ELECTROMAGNETIC SCATTERING
BY A
SHORT RIGHT CIRCULAR CONDUCTING CYLINDER

Thesis by

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ABSTRACT

Quasi-static solutions of the problem of a right circular perfectly conducting cylinder with closed plane ends subjected to longitudinal electric, transverse electric, longitudinal magnetic and transverse magnetic uniform applied fields are found for cylinders with diameter to length ratios of $\frac{1}{4}$, $\frac{1}{2}$, 1, 2 and 4. The corresponding electric and magnetic polarizabilities, which are necessary and sufficient for determining the total cross section and angular distribution of scattering when the cylinder is illuminated by a plane electromagnetic wave arriving at any angle of incidence and in any state of polarization provided that the wavelength is long compared with the greatest dimension of the cylinder, are obtained. The method for calculating the scattering is indicated and outstanding features of the angular distribution in elementary cases are discussed.

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I. INTRODUCTION

1.00. Definition of the Problem and Applications. The research reported in this thesis is concerned with the scattering produced by a short (non-infinite in length) right circular perfectly conducting cylinder with closed plane ends when the latter is illuminated by a plane electromagnetic wave whose wavelength is long compared with the greatest dimension of the cylinder. The assumption of long wavelength reduces the problem to one in which the applied fields in the neighborhood of the cylinder may be regarded as locally uniform and in which only the induced quasi-static electric and magnetic dipole moments, $\bar{M}_E$ and $\bar{M}_M$, are of consequence. The primary objective, therefore, is the calculation of the electric and magnetic tensor polarizabilities, α_{ij} and β_{ij}, of a short right circular cylinder under the influence of uniform applied fields.

The results obtained make possible not only the calculation of the total cross section and angular distribution of scattering by a single isolated cylinder but also the prediction of the properties of an artificial refractive medium composed of a lattice of small cylinders according to the methods of Kaprielian (1) and others (2), (3). An example of this type of calculation is to be found, for instance, in the derivation of the Clausius-Mossotti formula. Another situation in which the results of this research are useful is in the problem of the perturbation of resonant cavity frequencies by the introduction of small cylindrical objects. A formula due to Müller (4) and Slater (5), recently rederived in an elegantly simple way by Papas (6), gives the frequency shift in terms of the unperturbed fields and the values of the induced moments. In the process of finding the magnetic polarizability, the problem of designing a winding to be placed on the surface of a finite cylinder in order to produce a uniform magnetic field within is automatically solved; also solved is the hydrodynamic problem of the flow of a perfect fluid around such a cylinder since the magnetic lines of force of the total external field have the same form as the streamlines of flow.

For the short right circular cylinder with body axes chosen as in figure 1.00-1, the two tensor polarizabilities are simultaneously diagonal and each has the property that its xx and yy components are equal. Only two distinct components of each tensor must be determined; these are called $\alpha_{\ell\ell}$, α_{tt}, $\beta_{\ell\ell}$ and β_{tt}, where ℓ refers to the longitudinal (z) direction and t to the transverse (x or y) direction. The tensor components are defined in terms of the dimensionless forms

$$\left.\begin{aligned} \alpha_{\ell\ell} &= \frac{M_{e\ell}}{\epsilon E_\ell v_o}\,; \\ \alpha_{tt} &= \frac{M_{et}}{\epsilon E_t v_o}\,; \\ \beta_{\ell\ell} &= \frac{\mu M_{M\ell}}{B_\ell v_o}\,; \\ \beta_{tt} &= \frac{\mu M_{Mt}}{B_t v_o}\,; \end{aligned}\right\} \qquad (1)$$

by employing the geometrical volume, v_o, of the cylinder as a divisor. Rationalized M.K.S. units are used in these formulas and in all others appearing in this thesis. The problems of determining the four components listed above are called the longitudinal electric, the transverse electric, the longitudinal magnetic and the transverse magnetic problems respectively; the corresponding abbreviations, LE, TE, LM and TM, are employed frequently.

For the cylinders considered here and for all other perfectly conducting objects with rotational symmetry, the TE and LM problems are intimately related and are sometimes collectively designated as the TE-LM problem. As a result of this relationship

$$\beta_{\ell\ell} = -\frac{1}{2}\,\alpha_{tt} \tag{2}$$

and all such objects which are small compared with wavelength have identical relative angular distributions of scattering provided that the incident wave is polarized with its magnetic vector parallel to the axis of rotational symmetry. Van de Hulst comments upon the angular distribution of scattering by an ellipsoid under these conditions (7). The statement expressed by equation 2 is proved in Section IV.

A uniform system of nomenclature with regard to the physical features and dimensions of the cylinder is used throughout. This system is presented in figure 1.00-2.

1.01. Method of Solution. As an example of the method of solution, the longitudinal electric problem is discussed in some detail; the general features of this method are the same as those of the procedure used by Smythe for the solution of the freely charged right circular cylinder (8). In the longitudinal electric problem the cylinder is subjected to a uniform applied electric field in the z direction and the

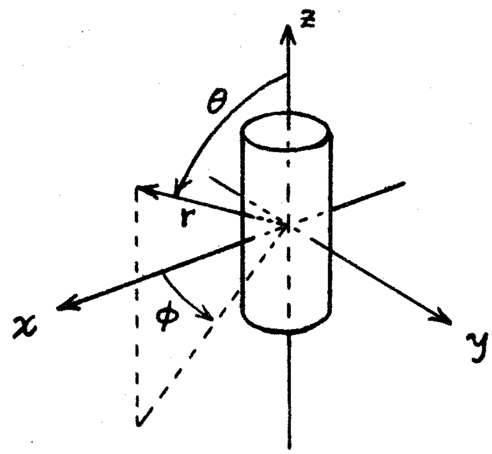

Figure 1.00-1. The Cylinder and its Body Axes.

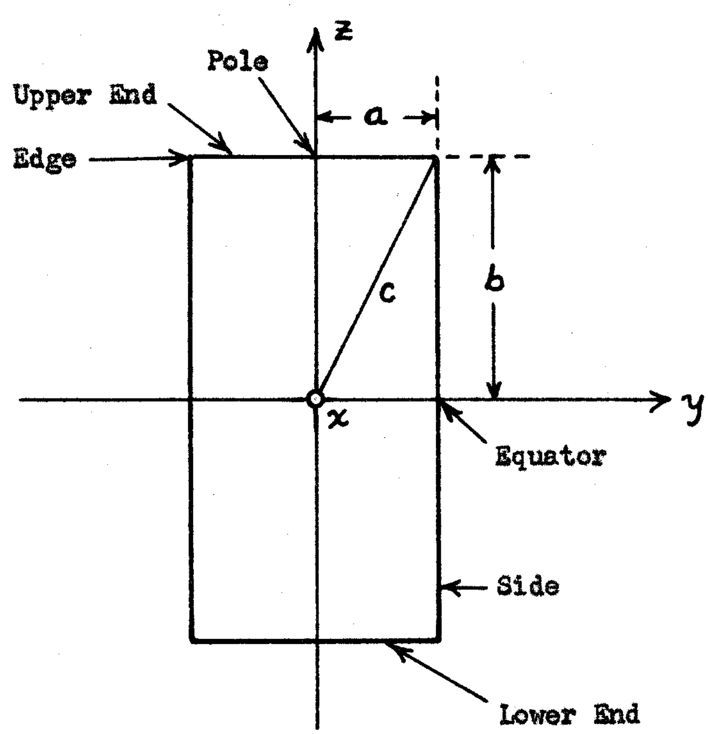

Figure 1.00-2. Cross section of the Cylinder with Nomenclature.

resulting electric dipole moment in the same direction is required. For a temporary expedient, the cylinder should be regarded merely as a geometrical surface upon which a variable charge distribution can be placed. This charge is the correct one if and only if the field which it generates is equal and opposite to the applied field at all points upon and within the surface. The technique employed has two salient aspects. First, the charge distribution is expressed in terms of a manageable number of variable coefficients in such a way that a good approximation to the true distribution is potentially attainable. Second, an equal number of mathematical conditions are imposed upon the internal (or surface) field of the charge so that, if satisfied, the applied field will indeed be cancelled to a good approximation within (or upon the surface of) the cylinder. The selection of coefficients and conditions must be such that a set of well-conditioned simultaneous linear equations is generated. The solution of these equations yields the values of the coefficients and the problem is solved.

Both aspects of the technique just described permit, at least in principle, a wide choice of methodology. In this thesis the charge distributions, whether on the side or end of the cylinder, are represented as finite weighted sums of certain biorthogonal functions based upon hypergeometric polynomials. Additional linear or constant terms, known as "basic" terms, are included to facilitate the approximation of the true distributions especially in cases involving very large or very small values of a/b. The biorthogonal functions employed here have an argument and three parameters; this gives them sufficient generality to serve for the description, not only of the charge distributions which occur in the LE problem, but of any of the charge or current distributions necessary for the solution of the other three problems. Two types of functions are defined. Only one of these plays a part in the charge or current expansions; the other is complementary to it in the orthogonality relation. The functions are

$$\bar{\Psi}_m(\gamma,\zeta,\nu,u) = a_m(-1)^m u^{\zeta}(1-u^2)^{\nu}{}_2F_1(-m, m+\nu+\gamma+\zeta; \gamma+\zeta; u^2) \quad (1)$$

and

$$\Psi_m(\gamma,\zeta,\nu,u) = b_m(-1)^m u^{\zeta}{}_2F_1(-m, m+\nu+\gamma+\zeta; \gamma+\zeta; u^2), \quad (2)$$

where

$$a_m = \frac{\Gamma(m+\gamma+\zeta)}{\Gamma(\gamma+\zeta)\,\Gamma(m+1+\nu)\,2^{\nu}} \quad (3)$$

and

$$b_m = \frac{2^{1+\nu}(2m+\nu+\gamma+\zeta)\,\Gamma(m+\nu+\gamma+\zeta)}{\Gamma(\gamma+\zeta)\,\Gamma(m+1)} . \quad (4)$$

The multiplicative constants, a_m and b_m, are defined as they are in order to simplify both the orthogonality relation and certain integral transforms (see Appendix A) which play an important part in the analysis. The orthogonality relation (which may be obtained from the Jacobi polynomial representations given in equations 6 and 7 below) is simply

$$\int_0^1 \bar{\Psi}_m \Psi_n \, u^{2\gamma-1}\, du = \delta_{mn} . \quad (5)$$

The parameter γ , which appears in this relation, creates the distinction between side and end functions since it is one-half for the former and unity for the latter; the dimensionless argument, u, is equal to z/b or to ρ/a in the two cases, respectively. The parameter ζ controls the parity of the function, being zero for even parity and unity for odd parity.

Finally, the parameter ν controls the type of singularity exhibited by the barred functions, $\bar{\psi}_m(u)$, as u tends to unity, that is as the edge of the cylinder is approached. For charge distributions and for current distributions in which the current flow is parallel to the edge, ν must equal minus one-third; for current distributions in which the flow is normal to the edge, however, ν takes the value of plus two-thirds. These values are dictated by the fact that, locally, the edge resembles a two dimensional right-angled corner, solutions for which may be obtained by the use of the Schwarz transformation. The various parameter values are summarized in Table 1.01-1.

Application	Parity	γ	ζ	$2\gamma - 1$
Side	even	$\frac{1}{2}$	0	0
Side	odd	$\frac{1}{2}$	1	0
End	even	1	0	1
End	odd	1	1	1

Table 1.01-1

All of the biorthogonal functions may be expressed in terms of Jacobi polynomials (9) as follows:

$$\bar{\psi}_m(\gamma,\zeta,\nu,u) = \frac{\Gamma(m+1)}{\Gamma(m+1+\nu)\,2^{\nu}}\, u^{\zeta}(1-u^2)^{\nu}\, P_m^{(\nu,\,\gamma+\zeta-1)}(2u^2-1)\,; \quad (6)$$

$$\psi_m(\gamma,\zeta,\nu,u) = \frac{2^{1+\nu}(2m+\nu+\gamma+\zeta)\,\Gamma(m+\nu+\gamma+\zeta)}{\Gamma(m+\gamma+\zeta)}\, u^{\zeta}\, P_m^{(\nu,\,\gamma+\zeta-1)}(2u^2-1). \quad (7)$$

In particular, the functions used on the side may also be expressed in terms of Gegenbauer polynomials (10):

$$\bar{\Psi}_m\left(\tfrac{1}{2},\zeta,\nu,u\right) = \frac{\Gamma(m+1)\,\Gamma(m+\frac{1}{2}+\zeta)\,\Gamma(\nu+\frac{1}{2})\,\pi^{-1/2}}{\Gamma(m+1+\nu)\,2^{\nu}\,\Gamma(m+\nu+\frac{1}{2}+\zeta)}(1-u^2)^{\nu}\, C_{2m+\zeta}^{\nu+1/2}(u); \quad (8)$$

$$\Psi_m\left(\tfrac{1}{2},\zeta,\nu,u\right) = \frac{2^{1+\nu}(2m+\nu+\frac{1}{2}+\zeta)\,\Gamma(\frac{1}{2}+\nu)}{\pi^{1/2}}\; C_{2m+\zeta}^{\nu+1/2}(u). \quad (9)$$

Returning to the specific consideration of the IE problem, the side and end charge distributions are written in terms of the dimensionless variable coefficients, r_m and t_m, as follows:

$$\frac{\sigma_s}{\epsilon E} = R(z) = \begin{cases} r_b \frac{z}{b} + \sum_{m=0}^{N_s-1} r_m \bar{\Psi}_m\left(\frac{1}{2},1,-\frac{1}{3},\frac{z}{b}\right) & \text{for } |z| \leq b \\ 0 & \text{for } |z| > b \end{cases}; \quad (10)$$

$$\frac{\sigma_e}{\epsilon E} = T(\rho) = \begin{cases} t_b + \sum_{m=0}^{N_e-1} t_m \bar{\Psi}_m\left(1,0,-\frac{1}{3},\frac{\rho}{a}\right) & \text{for } \rho \leq a \\ 0 & \text{for } \rho > a \end{cases}. \quad (11)$$

The terms involving r_b and t_b will be recognized at once as the basic terms described earlier; E is the magnitude of the applied electric field.

Attention is now directed to the mathematical conditions which the field of the charge must fulfill. Speaking of potential rather than of field, the potential of the charge distribution, $V(\rho,z)$ should ideally be equal to zE at all points upon and within the surface of the cylinder. If $\tilde{R}(k)$ and $\tilde{T}(k)$ are the Fourier and zeroth-order Bessel transforms of R(z) and T(ρ) respectively, it is shown in Section III that the potential of the charge distribution is given by

$$\frac{V}{E} = a\int_0^\infty \tilde{R}(k)\, K_o(ka)\, I_o(k\rho) \sin kz \; dk$$

$$+ a\int_0^\infty \tilde{T}(k)\, e^{-kb} J_o(k\rho) \sinh kz \; dk. \qquad (12)$$

The first integral converges within and upon the infinite cylinder $\rho = a$, the second between and upon the infinite parallel planes $z = \pm b$. The total expression therefore correctly represents the field at all points within and upon the surface of the cylinder.

A potential which is independent of ϕ and regular in a neighborhood of the origin, as is that given in equation 12, may be expanded in a series of the spherical harmonics $r^n P_n(\cos \theta)$. Since the nth harmonic reduces to z^n upon the z axis, the harmonic expansions for $I_o(k\rho) \sin kz$ and for $J_o(k\rho) \sinh kz$ have the same coefficients as the Maclaurin expansions of sin kz and sinh kz respectively, a fact noticed by Ramo and Whinnery (11). Thus the expansions are

$$I_o(k\rho) \sin kz = \sum_{p=0}^{\infty} \frac{(-1)^p}{(2p+1)!} (kr)^{2p+1} P_{2p+1}(\cos \theta) \qquad (13)$$

and

$$J_o(k\rho) \sinh kz = \sum_{p=0}^{\infty} \frac{1}{(2p+1)!} (kr)^{2p+1} P_{2p+1}(\cos \theta). \qquad (14)$$

Introducing c, the half-diagonal of the cylinder, at appropriate places in order to make certain quantities dimensionless, one finds that the potential of the charge distribution is

$$\frac{V}{cE} = \sum_{p=0}^{\infty} \Lambda_p \left(\frac{r}{c}\right)^{2p+1} P_{2p+1}(\cos\theta), \tag{15}$$

where the coefficients, Λ_p, are given by

$$\Lambda_p = \frac{1}{(2p+1)!} \frac{a}{c} \left\{ (-1)^p \int_0^\infty \tilde{R}(k)\, K_0(ka)\, (kc)^{2p+1} dk + \int_0^\infty \tilde{T}(k)\, e^{-kb} (kc)^{2p+1} dk \right\}. \tag{16}$$

Since $\tilde{R}(k)$ is a sum of functions involving the r_m coefficients and $\tilde{T}(k)$ a sum of functions involving the t_m coefficients, 16 becomes a set of simultaneous linear equations if a finite number of the Λ_p are set equal to desired values. Although there are N charge coefficients $(N = N_s + N_e + 2)$, only $N - 1$ of the Λ_p can be specified since another relationship called the "edge condition" must be included. This is explained in Section III. The matrix elements themselves are actually $_2F_1$ hypergeometric functions as is also shown in Section III. A potential which is "maximally linear" in a neighborhood of the origin and very nearly equal to zE is secured if

$$\Lambda_p = \begin{Bmatrix} 1 & \text{for} & p = 0 \\ 0 & \text{for} & 1 \leq p \leq N-2 \end{Bmatrix}. \tag{17}$$

The equations have been solved for $N = 18$ and for five different values ($\frac{1}{4}$, $\frac{1}{2}$, 1, 2 and 4) of a/b; the results are given in Section II. Similar techniques are used to solve the TE-LM problem and finally the TM problem which is by far the most difficult, and the most interesting of all.

It was originally planned to expand the potential of the charge distribution in the non-singular or unbarred biorthogonal functions, defined in equation 2, on the side and end surfaces of the cylinder itself and then to secure maximal linearity of the potential by setting appropriate coefficients equal to unity or to zero. This was, in fact, the real raison d'être of the biorthogonal functions. The plan was intended to exploit the fact that a solution of Laplace's equation which is regular within a given closed surface and is approximately linear, that is proportional to z for example, at all points upon this surface, will be even more accurately linear at points within. The matrix elements for the simultaneous linear equations which result when this method is used, however, involve infinite integrals of the products of four Bessel functions. Although these are reducible to convergent infinite series through the agency of Meijer's G-function (12), computational difficulty finally caused the abandonment of the method.

1.02. Checking the Solution. Checking of the solution of the IE problem is accomplished by calculating the total potential of both charge distribution and applied field at some point on the surface, for example the pole. If this total potential is not zero, it is possible to calculate a local deformation of the surface which, if actualized, would make the total potential on the deformed surface zero. The technique applicable at the equator is similar except for the fact that the z derivative of the potential rather than the potential itself must be used.

A modification of this method, employed for checking the TM problem at the equator, is described in Section V. The values of the local deformations at pole and equator, which are quite small compared with the cylinder dimensions, are given in Section II along with the other results.

An estimate of the accuracy of the results is obtained by solving the problem repeatedly with larger and larger numbers of equations and observing the limit toward which the calculated value of the polarizability tends. It is found that for a maximum number of equations equal to eighteen, the polarizability can be found to five significant figures with some doubt about the least significant figure. The accuracy is poorest for extreme ratios of a/b, best for a/b equal to unity. In the latter case, the least significant figure is either exact or in error by at most one unit.

II. RESULTS

2.00. Electric and Magnetic Polarizabilities. The two distinct tensor components of the electric and of the magnetic polarizability are given in Table 2.00-1.

a/b	LE Problem $\alpha_{\ell\ell}$	TE Problem α_{tt}	LM Problem $\beta_{\ell\ell}$	TM Problem β_{tt}
0	∞	2.0000	-1.0000	-2.0000
$\frac{1}{4}$	15.071	2.3151	-1.1575	-1.8508
$\frac{1}{2}$	7.0966	2.6115	-1.3057	-1.7352
1	3.8614	3.1707	-1.5853	-1.5793
2	2.4325	4.2173	-2.1087	-1.4131
4	1.7507	6.1814	-3.0907	-1.2716
∞	1.0000	∞	$-\infty$	-1.0000

Table 2.00-1

When a/b approaches infinity, the cylinder takes on the appearance of a disk and all four problems are susceptible to solution in closed form. For the TE and LM problems, these solutions are non-trivial and

$$\frac{M_{Et}}{\epsilon E_t} = -\frac{2\mu M_{M\ell}}{B_\ell} = \frac{16a^3}{3}, \qquad (1)$$

as was shown by Rayleigh (13). This means that as a/b tends to infinity,

$$\alpha_{tt} = -2\beta_{\ell\ell} \sim \frac{8}{3\pi}\frac{a}{b}. \qquad (2)$$

2.01. Scattering by Electric and Magnetic Dipoles; Total Cross Section. When a plane wave of angular frequency ω is incident upon the cylinder of figure 1.00-1, electric and magnetic dipoles varying sinusoidally in time are induced. In the general case, even in-phase components of these dipoles are not necessarily at right angles to one another in space. The total Hertz vector (14) in phasor* notation is

$$\check{\bar{\bar{Z}}} = \frac{e^{-ikr}}{4\pi\epsilon r}\left\{\check{\bar{M}}_E + \frac{1}{c}\left(\check{\bar{M}}_M \times \bar{e}_r\right)\left(1 - \frac{i}{kr}\right)\right\}, \quad (1)$$

where $k = 2\pi/\lambda = \omega/c$. The term "i/kr" becomes negligible in the far zone and will be dropped forthwith. As r tends to infinity

$$\check{\bar{\bar{E}}} = \nabla \times (\nabla \times \check{\bar{\bar{Z}}}) \sim k^2\left(\bar{e}_\theta \check{Z}_\theta + \bar{e}_\phi \check{Z}_\phi\right); \quad (2)$$

thus the complete expression for the far zone scattered electric field is

$$\check{E}_\theta = \frac{k^2 e^{-ikr}}{4\pi\epsilon r}\left\{\check{M}_{Ex}\cos\theta\cos\phi + \check{M}_{Ey}\cos\theta\sin\phi - \check{M}_{Ez}\sin\theta - \frac{1}{c}\check{M}_{Mx}\sin\phi + \frac{1}{c}\check{M}_{My}\cos\phi\right\}; \quad (3)$$

$$\check{E}_\phi = \frac{k^2 e^{-ikr}}{4\pi\epsilon r}\left\{-\check{M}_{Ex}\sin\phi + \check{M}_{Ey}\cos\phi - \frac{1}{c}\check{M}_{Mx}\cos\theta\cos\phi - \frac{1}{c}\check{M}_{My}\cos\theta\sin\phi + \frac{1}{c}\check{M}_{Mz}\sin\theta\right\}. \quad (4)$$

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* A quantity, A, which varies sinusoidally in time will be written in terms of its associated phasor, $\check{A}$, in the following way:

$$A = \mathrm{Re}\,\check{A}e^{i\omega t}.$$

The complex conjugate of $\check{A}$ will be written $\hat{A}$ and $|A|^2 = \check{A}\hat{A}$. If $\check{\bar{A}}$ is a vector, $|A|^2 = \check{\bar{A}} \cdot \hat{\bar{A}}$.

It is seen that the scattered field of the magnetic dipole is comparable in magnitude to that of the electric dipole only if $|M_M|/c$ is comparable to $|M_E|$. When the dipoles are induced by a plane wave field, this condition is realized and the two sets of fields interfere in the optical sense producing a total pattern which is in general significantly different from that of a single dipole alone.

The Poynting integral of the scattered field and the basis for the calculation of the scattering cross section is

$$P = \frac{1}{2\eta} \int |E|^2 \, r^2 \, d\Omega \, . \tag{5}$$

It is easy to evaluate this since every cross product term vanishes under integration. Thus

$$\begin{aligned} P = \frac{k^4}{2\eta(4\pi\epsilon)^2} \int \Big\{ & |M_{Ex}|^2 (\cos^2\theta \cos^2\phi + \sin^2\phi) \\ & + |M_{Ey}|^2 (\cos^2\theta \sin^2\phi + \cos^2\phi) \\ & + |M_{Ez}|^2 \sin^2\theta \\ & + \frac{1}{c^2} |M_{Mx}|^2 (\cos^2\theta \cos^2\phi + \sin^2\phi) \\ & + \frac{1}{c^2} |M_{My}|^2 (\cos^2\theta \sin^2\phi + \cos^2\phi) \\ & + \frac{1}{c^2} |M_{Mz}|^2 \sin^2\theta \Big\} \, d\Omega \, . \end{aligned} \tag{6}$$

Finally

$$P = \frac{k^4}{2\eta \cdot 6\pi\epsilon^2} \left\{ |M_E|^2 + \frac{1}{c^2} |M_M|^2 \right\} . \tag{7}$$

The total integral is simply the sum of the individual Poynting integrals of the electric dipole field alone and of the magnetic dipole field alone regardless of the spatial angles between in-phase components of the two dipoles. This interesting property is almost immediately deducible from the fact that electric dipole radiation has odd parity whereas magnetic dipole radiation has even parity.

Let two scalar constants of proportionality be defined as follows:

$$\alpha = \frac{|M_E|}{\epsilon |E| v_0} \tag{8}$$

and

$$\beta = \frac{-\mu |M_M|}{|B| v_0} = \frac{-|M_M|}{\epsilon c |E| v_0} , \tag{9}$$

where $|E|$ and $|B|$ are the magnitudes of the incident field vectors and v_o is the geometrical volume of the cylinder. Given that the incident field is a plane wave, the utmost in generality is attained if this wave is assumed to be elliptically polarized. In this case the incident electric field may be regarded as having complex direction cosines, $\check{\gamma}_i$, and

$$\alpha^2 = \sum_{j=1}^{3} \left(\sum_{i=1}^{3} \alpha_{ij} \check{\gamma}_i \right) \left(\sum_{i=1}^{3} \alpha_{ij} \hat{\gamma}_i \right) ; \tag{10}$$

a similar definition may be written for β^2. If equations 8 and 9 are substituted into 7, the total scattering cross section becomes

$$\boxed{\sigma = \frac{k^4 v_0^2}{6\pi} \left(\alpha^2 + \beta^2 \right)} \tag{11}$$

2.02. Discussion of the Angular Distribution. If all angles of incidence and all states of polarization of the incident wave field are considered, a wide variety of angular distributions is encountered. For this reason the present discussion is limited to cases in which the incident field is linearly polarized with the electric field parallel to one of the body axes of the cylinder as shown in figure 1.00-1 and the magnetic field parallel to another. Consider the coordinate system XYZ of figure 2.02-1 with origin at the center of the cylinder and oriented such that the X axis is parallel to the incident electric field, the Y axis to the incident magnetic field and the Z axis to the direction of propagation of the incident wave. In this system the angle Θ is the scattering angle in the usual sense. Three cases are distinguished. In case I, the cylinder axis is parallel to the incident electric field; in case II it is parallel to the direction of propagation and in case III it is parallel to the incident magnetic field. Formulas for the angular distributions in the three cases are as follows:

Case I; cylinder axis parallel to $\bar{E}$:

$$\frac{d\sigma}{d\Omega} = \frac{k^4 v_o^2}{16\pi^2}\left\{\left[\alpha_{\ell\ell}\cos\Theta + \beta_{tt}\right]^2 \cos^2\Phi + \left[\alpha_{\ell\ell} + \beta_{tt}\cos\Theta\right]^2 \sin^2\Phi\right\}. \quad (1)$$

Case II, cylinder axis parallel to $\bar{E} \times \bar{B}$:

$$\frac{d\sigma}{d\Omega} = \frac{k^4 v_o^2}{16\pi^2}\left\{\left[\alpha_{tt}\cos\Theta + \beta_{tt}\right]^2 \cos^2\Phi + \left[\alpha_{tt} + \beta_{tt}\cos\Theta\right]^2 \sin^2\Phi\right\}. \quad (2)$$

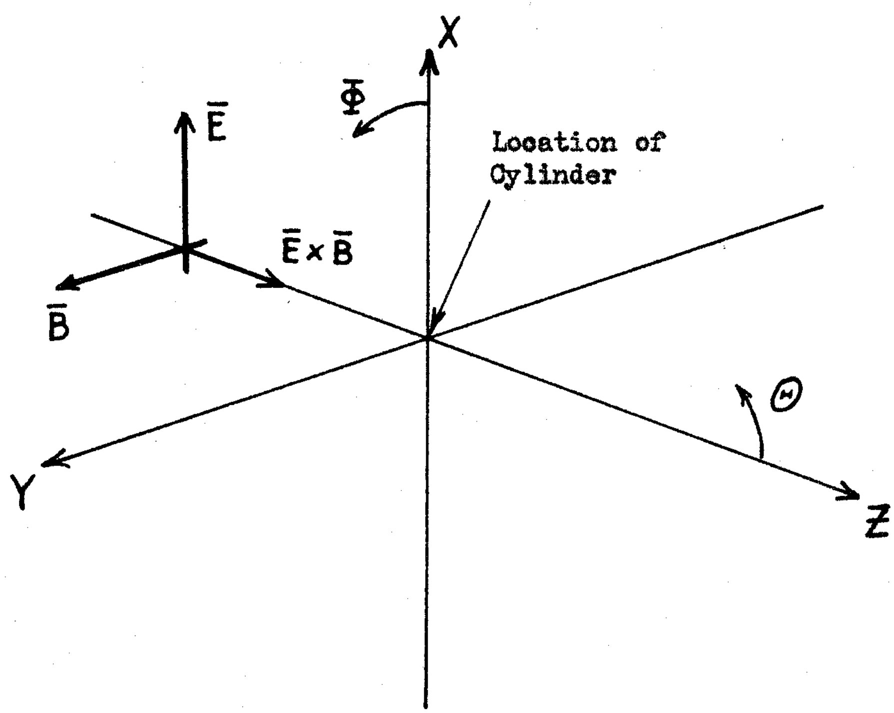

Figure 2.02-1. Coordinate System for Description of the Angular Distribution.

Case III; cylinder axis parallel to $\bar{B}$:

$$\frac{d\sigma}{d\Omega} = \frac{k^4 v_0^2}{16\pi^2} \alpha_{tt}^2 \left\{ \left[\cos\Theta - \tfrac{1}{2}\right]^2 \cos^2\Phi + \left[1 - \tfrac{1}{2}\cos\Theta\right]^2 \sin^2\Phi \right\}. \tag{3}$$

All of these are special cases of the general formula

$$\frac{d\sigma}{d\Omega} = \frac{k^4 v_0^2}{16\pi^2} \alpha^2 \left\{ \left[\cos\Theta + \frac{\beta}{\alpha}\right]^2 \cos^2\Phi + \left[1 + \frac{\beta}{\alpha}\cos\Theta\right]^2 \sin^2\Phi \right\} \tag{4}$$

and the characteristics of the relative angular distribution are dependent upon but one parameter, namely β/α, which invariably lies within the range $-1 < (\beta/\alpha) < 0$. When β/α tends to zero, as it does for a long cylinder with axis parallel to $\bar{E}$ or for a short cylinder with axis parallel to $\bar{E} \times \bar{B}$, the magnetic dipole becomes negligible and the scattering approaches what is known as <u>Rayleigh scattering</u>, or scattering by an electric dipole alone. On the other hand, when β/α tends to minus one, as in the case of a short cylinder with axis parallel to $\bar{E}$ or a long cylinder with axis parallel to $\bar{E} \times \bar{B}$, the magnitudes of the two dipoles tend to become equal. The writer takes the liberty of calling the limiting form of this type of scattering <u>Huygens scattering</u> since the angular distribution is the same as that of a Huygens source (15). As the scattering approaches the Huygens type, the total cross section simultaneously approaches zero.

All the angular distributions have two null directions which always occur in the XZ plane at an angle Θ_n given by $\Theta_n = \text{arc cos} - \beta/\alpha$. For Rayleigh scattering, $\Theta_n = \pi/2$; for

Huygens scattering the null directions coalesce and $\Theta_n = 0$. Of particular interest is the case in which the axis of the cylinder--or of any perfectly conducting object having rotational symmetry and dimensions small compared with wavelength--is parallel to $\bar{B}$. For this situation, β/α is always equal to minus one-half, $\Theta_n = \pi/3$ and the scattering is sphere-like or identical in relative angular distribution with that produced by a perfectly conducting sphere. Relative angular distributions for the three types of scattering just discussed are illustrated in figures 2.02-2 through 2.02-4.

2.03. Charge and Current Distributions; Local Surface Deformations. The charge distributions for the LE and TE problems are expressed as finite weighted sums of the barred biorthogonal functions, $\bar{\Psi}_m(\gamma, \zeta, -\frac{1}{3}, u)$, in a manner described briefly in Section I and in equations 3.00-1, 3.00-2, 4.01-1 and 4.01-2. The corresponding coefficients r_m, t_m, s_m and w_m, obtained by solving simultaneous linear equations 3.00-4 and 4.01-4, are given in Tables 2.03-1 through 2.03-5 which follow immediately. It should be noted that the ϕ-going current in the LM problem is related to the surface charge in the TE problem and a single tabulation suffices for both. In the TM problem, the "fundamental" current distributions j_{zs} on the side and $j_{\rho e}$ on the upper end are written as finite weighted sums of the functions $\bar{\Psi}_m(\gamma, \zeta, \frac{2}{3}, u)$ in equations 5.00-11 and 5.00-12. The coefficients f_m and g_m for these sums, also obtained by solving simultaneous linear equations, are tabulated beside the corresponding coefficients for the LE and TE-LM problems. In these tables, the notation "X(p)" means X times 10^p.

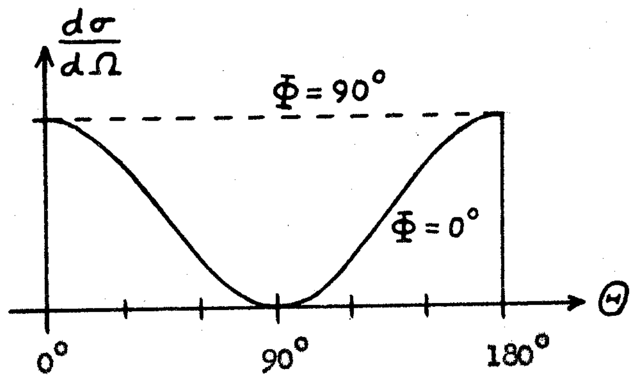

Figure 2.02-2. Rayleigh Scattering.

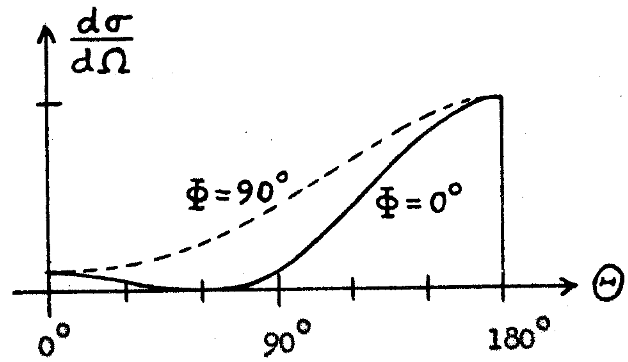

Figure 2.02-3. Sphere-Like Scattering.

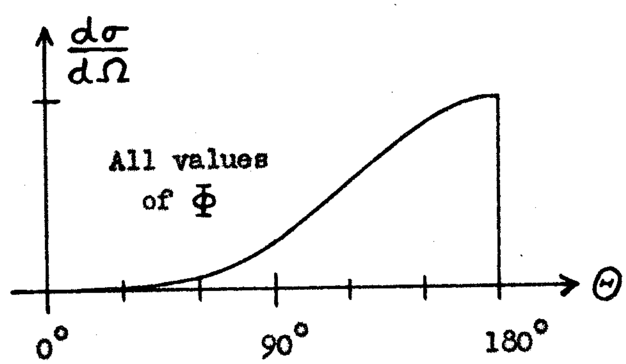

Figure 2.02-4. Huygens Scattering.

Detailed descriptions of the theory of the local surface deformations $\Delta\rho$ and Δz are given in Sections III, IV and V, under the appropriate subsection headings. These deformations apply to the equator and the pole of the cylinder respectively and indicate the amount by which the surface would have to be deformed locally in order to make it an equipotential in the LE and TE problems or a surface upon which normal B vanishes in the TM problem. The quantities beside the headings "Equator Check" and "Pole Check" are essential for the calculation of the deformations and are explained under the subsections mentioned. In each case they represent some potential or field quantity generated by the distribution on the cylinder and normalized with respect to the magnitude of the applied field. A successful technique for checking the TM problem at the pole was not found.

$a/b = \frac{1}{4}$			
	LE	TE-LM	TM
m	r_m	s_m	f_m
basic	+0.12905187 (+0)	+0.40387637 (+0)	+0.13801434 (+1)
0	+0.23603290 (+1)	+0.14690787 (+1)	+0.91240051 (+0)
1	−0.11393541 (+0)	−0.43239914 (+0)	+0.43020551 (+0)
2	−0.19185823 (−1)	−0.65706908 (−1)	+0.11880738 (+0)
3	−0.52174681 (−2)	−0.17221441 (−1)	+0.17780986 (−1)
4	−0.16484165 (−2)	−0.54219231 (−2)	−0.75448651 (−2)
5	−0.53665621 (−3)	−0.17807221 (−2)	−0.97450942 (−2)
6	−0.16813125 (−3)	−0.56407302 (−3)	−0.63465352 (−2)
7	−0.48283874 (−4)	−0.16353609 (−3)	−0.30670392 (−2)
8	−0.12195426 (−4)	−0.41591873 (−4)	−0.11642718 (−2)
9	−0.25981907 (−5)	−0.88990999 (−5)	−0.34677943 (−3)
10	−0.44409406 (−6)	−0.15241129 (−5)	−0.78764085 (−4)
11	−0.56727317 (−7)	−0.19470442 (−6)	−0.12874568 (−4)
12	−0.47918712 (−8)	−0.16423381 (−7)	−0.13517016 (−5)
13	−0.20011235 (−9)	−0.68402739 (−9)	−0.68553750 (−7)
Equator Check	E_z/E = −1.00000002	$V/aE \cos \phi$ = +1.00000002	$B_\rho/B \sin \phi$ = −1.00000001
$\Delta\rho/a$	negligible	negligible	negligible
m	t_m	w_m	g_m
basic	+0.15200248 (−1)	−0.17172000 (+0)	
0	+0.35624820 (+1)	+0.14693027 (+1)	−0.41228663 (+0)
1	−0.39128625 (−1)	+0.32011553 (−1)	−0.16349816 (+0)
Pole Check	V/bE +0.99999875	$E_\rho/E \cos \phi$ = −0.99902328	
$\Delta z/b$	−0.00000037	−0.00022077	

Table 2.03-1.

a/b = 1/2			
	LE	TE-LM	TM
m	r_m	s_m	f_m
basic	−0.19225150 (−4)	+0.38965487 (+0)	+0.13425268 (+1)
0	+0.18367917 (+1)	+0.16055212 (+1)	+0.76126544 (+0)
1	−0.48838691 (−1)	−0.35509127 (+0)	+0.22253894 (+0)
2	−0.54930270 (−2)	−0.43967119 (−1)	+0.50457558 (−1)
3	−0.10582670 (−2)	−0.10119125 (−1)	+0.10818778 (−1)
4	−0.23834773 (−3)	−0.28016839 (−2)	+0.11689120 (−2)
5	−0.54147255 (−4)	−0.78539136 (−3)	−0.66550668 (−3)
6	−0.11308514 (−4)	−0.20187101 (−3)	−0.55657526 (−3)
7	−0.20056279 (−5)	−0.43956298 (−4)	−0.23414408 (−3)
8	−0.27611488 (−6)	−0.74296836 (−5)	−0.63595086 (−4)
9	−0.25757920 (−7)	−0.85391338 (−6)	−0.10633429 (−4)
10	−0.12042175 (−8)	−0.49561337 (−7)	−0.84251433 (−6)
Equator Check	E_z/E = −1.00000003	$V/aE \cos \phi$ = +1.00000009	$B_\rho/B \sin \phi$ = −1.00001590
$\Delta\rho/a$	negligible	negligible	+0.00000827
m	t_m	w_m	g_m
basic	+0.46938430 (−1)	+0.50835705 (−2)	
0	+0.23357609 (+1)	+0.13797193 (+1)	−0.33634571 (+0)
1	−0.82663119 (−1)	+0.88406931 (−1)	−0.15216622 (+0)
2	−0.76227915 (−2)	+0.25064651 (−1)	−0.97479695 (−1)
3	−0.12692924 (−2)	+0.77264312 (−2)	−0.54630065 (−1)
4	−0.16584913 (−3)	+0.15415744 (−2)	−0.17509611 (−1)
Pole Check	V/bE = +0.99999997	$E_\rho/E \cos \phi$ = −1.00002004	
$\Delta z/b$	negligible	+0.00000890	

Table 2.03-2.

a/b = 1			
	LE	TE-LM	TM
m	r_m	s_m	f_m
basic	−0.37616817 (−1)	+0.25340314 (+0)	+0.13033492 (+1)
0	+0.14921036 (+1)	+0.19371617 (+1)	+0.53501366 (+0)
1	−0.83747159 (−2)	−0.32219422 (+0)	+0.79623388 (−1)
2	+0.23944987 (−2)	−0.38977600 (−1)	+0.37836744 (−2)
3	+0.12471808 (−2)	−0.93990090 (−2)	−0.82212558 (−2)
4	+0.48160777 (−3)	−0.25945051 (−2)	−0.73917111 (−2)
5	+0.14698364 (−3)	−0.64974220 (−3)	−0.38698448 (−2)
6	+0.31457070 (−4)	−0.12162489 (−3)	−0.12481735 (−2)
7	+0.34949152 (−5)	−0.12244613 (−4)	−0.19337628 (−3)
Equator Check	E_z/E = −1.00000001	V/aE cos ϕ = +0.99999999	B_ρ/B sin ϕ = −1.00000002
$\Delta\rho/a$	negligible	negligible	negligible
m	t_m	w_m	g_m
basic	+0.87113326 (−1)	−0.13425511 (+0)	
0	+0.16193448 (+1)	+0.14792105 (+1)	−0.30423830 (+0)
1	−0.11224086 (+0)	+0.58149961 (−1)	−0.12867214 (+0)
2	−0.14340444 (−1)	+0.17095607 (−1)	−0.77295716 (−1)
3	−0.34812949 (−2)	+0.59774893 (−2)	−0.47329430 (−1)
4	−0.95870761 (−3)	+0.20469643 (−2)	−0.25538463 (−1)
5	−0.23986408 (−3)	+0.59421158 (−3)	−0.10835430 (−1)
6	−0.44962281 (−4)	+0.12429137 (−3)	−0.31312670 (−2)
7	−0.45407822 (−5)	+0.13669068 (−4)	−0.45560872 (−3)
Pole Check	V/bE = +1.00000001	E_ρ/E cos ϕ = −0.99999996	
$\Delta z/b$	negligible	negligible	

Table 2.03-3.

a/b = 2			
	LE	TE–LM	TM
m	r_m	s_m	f_m
basic	−0.65161515 (−2)	+0.16191166 (+0)	+0.12426150 (+1)
0	+0.12482169 (+1)	+0.23744087 (+1)	+0.33052551 (+0)
1	+0.24818534 (−1)	−0.28135556 (+0)	+0.16573362 (−1)
2	+0.91060187 (−2)	−0.33694020 (−1)	−0.16308601 (−1)
3	+0.28707901 (−2)	−0.72222609 (−2)	−0.16129686 (−1)
4	+0.56748900 (−3)	−0.11581174 (−2)	−0.64332057 (−2)
Equator Check	E_z/E = −1.00000271	$V/aE \cos \phi$ = +0.99999988	$B_\rho/B \sin \phi$ = −0.99976251
$\Delta\rho/a$	+0.00000121	−0.00000005	−0.00009534
m	t_m	w_m	g_m
basic	+0.16025169 (+0)	−0.22721442 (+0)	
0	+0.11683321 (+1)	+0.15658591 (+1)	−0.25782519 (+0)
1	−0.12492981 (+0)	+0.42693596 (−1)	−0.10634017 (+0)
2	−0.17150631 (−1)	+0.12814464 (−1)	−0.59408680 (−1)
3	−0.42141901 (−2)	+0.43243348 (−2)	−0.34135506 (−1)
4	−0.12098644 (−2)	+0.14993639 (−2)	−0.18268058 (−1)
5	−0.34737952 (−3)	+0.49000507 (−3)	−0.85853189 (−2)
6	−0.90947390 (−4)	+0.14126899 (−3)	−0.33773169 (−2)
7	−0.20111927 (−4)	+0.33697692 (−4)	−0.10559328 (−2)
8	−0.34459642 (−5)	+0.61418057 (−5)	−0.24441984 (−3)
9	−0.40094338 (−6)	+0.75263582 (−6)	−0.37088616 (−4)
10	−0.23533528 (−7)	+0.46182535 (−7)	−0.27604425 (−5)
Pole Check	V/bE = +1.00000002	$E_\rho/E \cos \phi$ = −1.00000003	
$\Delta z/b$	negligible	negligible	

Table 2.03-4.

a/b = 4			
	LE	TE-LM	TM
m	r_m	s_m	f_m
basic	-0.91407486 (-1)	+0.36981774 (+0)	+0.11775220 (+1)
0	+0.11655470 (+1)	+0.27748758 (+1)	+0.18233431 (+0)
1	+0.92149445 (-2)	-0.13470990 (+0)	+0.78426568 (-3)
Equator Check	E_z/E = -0.99979406	V/aE cos Ø = +1.00001613	B_ρ /B sin Ø = -1.00377770
$\Delta\rho$ /a	-0.00005287	+0.00000530	+0.00094614
m	t_m	w_m	g_m
basic	+0.16831628 (+0)	-0.10545599 (+0)	
0	+0.92928755 (+0)	+0.15388757 (+1)	-0.21276529 (+0)
1	-0.15311162 (+0)	+0.82957625 (-1)	-0.98721873 (-1)
2	-0.25352751 (-1)	+0.25448502 (-1)	-0.61353937 (-1)
3	-0.71931540 (-2)	+0.96857682 (-2)	-0.39662601 (-1)
4	-0.23904452 (-2)	+0.39007948 (-2)	-0.24407099 (-1)
5	-0.81543454 (-3)	+0.15309282 (-2)	-0.13635446 (-1)
6	-0.26584526 (-3)	+0.55635828 (-3)	-0.67028961 (-2)
7	-0.78893956 (-4)	+0.18020312 (-3)	-0.28260460 (-2)
8	-0.20466149 (-4)	+0.50262768 (-4)	-0.99587013 (-3)
9	-0.44553523 (-5)	+0.11635018 (-4)	-0.28439526 (-3)
10	-0.77488570 (-6)	+0.21335661 (-5)	-0.63101622 (-4)
11	-0.10037598 (-6)	+0.28945170 (-6)	-0.10193176 (-4)
12	-0.85747540 (-8)	+0.25757752 (-7)	-0.10655254 (-5)
13	-0.36132625 (-9)	+0.11256670 (-8)	-0.54072219 (-7)
Pole Check	V/bE = +1.00000001	E_ρ /E cos Ø = -1.00000001	
Δz/b	negligible	negligible	

Table 2.03-5.

III. THE LONGITUDINAL ELECTRIC PROBLEM

3.00. Scope of the Longitudinal Electric Problem. The object of this section is the derivation of the expressions necessary for the calculation of the induced electric dipole moment in the cylinder of figure 1.00-1 when a uniform electric field of magnitude E is applied in the positive z direction. The induced surface charge densities on the side and ends of the cylinder are expressed in terms of the barred biorthogonal functions defined in Section I. On the side

$$\frac{\sigma_s}{\epsilon E} = R(z) = \begin{cases} r_b \frac{z}{b} + \sum_{m=0}^{N_s-1} r_m \bar{\Psi}_m\left(\frac{1}{2}, 1, -\frac{1}{3}, \frac{z}{b}\right); & |z| \leq b \\ 0; & |z| > b \end{cases}, \tag{1}$$

and on the upper end

$$\frac{\sigma_e}{\epsilon E} = T(\rho) = \begin{cases} t_b + \sum_{m=0}^{N_e-1} t_m \bar{\Psi}_m\left(1, 0, -\frac{1}{3}, \frac{\rho}{a}\right); & \rho \leq a \\ 0; & \rho > a \end{cases}. \tag{2}$$

The charge on the lower end is equal and opposite to the charge on the upper end since odd parity in z prevails. In a neighborhood of the origin, the potential of the total surface charge may be put into the form

$$\frac{V}{cE} = \sum_{p=0}^{\infty} \Lambda_p \left(\frac{r}{c}\right)^{2p+1} P_{2p+1}(\cos\theta). \tag{3}$$

It will become obvious that the Λ_p in this expression are linearly related to the r_b, r_m, t_b and t_m coefficients in the following way:

$$X_p^{sb} r_b + \sum_{m=0}^{N_s-1} X_p^{sm} r_m + X_p^{eb} t_b + \sum_{m=0}^{N_e-1} X_p^{em} t_m = \Lambda_p . \quad (4)$$

Note that the "b" index is used with the basic terms, the "m" with the non-basic terms. It is important to realize that the functions associated with the basic terms, namely z/b and unity, are really special cases of the $\bar{\Psi}_m$ functions with both ν and m set equal to zero. It is sufficient, therefore, to derive detailed expressions for only the non-basic terms; the corresponding expressions for the basic terms are obtainable from these simply by substituting zero for ν and m.

Evidently the primary task is the derivation of expressions for the matrix elements (the X's) in equation 4. There are a total of N charge coefficients $(N = N_s + N_e + 2)$; $N - 1$ of the Λ_p are set equal to the following values

$$\Lambda_p = \begin{Bmatrix} 1 & \text{for} & p = 0 \\ 0 & \text{for} & 1 \leq p \leq N-2 \end{Bmatrix} \quad (5)$$

and an additional relationship involving the charge coefficients, known as the "edge condition", is added. The edge condition is described in subsection 3.02. The final result is a set of simultaneous linear equations of order N which may be solved for the charge coefficients. Once the latter are known, the induced electric dipole moment is easily calculated according to the method of 3.03. Checking of the LE problem is described in subsections 3.04 through 3.06.

3.01. <u>The Matrix Elements</u>. With reference to figure 3.01-1, the internal and external potentials due to the side density are

$$\frac{V_{sI}}{E} = a\int_0^\infty \tilde{R}(k)\, K_o(ka)\, I_o(k\rho)\, \sin kz\ dk \tag{1}$$

and

$$\frac{V_{sII}}{E} = a\int_0^\infty \tilde{R}(k)\, I_o(ka)\, K_o(k\rho)\, \sin kz\ dk, \tag{2}$$

respectively. Here $\tilde{R}(k)$ is as yet undefined. The density is

$$\sigma_s = \epsilon \left[\frac{\partial V_{sI}}{\partial \rho} - \frac{\partial V_{sII}}{\partial \rho}\right]_{\rho = a} . \tag{3}$$

Invoking the appropriate Wronskian, one has:

$$\frac{\sigma_s}{\epsilon E} = R(z) = \int_0^\infty \tilde{R}(k)\, \sin kz\ dk . \tag{4}$$

Inverting:

$$\tilde{R}(k) = \frac{2}{\pi}\int_0^b R(z)\, \sin kz\ dz . \tag{5}$$

It is seen that $\tilde{R}(k)$ is the Fourier sine transform of $R(z)$ and therefore equation 1 expresses the internal potential of the side charge density by means of the transform of that density.

A similar technique is applied to the ends. With reference to figure 3.01-2, the internal and external potentials due to the end densities are

$$\frac{V_{eI}}{E} = a\int_0^\infty \tilde{T}(k)\, e^{-kb}\, J_o(k\rho)\, \sinh kz\ dk \tag{6}$$

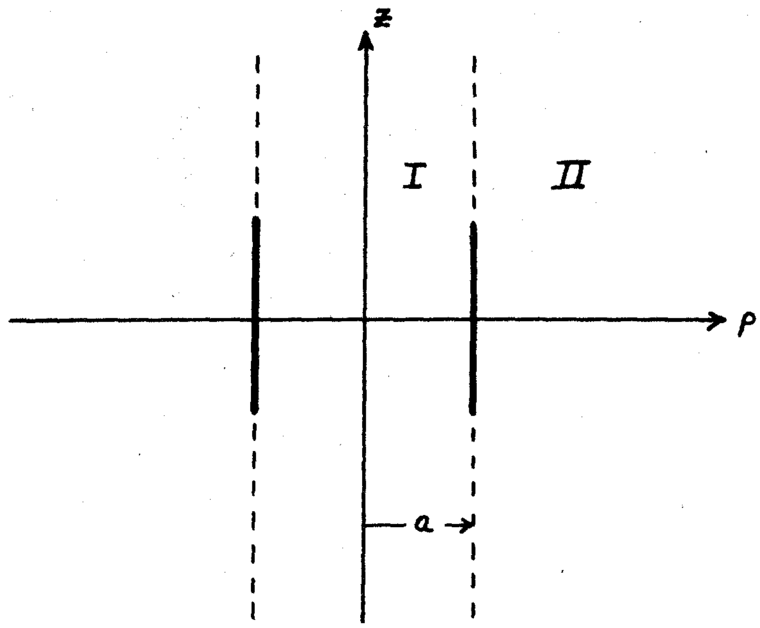

Figure 3.01-1. The Side of the Cylinder showing Internal (I) and External (II) Regions.

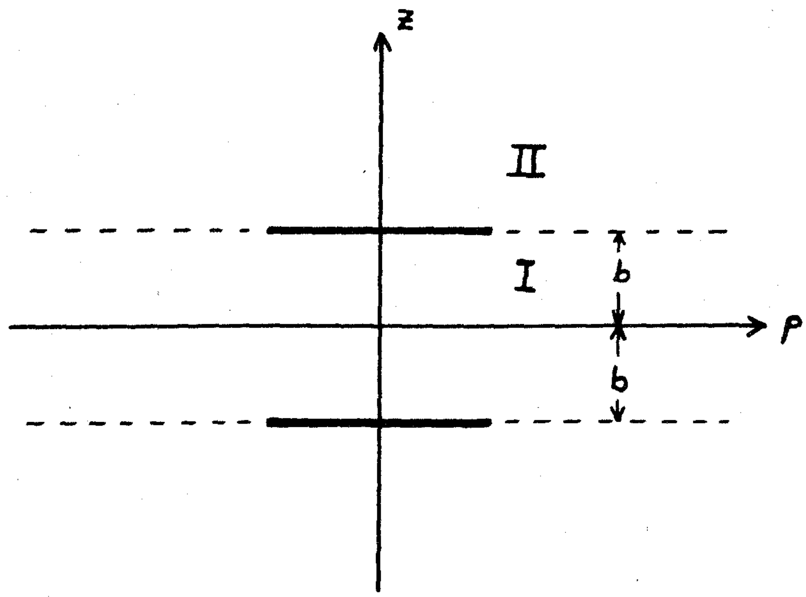

Figure 3.01-2. The Ends of the Cylinder showing Internal (I) and External (II) Regions.

and

$$\frac{V_{eII}}{E} = a\int_0^\infty \tilde{T}(k)\ \sinh kb\ \ J_0(k\rho)\ e^{-kz}\, dk. \tag{7}$$

Again, $\tilde{T}(k)$ is as yet undefined. The density on the upper end is

$$\sigma_e = \epsilon \left[\frac{\partial V_{eI}}{\partial z} - \frac{\partial V_{eII}}{\partial z}\right]_{z=b} \tag{8}$$

and

$$\frac{\sigma_e}{\epsilon E} = T(\rho) = a\int_0^\infty \tilde{T}(k)\ J_0(k\rho)\ k\, dk\,. \tag{9}$$

Inverting:

$$\tilde{T}(k) = \frac{1}{a}\int_0^\infty T(\rho)\ J_0(k\rho)\,\rho\, d\rho\,. \tag{10}$$

This time $\tilde{T}(k)$ is the zeroth-order Hankel transform of $T(\rho)$ and 6 is the expression of the internal potential of the ends via that transform.

The total internal potential, V, is evidently the sum of V_{sI} and V_{eI}. As was pointed out in Section I, it can be expressed in spherical harmonics through the agency of the two expansions

$$I_0(k\rho)\sin kz = \sum_{p=0}^{\infty} \frac{(-1)^p}{(2p+1)!}(kr)^{2p+1} P_{2p+1}(\cos\theta) \tag{11}$$

and

$$J_0(k\rho)\sinh kz = \sum_{p=0}^{\infty} \frac{1}{(2p+1)!}(kr)^{2p+1} P_{2p+1}(\cos\theta). \tag{12}$$

Combining 1, 6, 11 and 12, and using factors of "c" for dimensional reasons, one obtains

$$\frac{V}{cE} = \sum_{p=0}^{\infty} \frac{1}{(2p+1)!} \frac{a}{c} \left\{ (-1)^p \int_0^\infty \tilde{R}(k)\, K_0(ka)\,(kc)^{2p+1}\, dk \right.$$

$$\left. + \int_0^\infty \tilde{T}(k)\, e^{-kb}\,(kc)^{2p+1}\, dk \right\} \left(\frac{r}{c}\right)^{2p+1} P_{2p+1}(\cos\theta). \qquad (13)$$

The Λ_p of equation 3.00-3 are easily identified:

$$\Lambda_p = \frac{a/c}{(2p+1)!} \left\{ (-1)^p \int_0^\infty \tilde{R}(k)\, K_0(ka)(kc)^{2p+1} dk + \int_0^\infty \tilde{T}(k)\, e^{-kb}(kc)^{2p+1} dk \right\}. \qquad (14)$$

With the aid of Appendix A, the two transforms $\tilde{R}(k)$ and $\tilde{T}(k)$ are found to be

$$\tilde{R}(k) = \left(\frac{2}{\pi}\right)^{1/2} b \left\{ r_b \frac{J_{3/2}(kb)}{(kb)^{1/2}} + \sum_{m=0}^{N_s-1} r_m \frac{(-1)^m J_{2m+3/2+\nu}(kb)}{(kb)^{1/2+\nu}} \right\} \qquad (15)$$

and

$$\tilde{T}(k) = a \left\{ t_b \frac{J_1(ka)}{ka} + \sum_{m=0}^{N_e-1} t_m \frac{(-1)^m J_{2m+1+\nu}(ka)}{(ka)^{1+\nu}} \right\}. \qquad (16)$$

In this section, is always equal to minus one-third for the non-basic terms and to zero for the basic terms; the latter value is evident in the above expressions. The forms for the generalized matrix elements, applicable to either basic or non-basic terms, are obtained by substituting the expressions for the transforms into equation 14. Thus the side matrix elements are

$$X_p^{sm} = \left(\frac{2}{\pi}\right)^{1/2} \frac{(-1)^{m+p}}{(2p+1)!} \frac{ab}{c} \int_0^\infty K_0(ka) \frac{J_{2m+3/2+\nu}(kb)}{(kb)^{1/2+\nu}} (kc)^{2p+1} dk. \qquad (17)$$

Let kb = x, and

$$X_p^{sm} = \left(\frac{2}{\pi}\right)^{1/2} \frac{(-1)^{m+p}}{(2p+1)!} \frac{a}{c} \left(\frac{c}{b}\right)^{2p+1} \int_0^\infty K_0\left(\frac{a}{b}x\right) \frac{J_{2m+3/2+\nu}(x)}{x^{1/2+\nu}} x^{2p+1} dx. \qquad (18)$$

By (16), the infinite integral is

$$I\left(\frac{a}{b}\right) = \frac{2^{2p-\frac{1}{2}-\nu}\,\Gamma(m+p+\frac{3}{2})\,\Gamma(m+p+\frac{3}{2})}{(a/b)^{2p+2m+3}\;\Gamma(2m+\frac{5}{2}+\nu)} \tag{19}$$

$$F\left(m+p+\tfrac{3}{2},\, m+p+\tfrac{3}{2};\, 2m+\tfrac{5}{2}+\nu;\, -b^2/a^2\right).$$

The side matrix elements may be written finally in either of two forms depending upon whether the hypergometric function is allowed to remain as it is or whether it is transformed (17). These are

$$X_p^{sm} = \frac{(-1)^{p+m}\, 2^{2p-\nu}\,\Gamma(m+p+\frac{3}{2})\,\Gamma(m+p+\frac{3}{2})}{\Gamma(\frac{1}{2})\;\Gamma(2p+2)\,\Gamma(2m+\frac{5}{2}+\nu)}$$

$$\left\{\begin{array}{l} \left(\frac{c}{b}\right)^{2p}\left(\frac{b}{a}\right)^{2m+2p+2} F\left(m+p+\frac{3}{2}, m+p+\frac{3}{2}; 2m+\frac{5}{2}+\nu; -\frac{b^2}{a^2}\right) \qquad (20) \\ \\ \left(\frac{a}{c}\right)\left(\frac{b}{c}\right)^{2m+2} F\left(m+p+\frac{3}{2}, m-p+1+\nu; 2m+\frac{5}{2}+\nu; \frac{b^2}{c^2}\right) \qquad (21) \end{array}\right\}.$$

Using the same procedure for the end matrix elements, one obtains:

$$X_p^{em} = \frac{(-1)^m}{(2p+1)!}\,\frac{a^2}{c}\int_0^\infty e^{-kb}\,\frac{J_{2m+1+\nu}(ka)}{(ka)^{1+\nu}}\,(kc)^{2p+1}\,dk. \tag{22}$$

Now let $ka = x$, and recall that $e^{-kb} = (2/\pi)^{1/2}\,(kb)^{1/2}\,K_{1/2}(kb)$. Then

$$X_p^{em} = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{(-1)^m}{(2p+1)!}\,\frac{a}{c}\left(\frac{c}{a}\right)^{2p+1}\left(\frac{b}{a}\right)^{\frac{1}{2}}\int_0^\infty K_{\frac{1}{2}}\left(\frac{b}{a}x\right)\frac{J_{2m+1+\nu}(x)}{x^{1+\nu}}\,x^{2p+\frac{3}{2}}\,dx. \tag{23}$$

By (18), the infinite integral is

$$I\left(\frac{b}{a}\right) = \frac{2^{2p-\frac{1}{2}-\nu}\,\Gamma(m+p+\frac{3}{2})\,\Gamma(m+p+1)}{(b/a)^{2p+2m+\frac{5}{2}}\;\Gamma(2m+2+\nu)}$$

$$F\left(m+p+\tfrac{3}{2},\, m+p+1;\, 2m+2+\nu;\, -a^2/b^2\right). \tag{24}$$

Again the matrix element may be written in either of the two forms

$$\chi_p^{em} = \frac{(-1)^m \, 2^{2p-\nu} \, \Gamma(m+p+3/2)\, \Gamma(m+p+1)}{\Gamma(1/2)\, \Gamma(2p+2)\, \Gamma(2m+5/2+\nu)}$$

$$\left\{ \begin{array}{ll} \left(\frac{c}{a}\right)^{2p} \left(\frac{a}{b}\right)^{2m+2p+2} F\left(m+p+3/2,\, m+p+1;\, 2m+5/2+\nu;\, -\frac{a^2}{b^2}\right) & (25) \\ \\ \left(\frac{b}{c}\right) \left(\frac{a}{c}\right)^{2m+2} F\left(m+p+3/2,\, m-p+1+\nu;\, 2m+5/2+\nu;\, \frac{a^2}{c^2}\right) & (26) \end{array} \right\} .$$

It should be remarked that expressions 20 and 25 converge absolutely only for $0 \leq (b/a) < 1$ and $0 \leq (a/b) < 1$ respectively although they are susceptible to analytic continuation (19) into the ranges $1 < (b/a) < \infty$ and $1 < (a/b) < \infty$. The analytic continuations involve ψ functions* and are somewhat difficult to handle computationally; moreover both original series and continuations converge only conditionally or not at all for $a = b$, depending upon the value of p. The writer finds expressions 21 and 26 better adapted to computation in general; both series converge absolutely for all arguments and all indices in the entire range of interest. As might be expected, however, convergence is slow when the argument is near unity in either case.

3.02. The Edge Condition. It was mentioned in Section I that the edge of the cylinder locally resembles a two dimensional right-angled corner; in the IE problem this corner appears as if it were freely charged and

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* Here $\psi(z) = \frac{d}{dz} \log \Gamma(z)$.

the surface charge density a distance ℓ away from the edge in either direction must therefore be proportional to $\ell^{-1/3}$ for small ℓ. The selection of minus one-third for the value of ν insures that the individual $\bar{\Psi}_m(u)$ functions will have this desired behavior in a neighborhood of unit argument, that the total charge density on the side will be asymptotic to $C_1 \ell^{-1/3}$ and that that on the upper end will be asymptotic to $C_2 \ell^{-1/3}$ as ℓ tends to zero. The additional condition that $C_1 = C_2$, called the "edge condition" in this thesis, must also be imposed; it appears simply as another equation adjoined to the system of 3.00-4 as explained earlier.

Using the summation formula for the hypergeometric function of unit argument (20), it is seen that

$$\lim_{u\to 1} \frac{\bar{\Psi}_m(\gamma,\zeta,\nu,u)}{(1-u^2)^{\nu}} = \frac{a_m (-1)^m \Gamma(\gamma+\zeta)\, \Gamma(-\nu)}{\Gamma(m+\gamma+\zeta)\; \Gamma(-m-\nu)}$$

$$= \frac{(-1)^m \Gamma(-\nu)}{\Gamma(-m-\nu)\, \Gamma(m+1+\nu)\, 2^{\nu}}$$

$$= \frac{1}{\Gamma(1+\nu)\, 2^{\nu}} . \qquad (1)$$

In other words, as u tends to unity,

$$\bar{\Psi}_m(\gamma,\zeta,\nu,u) \sim \frac{(1-u^2)^{\nu}}{\Gamma(1+\nu)\, 2^{\nu}} \qquad (2)$$

It is convenient that this form is independent of γ, ζ and m. Evidently, as $z \to b$,

$$R(z) \sim \frac{b^{-\nu}(b-z)^{\nu}}{\Gamma(1+\nu)} \sum_{m=0}^{N_s-1} r_m \qquad (3)$$

and, as $\rho \to a$,

$$T(\rho) \sim \frac{a^{-\nu}(a-\rho)^{\nu}}{\Gamma(1+\nu)} \sum_{m=0}^{N_e-1} t_m . \tag{4}$$

The edge condition becomes

$$b^{-\nu} \sum_{m=0}^{N_s-1} r_m = a^{-\nu} \sum_{m=0}^{N_e-1} t_m . \tag{5}$$

Substituting minus one-third for ν, the additional equation which must be satisfied by the r_m and t_m coefficients is obtained. The basic terms, which remain finite at the edge, do not appear in this equation, which is

$$\boxed{\sum_{m=0}^{N_s-1} r_m - \left(\frac{a}{b}\right)^{1/3} \sum_{m=0}^{N_e-1} t_m = 0} \tag{6}$$

3.03. Calculation of the Dipole Moment. The z, and only, component of the induced electric dipole moment is calculated by taking the integral of z times the charge density over the entire surface. The derivation is as follows:

$$\frac{M_{Ez}}{\epsilon E} = 2\pi \int_{-b}^{b} z\, R(z)\, a\, dz + 4\pi b \int_{0}^{a} T(\rho)\, \rho\, d\rho . \tag{1}$$

Converting to dimensionless variables,

$$\frac{M_{Ez}}{\epsilon E} = 4\pi a b^2 \int_0^1 u\, R(u)\, du + 4\pi b a^2 \int_0^1 T(u)\, u\, du . \tag{2}$$

Note that u in the first integral is orthogonal to every term in R(u) except the basic and the first non-basic, since it may be identified as

$\psi_0(\frac{1}{2},1,\nu,u)/b_0(\frac{1}{2},1,\nu)$ in either case. Similiarly unity in the second integral may be identified as $\psi_0(1,0,\nu,u)/b_0(1,0,\nu)$. The geometrical volume, v_o, which is equal to $2\pi a^2 b$, may be divided out and the final expression for the polarizability is

$$\alpha_{\ell\ell} = \frac{2b}{a}\left[\frac{r_b}{b_0(\frac{1}{2},1,0)} + \frac{r_0}{b_0(\frac{1}{2},1,-\frac{1}{3})}\right] + 2\left[\frac{t_b}{b_0(1,0,0)} + \frac{t_0}{b_0(1,0,-\frac{1}{3})}\right] \quad (3)$$

Numerical values for $b_o(\gamma,\zeta,\nu)$ may be calculated from expression 1.01-4. It is obvious that the values of $b_o(\frac{1}{2},1,0)$ and $b_o(1,0,0)$ must be three and two respectively; the fact that expression 1.01-4 does indeed give these values constitutes a good check upon the integrity of the system of biorthogonal functions. The numerical values are given in Table 3.03-1.

$b_o(\frac{1}{2},1,0)$	= 3.0000000
$b_o(\frac{1}{2},1,-\frac{1}{3})$	= 1.9386755
$b_o(1,0,0)$	= 2.0000000
$b_o(1,0,-\frac{1}{3})$	= 1.4330188

Table 3.03-1

3.04. Local Surface Deformations. The charge distribution calculated for the cylinder is subject to certain inevitable errors and it follows that if the calculated charge were forcibly fixed to the cylinder, the latter would not quite be an equipotential surface under the joint action of this charge and of the applied field. Since the errors are small, however, at any given surface point it is possible to calculate the local normal outward deformation, Δn, which the surface would have to undergo in order to become an equipotential. Let V_t be the total potential, due to both charge and applied field, just outside the surface at a point where the calculated local charge density is σ . Then the differential coefficient $\partial V_t/\partial n$, which is insensitive to small positive or negative local surface displacements, will be very nearly equal to $-\sigma/\varepsilon$. The calculated potential at a given surface point will be the sum of the potential, V, due to the charge alone and the potential, V_a, due to the applied field. The surface deformation, then, must be such that

$$V + V_a - \frac{\sigma}{\epsilon}\Delta n = 0 \tag{1}$$

or

$$\Delta n = \frac{V + V_a}{\frac{\sigma}{\epsilon}} . \tag{2}$$

The calculation of surface deformations will be undertaken at the equator and at the pole of the cylinder. In the LE problem, equation 2 becomes indeterminate at the equator and both numerator and denominator must be differentiated with respect to z. Therefore

$$\Delta \rho = \frac{-E_z - E}{\frac{1}{\epsilon}\frac{\partial \sigma_s}{\partial z}} \tag{3}$$

where E_z represents the field of the charge alone and, as has been customary, E represents the applied field. It is convenient to divide by the latter and to introduce factors of a and b. The final result is

$$\boxed{\frac{\Delta \rho}{a} = -\frac{b}{a}\,\frac{\frac{E_z}{E}+1}{R'(0)}} . \tag{4}$$

Since

$$\bar{\Psi}_m'(\tfrac{1}{2},1,\nu,0) = (-1)^m \; a_m(\tfrac{1}{2},1,\nu), \tag{5}$$

$$R'(0) = a_b r_b + \sum_{m=0}^{N_s-1} (-1)^m a_m r_m . \tag{6}$$

At the pole, equation 2 applies directly, and

$$\Delta z = \frac{V - bE}{\frac{\sigma_e}{\epsilon}} \tag{7}$$

or

$$\boxed{\frac{\Delta z}{b} = \frac{\frac{V}{bE} - 1}{T(0)}} . \tag{8}$$

Since

$$\bar{\Psi}_m(1,0,\nu,0) = (-1)^m \; a_m(1,0,\nu), \tag{9}$$

$$T(0) = a_b t_b + \sum_{m=0}^{N_e-1} (-1)^m a_m t_m . \tag{10}$$

The problems of calculating E_z/E and V/bE are considered in the next subsection.

3.05. Derivation of the Checking Coefficients. At the equator, let the part of E_z due to the side charge alone be denoted E_{zs}. This quantity is obtained by differentiating the negative of 3.01-1, then setting z equal to zero and ρ equal to a. In normalized form, E_{zs} is

$$\frac{E_{zs}}{E} = -a\int_0^\infty \tilde{R}(k)\, K_0(ka)\, I_0(ka)\, k\, dk\,. \tag{1}$$

Since $\tilde{R}(k)$ is itself a sum, as given in 3.01-15, it is clear that E_{zs}/E is of the form

$$\frac{E_{zs}}{E} = A_b r_b + \sum_{m=0}^{N_s-1} A_m r_m \tag{2}$$

where the A_m are

$$A_m = -\left(\frac{2}{\pi}\right)^{1/2}(-1)^m\, a\, b\int_0^\infty \frac{J_{2m+3/2+\nu}(kb)}{(kb)^{1/2+\nu}}\, K_0(ka) I_0(ka)\, k\, dk\,. \tag{3}$$

Letting $kb = p$ and $a/b = \tau$,

$$A_m = \frac{-1}{2}\left(\frac{2}{\pi}\right)^{1/2}(-1)^m\, \tau\int_0^\infty \frac{J_{2m+3/2+\nu}(p)}{p^{1/2+\nu}}\, K_0(p\tau) I_0(p\tau)\, d(p^2)\,. \tag{4}$$

By (21)

$$K_0(p\tau) I_0(p\tau) = \frac{\pi^{-1/2}}{2}\, G_{13}^{21}\left(p^2\tau^2 \,\middle|\, \begin{matrix} 1/2 \\ 0,\ 0,\ 0 \end{matrix}\right) \tag{5}$$

and

$$p^{-(\frac{1}{2}+\nu)} J_{2m+3/2+\nu}(p) = 2^{-(\frac{1}{2}+\nu)}\, G_{02}^{10}\left(\frac{p^2}{4} \,\middle|\, m+1/2, -m-1-\nu\right). \tag{6}$$

The infinite integral of a product of G-functions is, under suitable conditions, another G-function (22). Thus the A_m coefficient is

$$\boxed{A_m = \frac{-(-1)^m 2^{-\nu}}{\pi} \tau \, G_{33}^{22}\left(4\tau^2 \middle| \begin{matrix} -m-\tfrac{1}{2}, & \tfrac{1}{2}, & m+1+\nu \\ 0, & 0, & 0 \end{matrix}\right)} \, . \tag{7}$$

Evidently A_b, the basic coefficient, is simply the A_o coefficient with ν set equal to zero.

The part of E_z at the equator due to the end charges alone is denoted E_{ze} and is obtained by differentiating the negative of 3.01-6, then setting z equal to zero and ρ equal to a. Thus

$$\frac{E_{ze}}{E} = -a \int_0^\infty \tilde{T}(k)\, e^{-kb} \, J_0(ka)\, k\, dk. \tag{8}$$

Again $\tilde{T}(k)$ is a sum, as given in 3.01-16, and E_{ze}/E is of the form

$$\frac{E_{ze}}{E} = B_b t_b + \sum_{m=0}^{N_e-1} B_m t_m \, . \tag{9}$$

The B_m are therefore

$$B_m = -(-1)^m a^2 \int_0^\infty \frac{J_{2m+1+\nu}(ka)}{(ka)^{1+\nu}} \, e^{-kb} \, J_0(ka)\, k\, dk. \tag{10}$$

Now let $ka = p$, and

$$B_m = -(-1)^m \int_0^\infty e^{-\frac{p}{\tau}} \, J_0(p) \, \frac{J_{2m+1+\nu}(p)}{p^\nu} \, dp \, . \tag{11}$$

By (23)

$$p^{-\nu} J_0(p)\, J_{2m+1+\nu}(p) = \pi^{-1/2}\, G_{24}^{12}\left(p^2 \,\middle|\, \begin{matrix} 1/2-\nu/2,\ -\nu/2 \\ m+1/2,-m-1/2-\nu, m+1/2, -m-1/2-\nu \end{matrix}\right). \qquad (12)$$

The infinite integral of an exponential and a G-function is another G-function (24). Therefore the B_m coefficients are

$$B_m = \frac{-(-1)^m}{\pi}\, \tau\, G_{44}^{14}\left(4\tau^2 \,\middle|\, \begin{matrix} 0,\ 1/2,\ 1/2-\nu/2,\ -\nu/2 \\ m+1/2,-m-1/2-\nu,\ m+1/2,-m-1/2-\nu \end{matrix}\right). \qquad (13)$$

Attention is now turned to the potential at the pole. That part due to the side charge alone is obtained by setting z equal to b and ρ equal to zero in 3.01-1. Thus

$$\frac{V_s}{bE} = \frac{a}{b}\int_0^\infty \tilde{R}(k)\, K_0(ka)\, \sin kb\, dk\,; \qquad (14)$$

$$\frac{V_s}{bE} = C_b r_b + \sum_{m=0}^{N_s-1} C_m r_m\,; \qquad (15)$$

$$C_m = \left(\frac{2}{\pi}\right)^{1/2} (-1)^m\, b\, \frac{a}{b}\int_0^\infty K_0(ka)\, \sin kb\, \frac{J_{2m+3/2+\nu}(kb)}{(kb)^{1/2+\nu}}\, dk. \qquad (16)$$

Now let $kb = p$, and note that $\sin p = (\pi/2)^{1/2}\, p^{1/2}\, J_{1/2}(p)$. Then

$$C_m = (-1)^m\, \frac{\tau}{2}\int_0^\infty K_0(p\tau)\, J_{1/2}(p)\, \frac{J_{2m+3/2+\nu}(p)}{p^{1+\nu}}\, d(p^2). \qquad (17)$$

Two factors of the integrand are written as G-functions:

$$p^{-1-\nu} J_{1/2}(p)\, J_{2m+3/2+\nu}(p) = \pi^{-1/2}\, G_{24}^{12}\left(p^2 \,\middle|\, \begin{matrix} -\nu/2,\ -1/2-\nu/2 \\ m+1/2, m, -m-1-\nu, -m-3/2-\nu \end{matrix}\right); \qquad (18)$$

$$K_0(p\tau) = \frac{1}{2}\, G_{02}^{20}\left(\frac{p^2\tau^2}{4}\,\middle|\, 0, 0\right). \tag{19}$$

The C_m coefficients are

$$\boxed{C_m = \frac{(-1)^m}{4\pi^{1/2}}\,\tau\, G_{44}^{41}\left(\frac{\tau^2}{4}\,\middle|\,\begin{matrix}-m-1/2, -m, m+1+\nu, m+3/2+\nu\\ \nu/2, 1/2+\nu/2, 0, 0\end{matrix}\right)}. \tag{20}$$

The part of the potential at the pole due to the ends alone is found by setting z equal to b and ρ equal to zero in 3.01-6. Thus

$$\frac{V_e}{bE} = \frac{a}{b}\int_0^\infty \tilde{T}(k)\, e^{-kb} \sinh kb \;dk\,; \tag{21}$$

$$\frac{V_e}{bE} = D_b t_b + \sum_{m=0}^{N_e-1} D_m t_m\,; \tag{22}$$

$$D_m = (-1)^m \frac{a^2}{b}\int_0^\infty \frac{1-e^{-2kb}}{2}\;\frac{J_{2m+1+\nu}(ka)}{(ka)^{1+\nu}}\,dk\,; \tag{23}$$

$$D_m = (-1)^m \frac{\tau}{2}\int_0^\infty \left(1-e^{-\frac{2p}{\tau}}\right)\frac{J_{2m+1+\nu}(p)}{p^{1+\nu}}\,dp. \tag{24}$$

This integral is simpler than those which have just been considered. It is convenient to split D_m into D_{mr}, the part due to the remote end, and D_{mp}, the part due to the proximate end:

$$D_{mr} = -(-1)^m \frac{\tau}{2}\int_0^\infty e^{-\frac{2p}{\tau}}\,\frac{J_{2m+1+\nu}(p)}{p^{1+\nu}}\,dp\,; \tag{25}$$

$$D_{mp} = -\lim_{\tau\to\infty} D_{mr}\,. \tag{26}$$

By (25) the final form for D_m is

$$D_m = (-1)^m \tau\, 2^{-2-\nu} \left\{ \frac{\Gamma(m+\frac{1}{2})}{\Gamma(m+\frac{3}{2}+\nu)} - \frac{\Gamma(2m+1)\, 2^{-2m}\, F\left(m+\frac{1}{2}, m+1+\nu; 2m+2+\nu; \left[1+\frac{4}{\tau^2}\right]^{-1}\right)}{\Gamma(2m+2+\nu)\left[1+\frac{4}{\tau^2}\right]^{m+\frac{1}{2}}} \right\}. \quad (27)$$

3.06. Numerical Values of the LE Checking Coefficients.

$a/b = \frac{1}{4}$				
m	A_m	B_m	C_m	D_m
basic	−0.29092770	−0.027496835	+0.12636348	+0.11721778
0	−0.33551515	−0.038094261	+0.20544521	+0.13959347
1	+0.34521422	+0.000636674	+0.02583489	−0.06445809
2	−0.32128356	−0.000007883	−0.00485845	+0.04463788
3	+0.29985009	+0.000000071	−0.00363311	−0.03524047
4	−0.28393154		−0.00079207	
5	+0.27219552		+0.00007923	
6	−0.26324614		+0.00011040	
7	+0.25613764		+0.00003220	
8	−0.25028169			
9	+0.24531356			
10	−0.24099940			
11	+0.23717830			
12	−0.23373160			

Table 3.06-1

$a/b = \frac{1}{2}$				
m	A_m	B_m	C_m	D_m
basic	−0.32601991	−0.08268658	+0.13328993	+0.21922359
0	−0.38941423	−0.11352682	+0.20515054	+0.25806868
1	+0.34362707	+0.00422630	−0.00091545	−0.12866229
2	−0.31022369	−0.00001874	−0.00494770	+0.08927276
3	+0.29056866	−0.00000726	−0.00035600	−0.07048091
4	−0.27754842	+0.00000047	+0.00015500	+0.05920399
5	+0.26794921		+0.00002700	−0.05156477
6	−0.26036741			+0.04599020
7	+0.25411243			
8	−0.24879877			
9	+0.24418865			
10	−0.24012402			

Table 3.06-2

a/b = 1				
m	A_m	B_m	C_m	D_m
basic	−0.32877249	−0.17868116	+0.10378375	+0.38196601
0	−0.39732877	−0.24711785	+0.15655223	+0.43883735
1	+0.33349897	+0.00421629	−0.00941805	−0.25383775
2	−0.30477222	+0.00278894	−0.00056000	+0.17839283
3	+0.28791833	−0.00019761	+0.00010880	−0.14095486
4	−0.27610605	−0.00002614		+0.11840765
5	+0.26706035	+0.00000406		−0.10312952
6	−0.25976597	+0.00000013		+0.09198040
7	+0.25367836			−0.08342408
8	−0.24847089			+0.07661395

Table 3.06-3

a/b = 2				
m	A_m	B_m	C_m	D_m
basic	−0.32342644	−0.28150102	+0.052946466	+0.58578644
0	−0.39053524	−0.41200798	+0.080432247	+0.64377273
1	+0.32964616	−0.04799989	−0.003582010	−0.47308746
2	−0.30350317	+0.01354131	+0.000089476	+0.35197905
3	+0.28734661	+0.00360955	+0.000000281	−0.28122666
4	−0.27578341	−0.00030521		+0.23671525
5	+0.26685358	−0.00020968		−0.20624399
6	−0.25962259	−0.00000542		+0.18395848
7		+0.00001011		−0.16684780
8		+0.00000128		+0.15322784
9		−0.00000039		−0.14208404
10		−0.00000011		+0.13276707

Table 3.06-4

a/b = 4				
m	A_m	B_m	C_m	D_m
basic	−0.32019622	−0.36248288	+0.018151740	+0.76393203
0	−0.38590956	−0.59374414	+0.027888457	+0.79345858
1	+0.32868365	−0.19822756	−0.000476891	−0.76063475
2	−0.30321292	−0.01102474	+0.000006597	+0.64208498
3	+0.28721018	+0.01862766	−0.000000078	−0.54203742
4		+0.00854478		+0.46662303
5		+0.00107944		−0.41018405
6		−0.00062555		+0.36712587
7		−0.00038019		−0.33342051
8		−0.00007223		+0.30635902
9		+0.00001903		−0.28413382
10		+0.00001663		+0.26552189
11		+0.00000420		−0.24967695
12		−0.00000043		+0.23599859

Table 3.06-5

IV. THE TRANSVERSE ELECTRIC AND LONGITUDINAL MAGNETIC PROBLEMS

4.00. Perfectly Conducting Objects with Rotational Symmetry.

It is desired to prove the theorem that for any perfectly conducting object with rotational symmetry, $\beta_{\ell\ell} = -\frac{1}{2}\alpha_{tt}$. An object of this type, formed by rotating the curve $\rho = \rho(z)$ about the z axis, is shown in figure 4.00-1. Let the unit vector $\bar{e}_\ell$ point along the tangent of this curve and the unit vector $\bar{e}_n$ point along the outward normal. The variable ℓ, associated with $\bar{e}_\ell$, measures distance along the curve and encompasses the object as it varies over the interval $0 \leq \ell \leq L$. It is sufficient if $\rho(z)$ has a uniquely defined tangent and normal at almost every point in this interval.

If the applied fields consist of a uniform magnetic field, B, in the z direction and a uniform electric field, E, in the x direction, the following must be proved:

$$\frac{\mu M_{Mz}}{B} = -\frac{1}{2}\frac{M_{Ex}}{\epsilon E}. \tag{1}$$

In the electric case, the scalar potential satisfies the conditions:

$$\left.\begin{aligned} &V = -E\, f_1(\rho, z) \cos\phi; \\ &f_1 = 0 \quad \text{on the surface of the object;} \\ &f_1 \sim \rho \quad \text{at infinity;} \\ &\nabla^2 V = 0. \end{aligned}\right\} \tag{2}$$

In the magnetic case, on the other hand, the vector potential fulfills the

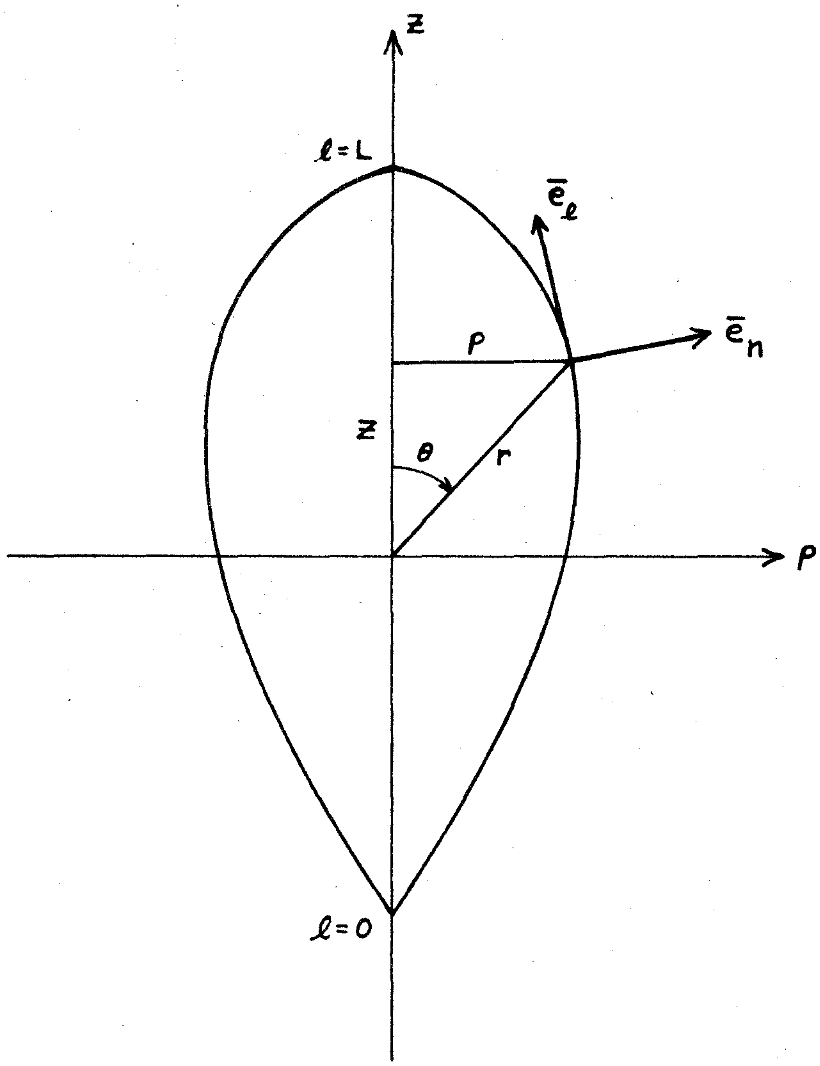

Figure 4.00-1. A Perfectly Conducting Object with Rotational Symmetry.

conditions:

$$\left.\begin{aligned} \bar{A} &= \bar{e}_\phi \frac{B}{2} f_2(\rho, z); \\ f_2 &= 0 \quad \text{on the surface of the object;} \\ f_2 &\sim \rho \quad \text{at infinity;} \\ \nabla^2 \bar{A} &= 0. \end{aligned}\right\} \tag{3}$$

Application of the Laplacian operator to either $\cos \phi\, f_1$ or to $\bar{e}_\phi f_2$ yields the same equation in ρ and z. In other words

$$\left\{ \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} - \frac{1}{\rho^2} + \frac{\partial^2}{\partial z^2} \right\} \left\{ \begin{matrix} f_1 \\ f_2 \end{matrix} \right\} = 0. \tag{4}$$

Since f_1 and f_2 obey the same differential equation, have the same boundary conditions at the surface of the object and have identical asymptotic behavior at infinity, it is clear that $f_1 = f_2 = f$. The surface charge density in the electric case is

$$\sigma = -\epsilon\, \bar{e}_n \cdot \nabla V = \epsilon E \cos\phi\, (\bar{e}_n \cdot \nabla f) \tag{5}$$

and the x component of the electric dipole moment is

$$M_{Ex} = \int_0^L \int_0^{2\pi} \sigma \rho \cos\phi \;\rho \, d\phi \, d\ell\,; \tag{6}$$

$$M_{Ex} = \epsilon E \pi \int_0^L (\bar{e}_n \cdot \nabla f)\, \rho^2 \, d\ell\,. \tag{7}$$

The surface current density in the magnetic case is

$$\bar{J} = \frac{1}{\mu}\, \bar{e}_n \times (\nabla \times \bar{A}) = \frac{B}{2\mu}\, \bar{e}_n \times (\nabla \times \bar{e}_\phi f). \tag{8}$$

By vector identity

$$\bar{e}_n \times (\nabla \times \bar{e}_\phi f) = \bar{e}_n \times (\nabla f \times \bar{e}_\phi + f \nabla \times \bar{e}_\phi). \tag{9}$$

Since $f = 0$ on the surface, the second term vanishes. Continuing,

$$\bar{e}_n \times (\nabla f \times \bar{e}_\phi) = \nabla f (\bar{e}_n \cdot \bar{e}_\phi) - \bar{e}_\phi (\bar{e}_n \cdot \nabla f). \tag{10}$$

Obviously $\bar{e}_n \cdot \bar{e}_\phi$ is equal to zero and therefore

$$\bar{J} = \frac{-B}{2\mu} \bar{e}_\phi (\bar{e}_n \cdot \nabla f). \tag{11}$$

The magnetic dipole moment is given in general by:

$$\bar{M}_M = \frac{1}{2} \int \bar{r} \times \bar{J} \, dS. \tag{12}$$

Since $\bar{r} = \bar{e}_z z + \bar{e}_\rho \rho$,

$$\bar{M}_M = \frac{-B}{4\mu} \int_0^L \int_0^{2\pi} (-\bar{e}_\rho z + \bar{e}_z \rho)(\bar{e}_n \cdot \nabla f) \rho \, d\phi \, d\ell. \tag{13}$$

By symmetry, the term containing $\bar{e}_\rho$ is destroyed upon integrating over 2π in ϕ. All that remains is a z component given by

$$M_{Mz} = \frac{-B}{2\mu} \pi \int_0^L (\bar{e}_n \cdot \nabla f) \rho^2 \, d\ell. \tag{14}$$

Evidently

$$\frac{\mu M_{Mz}}{B} = -\frac{1}{2} \frac{M_{Ex}}{\epsilon E} = -\frac{\pi}{2} \int_0^L (\bar{e}_n \cdot \nabla f) \rho^2 d\ell \tag{15}$$

and the asserted relation is true.

4.01. <u>Scope of the Transverse Electric and Longitudinal Magnetic Problems.</u> As is evident from the theorem of subsection 4.00, it is sufficient to consider only the TE problem with perhaps occasional reference to its dual, the LM problem. The objective, then, is to derive the expressions necessary for the calculation of the induced electric dipole moment in the cylinder of figure 1.00-1 when a uniform electric field of magnitude E is applied in the positive x direction. The induced surface charge densities on the side and ends of the cylinder, and their duals, are expressed in barred biorthogonal functions with indices chosen to give even parity on the side and odd parity on the ends. On the side

$$\frac{\sigma_s}{\epsilon E \cos\phi} = \frac{-2\mu j_{\phi s}}{B} = S(z) = \begin{cases} s_b + \sum_{m=0}^{N_s-1} s_m \bar{\Psi}_m\left(\frac{1}{2}, 0, -\frac{1}{3}, \frac{z}{b}\right) ; & |z| \leq b \\ 0 ; & |z| > b \end{cases} \quad (1)$$

and on the upper end

$$\frac{\sigma_e}{\epsilon E \cos\phi} = \frac{-2\mu j_{\phi e}}{B} = W(\rho) = \begin{cases} w_b \frac{\rho}{a} + \sum_{m=0}^{N_e-1} w_m \bar{\Psi}_m\left(1, 1, -\frac{1}{3}, \frac{\rho}{a}\right) ; & \rho \leq a \\ 0 ; & \rho > a \end{cases} . \quad (2)$$

The charge on the lower end is equal to that on the upper end. Actually $S(z)$ and $W(\rho)$ are equal to the $\bar{e}_n \cdot \nabla f$ of subsection 4.00 on side and end respectively. In a neighborhood of the origin, the potential of the total surface charge may be put into the form

$$\frac{V}{cE} = \cos\phi \sum_{p=0}^{\infty} \Lambda_p \left(\frac{r}{c}\right)^{2p+1} P^1_{2p+1}(\cos\theta) . \quad (3)$$

Again there exists a linear relationship connecting the charge coefficients to the Λ_p. It is

$$Y_p^{sb} s_b + \sum_{m=0}^{N_s-1} Y_p^{sm} s_m + Y_p^{eb} w_b + \sum_{m=0}^{N_e-1} Y_p^{em} w_m = \Lambda_p . \qquad (4)$$

Once more, N - 1 of the Λ_p will be set equal to the values given in 3.00-5 since this will yield a maximally linear internal potential very nearly equal to xE. The edge condition, described in subsection 4.03, supplies the N th equation. The calculation of the dipole moment is outlined in 4.04 and the method of checking in 4.05 through 4.07.

4.02. <u>The Matrix Elements</u>. With reference to figure 3.01-1, the internal and external potentials due to the side density are

$$\frac{V_{sI}}{E \cos\phi} = a \int_0^\infty \tilde{S}(k)\, K_1(ka)\, I_1(k\rho) \cos kz \; dk \qquad (1)$$

and

$$\frac{V_{sII}}{E \cos\phi} = a \int_0^\infty \tilde{S}(k)\, I_1(ka)\, K_1(k\rho) \cos kz \; dk, \qquad (2)$$

respectively. The density is related to the two potentials by 3.01-3 and, using a Wronskian similar to that used before,

$$\frac{\sigma_s}{\epsilon E \cos\phi} = S(z) = \int_0^\infty \tilde{S}(k) \cos kz \; dk . \qquad (3)$$

Inverting:

$$\tilde{S}(k) = \frac{2}{\pi} \int_0^b S(z) \cos kz \; dz . \qquad (4)$$

Thus equation 1 expresses the internal potential of the side density by means of the Fourier transform of that density.

A similar technique is applied to the ends. With reference to figure 3.01-2, the internal and external potentials due to the end densities are

$$\frac{V_{eI}}{E\cos\phi} = a\int_0^\infty \tilde{W}(k)\, e^{-kb} J_1(k\rho)\cosh kz \; dk \tag{5}$$

and

$$\frac{V_{eII}}{E\cos\phi} = a\int_0^\infty \tilde{W}(k)\cosh kb \;\; J_1(k\rho)\, e^{-kz}\, dk. \tag{6}$$

Using a now familiar procedure,

$$\frac{\sigma_e}{\epsilon E\cos\phi} = W(\rho) = a\int_0^\infty \tilde{W}(k)\;\; J_1(k\rho)\, k\, dk\,; \tag{7}$$

$$\tilde{W}(k) = \frac{1}{a}\int_0^a W(\rho)\;\; J_1(k\rho)\,\rho\, d\rho\,, \tag{8}$$

and equation 5 expresses the internal potential of the ends via the first-order Hankel transform of the end density.

Spherical harmonic expansions of $I_1(k\rho)$ cos kz cos ϕ and of $J_1(k\rho)$ cosh kz cos ϕ are now needed. Both of these represent potentials which are regular at the origin but since both vanish on the z axis, the method of subsection 1.01 is no longer applicable. It is possible to get the desired expansions, however, by letting $z = 0$ and using powers of ρ. These expansions are of the form

$$I_1(k\rho)\cos kz\ \cos\phi = \cos\phi \sum_{p=0}^{\infty} A_p\, r^{2p+1} P^1_{2p+1}(\cos\theta). \qquad (9)$$

At $z = 0$, that is on the xy plane, this becomes (26):

$$\cos\phi \sum_{p=0}^{\infty} \frac{k^{2p+1}\ \rho^{2p+1}}{2^{2p+1}\ p!\,(p+1)!} = \cos\phi \sum_{p=0}^{\infty} \frac{A_p \rho^{2p+1} (-1)^p (2p+1)!}{2^{2p}\ p!\ p!}. \qquad (10)$$

Equating the coefficients of ρ^{2p+1} shows that $A_p = (-1)^p\, k^{2p+1}/(2p+2)!$. Omitting the common factor of $\cos \phi$,

$$I_1(k\rho)\cos kz = \sum_{p=0}^{\infty} \frac{(-1)^p}{(2p+2)!} (kr)^{2p+1} P^1_{2p+1}(\cos\theta). \qquad (11)$$

Similarly

$$J_1(k\rho)\cosh kz = \sum_{p=0}^{\infty} \frac{1}{(2p+2)!} (kr)^{2p+1} P^1_{2p+1}(\cos\theta). \qquad (12)$$

Combining 1, 5, 11 and 12, and using factors of "c" for dimensional reasons, one obtains

$$\frac{V}{cE\cos\phi} = \sum_{p=0}^{\infty} \frac{1}{(2p+2)!} \frac{a}{c} \left\{ (-1)^p \int_0^{\infty} \tilde{S}(k) K_1(ka)(kc)^{2p+1} dk + \int_0^{\infty} \tilde{W}(k) e^{-kb} (kc)^{2p+1} dk \right\} \left(\frac{r}{c}\right)^{2p+1} P^1_{2p+1}(\cos\theta). \qquad (13)$$

The Λ_p of equation 4.01-3 are

$$\Lambda_p = \frac{a/c}{(2p+2)!} \left\{ (-1)^p \int_0^{\infty} \tilde{S}(k) K_1(ka)(kc)^{2p+1} dk + \int_0^{\infty} \tilde{W}(k) e^{-kb} (kc)^{2p+1} dk \right\}. \qquad (14)$$

With the aid of Appendix A, the two transforms $\tilde{S}(k)$ and $\tilde{W}(k)$ are found to be

$$\tilde{S}(k) = \left(\frac{2}{\pi}\right)^{1/2} b \left\{ s_b \frac{J_{1/2}(kb)}{(kb)^{1/2}} + \sum_{m=0}^{N_s-1} s_m \frac{(-1)^m J_{2m+1/2+\nu}(kb)}{(kb)^{1/2+\nu}} \right\} \tag{15}$$

and

$$\tilde{W}(k) = a \left\{ w_b \frac{J_2(ka)}{ka} + \sum_{m=0}^{N_e-1} w_m \frac{(-1)^m J_{2m+2+\nu}(ka)}{(ka)^{1+\nu}} \right\}. \tag{16}$$

In this section, as in Section III, ν is always equal to minus one-third for the non-basic terms and to zero for the basic terms. Substituting the transforms into equation 14, one obtains the matrix elements. For the side, these are

$$Y_p^{sm} = \left(\frac{2}{\pi}\right)^{1/2} \frac{(-1)^{m+p}}{(2p+2)!} \frac{ab}{c} \int_0^\infty K_1(ka) \frac{J_{2m+1/2+\nu}(kb)}{(kb)^{1/2+\nu}} (kc)^{2p+1} dk. \tag{17}$$

Let $kb = x$, and

$$Y_p^{sm} = \left(\frac{2}{\pi}\right)^{1/2} \frac{(-1)^{m+p}}{(2p+2)!} \frac{a}{c} \left(\frac{c}{b}\right)^{2p+1} \int_0^\infty K_1\left(\frac{a}{b}x\right) \frac{J_{2m+1/2+\nu}(x)}{x^{1/2+\nu}} x^{2p+1} dx. \tag{18}$$

By (27) the infinite integral is

$$I\left(\frac{a}{b}\right) = \frac{2^{2p-1/2-\nu}\, \Gamma(m+p+3/2)\,\Gamma(m+p+1/2)}{(a/b)^{2p+2m+2}\, \Gamma(2m+3/2+\nu)}$$

$$F\left(m+p+3/2,\ m+p+1/2;\ 2m+3/2+\nu;\ -b^2/a^2\right). \tag{19}$$

The side matrix elements may be written in either of two forms as before. These are

$$Y_p^{sm} = \frac{(-1)^{p+m}\, 2^{2p-\nu}\, \Gamma(m+p+3/2)\, \Gamma(m+p+1/2)}{\Gamma(1/2)\ \Gamma(2p+3)\, \Gamma(2m+3/2+\nu)}$$

$$\left\{ \begin{array}{ll} \left(\frac{c}{b}\right)^{2p} \left(\frac{b}{a}\right)^{2m+2p+1} F\left(m+p+3/2,\ m+p+1/2;\ 2m+3/2+\nu;\ -\frac{b^2}{a^2}\right) & (20) \\ \left(\frac{a^2}{bc}\right)\left(\frac{b}{c}\right)^{2m+2} F\left(m+p+3/2,\ m-p+1+\nu;\ 2m+3/2+\nu;\ \frac{b^2}{c^2}\right) & (21) \end{array} \right\}.$$

Using the same procedure for the end matrix elements, one obtains:

$$Y_p^{em} = \frac{(-1)^m}{(2p+2)!} \frac{a^2}{c} \int_0^\infty e^{-kb} \frac{J_{2m+2+\nu}(ka)}{(ka)^{1+\nu}} (kc)^{2p+1}\, dk. \quad (22)$$

Now let $ka = x$, and recall that $e^{-kb} = (2/\pi)^{1/2}\ (kb)^{1/2}\ K_{1/2}(kb)$. Then

$$Y_p^{em} = \left(\frac{2}{\pi}\right)^{1/2} \frac{(-1)^m}{(2p+2)!}\ \frac{a}{c} \left(\frac{c}{a}\right)^{2p+1} \left(\frac{b}{a}\right)^{1/2} \int_0^\infty K_{1/2}\left(\frac{b}{a}x\right) \frac{J_{2m+2+\nu}(x)}{x^{1+\nu}}\, x^{2p+3/2}\, dx. \quad (23)$$

By (28) the infinite integral is

$$I\left(\frac{b}{a}\right) = \frac{2^{2p-1/2-\nu}\, \Gamma(m+p+3/2)\, \Gamma(m+p+2)}{(b/a)^{2p+2m+3}\ \Gamma(2m+3+\nu)}$$

$$F\left(m+p+3/2,\ m+p+2;\ 2m+3+\nu;\ -b^2/a^2\right). \quad (24)$$

Again the matrix elements may be written in either of two forms

$$Y_p^{em} = \frac{(-1)^m \, 2^{2p-\nu} \, \Gamma(m+p+3/2)\, \Gamma(m+p+2)}{\Gamma(1/2)\, \Gamma(2p+3)\, \Gamma(2m+3+\nu)}$$

$$\left\{ \begin{array}{l} \left(\frac{c}{a}\right)^{2p} \left(\frac{a}{b}\right)^{2m+2p+3} F\left(m+p+3/2,\, m+p+2;\, 2m+3+\nu;\, -\frac{a^2}{b^2}\right) \\ \\ \left(\frac{a}{c}\right)\left(\frac{a}{c}\right)^{2m+2} F\left(m+p+3/2,\, m-p+1+\nu;\, 2m+3+\nu;\, \frac{a^2}{c^2}\right) \end{array} \right\} . \quad (25)$$

(26)

4.03. The Edge Condition. Since the asymptotic form of the function $\bar{\Psi}_m(\gamma, \zeta, \nu, u)$ as u tends to unity is a function of ν only, as is shown in subsection 3.02, the edge condition for the TE problem is the same as that for the LE. Therefore

$$\sum_{m=0}^{N_s-1} s_m - \left(\frac{a}{b}\right)^{1/3} \sum_{m=0}^{N_e-1} w_m = 0 \quad . \qquad (1)$$

4.04. Calculation of the Dipole Moment. The x, and only, component of the induced electric dipole moment is calculated by taking the integral of $\rho \cos \phi$ times the charge density over the entire surface. The derivation is as follows:

$$\frac{M_{Ex}}{\epsilon E} = \pi a \int_{-b}^{b} S(z)\, a\, dz + 2\pi \int_{0}^{a} \rho W(\rho) \rho\, d\rho. \qquad (1)$$

Converting to dimensionless variables

$$\frac{M_{Ex}}{\epsilon E} = 2\pi a^2 b \int_{0}^{1} S(u)\, du + 2\pi a^3 \int_{0}^{1} u W(u) u\, du. \qquad (2)$$

Note that unity in the first integral is orthogonal to every term in $S(u)$ except the basic and the first non-basic, since it may be identified as $\psi_o(\frac{1}{2}, 0, \nu, u)/b_o(\frac{1}{2}, 0, \nu)$ in either case. Similarly u in the second integral may be identified as $\psi_o(1,1,\nu,u)/b_o(1,1,\nu)$. The transverse electric polarizability and its dual, the longitudinal magnetic polarizability, are therefore

$$\boxed{\alpha_{tt} = -2\beta_{\ell\ell} = \left[\frac{s_b}{b_o(\frac{1}{2},0,0)} + \frac{s_o}{b_o(\frac{1}{2},0,-\frac{1}{3})}\right] + \frac{a}{b}\left[\frac{w_b}{b_o(1,1,0)} + \frac{w_o}{b_o(1,1,-\frac{1}{3})}\right]}. \qquad (3)$$

The necessary numerical values are given in Table 4.04-1.

$b_o(\frac{1}{2},0,0)$	= 1.0000000
$b_o(\frac{1}{2},0,-\frac{1}{3})$	= 0.83086092
$b_o(1,1,0)$	= 4.00000000
$b_o(1,1,-\frac{1}{3})$	= 2.3883647

Table 4.04-1

4.05. <u>Local Surface Deformations</u>. At the equator of the cylinder, equation 3.04-2 applies directly, and

$$\Delta\rho = \frac{V - aE\ \cos\phi}{\sigma_s/\epsilon} \tag{1}$$

or

$$\boxed{\frac{\Delta\rho}{a} = \frac{\dfrac{V}{aE\cos\phi} - 1}{S(0)}} . \tag{2}$$

The denominator is simply

$$S(0) = a_b s_b + \sum_{m=0}^{N_s-1} (-1)^m a_m s_m . \tag{3}$$

It is interesting to notice that although V is ϕ-dependent, $\Delta\rho$ is not.

At the pole, 3.04-2 becomes indeterminate and both numerator and denominator must be differentiated with respect to ρ . Thus

$$\Delta z = \frac{-E_\rho - E\ \cos\ \phi}{\dfrac{1}{\epsilon}\ \dfrac{\partial\sigma_e}{\partial\rho}} . \tag{4}$$

It is convenient to divide by $E\cos\phi$ and to introduce factors of a and b. Then

$$\boxed{\frac{\Delta z}{b} = -\frac{a}{b}\ \frac{\dfrac{E_\rho}{E\cos\phi} + 1}{W'(0)}} \tag{5}$$

where

$$W'(0) = a_b w_b + \sum_{m=0}^{N_e-1} (-1)^m a_m w_m . \tag{6}$$

The problems of calculating $V/aE\cos\phi$ and $E_\rho/E\cos\phi$ are considered in the next subsection.

4.06. Derivation of the Checking Coefficients. At the equator, let the part of V due to the side charge alone be denoted V_s. This quantity is obtained by setting z equal to zero and ρ equal to a in 4.02-1. Thus

$$\frac{V_s}{aE\cos\phi} = \int_0^\infty \tilde{S}(k)\, K_1(ka)\, I_1(ka)\, dk. \qquad (1)$$

Since $\tilde{S}(k)$ is itself a sum, as given in 4.02-15, it is clear that $V_s/aE\cos\phi$ is of the form

$$\frac{V}{aE\cos\phi} = A'_b s_b + \sum_{m=0}^{N_s-1} A'_m s_m \qquad (2)$$

where the A'_m are

$$A'_m = \left(\frac{2}{\pi}\right)^{1/2} (-1)^m\, b \int_0^\infty \frac{J_{2m+1/2+\nu}(kb)}{(kb)^{1/2+\nu}} K_1(ka) I_1(ka)\, dk. \qquad (3)$$

Letting $kb = p$ and $a/b = \tau$,

$$A'_m = \frac{1}{2}\left(\frac{2}{\pi}\right)^{1/2} (-1)^m \int_0^\infty \frac{J_{2m+1/2+\nu}(p)}{p^{3/2+\nu}} K_1(p\tau) I_1(p\tau)\, d(p^2). \qquad (4)$$

By (29)

$$K_1(p\tau) I_1(p\tau) = \frac{\pi^{-1/2}}{2} G_{13}^{21}\left(p^2\tau^2 \middle| \begin{matrix} 1/2 \\ 1,\ 0,\ -1 \end{matrix}\right) \qquad (5)$$

and

$$p^{-(3/2+\nu)} J_{2m+1/2+\nu}(p) = 2^{-(3/2+\nu)} G_{02}^{10}\left(\frac{p^2}{4} \middle| m-1/2,\ -m-1-\nu\right). \qquad (6)$$

The infinite integral of a product of G-functions is given by (30) and the A'_m coefficient is

$$\boxed{A'_m = \frac{(-1)^m 2^{-\nu}}{2\pi} G^{22}_{33}\left(4\tau^2 \middle| \begin{matrix} -m+\tfrac{1}{2}, \tfrac{1}{2}, m+1+\nu \\ 1, 0, -1 \end{matrix}\right)}. \tag{7}$$

The part of V at the equator due to the end charges alone is denoted V_e and is obtained by letting z equal zero and   equal a in 4.02-5. Thus

$$\frac{V_e}{a E \cos\phi} = \int_0^\infty \tilde{W}(k)\, e^{-kb} J_1(ka)\, dk. \tag{8}$$

Again $\tilde{W}(k)$ is a sum, as given in 4.02-16, and $V_e/aE \cos \phi$ is of the form

$$\frac{V_e}{a E \cos\phi} = B'_b w_b + \sum_{m=0}^{N_e-1} B'_m w_m. \tag{9}$$

The B'_m are therefore

$$B'_m = (-1)^m a \int_0^\infty \frac{J_{2m+2+\nu}(ka)}{(ka)^{1+\nu}} e^{-kb} J_1(ka)\, dk. \tag{10}$$

Now let $ka = p$, and

$$B'_m = (-1)^m \int_0^\infty \frac{J_{2m+2+\nu}(p)}{p^{1+\nu}} J_1(p)\, e^{-\frac{p}{\tau}}\, dp. \tag{11}$$

By (31)

$$p^{-(1+\nu)} J_1(p) J_{2m+2+\nu}(p) = \pi^{-1/2} G^{12}_{24}\left(p^2 \middle| \begin{matrix} -\nu/2, -1/2-\nu/2 \\ m+1, -m-1-\nu, m, -m-2-\nu \end{matrix}\right). \tag{12}$$

The infinite integral of an exponential and a G-function is another G-function (32). Therefore the B'_m coefficients are

$$B'_m = \frac{(-1)^m}{\pi} \tau \, G^{14}_{44}\left(4\tau^2 \left| \begin{matrix} 0, \ 1/2, \ -\nu/2, \ -1/2-\nu/2 \\ m+1, -m-1-\nu, \ m, \ -m-2-\nu \end{matrix} \right. \right) . \quad (13)$$

Attention is now turned to the electric field at the pole. That part due to the side charge alone is denoted $E_{\rho s}$ and is obtained by differentiating the negative of 4.02-1, then setting z equal to b and ρ equal to zero. Thus

$$\frac{E_{\rho s}}{E \cos\phi} = -\frac{a}{2}\int_0^\infty \tilde{S}(k)\, K_1(ka) \cos kb \; k\,dk \, ; \quad (14)$$

$$\frac{E_{\rho s}}{E \cos\phi} = C'_b s_b + \sum_{m=0}^{N_s-1} C'_m s_m \, ; \quad (15)$$

$$C'_m = -\left(\frac{2}{\pi}\right)^{1/2} (-1)^m \frac{ab}{2}\int_0^\infty K_1(ka) \cos kb \, \frac{J_{2m+1/2+\nu}(kb)}{(kb)^{1/2+\nu}} \, k\,dk . \quad (16)$$

Now let $kb = p$, and note that $\cos p = (\pi/2)^{1/2} p^{1/2} J_{-1/2}(p)$. Then

$$C'_m = -(-1)^m \frac{\tau}{4}\int_0^\infty K_1(p\tau)\, J_{-1/2}(p) \, \frac{J_{2m+1/2+\nu}(p)}{p^{\nu}} \, d(p^2) \, ; \quad (17)$$

$$p^{-\nu} J_{-1/2}(p)\, J_{2m+1/2+\nu}(p) = \pi^{-1/2} G^{12}_{24}\left(p^2 \left| \begin{matrix} 1/2-\nu/2, \ -\nu/2 \\ m, -m-1/2-\nu, m+1/2, -m-\nu \end{matrix} \right. \right) ; \quad (18)$$

$$K_1(p\tau) = \frac{1}{2} G^{20}_{02}\left(\frac{p^2\tau^2}{4} \left| \, \frac{1}{2}, -\frac{1}{2} \right. \right) . \quad (19)$$

The C'_m coefficients are

$$C'_m = \frac{-(-1)^m}{8\pi^{1/2}}\,\tau\, G^{41}_{44}\left(\frac{\tau^2}{4}\,\middle|\,\begin{matrix}-m,\ m+1/2+\nu,\ -m-1/2,\ m+\nu\\ -1/2+\nu/2,\ \nu/2,\ 1/2,\ -1/2\end{matrix}\right) \tag{20}$$

The part of the electric field at the pole due to the ends alone is found by differentiating the negative of 4.02-5, then setting z equal to b and ρ equal to zero. Thus

$$\frac{E_{\rho e}}{E\cos\phi} = -\frac{a}{2}\int_0^\infty \tilde{W}(k)\,e^{-kb}\cosh kb \quad k\,dk\,; \tag{21}$$

$$\frac{E_{\rho e}}{E\cos\phi} = D'_b w_b + \sum_{m=0}^{N_e-1} D'_m w_m\,; \tag{22}$$

$$D'_m = -(-1)^m\frac{a^2}{2}\int_0^\infty \frac{1+e^{-2kb}}{2}\,\frac{J_{2m+2+\nu}(ka)}{(ka)^{1+\nu}}\,k\,dk\,; \tag{23}$$

$$D'_m = -(-1)^m\frac{1}{4}\int_0^\infty \left(1+e^{-\frac{2p}{\tau}}\right)\frac{J_{2m+2+\nu}(p)}{p^{\nu}}\,dp\,. \tag{24}$$

It is convenient to split D'_m into D'_{mr}, the part due to the remote end, and D'_{mp}, the part due to the proximate end:

$$D'_{mr} = -(-1)^m\frac{1}{4}\int_0^\infty e^{-\frac{2p}{\tau}}\,\frac{J_{2m+2+\nu}(p)}{p^{\nu}}\,dp\,; \tag{25}$$

$$D'_{mp} = \lim_{\tau\to\infty} D'_{mr} \tag{26}$$

By (33), the final form for D'_m is

$$D'_m = -(-1)^m \, 2^{-2-\nu} \left\{ \frac{\Gamma(m+3/2)}{\Gamma(m+3/2+\nu)} + \frac{\Gamma(2m+3)\, 2^{-2m} F\left(m+3/2, m+1+\nu; 2m+3+\nu; \left[1+\frac{4}{T^2}\right]^{-1}\right)}{\Gamma(2m+3+\nu) \left[1+\frac{4}{T^2}\right]^{m+3/2}} \right\} . \quad (27)$$

4.07. Numerical Values of the TE Checking Coefficients.

$a/b = \frac{1}{4}$				
m	A'_m	B'_m	C'_m	D'_m
basic	+0.48567413	+0.0016906105	-0.24806947	-0.25012019
0	+0.46706690	+0.0028127304	-0.42241227	-0.30109362
1	-0.28985737	-0.0000460757	-0.24373313	+0.38686302
2	+0.20785455	+0.0000006550	-0.05589727	-0.44637931
3	-0.15664977	-0.0000000084	+0.01144904	+0.49336660
4	+0.12267493		+0.01338206	
5	-0.09920657		+0.00414450	
6	+0.08240599		-0.00014260	
7	-0.06998045		-0.00065000	
8	+0.06051847		-0.00025800	
9	-0.05312591			
10	+0.04722013			
11	-0.04241114			
12	+0.03843176			

Table 4.07-1

$a/b = \frac{1}{2}$				
m	A'_m	B'_m	C'_m	D'_m
basic	+0.45324694	+0.009664854	-0.24253563	-0.25091891
0	+0.45331629	+0.015895666	-0.34379359	-0.30242017
1	-0.22186504	-0.000721978	-0.11071474	+0.38689050
2	+0.13383088	+0.000023358	+0.00582310	-0.44637979
3	-0.09088765	-0.000000433	+0.00781273	+0.49336661
4	+0.06710064		+0.00055230	-0.53283593
5	-0.05250069		-0.00040600	+0.56721244
6	+0.04279397			-0.59787257
7	-0.03593609			
8	+0.03086104			
9	-0.02696777			
10	+0.02389470			

Table 4.07-2

a/b = 1				
m	A'_m	B'_m	C'_m	D'_m
basic	+0.38389461	+0.035531400	-0.22360680	-0.25623059
0	+0.40239953	+0.057711954	-0.27991450	-0.31104949
1	-0.14077125	-0.004428126	-0.02426401	+0.38755941
2	+0.07339133	+0.000048910	+0.00620624	-0.44642346
3	-0.04729610	+0.000022000	+0.00007680	+0.49336930
4	+0.03424682		-0.00011100	-0.53283609
5	-0.02657598			+0.56721245
6	+0.02157426			-0.59787257
7	-0.01807507			+0.62568060
8	+0.01550001			-0.65121858

Table 4.07-3

a/b = 2				
m	A'_m	B'_m	C'_m	D'_m
basic	+0.28895588	+0.08012455	-0.17677670	-0.28033009
0	+0.31447305	+0.13048760	-0.21161752	-0.34786258
1	-0.07786377	-0.00856820	+0.00225033	+0.39686370
2	+0.03759890	-0.00158380	+0.00045047	-0.44833458
3	-0.02386230	+0.00007759	-0.00003608	+0.49373502
4	+0.01720373	+0.00004910	+0.00000120	-0.53290393
5	-0.01332658	+0.00000140		+0.56722483
6	+0.01080855			-0.59787481
7				+0.62568100
8				-0.65121865
9				+0.67489927
10				-0.69702710

Table 4.07-4

a/b = 4				
m	A'_m	B'_m	C'_m	D'_m
basic	+0.19729197	+0.12665449	-0.11180340	-0.33541020
0	+0.22012863	+0.21079684	-0.13377119	-0.42300680
1	-0.04035999	-0.00124803	+0.00164845	+0.44541293
2	+0.01890936	-0.00509499	-0.00001084	-0.47197067
3	-0.01195678	-0.00132554	-0.00000012	+0.50412544
4		-0.00005575		-0.53726305
5		+0.00008590		+0.56900953
6		+0.00003338		-0.59859524
7		+0.00000350		+0.62596921
8		-0.00000180		-0.65133324
9				+0.67494462
10				-0.69704499
11				+0.71784091
12				-0.73750332

Table 4.07-5

V. THE TRANSVERSE MAGNETIC PROBLEM

5.00. Scope of the Transverse Magnetic Problem. The object of this section is the derivation of the expressions necessary for the calculation of the induced magnetic dipole moment in the cylinder of figure 1.00-1 when a uniform magnetic field of magnitude B is applied in the positive y direction. The structure of the induced current system, which must be such as to produce a magnetic field equal and opposite to the applied field at all points within the cylinder, merits some discussion. Certain components of this current, namely j_{zs} on the side and $j_{\rho e}$ on the upper end are regarded as fundamental whereas the ϕ components on side and end, constructed in such a way as to make the system solenoidal, are regarded as derived. The current on the lower end is equal and opposite to that on the upper end. A sketch of the current system appears in figure 5.00-1.

The fundamental current components are written in terms of the dimensionless and everywhere-positive functions, $F(z/b)$ and $G(\rho/a)$, as follows:

$$\frac{\mu j_{zs}}{B} = F(z/b) \cos\phi \; ; \tag{1}$$

$$\frac{\mu j_{\rho e}}{B} = -(a/\rho)\, G(\rho/a) \cos\phi \,. \tag{2}$$

The reason for the inclusion of the factor a/ρ in the definition of $G(\rho/a)$ becomes evident later. A graphical illustration of these fundamental components is shown in figure 5.00-2.

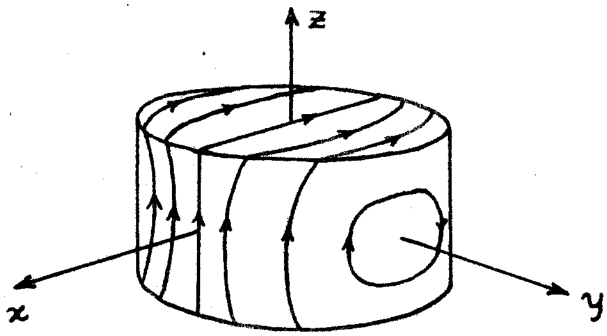

Figure 5.00-1. Current System for the TM Problem.

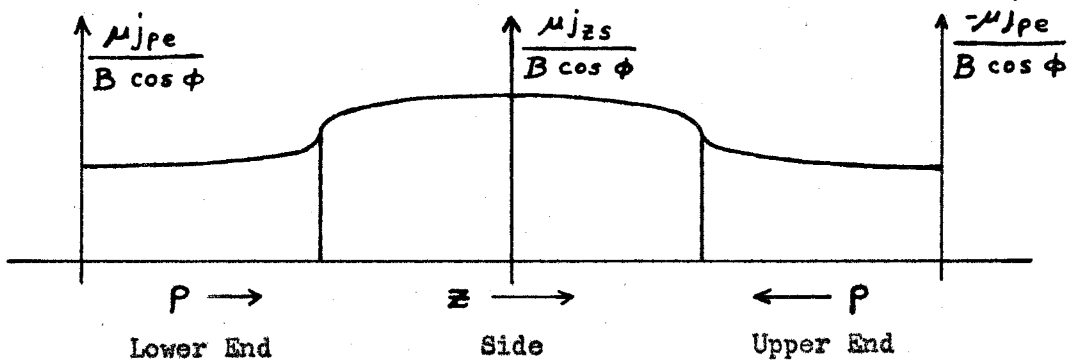

Figure 5.00-2. Sketch of Typical Fundamental Current Components.

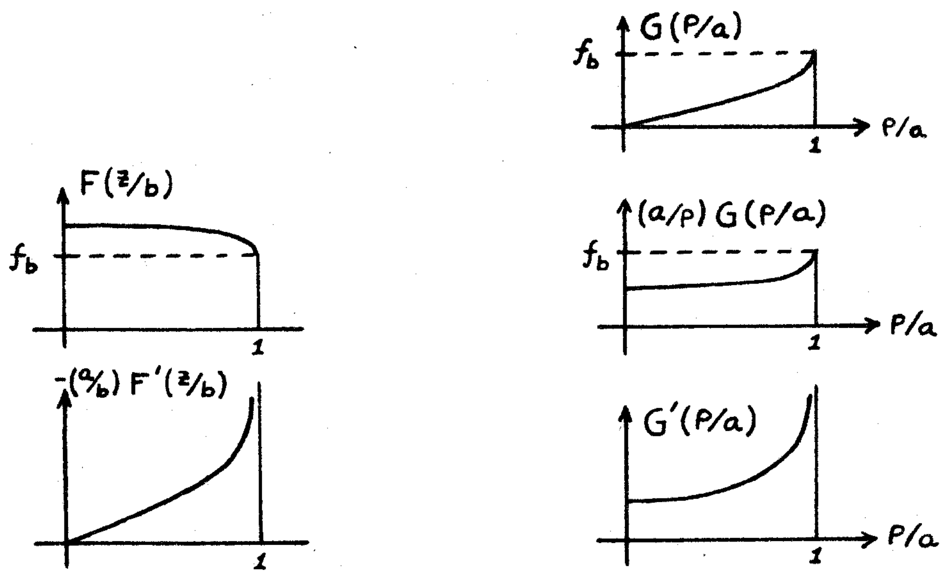

Figure 5.00-3. Typical Forms of the Functions which Describe the Current System.

The derived components must satisfy

$$\frac{1}{a}\frac{\partial j_{\phi s}}{\partial \phi} + \frac{\partial j_{zs}}{\partial z} = 0 \tag{3}$$

and

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho j_{\rho e}\right) + \frac{1}{\rho}\frac{\partial j_{\phi e}}{\partial \phi} = 0 \tag{4}$$

upon side and end respectively. Using a prime to denote differentiation with respect to the dimensionless argument, z/b or ρ/a, it is clear that the derived components must be

$$\frac{\mu j_{\phi s}}{B} = -\frac{a}{b} F'(z/b) \sin\phi \tag{5}$$

and

$$\frac{\mu j_{\phi e}}{B} = G'(\rho/a) \sin\phi . \tag{6}$$

The reason for the designation of the current components of equations 1 and 2 as fundamental should now be evident since the remaining components are obtained from these by differentiation. Had the opposite choice been made, integration would have been needed and arbitrary constants would have been involved.

Let ℓ be the distance of a given surface point from the edge of the cylinder on either side or end. Since the ϕ-going currents are parallel to the edge, it is clear that F' and G' must be asymptotic to constants times $\ell^{-1/3}$ as ℓ tends to zero. This, in turn, imposes certain requirements upon the functions F and G. Quantitatively,

$$F(z/b) \sim C_1 + C_2 (b-z)^{2/3} \quad \text{as} \quad z \to b; \tag{7}$$

$$G(\rho/a) \sim C_3 - C_4 (a-\rho)^{2/3} \quad \text{as} \quad \rho \to a; \tag{8}$$

$$-\frac{a}{b} F'(z/b) \sim \frac{2}{3} a\, C_2 (b-z)^{-1/3} \quad \text{as} \quad z \to b; \tag{9}$$

$$G'(\rho/a) \sim \frac{2}{3} a\, C_4 (a-\rho)^{-1/3} \quad \text{as} \quad \rho \to a. \tag{10}$$

It is seen that two edge conditions, $C_3 = C_1$ and $C_4 = C_2$, must be satisfied. The former of these involves the basic terms only and requires that the coefficients of these terms on side and end be equal. In other words, the basic terms correspond to a self-contained solenoidal system of very simple form with currents flowing in rectangular loops around the cylinder. Superposed upon this are the non-basic terms which give a gentle curvature to the current lines and which supply the necessary singular behavior at the edges. Thus the second edge condition, which concerns only the non-basic terms, is very similar to the edge conditions appearing in the LE and TE problems. It may be remarked that, although the basic terms could have been dispensed with in the LE and TE problems, they are absolutely necessary in the TM problem.

As in the two previous problems, F and G are represented by series of barred biorthogonal functions with the difference that ν is here equal to plus two-thirds rather than to minus one-third. On the side

$$\frac{\mu j'_{zs}}{B\cos\phi} = F(z/b) = \begin{cases} f_b + \sum_{m=0}^{N_s-1} f_m \bar{\Psi}_m\left(\frac{1}{2}, 0, \frac{2}{3}, \frac{z}{b}\right); & |z| \le b \\ 0; & |z| > b \end{cases} \tag{11}$$

and on the upper end

$$\frac{-\rho\mu j'_{\rho e}}{aB\cos\phi} = G(\rho/a) = \begin{cases} g_b \frac{\rho}{a} + \sum_{m=0}^{N_e-1} g_m \bar{\Psi}_m\left(1, 1, \frac{2}{3}, \frac{\rho}{a}\right); & \rho \le a \\ 0; & \rho > a \end{cases} \tag{12}$$

Clearly the first edge condition is satisfied if

$$g_b = f_b \tag{13}$$

and henceforth the single symbol, f_b, will be used as the coefficient of the basic term in both F and G. With apologies for repeating some features of figure 5.00-2, typical forms for the five functions F, $-(a/b)F'$, G, $(a/\rho)G$, and G', are sketched in figure 5.00-3.

5.01. The Matrix Elements. The equation

$$\left.\begin{aligned} A_i &= \frac{1}{4\pi}\int \frac{\mu j_i\, dS}{r}\,; \\ i &= x, y, z; \end{aligned}\right\} \tag{1}$$

asserts the existence of a unique relationship between any rectangular component of the vector potential and the corresponding rectangular component of the surface current density. Since this relationship is exactly the same as that which obtains between the scalar electric potential and the surface charge density, one may find any rectangular component of the vector potential by solving a boundary value problem using the following conditions at a current bearing surface, where the direction of increasing n is from region I into region II:

$$A_{iI} - A_{iII} = 0\,; \tag{2}$$

$$\frac{\partial A_{iI}}{\partial n} - \frac{\partial A_{iII}}{\partial n} = \mu j_i\,. \tag{3}$$

The currents on the side, in terms of their rectangular components, are

$$\frac{\mu j_{xs}}{B} = \frac{aF'}{2b} - \frac{aF'}{2b}\cos 2\phi\,; \tag{4}$$

$$\frac{\mu j_{ys}}{B} = - \frac{aF'}{2b} \sin 2\phi ; \tag{5}$$

$$\frac{\mu j_{zs}}{B} = F \cos \phi . \tag{6}$$

Considering figure 3.01-1, it is required to write down a set of vector potential components, for regions I and II respectively, which obey Laplace's equation, are equal at $\rho = a$ and are compatible with the current system given in equations 4 through 6. These components are

$$\frac{A_{xsI}}{B} = - a \int_0^\infty \frac{\tilde{L}(k)}{2} K_0(ka) I_0(k\rho) \sin kz \, dk$$
$$+ a \cos 2\phi \int_0^\infty \frac{\tilde{L}(k)}{2} K_2(ka) I_2(k\rho) \sin kz \, dk ; \tag{7}$$

$$\frac{A_{ysI}}{B} = a \sin 2\phi \int_0^\infty \frac{\tilde{L}(k)}{2} K_2(ka) I_2(k\rho) \sin kz \, dk ; \tag{8}$$

$$\frac{A_{zsI}}{B} = a \cos \phi \int_0^\infty \tilde{F}(k) K_1(ka) I_1(k\rho) \cos kz \, dk . \tag{9}$$

The corresponding components for region II may be obtained from these merely by writing "I_n" for "K_n" and vice versa. Applying equation 3 at $\rho = a$,

$$a \int_0^\infty \frac{\tilde{L}(k)}{2} \left(K_0 I_0' - I_0 K_0' \right) k \sin kz \, dk = - \frac{aF'}{2b} ; \tag{10}$$

$$a \int_0^\infty \frac{\tilde{L}(k)}{2} \left(K_2 I_2' - I_2 K_2' \right) k \sin kz \, dk = - \frac{aF'}{2b} ; \tag{11}$$

$$a\int_0^\infty \tilde{F}(k)\left(K_1 I_1' - I_1 K_1'\right) k \cos kz \; dk = F. \qquad (12)$$

The third of these yields

$$F(z/b) = \int_0^\infty \tilde{F}(k) \cos kz \; dk \qquad (13)$$

and

$$\tilde{F}(k) = \frac{2}{\pi}\int_0^b F(z/b) \cos kz \; dz . \qquad (14)$$

Thus $\tilde{F}(k)$ is the Fourier cosine transform of $F(z/b)$. Equations 10 and 11 both yield

$$-\frac{a}{b} F'(z/b) = \int_0^\infty \tilde{L}(k) \sin kz \; dk. \qquad (15)$$

Differentiation of 13 shows that

$$\tilde{L}(k) = ka \; \tilde{F}(k) \qquad (16)$$

and the description of the vector potential in terms of $\tilde{F}(k)$, and thus ultimately in terms of $F(z/b)$, is complete.

Since the basic term in $F(z/b)$ implicitly contains steps at $z = \pm b$, the differentiation of F generates oppositely signed impulses at these two points and $\tilde{L}(k)$ for the basic term becomes the Fourier sine transform of an odd impulse pair. These impulses correspond physically to four currents of infinitesimal cross section, two circulating around the upper edge and two circulating around the lower edge of an imaginary cylindrical shell carrying the side currents only. At the upper edge, these currents flow generally toward the negative x direction collecting near $\phi = 0$ the current which would normally leave the side and flow into the upper end and redistributing near $\phi = \pi$ the current which would normally leave the upper end and flow onto the side. At the lower edge, two other currents perform a similar function in the opposite sense. The internal vector potential expressions, 7 through 9, include the vector potentials of these impulse currents. Thus the side current system alone, and its vector potential, are solenoidal, as indeed they must be since the current system was deliberately constructed so as to obey equation 5.00-3. Similar remarks apply to the two imaginary disks carrying the end currents alone. When the two disks and the shell are assembled the impulse currents associated with each cancel one another and only the legitimate currents and their vector potentials remain, provided that the first edge condition, whereby the basic coefficients on side and end are equal, has been satisfied. These remarks become important when the matrix elements for the basic current system are calculated since some long expressions, which might not otherwise be expected to cancel, are found upon closer examination to cancel.

Attention is now turned to the end of the cylinder and the currents are written in terms of their rectangular components:

$$\frac{\mu \dot{J}_{xe}}{B} = -\frac{a}{\rho} G \cos^2\phi - G' \sin^2\phi ;$$

$$= -\frac{1}{2}\left[\frac{a}{\rho} G + G'\right] - \frac{1}{2}\left[\frac{a}{\rho} G - G'\right] \cos 2\phi ; \qquad (17)$$

$$\frac{\mu \dot{J}_{ye}}{B} = -\frac{1}{2}\left[\frac{a}{\rho} G - G'\right] \sin 2\phi . \qquad (18)$$

Considering figure 3.01-2, the following internal vector potential components suggest themselves:

$$\frac{A_{xeI}}{B} = -a\int_0^\infty \frac{ka\,\tilde{G}(k)}{2}\, e^{-kb} \sinh kz \;\; J_0(k\rho)\, dk$$

$$- a \cos 2\phi \int_0^\infty \frac{ka\,\tilde{G}(k)}{2}\, e^{-kb} \sinh kz \;\; J_2(k\rho)\, dk\,; \qquad (19)$$

$$\frac{A_{yeI}}{B} = - a \sin 2\phi \int_0^\infty \frac{ka\,\tilde{G}(k)}{2}\, e^{-kb} \sinh kz \; J_2(k\rho)\, dk. \qquad (20)$$

As before, the external vector potential components may be obtained from the internal by writing "sinh kb" for "e^{-kb}" and "e^{-kz}" for "sinh kz". Applying equation 3 at $z = b$,

$$a\int_0^\infty \frac{ka\,\tilde{G}(k)}{2} \left[e^{-kb} \cosh kb + e^{-kb} \sinh kb \right] J_0(k\rho)\, k\, dk$$

$$= \frac{1}{2}\left[\frac{a}{\rho}\, G + G' \right]; \qquad (21)$$

$$a\int_0^\infty \frac{ka\,\tilde{G}(k)}{2} \left[e^{-kb} \cosh kb + e^{-kb} \sinh kb \right] J_2(k\rho)\, k\, dk$$

$$= \frac{1}{2}\left[\frac{a}{\rho}\, G - G' \right]. \qquad (22)$$

Since

$$J_0(p) = J_1'(p) + \frac{J_1(p)}{p} \qquad (23)$$

and

$$J_2(p) = -J_1'(p) + \frac{J_1(p)}{p}, \qquad (24)$$

equations 21 and 22 become

$$a\int_0^\infty ka\,\tilde{G}(k) \left[J_1'(k\rho) + \frac{J_1(k\rho)}{k\rho} \right] k\, dk = \frac{a}{\rho}\, G + G'\,; \qquad (25)$$

$$a\int_0^\infty ka\,\tilde{G}(k) \left[-J_1'(k\rho) + \frac{J_1(k\rho)}{k\rho} \right] k\, dk = \frac{a}{\rho}\, G - G'. \qquad (26)$$

Evidently

$$G(\rho/a) = a\int_0^\infty \tilde{G}(k)\, J_1(k\rho)\; k\,dk \qquad (27)$$

and

$$\tilde{G}(k) = \frac{1}{a}\int_0^a G(\rho/a)\, J_1(k\rho)\,\rho\, d\rho\,. \qquad (28)$$

Again the vector potential is described in terms of an integral transform of the current distribution.

It is advantageous to replace the lengthy expressions used in the internal vector potential formulas by more compact symbols. Therefore let

$$-a\int_0^\infty \frac{ka\,\tilde{F}(k)}{2}\, K_0(ka)\, I_0(k\rho)\,\sin kz\;dk$$

$$-a\int_0^\infty \frac{ka\,\tilde{G}(k)}{2}\, e^{-kb}\,\sinh kz\;\; J_0(k\rho)\,dk \;=\; M_0(\rho,z)\,; \qquad (29)$$

$$a\int_0^\infty \frac{ka\,\tilde{F}(k)}{2}\, K_2(ka)\, I_2(k\rho)\,\sin kz\;dk$$

$$-a\int_0^\infty \frac{ka\,\tilde{G}(k)}{2}\, e^{-kb}\,\sinh kz\;\; J_2(k\rho)\,dk \;=\; M_2(\rho,z)\,; \qquad (30)$$

$$a\int_0^\infty \tilde{F}(k)\, K_1(ka)\, I_1(k\rho)\cos kz\;dk \;=\; M_1(\rho,z)\,. \qquad (31)$$

The total vector potential in the interior of the cylinder due to both side and ends is now written in a very concise form as follows:

$$\left.\begin{aligned}
\frac{A_x}{B} &= M_0 + M_2 \cos 2\phi ; \\
\frac{A_y}{B} &= M_2 \sin 2\phi ; \\
\frac{A_z}{B} &= M_1 \cos \phi ; \\
\frac{A_\rho}{B} &= (M_0 + M_2) \cos \phi ; \\
\frac{A_\phi}{B} &= (-M_0 + M_2) \sin \phi .
\end{aligned}\right\} \qquad (32)$$

The corresponding magnetic field is

$$\left.\begin{aligned}
\frac{B_\rho}{B} &= \left(-\frac{M_1}{\rho} + \frac{\partial M_0}{\partial z} - \frac{\partial M_2}{\partial z} \right) \sin \phi ; \\
\frac{B_\phi}{B} &= \left(\frac{\partial M_0}{\partial z} + \frac{\partial M_2}{\partial z} - \frac{\partial M_1}{\partial \rho} \right) \cos \phi ; \\
\frac{B_z}{B} &= \left(\frac{2 M_2}{\rho} + \frac{\partial M_2}{\partial \rho} - \frac{\partial M_0}{\partial \rho} \right) \sin \phi .
\end{aligned}\right\} \qquad (33)$$

Since this field is irrotational at all interior points of the cylinder, a scalar potential Φ is sought such that $\bar{B} = -\nabla \Phi$. Note that

$$\frac{2 I_2}{q} + \frac{\partial I_2}{\partial q} = I_1 (q) \qquad (34)$$

and

$$\frac{2 J_2}{q} + \frac{\partial J_2}{\partial q} = J_1 (q) . \qquad (35)$$

Therefore

$$\frac{B_z}{B} = a \sin\phi \int_0^\infty \left\{ \frac{ka\,\tilde{F}(k)}{2} K_2(ka)\, I_1(k\rho)\, \sin kz \right.$$
$$- \frac{ka\,\tilde{G}(k)}{2} e^{-kb} J_1(k\rho) \sinh kz$$
$$+ \frac{ka\,\tilde{F}(k)}{2} K_0(ka)\, I_1(k\rho) \sin kz$$
$$\left. - \frac{ka\,\tilde{G}(k)}{2} e^{-kb} J_1(k\rho) \sinh kz \right\} k\,dk. \qquad (36)$$

Now

$$K_2(q) = K_0(q) + \frac{2\,K_1(q)}{q} \qquad (37)$$

and therefore

$$\frac{B_z}{B} = a \sin\phi \int_0^\infty \left\{ \tilde{F}(k) \left[K_0(ka) + \frac{K_1(ka)}{ka} \right] I_1(k\rho) \sin kz \right.$$
$$\left. - \tilde{G}(k)\, e^{-kb} J_1(k\rho) \sinh kz \right\} ka\ \ k\,dk. \qquad (38)$$

Evidently a suitable scalar potential is

$$\boxed{\frac{\Phi}{B} = a \sin\phi \int_0^\infty \left\{ \tilde{F}(k) \left[K_0(ka) + \frac{K_1(ka)}{ka} \right] I_1(k\rho) \cos kz + \tilde{G}(k)\, e^{-kb} J_1(k\rho) \cosh kz \right\} ka\ \ dk} \qquad (39)$$

and it may be verified that $-\nabla\Phi/B$ does indeed yield all the terms of equations 33.

The problem is now clear; one must find an F and G such that Φ is maximally linear and as nearly as possible equal to yB. In other words if

$$\frac{\Phi}{cB} = \sin\phi \sum_{p=0}^{\infty} \Lambda_p \left(\frac{r}{c}\right)^{2p+1} P^{1}_{2p+1}(\cos\theta), \tag{40}$$

it is necessary to set Λ_0 equal to unity and as many as possible of the other Λ_p equal to zero. The symmetry of the potential under consideration is very similar to that of the electric potential in the TE problem and the results may be written down immediately. Thus

$$\Lambda_p = \frac{1}{(2p+2)!}\,\frac{a}{c}\left\{(-1)^p\int_0^{\infty}\tilde{F}(k)\;\; ka\,K_0(ka)+K_1(ka)\;\;(kc)^{2p+1}dk\right.$$

$$\left.+\int_0^{\infty} ka\,\tilde{G}(k)\,e^{-kb}\,(kc)^{2p+1}\,dk\right\}. \tag{41}$$

By Appendix A, the two transforms, $\tilde{F}(k)$ and $\tilde{G}(k)$, are

$$\tilde{F}(k) = \left(\frac{2}{\pi}\right)^{1/2} b\left\{f_b\,\frac{J_{1/2}(kb)}{(kb)^{1/2}} + \sum_{m=0}^{N_s-1} f_m\,\frac{(-1)^m J_{2m+1/2+\nu'}(kb)}{(kb)^{1/2+\nu'}}\right\}; \tag{42}$$

$$\tilde{G}(k) = a\left\{f_b\,\frac{J_2(ka)}{ka} + \sum_{m=0}^{N_e-1} g_m\,\frac{(-1)^m J_{2m+2+\nu'}(ka)}{(ka)^{1+\nu'}}\right\}. \tag{43}$$

In this section, ν' is always equal to plus two-thirds for the non-basic terms and to zero for the basic terms. Substituting the transforms into 41, one obtains the matrix elements. For the side, these are

$$Z_p^{sm} = \left(\frac{2}{\pi}\right)^{1/2} \frac{(-1)^{m+p}}{(2p+2)!}\,\frac{ab}{c}\int_0^{\infty}\left[ka\,K_0(ka)+K_1(ka)\right]$$

$$\frac{J_{2m+1/2+\nu'}(kb)}{(kb)^{1/2+\nu'}}\,(kc)^{2p+1}\,dk. \tag{44}$$

Let $kb = x$, and

$$Z_p^{sm} = \left(\frac{2}{\pi}\right)^{1/2} \frac{(-1)^{m+p}}{(2p+2)!} \frac{a}{c}\left(\frac{c}{b}\right)^{2p+1}$$

$$\int_0^{\infty} \left[\frac{a}{b} x K_0\left(\frac{a}{b}x\right) + K_1\left(\frac{a}{b}x\right)\right] \frac{J_{2m+1/2+\nu'}(x)}{x^{1/2+\nu'}} x^{2p+1}\, dx. \qquad (45)$$

For the non-basic elements, where $\nu' = 1 + \nu$,

$$\boxed{Z_p^{sm} = \frac{1}{(2p+2)} \frac{a}{b} X_p^{sm} + Y_p^{sm}(1+\nu)} \qquad \text{(non-basic).} \qquad (46)$$

For the end

$$Z_p^{em} = \frac{(-1)^m}{(2p+2)!} \frac{a^2}{c} \int_0^{\infty} e^{-kb} \frac{J_{2m+2+\nu'}(ka)}{(ka)^{1+\nu'}} (kc)^{2p+1}\, ka\, dk. \qquad (47)$$

Now let $ka = x$ and make a substitution for e^{-kb} as in the LE and TE problems. Then

$$Z_p^{em} = \left(\frac{2}{\pi}\right)^{1/2} \frac{(-1)^m}{(2p+2)!} \frac{a}{c}\left(\frac{c}{a}\right)^{2p+1}\left(\frac{b}{a}\right)^{1/2} \int_0^{\infty} K_{1/2}\left(\frac{b}{a}x\right) \frac{J_{2m+2+\nu'}(x)}{x^{1+\nu'}} x^{2p+5/2}\, dx. \qquad (48)$$

For the non-basic elements

$$Z_p^{em} = \left(\frac{2}{\pi}\right)^{1/2} \frac{(-1)^m}{(2p+2)!} \frac{a}{c}\left(\frac{c}{a}\right)^{2p+1}\left(\frac{b}{a}\right)^{1/2} \int_0^{\infty} K_{1/2}\left(\frac{b}{a}x\right) \frac{J_{2m+3+\nu}(x)}{x^{1+\nu}} x^{2p+3/2}\, dx. \qquad (49)$$

and

$$\boxed{Z_p^{em} = \frac{-1}{(2p+2)} X_p^{e,m+1}} \qquad \text{(non-basic).} \qquad (50)$$

The X's and Y's are matrix elements from the LE and TE problems, respectively, and it is fortunate that the present matrix elements can be so simply expressed in terms of these.

The basic elements are in a class by themselves. Letting $a/b = \tau$ in equation 45, one has for the side

$$Z_p^{sb} = \left(\frac{2}{\pi}\right)^{1/2} \frac{(-1)^p}{(2p+2)!} \left(\frac{c}{b}\right)^{2p} \tau \int_0^\infty \left[\tau x K_0(\tau x) + K_1(\tau x)\right] \frac{J_{1/2}(x)}{x^{1/2}} x^{2p+1} dx. \quad (51)$$

The integrals are readily evaluated by (34) and,

$$Z_p^{sb} = \left(\frac{2}{\pi}\right)^{1/2} \frac{(-1)^p}{(2p+2)!} \left(\frac{c}{a}\right)^{2p} \frac{1}{\tau} \frac{2^{2p+1/2}}{\Gamma(3/2)}$$

$$\Big[\Gamma(p+3/2)\,\Gamma(p+3/2)\; F(p+3/2, p+3/2; 3/2; -1/\tau^2)$$

$$+ \frac{1}{2}\Gamma(p+3/2)\,\Gamma(p+1/2)\; F(p+3/2, p+1/2; 3/2; -1/\tau^2) \Big]. \quad (52)$$

The end elements are obtained from 48; replacing J_2 by $2J_1/x - J_0$, one finds that

$$Z_p^{eb} = \left(\frac{2}{\pi}\right)^{1/2} \frac{1}{(2p+2)!} \left(\frac{c}{a}\right)^{2p} \tau^{-1/2} \int_0^\infty K_{1/2}\left(\frac{x}{\tau}\right) \left[\frac{2J_1(x)}{x} - J_0(x)\right] x^{2p+3/2} dx. \quad (53)$$

Evaluation of the integrals yields

$$Z_p^{eb} = \left(\frac{2}{\pi}\right)^{1/2} \frac{1}{(2p+2)!} \left(\frac{c}{a}\right)^{2p} 2^{2p+1/2}$$

$$\Big[\tau^{2p+2}\, \Gamma(p+3/2)\,\Gamma(p+1)\, F(p+3/2, p+1; 2; -\tau^2)$$

$$- \tau^{2p+2}\, \Gamma(p+3/2)\,\Gamma(p+1)\, F(p+3/2, p+1; 1; -\tau^2) \Big]. \quad (54)$$

Remembering that p is always a non-negative integer and applying the formulas for analytic continuation (35),

$$Z_p^{eb} = \left(\frac{2}{\pi}\right)^{1/2} \frac{1}{(2p+2)!} \left(\frac{c}{a}\right)^{2p} 2^{2p+1/2}$$

$$\left[\frac{1}{\tau} \frac{\Gamma(p+3/2)\Gamma(-1/2)}{\Gamma(-p+1/2)} F(p+3/2, p+1/2; 3/2; -1/\tau^2) \right.$$

$$+ \frac{\Gamma(p+1)\ \Gamma(1/2)}{\Gamma(-p+1)} F(p+1, p; 1/2; -1/\tau^2)$$

$$- \frac{1}{\tau} \frac{\Gamma(p+3/2)\Gamma(-1/2)}{\Gamma(-p-1/2)} F(p+3/2, p+3/2; 3/2; -1/\tau^2)$$

$$\left. - 0 \right]. \tag{55}$$

The contribution of the second hypergeometric function to Z_p^{eb} is unity when $p = 0$ and zero when $p \neq 0$; it will be denoted by the symbol $\left\{ {1 \atop 0} \right\}$. Furthermore, by (36)

$$\frac{\Gamma(-1/2)}{\Gamma(-p+1/2)} = \frac{-(-1)^p\ \Gamma(p+1/2)}{\Gamma(3/2)}; \tag{56}$$

$$\frac{\Gamma(-1/2)}{\Gamma(-p-1/2)} = \frac{(-1)^p\ \Gamma(p+3/2)}{\Gamma(3/2)}. \tag{57}$$

The final form for Z_p^{eb} becomes

$$Z_p^{eb} = \left\{ {1 \atop 0} \right\} + \left(\frac{2}{\pi}\right)^{1/2} \frac{(-1)^p}{(2p+2)!} \left(\frac{c}{a}\right)^{2p} \frac{1}{\tau} \frac{2^{2p+1/2}}{\Gamma(3/2)}$$

$$\left[-\Gamma(p+3/2)\Gamma(p+1/2)\ F(p+3/2, p+1/2; 3/2; -1/\tau^2) \right.$$

$$\left. -\Gamma(p+3/2)\Gamma(p+3/2)\ F(p+3/2, p+3/2; 3/2; -1/\tau^2) \right]. \tag{58}$$

The total basic matrix element, Z_p^b, is the sum of Z_p^{sb} and Z_p^{eb}. Thus

$$Z_p^b = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} - \frac{2^{2p+1}(-1)^p \Gamma(p+3/2)\Gamma(p+1/2)}{\pi \Gamma(2p+3)} \left(\frac{c}{a}\right)^{2p} \frac{b}{a}$$

$$F(p+3/2, p+1/2; 3/2; -b^2/a^2). \tag{59}$$

Transformation yields the polynomial form

$$Z_p^b = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} - \frac{(-1)^p \Gamma(p+1/2)}{2\Gamma(1/2)\Gamma(p+2)} \frac{b}{c} F(p+1/2, -p; 3/2; \frac{b^2}{c^2}); \quad \begin{Bmatrix} p=0 \\ p>0 \end{Bmatrix}; \tag{60}$$

(basic).

5.02. <u>The Second Edge Condition</u>. The first edge condition is implicitly satisfied if the basic terms on side and end are combined and governed by the same coefficient, f_b, as has been done. The second edge condition adds an equation to the matrix and it is this equation which must now be found. Since the condition involves derivatives, the asymptotic form of Ψ_m' as the argument tends to unity comes under investigation. Evidently

$$\Psi_m'(\gamma, \zeta, \nu', u) = a_m (-1)^m$$

$$\Big\{ \left[\zeta u^{\zeta-1}(1-u^2)^{\nu'} - 2\nu' u^{\zeta+1}(1-u^2)^{\nu'-1}\right] F(-m, m+\nu'+\gamma+\zeta; \gamma+\zeta; u^2)$$

$$+ u^\zeta (1-u^2)^{\nu'} \frac{d}{du} F(-m, m+\nu'+\gamma+\zeta; \gamma+\zeta; u^2) \Big\}. \tag{1}$$

The derivative of the hypergeometric polynomial is obviously finite for finite argument. Since ν' is equal to plus two-thirds, only one term survives and as u tends to unity,

$$\bar{\Psi}_m' \sim -2\nu' a_m (-1)^m (1-u^2)^{\nu'-1} F(-m, m+\nu'+\gamma+\zeta; \gamma+\zeta; 1). \qquad (2)$$

This becomes

$$\bar{\Psi}_m' \sim \frac{-2\nu'(1-u^2)^{\nu'-1}}{\Gamma(1+\nu')\, 2^{\nu'}} \qquad (3)$$

As z tends to b,

$$-\frac{a}{b} F'(z/b) \sim \frac{a}{b} \frac{\nu' \, b^{-(\nu'-1)} (b-z)^{\nu'-1}}{\Gamma(1+\nu')} \sum_{m=0}^{N_s-1} f_m . \qquad (4)$$

As ρ tends to a,

$$G'(\rho/a) \sim \frac{-\nu' \, a^{-(\nu'-1)} (a-\rho)^{\nu'-1}}{\Gamma(1+\nu')} \sum_{m=0}^{N_e-1} g_m . \qquad (5)$$

Then

$$b^{-\nu'} \sum_{m=0}^{N_s-1} f_m = - a^{-\nu'} \sum_{m=0}^{N_e-1} g_m \qquad (6)$$

and the edge condition becomes

$$\boxed{\sum_{m=0}^{N_s-1} f_m + \left(\frac{a}{b}\right)^{-2/3} \sum_{m=0}^{N_e-1} g_m = 0} \; . \qquad (7)$$

5.03. Calculation of the Dipole Moment. The y, and only, component of the induced magnetic dipole moment is obtained by taking the following integral over the entire surface of the cylinder:

$$M_{My} = \frac{1}{2} \int (z j_x - x j_z) \, dS . \tag{1}$$

Using 5.01-4, 6 and 17, this becomes

$$\frac{\mu M_{My}}{B} = \frac{1}{2} \int_0^{2\pi} \int_{-b}^{b} \left\{ z \left[\frac{aF'}{2b} - \frac{aF'}{2b} \cos 2\phi \right] - a \cos\phi \; F \cos\phi \right\} a \, dz \, d\phi$$

$$- b \int_0^{2\pi} \int_0^{a} \left\{ \frac{1}{2} \left[\frac{a}{\rho} G + G' \right] + \frac{1}{2} \left[\frac{a}{\rho} G - G' \right] \cos 2\phi \right\} \rho \, d\rho \, d\phi . \tag{2}$$

The terms containing 2ϕ vanish and

$$\frac{\mu M_{My}}{B} = \pi a \int_{-b}^{b} \left\{ z \frac{aF'}{2b} - a \frac{F}{2} \right\} dz - \pi b \int_0^{a} \left\{ \frac{a}{\rho} G + G' \right\} \rho \, d\rho . \tag{3}$$

Converting to dimensionless variables

$$\frac{\mu M_{My}}{B v_0} = \frac{1}{2} \int_0^1 (uF' - F) \, du - \frac{1}{2} \int_0^1 (uG' + G) \, du . \tag{4}$$

By parts

$$\frac{\mu M_{My}}{B v_0} = \frac{1}{2} \left[uF \right]_0^1 - \int_0^1 F \, du - \frac{1}{2} \left[uG \right]_0^1 . \tag{5}$$

The first edge condition affirms that the bracketed quantities cancel and an amazingly simple formula for the polarizability results. It is

$$\beta_{tt} = -\int_0^1 F\, du \; . \tag{6}$$

As in previous problems,

$$\boxed{\beta_{tt} = -\left[\frac{f_b}{b_o(\frac{1}{2},0,0)} + \frac{f_o}{b_o(\frac{1}{2},0,\frac{2}{3})} \right]} \; . \tag{7}$$

The numerical values are given in Table 5.03-1.

$b_o(\frac{1}{2}\,,\,0,\,0) = 1.0000000$
$b_o(\frac{1}{2}\,,\,0,\,\frac{2}{3}) = 1.9386755$

Table 5.03-1

5.04. Local Surface Deformations. Consider a given surface point, P, of an arbitrary perfectly conducting object and let the surface in a non-infinitesimal neighborhood of P be described by the equation q_1 = constant in a generalized curvilinear coordinate system characterized by the unit vectors $\bar{e}_i$, the coordinates q_i and the metrical coefficients h_i. The total magnetic field adjacent to the surface is related to the current density on the surface by

$$\mu \bar{J} = \bar{e}_1 \times \bar{B}_t , \tag{1}$$

or

$$\left.\begin{aligned} \mu j_2 &= -B_{3t} ; \\ \mu j_3 &= B_{2t} . \end{aligned}\right\} \tag{2}$$

Applying the condition $\nabla \cdot \bar{B}_t = 0$, one finds that

$$\frac{\partial}{\partial q_1}(h_2 h_3 B_{1t}) = \mu\left[\frac{\partial}{\partial q_3}(h_1 h_2 j_2) - \frac{\partial}{\partial q_2}(h_1 h_3 j_3)\right]. \qquad (3)$$

Evidently $\Delta n = h_1 \Delta q_1$. If, at P, $\bar{B}$ is the field due to the surface currents alone and $\bar{B}_a$ is the applied field, then

$$h_2 h_3 (B_1 + B_{1a}) + \frac{1}{h_1}\frac{\partial}{\partial q_1}(h_2 h_3 B_{1t})\Delta n = 0. \qquad (4)$$

Solving for Δn and substituting,

$$\Delta n = \frac{h_1 h_2 h_3 (B_1 + B_{1a})}{\mu\left[\frac{\partial}{\partial q_2}(h_1 h_3 j_3) - \frac{\partial}{\partial q_3}(h_1 h_2 j_2)\right]}. \qquad (5)$$

At the equator of the cylinder:

$$\left.\begin{array}{ll} q_1 = \rho; & h_1 = 1; \\ q_2 = \phi; & h_2 = \rho; \\ q_3 = z; & h_3 = 1, \end{array}\right\} \qquad (6)$$

and

$$\Delta\rho = \frac{a(B_\rho + B_{\rho a})}{\mu\left[\frac{\partial j_{zs}}{\partial \phi} - \frac{\partial}{\partial z}(\rho j_{\phi s})\right]}. \qquad (7)$$

Now replace $B_{\rho a}$ by its equivalent, $B \sin \phi$, and make appropriate normalizations:

$$\frac{\Delta\rho}{a} = \frac{\frac{B_\rho}{B \sin\phi} + 1}{\frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\frac{\mu j_{zs}}{B} - \frac{a}{\sin\phi}\frac{\partial}{\partial z}\frac{\mu j_{\phi s}}{B}}. \qquad (8)$$

Finally

$$\boxed{\frac{\Delta \rho}{a} = \frac{\dfrac{B_\rho}{B \sin\phi} + 1}{-F(0) + \dfrac{a^2}{b^2} F''(0)}} \quad . \tag{9}$$

The function F(u) is composed of even parity functions whose Maclauren expansions may be developed as follows:

$$\bar{\Psi}_m(\tfrac{1}{2}, 0, \nu', u) = a_m (-1)^m \left[1 - \nu' u^2 + \cdots\right] \left[1 - \frac{m(m + \nu' + \frac{1}{2})}{\frac{1}{2}} u^2 + \cdots\right]; \tag{10}$$

$$\bar{\Psi}_m(\tfrac{1}{2}, 0, \nu', u) = a_m (-1)^m \left[1 - (\nu' + 2m\{m + \nu' + \tfrac{1}{2}\}) u^2 + \cdots\right]. \tag{11}$$

Therefore

$$F(0) = a_b f_b + \sum_{m=0}^{N_s - 1} (-1)^m a_m f_m \tag{12}$$

and

$$F''(0) = -2 \sum_{m=0}^{N_s - 1} (-1)^m a_m f_m \left(\nu' + 2m\{m + \nu' + \tfrac{1}{2}\}\right). \tag{13}$$

5.05. Derivation of the Non-Basic Checking Coefficients.

At the equator, let the part of B_ρ due to the side currents alone be denoted $B_{\rho s}$. This quantity is most easily obtained by substituting into equation 5.03-33 those portions of equations 5.03-29 through 5.03-31 which contain $\tilde{F}(k)$, then setting ρ equal to a and z equal to zero. It may also be obtained by differentiating Φ, but considerable manipulation of Bessel functions is required in order to translate this form into the other. The desired expression is

$$\frac{B_{\rho s}}{B \sin\phi} = -\int_0^\infty \tilde{F}(k) \left[I_1 K_1 + \frac{(ka)^2}{2} (I_0 K_0 + I_2 K_2) \right] dk. \qquad (1)$$

Here all the Bessel functions have the argument ka. Since $\tilde{F}(k)$ is itself a sum, as given by 5.03-42, it is clear that $B_{\rho s}/B \sin\phi$ is of the form

$$\frac{B_{\rho s}}{B \sin\phi} = A''_b f_b + \sum_{m=0}^{N_s - 1} A''_m f_m \qquad (2)$$

where the A''_m are

$$A''_m = -\left(\frac{2}{\pi}\right)^{1/2} (-1)^m b \int_0^\infty \left[I_1 K_1 + \frac{(ka)^2}{2} (I_0 K_0 + I_2 K_2) \right]$$

$$(kb)^{-(1/2 + \nu')} J_{2m + 1/2 + \nu'}(kb)\ dk. \qquad (3)$$

Let $kb = p$ and $a/b = \tau$:

$$A''_m = -\left(\frac{2}{\pi}\right)^{1/2} (-1)^m \int_0^\infty \Big[\underset{\mathrm{I}}{I_1 K_1 (p\tau)} + \frac{p^2\tau^2}{2} \underset{\mathrm{II}}{I_0 K_0 (p\tau)}$$

$$+ \frac{p^2\tau^2}{2} \underset{\mathrm{III}}{I_2 K_2 (p\tau)} \Big] \frac{J_{2m+1/2+\nu'}(p)}{p^{1/2+\nu'}} \, dp. \qquad (4)$$

It is convenient to speak of A''_{mI}, A''_{mII} and A''_{mIII}, according to the Roman numerals under the terms in equation 4. With reference to earlier work, it is seen that

$$A''_{m\mathrm{I}} = - A'_m (\nu+1) ; \qquad (5)$$

$$A''_{m\mathrm{II}} = \frac{\tau}{2} A_m . \qquad (6)$$

The remaining integral is new:

$$A''_{m\mathrm{III}} = \frac{\tau}{2} \cdot \frac{-1}{2} \left(\frac{2}{\pi}\right)^{1/2} (-1)^m \tau \int_0^\infty I_2 K_2 (p\tau) \frac{J_{2m+1/2+\nu'}(p)}{p^{-1/2+\nu'}} \, d(p^2). \qquad (7)$$

The method of evaluation is parallel to that used in the LE and TE problems. Thus

$$I_2(p\tau) K_2(p\tau) = \frac{\pi^{-1/2}}{2} G^{21}_{13}\left(p^2\tau^2 \,\middle|\, \begin{matrix} & 1/2 & \\ 2, & 0, & -2 \end{matrix} \right); \qquad (8)$$

$$p^{1/2-\nu'} J_{2m+1/2+\nu'}(p) = 2^{1/2-\nu'} G^{10}_{02}\left(\frac{p^2}{4} \,\middle|\, m+1/2, -m-\nu' \right). \qquad (9)$$

The final result is

$$A''_m = -A'_m(\nu+1) + \frac{\tau}{2}$$

$$\left[A_m + \frac{-(-1)^m 2^{1-\nu'}}{\pi} \tau\, G^{22}_{33}\left(4\tau^2 \middle| \begin{matrix} -m-\frac{1}{2}, & \frac{1}{2}, & m+\nu' \\ 2, & 0, & -2 \end{matrix}\right)\right] \tag{10}$$

(non-basic).

The part of B_ρ due to the end currents alone is denoted $B_{\rho e}$. Using those portions of equations 5.03-29 through 5.03-31 which contain $\tilde{G}(k)$, one finds that

$$\frac{B_{\rho e}}{B \sin\phi} = -a\int_0^\infty \frac{ka\,\tilde{G}(k)}{2} e^{-kb}\left[J_0(ka) - J_2(ka)\right] k\,dk; \tag{11}$$

$$= -\int_0^\infty \tilde{G}(k) e^{-kb}\left[J_0(ka) - \frac{J_1(ka)}{ka}\right](ka)^2\,dk; \tag{12}$$

$$= B''_b f_b + \sum_{m=0}^{N_e-1} B''_m g_m; \tag{13}$$

$$B''_m = -(-1)^m a\int_0^\infty e^{-kb}\left[J_0(ka) - \frac{J_1(ka)}{ka}\right]$$

$$\frac{J_{2m+2+\nu'}(ka)}{(ka)^{1+\nu'}} (ka)^2\,dk. \tag{14}$$

This time let $ka = p$, then

$$B''_m = -(-1)^m\int_0^\infty e^{-\frac{p}{\tau}}\left[\underset{\text{I}}{p^2 J_0(p)} - \underset{\text{II}}{pJ_1(p)}\right]\frac{J_{2m+2+\nu'}(p)}{p^{1+\nu'}}\,dp. \tag{15}$$

The first integral is simply $-B_{m+1}$; the second is

$$B''_{m\text{II}} = (-1)^m\int_0^\infty e^{-\frac{p}{\tau}} J_1(p)\frac{J_{2m+2+\nu'}(p)}{p^{\nu'}}\,dp; \tag{16}$$

$$p^{-\nu'} J_1(p)\, J_{2m+2+\nu'}(p) = \pi^{-1/2}\, G_{24}^{12}\left(p^2 \middle| \begin{matrix} 1/2-\nu'/2,\ -\nu'/2 \\ m+3/2,\ -m-1/2-\nu',\ m+1/2,\ -m-3/2-\nu' \end{matrix}\right). \quad (17)$$

The final form of the B''_m coefficients is

$$B''_m = -B_{m+1} + \frac{(-1)^m}{\pi}\tau\, G_{44}^{14}\left(4\tau^2 \middle| \begin{matrix} 0,\ 1/2,\ 1/2-\nu'/2,\ -\nu'/2 \\ m+3/2,\ -m-1/2-\nu',\ m+1/2,\ -m-3/2-\nu' \end{matrix}\right) \quad (18)$$

(non-basic).

5.06. <u>Derivation of the Basic Checking Coefficients</u>. At the equator, that part of B_ρ due to the basic terms alone is denoted $B_{\rho b}$ and is given by

$$\frac{B_{\rho b}}{B \sin\phi} = A''_b f_b + B''_b f_b = \bar{A}''_b f_b. \quad (1)$$

From earlier work with the non-basic coefficients, it is evident that

$$A''_b = -\left(\frac{2}{\pi}\right)^{1/2} b \int_0^\infty \left\{ I_1 K_1(ka) + \frac{(ka)^2}{2}\left[I_0 K_0(ka) + I_2 K_2(ka) \right] \right\} \frac{J_{1/2}(kb)}{(kb)^{1/2}}\, dk \quad (2)$$

and

$$B''_b = -a \int_0^\infty \frac{(ka)^2}{2}\left[J_0(ka) - J_2(ka) \right] \frac{J_2(ka)}{ka}\, e^{-kb}\, dk. \quad (3)$$

The latter may be rewritten as

$$B_b'' = -a\int_0^\infty \left\{ka\, J_0J_1(ka) - \frac{(ka)^2}{2}\left[J_0J_0(ka) + J_2J_2(ka)\right]\right\} \frac{e^{-kb}}{ka}\, dk. \quad (4)$$

When A_b'' and B_b'' are added together, it can be shown that the integrals involving the quantities in the square brackets become equal to the same expression in G-functions and therefore cancel. The sum of the remaining terms is

$$\bar{A}_b'' = -\left(\tfrac{2}{\pi}\right)^{1/2} b\int_0^\infty I_1K_1(ka)\, \frac{J_{1/2}(kb)}{(kb)^{1/2}}\, dk$$

$$- a\int_0^\infty J_0J_1(ka)\, e^{-kb}\, dk. \quad (5)$$

It is useful to perform a parts integration on the second integral. Thus

$$\bar{A}_b'' = -\left(\tfrac{2}{\pi}\right)^{1/2}\int_0^\infty I_1K_1(p\tau)\; p^{-1/2} J_{1/2}(p)\, dp$$

$$+ \frac{1}{2}\int_0^\infty e^{-p}\; d\left[J_0J_0(p\tau)\right]; \quad (6)$$

$$\bar{A}_b'' = -\frac{1}{2} - \left(\tfrac{2}{\pi}\right)^{1/2}\int_0^\infty I_1K_1(p\tau)\; p^{-1/2}\, dp$$

$$+ \frac{1}{2}\int_0^\infty J_0J_0(p\tau)\, e^{-p}\, dp. \quad (7)$$

This becomes

$$\bar{A}_b'' = -\frac{1}{2} - \frac{1}{2\pi}\, G_{22}^{12}\left(4\tau^2 \,\middle|\, \begin{matrix} 1/2,\ 1/2 \\ 0,\ -1 \end{matrix}\right)$$

$$+ \frac{1}{2\pi}\, G_{22}^{12}\left(4\tau^2 \,\middle|\, \begin{matrix} 1/2,\ 1/2 \\ 0,\ 0 \end{matrix}\right); \quad (8)$$

$$\bar{A}''_b = -\frac{1}{2} - \frac{1}{2} F\left(\frac{1}{2}, \frac{1}{2}; 2; -4\tau^2\right) + \frac{1}{2} F\left(\frac{1}{2}, \frac{1}{2}; 1; -4\tau^2\right). \qquad (9)$$

By one of the Gauss recursion formulas (37)

$$F(a,b;c;z) - F(a,b;c+1;z) = \frac{abz}{c(c+1)} F(a+1,b+1;c+2;z). \qquad (10)$$

Therefore $\bar{A}''_b$ and two possible transformations thereof are

$$\bar{A}''_b = -\frac{1}{2} - \frac{\tau^2}{4} F\left(\frac{3}{2}, \frac{3}{2}; 3; -4\tau^2\right);$$

$$= -\frac{1}{2} - \frac{\tau^2}{4}\left(1+4\tau^2\right)^{-\frac{3}{2}} F\left(\frac{3}{2}, \frac{3}{2}; 3; \frac{4\tau^2}{1+4\tau^2}\right);$$

$$= -\frac{1}{2} - \frac{\tau^2}{4}\left(1+2\tau^2\right)^{-\frac{3}{2}} F\left(\frac{3}{4}, \frac{5}{4}; 2; \left[\frac{2\tau^2}{1+2\tau^2}\right]^2\right) \qquad (11)$$

(basic).

5.07. Numerical Values of the TM Checking Coefficients.

$a/b = \frac{1}{4}$		
m	$\bar{A}''_b$, A''_m	B''_m
basic	-0.51317097	
0	-0.37373614	-0.00026831
1	+0.13698592	+0.00000231
2	-0.09587221	
3	+0.08071083	
4	-0.07336844	
5	+0.06913581	
6	-0.06635005	
7	+0.06432259	
8	-0.06273584	
9	+0.06142956	
10	-0.06031557	
11	+0.05934058	
12	-0.05846917	
13	+0.05767260	
14	-0.05691443	

Table 5.07-1

$a/b = \frac{1}{2}$		
m	$\bar{A}''_b$, A''_m	B''_m
basic	-0.53593354	
0	-0.41660071	-0.00069169
1	+0.19334581	-0.00011725
2	-0.15901168	+0.00001080
3	+0.14624983	-0.00000049
4	-0.13908249	
5	+0.13409742	
6	-0.13024281	
7	+0.12708776	
8	-0.12441780	
9	+0.12210551	
10	-0.12006988	
11	+0.11825357	

Table 5.07-2.

a/b = 1		
m	$\bar{A}''_b$, A''_m	B''_m
basic	−0.56257577	
0	−0.53810001	+0.01347200
1	+0.33926002	−0.00360099
2	−0.30535060	+0.00015761
3	+0.28803091	+0.00003327
4	−0.27614116	−0.00000406
5	+0.26707480	−0.00000020
6	−0.25977300	+0.00000007
7	+0.25368218	
8	−0.24847313	

Table 5.07-3.

a/b = 2		
m	$\bar{A}''_b$, A''_m	B''_m
basic	−0.57045691	
0	−0.85893425	+0.09195852
1	+0.66042932	−0.01259618
2	−0.60707720	−0.00425556
3	+0.57470652	+0.00023736
4	−0.55157107	+0.00022391
5	+0.53370893	+0.00000885
6	−0.51924604	−0.00001034
7		−0.00000143
8		+0.00000039
9		
10		
11		

Table 5.07-4.

a/b = 4		
m	$\bar{A}''_b$, A''_m	B''_m
basic	−0.55977485	
0	−1.5839982	+0.26418332
1	+1.3149273	+0.02182087
2	−1.2128602	−0.01850443
3	+1.1488423	−0.00915989
4		−0.00128017
5		+0.00061064
6		+0.00039300
7		+0.00007805
8		−0.00001824
9		−0.00001693
10		−0.00000440
11		+0.00000039
12		+0.00000071
13		+0.00000023
14		

APPENDIX A

HANKEL TRANSFORMS OF THE $\bar{\Psi}_m$ FUNCTIONS

It will be shown that all of the integral transforms of the $\bar{\Psi}_m$ functions used in this thesis are special cases of the general transform

$$\int_0^1 u^{\sigma}(1-u^2)^{\nu} P_m^{(\nu,\sigma)}(2u^2-1)\, J_{\sigma}(tu)\, u\, du = \frac{(-1)^m \Gamma(m+1+\nu)\, 2^{\nu}}{\Gamma(m+1)} \cdot \frac{J_{2m+1+\sigma+\nu}(t)}{t^{1+\nu}}; \quad \text{Re}\,\sigma > -1;\ \text{Re}\,\nu > -1;\ t \text{ real and } >0; \tag{1}$$

which will now be proved. $P_m^{(\nu,\sigma)}(2u^2-1)$ is a Jacobi polynomial given by (39)

$$P_m^{(\nu,\sigma)}(2u^2-1) = \frac{\Gamma(m+1+\nu)}{\Gamma(m+1)\Gamma(\nu+1)} F(-m, m+\nu+\sigma+1; \nu+1; 1-u^2). \tag{2}$$

Notice that this is a different form from that of equation 1.01-1. Denote the integral in equation 1 by $f(t)$. Then

$$f(t) = \frac{\Gamma(m+1+\nu)}{\Gamma(m+1)\Gamma(\nu+1)} \sum_{r=0}^{m} \frac{(-m)_r\,(m+\nu+\sigma+1)_r}{(\nu+1)_r\,(1)_r} \int_0^1 u^{\sigma}(1-u^2)^{r+\nu} J_{\sigma}(tu)\, u\, du. \tag{3}$$

By (40)

$$\int_0^1 u^{\sigma}(1-u^2)^{r+\nu} J_{\sigma}(tu)\,u\,du = 2^{r+\nu}\Gamma(r+\nu+1)\,\frac{J_{\sigma+r+\nu+1}(t)}{t^{r+\nu+1}} \tag{4}$$

under the conditions Re $\sigma > -1$; Re $\nu > -1$; t real and > 0. Therefore $f(t)$ becomes

$$f(t) = \frac{\Gamma(m+1+\nu)}{2\,\Gamma(m+1)} \sum_{r=0}^{m} \frac{(-m)_r\,(m+\nu+\sigma+1)_r}{(1)_r} \cdot \frac{J_{\sigma+r+\nu+1}(t)}{(t/2)^{r+\nu+1}} \tag{5}$$

and it remains to evaluate the <u>finite</u> sum in 5. Thus

$$f(t) = \frac{\Gamma(m+1+\nu)}{2\,\Gamma(m+1)} \left(\frac{t}{2}\right)^{\sigma}$$

$$\sum_{r=0}^{m} \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}\left(\frac{t}{2}\right)^{2\ell} (-m)_r\,(m+\nu+\sigma+1)_r}{\Gamma(\ell+1)\,\Gamma(\sigma+\nu+r+\ell+2)\,(1)_r} \,. \tag{6}$$

The sum over r is merely a $_2F_1$ hypergeometric polynomial of unit argument, and

$$\frac{1}{\Gamma(\sigma+\nu+\ell+2)} \sum_{r=0}^{m} \frac{(-m)_r\,(m+\nu+\sigma+1)_r}{(\sigma+\nu+\ell+2)_r\,(1)_r} = \frac{\Gamma(\ell+1)}{\Gamma(\sigma+\nu+\ell+2+m)\,\Gamma(\ell+1-m)}\,; \tag{7}$$

$$f(t) = \frac{\Gamma(m+1+\nu)}{2\,\Gamma(m+1)} \left(\frac{t}{2}\right)^{\sigma} \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell}\left(\frac{t}{2}\right)^{2\ell}}{\Gamma(\ell+1-m)\,\Gamma(\sigma+\nu+\ell+2+m)} \,. \tag{8}$$

Make a change of variable in the summation; let $\ell = m + k$:

$$f(t) = \frac{(-1)^m\,\Gamma(m+1+\nu)}{2\,\Gamma(m+1)} \left(\frac{t}{2}\right)^{\sigma+2m} \sum_{k=0}^{\infty} \frac{(-1)^k\left(\frac{t}{2}\right)^{2k}}{\Gamma(1+k)\,\Gamma(2m+2+\sigma+\nu+k)} \,. \tag{9}$$

This is

$$f(t) = \frac{(-1)^m \Gamma(m+1+\nu)}{2\Gamma(m+1)} \left(\frac{t}{2}\right)^{-1-\nu} J_{2m+1+\sigma+\nu}(t) \tag{10}$$

and the transform of equation 1 is true.

If σ is set equal to $\gamma + \zeta - 1$, the transform becomes

$$\int_0^1 \bar{\Psi}_m(\gamma,\zeta,\nu,u)\; J_{\gamma+\zeta-1}(tu)\;(tu)^\gamma\, du$$
$$= \frac{(-1)^m J_{2m+\gamma+\zeta+\nu}(t)}{t^{1-\gamma+\nu}}\;. \tag{11}$$

If the parameters γ and ζ are allowed to take the values in all of the combinations of Table 1.01-1, one obtains

Side; even	$\int_0^1 \bar{\Psi}_m(\tfrac{1}{2},0,\nu,u)\cos tu\, du = \left(\frac{\pi}{2}\right)^{1/2}(-1)^m \frac{J_{2m+1/2+\nu}(t)}{t^{1/2+\nu}}$;	(12)
Side; odd	$\int_0^1 \bar{\Psi}_m(\tfrac{1}{2},1,\nu,u)\sin tu\, du = \left(\frac{\pi}{2}\right)^{1/2}(-1)^m \frac{J_{2m+3/2+\nu}(t)}{t^{1/2+\nu}}$;	(13)
End; even	$\int_0^1 \bar{\Psi}_m(1,0,\nu,u)\, J_0(tu)\, u\, du = (-1)^m \frac{J_{2m+1+\nu}(t)}{t^{1+\nu}}$;	(14)
End; odd	$\int_0^1 \bar{\Psi}_m(1,1,\nu,u)\, J_1(tu)\, u\, du = (-1)^m \frac{J_{2m+2+\nu}(t)}{t^{1+\nu}}$;	(15)

Re $\nu > -1$; t not restricted.

Because of the special values taken by γ and ζ, the singularity at the origin which might have occurred in $(tu)^\gamma\; J_{\gamma+\zeta-1}(tu)$ does not appear. Both sides of equations 12 through 15 become entire functions of t and all restrictions on t may be removed.

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