EQUILIBRIUM MODELS OF MULTIPLE-OBJECT AUCTIONS

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ABSTRACT

This dissertation uses two different game-theoretic models to explore properties of equilibria in multiple-object auctions and presents the results of an empirical test of one of them. The first chapter surveys the most important contributions to auction and bidding theory, discusses some questions which have not yet been answered satisfactorily and outlines some of the specific problems which must be addressed when studying multiple-object auctions as opposed to single-object auctions.

Chapter two examines the existence and characterization of pure strategy Nash equilibria in multiple-object auction games in which buyers face a binding constraint on exposure. There are five major results. First, symmetric Nash equilibria exist if and only if there are two or less buyers and two or less objects. Second, a Nash equilibrium may not exist if the seller sets a positive reservation bid. Third, asymmetric solutions to symmetrically parametrized games typically involve "high-low" strategies: buyers submit positive bids only on some restricted subset of the objects. Fourth, Nash equilibria typically generate zero "profits" to the buyers. Fifth, when asymmetric solutions exist and the buyers are identical, these solutions are never unique.

Chapter three examines the bundling decisions by a multiproduct monopolist with incomplete information about demand.

Previously the bundling problem has been analyzed only in a world of perfect and complete information in which the monopolist uses a standard take-it-or-leave-it pricing scheme. The model in chapter three shows that tied-in sales are sometimes *ex ante* optimal under a reasonable set of assumptions about a world in which there are no production economies or diseconomies and no demand interdependencies. A number of additional results were obtained deriving general sufficient conditions for buyers to prefer bundling, as well as conditions under which bundling is optimal in terms of maximizing expected consumer plus producer surplus.

Chapter four reports the results of an empirical examination of the predictions made in chapter three. Testable hypotheses were developed in that chapter which addressed questions about seller revenues, market efficiency, buyer behavior and distributional consequences of a monopolistic seller's bundling decision in multiple object auctions. The data provide strong support for these theoretically-based hypotheses.

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CHAPTER ONE

Single-object and Multiple-object Auctions: An Overview

I. INTRODUCTION

Auctions have played a key role in the history of economics by virtue of being one of the most commonly used mechanisms for organizing markets and allocating resources for thousands of years. Therefore, it is rather surprising that only in recent years have economists carefully studied this class of allocation mechanisms. If one searches the literature quite thoroughly one can find only occasional mention of auctions in the classic treatises[1] on economics. On the other hand, in the last twenty years literally hundreds of papers on auctions have been written and subsequently published in most of the professional journals of economics and operations research.[2] This recent interest can be viewed as part of the shift of direction in economic theory which one economist[3] has referred to as the "(new)^2 welfare economics," in which great emphasis is placed on questions about the role of particular institutions.

There is no question that auctions are important and universally observed market institutions. Cassady(1967) describes in great detail the diversity of auctions which have taken place from ancient times to the present all over the globe. The primary element shared in common by all these auction mechanisms is that one side of the market is relatively passive and each of several competing members of the other side of the market reveals solicited revelations of their respective willingness-to-pay (or costs, if the passive side of the market is a purchaser) for whatever set of products or services is being sold (or purchased).[4]

Most of the recent work on auctions has been motivated by three important auction markets:

(1) the sale by the U.S. government of treasury bills;

(2) the sale by the U.S. government of oil and gas development leases on the outer continental shelf; and

(3) the procurement of contracts by various governmental agencies (e.g. construction and defense contracts).

This apparent narrowness of motivation is a bit misleading. The applicability of predictions based on current theoretical models of acutions is by no means limited to these three markets. Furthermore, pure theoretical interest should probably be listed as an additional motivation for much of this research.

The types of questions which have been most frequently addressed fall into a number of different categories. The most common approach has been to show how the specific auction rules affect the strategies of competing bidders and the seller's revenue. Other efforts have tried to understand why auction institutions are observed in some markets and different pricing institutions in other markets.

One of the limitations which is shared by most, if not all, of the models is that they address auctions in which only one good is being sold. More typically a number of different goods are sold by an auctioneer to the same set of bidders. To the extent that characteristics of the goods or the preferences of the bidders create an interdependence between the auctions, models which do not explicitly take this interdependence into account may yield misleading conclusions. This dissertation presents two different game-theoretic models of multiple-object auctions and presents the results of an empirical test of one of them.

II. SINGLE-OBJECT AUCTIONS

Auctions are generally conducted either in an open or closed format. In "closed" auctions, such as sealed-bid auctions, prospective buyers submit private bids for the-different "lots" being offered. The private bids are then simultaneously revealed at a prespecified time and place and the highest bidder for each lot is awarded that lot for a price determined by the auction rules. In "open" auctions, all of the prospective buyers and the seller gather together at some prespecified time and place. When a lot is auctioned off, the floor is opened for bidding, bidding occurs for a period of time and then is terminated according to the rules of the auction. The last active bidder at the time of termination is awarded the item at whatever price he bid. The key difference is that in closed auctions bids are private, while in open auctions bids are public.

Two types of closed auction rules which are commonly used are the first-price sealed-bid auction and the second-price sealed-bid auction. In a first-price auction the winning bidder pays his bid and in a second- price auction the highest bidder pays the second-highest bid. Two common open auctions are the English (ascending-bid) type, in which buyers sequentially announce higher and higher bids until only one buyer is bidding, and the Dutch (descending-bid) type, in which the seller announces lower and lower prices until one of the buyers accepts (bids) the price.

An important observation by Vickrey(1961) and later generalized by others is that the first-price sealed-bid auction is strategically isomorphic to the Dutch auction and the second-price sealed-bid auction is isomorphic to the English auction. In order to explain exactly what is meant by strategically isomorphic it is necessary to outline the Vickrey model.

Suppose there are n competing buyers and a single object is being sold. Each buyer knows with certainty exactly what the object is worth to him. The information each buyer has about the valuations of the other buyers is much more limited. Specifically, each buyer knows only that each of the other buyers' valuations are independent random draws from some known probability distribution. The seller also knows the distribution but does not the exact valuation of any buyer. If each buyer maximizes the difference between his valuation and bid times the probability of winning,[5] given the other bidders' strategies, then the Bayesian equilibrium[6] in a first-price auction has each buyer bidding the expected second-highest valuation of the other (n-1) bidders conditional on his value being the highest. This is also the price at which the buyer should accept in a Dutch auction. Thus we obtain the first equivalence.

In a second-price auction each bidder has a dominant strategy[5] to bid his own valuation, just as in the English auction in which each bidder is always best off to keep raising the current bid until either it reaches his valuation or he is the sole remaining bidder. Thus we obtain the second equivalence.

Neither of these two isomorphisms depends on the assumption that buyers maximize expected profits rather than the expected utility of their profits. However, in this risk neutral case, a third equivalence is obtained, namely all four of the auction mechanisms will yield the same revenue to the seller. This last equivalence does not hold unless the bidders are risk neutral.

A very large fraction of the work which has been done since this observation was first made has concentrated on extending this result. For example, Ortega-Reichert(1968) shows that if a seller sets a reservation price at a level which maximizes expected profit, then he will set the same reserve level for a first-price auction and a Dutch auction and set the same level for second-price and English auctions. Thus the isomorphism still holds even when the seller sets reservation prices.

A related part of this literature addresses the revenue-generating properties of auctions. Harris and Raviv(1979a), Riley and Samuelson(1979), Myerson(1978) and Maskin and Riley(1979a) have all demonstrated that if all buyers are risk neutral there is no efficient auction mechanism which generates strictly more revenue for the seller than the four mechanisms discussed above. Holt(1979), Harris and Raviv(1979), Samuelson(1978) and Cox(1979) have verified an earlier conjecture by Vickrey(1961) that first-price auctions generate more revenue than second-price auctions when bidders are risk averse. Matthews(1979b) takes the analysis a step further, demonstrating that a risk neutral seller's profit maximizing reservation bid for a first-price auction is less than the corresponding reservation bid in a second-price auction. Consequently a first-price auction is more efficient ex ante than a second-price auction if the buyers are risk averse and the seller is risk neutral. This is also true if the buyers are risk neutral and the seller is risk averse.

The Vickrey model has also been used by Harris and Raviv(1979b) to explain the behavior of a monopolist facing uncertain demand. Suppose that the monopolist has q indivisible units of the same item and there are $n>q$ bidders, each of whom has a demand of exactly one unit.[8] Consider the following two auction mechanisms which are analogous to the first- and second-price auctions. According to the first mechanism, the q highest bidders are each awarded one unit of the item and are charged their actual bid; according to the second mechanism, the q highest bidders each pay the highest unaccepted bid. The former is often referred to as discriminatory and the latter competitive for obvious reasons. In either mechanism the same revenue is generated if the buyers are risk neutral. Moreover, if, in addition, the monopolist sets an optimal reservation price, then the expected revenue generated by either type of auction exceeds the expected revenue which could be generated by any other allocation mechanism. On the other hand, if $n<q$ the monopolist may earn more revenue by simply fixing a price and filling any orders which are requested at that price. Thus it is not surprising that we find in the real world auction markets existing for land, rare collectables, oil and gas drilling rights, and other commodities which have a fixed, low level of supply and are not easily divisible.

Using a similar approach, Maskin and Riley(1979b) show that if a commodity is perfectly divisible and buyers have linear demand curves with identical (known) slopes but random intercepts, then the seller's expected profit maximizing auction rule is to announce a specific non-linear price schedule ("quantity discounts") and solicit reported intercepts from each buyer. Naturally the functional form of the price schedule depends upon the distribution of buyers' intercepts. This model is of interest because it provides a convincing argument that sellers can price discriminate very effectively even without much information about buyer demand. In addition they were able to establish theoretically the frequently-heard claim that quantity discounts are just another tool which a monopolist uses to price discriminate.

All of the above models posit a specific type of uncertainty in which every bidder knows his own valuation with certainty. Wilson(1967) suggests that one of the important features in many auctions is that bidders do not even know their own true valuations with certainty.[9] However, prior to the auction bidders may have an opportunity to gather information which can improve their estimate of the true valuation of the object being sold. This introduces a type of dependence between buyers' valuations of the object which is not present under the Vickrey informational assumptions. Unfortunately, the model is much more complicated and properties of equilibria are more difficult to analyze. Nonetheless, in Wilson(1977), Matthews(1979) and Milgrom(1979) some results are obtained about the asymptotic properties of the auctions and about the effect of the information-gathering incentives on efficiency. Wilson(1977) and

Milgrom(1979) show that as the number of bidders becomes arbitrarily large, the winning bid in a first-price auction converges almost surely to the value of the object. Matthews(1979) shows that there is overinvestment in pre-auction information-gathering by the competing bidders.[10]

Wilson has also applied his formulation to analyze the revenue-generating properties of a number of mechanisms which the government has either considered using or actually used in its sale of outer continental shelf oil and gas leases. In particular he has examined royalty bidding, in which the winning firm will pay the government a share of its future stream of revenues, and share auctions (the so-called Phillips Plan) in which each of the bidders wins a portion of the object, the size of a buyer's portion being determined by the relative size of that buyer's bid. Wilson(1977c) offers some arguments suggesting that efficiency losses due to overinvestment in pre-auction information may be ameliorated under alternative auction procedures such as these. Whether or not this compensates for the revenue loss from the alternative procedures (see Wilson [1977b] and Wilson[1976b]) is an open question.[11]

There are a huge number of unanswered questions about even the simplest auctions involving either single units of a commodity or multiple units of a commodity with each buyer demanding at most one unit. For example, no one has come up with a compelling explanation for the existence of open auctions, which one could argue incur greater transaction costs to the seller. In fact, some empirical investigations (e.g. Coppinger, Smith, and Titus [1980]) suggest that revenues in first-price sealed-bid auctions generally exceed by a significant amount the revenues in and English or Dutch open auctions. No one has a good explanation for this either.

There are also many open questions about asymmetric auction games. Aside from some examples (Wilson[1967],Vickrey[1961]), few results have been obtained when buyers are not identical. This is partly due to the fact that asymmetric equilibria are very difficult to solve for. When buyers are not identical perverse equilibrium outcomes can occur where the highest bidder does not have the highest valuation. Cox(1979) has investigated a model in which buyers are identical except for different levels of constant relative risk aversion. Each buyer knows only the distribution of his competitors' risk parameters.[12] Not surprisingly, he finds that if buyers have different risk parameters the buyer with the highest valuation may not win, since more risk averse buyers will bid higher than less risk averse buyers.

Another open question involves markets in which both auctions and standard fixed-price schemes are used. For example, in markets for stamps and coins, art, rugs, antiques and other rare collectibles, one observes both auctions and standard fixed-price mechanisms. Moreover these markets are not generally considered to be monopolized--they are competitive. Thus one would expect auction prices and fixed-prices to be codetermined. In one direction of determination, prices in auctions are used as a signal for dealers to decide what fixed-price to charge. On the other hand many dealers purchase at auction and decide how much to bid on the basis of what they expect the resale value to be. Nowhere in the auction literature are questions about the effects of subsequent resale ever addressed carefully. Auctions have been modelled as if these later resale markets can be ignored.

Finally, it is only very recently that research effort has been directed toward the study of interdependencies between auctions. One of the reasons for this is that it is much more difficult to model several auctions together than to model each auction as a separate independent event. The next section discusses the complications which can arise in a multiple-object auction setting, briefly surveys the literature in the area and outlines some fruitful directions of new research on the subject.

III. MULTIPLE-OBJECT AUCTIONS

A seller may have several objects which are being sold by an auction mechanism. An auction of this sort will be referred to as a multiple-object auction. Except in very special cases which were discussed in the previous section,[13] the properties of these auctions are much different from the properties of single-object auctions. These differences arise from at least two sources. First, the number of mechanisms a seller may choose to use is considerably greater. Second, the equilibrium behavior of buyers may be different. As a result, the analysis of allocations resulting from different mechanisms is a more complicated task. This is probably the most obvious explanation for the scarcity of published research on multiple-object auctions.

If a seller wishes to auction off several objects, he must make a number of choices which are not made in single-object auctions. One of these decisions is whether to sell the objects sequentially or simultaneously. If they are sold sequentially, the buyers and the seller may gain information in earlier auctions which can affect decisions in later auctions. If they are sold simultaneously, neither the seller nor the buyers gain any information beyond that which each one was assumed to have started with. Thus one expects that quite generally the seller's revenue in a simultaneous multiple-object auction will differ substantially from the revenue in a sequentially operated multiple-object auction. A seller may adopt a selling strategy which is neither purely sequential nor purely simultaneous.

Some subset of the objects may be sold before other subsets of the items, each object in a given subset being sold simultaneously with every other object in that subset. Thus the seller must choose exactly how to partition the set of all objects into a sequence of subsets. Such decisions will be called sequencing decisions.

A similar type of choice the seller faces involves another type of partition, called a bundling decision. Objects may be sold as a bundle, or a "package deal", in which case bundles of objects are sold in an auction together rather than being sold separately in several auctions.

A somewhat different type of decision a seller must make is a message decision. The seller must specify what messages the buyers should report. In a single-object auction, the seller always specifies that the buyers report a scalar, which is just a (perhaps false) revelation of their willingness-to-pay for the object. In a multiple-object auction the seller may in principle solicit bids on every subset of the set of objects he is selling. If buyers' values for a bundle are not equal to a simple function of their values for each object in the bundle, then such messages will provide the seller with more information than could be gained just by soliciting bids for each single object. In such cases the allocation rule of the auction may depend upon the bids from this larger message space.

The buyers' decision problem is much more complicated in multiple-object auctions, and equilibrium strategies are generally quite different. One reason for this is that if a buyer's valuation for a bundle does not equal the sum of valuations for the objects in the bundle, then his decisions in one auction must be made contingent on his success in other auctions. This is the <u>non-additivity problem</u>. Therefore strategies will depend upon the sequencing decision, the bundling decision and the message decision of the seller. In the case of sequential multiple-object auctions, buyers must solve a dynamic program under uncertainty. In the case of simultaneous auctions, the value of winning one auction is dependent upon whether or not the buyer wins in other auctions. Because of this dependence, if the message space does not include reported valuations for all subsets of items, then dominant strategy auction mechanisms, such as some analogue of the second-price auction, will not generally exist.[14] This is the <u>dominant strategy problem</u>.

On the other hand, if the message space does include bids on all subsets of objects, then the number of bids each buyer submits becomes very large very quickly, since the number of subsets of objects increases exponentially with the total number of objects.[15] This is an improtant practical consideration and, for lack of a better term, it is called the <u>implementation problem</u>.

From the above, one can see that a multiple-object auction is a very much more complicated market to model than a single-object auction. Yet practically the only auctions ever observed are multiple-object auctions. Thus there is a good reason to question whether or not single-object auction models, which ignore sequencing decisions, bundling decisions, message decisions, and the non-additivity problem, can provide accurate predictions about auction markets.

A few efforts have been made to understand behavior in multiple-object auctions. Agnew(1972), Kortanek, Soden and Sodaro(1973), Cook, Kirby and Mehndiratta(1975) and Oren and Rothkopf(1975) have investigated one consequence of a seller's sequencing decision. Oren and Rothkopf(1975) set up a model in which the seller plays no active role and they use a decision-theoretic approach to solve for a single strategic buyer's optimal sequential bid strategy. The other buyers have fixed strategies which are simply a set of reaction functions which are responsive in a specific functional way to the strategic bidder's choice of bids. They are not optimizers, so equilibrium strategies cannot be explored using this model.[16] Agnew(1972) presents a learning model in which a single buyer uses information from earlier auctions to improve his prediction about the bidding behavior of his competitors. All of the above authors limit the strategy space to multiplicative strategies. That is, the lone strategic bidder always bids so that if he wins his profit is a constant multiple of his bid. Cook, Kirby and Mehndiratta(1975) and Kortanek, Soden and Sodaro(1973) investigate optimal sequential bidding behavior when the bidders face constraints limiting the number of auctions they can win. Kortanek, Soden and Sodaro(1973) use a decision-theoretic approach in which only one bidder is strategic. Their major result, as interpreted by Attanasi(1974), is that a bidder bids so that the margin of profit he makes if he wins is proportional to the reciprocal of the hazard rate of the distribution of his opponents' bids. Unfortunately, there is no reason to expect equilibrium bids to take this form since strategy spaces of the bidders are intertwined. This is the key observation made by Cook, Kirby and Mehndiratta(1975) and they attempt to characterize equilibrium strategies when there are exactly two competing bidders. They use a model in which the constraint may be violated sometimes as long as it is not violated "on average".

Sakaguchi(1962), Griesmer and Shubik(1963) and Rothkopf(1977) have studied simultaneous multiple-object auctions in which buyers face the constraint that the sum of their bids cannot exceed a certain number. This introduces an interdependence between auctions. Rothkopf(1977) uses a decision-theoretic approach and does not investigate equilibrium strategies. Stark and Mayer(1971) also provide a discussion of a number of decision-theoretic models along this same vein. Sakaguchi(1962) and Griesmer and Shubik(1963) investigate two-bidder Nash equilibria of simultaneous auctions with bidding constraints.

Engelbrecht-Wiggans and Weber(1979) have examined equilibria in a very special example of a multiple-object auction. In their model, several different auctions are conducted simultaneously for "copies" of the same product or service. Buyers enter exactly two of these auctions. Each buyer has a positive valuation for the first copy he wins and has a zero valuation for each duplicate copy. Thus, a specific form of non-additivity is introduced which is substantially different from the non-additivity induced by bidding constraints.

They find that pure strategy Nash equilibria fail to exist in general and they solve for a symmetric mixed strategy equilibrium.

Of all the above papers, only four have explored equilibria in multiple-object auctions. Three of these examined Nash equilibria in two-buyer auctions when buyers face a constraint on exposure (Cook, Kirby and Mehndiratta[1975], Griesmer and Shubik[1963] and Sakaguchi[1962]) and the fourth (Engelbrecht-Wiggans and Weber[1979]) investigated a very special example of an unusual auction market. None of the literature has addressed questions about strategic behavior on the part of the seller. Clearly there is a great deal of research remaining to be done about multiple-object auctions.

The rest of this dissertation addresses two problems involving multiple-object auctions. The first problem deals with existence and symmetry properties of Nash equilibria when an arbitrary number of identical buyers who face identical bidding constraints compete in an arbitrary number of simultaneous auctions. This approach is a generalization of Sakaguchi(1962) and is similar in spirit to Griesmer and Shubik (1963) and Cook, Kirby and Mehndiratta(1975). The remarkable result is that symmetric pure strategy Nash equilibria generally fail to exist. Thus there is either non-existence of a pure Nash equilibria or else existence of non-unique asymmetric Nash equilibria. These are fully characterized in chapter two.

The second problem investigated in this dissertation is the bundling decision of the seller. In chapter three the Vickrey informational structure is used to analyze the economic consequences of the seller's revenue maximizing bundling decision when buyer valuations are additive and uncorrelated across items. Additivity and noncorrelation were imposed so that the bundling question could be explored independently of the sequencing problem and the non-additivity problem. A number of interesting theoretical results were obtained. First, the seller will choose to bundle if the number of bidders is small and will not bundle if the number of bidders is large. Second, except for buyers with very high valuations for all objects, buyers are better off if the seller does not bundle. Third, if the seller requires no minimum bid then bundling is less efficient than selling the objects in separate auctions. Fourth, if the seller sets a revenue maximizing minimum bid requirement, then in some cases bundling is more efficient than selling the objects in separate auctions.

These theoretical predictions are tested empirically in chapter four using auction data from controlled laboratory experiments. The predictions were strongly supported by the data.

NOTES

1. A rare example is the horse auction which is discussed in Bohm-Bawerk(1959),

2. Stark and Rothkopf(1979) have compiled a lengthy bibliography including most of these papers.

3. See Reiter(1977).

4. Henceforth in this dissertation discussion is limited to auctions with a passive seller and several competing buyers. This is done for expository convenience only. All important results extend quite naturally to the case of a passive purchaser and many competing sellers.

5. In other words, buyers are risk neutral.

6. A Bayesian equilibrium is a Nash equilibrium for a game with incomplete information played by Bayesian players. See Harsanyi(1967-1968) for a complete discussion of this equilibrium concept.

7. A strategy is said to be dominant if it maximizes a player's objective function regardless of the strategies played by the other players of the game.

8. Since each buyer demands only one unit, this problem essentially reduces to the single-object, single-unit case. For this reason, such models are not to be considered as multiple-object auctions.

9. Much of Wilson's work is motivated by the auction market for outer continental shelf oil and gas leases.

10. Unfortunately, it has never been proved that there exists an information-gathering equilibrium, so his analysis proceeded by

assuming existence.

11. Properties of royalty auctions and share auctions are not rigorously proved, rather attempts are made to draw inferences on the basis of a few examples.

12. Since buyers' risk parameters are independently drawn from the same distribution, the problem reduces to a symmetric one.

13. The reference here is to the multiple-unit auctions in which each buyer demands only one unit of a single good. See note 8.

14. See Forsythe and Isaac(1980).

15. This problem has been acknowledged in a different setting, in the study of mechanisms for choosing public goods. See Ferejohn, Forsythe, Noll and Palfrey(1979) for a brief discussion of this.

16. Most of the models of auctions can be classified as either decision-theoretic or equilibrium models. The weakness of decision-theoretic models is that they do not describe market behavior, but instead describe the maximizing behavior of an isolated individual. As a result, they lack the predictive power of equilibrium models. Equilibrium models may also be based on individual maximizing behavior, but require additionally that the behavior of all players satisfy some sort of mutual consistency requirement. Naturally, a weakness of equilibrium models is that they do not perform well in environments where players' behavior is somehow not mutually consistent, that is, in disequilibrium environments.

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CHAPTER TWO

Multiple-object, Discriminatory Auctions with Bidding Constraints: A Game-theoretic Analysis

I. INTRODUCTION

This paper has two major objectives.* The first is to show that the incorporation of exposure constraints into the decision problem of bidding agents may dramatically change the character of "solutions" to an auction game. A second, more fundamental objective is to demonstrate that a Nash equilibrium is of questionable value as a solution concept for auction games. It is shown that Nash equilibria exist only occasionally and, when they do, often exist non-uniquely. The non-uniqueness or non-existence problem indicates that alternative solution concepts might be more accurate predictors of behavior in many auction situations. Because auctions are frequently used allocation mechanisms, one of the more important characteristics of theoretical models of auctions is that the results be reasonably consistent with observed data. Though more empirical work remains to be done, the preliminary indications are that the model developed in this paper compares favorably with previous work in this regard.

II. MULTIPLE-OBJECT AUCTIONS

Multiple-object auctions, such as those conducted by the United States Geological Survey for outer continental shelf oil and gas leases, represent an important type of market structure for the allocation of scarce, lumpy objects. Surprisingly, the economics literature on this subject is virtually nonexistent. Before discussing briefly the work that has been done on this subject, a few distinctions should be made about specific auction institutions.

First of all, this paper examines sealed-bid auctions rather than oral auctions. Second, it examines multiple-object rather than single-object auctions. In a single-object auction, each participant submits at most one bid for whatever item is being sold. A participant either submits the highest bid and is a winner, or goes home empty-handed.

A multiple-object auction consists of at least two single-object subauctions in which a bidder may submit losing bids in several subauctions yet still win something. A multiple-object auction can be conducted either sequentially or simultaneously. In case it is sequential, participants may use information from earlier subauctions in deciding strategies for later subauctions. This information might be very useful. For example two items for sale may be highly complementary, such as a right shoe and a left shoe. The information that the bidder has at the beginning of the second auction -- specifically, knowledge of who won the first auction -- is of great value to him. Ruling out secondary markets,[1] the second shoe has no value to a bidder who has lost the first auction, but it is of considerable value to the winner of the first auction. In a simultaneous auction, all bids are submitted before the outcome on any subauction has been revealed. Bids in the auction are vectors, with each component corresponding to a subauction.

Auctions can also be distinguished by the message space of the bidders. For example, in many contractual agreements bids are submitted in which the cost, time of completion, product quality and other variables may jointly determine the winning competitive bid. This paper ignores multi-variate bidding and deals only with auctions in which a bid is a scalar, price. The message space of an auction must also specify the subsets of the set of items to be sold on which bids are to be submitted. For example, bidders may be permitted to submit sealed tenders for every subset of the set of items being auctioned. The message space considered here is one in which bids may be submitted only on singleton subsets. Thus, the auction institution examined in this paper is a simultaneous, multiple-object discriminatory auction.

The focus of much of the bidding literature in the past has been on the choice of bids in single-object auctions. Implicit in such analysis is the view that little is lost by analyzing a multiple-object auction as a series of independent single-object auctions. But single-object auction models fail to explain a number of empirical phenomena. An example is the bidding behavior in auctions for offshore and outer continental shelf drilling rights.

One of the most striking observations is that two companies with identical information often bid much differently. Capen, Clapp and Campbell (1971) document this phenomenon in the 1969 North Slope auction. They found that two joint explorers, Humble and ARCO, bid much differently on individual tracts, although neither <u>consistently</u> bid much greater than the other on all 55 tracts.[2] Some authors interpret this to mean that these two companies simply imputed much different value estimates to the same information. While different companies will interpret the same exploration data as indicating different amounts of oil and gas, it is hard to believe that this can explain as much variation in bids as was observed. On one tract Humble bid 17 times as much as ARCO bid and on another the ARCO bid was 33 times as much as Humble's.

One possible alternative explanation of such divergent bids is that the two companies used randomized, or mixed, strategies. This could result in significantly different bids, even if the companies made identical value estimates. A second explanation is that the companies faced bidding constraints. For example each company might have a "target" number of tracts it wants to win, or a maximum total bonus it can afford to pay, or some mix of these two objectives. This emphasizes an important limitation in previous models of optimal bidding in auctions. When more than one item is being sold, the objective function of a firm may not be simply the cross product of the expected net values of each of the items and the probabilities of winning each. In many situations the net value of an item is linked with the total number of items that are won.

Engelbrecht-Wiggans and Weber (1979) have constructed a "garage sale" model for the amount one should bid in each of several simultaneous auctions for identical goods, where the value of the first item won is $X > 0$ and the value of each subsequent item an agent wins is 0. Their analysis assumes an oral English auction, rather than sealed bids. The questions they ask are how many auctions to enter and how much to bid in each auction. In particular, they search for a symmetric Nash equilibrium. In one special case in which the number of bidders and the number of items are equal to n, Engelbrecht-Wiggans and Weber claim that if the number of auctions an agent can enter is limited to two, then as n goes to infinity the optimal Nash strategy is bid high on one randomly selected item and low on another.

There are a number of modifications of their model which are worth examining. These include:

(1) looking at the case where n is finite, rather than focusing on asymptotic results;

(2) examining the case where the number of agents does not equal the number of items;

(3) allowing agents to bid on as many items as they wish; and

(4) introducing an explicit constraint on the total value of the bids.

A few attempts have been made by past authors to incorporate explicit bidding constraints. Sakaguchi (1961) makes some progress characterizing Nash equilibrium pure strategy solutions when there are two items and two bidders. He offers an incomplete proof of a proposition which is presented (and correctly proved) as theorem 1 in this paper. Rothkopf (1977) formulates a decision-theoretic model of a bidder's optimal strategy in simultaneous auctions with a constraint on exposure, given a known expected payoff function, the only argument of which is an agent's own bid. The payoff function is also additively separable in the n objects at auction. Equilibrium strategies are not discussed.

Griesmer and Shubik (1963) and Cook, Kirby and Mehndiratta (1975) deal with constrained, simultaneous multi-object auctions in a slightly different context. Both papers assume that the bidders are bidding to sell (lowest bid wins), and have a resource constraint which limits the number of auctions they can win. Cook, Kirby, and Mehndiratta (1975) use an "expected" exposure constraint rather than a certain exposure constraint. Griesmer and Shubik (1963) deal primarily with the case in which agents face the constraint that they must bid identically in all subauctions. The authors speculate that solutions often do not exist if different bids are allowed.

III. BIDDING CONSTRAINTS

The above authors have made an important contribution to the theory of competitive bidding by suggesting that solutions to an auction game can change if the total net payoff to an agent is not simply the sum of the net payoffs in each separate auction.

Nonlinear payoff functions apparently are present among agents bidding for outer continental shelf gas and oil leases. There is evidence that firms face constraints that limit the number of tracts they want to win. For example oil companies which win a substantial number of leases in a sale, sometimes resell some of them to other companies.[3] One can imagine a number of internal and external forces which might lead a firm to limit the number of tracts it bids on or the total amount of its winning bids. Because a firm does not have perfect instantaneous access to an infinite supply of capital at a constant rate of interest, the leasing division of a firm is likely to face a budget constraint. For this reason, one would expect that both the cash outlay for winning bids and the total development expenditures on all tracts won must be constrained. One might object by saying that although they face this constraint, it should not affect their bidding strategy so long as there is a secondary market in leases. Unfortunately there is a fault in that logic. If a firm submits the winning bid, it probably means that its estimate of the value of the tract was greater than all other valuations. Thus it is unlikely that the firm will receive as high a price in the secondary market as what it paid for the tract. Compounding this problem is that the attempt to resell the tract sends a signal to other firms that the tract is not worth as much to the firm as was originally believed.[4] For all the other firms know, the winning firm might have just noticed an error in its value estimate and for this reason wants to unload the tract. Hence a firm would not expect to obtain a price for the item as great as the bonus it paid in the auction.

There are several ways one might wish to formalize the budget constraint. Perhaps the most realistic is to postulate that each firm has a loss function, L(C), in which C is the total amount of capital used to extract value from the items it wins. In the case of oil tracts the cost of exploration, purchase and development of each tract is

$$C = \sum_{j=1}^{J} \left[E_j + \delta_j \, (b_j + D_j) \right]$$

where

E_j = pre-auction exploration costs of tract j

b_j = bid on tract j

δ_j = 1 if b_j was the unique winning bid.

= 0 if b_j was a losing bid.

D_j = development costs of the tracts won. (Alternatively, $-D_j$ can be thought of as the capital obtained for j on a secondary market, if the item is resold instead of developed.)

One would expect that $L'(C) > 0$ and $L''(C) > 0$, to reflect costs of rapid expansion, increasing cost of capital in the lending market, and other costs which are not directly incorporated in C. If L is a smooth convex function of C then it can be loosely interpreted as a "soft" budget constraint, in the sense that the money cost of exploration purchase and development of tracts understates the true cost to the firm.

A "hard" budget constraint is an extreme case in which there exists some M such that $L(C) = \infty$ for $C > M$. Thus a second representation of the budget constraint might require that <u>in equilibrium</u> (or in the case of a mixed strategy equilibrium, <u>expected</u> in equilibrium)[5] a constraint $C \leq M$, can be satisfied.

Under the above formulation, one could model the firm's decision problem as being either static or sequential. In the sequential case, the firm first makes exploration decisions, then bidding decisions, and finally development decisions.

The model used in this paper postulates a "hard" budget constraint for reasons of analytical convenience. Exploration and development costs are ignored and thus the decision problem of a firm in static rather than sequential. The constraint faced by each agent, i, is: $\sum_{j=1}^{n} b_j^i \leq M^i$. This has been referred to in the literature as a constraint on exposure. This constraint makes the problem at hand a special case of a Colonel Blotto game.[6] Blackett (1954) describes this type of game the following way:

> Two players contending N independent battlefields distribute their forces to the battlefields before knowing the opposing deployment. The payoff on the i^{th} battlefield is given by a function $P_i(x,y)$ depending only on the battlefield and the opposing forces x and y committed to the battlefield by A and B. The payoff of the game as a whole is the sum of the payoffs on the individual battlefields.[7]

In our case, armies are dollars and battlefields are items. What makes our game a rather perverse Blotto game from the military standpoint is the particular nature of the payoff function. If you win a battle, you lose all your forces -- but gain the fort. If you lose a battle, you lose no forces, but fail to gain the fort. While this may not seem realistic on a battlefield it describes an auction quite adequately.

IV. THE MODEL

In general, participants will be indexed by the superscript i, items will be indexed by the subscript j. Let:

V_j^i = the value of item j to participant i

M^i = the budget of participant i

b_j^i = the bid of participant i on item j

I = the number of participants

J = the number of items

The budget constraint imposed in this model is that

(1) $$\sum_{j=1}^{J} b_j^i \leq M^i \quad \forall i.$$

A <u>pure strategy</u> for agent i, σ^i, is a J-vector of positive numbers. Thus the strategy space for any participant is

$$R_j^+ = \{\sigma^i = (b_j^i, \ldots, b_j^i) \mid b_j^i \geq b_j^* \ \forall\ j = i, \ldots, J\}$$

where b_j^* is the seller's reservation bid. It will be demonstrated below that the presence of a strictly positive reservation bid may alter bidding strategies profoundly.

A pure strategy $\sigma^i \in \Sigma^i$ is <u>feasible</u> if $\sigma^i \cdot 1 \leq M^i$

The subset of R_J^+ which includes only and all feasible pure strategies is Σ^i.

A pure strategy is <u>full</u> if $\sigma^i \cdot 1 = M^i$. The subset of Σ^i which includes only and all full pure strategies is $\bar{\Sigma}^i$.

<u>A feasible mixed strategy</u> of participant i is a distribution function $F^i(\cdot)$ defined over Σ^i.

A mixed strategy is <u>full</u> if the domain of F is $\bar{\Sigma}^i$.

The payoff function, $\pi^i(\vec{V}^i, M^i, \sigma^1, \ldots, \sigma^I)$, for individual i, where $\sigma^1, \ldots, \sigma^I$ are strategies of <u>all</u> the players is assumed to be:

$$\pi^i = M^i + \sum_{j=1}^{J} \delta_j^i (b_j^1, \ldots, b_j^I) \left[V_j^i - b_j^i\right]$$

where

$$\delta^i_j = 1 \text{ if } b^i_j > b^k_j \quad \forall\, k \neq i$$

$$= 0 \text{ if } b^i_j < b^k_j \quad \text{some } k \neq i$$

Tie-breaking Rule: If several participants in the auction tie for the winning bid on an item, they evenly divide the cost and ownership of the item:

$$b^i_j \geq b^k_j \quad \forall\, k \neq i \quad \rightarrow \quad \delta^i_j = \frac{1}{K}$$

where K is the number of agents submitting identical winning bids, equal to b^i_j.

Throughout the remaining analysis, four assumptions are maintained. The first assumption is implicit in the definition of $\delta(\cdot)$.

Assumption 1: Values are linear in the object, in the sense that if an individual receives a share α^i of item j, the value of that share to him is $\alpha^i\, V^i_j$.

Assumption 2: Values are constant and known with certainty.

This assumption limits the comparability of the results of this paper with the results found in the standard bidding literature (e.g. Wilson [1977]). However, insight into the case where values are uncertain may well require a full understanding of the certainty case if budget constraints exist.

The assumption that values are constant precludes the possibility that winning item 1 affects the value of item 2.

This restriction is strong, for it excludes auctions for complementary items (e.g. bidding on a left shoe and a right shoe) and "duplicates" such as the extreme case in which a bidder attaches positive value only to the first item won, and resale is impossible or costly.[8]

<u>Assumption 3</u>: $I \geq 2$, $J \geq 2$, and all items $j = 1, \ldots, J$ are auctioned simultaneously.

The first part of this assumption merely rules out trivial cases. The second part of the assumption rules out sequential auctions. For interesting examples of sequential auctions and some analysis about their characteristics see Engelbrecht-Wiggans (1977).[9]

<u>Assumption 4</u>: $M^i < \sum_{j=1}^{J} V_j^i$.[10]

This assumption is a necessary condition for the budget constraint to be binding.

In the analysis that follows, three types of symmetries appear. They provide a convenient classification of the cases which must be examined.

The first type of symmetry, (S1), is between values of the items to each person. Are the items the same? In this situation, for each individual, i,

$$V_j^i = V_k^i \quad \forall\ j,k \qquad \text{(S1)}$$

although it may be the case that

$$V_j^i \neq V_j^k \quad \text{for some } i,j,k.$$

A second symmetry, (S2), exists if the value of each item j, is the same for all individuals. That is

$$V_j^i = V_j^k \quad \forall\ i,k \qquad \text{(S2)}$$

although it may be the case that

$$V_j^i \neq V_k^i \quad \text{for some } i,j,k.$$

The third type of symmetry, (S3), exists if all individuals have the same budget constraint. That is

$$M^i = M^k \quad \forall\ i,k. \qquad \text{(S3)}$$

A second mode of classification is the scope of the market. How many items are auctioned off simultaneously? Finally, a third mode of classification is the depth of the market. How many participants are involved in the market? As will become apparent, these last two characteristics of the auction market, scope and depth, interact in very interesting ways and largely determine whether solutions to the auction game exist.

In what follows, an attempt is made to specify exactly when Nash equilibria exist and to characterize these Nash equilibria in terms of symmetry, profitability, and other criteria.

V. SYMMETRIC BIDDING STRATEGIES

<u>Theorem 1</u>:[11] Assume I = 2, J = 2 and conditions (S2), (S3) are satisfied. There will always exist a unique Nash equilibrium pure strategy at

$$\sigma^1 = \sigma^2 = (A_1, A_2)$$

where

$$A_i = \max\left\{0, \min\left[\frac{M + V_i - V_j}{2}, M\right]\right\} . \qquad (*)$$

The proof of Theorem 1 requires a number of initial observations to be made.

Lemma 1: If $I = 2$, $J \geq 2$ and (S2), (S3) are satisfied, then in equilibrium, $\sigma^1 = \sigma^2$.

Proof: Suppose $\sigma^1 \neq \sigma^2$. We can assume without loss of generality that $b_1^1 < b_1^2$. This implies that bidder 2 could be better off bidding $b_1^2 - \varepsilon$ on item 1. Hence, if $\sigma^1 \neq \sigma^2$, then (σ^1, σ^2) is not a Nash strategy pair.

□

Lemma 2: If $I = 2$, $J \geq 2$ and (S2), (S3) are satisfied, then in equilibrium $\sum_{j=1}^{J} b_j^i = M^i$

Proof: Suppose that $\sum_{j=1}^{J} b_j^1 = \sum_{j=1}^{J} b_j^2 < M$, which we can assume from Lemma 1. The payoff for each individual is

$$\pi = M + \frac{\sum_{j=1}^{J} (V_j - b_j)}{2}$$

since they tie on each item. (Note that because bids, values and budgets are identical the superscripts can be ignored.)

By Assumption 4 we know that $b_k < V_k$ for some k. Hence, because budgets are not exhausted, player 1 can bid $b_k + \varepsilon$ on that item, and receive a new profit of

$$\pi' = M + V_k - b_k - \varepsilon + \frac{\sum_{\substack{j=1 \\ j\neq k}}^{J} (V_j - b_j)}{2} .$$

The effect on profits of raising b_k by ε is:

$$\pi' - \pi = V_k - b_k - \varepsilon - \frac{V_k - b_k}{2}$$

$$= \frac{V_k - b_k}{2} - \varepsilon .$$

Since $V_k - b_k > 0$, there exists an ε small enough so that $\pi^1 - \pi > 0$. Hence the original bid configuration was not a Nash equilibrium, and so Nash strategies must be full.

□

<u>Lemma 3</u>: Assume I = 2, J = 2 and conditions (S2), (S3) are satisfied. Let $\sigma^1 = (b_1^1, b_2^1)$ and $b_2^1 \leq b_1^1$. If (σ^1,σ^1) is an equilibrium strategy pair, then

$$(V_1^1 - b_1^1)b_2^1 = (V_2^1 - b_2^1)b_2^1.$$

<u>Proof</u>: Since, by assumption, (σ^1,σ^1) is an equilibrium, it must be the case that no unilateral bid change can make that agent better off. Since equilibrium strategies are full, and strategies are symmetric, any unilateral change of bid necessarily means that each individual wins exactly one item. Assume that the conclusion of Lemma 3 is not true, e.g., that

b_2 is not equal to zero and that $(V_1 - b_1) \neq (V_2 - b_2)$. Without loss of generality, let $V_1 - b_1 > V_2 - b_2$. The profit resulting from σ^1 is given by

$$\pi(\sigma^1) = \frac{V_1 + V_2 - M}{2} .$$

If a bidder now bids $\sigma^2 = (b_1 + \varepsilon, b_2 - \varepsilon)$, the profit becomes

$$\pi(\sigma^2) = V_1 - b_1 - \varepsilon > \frac{V_1 - b_1}{2} + \frac{V_2 - b_2}{2} = \pi(\sigma^1)$$

for small enough ε. Hence (σ^1,σ^2) is not an equilibrium strategy pair. Thus, for σ^1 to be an equilibrium, either b_2 equals zero or $(V_1^1 - b_1^1) = (V_2^1 - b_2^1)$, hence

$$(V_1 - b_1)b_2^1 = (V_2^1 - b_2^1)b_2^1 .$$

□

The proof of Theorem 1 follows immediately from Lemma 2 and Lemma 3, which produce two equations in two unknowns, b_1 and b_2, subject to the constraint that b_1, b_2 are nonnegative:

$$b_1 + b_2 = M$$

$$(V_1 - b_1)b_2 = (V_2 - b_2)b_2 .$$

This establishes that if a Nash equilibrium pure strategy exists, it must satisfy (*).

To show that (*) in fact is a Nash equilibrium can be demonstrated by showing that if one agent uses (*), the other can make himself no better off by using a strategy other than (*). We need consider only the following two cases:

Case 1: Nonnegativity constraints are not binding. Fix bidder 1's strategy at (*). If bidder 2 bids something different, he wins at most one item. His profit then is:

$$\pi' = v_j - b_j \text{ where } b_j > b_j^*.$$

Hence $$\pi' = V_j - b_j < 2\,\frac{V_j - b_j^*}{2} = \pi^*.$$

Case 2: Some nonnegativity constraint is binding. Player A bids (0,M). Suppose player B bids (a,b) where $a > 0$, $b < M$. Then his profit is $[V_1 - a]$. Since the nonnegativity constraint is binding, it must be the case that

$$\frac{M + V_1 - V_2}{2} < 0$$

which implies that

$$\frac{V_1 + V_2 - M}{2} > V_1$$

If $a > 0$, then

$$\frac{V_1 + V_2 - M}{2} > V_1 - a$$

The LHS of the last inequality is player B's profit if strategy (*) is adopted, so (*) is an equilibrium.

Thus we have shown that (*) characterizes the unique Nash equilibrium strategy configuration to this bidding game.

□

Theorem 2: Assume $I > 2$, $J > 2$ and conditions (S2), (S3) are satisfied. If there are at least 3 items, $j = 1,2,3$ such that it is possible to have

$$V_1 - b_1 = V_2 - b_2 = V_3 - b_3$$

where b_1, b_2, b_3 are all non-negative and

$$b_1 + b_2 + b_3 = M$$

then a symmetric pure strategy Nash equilibrium cannot exist.

Proof: By Lemma 3,[12] we know that a necessary condition for a Nash equilibrium when the bid nonnegativity constraints are not binding is:

$$V_j - b_j^* = V_k - b_k^*.$$

Assume that $\sigma^1 = \sigma^2 = \ldots = \sigma^I = (b_1, \ldots, b_J)$ is a Nash equilibrium strategy I-tuple. Let $\Sigma = (\sigma^1, \ldots, \sigma^I)$. Without loss of generality, assume that $b_1, \ldots, b_K$ are all positive, where $3 \leq K \leq J$, and $\sum_{j=1}^{J} b_j = M > 0$.

Let π^* be the payoff each participant receives under this strategy I-tuple:

$$\pi^* = \sum_{j=1}^{J} \frac{1}{I} (V_j - b_j) + M .$$

Let

$$\pi_K^* = \sum_{j=1}^{K} \frac{1}{I} (V_j - b_j)$$

and

$$\pi_J^* = \sum_{j=K+1}^{J} \frac{1}{I} (V_j - b_j)$$

so

$$\pi^* = \pi_K^* + \pi_J^* + M .$$

Since Σ is a Nash equilibrium I-tuple, it must be the case that no agent can unilaterally receive a larger profit by departing from Σ. In particular, an agent cannot reduce the bid on one item, redistribute it over other items and receive a greater profit. Suppose an agent bids $b_k - \varepsilon$ on item $K < J$ and increases the bids on items 1 through $K - 1$ by $\frac{\varepsilon}{K - 1}$. The new profit is:

$$\hat{\pi} = \sum_{j=1}^{K-1} \left[V_j - (b_j + \frac{\varepsilon}{K - 1}) \right] + \sum_{j=K+1}^{J} \frac{1}{I} (V_j - b_j) + M .$$

This fails to improve the payoff associated with Σ if and only if

$$\sum_{j=1}^{K-1} \left[V_j - (b_j + \frac{\varepsilon}{K - 1}) \right] \leq \frac{1}{I} \sum_{j=1}^{K} (V_j - b_j) .$$

This is true for all $\varepsilon > 0$ if and only if

$$\sum_{j=1}^{K-1} [V_j - b_j] \leq \frac{1}{I} \sum_{j=1}^{K} (V_j - b_j) .$$

However, in order for Σ to have been a Nash equilibrium, this condition must hold not only for item K, but for all items $k \in \{1, \ldots K\}$. Thus we have

$$\sum_{\substack{j=1 \\ j \neq k}}^{K} [V_j - b_j] \leq \frac{1}{I} \sum_{j=1}^{K} (V_j - b_j) \quad k=1, \ldots, K$$

These can be rewritten as

$$(I - 1) \sum_{\substack{j=1 \\ j \neq k}}^{K} [V_j - b_j] \leq V_k - b_k \quad k=1, \ldots, K$$

Summing these K inequalities, we obtain

$$(I - 1) \sum_{k=1}^{K} \sum_{\substack{j=1 \\ j \neq k}}^{K} [V_j - b_j] \leq \sum_{j=1}^{K} [V_j - b_j]$$

$$\Rightarrow (K - 1)(I - 1) \sum_{j=1}^{K} [V_j - b_j] \leq \sum_{j=1}^{K} [V_j - b_j]$$

$$\Rightarrow (K - 1)(I - 1) \leq 1$$

Hence, in multiple-object auctions if either $K > 2$ or $I > 2$, symmetric Nash equilibrium pure strategies cannot exist.

□

Example: Bidding With a Reservation Bid Requirement

In this example the concept of a reservation price is introduced. The auctioneer requires a minimum bid he will accept. In this situation, Nash equilibrium bidding strategies may not exist.

Let $I = 2$, $J = 2$

$B_R \equiv$ reservation bid

Assume S2 and S3 hold.

Theorem 3: Nash equilibrium pure strategies do not exist if $B_R < \frac{V_2}{2}$ and $M + V_2 < V_1 < M + V_2 + (V_2 - 2B_R)$.

Proof: The strategy of the proof is to look at the boundary solutions. By the earlier theorem we know that the unconstrained problem has a solution:

$$\sigma^1 = \sigma^2 = (M,0).$$

Under this strategy pair each individual payoff is equal to

$$\pi_1 = \pi_2 = \pi = M + \frac{V_1 + V_2 - M}{2} = \frac{M + V_1 + V_2}{2}$$

under the old rule (i.e. $B_R = 0$).

However, if $B_R > 0$, then

$$\pi_1 = \pi_2 = \pi = M + \frac{V_1 - M}{2} = \frac{M + V_1}{2} .$$

Now, by assumption

$$V_1 < M + V_2 + (V_2 - 2B_R)$$

which implies that

$$V_2 - B_R > \frac{V_1 - M}{2}$$

$$\Rightarrow M + V_2 - B_R > \frac{M + V_1}{2} .$$

The left hand side of the inequality can be achieved unilaterally by either player (say player 1) simply by changing his bid to

$$\hat{\sigma}^1 = (A, B_R)$$

where

$$A \in [0, M - B_R].$$

Player two now has an incentive to bid less than M on item 1, in fact he will want to bid as low as A. Now, we need to check if

$$\hat{\sigma}^1 = \hat{\sigma}^2 = (M - B_R, B_R)$$

is a Nash equilibrium pure strategy.[13] This is not a Nash strategy pair because

$$V_2 - b_1 < V_1 - b_2$$

so by lemma 3, the bidder has an incentive to cut his bid on item 2 and increase his bid on item 1. In fact, this will be the case whenever $b_1 \geq 0$. Since the boundary solution ($b_1 = 0$) does not support a Nash equilibrium, the claim is demonstrated.

□

VI. ASYMMETRIC BIDDING STRATEGIES

Two questions immediately arise. When do asymmetric Nash equilibrium pure strategies exist, and what form do they take when they do exist. A first observation, that there are conditions where asymmetric Nash equilibria exist, can be made with reference to an example. This case is rather trivial, in the sense that no agent earns a profit.

Example 1: Suppose there are three agents and three items. The agents all have identical budget constraints and identical values

$$M^1 = M^2 = M^3 = 10.0$$

$$V_j^i = 4.0 \quad i = 1,2,3; \quad j = 1,2,3.$$

The following bid configuration is a Nash equilibrium.

Item / Bidder	1	2	3
1	4	4	2
2	2	4	4
3	4	2	4

One can immediately see that no agent earns a profit, because all winning bids are at the value of the item. Furthermore, one can see why it is a Nash equilibrium. Clearly, for any Nash equilibrium, there have to be at least two bidders tied for the highest bid on eath item. Otherwise the winning bidder would have an incentive to cut the winning bid to just barely above the bid of the nearest competitor. This is stated more clearly below.

Lemma 4: Assume S2, $I \geq 2$.

If $\Sigma = (\sigma_1, \ldots, \sigma_I)$ is a Nash equilibrium pure strategy I-tuple, then there must be at least two bidders tied for the highest bid on each item.

Proof: Let Σ be a Nash equilibrium pure strategy K-tuple.
Further suppose that for some (i,j), $b_j^i > b_j^k \quad \forall\ k \neq i$. Then the payoff for item j to individual i is

$$V_j^i - b_j^i = \pi_{ij} \ .$$

Agent i can earn more profit on item j by bidding $b_j^i - \varepsilon$, for some ε small enough so that $b_j^i - \varepsilon$ is still the winning bid on item j. Agent i then receives

$$\pi'_{ij} = V^i_j - b^i_j + \varepsilon > \pi_{ij} .$$

This contradicts the assumption that Σ is an equilibrium.

□

Example 1 demonstrates the "high-low" class of strategies, in which people bid up to their value on some items and very little on the other items. In fact, bidder 1 is indifferent between bidding 2 on the third item and bidding any number between 0 and 2 on that item. Thus the example has an infinite number of pure strategy equilibria, of the form $b_1 = b_j = 4$, $b_k = a$ where $a \in [0,2]$.

One wonders if such "high-low" equilibrium strategies ever exist which support positive profits. This question can be answered in the affirmative, by giving an example.

<u>Example 2</u>:

$$M^1 = M^2 = M^3 = 6.0$$

$$V^i_j = 4.0$$

Consider the strategies:

Bidder \ Items	1	2	3
1	3	3	0
2	0	3	3
3	3	0	3

One can see that no player can make larger profits by departing from this strategy. On the other hand, we can easily construct an example in which no such equilibrium exists. For instance if the auction includes a fourth player with the same parameters as the other agents, a Nash equilibrium no longer exists.

Another feature of asymmetric solutions to symmetrically parametrized auctions is that no agent can win more than 2 of the items for which both the winning bid and the net profit are strictly positive.

Lemma 5: Assume S2, S3, $I \geq 2$, $J > 3$. Let $\Sigma = (\sigma_1, \ldots, \sigma_I)$ be an equilibrium. If an agent, i, earns profits greater than 0 under Σ, then i can win at most two of the items i submitted positive bids on.

Proof: Suppose that agent i is tied with other bidders on three items, 1, 2, and 3, earning profits at most:

$$\pi_i = \sum_{j=1}^{3} (V_j^i - b_j^i) \frac{1}{2} .$$

Suppose, without loss of generality, that $V_1^i - b_1^i \leq V_2^i - b_2^i \leq V_3^i - b_3^i$. If i bids $b_1^i - \varepsilon$ on the first item and $b_2^i + \frac{\varepsilon}{2}$, $b_3^i + \frac{\varepsilon}{2}$ on the second and third items, then profits become:

$$\pi_i = \sum_{j=2}^{3} (V_i^j - b_i^j) - \varepsilon > \sum_{j=1}^{3} (V_i^j - b_i^j) \frac{1}{2}$$

for some $\varepsilon > 0$.

□

An additional proposition is demonstrated below:

Lemma 6: Assume S2, S3, $I \geq 2$, $J \geq 2$. Let $\Sigma = (\sigma_1, \ldots, \sigma_I)$ be an equilibrium. If i earns positive profits under Σ, then if $b_j^i < b_j^k$ for some $k \neq i$, then $b_j^i = 0$.

Proof: If $b_j^i > 0$, then agent i can bid $b_j^i - \varepsilon$ on j and $b_k^i + \varepsilon$ on some item k for which b_k^i is tied for the winning bid. Agent i will then be the sole highest bidder on item k thereby capturing all of its value.

□

In addition, from Lemma 3 an agent i must be earning equal profits on the items i wins with a positive bid. That is

$$V_j^i - b_j^i = V_k^i - b_k^i$$

whenever i is a winner in the $j^{\underline{th}}$ and $k^{\underline{th}}$ items. These restrictions are really quite strong, and seem to limit to only a few special cases the situations in which asymmetric Nash equilibria exist that generate positive profits when 3 or more people bid for 3 or more items.

As stated above, Nash solutions generally are not unique. In particular, example 1 has six permutations of the given individual strategies, all of which are Nash equilibria. Given that this is the case, how would an individual decide which strategy to use? In this situation one can hardly expect a Nash equilibrium to be achieved by non-cooperative behavior, because the agents must, in a sense, agree beforehand which equilibrium strategy I-tuple to play.

The preceding discussion assumes symmetry in agents and values of items. The following example shows that if budget constraints are "not too binding," it is the symmetry between individuals, rather than symmetry between items, which leads to the existence of non-unique Nash equilibria. Suppose that the individuals are identical and V_1, V_2 have the greatest value. Furthermore, suppose that there are fewer items then there are bidders. Also, assume that $M \geq V_1 + V_2$. Then a Nash equilibrium exists. But if there are "too many" more items than bidders, so that the budget constraint makes it impossible for at least 2 agents to bid their value on each item, then Nash equilibria may not exist. The following sequence of examples illustrates these points.

Example 3a: I = 3 J = 3

$$V_1^i = 5 \qquad \forall i$$

$$V_2^i = 3 \qquad \forall i$$

$$V_3^i = 2 \qquad \forall i$$

$$M_i = 8 \qquad \forall i$$

One permutation of Nash strategy triples is shown below:

Bidder \ Items	1	2	3
1	5	1	2
2	3	3	2
3	5	3	0

Example 3b: Same parameters as example 3a, except that

$M = 7$.

By reducing the budgets no Nash equilibrium exists. The reason is that it is no longer possible for any participant to bid the full value of items one and two simultaneously.

On the other hand, if there are four bidders with these parameters, Nash equilibria do exist. One such bid configuration is shown in the following table.

Example 3c:

Bidder \ Items	1	2	3
1	5	0	2
2	5	0	2
3	2	3	2
4	2	3	2

In both 3a and 3c the Nash equilibria all generated zero profits. This will always be the case when values are different across items (but identical across individuals) budgets are identical, and both the number of items and the number of bidders is greater than two. This is stated more precisely in the next theorem.

Theorem 4: Let $I \geq 3$, $J \geq 3$. Assume S2, S3

If

(1) $\exists\, j_1, j_2$ s.t. $V_{j_1} \neq V_{j_2}$;

(2) everyone submits positive bids on at least 2 items;

(3) all items are bid on,

then at any Nash equilibrium pure strategy I-tuple, $\Sigma = (\sigma^1, \ldots, \sigma^I)$,

$$\sum_{j \varepsilon A^i} V_j^i - b_j^i = 0 \quad i = 1, \ldots, I$$

where

$$A^i = \{j \mid b_j^i \geq b_j^k \quad k=1, \ldots, I\}$$

Proof: Suppose that some agent, i, makes a positive profit. First we show that in equilibrium this agent must make an equal profit on all items for which he submitted a positive winning bid.

Suppose that agent i has submitted winning bids on K items, and does not earn an equal profit on all K items. From lemma 4, in equilibrium at least two agents must have submitted winning bids on each of these items. From lemma 5 and lemma 6 we know that agent i submitted strictly positive winning bids on at least two items, say j and k. Suppose, without loss of generality,

$$V_j^i - b_j^i > V_k^i - b_k^i .$$

Then i can earn greater profits by bidding slightly more on j and slightly less on k, since $V_j^i - b_j^i > \frac{1}{2} (V_j^i - b_j^i + V_k^i - b_k^i)$.

In fact, in equilibrium, if the budget constraint is binding, all strictly positive bids must be winning bids, by lemma 6. Since all agents have the same budget and the same values, it must be the case that

$$V_j - b_j^* = V_k - b_k^* \quad \forall\ j,k$$

where b_j^* and b_k^* are the winning bids of j and k, respectively.

Therefore, if A and B are the two least valuable items, then

$$b_A^* + b_B^* < M$$

where b_A^* and b_B^* are the winning bids on items A and B. In this case any agent can obtain greater profits by bidding $b_A^* + \varepsilon$ on A, $b_B^* + \varepsilon$ on B, and 0 on all other items. This contradicts the Nash assumption, so the theorem is proven.

□

Assumption (2) in the statement of the theorem is actually stated just to rule out two special cases. One such case occurs if one item is so much less valuable than the other items that nobody bids on it. Referring back to example 4, if there were a fourth item valued at $\frac{1}{8}$, then the following bid configuration is a Nash equilibrium.

Example 4:

Bidder \ Items	1	2	3	4
1	3	3	0	0
2	0	3	3	0
3	3	0	3	0

Each agent is bidding on at least two items. However, all items receiving positive bids have the same value, so we are essentially back in the "identical value" type of auction. An alternative assumption to avoid this special case is that

$$V^{\alpha} - V^{\beta} < M$$

where V^{α} os the highest value of all items and V^{β} is the second highest value ($V^{\alpha} \neq V^{\beta}$). A third possibility is to require that

$$i \neq j \Rightarrow V_k \neq V_j \quad \forall i, j.$$

The second special case to rule out is when some players bid on only one very valuable item. One way this can occur is if every item is more valuable than the budget of each agent. In this case a Nash equilibrium may exist in which each agent's entire budget is bid on one item, as in the following example.

Example 5:

$$M = 6 \quad V_1 = 18 \quad V_2 = 16 \quad V_3 = 15$$

A Nash equilibrium strategy 6-tuple is:

Bidder \ Items	1	2	3
1	6	0	0
2	6	0	0
3	0	6	0
4	0	0	6
5	0	6	0
6	0	0	6

This special case is ruled out by considering only cases in which everyone bids on at least two items. It is actually only necessary to require that at least one individual bid on more than one item.

The point of this discussion has been to demonstrate that assumption (2) in the theorem is not as strong as it may at first appear. There exist fairly weak sufficient conditions for (2) to hold. Furthermore, it rules out cases which are, for the most part uninteresting.

VII. CONCLUSION

This paper has demonstrated several properties of Nash equilibria in multiple-object simultaneous sealed-bid auctions in which the participants face a constraint on exposure. First of all it has been shown that if there are more than two bidders and more than two objects, symmetric pure strategy Nash equilibria do not exist. Second, the presence of a reservation bid requirement can also result in the nonexistence of Nash equilibria, even if there are only two bidders and two objects. Third, when there are more than two bidders and two objects, sufficient conditions were derived for Nash equilibria to result in zero profits to the buyers. The conditions were fairly weak, indicating that when a Nash equilibrium exists, profits will often be zero.

The lack of symmetric pure strategy Nash equilibria is particularly interesting. The implication is that Nash equilibria, when they exist, can be realistically achieved only if the bidders cooperate with each other. Referring back to example 3, bidder 1 will submit (4,4,2) only if he knows bidders 2 and 3 will submit (4.2.4) and 2.4.4). Otherwise, the first agent's optimal response will be something else. Collusion is required for the buyers to coordinate their bids. Such collusion, unlike prisoner's dilemma situations which characterize many collusive arrangements such as cartels, is stable, for the point of collusion is a Nash equilibrium.

NOTES

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1. In the analysis that follows, it is assumed that secondary markets do not exist.

2. Capen, Clapp, and Campbell (1971), pp. 642-643.

3. Engelbrecht-Wiggans and Weber (1979) cite a congressional study (1976), "An Analysis of the Economic Inpact of the Current OCS Bidding System," prepared for Representative Hughes (D. N.J.).

4. This is perhaps analogous to the "market for lemons" problem in which the bad drives out the good. In this case, one would expect resale value for even very good tracts to be far below the cash bonus originally paid for them.

5. See Engelbrecht-Wiggans (1979), p. 37; and Cook, Kirby, and Mehndiratta (1975), p. 729 ff.

6. This connection was first noticed by Sakaguchi (1962).

7. Blackett (1954), p. 55.

8. Engelbrecht-Wiggans and Weber (1979) deal with auctions in which the items were duplicates.

9. Of particular interest is his brief discussion of horse auction. See also Schotter (1974), Engelbrecht-Wiggans (1977), and Brams and Straffin (1979).

10. This merely rules out trivial cases.

11. This theorem can be found in Sakaguchi (1962), where it was first stated.

12. Although lemma 3 was proved for the two-bidder, two-item case, one can easily see that the "equal profits condition" is necessary for an equilibrium in the n-bidder, m-item case as long as non-negativity constraints on bids are not binding. If profits are not equal between two items which an agent submits positive bids on, then the agent has an incentive to bid slightly lower on the item with less profit and slightly higher on the item which has a greater profit.

13. We need not examine (A,B_R) where

$$A < M - B_R$$

because either agent could unilaterally bid $A + \varepsilon$ and be better off.

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CHAPTER THREE

Bundling Decisions by a Multiproduct Monopolist with Incomplete Information

I. INTRODUCTION

An important topic in the theory of monopolistic behavior involves one aspect of the marketing strategy of a monopolist, the "bundling" of output for sale.* For example, tying contracts have been the subject of both academic research and landmark court cases.[1] The basic problem is the following. The monopolist is the only provider of n types of products or services. Barring legal restrictions, the seller may refuse to sell one of the items, unless the buyer also agrees to purchase another product or service. Such an arrangement is called a tying contract.[2]

Typical explanations for such marketing behavior include the following:

1. The items named in a tying contract are complementary products or services.
2. Monopoly power in one product line can provide leverage in related markets in which the seller does not have an exclusive position.

3. There are economies of scope in producing or distributing the quantities together.

4. Tied sales may be a convenient way to avoid price controls. A recent example of this occurred during the May 1979 gasoline shortage, during which some gasoline stations offered gasoline only to customers who also paid for a carwash.

Recent work by Adams and Yellen (1976) and Phillips (1979) have sought alternative explanations for commodity bundling which are not based on demand or production interdependencies or distortions caused by regulatory controls. Adams and Yellen demonstrate the plausibility of commodity bundling in a two-good world in which there is no uncertainty, all agents have complete and perfect information, demand for the two goods are independent and production of the two goods exhibits no joint economies. In their model, the allocation mechanism involves the seller setting a take-it-or-leave-it price at which he is willing to deliver any quantity a buyer demands. In a similar context, Phillips obtains conditions on cross-price relationships between the two commodities under which a monopolist would choose one bundling strategy as opposed to another. Both find that some form of bundling may occur under these circumstances.

This paper also focuses on a monopolist selling two independent goods. However, in the framework used here, the buyers and seller possess incomplete and asymmetric information about the preferences of the participants in the market.

Each individual knows with certainty his/her own preferences but is uncertain of the other agents' preferences. The market mechanism for determining a price in both markets is an auction.[3]

The paper is organized in four sections. Section II sets up the general model for n buyers and one seller. Section III analyzes the case in which the seller does not set a reservation bid for any of the auctions. Conditions under which the seller prefers a single bundled auction to separate auctions are derived. Efficiency, both *ex post* and *ex ante*, of the resulting equilibrium is discussed. Section IV analyzes the case in which the seller chooses an optimal pricing scheme, which in this case is a second price auction with a reserve bid requirement. Again efficiency of the resulting equilibrium is discussed from both *ex post* and *ex ante* perspectives. In Section V, the results of the paper are discussed in light of some additional unanswered questions it raises, and some possible extensions which could be made.

II. THE GENERAL MODEL

A single seller has one unit of each of J items to sell. There are n buyers. Each buyer has a utility function $U^i = M^i + \sum_{j=1}^{J} v^i_j$. The seller has incomplete information about demand in the sense that the seller knows only that for any i and j, the reservation value of item j to buyer i, v^i_j, is a random variable which is distributed on the interval $[0,\bar{v}]$ according to the cumulative distribution function (CDF), $F^i_j(\cdot)$ with a continuous probability density function (PDF), $f_j^i(\cdot)$, which is strictly positive on $(0,\bar{v})$. The holding of the numeraire commodity by buyer i is symbolized by M^i. Assume that all nJ random variables are distributed identically and independently of each other. The set of reservation values $\{v_1^i,\ldots,v_J^i\}$ is known to buyer i, but i has only the same incomplete information as the seller about each of the other buyers.

The two auction mechanisms considered here are the generalized Vickrey (competitive) auction and the generalized first-price discriminatory auction.[4] In both auctions the seller may set some reserve bid, b^s, below which bids will not be accepted. In the Vickrey auction, the winning bidder wins the item, and pays the seller either the second highest bid or b^s, whichever is larger. In the discriminatory auction, the payment is the winning bid. This, of course, presupposes that there will be a winning bidder. This is not necessarily the case; it is possible in either type of auction to have no winning bid, since there is a reserve bid requirement. This paper will show that this is an important source of inefficiency in auctions, although it is not the only source.[5]

In the Vickrey auction, bidders are commonly assumed to use dominant strategies, which exist under this particular arrangement. In the discriminatory auction, buyers are assumed to follow Bayesian equilibrium strategies [Harsanyi (1967-8)], since dominant strategies do not exist. According to this equilibrium concept, there is a bidding function, $b^*(v)$, such that if a buyer has a reservation value v for an item, or a package of items, he can do no better in expected value terms than submitting a bid of $b^*(v)$, assuming that every other buyer also bids according to the same bidding function.

The "auction game" is analyzed in two different settings, depending upon the role of the seller. In one setting, the seller is assumed to set no reserve bid. In the other setting the seller strategically sets a reserve bid requirement in such a way as to maximize expected profits. In either setting, however, the seller is assumed to bundle the items in such a way as to maximize expected profits. A <u>bundling decision</u>, p, is defined as a partition of the set of items into subsets $J_{p1},\ldots,J_{pK_p}$, where K_p is the cardinality of p. We will refer to the subsets as <u>bundles</u>. In other words:

$$p = \{J_{p1}, J_{p2}, J_{p3}, \ldots, J_{pK_p}\}$$

Such that: (1) $J_{pi} \cap J_{pj} = \phi$

(2) $\bigcup_{k=1}^{K_p} J_{pk} = \{1,\ldots,J\}$

(3) $J_{pk} \neq \phi$ for $k = 1,\ldots,K_p$

The set of all partitions will be donated P. The seller then sells at auction each of these bundles separately, using either a Vickrey or discriminatory sealed-bid auction.

Accordingly, the buyers have induced reservation values for each element, J_{pk}, of p. This valuation, v^i_{pk}, is simply equal to $\sum_{j \epsilon J_{pk}} v^i_j$. In other words, reservation values are assumed to be additive.

Accordingly, the valuation, v^i_{pk}, is treated as a random variable by all other buyers and by the seller. Since the random variables, $\{v^i_j\}$, are all independent and the values of items are additive for any individual, the random variable for the value of a package is naturally the sum of the random variables of the items in the package. Therefore, the CDF of this new random variable, H_{pk}, is the convolution of the distribution functions of each of the items in bundle J_{pk}.

This section closes with a summary of notation and assumptions:

n: Number of buyers

J: Number of items being sold

v^i_j: Value of item j to individual i

f: Probability density function for the reservation value of an item to an arbitrary buyer

F: Cumulative distribution function for the reservation value of an item to an arbitrary buyer

$\mathcal{P}$: The set of all partitions of the set $\{1,\ldots,J\}$

p: The p^{th} element of $\mathcal{P}$. Note that p is a set of subsets, or bundles.

J_{pk}: The k^{th} element of p. Note that J_{pk} is a set of items.

v^i_{pk}: Value of bundle J_{pk} to individual i

K_p: Cardinality of p

h_{pk}: Probability density function for the value of bundle J_{pk} to an arbitrary buyer. Note that if two bundles contain the same number of items, then they will have the same distribution of values, because of the iid assumption about the reservation values of items

H_{pk}: CDF of J_{pk} for an arbitrary buyer

R_{pk}: The support of h_{pk}

b^i_{pk}: Buyer i's bid on bundle pk

b^s_{pk}: Seller's reserve bid requirement for bundle pk

$b^i_{pk}(v^i_{pk}, b^s_{pk})$: Buyer i's bidding function for bundle pk

$b^*_{pk}(\cdot, b^s_{pk})$: Bayesian equilibrium bidding function for bundle pk

To reiterate, the assumptions maintained throughout the paper are:

(A1) All agents are risk neutral.

(A2) Values are additive. (The value of a bundle of items equals the sum of the separate values of each item.)

(A3) All bidders' values are independent samples from random variables which are identically distributed on the closed real interval $[0,\bar{v}]$, according to a distribution function, $F(\cdot)$, which has the following properties:

(a) $F(0) = 0$, $F(\bar{v}) = 1$

(b) $x \in (0,\bar{v}) \Rightarrow F(x) \in (0,1)$

(c) $f(x) = F'(x)$ exists and is a strictly positive, bounded, continously differentiable function on the open interval $(0,\bar{v})$.

III. BUNDLING DECISIONS AND WELFARE IMPLICATIONS WHEN THE SELLER DOES NOT SET A RESERVE BID REQUIREMENT.

The following assumption is posited for this entire section.

(A4) $b^{s}_{pk} = 0$ for all pk.

Buyers' Decision Problem

The decision problem faced by buyer i is the following. Suppose buyer i knows that every other buyer $\ell, \ell \neq i$, bids for package J_{pk} according to the bidding function $\hat{b}_{pk}(v^{\ell}_{pk})$.

It is assumed that $\dfrac{\partial \hat{b}_{pk}}{\partial v^{\ell}_{pk}} > 0$, so that $\hat{b}_{pk}(\cdot)$ is

is an "increasing" bidding function. If buyer i has a value v^{i}_{pk}, the objective of buyer i is to maximize:

$$E\Pi^{i}_{pk}\left(v^{i}_{pk}, b^{i}_{pk} \mid \hat{b}_{pk}(\cdot)\right) = \left(v^{i}_{pk} - b^{i}_{pk}\right)\left(\text{prob.}\left\{b^{i}_{pk} \geq b^{\ell}_{pk},\ \forall \ell \neq i\right\}\right).$$

Using a well-known result,[6] the bidding function that is a symmetric Bayesian equilibrium is as follows:

$$b^*_{pk}\ (v_{pk}) = v_{pk} - \int_0^{v_{pk}} \left[\frac{H_{pk}(x)}{H_{pk}(v_{pk})}\right]^{n-1} dx\ . \qquad (1)$$

The expected surplus in equilibrium for a buyer with valuation v_{pk} is simply:

$$E\Pi_{pk}\ (v_{pk}) = [v_{pk} - b^*_{pk}\ (v_{pk})][H_{pk}(v_{pk})]^{n-1}\ .$$

This reduces to

$$E\Pi^i_{pk} = \int_0^{v^i_{pk}} [H_{pk}(x)]^{n-1}\ dx\ , \qquad (2)$$

The expected surplus to a buyer for the auction of all J items under an arbitrary packaging decision p is:

$$E\Pi^i_p = \sum_{k=1}^{K_p} E\Pi^i_{pk} = \sum_{k=1}^{K_p} \int_0^{v^i_{pk}} [H_{pk}(x)]^{n-1}\ dx\ . \qquad (3)$$

If p and q are two bundling decisions, the <u>ex ante</u> preference relation for buyer i, $\overset{i}{\geq}$, is defined as:

$$p \overset{i}{\geq} q \iff E\Pi^i_p \geq E\Pi^i_q\ .$$

One should note that this is *ex ante* notion is not always used in the auction literature. matthews (1979), for example, means by *ex ante* that the buyers do not even know their *own* valuations, so that the *ex ante* expected surplus function for an agent would be:

$$E\Pi_p^i = \sum_{k=1}^{K_p} \int_{R_{pk}} \left[\int_0^t \left[H_{pk}(x) \right]^{n-1} dx \right] h_{pk}(t) dt. \qquad (4)$$

Sellers Bundling Decision

The seller's preferences over packaging decisions are somewhat more complicated. This complication arises from the problem of specifying precisely the seller's objective function. Consider the case in which the seller does not set a reserve bid. This presents no problem because, by the assumption that buyers have positive valuations with probability 1, the object will always be sold. Other cases in which the seller sets an optimal reserve bid will be discussed at length in section IV.

In the zero reserve bid case, the seller's expected profit for a given package is:

$$E\Pi^s_{pk} = \int_{R_{pk}} \left(x - \int_0^x \left[\frac{H_{pk}(t)}{H_{pk}(x)}\right]^{n-1} dt \right) n[H_{pk}(x)]^{n-1} h_{pk}(x)dx .$$

Because the term

$$n[H_{pk}(x)]^{n-1} h_{pk}(x)$$

is the density function of the highest valuation out of a sample of size n, this can be simplified to:

$$E\Pi^s_{pk} = \int_{R_{pk}} \left(x[H_{pk}(x)]^{n-1} - \int_0^x [H_{pk}(t)]^{n-1} dt \right) nh_{pk}(x)dx . \qquad (5)$$

The seller's expected profit from packaging decision p is simply:

$$E\Pi^s_p = \sum_{k=1}^{K_p} \int_{R_{pk}} \left(x[H_{pk}(x)]^{n-1} - \int_0^x [H_{pk}(t)]^{n-1} dt \right) nh_{pk}(x)dx. \qquad (6)$$

Therefore the seller's <u>ex ante</u> preference relation, $\overset{s}{\succsim}$, is defined by:

$$p \overset{s}{\succsim} q \iff E\Pi^s_p \geq E\Pi^s_q .$$

Welfare Measures

A third and a fourth preference relation, which represent two alternative welfare measures, or "policy-maker preferences" can now be defined. The policy-maker knows only the probability distribution of the buyers' valuations. One welfare ranking which is complete and transitive orders the packaging decisions according to expected consumer plus producer surplus. Because the policy-maker does not know each buyer's valuations, only the naive ex ante expected surplus value given in equation (4) can be used to evaluate a buyer's expected profit. Thus, the expected total surplus from bundling decision p, combining (4) and (6), is:

$$ETS_p = n \sum_{k=1}^{K_p} \int_{R_{pk}} \left[\int_0^t [H_{pk}(x)]^{n-1} dx \right] h_{pk}(t)\ dt + \sum_{k=1}^{K_p} \int_{R_{pk}} \Big(t[H_{pk}(t)]^{n-1} - \int_0^t [H_{pk}(x)]^{n-1} dx \Big) n h_{pk}(t)\ dt$$

$$= n \sum_{k=1}^{K_p} \int_{R_{pk}} t[H_{pk}(t)]^{n-1} h_{pk}(t)\ dt\ . \qquad (7)$$

The fourth preference relation, although not necessarily a complete relation is the ex post dominance relation defined over bundling decisions. According to Vickrey (1961), Matthews (1979), Cox (1979 and others, ex post Pareto optimality occurs if and only if there are no individually rational recontracting possibilities after the auction. In other words, for each item, whoever purchased that item in the auction is the buyer whose valuation was highest.

The dominance relation is defined in the following way. Bundling decision p_1 weakly dominates bundling decision p_2 ex post if, in every state of the world which can possibly obtain, the total surplus generated by p_1 is at least as great as the surplus generated by p_2; p_1 dominates p_2 ex post if p_1 weakly dominates p_2 ex post and in at least one state of the world which can possibly occur the total surplus generated by p_1 is strictly greater than the total surplus generated by p_2.

In the next part of this section, two additional assumptions are imposed for analytical convenience. These will be relaxed later.

(A5) There are 2 bidders.

(A6) There are 2 items.

Three results are obtained under A1-A6. First, buyers unanimously prefer two separate auctions to a "bundled" auction ex ante. That is, no matter what pair of values a bidder observes for the two items, the bidder will receive a higher profit in expected value in two separate auctions than in one single auction. The seller, on the other hand, prefers to bundle the two items and sell them as a lot in a single auction. Third, the seller's profit-maximizing bundling decision leads to inefficient allocations both ex ante and ex post.

Given assumptions A1-A6 and the structure of the general model outlined in section II, there are two possible packaging decisions. The seller can sell items one and two either separately or together. Consistent with earlier notation, we define the distribution of values of the bundle, $H = F * F$, which is simply the convolution of the two value distributions of the separate items.

The following lemma specifies a necessary and sufficient condition under which all buyers prefer separate auctions.

Lemma 1.1: Assume A1-A6. Buyers unanimously prefer two separate auctions to a single bundled auction if and only if:

$$\int_0^t [F(x) - H(2x)]\, dx \geq 0 \quad \forall\, t \in [0, \bar{v}],$$

where $H = F*F$

Proof: Suppose a buyer observes two identical values, $v_1 = v_2 = v$. That buyer will prefer two separate auctions to a single bundled auction if and only if the expected surplus earned in the two separate auctions is greater than or equal to the expected surplus earned in a bundled auction. From equation (1), this condition is:

$$2\left[v - \left\{v - \int_0^v \frac{F(x)}{F(v)}\, dx\right\}\right] F(v) > \left[2v - \left\{2v - \int_0^{2v} \frac{H(z)}{H(2v)}\, dz\right\}\right] H(2v),$$

which reduces to:

$$2\int_o^v F(x)\,dx > \int_o^{2v} H(z)\,dz\ .$$

With a simple change of variable, we obtain:

$$\int_o^v [F(x) - H(2x)]\,dx > 0. \qquad (8)$$

Thus we have shown that a buyer who observes two identical values will always prefer separate auctions if and only if (8) holds. We must now show that this is also true for a buyer who observes different values, $v_1 \neq v_2$. With this in mind, we demonstrate that the expected profit in separate auctions for a buyer who observes values v_1, v_2 is always greater than or equal to the expected profit of a buyer who observes values $\hat{v}_1 = \frac{v_1 + v_2}{2}$, $\hat{v}_2 = \frac{v_1 + v_2}{2}$. In other words, for every $v \varepsilon [0, \frac{\bar{v}}{2}]$, and for every a such that $0 < a < v$:

$$2\int_0^v F(x)\,dx < \int_0^a F(y)\,dy + \int_0^{2v-a} F(z)\,dz.$$

Since, by assumption, $a < v$ (an assumption which costs no generality), then a trivial observation is that:

$$\int_a^v F(x)\,dx < \int_a^v F(v-a+t)\,dt$$

$$\Rightarrow \qquad \int_0^v F(x)\,dx < \int_v^{2v-a} F(z)\,dz$$

by a simple change of variable. Adding $\int_0^a F(t)dt$ to both sides of the inequality yields:

$$\int_0^v F(x)\,dx < \int_0^a F(y)\,dy + \int_v^{2v-a} F(z)\,dz$$

and, finally, by adding $\int_0^v F(t)dt$ to both sides of the equations, we obtain

$$2\int_0^v F(x)\,dx < \int_0^a F(y)\,dy + \int_0^{2v-a} F(z)\,dz,$$

which is the desired result. This completes the proof.

□

Based on this Lemma, the following theorem becomes an application of Rothschild and Stiglitz(1970).

<u>Theorem 1</u>: Assume A1–A6. Buyers unanimously prefer separate auctions.

<u>Proof:</u> What we must show is that:

$$\int_0^t [F(x) - H(2x)]\,dx \geq 0$$

for all t $\varepsilon[0,v]$ and for all F satisfying A3. To start, note that:

$$H(2z) = \text{prob.}\ \{\frac{x+y}{2} \leq z\}\ ,$$

where x and y are in the domain of F.

Thus H(2z) is just the distribution function of the mean of a sample of size 2 from the distribution of values of separate items. If X and Y are the random variables of valuations of items 1 and 2, respectively, and Z is the random variable of the average valuation of items 1 and 2, we can write:[7]

$$X = \frac{X+Y}{2} + \frac{X-Y}{2} = Z + \frac{X-Y}{2}.$$

Because X and Y are independent, $E\left(\frac{X-Y}{2} \mid Z\right) = 0$. Therefore, according to Rothschild and Stiglitz (1970), X (or Y) is a mean preserving spread of Z. If follows immediately that:

$$\int_o^t [F(x) - H(2x)] \geq 0 \; \forall \; t\varepsilon[o,\bar{v}] .$$

□

Next we demonstrate that the seller always bundles two commodities, contrary to the preferences of all buyers.

Theorem 2: Assume A1-A6. The seller prefers a bundled auction.

Proof: The seller's expected profit in a second price auction is the expected second highest valuation of whatever is being sold. For an arbitrary distribution G(t), the expected second highest value of n samples, $E[G_2^n]$, is simply:

$$E[G_2^n] = \int_o^{\bar{v}} tn(n-1)G(t)^{n-2}(1-G(t))g(t)dt.$$

Similarly, the expected highest valuation, $E[G_1^n]$, is

$$E[G_1^n] = \int_o^{\bar{v}} tn(F(t)^{n-1}g(t)dt,$$

because the distribution function of the first order statistic is equal to:

$$G_{1,n}(t) = G(t)^n$$

with pdf:

$$g_{1,n}(t) = nG(t)^{n-1}\ g(t).$$

With a little algebraic manipulation, $E[G_2^n]$ can be expressed as a function of n, $E(G_1^n)$ and $E(G_1^{n-1})$.

$$E\ (G_2^n) = \int_0^{\bar{v}} tn(n-1)\ G(t)^{n-2}\ (1-G(t))\ g(t)\ dt$$

$$= \int_0^{\bar{v}} tn(1-G(t))\ g_{1,n-1}(t)\ dt$$

$$= n\int_0^{\bar{v}} tg_{1,n-1}(t)dt - n\int_0^{\bar{v}} tG(t)g_{1,n-1}(t)\ dt$$

$$= n\int_0^{\bar{v}} tg_{1,n-1}(t)dt - (n-1)\int_0^{\bar{v}} tg_{1,n}(t)\ dt$$

$$= nE(G_1^{n-1}) - (n-1)\,E\ (G_1^n). \qquad (10)$$

With two buyers, the seller will prefer to bundle the two items rather than auction them separately if and only if the expected value of the second order statistic of H, $E(H_2^2)$, is greater than twice the expected value of the second order statistic of F, since these are the seller's expected profits in bundled and separate auctions, respectively. According to (10), this means that the seller prefers to bundle if and only if:

$$\Delta_2 = \left[2\ E(H_1^1) - E(H_1^2)\right] - 2\left[2E(F_1^1) - E(F_1^2)\right] > 0 \ . \qquad (11)$$

But Δ_2 reduces to:

$$\Delta_2 = 2E(F_1^2) - E(H_1^2) \qquad (12)$$

because $E(H_1^1) = 2E(F_1^1)$ by the linearity of the expectation operator. By Lemma A and Lemma B in Appendix A, Δ_2 must be positive, so the seller prefers to bundle.

□

Next we turn to the question of the relative efficiency of separate versus bundled auctions. The key to understanding the relative efficiency of the two bundling decisions is the recognition that if the seller "produces" at zero cost and does not set a reserve bid, an inefficient allocation occurs if and only if an item is allocated to a bidder who does not have the highest valuation for that item. A rather obvious theorem can be proved about the *ex post* efficiency of bundling decisions when the seller does not set a reserve bid.

Theorem 3: Assume A1-A6. Two separate auctions dominate a single bundled auction on the criterion of *ex post* efficiency.

Proof: Since $b^*_{pk}(\cdot)$ is an increasing bidding function and no reserve bid is set, whichever bidder values bundle J_{pk} the most also submits the highest reported valuation. Therefore if each bundle is a singleton, that is $p = \{\{1\}, \{2\}\}$, and if buyer i values the item the highest, buyer i will purchase that item. Therefore no misallocation can occur with separate auctions. However, if items 1 and 2 are sold as a bundle, the agent who has the highest sum of valuations for the two items will be the winning bidder. Of course, a bidder may have the highest sum of valuations without having the highest valuation of each item in the bundle. This would result in a misallocation.

□

The question of ex ante efficiency can be approached by interpreting the following expression in equation (7):

$$\int_{R_{pk}} tn \left[H_{pk}(t) \right]^{n-1} h_{pk}(t)\ dt$$

which is the expected maximum value after n samples of the random variable v_{pk} (i.e. it is the expected value of the first order statistic).

We immediately observe the following result:

Theorem 4: The expected total surplus generated by a set of first or second price auctions of n objects, with packaging decision p, is equal to the sum of the expected values of the first order statistics of valuations of each of the packages, J_{pk}, $k = 1,\ldots,K_p$.

Proof: see above.

From Theorem 3 we can easily verify that the sum of expected values of the first order statistics of the valuation of each of the two items exceeds the expected value of the first order statistic of the sum of valuations of the two items. Consequently, two separate auctions is *ex ante* more efficient than a single bundled auction. This is demonstrated more systematically in Theorem 5, but the intuition is clear: *ex post* efficiency implies *ex ante* efficiency, according to our definitions of the two concepts.

Theorem 5: Assume A1-A6. The expected total surplus in two separate auctions is greater than the expected total surplus in one bundled auction.

Proof: The idea of the proof is given in the preceeding paragraph, and is shown formally in Appendix A.

□

The results in this section, although specifically limited by assumptions A1-A6, do generalize somewhat. One generalization is that the results hold in second-price sealed-bid auctions as well as first-price sealed-bid auctions. Harris and Raviv [1979a] and Ortega-Reichert [1968] demonstrate that if buyers and sellers are risk neutral, first and second price auctions yield the same expected profit to the seller and the same expected profit to buyers.

Furthermore, Vickrey (1961) and others have established an isomorphism between English oral progressive auctions and second-price sealed-bid auctions and an isomorphism between Dutch descending-bid auctions and first-price sealed-bid auctions, under conditions of bidder symmetry and risk neutrality . Therefore the results obtained here apply equally to all four types of auctions.

Other extensions are easily made for the cases in which the number of bidders or the number of items are greater than two. For example, (A6) can be dropped without affecting any of the results. These and other generalizations are set forth more formally in Section V and proved in Appendix B.

Summarizing this section, it has been shown that in a first-price auction with independent goods, using a 2-bidder Bayesian equilibrium, the seller always prefers to bundle. The results were shown to generalize to a number of other pricing mechanisms. Under very general conditions, the two buyers will unanimously prefer that the monopolist does not impose tying conditions to the sale.

The *ex ante* measure of optimality, expected total surplus, indicates that bundling is not optimal. This is also the conclusion one reaches from an *ex post* analysis. Under the assumptions made in this section, a monopolist will always choose a suboptimal packaging decision. It is important to note that the tied sale is not a result of price controls, complementarity in consumption, or economies in production or distribution. The goods are entirely independent, and the result is driven by the lack of complete information on the part of the participants in the market.

IV. EXPECTED PROFIT MAXIMIZING CHOICE OF A RESERVE BID BY THE SELLER

Frequently, the seller sets a reserve bid, which is a requirement that only bids above a specified level will be accepted. In this section we assume that the seller knows that the buyers have values which are random variables. Furthermore the seller knows that these random variables are distributed independently and identically across bidders and items, according to some CDF, $F(\cdot)$, with PDF, $f(\cdot)$. These distributions are the same for the seller as they are for each of the buyers. The seller assumes that in a first price auction buyers will use Bayesian equilibrium bidding functions which depend on the seller's choice of a reserve bid. In a second price auction, the seller assumes buyers bid their true valuations if they exceed the reserve bid. Furthermore, we assume that the seller faces a set of costs, $\{c_1,\ldots,c_J\}$ for providing each item. These costs are assumed to be known to all buyers and the seller before the auction occurs. Consequently, buyers' strategies are the same regardless of whether the seller reveals the reserve requirement before buyers submit their bids. Furthermore, it is assumed that the seller incurs the cost if and only if the item is sold (i.e. if and only if a bid is submitted which is greater than or equal to the reserve bid). This is an assumption which is a bit strong because one can think of many cases when it is not true. In simple language, the seller "produces-to-order." Alternative scenarios include the following: an owner selling an object which has some value to him if it is not sold; an auctioneer whose profit is based on a percentage of the winning bid if the item is sold; a seller who must produce before the auction is conducted. These other types of sellers will be considered later.

In sections II and III, the analysis concentrated on the first price auction, although the results apply equally to a second price auction because of risk neutrality and symmetry assumptions. Because those assumptions are also made in this section the second price auction will be analyzed in this section. As it turns out, this facilitates the analysis considerably.

The specific second-price mechanism is the following. The seller sets a reserve level, b^s_{pk}, for the bundle pk. Each buyer either submits a bid greater than or equal to b^s_{pk}, or else submits no bid at all. If at least one buyer submits a bid greater than or equal to b^s_{pk}, the buyer who bids highest receives the bundle and pays the seller an amount equal to the second highest bid, or in case he was the only buyer to submit a positive bid, b^s_{pk}. One can easily verify that each buyer has a dominant strategy which simply involves bidding one's value if it exceeds b^s_{pk}.

In the following analysis, in addition to A1-A2, A4-A6, the following assumption is made.

Assumption 7 (A7): All values are independent draws from a random variable which is distributed uniformly on the unit interval.

These samples, once again, are denoted v^i_1 and v^i_2 for each buyer i.

The density and distribution functions of the sum of two unit uniform distributions is easily derived below. First we have $H(z) = 0$ for $z\varepsilon[-\infty,0]$. Less trivially, for $z\varepsilon[0,1]$,

$$H(z) = \int_0^z f(x)\ F(z-x)dx = \frac{z^2}{2}$$

since $f(x) = 1$ and $F(z-x) = z-x$.

If $z > 1$, then due to the restricted support of (v_1,v_2), H(z) can no longer be calculated using (11). Rather, it is

$$H(z) = F(z-1) + \int_{z-1}^{1} f(x)\ F(z-x)dx \quad \text{for} \quad z\varepsilon[1,2].$$

Notice that the support of z is [0,2], while the supports of v_1 and v_2 were both [0,1]. Substituting into the above equation, $F(z-1) = z-1$, $f(x) = 1$, and $F(z-x) = z-x$, so:

$$H(z) = z-1 + \left(zx - \frac{x^2}{2}\right)\Big|_{z-1}^{1} = \frac{z^2}{2} - (z-1)^2$$

for $z\varepsilon[1,2]$.

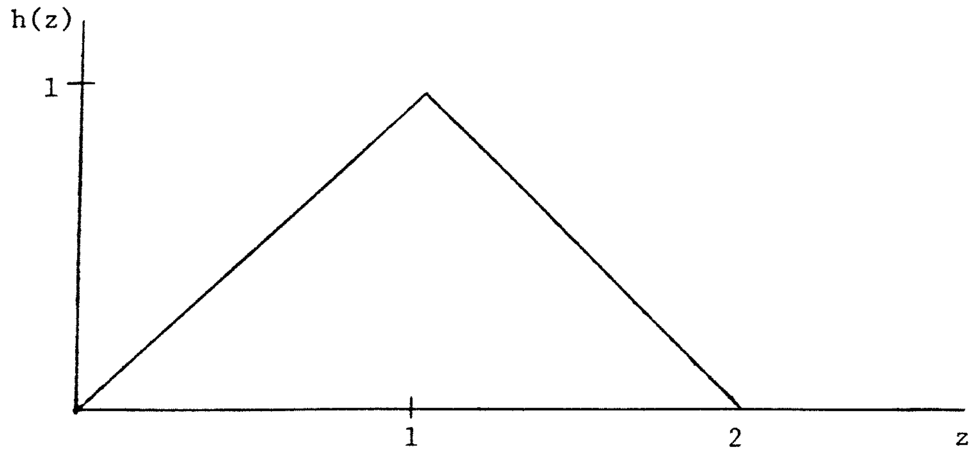
$$\text{Finally,}\quad H(z) = 1 \qquad \text{for} \quad z\varepsilon(2,\infty)$$
$$= 0 \qquad \text{for} \quad z\varepsilon(-\infty,0)$$

From above, h(z) is calculated quite easily (see Figure 1):

$$h(z) = H'(z) = z \qquad \text{for } x\varepsilon[0,1]$$
$$= 2-z \qquad \text{for } x\varepsilon[1,2] \quad .$$

FIGURE 1

DENSITY OF THE CONVOLUTION OF TWO UNIT RECTANGULAR DISTRIBUTIONS



The seller has some cost c_1 which will be incurred if the first item is sold and another cost, c_2, which will be incurred if the second item is sold. Therefore, if the seller conducts separate second-price auctions the expected profit in auction 1 as a function of $b_1^s = r_1$, is:

$$E\Pi_1(r_1) = \int_{r_1}^{1} (x-c_1)\ 2(1-x)dx + (r_1-c_1)\ 2r_1(1-r_1)\ .$$

The first term corresponds to the cases where both buyers submit acceptable bids. The second term corresponds to the case where only one buyer submits an acceptable bid. Similarly, the expected profit in auction 2 is:

$$E\Pi_2(r_2) = \int_{r_2}^{1} (y-c_2)\ 2(1-y)\,dy + (r_2-c_2)\ 2r_2(1-r_2). \tag{14}$$

These functions are maximized when the seller sets:

$$r_1^* = \frac{1+c_1}{2} \tag{14a}$$

$$r_2^* = \frac{1+c_2}{2}\ . \tag{14b}$$

Assuming the seller chooses a profit-maximizing reserve bid, the expected profits in separate auctions are:

$$E\Pi_1(c_1) = 2\int_{\frac{1+c_1}{2}}^{1} (x-c_1)(1-x)\,dx + 2\left(\frac{1-c_1}{2}\right)^2\left(\frac{1+c_1}{2}\right) = \frac{1}{3} + \frac{(1+c_1)^3}{12} - c_1$$

$$E\Pi_2(c_2) = 2\int_{\frac{1+c_2}{2}}^{1} (y-c_2)(1-y)\,dy + 2\left(\frac{1-c_2}{2}\right)^2\left(\frac{1+c_2}{2}\right) = \frac{1}{3} + \frac{(1+c_2)^3}{12} - c_2\ .$$

Therefore, if the seller strategically sets a reserve bid, and

$$E\Pi_2(r_2) = \int_{r_2}^{1} (y-c_2\ 2(1-y)dy + (r_2-c_2)\ 2r_2(1-r_2). \tag{14}$$

These functions are maximized when the seller sets:

$$r_1^* = \frac{1+c_1}{2} \tag{14a}$$

$$r_2^* = \frac{1+c_2}{2} \quad . \tag{14b}$$

As a next step, we calculate expected profits for the seller in separate auctions as functions of c_1 and c_2, assuming the seller chooses a profit maximizing reserve bid.

$$E\Pi_1(c_1) = 2\int_{\frac{1+c_1}{2}}^{1} (x-c_1)(1-x)dx + 2\left(\frac{1-c_1}{2}\right)^2\left(\frac{1+c_1}{2}\right)$$

$$E\Pi_2(c_2) = 2\int_{\frac{1+c_2}{2}}^{1} (y-c_2)(1-y)dy + 2\left(\frac{1-c_2}{2}\right)^2\left(\frac{1+c_2}{2}\right)$$

These reduce to

$$E\Pi_1(c_1) = \frac{1}{3} + \frac{(1+c_1)^3}{12} - c_1$$

and

$$E\Pi_2(c_2) = \frac{1}{3} + \frac{(1+c_2)^3}{12} - c_2 .$$

Therefore, if the seller strategically sets a reserve bid, and sells the two items in separate auctions, the total expected profit from that packaging decision is:

$$E\Pi_1 + E\Pi_2 = \frac{2}{3} + \frac{1}{12}\left[(1+c_1)^3 + (1+c_2)^3\right] - (c_1+c_2) \quad . \tag{15}$$

Next, we calculate the expected profit if the seller sells the two items as a bundle in a single auction. The seller incurs costs $c = c_1 + c_2$ if the bundle is sold, zero otherwise. The expected profit to the seller, as a function of the reserve bid, $r_{12} \equiv b^s_{12}$, for a general distribution $F(\cdot)$, of values with support $[\underline{v},\bar{v}]$ is given in equation (16).

$$E\Pi_{12}(r_{12}) = \int_{r_{12}}^{\bar{v}} (x-c)\cdot 2\cdot f(x)(1-F(x))dx + (r_{12}-c)\cdot 2\cdot F(r_{12})(1-F(r_{12})) \ . \tag{16}$$

The first order condition for a maximum is:

$$\frac{1}{r_{12}-c} = \frac{f(r_{12})}{1-F(r_{12})} \ . \tag{17}$$

Equation (14) is in fact a sufficient condition for an interior maximum if the hazard rate of the distribution, $\frac{f(\cdot)}{1-F(\cdot)}$, is nondecreasing in its argument, and in addition $f > 0$ if $f(x) \epsilon (0,\bar{v})$.[8] In our special case, the triangular distribution, these conditions hold. If we substitute the following equations:

$$F(r_{12}) = \frac{r_{12}^2}{2} \qquad \text{for} \quad r_{12} \varepsilon [0,1]$$

$$= 1 - \frac{(2-r_{12})^2}{2} \qquad r_{12} \varepsilon [1,2]$$

$$f(r_{12}) = r_{12} \qquad \text{for} \quad r_{12} \varepsilon [0,1]$$
$$= 2 - r_{12} \qquad r_{12} \varepsilon [1,2]$$

into equation (14), we are left with two possibilities. Either $r^*_{12} < 1$ or $r^*_{12} \geq 1$. In the first case we obtain:

$$r^*_{12} = \frac{c + \sqrt{c^2 + 6}}{3} \qquad \text{for} \qquad r_{12} \varepsilon [0,1] \, . \tag{18}$$

In other words, $r^*_{12} = 1$ when $c = \frac{1}{2}$. For values of c greater than $\frac{1}{2}$:

$$r^*_{12} = \frac{2}{3}(1+c) \qquad \text{for} \qquad c \varepsilon [\tfrac{1}{2}, 2] \, . \tag{19}$$

Now we are ready to calculate $E\Pi_{12}(c)$. Two cases will be considered:

Case 1: $c = c_1 + c_2 < \frac{1}{2}$

$$E\Pi_{12}(c) = 2\int_{r_{12}(c)}^{2} (x-c) f(x)(1-F(x))dx + 2(r_{12}(c)-c)F(r_{12}(c))(1-F(r_{12}(c)))$$

where $r_{12}(c) = \dfrac{c + \sqrt{c^2 + 6}}{3}$.

Substituting for f(x), F(x), we obtain (denoting $r = r_{12}$)

$$E\Pi_{12}(c) = 2\int_r^1 (x-c)x(1-\frac{x^2}{2})dx + (r-c)r^2(1-\frac{r^2}{2}) + 2\int_1^2 (x-c)(2-x)\frac{(2-x)^2}{2}dx$$

$$= \frac{2}{3}x^3 - cx^2 - \frac{1}{5}x^5 + \frac{c}{4}x^4 \Big|_r^1 + r^3 - cr^2 - \frac{r^5}{2} + \frac{cr^4}{2}$$

$$+ \frac{2-c}{4}x^4 - \frac{1}{5}x^5 \Big|_0^1$$

$$= \frac{3}{10} - \frac{c}{4} + \frac{c}{4} - \frac{1}{5} + \frac{2}{3} - c + \frac{r^3}{3} - \frac{3}{10}r^5 + \frac{cr^4}{4}$$

$$= \frac{1}{10} + \frac{2}{3} - c + r^3\left[\frac{1}{3} - \frac{3}{10}r^2 + \frac{c}{4}r\right]. \tag{20}$$

The difference in expected profits from the packaging decisions if $c_1 + c_2 < \frac{1}{2}$ is simply the difference between (15) and (20):

$$E\Pi_1 + E\Pi_2 - E\Pi_{12} = \frac{1}{12}\left((1+c_1)^3 + (1+c_2)^3\right) - \frac{1}{10} - r_{12}^3\left(\frac{1}{3} - \frac{3}{10}r_{12}^2 + \frac{cr_{12}}{4}\right). \tag{21}$$

<u>Case 2</u>: $c_1 + c_2 \geq \frac{1}{2}$

Note that in this case, $r_{12} \geq 1$ since $c \geq \frac{1}{2}$.

In this case, the expected profit to a seller employing a bundled auction is:

$$E\Pi_{12}(c) = \int_r^2 (x-c)(2-x)^3 dx + (r-c)\left(1 - \frac{(2-r)^2}{2}\right)(2-r)^2$$

$$= \frac{(2-c)(2-r)^4}{4} - \frac{(2-r)^5}{5} - \frac{(r-c)}{2}(2-r)^4 + (r-c)(2-r)^2. \tag{22}$$

From (25), we obtain:

$$2 - r = \frac{2}{3}(2-c)$$

$$r - c = \frac{1}{3}(2-c) \quad .$$

So (28) reduces to

$$E\Pi_{12}(c) = (2-c)^3 \left[\frac{4}{27} - \frac{4}{405}(2-c)^2\right] \quad . \qquad (23)$$

At this point, we can summarize the above observations about a seller who is "strategic," in the sense that the seller sets an optimal reserve bid.

Theorem 6: Assume A1-A2, A5-A7. If a seller observes costs c_1, c_2, then the optimal reserve bids in separate auction are:

$$r_1^* = \frac{1}{2}(1+c_1)$$

$$r_2^* = \frac{1}{2}(1+c_2) \quad .$$

The seller's total expected profit in separate auctions is:

$$E\Pi_1 + E\Pi_2 = \frac{2}{3} + \frac{1}{12}\left[(1+c_1)^3 + (1+c_2)^3\right] - (c_1+c_2).$$

The optimal reserve bid in a bundled auction if the same costs are observed is:

$$r^*_{12} = \begin{cases} \dfrac{(c_1+c_2) + \sqrt{(c_1+c_2)^2 + 6}}{3} & \text{if } c_1 + c_2 \, \varepsilon[0,\tfrac{1}{2}] \\ \tfrac{2}{3}(1+c_1+c_2) & \text{if } c_1 + c_2 \, \varepsilon[\tfrac{1}{2},2] \end{cases}$$

The seller's total expected profit in separate auctions is

$$E\Pi_{12} = \begin{cases} \dfrac{23}{30} - c_1 - c_2 + r^{*3}_{12}\left[\dfrac{1}{3} - \dfrac{3}{10} r^{*2}_{12} + \dfrac{(c_1+c_2)r^*_{12}}{4}\right] & \text{if } c_1 + c_2 \varepsilon[0,\tfrac{1}{2}] \\ (2-c_1-c_2)^3\left[\dfrac{4}{27} - \dfrac{4}{405}(2-c_1-c_2)^2\right] & \text{if } c_1 + c_2 \varepsilon[\tfrac{1}{2},2]. \end{cases}$$

Proof: Proof is summarized in previous pages. For details, simply perform the algebra and the necessary substitutions.

A number of interesting corollaries follow from Theorem 6.

Corollary 6.1: Assume A1-A2, A5-A7. If the seller has zero cost for both items, the profit maximizing packaging decision is to have a single bundled auction.

Proof: Let $\Delta = E\Pi_1 + E\Pi_2 - E\Pi_{12}$. From (21),

$$\Delta = \frac{1}{6} - \frac{1}{10} - r^3_{12}\left(\frac{1}{3} - \frac{3}{10} r^2_{12}\right)$$

where $r^3_{12} = \dfrac{c_1 + c_2 + \sqrt{(c_1+c_2)^2 + 6}}{3}$

$$= \sqrt{\frac{2}{3}}$$

if $c_1 = c_2 = 0$.

Therefore $$\Delta = \frac{1}{15} - \sqrt{\frac{8}{27}}\left(\frac{1}{3} - \frac{1}{5}\right)$$

$$= \frac{1}{15} - \frac{2}{15}\sqrt{\frac{8}{27}}$$

since $\frac{8}{27} > \frac{1}{4}$, $\Delta < 0$ which implies that the seller prefers to bundle.

□

<u>Corollary 6.2</u>: Assume A1-A.2, A5-A7. If the sum of the seller's costs exceeds $\frac{1}{2}$, the seller's profit maximizing bundling decision is to have two separate auctions.

<u>Proof</u>: This proof consists of demonstrating four things:

(1) If $c_1 + c_2 = \bar{c}$, then the seller's expected profits are an increasing function of $\left|\frac{\bar{c}}{2} - c_1\right|$. In other words, if a seller had a choice <u>ex ante</u> of buying (conditionally on resale in the auction) the two items for $\frac{\bar{c}}{2}$ and $\frac{\bar{c}}{2}$ or $\bar{c}$ and 0, the seller would prefer the latter contractual arrangement.

(2) If $c_1 = \frac{1}{4}$, $c_2 = \frac{1}{4}$, then the seller prefers separate auctions.

(3) If $c_1 = 1$, $c_2 = 1$, then the seller is indifferent between the two packaging arrangements.

(4) Let $\Delta(\bar{c}) = E\Pi_1(\frac{\bar{c}}{2}) + E\Pi_2(\frac{\bar{c}}{2}) - E\Pi_{12}(\bar{c})$.
Then Δ is increasing over some range $[\frac{1}{2}, \hat{c})$ and decreasing over the range $(\hat{c}, 2]$

First we will demonstrate (1).

Fix $\bar{c}$. Then the expected profits are

$$E\Pi_1 + E\Pi_2 = \frac{2}{3} - \bar{c} + \frac{1}{12}\left((1+c_1)^3 + (1+\bar{c}-c_1)^3\right)$$

so

$$\frac{\partial(r_{s\ 1} + F_{\ 2})}{\partial c_1} = \frac{1}{12}\left[3(1+c_1)^2 - 3(1+\bar{c}-c_1)^2\right].$$

Therefore $c_1 = \frac{\bar{c}}{2}$ is a critical point of $E\Pi_1 + E\Pi_2$. Looking at the second order equation:

$$\frac{\partial^2(E\Pi_1+E\Pi_2)}{\partial c_1^2} = \frac{1}{12}\left[6(1+c_1) + 6(1+\bar{c}-c_1)\right]$$

we observe that it is always positive, so $E\Pi_1 + E\Pi_2$ reaches a minimum when $c_1 = c_2$, keeping $\bar{c}$ fixed. Therefore (1) is demonstrated.

To show (2), simply perform the calculations. Using equation (18) or equation (19) we obtain $r_{12}(\frac{1}{2}) = 1$. So, from equation (21):

$$\Delta(\tfrac{1}{2}) = \frac{1}{12}\left(\frac{250}{64}\right) - \frac{1}{10} - \left(\frac{1}{3} - \frac{3}{10} + \frac{1}{8}\right)$$

$$> 0.$$

From equations (15), (23) we know that profits under any packaging choice are equal to zero if $c_1 = c_2 = 1$. This establishes (3).

Finally, we must show (4). From (15) and (23),

$$\Delta(\bar{c}) = \frac{2}{3} - \bar{c} + \frac{1}{6}\left(1+\frac{\bar{c}}{2}\right)^2 - \frac{4}{9}(2-\bar{c})^3 + \frac{4}{405}(2-\bar{c})^5.$$

Therefore,

$$\Delta'(\bar{c}) = (2-\bar{c})\left[\frac{-1}{4}\left(\frac{3}{2}+\frac{\bar{c}}{4}\right) + \frac{4}{9}(2-\bar{c}) - \frac{4}{81}(2-\bar{c})^3\right]. \qquad (24)$$

Since $(2-\bar{c})$ is always positive on the range $[\frac{1}{2},2)$, then $\frac{\partial\Delta}{\partial\bar{c}}$ is positive (negative) if and only if the second factor of (24) is positive (negative). By making the necessary substitutions, one may observe that

$\Delta'(\frac{1}{2}) > 0$. Next we observe that $\Delta'(\bar{c})$ is continuous in $\bar{c}$. Let

$$\hat{\Delta}'(\bar{c}) = \frac{-1}{4}\left(\frac{3}{2} + \frac{\bar{c}}{4}\right) + \frac{4}{9}(2-\bar{c}) - \frac{4}{81}(2-\bar{c})^3.$$

Note that $\Delta'(\frac{1}{2}) > 0$ and $\Delta'(1) < 0$. These facts have the following interpretation: $\Delta'(\frac{1}{2}) > 0$ and $\Delta'(\bar{c})$ continues to be positive for some interval. By differentiating $\Delta'(\cdot)$ we observe that Δ is concave on this interval. Since $\Delta'(1) < 0$, we know that at some point $\hat{c} \varepsilon [\frac{1}{2},1]$, Δ reaches a local maximum; after that point, since $\hat{\Delta}'(\bar{c}) < 0$, $\Delta'(\bar{c})$ is negative, until $\bar{c} = 2$, at which point it is zero. Figure 1 illustrates Δ as a function of $\bar{c}$.

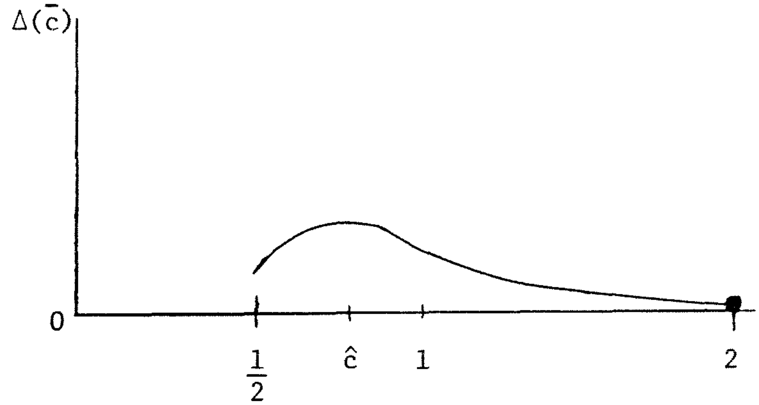


FIGURE 2

From (1), (2), (3), (4), we can conclude that the corollary is true.

□

So far in this section, we have considered only the case where the seller produces to order. We will briefly consider four other types of sellers:

(1) The seller has incurred costs prior to the auction;

(2) The seller is the owner of the items being sold -- these items each have some scrap value, $c_1,\ldots,c_J$.

(3) The seller is an auctioneer, working as an agent for a principal who pays the agent on a commission basis.

(4) The seller incurs costs after the auction, but incurs costs for the items as a bundle. For example, the items may be joint products, in which case production of one item inevitably means production of the other items. Alternatively, the seller may be a middleman who purchases "lots" and resells the items in the lots by auction.

In case (1), with n buyers, if the two items cost c_1 and c_2, the seller tries to maximize $L_1^1 + L_2^1$ in the case of separate auction, where:

$$L_1^1(r_1) = \int_{r_1}^{\bar{v}} (x-c_1)g_{2,n}(x)dx + (r_1-c_1)\left[G_{2,n}(r_1) - G_{1,n}(r_1)\right] - c_1G_{1,n}(r_1) \tag{25a}$$

$$L_2^1(r_2) = \int_{r_2}^{\bar{v}} (x-c_2)g_{2,n}(x)dx + (r_2-c_2)\left[G_{2,n}(r_2) - G_{1,n}(r_2)\right] - c_2G_{1,n}(r_2), \tag{25b}$$

where $g_{1,n}$ and $g_{2,n}$ are the PDF's for the first and second order statistics of F, respectively, and $G_{1,n}$, $G_{2,n}$ are the CDF's of the first and second order statistics of F.

The resulting first order conditions are

$$\frac{1}{r_1} = \frac{f(r_1)}{1-f(r_1)} \quad \text{and} \quad \frac{1}{r_2} = \frac{f(r_2)}{1-f(r_2)} . \tag{26}$$

Therefore, we get the expected result that the seller ignores sunk costs. As a corollary if values are distributed uniformly, the seller will choose to sell the items as one bundle.

In case (2), the seller is the owner of the items, with scrap values c_1 and c_2. If the seller is risk neutral, the same reserve bid will be set as in the case in which the seller produces "to order." This can be demonstrated more formally by considering the objective function of this type of seller. The expression analogous to (25a) is:

$$L_1^2(r_1) = \int_{r_1}^{\bar{v}} x g_{2,n}(x)\,dx + r_1[G_{1,n}(r_1) - G_{2,n}(r_1)] + c_1 G_{1,n}(r_2) \tag{27}$$

where c_1 is the scrap value to the seller. The first order conditions for (27) reduce to:

$$\frac{1}{r_1 - c_1} = \frac{f(r_1)}{1 - F(r_1)}$$

which is identical to (17).

In case (3), the seller earns a commission, which is some fixed proportion, ρ of the sale price. Again the expression analogous to (25a) is:

$$L_1^3(r_1) \quad = \rho\left\{\int_{r_1}^{\bar{v}} x g_{2,n}(x)\,dx + r_1[G_{1,n}(r_1) - G_{2,n}(r_1)]\right\} . \tag{28}$$

This first order necessary condition for a maximum reduces to (26). In otherwords, a principal's agent will set a reserve price as if the cost or scrap value of the item or bundle were zero. As a result, we conclude that the percentage commission institution, which is pervasive in the auction industry, does not result in the agent acting in the principal's best interest. Not only will the particular choice of reserve bids be different from the principal's optimal choice, but in the case of uniformly distributed values, the agent will sell the two items as a bundle, whereas the principal, if the sum of the scrap values (or costs) exceeds $\frac{1}{2}$, would have preferred that the agent sold the two items in separate auctions.

Finally, we consider case (4). The objective function for the seller is considerably different in this model compared to the previous ones. The seller has to produce both items, even if only one item is sold, which suggests that both the bundling decision and the reserve bid will be affected. As we will see, the latter is not the case. In a bundled auction, the objective function of the seller is simply (16). In separate auctions, the objective function is:

$$L^4(r_1,r_2) = \int_{r_1}^{\bar{v}} 2x\ f(x)(1-F(x))(1-F(r_2))dx + \int_{r_2}^{\bar{v}} 2y\ f(y)(1-F(y))(1-F(r_1))dy$$

$$+ 2[1-F(r_2)]r_1\ F(r_1)(1-F(r_1)) + 2[1-F(r_1)]r_2\ F(r_2)(1-F(r_2))$$

$$- [1-F(r_2)][1-F(r_1)]c$$

$$+ \int_{r_1}^{\bar{v}} 2xf(x)(1-F(x))F(r_2)dx + 2F(r_2)r_1F(r_1)(1-F(r_1)) - [1-F(r_1)]F(r_2)c$$

$$+ \int_{r_2}^{\bar{v}} 2yf(y)(1-F(y))F(r_1)dy + 2F(r_1)r_2F(r_2)(1-F(r_2)) - [1-F(r_2)]F(r_1)c,$$

where c ≡ cost of the bundle. This conveniently reduces to:

$$L^4(r_1,r_2) = \int_{r_1}^{\bar{v}} 2x\ f(x)(1-F(x))dx + \int_{r_2}^{\bar{v}} 2yf(y)(1-F(y))dy + 2r_1F(r_1)(1-F(r_1))$$

$$+ 2r_2F(r_2)(1-F(r_2)) - [1-F(r_1)F(r_2)]c. \qquad (30)$$

The first order conditions for a maximum of $L^4(r_1,r_2)$ are:

$$\frac{\partial L^4}{\partial r_1} = 2F(r_1)(1-F(r_1)) - 2r_1F(r_1)f(r_1) + f(r_1)F(r_2)c = 0 \qquad (31)$$

$$\frac{\partial L^4}{\partial r_2} = 2F(r_2)(1-F(r_2)) - 2r_2F(r_2)f(r_2) + f(r_2)F(r_1)c = 0. \qquad (32)$$

It is easily verified that if r_1 and r_2 satisfy:

$$\frac{1}{r_2 - \frac{c}{2}} = \frac{f(r_1)}{1 - F(r_1)} \tag{33}$$

$$\frac{1}{r_1 - \frac{c}{2}} = \frac{f(r_2)}{1 - F(r_2)} \tag{34}$$

then they also satisfy (31) and (32). This has a very intuitive interpretation. The seller would choose reserve bids in separate auctions as if each item had a cost of $\frac{c}{2}$. The seller is essentially allocating a fixed cost to the items in a manner that will maximize profits. It is interesting that it in fact <u>does matter</u> how the seller internally allocates these costs.

Another interesting observation about this case is that in a separate auction, the seller will expect to earn less profit than if he could "produce to order" each item individually at a cost of $\frac{c}{2}$ each. In fact the difference between the two expected profits is equal to $2F(r)c$. Not surprisingly, this would reverse the conclusion of Corollary 6.2, that is, the seller will prefer to bundle if $c > \frac{1}{2}$ and costs must be incurred jointly.

So far in this section, only the seller's preferences over packaging decisions have been discussed. We now turn our attention to two other key questions. First, what is the nature of buyers' preferences over packaging decision when the seller sets an optimal reserve bid? Second, will the seller's choice result in an efficient allocation of the items from either an ex ante or an ex post perspective?

Because the seller sets a reserve bid in each auction, it is quite difficult to specify an arbitrary buyer's preference between a single bundled auction and two separate auctions. A first observation is that unanimity among buyers is highly unlikely, and given the additional structure of uniformly distributed values, is impossible. This is demonstrated in the following way. Suppose the seller observes costs c_1, c_2. The seller will then choose reservation prices r_1^* and r_2^* according to (14a) and (14b) if there are two separate auctions, or r_{12}^* as in (18) and (19) if there is a single bundled auction. Given the values of r_1^*, r_2^* and r_{12}^* in the uniform, two-buyer case, it is always possible for a buyer to have v_1^i, v_2^i such that $v_1^i + v_2^i > r_{12}^*$ but $v_1^i < r_1^*$ and $v_2^i < r_2^*$, since $r_1^*(c_1) + r_2^*(c_2) > r_{12}^*(c_1+c_2)$ for $c_1 + c_2 \,\varepsilon[0,2]$. In such a case the buyer will clearly prefer the bundled auction. Similarly, it is possible for some other buyer to have values v_1^i, v_2^i such that $v_1^i + v_2^i < r_{12}^*$ but $v_1^i > r_1^*$. In this case, the buyer will prefer separate auctions.

In view of the above difficulty, we will consider auctions corresponding to sellers of type (1) or type (3). In other words, we assume the seller faces zero costs. Those buyers whose values are greater than $\frac{1}{2}$ for each item are never "cut out" of the auction by the sellers reservation bid. This set of buyers unanimously prefers a bundled auction.

Proposition: Assume A1-A3, A5-A7. If a buyer has values $v_1 > \frac{1}{2}$, $v_2 > \frac{1}{2}$ and $c_1 = c_2 = 0$, then that buyer will prefer the bundled auction to separate auctions.

Proof: The buyer's expected profit in separate auctions, given v_1, v_2 is:

$$E\Pi_1 + E\Pi_2 = F(r_1)(v_1 - r_1) + \int_{r_1}^{v_1} (v_1 - x)dx + F(r_2)(v_2 - r_2) + \int_{r_2}^{v_2} (v_2 - y)dy.$$

This expression reduces to:

$$E\Pi_1 + E\Pi_2 = \frac{1}{2} v_1^2 + \frac{1}{2} v_2^2 - \frac{1}{4}.$$

One immediately notes that for $v_1 + v_2 = \bar{v}$, $E\Pi_1 + E\Pi_2$ is maximized when $v_1 = v_2 = \frac{\bar{v}}{2}$. Therefore, we will just need to show that $E\Pi_{12}(\bar{v})$, expected profit in a bundled auction as a function of $\bar{v} = v_1 + v_2$, satisfies:

$$E\Pi_{12}(\bar{v}) > E\Pi_1(\tfrac{\bar{v}}{2}) + E\Pi_2(\tfrac{\bar{v}}{2}) = \frac{\bar{v}^2}{4} - \frac{1}{4}. \tag{35}$$

The expected profit in a bundled auction is

$$E\Pi_{12}(\bar{v}) = F(r_{12})(\bar{v} - r_{12}) + \int_{r_{12}}^{\bar{v}} (\bar{v} - x) f(x) dx.$$

This reduces to:

$$E\Pi_{12}(\bar{v}) = \bar{v}^2 - \frac{1}{6}\bar{v}^3 - \bar{v} + \frac{1}{3} - \frac{1}{9}\sqrt{\frac{2}{3}} \quad . \tag{36}$$

Subtracting (35) from (36) we obtain:

$$\Delta(\bar{v}) = E\Pi_1(\frac{\bar{v}}{2}) + E\Pi_2(\frac{\bar{v}}{2}) - E\Pi_{12}(\bar{v}) = \frac{3}{4}\bar{v}^2 - \frac{1}{6}\bar{v}^3 - \bar{v} + \frac{7}{12} - \frac{1}{9}\sqrt{\frac{2}{3}} \quad .$$

If $\bar{v} = 1$, then:

$$\Delta(1) = \frac{1}{6} - \frac{1}{9}\sqrt{\frac{2}{3}} > 0$$

and

$$\Delta'(\bar{v}) = \frac{3}{2}\bar{v} - \frac{1}{2}\bar{v}^2 - 1 = (\bar{v}-1)(1-\frac{\bar{v}}{2}) \geq 0 \quad \text{for} \quad \bar{v}\,\varepsilon[1,2] \; .$$

Therefore, if $v_1 > \frac{1}{2}$, $v_2 > \frac{1}{2}$, the buyer will prefer a bundled auction. □

The relative efficiency of separate auctions and a bundled auction depends upon the following factors. There are three sources of ex post inefficiency:

(1) All buyers can be shut out of an auction because the value of the bundle to every buyer is exceeded by the reserve bid, strategically set by the seller. In this case inefficiency only results if the maximum valuation of all buyers is between the seller's cost and the reservation bid.

(2) The "wrong" buyer can end up winning an item. This sort of misallocation can only occur in a bundled auction, because in a separate auction, either no buyer wins the item or else the buyer with the highest valuation wins the item.

(3) The seller may produce one of the items "at a loss." Again, this can only occur if the seller uses a bundled auction. This happens when the valuation of an item to the winning bidder is less than the seller's cost of producing that item.

One may notice that two of these types of inefficiency occur only when the seller bundles. However, the first type of misallocation can occur regardless of the packaging decision of the seller. One *ex post* criterion originally employed by Vickrey (1961) and later modified by Matthews (1979) can be extended in a natural way so that one packaging decision is said to be more efficient if the set of states of the world in which one of the above three inefficiencies occurs strictly includes the set of states in which the other results in an *ex post* inefficiency. Under this criterion, neither dominates the other in an *ex post* fashion.

It is quite straightforward to calculate which packaging decision is optimal *ex ante*, using the *ex ante* notion developed in section 1, with expression (4) as the buyer's expected profit, if the seller incurs zero costs, or alternatively these are sunk costs (type 1 seller) or the seller is an agent for some principal and is paid on a percentage commission basis (type 3 seller). In this context, buyer i who has values v_1^i and v_2^i has an expected profit in either of the two separate auctions which is equal to:

$$E\Pi^i = 0 \qquad \text{if} \quad v_j^i \le \frac{1}{2}$$

$$E\Pi^i = \frac{1}{2}\,(v_j^i - \frac{1}{2}) + \int_{\frac{1}{2}}^{v_j^i} (v_j^i - x)dx \qquad \text{if} \quad v_j^i \varepsilon [\frac{1}{2}, 1]$$

$$= \frac{1}{2}\, {v_j^i}^2 - \frac{1}{8}.$$

Therefore, if a buyer has not yet observed values (v_1^i, v_2^i), _ex ante_ expected profit is equal to:

$$E\Pi_j = \int_{\frac{1}{2}}^{1} (\frac{1}{2}\, x^2 - \frac{1}{8})dx = \frac{1}{24}.$$

So, any buyer's expected profit in the two separate auctions is equal to $\frac{1}{24} + \frac{1}{24} = \frac{1}{12}$.

Calculating a buyer's expected profit in the bundled auction is no less straightforward. Buyer i, who has observed the sum of values equal to v, has expected profit equal to:

$$E\Pi_j^i = 0 \qquad \text{if } v < \sqrt{\tfrac{2}{3}}$$

$$= \frac{1}{3}\left(v - \sqrt{\tfrac{2}{3}}\right) + \int_{\sqrt{\frac{2}{3}}}^{v} (v-x)x\,dx \qquad \text{if } v \in \left[\sqrt{\tfrac{2}{3}}, 1\right] \qquad (37)$$

$$= \frac{1}{3}\left(v - \sqrt{\tfrac{2}{3}}\right) + \int_{\sqrt{\frac{2}{3}}}^{1} (v-x)x\,dx + \int_{1}^{2} (v-x)(2-x)\,dx \qquad v \in [1,2]. \qquad (38)$$

So if $v \in \left[\sqrt{\frac{2}{3}}, 1\right]$, (37) reduces to

$$E\Pi^i = \frac{1}{6}v^3 - \frac{1}{9}\sqrt{\tfrac{2}{3}} \qquad \text{for } v \in \left[\sqrt{\tfrac{2}{3}}, 1\right] \qquad (37a)$$

and if $v \in [1,2]$, (38) reduces to

$$E\Pi^i = v^2 - \frac{1}{6}v^3 - v + \frac{1}{3} - \frac{1}{9}\sqrt{\tfrac{2}{3}} \qquad \text{for } v \in [1,2]. \qquad (38a)$$

If a buyer has not yet observed values, then <u>ex</u> <u>ante</u> profit in a bundled auction is:

$$E\Pi = \int_{\sqrt{\frac{2}{3}}}^{1} \left[\frac{1}{6}v^3 - \frac{1}{9}\sqrt{\tfrac{2}{3}}\right] v\,dv + \int_{1}^{2} \left[v^2 - \frac{1}{6}v^3 - v + \frac{1}{3} - \frac{1}{9}\sqrt{\tfrac{2}{3}}\right](2-v)\,dv$$

$$= \frac{1}{30}\left(1 - \sqrt{\tfrac{2}{3}}\right) + \frac{1}{6} + \frac{1}{15} - \frac{1}{30} - \frac{1}{18}\sqrt{\tfrac{2}{3}}$$

$$= \frac{1}{12} + \left(\frac{1}{12} - \frac{1}{18}\sqrt{\tfrac{2}{3}}\right) + \left(\frac{1}{15} - \frac{1}{30}\sqrt{\tfrac{2}{3}}\right) > \frac{1}{12}.$$

Therefore, ex ante a buyer will prefer that the items are sold as a bundle rather than separately. Since we already know that a type 1 or type 3 seller or a seller who has zero cost will prefer to bundle, we can conclude that the bundled auction is ex ante more efficient than the two separate auctions in these cases.

V. EXTENSIONS WHEN THERE ARE MORE THAN 2 BIDDERS OR MORE THAN 2 ITEMS.

This paper has discussed bundling decisions by a monopolist under incomplete information. The price mechanism used by the monopolist was a first or a second price auction. Because of the analytical complexity involved in a completely general formulation of the problem, some additional structure was added to the model which yielded some interesting results. The natural question to ask is what happens when we relax some of the assumptions employed throughout most of Sections III and IV.

One of the first assumptions we would like to relax is (A6): There are only 2 items. Theorems 1,2,3, and 5 are completely unaffected by the total number of objects being auctioned. Buyers still unanimously prefer separate auctions. The seller will prefer to bundle all the objects together and sell them as a lot in a single auction. It is ex ante and ex post efficient to have each item sold in a separate auction. Therefore, we have the following Lemma and Theorems.

Lemma 1.1 J: Assume A1-A5. Buyers unanimously prefer J separate auctions to a single bundled auction of all J items if and only if

$$\int_0^t \left[F(x) - H(Jx)\right] dx \geq 0 \; \forall \, t \; [o,v],$$

where H is the J-fold convolution of F with itself.

Proof: see Appendix B.

Theorem 1 J: Assume A1-A5. Suppose there are J items. No matter what valuations a buyer observes for the items, the buyer will prefer J separate auctions to any other bundling decision by the seller.

Proof: See Appendix B.

Theorem 2 J: Assume A1-A5. If there are J items, the seller will maximize his expected profits by selling all J items as a lot in a single bundled auction.

Proof: See Appendix B.

Theorem 3 J: Assume A1-A5. If there are J items, then J separate auctions dominate any other bundling decision according to the criterion of *ex post* efficiency.

Proof: This proof is identical to the proof of Theorem 3.

Theorem 5 J: Assume A1-A5. If there are J items, then J separate auctions dominate any other bundling decision according to the criterion of *ex ante* efficiency.

Proof: This proof is identical to the proof of Theorem 5.

Another assumption one would like to relax is A5: There are only two bidders. If there are n bidders, Theorems 1 and 2 do not generalize, although we obtain new theorems characterizing which buyers prefer separate auctions and conditions under which sellers will prefer bundled auctions. Theorems 3 and 5 do generalize to n bidders.

Lemma 1.1 N: Assume A1-A4, A6. There are n bidders. Bidders unanimously prefer two separate auctions to a single bundled auction if and only if:

$$\int_o^t \left\{ \left[F(x)\right]^{n-1} - \left[H(2x)\right]^{n-1} \right\} dx \geq 0 \;\; \forall\, t \in \left[o, \bar{v}\right].$$

Proof: See Appendix B.

Theorem 1 N: Assume A1-A4, A6. If there are n bidders, $n \geq 3$, then buyers never unanimously prefer separate auctions.

Proof: See Appendix B.

Corollary 1.1 N: Assume A1-A4, A6, $n \geq 3$. If a buyer observes sufficiently high values on both items, then that buyer will prefer a bundled auction.

Proof: See Appendices B and C.

Theorem 2N is easier to state and interpret if we give Theorem 3N and 5N first.

Theorem 3 N: Assume A1-A4, $n \geq 3$. Separate auctions always dominate any other bundling decision, *ex post*.

Proof: This proof is, again, identical to the proof of Theorem 3.

Theorem 5 N: Assume A1-A4, $n \geq 3$. Separate auctions always dominate any other bundling decision *ex ante*.

Proof: This proof is, again, identical to the proof of Theorem 5.

It is useful now to introduce a concept of surplus loss. The surplus loss of a bundled auction is equal to the absolute value of the difference between the expected surplus generated by bundled and separate auctions.

Theorem 2 N: If $I = n$, $J = 2$, then the seller will prefer 2 separate auctions to a single bundled auction whenever the surplus loss with n buyers is less than the surplus loss with (n-1) buyers.

This Theorem provides a sufficient, but not necessary, condition for compatibility of the seller's bundling decision and *ex ante* efficiency This condition must hold for some n, and in fact it will hold for infinitely many n for the following reason

Remark: The expected loss from a bundled auction vanishes to zero as n becomes large.

Proof: $\lim_{n \to \infty} E(H_1^n) = 2\bar{v} = \lim_{n \to \infty} 2E(F_1^n)$. That is, as the sample size becomes greater and greater, the expected highest value approaches the maximum value in the support.

□

Note also, however, that the surplus loss for n buyers produces an infinite sequence of positive numbers which converge to zero, but only in the limit. That is we have a sequence $\{a_2, \ldots, a_n, \ldots\}$ such that

$$a_n > 0 \quad \forall n = 2,3,\ldots$$

and

$$\lim_{n \to \infty} a_n = 0 .$$

This implies that there is an infinite subsequence which monotonically converges to 0 from above. Thus there is an infinity of values, n, such that the seller prefers separate auctions. [Note: One suspects that for any F, there exists some number $n_F^* > 2$ such that $n > n_F^*$ implies that the seller prefers separate auctions.

In other words, sellers prefer to bundle with a small number of buyers, but not with a large number of buyers.]

There are a couple of other desirable directions for generalization. We have assumed throughout the analysis that the seller and all buyers are risk neutral. Studies by Cox (1979), Matthews (1979), Harris and Raviv (1979a), and others suggest that risk aversion influences equilibrium behavior in auctions and affects the relative desirability of different auction mechanisms. One suspects that risk attitudes may also affect the seller's bundling choice and the optimality of packaging decisions.

Another assumption maintained in this paper is that the goods are independent, in the sense that v_j^i and v_k^i are independently distributed. One of the results obtained in this paper under the assumption that the values were additive is that a seller has a strong preference to bundle if there is no reservation bid or if the seller has zero cost. This suggests that in some cases in which 1 and 2 items are "duplicates" (or subadditive) in the sense that $v_1^i + v_1^i > v_{12}^i$ for all bidders, the seller will still prefer to sell the items together. Moreover, it is _ex ante_ efficient for "slight" duplicates to be sold as a bundle if the seller faces zero cost and sets an optimal reserve bid. On the other hand, if the seller does not set a reserve bid, it may be _ex ante_ efficient for items with a small degree of superadditivity ($v_1 + v_2 < v_{12}$) to be sold separately. Both of these possibilities seem nonintuitive and deserve further examination.

Along the above lines, the independence assumption implies that the two goods are completely different goods. Typically a given seller offers items for which an individual may have correlated values. This is the case which Adams and Yellen (1976) discussed in a different context. Additionally, it is not uncommon for some correlation to exist across individuals.

Another set of problems one might wish to pursue involves the symmetry assumptions about the buyers. Although they do have different values and have asymmetric information in the sense that each buyer knows his own value but only the distribution from which competitors' values are drawn, they all draw values from the same distribution; they all have the same perfect information about each of the other buyer's sample distributions as well as the seller's costs.

VI. CONCLUSION

This paper has discussed a set of decisions faced by a monopolist with incomplete information. How will a monopolist choose to package the items being sold and what are the efficiency implications of these decisions? An attempt is made to bridge the gap between recent theoretical work on specific types of demand revealing mechanisms commonly employed by monopolists (auctions) and a topic which heretofore has typically been analyzed in a world of perfect and complete information in which the monopolist used a standard take-it-or-leave-it pricing scheme. It was found that tied-in sales are sometimes *ex ante* optimal under a reasonable set of assumptions about a world in which there are no production economies or diseconomies and no demand interdependence. A number of additional results were obtained giving general sufficient conditions for buyers to prefer bundling, as well as conditions under which bundling is optimal in terms of maximizing expected consumer plus producer surplus.

APPENDIX A

Lemma A: Assume A1-A4. There are $n \geq 2$ bidders and $J \geq 2$ items. The expected total surplus in J separate auctions is greater than the expected total surplus in one bundled auction if and only if

$$\int_0^{\bar{v}} \left\{ \left[H\ (Jx) \right]^n - \left[F(x) \right]^n \right\} dx > 0 .$$

Proof: The expected total surplus (ETS) in an auction is simply the expected highest value. Therefore, for the bundled auction:

$$ETS_{Bundle} = \int_0^{J\bar{v}} nxh(x) \left[H(x) \right]^{n-1} dx$$

$$= x \left[H(x) \right]^n \Big|_0^{J\bar{v}} - \int_0^{J\bar{v}} \left[H(x) \right]^n dx$$

$$= J\bar{v} - \int_0^{\bar{v}} J \left[H(Jx) \right]^n dx .$$

Similarly, for each separate auction, j, $j = 1, \ldots, J$ we obtain:

$$ETSj = \bar{v} - \int_0^{\bar{v}} \left[F(x) \right]^n dx \quad .$$

Therefore:

$$\left[\sum_{j=1}^{J} ETSj\right] - ETS_{Bundle} = \int_{0}^{\bar{v}} J\left\{\left[H\ (Jx)\right]^{n} - \left[F\ (x)\right]^{n}\right\} dx\ .$$

□

Lemma B: Assume A1-A3, $n \geq 2$, $J \geq 2$. It is always the case that:

$$\int_{0}^{\bar{v}} \left\{\left[H(Jx)\right]^{n} - \left[F(x)\right]^{n}\right\} dx > 0\ .$$

Proof: This follows immediately from Lemma A, Theorem 5, Theorem 5J, Theorem 5N.

□

Proof of Theorem 5: *Ex ante* efficiency of separate auctions follows from the *ex post* efficiency shown in Theorem 3. In every possible state of the world, the total surplus generated by separate auctions equals the sum of the highest valuations on two items. This is always at least as great as the surplus generated by the bundled auction, which is simply the maximum sum of valuations.

Let $TS_1(v_1^1, v_2^1, v_1^2, v_2^2)$ be the total surplus, given values v_1^1, v_2^1, v_1^2, v_2^2 in the separate auction for item 1,

$TS_2(v_1^1, v_2^1, v_1^2, v_2^2)$ be the total surplus in the separate auction for item 2 and $TS_{12}(v_1^1, v_2^1, v_1^2, v_2^2)$ be the total surplus in the bundled auction. Then:

$$TS_1(v_1^1, v_2^1, v_1^2, v_2^2) = \max(v_1^1, v_1^2);$$

$$TS_1(v_1^1, v_2^1, v_1^2, v_2^2) = \max(v_2^1, v_2^2); \text{ and}$$

$$TS_{12}(v_1^1, v_2^1, v_1^2, v_2^2) = \max(v_1^1 + v_2^1, v_1^2 + v_2^2).$$

Now $Ts_1 + TS_2 \geq TS_{12}$ if $\omega = (v_1^1, v_2^1, v_1^2, v_2^2)$ is any realization of the random variables. Therefore, expecting over all possible states, ω, we obtain

$$ETS_1 + ETS_2 \geq ETS_{12}.$$

To show that this inequality is strict requires demonstrating that there is some set of states, Ω', which can occur with positive probability in which:

$$TS_1(\omega) + TS_2(\omega) > TS_{12}(\omega) \; \forall \; \omega \in \Omega'.$$

Since values are distributed independently on the cube $[o, \bar{v}]^4$ and the joint density is positive everywhere in the interior of the cube, if we can find some such set Ω' in the interior of this cube, the proof will be completed. This is easily done. Let $\omega = (\frac{2}{3}\bar{v}, \frac{1}{4}\bar{v}, \frac{1}{2}\bar{v}, \frac{1}{2}\bar{v})$ and observe that $TS_1(\omega) + TS_2(\omega) > TS_{12}(\omega)$. This is also true in a neighborhood of radius $\frac{1}{12}$ or less around ω. Hence

$$ETS_1(\omega) + ETS_2(\omega) > ETS_{12}.$$

□

APPENDIX B

Proof of Lemma 1.1 J: The proof is very similar to Lemma 1.1. If a buyer observes $v_1 = v_2 = \ldots = v_J = v$ then that buyer will prefer separate auctions if and only if :

$$J\left[v - \{v - \int_0^v \left[\frac{F(x)}{F(v)}\right] dx\}\right] F(v) > \left[Jv - \{Jv - \int_0^{Jv} \left[\frac{H(z)}{H(Jv)}\right] dz\}\right] H(Jv),$$

which reduces to

$$\int_0^v \left[F(x) - H(Jx)\right] dx > 0,$$

It is also simple to show by an argument virtually identical to the one used in Lemma 1.1, that if $a_1, \ldots a_{J-1}$ are all positive and $\sum_{j=1}^{J-1} a_j \leq Jv$, and Jv is in the support of F, then

$$J \int_0^v F(x)\, dx < \sum_{j=1}^{J-1} \int_0^{a_j} F(y_j) dy_j + \int_0^{Jv - \sum_{j=1}^{J-1} a_j} F(z) dz .$$

This proves the lemma. □

Proof of Theorem 1 J: This proof involves showing that the distributions of J independent and identical random variables, $X_1, \ldots, X_J$, are each mean-perserving-spreads of the distribution of the average, Z, of these random variables.

We can represent the mean, Z, of $X_1, \ldots, X_J$ as

$$Z = \frac{X_1 + X_2 + \ldots X_J}{J} .$$

Now for any j = 1,..., J, we can write:

$$X_j = \frac{X_j + \sum_{\substack{k=1 \\ k \neq j}}^{J} X_k}{J} + \frac{(J-1)X_j - \sum_{\substack{k=1 \\ k \neq j}}^{J} X_k}{J} = Z + \frac{(J-1)X_j - \sum_{\substack{k=1 \\ k \neq j}}^{J} X_k}{J} .$$

The second term of the right hand side of this expression has a mean of zero, conditional on Z, so that X_j is in fact a mean perserving spread of Z, for all j = 1,...,J. Hence the Theorem 1J is proved.

□

Proof of Theorem 2 J:

Let p and q be two alternative bundling decisions:

$$p = \{p_1 p_2, \ldots, p_k, \ldots, p_{Kp}\}$$

$$q = \{q_1, q_2, \ldots q_k, \ldots, q_{Kq}\} .$$

From equation (10) we obtain an expression for Δ_2 analogous to expression (11):

$$\Delta_2 = \left[2 \sum_{k=1}^{K}{}^{p} E(H^1_{pk}) - \sum_{k=1}^{K}{}^{p} E(H^2_{pk}) \right] - \left[2 \sum_{k=1}^{K}{}^{q} E(H^1_{qk}) - \sum_{k=1}^{K}{}^{q} E(H^2_{qk}) \right] \qquad (11')$$

$$= \left[\sum_{k=1}^{K}{}^{p} E(H^1_{pk}) - \sum_{k=1}^{K}{}^{q} E(H^1_{qk}) \right] + \left[\sum_{k=1}^{K}{}^{q} E(H^2_{qk}) - \sum_{k=1}^{K}{}^{p} E(H^2_{pk}) \right] .$$

Now the first term in brackets equals zero since each of the sums inside the brackets equals $J \int_0^{\bar{v}} xf(x)\, dx$, or J times the expected value of a single item. We are left with an expression analogous to (12):

$$\Delta_2 = \sum_{k=1}^{K}{}^{q} E(H^2_{qk}) - \sum_{k=1}^{K}{}^{p} E(H^2_{pk}) . \qquad (12')$$

The first sum is simply the total expected surplus under packaging decision q and the second term is the total expected surplus from packaging decision p . Therefore, the seller prefers p to q (i.e., $\Delta_2 > o$) if and only if p generates <u>less</u> surplus than q <u>ex ante</u>. Using an argument virtually identical to the one used in Theorem 3, selling all items in a single bundled auction generates the least expected total surplus. Therefore the seller prefers that bundling decision above all others, regardless of how many items there are.

□

In other words, the seller makes the least efficient bundling decision.

<u>Proof of Lemma 1.1N</u>: The proof here is virtually identical to Lemma

1.1. If a buyer observes $v_1 = v_2 = v$, then that buyer will prefer separate auctions if and only if:

$$2[v - \{v - \int_0^v [\frac{F(x)}{F(v)}]^{n-1}dx\}]F(v)^{n-1} \geq [2v - \{2v - \int_0^{2v} [\frac{H(x)}{H(2v)}]^{n-1}dz\}]H(2v)^{n-1},$$

which reduces to:

$$\int_0^v [F(x)^{n-1} - H(2x)^{n-1}]dx \geq 0. \tag{B1}$$

It is also simple to show by an argument virtually identical to the one used to prove Lemma 1.1, that:

$$2\int_0^v [F(x)]^{n-1}dx \leq \int_0^a [F(y)]^{n-1}dy + \int_0^{2v-a} [F(x)]^{n-1}dz \tag{B2}$$

for all $v \varepsilon [0, \frac{\bar{v}}{2}]$, and for any $a \varepsilon [0,2v]$. This proves the theorem.

□

Note that if $\frac{v_1^i + v_2^i}{2} = v$ and $v_1^i \neq v_2^i$, then (B1) is not a sufficient condition for i to prefer separate auctions, in light of (B2). However, if the inequality of (B1) is reversed, then this new inequality is a sufficient condition for any buyer i, such that $\frac{v_1^i + v_2^i}{2} = v$, to prefer a bundled auction.

<u>Proof of Theorem 1N</u>: From lemma B, we have a strict inequality which is exactly the opposite of (B1). Therefore, a buyer who observes $v_1 = v_2 = \bar{v}$ will prefer a bundled auction.

Proof of Corollary 1.1 N: Let:

$$\theta(v) = \int_0^v \left\{ \left[F(x)\right]^{n-1} - \left[H(2x)\right]^{n-1} \right\} dx \quad .$$

From assumption A3, $\theta(\cdot)$ is continous in v. By lemma B, $\theta(\bar{v}) > o$. Therefore, there exists some $t_F < \bar{v}$ such that $\theta(t_F) > o$ and $\theta(v) > o$ for all v such that $t_F \le v \le \bar{v}$. From B2 we can conclude that any buyer who observes values $v_1^i = v_2^i = v$ such that $t_F \le v = \bar{v}$ will prefer a bundled auction.

□

[Note that if the distribution of each of the two values is a simple mean preserving spread (see Diamond and Stiglitz pp. 338-339) of the distribution of the mean of the two values then there exists a unique t_F^* such that all buyers having valuations v_1^i, v_2^i such that $\frac{v_1^i + v_2^i}{2} \le t_F^*$ will prefer separate auctions, but for every $v > t$, there will always be at least one potential buyer who will prefer bundled auctions. An example of this is given in Appendix C.]

Proof of Theorem 2 N:

Proof: From (11), generalizing for the case of $n>2$, the seller prefers separate auctions if and only if:

$$nE(H_1^{n-1}) - (n-1)E(H_1^n) < 2\,[n\,E(F_1^{\,n-1}) - (n-1)E(F_1^{\,n})]\ ,$$

which can be rewritten as:

$$n[E(N_1^{n-1}) - 2\ E(F_1^{n-1})] < (n-1)[E(N_1^n) - 2\ E(F_1^n)]\ .$$

If the surplus loss with n buyers is less than the surplus loss with (n-1) buyers, then:

$$E(H_1^{n-1}) - 2\ E(F_1^{n-1}) < E(H_1^n) - 2\ E(F_1^n),$$

so

$$n[(E(H_1^{n-1}) - 2\ E(F_1^{n-1})] < (n-1)[E(H_1^n) - 2\ E(F_1^n)].$$

□

APPENDIX C

This is an example, showing that distributions which satisfy a symmetry property will lead buyers with "sufficiently high" valuations for both objects to prefer a bundled auction.

Proposition: Suppose there are 2 items, n buyers, and assumptions A1-A4 hold. Furthermore, assume that $F(\bar{v} - x) = 1 - F(x)$ for all $x \in [o,\bar{v}]$. Then there is a unique value, $\hat{v}_n$, for each n, such that if a buyer has valuations v_1, v_2 such that $v_1 + v_2 < 2\hat{v}_n$, then the buyer always prefers separate auctions and for every $v < \frac{\hat{v}n}{2}$, a buyer who has values $v_1 = v_2 = v$ will prefer a bundled auction. In addition:

(1) $\hat{v}_2 = \bar{v}$

(2) $\hat{v}_n \in (\frac{\bar{v}}{2},\bar{v})$ for $n = 3,4,\ldots$

Proof: Since $F(\bar{v} - x) = 1\text{-}F(x)$, we also know that $H(2(\bar{v} - x)) = 1\text{-}H(2x)$. We first use the fact that $H(2(\bar{v}-x)) = 1\text{-}H(2x)$ and $F(\bar{v}-x) = 1\text{-}F(x)$ for $x \in [0,\bar{v}]$. This gives us:

$$\int_0^{\bar{v}} [F(x)^{n-1} - H(2x)^{n-1}]dx =$$

$$\int_0^{\frac{\bar{v}}{2}} \{F(x)^{n-1} - H(2x)^{n-1} - [(1\text{-}H(2x))^{n-1} - (1\text{-}F(x))^{n-1}]dx. \qquad \text{(C1)}$$

The following lemma is useful.

Lemma 4N.1: Let $t > 0$, $A < t$, $n > 2$. Then $F(t) = t^{n-1} - (t-A)^{n-1}$ is an increasing function of t.

Proof: $F'(t) = (n-1)\ [t^{n-2} - (t-A)^{n-2}]$

$> 0.$ □

Notice that by letting $A = F(x) - H(2x)$, $t_1 = F(x)$, $t_2 = 1-H(2x)$ we can write (C1) in the following way:

$$\int_0^{\frac{\bar{v}}{2}} \{t_1^{n-1} - (t_1-A)^{n-1} - [t_2^{n-1} - (t_2-A)^{n-1}]\}dx\,.$$

Since $t_2 > t_1$ for all x, the integrand is always negative, so:

$$\int_0^{\bar{v}} [F(x)^{n-1} - H(2x)^{n-1}]dx < 0.$$

Therefore, by Lemma 1.1N., if there are n buyers, $n > 2$, a buyer with value $v_1 = v_2 = \bar{v}$ will prefer a bundled auction. In addition, it is easily verified that:

$$F(x) > H(2x) \qquad \text{for} \quad x\varepsilon(0,\tfrac{\bar{v}}{2})$$

and

$$H(2x) > F(x) \qquad \text{for} \quad x\epsilon(\tfrac{\bar{v}}{2},\bar{v}).$$

Because $\int_0^t [F(x)^{n-1} - H(2x)^{n-1}]dx = G(t)$ is continuous in t,

$G'(t) > 0$ for $t\varepsilon(0,\frac{\bar{v}}{2})$, $G'(t) < 0$ for $t\varepsilon(\frac{\bar{v}}{2},\bar{v})$, and $G(0) = 0$, $G(\bar{v}) < 0$, we can conclude that $G(t) > 0$ for $t\varepsilon(0,\frac{\bar{v}}{2})$ and there exists a unique $t = \hat{v}_n \ \varepsilon(0,\frac{\bar{v}}{2})$ such that $G(t) = 0$. If $n = 2$, (C1) reduces to 0. Thus we have established the desired result. □

FOOTNOTES

* The author thanks G. Chamberlain, R. Noll, M. Rothschild, C. Wilson and participants of the University of Chicago Public Policy Workshop for helpful comments and suggestions. All errors in the chapter are mine.

1. See, for example, Scherer(1970) pp. 505-512, Bailey(1954),Phillips(1979), Adams and Yellen(1976), Burstein(1960), Telser(1979) and Stigler(1968).
2. They may also be referred to as tied-in sales, bundling decisions or packaging decisions. An important point here is that tying contracts occur frequently in highly competitive markets as well, and may often be an efficient method of allocating resources. In other words, "tying contract" should carry no particular positive or negative connotation.
3. In a recent paper by Harris and Raviv(1979b), they demonstrate that under conditions of demand uncertainty, the seller will often use an auction mechanism rather than a take-it-or-leave-it pricing scheme, such as the one analyzed by Phillips(1979) and Adams and Yellen(1976). This is always true if the goods are "rare" in the sense that potential demand exceeds the capacity to supply and supply is fixed. In Phillips and Adams and Yellen, the producer has a constant marginal cost, so that an infinite amount could be produced. In this paper, we follow the assumption used in Harris and Raviv(1979b) that capacity is fixed. Therefore, this analysis is applicable to the wide variety of situations in which potential

(or zero-price) demand exceeds capacity to produce. An additional application involves the letting of incentive contracts in which a purchaser pays the lowest bidding contractor to fulfill some contractual obligation. Because of the specific nature of these contracts, this type of monopsonist situation typically involves some type of auction mechanism. See Stark and Rothkopf(1979), Holt(1979), and Baron(1972) for examples along these lines.

4. Because all agents are risk neutral, all results apply equally to first- and second-price auctions. Because it facilitates analysis, sections 1 and 2 are written as if only first-price auctions were being considered and section 3 seems to be addressing primarily second price auctions. This note is to alert the reader that these results generalize to both types of auctions.

5. This efficiency problem is discussed elsewhere, as well. See, for example, Vickrey(1961), Matthews(1979).

6. For more thorough treatments of the derivation of equilibrium bidding functions, the reader is referred to Wilson(1977), Ortega-Reichard(1968), Vickrey(1961), Harris and Raviv(1979a), Riley and Samuelson(1979).

7. This was pointed out to me by Gary Chamberlain.

8. See Matthews (1979).

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CHAPTER FOUR

An Empirical Examination of the Economic Consequences of Bundling Decisions by a Monopolist Facing Uncertain Demand

I INTRODUCTION

This chapter reports the results of an empirical examination of the predictions made in chapter three. Testable hypotheses were developed in that chapter which addressed questions about seller revenues, market efficiency, buyer behavior, and distributional consequences of a monopolistic seller's bundling decision in multiple object auctions. The data presented in this chapter provide strong support for these theoretically-based hypotheses.

Recall the scenario discussed in chapter three. There are several buyers and one seller. The seller has a set of indivisible items to sell. The seller makes a bundling decision, which is a partition of the set of items into mutually exclusive and collectively exhaustive subsets, called bundles. The seller places the bundles for sale using a first-price sealed-bid auction. In each such auction, the seller solicits private, written bids for a bundle from each buyer and sells that bundle to the highest bidder at a price equal to the highest bid.

The theoretical model presented earlier postulates a specific type of information structure in these markets in order to incorporate uncertainty in a precise way. The information structure is the following. Each buyer is supposed to have a fixed "valuation" for each item, which is known to no other buyer. Each buyer knows the probability distribution from which the valuations of each of the items for each of the other bidders were independently drawn. Buyers all know their own valuations with certainty. A buyer's valuation for a bundle simply equals the sum of his valuations of the items contained in the bundle. Buyers also know how many other buyers there are.

Each auction is modelled as a game with incomplete information, in which each buyer is a player. A strategy of a player is simply a function which maps valuations into bids. These strategies in general may depend on the distribution of values for the bundle being auctioned and the number of competing buyers in that auction. Testable hypotheses are then derived based on noncooperative behavior on the part of the buyers.

In order to provide a clear test of this theory, a number of variables must be measurable and controllable. In particular, reasonable comparisons with the theoretical predictions require a knowledge of the relevant probability distributions, buyer valuations and the number of bidders competing. In addition, four key assumptions are made in the theoretical model which are particularly difficult to control for. One of these assumptions is that a buyer's valuation for a lot equals the sum of his valuations for the bundle. A second assumption is that there is no after market in which the items may be resold by the winning bidder to the other buyers who participated in the auction. A third assumption is that all of the buyers are risk neutral. A fourth assumption is that the postulated information structure accurately describes buyer information.

The type of data available from real estate auctions, art auctions and other frequently held auctions can be obtained easily enough, but do not provide enough information to measure and control for all of the parameters and assumptions of the model. In other words, such data would provide at best a very rough test of the theory. For this reason, a series of experimental auctions were designed and carried out in a carefully controlled laboratory environment. The buyers in these laboratories were given complete and accurate information about the distribution from which values were drawn and the number of bidders in each auction. All four key assumptions were met. One of these assumptions, that all buyers are risk neutral, normally would be impossible to control for. However even though buyers faced risk in each particular auction, each buyer participated in a sufficiently large number of auctions that any risk which existed was reduced to a nominal level through diversification. One of the more useful aspects of using experimental auctions is that it is possible to sell an item twice, once as a single-item bundle and once as part of a two-item bundle. This allows one to directly compare seller revenues, distributional consequences, buyer strategies and market efficiency in separate as opposed to bundled auctions. This intriguing possibility is taken full advantage of in this series of experiments and is one of the major reasons that such powerful tests of the theoretical hypotheses could be made.

The use of experiments to test theories about sealed bid auctions is not new. Frahm and Schrader (1970), Smith (1974), Miller and Plott (1979), Belovicz (1977), and Coppinger, Smith and Titus (1980) have presented results relating to the comparative revenue-generating power of various auction mechanisms. These mechanisms include first-price sealed- bid auctions, second-price sealed-bid auction, English (oral progression) auctions and Dutch (descending bid auctions). The approach used by Coppinger et al. most closely resembled the design of the experiments presented here. Not surprisingly, some of their observations about buyer behavior in first-price sealed bid auctions were also observed in the experiments discussed below. The other four previous studies examined markets in which multiple units of a homogeneous commodity were allocated to several buyers in each auction. Therefore their results are not directly comparable to the results of the experiments reported here.

II. EXPERIMENTAL DESIGN

Three series of experiments, using a total of 24 different subjects, were designed and carried out.[4] Each experiment consisted of 240 different auctions in which experimental subjects were buying items from the experimenter. In 120 of these auctions there were two competing buyers and in 120 of the auctions there were four competing buyers. This divided the auctions into two sets of market sizes according to the number of competing bidders. For such market size there were forty auctions selling a single item, forty auctions selling two items bundled together and forty auctions selling four items bundled together. Thus the auctions are divided into three sets along the dimension of bundle size. This 2 X 3, or 6-cell, design is summarized in Table 1. The entry in each cell of the table indicates the number of auctions of that type in an experiment. Henceforth a cell will be referred to by the Roman numeral in the upper right hand corner of that cell in Table 1. The buyers in an experiment will be referred to by Arabic numerals 1 through 8.

As one can easily deduce, no buyer competed in all auctions. In particular, each buyer competed in 10 of the 40 auctions in each of cells I, II, and III and 20 of the 40 auctions in each of cells IV, V, and VI. Specification of exactly which auctions a buyer participated in is given in detail later on in this section. However, at this point it will be helpful to describe how buyers' valuations for items were induced.

For each auction in cells I and IV, each participating buyer

		MARKET SIZE	
		2 Bidders	4 Bidders
BUNDLE SIZE	1 Item	I 40	IV 40
	2 Items	II 40	V 40
	4 Items	III 40	VI 40

Table 1. The Basic 2 x 3 Experimental Design.

was given a valuation which was independently drawn at random from the interval $0.00 to $1.99. All valuations were in penny increments. The values of the bundles for which buyers competed in cells II and V were determined by adding together values which had been randomly drawn for items in cells I and IV. Whichever subset of the buyers competed in two single-item auctions also competed in the corresponding two-item bundled auction. Similarly, the values of the bundles for which buyers competed in cells III and VI were obtained by adding together values of certain pairs of two-item bundles from cells II and V. Again, whichever subset of the buyers competed in a pair of two-item bundled auctions also competed in the corresponding four-item bundled auction.

In order to facilitate smooth operation of the experiment each series of 240 auction experiments was conducted over a sequence of 5 "market years". This meant that in each year buyers had to make simultaneous bidding decisions in only 18 auctions. Conducting the experiment in this fashion also allowed buyers to make adjustments in their strategies after each year if they wished. Of the 18 experiments in which each buyer participated during a market year there were two each in cells I, II and III and four each in cells IV, V and VI. Which auctions were conducted in which year was randomized for each cell.

A potential problem with this particular design is that if a buyer can figure out which bundled auctions correspond to which unbundled auctions that buyer will be able to use information from previous auctions to update his priors about the distribution of competitors' valuations in corresponding auctions which may occur in later years. This problem was avoided in the following way. For each bundle in cells II and V for which a buyer would be competing that buyer was given two new values which were randomly chosen subject to the constraint that the sum of the two values equaled the sum of the two values of the items in the corresponding single-item auctions of cells I and IV. The buyers were not informed that these draws were dependent in this way on earlier draws. This prevented buyers from figuring out that there was a connection between the auctions in different cells. Buyers viewed each auction as a completely independent event.

The next problem to overcome involved setting up the auctions so that bidders who competed with each other in a bundled auction also competed with each other in the corresponding separate auctions. This was done in the following way. In cells I, II and III buyers only competed in the following pairs: [1,2]; [1,3]; [2,3]; [4,5]; [4,6]; [5,6]; [7,8]. The first six of these groups competed in five auctions in each of cells I, II and III while the last group, [7,8], competed in ten auctions in each of these cells. In cells IV, V and VI the buyers were divided into two groups, [1,3,5,7] and [2,4,6,8]. These two groups each participated in twenty auctions in each of cells IV, V and VI.

Each experiment was conducted in the following way. The eight experimental subjects were each given a folder containing a list of valuations, 5 information and record sheets (one for each market year), 5 bidding forms (one for each market year) and a three page instruction booklet. At the beginning of the experiment, the experimenter read the instructions while the subjects followed along. In these instructions, the subjects were told that the values of the 210 items on their list of valuations had been randomly chosen in the range from $0.00 - $1.99. In addition they were told how to figure out the value of a bundle, how to keep records, how to bid and how to calculate their profits. The instructions are given in the appendix. After these instructions were read and questions were answered, a "practice" year took place which allowed the subjects to become accustomed to the rules and the recording format. No payoffs were made on the basis of outcomes of this practice year. During each real market year each buyer privately submitted to the experimenter written bids for each of the 18 auctions in which the buyer was competing during that year. When all eight buyers had submitted their bidding forms, the experimenter announced the highest and the second highest bids in each of the 48 auctions that year. This information was posted so that all subjects could study the information if they wished. Subjects recorded their profit for each auction in which they had participated and then proceeded to the next market year.

Summarizing the design, there were three experiments in which a total of 720 auctions were conducted. Perhaps the most important aspect of the design is that each bundled auction corresponds in a very carefully planned way with some set of separate auctions. Because of this, items are essentially sold three times: once in a single-item auction; once in a two-item bundled auction; and once in a four-item bundled auction. This facilitates the analysis of the experimental data tremendously by making it possible to use relatively simple statistical techniques to test the predictions. These techniques, along with the results of the statistical tests, are described in the next two sections.

III. THEORETICAL PREDICTIONS AND TESTABLE HYPOTHESES

This section discusses the predictions which are derived from the theoretical model detailed in chapter three. Following each prediction, a number of specific testable hypotheses are stated in a form which allows the data from the experimental auctions to be brought to bear directly on the validity of the predictions.

There are four different types of predictions which are made:

(1) predictions about seller revenues;

(2) predictions about buyer surplus;

(3) predictions about buyers′ strategies; and

(4) predictions about total surplus (efficiency).

In all, eleven predictions are made. The first four predictions, (P1) - (P-4), deal with the effect of bundling and number of bidders on the seller′s revenue. The next five predictions, (P5) - (P9), address the question of buyers′ surplus and the distributional consequences of bundling. The next prediction, (P10), asserts that bidding strategies are superadditive. The final prediction, (P11), states that bundling creates inefficiencies. Each prediction is discussed in detail below. To facilitate this discussion, direct reference is made to theorems, corollaries and lemmata from chapter three.

For each hypothesis, two types of tests are made. First, three comparisons are made, one for each of the three series of experiments, between the results of bundled and separate auctions.

Second, a ggregate comparisons are also made by pooling the data points from all three series of experiments.

Theorems 2,2J and 2N lead to specific predictions about seller revenues. Theorem 2 states that if there are two bidders and two items, the seller can earn more money selling the two items in a bundled auction than in two separate auctions. Theorem 2J extends this to more than two items. Thus our first prediction:

> (P1) If there are two buyers competing in an auction, then the expected revenue per item generated in a bundled auction is an increasing function of the number of items in the bundle.

This prediction is robust with respect to the distribution from which buyers' values are drawn. In particular, P1 is true for uniformly distributed values. Recall that the uniform distribution was used in all laboratory auctions because it was the simplest to explain to the participants of the auctions.

In order to test this prediction using the experimental data, comparisons are made between average revenues from auctions in each of the six cells of Table 1. Specifically, the statistical hypotheses for prediction (P1), which are tested using both aggregated and disaggregated data are the following.

> (H1.1): The sample mean of differences between revenues generated by two item auctions in cell II and sums of revenues generated in the corresponding single-item auctions of cell I

is significantly greater than zero.

(H1.2): The sample mean of differences between revenues generated by four-item auctions in cell III and the sums of revenues generated in the corresponding single-item auctions of cell I is significantly greater than zero.

(H1.3): The sample mean of differences between revenues generated by four-item auctions in cell III and sums of revenues generated in the corresponding single item auctions of cell I is significantly greater than the sample mean of differences between revenues generated by two-item auctions in cell II and sums of revenues generated in the corresponding single-item auctions of cell I.

The second prediction about seller revenues is derived from a corollary of theorem 2N.

(P2) If there are more than three buyers, then the expected revenue per item generated in a bundled auction is a decreasing function of the number of items in the bundle.

The following specific hypotheses are used to test prediction (P2).

(H2.1): The sample mean of differences between revenues generated by two-item auctions in cell V and sums of revenues generated in the corresponding single-item auction of cell IV is significantly less than zero.

(H2.2): The sample mean of differences between revenues generated by four-item auctions in cell VI and sums of revenues generated in the corresponding single-item auctions of cell IV is significantly less than zero.

(H2.3): The sample mean of differences between revenues generated by four-item auctions in cell VI and sums of revenues generated in the corresponding two-item auctions of cell V is significantly less than zero.

The intuition behind (P1) and (P2) is that the expected second highest sample out of N draws from the distribution is the seller's expected revenue, where N is the number of bidders. Since the distribution of the value of a given item is a simple mean preserving spread of the distribution of the average value of a bundle of at least two items, then if the expected second highest sample of N draws from the distribution is greater than the mean of the distribution, it will also be greater than the expected second highest sample of N draws from a distribution of the average value of a bundle containing at least two items. Similarly, if the expected second highest value is less than the mean, the opposite will be true.

In (P1), there are only two bidders, so the expected second highest value is the expected lowest value which is always less than the mean. In (P2), the expected second highest value with more than three bidders is always greater than the mean for any symmetric distribution such as the uniform distribution. Hence bundling will hurt the seller in this case.

Combining (P1) and (P2) and using theorem 2J, one obtains the following result.

(P3) The absolute difference between the expected revenue per item in a bundled auction and the expected revenue per item in separate auctions is an increasing function of the number of items being bundled.

Prediction (P3) is just a weaker form of predictions (P1) and (P2) and is tested using the same statistical hypotheses.

Another prediction about the seller's revenue is that more buyers increase the seller's revenue.[2]

(P4) The revenue from an auction is an increasing function of the number of buyers.

This is a very well known and intuitively evident theoretical result. Prediction (P4) is tested by the following three hypotheses.

(H4.1) The sample mean of revenues generated by auctions in cell I is significantly less than the sample mean of revenues generated by auctions in cell IV.

(H4.2) The sample mean of revenues generated by auctions in cell II is significantly less than the sample mean of revenues generated by auctions in cell V.

(H4.3) The sample mean of revenues generated by auctions in cell III is significantly less than the sample mean of revenues generated by auctions in cell VI.

Theorems 1, 1N, 1J, corollary 1.1, the proposition in the appendix and the proofs of lemma 1.1, lemma 1.1J and lemma 1.IN lead to specific hypotheses about buyer surplus and the distributional consequences of the seller's bundling decision. Theorem 1J states that when there are two buyers then all buyers will prefer separate auctions ex ante. Note that it does not imply that a buyer will prefer separate auctions in every state of the world.

(P5) If there are two buyers, then no matter what values a buyer has for the items, the expected surplus to that buyer is greater in separate auctions than in a bundled auction.

The testing of this and the next four predictions about the distributional consequences of bundling require a somewhat more involved statistical analysis. In each auction in each of cells II, III, V and VI, each buyer is categorized according to the magnitude of his valuation of items in the bundle and the variation of his valuations of items in the bundle. Specifically, there are three categories of magnitude (high, medium and low) and three categories of variation (high, medium and low). This divides the buyers in each auction in cells II, III, V and VI into nine categories. This is illustrated in Table 2. Each category is labelled in Table 2 for future reference. Entries for each category are the average difference between the average surplus for buyers in that category in a bundled auction and the average sum of surpluses for buyers in that category in the corresponding separate auctions.

The cutoff points to determine in which category a buyer belong

		VARIATION OF VALUATIONS		
		HIGH	MEDIUM	LOW
MAGNITUDE OF VALUATIONS	HIGH	A	D	G
	MEDIUM	B	E	H
	LOW	C	F	J

Table 2. Categorization of buyers according to magnitude and variation of valuations for items in a bundle.

are chosen so that the expected number of cases in each category were approximately equal[5]. Unfortunately, due to correlation between magnitude and variation, it is not possible to cross-categorize sets of values in such a way that the expected number of cases is exactly the same across all the nine categories. In statistical terms, this results in a small sacrifice of the efficiency of our statistical tests. This means that in the tests which we will be using, it may be particularly difficult to establish statistical significance.

The following statistical hypothesis is tested to determine the validity of the prediction (P5). Again, as in earlier hypotheses, each buyer in each auction is treated as a separate case.

> (H5.1) In auctions from cells II and III, entries for all categories, A, B,C,D,E,F,G,H and J are significantly greater than zero.

The intuition behind prediction (P5) is that with a small number of buyers, all buyers are better off as the distribution becomes more spread out. The distribution of values for a bundle is less spread out than the distribution of the component items of the bundle. However, if there are more than two buyers, then a buyer who has extremely high values for a bundle prefers a less dispersed distribution. The logic behind this is that in a less dispersed distribution, the probability that such a buyer would have the highest value is greater. Hence, for large numbers of bidders one loses the "unanimity" result of theorem 1J. An even stronger statement of (P5) is the following prediction which is also an immediate consequence of theorem 1J.

(P6) If there are two buyers, then the expected surplus to a buyer is a decreasing function of the number of items bundled.

The following statistical hypothesis is tested to determine the validity of prediction (P6).

(H6.1) Entries for all categories in cell II are each significantly less than the corresponding entries for all categories in cell III.

Corollary 1.1N and the proposition in appendix A of chapter three yield specific predictions about which buyers will prefer a bundled auction, if there are more than two buyers.

(P7) If there are more than three buyers, then buyers with relatively high valuations on all items and relatively small variation in valuation will prefer a bundled auction to several separate auctions.

The following statistical hypotheses are tested to determine the validity of prediction (P7).

(H7.1) Entries for categories A,B,C,D,E,F,H and J in cells V and VI are are each significantly greater than zero.

(H7.2) The entry for category G in cells V and VI is significantly less than zero.

(H7.3) Entries for categories A,B,C,D,E,F,H and J in cell V are each significantly less than the corresponding entires in cell VI.

(H7.4) Entries for category G in cell V are significantly greater (less negative) than the corresponding entry in cell VI.

This prediction, (P7), expresses the rather obvious notion that in general different buyers are differently affected by the seller's bundling decision. In particular, theorem 1.1N, in conjunction with the proposition proved in appendix 3 and the proof of lemma 1.1 lead to the following two predictions.

(P8) Buyers with relatively high variation in valuations are relatively worse off when the seller bundles compared to buyers who have the same average valuation but less variation in their values.

(P9) Buyers with relatively low average valuations are relatively better off when the seller bundles compared to buyers who have the same variation in valuation, but have medium valuations.

An example which illustrates the intuition behind (P8) is the following. Suppose a buyer has a valuation of 0 on one item and A (the maximum possible value) on the second item. If these two items are sold separately, the bidder will (in equilibrium) win the second item with probability 1. However if the two items are bundled and sold together in a single auction, this bidder will probably win neither item. This is particularly obvious if there are many bidders. In addition, in equilibrium the profit he makes in the bundled auction if he wins is less than the equilibrium profit if he wins the second of two separate auctions. Suppose instead that this buyer's valuations were A/2 and A/2 instead. In such a case the bidder is affected not nearly as much by a seller who chooses to bundle.

The logic behind (P9) is the following. A buyer with valuations for the two items equal to 0 and 0 will earn 0 regardless of the seller's bundling decision. Buyers with very high valuations are either unaffected by the seller's bundling decisions or else are better off in a bundled auction than in separate auctions. However, from corollary 1.1N, we know that buyers with medium valuations are worse off in bundled auctions. In particular, for the uniform distribution, and for bidders with no variation in values, the difference between the expected profit in a bundled auction and expected profit in separate auctions starts at 0 for $v_1 = v_2 = 0$ monotonically decreases up to the point where $v_1 = v_2 = A/2$ and then monotonically increases up to some non-negative number as $v_1 = v_2 = v$ increases from $A/2$ to A.

The hypotheses designed to test prediction (P8) are the following:

(H8.1) In cells II, III, V and VI, the entries in categories D,E,F are significantly less than the entries in categories A,B,C, respectively.

(H8.2) In cells II,III,V and VI, the entries in categories

G,H and J.are significantly less than the entries in categories A,B and C, respectively.

(H8.3) In cells II,III,V and VI, the entries in categories G,H and J are significantly less than the entries in categories D,E and F, respectively.

The hypotheses designed to test prediction (P9) are the following.

(H9.1) In cells II,III,V and VI, the entries in categories B,E and H are significantly greater than the entries in categories A,D and G, respectively.

(H9.2) In cells II,III,V and VI, the entries in categories B,E and H are significantly greater than the entries in categories C,F and J, respectively.

The next prediction is easily derived theoretically by using equation (2) in chapter three to calculate the bidding functions in single-item and bundled auctions when values of items are distributed uniformly. This prediction is not as important for the sake of its economic implication, rather it is important because it is perhaps the most crucial prediction about bidding behavior upon which many of the other predictions may live or die.

(P10) A bidder with a given set of values for a given set of items will bid an amount in a bundled auction which is greater than the sum of his bids if the items were sold in separate auctions.[3]

As in some of the earlier predictions, this is due to the fact that the distribution of values for a single item is a mean-preserving spread of the distribution of sums of values for a bundle of items.

To facilitate analysis of prediction (P10), "bid differences" are calculated. A bid difference equals a buyer's bid for a bundle minus the sum of that buyer's bids in the separate single-item auctions for the items contained in the bundle. The hypotheses designed to test prediction (P10) are the following.

(H10.1) The sample mean of the bid differences between cells I and II is significantly greater than zero.

(H10.2) The sample mean of the bid differences between cells I and III is significantly greater than zero.

(H10.3) The sample mean of the bid differences between cells I and II is significantly less than the sample mean of the bid differences between cells I and III.

(H10.4) The sample mean of the bid differences between cells IV and V is significantly greater than zero.

(H 10.5) The sample mean of the bid differences between cells IV and VI is significantly greater than zero.

(H10.6) The sample mean of the bid differences between cells IV and V is significantly less than the sample mean of the bid differences between cells IV and VI.

Hypotheses H10.1 through H10.6 are tested at three different levels of aggregation: the individual level; the experiment level (aggregating the behavior of 8 buyers); and the fully aggregated level (aggregating the behavior of 24 buyers).

The final prediction is given by theorems 3, 3J and 3N.

(P11) The total surplus per item generated by an auction is a decreasing function of the number of items sold as a bundle in that auction.

In other words, bundling creates inefficiences. The intuition behind this is simple. If items are sold separately, in equilibrium the highest bidder in a given auction will have the highest valuation for the corresponding item. This is the perfectly efficient case. If several items are sold as a bundle, then in equilibrium the highest bidder for that auction will have the highest valuation for the bundles. However, that buyer will not in general have the highest valuation for each separate item in the bundle. This creates inefficiencies.

For each auction, efficiency is measured by the valuation of the winning bidder divided by the sum of the highest valuation of each of the separate items being sold in the auction. In other words, we measure efficiency as the percent of maximum possible total surplus. For each auction in cells II, III, V, VI, an efficiency difference is calculated, which equals the percent of maximum total surplus in separate auctions minus the percentage of maximum total surplus in the corresponding bundled auction The following hypotheses were designed to test prediction (P11).

(H11.1) The sample mean efficiency differences in cells II, III, V and VI are each significantly greater than zero.

(H11.2) The sample mean efficiency difference in cell III is significantly greater than the sample mean efficiency difference in cell II.

(H11.3) The sample mean efficiency difference in cell VI is significantly greater than the sample mean efficiency difference in cell V.

In this section, 29 testable hypotheses were outlined. The next section of this paper presents the results of the experimental auctions which were designed as an empirical test of these hypotheses. Some final concluding remarks are made in section V.

IV. DATA

The data provide support for the hypotheses concerning the effect of bundling on seller revenue. The mean revenue differences for each experiment as well as the pooled mean revenue differences for all three experiments combined are given in Table 3. There is weak support for hypotheses (H1.1), (H1.2), and (H1.3) but the statistical tests for the pooled mean revenue differences between cells I,II and III are not significant at the ten percent level.[6] However, these pooled means have the right sign in each case. So in the three experiments conducted, on average the seller was better off bundling when there were 2 buyers, but not significantly better off.

The data offer very strong evidence that supports hypotheses (H2.1) (H2.2) and (H2.3). Using a one-tailed test all three of the pooled mean differences have the right signs[7] and all are significant at the one percent level. In fact all of these mean differences in each of the three experiments are also significantly less than zero at either the five percent or the one percent level. Thus we can state with a great deal of confidence that with four bidders the seller is worse off bundling.

The prediction that seller revenue increases as a function of the number of bidders is also borne out by the data. Table 4 contains the revenue differences.[8] In each experiment, the revenues in cells IV, V, and VI were significantly greater than the revenues in cells I,II and III respectively. The significance level was one percent. Thus hypotheses (H4.1), (H4.2) and (H4.3) cannot be rejected.

	Experiment 1	Experiment 2	Experiment 3	Experiment 1-3 (pooled)
cell II - cell I	1.9 (5.0)	13.9*** (3.8)	-8.5 (5.6)	2.5 (2.9)
cell III - cell I	12.9* (9.3)	11.5** (6.3)	-13.3 (6.2)	3.7 (4.3)
cell III - cell II	11.0 (10.6)	-2.4 (7.4)	-4.8 (8.4)	1.2 (5.2)
cell V - cell IV	-23.2*** (4.4)	-13.2** (6.0)	-19.0*** (4.0)	-18.5*** (2.8)
cell VI - cell IV	-57.7*** (5.7)	-51.2*** (10.6)	-58.5*** (6.7)	-55.8*** (4.6)
cell VI - cell V	-34.5*** (7.2)	-39.0*** (12.2)	-39.5*** (7.8)	-37.3*** (5.4)

Table 3. Revenue differences attributable to bundling. Standard errors are in parentheses

* Significant at 10% level.
** Significant at 5% level.
*** Significant at 1% level.

	Experiment I	Experiment 2	Experiment 3	Experiments 1-3 (pooled)
cell I - cell IV	-45.925***	-29.825***	-18.775***	-31.250***
cell II - cell V	-62.725***	-29.025***	-27.025***	-41.598***
cell III - cell VI	-113.100***	-53.500***	-29.950***	-65.517***
t-value for I-IV	(6.6)	(7.5)	(5.8)	(3.9)
t-value for II-V	(9.9)	(11.2)	(9.8)	(6.2)
t-value for III-VI	(13.6)	(14.5)	(11.8)	(8.3)

Table 4. Revenue differences attributable to number of competing buyers.

* Significant at 10% level
** Significant at 5% level
*** Significant at 1% level

With few exceptions, the experiments supported the hypotheses about the distributional consequences of bundling. On average, most buyers were made significantly worse off from bundling. Data for the individual experiments are given in Table 5. These differences[9] for all of the experiments pooled are summarized in Table 6.

In cell II, the major exceptions to this are those buyers who have relatively little variation in valuations for items in a bundle. In these exceptions, the signs of the observed mean surplus differences were almost always positive, but not significantly greater than zero at the ten percent level. In cell III, only categories G and H (again "low variation" categories) failed to have significantly positive surplus differences, but again the signs of these means were positive. Each individual experiment also supports these hypotheses although due to fewer data points (i.e. fewer auctions), fewer categories showed significantly positive mean differences, although most of these sample mean differences had the correct sign. We conclude that the experiments strongly support hypothesis (H5.1) which states that if there are two buyers, a buyer is better off on average when the items are sold in separate auctions regardless of his valuations.

The next hypothesis, (H6.1), states that when there are two buyers, on average buyers are worse off the more items that are bundled together. Table 7 displays the sample mean surplus differences between cells II and III. In six of the nine categories, the hypothesis is supported at a one percent significance level. The three exceptions are cells D, G and H. One should note that these are

	Experiment 1				Experiment 2				Experiment 3			
	Cell II	Cell III	Cell V	Cell VI	Cell II	Cell III	Cell V	Cell VI	Cell II	Cell III	Cell V	Cell VI
A	3.46 (7.87)	41.70* (28.22)	-.93 (8.24)	3.39 (12.26)	23.7** (11.1)	61.8*** (11.4)	17.5*** (6.8)	20.4*** (7.2)	6.6 (15.6)	37.3*** (14.0)	-4.9 (10.9)	28.7** (16.6)
B	20.00*** (7.70)	57.00*** (12.28)	20.65*** (5.17)	33.22*** (4.22)	26.5*** (7.1)	63.3*** (7.1)	18.7*** (3.6)	41.3*** (5.7)	23.0*** (8.3)	51.6*** (6.3)	15.2*** (3.5)	29.9*** (6.8)
C	7.67 (6.29)	28.21*** (8.06)	10.89*** (4.16)	34.64*** (9.12)	11.1** (4.8)	35.0*** (9.2)	12.3*** (4.5)	24.1*** (4.6)	11.0** (6.5)	61.6*** (7.7)	8.2*** (2.7)	15.9*** (3.5)
D	16.14 (17.70)	9.86 (27.14)	-.28 (8.11)	-7.71 (7.83)	25.6** (13.2)	-1.5 (10.5)	-15.8 (10.4)	29.6** (15.2)	-.5 (9.2)	27.1** (12.1)	-32.1 (13.7)	-29.4 (17.0)
E	0.0 (0.0)	51.14** (24.25)	6.69** (3.73)	24.67** (12.39)	1.7 (1.7)	32.3*** (10.5)	9.0 (9.0)	25.6*** (8.1)	6.7 (6.7)	21.0 (28.0)	2.3 (2.8)	9.3* (6.8)
F	4.10*** (1.43)	25.80*** (10.53)	3.33* (2.32)	7.29*** (2.30)	4.5* (2.9)	42.3*** (15.8)	1.4*** (.6)	1.5 (1.5)	2.5** (1.4)	10.0 (10.0)	0.0 (0.0)	7.4** (3.5)
G	-11.25 (25.77)	27.25 (65.71)	-16.00* (10.43)	-24.64* (17.82)	17.3** (7.7)	12.5 (13.9)	2.3 (7.7)	-13.0* (10.1)	3.3 (9.9)	-6.1 (13.7)	-9.8* (6.3)	-6.2 (19.2)
H	1.20 (19.70)	7.00*** (0)	-10.25 (5.38)	16.83*** (5.58)	0.0 (8.2)	-22.0 (0.0)	-2.4 (2.0)	10.1* (6.7)	-35.0 (35.0)	24.0*** (4.0)	3.3 (2.7)	6.4 (5.9)
J	-.08 (.40)	3.92*** (1.43)	0.0 (0.0)	7.82*** (2.75)	-1.0 (1.0)	14.0** (8.1)	0.0 0.0	1.5** (.7)	2.6* (2.0)	10.1*** (2.8)	0.0 (0.0)	4.2* (2.8)

Table 5. Buyers' mean surplus differences attributable to bundling (experiments 1, 2, 3). Standard errors are in parentheses.

* Significant at 10% level.
** Significant at 5% level.
*** Significant at 1% level.

	Cell II	Cell III	Cell V	Cell IV
A	11.2** (6.3)	48.3*** (10.1)	6.7* (4.9)	17.1*** (6.7)
B	22.9*** (4.4)	61.7*** (5.3)	18.5*** (2.5)	35.2*** (3.1)
C	10.3*** (3.2)	41.6*** (5.6)	10.3*** (2.1)	22.2*** (2.9)
D	9.8* (7.4)	16.0* (12.8)	-12.3 (6.2)	-4.4 (8.0)
E	3.6 (2.8)	38.4*** (10.9)	4.9** (2.3)	19.2*** (5.0)
F	3.9*** (1.1)	28.9*** (7.9)	1.4** (.7)	6.0*** (1.7)
G	6.5 (6.6)	2.8 (11.3)	-7.8** (4.5)	-13.4* (10.3)
H	-4.9 (9.6)	8.3 (11.0)	-3.1 (2.4)	10.5*** (3.6)
J	.6 (.7)	8.1*** (2.1)	0.0 (0.0)	4.7*** (1.4)

Table 6. Buyers' mean surplus differences attributable to bundling (all experiments pooled). Standard errors are in parentheses.

* Significant at 10% level.
** Significant at 5% level.
*** Significant at 1% level.

	Experiment 1		Experiment 2		Experiment 3		Experiment 1-3 (pooled)	
	Cell II- Cell III	Cell V- Cell VI	Cell II- Cell III	Cell V- Cell VI	Cell II- Cell III	Cell V Cell VI	Cell II- Cell III	Cell VI- Cell V
A	38.2* (29.3)	4.3 (14.8)	38.1*** (15.9)	2.9 (9.9)	30.7* (21.0)	33.6** (19.9)	37.1*** (11.9)	8.4 (8.3)
B	37.0*** (14.5)	12.6** (6.7)	36.8*** (10.0)	22.6*** (6.7)	28.6*** (10.4)	14.7** (7.6)	38.8*** (6.9)	16.7*** (4.0)
C	20.5** (10.3)	23.7*** (10.0)	23.9** (10.4)	11.8** (6.4)	50.6*** (10.1)	7.7** (4.4)	31.3*** (6.4)	11.9*** (3.6)
D	6.2 (33.0)	-7.4 (11.2)	-27.1 (16.9)	45.4*** (18.4)	27.6** (15.6)	2.7 (21.8)	6.2 (14.8)	7.9 (10.1)
E	51.1** (24.3)	18.0* (12.9)	30.6*** (10.6)	16.6* (12.1)	14.3 (28.8)	7.0 (7.4)	34.8*** (11.3)	14.3*** (5.5)
F	21.7** (10.6)	4.0 (3.3)	37.8*** (16.1)	.1 (1.6)	7.5 (10.1)	7.4** (3.5)	25.0*** (8.0)	4.6*** (1.8)
G	38.0 (70.6)	-8.6 (20.6)	-4.8 (15.9)	-15.3 (12.7)	-9.4 (16.9)	3.6 (20.2)	-3.7 (13.1)	5.6 (11.2)
H	5.8 (19.7)	27.1*** (7.8)	-22.0 (8.2)	12.5** (7.0)	59.0** (35.2)	3.1 (6.5)	13.2 (14.6)	13.6*** (4.3)
J	4.0*** (1.5)	7.8*** (2.8)	15.0** (8.2)	1.5** (.7)	7.5** (3.4)	4.2* (2.8)	7.5*** (2.2)	4.7*** (1.4)

Table 7. Comparison of buyers' mean surplus differences with 2-item bundles and 4-item bundles. Standard errors are in parentheses.

* Significant at 10% level.
** Significant at 5% level.
*** Significant at 1% level.

three categories for which these differences are predicted to be quite small. Even so, two of these three categories had means of the predicted sign. Again the results strongly support the theoretical prediction.

Hypotheses (H7.1), (H7.2) and (H7.3) are very strongly supported by the data. In Table 6, categories A,B,C,E and F have significantly positive entries in cell V. In category J there were no observations other than 0 since none of these buyers won any separate or bundled auctions. Category D is a bit of an anomaly since it shows a significantly negative sign. In cell VI, all categories except D have positive mean surplus differences and they are significant. Category D is negative but not significant. The data from individual experiments (Table 5) are also supportive of (H7.1), but fewer categories are significant due to larger standard errors resulting from a smaller sample size. Thus we conclude that (H7.1) is strongly supported by the data with the exception of category D.

Hypothesis (H7.2) predicts that in cells V and VI the entry for category G should be significantly less than zero. In the pooled data, this is confirmed. This is also confirmed in the individual experiments (Table 5) in the four cases in which the sign of the sample mean is significant.

Hypothesis (H7.3) is supported in all categories except A and D and Hypothesis (H7.4) is not supported (refer to Table 7). Again, with few exceptions this indicates that predictions for most categories are qualitatively very precise. This also appears to be true when using the data from the three experiments individually instead of pooled. The signs are generally correct in the individual experiments, but are not always significant because of a small sample problem.

Tables[10] 8 and 9 display the summarized data used to test (H8.1) (H8.2) and (H8.3). Using the pooled data (Table 9), (H8.1) was strongly supported in all cells and for all categorical comparisons except for the comparison of categories A and D in cell II. Once again the sign was correct but not significant at the ten-percent level. Hypothesis (H8.2) was similarly supported with one exception: the sign for the comparison between categories A and G was correct but not significant. Hypothesis (H8.3) was also supported by the pooled data but the evidence was not as convincing as the evidence supporting (H8.1) and (H8.2). In all four cells, the mean surplus difference between categories D and G were not significantly different. This was also true for the comparison of categories E and H in cell III and the comparison of F and J in cell VI. All other signs were correct and significant at the ten-percent level.

Table 8 shows the sample means and standard errors used to test (H8.1), (H8.2) and (H8.3) in each of the three experiments. These data also offer similar support for these three hypotheses, but the support is not as strong because fewer signs are significant due to the smaller sample size. From Tables 8 and 9 we conclude that H8.1 H8.2 and H8.3 cannot be rejected.

The next set of hypotheses, (H9.1) and (H9.2), are tested by

	Experiment 1				Experiment 2				Experiment 3			
	Cell II	Cell III	Cell V	Cell VI	Cell II	Cell III	Cell V	Cell VI	Cell II	Cell III	Cell V	Cell VI
A-D	-12.7 (19.4)	31.8 (39.1)	-.6 (11.5)	10.1 (14.6)	-1.9 (17.2)	63.3*** (15.5)	33.3*** (12.4)	-9.3 (16.8)	7.1 (18.1)	10.1 (18.8)	27.2** (17.5)	58.2*** (23.8)
B-E	20.0*** (7.7)	5.9 (27.2)	13.9** (6.4)	8.5 (13.1)	24.8*** (7.3)	47.0*** (12.7)	9.7 (9.7)	15.7* (9.9)	16.3* (10.7)	30.7 (28.7)	12.9*** (4.5)	20.5** (9.6)
C-F	3.6 (6.5)	2.4 (13.3)	7.6* (4.8)	27.3*** (9.4)	6.6 (5.6)	-7.3 (18.3)	10.9*** (4.5)	22.6*** (4.8)	8.5 (6.6)	51.6*** (12.6)	8.2*** (2.7)	8.5** (4.9)
A-G	14.7 (27.0)	14.4 (71.5)	15.1 (13.2)	28.0 (21.6)	6.5 (13.5)	49.3*** (18.0)	15.2* (10.2)	33.4*** (12.4)	3.3 (18.5)	43.4** (19.6)	4.8 (12.6)	35.0* (25.4)
B-H	18.8 (21.2)	50.0*** (12.3)	30.9*** (7.5)	16.4*** (7.0)	26.5*** (10.8)	91.3*** (7.1)	21.1*** (4.1)	31.2*** (9.0)	58.0* (36.0)	27.7*** (7.5)	11.8*** (4.4)	23.4*** (9.0)
C-J	7.8 (6.3)	24.3*** (8.2)	10.9 (4.2)	26.8*** (9.5)	12.1*** (4.9)	21.0** (12.3)	12.3*** (4.5)	22.6*** (4.7)	8.4 (6.8)	51.5*** (8.2)	8.2*** (2.7)	11.7*** (4.5)
D-G	27.4 (31.3)	-17.35 (71.1)	15.7 (13.2)	16.9 (19.4)	8.4 (15.3)	-14.0 (17.4)	-18.1 (12.9)	42.6*** (18.2)	-3.8 (13.5)	33.3** (18.6)	-22.3 (15.1)	-23.2 (25.6)
E-H	-1.2 (19.7)	44.1** (24.3)	17.0*** (6.5)	7.9 (13.6)	1.7 (8.4)	54.0*** (10.5)	11.4 (9.2)	15.5* (10.7)	41.7 (35.6)	-3.0 (28.3)	-1.1 (3.9)	2.9 (9.0)
F-J	4.2*** (1.5)	21.9** (10.6)	3.3* (2.3)	-.5 (3.6)	5.5** (3.1)	28.3* (17.8)	1.4*** (.6)	0.0 (1.7)	-.1 (2.4)	-.1 (10.4)	0.0 (0.0)	3.2 (4.5)

Table 8. Comparison of buyers' mean surplus differences along the dimension of variation of values (Experiment 1, 2, 3). Standard errors are in parentheses.

* Significant at 10% level.
** Significant at 5% level.
*** Significant at 1% level.

	Cell II	Cell III	Cell V	Cell VI
A-D	1.3 (9.7)	32.3** (16.3)	19.0*** (7.9)	21.5** (10.4)
B-E	19.4*** (5.2)	23.3** (12.1)	13.5*** (3.4)	16.0*** (5.9)
C-F	6.4** (3.4)	12.7* (9.7)	8.9*** (2.2)	16.3*** (3.2)
A-G	4.7 (9.1)	45.6*** (15.2)	14.6** (6.7)	30.5*** (12.3)
B-H	27.9*** (10.6)	53.5*** (12.2)	21.6*** (3.5)	24.8*** (4.8)
C-J	9.7*** (3.3)	33.5** (6.0)	10.3*** (2.1)	17.6*** (3.2)
D-G	3.3 (9.9)	13.3 (17.1)	-4.5 (7.7)	9.1 (13.0)
E-H	8.5 (10.0)	30.1** (15.5)	8.1*** (3.3)	8.8* (6.2)
F-J	3.3*** (1.3)	20.8*** (8.2)	1.4** (.7)	1.3 (2.2)

Table 9. Comparison of buyers' mean surplus differences along the dimension of variation of values (all experiments pooled). Standard errors are in parentheses.

* Significant at 10% level.
** Significant at 5% level.
*** Significant at 1% level.

comparing sample means of buyers′ surplus differences along the dimension of the magnitude of the buyers′ valuations for items in a bundle. Table 10 contains this data for each separate experiment and Table 11 contains the data pooled from all three experiments. For all experiments combined, the signs of the differences were significantly positive, as predicted, for all categories in cell VI. In cell V, four of the six categorical comparisons had signs which were significant and consistent with hypothesis. The exceptions were the differences between categories G and H and the differences between categories H and J. Considering that the entries in Table 2 for each of the three, G,H and J, are predicted by theory to be quite small, the fact that the differences between the entries in these three categories were insignificantly different from zero is not particularly surprising. In cells II and III combined, only four of the twelve signs were significant. All of these signs conformed with the predicted signs. Table 10 also shows that if each experiment is analyzed separately, an overwhelming proportion of the categorical comparisons have the predicted sign and a large number have not only the predicted sign but are also significant.

In all of the hypotheses making comparisons between the nine categories (H5.1 - H9.2), the tests have shown more significant comparisons when there are four bidders (cells V and VI) than when there are two bidders (cells II and III). There is a good reason for this. The comparisons in cells V and VI have twice as many observations as the comparisons in cells II and III since there are four bidders instead of two bidders. This results in larger standard

	Experiment 1				Experiment 2				Experiment 3			
	Cell II	Cell III	Cell V	Cell VI	Cell II	Cell III	Cell V	Cell VI	Cell II	Cell III	Cell V	Cell VI
B-A	16.5* (11.0)	15.3 (30.8)	21.5** (9.7)	29.8** (13.0)	2.8 (13.2)	7.6 (13.4)	1.2 (7.7)	21.0** (9.2)	16.4 (17.7)	14.4 (15.4)	20.1** (11.4)	58.5*** (17.9)
E-D	-16.1 (17.7)	41.2 (36.4)	7.0 (8.9)	32.4** (14.6)	-23.9 (13.3)	33.8** (14.8)	24.8** (13.8)	-4.0 (17.2)	7.2 (11.4)	-6.1 (30.7)	34.3*** (14.0)	38.7*** (18.3)
H-G	12.5 (32.5)	-20.2 (65.7)	5.8 (11.7)	41.4** (18.7)	-17.3 11.2	-34.5 (13.9)	-4.7 (8.0)	23.1** (12.3)	-38.3 (36.4)	30.1** (14.3)	13.1** (6.9)	12.6 (20.1)
B-C	12.3 (9.9)	28.8** (14.7)	9.7* (6.7)	-1.4 (10.0)	15.4** (8.0)	34.3*** (11.6)	6.4 (5.8)	17.2*** (7.3)	12.0 (10.5)	-9.9 (9.9)	6.9* (4.4)	13.9** (7.6)
E-F	-4.1 (1.4)	25.3 (26.5)	3.4 (4.4)	17.4* (12.6)	-2.8 (3.4)	-9.9 (19.0)	7.6 (9.0)	24.1*** (8.2)	4.2 (6.8)	11.0 (29.7)	2.3 (2.8)	1.9 (7.6)
H-J	1.3 (19.7)	3.1** (1.4)	-10.25 (5.4)	9.0* (6.3)	1.0 (8.3)	-36.0 (8.1)	-2.4 (2.0)	8.6 (7.0)	-37.6 (35.1)	13.9*** (4.9)	3.3 (2.7)	2.2 (6.5)

Table 10. **Comparison of buyers' mean surplus differences along the** dimension of magnitude of values (experiments 1, 2, 3). Standard errors are in parentheses.

* Significant at 10% level.
** Significant at 5% level.
*** Significant at 1% level.

	Cell II	Cell III	Cell V	Cell VI
B-A	11.7* (7.7)	13.4 (11.4)	11.7** (5.5)	18.1*** (7.4)
E-D	-6.2 (7.9)	22.4* (16.8)	17.3*** (6.6)	23.6*** (9.4)
H-G	-10.4 (11.6)	5.5 (15.8)	4.7 (5.1)	23.9** (10.9)
B-C	12.6** (5.4)	20.1*** (7.7)	8.2*** (3.3)	13.0*** (4.2)
E-F	-.3 (3.0)	9.5 (13.5)	3.6* (2.4)	13.3*** (5.3)
H-J	-5.5 (9.6)	.1 (11.2)	-3.1 (2.4)	5.8* (3.9)

Table 11. Comparison of buyers' mean surplus differences along the dimension of magnitude of values (all experiments pooled). Standard errors are in parentheses

* Significant at10% level.
** Significant at 5% level.
*** Significant at 1% level.

errors of the estimated mean differences in cells II and III than in cells V and VI, so that one would expect fewer of the signs to be significant.

The next set of hypotheses, H10.1, H10.2, H10.3, H10.4, H10.5 and H10.6, addresses the question of superadditivity of buyers' bidding strategies. Table 12 displays these data for individual buyers. Since there were three experiments, there were twenty-four buyers in all. Each entry is an average measure of superadditivity from a specific buyer. Each row corresponds to a different buyer and each column corresponds to an average difference between an individual's bids in bundled auctions from one cell and sums of that individual's bids in the corresponding unbundled auctions in another cell. Positive entries indicate superadditivity and negative entries indicate subadditivity. As one can see, the preponderance of entries are positive as predicted. In fact 83 of the 144 entries in the table are significantly positive at the one percent level. Note that column one corresponds to (H10.1), column two corresponds to (H10.2) and so forth. Referring to column one, thirteen of the twenty-four buyers satisfy (H10.1) at the ten percent significance level, and all but one of the remaining eleven buyers bid superadditively, but not significantly superadditively. The other hypotheses are even more strongly supported by this table. There is not one single entry which is significantly less than zero even at a significance level of fifteen percent. What is remarkable is that so many entries in this table are significantly positive even though the sample size for entries in the first three columns is only ten and the sample size in

	Cell II-Cell I	Cell III-Cell I	Cell III-Cell II	Cell V-Cell IV	Cell VI-Cell IV	Cell VI-Cell V
Experiment 1	7.9*** (2.1)	33.2*** (4.6)	25.3*** (5.1)	4.8*** (2.0)	29.7*** (3.3)	24.9*** (3.9)
Experiment 2	19.4*** (3.1)	65.9*** (5.8)	46.5*** (6.6)	15.3*** (2.0)	51.6*** (3.6)	36.3*** (4.1)
Experiment 3	9.9*** (3.4)	39.4*** (4.2)	29.5*** (5.4)	4.3*** (1.5)	24.8*** (2.9)	20.5*** (3.3)

Table 13. Mean bid differences (by experiment). Standard errors are in parentheses.

* Significant at 10% level.
** Significant at 5% level.
*** Significant at 1% level.

Buyer Number	Cell II-Cell I	Cell III Cell I	Cell III-Cell II	Cell V-Cell IV	Cell VI-Cell IV	Cell VI-Cell V
1	5.6 (5.6)	55.3*** (8.0)	49.7*** (9.8)	16.5*** (4.7)	79.0*** (4.5)	62.5*** (6.5)
2	16.5** (7.7)	9.3 (11.0)	-7.2 (13.4)	6.1* (4.0)	31.1*** (5.5)	25.0*** (6.8)
3	2.1* (1.3)	3.6*** (1.4)	1.5 (1.9)	1.8 (7.0)	-.4 (.4)	-2.1 (7.0)
4	.8 (2.0)	8.6 (9.2)	7.8 (9.4)	-3.3 (6.0)	9.5 (10.0)	12.8 (11.7)
5	14.4*** (4.0)	56.2*** (7.3)	41.8*** (8.3)	13.0*** (3.1)	37.4*** (6.2)	24.4*** (6.9)
6	6.8 (11.0)	49.6*** (19.3)	42.8** (22.2)	1.7 (4.4)	22.2*** (7.0)	20.5*** (8.3)
7	4.4 (5.3)	32.8*** (7.2)	28.4*** (8.9)	.6 (5.6)	41.6*** (7.4)	41.0*** (9.3)
8	12.8*** (4.9)	50.0*** (18.7)	37.2** (19.3)	2.0 (8.0)	17.0 (14.4)	15.0 (16.5)

the second three columns is only twenty. In Table 13, bid differences are averaged across all bidders, by experiment. When this aggregation is performed, all entries are significantly positive at the one percent level. The data clearly confirm (H10.1) - (H10.6) beyond much doubt.

A final observation about bidding behavior can be made. Buyers did not bid according to the bidding strategies predicted by theory. In fact, with few exceptions, bids exceeded the equilibrium predictions. This observation was also made by Coppinger, et al. on the basis of their experimental data. Apparently this phenomenon is a persistent one. The fact that, despite this, the predictions from the theory about the effects of bundling decisions were strongly supported by the experiments described in this paper indicates that predictions of this sort are quite robust. Nonetheless, this phenomenon points to a weakness in the theory of bidding behavior which deserves to be explored in future research endeavors.

The final set of hypotheses, (H11.1), (H11.2), (H11.3), addresses the loss of efficiency due to bundling. Table 14 presents average efficiency differences between cells for each experiment and for all experiments pooled. The efficiency measure used was the percent of maximum total surplus that was generated by the auction. The average efficiency difference between two cells is equal to the average difference between the surplus generated in separate auctions and the surplus generated in the corresponding bundled auction. The evidence presented in Table 14 overwhelmingly supports (H11.1),(H11.2) and (H11.3). Undeniably bundling creates inefficiences ex post.

	Cell I- Cell II	Cell I- Cell III	Cell II- Cell III	Cell IV- Cell V	Cell IV- Cell VI	Cell V- Cell VI
Experiment 1	4.6*** (1.7)	9.1*** (2.1)	4.5** (2.7)	13.7*** (2.1)	18.5*** (1.8)	4.8** (2.8)
Experiment 2	6.0*** (2.0)	14.4*** (1.7)	8.4*** (2.6)	13.7*** (2.7)	21.3*** (1.9)	7.6*** (3.3)
Experiment 3	8.5*** (2.2)	11.6*** (1.7)	3.3 (2.8)	6.4*** (1.7)	16.1*** (1.7)	9.7*** (2.4)
All Experiments pooled	6.4*** (1.1)	11.7*** (1.1)	5.3*** (1.5)	11.3*** (1.3)	18.7*** (1.0)	7.4*** (1.6)

Table 14. Mean efficiency differences. Standard errors are in parentheses.

* Significant at 10% level.
** Significant at 5% level.
*** Significant at 1% level.

V CONCLUSIONS

A series of experiments was designed and carried out with the purpose in mind being to provide data to statistically test a number of specific predictions generated by a theoretical model of the behavior of buyers in first-price auctions when uncertainty is present. The predictions were qualitative in nature and addressed questions of the effects of bundling and the effect of the number of competing bidders on seller revenues, the ex ante distribution of buyer surplus, the strategies of the buyers, and the ex post distribution of total surplus (i.e. efficiency). The predictions which held out quite well under statistical scrutiny were:

(1) Seller revenues increase as a function of the number of bidders.

(2) Buyers bid superadditively.

(3) When there are a large number (four or more) of bidders, the seller is better off not bundling.

(4) Bundling creates significant inefficiences.

(5) On average, buyers with relatively more dispersed valuations are affected more adversely under bundling than buyers with relatively less dispersed valuations.

(6) Ex ante, buyers are affected adversely by bundling.

(7) If there are a large number (four or more) of bidders, then buyers with exceptionally high valuation benefit, on average,

from bundling.

Two of the predictions were supported by the data, but not so convincingly as the above predictions. These two are:

(1) With two competing bidders, the seller is better off bundling.

(2) On average, buyers with mediocre valuations are affected more adversely under bundling than buyers with relatively high or relatively low valuations.

In no way can one claim that these results are absolutely conclusive. However, the statistical tests indicate that they are almost certainly replicable. Such attempts to replicate the results would be welcome.

In conclusion, the model developed in chapter 3 as judged by the success of its testable predictions in these three experiments, provides a theoretical framework which can be useful in predicting the effects of a seller's bundling decision on market outcomes. Thus it is a reasonable approach which should be developed and embellished in such a way as to extend the set of economic environments to which it can be applied.

APPENDIX

Instructions

General Instructions:

This is an experiment in the economics of market decision-making. Various research foundations have provided funds for this research. The instructions are simple. If you follow them carefully and make good decisions, you might earn a considerable amount of money. Your earnings will be paid to you in cash at the end of the experiment. In addition, you will also be paid $3.00 at the end of the experiment for your participation.

In this experiment we are going to conduct auctions in which you will all be buying items from the experimenter. You will participate in several such auctions in a sequence of 5 market years. In your folder you will find an information and record sheet for each market year, as well as a "list of valuations." These will determine the amount you will be paid if you win an auction. You are not to reveal this information to anyone. It is your own private information.

Specific Instructions:

During each market year, several auctions are conducted for several lots. Each lot consists of one or more items. At the beginning of each year, you will be asked to submit private, written bids for each of the lots listed in the first column of your information and record sheet. These bids must be in penny increments. You have been provided with bidding forms for this purpose. Each lot consists of the set of items listed in column two. You will also be told how many of the peop;e in the room (including yourself) will be bidding on each lot. This number is listed in the fourth column of your information and record sheets.

Your redemption value for each item may be found in your "list of valuations." You will notice that all of your redemption values are between $0.00 and $1.99. Each of these values was drawn perfectly randomly in the range from $0.00 to $1.99 for each bidder. Each value for each bidder for each item is equally likely to be anywhere from $0.00 to $1.99. Therefore, different bidders will almost certainly have different values for each item. All values are in penny increments. The only values you know for sure are your own and you are not to reveal any information about these to anyone else.

Remember, each lot may consist of several items. The total value to you of a lot equals the sum of your redemption values for the items in the lot. For your convenience, your total redemption value for each lot has been calculated for you and is listed in column three of your information and record sheet.

Your Profit:

If someone else submits a higher bid than yours for a particular lot, you neither receive nor pay any money. Your profit for that lot is zero.

If your bid for a lot is higher than any other bid for that lot, then the experimenter will pay you your redemption values for all items in that lot minus your bid for that lot. Your profit for that lot, if you win, is equal to the difference between your total redemption value for the lot and your bid for the lot. For example, suppose that lot #2 consisted of items 48, 53, and 117, and your values for these items were \$0.91, \$1.45, and \$0.61, respectively. Then your total redemption value for lot #2 is:

$$\$0.91 + \$1.45 + \$0.61 = \$2.97.$$

If you submitted a bid of \$1.29 for lot #2 and this was the highest bid submitted by any bidder for lot #2, then your profit for this lot would be

$$\$2.97 - \$1.29 = \$1.68.$$

If you and at least one other bidder tie for the highest bid on a lot then your profit equals your total redemption value minus your bid divided by the number of winning bidders. In the example above, if one other bidder also submitted a bid of \$1.29, and no one submitted a bid higher than \$1.29, then your profit would be

$$\frac{\$1.68}{2} = \$0.84.$$

Your total profits for the experiment will be the sum of your profits in each auction plus a payment of \$3.00 for your participation.

Recording Instructions:

Each market year, your bids should be recorded in column five of your information and record sheet, and on the bidding forms which have been provided for you by the experimenter. When you have completed your bidding form raise your hand and the experiment will collect it.

After everyone has submitted their bidding forms for that year, the experimenter will announce the highest bid and the second highest bid for each lot. Please record the highest bid in column six of your information and record sheet. The experimenter will also announce whether there were any ties. When the experimenter has finished this, you should record your profit for each lot in the last column of your information and record sheet. Your total profit for the market year is computed by adding rows one through eighteen. Please record this number in the box at the bottom of the page. When everyone has done this, we will proceed to the next market year.

Are there any questions?

NOTES

1. This model of the bidding process was originally formulated in Vickrey (1961).

2. This is a well-known prediction of Vickrey's bidding model, and in fact is always true if bidders use an increasing bidding function.

3. This will be called superadditive bidding.

4. All subjects were undergraduate students at the California Institute of Technology.

5. For cells II and V the categories were divided in the following way: m = magnitude of valuation = (sum of valuations of items in the lot)/ (number of items in the lot); s = variation of valuations = (sum of absolute differences between the values of items in the lot and m for that lot)/ (number of items in the lot).

A	$118.3 < m < 199$	$s > 30$
B	$81.7 < m < 188.3$	$s > 30$
C	$0 < m < 81.7$	$s > 30$
D	$118.3 < m < 199$	$15 < s < 30$
E	$81.7 < m < 118.3$	$15 < s < 30$
F	$0 < m < 81.7$	$15 < s < 30$
G	$118.3 < m < 199$	$0 < s < 15$
H	$81.7 < m < 118.3$	$0 < s < 15$
J	$0 < m < 81.7$	$0 < s < 15$

For cells III and VI, the categories were divided in the following way:

A	$115 < m < 199$	$s > 40$
B	$85 < m < 115$	$s > 40$
C	$0 < m < 85$	$s > 40$
D	$115 < m < 199$	$30 < s < 40$
E	$85 < m < 115$	$30 < s < 40$
F	$0 < m < 85$	$30 < s < 40$
G	$115 < m < 199$	$0 < s < 30$
H	$85 < m < 115$	$0 < s < 30$
I	$0 < m < 85$	$0 < s < 30$

6. These entries are hypothesized to be significantly greater than zero. (In all tables, standard errors are in parentheses below the appropriate sample mean.)

7. These entries are hypothesized to be significantly less than zero.

8. These entries are hypothesized to be significantly greater than zero.

9. All entries in Table 5, 6, and 7 are hypothesized to be significantly greater than zero except for entries for category G in the "Cell V" columns, "Cell VI" columns and "Cell V-Cell VI" columns, which are hypothesized to be significantly less than zero.

10. All entries in Tables 8, 9, 10, 12, 13 and 14 are hypothesized to

be significantly greater than zero.

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