P_i^* = P_i' \cup W_i$.

Thus, $h_1''h_0''P_i = h_0''P_i^*$, where $P_i^*\cap\beta = P_i\cap\beta$. We set $q = \langle P_1^*,\dots,P_n^*;A\rangle$. Then $\hat{h}_0 q = \hat{h}_1\hat{h}_0 p$ and if x appears in p as P_k, $x\cap\beta = P_k^*\cap\beta$.

Consequently,

$$\langle (x\cap\beta)\check{},\hat{h}_1\hat{h}_0 p\rangle \in \hat{h}_0\dot{g}_\beta.$$

Therefore, $\hat{h}_1\hat{h}_0\dot{g}_\beta \subseteq \hat{h}_0\dot{g}_\beta$.

Conversely, if $\langle (x\cap\beta)\check{},\hat{h}_0 p\rangle \in \hat{h}_0\dot{g}_\beta$, where $p = \langle P_1,\dots,P_n;A\rangle$ and $x = P_k$ for some $k \in \{1,\dots,n\}$, then we have to find $q \in \mathbf{P}_U$ such that $\langle (x\cap\beta)\check{},\hat{h}_0 p\rangle = \langle (y\cap\beta)\check{},\hat{h}_1\hat{h}_0 p\rangle$, where y appears in q. As before, without loss of generality, we assume that for every P in A $h_0''P = h_1''P = P$. We set $P_i'' \subseteq P_i$ such that $h_0''P_i'' = (h_0''P_i) \cap \delta$. Then, $h_1''h_0''P_i' = h_0''P_i$. Let W_i be such that $(h_0''P_i) \setminus \delta = h_1''h_0''W_i$. Then, every $\zeta \in P_i' \cap W_i$ satisfies $h_0\zeta < \delta$ and so $h_1h_0\zeta = h_0\zeta < \delta$ and as $\zeta \in W_i$, then $h_1h_0\zeta \in (h_0''P_i) \setminus \delta$, i.e., $h_1h\zeta \geq \delta$ which is a contradiction.

Moreover, if $\zeta \in \beta \cap W_i$ has the property $h\zeta < \delta$ and as $\delta = \mathrm{suppt}(h_1)$, then $h_1h_0\zeta < \delta$ and since $h_1''h_0''W_i \subseteq \lambda \setminus \delta$, then $h_1h_0\zeta \geq \delta$. Thus, $W_i \cap \beta = 0$. Next, we set $P_i^* = P_i' \cup W_i$. Then, $P_i^* \cap \beta = P_i \cap \beta$ and $h_0''P_i = h_1''h_0''P_i^*$. Let $q = \langle P_1^*, \ldots, P_n^*; A\rangle$. So $\hat{h}p = \hat{h}_1\hat{h}_0q$ and if x is P_i, then $P_i^* \cap \beta = x \cap \beta$. Therefore, $\langle (x \cap \beta)\check{}, \hat{h}_0p\rangle \in \hat{h}_1\hat{h}_0\dot{g}_\beta$ and so $\hat{h}_0\dot{g}_\beta \subseteq \hat{h}_1\hat{h}_0\dot{g}_\beta$.

Hence, $\hat{h}_1\hat{h}_0\dot{g}_\beta = \hat{h}_0\dot{g}_\beta$.

2.A.7 Lemma □

2.A.8 Definition *Let* $\langle x;A\rangle \in \mathbf{P}_U$. *A subset* A′ *of* $\mathcal{P}_\kappa(\delta)$ *is called* $\delta -$ good *for* $\langle x;A\rangle$ *iff*

$$A' \in U_\delta \quad \& \quad \forall z \in [A']^{<\omega}\Big(x\restriction\delta \underset{\sim}{\subset} z \Rightarrow \exists w \in [A]^{<\omega}(x \underset{\sim}{\subset} w \ \&\ w\restriction\delta = z)\Big).$$

2.A.8 Definition □

2.A.9 Lemma *Let* $\langle x;A\rangle \in \mathbf{P}_U$. *Then, for every* $\delta \in [\kappa, \lambda)$ *there exists* A′ *which is* $\delta -$ good *for* $\langle x;A\rangle$.

Proof: We assume that $x = \{P_1, \ldots, P_n\}$. Let us take a $B \in U$ such that $B \subseteq A$ and $\forall P \in B(P_n \underset{\sim}{\subset} P)$. We define the following partition of $B \restriction \delta (\in U_\delta)$

$$f : [B \restriction \delta]^{<\omega} \to 2$$

such that

$$f(z) = \begin{cases} 1, & \text{if } \exists w \in [B]^{<\omega}(w \restriction \delta = z) \\ 0, & \text{if otherwise.} \end{cases}$$

Since U_δ satisfies the usual partition property, then there exists an $A' \subset B \restriction \delta$ such that $A' \in U_\delta$ which is homogeneous with respect to f, i.e.,

$$(\forall n < \omega)(|f''[A']^n| = 1).$$

We show that $(\forall n < \omega)(|f''[A']^n| = \{1\})$.

We fix $n \in \omega$ and without loss of generality we assume that $1 \leq n < \omega$. Since g_δ and g are the $\mathbf{P}_{U_\delta}$ and $\mathbf{P}_U$ – generic sequences respectively induced by the $\mathbf{P}_U$ – generic filter G, then $\exists m_1, m_2 < \omega$ such that

$$g_\delta \restriction [m_1, \omega) \subseteq A' \text{ and } g \restriction [m_1, \omega) \subseteq B.$$

Notice that $g_\delta = \langle g(i) \cap \delta | \, i < \omega \rangle$. Let $m_0 = \max\{m_1, m_2\}$. Then,

$$g_\delta \restriction [m, \omega) \subseteq A' \quad \& \quad g \restriction [m, \omega) \subseteq B.$$

Consider $x = \langle g_\delta(m), \ldots, g_\delta(m+n) \rangle$.

Then, $x \in [A']^{<\omega}$ and for $w = \langle g_\delta(m), \ldots, g_\delta(m+n) \rangle$, $w \in [B]^{<\omega}$ and $w \restriction \delta = x$.

Hence, $f(x) = 1$ and by the homogeneity of A' we conclude that

$f''[A']^n = \{1\}$.

Therefore, $f''[A']^{<\omega} = \{1\}$ and this shows that A' is δ – good for $\langle x; A \rangle$.

2.A.9 Lemma □

2.A.10 Definition *A condition* $\langle x;A\rangle \in \mathbf{P}_U$ *is called* $\delta -$ nice *iff* $\delta \in [\kappa, \lambda)$ *and* $A\restriction\delta$ *is* $\delta -$ good *for* $\langle x;A\rangle$.

2.A.10 Definition □

2.A.11 Corollary *For every* $\langle x;A\rangle \in \mathbf{P}_U$ *there exists a condition* $\langle x;A^*\rangle \leq \langle x;A\rangle$ *so that* $\langle x;A^*\rangle$ *is* $\delta -$ nice. *This shows that* $D_s = \{p \in \mathbf{P}_U \mid p\ \delta - \text{nice}\}$ *is dense. Hence* $(\forall p \in G)(\exists q \leq p)(q \text{ is } \delta - \text{nice} \ \&\ q \in G)$.

Proof: We are using **2.A.9** to find a set A' which is $\delta -$ good for $\langle x;A\rangle$. Then, we define $A^* = \left\{P \in A \mid P \cap \delta \in A'\right\}$. By definition of A^* we have that $A^*\restriction\delta \subseteq A'$. In addition, if $P \in A'$, then $\exists Q \in B$ such that $Q \cap \delta = P$, i.e., $Q \in A^*$ and so $P \in A^*\restriction\delta$. Hence, $A^*\restriction\delta = A'$ and $A^* \subseteq B \subseteq A$. Obviously, $A^*\restriction\delta$ is $\delta -$ good for $\langle x;A^*\rangle$.

Therefore, for every $\delta \in [\kappa, \lambda)$ and $\langle x;A\rangle \in \mathbf{P}_U$ there exists $A^* \subseteq A$ such that $\langle x;A^*\rangle$ is $\delta -$ nice.

2.A.11 Corollary □

By taking λ to be the least innaccessible above κ, it is easy to show that the measure U concentrates on a set $\mathcal{X}$ with the following property:

For every $P, Q \in \mathcal{X}$

$$P \cap \kappa = Q \cap \kappa \Rightarrow |P| = |Q|.$$

Then a priori we take the forcing $\mathbf{P}_U$ to have all the conditions below $\langle \emptyset;\mathcal{X}\rangle$.

2.A.12 Lemma *Assume that* $\hat{\mathrm{h}} \in \mathcal{A}$ *with* $\mathrm{suppt(h)} = \gamma$, $\dot{\mathrm{F}}$ *is a* $\mathbf{P}_{\mathrm{U}}$ – name *which is invariant under* $\mathcal{A}$, $\delta = \sup(\mathrm{h}''\beta \cup \mathrm{h}^{-1}\beta \cup \{\beta, \gamma\}) + 1$ *and* p *is a* δ – nice *condition. If* $\mathrm{p} \Vdash_{\mathbf{P}_{\mathrm{U}}} \phi(\check{\alpha}, \check{\mathrm{u}}, \hat{\mathrm{h}}\dot{\mathrm{g}}_{\beta}, \dot{\mathrm{F}})$ *and* $\mathrm{q} \in \mathbf{P}_{\mathrm{U}}$ *such that* $\mathrm{q}\restriction\delta = \mathrm{p}\restriction\delta$, *then* $\mathrm{q} \Vdash_{\mathbf{P}_{\mathrm{U}}} \phi(\check{\alpha}, \check{\mathrm{u}}, \hat{\mathrm{h}}\dot{\mathrm{g}}_{\beta}, \dot{\mathrm{F}})$.

Proof: Assume that $\mathrm{p} = \langle \mathrm{P}_1, \ldots, \mathrm{P}_n; \mathrm{A} \rangle$ and $\mathrm{q} = \langle \mathrm{Q}_1, \ldots, \mathrm{Q}_n; \mathrm{B} \rangle$. Since p is assumed to be δ – nice then $\mathrm{A}\restriction\delta$ is δ – good for p. Assume towards a contradiction that

$$\mathrm{q} \not\Vdash_{\mathbf{P}_{\mathrm{U}}} \phi(\check{\alpha}, \check{\mathrm{u}}, \hat{\mathrm{h}}\dot{\mathrm{g}}_{\beta}, \dot{\mathrm{F}}).$$

Then, there exists

$$\mathrm{q}' = \langle \mathrm{Q}_1, \ldots, \mathrm{Q}_n, \mathrm{R}_1, \ldots, \mathrm{R}_s; \mathrm{B}' \rangle \leq \mathrm{q}$$

such that $\mathrm{q}' \Vdash_{\mathbf{P}_{\mathrm{U}}} \neg\phi(\check{\alpha}, \check{\mathrm{u}}, \hat{\mathrm{h}}\dot{\mathrm{g}}_{\beta}, \dot{\mathrm{F}})$. Since $\mathrm{q}' \leq \mathrm{q}$, then

$$\langle \mathrm{R}_1, \ldots, \mathrm{R}_s \rangle \in [\mathrm{B}]^{<\omega}$$

and so

$$\langle \mathrm{R}_1 \cap \delta, \ldots, \mathrm{R}_s \cap \delta \rangle \in [\mathrm{B}\restriction\delta]^{<\omega}.$$

Since $\mathrm{B}\restriction\delta = \mathrm{A}\restriction\delta$, then $\langle \mathrm{R}_1 \cap \delta, \ldots, \mathrm{R}_s \cap \delta \rangle \in [\mathrm{A}\restriction\delta]^{<\omega}$. By the assumption that $\mathrm{A}\restriction\delta$ is δ – good for A, then there exists $\langle \mathrm{T}_1, \ldots, \mathrm{T}_s \rangle \in [\mathrm{A}]^{<\omega}$ such that $\mathrm{P}_n \subsim \mathrm{T}_1$ and

$$\langle \mathrm{R}_1 \cap \delta, \ldots, \mathrm{R}_s \cap \delta \rangle = \langle \mathrm{Y}_1 \cap \delta, \ldots, \mathrm{Y}_s \cap \delta \rangle.$$

We set $\bar{\mathrm{A}} = \mathrm{A} \setminus \{\mathrm{T}_1, \ldots, \mathrm{T}_s\}$ and $\mathrm{C} = \bar{\mathrm{A}} \cap \mathrm{B}'$. Then, set

$$\mathrm{p}_1 = \langle \mathrm{P}_1, \ldots, \mathrm{P}_n, \mathrm{T}_1, \ldots, \mathrm{T}_s; \mathrm{C} \rangle$$

and

$$q_1 = \langle Q_1, \ldots, Q_n, R_1, \ldots, R_s; C\rangle.$$

Since $p_1 \leq p$ and $q_1 \leq q$, then

$$p_1 \underset{\mathbf{P}_U}{\Vdash} \phi(\check{\alpha}, \check{u}, \hat{h}\dot{g}_\beta, \dot{F}) \text{ and } q_1 \underset{\mathbf{P}_U}{\Vdash} \neg\phi(\check{\alpha}, \check{u}, \hat{h}\dot{g}_\beta, \dot{F}) \text{ and } p_1\restriction\delta = q_1\restriction\delta.$$

At this point we define a permutation h_1 of λ such that $\mathrm{suppt}(h_1) = \delta$ and $h_1''S_i = S_1'$ where $p_1 = \langle S_1, \ldots, S_{n+s}; C\rangle$ and $q_1 = \langle S'_1, \ldots, S'_{n+s}; C\rangle$. The construction of such an automorphism has been influenced by similar arguments that appear in [Magidor 77] and [Apter 85].

The definition of such an h_1 is as follows:

We recall that the measure U has the property which implies that for every P, Q appearing in some condition for which $P \cap \kappa = Q \cap \kappa$, then $|P| = |Q|$. Since $S_i \cap \delta = S_i' \cap \delta$ and $\delta \geq \kappa$, then $|S_i| = |S_i'|$. Therefore, $|S_i \setminus \delta| = |S_i' \setminus \delta|$ and we set $h_{1,1} :\overset{\text{def}}{=} (\mathrm{id}\restriction\delta) \cup k_1$ where

$$k_1 : S_1 \setminus \delta \xrightarrow[\text{onto}]{1-1} S_1' \setminus \delta.$$

If

$$k_{i+1} : S_{i+1} \setminus S_i \xrightarrow[\text{onto}]{1-1} S_{i+1}' \setminus S_i',$$

then we define as

$$h_{1,i+1} :\overset{\text{def}}{=} h_{1,i} \cup k_{i+1}$$

and we observe that

$$h_{1,1} \subseteq \ldots \subseteq h_{1,i+1}$$

and $h_{1,i+1} : S_{i+1} \to S'_{i+1}$ is a bijection with $h_{1,i+1}\restriction \delta = \mathrm{id}\restriction \delta$. Finally, we set

$$h_{1,n+s+1} : (\lambda \setminus S_{n+s}) \setminus \delta \xrightarrow[\text{onto}]{1-1} (\lambda \setminus S'_{n+s}) \setminus \delta$$

and

$$h_1 :\stackrel{\text{def}}{=} \bigcup_{1\leq i\leq n+s+1} h_{1,i}.$$

By the construction of the $h_{1,i}$'s, h_1 is a permutation of λ with $h_1\restriction \delta = \mathrm{id}\restriction \delta$ and $h_1'' S_i = S'_i$.

Hence, we consider the automorphism $\hat{h}_1$ of $\mathbf{P}_U$ induced by h_1 and we observe that

$$\hat{h}_1(p_1) = \langle S'_1, \ldots, S'_{n+s}; h_1'' C\rangle$$

is compatible with q_1. Let $r \leq \hat{h}_1(p_1), q_1$.

As

$$p_1 \Vdash_{\mathbf{P}_U} \phi(\check{\alpha}, \check{u}, \hat{h}\dot{g}_\beta, \dot{F}),$$

then

$$\hat{h}_1(p_1) \Vdash_{\mathbf{P}_U} \phi(\check{\alpha}, \check{u}, \hat{h}_1(\hat{h}\dot{g}_\beta), \dot{F}).$$

By **2.A.9** $\hat{h}_1(\hat{h}\dot{g}_\beta) = \hat{h}\dot{g}_\beta$ and by the definition of $\dot{F}$ we conclude that $\hat{h}_1(\dot{F}) = \dot{F}$ due to the fact that $\mathrm{suppt}(\hat{h}_1 \circ h) = \min\{\mathrm{suppt}(h_1), \mathrm{suppt}(h)\} \geq \kappa$ for any h with $\mathrm{suppt}(h) \geq \kappa$ (i.e., $\hat{h} \circ h \in \mathcal{A}$).

Therefore,

$$\hat{h}_1(p_1) \Vdash_{\mathbf{P}_U} \phi(\check{\alpha}, \check{u}, \hat{h}\dot{g}_\beta, \dot{F})$$

and as $\mathrm{r} \leq \hat{\mathrm{h}}_1(\mathrm{p}_1), \mathrm{q}_1$, then

$$\mathrm{r} \underset{\mathbf{P}_{\mathrm{U}}}{\Vdash} \phi(\check{\alpha}, \check{\mathrm{u}}, \hat{\mathrm{h}}\dot{\mathrm{g}}_\beta, \dot{\mathrm{F}})$$

and at the same time

$$\mathrm{r} \underset{\mathbf{P}_{\mathrm{U}}}{\Vdash} \neg\phi(\check{\alpha}, \check{\mathrm{u}}, \hat{\mathrm{h}}\dot{\mathrm{g}}_\beta, \dot{\mathrm{F}})$$

which is a contradiction.

Therefore, $\mathrm{q} \underset{\mathbf{P}_{\mathrm{U}}}{\Vdash} \phi(\check{\alpha}, \check{\mathrm{u}}, \hat{\mathrm{h}}\dot{\mathrm{g}}_\beta, \dot{\mathrm{F}})$.

2.A.12 Lemma □

The next result is in the spirit of [Apter 85].

2.A.13 Lemma *Let* $\underset{\sim}{\mathrm{H}} :\overset{\mathrm{def}}{=} \bigcup\{\mathcal{P}(\kappa) \cap \mathbf{V}[(\hat{\mathrm{h}}\dot{\mathrm{g}}_\beta)^{\mathbf{V}[\mathrm{G}]}] \mid \alpha \in [\kappa, \lambda)\}$. *If* $\mathrm{z} \in \mathbf{V}$ *and* $\mathrm{x} \in \mathcal{P}(\mathrm{z}) \cap \mathbf{V}(\underset{\sim}{\mathrm{H}})$, *then there exists a* $\delta \in [\kappa, \lambda)$ *such that* $\mathrm{x} \in \mathcal{P}(\mathrm{z}) \cap \mathbf{V}[\mathrm{G}\restriction \delta]$.

Proof: Let $\mathrm{x} \in \mathcal{P}(\mathrm{z}) \cap \mathbf{V}(\underset{\sim}{\mathrm{H}})$. We fix $\dot{\mathrm{F}}$ to be a $\mathbf{P}_{\mathrm{U}}$ – name for $\underset{\sim}{\mathrm{H}}$ in $\mathbf{V}[\mathrm{G}]$ which is invariant under $\mathcal{A}$. Then, there exists a formula ϕ ,v, $\beta \in [\kappa, \lambda)$, and $\hat{\mathrm{h}} \in \mathcal{A}$ with $\mathrm{suppt(h)} = \gamma$ and $\mathrm{v} \in \mathcal{P}(\kappa) \cap \mathbf{V}[(\hat{\mathrm{h}}\dot{\mathrm{g}}_\beta)^{\mathbf{V}[\mathrm{G}]}]$ such that for all $\alpha \in \mathrm{z}$,

$$\alpha \in \mathrm{x} \leftrightarrow \mathbf{V}(\underset{\sim}{\mathrm{H}}) \models \phi(\alpha, \mathrm{u}, \mathrm{v}, (\dot{\mathrm{F}})^{\mathbf{V}[\mathrm{G}]}).$$

Using the formula ϕ we can find another formula Ψ such that for all $\alpha \in \mathrm{z}$

$$\alpha \in \mathrm{x} \leftrightarrow \mathbf{V}[\mathrm{G}] \models \Psi(\alpha, \mathrm{u}, (\hat{\mathrm{h}}\dot{\mathrm{g}}_\beta)^{\mathbf{V}[\mathrm{G}]}, (\dot{\mathrm{F}})^{\mathbf{V}[\mathrm{G}]}).$$

Let $\dot{\mathrm{x}} \in \mathbf{V}^{\mathbf{P}_{\mathrm{U}}}$ such that $\dot{\mathrm{x}}^{\mathbf{V}[\mathrm{G}]} = \mathrm{x}$ and we fix a condition $\mathrm{p}_0 \in \mathrm{G}$ such that

$$\mathrm{p}_0 \underset{\mathbf{P}_{\mathrm{U}}}{\Vdash} \forall\alpha \in \check{\mathrm{z}}(\alpha \in \check{\mathrm{x}} \leftrightarrow \Psi(\check{\alpha}, \check{\mathrm{u}}, \hat{\mathrm{h}}(\dot{\mathrm{g}}_\beta), \dot{\mathrm{F}}).$$

We let $\delta = \sup(h''\beta \cup h^{-1}\beta \cup \{\beta, \gamma\}) + 1$.

Next, we define the following subset of z:

$$y = \{\alpha \in z | \exists p \leq p_0 (p \text{ is } \delta - \text{nice} \,\&\, p\restriction\delta \in G\restriction\delta \,\&\, p \Vdash_{\mathbf{P_U}} \Psi(\check{\alpha}, \check{u}, \hat{h}\dot{g}_\beta, \dot{F}))\}.$$

We observe that $y \in \mathbf{V}[G\restriction\delta]$ and we claim that $y = x$.

Proof of "$y \subseteq x$" :

Let $\alpha \in y$. We fix a δ − nice p such that $p\restriction\delta \in G\restriction\delta$ and

$$p \Vdash_{\mathbf{P_U}} \Psi(\check{\alpha}, \check{u}, \hat{h}\dot{g}_\beta, \dot{F}).$$

Since $p\restriction\delta \in G\restriction\delta$, then there exists a $q \in G$ such that $p\restriction\delta = q\restriction\delta$.

By **2.A.7** we get that $q \Vdash_{\mathbf{P_U}} \Psi(\check{\alpha}, \check{u}, \hat{h}\dot{g}_\beta, \dot{F})$. Since $q \in G$, then

$$\mathbf{V}[G] \models \Psi(\alpha, u, (\hat{h}\dot{g}_\beta)^{\mathbf{V}[G]}, (\dot{F})^{\mathbf{V}[G]}),$$

i.e., $\alpha \in x$. So $y \subseteq x$.

Proof of "$x \subseteq y$" :

Let $\alpha \in x$. Then we can find $p = \langle \sigma\restriction n; A \rangle \in G$ such that

$$p \leq p_1 \quad \text{and} \quad p \Vdash_{\mathbf{P_U}} \Psi(\check{\alpha}, \check{u}, \hat{h}\dot{g}_\beta, \dot{F}).$$

By the **2.A.11** we can find a $q \in G$, $q \leq p$ such that q is δ − nice. Hence, $q \Vdash_{\mathbf{P_U}} \Psi(\check{\alpha}, \check{u}, \hat{h}\dot{g}_\beta, \dot{F})$ and as $q \in G$ this implies that $q\restriction\delta \in G\restriction\delta$. Thus, $\alpha \in y$. Hence, $x \subseteq y$.

This concludes the proof that $x = y$.

Consequently, $x \in \mathbf{V}[G\restriction\delta]$.

2.A.13 Lemma □

2.A.14 Corollary *Let* $\underset{\sim}{\mathrm{H}} :\stackrel{\text{def}}{=} \bigcup\{\mathcal{P}(\kappa) \cap \mathbf{V}[(\hat{\mathrm{h}}\dot{\mathrm{g}}_\beta)^{\mathbf{V}[\mathrm{G}]}] \mid \alpha \in [\kappa, \lambda)\}$. *If* $\mathrm{z} \in \mathbf{V}[\mathrm{G}\restriction \alpha]$ *for* $\alpha \in [\kappa, \lambda)$ *and* $\mathrm{x} \in \mathcal{P}(\mathrm{z}) \cap \mathbf{V}(\underset{\sim}{\mathrm{H}})$, *then there exists a* $\delta \in [\kappa, \lambda)$ *such that* $\mathrm{x} \in \mathcal{P}(\mathrm{z}) \cap \mathbf{V}[\mathrm{G}\restriction \delta]$.

Proof: We repeat the proof of **2.A.9** with the following changes. We find the $\mathbf{P}_\mathrm{U}$ – names $\dot{\mathrm{z}}, \dot{\mathrm{x}}$ such that $(\dot{\mathrm{z}})^{\mathbf{V}[\mathrm{G}]} = \mathrm{z}$ and $(\dot{\mathrm{x}})^{\mathbf{V}[\mathrm{G}]} = \mathrm{x}$ and a condition $\mathrm{p}_0 \in \mathrm{G}$ such that

$$\mathrm{p}_0 \Vdash_{\mathbf{P}_\mathrm{U}} \forall \alpha \in \dot{\mathrm{z}}(\alpha \in \check{\mathrm{x}} \leftrightarrow \Psi(\check{\alpha}, \check{\mathrm{u}}, \hat{\mathrm{h}}(\dot{\mathrm{g}}_\beta), \dot{\mathrm{F}}).$$

We define $\delta = \sup(\mathrm{h}''\beta \cup \mathrm{h}^{-1}\beta \cup \{\alpha, \beta, \gamma\}) + 1$ and the set y is defined exactly in the same way.

$$\mathrm{y} = \{\alpha \in \mathrm{z} \mid \exists \mathrm{p} \leq \mathrm{p}_0(\mathrm{p} \text{ is } \delta - \text{nice} \;\&\; \mathrm{p}\restriction \delta \in \mathrm{G}\restriction \delta \;\&\; \mathrm{p} \Vdash_{\mathbf{P}_\mathrm{U}} \Psi(\check{\alpha}, \check{\mathrm{u}}, \hat{\mathrm{h}}\dot{\mathrm{g}}_\beta, \dot{\mathrm{F}}))\}.$$

Then, the proof of $\mathrm{y} = \mathrm{x}$ goes through and as $\mathrm{z} \in \mathbf{V}[\mathrm{G}\restriction \delta]$, then $\mathrm{x} = \mathrm{y} \in \mathbf{V}[\mathrm{G}\restriction \delta]$, i.e., $\mathrm{x} \in \mathbf{V}[\mathrm{G}\restriction \delta]$.

2.A.14 Corollary □

2.A.15 Lemma *If* $\underset{\sim}{\mathrm{H}} = \bigcup\{\mathcal{P}(\kappa) \cap \mathbf{V}[(\hat{\mathrm{h}}\dot{\mathrm{g}}_\beta)^{\mathbf{V}[\mathrm{G}]}] \mid \hat{\mathrm{h}} \in \mathcal{A} \;\&\; \beta \in [\kappa, \lambda)\}$, *then* $\underset{\sim}{\mathrm{H}} = \mathrm{H}$ *and* $\mathrm{H} = \mathcal{P}(\kappa) \cap \mathbf{V}(\mathrm{H})$.

Proof: Since

$$\underset{\sim}{\mathrm{H}} = \bigcup\{\mathcal{P}(\kappa) \cap \mathbf{V}[(\hat{\mathrm{h}}\dot{\mathrm{g}}_\beta)^{\mathbf{V}[\mathrm{G}]}] \mid \hat{\mathrm{h}} \in \mathcal{A} \;\&\; \beta \in [\kappa, \lambda)\} \subseteq \mathcal{P}(\kappa) \cap \mathbf{V}(\underset{\sim}{\mathrm{H}}),$$

then $\mathrm{H} \subseteq \underset{\sim}{\mathrm{H}}$ is obvious.

Conversely, if $x \in \underset{\sim}{\mathrm{H}}$ then $x \in \mathbf{V}(\underset{\sim}{\mathrm{H}}) \cap \mathcal{P}(\kappa)$ and by **2.A.13** there exists some $\delta \in [\kappa, \lambda)$ such that

$$x \in \mathcal{P}(\kappa) \cap \mathbf{V}[\mathrm{G}\restriction \delta] = \mathbf{V}[(\hat{\mathrm{h}}\dot{\mathrm{g}}_\beta)^{\mathbf{V}[\mathrm{G}]}].$$

Therefore, $x \in \mathrm{H}$. So $\mathrm{H} = \underset{\sim}{\mathrm{H}}$ and $\mathrm{H} = \mathcal{P}(\kappa) \cap \mathbf{V}(\underset{\sim}{\mathrm{H}}) = \mathcal{P}(\kappa) \cap \mathbf{V}(\mathrm{H})$.

2.A.15 Lemma □

2.A.16 Corollary *If G is the fixed* $\mathbf{P}_{\mathrm{U}}$ – generic *over* $\mathbf{V}$ *and*

$$\mathrm{H} = \mathrm{H}^{\mathrm{G}} = \bigcup\{\mathcal{P}(\kappa) \cap \mathbf{V}[\mathrm{G}\restriction \alpha] \mid \alpha \in [\kappa, \lambda)\},$$

then $\mathbf{V}(\mathrm{H}) \subsetneq \mathbf{V}[\mathrm{G}]$ *and moreover,*

$$\mathbf{V}(\mathrm{H}) \models \lambda = \kappa^+.$$

Proof: We have already seen that for every $\beta \in [\kappa, \lambda)$

$$\mathbf{V}[\mathrm{G}\restriction \beta] \models |\beta| = \kappa.$$

So in each $\mathbf{V}[\mathrm{G}\restriction \beta]$ there exists the collapsing map f_β of λ onto κ, i.e.,

$$\mathrm{f}_\beta : \kappa \longrightarrow \lambda.$$

By the usual arguments there exists $\mathrm{E}_\beta \subseteq \kappa \times \kappa$, $\mathrm{E}_\beta \in \mathbf{V}[\mathrm{G}\restriction \beta]$ such that f_β is the Mostowski collapsing map of $\langle \kappa, \mathrm{E}_\beta \rangle$.

Using the bijection of κ onto $\kappa \times \kappa$ in $\mathbf{V}$ E_β is coded by a subset of κ E^*_β which is of course in $\mathbf{V}[\mathrm{G}\restriction\beta]$. Similarly, the sequence g_β itself is coded in a subset of κ in $\mathbf{V}[\mathrm{G}\restriction\beta]$. Therefore,

$$\mathbf{V}[\mathrm{G}\restriction\beta] \subseteq \mathbf{V}(\mathrm{H})$$

and

$$\mathbf{V}(\mathrm{H}) \models \forall\beta < \lambda(|\beta| = \kappa).$$

Since λ is inaccessible and $|\mathbf{P}_{\mathrm{U}_\beta}| < \lambda$, then

$$\mathbf{V}[\mathrm{G}\restriction\beta] \models \lambda \text{ is inaccessible.}$$

If

$$\mathbf{V}(\mathrm{H}) \models |\lambda| = \kappa,$$

then we could code the collapsing map of λ on κ by a subset E^*_λ of κ as above. But by **2.A.13** for some $\beta \in [\kappa, \lambda)$ we would have that

$$\mathrm{E}^*_\lambda \in \mathbf{V}[\mathrm{G}\restriction\beta].$$

Since the coding is absolute, then that would imply

$$\mathbf{V}[\mathrm{G}\restriction\beta] \models |\lambda| = \kappa.$$

But this is contrary to the fact that $\mathbf{V}[\mathrm{G}\restriction\beta] \models \lambda$ is inaccessible.

Therefore,

$$\mathbf{V}(\mathrm{H}) \models \lambda \text{ is a cardinal } \& \ (\forall\beta < \lambda)(|\beta| = \kappa).$$

Moreover, as

$$\mathbf{V}[G\restriction\beta] \subseteq \mathbf{V}(H) \subseteq \mathbf{V}[G],$$

then

$$\forall\mu < \kappa(V_\mu \subseteq (V_\mu)^{\mathbf{V}(H)} \subseteq V_\mu^{\mathbf{V}[G]} = V_\mu).$$

Since $\mathbf{V}[G] \models [\kappa$ is strong limit], then

$$\mathbf{V}(H) \models \kappa \text{ is a strong limit } \& \ \kappa^+ = \lambda.$$

But

$$\mathbf{V}[G] \models |\lambda| = \kappa$$

and consequently, $\mathbf{V}(H) \underset{\neq}{\subset} \mathbf{V}[G]$.

2.A.16 Corollary □

To summarize, in this section we establish that $\mathbf{V}(H)$ is a proper inner submodel of $\mathbf{V}[G]$ which believes in the regularity of λ. Moreover, the power set of κ is H itself.

2.B The Prikry-generic sequences in V(H)

In this section we will show that there exist many canonical forcings in $\mathbf{V}[G\upharpoonright\beta]$ so that $\mathbf{V}[G]$ is a generic extension via these forcings of $\mathbf{V}[G\upharpoonright\beta]$. We recall that A' is called β – good for $\langle x;A\rangle \in \mathbf{P}_U$ iff

$$\forall z \in [A']^{<\omega}(x\upharpoonright\beta \underset{\sim}{\subset} z \rightarrow \exists y \in [A]^{<\omega}(x \underset{\sim}{\subset} y \;\&\; z = y\upharpoonright\beta)).$$

Also we recall that a condition $\langle x;A\rangle \in \mathbf{P}_U$ is called β – nice iff

$$A\upharpoonright\beta \text{ is } \beta - \text{good for } \langle x;A\rangle.$$

2.B.1 Definition *If* $\alpha \in [\kappa,\lambda)$ *and* τ *is a* $\mathbf{P}_{U_\alpha}$ – generic *sequence over* $\mathbf{V}$, *then we define* $\mathcal{Q}_\alpha(\tau)$ *to be the following forcing:*
For any $\langle x;A\rangle \in \mathbf{P}_U$

$$\langle x;A\rangle \in \mathcal{Q}_\alpha(\tau) :\overset{\text{def}}{\Longleftrightarrow} (\exists B \subseteq A)\Big[\langle x;B\rangle \text{ is } \delta - \text{nice } \& \ \langle x\upharpoonright\alpha;B\upharpoonright\alpha\rangle \in \mathcal{F}^\alpha_\tau\Big]$$

where

$$\mathcal{F}^\alpha_\tau :\overset{\text{def}}{=} \{\langle z;C\rangle \in \mathbf{P}_{U_\alpha} \mid z \subset \tau \subset z \cup C\}$$

is the $\mathbf{P}_{U_\alpha}$ – generic *filter over* $\mathbf{V}$ *that induces the* $\mathbf{P}_{U_\alpha}$ – generic *sequence* τ *(*$\tau \in [\mathcal{P}_\kappa(\delta)]^\omega$*). The order on* $\mathcal{Q}_\alpha(\tau)$ *is* $\leq \cap \mathcal{Q}_\alpha(\tau) \times \mathcal{Q}_\alpha(\tau)$ *where* $\leq$ *is the* $\mathbf{P}_U$ – order.

2.B.1 Definition □

2.B.2 Proposition *Let* σ^α *be the* $\mathbf{P}_{U_\alpha}$ – generic *sequence induced by* $G\upharpoonright\alpha$, *i.e.,* $\mathcal{F}^\alpha_{\sigma^\alpha} = G\upharpoonright\alpha$. *If* $H_\alpha = \mathcal{Q}_\alpha(\sigma_\alpha) \cap G$, *then* H_α *is* $\mathcal{Q}_\alpha(\sigma^\alpha)$ – generic *over* $\mathbf{V}[G\upharpoonright\alpha]$ *and* $\mathbf{V}[G\upharpoonright\alpha][H_\alpha] = \mathbf{V}[G]$.

Proof: We denote by σ the $\mathbf{P}_{\mathrm{U}}$ – generic sequence by G, i.e., $\mathcal{F}_\sigma^\lambda = \mathrm{G}$.

First, we show that H_α is a $\mathcal{Q}_\alpha(\sigma^\alpha)$ – filter.

Assume that $q_1, q_2, \in \mathrm{H}_\alpha$. Since $q_1, q_2 \in \mathrm{G}$, then $\exists q \in \mathrm{G}(q \leq q_1, q_2)$.

By **2.A.11**

$$(\exists q' \in \mathrm{G})(q' \text{ is } \alpha - \text{nice } \& \ q' \leq q \leq q_1, q_2)$$

and so $q' \in \mathrm{H}_\alpha$. If

$$q \in \mathrm{H}_\alpha \ \& \ q \leq p \ \& \ p \in \mathcal{Q}_\alpha(\sigma^\alpha),$$

then $p \in \mathrm{G}$ as $q \in \mathrm{G}$ and so $p \in \mathrm{H}_\alpha$.

Therefore, H_α is a $\mathcal{Q}_\alpha(\sigma^\alpha)$ – filter.

It remains to be seen that H_α is $\mathcal{Q}_\alpha(\sigma^\alpha)$ – generic over $\mathbf{V}[\mathrm{G}\restriction \alpha]$.

Let D be $\mathcal{Q}_\alpha(\sigma^\alpha)$ – dense in $\mathbf{V}[\mathrm{G}\restriction \alpha]$. Then there exists $\mathbf{V}^{\mathbf{P}_{\mathrm{U}_\alpha}}$ – name $\dot\tau$ such that

$$\mathrm{D} = (\dot\tau)^{\mathbf{V}[\mathrm{G}\restriction \alpha]}.$$

We define $\bar{\mathrm{D}} \subseteq \mathbf{P}_{\mathrm{U}}$ such that

$$\bar{\mathrm{D}} = \{p \in \mathbf{P}_{\mathrm{U}} \mid (\exists r \geq p)(p \Vdash_{\mathbf{P}_{\mathrm{U}}} \check{r} \in (\dot\tau)^{\mathbf{V}[\dot{g}_\alpha]})\}.$$

We observe that if

$$q \in \bar{\mathrm{D}} \cap \mathrm{G},$$

then

$$(\exists r \geq p)(\ p \Vdash_{\mathbf{P}_{\mathrm{U}}} \check{r} \in (\dot\tau)^{\mathbf{V}[\dot{g}_\alpha]}$$

and so

$$\mathbf{V}[\mathrm{G}] \models r \in \mathrm{D} \cap \mathrm{G}, \qquad \text{i.e., } \mathrm{D} \cap \mathrm{H}_\alpha \neq \emptyset.$$

Hence, it suffices to show that $\bar{D} \cap G \neq 0$. In order to show that, we prove that for every $p \in G$ there exists $q \leq p$ such that $q \in \bar{D}$.

So let $p = \langle \sigma\restriction \kappa; A\rangle \in G$.

We find $p^* = \langle \sigma\restriction m; A^*\rangle$ which is $\alpha-$nice and belongs in H_α such that $p^* \leq p$.

Since D is $\mathcal{Q}_\alpha(\sigma^\alpha)$ – dense, then we can find

$$r = \langle \sigma\restriction m^\frown\langle P_1, \dots, P_n\rangle; B\rangle \in D$$

such that $r \leq p^*$ and r is α – nice. Let

$$s = \langle \sigma\restriction m^\frown\langle \sigma(m+1), \dots, \sigma(m+n)\rangle^\frown\langle \sigma(m+n+1), \dots, \sigma(m+n+l)\rangle; B'\rangle \in G$$

such that $s \leq p^*$ and

$$s \underset{\mathbf{P}_U}{\Vdash} \check{r} \in (\dot{\tau})^{\mathbf{V}[\dot{g}_\alpha]}.$$

Without loss of generality, we assume that s is α – good and $s \in G$. Since

$$r\restriction \alpha \in \mathcal{F}^\alpha_{\sigma^\alpha} = G\restriction \alpha,$$

then

$$P_i \cap \alpha = \sigma(m+1) \cap \alpha$$

for all i such that $1 \leq i \leq n$ and since $r\restriction \alpha \in g\restriction \alpha$, $s\restriction \alpha \in g\restriction \alpha$ and $lh(s\restriction \alpha) \geq lh(r\restriction \alpha)$, then

$$\sigma(m+n+1) \cap \alpha \in B\restriction \alpha \quad \text{for} \quad 1 \leq i \leq n.$$

But then, as r is α – nice, there exists

$$\langle P_1, \dots, P_{n+l}\rangle \in [B]^{<\omega}$$

so that $P_n \underset{\sim}{\subset} P_{n+1}$ and

$$P_{n+i} \cap \alpha = \sigma(m+n+i) \cap \alpha \quad \text{for} \quad 1 \leq i \leq l.$$

We let

$$E = \{P \in B \cap B' \mid P_{n+l} \underset{\sim}{\subset} P \ \& \ \sigma(m+n+l) \underset{\sim}{\subset} P\}.$$

Next we find E′ α – good for E and

$$E^* = \{P \in E \mid P \cap \alpha \in E'\}.$$

Then, $E^* \subseteq E$ and $s^* = \langle \sigma\restriction(m+n+l+1); E^*\rangle \leq s$ and s^* is α – nice. So

$$s^* \underset{\mathbf{P}_U}{\Vdash} \check{r} \in (\dot{\tau})^{\mathbf{V}[\dot{g}_\alpha]}$$

and

$$q = \langle \sigma\restriction m^\frown\langle P_1,\dots,P_n,\dots,P_{n+l}\rangle; E^*\rangle \leq r \quad \text{and} \quad q\restriction\alpha = s^*\restriction\alpha.$$

Since S^* is α – nice and $q\restriction\alpha = s^*\restriction\alpha$, then by **2.A.12**

$$q \underset{\mathbf{P}_U}{\Vdash} \check{r} \in (\dot{\tau})^{\mathbf{V}[\dot{g}_\alpha]} \quad \& \quad q \leq r.$$

Therefore, $q \in \bar{D}$. Since $r \leq p^* \leq p$, then $q \leq p$. Hence, we have shown that

$$(\forall p \in G)(\exists q \in \bar{D})(q \leq p).$$

Therefore, $\bar{D} \cap G \neq \emptyset$ and as we have shown this implies that $D \cap H_\alpha \neq \emptyset$.

Thus, H_α is a $\mathcal{Q}_\alpha(\sigma^\alpha)$ – generic filter.

Obviously, $H_\alpha \in \mathbf{V}[G]$ and so $\mathbf{V}[G\restriction\alpha][H_\alpha] \subseteq \mathbf{V}[G]$. Since

$$(\forall p \in G)(\exists q \in H_\alpha)(q \leq p),$$

then

$$\sigma' = \bigcup\{x \mid \exists A\langle x;A\rangle \in H_\alpha\} = \sigma$$

and so $G \in \mathbf{V}[G\restriction\alpha][H_\alpha]$.

Therefore, $\mathbf{V}[G\restriction\alpha][H_\alpha] = \mathbf{V}[G]$.

2.B.2 Proposition □

2.B.3 Lemma *Let U be a* $\mathcal{P}_\kappa(\beta)$ – normal *measure for* $\beta \in [\kappa,\lambda)$ *and for* $\alpha \in [\kappa,\beta)$ *we let* $U_\alpha = \{A\restriction\alpha \mid A \in U\}$ *and* $\mathbf{P}_U, \mathbf{P}_{U_\alpha}$ *be the corresponding Supercompact Prikry forcings. Then there exists a complete embedding* $i : \mathbf{P}_{U_\alpha} \to \mathbf{r.o.}(\mathbf{P}_U)$.

Proof: We define $i : \mathbf{P}_{U_\alpha} \to \mathbf{r.o.}(\mathbf{P}_U)$ as follows:

For $p \in \mathbf{P}_{U_\alpha}$ we set,

$$i(p) :\stackrel{\text{def}}{=} \sum\{q \in \mathbf{P}_U \mid q\restriction\alpha \leq p\}.$$

Recall that if $q = \langle x;A\rangle$, then $q\restriction\alpha = \langle x\restriction\alpha;A\restriction\alpha\rangle$. We set

$$\mathcal{X}(p) = \{q \in \mathbf{P}_U \mid q\restriction\alpha \leq p\}.$$

Hence,

$$\forall p \in \mathbf{P}_{U_\alpha}[i(p) = \sum\mathcal{X}(\phi)].$$

Next, we show that i is a complete embedding.

(1) We assume that $q_1 \leq_\alpha q_2$ where $\leq_\alpha$ is the $\mathbf{P}_{U_\alpha}$ – order and $\leq$ is the $\mathbf{P}_U$ – order. If $r \in \mathcal{X}(q_1)$, then $r\restriction\alpha \leq_\alpha q_1 \leq_\alpha q_2$. Hence, $r\restriction\alpha \leq q_2$, i.e.,

$r \in \mathcal{X}(q_2)$.

As $\mathcal{X}(q_1) \subseteq \mathcal{X}(q_2)$, then

$$\sum \mathcal{X}(q_1) \leq \sum \mathcal{X}(q_2)$$

and so $i(q_1) \leq i(q_2)$.

(2) Assume that $q_1 \perp q_2$ in $\mathbf{P}_{U_\alpha}$. We take $s \in \mathcal{X}(q_1)$ and $t \in \mathcal{X}(q_2)$. If $s' \leq s, t$, then $s' \restriction \alpha \leq_\alpha t \restriction \alpha \leq_\alpha q_2$, i.e., $q_1 \not\perp q_2$. So

$$(\forall s \in \mathcal{X}(q_1))(\forall t \in \mathcal{X}(q_2))(s \perp t).$$

Hence, $i(q_1) \perp i(q_2)$. This shows that if $i(q_1) \perp i(q_2)$ in $\mathbf{P}_{U_\alpha}$, then $i(q_1) \perp i(q_2)$ in $\mathbf{r.o.}(\mathbf{P}_U)$.

(3) Next we consider $p = \langle z;A \rangle \in \mathbf{P}_U$ and we have to find a reduction q of p in $\mathbf{P}_U$. For that we find A' α – good for $\langle z;A \rangle$ and we set $\langle z \restriction \alpha;A' \rangle$. We claim that $q \in \mathbf{P}_{U_\alpha}$ is a reduction of p in $\mathbf{P}_{U_\alpha}$. Let

$$r = \langle z \restriction \alpha ^\frown w';A'' \rangle \leq q.$$

Since $w' \in [A']^{<\omega}$, then we find $w \in [A]^{<\omega}$ such that $w' = w \restriction \alpha$. We set

$$A^* = \{p \in A \mid \; P \underset{\sim}{\subset} w \quad \& \quad P \cap \alpha \in A''\}$$

and $p' = \langle z ^\frown w;A^* \rangle$.

Then $p' \leq p$ and $p' \restriction \alpha \leq r$. Hence,

$$\exists x \in \mathcal{X}(r)(x \parallel p)$$

which implies that $p \parallel i(x)$.

This shows that q is indeed a reduction of p in $\mathbf{P}_{U_\alpha}$.

The parts (1),(2),(3) complete the proof that $i : \mathbf{P}_{U_\alpha} \to \mathbf{r.o.}(\mathbf{P}_U)$ is a complete embedding.

2.B.3 Lemma □

2.B.4 Lemma *Let U be a* $\mathcal{P}_\kappa(\beta)$ – normal *measure for* $\beta \in [\kappa, \lambda)$ *and for* $\alpha \in [\kappa, \beta)$ *we let* $U_\alpha = \{A \restriction \alpha \mid A \in U\}$ *and* $\mathbf{P}_U, \mathbf{P}_{U_\alpha}$ *be the corresponding Supercompact Prikry forcings. Assume that h is a* $\mathbf{P}_U$ – generic *sequence and* $\mathcal{F}_h^\beta$ *is the corresponding* $\mathbf{P}_U$ – generic *filter. If*

$$K = \{p \in \mathbf{P}_{U_\alpha} \mid i(p) \in \mathcal{F}_h^\beta\},$$

then K is $\mathbf{P}_{U_\alpha}$ – generic *and its corresponding* $\mathbf{P}_{U_\alpha}$ – generic *sequence* σ_K *satisfies the property* $\sigma_K = h \restriction \alpha$.

Proof: By **2.B.3** K is a $\mathbf{P}_{U_\alpha}$ – generic filter, as i is a complete embedding. Next we fix $p \in K$. Since $i(p) = \sum \mathcal{X}(p) \in K$ and K is $\mathbf{P}_U$ – generic, then there exists an r

$$r \in \mathcal{X}(p) \cap K.$$

Let $r = x \restriction A$. Since $h = \sigma_{\mathcal{F}_h^\beta}$, then $x \subset h \subset x \cup A$ and as $r \in \mathcal{X}(p)$, then

$$r \restriction \alpha = \langle x \restriction \alpha; A \restriction \alpha \rangle \le p = \langle z; B \rangle.$$

Therefore,

$$x \restriction \alpha \subset h \restriction \alpha \subseteq x \restriction \alpha \cup A \restriction \alpha$$

and so $\langle z;B\rangle \in \mathcal{F}^{\alpha}_{h\restriction\alpha}$. Hence, $K \subseteq \mathcal{F}^{\alpha}_{h\restriction\alpha}$.

Since K and $\mathcal{F}^{\alpha}_{h\restriction\alpha}$ are $\mathbf{P}_{U_\alpha}$ – generic filters over $\mathbf{V}$ and $K \subseteq \mathcal{F}^{\alpha}_{h\restriction\alpha}$, then

$$\mathcal{F}^{\alpha}_{h\restriction\alpha} = K.$$

Therefore, $\sigma_K = h\restriction\alpha$.

2.B.4 Lemma □

The following theorem was proved by K. Prikry [Prikry 70] for the Prikry forcing on a measurable cardinal κ. Exactly the same arguments work for any $\mathcal{P}_\kappa(\delta)$ Supercompact Prikry forcing. So, we mention the result without a proof.

2.B.5 Theorem *We let $\beta > \kappa$ where κ is a Supercompact cardinal, β is a regular cardinal, U is a normal measure on $\mathcal{P}_\kappa(\beta)$ and $\mathbf{P}_U$ is the corresponding Supercompact Prikry forcing and G is a $\mathbf{P}_U$ –* generic *over* $\mathbf{V}$.

1. *If $\dot{\tau}$ is a $\mathbf{P}_U$ –* name *such that*

$$\Vdash_{\mathbf{P}_U} [\dot{\tau} : \check{\mu} \to \nu \quad \& \quad \mathrm{range}(\dot{\tau}) = \mathrm{E}]$$

where $\mu < \kappa$ and $(\dot{\mathrm{E}})^{\mathbf{V}[G]} = \mathrm{E}$ is a set of ordinals, then there exist sets $\{\mathrm{E}_n \mid n \in \omega\} \subset \mathbf{V}$ such that $(|\mathrm{E}_n| \leq \mu)^{\mathbf{V}}$ and $\mathrm{E} = \bigcup_{n<\omega} \mathrm{E}_n$.

2. *If $\mathcal{X} \in \mathbf{V}$ and $\mathrm{E} \in \mathbf{V}[G]$, $\mathrm{E} \subset \mathcal{X}$ such that $(|\mathrm{E}| < \kappa)^{\mathbf{V}[G]}$, then there exists a set $\{\mathrm{E}_n \mid n \in \omega\} \subset \mathbf{V}$ such that $(|\mathrm{E}_n| < \kappa)^{\mathbf{V}}$ and $\mathrm{E} = \bigcup_{n<\omega} \mathrm{E}_n$.*

3. If $\mathcal{X} \in \mathbf{V}$, $\mathrm{E} \subset \mathcal{X}$, $\mathrm{E} \in \mathbf{V}[\mathrm{G}]$ *and* $(|\mathrm{E}| = \kappa)^{\mathbf{V}[\mathrm{G}]}$, *then there exist sets* $\mathrm{E_n} \in \mathbf{V}$ $(|\mathrm{E_n}| < \kappa)^{\mathbf{V}}$ *such that* $\mathrm{E} = \bigcup_{\mathrm{n}<\omega} \mathrm{E_n}$.

2.B.5 Theorem □

2.B.6 Theorem *We let* $\lambda > \kappa$ *where* κ *is a Supercompact cardinal,* β *is a strong limit cardinal, U is a normal measure on* $\mathcal{P}_\kappa(\lambda)$, $\mathbf{P}_{\mathrm{U}}$ *is the corresponding Supercompact Prikry forcing and G is a* $\mathbf{P}_{\mathrm{U}}$ – generic *over* $\mathbf{V}$. *Let* $\mathcal{U}$ *be any normal measure on* $\mathcal{P}_\kappa(\beta)$ *for* $\beta \in [\kappa, \lambda)$. *Then, in* $\mathbf{V}(\mathrm{H})$, *we can find a* $\mathbf{P}_{\mathcal{U}}$ – generic.

Proof: Since $\beta \in [\kappa, \lambda)$ and λ is strong limit, then we can find a cardinal $\gamma \in [\kappa, \lambda)$ such that $(|\mathcal{U}| = \gamma)^{\mathbf{V}}$. Since

$$\mathbf{V}[\mathrm{G}\restriction\gamma] \models |\gamma| = \kappa,$$

then we conclude that

$$\mathbf{V}[\mathrm{G}\restriction\gamma] \models (\exists \mathrm{f})(\mathrm{f} : \kappa \xrightarrow[\text{onto}]{1-1} \mathcal{U}) \quad \& \quad \mathcal{U} \subset (\mathcal{P}_\kappa(\beta))^{\mathbf{V}}$$

and

$$(\mathcal{P}_\kappa(\beta))^{\mathbf{V}} \in \mathbf{V}.$$

According to **2.B.5** we can find sets

$$\mathrm{E_n} \in \mathbf{V}$$

for $\mathrm{n} \in \omega$ such that

$$|\mathrm{E_n}| < \kappa \quad \text{with} \quad \mathcal{U} = \bigcup_{\mathrm{n}<\omega} \mathrm{E_n}.$$

By taking

$$E^*_m = \bigcup_{n \leq m} E_k$$

then, we have that

$$\{E^*_m \mid m \in \omega\} \in \mathbf{V}[G\restriction\gamma] \quad \& \quad E^*_m \in \mathbf{V} \quad \& \quad |E^*_m| < \kappa \quad \& \quad \mathcal{U} = \bigcup_{m<\omega} E^*_m.$$

We let

$$A^*_m = \cap E^*_m = \cap\{A \in \mathcal{U} \mid A \in E^*_m\}.$$

By the κ-completeness of the measure $\mathcal{U}$ and as

$$E^*_m \in \mathbf{V} \quad \& \quad (|E^*_m| < \kappa)^{\mathbf{V}},$$

then each $A^*_m \in \mathcal{U}$. We set $P_0 \in A^*_m$. Then we find $P_n \underset{\sim}{\subset} P_{n+1}$ such that $P_{n+1} \in A^*_{n+1}$ and

$$\forall m \in \omega \quad \{P_\kappa \mid \kappa \geq m\} \subset A^*_m.$$

Therefore,

$$\langle P_n \mid n \in \omega \rangle \in \mathbf{V}[G\restriction\gamma]$$

and $\langle P_n \mid n \in \omega \rangle$ is a $\mathbf{P}_{\mathcal{U}}$ – generic sequence over $\mathbf{V}$. Since

$$\langle P_n \mid n \in \omega \rangle \in \mathbf{V}[G\restriction\gamma] \subset \mathbf{V}(H),$$

then we found a $\mathbf{P}_{\mathcal{U}}$ – generic filter in $\mathbf{V}(H)$.

2.B.6 Theorem □

2.B.7 Lemma *We let* $\mathbf{P}_{\mathcal{U}}$ *be the* $\mathcal{P}_\kappa(\beta)$ – Supercompact *Prikry forcing with respect to* $\mathcal{U}$. *We take* $\mathcal{Q} \subset_c \mathbf{r.o.}(\mathbf{P}_{\mathcal{U}})$ *and* $g \in \mathbf{V}(H)$ *be a* $\mathcal{Q}$ – generic *over* $\mathbf{V}$ *filter. Then there exists in* $\mathbf{V}(H)$ *a* K *which is* $\mathbf{P}_{\mathcal{U}}$ – generic *over* $\mathbf{V}$ *such that* $K \cap \mathcal{Q} = g$.

Proof: We use the notation of **2.B.6** and for some $\beta, \gamma \in [\kappa, \lambda)$ we find

$$\langle A^*_n \mid n \in \omega \rangle \in \mathbf{V}[G\restriction \gamma]$$

such that for every $A \in \mathcal{U}$

$$(\exists n < \omega)(A \supseteq A^*_n).$$

We set $V_1 = \mathbf{V}[G\restriction \gamma]$. Then, by recursion on ω, we define $\langle \mathcal{T}^*_n \mid n \in \omega \rangle \in V_1$ as follows:

$$\begin{aligned} \langle P_0, \ldots, P_m \rangle \in \mathcal{T}^*_0 \quad & :\overset{\text{def}}{\Longleftrightarrow} \langle P_0, \ldots, P_m \rangle \in [\mathcal{P}_\kappa(\delta)]^{<\omega} \quad \& \\ & \& \quad (\forall k < m)(P_k \notin A^*_0) \quad \& \quad P_m \in A^*_0 \quad \& \\ & \& \quad (\forall x \in g)(\langle P_0, \ldots, P_{m-1}; A^* \rangle \parallel x). \end{aligned}$$

If we assume that T^*_n has been constructed, then we define

$$\begin{aligned} \langle P_0, \ldots, P_m \rangle \in \mathcal{T}^*_{n+1} \quad & :\overset{\text{def}}{\Longleftrightarrow} \langle P_0, \ldots, P_k \rangle \in \mathcal{T}^*_n \quad \text{or} \\ & \& \quad P_k \ldots P_{m-1} \in A^*_n \quad \& \quad P_m \in A^*_{n+1} \quad \& \\ & \& \quad (\forall x \in g)(\langle P_0, \ldots, P_{m-1}; A^*_{n+1} \rangle \parallel x). \end{aligned}$$

Then $\langle \mathcal{T}_n^* \mid n \in \omega \rangle \in V_1$.

We set

$$\mathcal{T}^* :\stackrel{\mathrm{def}}{=} \bigcup_{n<\omega} \mathcal{T}_n^*$$

and we define a tree $\mathcal{T} \subset [\mathcal{P}_\kappa(\delta)]^{<\omega}$ such that

$$\forall x \in [\mathcal{P}_\kappa(\delta)]^{<\omega} \Big(x \in \mathcal{T} :\stackrel{\mathrm{def}}{\Longleftrightarrow} \exists y \in \mathcal{T}^*(x \subset y) \Big).$$

Obviously, $\mathcal{T} \in V_1$.

In V_1, we construct the quotient forcing $\mathbf{P}_{\mathcal{U}}/g$ and we take K to be $\mathbf{P}_U/g$ – generic over V_1. Then K is $\mathbf{P}_{\mathcal{U}}$ – generic over $\mathbf{V}$ and $K \cap \mathcal{Q} = g$. Let σ be the $\mathbf{P}_U$ – generic sequence induced by K. Then, by induction on n, we can show that

$$(\forall n)(\exists k_n)(\sigma \restriction k_n \in \mathcal{T}_n^*).$$

Hence, σ is a branch through the tree $\mathcal{T}$, i.e.,

$$V_1[K] \models [\mathcal{T} \text{ is not well – founded}].$$

But by absoluteness of the "well – foundedness"

$$V_1 \models [\mathcal{T} \text{ is not well – founded}]$$

and as $V_1 \subset \mathbf{V}(H)$, then

$$\mathbf{V}(H) \models [\mathcal{T} \text{ is not well – founded}].$$

Hence, $\mathcal{T}$ has a branch in $\mathbf{V}(H)$.

Obviously, any branch b through $\mathcal{T}$ is a $\mathbf{P}_U$ – generic sequence and if $\mathcal{F}_b^\beta$ is the generic filter induced by the sequence b, then

$$\mathcal{F}_{\mathrm{b}}^{\beta} \in \mathbf{V}(\mathrm{H})$$

and $\mathcal{F}_{\mathrm{b}}^{\beta} \cap \mathcal{Q} = \mathrm{g}$.

2.B.7 Lemma □

2.B.8 Lemma *We assume that* $\langle \bar{\mathrm{z}};\mathrm{A}\rangle \in \mathbf{P}_{\mathrm{U}}$ *and that* $\mathrm{x} \in \mathbf{V}(\mathrm{H})$ *is a* $\mathbf{P}_{\mathcal{U}_\alpha}$ – generic *sequence, where* $\mathcal{U}$ *is a* $\mathcal{P}_\kappa(\beta)$ – normal *measure measure in* $\mathbf{V}$, $\alpha < \beta$,

$$\mathcal{U}_\alpha = \mathcal{U}\restriction\alpha = \{\mathrm{A}\restriction\alpha \mid \mathrm{A} \in \mathcal{U}\}$$

and $\bar{\mathrm{z}}\restriction\alpha \subset \mathrm{x} \subset \bar{\mathrm{z}}\restriction\alpha \cup \mathrm{A}\restriction\alpha$. *Then, there exists a* $\mathrm{y} \in \mathbf{V}(\mathrm{H})$ *such that* y *is* $\mathbf{P}_{\mathcal{U}}$ – generic *over* $\mathbf{V}$, $\bar{\mathrm{z}} \subset \mathrm{y} \subset \bar{\mathrm{z}} \cup \mathrm{A}$ *and* $\mathrm{y}\restriction\alpha = \mathrm{x}$.

Proof: We use **2.B.3** to find the canonical embedding

$$\mathrm{i} : \mathbf{P}_{\mathcal{U}_\alpha} \subset_c \mathbf{r.o.}(\mathbf{P}_{\mathcal{U}}).$$

If $\mathcal{F}_{\mathrm{x}}^{\alpha}$ is the $\mathbf{P}_{\mathcal{U}_\alpha}$ –generic filter induced by x, then we let $\mathcal{Q}$ be the forcing $\mathrm{i}''\mathbf{P}_{\mathcal{U}_\alpha}$ and obviously $\mathrm{i}''\mathbf{P}_{\mathcal{U}_\alpha} \subset_c \mathbf{r.o.}(\mathbf{P}_{\mathcal{U}})$. Also, we consider g to be the $\mathcal{Q}$ – generic filter over $\mathbf{V}$ induced by $\mathrm{i}''\mathcal{F}_{\mathrm{x}}^{\alpha}$ in $\mathcal{Q}$. We have proved in **2.B.4** that if K is $\mathbf{P}_{\mathcal{U}}$ – generic over $\mathbf{V}$ such that $\mathcal{Q} \cap \mathrm{K} = \mathrm{g}$, then

$$\sigma_{\mathrm{K}}\restriction\alpha = \mathrm{x}.$$

Next, we do the construction of $\langle \mathrm{T}_{\mathrm{n}}^{*} \mid \mathrm{n} \in \omega\rangle$ in some $\mathbf{V}[\mathrm{G}\restriction\gamma]$ for $\gamma > \beta$ where A_0^{*} is replaced by $\mathrm{A}_0^{*} \cap \mathrm{A}$ and $\mathcal{T}_{\mathrm{n}}^{*}$ is defined as follows:

$$\langle \mathrm{P}_0,\ldots,\mathrm{P}_{\mathrm{l}}\rangle \in \mathcal{T}_0^{*} :\overset{\mathrm{def}}{\Longleftrightarrow} \quad \langle \mathrm{P}_0,\ldots,\mathrm{P}_{\mathrm{l}}\rangle \in [\mathcal{P}_\kappa(\beta)]^{<\omega} \quad \&$$

$$\& \quad \bar{z} \subset \langle P_0, \ldots, P_l \rangle \quad \&$$

$$\& \quad (\forall i)(lh(\bar{z}) \leq i \leq l \quad P_i \in A) \quad \&$$

$$\& \quad \forall i < l \quad P_i \notin A_0^* \quad \& \quad P_l \in A_0^* \quad \&$$

$$\& \quad \forall x \in g(\langle 0_0, \ldots, 0_{l-1}; A_0^* \rangle \parallel x).$$

If we assume that T_n^* has been constructed, then we define

$$\langle P_0, \ldots, P_m \rangle \in \mathcal{T}_{n+1}^* : \overset{\text{def}}{\Longleftrightarrow} \quad \langle P_0, \ldots, P_k \rangle \in \mathcal{T}_n^* \quad \text{or}$$

$$[P_k, \ldots, P_{m-1} \in A_n^* \quad \&$$

$$\& \quad P_m \in A_{n+1}^* \quad \& \quad (\forall x \in g)$$

$$(\langle P_0, \ldots, P_{m-1}; A_{n+1}^* \rangle \parallel x)].$$

As before, we find a $\mathbf{P}_{\mathcal{U}}$ – generic K in $\mathbf{V}$(H) and if σ_K is the induced $\mathbf{P}_{\mathcal{U}}$ – generic sequence which is a branch through $\mathcal{T}$, then $\sigma_K \restriction \alpha = x$ and $\bar{z} \subset \sigma_K \subset \bar{z} \cup A$. This concludes the proof of the lemma.

2.B.8 Lemma □

2.B.9 Lemma *Let U be our fixed $\mathcal{P}_\kappa(\lambda)$ –* normal *measure in* $\mathbf{V}, \mathbf{P}_U$ *is the corresponding Supercompact Prikry forcing and* $\mathbf{P}_{U_\alpha}$ where $U_\alpha = U \restriction \alpha$. *Assume that y is a* $\mathbf{P}_{U_\beta}$ – generic *sequence and* $\langle \bar{z}; A \rangle \in \mathbf{P}_U$ *for some* $\beta \in [\kappa, \lambda)$. *Then there exists a forcing in* $\mathbf{V}[y]$ *of size* $\leq \lambda^+$ *that forces a* $\mathbf{P}_U$ – generic *filter K such that* $\sigma_K \restriction \beta = y$ *and* $\bar{z} \subset \sigma_K \subset \bar{z} \cup A$.

Proof: We have shown that since λ is an inaccessible above the Supercompact κ, then $|U| \leq \lambda^+$. Let $C = \text{Coll}(\omega, \lambda^+)$ and L be C – generic over

$\mathbf{V}[y]$. Then

$$\mathbf{V}[y][L] \models |U| = \omega$$

and let $\langle A_n \mid n \in \omega \rangle$ be the enumeration of U in $\mathbf{V}[y][L]$. We set

$$A_0^* = A_0 \cap A \quad \& \quad A_k^* = \bigcap_{n \leq k} A_n,$$

i.e., $A_0^* \supseteq \ldots \supseteq A_k^* \supseteq \ldots$ and each $A_n^* \in U$.

Then we construct the tree $\mathcal{T}$, as in **2.B.8**, where $\beta = \lambda$ and $g = \mathcal{F}_y^\beta$. Then, by DC, we can find in $\mathbf{V}[y][L]$ a branch b through $\mathcal{T}$.

Let K be the $\mathbf{P}_U$ – generic $\mathcal{F}_b^\lambda$. Then

$$\bar{z} \subset b \subset \bar{z} \cup A \quad \text{and} \quad b \restriction \beta = y.$$

We consider the forcing $\mathcal{Q}_\beta(y)$ as we defined it in the **2.B.1**. Since $b \restriction \beta = y$, then we let

$$H_\beta = \mathcal{Q}_\beta(y) \cap K.$$

Hence, by **2.B.2** H_β is $\mathcal{Q}_\beta(y)$ – generic over $\mathbf{V}$ and

$$\mathbf{V}[y][H_\beta] = \mathbf{V}[b] \quad \text{and} \quad b \restriction \beta = y.$$

Moreover,

$$\mathcal{Q}_\beta(y) \subseteq \mathbf{P}_U \quad \text{and} \quad |\mathcal{Q}_\beta(y)| \leq \lambda^+.$$

A different way to obtain this result is as follows:

We let $\mathcal{Q} = i''\mathbf{P}_{U_\beta} \subset_c \mathbf{r.o.}(\mathbf{P}_U)$ where i is the canonical complete embedding defined in the proof of **2.B.3**

$$i : \mathbf{P}_{U_\beta} \subset_c \mathbf{r.o.}(\mathbf{P}_U).$$

We let g be the $\mathcal{Q}$ – generic over **V** filter induced by $i''\mathcal{F}^{\beta}_{y}$. Then, we consider the forcing

$$\mathbf{r.o.}(\mathbf{P}_U)/g \in \mathbf{V}[G] = \mathbf{V}[y].$$

We let K be a $\mathbf{r.o.}(\mathbf{P}_U)/g$ – generic filter over **V**. Then K is $\mathbf{r.o.}(\mathbf{P}_U)$ – generic over **V** such that

$$K \cap \mathbf{P}_U = g.$$

Therefore, $\sigma_K \restriction \beta = y$. Now again, H_β is $\mathcal{Q}_\beta(y)$ – generic over **V** and so

$$\mathbf{V}[y][H_\beta] = \mathbf{V}[y]_{\mathbf{r.o.}(\mathbf{P}_U)/g}[K] = \mathbf{V}[K].$$

2.B.9 Lemma □

2.B.10 Lemma *Let* $\dot\tau$ *be a* $\mathbf{P}_{U_\beta}$ – name *such that*

$$\Vdash_{\mathbf{P}_{U_\beta}} [(\dot\tau \text{ is } \mathbf{P}_{U_\alpha} - \text{generic over } \mathbf{V}) \ \& \ (x^\alpha \restriction n \subset \dot\tau \subset x^\alpha \restriction n \cup A \restriction \alpha)]$$

and $\kappa \leq \alpha < \beta < \lambda$. *If* G^* is a $\mathbf{P}_U$ – generic *filter such that* $\mathbf{V}[G^*] = \mathbf{V}[G]$ *and* $\langle x \restriction n; A \rangle \in G^*$, *then in* **V**(H) *there exists a* $\mathbf{P}_{U_\beta}$ – generic *sequence h such that* $\dot\tau^{\mathbf{V}[h]} = G^* \restriction \alpha$.

Proof: Without loss of generality we identify the forcing $\mathbf{P}_{U_\beta}$ with

$$(\mathbf{P}_{U_\beta})_{\langle x^\beta \restriction n; A \restriction \beta \rangle} = \{r \in \mathbf{P}_{U_\beta} \mid r \leq \langle x^\beta \restriction n; A \restriction \beta \rangle\}$$

and $\mathbf{P}_{U_\alpha}$ with $(\mathbf{P}_{U_\alpha})_{\langle x^\alpha \restriction n; A \restriction \alpha \rangle}$. Then we still have that

$$\Vdash_{\mathbf{P}_{U_\beta}} [(\dot\tau \text{ is } \mathbf{P}_{U_\alpha} - \text{generic over } \mathbf{V}].$$

We use the standard argument (see [Jech 78]) to show that there exists a $p_0 \in \mathbf{P}_{U_\alpha}$ such that

$$\forall p \leq p_0 [\![\check{p} \in \dot{\tau}]\!]_{\mathbf{r.o.}(\mathbf{P}_{U_\beta})} \neq 0.$$

Following that argument we define

$$Y = \{s \in \mathbf{r.o.}(\mathbf{P}_{U_\alpha}) \mid \quad [\![\check{s} \in \dot{\tau}]\!]_{\mathbf{r.o.}(\mathbf{P}_{U_\beta})} = 0\}.$$

We let $s_0 = \sup \mathcal{Y} \in \mathbf{r.o.}(\mathbf{P}_{U_\alpha})$. But then,

$$\begin{aligned} [\![\check{s} \in \dot{\tau}]\!]_{\mathbf{r.o.}(\mathbf{P}_{U_\beta})} &= [\![\exists x \in \check{\mathcal{Y}}(\check{x} \in \dot{\tau})]\!]_{\mathbf{r.o.}(\mathbf{P}_{U_\beta})} \\ &= \sum\{[\![\check{x} \in \dot{\tau}]\!]_{\mathbf{r.o.}(\mathbf{P}_{U_\beta})} \mid x \in \mathcal{Y}\} \\ &= 0. \end{aligned}$$

We let $p_0 \in \mathbf{P}_{U_\alpha}$ such that $p_0 \leq \neg s_0$. Then

$$\forall p \leq p_0 [\![\check{p} \in \dot{\tau}]\!]_{\mathbf{r.o.}(\mathbf{P}_{U_\beta})} \neq 0.$$

The above argument shows that the map

$$e : (\mathbf{r.o.}(\mathbf{P}_{U_\alpha})) \to \mathbf{r.o.}(\mathbf{P}_{U_\beta}) \text{ defined by } e(p) = [\![\check{p} \in \dot{\tau}]\!]_{\mathbf{r.o.}(\mathbf{P}_{U_\beta})}$$

is a complete embedding.

Without loss of generality we assume that $p_0 = \mathbf{1}_{\mathbf{r.o.}(\mathbf{P}_{U_\alpha})}$.

Then, we get that $G^*\restriction\alpha$ is $\mathbf{P}_{U_\alpha}$ – generic over $\mathbf{V}$. We set $Q :\stackrel{\text{def}}{=} e''\mathbf{r.o.}(\mathbf{P}_{U_\alpha})$ and k^* to be the $\mathcal{Q}$ – generic filter over $\mathbf{V}$ generated in $\mathcal{Q}$ by $e''(G^*\restriction\alpha)$.

By **2.B.7** we find a $\mathbf{P}_{U_\beta}$ – generic sequence in $\mathbf{V}(H)$ such that $\mathcal{F}^\beta_h \cap \mathcal{Q} = k^*$, i.e.,

$$\begin{aligned} p \in G^*\restriction\alpha &\iff e(p) \in \mathcal{F}_h^\beta \\ &\iff [\![\check{p} \in \dot{\tau}]\!]_{\mathbf{r.o.}(\mathbf{P}_{U_\beta})} \in \mathcal{F}_h^\beta \\ &\iff p \in (\dot{\tau})^{\mathbf{V}[h]}. \end{aligned}$$

Therefore, $\dot{\tau}^{\mathbf{V}[h]} = G^*\restriction\alpha$ and $h \in \mathbf{V}(H)$.

2.B.10 Lemma □

Chapter 3

The partition relation in V(H)

3.A The way from V(H) to V[G]

In the following section we are going to prove that for every $\mathbf{P}_{U_\beta}$ – generic sequence τ $\mathbf{V}$ with $\tau \in \mathbf{V}(H)$ we can find a $\mathbf{P}_{U_\alpha}$ – generic filter G^* over $\mathbf{V}$ such that $\sigma_{G^*}\restriction\beta = \tau$ and $H = \bigcup\{\,\mathcal{P}(\kappa) \cap \mathbf{V}[G^*\restriction\alpha] \mid \alpha \in [\kappa, \lambda)\,\}$.

In this first section we are defining the following forcing $\mathcal{Q}$.

3.A.1 Definition *We define, in* $\mathbf{V}(H)$, *the following forcing* $\mathcal{Q}$*:*

$$\begin{aligned}\langle x,n,\alpha,A\rangle \in \mathcal{Q} :\overset{\text{def}}{\Longleftrightarrow}\quad & \langle x\restriction n;A\rangle \in \mathbf{P}_U \quad \& \quad (x^\alpha\restriction n)^\frown x\restriction[n,\omega) \\ & \text{is a } \mathbf{P}_{U_\alpha} - \text{generic over } \mathbf{V} \quad \& \quad x \in \mathbf{V}(H) \quad \& \\ & \& \quad \underset{\mathcal{Q}_\lambda}{\overset{\mathbf{V}[x]}{\Vdash}} [\exists y(y \text{ is } \mathbf{P}_U - \text{generic over } \mathbf{V} \quad \& \\ & \& \quad x\restriction n \subset y \subset x\restriction n \cup A \quad \& \quad y\restriction\alpha = x\restriction\alpha)]\end{aligned}$$

where $\mathcal{Q}_\lambda :\overset{\text{def}}{=} \mathrm{Coll}(\omega, (2^{2^\lambda})^+)$ *and* $x^\alpha\restriction n\langle x(k) \cap \alpha \mid k < n\rangle$. *The partial order on* *the set $\mathcal{Q}$ is defined as follows:*

$$\text{For} \quad \langle x,n,\alpha,A\rangle, \langle y,m,\beta,B\rangle \in \mathcal{Q}$$

$$\langle x,n,\alpha,A\rangle \leq_{\mathcal{Q}} \langle y,m,\beta,B\rangle :\stackrel{\text{def}}{\Longleftrightarrow} \langle x\restriction n;A\rangle \leq \langle y\restriction m;B\rangle \quad \& \quad x\restriction\beta = y\restriction\beta.$$

3.A.1 Definition □

3.A.2 Proposition *Let $\langle x,n,\alpha,A\rangle \in \mathcal{Q}$ and $\beta \in (\alpha,\lambda)$. Then, there exists $\langle y,m,\beta,B\rangle \in \mathcal{Q}$ such that $\langle y,m,\beta,A^*\rangle \leq_{\mathcal{Q}} \langle x,n,\alpha,A\rangle$.*

Proof: Since $\langle x,n,\alpha,A\rangle \in \mathcal{Q}$, then

$$\underset{\mathcal{Q}_\lambda}{\overset{\mathbf{V}[x]}{\Vdash}} [\exists y(y \text{ is } \mathbf{P}_U - \text{generic over } \mathbf{V} \quad \& \quad x\restriction n \subset y \subset x\restriction n \cup A \quad \& \quad y\restriction\alpha = x\restriction\alpha)].$$

As in the proof of **2.B.7** we find some $\gamma \in [\beta,\lambda)$ and we construct a tree $\mathcal{T}_\beta$ on $\mathcal{P}_\kappa(\beta)$ in $\mathbf{V}[G\restriction\gamma]$ such that every infinite branch of $\mathcal{T}_\beta$ is a $\mathbf{P}_{U_\beta}$ – generic sequence z and this branch z has the following properties:

1) $x^\beta\restriction n \subset x^\beta\restriction n \cup A\restriction\beta$ and

2) $z\restriction\alpha = x\restriction\alpha (= x^\alpha\restriction n \cup x\restriction[n,\omega))$.

Since $x, \mathcal{T}_\beta \in \mathbf{V}[G\restriction\gamma]$ and $\mathbf{V}[G\restriction\gamma]$ is a generic extension of $\mathbf{V}[x]$ via a forcing $\mathbf{R}$ of size $\leq (2^{2^\lambda})^+$, then $\mathbf{R} \times \mathcal{Q}_\lambda$ collapses $(2^{2^\lambda})^+$ when we force with it over $\mathbf{V}[x]$ and $\mathbf{R} \times \mathcal{Q}_\lambda$ has size $\leq (2^{2^\lambda})^+$.

Hence, by Solovay's result $\mathbf{R} \times \mathcal{Q}_\lambda$ is equivalent with $\mathcal{Q}_\lambda$ in $\mathbf{V}[x]$, i.e.,

$$\mathbf{r.o.}(\mathbf{R} \times \mathcal{Q}_\lambda) = \mathbf{r.o.}(\mathcal{Q}_\lambda).$$

We let K^* be a $\mathcal{Q}_\lambda$ – generic over $\mathbf{V}[\mathrm{G}\restriction\gamma]$.

Then $\mathrm{G}\restriction\gamma\times\mathrm{K}^*$ is $\mathbf{R}\times\mathcal{Q}_\lambda$ – generic over $\mathbf{V}[\mathrm{x}]$ and so there exists a K which is $\mathcal{Q}_\lambda$ – generic over $\mathbf{V}[\mathrm{x}]$ and satisfies the following:

$$\mathbf{V}[\mathrm{G}\restriction\gamma][\mathrm{K}^*]=\mathbf{V}[\mathrm{x}][\mathrm{K}].$$

Since

$$\underset{\mathcal{Q}_\lambda}{\overset{\mathbf{V}[\mathrm{x}]}{\Vdash}}[\exists \mathrm{y}(\mathrm{y}\text{ is }\mathbf{P}_{\mathrm{U}}-\text{generic over }\mathbf{V}\quad \&\quad \mathrm{x}\restriction\mathrm{n}\subset\mathrm{y}\subset\mathrm{x}\restriction\mathrm{n}\cup\mathrm{A}\quad \&\quad \mathrm{y}\restriction\alpha=\mathrm{x}\restriction\alpha)]$$

holds, then

$$\mathbf{V}[\mathrm{x}][\mathrm{K}]\models\exists\mathrm{y}\Big(\mathrm{y}\text{ is }\mathbf{P}_{\mathrm{U}_\beta}-\text{generic over }\mathbf{V}\quad \&\quad \mathrm{y}\restriction\alpha=\mathrm{x}\restriction\alpha\quad \&\quad \mathrm{x}^\beta\restriction\mathrm{n}\subset\mathrm{y}\subset\mathrm{x}^\beta\restriction\mathrm{n}\cup\mathrm{A}\restriction\beta\Big).$$

Thus $\mathbf{V}[\mathrm{x}][\mathrm{K}]\models[\mathcal{T}_\beta$ is not well – founded].

By the absoluteness of the well – foundedness property, we conclude that

$$\mathbf{V}[\mathrm{G}\restriction\gamma]\models[\mathcal{T}_\beta\text{ is not well – founded}],$$

i.e.,

$$\mathbf{V}[\mathrm{G}\restriction\gamma]\models[\mathcal{T}_\beta\text{ has an infinite branch}].$$

Let y be such a branch. As we have seen in **2.B.8** the construction of the tree $\mathcal{T}_\beta$ forces that y is a $\mathbf{P}_{\mathrm{U}_\beta}$ – generic sequence such that

$$\mathrm{y}\restriction\alpha=\mathrm{x}\restriction\alpha\quad \&\quad \mathrm{x}^\beta\restriction\mathrm{n}\subset\mathrm{y}\subset\mathrm{x}^\beta\restriction\mathrm{n}\cup\mathrm{A}.$$

In addition, y is in $\mathbf{V}(H)$ as $\mathbf{V}[G\restriction\gamma] \subset \mathbf{V}(H)$.

Next we have to show that $\langle x\restriction n^\frown y\restriction[n,\omega),n,\beta,A\rangle$ is a condition of the forcing $\mathcal{Q}$. In **2.B.9** we defined a forcing $\mathcal{Q}_\beta(y) \subset \mathbf{P}_U$ and $\mathcal{Q}_\beta(y) \in \mathbf{V}[y]$ so that if we force with $\mathcal{Q}_\beta(y)$ over $\mathbf{V}[y]$, then we add generically to $\mathbf{V}[y]$ a $\mathbf{P}_U$ – generic sequence w with the property that

$$x\restriction n \subset x\restriction n \cup A \ \&\ y\restriction\beta = w\restriction\beta.$$

Then, in $\mathbf{V}[y]$ the forcing $\mathcal{Q}_\beta(y) \times \mathcal{Q}_\lambda$ has size $\leq (2^{2^\lambda})^+$ and collapses $(2^{2^\lambda})^+$ so it is equivalent to $\mathcal{Q}_\lambda$.

If K^* is a $\mathcal{Q}_\lambda$ – generic filter over $\mathbf{V}[y][w]$, then we can find a K which is $\mathcal{Q}_\lambda$ – generic over $\mathbf{V}[y]$ such that

$$\mathbf{V}[y][w][K^*] = \mathbf{V}[y][K].$$

Therefore,

$$\Vdash^{\mathbf{V}[y]}_{\mathcal{Q}_\lambda} \exists w(w \text{ is } \mathbf{P}_U - \text{generic over } \mathbf{V} \quad \& \quad x\restriction n \subset w \subset x\restriction n \cup A \quad \& \quad w\restriction\beta = y\restriction\beta).$$

Set $\bar{y} = x\restriction n^\frown y\restriction[n,\omega)$. Then,

$$\langle\bar{y},n,\beta,A\rangle \in \mathcal{Q}$$

and

$$\langle\bar{y},n,\beta,A\rangle \leq_{\mathcal{Q}} \langle x,n,\alpha,A\rangle.$$

Hence, the proposition has been proven.

3.A.2 Proposition □

The next result shows that any Prikry generic sequence in **V**(H) can be lifted high enough in **V**(H) so that it captures any fixed restriction of the generic sequence induced by G.

3.A.3 Lemma *Let* $\alpha' \in [\kappa, \lambda)$ *and* $\langle x, n, \alpha, A\rangle \in \mathcal{Q}$. *Then, there exists a* $\delta \in [\kappa, \lambda)$ *and a* $y^* \in \mathbf{V}(H)$ *so that*

$$x^\delta \restriction n = y^* \restriction n \quad \& \quad y^* \restriction \alpha = x \restriction \alpha \quad \& \qquad x^\delta \restriction n \subset y^* \subset x^\delta \restriction n \cup A \restriction \delta \quad \& \quad \& \quad G \restriction \alpha' \in \mathbf{V}[y'].$$

Proof: We assume towards a contradiction that the conclusion of the assertion of the Lemma fails, i.e., we assume that for some fixed $\langle x, n, \alpha, A\rangle \in \mathcal{Q}$ and

$$\alpha' \in [\kappa, \lambda)$$

$$\mathbf{V}(H) \models \neg \exists y^* (x^\delta \restriction n \subset y^* \subset x^\delta \restriction n \cup A \restriction \delta \qquad \& \quad y^* \restriction \alpha = x \restriction \alpha \quad \& \quad G \restriction \alpha' \in \mathbf{V}[y^*]).$$

Then, we make the following claim:

3.A.3.1 Claim

$$\mathbf{V}[x, G \restriction \alpha'] \models (\forall R \text{ non-atomic forcings}) \quad \neg(\exists p \in R) \quad [p \Vdash_{R} \exists \delta \exists g \phi(\check{\delta}, g, \check{G} \restriction \alpha, \check{x}, \check{n})]$$

where ϕ is the formula

$$\phi(\delta, g, G\restriction\alpha', x, n) :\stackrel{\text{def}}{\Longleftrightarrow} g \text{ is } \mathbf{P}_{U_\delta}-\text{generic over } \mathbf{V} \;\;\&\;\; G\restriction\alpha' \in \mathbf{V}[g] \;\;\& \\ \&\;\; g\restriction\alpha = x\restriction\alpha \;\;\& \\ \&\;\; x^\delta\restriction n \subset g \subset x^\delta\restriction n \cup A\restriction\delta.$$

Proof of the Claim: If the claim fails, then there exists a forcing

$$R \in \mathbf{V}[x, G\restriction\alpha]$$

and for some K which is R – generic over $\mathbf{V}[x, G\restriction\alpha]$ there exist δ, g so that

$$\mathbf{V}[x, G\restriction\alpha'][K] \models \phi(\delta, g, G\restriction\alpha', x, n).$$

Since $G\restriction\alpha' \in \mathbf{V}[g]$, then we find a $\mathbf{P}_{U_\delta}$ – name $\dot{\tau}$ such that

$$(\dot{\tau})^{\mathbf{V}[G]} = G\restriction\alpha'.$$

We let

$$\mathcal{X}_0 = \{[\![\check{p} \in \dot{\tau}]\!]_{\mathbf{r.o.}(\mathbf{P}_{U_\delta})} \mid p \in \mathbf{P}_{U_{\alpha'}}\}$$

and $\mathcal{B}_{\dot{\tau}}$ the Boolean subalgebra of $\mathbf{r.o.}(\mathbf{P}_{U_\delta})$ generated by $\mathcal{X}_0$.

Then

$$\mathcal{B}_{\dot{\tau}} \cap g \text{ is } \mathcal{B}_{\dot{\tau}} - \text{generic over } \mathbf{V}$$

and moreover, if h is any $\mathbf{P}_{U_\delta}$ – generic over $\mathbf{V}$ such that $\mathcal{B}_{\dot{\tau}} \cap g = \mathcal{B}_{\dot{\tau}} \cap h$, then

$$(\dot{\tau})^{\mathbf{V}[h]} = G\restriction\alpha' = (\dot{\tau})^{\mathbf{V}[g]}.$$

Let

$$\mathrm{i} : \mathbf{P}_{\mathrm{U}_\alpha} \to \mathbf{r.o.}(\mathbf{P}_{\mathrm{U}_\delta})$$

be the canonical embedding which we defined in **2.B.3.**

Since $\mathrm{x}\restriction\alpha$ is $\mathbf{P}_{\mathrm{U}_\alpha}$ – generic over **V**, we let k^* be the $\mathrm{i}''\mathbf{P}_{\mathrm{U}_\alpha}$ – generic filter generated by $\mathrm{i}''\mathcal{F}^{\alpha}_{\mathrm{x}\restriction\alpha}$ in $\mathrm{i}''\mathbf{P}_{\mathrm{U}_\alpha}$.

By the definition of i, if h is a $\mathbf{P}_{\mathrm{U}_\delta}$ – generic ω – sequence over **V** such that $\mathrm{k}^* = \mathrm{i}''\mathbf{P}_{\mathrm{U}_\alpha} \cap \mathcal{F}^{\delta}_{\mathrm{h}}$, then $\mathrm{h}\restriction\alpha = \mathrm{x}\restriction\alpha$. In addition, we find $\eta \in (\delta, \lambda)$ such that

$$\mathbf{V}[\mathrm{G}\restriction\eta] \models |\mathrm{U}_\delta| = \kappa.$$

Hence, as in **2.B.6** we can find in $\mathbf{V}[\mathrm{G}\restriction\eta]$ a sequence $\langle \mathrm{A}_\mathrm{n} \mid \mathrm{n} \in \omega\rangle$ such that

$$(\forall \mathrm{n} < \omega)(\mathrm{A}_\mathrm{n} \in \mathbf{V} \quad \& \quad (|\mathrm{A}_\mathrm{n}| < \kappa)^{\mathbf{V}})$$

and

$$\mathrm{U}_\delta = \bigcup_{\mathrm{n}<\omega} \mathrm{A}_\mathrm{n}.$$

We assume that $\mathrm{A}\restriction\delta \in \mathrm{A}_0$ and we define

$$\mathrm{A}^*_\mathrm{n} = \cap\{\mathrm{B} \mid \exists \mathrm{k} \leq \mathrm{n}(\mathrm{B} \in \mathrm{A}_\mathrm{k})\}$$

and so,

$$\langle \mathrm{A}^*_\mathrm{n} \mid \mathrm{n} \in \omega\rangle \in \mathrm{U}^{\omega}_{\delta} \cap \mathbf{V}[\mathrm{G}\restriction\eta]$$

and moreover,

$$\forall \mathrm{B} \in \mathrm{U}_\delta \exists \mathrm{n} < \omega(\mathrm{A}^*_\mathrm{n} \subseteq \mathrm{B}).$$

Then, in $\mathbf{V}_1 :\overset{\text{def}}{=} \mathbf{V}[G\restriction \eta]$ we define by the recursion on ω a sequence $\langle \mathcal{T}^*_n \mid n \in \omega \rangle$ as follows:

$$\begin{aligned} \langle P_0, \dots, P_{n'} \rangle \in \mathcal{T}^*_0 \quad & :\overset{\text{def}}{\Longleftrightarrow} \langle P_0, \dots, P_{n'} \rangle \in [\mathcal{P}_\kappa(\delta)]^{<\omega} \quad \& \\ & \& \quad (\forall k < n')(P_k \notin A^*_0) \\ & \& \quad x^\delta \restriction n \subset \langle P_0, \dots, P_{n'-1} \rangle \quad \& \quad P'_n \in A^*_0 \quad \& \\ & \& \forall z \in k^* \forall w \in \mathcal{B}_{\dot{T}} \cap g(\langle P_0, \dots, P_{n-1}; A^*_0 \rangle \parallel z, w) \end{aligned}$$

and

$$\begin{aligned} \langle P_0, \dots, P_{m'} \rangle \in \mathcal{T}^*_{m+1} \quad & :\overset{\text{def}}{\Longleftrightarrow} \langle P_0, \dots, P_{m'} \rangle \in \mathcal{T}^*_m \quad \text{or} \\ & \text{or} \quad [(\exists k < m')(\langle P_0, \dots, P_k \rangle \in \mathcal{T}^*_m) \quad \& \\ & \& P_k, \dots, P_{m'-1} \in A^*_m \quad \& \quad P'_m \in A^*_{m+1} \quad \& \\ & \& \forall z \in k^* \forall w \in \mathcal{B}_{\dot{T}} \cap g \\ & (\langle P_0, \dots, P_{m'-1}; A^*_{m+1} \rangle \parallel z, w)]. \end{aligned}$$

We set $\mathcal{T}^* = \bigcup_{n<\omega} \mathcal{T}^*_n$ and we define a tree $\mathcal{T}$ on $\mathcal{P}_\kappa(\delta)$,
i.e., $\mathcal{T} \subset [\mathcal{P}_\kappa(\delta)]^{<\omega}$ and

$$\forall z \in [\mathcal{P}_\kappa(\delta)]^{<\omega} \quad \left[z \in \mathcal{T} : \overset{\text{def}}{\Longleftrightarrow} \exists y \in \mathcal{T}^*(z \subset y) \right].$$

Obviously, $\mathcal{T} \in \mathbf{V}_1$.

Moreover, if y is a $\mathbf{P}_{U_\delta}$ – generic sequence over $\mathbf{V}$ such that $G\restriction \alpha' \in \mathbf{V}[y]$ and $y\restriction \alpha = x\restriction \alpha$, then

$$\mathcal{F}^\delta_y \cap \mathcal{B}_{\dot{T}} = g \cap \mathcal{B}_{\dot{T}} \quad \& \quad k^* = \mathcal{F}^\delta_y \cap i'' \mathbf{P}_{U_\alpha}$$

and y is a branch through $\mathcal{T}$.

Conversely, if y is a branch through $\mathcal{T}$, then y is a $\mathbf{P}_{\mathrm{U}_\delta}$ – generic sequence and in addition $\mathcal{F}^\delta_{\mathrm{y}} \cap \mathcal{B}_{\dot\tau}$, $\mathcal{F}^\delta_{\mathrm{y}} \cap \mathrm{i}''\mathbf{P}_{\mathrm{U}_\alpha}$ are $\mathcal{B}_{\dot\tau}$ and $\mathrm{i}''\mathbf{P}_{\mathrm{U}_\alpha}$ – generic (respectively) filters over $\mathbf{V}$ with the property:

$$(\forall \mathrm{w} \in \mathrm{g} \cap \mathcal{B}_{\dot\tau})(\forall \mathrm{z} \in \mathrm{k}^*)(\forall \mathrm{z}' \in \mathcal{F}^\delta_{\mathrm{y}})\Big[\mathrm{z}' \parallel \mathrm{w}, \mathrm{z}\Big].$$

Therefore,

$$\mathrm{g} \cap \mathcal{B}_{\dot\tau} = \mathcal{F}^\delta_{\mathrm{y}} \cap \mathcal{B}_{\dot\tau}$$

and

$$\mathrm{k}^* = \mathcal{F}^\delta_{\mathrm{y}} \cap \mathrm{i}''\mathbf{P}_{\mathrm{U}_\alpha}.$$

This implies that

$$(\dot\tau)^{\mathbf{V}[\mathrm{y}]} = \mathrm{G}\restriction \alpha' \in \mathbf{V}[\mathrm{y}]$$

and

$$\mathrm{y}\restriction \alpha = \mathrm{x}\restriction \alpha \quad \& \quad \mathrm{x}^\delta\restriction \mathrm{n} \subset \mathrm{y} \subset \mathrm{x}^\delta\restriction \mathrm{n} \cup \mathrm{A}.$$

If we have chosen K to be $\mathbf{R}$ – generic over $\mathbf{V}_1$, then we would have

$$\mathbf{V}[\mathrm{K}] \models \mathcal{T} \text{ is not well – founded}$$

and so

$$\mathbf{V}_1 \models \mathcal{T} \text{ is not well – founded}.$$

Hence,

$$\mathbf{V}[\mathrm{G}\restriction \eta] \models [\mathcal{T} \text{ has an infinite branch}]$$

and since $\mathbf{V}[G\restriction\eta] \subseteq \mathbf{V}(H)$, then

$$\mathbf{V}(H) \models \exists y\Big[x^{\delta}\restriction n \subset y \subset x^{\delta}\restriction n \cup A \quad \& \quad y\restriction\alpha = x\restriction\alpha \quad \& \quad G\restriction\alpha' \in \mathbf{V}[y]\Big].$$

But the latest contradicts our assumption that no such y exists in $\mathbf{V}(H)$. Therefore, our claim has been proven.

3.A.3.1 Claim □

Since our latest claim holds, if $x, G\restriction\alpha' \in \mathbf{V}[G\restriction\beta]$ for some $\beta > \alpha, \alpha'$, then we can find a $\mathbf{P}_{U_\beta}$ −name $\dot{\tau}$ and a $\mathbf{P}_U$ −generic filter G^* such that $\langle x\restriction n;A\rangle \in G^*$ and $\mathbf{V}[G] = \mathbf{V}[G^*]$. This is possible by the homogeneity of the Supercompact Prikry forcing $\mathbf{P}_U$. Moreover, the following holds:

$$\Vdash_{\mathbf{P}_{U_\beta}} \Big[(\dot{\tau} \text{ is } \mathbf{P}_{U_\alpha} - \text{generic over } \dot{\mathbf{V}}) \quad \& \quad (x^{\alpha}\restriction n \subset \dot{\tau} \subset x^{\alpha}\restriction n \cup A\restriction\alpha) \quad \& \quad (\mathbf{V}[\dot{\tau}, \dot{\Gamma}\restriction\check{\alpha}'] \models \Psi(\dot{\Gamma}\restriction\check{\alpha}', \dot{\tau}))\Big]$$

where

$$\Psi(\dot{\Gamma}\restriction\check{\alpha}', \dot{\tau}) :\overset{\text{def}}{\Longleftrightarrow} \quad (\forall R \text{ non-atomic forcings})\neg(\exists p \in R)\ p \Vdash_{R} \exists\delta\exists y\phi(\delta, y, (\dot{\Gamma}\restriction\check{\alpha}')\check{}, (\dot{\tau})\check{}, \check{n})$$

and ϕ is the formula which appears in **3.A.3.1**. We apply **2.B.10** to find a $\mathbf{P}_{U_\beta}$ − generic sequence $h \in \mathbf{V}(H)$ such that

$$(\dot{\tau})^{\mathbf{V}[h]} = G\restriction\alpha.$$

We let δ be an ordinal in (β,λ) such that $\mathrm{h}\in\mathbf{V}[\mathrm{G}^*\restriction\delta]$. We consider $\mathbf{R}$ to be the forcing so that $\mathbf{V}[\mathrm{G}^*\restriction\delta]$ is an $\mathbf{R}$ – generic extension of $\mathbf{V}[\mathrm{G}^*\restriction\alpha,\mathrm{h}\restriction\alpha]$. Since

$$\mathbf{V}[\mathrm{G}^*\restriction\delta]=\mathbf{V}[\mathrm{G}^*\restriction\alpha,\mathrm{h}\restriction\alpha'][\mathrm{L}]$$

and

$$\begin{aligned}\mathbf{V}[\mathrm{G}^*\restriction\alpha,\mathrm{h}\restriction\alpha'][\mathrm{L}]\models\quad&(\mathrm{G}^*\restriction\delta)\restriction\alpha=\mathrm{G}^*\restriction\alpha\quad\&\\&\&\quad\mathrm{h}\restriction\alpha'\in\mathbf{V}[\mathrm{G}^*\restriction\delta]\quad\&\\&\&\quad\langle\mathrm{x}^\delta\restriction\mathrm{n};\mathrm{A}\delta\rangle\in\mathrm{G}^*\restriction\delta,\end{aligned}$$

then

$$\mathbf{V}[\mathrm{h}]\models[\mathbf{V}[(\dot\tau)^{\mathbf{V}[\mathrm{h}]},\mathrm{h}\restriction\alpha']\models\Psi(\mathrm{h}\restriction\alpha'\,(\dot\tau)^{\mathbf{V}[\mathrm{h}]})],$$

i.e.,

$$\begin{aligned}\mathbf{V}[\mathrm{G}^*\restriction\alpha,\mathrm{h}\restriction\alpha'][\mathrm{L}]\quad&\models\exists\delta\exists\mathrm{y}(\mathrm{y}\text{ is }\mathbf{P}_{\mathrm{U}_\delta}-\text{generic over }\mathbf{V}\quad\&\\&\&\quad\mathrm{y}\restriction\alpha=\mathrm{G}^*\restriction\alpha\quad\&\quad\mathrm{x}^\delta\restriction\mathrm{n}\subset\mathrm{y}\subset\mathrm{x}^\delta\restriction\mathrm{n}\cup\mathrm{A}\restriction\delta\quad\&\\&\&\quad\mathrm{h}\restriction\alpha'\in\mathbf{V}[\mathrm{y}]).\end{aligned}$$

But then, there exists a $\mathrm{p}\in\mathbf{P}_{\mathrm{U}_\beta}$ such that

$$\begin{aligned}\mathrm{p}\underset{\mathbf{P}_{\mathrm{U}_\beta}}{\Vdash}\Big(\mathbf{V}[\dot\tau,\dot\Gamma\restriction\alpha']\models\quad&(\exists\mathbf{R}\text{ a non – atomic forcing})(\exists\mathrm{q}\in\mathbf{R})\\&\mathrm{q}\underset{\mathbf{R}}{\Vdash}\exists\delta\exists\mathrm{y}\phi(\delta,\mathrm{y},(\dot\Gamma\restriction\dot\alpha)\check{},(\dot\tau)\check{},\check{\mathrm{n}})\Big),\end{aligned}$$

which is contrary to

$$\underset{\mathbf{P}_{\mathrm{U}_\beta}}{\Vdash}[(\dot\tau\text{ is }\mathbf{P}_{\mathrm{U}_\alpha}-\text{generic over }\dot{\mathbf{V}})\qquad\&\quad(\mathrm{x}^\alpha\restriction\mathrm{n}\subset\dot\tau\subset\mathrm{x}^\alpha\restriction\mathrm{n}\cup\mathrm{A}\restriction\alpha)\quad\&$$

$$\& \quad (\mathbf{V}[\dot{\tau}, \dot{\Gamma}\restriction \check{\alpha}'] \models \Psi(\dot{\Gamma}\restriction \check{\alpha}', \dot{\tau}))].$$

The above argument shows that for any $\langle x,n,\alpha,A\rangle \in \mathcal{Q}$ and any $\alpha' \in [\kappa, \lambda)$ there exists a $\delta \in [\kappa, \lambda)$ and a $y^* \in \mathbf{V}(H)$, y^* which is a $\mathbf{P}_{U_\delta}$ – generic sequence over $\mathbf{V}$ such that

$$y^*\restriction \alpha = x\restriction \alpha \quad \& \quad \langle x^\delta\restriction n; A\restriction \delta\rangle \in \mathcal{F}^\delta_{y^*}$$

with $G\restriction \alpha' \in \mathbf{V}[y^*]$.

Therefore, the proof of our lemma is complete.

3.A.3 Lemma □

However **3.A.3** can be generalized in terms of the following result:

3.A.4 Corollary *Let* $\langle x,n,\alpha,A\rangle \in \mathcal{Q}$ *and* $Y \in H$. *Then,* $\exists\delta\exists y^* \in \mathbf{V}(H)$ *such that* $x\restriction \alpha = y^*\restriction \alpha$ *and*

$$x^\delta\restriction n \subset y^* \subset x^\delta\restriction n \cup A\restriction \delta$$

and $Y \in \mathbf{V}[y^*]$.

Proof: Since $Y \in H$, then there exists an $\alpha' \in [\kappa, \lambda)$ such that

$$Y \in \mathcal{P}(\kappa) \cap \mathbf{V}[G\restriction \alpha'].$$

By **3.A.3** we can find δ, y^* such that

$$\delta \in [\kappa, \lambda) \quad \& \quad y^* \text{ is a } \mathbf{P}_{U_\delta} \text{ – generic sequence over } \mathbf{V}$$
$$\& \quad y^* \in \mathbf{V}(H) \quad \& \quad x^\delta\restriction n \subset y^* \subset x^\delta\restriction n \cup A\restriction \delta \quad \&$$
$$\& \quad x\restriction \alpha = y^*\restriction \alpha \quad \& \quad G\restriction \alpha' \in \mathbf{V}[y^*].$$

But then, $\mathcal{P}(\kappa) \cap \mathbf{V}[G\restriction \alpha'] \in \mathbf{V}[y]$ and so $Y \in \mathbf{V}[y^*]$.

3.A.4 Corollary □

3.A.5 Proposition *Let* $\langle x,n,\alpha,A\rangle \in \mathcal{Q}$ *and* $Y \in H$.

Then, the following holds:

$$\exists \delta \in [\kappa, \lambda) \exists y \in \mathbf{V}(H)(Y \in \mathbf{V}[y] \quad \& \quad \langle y,n,\delta,B\rangle \leq_{\mathcal{Q}} \langle x,n,\alpha,A\rangle).$$

Proof: We use **3.A.4** to find a $y^* \in \mathbf{V}(H)$ so that y^* is a $\mathbf{P}_{U_\delta}$ – generic sequence over $\mathbf{V}$ and

$$y^*\restriction n = x^\delta\restriction n \quad \& \quad y^*\restriction \alpha = x\restriction \alpha \quad \& \quad y^*\restriction [n,\omega) \subset A\restriction \delta \quad \& \quad Y \in \mathbf{V}[y^*].$$

Then, as in **3.A.2** we consider the forcing $\mathcal{Q}_\delta(y^*)$ in $\mathbf{V}[y^*]$ and by forcing with it we add generically to $\mathbf{V}[y^*]$ a $\mathbf{P}_{U_\delta}$ – generic over $\mathbf{V}$ sequence w such that

$$w\restriction \delta = y^*\restriction \delta \quad \& \quad \langle x,n,\alpha,A\rangle \in \mathcal{F}^\lambda_w.$$

Then, in $\mathbf{V}[y^*]$ we consider the forcing $\mathcal{Q}_\lambda$ and the product forcing $\mathcal{Q}_\delta(y^*) \times \mathcal{Q}_\lambda$ in $\mathbf{V}[y^*]$. As before, $\mathcal{Q}_\delta(y^*) \times \mathcal{Q}_\lambda$ is equivalent to $\mathcal{Q}_\lambda$ and every K^* which is $\mathcal{Q}_\lambda$ – generic over $\mathbf{V}[y^*][w]$ induces a K which is $\mathcal{Q}_\lambda$ – generic over $\mathbf{V}[y^*]$ so that

$$\mathbf{V}[y^*][w][K^*] = \mathbf{V}[y^*][K].$$

Moreover,

$$\mathbf{V}[y^*][K] \models \exists w \quad \Big[w \text{ is } \mathbf{P}_U - \text{generic over } \mathbf{V} \ \& \ \& \ w\restriction\delta = y^*\restriction\delta \ \& \ \& \ \langle x\restriction n;A\rangle \in \mathcal{F}^\lambda_w \Big].$$

Hence,

$$\underset{\mathcal{Q}_\lambda}{\overset{\mathbf{V}[y^*]}{\Vdash}} \exists w \quad \Big[w \text{ is } \mathbf{P}_U - \text{generic over } \mathbf{V} \ \& \ w\restriction\delta = y^*\restriction\delta \ \& \ \langle x\restriction n;A\rangle \in \mathcal{F}^\lambda_w) \Big].$$

This shows that $\langle y,n,\delta,A\rangle \in \mathcal{Q}$ where $y = x\restriction n {}^\frown y^*\restriction [n,\omega)$ and in addition

$$\langle y,n,\delta,A\rangle \leq_{\mathcal{Q}} \langle x,n,\alpha,A\rangle \quad \& \quad Y \in \mathbf{V}[y](= \mathbf{V}[y^*]).$$

This completes the proof the above proposition.

3.A.5 Proposition □

3.A.6 Corollary *Let* Y ∈ H *and*

$$D_Y = \{\langle x,n,\alpha,A\rangle \in \mathcal{Q} \mid Y \in \mathbf{V}[x]\}.$$

Then, D_Y *is a dense subset of the forcing* $\mathcal{Q}$ *and* $D_Y \in \mathbf{V}(H)$.

Proof: It follows easily from **3.A.5**.

3.A.6 Corollary □

3.A.7 Proposition *Let* $\mathcal{G}^*$ *be* $\mathcal{Q}$ – generic *over* $\mathbf{V}(\mathrm{H})$. *Then* $\mathcal{G}^*$ *induces a* $\mathbf{P}_\mathrm{U}$ – generic *filter over* $\mathbf{V}$.

Proof: Let

$$\mathrm{G}^* = \{\langle \mathrm{w;A}\rangle \in \mathbf{P}_\mathrm{U} \mid \exists \mathrm{x}, \mathrm{n}, \alpha, \mathrm{A}'(\langle \mathrm{x,n,}\alpha\mathrm{,A}'\rangle \in \mathcal{G}^* \quad \& \quad \mathrm{x}\restriction \mathrm{lh(w)} = \mathrm{w} \quad \& \quad \& \quad \mathrm{A}' \subseteq \mathrm{A})\}.$$

This is the subset of $\mathbf{P}_\mathrm{U}$ that $\mathcal{G}^*$ induces. We claim that G^* is $\mathbf{P}_\mathrm{U}$ – generic over $\mathbf{V}$. We define

$$\sigma^* = \bigcup\{\mathrm{w} \mid \exists\langle \mathrm{x,n,}\alpha\mathrm{,A}\rangle \in \mathrm{G}(\mathrm{w} = \mathrm{x}\restriction \mathrm{n})\}.$$

Then, we make the following claim:

3.A.7.1 Claim

$$\mathrm{G}^* = \{\langle \mathrm{w;A}\rangle \mid \mathrm{w} \subset \sigma^* \subset \mathrm{w} \cup \mathrm{A}\}$$

and

$$\sigma^* = \bigcup\{\mathrm{w} \mid \exists \mathrm{B}(\langle \mathrm{w;B}\rangle \in \mathrm{G}^*)\},$$

i.e., σ^* *is the* $\mathbf{P}_\mathrm{U}$ – generic *sequence induced by* G^* *and* G^* *is the* $\mathbf{P}_\mathrm{U}$ – generic *filter induced by* σ^*, *provided that* G^* *is* $\mathbf{P}_\mathrm{U}$ – generic *over* $\mathbf{V}$.

Proof of the Claim: We assume that $\langle \mathrm{w;A}'\rangle \in \mathrm{G}^*$. Then there exists a condition $\langle \mathrm{x,n,}\alpha\mathrm{,A}\rangle \in \mathcal{G}^*$ such that $\langle \mathrm{x}\restriction \mathrm{n;A}\rangle \leq \langle \mathrm{w;A}'\rangle$. We fix $\mathrm{B} \in \mathrm{U}$ and $\mathrm{m} \in \omega$. Let

$$\mathrm{D}_{\mathrm{m,B}} = \{\langle \mathrm{x,n,}\alpha\mathrm{,A}\rangle \in \mathcal{Q} \mid \mathrm{A} \subseteq \mathrm{B} \quad \& \quad \mathrm{n} \geq \mathrm{m}\}.$$

We show that $D_{m,B}$ is a $\mathcal{Q}$ – dense set. Let $\langle y,k,\alpha,A\rangle \in \mathcal{Q}$ and $k < m$. Since

$$\langle y,k,\alpha,A\rangle \in \mathcal{Q},$$

then

$$\Vdash^{\mathbf{V}[y]}_{\mathcal{Q}_\lambda} [\exists y^* \quad (y^* \text{ is } \mathbf{P}_U \text{ – generic over } \mathbf{V} \quad \& \quad \& \quad \langle y\restriction k;A\rangle \in y^* \quad \& \quad y^*\restriction\alpha = y\restriction\alpha)].$$

We fix a $\mathcal{Q}_\lambda$ – name $\dot{y}^*$ such that

$$\Vdash^{\mathbf{V}[y]}_{\mathcal{Q}_\lambda} [\dot{y}^* \text{ is } \mathbf{P}_U \text{ – generic over } \mathbf{V} \quad \& \quad \langle y\restriction k;A\rangle \in \dot{y}^* \quad \& \quad \dot{y}^*\restriction\alpha = (y\restriction\alpha)\check{}\,].$$

Then $\exists n \geq m$ such that

$$\Vdash^{\mathbf{V}[y]}_{\mathcal{Q}_\lambda} [\dot{y}^* \text{ is } \quad \mathbf{P}_U \text{ – generic over } \mathbf{V} \quad \& \quad \langle y\restriction k;A\rangle \in \dot{y}^* \quad \& \quad \dot{y}^*\restriction\alpha = (y\restriction\alpha)\check{} \quad \& \quad y^*\restriction[n,\omega) \subseteq B\cap A].$$

Let us find a $q \in \mathcal{Q}_\lambda$ and a $w \in [\mathcal{P}_\kappa(\lambda)]^{n-k}$ such that

$$q \Vdash^{\mathbf{V}[y]}_{\mathcal{Q}_\lambda} [\dot{y}^*\restriction[k,n) = w].$$

Then, we set

$$\bar{y} = y\restriction k\,^\frown w\,^\frown y\restriction[n,\omega).$$

Obviously,

$$\langle \bar{y},k,\alpha,A\cap B\rangle \in \mathcal{Q}\cap D_{m,B}.$$

Therefore, $D_{m,B}$ is a $\mathcal{Q}$ -dense set. Hence, if $\langle w;A'\rangle \in G^*$ and $\langle x,n,\alpha,A\rangle \in \mathcal{G}^*$ such that $\langle x\restriction n;A\rangle \leq \langle w;A'\rangle$, then we can find a $\langle z,m,\alpha,C\rangle \in \mathcal{G}^*$ such that

$$m \geq n \quad \& \quad C \subseteq A' \quad \& \quad \langle z,m,\alpha,C\rangle, \langle x,n,\alpha,A\rangle \in \mathcal{G}^*.$$

Since

$$\langle z,m,\alpha,C\rangle, \langle x,n,\alpha,A\rangle \in \mathcal{G}^*$$

and as $m \geq n$, then $z\restriction n = x\restriction n$ and $z\restriction \alpha = x\restriction \alpha$.

However, as

$$(\forall m')(D_{m',A'} \text{ is } \mathcal{Q} - \text{dense}),$$

then

$$w \subset \sigma^* \subset w \cup A'.$$

Conversely, if $w \subset \sigma^* \subset w \cup A$, then $\exists\langle x,n,\alpha,A'\rangle \in \mathcal{G}^*$ with $n \geq lh(w)$ such that $w \subset x\restriction n$ and $A' \subseteq A$. Then, $\langle x\restriction n;A'\rangle \leq \langle w;A\rangle$,

i.e.,

$$G^* = \{\langle w;A\rangle \mid w \subset \sigma^* \subset w \cup A\}.$$

So $\sigma^* = \sigma_{G^*}$ according to the notation established in Chapter 2.

Next we will show that $G^* = \mathcal{F}^\lambda_{\sigma^*}$. Towards this goal, we take $w \subset \sigma^*$. Hence, $\exists\langle x,n,\alpha,A\rangle \in \mathcal{G}^*$ such that $w \subset x\restriction n$. Then

$$\langle x\restriction n;A\rangle \in G^* \Rightarrow w \subset \bigcup\{w \mid \exists B(\langle w;B\rangle \in G^*)\},$$

i.e.,

$$\sigma^* \subseteq \cup\{w \mid \exists B(\langle w;B\rangle \in G^*)\}.$$

Since

$$dom(\sigma^*) = \omega = dom(\sigma_{G^*}),$$

then $\sigma^* = \sigma_{G^*}$. This completes the proof of the above claim.

3.A.7.1 Claim □

In order to finish the proof of our proposition, we have to show that σ^* is a $\mathbf{P}_{\mathrm{U}}$ – generic over $\mathbf{V}$ sequence in $[\mathcal{P}_\kappa(\lambda)]^\omega$. So we let $A \in U$. Since

$$(\forall m)(D_{m,A} \text{ is } \mathcal{Q} - \text{dense}),$$

then

$$\exists k(\sigma^* \restriction [k, \omega) \subset A)$$

and so σ^* is a $\mathbf{P}_{\mathrm{U}}$ – generic sequence and G^* is the corresponding $\mathbf{P}_{\mathrm{U}}$ – generic filter induced by $\mathcal{G}^*$.

3.A.7 Proposition □

3.A.8 Proposition *Let $\mathcal{G}^*$ be a $\mathcal{Q}$ –* generic *filter over* $\mathbf{V}(H)$ *and* G^* *is the* $\mathbf{P}_{\mathrm{U}}$ – generic *over* $\mathbf{V}$ *filter induced by* $\mathcal{G}^*$ *as it was defined in* **3.A.7**. *We set*

$$H^{G^*} = \bigcup\{\mathcal{P}(\kappa) \cap \mathbf{V}[G^* \restriction \alpha] \mid \alpha \in [\kappa, \lambda)\}.$$

Then, $H = H^{G^*}$ *where* $H = H^G$.

Proof: Let $Y \in H$. By **3.A.6**, we have that

$$D_Y = \{\langle x,n,\alpha,A\rangle \in \mathcal{Q} \mid Y \in \mathbf{V}[x]\}$$

is $\mathcal{Q}$ – dense and $D_Y \in \mathbf{V}(H)$. So by the genericity of $\mathcal{G}^*$ over $\mathbf{V}(H)$ we have that

$$\mathcal{G}^* \cap D_y \neq \emptyset.$$

If

$$\langle \mathrm{x,n,}\alpha\mathrm{,A}\rangle \in \mathcal{G}^* \cap \mathrm{D},$$

then $\mathrm{x}{\restriction}\,\alpha = \sigma^*{\restriction}\,\alpha$ and

$$\mathrm{Y} \in \mathbf{V}[\mathrm{x}] = \mathbf{V}[\sigma^*{\restriction}\,\alpha] = \mathbf{V}[\mathrm{G}{\restriction}\,\alpha].$$

Therefore, $\mathrm{Y} \in \mathrm{H}^{\mathrm{G}^*}$. Hence, $\mathrm{H} \subseteq \mathrm{H}^{\mathrm{G}^*}$.

Conversely, as $\mathcal{Q} \in \mathbf{V}(\mathrm{H})$, then

$$\forall\alpha \in [\kappa, \lambda)(\mathrm{G}^*{\restriction}\,\alpha \in \mathbf{V}(\mathrm{H}))$$

and so

$$\mathcal{P}(\kappa) \cap \mathbf{V}[\mathrm{G}^*{\restriction}\,\alpha] \subseteq \mathrm{H}.$$

Thus $\mathrm{H}^{\mathrm{G}^*} \subseteq \mathrm{H}$ and so $\mathrm{H} = \mathrm{H}^{\mathrm{G}^*}$.

3.A.8 Proposition □

3.B From V(H) to V[G]: Another Forcing

The purpose of this section is to demonstrate that there exists another, more combinatorial forcing, which can produce the same effects as the forcing $\mathcal{Q}$ of the previous section. As usual, we begin with a definition.

3.B.1 Definition A′ *be a set in* $\mathcal{U}_\delta$ *where* $\mathcal{U}$ *is a* $\mathcal{P}_\kappa(\beta)$ *normal measure and*

$$\mathcal{U}_\delta :\overset{\text{def}}{=} \mathcal{U}\restriction\delta = \{A\restriction\delta \mid A \in \mathcal{U}\}$$

and $\kappa \leq \delta < \beta \leq \lambda$. *Then,* A′ *is* $\delta -$ very *good for* $A :\overset{\text{def}}{\Longleftrightarrow}$

$$\forall \bar{x} \in [A']^{<\omega} \forall P \in A(P \cap \delta \underset{\sim}{\subset} \bar{x} \rightarrow \exists \bar{w} \in [A]^{<\omega}(P \underset{\sim}{\subset} \bar{w} \quad \& \quad \bar{w}\restriction\delta = \bar{x})).$$

3.B.1 Definition □

3.B.2 Definition *Let* $\mathcal{U}$ *be a* $\mathcal{P}_\kappa(\beta)$ *normal measure. A condition* $p = \langle x;A\rangle \in \mathbf{P}_\mathcal{U}$ *is called* $\delta -$ very *nice iff* $A\restriction\delta$ *is* $\delta -$ very *good for* A.

3.B.2 Definition □

Remark: We recall that the forcing $\mathbf{P}_\mathcal{U}$ is defined as follows :

$$p = \langle x;A\rangle \in \mathbf{P}_\mathcal{U} :\overset{\text{def}}{\Longleftrightarrow} \quad x \in [\mathcal{P}_\kappa(\beta)]^{<\omega} \quad \&$$

$$A \in \mathcal{U} \quad \& \quad (\forall P \in A)(x \underset{\sim}{\subset} P),$$

where $x \underset{\sim}{\subset} P$ means that if $Q = x(\mathrm{lh}(x) - 1)$, then $Q \underset{\sim}{\subset} P$,

i.e.,

$$Q \subsetneq P \quad \& \quad |Q| < |P \cap \kappa|.$$

□

3.B.3 Lemma *Let $\mathcal{U}$ be a $\mathcal{P}_\kappa(\beta)$ normal measure. For every $\delta \in [\kappa, \beta)$ and for every $A \in \mathcal{U}$ there exists a set A' which is δ –* very *good for A.*

Proof: We recall that the measure $\mathcal{U}_\delta$ satisfies the following partition property: For every coloring

$$f : \mathcal{P}_\kappa(\delta)^{<\omega} \to 2$$

there exists a set $A' \in \mathcal{U}_\delta$ such that $\forall n \in \omega |f''[A']^n| = 1$ where

$$[A']^n = \{\langle P_0, \dots, P_n\rangle \mid P_i \in A' \text{ for } i \leq n \quad \& \quad P_0 \underset{\sim}{\subset} \dots P_n\}.$$

So we define the following coloring of $A\restriction\delta (\in \mathcal{U}_\delta)$. For $\bar{x} \in [A\restriction\delta]^{<\omega}$

$$f(\bar{x}) = \begin{cases} 0, & \text{if } \forall\langle P\rangle \in [A]^{<1}(\langle P \cap \delta\rangle \underset{\sim}{\subset} x \to \exists \bar{w} \in [A]^{<\omega}(P \underset{\sim}{\subset} w \,\&\, \bar{w}\restriction\delta = x)) \\ 1, & \text{if otherwise.} \end{cases}$$

Since f is a coloring of $[A\restriction\delta]^{<\omega}$ in **V** and $A\restriction\delta \in \mathcal{U}_\delta$, then there exists a set A" $\in \mathcal{U}_\delta$ such that $A' \subset A\restriction\delta$ and A' is homogeneous for f.

We show that $(\forall n < \omega)(f''[A']^n = \{0\})$.

Let us fix $n \in \omega$. In addition we fix a $\mathbf{P}_\mathcal{U}$ – generic filter over **V**. Then, as $A' \in \mathcal{U}_\delta$ and σ^δ_G is the $\mathbf{P}_{\mathcal{U}_\delta}$ – generic over **V** sequence induced by $G\restriction\delta$, we can find $m \in \omega$ such that

$$\sigma^\delta_G\restriction[m, \omega) \subset A'.$$

Let

$$\bar{x} = \sigma^\delta_G\restriction[m, m + n].$$

We are going to show that $f(\bar{x}) = 0$. Let $\langle P \rangle \in [A]^{<1}$ such that $P \cap \delta \underset{\sim}{\subset} \bar{x}$, i.e., $P \cap \delta \underset{\sim}{\subset} \bar{x}(0)$.

Then,

$$\langle P \cap \delta \rangle ^\frown \sigma_G^\delta \restriction [m, \omega)$$

is a $\mathbf{P}_{\mathcal{U}_\delta}$ – generic over $\mathbf{V}$ sequence.

We set

$$\bar{A} = \{Q \in A \mid P \underset{\sim}{\subset} Q\}$$

and we consider the following condition:

$$\langle \langle P \rangle ; \bar{A} \rangle \in \mathbf{P}_{\mathcal{U}}.$$

By **2.B.10** we can force to find a z which is $\mathbf{P}_{\mathcal{U}}$ -generic over $\mathbf{V}$ sequence such that

$$z \restriction \delta = \langle P \cap \delta \rangle ^\frown \sigma_G^\delta \restriction [m, \omega)$$

and $P \subset z \cup P \subset \bar{A}$.

Next, we let $\bar{w} = z \restriction [1, 1 + k]$.

Since $\bar{w} \in [\mathcal{P}_\kappa(\beta)]^{<\omega}$, then $\bar{w} \in \mathbf{V}$ and $\bar{w} \in [\bar{A}]^{<\omega} \subset [A]^{<\omega}$ and $P \underset{\sim}{\subset} \bar{w}$ & $\bar{w} \restriction \delta = \bar{x}$. Thus

$$\forall \langle P \rangle \in [A]^{<1} (\langle P \cap \delta \rangle \underset{\sim}{\subset} \bar{x} \rightarrow \exists \bar{w} \in [A]^{<\omega} (P \underset{\sim}{\subset} \bar{w} \quad \& \quad w \restriction \delta = \bar{x})).$$

In particular, $f(\bar{x}) = 0$. Since $\bar{x} \in [A']^n$ and A' is f-homogeneous, then $f''[A']^n = \{0\}$. As n was arbitrary then, $f''[A']^{<\omega} = \{0\}$. This shows that A' is δ – very good for A. Therefore, the lemma has been proven.

3.B.3 Lemma □

3.B.4 Corollary *Let $\mathcal{U}$ be a $\mathcal{P}_\kappa(\beta)$ normal measure. For every $\delta \in [\kappa, \beta)$ and for every $p \in \mathbf{P}_\mathcal{U}$ there exists a $q \in \mathbf{P}_\mathcal{U}$ such that $q \leq p$ and q is a δ-very nice condition.*

Proof: Assume that $p = \langle x;A \rangle$.

By the previous lemma we find a set A' which is δ -very good for A. Then, we consider the set

$$A^* = \{P \in A \mid P \cap \delta \in A'\}.$$

Obviously, for $q = \langle x;A^* \rangle$ we have that $q \in \mathbf{P}_\mathcal{U}$ and $q \leq p$. Moreover, we claim that $A^* \restriction \delta = A'$ and that A' is δ -very good for A^*. First we show that $A^* \restriction \delta = A'$.

Obviously, $A^* \restriction \delta \subset A'$. Conversely, if $Q \in A'$, then $\emptyset \in [A]^{<1}$ and since $\emptyset \cap \delta \underset{\sim}{\subset} Q$, then $\exists P \in A$ such that $P \cap \delta = Q$ and in particular $P \in A^*$.

So $A^* \restriction \delta = A'$. Similarly, if $\langle P \rangle \in [A]^{<1}$ and $\bar{x} \in [A']^{<\omega}$ and $\langle P \cap \delta \rangle \underset{\sim}{\subset} \bar{x}$, then as A' is δ – very good for A, there exists a $\bar{w} \in [A]^{<\omega}$ such that

$$\bar{w} \restriction \delta = \bar{x} \quad \& \quad \langle P \rangle \underset{\sim}{\subset} \bar{w}.$$

Since $\bar{w} \restriction \delta = \bar{x} \in [A']^{<\omega}$, then $\bar{w} \in [A^*]^{<\omega}$ and therefore, $A' = A^* \restriction \delta$ is δ – very good for A^*.

This shows that q is a δ – very nice condition of $\mathbf{P}_\mathcal{U}$ below p.

3.B.4 Corollary □

3.B.5 Definition *Let U be our fixed normal measure over $\mathcal{P}_\kappa(\lambda)$ and $\mathbf{P}_U$ be the corresponding supercompact Prikry forcing. We define the following forcing $\mathcal{Q}^*$ in* $\mathbf{V}(\mathrm{H})$

$$\langle x,n,\alpha,A\rangle \in \mathcal{Q}^* :\overset{\text{def}}{\Longleftrightarrow} \quad \langle x\restriction n;A\rangle \text{ is } \alpha-\textbf{very nice} \quad \&$$
$$\& \quad x^\alpha\restriction n{}^\frown x\restriction[n,\omega) \text{ is } \mathbf{P}_U-\text{generic} \quad \&$$
$$\& \quad x\restriction[n,\omega) \subset A\restriction\alpha.$$

3.B.5 Definition □

3.B.6 Proposition *The following set* D_β *is* $\mathcal{Q}^*$ – dense.

$$D_\beta = \{\langle x,n,\alpha,A\rangle \in \mathcal{Q}^* | \alpha \geq \beta\}.$$

Proof: Let $\langle x,n,\alpha,A\rangle \in \mathcal{Q}$ and assume that $\alpha < \beta$. We let C_1 be β–very good for A and C_2 be α-very good for C_1 and we find $m \geq n$ such that $x\restriction[m,\omega) \subset C_2$. Since

$$x\restriction[n,m) \subset A\restriction\alpha$$

and $A\restriction\alpha$ is α -very good for A, then we can lift $x\restriction[n,m)$ to some $x' \subset A$ such that $x'\restriction\alpha = x\restriction[n,m) \quad \& \quad x \subset x'$. We set

$$A^* = \{P \in A \mid x' \underset{\sim}{\subseteq} P\}$$

and

$$C_1^* = \{P \in C_1 \mid x' \restriction \beta \underset{\sim}{\subset} P\}.$$

We set

$$y = x \restriction n ^\frown x' ^\frown x \restriction [m, \omega).$$

Then

$$\langle y^\beta \restriction m; C_1^* \rangle \in \mathcal{F}^\alpha_{y^\alpha}$$

and we can find in $\mathbf{V}[h]$ a z which is $\mathbf{P}_{U_\beta}$ – generic over $\mathbf{V}$ such that

$$z \restriction \alpha = y^\alpha \quad \& \quad \langle y^\beta \restriction m; C_1^* \rangle \in z.$$

We have to show that C_1^* is β – very good for A^*. For that, we let

$$\bar{x} \in [C_1^*]^{<\omega} \text{ and } \bar{y} \in [A_1^*]^{<\omega}$$

such that $\bar{y} \restriction \beta \underset{\sim}{\subset} \bar{x}$. Since $\bar{x} \in [C_1^*]^{<\omega} \quad \& \quad \bar{y} \in [A_1^*]^{<\omega}$, then

$$\exists \bar{w} \in [A]^{<\omega} (\bar{w} \restriction \delta = \bar{x} \quad \& \quad \bar{y} \underset{\sim}{\subset} \bar{w}).$$

Since $\bar{y} \in [A^*]^{<\omega}$, then $\bar{x}' \underset{\sim}{\subset} \bar{y} \rightarrow \bar{x}' \underset{\sim}{\subset} \bar{w} \quad \& \quad \bar{w} \in [A]^{<\omega}$, i.e., $\bar{w} \in [A^*]^{<\omega}$. Hence, C_1^* is β – very good for A^*. Set

$$C = \{P \in A^* \mid P \cap \beta \in C_1^*\}.$$

Then, $C \restriction \beta = C_1^*$ and $\langle z, m, \beta, C \rangle$ is in $\mathcal{Q}^*$ and

$$\langle z, m, \beta, C \rangle \leq \langle x, n, \alpha, A \rangle.$$

3.B.6 Proposition □

Next we prove a proposition analogous to **3.A.5** for the forcing $\mathcal{Q}^*$.

3.B.7 Proposition *Let* $\langle x,n,\alpha,A\rangle \in \mathcal{Q}^*$ *and* $Y \in H$.

Then, the following holds:

$$\exists\delta \in [\kappa,\lambda)\exists y \in \mathbf{V}(H)(Y \in \mathbf{V}[y] \quad \& \quad \langle y,n,\delta,B\rangle \leq_{\mathcal{Q}^*} \langle x,n,\alpha,A\rangle).$$

Proof: We use **3.A.4** to find a $y^* \in \mathbf{V}(H)$ so that y^* is a $\mathbf{P}_{U_\delta}$ – generic sequence over $\mathbf{V}$ and

$$y^*\restriction n = x^\delta\restriction n \quad \& \quad y^*\restriction\alpha = x\restriction\alpha \quad \& \quad y^*\restriction[n,\omega) \subset A\restriction\delta \quad \& \quad Y \in \mathbf{V}[y^*].$$

Then, we observe that $\langle y^*,n,\delta,A\rangle \leq_{\mathcal{Q}^*} \langle x,n,\alpha,A\rangle$. This completes the proof of this proposition.

3.B.7 Proposition □

After this result we can duplicate the arguments of the previous section, in order to show the analogues of **3.A.6**, **3.A.7**, **3.A.8**. We mention these results without proofs.

3.B.8 Corollary *Let* $Y \in H$ *and*

$$D^*_Y = \{\langle x,n,\alpha,A\rangle \in \mathcal{Q}^* \mid Y \in \mathbf{V}[x]\}.$$

Then, D^*_Y *is a dense subset of the forcing* $\mathcal{Q}^*$ *and* $D^*_Y \in \mathbf{V}(H)$.

3.B.8 Corollary □

3.B.9 Proposition *Let* $\mathcal{G}^*$ *be* $\mathcal{Q}^*$ *-generic over* $\mathbf{V}(\mathrm{H})$. *Then,* $\mathcal{G}^*$ *induces a* $\mathbf{P}_{\mathrm{U}}$ – generic *filter over* $\mathbf{V}$.

3.B.9 Proposition □

3.B.10 Proposition *Let* $\mathcal{G}^*$ *be a* $\mathcal{Q}^*$ – generic *filter over* $\mathbf{V}(\mathrm{H})$ *and* G^* *is the* $\mathbf{P}_{\mathrm{U}}$ – generic *over* $\mathbf{V}$ *filter induced by* $\mathcal{G}^*$ *as it was defined in* **3.A.7**. *We set*

$$\mathrm{H}^{\mathrm{G}^*} = \bigcup\{\mathcal{P}(\kappa) \cap \mathbf{V}[\mathrm{G}^* \restriction \alpha] \mid \alpha \in [\kappa, \lambda)\}.$$

Then, $\mathrm{H} = \mathrm{H}^{\mathrm{G}^*}$ *where* $\mathrm{H} = \mathrm{H}^{\mathrm{G}}$.

3.B.10 Proposition □

3.C OD partitions in the model V(H)

3.C.1 Theorem *We assume that ϕ is a formula and $\mu < \kappa$ so that*

$$\mathbf{V}(\mathrm{H}) \models \forall \mathrm{s} \in [\kappa]^{\omega} \exists ! \mathrm{x} \in \mathbf{V}_{\mu} \phi(\mathrm{s}, \mathrm{x}, \vec{\delta}, \mathrm{H}).$$

Then, there exists an $\mathrm{s} \in [\kappa]^{\omega} \cap \mathbf{V}(\mathrm{H})$ *and a* $\mathrm{x}_0 \in \mathbf{V}_{\mu}$ *such that*

$$\mathbf{V}(\mathrm{H}) \models \forall \mathrm{t} \in [\mathrm{s}]^{\omega} \phi(\mathrm{t}, \mathrm{x}_0, \vec{\delta}, \mathrm{H}).$$

In particular,

$$\mathbf{V}(\mathrm{H}) \models (\kappa \text{ is a strong limit}) \quad \& \quad \forall \mu < \kappa (\kappa \underset{\mathbf{OD}}{\rightarrow} (\omega)^{\omega}_{\mathbf{V}_{\mu}}).$$

Proof: Since

$$\mathbf{V}_{\mu}^{\mathbf{V}(\mathrm{H})} = \mathbf{V}_{\mu}^{\mathrm{V}} (= \mathbf{V}_{\mu})$$

and $|\mathbf{V}_{\mu}| < \kappa$, then for every $\mathrm{x} \in \mathbf{V}_{\mu}$ there exists an $\mathrm{A}_{\mathrm{x}} \in \mathrm{U}$ such that

$$\langle \emptyset ; \mathrm{A}_{\mathrm{x}} \rangle \Vdash_{\mathbf{P}_{\mathrm{U}}} [\dot{\mathbf{V}}(\mathrm{H}^{\dot{\Gamma}}) \models \phi(\dot{\Gamma} \restriction \check{\kappa}, \check{\mathrm{x}}, \check{\vec{\delta}}, \mathrm{H}^{\dot{\Gamma}})]$$

where $\dot{\Gamma}$ is the canonical $\mathbf{P}_{\mathrm{U}}$ – name for the $\mathbf{P}_{\mathrm{U}}$ – generic filter. The above is possible since $\mathbf{P}_{\mathrm{U}}$ satisfies the **Prikry property**. As

$$\{\mathrm{A}_{\mathrm{x}} \mid \mathrm{x} \in \mathbf{V}_{\mu}\} \in \mathbf{V}$$

and

$$|\{\mathrm{A}_{\mathrm{x}} \mid \mathrm{x} \in \mathbf{V}_{\mu}\}| < \kappa,$$

then the κ – completeness of the measure U implies that

$$\mathrm{A} :\stackrel{\mathrm{def}}{=} \bigcap\{\mathrm{A_x} \mid \mathrm{x} \in \mathbf{V}_\mu\} \in \mathrm{U}.$$

If for all $\mathrm{x} \in \mathbf{V}_\mu$

$$\langle\emptyset;\mathrm{A}\rangle \Vdash_{\mathbf{P}_\mathrm{U}} \neg\dot{\mathbf{V}}(\mathrm{H}^{\dot{\Gamma}}) \models \phi(\dot{\Gamma}\restriction\check{\kappa}, \check{\mathrm{x}}, \check{\vec{\delta}}, \mathrm{H}^{\dot{\Gamma}}),$$

then we get a contradiction as follows:

By

$$\mathbf{V}(\mathrm{H}) \models \forall \mathrm{s} \in [\kappa]^\omega \exists! \mathrm{x} \in \mathbf{V}_\mu \phi(\mathrm{s}, \mathrm{x}, \vec{\delta}, \mathrm{H})$$

there exists a $\mathrm{p} \in \mathbf{P}_\mathrm{U}$ such that $\mathrm{p} \leq \langle\emptyset;\mathrm{A}\rangle$ and $\mathrm{x}_0 \in \mathbf{V}_\mu$ such that

$$\mathrm{p} \Vdash_{\mathbf{P}_\mathrm{U}} [\dot{\mathbf{V}}(\mathrm{H}^{\dot{\Gamma}}) \models \phi(\dot{\Gamma}\restriction\check{\kappa}, \check{\mathrm{x}}_0, \check{\vec{\delta}}, \mathrm{H}^{\dot{\Gamma}})].$$

Also, since

$$(\forall \mathrm{x} \in \mathbf{V}_\mu)[\langle\emptyset;\mathrm{A}\rangle \Vdash_{\mathbf{P}_\mathrm{U}} \neg\dot{\mathbf{V}}(\mathrm{H}^{\dot{\Gamma}}) \models \phi(\dot{\Gamma}\restriction\check{\kappa}, \check{\mathrm{x}}, \check{\vec{\delta}}, \mathrm{H}^{\dot{\Gamma}})]$$

holds, then we conclude that

$$\mathrm{p} \Vdash_{\mathbf{P}_\mathrm{U}} [\neg\dot{\mathbf{V}}(\mathrm{H}^{\dot{\Gamma}}) \models \phi(\dot{\Gamma}\restriction\check{\kappa}, \check{\mathrm{x}}, \check{\vec{\delta}}, \mathrm{H}^{\dot{\Gamma}})].$$

But this is a contradiction. Thus for some $\mathrm{x}_0 \in \mathbf{V}_\mu$ such that

$$\langle\emptyset;\mathrm{A}\rangle \Vdash_{\mathbf{P}_\mathrm{U}} [\dot{\mathbf{V}}(\mathrm{H}^{\dot{\Gamma}}) \models \phi(\dot{\Gamma}\restriction\check{\kappa}, \check{\mathrm{x}}_0, \check{\vec{\delta}}, \mathrm{H}^{\dot{\Gamma}})].$$

We use the homogeneity of the forcing $\mathbf{P}_\mathrm{U}$ to find a $\mathbf{P}_\mathrm{U}$ – generic $\bar{\mathrm{G}}$ such that

$$\langle\emptyset;\mathrm{A}\rangle \in \bar{\mathrm{G}}$$

and

$$\mathbf{V}[\mathrm{G}] = \mathbf{V}[\bar{\mathrm{G}}].$$

We let

$$\tau = \dot{\sigma}^{\kappa}_{\bar{\mathrm{G}}} = \sigma_{\bar{\mathrm{G}}} \restriction \kappa = \sigma_{\bar{\mathrm{G}} \restriction \kappa}.$$

We claim that it is homogeneous for the OD coloring ϕ.

In order to verify the above claim, we let $s \in [\tau]^{\omega}$.

By the **geometric property** of the Prikry type forcings, s is a $\mathbf{P}_{U_\kappa}$ – generic over $\mathbf{V}$ sequence with $s \subset A \restriction \kappa$. Then, we consider $\mathcal{G}^*$ a $\mathcal{Q}$ – generic over $\mathbf{V}$ such that $\langle s,0,k,A \rangle \in \mathcal{G}^*$. Notice that $\langle s,0,k,A \rangle$ belongs to $\mathcal{Q}$ since for some $\alpha \in [\omega]^{\omega}$ with $s = \tau \circ \alpha$, then $y = \sigma_{\bar{\mathrm{G}}} \circ \alpha$ is a $\mathbf{P}_{\mathrm{U}}$ – generic over $\mathbf{V}$ sequence with $\langle \emptyset;A \rangle \in y$ and with $y \restriction \kappa = s$.

If G^* is the $\mathbf{P}_{\mathrm{U}}$ – generic filter induced by $\mathcal{G}^*$, then

$$\mathrm{G}^* \restriction \kappa = s \quad \text{and} \quad \langle \emptyset;A \rangle \in \mathrm{G}^*.$$

Consequently,

$$\mathbf{V}[\mathrm{G}^*] \models [\mathbf{V}(\mathrm{H}^{\mathrm{G}^*}) \models \phi(\mathrm{G}^* \restriction \kappa, \check{x}_0, \check{\vec{\delta}}, \mathrm{H}^{\mathrm{G}^*})],$$

i.e.,

$$\mathbf{V}(\mathrm{H}^{\mathrm{G}^*}) \models \phi(s, \check{x}_0, \vec{\delta}, \mathrm{H}^{\mathrm{G}^*}).$$

But then, by **3.A.8** $\mathrm{H}^{\mathrm{G}^*} = \mathrm{H}$.

Therefore, for all $s \in [\tau]^{\omega}$

$$\mathbf{V}(\mathrm{H}) \models \phi(s, x_0, \vec{\delta}, \mathrm{H})$$

and so we have proven that

$$\mathbf{V}(\mathrm{H}) \models [\forall \mu < \kappa(\kappa \underset{\mathbf{OD}}{\rightarrow} (\omega)^{\omega}_{\mathbf{V}_\mu}) \quad \& \quad \kappa \text{ is strong limit}].$$

3.C.1 Theorem □

3.C.2 Lemma *Assume that ϕ is a formula $\alpha \in [\kappa, \lambda)$ and $\mathrm{u} \in \mathbf{V}$. Then,*

$$[\mathbf{V}(\mathrm{H}) \models \phi(\mathrm{u}, \mathrm{G}\restriction \alpha, \mathrm{H})] \iff \quad [(\exists \mathrm{p} \in \mathbf{P}_{\mathrm{U}})(\mathrm{p} \text{ is an } \alpha - \text{nice condition} \quad \&$$
$$\& \quad \mathrm{p}\restriction \alpha \in \mathrm{G}\restriction \alpha \quad \&$$
$$\& \quad \mathrm{p} \underset{\mathbf{P}_{\mathrm{U}}}{\Vdash} [\dot{\mathbf{V}}(\mathrm{H}^{\dot{\Gamma}}) \models \phi(\check{\mathrm{u}}, \dot{\Gamma}\restriction \check{\alpha}, \mathrm{H}^{\dot{\Gamma}})] \quad].$$

In particular, there exists a formula $\bar{\phi}$ such that

$$[\mathbf{V}(\mathrm{H}) \models \phi(\mathrm{u}, \mathrm{G}\restriction \alpha, \mathrm{H})] \iff [\mathbf{V}[\mathrm{G}\restriction \alpha] \models \bar{\phi}(\mathrm{u}, \mathrm{G}\restriction \alpha, \mathrm{v})]$$

for some $\mathrm{v} \in \mathbf{V}$.

Proof: Assume that

$$\mathbf{V}(\mathrm{H}) \models \phi(\mathrm{u}, \mathrm{G}\restriction \alpha, \mathrm{H}).$$

Then, there exists $\mathrm{p} \in \mathrm{G}$ and p is an α – nice condition such that

$$\mathrm{p} \underset{\mathbf{P}_{\mathrm{U}}}{\Vdash} [\dot{\mathbf{V}}(\mathrm{H}^{\dot{\Gamma}}) \models \phi(\check{\mathrm{u}}, \dot{\Gamma}\restriction \check{\alpha}, \mathrm{H}^{\dot{\Gamma}})].$$

But then, $\mathrm{p}\restriction \alpha \in \mathrm{G}\restriction \alpha$ and so

$$\exists \mathrm{p} \in \mathbf{P}_{\mathrm{U}}(\mathrm{p} \text{ is an } \alpha - \text{nice} \quad \text{condition} \quad \& \quad \mathrm{p}\restriction \alpha \in \mathrm{G}\restriction \alpha \quad \&$$
$$\& \quad \mathrm{p} \underset{\mathbf{P}_{\mathrm{U}}}{\Vdash} [\dot{\mathbf{V}}(\mathrm{H}^{\dot{\Gamma}}) \models \phi(\check{\mathrm{u}}, \dot{\Gamma}\restriction \check{\alpha}, \mathrm{H}^{\dot{\Gamma}})].$$

Conversely, if there exists a $p \in \mathbf{P}_U$ such that p is an α – nice condition and

$$p\restriction\alpha \in G\restriction\alpha \quad \& \quad \& \quad p \Vdash_{\mathbf{P}_U} [\dot{\mathbf{V}}(H^{\dot{\Gamma}}) \models \phi(\check{u}, \dot{\Gamma}\restriction\check{\alpha}, H^{\dot{\Gamma}})],$$

then there exists a $q \in G$ such that

$$q\restriction\alpha = p\restriction\alpha.$$

Consequently

$$q \Vdash_{\mathbf{P}_U} [\dot{\mathbf{V}}(H^{\dot{\Gamma}}) \models \phi(\check{u}, \dot{\Gamma}\restriction\check{\alpha}, H^{\dot{\Gamma}})]$$

and so

$$\mathbf{V}[G] \models \mathbf{V}[H^G] \models \phi(u, G\restriction\alpha, H^G),$$

i.e.,

$$\mathbf{V}(H) \models \phi(u, G\restriction\alpha, H).$$

Obviously, if we set

$$\begin{aligned}\bar{\phi}(x, y, z) :\overset{\text{def}}{\Longleftrightarrow} \quad & (\exists p \in \mathbf{P}_U)[p \text{ is an } \alpha \text{ – nice condition } \quad \& \\ & \& \quad p\restriction\alpha \in y\restriction\alpha \quad \& \\ & \& \quad p \Vdash_{\mathbf{P}_U} \dot{\mathbf{V}}(H^{\dot{\Gamma}}) \models \phi(x, \dot{\Gamma}\restriction\check{\alpha}, H^{\dot{\Gamma}})],\end{aligned}$$

then

$$\mathbf{V}(H) \models \phi(u, G\restriction\alpha, H) \Longleftrightarrow \mathbf{V}[G\restriction\alpha] \models \bar{\phi}(u, G\restriction\alpha, \langle\alpha, U, \mathbf{P}_U, \lambda\rangle).$$

3.C.2 Lemma □

3.C.3 Lemma *Assume that* $\mathrm{b} \in \mathrm{H}$ *and* $\mathrm{H} = \mathrm{H}^{\mathrm{G}}$ *where G is our fixed* $\mathbf{P}_{\mathrm{U}}$ – generic *filter over* $\mathbf{V}$. *For every formula* ϕ *and parameters* $\mathrm{a} \in \mathbf{V}$ *there exists a formula* ϕ^* *such that* $\mathbf{V}(\mathrm{H}) \models \phi(\mathrm{a}, \mathrm{b}) \Longleftrightarrow \mathbf{V}[\mathrm{b}] \models \phi^*(\mathrm{a}, \mathrm{U}, \kappa, \lambda, \mathrm{b})$.

Proof: We define the formula ϕ^* as follows:

$$\phi^*(\mathrm{a}, \mathrm{U}, \kappa, \lambda, \mathrm{b}) : \overset{\mathrm{def}}{\Longleftrightarrow} \quad \underset{\mathcal{Q}_\lambda}{\Vdash} \exists \mathrm{y}(\mathrm{y} \text{ is a } \mathbf{P}_{\mathrm{U}} - \text{generic over } \mathbf{V}$$
$$[\mathbf{V}(\mathrm{H}^{\mathrm{y}}) \models \phi(\mathrm{a}, \mathrm{b})].)$$

We claim that $\mathbf{V}(\mathrm{H}) \models \phi(\mathrm{a}, \mathrm{b}) \Longleftrightarrow \mathbf{V}[\mathrm{b}] \models \phi^*(\mathrm{a}, \mathrm{U}, \kappa, \lambda, \mathrm{b})$.

Assume that $\mathbf{V}[\mathrm{b}] \models \phi^*(\mathrm{a}, \mathrm{U}, \kappa, \lambda, \mathrm{b})$. Let K be $\mathcal{Q}_\lambda$ – generic over $\mathbf{V}[\mathrm{b}]$ and we find $\bar{\mathrm{G}}$ $\mathbf{P}_{\mathrm{U}}$ – generic such that $\mathbf{V}(\mathrm{H}^{\bar{\mathrm{G}}}) \models \phi(\mathrm{a}, \mathrm{b})$. Since $\mathrm{b} \in \mathrm{H}^{\bar{\mathrm{G}}}$, then we find $\eta \in (\kappa, \lambda)$ such that $\mathrm{b} \in \mathbf{V}[\bar{\mathrm{G}} \restriction \eta]$. So we find a $\mathbf{P}_{\mathrm{U}_\eta}$ – name $\dot{\tau}$ such that $(\dot{\tau})^{\mathbf{V}[\bar{\mathrm{G}} \restriction \eta]} = \mathrm{b}$ and we consider the Boolean algebra $\mathcal{B}_{\dot{\tau}}$ generated by $\dot{\tau}$.

Then, for

$$\mathcal{X} = \{ \| \check{\alpha} \in \dot{\tau} \|_{\mathbf{r.o.}(\mathbf{P}_{\mathrm{U}_\eta})} | \; \alpha < \mathrm{b} \}$$

we know that $\mathbf{V}[\mathrm{b}] = \mathbf{V}[\bar{\mathrm{G}} \restriction \eta \cap \mathcal{X}]$. Since $\bar{\mathrm{G}} \restriction \eta \cap \mathcal{X} = \{ \| \check{\alpha} \in \dot{\tau} \|_{\mathbf{r.o.}(\mathbf{P}_{\mathrm{U}_\eta})} | \; \alpha \in \mathrm{b} \}$, then $\bar{\mathrm{G}} \restriction \eta \cap \mathcal{X} \in \mathbf{V}(\mathrm{H})$. Since $\mathcal{B}_{\dot{\tau}} \subset_{\mathrm{c}} \mathbf{r.o.}(\mathbf{P}_{\mathrm{U}_\eta})$, then by the **2.B.7** we can find a $\mathbf{P}_{\mathrm{U}_\eta}$ – generic over $\mathbf{V}$ filter Z such that

$$\mathrm{Z} \cap \mathcal{B}_{\dot{\tau}} = \bar{\mathrm{G}} \restriction \eta \cap \mathcal{X}.$$

Hence, $(\dot{\tau})^{\mathbf{V}[\mathrm{Z}]} = (\dot{\tau})^{\mathbf{V}[\bar{\mathrm{G}} \restriction \eta]} = \mathrm{b}$. Then, as in the proof of **3.C.1** we find a $\mathcal{Q}$ – generic filter $\mathcal{G}^*$ over $\mathbf{V}(\mathrm{H})$ such that for the induced $\mathbf{P}_{\mathrm{U}}$ – generic filter G^* we have that

$$\mathrm{G}^* \restriction \eta = \mathrm{Z}.$$

Therefore, $(\dot{\tau})^{\mathbf{V}[G^*\upharpoonright\eta]} = (\dot{\tau})^{\mathbf{V}[\bar{G}\upharpoonright\eta]} = \mathrm{b}$. Since

$$\mathbf{V}(\mathrm{H}^{\bar{\mathrm{G}}}) \models \phi(\mathrm{a},\mathrm{b}),$$

then we might choose the $\mathbf{P}_{\mathrm{U}_\eta}$ – name $\dot{\tau}$ such that

$$\Vdash_{\mathbf{P}_\mathrm{U}} \mathbf{V}(\mathrm{H}^{\dot{\Gamma}} \models \phi(\check{\mathrm{a}}, (\dot{\tau})^{\mathbf{V}[G^*\upharpoonright\eta]})$$

and consequently,

$$\mathbf{V}(\mathrm{H}^{\mathrm{G}^*}) \models \phi(\mathrm{a}^*,\mathrm{b}).$$

Now we apply **3.A.8** to conclude that $\mathbf{V}(\mathrm{H}^{\mathrm{G}^*}) = \mathbf{V}(\mathrm{H})$ and so $\mathbf{V}(\mathrm{H}) \models \phi(\mathrm{a},\mathrm{b})$. Conversely, if $\mathbf{V}(\mathrm{H}) \models \phi(\mathrm{a},\mathrm{b})$, then there exists a generic extension of $\mathbf{V}[\mathrm{b}]$ namely $\mathbf{V}[\mathrm{b}]$ such that $\mathbf{V}(\mathrm{H}^{\mathrm{G}}) \models \phi(\mathrm{a},\mathrm{b})$. But then, using the collapsing algebra $\mathcal{Q}_\lambda (= \mathrm{Coll}(\omega, ((2^{2^\lambda})^+))$ we should have that $\mathbf{V}[\mathrm{b}] \models \phi^*(\mathrm{a},\mathrm{U},\lambda,\mathrm{b})$. This completes the proof of the lemma.

3.C.3 Lemma □

Chapter 4

The partition relation in a ZFC model

4.A $< \lambda -$ DC in the inner model $\mathbf{V}$(H)

In this section we are going to prove that the inner model $\mathbf{V}$(H) satisfies $< \lambda -$ DC.

$< \lambda -$ DC is the following statement:

> *"For every cardinal $\mu < \lambda$ and every partial order* $\mathbf{R}$
> *either there exists an* $\mathbf{R}$*-descending chain of length μ*
> *or there exists a maximal* $\mathbf{R}$*-descending chain of length $< \mu$."*

Obviously, the Axiom of Dependent Choices (DC) is the statement $< \omega_1 -$DC. If A is a set, then $< \lambda -$ DC differs from $< \lambda -$ DC only in the requirement that $\mathbf{R}$ should be a partial order on A, i.e., $\mathbf{R} \subseteq$ A $\times$ A.

First, we are going to show that for any ordinal θ the model $\mathbf{V}$(H) satisfies $< \lambda - \mathrm{DC}_{\theta \times \mathrm{H}}$.

In the following arguments we fix an ordinal θ and a partial order $\mathbf{R}$, where

$\mathbf{R} \subseteq (\theta \times \mathrm{H}) \times (\theta \times \mathrm{H})$ and $\mathbf{R} \in \mathbf{V}(\mathrm{H})$. As we have seen, $\mathbf{R}$ is definable over $\mathbf{V}(\mathrm{H})$ by some formula ϕ with parameters u and $\mathrm{G}\restriction\alpha_0$ where $\mathrm{u} \in \mathbf{V}$, $\alpha_0 \in [\kappa, \lambda)$ and G is our fixed $\mathbf{P}_{\mathrm{U}}$ – generic over $\mathbf{V}$ filter with $\mathrm{H} = \mathrm{H}^{\mathrm{G}}$.

We recall that we have already proven that $\mathrm{H} = \mathrm{H}^{\mathrm{G}} = \mathcal{P}(\kappa) \cap \mathbf{V}(\mathrm{H})$.

Using **3.C.3**, we fix a formula ϕ^* such that

$$\mathrm{p}\mathbf{R}\mathrm{q} \iff \mathbf{V}[\mathrm{G}\restriction\alpha_0, \mathrm{p}, \mathrm{q}] \models \phi^*(\mathrm{p}, \mathrm{q}, \mathrm{G}\restriction\alpha_0).$$

Having $\mathbf{R}, \mathrm{G}\restriction\alpha_0, \phi^*$ and H fixed, we are preparing for the proof of $< \lambda - \mathrm{DC}_{\theta\times\mathrm{H}}$ by giving the following definitions.

4.A.1 Definition *Let* $\mathrm{z} \in \mathbf{V}(\mathrm{H})$ *be a* $\mathbf{P}_{\mathrm{U}_\alpha}$ – generic *sequence over* $\mathbf{V}$ *such that* $\mathrm{z}\restriction\alpha_0 = \mathrm{G}\restriction\alpha_0$ *and* $\alpha \in (\alpha_0, \lambda)$.

If $\xi < \lambda$, *then we define, in* $\mathbf{V}(\mathrm{H})$, *the set* $\mathcal{M}_\xi(\mathrm{z})$ *of all* $\mathbf{R}$ – maximal *descending chains of length* $< \xi$ *which are elements of* $\mathbf{V}[\mathrm{z}]$, *i.e.*,

$$\begin{aligned} \mathrm{x} \in \mathcal{M}_\xi(\mathrm{z}) \quad &:\overset{\mathrm{def}}{\iff} \mathrm{x} \in \mathbf{V}[\mathrm{z}] \;\;\&\;\; \mathrm{x} \in (\theta \times \mathcal{P}(\kappa) \cap \mathbf{V}[\mathrm{z}])^{<\xi} \;\;\&\;\; (\forall \eta < \xi) \\ &\& \;\; (\mathbf{V}[\mathrm{G}\restriction\alpha_0, \mathrm{x}(\eta+1), \mathrm{x}(\eta)] \models \phi^*(\mathrm{x}(\eta+1), \mathrm{x}(\eta), \mathrm{G}\restriction\alpha_0)) \\ &\& \;\; \neg(\exists \mathrm{p} \in \theta \times \mathcal{P}(\kappa) \cap \mathbf{V}[\mathrm{z}]) \\ &\qquad (\forall \mathrm{q} \in \mathrm{ran}(\mathrm{x}))(\mathbf{V}[\mathrm{G}\restriction\alpha_0, \mathrm{p}, \mathrm{q}] \models \phi^*(\mathrm{p}, \mathrm{q}, \mathrm{G}\restriction\alpha_0)). \end{aligned}$$

4.A.1 Definition □

4.A.2 Definition *If* $\mathrm{z} \in \mathbf{V}(\mathrm{H})$ *is a* $\mathbf{P}_{\mathrm{U}_\alpha}$ – generic *sequence over* $\mathbf{V}$ *such that* $\mathrm{z}\restriction\alpha_0 = \mathrm{G}\restriction\alpha_0$ *for* $\alpha \in (\alpha_0, \lambda)$ *and if* $\mathcal{M}_\xi(\mathrm{z})$ *is the set defined in* **4.A.1**, *then we* *define* $\delta(\alpha, z, \xi)$ *as follows:*

If

$$\mathcal{D}(\alpha, z, \xi) \quad :\stackrel{\text{def}}{=} \Big\{\gamma \in (\alpha,\lambda) \mid \exists w(w \text{ is } \mathbf{P}_{U_\gamma} - \text{generic over } \mathbf{V} \quad \& \\ \& \quad w \in \mathbf{V}(H) \quad \& \quad w\restriction\alpha = z \quad \& \\ \& \quad \forall x \in \mathcal{M}_\xi(z)(\exists p \in \theta \times \mathcal{P}(\kappa) \cap \mathbf{V}[w])(\forall q \in \operatorname{ran}(x)) \\ [\mathbf{V}[G\restriction\alpha_0, p, q] \models \phi^*(p, q, G\restriction\alpha_0)]\Big\},$$

then we let

$$\delta(\alpha, z, \xi) = \begin{cases} \bigcap \mathcal{D}(\alpha, z, \xi), & \text{if } \mathcal{D}(\alpha, z, \xi) \neq \emptyset \\ 0, & \text{if otherwise.} \end{cases}$$

4.A.2 Definition □

In the following two propositions we will show that under some appropriate assumptions $\mathcal{D}(\alpha, z, \xi)$ is non-empty and that $\delta(\alpha, z, \xi)$ is independent from the $\mathbf{P}_{U_\alpha}$ – generic sequence z.

4.A.3 Proposition *Assume that* $z \in \mathbf{V}(H)$ *is a* $\mathbf{P}_{U_\alpha}$ – generic *sequence over* **V** *such that* $z\restriction\alpha_0 = G\restriction\alpha_0$ *for* $\alpha \in (\alpha_0, \lambda)$ *and that in* **V**(H) *there exists neither an* **R** – descending *sequence of length* ξ *nor a maximal (in* **V**(H)*)* **R** – descending *sequence of length* $<\xi$. *Then* $\mathcal{D}(\alpha, z, \xi)$ *is non-empty and so* $\delta(\alpha, z, \xi) \in (\alpha, \lambda)$.

Proof: First we observe that $\mathcal{M}_\xi(z)$ is non-empty.

The reason is the following:

If in $\mathbf{V}[z]$ there exists an $x \in (\theta \times \mathcal{P}(\kappa) \cap \mathbf{V}[z])^{\xi} \cap \mathbf{V}[z]$ such that

$$(\forall \eta < \xi)\Big(\mathbf{V}[G\restriction \alpha_0, x(\eta+1), x(\eta)] \models \phi^*(x(\eta+1), x(\eta), G\restriction \alpha_0)\Big),$$

then by definition of ϕ^* this will imply that

$$(\forall \eta < \xi)[x(\eta+1)\mathbf{R}x(\eta)]$$

and as $x \in \mathbf{V}[z]$, then $x \in \mathbf{V}(H)$. But this proves that $\mathbf{V}(H)$ contains an $\mathbf{R}$ – descending chain of length ξ, contrary to our hypothesis.
Therefore, in $\mathbf{V}[z]$, there is no $\mathbf{R}$ – descending chain of length ξ.
As $\mathbf{V}[z] \models \mathrm{AC}$, then $\mathbf{V}[z]$ contains an $\mathbf{R}$ – descending chain x of length $< \xi$, so that $\mathbf{V}(H)$ thinks that x is maximal for $\mathbf{R}\restriction \theta \times \mathcal{P}(\kappa) \cap \mathbf{V}[z]$. Hence, $\mathcal{M}_{\xi}(z)$ is non-empty.

Moreover as $\mathcal{M}_{\xi}(z) \subset (\theta \times \mathcal{P}(\kappa) \cap \mathbf{V}[z])^{<\xi}$ and since

$$\mathbf{V}[z] \models [\lambda \text{ is inaccessible}],$$

then there exists a cardinal $\mu < \lambda$ and a bijection π so that

$$\pi : \mathcal{M}_{\xi}(z) \xrightarrow[\text{onto}]{1-1} \mu.$$

Since, according to our hypothesis, every $x \in \mathcal{M}_{\xi}(z)$, although a maximal $\mathbf{R}$ – descending sequence of length $< \xi$ in $\mathbf{V}[z]$, cannot be maximal in $\mathbf{V}(H)$, then there exists a $\mathbf{P}_{U_\eta}$ – generic sequence $y \in \mathbf{V}(H)$, for some $\eta \in (\alpha, \lambda)$, so that $y\restriction \alpha = z$ and

$$(\exists p \in \theta \times \mathcal{P}(\kappa) \cap \mathbf{V}[y])(\forall q \in \mathrm{ran}(x))\Big[\mathbf{V}[G\restriction \alpha_0, p, q] \models \phi^*(p, q, G\restriction \alpha_0)\Big].$$

Hence we can define in $\mathbf{V}(\mathrm{H})$ a map $\bar{\pi}:\mu\rightarrow\lambda$ as follows:

$$\bar{\pi}(\beta) \stackrel{\text{def}}{=} \bigcap\Big\{\eta\in[\kappa,\lambda) \quad \mid (\exists \mathrm{y}\in\mathbf{V}(\mathrm{H}))(\mathrm{y} \text{ is } \mathbf{P}_{\mathrm{U}_\eta}-\text{generic over } \mathbf{V} \quad \& \quad \& \quad \mathrm{y}\restriction\alpha=\mathrm{z} \quad \& \quad (\exists \mathrm{p}\in\theta\times\mathcal{P}(\kappa)\cap\mathbf{V}[\mathrm{y}])(\forall \mathrm{q}\in\operatorname{ran}(\pi^{-1}(\beta))) \Big[\mathbf{V}[\mathrm{G}\restriction\alpha_0,\mathrm{p},\mathrm{q}]\models\phi^*(\mathrm{p},\mathrm{q},\mathrm{G}\restriction\alpha_0)\Big]\Big\}.$$

By the above remarks, we conclude that $\bar{\pi}$ is a well-defined map in $\mathbf{V}(\mathrm{H})$ with $\operatorname{ran}(\bar{\pi})\subseteq\lambda$.

Since $\mu<\lambda$ and $\mathbf{V}(\mathrm{H})\models[\lambda \text{ is regular}]$, then $\sup\bar{\pi}''\mu<\lambda$.

We set $\gamma=\sup\bar{\pi}''\mu<\lambda$.

Next, we show that if w is any $\mathbf{P}_{\mathrm{U}_\gamma}-$ generic in $\mathbf{V}(\mathrm{H})$ such that $\mathrm{w}\restriction\alpha=\mathrm{z}$, then

$$(\forall \mathrm{x}\in\mathcal{M}_\xi(\mathrm{z}))(\exists \mathrm{p}\in\mathcal{P}(\kappa)\cap\mathbf{V}[\mathrm{w}])(\forall \mathrm{q}\in\operatorname{ran}(\mathrm{x}))\Big[\mathbf{V}[\mathrm{G}\restriction\alpha_0,\mathrm{p},\mathrm{q}]\models\phi^*(\mathrm{p},\mathrm{q},\mathrm{G}\restriction\alpha_0)\Big].$$

In order to prove the above claim, we fix an $\mathrm{x}\in\mathcal{M}_\xi(\mathrm{z})$. By definition of γ, we know that there exists a $\mathbf{P}_{\mathrm{U}_\gamma}-$ generic y such that

$$\mathrm{y}\restriction\alpha=\mathrm{z},$$

and moreover,

$$\mathbf{V}[\mathrm{y}]\models(\exists \mathrm{p}\in\theta\times\mathcal{P}(\kappa))(\forall \mathrm{q}\in\operatorname{ran}(\mathrm{x}))\Big[\mathbf{V}[\mathrm{G}\restriction\alpha_0,\mathrm{p},\mathrm{q}]\models\phi^*(\mathrm{p},\mathrm{q},\mathrm{G}\restriction\alpha_0)\Big].$$

Let us fix w an arbitrary $\mathbf{P}_{\mathrm{U}_\gamma}-$ generic in $\mathbf{V}(\mathrm{H})$ such that $\mathrm{w}\restriction\alpha=\mathrm{z}$. Also we fix a $\mathbf{P}_{\mathrm{U}_\alpha}-$name $\dot{\mathrm{x}}$ so that $(\dot{\mathrm{x}})^{\mathbf{V}[\mathrm{z}]}=\mathrm{x}$. Then, we can find an $\alpha-$nice condition (as we defined it in **2.A.10**) $q \in \mathcal{F}^{\gamma}_{y}$ such that

$$q \Vdash_{\mathbf{P}_{U_\gamma}} [\exists p \in \theta \times \mathcal{P}(\kappa) \forall q \in (\dot{x})^{\dot{\mathbf{V}}[\dot{\Gamma}\restriction \alpha]} \dot{\mathbf{V}}[\dot{\Gamma}\restriction \alpha_0, p, q] \models \phi^*(p, q, \dot{\Gamma}\restriction \alpha_0)].$$

Since

$$q\restriction \alpha \in \mathcal{F}^{\gamma}_{y} \quad \& \quad \mathcal{F}^{\gamma}_{w}\restriction \alpha = \mathcal{F}^{\alpha}_{z},$$

then there exists a condition $s \in \mathcal{F}^{\gamma}_{w}$ such that $q\restriction \alpha = s\restriction \alpha$. At this point we use the fact that q is an α – nice condition and we apply **2.A.12** to conclude that

$$s \Vdash_{\mathbf{P}_{U_\gamma}} [\exists p \in \theta \times \mathcal{P}(\kappa) \forall q \in (\dot{x})^{\dot{\mathbf{V}}[\dot{\Gamma}\restriction \alpha]} \dot{\mathbf{V}}[\dot{\Gamma}\restriction \alpha_0, p, q] \models \phi^*(p, q, \dot{\Gamma}\restriction \alpha_0).$$

Since $s \in \mathcal{F}^{\gamma}_{w}$, then

$\mathbf{V}[w] \models [(\exists p \in \theta \times \mathcal{P}(\kappa))(\forall q \in \mathrm{ran}((\dot{x})^{\mathbf{V}[w\restriction \alpha]}))[\mathbf{V}[w\restriction \alpha_0, p, q] \models \phi^*(p, q, w\restriction \alpha_0)].$

But then we recall that $(\dot{x})^{\mathbf{V}[w\restriction \alpha]} = x$ and that $\mathcal{F}^{\gamma}_{w}\restriction \alpha_0 = G\restriction \alpha_0$.

Therefore,

$$(\exists p \in \theta \times \mathcal{P}(\kappa) \cap \mathbf{V}[w])(\forall q \in \mathrm{ran}(x))\Big[\mathbf{V}[G\restriction \alpha_0, p, q] \models \phi^*(p, q, G\restriction \alpha_0)\Big]$$

and this shows that the claim holds, i.e.,

if w is any $\mathbf{P}_{U_\gamma}$ – generic in $\mathbf{V}(H)$ such that $w\restriction \alpha = z$, then

$$(\forall x \in \mathcal{M}_\xi(z)) \qquad (\exists p \in \theta \times \mathcal{P}(\kappa) \cap \mathbf{V}[w])(\forall q \in \mathrm{ran}(x)) \Big[\mathbf{V}[G\restriction \alpha_0, p, q] \models \phi^*(p, q, G\restriction \alpha_0)\Big].$$

In particular, we have shown that

$$\exists\gamma\exists w\Big(w \text{ is } \quad \mathbf{P}_{U_\gamma} - \text{generic} \quad \& \quad w \in \mathbf{V}(H) \quad \& \quad w\restriction \alpha = z\Big) \quad \&$$

$$\& \quad \forall x \in \mathcal{M}_\xi(z)(\exists p \in \theta \times \mathcal{P}(\kappa) \cap \mathbf{V}[w])(\forall q \in \operatorname{ran}(x))$$
$$\Big[\mathbf{V}[G \restriction \alpha_0, p, q] \models \phi^*(p, q, G \restriction \alpha_0)\Big]\Big),$$

and

$$\exists \gamma \forall w[(w \text{ is } \quad \mathbf{P}_{U_\gamma} - \text{generic} \quad \& \quad w \in \mathbf{V}(\mathrm{H}) \quad \& \quad w \restriction \alpha = z) \Longrightarrow$$
$$\Longrightarrow (\forall x \in \mathcal{M}_\xi(z))(\exists p \in \theta \times \mathcal{P}(\kappa) \cap \mathbf{V}[w])(\forall q \in \operatorname{ran}(x))$$
$$\mathbf{V}[G \restriction \alpha_0, p, q] \models \phi^*(p, q, G \restriction \alpha_0)].$$

Therefore, the set $\mathcal{D}(\alpha, z, \xi)$ is non-empty and consequently $\delta(\alpha, z, \xi)$ is a well–defined ordinal in (α, λ).

4.A.3 Proposition □

4.A.4 Proposition *Assume that* $z \in \mathbf{V}(\mathrm{H})$ *is a* $\mathbf{P}_{U_\alpha}$ – generic *sequence over* $\mathbf{V}$ *such that* $z \restriction \alpha_0 = G \restriction \alpha_0$ *for* $\alpha \in (\alpha_0, \lambda)$ *and that in* $\mathbf{V}(\mathrm{H})$ *there exists neither an* $\mathbf{R}$*-descending sequence of length* ξ *nor a maximal (in* $\mathbf{V}(\mathrm{H})$*)* $\mathbf{R}$–descending *sequence of length* $< \xi$. *Let* $\sigma^\alpha = \sigma_G \restriction \alpha$ *is the* $\mathbf{P}_{U_\alpha}$–generic *sequence generated by* $G \restriction \alpha$. *Then* $\delta(\alpha, z, \xi) = \delta(\alpha, \sigma^\alpha, \xi)$.

Proof: We fix an ordinal $\beta \in (\alpha, \lambda)$ such that $\beta \in \mathcal{D}(\alpha, z, \xi)$ with witness w, i.e., w is a $\mathbf{P}_{U_\beta}$ – generic in $\mathbf{V}(\mathrm{H})$ such that $w \restriction \alpha = z$ and

$$\forall x \in \mathcal{M}_\xi(z)(\exists p \in \theta \times \mathcal{P}(\kappa) \cap \mathbf{V}[w])(\forall q \in \operatorname{ran}(x))\Big[\mathbf{V}[G \restriction \alpha_0, p, q] \models \phi^*(p, q, G \restriction \alpha_0)\Big].$$

Since $w \restriction \alpha = z$ and $z \restriction \alpha_0 = \sigma^\alpha \restriction \alpha_0$, then we can find a $\mathbf{P}_{U_\beta}$ – name $\dot{w}$ such that

$$\Vdash_{\mathbf{P}_{U_\beta}} \quad [\dot{w} \text{ is a } \mathbf{P}_{U_\beta} - \text{generic sequence } \& \ \forall x \in \mathcal{M}_\xi(\dot{w} \restriction \alpha)(\exists p \in \theta \times \mathcal{P}(\kappa) \cap \dot{\mathbf{V}})[\dot{w}]$$
$$(\forall q \in \operatorname{ran}(x))[\dot{\mathbf{V}}[\dot{w} \restriction \alpha_0, p, q] \models \phi^*(p, q, \dot{w} \restriction \alpha_0)].$$

Then, we use **2.B.10** to find a $\mathbf{P}_{U_\beta}$ – generic filter K in $\mathbf{V}(H)$, so that $(\dot{w}^{\mathbf{V}[K]}) = G\restriction\beta$.

Therefore,

$$(\forall x \in \mathcal{M}_\xi((G\restriction\beta)\restriction\alpha)) \quad (\exists p \in \theta \times \mathcal{P}(\kappa) \cap \mathbf{V}[G\restriction\beta])(\forall q \in \operatorname{ran}(x))$$
$$\mathbf{V}[G\restriction\alpha_0, p, q] \models \phi^*(p, q, G\restriction\alpha_0)$$

Hence, $G\restriction\beta$ is a $\mathbf{P}_{U_\beta}$ – generic such that $(G\restriction\beta)\restriction\alpha = G\restriction\alpha$ and

$$(\forall x \in \mathcal{M}_\xi(G\restriction\alpha)) \quad (\exists p \in \theta \times \mathcal{P}(\kappa) \cap \mathbf{V}[G\restriction\beta])(\forall q \in \operatorname{ran}(x))$$
$$\mathbf{V}[G\restriction\alpha_0, p, q] \models \phi^*(p, q, G\restriction\alpha_0).$$

This implies that $\delta(\alpha, z, \xi) = \delta(\alpha, \sigma^\alpha, \xi)$.

4.A.4 Proposition □

4.A.5 Theorem *For every ordinal θ, the inner model* $\mathbf{V}(H)$ *satisfies* $< \lambda - DC_{\theta\times H}$.

Proof: Let $\mathbf{R}$ be a partial order on $\theta \times H$, i.e., $\mathbf{R} \subseteq (\theta \times H) \times (\theta \times H)$. Let us fix ξ a regular cardinal below λ. As in the beginning of this section, we fix an ordinal

$\alpha_0 \in [\kappa, \lambda)$ and a formula ϕ^* such that

$$p\mathbf{R}q \iff \mathbf{V}[G\restriction\alpha_0, p, q] \models \phi^*(p, q, G\restriction\alpha_0).$$

We assume towards a contradiction that in $\mathbf{V}(H)$ there exists

neither an $\mathbf{R}$ – descending sequence of length ξ nor a maximal (in $\mathbf{V}(H)$)

$\mathbf{R}$ – descending sequence of length $< \xi$.

We set δ_0^* to be the least ordinal γ above α_0 so that

$(\exists \mathrm{p}, \mathrm{q} \in \theta \times \mathcal{P}(\kappa) \cap \mathbf{V}[\mathrm{G}\restriction\gamma])\mathbf{V}[\mathrm{G}\restriction\alpha_0, \mathrm{p}, \mathrm{q}] \models \phi^*(\mathrm{p}, \mathrm{q}, \mathrm{G}\restriction\alpha_0)$.

Then, we define by recursion on $\zeta < \xi$ inside the model $\mathbf{V}(\mathrm{H})$ a ξ – sequence of ordinals

$$\langle \delta_\zeta \mid \zeta < \xi \rangle$$

as follows:

Set $\delta_0 = \delta_0^*$.

If $\langle \delta_\zeta \mid \zeta < \beta \rangle$ has been already constructed, for some $\beta < \xi$, then to define δ_β we consider two cases.

CASE I: $\beta = \alpha + 1$

In this case we set

$$\delta_\beta = \gamma :\stackrel{\text{def}}{\Longleftrightarrow} \exists \mathrm{y} \in \mathbf{V}(\mathrm{H}) \quad (\mathrm{y} \text{ is a } \mathbf{P}_{\mathrm{U}_{\delta_\alpha}} - \text{generic over } \mathbf{V} \quad \& \quad \& \quad \gamma = \delta(\delta_\alpha, \mathrm{y}, \xi)).$$

CASE II: β is a limit ordinal.

Since the construction of $\langle \delta_\zeta \mid \zeta < \beta \rangle$ took place in $\mathbf{V}(\mathrm{H})$, then by the regularity of λ in $\mathbf{V}(\mathrm{H})$ we conclude that $\langle \delta_\zeta \mid \zeta < \beta \rangle$ is bounded below λ. Assume that $\delta^* :\stackrel{\text{def}}{=} \sup\{\delta_\zeta \mid \zeta < \beta\}$. Then we set

$$\delta_\beta = \gamma :\stackrel{\text{def}}{\Longleftrightarrow} \exists \mathrm{y} \in \mathbf{V}(\mathrm{H})(\quad \mathrm{y} \text{ is a } \mathbf{P}_{\mathrm{U}_{\delta^*}} - \text{generic over } \mathbf{V} \quad \& \quad \& \quad \gamma = \delta(\delta^*, \mathrm{y}, \xi)).$$

This completes the recursive definition of $\langle \delta_\zeta \mid \zeta < \xi \rangle$.

Since the construction of this sequence took place in $\mathbf{V}(\mathrm{H})$, then it belongs to $\mathbf{V}(\mathrm{H})$ and by the regularity of λ we conclude that $\langle \delta_\zeta \mid \zeta < \xi \rangle$ is bounded below λ.

Let $\delta^\star :\stackrel{\mathrm{def}}{=} \sup\{\delta_\zeta \mid \zeta < \xi\}$.

Next, we consider the $\mathbf{P}_{\mathrm{U}_{\delta^\star}}$ – generic over $\mathbf{V}$ filter $\mathrm{G}\restriction \delta^\star$ and in $\mathbf{V}[\mathrm{G}\restriction \delta^\star]$ we construct an $\mathbf{R}$ – descending chain of length ξ to get a contradiction.

Using **4.A.3** and **4.A.4** we conclude that for any $\beta < \xi$

$$\delta_{\alpha+1} = \delta(\delta_\alpha, \sigma^{\delta_\alpha}, \xi)),$$

and for any limit ordinal $\beta < \xi$

$$\delta_\beta = \delta(\delta^*, \sigma^{\delta^*}, \xi)),$$

where $\delta^* = \sup\{\delta_\zeta \mid \zeta < \beta\}$. In $\mathbf{V}$, we fix a wellordering $\lhd$ of all $\mathbf{P}_{\mathrm{U}_\alpha}$ – good names (for $\alpha < \delta^*$) of elements of $(\theta \times \mathcal{P}(\kappa))^{<\xi}$. By recursion on $\zeta < \xi$, we construct a ξ – sequence

$$\langle \mathrm{x}_\zeta \mid \zeta < \xi \rangle,$$

which is an $\supset$-descending chain of $\mathbf{R}$ – descending chains of length $< \xi$. Our goal is to make the above construction in $\mathbf{V}[\mathrm{G}\restriction \delta^\star]$.

We set

$\dot{\mathrm{x}}_0 \quad :\stackrel{\mathrm{def}}{=} \lhd$ – least $\mathbf{P}_{\mathrm{U}_{\delta_0}}$ – good name $\dot{\mathrm{x}}$ for an element of $(\theta \times \mathcal{P}(\kappa))^{<\xi} \cap \mathbf{V}[\mathrm{G}\restriction \delta_0]$ such that $(\dot{\mathrm{x}})^{\mathbf{V}[\mathrm{G}\restriction \delta_0]} \in \mathcal{M}_\xi(\mathrm{G}\restriction \delta_0)$.

Then, we define

$$x_0 :\stackrel{\text{def}}{=} (\dot{x}_0)^{\mathbf{V}[G\restriction\delta_0]}.$$

We assume that $\langle x_\zeta \mid \zeta < \beta\rangle$ - for some $\beta < \xi$- has been already constructed in $\mathbf{V}[G\restriction\delta^*]$, and that it satisfies the following inductive hypothesis:

1) $(\forall\zeta < \beta)(\exists\eta)(x_\zeta \in \mathcal{M}_\xi(G\restriction\delta_\eta))$

2) $(\forall\gamma < \gamma' < \beta)(x_\gamma \subset x_{\gamma'})$.

In order to define x_β, we consider two cases.

CASE I : $\beta = \alpha + 1$

By the inductive hypothesis, x_α belongs to some $\mathcal{M}_\xi(G\restriction\delta_\eta)$. Using the definition of $\delta_{\eta+1}$ we set

$$\begin{aligned}\dot{x}_\beta \quad &:\stackrel{\text{def}}{=} \lhd - \text{least } \mathbf{P}_{U_{\delta_{\eta+1}}} - \text{good name } \dot{x} \text{ for an element of}\\ &(\theta\times\mathcal{P}(\kappa))^{<\xi}\cap\mathbf{V}[G\restriction\delta_{\eta+1}] \text{ such that } (\dot{x})^{\mathbf{V}[G\restriction\delta_{\eta+1}]}\in\mathcal{A}\end{aligned}$$

where

$$\mathcal{A} :\stackrel{\text{def}}{=} \{x \in \mathcal{M}_\xi(G\restriction\delta_{\eta+1}) \mid x_\alpha \subset x\}.$$

Then, we set

$$x_\beta :\stackrel{\text{def}}{=} (\dot{x}_\beta)^{\mathbf{V}[G\restriction\delta_\beta]}.$$

CASE II : β is a limit ordinal.

By the choice of $\lhd$ and the above construction, we conclude that

$$\langle x_\alpha \mid \alpha < \beta\rangle \in \mathbf{V}[G\restriction\delta^*],$$

where $\delta^* = \sup\{\delta_{\eta_\alpha} \mid \alpha < \zeta\}$.

So, we consider $x = \bigcup_{\zeta<\beta} x_\zeta$. According to the inductive hypothesis, z is a well – defined $\mathbf{R}$ – descending chain in $\mathbf{V}[G\restriction \delta^\star]$.

Let $\eta_\beta :\stackrel{\text{def}}{=} \sup\{\eta\zeta \mid \zeta < \beta\}$.

Using the definition of $\langle \delta_\zeta \mid \zeta < \xi \rangle$, we can define the following:

$$\dot{x}_\beta \quad :\stackrel{\text{def}}{=} \triangleleft - \text{least } \mathbf{P}_{U_{\delta_{\eta_\beta}}} - \text{good name } \dot{x} \text{ for an element of}$$
$$(\theta \times \mathcal{P}(\kappa))^{<\xi} \cap \mathbf{V}[G\restriction \delta_{\eta_\beta}] \text{ such that } (\dot{x})^{\mathbf{V}[G\restriction \delta_{\eta_\beta}]} \in \mathcal{B},$$

where

$$\mathcal{B} :\stackrel{\text{def}}{=} \{z \in \mathcal{M}_\xi(G\restriction \delta_{\eta_\beta}) \mid x \subset z\}.$$

Then, we set

$$x_\beta :\stackrel{\text{def}}{=} (\dot{x}_\beta)^{\mathbf{V}[G\restriction \delta_{\eta_\beta}]}.$$

It is easy to verify that in both cases the x_β satisfies the inductive hypothesis.

Therefore, we can construct in $\mathbf{V}[G\restriction \delta^\star]$ the sequence

$$\langle x_\zeta \mid \zeta < \xi \rangle$$

with the following two properties:

(A) $(\forall \zeta < \xi)(\exists \eta)(x_\zeta \in \mathcal{M}_\xi(G\restriction \delta_\eta))$.

(B) $(\forall \gamma < \gamma' < \xi)(x_\gamma \subset x_{\gamma'})$.

Thus, if we set

$$w :\stackrel{\text{def}}{=} \bigcup_{\zeta<\xi} x_\zeta,$$

using the properties (A) and (B) we would have that w is an $\mathbf{R}-$ descending chain of length ξ in $\mathbf{V}[G\restriction\delta^\star]$ and consequently is in $\mathbf{V}$(H). But the latest is contrary to our hypothesis that no such chain exists in $\mathbf{V}$(H) and so we get a contradiction.

Therefore, $<\lambda-\mathrm{DC}_{\theta\times\mathrm{H}}$ holds in $\mathbf{V}$(H).

4.A.5 Theorem □

4.A.6 Theorem *The model* $\mathbf{V}$(H) *satisfies* $<\lambda-$ DC.

Proof: Let $\mathbf{R}$ be a partial order in $\mathbf{V}$(H). If $\mathrm{S}:\overset{\mathrm{def}}{=}$ field($\mathbf{R}$), then by the definition of $\mathbf{V}$(H) we conclude that for every element x of S there exists an element z of $\mathbf{V}$, a member w of H and a formula ϕ such that x is definable in $\mathbf{V}$(H) by ϕ with parameters z and w. Then, using Replacement in $\mathbf{V}$(H), we can find an ordinal α with the property that each element of S is definable over $\mathbf{V}$(H) with parameters from $\mathbf{V}$, which appear in $\mathrm{V}_\alpha^{\mathbf{V}(\mathrm{H})}$. But then all the $\mathbf{V}-$ parameters necessary for the definitions of the elements of S appear in $\mathrm{V}_\alpha^{\mathbf{V}}$. Using this fact we define in $\mathbf{V}$(H) a surjection $\pi:\mathrm{V}_\alpha^{\mathbf{V}}\times\mathrm{H}\overset{\mathrm{onto}}{\longrightarrow}\mathrm{S}$. Since $\mathbf{V}\models\mathrm{AC}$, then we find in $\mathbf{V}$ a well – ordering of $\mathrm{V}_\alpha^{\mathbf{V}}$, say $\langle \mathrm{x}_\alpha \mid \alpha<\theta\rangle$ is an enumeration of $\mathrm{V}_\alpha^{\mathbf{V}}$ in $\mathbf{V}$ for some ordinal θ.

Next, we define a partial order $\mathbf{R}^*$ on $\theta\times\mathrm{H}$ by lifting $\mathbf{R}$ via π,

i.e., $\langle\gamma,\mathrm{a}\rangle\mathbf{R}^*\langle\delta,\mathrm{b}\rangle:\overset{\mathrm{def}}{\Longleftrightarrow}\pi(\langle\mathrm{x}_\gamma,\mathrm{a}\rangle)\mathbf{R}\pi(\langle\mathrm{x}_\delta,\mathrm{b}\rangle)$. Since π is a surjection, then for every $\mu<\lambda$, every maximal $\mathbf{R}^*-$descending chain of length $<\mu$ induces via π a maximal $\mathbf{R}-$descending chain of length $<\mu$. Then, we apply $<\lambda-\mathrm{DC}_{\theta\times\mathrm{H}}$

in $\mathbf{V}(\mathrm{H})$ for $\mathbf{R}^*$ to prove that $< \lambda -$ DC holds for $\mathbf{R}$ in $\mathbf{V}(\mathrm{H})$. This is possible, since the $\pi-$image of an $\mathbf{R}^*-$descending chain of length μ is an $\mathbf{R}-$descending chain of length μ.

Therefore, $< \lambda -$ DC holds in $\mathbf{V}(\mathrm{H})$.

4.A.6 Theorem □

Next, we are going to generically force choice over the model $\mathbf{V}(\mathrm{H})$. *Since this is equivalent to adding generically a* well – ordering *of the set H, we use in* $\mathbf{V}(\mathrm{H})$ *the following canonical forcing* $\mathcal{P}$.

4.A.7 Definition *For all* $\mathcal{A}$, *we define in* $\mathbf{V}(\mathrm{H})$

$$\mathcal{A} \in \mathcal{P} :\overset{\text{def}}{\Longleftrightarrow} \mathcal{A} \text{ is a function} \quad \& \quad \mathrm{dom}(\mathcal{A}) < \lambda \quad \& \quad \mathrm{ran}(\mathcal{A}) \subsetneq \mathrm{H}.$$

The order on the set $\mathcal{P}$ *is the reverse inclusion, i.e.,*

$\mathcal{A} \leq \mathcal{B} :\overset{\text{def}}{\Longleftrightarrow} \mathcal{B} \subseteq \mathcal{A}$, *and we denote the poset* $\langle \mathcal{P}, \leq \rangle$ *again by* $\mathcal{P}$.

4.A.1 Definition □

4.A.8 Lemma *Let* $\mathrm{x} \in \mathcal{P}(\kappa) \cap \mathbf{V}(\mathrm{H}) (= \mathrm{H})$.

We set $\mathrm{D_x} = \{\mathcal{A} \in \mathcal{P} \mid \mathrm{x} \in \mathrm{ran}\mathcal{A}\}$ *and for* $\beta < \lambda$

$\mathrm{E}_\beta = \{\mathcal{A} \in \mathcal{P} \mid \beta \geq \mathrm{lh}(\mathcal{A})\}$. *Then,* $\mathrm{D_x}$ *and* E_β *are* $\mathcal{P} -$ dense *in* $\mathbf{V}(\mathrm{H})$.

Proof: Let $\mathcal{B} \in \mathcal{P}$ and assume that $\mathrm{x} \notin \mathrm{ran}(\mathcal{B})$. Since $\alpha :\overset{\text{def}}{=} \mathrm{dom}(\mathcal{B}) < \lambda$, we set $\mathcal{A} = \mathcal{B} \cup \{\langle \alpha, \mathrm{x} \rangle\}$. Obviously, $\mathcal{A} \in \mathcal{P}$, $\mathcal{A} \leq \mathcal{B}$ and $\mathrm{x} \in \mathrm{field}(\mathcal{A})$. Therefore, $\mathrm{D_x}$ is $\mathcal{P} -$ dense.

In order to show that E_β is $\mathcal{P}$ – dense for a fixed β below λ, we consider a condition $\mathcal{B} = \langle \mathrm{x}_\xi \mid \xi < \alpha \rangle$. Since $\alpha < \lambda$ then $|\alpha| \leq \kappa$. We pick an injection g from α into κ. Using g we code $\mathcal{B}$ by the set $\mathrm{a} = \bigcup_{\xi < \alpha} \{\mathrm{g}(\xi)\} \times \mathrm{x}_\xi$.
Then $\mathrm{a} \subset \kappa \times \kappa$ and $\mathrm{a} \in \mathbf{V}(\mathrm{H})$. This implies that $\mathrm{a} \in \mathbf{V}[\mathrm{G}\restriction\gamma]$ for some $\gamma \in [\kappa, \lambda)$. But then we can find a large enough $\delta \in [\kappa, \lambda)$ above γ such that $\mathbf{V}[\mathrm{G}\restriction\delta] \models (|\beta| = \kappa)$. Therefore, in $\mathbf{V}[\mathrm{G}\restriction\delta]$ we can find a sequence b of sets in $\mathcal{P}(\kappa) \setminus \mathrm{ran}(\mathcal{B})$ of length β. Then $\mathcal{B}^\frown\mathrm{b}$ defines a condition $\mathcal{A}$ in $\mathcal{P}$ of length $\geq \beta$. Moreover, $\mathcal{A} \leq \mathcal{B}$ and $\mathcal{A} \in \mathrm{E}_\beta$. Hence, E_β is $\mathcal{P}$ – dense.

4.A.8 Lemma □

Let $\mathcal{G}$ be a $\mathcal{P}$ – generic filter over $\mathbf{V}(\mathrm{H})$.
Then, $(\forall \mathrm{x} \in \mathcal{P}(\kappa) \cap \mathbf{V}(\mathrm{H}))(\mathrm{D}_\mathrm{x} \cap \mathcal{G} \neq \emptyset)$. We set $\mathbf{g} \stackrel{\mathrm{def}}{=} \bigcup \mathcal{G}$. Since $\mathcal{G}$ is a filter, then $\mathbf{g}$ is a well-defined map with $\mathrm{ran(g)} \subseteq \mathcal{P}(\kappa)$ and domain contained in λ. By **4.A.8**, we have that $\mathrm{ran(g)} = \mathcal{P}(\kappa) \cap \mathbf{V}(\mathrm{H})$ & $\mathrm{dom(g)} = \lambda$. Thus any $\mathcal{P}$ – generic over $\mathbf{V}(\mathrm{H})$ filter $\mathcal{G}$ induces a surjection $\mathbf{g}$ such that $\mathbf{g} : \lambda \to \mathrm{H}$. Therefore, H is well – ordered in $\mathbf{V}(\mathrm{H})[\mathcal{G}]$.

Next, we will show that $\mathcal{P}$ is an almost homogeneous forcing. If $\mathcal{A}, \mathcal{B} \in \mathcal{P}$ with domains μ and ν respectively, then $\mathcal{A}^\frown\mathcal{B}$ denotes the concatenation of the two sequences, i.e.,

$$\mathcal{A}^\frown\mathcal{B}(\alpha) = \begin{cases} \mathcal{A}(\alpha), & \text{if } \alpha < \mu \\ \mathcal{B}(\beta), & \text{if } \alpha = \beta + \mu. \end{cases}$$

4.A.9 Proposition *The forcing $\mathcal{P}$ is almost homogeneous.*

Proof: The almost homogeneity of the forcing $\mathcal{P}$ is equivalent to the following fact:

> *For every condition* $\rho \in \mathcal{P}$ *and every* $\mathcal{P}-$generic *filter* $\mathcal{G}$ *there exists another* $\mathcal{P}$ – generic *filter* $\mathcal{G}^*$, *with the property that* $\mathbf{V}[\mathcal{G}] = \mathbf{V}[\mathcal{G}^*]$ *and* $\mathrm{p} \in \mathcal{G}^*$.

Following the above characterization of the homogeneity, we consider $\mathcal{A}$ to be a condition in $\mathcal{P}$ and we fix $\mathcal{G}$ to be a $\mathcal{P}$ – generic filter over $\mathbf{V}(\mathrm{H})$. Let $\mathbf{g}$ denote the generic surjection $\mathbf{g} : \lambda \xrightarrow{\text{onto}} \mathcal{P}(\kappa) \cap \mathbf{V}(\mathrm{H})$, induced by $\mathcal{G}$. Obviously, $\mathcal{G}$ is the set of all initial segments of $\mathbf{g}$. Let $\mu = \mathrm{dom}(\mathcal{A})$. Using $\mathbf{g}$ and $\mathcal{A}$, we define a surjection $\mathbf{g}^*$ of length λ onto H, as follows:

$$\mathbf{g}^*(\alpha) = \begin{cases} \mathcal{A}(\alpha), & \text{if } \alpha < \mu \\ \mathbf{g}(\beta), & \text{if } \alpha = \beta + \mu. \end{cases}$$

Since $\mathbf{g}$ is a surjection of λ onto H, then so is $\mathbf{g}^*$ and moreover if $\mathcal{G}^*$ is the $\mathcal{P}$ – filter induced by $\mathbf{g}^*$, then $\mathcal{A} \in \mathcal{G}^*$. It remains to be seen that $\mathcal{G}^*$ is $\mathcal{P}$ – generic over $\mathbf{V}(\mathrm{H})$. We argue towards a contradiction and we assume that $\mathcal{G}^*$ is not $\mathcal{P}$ – generic.

First, we define $\dot{\tau} \stackrel{\text{def}}{=} \left\{ \langle (\mathcal{A}^\frown\mathcal{B})\check{}, \mathcal{B} \rangle \mid \mathcal{B} \in \mathcal{P} \right\}$. Then, $(\dot{\tau})^{\mathbf{V}(\mathrm{H})[\mathcal{G}]} = \mathcal{G}^*$.

If D is a $\mathcal{P}$ – dense set, such that $\mathcal{G}^* \cap \mathrm{D} = \emptyset$, then we can find a condition $\mathcal{B} \in \mathcal{G}^*$, such that $\mathcal{B} \Vdash_{\mathcal{P}} \dot{\tau} \cap \check{\mathrm{D}} = \emptyset$. Then, we consider the condition $\mathcal{A}^\frown\mathcal{B} \in \mathcal{P}$ and by the density of D we can find a sequence $\mathcal{C}$, with the property that $\mathcal{A}^\frown\mathcal{B}^\frown\mathcal{C} \in \mathrm{D}$. Using the definition of $\dot{\tau}$, we verify that for every $\mathcal{P}$ – generic filter $\mathcal{G}'$ that contains $\mathcal{B}^\frown\mathcal{C}$ we have that $\mathcal{A}^\frown\mathcal{B}^\frown\mathcal{C} \in (\dot{\tau})^{\mathbf{V}(\mathrm{H})[\mathcal{G}']} \cap \mathrm{D}$. Hence,

$\mathcal{B}^\frown\mathcal{C} \underset{\mathcal{P}}{\Vdash} \dot\tau \cap \check{\mathrm{D}} \neq \emptyset$. However, as $\mathcal{B}^\frown\mathcal{C} \leq \mathcal{B}$ then $\mathcal{B}^\frown\mathcal{C} \underset{\mathcal{P}}{\Vdash} \dot\tau \cap \check{\mathrm{D}} = \emptyset$, which is a contradiction.

Therefore, $\mathcal{G}^*$ is $\mathcal{P}-$generic and $\mathcal{A} \in \mathcal{G}^*$. In addition, $\mathbf{V}[\mathcal{G}] = \mathbf{V}[\mathcal{G}^*]$ because $\mathcal{G}$ and $\mathcal{G}^*$ easily describe one another. Thus $\mathcal{P}$ is an almost homogeneous forcing.

4.A.9 Proposition □

Next we show that forcing with $\mathcal{P}$ over $\mathbf{V}(\mathrm{H})$ does not add any new $<\kappa-$sequences of κ. First, using the regularity of λ in $\mathbf{V}(\mathrm{H})$ and the definition of the $\mathcal{P}-$ forcing conditions, we prove the next proposition.

4.A.10 Proposition *The forcing* $\mathcal{P}$ *is* $<\lambda -$ closed *in* $\mathbf{V}(\mathrm{H})$.

4.A.10 Proposition □

4.A.11 Proposition *Let* $\mathcal{G}$ *be* $\mathcal{P}-$generic *over* $\mathbf{V}(\mathrm{H})$ *and set* $\mathbf{V}_1 :\overset{\mathrm{def}}{=} \mathbf{V}(\mathrm{H})[\mathcal{G}]$. *Then, for every regular cardinal* $\mu < \kappa$ *we have that* $\kappa^\mu \cap \mathbf{V}(\mathrm{H}) = \kappa^\mu \cap \mathbf{V}_1$.

Proof: Let $\mathrm{f} \in \kappa^\mu \cap \mathbf{V}_1$. We show that such f belongs to the ground model $\mathbf{V}(\mathrm{H})$. Let $\dot\tau$ be a $\mathcal{P}$ -name for f and x* be a condition such that

$$\mathrm{x}^* \underset{\mathcal{P}}{\Vdash} [\dot\tau \text{ is a function from } \mu \text{ into } \kappa].$$

We define $\mathbf{R} \subset (\mathcal{P} \times \mu) \times (\mathcal{P} \times \mu)$ to be the following relation:

$$\mathbf{R}(\langle \mathrm{x}, \alpha\rangle, \langle \mathrm{y}, \beta\rangle) :\overset{\mathrm{def}}{\Longleftrightarrow} \quad \mathrm{x} \leq \mathrm{y} \ \ \& \ \ \neg(\mathrm{y} \leq \mathrm{x} \leq \mathrm{x}^*) \ \ \& \ \ \alpha > \beta \ \ \& \\ \& \ \ (\forall \eta < \alpha)(\exists \gamma < \kappa)(\mathrm{x} \underset{\mathcal{P}}{\Vdash} \dot\tau(\check\eta) = \check\gamma).$$

By the definability of the forcing $\mathcal{P}$ in $\mathbf{V}(\mathrm{H})$, we conclude that $\mathbf{R} \in \mathbf{V}(\mathrm{H})$. We also claim that there exists no maximal $\mathbf{R}$ – descending chain of length $< \mu$. To verify this claim, we argue as follows:

If $\mathrm{a} = \langle \mathrm{a}_\xi \mid \xi < \eta \rangle$ is any $\mathbf{R}$ – descending chain for $\eta < \mu$ and if $\mathrm{a}_\xi = \langle \mathrm{x}_\xi, \alpha_\xi \rangle$, then by the regularity of μ we find $\beta = \sup\{\alpha_\xi \mid \xi < \eta\}$ below μ. Also, since $\mathcal{P}$ is $< -\lambda$ closed, then there exists a condition x in $\mathcal{P}$ below every x_ξ for $\xi < \eta$. Since $\mathrm{x}^* \Vdash_{\mathcal{P}} [\dot{\tau}$ is a function from μ into $\kappa]$, then there exists a $\mathrm{y} \in \mathcal{P}$ $\mathrm{y} \leq \mathrm{x}$ and a $\delta < \kappa$ such that $(\mathrm{y} \Vdash_{\mathcal{P}} \dot{\tau}(\check{\beta}) = \check{\delta})$. Hence,

$$(\forall \xi < \eta)(\mathbf{R}(\langle \mathrm{y}, \beta + 1 \rangle, \mathrm{a}_\xi).$$

This shows that $\mathrm{a} = \langle \mathrm{a}_\xi \mid \xi < \eta \rangle$ is not an $\mathbf{R}$-descending chain.

Therefore, $\mathbf{R}$ has no maximal descending chain of length $< \mu$.

Since $< \lambda$ – DC holds, then in $\mathbf{V}(\mathrm{H})$ there exists an $\mathbf{R}$ – descending chain of length μ. Let $\mathcal{D} = \langle \langle \mathrm{x}_\xi, \alpha_\xi \rangle \mid \xi < \mu \rangle$ be any such chain. Since λ, μ are regular in $\mathbf{V}(\mathrm{H})$, then the sequence $\langle \alpha_\xi \mid \xi < \mu \rangle$ is cofinal in μ.

Then there exists a sequence $\langle \beta_\xi \mid \xi < \mu \rangle$ of ordinals below κ in $\mathbf{V}(\mathrm{H})$, such that

$$(\forall \xi < \mu)(\mathrm{x}_\xi \Vdash_{\mathcal{P}} \dot{\tau}(\check{\xi}) = \check{\beta}_\xi).$$

If x is a condition of $\mathcal{P}$ below every x_ξ (it is is possible to find such x since $\mathcal{P}$ is $< \lambda$-closed), then

$$(\forall \xi < \mu)(\mathrm{x} \Vdash_{\mathcal{P}} \dot{\tau}(\check{\xi}) = \check{\beta}_\xi).$$

If we repeat this argument below any condition of $\mathcal{P}$, then we show that

$$\forall \mathrm{p} \leq \mathrm{x}^* \exists \mathrm{q} \in \mathcal{P}(\mathrm{q} \leq \mathrm{p} \quad \& \quad \exists \mathrm{h} \in \kappa^\mu \cap \mathbf{V}(\mathrm{H})\ \mathrm{q} \Vdash_{\mathcal{P}} \dot{\tau} = \check{\mathrm{h}}).$$

Therefore,

$$\{\mathrm{q} \mid \mathrm{q} \Vdash_{\mathcal{P}} \dot{\tau} \in \dot{\mathbf{V}}\}$$

is pre – dense below x^* in $\mathcal{P}$ and this implies that

$$\dot{\tau}^{\mathbf{V}(\mathrm{H})[\mathcal{G}]} \in \mathbf{V}(\mathrm{H}),$$

i.e., $\mathrm{f} \in \mathbf{V}(\mathrm{H})$. Hence, $\kappa^\mu \cap \mathbf{V}(\mathrm{H}) = \kappa^\mu \cap \mathbf{V}_1$ for every regular cardinal below κ.

4.A.11 Proposition □

4.A.12 Corollary *If* $\mathbf{V}_1 :\stackrel{\text{def}}{=} \mathbf{V}(\mathrm{H})[\mathcal{G}]$, *then for every* $\mu < \kappa$ $\mathrm{V}_\mu^{\mathbf{V}(\mathrm{H})} = \mathrm{V}_\mu^{\mathbf{V}_1}$.

Proof: This follows by induction on $\xi < \kappa$, by using **4.A.10** and the fact that $(\forall \zeta < \kappa)[\mathrm{V}_\zeta^{\mathbf{V}} = \mathrm{V}_\zeta^{\mathbf{V}(\mathrm{H})}]$.

4.A.12 Corollary □

4.B A ZFC model of the partition relation

In the following section we are going to show that for any $\mathcal{G}$ which is $\mathcal{P}-$generic over $\mathbf{V}(\mathrm{H})$, $\mathbf{V}(\mathrm{H})[\mathcal{G}]$ is a model of **ZFC** which satisfies the partition relation:

$$\forall\mu < \kappa(\kappa \underset{\mathrm{OD}}{\rightarrow} (\omega)^{\omega}_{\mathbf{V}_\mu}) \quad \& \quad \kappa \text{ is strong limit.}$$

4.B.1 Proposition *Let* $\mathbf{V}$ *be a transitive model of* **ZFC**, *in which* κ *is a supercompact cardinal and* λ *is an inaccessible cardinal above* κ. *Let* $\mathcal{G}$ *be a fixed* $\mathcal{P}$ – generic *filter over* $\mathbf{V}(\mathrm{H})$, *where* $\mathcal{P}$ *is the forcing defined in* **4.A.7**. *Let* $\mathbf{V}_1 :\stackrel{\mathrm{def}}{=} \mathbf{V}(\mathrm{H})[\mathcal{G}]$. *Then,*

$$\mathbf{V}_1 \models \mathbf{ZFC} + \kappa \text{ is a strong limit.}$$

Proof: Since $\mathbf{V}_1$ is a generic extension of a model of ZF, namely $\mathbf{V}(\mathrm{H})$, then $\mathbf{V}_1$ is itself a model of ZF. Moreover, as we have shown in **4.A.12**

$$(\forall\mu < \kappa)(\mathrm{V}_\mu^{\mathbf{V}_1} = \mathrm{V}_\mu^{\mathbf{V}(\mathrm{H})})$$

and as κ is a strong limit cardinal in $\mathbf{V}(\mathrm{H})$, then

$$(\forall\mu < \kappa)(|\mathrm{V}_\mu^{\mathbf{V}_1}| < \kappa),$$

i.e., κ is a strong limit cardinal in $\mathbf{V}_1$.

So it remains to be seen that $\mathbf{V}_1 \models$ AC. The proof of $\mathbf{V}_1 \models$ AC is based upon the following three claims.

4.B.1.1 Claim *If $\dot\tau \in \mathbf{V}(\mathrm{H})^{\mathcal{P}}$ and $\mathcal{S} = \mathrm{dom}(\dot\tau) \subseteq \mathbf{V}(\mathrm{H})^{\mathcal{P}}$, then $\mathrm{e}\restriction\mathcal{S} : \mathcal{S} \to \mathrm{e}(\dot\tau)$ is onto and belongs to $\mathbf{V}_1$, where e is the interpretation map*

$$\mathrm{e}(\dot{\mathrm{x}}) = \dot{\mathrm{x}}^{\mathbf{V}(\mathrm{H})[\mathcal{G}]} = \dot{\mathrm{x}}^{\mathbf{V}_1}$$

for any $\dot{\mathrm{x}} \in \mathbf{V}(\mathrm{H})^{\mathcal{P}}$.

Proof of the Claim: This is a well-known fact that appears also in [Jech 78].

4.B.1.1 Claim □

4.B.1.2 Claim *If $\mathcal{M}$ is a transitive model of ZF and f is a surjection* $\mathrm{f} : \mathrm{S} \to \mathrm{T}$ *and* $\mathrm{S}, \mathrm{T} \in \mathcal{M}$ *with S being* well – ordered, *then T is* well – orderable.

Proof of the Claim: For any $\mathrm{x} \in \mathrm{T}$ let $\mathcal{Y}_\mathrm{x} = \mathrm{f}^{-1}\{\mathrm{x}\}$. Since $\mathrm{f}, \mathrm{T} \in \mathcal{M}$ then, $\langle \mathcal{Y}_\mathrm{x} \mid \mathrm{x} \in \mathrm{T}\rangle \in \mathcal{M}$. Let $\lhd$ be a well – ordering on S.
Let $\mathrm{z}_\mathrm{x} :\stackrel{\mathrm{def}}{=}$ the $\lhd$ –least element of Y_x. So $\langle \mathrm{z}_\mathrm{x} \mid \mathrm{x} \in \mathrm{T}\rangle \in \mathcal{M}$. Next, we well – order T by $\leq$ such that for $\mathrm{x}, \mathrm{x}' \in \mathrm{T}$ $\mathrm{x} \leq \mathrm{x}' :\stackrel{\mathrm{def}}{\Longleftrightarrow} \mathrm{z}_\mathrm{x} \lhd \mathrm{z}'_\mathrm{x}$. Obviously, $\leq$ is a well – order of T.

4.B.1.2 Claim □

4.B.1.3 Claim *For every* $\mathrm{S} \in \mathbf{V}(\mathrm{H})$, $\mathbf{V}_1 \models \mathrm{S}$ is well – orderable.

Proof of the Claim: We fix an $\mathrm{S} \in \mathbf{V}(\mathrm{H})$ and by the definition of $\mathbf{V}(\mathrm{H})$ we conclude that for every element x of S there exists an element z of $\mathbf{V}$, a member w of H and a formula ϕ such that x is definable in $\mathbf{V}(\mathrm{H})$ by ϕ

with parameters z and w. Then, using Replacement in $\mathbf{V}(\mathrm{H})$, we can find an ordinal α with the property that each element of S is definable over $\mathbf{V}(\mathrm{H})$ with parameters from $\mathbf{V}$, which appear in $\mathrm{V}_\alpha^{\mathbf{V}(\mathrm{H})}$. But then all the $\mathbf{V}$ – parameters necessary for the definitions of the elements of S appear in $\mathrm{V}_\alpha^{\mathbf{V}}$. Using this fact we define in $\mathbf{V}(\mathrm{H})$ a surjection $\pi : \mathrm{V}_\alpha^{\mathbf{V}} \times \mathrm{H} \stackrel{\text{onto}}{\longrightarrow} \mathrm{S}$. Since $\mathbf{V}$ satisfies the Axiom of Choice and since H is well – ordered in $\mathbf{V}_1$, then we conclude that $\mathrm{V}_\alpha^{\mathbf{V}} \times \mathrm{H}$ is well – ordered in $\mathbf{V}_1$. Then, by **4.B.1.2**, we get that S is well – ordered in $\mathbf{V}_1$.

4.B.1.3 Claim □

To finish the proof of the proposition, we consider an $\mathcal{X} \in \mathbf{V}_1$ and we show that $\mathcal{X}$ can be well – ordered. We let $\dot{\mathcal{X}}$ be a $\mathbf{V}(\mathrm{H})^{\mathcal{P}}$-name for $\mathcal{X}$, i.e., $\mathrm{e}(\dot{\mathcal{X}}) = \dot{\mathcal{X}}^{\mathbf{V}_1} = \mathcal{X}$ and we set $\mathcal{S} :\stackrel{\text{def}}{=} \mathrm{dom}(\dot{\mathcal{X}})$.
Then, $\mathcal{S} \in \mathbf{V}(\mathrm{H})$ and by **4.B.1.1** we have that $\mathrm{e}{\restriction}\mathcal{S} \in \mathbf{V}_1$ and

$$\mathrm{e}{\restriction}\mathcal{S} : \mathcal{S} \to \mathcal{X} \text{ is onto.}$$

Since $\mathbf{V}_1 \models \mathrm{ZF}$ and $\mathcal{S} \in \mathbf{V}(\mathrm{H})$, then by **4.B.1.2** we get that $\mathcal{S}$ is well–ordered in $\mathbf{V}_1$ and thus according to **4.B.1.3** $\mathcal{X}$ is well – ordered in $\mathbf{V}_1$. So, this concludes the proof of $\mathbf{V}_1 \models \mathrm{AC}$. Therefore,

$$\mathbf{V}_1 \models \mathrm{ZFC} + \kappa \text{ is a strong limit cardinal.}$$

4.B.1 Proposition □

4.B.2 Theorem *Let* $\mathbf{V}$ *be a transitive model of* **ZFC**, *in which* κ *is a supercompact cardinal and* λ *is an inaccessible cardinal above* κ. *Let* $\mathcal{G}$ *be a fixed* $\mathcal{P}$ – generic *filter over* $\mathbf{V}(\mathrm{H})$, *where* $\mathcal{P}$ *is the forcing defined in* **4.A.7**. *If*

$$\mathbf{V}_1 :\stackrel{\mathrm{def}}{=} \mathbf{V}(\mathrm{H})[\mathcal{G}],$$

then

$$\mathbf{V}_1 \models [\mathrm{ZFC} + \kappa \text{ is a strong limit } + \forall \mu < \kappa(\kappa \underset{\mathbf{OD}}{\rightarrow} (\omega)^{\omega}_{\mathrm{V}_\mu})].$$

Proof: We fix $\mu < \kappa$ and we consider an OD coloring $\mathrm{F} \in \mathbf{V}_1$

$$\mathrm{F} : [\kappa]^\omega \to \mathrm{V}_\mu^{\mathbf{V}_1}.$$

We have already seen that

$$[\kappa]^\omega \cap \mathbf{V}_1 = [\kappa]^\omega \cap \mathbf{V}(\mathrm{H})$$

and

$$\mathrm{V}_\mu^{\mathbf{V}_1} = \mathrm{V}_\mu^{\mathbf{V}(\mathrm{H})}.$$

So we assume that there exists a formula ϕ and ordinal parameters $\vec{\mathrm{s}}$ such that for all $\mathrm{s} \in [\kappa]^\omega$ and $\mathrm{x} \in \mathrm{V}_\mu^{\mathbf{V}(\mathrm{H})}$

$$\mathrm{F}(\mathrm{s}) = \mathrm{x} \iff \mathbf{V}(\mathrm{H})[\mathcal{G}] \models \phi(\mathrm{s}, \mathrm{x}, \vec{\delta}).$$

Since $\mathrm{s}, \mathrm{x} \in \mathbf{V}(\mathrm{H})$ and $\mathcal{P}$ is homogeneous, then

$$\begin{aligned} \mathrm{F}(\mathrm{s}) = \mathrm{x} \quad &\iff \mathbf{V}(\mathrm{H})[\mathcal{G}] \models \phi(\mathrm{s}, \mathrm{x}, \vec{\delta}) \\ &\iff \mathbf{1}_{\mathcal{P}} \underset{\mathcal{P}}{\Vdash} \phi(\check{\mathrm{s}}, \check{\mathrm{x}}, \check{\vec{\delta}}). \end{aligned}$$

Since the forcing $\mathcal{P}$ is definable by H in $\mathbf{V}(\mathrm{H})$ and by the definability of the forcing relation $\underset{\mathcal{P}}{\Vdash}$, then there exists a formula ϕ^* such that

$$\begin{aligned} \mathrm{F}(\mathrm{s}) = \mathrm{x} \quad & \Longleftrightarrow \mathbf{1}_{\mathcal{P}} \underset{\mathcal{P}}{\Vdash} \phi(\check{\mathrm{s}}, \check{\mathrm{x}}, \vec{\delta}) \\ & \Longleftrightarrow \mathbf{V}(\mathrm{H}) \models \phi^*(\mathrm{s}, \mathrm{x}, \vec{\delta}, \mathrm{H}) \end{aligned}$$

and

$$\mathbf{V}(\mathrm{H}) \models (\forall \mathrm{s} \in [\kappa]^\omega)(\exists! \mathrm{x})\phi^*(\mathrm{s}, \mathrm{x}, \vec{\delta}, \mathrm{H}).$$

By **3.C.1** there exists $\sigma \in [\kappa]^\omega$ and $\mathrm{x}_0 \in \mathbf{V}_\mu^{\mathbf{V}(\mathrm{H})}$ such that

$$\forall \tau \in [\sigma]^\omega \ \mathrm{F}(\tau) = \mathrm{x}_0.$$

This shows that F has an homogeneous set in $\mathbf{V}_1$. Hence,

$$\mathbf{V}_1 \models \kappa \underset{\mathrm{OD}}{\rightarrow} (\omega)^\omega_{\mathrm{V}_\mu}.$$

Then by **4.B.1** we get that

$$\mathbf{V}_1 \models \mathbf{ZFC} + \kappa \text{ is strong limit } + \forall \mu < \kappa (\kappa \underset{\mathrm{OD}}{\rightarrow} (\omega)^\omega_{\mathrm{V}_\mu}).$$

4.B.2 Theorem □

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