SOME ASPECTS OF OPEN

STRING FIELD THEORIES

Thesis by

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Abstract

The construction of covariant string field theories is an important step toward a deeper understanding of string theories. This thesis discusses the general formulation of bosonic and supersymmetric covariant open string field theories and their second quantization. A particular emphasis is given to the perturbative calculation in the framework of string field theory. The use of string wave functional and the technique of conformal field theory are illustrated by explicit calculations of on- and off-shell string amplitudes. The background independent cubic actions for open strings are described briefly.

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Introduction and Summary

The quest for a coherent understanding of the fundamental building blocks of the universe and the interactive forces among them has been a very active area of research. The renewed interest in string theory in recent years has generated a realistic hope that this most ambitious goal in physics may eventually be achievable in the framework of string theory [1]. At the early stage of string theory all the formulations and the subsequent calculations were made in the first quantized formalism and restricted in the light-cone gauge [2]. There a flat vacuum is assumed and string amplitudes were calculated perturbatively according to a set of intuitive Feynman rules. As space-time symmetry is not preserved, all the string states must be restricted on the mass-shell in a thus obtained string amplitude. Furthermore, the relation between gauge symmetry and dynamics in string theory is also hidden in this formalism. It was then obvious that a more appropriate formulation of string theory is necessary.

The would-be dynamic theory of strings must have the following required properties: The space-time symmetry is preserved, the gauge symmetry is explicitly realized, vacuum structure would come out as dynamic solutions of the theory, instead of something presupposed, perturbative Feynman rules should be derived systematically and off-mass-shell string amplitudes calculated.

In parallel to point particle field theory, it was realized [2] that a covariant field theory of strings would fulfill all the requirements. But a covariant string field theory is very difficult to develop, since it involves an infinite number of gauge invariances. The string field theory constructed in early 1970s was still prefixed in the light-cone gauge without space-time symmetry [3]. Later Siegel initiated a study [4] of string field theory from a BRST invariant string field action. Subsequently, using the concept of noncommutative geometry, Witten achieved a formulation of a covariant interacting field theory for both bosonic [5] and supersymmetric [6] open strings.

However, a covariant field theory for neither bosonic nor supersymmetric closed strings has been constructed.

To compute string amplitudes perturbatively, Witten chose a gauge-fixing condition for his bosonic open string field theory and conjectured a gauge-fixed string field action. Then it was argued that the correct string amplitude could be obtained [7] from the Feynman rules derived from the conjectured gauge-fixed string field action. The explicit calculation of string amplitudes from Witten's bosonic string field theory, however, is a nontrivial matter. Giddings succeeded in calculating on-shell four-tachyon amplitude [8].

The calculation of string amplitudes in the dual model [2] and the Polyakov path integral approach [9] are restricted to on-shell string states. For a better understanding of string theories the off-shell information is important. For example, just as in point particle field theories, off-shell amplitudes are necessary to derive a low energy effective theory of strings and may provide the necessary means to explore the short distance behavior of string theories [10]. Early attempts to obtain off-shell amplitudes in string theories were not very successful. The covariant string field theories suggest another way to approach the problem.

The off-shell four-tachyon and two-tachyon two-vector particle amplitudes are calculated explicitly [11–13]. The off-shell four-tachyon amplitude is calculated in reference [12] using the momentum space wave functional. In that approach, it appears difficult to obtain amplitudes involving higher mass level states. In reference [13], the off-shell four-tachyon and two-tachyon two-vector particle amplitudes were calculated in the operator formalism [14], where an assumption was made about the value of the infinite sum of contributions from intermediate particles which was too difficult to evaluate directly in that formalism. The methods in reference [11] is more powerful and is also explicitly world-sheet reparametrization-independent, which will be discussed in detail in this thesis.

The bosonic string field theory has been formally second quantized [15–17]. A gauge-fixed BRST invariant string field action and the BRST transformation that

leaves the action invariant were found. The correct Feynman rules have been derived from the gauge-fixed action. As far as the perturbative calculation of string amplitudes are concerned, the classical string fields for on-shell or off-shell string states suffice.

Witten's superstring field theory has some difficulties however, and it was found that the superstring multiplication has an associative anomaly [31]. This implies a violation of gauge invariance of the theory and a divergence in tree level four-boson amplitudes. All those problems are related to the insertion of a picture-changing operator [29] at the string midpoint in the definition of superstring multiplication. Since the operator product of two picture-changing operators is divergent, when two or more picture-changing operators overlap at the same point in superstring multiplications, the product is not well defined. A possible resolution is to modify the superstring action and the gauge transformation by introducing an infinite number of higher order contact terms such that the divergences are canceled order by order. This is obviously very difficult to implement.

A different approach is taken in reference [32], the key is the realization that the conformal and superconformal ghost numbers must be counted separately. A general physical superstring field is the linear combination of component fields of conformal ghost number $-1/2$ and various superconformal ghost numbers. A component field could be an NS field or an R field depending on whether its superconformal ghost number is an integer or half-integer. It is then natural to define superstring multiplication and integration as the generalization of the corresponding bosonic string operations without the insertion of the picture-changing operator and its inverse. The new superstring action is still an integral of the Chern-Simons three form. The action is gauge invariant and supersymmetric without additional contact terms. The second quantization and some sample calculations have been carried out [32].

The open string action as an integral of the Chern-Simons three form has dependence on space-time background through the BRST charge. If the string field theory were to give a complete description of a dynamic process, the background

should come out as a solution of the theory instead of something being put in at the beginning. In an attempt to eliminate background dependence, Witten's open bosonic string field theory action has been written in a purely cubic form [38]. The original open string action can be recovered by expanding around a particular classical solution of the equation of motion. Moreover the cubic form of open bosonic string field theory also contains closed string states [39], this gives a hope that the difficulties in formulating closed string field theory may be sidestepped. The extension to Witten's open superstring field theory has been made [40]. However, the associative anomaly of Witten's superstring multiplication would also render the superstring cubic action not well defined. In the light of the modified superstring field theory [32], a well-defined superstring cubic action is proposed [41].

The main subject of this thesis will be the covariant field theories for bosonic and supersymmetric open strings. The first chapter starts with a brief review of bosonic string field theory and the calculation of the on-shell four-tachyon amplitude. Then a more detailed world-sheet reparametrization-independent calculation of off-shell four-tachyon and two-tachyon two-vector particle amplitudes is presented. The conformal field theory method for off-shell string amplitude calculation is also described.

The second chapter begins with the discussion of Witten's open superstring field theory and the difficulties in its original formulation. A proposal for a modified superstring field theory is given with particular emphasis on the role played by the superconformal ghost. The canonical structure of the open superstring in the Neveu–Schwarz and Ramond sectors is shown to be the consequence of the gauge-fixed string field equations of motion. The theory is second quantized and some string amplitudes for massless string states are calculated.

The third chapter is devoted to the exposition of the purely cubic action for the bosonic and the supersymmetric open strings.

1. Bosonic String Field Theory

1.1 General formulation

A string field $A[X^\mu(\sigma), c(\sigma), b(\sigma)]$ in Witten's string field theory [5] is a functional of string coordinates $X^\mu(\sigma)$ and ghost fields $c(\sigma)$ and $b(\sigma)$ on the string world-sheet. $X^\mu(\sigma)$, $c(\sigma)$, and $b(\sigma)$ can be expanded into modes a_n^μ, c_n and b_n. They satisfy commutation relations

$$
\begin{aligned}
&[a_m^\mu, a_n^\nu] = m\eta^{\mu\nu}\delta_{m+n,0}\,, \\
&\{c_m, b_n\} = \delta_{m+n,0}\,, \\
&\{c_m, c_n\} = \{b_m, b_n\} = 0, \\
&[a_m^\mu, c_n] = [a_m^\mu, b_n] = 0,
\end{aligned}
\tag{1.1}
$$

and hermiticity conditions

$$
a_n^{\mu\dagger} = a_{-n}^\mu, \quad c_n^\dagger = c_{-n}, \quad b_n^\dagger = b_{-n},
\tag{1.2}
$$

where $\eta^{\mu\nu} = (-1, 1, \cdots, 1)$ is the space-time metric and $\mu = 0, 1, 2, \cdots, D-1$; D is the dimension of space-time, and $m, n = 0, \pm 1, \pm 2, \cdots$.

The world-sheet BRST operator Q [18] can be decomposed as

$$
Q = c_0(L_0 - 1) + b_0 T_+ + Q_+
\tag{1.3}
$$

where $L_0 - 1$, T_+, and Q_+ do not contain ghost zero modes c_0 or b_0. And Q is nilpotent, $Q^2 = 0$, provided $D = 26$. The ghost number operator is defined by

$$
N = \frac{1}{2}(c_0 b_0 - b_0 c_0) + \sum_{n=1}^\infty (c_{-n} b_n - b_{-n} c_n)\,.
\tag{1.4}
$$

The BRST operator carries the ghost number $+1$.

A general string field Φ in the first-quantized Fock-space can be thought of as a linear composition of string states with certain ghost numbers

$$|\Phi\rangle = \sum_\alpha \phi_\alpha(x) |\alpha\rangle ,$$

where $\phi_\alpha(x)$ is a number field and $|\alpha\rangle$ carries the ghost number N_α. $|\alpha\rangle$ is assigned Grassmann-odd (or -even) if $N_\alpha = -\frac{1}{2} + 2n$ (or $\frac{1}{2} + 2n$), where n is an integer.

It is convenient to bosonize the ghost and antighost field on the string world-sheet,

$$c^w = e^{i\phi(w,\bar{w})} , \quad b_{ww} = e^{-i\phi_+(w)} , \qquad (1.5a,b)$$

where $\phi(w,\bar{w})$, the bosonized ghost, is a scalar on the string world-sheet. The world-sheet action for string coordinates $X^\mu(\sigma)$ and scalar ghost $\phi(\sigma)$ are [5]

$$S_X = \frac{1}{4\pi} \int d^2\sigma \sqrt{g}\, g^{ab}\, \partial_a X \cdot \partial_b X , \qquad (1.6a)$$

$$S_\phi = \frac{1}{2\pi} \int d^2\sigma \sqrt{g}\, g^{ab}\, \partial_a\phi\partial_b\phi + \frac{3i}{2\pi} \int d^2\sigma \sqrt{g}\, R\, \phi , \qquad (1.6b)$$

where R is the curvature scalar. The conventions for the fields and propagators are

$$X^\mu(w,\bar{w}) = X^\mu(w) + X^\mu(\bar{w}), \quad \phi(w,\bar{w}) = \frac{1}{2}[\phi_+(w) + \phi_-(\bar{w})],$$

$$\langle X^\mu(w_1,\bar{w}_1) X^\nu(w_2,\bar{w}_2)\rangle = -\eta^{\mu\nu}\left(\ln|w_1 - w_2| + \ln|w_1 - \bar{w}_2|\right),$$

$$\langle \phi(w_1,\bar{w}_1)\phi(w_2,\bar{w}_2)\rangle = -\frac{1}{2}\left(\ln|w_1 - w_2| + \ln|w_1 - \bar{w}_2|\right)$$

on the upper half-plane.

The string multiplication

$$A = B * C$$

is defined as a path integral

$$A[X(\sigma_1), \phi(\sigma_1)] \equiv \int DX\, D\phi\, e^{-S_X} e^{-S_\phi}\, B[X(\sigma_2), \phi(\sigma_2)]\, C[X(\sigma_3), \phi(\sigma_3)] \qquad (1.7)$$

on a piece of open string world-sheet sketched in Fig. 1. Where A is defined on boundary F_1, and B and C on F_2 and F_3 respectively. The limit $\delta \to 0$ is taken at the end. The string integration, a c-number, is defined as a path integral

$$\int A \equiv \int DX\, D\phi\, e^{-S_X} e^{-S_\phi}\, A[X(\sigma), \phi(\sigma)] \qquad (1.8)$$

on a piece of world-sheet sketched in Fig. 2. Where A is defined on the boundary F, and the limit $\delta \to 0$ is implied.

The multiplication $*$ and integration $\int$ can further be shown [5] to satisfy

$$Q(A * B) = (QA) * B + (-)^A A * (QB) \qquad (1.9a)$$

$$\int A * B = (-)^{AB} \int B * A \qquad (1.9b)$$

$$\int QA = 0 \qquad (1.9c)$$

$$(A * B) * C = A * (B * C) \qquad (1.9d)$$

where A, B, and C are states in the first-quantized Fock-space. $(-)^A$ is -1 (or $+1$) if A is odd (or even), and $(-)^{AB}$ is -1 if both A and B are odd, $+1$ otherwise. The operations $*$ and $\int$ have a ghost number anomaly $3/2$ and $-3/2$ respectively, due to the second term in the scalar ghost action $(1.6b)$ which couples the ghost number to world-sheet curvature. Eqs.(1.9a)–(1.9c) have also been realized in an explicit mode expansion [14].

Witten's action for bosonic open string field is

$$I = \frac{1}{2} \int \left(A * QA + \frac{2}{3} A * A * A \right),$$ (1.10)

where A is required to have ghost number $-1/2$ and to be Grassmann-odd. The action is therefore invariant under gauge transformation

$$\delta A = Q\Lambda + A * \Lambda - \Lambda * A$$ (1.11)

where Λ is required to have ghost number $-3/2$ and to be Grassmann-even. In order to do perturbative calculations of string amplitudes, a gauge-fixing condition must be imposed. The condition

$$b_0 A = 0$$ (1.12)

fixes the gauge in linearized theory completely, and it is also used in the interaction theory. In principle, one can calculate the scattering amplitudes of various open string states [7]. Four-tachyon on-shell amplitudes [8] and four-point off-shell amplitudes involving tachyons and massless vector particles [11–13] have been calculated. We will give a detailed account of the methods in reference [11] later in this chapter.

The linearized equations of motion for the classical physical string field is

$$QA = 0.$$ (1.13)

Using the expression in eq. (1.3) for Q we find that eqs. (1.12) and (1.13) are equivalent to

$$(L_0 - 1)A = 0, \quad Q_+ A = 0,$$ (1.14a, b)

which is the desired equation of motion and constraint for the physical string field.

It is known that the gauge fixing of the linearized action leads to the introduction of a Faddeev–Popov ghost string field, ghost of ghost field, and so on [19]. The same procedure has been carried out [15] for the action in eq. (1.10). There an infinite number of ghost string fields

$$\Phi_{-3/2}, \ \Phi_{-5/2}, \ \Phi_{-7/2}, \cdots$$

and antighost string fields

$$\Phi_{1/2}, \ \Phi_{3/2}, \ \Phi_{5/2}, \cdots$$

are introduced. The subscripts indicate the ghost numbers carried by the string fields. The physical string field A in eq. (1.10) now corresponds to $\Phi_{-1/2}$. All the string fields can be written in a compact form

$$\Phi = \Phi_{-1/2} + \Phi_{-3/2} + \Phi_{-5/2} + \cdots + \Phi_{1/2} + \Phi_{3/2} + \cdots, \tag{1.15}$$

where each component field is assumed to be Grassmann-odd, therefore Φ is overall odd. The gauge-fixed BRST-invariant action is found to be

$$I_{GF} = \frac{1}{2} \int (\Phi * Q\Phi + \frac{2}{3}\Phi * \Phi * \Phi - 2(b_0\beta) * \Phi), \tag{1.16}$$

where β, having all ghost numbers, is a Lagrange multiplier imposing the gauge-fixing condition $b_0\Phi = 0$. The equation of motion from eq. (1.16) is

$$Q\Phi + \Phi * \Phi - (b_0\beta) = 0,$$

$$b_0\Phi = 0. \tag{1.17}$$

The BRST transformation has the form

$$\delta\Phi_+ = (b_0\beta)_+,$$

$$\delta\Phi_- = (Q\Phi + \Phi * \Phi)_-\,,$$

$$\delta\beta = 0\,,\tag{1.18}$$

where $+$ (or $-$) means the component fields of positive (or negative) ghost numbers. One can check that the action in eq. (1.16) is invariant under the BRST transformation. The Grassmann-odd property of Φ is essential for the invariance. It can also be shown [15] that the BRST transformation δ is nilpotent,

$$\delta^2\Phi_+ = 0,$$

$$\delta^2\Phi_- = 0,$$

$$\delta^2\beta = 0,\tag{1.19}$$

by using the equation of motion (1.17). The correct Feynman rules for perturbative calculation can therefore be derived from the gauge-fixed action.

The string field propagator for the gauge-fixed theory is [5]

$$b_0\,\frac{1}{L_0-1} = b_0 \int_0^\infty d\tau\, e^{-\tau(L_0-1)}\,,\tag{1.20}$$

where $e^{-\tau(L_0-1)}$ has a path integral representation on a strip of world-sheet with length τ shown in Fig. 3. The antighost zero mode

$$b_0 \equiv \frac{1}{\pi} \int_0^\pi d\sigma\, b(\sigma) = \int_l \left(\frac{dw}{2\pi i} b_{ww} + \frac{d\bar{w}}{2\pi i} b_{\bar{w}\bar{w}} \right)\tag{1.21}$$

can be represented as a line integral across the world-sheet in Fig. 3 as indicated. The three-string vertex is just that of Fig. 1. The appropriate world-sheet Feynman diagram for a string amplitude can be formed by gluing together vertices and propagators.

1.2 The on-shell four-tachyon amplitude

The four-tachyon amplitude in Witten's bosonic string field theory has the world-sheet diagram sketched in Fig. 4a, where each external on-shell state is represented by a semi-infinite strip with an appropriate vertex operator inserted at infinity. The propagator is represented by the strip of length τ and then integrated over the parameter τ. The path integral representation of the amplitude is

$$A_s = \int_0^\infty d\tau \left\langle \left(\prod_j V(w_j, \bar{w}_j) \right) \left(\int_l \left(\frac{dw}{2\pi i} b_{ww} + \frac{d\bar{w}}{2\pi i} b_{\bar{w}\bar{w}} \right) \right) \right\rangle \tag{1.22}$$

where

$$\langle (\cdots) \rangle = \int DX \, D\phi \, (\cdots) e^{-S_X} e^{-S_\phi}.$$

S_X and S_ϕ are world-sheet actions (eqs. (1.6a) and (1.6b)), and the tachyon vertex operator is

$$V(w_j, \bar{w}_j) = c^w(w_j, \bar{w}_j) e^{iP_j \cdot X(w_j, \bar{w}_j)} \tag{1.23}$$

with $P_j^2 = 1$, $j = 1, \ldots, 4$. $c^w = e^{i\phi(w, \bar{w})}$ and $b_{ww} = e^{-i\phi_+(w)}$ are ghost and antighost fields.

The string world-sheet can be mapped to the upper half-plane by a Schwarz–Christoffel transformation [8],

$$\frac{dw}{dz} = N \frac{\sqrt{z^2 + \gamma^2}\sqrt{z^2 + 1/\gamma^2}}{(z^2 - \alpha^2)(z^2 - 1/\alpha^2)}, \tag{1.24}$$

where

$$N = \frac{2(1 - \alpha^4)}{\alpha\sqrt{\alpha^2 + \gamma^2}\sqrt{\alpha^2 + 1/\gamma^2}}.$$

The parameters in the conformal mapping in eq. (1.24) are required to satisfy the constraints,

$$\frac{1}{2} = \Lambda_0(\beta_1, k) - \Lambda_0(\beta_2, k), \tag{1.25a}$$

$$\frac{1}{2}\tau = K(k')[Z(\beta_2, k') - Z(\beta_1, k')], \tag{1.25b}$$

where

$$k^2 = \gamma^4 = 1 - k'^2, \qquad \sin^2\beta_1 = \frac{1}{1 + \alpha^2\gamma^2}, \qquad \sin^2\beta_2 = \frac{\alpha^2}{\alpha^2 + \gamma^2}.$$

Λ_0 and Z are Heuman's lambda function and the jacobian zeta function,

$$\Lambda_0(\beta, k) = \frac{2}{\pi}[E(k)F(\beta, k') + K(k)E(\beta, k') - K(k)F(\beta, k')],$$

$$Z(\beta, k) = E(\beta, k) - \frac{E(k)}{K(k)}F(\beta, k),$$

where $K(k)$ and $E(k)$ are complete elliptic integrals of the first and second kinds and $F(\beta, k)$ and $E(\beta, k)$ are incomplete elliptic integrals of the first and second kinds. The constraints in eqs. (1.25a) and (1.25b) implicitly determine γ and τ in terms of α. From eq. (1.25b) it follows that

$$d\tau = -\frac{2\pi}{K(k)}\frac{d\alpha}{\sqrt{1 + \alpha^2\gamma^2}\sqrt{\alpha^2 + \gamma^2}}. \tag{1.26}$$

Upon the conformal coordinate change to the upper half-plane as shown in Fig. 4b. the ghost part of the path integral in eq. (1.22) can be carried out to give

$$A_{\mathrm{g}} = \frac{1}{2\pi\alpha^3}\sqrt{\alpha^2 + \gamma^2}\sqrt{1 + \alpha^2\gamma^2}(1 - \alpha^4)K(k). \tag{1.27}$$

After evaluating the correlation function involving $X^\mu(w, \bar{w})$ and the change of variable in eq. (1.26), the s-channel on-shell four-tachyon amplitude is

$$A_s = -2\int_{\alpha_0}^{0} d\alpha\frac{1 - \alpha^4}{\alpha^3}\left(\alpha - \frac{1}{\alpha}\right)^{2P_1 \cdot P_4 + 2P_2 \cdot P_3}$$

$$\times \left(\alpha + \frac{1}{\alpha}\right)^{2P_1 \cdot P_3 + 2P_2 \cdot P_4}(2\alpha)^{2P_1 \cdot P_2}\left(\frac{2}{\alpha}\right)^{2P_3 \cdot P_4}, \tag{1.28}$$

where $\alpha_0 = \sqrt{2} - 1$. The change of variable

$$x = \left(\frac{1 - \alpha^2}{1 + \alpha^2}\right)^2 \tag{1.29}$$

then gives

$$A_s = \frac{1}{4} \int_{1/2}^{1} dx \, x^{-t-2}(1 - x)^{-s-2} \,, \tag{1.30}$$

where $s = -(P_1 + P_2)^2$, $t = -(P_2 + P_3)^2$. The Koba–Nielsen amplitude is the sum of s- and t-channel contributions

$$A_{\mathrm{KN}} = A_s + A_t = \frac{1}{4} \int_{0}^{1} dx \, x^{-t-2}(1 - x)^{-s-2}. \tag{1.31}$$

1.3. The off-shell four-tachyon amplitude

In the path integral approach to on-shell string amplitudes, the prescription is to integrate over all embeddings of the world-sheet into space-time and all possible world-sheet metrics [9]. This includes integration over all possible world-sheet geometries. The Weyl invariance on the string world-sheet ensures that the resulting on-shell amplitudes are independent of the world-sheet geometry chosen in calculations. If the external momenta are extended off the mass-shell, the Weyl invariance is lost. It is not clear how to proceed in this case.

In string field theory, however, the three-string interaction vertex determines the world-sheet geometry. For example, in Witten's string field theory, the three-string interaction is specified by overlapping half of one string to half of the other to form the third string (see Fig. 1). This corresponds to a particular world-sheet geometry like the one in Figs. 4a and 6a. To calculate the string amplitudes, only the embeddings of the world-sheet into space-time are integrated, not the metric on the world-sheet, namely, the geometry of the world-sheet is held fixed. In the light-cone-like string field theory [20], the three-string interaction is described by two

strings joining ends to form the third string, which also fixes a particular world-sheet geometry. So this means that in a given string field theory it is the string dynamics, particularly the string interaction, that specifies the world-sheet geometry.

Although different string field theories specify different world-sheet geometries, they give the same on-shell string amplitudes. However, the off-shell amplitudes in one theory may differ from those in another in general. By studying the off-shell behaviors of the two theories, we may be able to determine whether one is physically more sensible than the other.

In the remaining three sections of this chapter we use Witten's bosonic string field theory to calculate off-shell string amplitudes. In this section, we calculate the off-shell four-tachyon amplitude and extract the first few poles in the s-channel. Our calculation will be formulated such that it is explicitly independent of world-sheet parametrization.

An on-shell string state is represented by a vertex operator with conformal dimension zero. The off-shell extension of the vertex operators should be world-sheet scalars, therefore well-defined world-sheet reparametrization-independent off-shell amplitudes can be obtained. We will first find a vertex operator for the off-shell tachyon state which is world-sheet reparametrization-independent. This means that the off-shell tachyon vertex operator should be a world-sheet scalar. The vertex operator

$$
\begin{aligned}
e^{iP\cdot X(\sigma)} &= \left\langle e^{iP\cdot X(\sigma)} \right\rangle : e^{iP\cdot X(\sigma)} : \\
&= \lim_{\sigma' \to \sigma} \exp\left(-\frac{1}{2} P_\mu P_\nu \left\langle X^\mu(\sigma') X^\nu(\sigma) \right\rangle \right) : e^{iP\cdot X(\sigma)} : \qquad (1.32)
\end{aligned}
$$

is a world-sheet scalar, but the first factor is singular. So we introduce an invariant distance cutoff [21],

$$
\epsilon^2 = g_{ab} \triangle\sigma^a \triangle\sigma^b, \quad \triangle\sigma^a = \sigma'^a - \sigma^a.
$$

Since the expression (1.32) is coordinate-independent, for convenience we can choose a parametrization so that the world-sheet metric is conformally flat and use complex

variables,

$$z = \sigma^1 + i\sigma^2, \quad \bar{z} = \sigma^1 - i\sigma^2,$$

$$\epsilon^2 = 2g_{z\bar{z}}\triangle z \triangle \bar{z}, \quad \triangle z = z' - z, \tag{1.33}$$

$$\langle X^\mu(z', \bar{z}')X^\nu(z, \bar{z}) \rangle = g^{\mu\nu}G(z', \bar{z}'; z, \bar{z}). \tag{1.34}$$

The vertex operator will be put on world-sheet boundaries. In this case, the correlation function is

$$G(z', \bar{z}'; z, \bar{z}) = -\ln \left| \frac{E^{\sqrt{z'}, \sqrt{z}}(z', z)}{\sqrt{dz'}\sqrt{dz}} \right|^2,$$

where

$$E(z', z) = \frac{E^{\sqrt{z'}, \sqrt{z}}(z', z)}{\sqrt{dz'}\sqrt{dz}}$$

is the prime form [22], and it transforms like a $(-1/2, -1/2)$ form with the limit

$$\lim_{z' \to z} \ln E^{\sqrt{z'}, \sqrt{z}}(z', z) = \ln(z' - z) + \text{finite}.$$

So we have

$$G(z', \bar{z}'; z, \bar{z}) = -\ln |z' - z|^2 + \text{finite}.$$

Then the regularized vertex operator

$$V_X = \lim_{\epsilon \to 0} e^{-P^2 \ln \epsilon} e^{iP \cdot X(z, \bar{z})}$$

$$= \lim_{z' \to z} \left\{ e^{-P^2 \ln|z'-z|} \exp\left[-\frac{1}{2}P^\mu P^\nu \langle X^\mu(z', \bar{z}')X^\nu(z, \bar{z}) \rangle \right] \right\}$$

$$\times \left(e^{-P^2 \ln(2g_{z\bar{z}})^{\frac{1}{2}}} : e^{iP \cdot X(z, \bar{z})} : \right) \tag{1.35}$$

is also a world-sheet scalar.

From eq. (1.6b) we see that

$$\langle \phi(z, \bar{z}) \rangle = \widetilde{\phi}(z, \bar{z}) = -\frac{3i}{4\pi} \int \left(\frac{dz' d\bar{z}'}{i} \right) \sqrt{g} \ G(z, \bar{z}; z', \bar{z}') R(z', \bar{z}'). \qquad (1.36)$$

We redefine the scalar field,

$$\widehat{\phi}(z, \bar{z}) = \phi(z, \bar{z}) - \widetilde{\phi}(z, \bar{z}), \qquad (1.37a)$$

then

$$\left\langle \widehat{\phi}(z, \bar{z}) \right\rangle = 0, \quad \left\langle \widehat{\phi}(z, \bar{z}) \widehat{\phi}(z', \bar{z}') \right\rangle = \frac{1}{2} G(z, \bar{z}; z', \bar{z}'). \qquad (1.37b)$$

The vertex operator

$$e^{i\phi(z,\bar{z})} = e^{i\widehat{\phi}(z,\bar{z})} e^{i\widetilde{\phi}(z,\bar{z})}$$

$$= \left\langle e^{i\widehat{\phi}(z,\bar{z})} \right\rangle : e^{i\widehat{\phi}(z,\bar{z})} : \ e^{i\widetilde{\phi}(z,\bar{z})}$$

is a scalar. After the invariant distance cutoff in eq. (1.33), we have

$$V_\phi = \lim_{\epsilon \to 0} e^{-\frac{1}{2} \ln \epsilon} e^{i\phi(z,\bar{z})}$$

$$= \lim_{z' \to z} \left[e^{-\frac{1}{2} \ln |z' - z|} \exp \left(-\frac{1}{2} \left\langle \widehat{\phi}(z', \bar{z}') \widehat{\phi}(z, \bar{z}) \right\rangle \right) \right]$$

$$\times \left(e^{-\frac{1}{2} \ln (2 g_{zz})^{\frac{1}{2}}} : e^{i\widehat{\phi}(z,\bar{z})} : \ e^{i\widetilde{\phi}(z,\bar{z})} \right) \qquad (1.38)$$

which is a world-sheet scalar as well. In eq. (1.38) there is an extra factor $e^{i\widetilde{\phi}(z,\bar{z})}$; to obtain the ghost part of tachyon vertex operator this factor must be removed. In the following, we will find a scalar factor such that the ghost part of the tachyon vertex operator is the product of V_ϕ in eq. (1.38) with this factor. In Witten's open string field theory, the world-sheets are flat except at a few points where curvature

singularities have the value $\pm\pi$, therefore the volume form can be factorized into a product of a holomorphic and an antiholomorphic one-form,

$$2g_{z\bar{z}}dz\,d\bar{z} = d\omega(z)d\bar{\omega}(\bar{z}),$$

(1.39a)

$$d\omega(z) \equiv \omega_z(z)dz, \quad d\bar{\omega}(\bar{z}) \equiv \bar{\omega}_{\bar{z}}(\bar{z})d\bar{z}.$$

(1.39b)

It should be noticed that in defining the string state the vertex operator will be placed on a semi-infinite strip (Fig. 5), and the total curvature on the strip is 2π. From eq. (1.36) we find that the scalar factor

$$e^{-i\widetilde{\phi}(z,\bar{z})}e^{\frac{3}{2}\ln(\omega_z dz\bar{\omega}_{\bar{z}}d\bar{z})^{\frac{1}{2}}}$$

(1.40)

is the one needed. Then the vertex in eq. (1.38) becomes, after multiplication by the factor in eq. (1.40),

$$V_c = \lim_{z'\to z}\left[e^{-\frac{1}{2}\ln|z'-z|}\exp\left(-\frac{1}{2}\left\langle\widehat{\phi}(z',\bar{z}')\widehat{\phi}(z,\bar{z})\right\rangle\right)\right]$$

$$\times\left(e^{-\frac{1}{2}\ln(2g_{z\bar{z}})^{\frac{1}{2}}}:e^{i\widehat{\phi}(z,\bar{z})}:e^{\frac{3}{2}\ln(\omega_z dz\bar{\omega}_{\bar{z}}d\bar{z})^{\frac{1}{2}}}\right).$$

(1.41)

Now the tachyon vertex operator is

$$V_P(z,\bar{z}) \equiv V_X(z,\bar{z})V_c(z,\bar{z}),$$

(1.42)

where V_X and V_c are as in eqs. (1.35) and (1.41).

In section 1.2, the on-shell amplitude has an antighost line integral across the string propagator,

$$\int_l\left(\frac{dw}{2\pi i}b_{ww} + \frac{d\bar{w}}{2\pi i}b_{\bar{w}\bar{w}}\right).$$

(1.43)

Now we need a one-form having a line integral equal to the antighost line integral in a given coordinate system. When z is away from curvature singularities, we have,

from eqs. (1.6b) and (1.36),

$$\partial_z \partial_{\bar{z}} \phi(z, \bar{z}) = 0, \quad \partial_z \partial_{\bar{z}} \widetilde{\phi}(z, \bar{z}) = 0, \tag{1.44}$$

which implies

$$\phi(z, \bar{z}) = \frac{1}{2} \left[\phi_+(z) + \phi_-(\bar{z}) \right], \quad \widetilde{\phi}(z, \bar{z}) = \frac{1}{2} \left[\widetilde{\phi}_+(z) + \widetilde{\phi}_-(\bar{z}) \right]. \tag{1.45}$$

Let

$$\widehat{\phi}_+(z) \equiv \phi_+(z) - \widetilde{\phi}_+(z),$$

and the correlation function is

$$\left\langle \widehat{\phi}_+(z) \widehat{\phi}_+(z') \right\rangle = -\ln \frac{E^{\sqrt{z}, \sqrt{z'}}(z, \bar{z}')}{\sqrt{dz}\sqrt{dz'}}. \tag{1.46}$$

The vertex operator

$$e^{-i\phi_+(z)} = \left\langle e^{-i\widehat{\phi}_+(z)} \right\rangle : e^{-i\widehat{\phi}_+(z)} : e^{-i\widetilde{\phi}_+(z)} \tag{1.47}$$

is a world-sheet scalar. We can make a parametrization-independent regularization to obtain, from eqs. (1.46) and (1.47),

$$
\begin{aligned}
V_{\phi_+} &= \lim_{z' \to z} e^{-\frac{1}{2} \ln \omega_z(z) E^{\sqrt{z'}, \sqrt{z}}(z', z)} e^{-i\phi_+(z)} \\
&= \lim_{z' \to z} \left[e^{-\frac{1}{2} \ln \omega_z(z) E^{\sqrt{z'}, \sqrt{z}}(z', z)} \exp \left(\frac{1}{2} \ln \frac{E^{\sqrt{z'}, \sqrt{z}}(z', z)}{\sqrt{dz'}\sqrt{dz}} \right) \right] \\
&\quad \times \left(: e^{-i\widehat{\phi}_+(z)} : e^{-i\widetilde{\phi}_+(z)} \right) \\
&= e^{-\frac{1}{2} \ln \omega_z(z) dz} : e^{-i\widehat{\phi}_+(z)} : e^{-i\widetilde{\phi}_+(z)}, \tag{1.48}
\end{aligned}
$$

where $\omega_z(z)$ is as in eq. (1.39). As in eq. (1.38), $e^{-i\widetilde{\phi}_+(z)}$ should be removed by multiplying a scalar factor. Since the vertex operator will be put on the open string

propagator which is a strip of length τ and total curvature 2π, from eqs. (1.36), (1.45) and (1.48), the scalar factor is found to be

$$e^{i\widetilde{\phi}_+(z)}e^{-\frac{3}{2}\ln(\omega_z dz)}.$$ (1.49)

Multiplied by the factor in eq. (1.49), the vertex operator in eq. (1.48) becomes

$$V_b(z) = e^{-2\ln(\omega_z dz)} : e^{-i\widehat{\phi}_+(z)} :,$$ (1.50)

where ω_z is as in eq. (1.39b). Since V_b is a scalar, $V_b d\omega(z)$ is a one-form. The covariant version of the antighost line integral in eqs. (1.22) and (1.43) is

$$\int\limits_l \frac{dw}{2\pi i} b_{ww} = \int\limits_l \frac{1}{2\pi i} V_b(z) d\omega(z),$$ (1.51a)

and similarly

$$\int\limits_l \frac{d\bar{w}}{2\pi i} b_{\bar{w}\bar{w}} = \int\limits_l \frac{1}{2\pi i} \bar{V}_b(\bar{z}) d\bar{\omega}(\bar{z}).$$ (1.51b)

The formulae in eqs. (1.35), (1.41), (1.42) and (1.51) are completely general, they can be used in any coordinate system on the world-sheet. For the parametrization used in section 1.2, $2g_{w\bar{w}} = 1$ and $\omega_w = \bar{\omega}_{\bar{w}} = 1$. After the conformal coordinate change from the natural coordinate w on the world-sheet to the coordinate z in eq. (1.24), we have

$$2g_{z\bar{z}} = \left(N\frac{\sqrt{z^2 + \gamma^2}\sqrt{z^2 + 1/\gamma^2}}{(z^2 + \alpha^2)(z^2 + 1/\alpha^2)}\right)\left(N\frac{\sqrt{\bar{z}^2 + \gamma^2}\sqrt{\bar{z}^2 + 1/\gamma^2}}{(\bar{z}^2 - \alpha^2)(\bar{z}^2 - 1/\alpha^2)}\right),$$ (1.52a)

and

$$\omega_z(z) = N\frac{\sqrt{z^2 + \gamma^2}\sqrt{z^2 + 1/\gamma^2}}{(z^2 - \alpha^2)(z^2 - 1/\alpha^2)}, \quad \bar{\omega}_{\bar{z}}(\bar{z}) = N\frac{\sqrt{\bar{z}^2 + \gamma^2}\sqrt{\bar{z}^2 + 1/\gamma^2}}{(\bar{z}^2 - \alpha^2)(\bar{z}^2 - 1/\gamma^2)}.$$ (1.52b)

On the upper half-plane, the correlation function has the simple form

$$G(z, \bar{z}; z', \bar{z}') = -\frac{1}{2} \ln \left| \frac{E^{\sqrt{z}, \sqrt{z'}}(z, z')}{\sqrt{dz}\sqrt{dz'}} \right|^2 - \frac{1}{2} \ln \left| \frac{E^{\sqrt{z}, \sqrt{\bar{z}'}}(z, \bar{z}')}{\sqrt{dz}\sqrt{d\bar{z}'}} \right|^2 \qquad (1.53)$$

with

$$E^{\sqrt{z}\sqrt{z'}}(z, z') = (z - z').$$

The vertex operators in eqs. (1.35) and (1.41) now take the form

$$V_X(z, \bar{z}) = e^{-P^2 \ln(\omega_z dz \bar{\omega}_{\bar{\omega}} d\bar{z})^{\frac{1}{2}}} : e^{iP \cdot X(z, \bar{z})} :, \qquad (1.54a)$$

$$V_c(z, \bar{z}) = e^{\ln(\omega_z dz \bar{\omega}_z d\bar{z})^{\frac{1}{2}}} : e^{i\widehat{\phi}(z, \bar{z})} :, \qquad (1.54b)$$

where ω_z and $\bar{\omega}_{\bar{z}}$ are as given in eq. (1.52b).

The vertex operators (1.54a) and (1.54b) contain dz's and $d\bar{z}$'s in the exponentials. But in calculating string amplitudes, the total momentum and the total ghost number on the string world-sheets are zero. This ensures that all dz's and $d\bar{z}$'s drop out.

Now we want to find the representation of the off-shell tachyon state. The path integral with vertex operator $V_X V_c$ placed at distance τ from the boundary EF, as shown in Fig. 5, is proportional to $e^{-(P^2-1)\tau}$ at large τ. To define a finite wave functional we must multiply the path integral by the factor $e^{(P^2-1)\tau}$. So the wave functional for the off-shell tachyon state can be written as

$$A[X^\mu(\sigma), \phi(\sigma)] \equiv \lim_{\tau \to \infty} \langle V_X V_c \rangle_{X(\sigma), \phi(\sigma), \tau} \, e^{(P^2-1)\tau}, \qquad (1.55)$$

where $X^\mu(\sigma)$ and $\phi(\sigma)$ are string coordinates and the ghost field, and

$$\langle V_X V_c \rangle = \int_{X(\sigma), \phi(\sigma), \tau} DX D\phi \, e^{-S_X} e^{-S_\phi} V_X V_c$$

is the path integral on the strip in Fig. 5 with the boundary condition $X(\sigma)$ and $\phi(\sigma)$ on EF and the operator $(V_X V_c)$ placed at distance τ from EF. This will become more clear as we actually do the calculation later.

At this point one might worry about the uniqueness of the off-shell tachyon state, since scalar particle states at higher mass levels may mix into the off-shell tachyon state. However, this will not happen. For example, if a term proportional to the vertex operator of the massive scalar at the fourth mass level is added to the tachyon vertex operator in eq .(1.55), this additional term vanishes like $e^{-3\tau}$ in the limit $\tau \to \infty$. So there is no mixing from the higher mass level states. Similarly, if one demands the off-shell wave functional for the higher mass level states to be finite, there is no mixing from the lower mass level states either (see section 1.4 for the vector particle case).

Similar to the on-shell amplitude calculation, but using $A[X(\sigma), \phi(\sigma)]$ defined in eq. (1.55) as the off-shell tachyon wave functional, the s-channel contribution to the off-shell four-tachyon scattering amplitude is as the world-sheet diagram sketched in Fig. 6a. The path integral representation of the amplitude is

$$
A_s = \int\limits_0^\infty d\tau \left\langle \left(\prod_j V_X(z_j, \bar{z}_j) V_c(z_j, \bar{z}_j) \right) \right.
$$

$$
\left. \times \left(\int\limits_l \left(\frac{1}{2\pi i} V_b(z) d\omega(z) + \frac{1}{2\pi i} \bar{V}_b(\bar{z}) d\bar{\omega}(\bar{z}) \right) \right) \right\rangle \prod_j e^{(P_j^2 - 1)\tau_j} \qquad (1.56)
$$

where $V_X(z_j, \bar{z}_j)$, $V_c(z_j, \bar{z}_j)$, $V_b(z)$ and $d\omega(z)$ are defined in eqs. (1.35), (1.41), (1.50) and (1.39), and $j = 1, \ldots, 4$. In eq. (1.56) the limit $\tau_j \to \infty$ is implicit, moreover the coordinate z is completely general.

The string world-sheet in Fig. 6a can be mapped to the upper half-plane by the conformal coordinate change in eq. (1.24). Under this transformation, (w_1, w_2, w_3, w_4) corresponds to (z_1, z_2, z_3, z_4); (A_1, A_2, A_3, A_4) corresponds to $(\alpha, -\alpha, -1/\alpha, 1/\alpha)$; and (B, D, E, F) corresponds to $(0, i\gamma, i/\gamma, \infty)$. The limit $\tau_j \to \infty$ is equivalent to the limit $(z_1, z_2, z_3, z_4) \to (\alpha, -\alpha, -1/\alpha, 1/\alpha)$. On the other hand,

$$
(d\tau_j)^2 = g_{z\bar{z}}(z_j, \bar{z}_j) dz_j d\bar{z}_j = |\omega_z(z_j) dz_j|^2, \qquad (1.57)
$$

where $j = 1, 2, 3, 4$. On the upper half-plane, z_j's are on the real axis and $\omega_z(z_j)$ is

as in eq. (1.52b). In the limit $\tau_j \to \infty$, eq. (1.57) becomes

$$
\begin{aligned}
d\tau_1 &= -\frac{dz_1}{z_1 - \alpha}, & d\tau_2 &= -\frac{dz_2}{z_2 + \alpha}, \\
d\tau_3 &= -\frac{dz_3}{z_3 + 1/\alpha}, & d\tau_4 &= -\frac{dz_4}{z_4 - 1/\alpha}.
\end{aligned}
\tag{1.58}
$$

The above equations can be solved to give

$$
\begin{aligned}
\alpha - z_1 &= C_1 e^{-\tau_1}, & z_2 + \alpha &= C_2 e^{-\tau_2}, \\
-1/\alpha - z_3 &= C_3 e^{-\tau_3}, & z_4 - 1/\alpha &= C_4 e^{-\tau_4},
\end{aligned}
\tag{1.59}
$$

where $C_1, \ldots, C_4$ are integration constants depending on α and γ. By the symmetry of the conformal mapping, we have the relations

$$
C_1 = C_2, \qquad C_3 = C_4.
\tag{1.60}
$$

In the appendix it is shown that (see eqs. $(A.10)$ and $(A.12)$),

$$
C_1 = \alpha C, \qquad C_4 = \frac{1}{\alpha} C,
\tag{1.61a, b}
$$

and

$$
\ln C = -\frac{\pi}{2K} F(\beta, k') + \ln \frac{\theta_1[(i\pi/K)F(\beta, k')]\theta_0[(i\pi/K)F(\beta, k')]2K}{\theta'_1[0]\theta_0[0]N_1 i\pi}
\tag{1.62}
$$

where

$$
k'^2 = 1 - k^2, \quad k = \gamma^2, \quad \sin\beta = \frac{\alpha}{\sqrt{\alpha^2 + \gamma^2}}, \quad N_1 = \frac{\alpha}{\sqrt{\alpha^2 + \gamma^2}\sqrt{1 + \alpha^2\gamma^2}};
$$

$K = K(k)$ and $F(\beta, k')$ are the complete and incomplete elliptic integrals of the first kind; θ_0 and θ_1 are the jacobian θ functions (for details, see reference [23]). γ depends on α through the relation in eq. (1.25a).

Using eqs. (1.50), (1.52), (1.54), and (1.58)-(1.61), the off-shell four-tachyon amplitude in eq. (1.56) becomes the path integral on the upper half-plane:

$$A_s = \int\limits_0^\infty d\tau \left\langle \exp\left(\sum_j iP_j \cdot X(z_j, \bar{z}_j)\right)\right\rangle_X$$

$$\times \left\langle \int\limits_l \frac{(\omega_z)^{-1}dz}{2\pi i} \exp\left(i\sum_j \hat{\phi}(z_j, \bar{z}_j)\right) e^{-i\hat{\phi}_+(z)}\right\rangle_\phi C^{\Sigma-4}\alpha^{P_1^2+P_2^2-P_3^2-P_4^2} , \quad (1.63)$$

where $\Sigma = \sum_j P_j^2$, and C is defined in eq. (1.62). The ghost part of the correlation function in eq. (1.63) is exactly the same as in the case of the on-shell four-tachyon amplitude which is given in eq. (1.27). After the change of variable from τ to α (see eq. (1.26)), we find

$$A_s = -2\int d\alpha \frac{1-\alpha^4}{\alpha^3}(\alpha - 1/\alpha)^{2P_2\cdot P_3+2P_1\cdot P_4}(\alpha + 1/\alpha)^{2P_1\cdot P_3+2P_2\cdot P_4}$$

$$\cdot (2\alpha)^{2P_1\cdot P_2}(2/\alpha)^{2P_3\cdot P_4}(C(\alpha))^{\Sigma-4}\alpha^{P_1^2+P_2^2-P_3^2-P_4^2}. \quad (1.64)$$

To convert eq. (1.64) into a more familiar form, we make a change of variable $x = \left[(1-\alpha^2)/(1+\alpha^2)\right]^2$. The integration regions for the different variables are

$$\tau \in (0, \infty), \quad \alpha \in (\sqrt{2}-1, 0), \quad x \in (1/2, 1).$$

The final expression for the s-channel off-shell four-tachyon amplitude is

$$A_s = \frac{1}{4}\int\limits_{1/2}^1 dx\, x^{-t-2}(1-x)^{-s-2}\left(\frac{C(x)}{2\sqrt{x}}\right)^{\Sigma-4} , \quad (1.65)$$

where $s = -(P_1+P_2)^2$, $t = -(P_2+P_3)^2$, $\Sigma = \sum_j P_j^2$ and $C(x)$ is defined in eq. (1.62). The t-channel contribution is obtained by permuting the states $1 \to 2 \to 3 \to 4 \to 1$

in A_s,

$$A_t = \frac{1}{4} \int_{1/2}^{1} dx \, x^{-s-2}(1-x)^{-t-2} \left(\frac{C(x)}{2\sqrt{x}}\right)^{\Sigma-4}. \qquad (1.66)$$

If $P_j^2 = 1$, $A_s + A_t$ reduces to the Koba–Nielsen amplitude for four on-shell tachyons.

In what follows we discuss the pole structure of the off-shell amplitude in the s-channel. In the appendix, $C(x)$ is expanded in power of $\epsilon = 1 - x$ (see $(A.17)$),

$$C(x) = \left(\frac{4}{3}\right)^{3/2} \left(1 - \frac{5}{32}(1-x)^2 + \cdots\right).$$

The integrand in eq. (2.65) has the expression

$$x^{-t-2}(1-x)^{-s-2} \left(\frac{C(x)}{2\sqrt{x}}\right)^{\Sigma-4}$$

$$= \left(\frac{16}{27}\right)^{(\Sigma-4)/2} \epsilon^{-s-2} \left\{1 + \left(t + \frac{\Sigma}{2}\right)\epsilon\right.$$

$$\left. + \left[\frac{1}{2}(t+2)(t+3) + \Sigma - 4\right)\left(\frac{t}{2} + \frac{\Sigma}{8} + \frac{19}{32}\right)\right] \epsilon^2 \right\}. \qquad (1.67)$$

The logarithmically divergent terms in eq. (1.65) correspond to intermediate particle poles in the s-channel. Now the residues of the poles are

$$A_s \underset{s\to-1}{\sim} \frac{1}{-s-1} \left(\frac{16}{27}\right)^{\frac{(\Sigma-4)}{2}}, \qquad (1.68a)$$

$$A_s \underset{s\to0}{\sim} \frac{1}{-s} \left(\frac{16}{27}\right)^{\frac{(\Sigma-4)}{2}} \left(t + \frac{1}{2}\Sigma\right), \qquad (1.68b)$$

$$A_s \underset{s\to1}{\sim} \frac{1}{1-s} \left(\frac{16}{27}\right)^{\frac{(\Sigma-4)}{2}} \left[\frac{1}{2}(t+2)(t+3) + (\Sigma - 4)\left(\frac{1}{2}t + \frac{1}{8}\Sigma + \frac{19}{32}\right)\right]. (1.68c)$$

Eqs. $(1.68a)$-$(1.68c)$ agree with the results of reference [15], where the Neumann function method [14] is used to calculate the individual intermediate particle contributions to the s-channel four-tachyon amplitude.

1.4. The off-shell four-point amplitudes involving vector particles

The method used in calculating the off-shell four-tachyon amplitude can be applied to the off-shell four-point amplitudes concerning other higher mass level particles. In this section, we consider the four-point amplitudes involving vector particles and tachyons; after this is done, the generalization to other cases should be clear.

The extension of the on-shell vector particle vertex operator

$$c^z : \xi \cdot \partial_z X(z, \bar{z}) e^{iP \cdot X(z, \bar{z})} :$$

is

$$V(z, \bar{z}, P, \xi) = (\omega_z)^{-1} V_c(z, \bar{z}) \lim_{z' \to z} \left[\xi \cdot \nabla_{z'} X(z', \bar{z}') V_X(z, \bar{z}) \right.$$

$$\left. -i\xi \cdot P \nabla_{z'} G(z', \bar{z}'; z, \bar{z}) V_X(z, \bar{z}) \right], \tag{1.69}$$

where V_X, V_c, $G(z', \bar{z}'; z, \bar{z})$ and ω_z are defined in eqs. (1.35), (1.41), (1.53) and (1.39), and ∇_z is the covariant derivative (described, for example, in reference [24]). One can see immediately that $V(z, \bar{z}, P, \xi)$ is a world-sheet scalar. The off-shell wave functional for the vector particle can be defined by the path integral like that in eq. (1.55),

$$A[X(\sigma), \phi(\sigma)] \equiv \lim_{\tau \to \infty} \Big\langle V(z, \bar{z}, P, \xi)$$

$$-i\xi \cdot P \left((\omega_z)^{-1} \partial_z \tau \right) V_c(z, \bar{z}) V_X(z, \bar{z}) \Big\rangle_{X(\sigma), \phi(\sigma), \tau} e^{P^2 \tau} \tag{1.70}$$

where $V(z, \bar{z}, P, \xi)$ is given in eq. (1.69). The second term in eq. (1.70) cancels the divergence in the first term. The parameter τ is the invariant distance on the world-sheet from the boundary EF (see Fig. 5) to the point where the vertex operator is inserted, so $(\omega_z)^{-1} \partial_z \tau$ is reparametrization independent. Since the path integral in eq. (1.70) is proportional to $e^{-P^2 \tau}$ at large τ, the factor $e^{P^2 \tau}$ is introduced to make the wave functional well defined. The definition in eq. (1.70) for the off-shell vector particle state is a natural extension of the on-shell state.

The two-vector particle two-tachyon amplitude (Fig. 7a) has the same string world-sheet diagram as the four-tachyon amplitude (see Fig. 6a). So most of the calculations here are parallel to that of the four-tachyon amplitude. The amplitude under consideration has the same path integral representation as that in eq. (1.56), except the tachyon vertex operators $V(z_1, \bar{z}_1)$ and $V(z_2, \bar{z}_2)$ are replaced by vector particle vertex operators $V(z_1, \bar{z}_1, P_1, \xi_1)$ and $V(z_2, \bar{z}_2, P_2, \xi_2)$ defined in eq. (1.69). It should be remembered that the vertex operators are on the world-sheet boundary where $z = \bar{z}$ and $z' = \bar{z}'$. From eqs. (1.39) and (1.53) we have

$$
\begin{aligned}
\nabla_z E^{\sqrt{z}, \sqrt{z'}} &= (g_{z\bar{z}})^{-\frac{1}{2}} \partial_z \left[(g_{z\bar{z}})^{\frac{1}{2}} E^{\sqrt{z}, \sqrt{z'}} \right] \\
&= \partial_z E^{\sqrt{z}, \sqrt{z'}} + (\omega_z)^{-1} \left(\frac{d\omega_z}{dz} \right) E^{\sqrt{z}, \sqrt{z'}},
\end{aligned}
\tag{1.71a}
$$

and

$$
\begin{aligned}
\nabla_z G(z, \bar{z}; z', \bar{z}') &= -2 \left(\frac{\nabla_z E^{\sqrt{z}, \sqrt{z'}}}{E^{\sqrt{z}, \sqrt{z'}}} \right) \\
&= -2 \left[\frac{\partial_z E^{\sqrt{z}, \sqrt{z'}}}{E^{\sqrt{z}, \sqrt{z'}}} + (\omega_z)^{-1} \left(\frac{d\omega_z}{dz} \right) \right].
\end{aligned}
\tag{1.71b}
$$

Using the tachyon wave functional in eq. (1.55) and the vector particle wave functional in eq. (1.70), we find that the path integral representation of the two-tachyon two-vector particle amplitude with the diagram in Fig. 7a is

$$
A_{7a} = \int_0^\infty d\tau \left\langle \left(\prod_j V_X(z_j, \bar{z}_j) V_c(z_j, \bar{z}_j) \right) \left(\prod_k (\omega_{z_k})^{-1} V_c(z_k, \bar{z}_k) \right. \right.
$$

$$
\times \left(: \xi_k \cdot \nabla_{z_k} X(z_k, \bar{z}_k) V_X(z_k, \bar{z}_k) : -i\xi_k \cdot P_k (\partial_{z_k} \tau_k) V_X(z_k, \bar{z}_k) \right) \Big)
$$

$$
\times \left[\int_l \left(\frac{1}{2\pi i} V_b(z) d\omega(z) + \frac{1}{2\pi i} \bar{V}_b(\bar{z}) d\bar{\omega}(\bar{z}) \right) \right] \right\rangle \left(\prod_j e^{(P_j^2 - 1)\tau_j} \right) \left(\prod_k e^{P_k^2 \tau_k} \right),
\tag{1.72}
$$

where $k = 1, 2$; $j = 3, 4$, and

$$: \xi \cdot \nabla_z X(z, \bar{z}) V_X(z, \bar{z}) := \lim_{z' \to z} \left[\xi \cdot \nabla_{z'} X(z', \bar{z}') V_X(z, \bar{z}) - i\xi \cdot P \nabla_{z'} G(z', \bar{z}'; z, \bar{z}) \right].$$

The ghost part of A_{7a} is exactly the same as that in the four-tachyon amplitude. Making use of eqs. (1.53) and (1.71), the amplitude is found to be

$$A_{7a} = 2 \int d\alpha \frac{1 - \alpha^4}{\alpha^3} \left\{ \left(\frac{1 - \alpha}{\alpha} \right)^{2P_1 \cdot P_4 + 2P_2 \cdot P_3} \right.$$

$$\left(\frac{1 + \alpha}{\alpha} \right)^{2P_1 \cdot P_3 + 2P_2 \cdot P_4} (2\alpha)^{2P_1 \cdot P_2} \left(\frac{2}{\alpha} \right)^{2P_3 \cdot P_4} \right\}$$

$$\times \frac{4}{(2\alpha)^2} \left(\frac{\xi_1 \cdot \xi_2}{2} + \left\{ \xi_1 \cdot \left[P_4 \left(\frac{2\alpha^2}{1 - \alpha^2} \right) - P_3 \left(\frac{2\alpha^2}{1 + \alpha^2} \right) - P_2 - 2P_1 \alpha B_1 \right] \right\} \right.$$

$$\times \left\{ \xi_2 \cdot \left(P_4 \left(\frac{2\alpha^2}{1 + \alpha^2} \right) - P_3 \left(\frac{2\alpha^2}{1 - \alpha^2} \right) + 2P_2 \alpha B_1 + P_1 \right) \right\} \right)$$

$$\times \left\{ [C(\alpha)]^{\Sigma - 2} \alpha^{P_1^2 + P_2^2 - P_3^2 - P_4^2} \right\}, \tag{1.73}$$

where

$$B_1 = \alpha \left(\frac{1}{2\alpha^2} + \frac{2}{\alpha^2 - 1/\alpha^2} - \frac{1}{\alpha^2 + \gamma^2} - \frac{1}{\alpha^2 + 1/\gamma^2} \right).$$

The change of variable $x = \left[(1 - \alpha^2)/(1 + \alpha^2) \right]^2$ gives

$$A_{7a} = - \int_{1/2}^{1} dx \, x^{-t-1} (1 - x)^{-s-2} \left[\frac{C(x)}{2\sqrt{x}} \right]^{\Sigma - 2}$$

$$\times \left\{ \frac{\xi_1 \cdot \xi_2}{2} + \left[\xi_1 \cdot \left(P_4 \frac{1 - \sqrt{x}}{\sqrt{x}} - P_3(1 - \sqrt{x}) - P_2 - 2P_1 \alpha B_1 \right) \right] \right.$$

$$\times \left[\xi_2 \cdot \left(P_4(1 - \sqrt{x}) - P_3 \frac{1 - \sqrt{x}}{\sqrt{x}} + P_2(1 + \lambda)\alpha B_1 + P_1 \right) \right] \Bigg\}, \qquad (1.74)$$

where

$$s = -(P_1 + P_2)^2, \quad t = -(P_2 + P_3)^2, \quad \Sigma = \sum_j P_j^2,$$

$$B_1 = \alpha \left(\frac{1}{2\alpha^2} + \frac{2}{\alpha^2 - 1/\alpha^2} - \frac{1}{\alpha^2 + \gamma^2} - \frac{1}{\alpha^2 + 1/\gamma^2} \right),$$

and $C(x)$ is given in eq. (1.62).

The diagram in Fig. 7g is identical to the diagram in Fig. 7a, so the corresponding amplitudes should be equal. Following the same procedure as described previously, we find

$$A_{7g} = - \int\limits_{1/2}^{1} dx\, x^{-t-1}(1 - x)^{-s-2} \left[\frac{C(x)}{2\sqrt{x}} \right]^{\Sigma - 2}$$

$$\times \left\{ \frac{\xi_1 \cdot \xi_2}{2} - \left[\xi_1 \cdot \left(P_4 \frac{1 + \sqrt{x}}{\sqrt{x}} + P_3(1 + \sqrt{x}) + P_2 + 2P_1 \frac{1}{\alpha} B_2 \right) \right] \right.$$

$$\times \left[\xi_2 \cdot \left(P_4(1 + \sqrt{x}) + P_3 \frac{1 + \sqrt{x}}{\sqrt{x}} + 2P_2 \frac{1}{\alpha} B_2 + P_1 \right) \right] \Bigg\}, \qquad (1.75)$$

where

$$B_2 = \frac{1}{\alpha} \left(\frac{\alpha^2}{2} + \frac{2}{1/\alpha^2 - \alpha^2} - \frac{1}{1/\alpha^2 + \gamma^2} - \frac{1}{1/\alpha^2 + 1/\gamma^2} \right).$$

With the help of an identity

$$\alpha B_1 = 1 - \frac{1}{\alpha} B_2,$$

it is not hard to show that A_{7a} and A_{7g} are equal as expected.

In the same manner, the amplitudes for diagrams in Figs. 7*b* and 7*c* are found to be

$$A_{7b} = -\int_{1/2}^{1} dx\, x^{-s-1}(1-x)^{-t-2}\left[\frac{C(x)}{2\sqrt{x}}\right]^{\Sigma-2}$$

$$\times\left\{\frac{\xi_1\cdot\xi_2}{2}\left(\frac{1-x}{x}\right) + \left[\xi_1\cdot\left(P_4 + P_3(1-\sqrt{x}) - P_2\frac{1-\sqrt{x}}{\sqrt{x}} + 2P_1\alpha B_1\right)\right]\right.$$

$$\left.\times\left[\xi_2\cdot\left(P_4(1+\sqrt{x}) + P_3 + P_2\frac{1}{\alpha}B_2 + P_1\frac{1+\sqrt{x}}{\sqrt{x}}\right)\right]\right\},\qquad(1.76)$$

and

$$A_{7c} = -\frac{1}{4}\int_{1/2}^{1} dx\, x^{-t-1}(1-x)^{-u-2}\left[\frac{C(x)}{2\sqrt{x}}\right]^{\Sigma-2}$$

$$\times\left\{\frac{\xi_1\cdot\xi_2}{2}(1-x) - \left[\xi_1\cdot\left(-P_4\frac{1-\sqrt{x}}{\sqrt{x}} + P_3 + P_2(1-\sqrt{x}) + 2P_1\alpha B_1\right)\right]\right.$$

$$\left.\times\left[\xi_2\cdot\left(P_4 + P_3\frac{1+\sqrt{x}}{\sqrt{x}} + 2P_2\frac{1}{\alpha}B_2 + P_1(1+\sqrt{x})\right)\right]\right\},\qquad(1.77)$$

where

$$s = -(P_1 + P_2)^2,\quad t = -(P_2 + P_3)^2,\quad u = -(P_1 + P_3)^2,$$

$$\Sigma = P_1^2 + P_2^2 + P_3^2 + P_4^2.$$

The amplitudes A_{7d}, A_{7e}, and A_{7f} can be obtained from A_{7a}, A_{7b}, and A_{7c} by switching the labels 1 and 2. It is easily seen that the off-shell amplitudes $A_{7a}, \ldots, A_{7f}$ are reduced to the corresponding on-shell amplitudes, when the conditions

$$\xi_1\cdot P_1 = \xi_2\cdot P_2 = 0,\quad P_1^2 = P_2^2 = 0,\quad P_3^2 = P_4^2 = 1$$

are imposed.

The calculation of the off-shell four-vector particle amplitude is straightforward. The method used in this section is also applicable to the off-shell amplitudes involving other states.

1.5. The conformal field theory method

In section 1.3 and 1.4, off-shell string amplitudes involving tachyons and vector particles are calculated in a way which is explicitly independent of world-sheet parametrization. Now we briefly describe how to calculate off-shell amplitudes in Witten's bosonic string field theory by the conformal field theory method [25]. Although this method is not explicitly world-sheet reparametrization-independent, the final results agree with those obtained in previous sections.

The prescription for conformal field theory calculation is the following. Instead of using eqs. (1.35), (1.41) and (1.50), we take vertex operators,

$$V_X(w, \bar{w}) =: e^{iP \cdot X(w, \bar{w})} :, \qquad (1.78a)$$

$$V_c(w, \bar{w}) =: e^{i\widehat{\phi}(w, \bar{w})} :, \qquad (1.78b)$$

$$b_{ww} = V_b(w) =: e^{-i\widehat{\phi}_+(w)} : \qquad (1.78c)$$

with conformal dimensions P^2, -1 and 2, respectively. Choosing the natural coordinate system, with $2g_{w\bar{w}} = 1$, on the world-sheet [5, 8], we define the off-shell tachyon wave functional as that in eq. (1.55), but the vertex operators in there are replaced by those in eqs. (1.78a) and (1.78b). The path integral for the off-shell four-tachyon amplitude in s-channel is

$$A_s = \int\limits_0^\infty d\tau \left\langle \left(\prod_j V_X(w_j, \bar{w}_j) V_c(w_j, \bar{w}_j) \right) \right.$$

$$\left. \left(\int\limits_l \left(\frac{dw}{2\pi i} b_{ww} + \frac{d\bar{w}}{2\pi i} b_{\bar{w}\bar{w}} \right) \right) \right\rangle \prod_j e^{(P_j^2 - 1)\tau_j} . \qquad (1.79)$$

Then we make the conformal coordinate change in eq. (1.24) and note that

$$V_X(w, \bar{w}) = \left| \frac{dz}{dw} \right|^{P^2} V_X(z, \bar{z}),$$

$$V_c(w, \bar{w}) = \left| \frac{dz}{dw} \right|^{-1} V_c(z, \bar{z}),$$

$$b_{ww} = \left(\frac{dz}{dw} \right)^2 b_{zz}. \tag{1.80}$$

Inserting eq. (1.80) into eq. (1.79) and doing the calculation similar to that in section 1.3, we obtain the same result as that in eq. (1.65).

For the vector particle, the vertex operator is

$$V(w, \bar{w}, P, \xi) = V_c(w, \bar{w}) \lim_{w' \to w} \Big[\xi \cdot \nabla_{w'} X(w', \bar{w}') V_X(w, \bar{w})$$

$$-i\xi \cdot P \nabla_{w'} G(w', \bar{w}'; w, \bar{w}) V_X(w, \bar{w}) \Big]$$

$$= V_c(w, \bar{w}) : \xi \cdot \nabla_w X(w, \bar{w}) V_X(w, \bar{w}) :, \tag{1.81}$$

where $V_X(w, \bar{w})$ and $V_c(w, \bar{w})$ are as defined in eq. (1.78). The vertex operator in eq. (1.81) has the conformal dimension P^2. Analogous to eq. (1.70), the wave functional for off-shell vector particle is

$$A[X(\sigma), \phi(\sigma)] \equiv \lim_{\tau \to \infty} \Big\langle V(w, \bar{w}, P, \xi)$$

$$-i\xi \cdot P(\partial_w \tau) V_c(w, \bar{w}) V_X(w, \bar{w}) \Big\rangle_{X(\sigma), \phi(\sigma), \tau} e^{P^2 \tau} \tag{1.82}$$

where $V(w, \bar{w}, P, \xi)$, $V_c(w, \bar{w})$ and $V_X(w, \bar{w})$ are given in eqs. (1.81), (1.78a) and (1.78b). The amplitude for the diagram in Fig. 7a has the same expression as eq. (1.79) except two of the tachyon states are replaced by two vector particle states. Then performing calculations similar to what follows eqs. (1.72) and (1.79), we obtain the result in eq. (1.74). Other diagrams in Fig. 7 can be done similarly.

2. Supersymmetric String Field Theory

2.1 Reformulation of the superstring field theory

We begin by reviewing Witten's open superstring field theory [6] and some of the difficulties associated with it. We will then propose modifications that circumvent these problems.

In Witten's theory, a superstring field $A = (\phi, \psi)$ contains a Neveu–Schwarz field ϕ and a Ramond field ψ, where both ϕ and ψ are functional of string coordinates $X^\mu(\sigma)$ and $\psi^\mu(\sigma)$, conformal ghosts $c(\sigma)$ and $b(\sigma)$, and superconformal ghosts $\gamma(\sigma)$ and $\beta(\sigma)$. Each of those fields corresponds to a set of harmonic oscillators:

$$X^\mu(\sigma): \quad a_n^\mu, \qquad a_n^{\mu\dagger} = a_{-n}^\mu,$$

$$\psi^\mu(\sigma): \quad d_n^\mu, \quad d_n^{\mu\dagger} = d_{-n}^\mu; \qquad \text{or } d_r^\mu, \quad d_r^{\mu\dagger} = d_{-r}^\mu,$$

$$c(\sigma): \quad c_n, \quad c_n^\dagger = c_{-n}; \qquad b(\sigma): \quad b_n, \quad b_n^\dagger = b_{-n},$$

$$\gamma(\sigma): \quad \gamma_n, \quad \gamma_n^\dagger = \gamma_{-n}; \qquad \text{or } \gamma_r, \quad \gamma_r^\dagger = \gamma_{-r},$$

$$\beta(\sigma): \quad \beta_n, \quad \beta_n^\dagger = -\beta_{-n}; \qquad \text{or } \beta_r, \quad \beta_r^\dagger = -\beta_{-r},$$

where n is an integer and r is a half-integer, and $\mu = 0, \cdots, D-1$. $D = 10$ is the dimension of space-time. The oscillators satisfy the following (anti-)commutation relations [26, 27, 28].

In the Neveu–Schwarz (NS) sector:

$$[a_m^\mu, a_n^\nu] = m\eta^{\mu\nu}\delta_{m+n,0}, \tag{2.1a}$$

$$\{d_r^\mu, d_s^\nu\} = \eta^{\mu\nu}\delta_{r+s,0}, \tag{2.1b}$$

$$\{c_m, b_n\} = \delta_{m+n,0}\,, \tag{2.1c}$$

$$[\gamma_r, \beta_s] = \delta_{r+s,0}\,, \tag{2.1d}$$

and other (anti-)commutators are zero. Here m and n are integers, while r and s are half-integers. $\eta^{\mu\nu} = (-1, 1, \ldots 1)$ is the metric of ten-dimensional space-time. The BRST charge is

$$Q^{\mathrm{NS}} = (L_0 - 1/2)c_0 + M^{\mathrm{NS}}b_0 + \bar{Q}^{\mathrm{NS}}, \tag{2.2}$$

where M^{NS} contains only conformal and superconformal ghost modes, and

$$\bar{Q}^{\mathrm{NS}} = \sum_{n \neq 0} L_n c_{-n} + \sum_r G_r \gamma_{-r}\,, \tag{2.3}$$

where L_n and G_r are generators of super-Virasoro algebra. The BRST charge Q^{NS} is nilpotent,

$$(Q^{\mathrm{NS}})^2 = 0\,,$$

in ten-dimensional space-time. The conformal and superconformal ghost number operators are

$$N_c = \frac{1}{2}(c_0 b_0 - b_0 c_0) + \sum_{n=1}^{\infty} (c_{-n}b_n - b_{-n}c_n), \tag{2.4a}$$

$$N_{sc} = -\sum_{s=1/2}^{\infty} (\gamma_{-s}\beta_s + \beta_{-s}\gamma_s). \tag{2.4b}$$

In the Ramond (R) sector:

$$[a_m^\mu, a_n^\nu] = m\eta^{\mu\nu}\delta_{m+n,0}\,, \tag{2.5a}$$

$$\{d_m^\mu, d_n^\nu\} = \eta^{\mu\nu}\delta_{m+n,0}\,, \tag{2.5b}$$

$$\{c_m, b_n\} = \delta_{m+n,0}\,, \tag{2.5c}$$

$$[\gamma_m, \beta_n] = \delta_{m+n,0}\,, \tag{2.5d}$$

and other (anti-)commutators vanish. Both m and n are integers. The BRST charge is given by

$$Q^{\mathrm{R}} = L_0 c_0 + (M^{\mathrm{R}} - \gamma_0^2)b_0 + F_0 \gamma_0 + K\beta_0 + \bar{Q}^{\mathrm{R}}. \tag{2.6}$$

In eq. (2.6) the dependence on conformal and superconformal ghost zero modes is made explicit. M^{R} and K contain ghost modes only, and

$$\bar{Q}^{\mathrm{R}} = \sum_{n \neq 0} L_n c_{-n} + \sum_{n \neq 0} F_n \gamma_{-n}\,, \tag{2.7}$$

where L_n and F_n are generators of super-Virasoro algebra. We also have

$$(Q^{\mathrm{R}})^2 = 0,$$

when $D = 10$. The ghost number operators in the R sector are

$$N_c = \frac{1}{2}(c_0 b_0 - b_0 c_0) + \sum_{n=1}^{\infty}(c_{-n} b_n - b_{-n} c_n), \tag{2.8a}$$

$$N_{sc} = -\frac{1}{2}(\gamma_0 \beta + \beta_0 \gamma_0) - \sum_{n=1}^{\infty}(\gamma_{-n}\beta_n + \beta_{-n}\gamma_n). \tag{2.8b}$$

The conformal and superconformal ghost system can be bosonized [29],

$$c = e^{i\rho}, \qquad b = e^{-i\rho}, \tag{2.9}$$

$$\gamma = e^{\phi}\eta, \qquad \beta = e^{-\phi}\partial\xi, \tag{2.10}$$

where σ and ϕ are two scalar fields having correlation functions

$$\langle \rho(z_1)\rho(z_2) \rangle = -\ln(z_1 - z_2), \quad \langle \phi(z_1)\phi(z_2) \rangle = -\ln(z_1 - z_2) \tag{2.11a,b}$$

on the world-sheet, and η and ξ are anticommuting. The conformal dimensions of the fields are

$$e^{in\rho} \; : \; \frac{1}{2}n(n-3), \quad e^{q\phi} \; : \; -\frac{1}{2}q(q+2),$$

$$\eta \; : \; 1 \, , \qquad \xi \; : \; 0 \, . \tag{2.12}$$

The ghost number currents are

$$j^c = cb = i\partial_z\rho \, , \qquad j^{sc} = \gamma\beta = \partial_z\phi \, . \tag{2.13}$$

So η and ξ have a ghost number zero. Then the picture-changing operator

$$X = \{Q, \xi\} \tag{2.14a}$$

and inverse picture-changing operator

$$Y = c\partial_z\xi e^{-2\phi} \tag{2.14b}$$

have a conformal dimension zero and ghost number $+1$ and -1 respectively. Y is the inverse of X in the sense that

$$\lim_{z_1 \to z_2} Y(z_1)Y(z_2) = 1 \, . \tag{2.15}$$

The simple extension of string field multiplication $*$ in eqs. (1.9a)–(1.9d) to the superstring field theory now has ghost number anomalies $3/2$ from the conformal

ghost and -1 from the superconformal ghost, whereas the extension of string state integration $\int$ has ghost number anomalies $-3/2$ and 1 respectively. The operators $*$ and $\int$ for superstring fields (we still use the same notation as in bosonic string field theory) satisfy the relations $(1.9a)$–$(1.9d)$ if A, B and C there are interpreted as superstring fields. In order to have a desired ghost number counting, Witten defined the superstring multiplication and integration [6] as

$$
\begin{aligned}
A_1 \star A_2 &= (\phi_1, \psi_1) * (\phi_2, \psi_2) \\
&= (X(\phi_1 * \phi_2) + \psi_1 * \psi_2, X(\phi_1 * \psi_2 + \psi_1 * \phi_2))
\end{aligned}
\tag{2.16}
$$

and

$$
\oint A = \oint (\phi, \psi) = \left(\int Y(\pi/2)\phi, 0 \right)
\tag{2.17}
$$

where X and Y are inserted at the middle point of the string and the integration of the Ramond field ψ is zero. The operations $\star$ and $\oint$ so defined also satisfy the superstring version of eqs. $(1.9a)$–$(1.9d)$. Witten's gauge-invariant superstring action takes the form

$$
I = \frac{1}{2} \oint \left(A \star QA + \frac{2}{3} A \star A \star A \right),
\tag{2.18}
$$

where $A = (\phi, \psi)$ has ghost number $(-1/2, 0)$. The gauge transformation is

$$
\delta A = Q\Lambda + A \star \Lambda - \Lambda \star A,
\tag{2.19}
$$

where $\Lambda = (\epsilon, \chi)$ has ghost number $(-3/2, -1)$. The supersymmetry charge is

$$
Q_\alpha = \int d\sigma e^{-\phi/2} S_\alpha,
\tag{2.20}
$$

where S_α is the spin field on the world-sheet [27]. The supersymmetry transformation of the string field is defined as

$$
\delta A = UA = (\delta\phi, \delta\psi) = (Q_\alpha\psi, Y(\pi/2)Q_\alpha\phi).
\tag{2.21}
$$

In reference [5], it was argued on general grounds that a linear transformation U like that in eq. (2.21) is a global symmetry of the action in eq. (2.18) if it has the

following three properties,

$$\oint UA = 0, \qquad (2.22a)$$

$$U(A \star B) = (UA) \star B + A \star (UB), \qquad (2.22b)$$

$$[Q, U] = 0. \qquad (2.22c)$$

And the supersymmetry transformation in eq. (2.21) indeed satisfies eqs. (2.22a)–(2.22c), therefore it is a global symmetry.

Using eqs. (2.15)–(2.17), one can find that the quadratic part of eq. (2.18) is

$$\begin{aligned}
I &= \int \phi * Q^{\mathrm{NS}}\phi + \int \psi * Y(\pi/2)Q^{\mathrm{R}}\psi \\
&= (\phi|Q^{\mathrm{NS}}\phi) + (\psi|Y(\pi/2)Q^{\mathrm{R}}\psi).
\end{aligned} \qquad (2.23)$$

The derived equations of motion have the standard form

$$Q^{\mathrm{NS}}\phi = 0, \qquad Q^{\mathrm{R}}\psi = 0. \qquad (2.24)$$

The gauge-fixing conditions were chosen as

$$b_o \phi = 0, \qquad b_0 \psi = \beta_0 \psi = 0. \qquad (2.25)$$

Eqs. (2.24) and (2.25) determine the spectrum of physical excitations in the NS and R sectors [28]. It was shown in detail [6] that the linearized gauge-fixed action is

$$I = (\phi'|(L_0 - 1/2)|\phi') + (\psi'|F_0|\psi'), \qquad (2.26)$$

where ϕ' and ψ' contain no ghost zero modes.

The four-massless-particle amplitudes were calculated schematically in reference [30]. More careful calculation [31] revealed that the four-boson amplitude is divergent, because the two picture-changing operators inserted on the world-sheet overlap. For the same reason the gauge invariance of the action in eq. (2.18) is also spoiled. To fix this problem, a four-point contact term is added to the action in eq. (2.18) as a counter term [31]. It makes the action gauge invariant to order g^2 of the coupling constant and, at the same time, cancels the divergence in the four-vector boson amplitude. Once a four-point contact term is introduced, higher-point contact terms are inevitable in order to ensure the gauge invariance of the action at higher orders of the coupling constant g. This will continue order by order. On the other hand, divergences are expected in higher-point amplitudes, and it is not clear that the same sort of cancellation occurs at higher orders, namely that the same counter term cancels the divergence in amplitudes and ensures gauge invariance of the action at each order simultaneously.

Having the aforementioned difficulties, we are forced to look for an alternative formulation of open superstring field theory such that those problems can be by-passed. Moreover, just like the bosonic string field theory, string amplitudes should be derived from the gauge-fixed BRST-invariant action like that in section 2.1. But it is not clear how to proceed on gauge-fixing explicitly in the above formulation of superstring field theory. Finally, in the above formulation, no distinction has been made between conformal and superconformal ghosts in ghost number counting. Since the two ghost systems are independent, one would like the ghost numbers of the two different systems to be counted separately. Fortunately, all those problems can be solved very naturally in a modified theory [32]. The rest of this chapter is devoted to this purpose.

Based on the discussion of bosonic string field theory in Chapter 1 and the observation of picture-changing phenomenon [29], one naturally expects that a generic superstring field should contain all conformal and superconformal ghost numbers,

$$\Phi = \sum_{s,n} \Phi_{s,n} \,, \qquad\qquad (2.27)$$

where $\Phi_{s,n}$ is a string field with the conformal ghost number s which takes half-integer values and the superconformal ghost number n which could be integers or half-integers. The component field $\Phi_{s,n}$ is a Neveu–Schwarz (or Ramond) field, if n is an integer (or half-integer). The physical states are contained in string fields of conformal ghost number $-1/2$, namely

$$\Phi_{\text{physical}} \equiv \Phi_{-1/2} \equiv \sum_{n} \Phi_{-1/2,n} \,. \qquad\qquad (2.28)$$

String fields of other conformal ghost numbers are ghost and antighost fields. In a "canonical choice of picture," we identify $\Phi_{-1/2,0}$ as physical NS fields and $\Phi_{-1/2,1/2}$ as physical R fields. In the more familiar language, this corresponds to the F_2-picture. We also require that each individual superstring field $\Phi_{s,n}$ is Grassmann-odd, consequently Φ is overall odd. All this will become clear as we get more explicit later.

The integration and multiplication we will use for superstring fields are just the simple extensions of their bosonic string counterparts. The multiplication $*$ has conformal and superconformal ghost number anomalies $(3/2, -1)$ and the integration $\int$ has anomalies $(-3/2, 1)$. Thus the product of two NS fields or two R fields is an NS field, and the product of an NS and an R field is an R field, as expected. One can show that $*$ and $\int$ for superstring theory obey the superstring version of eqs. $(1.9a)$–$(1.9d)$. The integration of an R field is zero by superconformal ghost number counting. Now we can write down an action for superstring field theory,

$$I = \frac{1}{2} \int \left(\Phi_{-1/2} * Q\Phi_{-1/2} + \frac{2}{3}\Phi_{-1/2} * \Phi_{-1/2} * \Phi_{-1/2} \right), \qquad (2.29)$$

where $\Phi_{-1/2}$ is defined in eq. (2.28). Evidently, this action is invariant under the

gauge transformation,

$$\delta\Phi_{-1/2} = Q\Omega_{-3/2} + \Phi_{-1/2} * \Omega_{-3/2} - \Omega_{-3/2} * \Phi_{-1/2} \, , \qquad (2.30)$$

where $\Omega_{-3/2}$ is Grassmann-even, and

$$\Omega_{-3/2} \equiv \sum_n \Omega_{-3/2,n} \, . \qquad (2.31)$$

We do not need to worry about the possible associative anomaly in the multiplication $*$, as long as $\Phi_{-1/2}$ and $\Omega_{-3/2}$ are open string Fock-space states, which are the states that will concern us in this paper.

The supersymmetry transformation is now generated by the supersymmetry charge in eq. (2.20) without any inverse picture-changing operator insertion,

$$\delta\Phi_{-1/2} = Q_\alpha \Phi_{-1/2}. \qquad (2.32)$$

The supersymmetry charge Q_α has ghost number $(0, -1/2)$, so it interchanges NS states which have integer superconformal ghost numbers and R states which have half-integer superconformal ghost numbers. Following the argument given in reference [5], it is not difficult to show that

$$\int Q_\alpha \Phi = 0 \, , \qquad (2.33a)$$

$$Q_\alpha(\Phi_1 * \Phi_2) = (Q_\alpha \Phi_1) * \Phi_2 + \Phi_1 * (Q_\alpha \Phi_2) \, , \qquad (2.33b)$$

$$[Q, Q_\alpha] = 0 \, . \qquad (2.33c)$$

It then follows that eq. (2.32) is a global symmetry of the action in eq. (2.29). The perturbative vacuum is described by $\Phi_{-1/2} = 0$, which is obviously invariant under eq. (2.32).

2.2. Equations of motion and physical states

Since the new superstring action (eq. (2.29)) has the gauge invariance (eq. (2.30)), in order to derive the equations of motion that define physical states and perform the perturbative calculation of string amplitudes, we must fix the gauge.

As far as conformal ghost zero modes c_0 and b_0 are concerned, the BRST charges Q^{NS} and Q^{R} in eqs. (2.2) and (2.6) have the same structure as the bosonic case in eq. (1.3). Following the same argument used in bosonic string field theory, we see that

$$b_0 \Phi_{-1/2} = 0 \qquad (2.34)$$

fixes the gauge for the linearized version of eq. (2.29) completely. From eq. (2.20), we have

$$[b_0, Q_\alpha] = 0, \qquad (2.35)$$

where Q_α is the generator of the supersymmetry transformation in eq. (2.32). It follows that the gauge-fixing condition in eq. (2.34) is supersymmetry-covariant.

Now we will study the linearized equations of motion more carefully to see that they give rise to the correct physical state spectrum in the NS and R sectors. As in eq. (2.28), the physical superstring field is the sum of the NS and R fields,

$$\Phi_{-1/2} = \Phi_{\mathrm{NS}} + \Phi_{\mathrm{R}} \equiv \sum_{n=\mathrm{integer}} \Phi_{-1/2,n} + \sum_{n=\mathrm{half-integer}} \Phi_{-1/2,n}. \qquad (2.36)$$

We can write the linearized equations of motion separately,

$$Q^{\mathrm{NS}} \Phi_{\mathrm{NS}} = 0, \quad b_0 \Phi_{\mathrm{NS}} = 0 \qquad (2.37a,b)$$

for the NS sector, and

$$Q^{\mathrm{R}} \Phi_{\mathrm{R}} = 0, \quad b_0 \Phi_{\mathrm{R}} = 0 \qquad (2.38a,b)$$

for the R sector. In addition, the string fields in the NS sector are required to be

invariant under the generalized G-parity projection [27, 33],

$$P_{\rm NS}\Phi_{\rm NS} \equiv \frac{1}{2}(1 - (-1)^{N_d+N_{sc}})\Phi_{\rm NS} = \Phi_{\rm NS}, \qquad (2.39)$$

where $N_d = \sum_{r=1/2}^{\infty} d_{-r} \cdot d_r$, and N_{sc} is defined in eq. (2.4b). The string fields in the R sector are also required to be invariant under the generalized Weyl-projection [27, 33],

$$P_R\Phi_R \equiv \frac{1}{2}(1 + \gamma_{11}(-1)^{N_d+N_{sc}-1/2})\Phi_R = \Phi_R, \qquad (2.40)$$

where $N_d = \sum_{n=1}^{\infty} d_{-n} \cdot d_n$, and N_{sc} is defined in eq. (2.8b).

In order to expand the physical string field into component fields, we must find the Fock-space vacuum state in the NS sector. To do so, we first define vacuum states for each set of oscillators in eq. (2.1). Because the conformal (anti-)ghost has zero modes, there are two vacuum states in its Fock-space [18], which satisfy

$$c_0 \left|+\right\rangle_c = c_n \left|+\right\rangle_c = b_n \left|+\right\rangle_c = 0, \quad n > 0,$$

$$b_0 \left|-\right\rangle_c = c_n \left|-\right\rangle_c = b_n \left|-\right\rangle_c = 0, \quad n > 0 \qquad (2.41)$$

and have ghost number

$$\left|+\right\rangle_c \ : \ N_c = 1/2,$$

$$\left|-\right\rangle_c \ : \ N_c = -1/2. \qquad (2.42)$$

Following the Z_2-grading for string field in reference [5], $\left|+\right\rangle_c$ is Grassmann-even, and $\left|-\right\rangle_c$ is odd. The vacuum state for the superconformal (anti-)ghost is defined as

$$\gamma_s \left|0\right\rangle_{sc} = \beta_s \left|0\right\rangle_{sc} = 0, \quad s > 0 \qquad (2.43)$$

with ghost number

$$\left|0\right\rangle_{sc} \ : \ N_{sc} = 0. \qquad (2.44)$$

We require $\left|0\right\rangle_{sc}$ to be Grassmann-even. The harmonic oscillators for bosonic string

coordinates have, as usual, the vacuum state

$$a_n^\mu |0\rangle_a = 0, \quad n \geq 0, \tag{2.45}$$

where $|0\rangle_a$ is Grassmann-even. The vacuum state for fermionic coordinates is defined by

$$d_s^\mu |0\rangle_d = 0, \quad s > 0. \tag{2.46}$$

We require $|0\rangle_d$ to be Grassmann-odd.

Then the Fock-space vacuum state in the NS sector is

$$|0\rangle_{\mathrm{NS}} \equiv |0\rangle_a |0\rangle_d |-\rangle_c |0\rangle_{sc}, \tag{2.47}$$

so $|0\rangle_{\mathrm{NS}}$ is Grassmann-even with ghost number

$$|0\rangle_{\mathrm{NS}} : N_c = -1/2, \quad N_{sc} = 0. \tag{2.48}$$

The physical string field that satisfies eq. (2.37b) and the projection condition in eq. (2.39) now has the component expansion,

$$\Phi_{-1/2,0} = [A_\mu(x)d_{-1/2}^\mu + \widetilde{A}_\mu(x)d_{-3/2}^\mu + A_{\mu\nu}(x)a_{-1}^\mu d_{-1/2}^\nu$$

$$+A_{\mu\nu\lambda}(x)d_{-1/2}^\mu d_{-1/2}^\nu d_{-1/2}^\lambda + N_\mu(x)d_{-1/2}^\mu \gamma_{-1/2}\beta_{-1/2} + \cdots] |0\rangle_{\mathrm{NS}}, \tag{2.49}$$

where A_μ is a massless spin 1 field, $\widetilde{A}_\mu$, $A_{\mu\nu}$, $A_{\mu\nu\lambda}$ are massive physical modes, and N_μ is an auxiliary field. The vector field A_μ is even, since $|0\rangle_{\mathrm{NS}}$ is even, and $\Phi_{-1/2,0}$ is odd. Similarly, the higher mass level component fields in eq. (2.49) all have correct statistics.

The vacua for conformal (anti-)ghost and bosonic coordinate oscillators in the R sector are exactly the same as those in the NS sector, eqs. (2.41) and (2.45). Since the superconformal (anti-)ghosts have zero modes, the vacuum states are more complicated here; they are

$$\beta_0 \left|+\right\rangle_{sc} = \beta_n \left|+\right\rangle_{sc} = \gamma_n \left|+\right\rangle_{sc} = 0, \quad n > 0,$$

$$\gamma_0 \left|-\right\rangle_{sc} = \beta_n \left|-\right\rangle_{sc} = \gamma_n \left|-\right\rangle_{sc} = 0, \quad n > 0 \qquad (2.50)$$

with ghost number

$$\left|+\right\rangle \; : \; N_{sc} = 1/2,$$

$$\left|-\right\rangle \; : \; N_{sc} = -1/2, \qquad (2.51)$$

and both are taken to be Grassmann-even.

The zero modes of fermionic coordinates have the same algebra as that of the ten-dimensional γ-matrices, eq. (2.5b), so the vacua have spinor structure. In ten-dimensional space, there are two independent Majorana–Weyl spinors of opposite chiralities. Then we have

$$d_n^\mu \left|S_L\right\rangle = 0, \quad d_n^\mu \left|S_R\right\rangle = 0, \quad n > 0, \qquad (2.52)$$

$$d_0^\mu \left|S_L\right\rangle = (1/\sqrt{2})\gamma^\mu \left|S_R\right\rangle, \quad d_0^\mu \left|S_R\right\rangle = (1/\sqrt{2})\gamma^\mu \left|S_L\right\rangle, \qquad (2.53)$$

where the spinor indices are implicit. Following reference [34], we adopt the convention that $\left|S_L\right\rangle$ is Grassmann-odd, and $\left|S_R\right\rangle$ is even. Now we can define Fock-space

vacuum states in the R sector,

$$|S_L\rangle_{\rm R} \equiv |0\rangle_a \, |S_L\rangle \, |-\rangle_c \, |+\rangle_{sc} \,, \tag{2.54a}$$

$$|S_R\rangle_{\rm R} \equiv |0\rangle_a \, |S_R\rangle \, |-\rangle_c \, |+\rangle_{sc} \tag{2.54b}$$

with ghost numbers

$$|S_L\rangle_{\rm R} \; : \; N_c = -1/2, \quad N_{sc} = 1/2, \tag{2.55a}$$

$$|S_R\rangle_{\rm R} \; : \; N_c = -1/2, \quad N_{sc} = 1/2. \tag{2.55b}$$

It also follows that $|S_L\rangle_{\rm R}$ is Grassmann-even, consequently $|S_R\rangle_{\rm R}$ is Grassmann-odd.

The physical string field that satisfies eq. (2.38b) and the projection condition in eq. (2.40) can be expanded as

$$\Phi_{-1/2,1/2} = \psi(x) \, |S_L\rangle_{\rm R} + \psi_1^\mu(x) a_{-1}^\mu \, |S_L\rangle_{\rm R} + \psi_2^\mu(x) d_{-1}^\mu \, |S_R\rangle_{\rm R}$$

$$+ L(x)\gamma_0\beta_{-1} \, |S_L\rangle_{\rm R} + \cdots, \tag{2.56}$$

where $\psi(x)$ is a massless left-handed spin 1/2 field, $\psi_1^\mu(x)$ is a massive left-handed spin 3/2 field, $\psi_2^\mu(x)$ is a massive right-handed spin 3/2 field, and $L(x)$ is an auxiliary field. All the spinor indices are implicit in eq. (2.56). Because $\Phi_{-1/2,1/2}$ is Grassmann-odd, the three fields $\psi(x)$, $\psi_1^\mu(x)$, and $\psi_2^\mu(x)$, as well as higher mass level fields including auxiliary fields in expansion (2.56) are all Grassmann-odd, therefore, they have correct statistics.

The equations for physical string fields in (2.37) and (2.38) are

$$Q^{\mathrm{NS}}\Phi_{-1/2,0} = 0\,, \quad b_0\Phi_{-1/2,0} = 0\,, \qquad (2.57a,b)$$

and

$$Q^{\mathrm{R}}\Phi_{-1/2,1/2} = 0\,, \quad b_0\Phi_{-1/2,1/2} = 0\,. \qquad (2.58a,b)$$

We can expand the string field in NS sector as

$$\Phi_{-1/2,0} = \phi^{\mathrm{NS}}\,|0\rangle_{\mathrm{NS}} + \widetilde{\phi}^{\mathrm{NS}}c_0\,|0\rangle_{\mathrm{NS}}\,, \qquad (2.59)$$

where ϕ^{NS} has ghost number $(0, 0)$. Plugging eq. (2.59) into eq. (2.57a), we obtain

$$(L_0 - 1/2)\phi^{\mathrm{NS}} = 0\,, \quad \bar{Q}^{\mathrm{NS}}\phi^{\mathrm{NS}} = 0\,, \quad \widetilde{\phi}^{\mathrm{NS}} = 0\,, \qquad (2.60a,b,c)$$

which determine the spectrum of physical excitations in the NS sector contained in ϕ^{NS} [30]. For the R sector the expansion is

$$\Phi_{-1/2,1/2} = |\Psi\rangle + |\widetilde{\Psi}\rangle\,, \qquad (2.61)$$

where

$$|\Psi\rangle = \Psi_L\,|S_L\rangle_{\mathrm{R}} + \Psi_R\,|S_R\rangle_{\mathrm{R}}\,, \quad |\widetilde{\Psi}\rangle = \widetilde{\Psi}_Lc_0\,|S_L\rangle_{\mathrm{R}} + \widetilde{\Psi}_Rc_0\,|S_R\rangle_{\mathrm{R}}\,, \qquad (2.62a,b)$$

and

$$\Psi_L = \sum_n (\gamma_0)^n\Psi_L^{(n)}\,, \quad \Psi_R = \sum_n (\gamma_0)^n\Psi_R^{(n)}\,. \qquad (2.63a,b)$$

In eq. (2.63), $\Psi_L^{(0)}$ and $\Psi_R^{(0)}$ have ghost number $(0, 0)$, and all other component fields have nonzero ghost numbers. Therefore the physical excitations in the R sector

should have the form

$$\Psi_L^{(0)} \neq 0, \quad \Psi_R^{(0)} \neq 0, \quad \Psi_L^{(n)} = \Psi_R^{(n)} = 0, \quad n \neq 0. \tag{2.64}$$

Inserting eqs. (2.61)–(2.64) into eq. (2.58b), we have

$$F_0|\Psi^{(0)}\rangle = 0, \quad \bar{Q}^{\mathrm{R}}|\Psi^{(0)}\rangle = 0, \tag{2.65}$$

where

$$|\Psi^{(0)}\rangle = \Psi_L^{(0)}|S_L\rangle_{\mathrm{R}} + \Psi_R^{(0)}|S_R\rangle_{\mathrm{R}} .$$

Equations in (2.65) are the conditions satisfied by the physical excitations in the R sector [28].

From the previous discussions we see that the generalized GSO projections (see eqs. (2.39) and (2.40)) not only lead to the same number of physical states at each mass level in the NS and R sectors, they also ensure that each field has the correct statistics. So far we have only discussed superstring field theory at the classical level. Just like in bosonic string field theory [8, 11–13], it would suffice to use classical superstring field if we only want to calculate superstring amplitudes perturbatively. We will do some explicit calculations in section 2.4. However, the superstring field theory in eq. (2.29) can be second quantized in very much the same way as that of bosonic string field theory, which will be the subject of the next section.

2.3. Gauge-fixing and second quantization

Witten's original superstring field theory has been formally second quantized [35] under the assumption that superstring multiplication is well defined and the action is gauge invariant. Since the modified theory in eq. (2.29) satisfies these two conditions, the procedure in [35] is actually applicable here. In what follows, however, an approach parallel to that of section 2.1 will be taken. We observe first that the physical superstring field in eq. (2.28) is formally similar to the physical

string field $\Phi_{-1/2}$ of bosonic string theory (the same notation is used for both cases) except the field content is different. Furthermore the superstring action (eq. (2.29)), gauge transformation (eq. (2.30)) and gauge-fixing condition (eq. (2.34)) all have corresponding similarities to their counter part in bosonic string field theory (see section 2.1). We can just borrow the relevant formulae from there. We introduce ghost superstring fields

$$\Phi_{-3/2}, \ \Phi_{-5/2}, \ \Phi_{-7/2}, \cdots \tag{2.66}$$

as well as antighost superstring fields

$$\Phi_{1/2}, \ \Phi_{3/2}, \ \Phi_{5/2}, \cdots , \tag{2.67}$$

where each field has conformal ghost number as indicated and all superconformal ghost numbers, namely,

$$\Phi_s = \sum_n \Phi_{s,n} \quad \text{for} \quad s = -\frac{3}{2}, \ -\frac{5}{2}, \cdots, \frac{1}{2}, \frac{3}{2}, \cdots . \tag{2.68}$$

All of the string field is required to be Grassmann-odd. The succession of ghost and antighost string fields are necessary for complete gauge fixing [15, 19]. As has already been done in eq. (2.27) we can use a more compact notation

$$\Phi = \sum_s \Phi_s ,$$

where Φ_s is defined in eq. (2.68).

We can adopt a procedure similar to that of the bosonic string field theory and find the gauge-fixed BRST-invariant action

$$I_{\mathrm{GF}} = \frac{1}{2} \int \left[\Phi * Q\Phi + \frac{2}{3} \Phi * \Phi * \Phi - 2(b_0 B) * \Phi \right] , \tag{2.69}$$

where B is a Lagrange multiplier enforcing the gauge condition

$$b_0 \Phi = 0, \tag{2.70}$$

and B has all conformal and superconformal ghost numbers. The equations of motion

from eq. (2.69) is

$$Q\Phi + \Phi * \Phi - b_0 B = 0 \,, \tag{2.71}$$

which have the same form as the equation for the bosonic open string field theory. The BRST transformation that leaves gauge-fixed action (eq. (2.69)) invariant is found to be

$$\delta\Phi_+ = (b_0 B)_+ \,,$$
$$\delta\Phi_- = (Q\Phi + \Phi * \Phi + b_0 B)_- \,,$$
$$\delta B = 0 \,, \tag{2.72}$$

which is also nilpotent

$$\delta^2\Phi_+ = 0, \quad \delta^2\Phi_- = 0, \quad \delta^2 B = 0 \tag{2.73}$$

by iterating eq. (2.72) and using the equation of motion (2.71).

2.4. The Feynman rules and string amplitudes

In this section we will find the propagators for NS and R states and interaction vertices from the gauge-fixed action obtained in section 2.3. In principle, all the string amplitudes can thus be calculated perturbatively in string field theory. Four-point amplitudes for massless states are worked out as examples.

The kinetic term in the gauge-fixed action (eq. (2.69)) is

$$I_{\rm K} = \int (\Phi_{\rm NS} * Q^{\rm NS}\Phi_{\rm NS} + \Phi_{\rm R} * Q^{\rm R}\Phi_{\rm R}). \tag{2.74}$$

After imposing the gauge-fixing condition (eq. (2.70)), we have

$$\Phi_{\rm NS} = b_0 c_0 \Phi_{\rm NS}, \quad \Phi_{\rm R} = b_0 c_0 \Phi_{\rm R}. \tag{2.75}$$

Eqs. (2.2) and (2.6) imply

$$\{Q^{\text{NS}}, b_0\} = L_0 - 1/2, \quad \{Q^{\text{R}}, b_0\} = L_0. \tag{2.76}$$

From eqs. (2.75) and (2.76), we can derive the gauge-fixed form of eq. (2.74),

$$I_{\text{K}} = \int [\Phi_{\text{NS}} * c_0(L_0 - 1/2)\Phi_{\text{NS}} + \Phi_{\text{R}} * c_0 L_0 \Phi_{\text{R}}]. \tag{2.77}$$

The propagators are just the inverse of kinetic operators $c_0(L_0 - 1/2)$ and $c_0 L_0$ when acting on states (eq. (2.75)). They are easily found to be

$$T_{\text{NS}} = b_0 \frac{1}{L_0 - 1/2} = b_0 \int_0^\infty d\tau e^{-\tau(L_0 - 1/2)} \tag{2.78}$$

for NS states and

$$T_{\text{R}} = b_0 \frac{1}{L_0} = b_0 \int_0^\infty d\tau e^{-\tau L_0} \tag{2.79}$$

for R states. Analogous to the bosonic string field theory, the propagators T_{NS} and T_{R} have representations as a path integral on a strip of length τ with b_0 as a line integral across the strip. Then the parameter τ is integrated from 0 to ∞ at the end.

The three-string interaction vertex in the action (eq. (2.69)) is

$$I_{\text{int}} = \frac{2}{3} \int \Phi * \Phi * \Phi = \frac{2}{3} \int (\Phi_{\text{NS}} * \Phi_{\text{NS}} * \Phi_{\text{NS}} + 3\Phi_{\text{NS}} * \Phi_{\text{R}} * \Phi_{\text{R}}). \tag{2.80}$$

Like bosonic string field theory [5, 8, 11], an external physical state in superstring field theory is a path integral on a semi-infinite strip with the appropriate vertex operator inserted at infinity,

$$A[X(\sigma), \psi(\sigma), \rho(\sigma), \xi(\sigma), \eta(\sigma), \phi(\sigma)] =$$

$$\int DX D\psi D\rho D\xi D\eta D\phi \, e^{-(S_X + S_\psi + S_\rho + S_{\xi - \eta} + S_\phi)} V, \tag{2.81}$$

where $X(\sigma), \cdots, \phi(\sigma)$ are the boundary conditions for the corresponding fields on EF in Fig. 5. The string field for an external state must satisfies eq. (2.37) or (2.38).

Because the BRST current is conserved on the string world-sheet, a string field of the form (eq. (2.81)) is a solution of eq. (2.37) or (2.38) if the vertex operator V anticommutes with the BRST charge,

$$\{Q, V\} = 0 \,. \tag{2.82}$$

In the following, we will find the solutions of eqs. (2.37) and (2.38) that represent various physical string states. The string field $A^{\rm v}_{-1/2,0}$ which represent a massless vector particle in the NS sector is

$$A^{\rm v}_{-1/2,0}[X(\sigma), \cdots, \phi(\sigma)] = \int DX \cdots D\phi \, e^{-(S_X + \cdots + S_\phi)}$$

$$\times \left(\xi_\mu c^w e^{-\phi(w)} \psi^\mu(w) e^{2ip \cdot X(w)} \right) \tag{2.83}$$

with $\xi \cdot P = 0 \,, P^2 = 0$. Since

$$\left\{ Q^{\rm NS} \,, \left(\xi_\mu c^w e^{-\phi(w)} \psi^\mu(w) e^{2ip \cdot X(w)} \right) \right\} = 0 \,,$$

we find that $A^{\rm v}_{-1/2,0}$ satisfies the equations for physical string fields,

$$Q^{\rm NS} A^{\rm v}_{-1/2,0} = 0 \,, \quad b_0 A^{\rm v}_{-1/2,0} = 0 \,. \tag{2.84}$$

The string field $A^{\rm v}_{-1/2,0}$ is in the F_2-picture in the usual language. Up until now we have not made any use of the picture-changing operator [29]. In our formulation, the effect of picture change is that for a given string state there is a multitude of string fields with different superconformal ghost numbers; they are the solutions of the eqs. (2.37) or (2.38) and correspond to this same string state. For example,

$A^{\text{v}}_{-1/2,0}$ in eq. (2.83) is a representation of vector particle state, we also have

$$A^{\text{v}}_{-1/2,1}[X(\sigma),\cdots,\phi(\sigma)] = \int DX \cdots D\phi\, e^{-(S_X+\cdots+S_\phi)}$$

$$\times \left\{ Q\,,\xi\left(i\zeta_\mu c^w e^{-\phi(w)}\psi^\mu(w)e^{2iP\cdot X(w)}\right) \right\}\,, \tag{2.85}$$

with $\zeta\cdot P = P^2 = 0$, as a solution of equations

$$Q^{\text{NS}} A^{\text{v}}_{-1/2,1} = 0\,, \quad b_0 A^{\text{v}}_{-1/2,1} = 0\,, \tag{2.86}$$

where

$$\left\{ Q\,,\xi\left(i\zeta_\mu c^w e^{-\phi(w)}\psi^\mu(w)e^{2iP\cdot X(w)}\right) \right\} =$$

$$i\zeta_\mu c^w\left(\partial_w X(w) + 2iP\cdot\psi\psi^\mu\right)e^{2iP\cdot X(w)}\,.$$

The string field $A^{\text{v}}_{-1/2,1}$ is another representation of the vector particle state in a different picture.

Similarly, the string field that represents the massless fermion in the R sector is

$$A^{\text{f}}_{-1/2,1/2}[X(\sigma),\cdots,\phi(\sigma)] = \int DX \cdots D\phi\, e^{-(S_X+\cdots+S_\phi)}$$

$$\times \left(u^\alpha(P)c^w e^{-\phi(w)/2} S_{L\alpha}(w)e^{2iP\cdot X(w)} \right) \tag{2.87}$$

with $P\cdot\gamma u(P) = P^2 = 0$, which is a solution of equations,

$$Q^{\text{R}} A^{\text{f}}_{-1/2,1/2} = 0\,, \quad b_0 A^{\text{f}}_{-1/2,1/2} = 0\,. \tag{2.88}$$

In the NS and R sectors, for each physical state, we can find the corresponding string fields as the solutions of eq. (2.37) or (2.38) with various superconformal ghost numbers (namely, in different pictures). It should be noticed, as is clear from the discussion thus far, that the picture-changing effect in reference [29] is actually achieved in the superstring field theory described here by the solutions, with various superconformal ghost numbers, to the equations for physical string fields.

Taking into account ghost number conservations, especially superconformal ghost number conservation, we find, from eqs. (2.80), (2.83), (2.85) and (2.87), the three-vector particle vertex and the two-fermion one-vector particle vertex to be

$$\int \Phi_{NS} * \Phi_{NS} * \Phi_{NS} = 3 \int A^{v}_{-1/2,0} * A^{v}_{-1/2,0} * A^{v}_{-1/2,1} \qquad (2.89a)$$

and

$$3 \int \Phi_{NS} * \Phi_R * \Phi_R = 3 \int A^{v}_{-1/2,0} * A^{f}_{-1/2,1/2} * A^{f}_{-1/2,1/2}. \qquad (2.89b)$$

The three-string vertex has the world-sheet diagram sketched in Fig. 8a. The calculation is nearly the same as the bosonic theory. Inserting eqs. (2.83), (2.85) and (2.87) into eqs. (2.89a) and (2.89b), then followed by a conformal coordinate change,

$$\frac{dw}{dz} = \sqrt{3} \frac{\sqrt{z^2 + 1/3}}{z(z^2 - 1)},$$

to the upper half-plane as shown in Fig. 8b, we obtain the three-vector particle vertex

$$V(\xi_1, \xi_2, \xi_3) = (\xi_1 \cdot P_2 \xi_2 \cdot \xi_3 + \xi_2 \cdot P_3 \xi_3 \cdot \xi_1 + \xi_3 \cdot P_1 \xi_1 \cdot \xi_2),$$

and two-fermion one-vector particle vertex

$$V(u_1, \xi, u_2) = \bar{u}_1 \xi \cdot \gamma u_2,$$

which agree with the results in reference [36].

The Feynman rules for superstring field theory are similar to the bosonic case [5, 7, 15, 16]. In superstring field theory, however, the interaction vertex has conformal and superconformal ghost number anomalies $(3/2, -1)$, and the NS and R propagators both have ghost numbers $(-1, 0)$. In a Feynman diagram one must choose external states such that the total ghost numbers of the diagram are zero. For instance, to calculate the four-vector particle amplitude in superstring field theory, we choose four NS states that represent massless vector particles: $A_{-1/2,1}(1)$,

$A_{-1/2,0}(2)$, $A_{-1/2,0}(3)$, and $A_{-1/2,1}(4)$ as given in eqs. (2.83) and (2.85). Two of the Feynman diagrams are shown in Figs. 9a and 9b, there the dashed lines represent external NS states or propagators, while the solid lines represent external R states or propagators. In Figs. 9a and 9b the total conformal and superconformal ghost numbers from propagators, interaction vertices and external states add up to zero. Since external states and propagators are represented by strips, the superstring field theory diagrams have the same world-sheet geometry as the diagrams in the bosonic string field theory, the calculations are almost parallel to the bosonic case. The diagrams in Figs. 9a and 9b each contribute to half of the Koba–Nielsen integration range with the same integrand. Their sum is

$$A_{4\mathrm{B}} = A_{9a} + A_{9b} = \frac{\Gamma(2s)\Gamma(2t)}{\Gamma(1+2s+2t)} K(\xi_1, p_1; \xi_2, p_2; \xi_3, p_3; \xi_4, p_4), \qquad (2.90)$$

where $s = (p_1 + p_2)^2$, $t = (p_2 + p_3)^2$ and K is the kinetic factor. The amplitude agrees with the result in reference [36]. In the above calculation we have set $\alpha' = 2$. Since no picture-changing operator has been inserted in the interaction vertex, tree-level amplitudes are finite in this modified theory, therefore contact terms are no longer needed.

The four-fermion amplitude can be calculated in a similar fashion. We take four R states that correspond to the massless spin 1/2 fermion as is given in eq. (2.87). Two of the Feynman diagrams are shown in Figs. 10a and 10b. The sum of the two diagrams gives

$$A_{4\mathrm{F}} = A_{10a} + A_{10b} = \frac{\Gamma(2s)\Gamma(2t)}{\Gamma(1+2s+2t)} [t\bar{u}_1\gamma^\mu u_2\bar{u}_3\gamma_\mu u_4 - s\bar{u}_2\gamma^\mu u_3\bar{u}_1\gamma_\mu u_4], \quad (2.91)$$

where correlation functions of spin fields derived in reference [37] have been used.

Finally, we consider two-fermion two-boson amplitudes. The four external states are $A_{-1/2,1/2}(1)$, $A_{-1/2,1/2}(2)$, $A_{-1/2,0}(3)$, and $A_{-1/2,1}(4)$. Four of the diagrams are shown in Fig. 11a–Fig. 12b. The contributions of Figs. 11a and 11b are added up to

give

$$A_{2\text{B}-2\text{F}} = A_{11a} + A_{11b} = \frac{\Gamma(2s)\Gamma(2t)}{\Gamma(1 + 2s + 2t)}[-s\bar{u}_2\gamma \cdot \xi_3\gamma \cdot (p_1 + p_4)\gamma \cdot \xi_4 u_1$$

$$+2t(\xi_3 \cdot p_4\bar{u}_2\gamma \cdot \xi_4 u_1 - \xi_4 \cdot p_3\bar{u}_2\gamma \cdot \xi_3 u_1 - \xi_3 \cdot \xi_4\bar{u}_2\gamma \cdot p_4 u_1). \quad (2.92)$$

Similarly, the sum of diagrams in Figs. 12a and 12b is

$$A_{\text{FB}-\text{FB}} = A_{12a} + A_{12b} = \frac{\Gamma(2u)\Gamma(2t)}{\Gamma(1 + 2u + 2t)}[t\bar{u}_1\gamma \cdot \xi_3\gamma \cdot (p_2 + p_4)\gamma \cdot \xi_4 u_2+$$

$$u\bar{u}_1\gamma \cdot \xi_4\gamma \cdot (p_3 + p_2)\gamma \cdot \xi_3 u_2]. \quad (2.93)$$

where $u = (p_1 + p_3)^2$.

We have seen, from the explicit calculations, that all the massless four-particle amplitudes obtained from the modified open superstring field theory coincide with the previously known results [36]. Along the same line, it is straightforward to compute other tree amplitudes of the theory. It is still remained to be checked by explicit calculations, however, that whether the loop amplitudes come out correctly.

3. Cubic Actions for Open Strings

3.1 Cubic action for bosonic strings

In Witten's bosonic string action (eq. (1.10)) there is a background dependence since the BRST charge depends on background fields. A theory intended for a complete description of a physical process must include the dynamics of the background as well. The attempt to eliminate explicit background dependence of Witten's open bosonic string field theory has led to the discovery of purely cubic action [38]

$$S = \frac{1}{3} \int A * A * A, \qquad (3.1)$$

where the notation is as in Chapter 1. The field A with ghost number $-1/2$ is Grassmann-odd. The action is invariant under gauge transformation

$$A' = U^{-1} * A * U \qquad (3.2)$$

where

$$U = e^{\Lambda} = I + \Lambda + \frac{1}{2}\Lambda * \Lambda + \cdots. \qquad (3.3)$$

The fields U and Λ are Grassmann-even and have ghost number $-3/2$. The identity I in eq. (3.3) is a string field satisfying

$$I * B = B * I = B \quad \forall B, \qquad (3.4a)$$

$$QI = 0. \qquad (3.4b)$$

The field I has a path integral representation, in the sense of section 1.3, on the world-sheet shown in Fig. 2 without any vertex insertion. For an infinitesimal Λ,

eq. (3.2) has the form

$$\delta A = A' - A = A * \Lambda - \Lambda * A. \tag{3.5}$$

The equation of motion from the action (eq. (3.1)) is

$$\bar{A} * \bar{A} = 0. \tag{3.6}$$

For any solution $\bar{A}$, an operator $D_{\bar{A}}$ can be defined by

$$D_{\bar{A}} B \equiv \bar{A} * B - (-)^B B * \bar{A} \tag{3.7}$$

where $(-)^B = -1$ if B is odd and $+1$ if it is even. Then it follows from eqs. (1.9b), (3.6) and (3.7) that

$$\begin{aligned} (D_{\bar{A}})^2 &= 0, \\ \int D_{\bar{A}} B &= 0 \quad \forall B, \\ D_{\bar{A}}(A * B) &= (D_{\bar{A}} A) * B + (-)^A A * (D_{\bar{A}} B). \end{aligned} \tag{3.8}$$

To make a connection between the action in eqs. (3.1) and (1.10) we will first find a special solution of eq. (3.6). Let us consider the decomposition

$$Q = Q_{\mathrm{L}} + Q_{\mathrm{R}},$$

where

$$Q_{\mathrm{L}} = \int_0^{\pi/2} j_w \, dw, \quad Q_R = \int_{\pi/2}^{\pi} j_w \, dw$$

are the BRST current j_w integrated over the left- and right-half of the string.

Eq. (3.4b) then implies

$$Q_{\mathrm{R}} I = -Q_{\mathrm{L}} I \,.$$

(3.9)

From the operator product $j_z j_w$ of the BRST current, it can be checked that

$$\{Q,\, j_w\} = 0,$$

(3.10a)

therefore

$$\{Q,\, Q_{\mathrm{L}}\} = 0 \,.$$

(3.10b)

By the definition of the multiplication * (see Fig. 1) and the conservation of the BRST current on the string world-sheet, we can derive

$$(Q_{\mathrm{R}} A) * B = -(-)^A A * (Q_{\mathrm{L}} B) \quad,$$

(3.11)

as the consequence of the fact that the integral of the BRST current around a closed curve vanishes.

From eqs. (3.9) and (3.11), we obtain

$$(Q_{\mathrm{L}} I) * (Q_{\mathrm{L}} I) = Q_{\mathrm{R}} Q_{\mathrm{L}} I$$

and

$$(Q_{\mathrm{L}} I) * (Q_{\mathrm{L}} I) = -(Q_{\mathrm{R}} I) * (Q_{\mathrm{L}} I) = Q_{\mathrm{L}}^2 I$$

which implies

$$(Q_{\mathrm{L}} I) * (Q_{\mathrm{L}} I) = \frac{1}{2} Q Q_{\mathrm{L}} I = \frac{1}{2} \{Q, Q_{\mathrm{L}}\} I = 0 \quad.$$

(3.12)

Eq. (3.12) simply says that $Q_L I$ is a solution of eq. (3.6). Using eqs. (3.9) and (3.11)

again, we have

$$D_{Q_L I} A = (Q_L I) * A - (-)^A A * (Q_L I) = QA \quad , \quad \forall A \quad . \tag{3.13}$$

Now we can expand around the classical solution

$$A = \bar{A} + \tilde{A} = Q_L I + \tilde{A}$$

and insert it into eq. (3.1), the result is

$$\begin{aligned}
S &= \frac{1}{2} \int (\tilde{A} * D_{Q_L I} \tilde{A} + \frac{2}{3} \tilde{A} * \tilde{A} * \tilde{A}) \\
&= \frac{1}{2} \int (\tilde{A} * Q\tilde{A} + \frac{2}{3} \tilde{A} * \tilde{A} * \tilde{A})
\end{aligned} \tag{3.14}$$

where the eq. (3.13) has been used. The gauge invariances in eqs. (3.5) and (3.2) become

$$\delta \tilde{A} = Q\Lambda + \tilde{A} * \Lambda - \Lambda * \tilde{A} \quad ,$$
$$A' = U^{-1} * QU + U^{-1} * A * U \quad .$$

So we have derived the Chern–Simon-type action (see eq. (1.10)) from the cubic action of bosonic strings. The action in eq. (3.1) was also taken as a starting point to discuss the closed string states in bosonic open string field theory [39].

3.2 Cubic action for superstrings

The extension of the cubic action to superstring theory were made in reference [40] based on Witten's original formulation of the open superstring theory. As the superstring multiplication was not well defined [31] in its original form, neither was the superstring cubic action. However, a well-defined superstring cubic action

[41] can be constructed in the modified open superstring field theory [32]. The superstring cubic action is

$$S = \frac{1}{3} \int \Phi_{-1/2} * \Phi_{-1/2} * \Phi_{-1/2} \qquad (3.15)$$

where the notation of section 2.1 is used. The identity $\hat{I}$ of the superstring field theory is simply a generalization of the bosonic case,

$$\hat{I} * \Phi = \Phi * \hat{I} = \Phi \quad \forall \Phi, \qquad (3.16a)$$

$$Q^{\mathrm{NS}} \hat{I} = 0 . \qquad (3.16b)$$

Since $\hat{I}$ has conformal ghost number $-3/2$ and superconformal ghost number 1, it is a NS field. The gauge invariance of the action (eq. (3.15)) is

$$\Phi'_{-1/2} = U^{-1} * \Phi_{-1/2} * U \qquad (3.17a)$$

with

$$U = \hat{I} + \Omega_{-3/2} + \frac{1}{2} \Omega_{-3/2} * \Omega_{-3/2} + \cdots . \qquad (3.17b)$$

If $\Omega_{-3/2}$ is infinitesimal, eq. (3.17a) reduces to

$$\delta \Phi_{-1/2} = \Phi'_{-1/2} - \Phi_{-1/2} = \Phi_{-1/2} * \Omega_{-3/2} - \Omega_{-3/2} * \Phi_{-1/2} . \qquad (3.18)$$

In addition to gauge invariance the action is invariant under supersymmetry transformation

$$\delta \Phi_{-1/2} = Q_\alpha \Phi_{-1/2} \qquad (3.19)$$

with the supersymmetry charge Q_α given in eq. (2.20). The equation of motion is

$$\hat{\Phi}_{-1/2} * \hat{\Phi}_{-1/2} = 0. \qquad (3.20)$$

For any solution $\hat{\Phi}_{-1/2}$ we can define a superstring operation

$$D_{\hat{\Phi}_{-1/2}} \Phi \equiv \hat{\Phi}_{-1/2} * \Phi - (-)^{\Phi} \Phi * \hat{\Phi}_{-1/2} \qquad (3.21)$$

where $(-)^{\Phi} = -1$ (or $+1$) if Φ is odd (or even), The relations (3.8), (3.10), (3.11)

and (3.12) as well as their derivations can all be generalized to superstring theory. So we have

$$(D_{\hat{\Phi}_{-1/2}})^2 = 0\,,$$

$$\int D_{\hat{\Phi}_{-1/2}} \Phi = 0 \quad \forall \Phi,$$

$$D_{\hat{\Phi}_{1/2}}(\Phi * \Phi') = (D_{\hat{\Phi}_{-1/2}} \Phi) * \Phi' + (-)^{\Phi}\Phi * (D_{\hat{\Phi}_{-1/2}} \Phi')\,, \tag{3.22}$$

$$\{Q, Q_{\mathrm{L}}\} = 0, \tag{3.23}$$

$$(Q_{\mathrm{R}}\Phi) * \Phi' = -(-)^{\Phi}\Phi * (Q_{\mathrm{L}}\Phi'), \tag{3.24}$$

and

$$(Q_{\mathrm{L}}^{\mathrm{NS}}\hat{I}) * (Q_{\mathrm{L}}^{\mathrm{NS}}\hat{I}) = -\frac{1}{2}Q^{\mathrm{NS}}Q_{\mathrm{L}}^{\mathrm{NS}}\hat{I} = \{Q^{\mathrm{NS}}, Q_{\mathrm{L}}^{\mathrm{NS}}\}\hat{I} = 0. \tag{3.25}$$

In eq. (3.25), $Q_{\mathrm{L}}^{\mathrm{NS}}\hat{I}$ is an NS field and a solution of eq. (3.20). From eq. (3.24) it follows that

$$D_{Q_{\mathrm{L}}^{\mathrm{NS}}\hat{I}}\Phi = (Q_{\mathrm{L}}^{\mathrm{NS}}\hat{I}) * \Phi - (-)^{\Phi}\Phi * (Q_{\mathrm{L}}^{\mathrm{NS}}\hat{I}) = Q\Phi. \tag{3.26}$$

The expansion

$$\Phi_{-1/2} = \hat{\Phi}_{-1/2} + \tilde{\Phi}_{-1/2} = Q_{\mathrm{L}}^{\mathrm{NS}}\hat{I} + \tilde{\Phi}_{-1/2} \tag{3.27}$$

then gives

$$\begin{aligned} S &= \frac{1}{2}\int \left(\tilde{\Phi}_{-1/2} * D_{Q_{\mathrm{L}}^{\mathrm{NS}}\hat{I}}\tilde{\Phi}_{-1/2} + \frac{2}{3}\tilde{\Phi}_{-1/2} * \tilde{\Phi}_{-1/2} * \tilde{\Phi}_{-1/2}\right) \\ &= \frac{1}{2}\int \left(\tilde{\Phi}_{-1/2} * Q\tilde{\Phi}_{-1/2} + \frac{2}{3}\tilde{\Phi}_{-1/2} * \tilde{\Phi}_{-1/2} * \tilde{\Phi}_{-1/2}\right) \end{aligned} \tag{3.28}$$

with gauge invariance

$$\delta\tilde{\Phi}_{-1/2} = Q\Omega_{-3/2} + \tilde{\Phi}_{-1/2} * \Omega_{-3/2} - \Omega_{-3/2} * \tilde{\Phi}_{-1/2}\,, \tag{3.39a}$$

$$\Phi'_{-1/2} = U^{-1} * QU + U^{-1} * \Phi_{-1/2} * U. \tag{3.29b}$$

From the operator product $j_\alpha(z)\, j_{\mathrm{BRST}}(w)$ of supersymmetry charge current $j_\alpha(z)$ and BRST charge current $j_{\mathrm{BRST}}(w)$ it can be checked that

$$[Q_\alpha,\, j_{\mathrm{BRST}}(w)] = 0, \tag{3.30a}$$

and consequently

$$[Q_\alpha,\, Q_{\mathrm{L}}] = 0. \tag{3.30b}$$

Since the supersymmetry charge current is conserved on world-sheet [6, 29], we have

$$Q_\alpha \hat{I} = 0\,. \tag{3.31}$$

Eqs. (3.30b) and (3.31) imply

$$Q_\alpha(Q_{\mathrm{L}}^{\mathrm{NS}}\hat{I}) = [Q_\alpha,\, Q_{\mathrm{L}}]\hat{I} = 0, \tag{3.32}$$

which means that the background $Q_{\mathrm{L}}^{\mathrm{NS}}\hat{I}$ is invariant under supersymmetry transformation (eq. (3.19)). From eqs. (3.32), (3.19) and (3.27) we obtain

$$\delta\tilde{\Phi}_{-1/2} = Q_\alpha\tilde{\Phi}_{-1/2} \tag{3.33}$$

as a global symmetry of the action (3.28).

It is clear, by comparing eqs. (3.28), (3.29) and (3.33) with eqs. (2.29), (2.30) and (2.32), that we have derived the results of section 2.1 from our cubic action in eq. (3.15) by expanding around the background field $Q_{\mathrm{L}}^{\mathrm{NS}}\hat{I}$. Further progress in understanding open string field theory can be made by exploiting the cubic action in eqs. (3.1) and (3.15). For example, it would be important to find and classify other nontrivial solutions of eqs. (3.6) and (3.20), and study the closed string states in eq. (3.15). It would be very interesting to find out how far this can lead us.

Appendix

In this appendix we give a brief derivation of eqs. (1.61b) and (1.62). All the formulae involving elliptic integrals can be found in reference [23].

Bw_1 in Fig. 6a corresponds to Bz_1, in Fig. 6b. Using eqs. (1.52b) and (1.57), we have the integral

$$
\tau_1 = N \int_0^{z_1} dz \frac{\sqrt{z^2 + \gamma^2}\sqrt{z^2 + 1/\gamma^2}}{(z^2 - \alpha^2)(z^2 - 1/\alpha^2)}
$$

$$
= N \int_0^{z_1} \frac{dz}{\sqrt{z^2 + \gamma^2}\sqrt{z^2 + 1/\gamma^2}} \left[1 + \frac{(\alpha^2 + \gamma^2)(\alpha^2 + 1/\gamma^2)}{1 - \alpha^4} \right.
$$

$$
\left. \left(\frac{\alpha^2}{\alpha^2 - z^2} \frac{1/\alpha^2}{1/\alpha^2 - z^2} \right) \right] \tag{A.1}
$$

where

$$
N = \frac{2(1 - \alpha^4)}{\alpha\sqrt{\alpha^2 + \gamma^2}\sqrt{\alpha^2 + 1/\gamma^2}}.
$$

To evaluate eq. (A.1), we first work out the following integrals from the tables in reference [23].

$$
I_1 = \int_0^{z_1} \frac{dz}{\sqrt{z^2 + \gamma^2}\sqrt{z^2 + 1/\gamma^2}} = \gamma F(\beta, k'), \tag{A.2}
$$

where $\tan \beta = z_1/\gamma$, $k'^2 = 1 - k^2$, $k = \gamma^2$.

$$
I_2 = \int_0^{z_1} \frac{dz}{\sqrt{z^2 + \gamma^2}\sqrt{z^2 + 1/\gamma^2}} \left(\frac{\alpha^2}{\alpha^2 - z^2} \right)
$$

$$
= (-i\gamma) \int_0^t \frac{dt}{\sqrt{1 - t^2}\sqrt{1 - k^2 t^2}} \left(\frac{1}{1 + \alpha_1^2 t^2} \right)
$$

$$= (-i\gamma)\left[\frac{\alpha^2}{\alpha^2 + \gamma^2}F(i\phi, k) - \frac{\pi}{2K}N_1\right.$$

$$\left.\times\left(F(i\phi, k)\Lambda_0(\beta_2, k) - \frac{iK}{\pi}\ln\frac{\theta_1(\nu + iw)}{\theta_1(-\nu + iw)}\right)\right] \qquad (A.3)$$

where

$$z = -i\gamma t, \quad t_1 = i\frac{z_1}{\gamma}, \quad \alpha_1^2 = \frac{\gamma^2}{\alpha^2},$$

$$N_1 = \frac{\alpha}{\sqrt{\alpha^2 + \gamma^2}\sqrt{\alpha^2 + 1/\gamma^2}}, \quad \sinh\phi = z_1/\gamma, \quad \sin\beta_2 = \frac{\alpha}{\sqrt{\alpha^2 + \gamma^2}},$$

$$K = K(k), \quad \nu = \frac{\pi}{2K}F(i\phi, k), \quad w = \frac{\pi}{2K}F(\beta_2, k').$$

We have two identities

$$F(i\phi, k) = iF(\beta, k'),$$

$$\Lambda_0(\beta_1, k) + \Lambda_0(\beta_2, k) = 1 + \frac{2}{\pi}(1 - k^2)KN_1,$$

and a constraint from eq. (1.25a)

$$\Lambda_0(\beta_1, k) - \Lambda_0(\beta_2, k) = \frac{1}{2},$$

where

$$\sin\beta_1 = \frac{1}{(1 + \alpha^2\gamma^2)^{1/2}}, \quad \sin\beta_2 = \frac{\alpha}{(\alpha^2 + \gamma^2)^{1/2}}, \quad \tan\beta = \sinh\phi = z_1/\gamma.$$

Using the identities and the constraint, I_2 can be rewritten as

$$I_2 = \gamma\left[\frac{\alpha^2}{\alpha^2 + \gamma^2} - \frac{1}{2}(1 - k^2)N_1\right]F(\beta, k') - \frac{\pi\gamma}{8K}N_1F(\beta, k')$$

$$+ \frac{\gamma N_1}{2}\ln\frac{\theta_1\left[\frac{i\pi}{2K}(F(\beta_2, k') + F(\beta, k'))\right]}{\theta_1\left[\frac{i\pi}{2K}(F(\beta_2, k') - F(\beta, k'))\right]}. \qquad (A.4)$$

Similarly,

$$I_3 = \int\limits_0^{z_1} \frac{dz}{\sqrt{z^2 + \gamma^2}\sqrt{z^2 + 1/\gamma^2}} \left(\frac{1/\alpha^2}{1/\alpha^2 - z^2} \right)$$

$$= \alpha \left[\frac{\gamma^2}{\alpha^2 + \gamma^2} + \frac{1}{2}(1 - k^2)N_1 \right] F(\beta, k') + \frac{\pi\gamma}{8K}N_1 F(\beta, k')$$

$$+ \frac{\gamma N_1}{2} \ln \frac{\theta_0 \left[\frac{i\pi}{2K}(F(\beta_2, k') - F(\beta, k')) \right]}{\theta_0 \left[\frac{i\pi}{2K}(F(\beta_2, k') + F(\beta, k')) \right]} . \tag{A.5}$$

From eqs. $(A.1)$–$(A.5)$ we have

$$\tau_1 = N \left[I_1 + \frac{(\alpha^2 + \gamma^2)(\alpha^2 + 1/\gamma^2)}{1 - \alpha^4}(I_2 - I_3) \right]$$

$$= -\frac{\pi}{2K}F(\beta, k')$$

$$+ \ln \left\{ \frac{\theta_1 \left[\frac{i\pi}{2K}(F(\beta_2, k') + F(\beta, k')) \right]}{\theta_1 \left[\frac{i\pi}{2K}(F(\beta_2, k') - F(\beta, k')) \right]} \cdot \frac{\theta_0 \left[\frac{i\pi}{2K}(F(\beta_2, k') + F(\beta, k')) \right]}{\theta_0 \left[\frac{i\pi}{2K}(F(\beta_2, k') - F(\beta, k')) \right]} \right\} . \tag{A.6}$$

The limit $z_1 \to \alpha$ implies $\beta \to \beta_2$, and

$$F(\beta_2, k') - F(\beta, k') = \frac{(\alpha - z_1)}{\sqrt{\alpha^2 + \gamma^2}\sqrt{1 + \alpha^2\gamma^2}}. \tag{A.7}$$

Inserting eq. $(A.7)$ into eq. $(A.6)$, we obtain

$$\tau_1 = -\frac{\pi}{2K}F(\beta, k') + \ln \frac{\theta_1 \left[(\frac{i\pi}{K})F(\beta, k') \right] \theta_0 \left[(\frac{i\pi}{K})F(\beta, k') \right] 2\alpha K}{\theta_0[0]\theta'_1[0]i\pi N_1} - \ln(\alpha - z_1) , \tag{A.8}$$

Comparison of eq. $(A.8)$ with eqs. (1.59) and $(1.61a)$ then yields

$$\tau_1 = \ln C_1 - \ln(\alpha - z_1) \tag{A.9}$$

and

$$\ln C = -\frac{\pi}{2K}F(\beta, k') + \ln \frac{\theta_1\left[\left(\frac{i\pi}{K}\right)F(\beta, k')\right]\theta_0\left[\left(\frac{i\pi}{K}\right)F(\beta, k')\right]2K}{\theta_0[0]\theta_1'[0]N_1 i\pi}, \qquad (A.10)$$

where

$$N_1 = \frac{\alpha}{\sqrt{\alpha^2 + \gamma^2}\sqrt{\alpha^2 + 1/\gamma^2}}, \quad \sin\beta = \frac{\alpha}{\sqrt{\alpha^2 + \gamma^2}}.$$

C is a function of α only, since γ depends on α through the constraint (eq. $(4a)$).

In the same way,

$$\tau_4 = -N \int\limits_{\infty}^{z_4} dz \frac{\sqrt{z^2 + \gamma^2}\sqrt{z^2 + 1/\gamma^2}}{(z^2 - \alpha^2)(z^2 - 1/\alpha^2)}.$$

Changing the variable $z \to 1/z$ and then comparing with eqs. $(A.1)$ and $(A.9)$, we have, in the limit $z_4 \to 1/\alpha$,

$$\tau_4 = N \int\limits_{0}^{1/z_4} dz \frac{\sqrt{z^2 + \gamma^2}\sqrt{z^2 + 1/\gamma^2}}{(z^2 - \alpha^2)(z^2 - 1/\alpha^2)} = \ln C_1 - \ln(\alpha - 1/z_4)$$

$$= \ln C_1 - \ln\alpha^2 - \ln(z_4 - 1/\alpha). \qquad (A.11)$$

This actually implies

$$\ln C_4 = \ln C_1 - \ln\alpha^2,$$

and

$$C_4 = \frac{1}{\alpha}C. \qquad (A.12)$$

In the remainder of this appendix, we evaluate C in the limit $\alpha \to 0$. After

changing variable $z = \gamma t$ and expanding to order k^2, eq. $(A.1)$ becomes

$$\tau_1 = \gamma N \left[\int\limits_0^{P_1} dt \frac{\sqrt{1 + t^2}}{1 - t^2/\rho}(1 + N_2) - N_2 \int\limits_0^{P_1} dt\sqrt{1 + t^2} \right], \qquad (A.13)$$

where $\rho = z/\gamma$, $\rho_1 = z_1/\gamma$, $N_2 = (1/2 + \rho^2)\rho^2 k^2$. Eq. $(A.13)$ can be evaluated to give

$$\ln C = \frac{-2\rho}{\sqrt{1 + \rho^2}}(1 + 1/2(1/2 + \rho^2)k^2)\ln(\rho + \sqrt{1 + \rho^2})$$

$$+ \ln(2 + 2\rho^2) - (1/2 + \rho^2)\rho^2 k^2. \qquad (A.14)$$

The constraint (eq. $(1.25a)$) is actually an integral (see reference [8]),

$$\frac{1}{2}\pi i = N \int\limits_0^{i\gamma} dz \frac{\sqrt{z^2 + \gamma^2}\sqrt{z^2 + 1/\gamma^2}}{(z^2 - \alpha^2)(z^2 - 1/\alpha^2)}. \qquad (A.15)$$

We change the variable by $z = i\gamma t$ and evaluate the integral in eq. $(A.15)$ to order k^2, then the result is

$$\frac{1}{2} = N\gamma\rho^2 \left[-\frac{1}{4}N_3 - \frac{1}{2}(1 + N_3\rho^2) + (1 + N_3\rho^2)(1 + \rho^2)^{1/2}\frac{1}{2\rho} \right], \qquad (A.16)$$

where $N_3 = (1/2 + \rho^2)k^2$. Substituting $\rho = \rho_0(1 + \eta k^2)$ into eq. $(A.16)$, the k^0 and k^2 terms yield

$$\rho_0 = \frac{1}{\sqrt{3}} \quad \text{and} \quad \eta = -\frac{5}{9}.$$

Inserting

$$\rho = \frac{1}{\sqrt{3}} \left(1 - \frac{5}{9}k^2 \right)$$

and

$$1 - x = 4\alpha^2 = 4\rho^2 k \simeq \frac{4}{3}k$$

into eq. $(A.14)$, we obtain

$$\ln C = \frac{3}{2}\ln\left(\frac{4}{3}\right) - \frac{5}{32}(1-x)^2 + \cdots,$$

$$C(x) = \left(\frac{4}{3}\right)^{3/2}\left[1 - \frac{5}{32}(1-x)^2 + \cdots\right].$$
\hfill $(A.17)$

References

1. M.B. Green, J.H. Schwarz and E. Witten, *Superstring Theory*, Vol. 1 & 2, (Cambridge University Press, Cambridge, 1987).

2. J.H. Schwarz, ed., *Superstrings: the first 15 years of superstring theory*, Vol. 1 & 2, (World Scientific, Singapore, 1985).

3. S. Mandelstam, *Nucl. Phys.* **B64** (1973) 205; **B69** (1974) 209;
   M. Kaku and K. Kikkawa, *Phys. Rev.* **D 10** (1974) 1110, 1823;
   E. Gremmer and J.L. Gervais, *Nucl. Phys.* **B74** (1974).

4. W. Siegel, *Phys. Lett.* **149B** (1984) 157, 162; **151B** (1985) 391, 396.

5. E. Witten, *Nucl. Phys.* **B268**(1986) 253.

6. E. Witten, *Nucl. Phys.* **B276** (1986) 291.

7. S. Giddings and E. Martinec, *Nucl. Phys.* **B278** (1986) 91;
   S. Giddings, E. Martinec and E. Witten, *Phys. Lett.* **176B** (1986) 362.

8. S. Giddings, *Nucl. Phys.* **B278** (1986) 242.

9. A.M. Polyakov, *Phys. Lett.* **B103** (1981) 207.

10. D.J. Gross and P.F. Mende, *Phys. Lett.* **B197** (1987) 129; *Nucl. Phys.* **B303** (1988) 407.

11. J. Feng, Caltech preprint CALT-68-1479 (revised), to appear in *Nucl. Phys.* B.

12. J. Sloan, *Nucl. Phys.* **B302** (1988) 349.

13. S. Samuel, *Nucl. Phys.* **B308** (1988) 285.

14. D. Gross and A. Jevicki, *Nucl. Phys.* **B283** (1987) 1; **B287** (1987) 225; **B293** (1987) 29;
    E. Cremmer, A. Schwimmer, and C. Thorn, *Phys. Lett.* **B179** (1986) 57;
    S. Samuel, *Phys. Lett.* **B181** (1986) 255.

15. C.B. Thorn, *Nucl. Phys.* **B287** (1987) 61.

16. M. Bochicchio, *Phys. Lett.* **188B** (1987) 330; *Phys. Lett.* **193B** (1987) 31.

17. A.R. Bogojevic, *Phys. Lett.* **198B** (1987) 479.

18. M. Kato and K. Ogawa, *Nucl. Phys.* **B212** (1983) 443.

19. W. Siegel and B. Zwiebach, *Nucl. Phys.* **B263** (1986) 105;
    T. Banks and M. Peskin, *Nucl. Phys.* **B264** 91986) 513;
    A. Neveu, H. Nicolai, and P.C. West, *Phys. Lett.* **167B** (1986) 307;
    M. Kaku, *Phys. Lett.* **162** (1985) 97.

20. H. Hata, K. Itoh, T. Kugo, H. Kumitomo and K. Ogawa, Phys. Lett. **B172** (1986) 186; *Phys. Rev.* **D34** (1986) 2306;
    A. Neveu and P. West, *Phys. Lett.* **B168** (1986) 192.

21. W.I. Weisberger, *Nucl. Phys.* **B284** (1987) 171.

22. D. Mumford, *Tata Lecture On Theta II*, Progress in Mathematics, **Vol. 43** (Birkhauser, Boston, 1984).

23. P.F. Byrd and M.D. Friedman, *Handbook of elliptic integrals for engineers and scientists*, 2nd ed., revised (Springer-Verlag, Berlin, 1971).

24. D. Friedan, Introduction to Polyakov's string theory, in *Recent advances in field theory and statistical mechanics*, eds. J.B. Zuber and R. Stora (North Holland, Amsterdam, 1983);
    O. Alvarez, *Nucl. Phys.* **B216** (1983) 125.

25. A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov, *Nucl. Phys.* **B241** (1984) 333.

26. P. Ramond, *Phys. Rev.* **D3** (1971) 2415;
    A. Neveu and J.H. Schwarz, *Nucl. Phys.* **B31** 91971) 86.

27. H. Terao and S. Uehara, *Phys. Lett.* **168B** (1986) 70;
    H. Ooguri, *Phys. Lett.* **B172** (1986) 204.

28. N. Ohta, *Phys. Rev.* **D33** (1986) 1681;
    M. Ito, T. Morozumi, S. Nojiri and S. Uehara, Prog. Theor. Phys. **75** (1986) 934.

29. D. Friedan, E. Martinec and S. Shenker, *Nucl. Phys.* **B271** (1986) 93.

30. Y. Cai and A. He, *Phys. Lett.* **196B** (1987) 60.

31. C. Wendt, *Nucl. Phys.* **B314** (1989) 209.

32. J. Feng, Caltech preprint CALT-68-1507 (revised, 1989).

33. F. Gliozzi, J. Scherk, and D. Olive, *Nucl. Phys.* **B122** (1977) 253.

34. A. LeClair, *Phys. Lett.* **168B** (1986) 53;
    J.P. Yamaron, *Phys. Lett.* **B174** (1986) 69.

35. M. Bochicchio, *Phys. Lett.* **198B** (1987) 475.

36. J.H. Schwarz, Phys. Rep. **89** (1982) 223.

37. V.G. Knizhnik, *Phys. Lett.* **160B** (1985) 403;
    J. Cohn, D. Friedan, Z. Qiu and S. Shenker, *Nucl. Phys.* **B278** (1986) 577.

38. G.T. Horowitz, J. Lykken, R. Rohm and A. Strominger, *Phys. Rev. Lett.* **57** (1986) 283.

39. A. Strominger, *Nucl. Phys.* **B294** (1987) 93;
    M. Srednicki and R. P. Woodard, *Nucl. Phys.* **B293** (1987) 602.

40. M. Awada, *Phys. Lett.* **B188** (1987) 421;
    Z. Qiu and A. Strominger, *Phys. Rev.* **D36** (1987) 1794.

41. J. Feng, Caltech preprint CALT-68-1543 (1989).
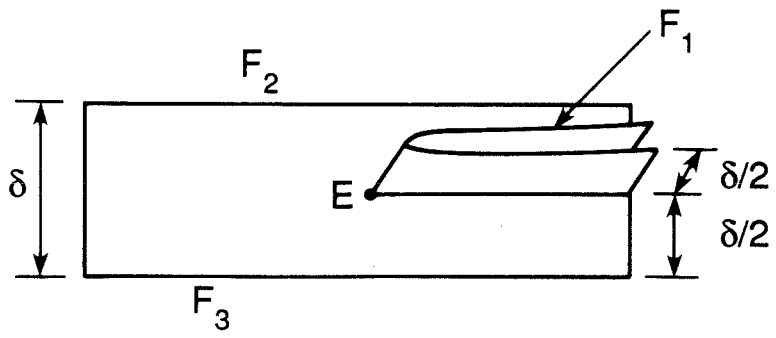
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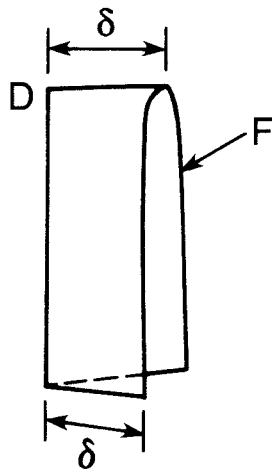
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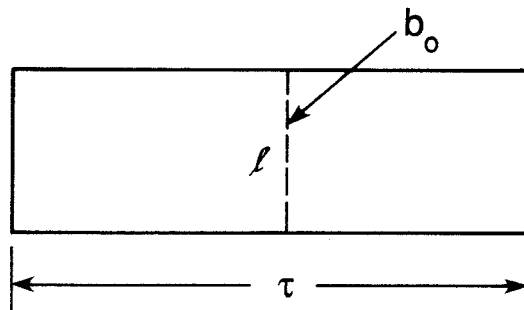
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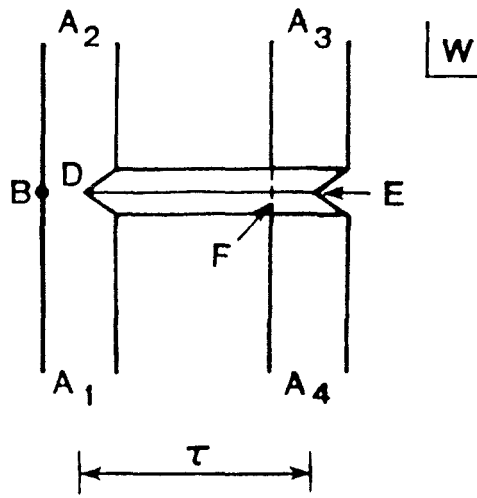
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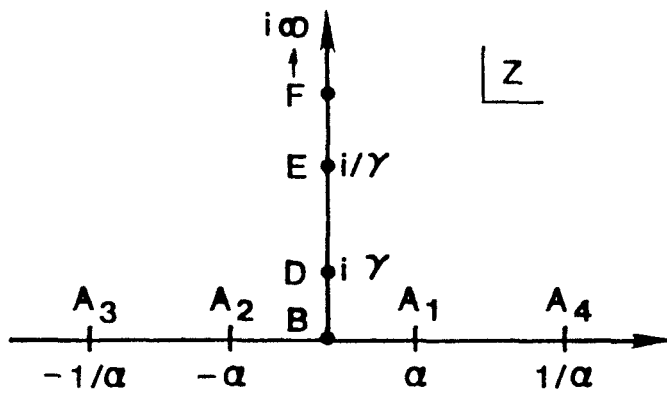
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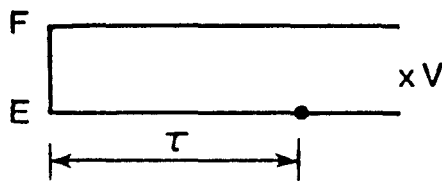
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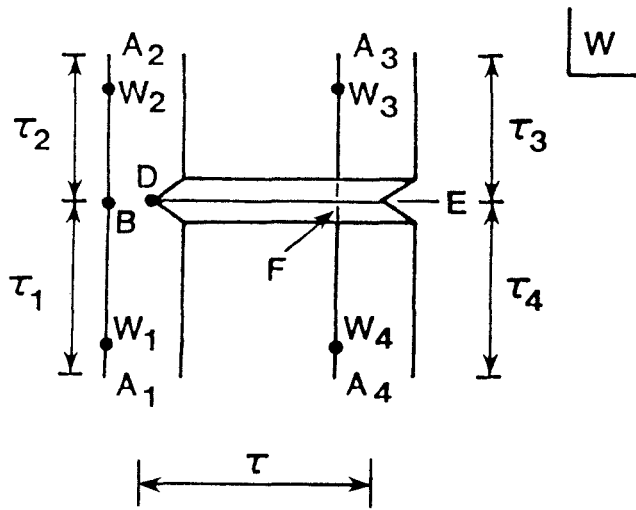
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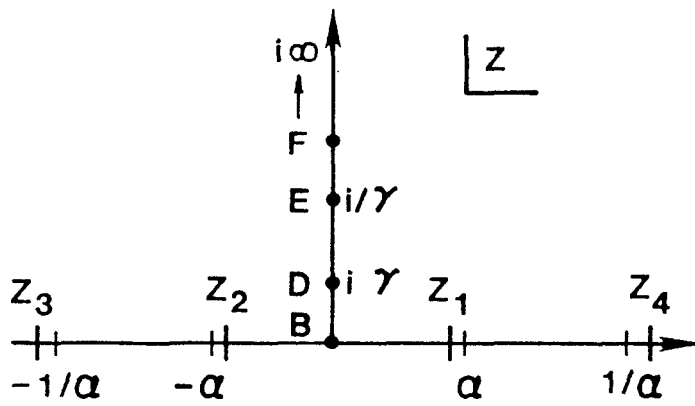
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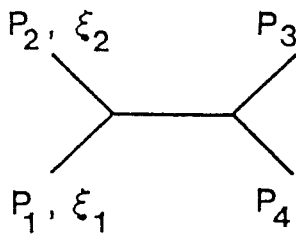
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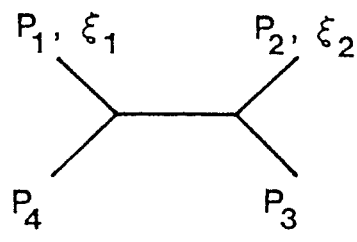
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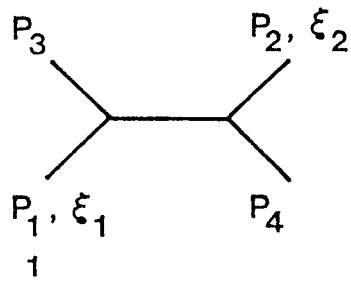
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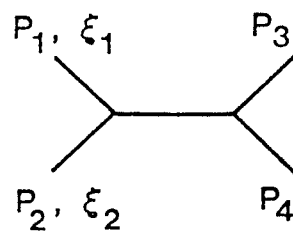
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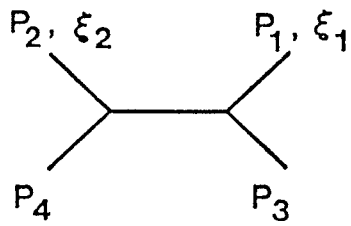
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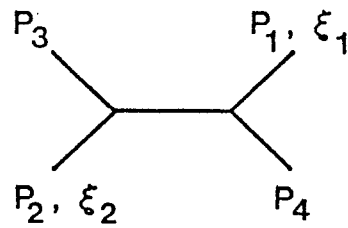
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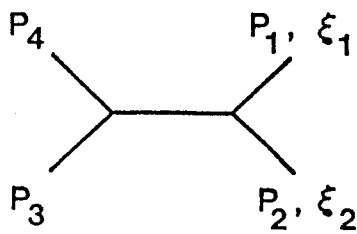
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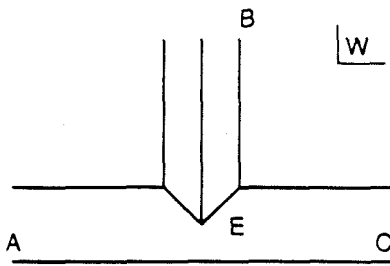
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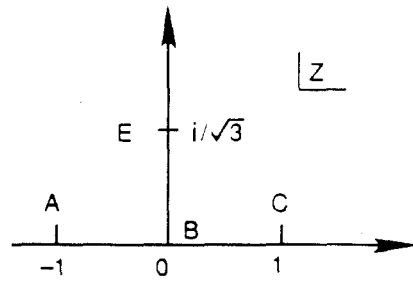
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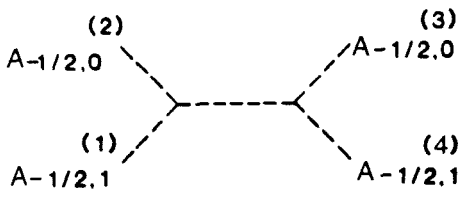
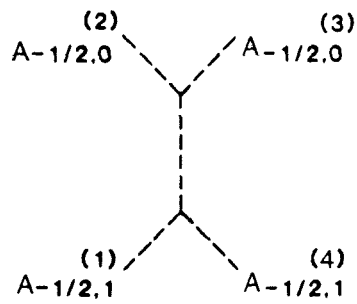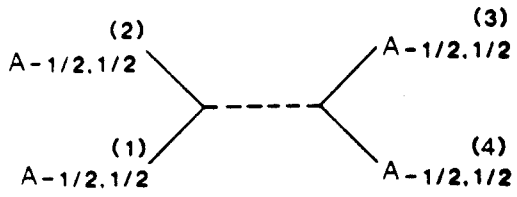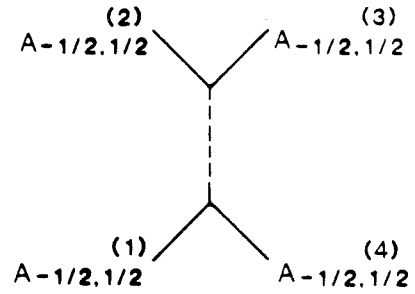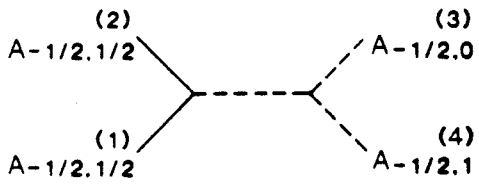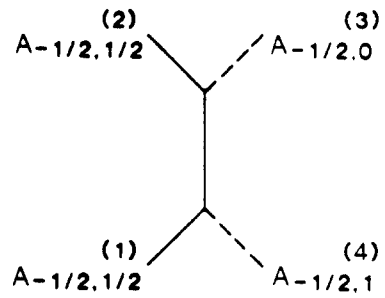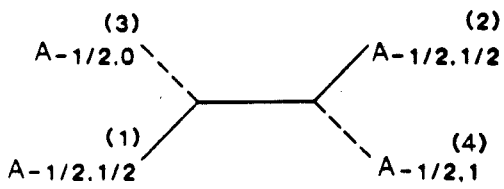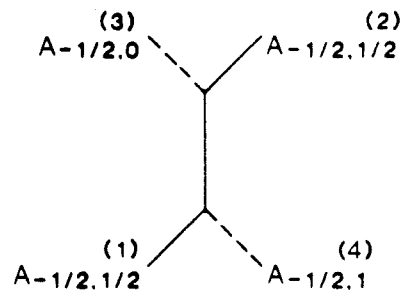

Fig. 9a

Fig. 9b

Fig. 10a

Fig. 10b

Fig. 11a

Fig. 11b

Fig. 12a

Fig. 12b