QUANTUM ASPECTS OF GRAVITY

Thesis by
Arun K. Gupta

In Partial Fulfillment of the Requirements
for the Degree of
Doctor of Philosophy

California Institute of Technology
Pasadena, California.

1990
(Submitted May 25, 1990)

ACKNOWLEDGEMENTS

Were I to recount here all that has been given me by the wonderful people around me, then these pages would not suffice. Let me here just thank my advisor Mark Wise, and John Preskill for their advice, support and much interesting physics.

ABSTRACT

We study here some models of quantum gravity. In Euclidean quantum gravity, some of the possible consequences of including topology changes in the path integral are studied in the semiclassical approximation. The effects of wormhole interactions on the semiclassical sum are considered. The effects of wormholes in the Yang-Mills-Einstein system on the phase structure of these theories is discussed. Also, we perform the computation of some partition and correlation functions in conformal gauge, in a two dimensional model of quantum gravity, i.e., the sub-critical Polyakov string.

Table of Contents

Introduction

One of the deep problems in physics today is the formulation of a quantum theory of gravity. Gravity is well described at the classical level by Einstein's General Theory of Relativity . In this theory, the dynamical quantity is the metric of space-time $g_{\mu\nu}$, and the dynamics is to be derived from the Hilbert action:

$$S = \int d^4x\sqrt{g}\left[-\frac{1}{16\pi G} R + \Lambda\right] + S_{\mathrm{matter}}$$

where G is Newton's constant, Λ is the cosmological constant, and R is the curvature scalar.

Quantum versions of this theory are unsatisfactory. The theory is perturbatively non-renormalizable. Additional terms involving the curvature tensor, resummation procedures, enlargement of the symmetry group of the action — none of these prescriptions improves the behavior of the theory [1]. It appears that complete theory takes us far beyond Einstein gravity. However, as far as low energy phenomena are concerned, the Einstein-Hilbert action defines an effective quantum theory with a cut-off at the scale set by $G^{-1/2} \sim M_{Planck} = 10^{19} GeV$. Perturbative quantum corrections to classical gravity become important at this energy scale. This energy scale was probably directly relevant to our universe only during the initial instant of the Big Bang.

The clues that nature yields, if any, then are very subtle. One may be that the cosmological constant is very small in natural units. A theory of everything such as superstring theory may mesh the description of gravity with that of everyday matter, and then a fact requiring explanation is the very small masses of everyday particles on the Planck scale.

We, therefore, study Einstein gravity as an effective theory. We search non-perturbative consequences that might carry over to the correct theory, and may perhaps lead to observable consequences.

We study the Euclidean path integral version of gravity [2]. We compute quantum mechanical amplitudes for transitions between states of the universe. These are functionals of the 3-geometries of the 3-manifolds. The path integral is to be computed by summing over all 4-manifolds that have these 3-geometries as boundaries. We may have any number of disconnected 3-geometries in the amplitude, meaning that we may create and annihilate universes. When we have more than two components in the boundary, the corresponding amplitudes are for transitions between varying numbers of initial and final universes. This is what we mean by topology change. (Nobody, so far, has proved either the necessity or the inconsistency of topology change.) We choose the Euclidean version of the path integral over the Minkowskian because naively the convergence of the path integral is improved and we can do an expansion around stationary points of the action. (This hope is not justified [2]. The Einstein action is unbounded from below. Given any metric on a manifold with a certain action, we can obtain a metric conformal to it with a lower action. This is usually dealt with by complexifying the conformal factor (in some gauge) and rotating the contour of integration [2,3]. There is no natural prescription for this and this is one of the weaknesses of the formalism.)

How to interpret what we compute is another problem. Rotation of a Euclidean space to a Minkowskian space may not be possible. We, however, interpret Euclidean amplitudes computed between asymptotically large and smooth spaces as quantum tunnelling amplitudes for classically forbidden processes in Minkowskian space-time. The question then is :

(i) are there stationary points of the action that correspond to topology changes? (Stationary points correspond to classical solutions.)

(ii) is there a consistent semi-classical expansion about these critical points?

The answer to (i) is yes. Einstein gravity by itself does not have any such stationary points. However in a cut-off theory we may place a constraint on the size of features in the solution, and there are indeed constrained "wormholes" [3]. Secondly, such solutions arise in theories of gravity with the correct matter content:

axion fields, scalar fields, Yang-Mills fields, etc. These solutions typically have a length scale governed by the Planck scale.

The answer to (ii) is that the problem of the conformal factor makes this non-trivial. If there exists a suitable deformation of the contour of integration of the conformal factor, then the answer is yes.

There are then non-zero amplitudes for a single universe to evolve into a multi-verse through quantum tunnelling. What might the consequences of these be?[4]

An observer who went from universe one to universe two would not be able to observe disconnected 3-geometries and hence would have to average his amplitudes over the unobservable geometries. Thus a pure state would evolve into a mixed state and quantum coherence would be lost.

The resolution of this problem [5] lies in the realization that the states with definite number of baby universes attached are not the correct basis elements to work with. Just as Q.C.D. is in a θ vacuum that is a superposition of eigenstates of instanton number, and no local operator can change the value of θ, so the universe is in a superposition of baby universe number eigenstates and is characterized by a number of vacuum angles, one for each kind of wormhole. Local operators are diagonal in this basis. As far as the low energy physics is concerned, the attachment of the baby universe is a point event and will be summarized by a local operator (like vertex operators in string theory). This operator will carry the same quantum numbers as the baby universe. Since closed universes can carry no gauge charges, or energy or momentum, or angular momentum, our vertex operators will be gauge invariant Lorentz scalars, and will generically break all ungauged symmetries. Each operator is multiplied by a coefficient that reflects the cost in action of making the corresponding baby universe, and the vacuum angle, which is seemingly completely arbitrary. This means that we can no longer predict the fundamental constants as they appear in the effective low energy Lagrangian! Large wormholes that would be suppressed by their large actions, to be as relevant as small ones, because of multiplication by a large vacuum angle. This problem of large wormholes is still unresolved.

Now the fundamental couplings have become random variables, determined by the initial conditions at the Big Bang and by the collapse of the wave-function of the universe. Our best hope is that, for some reason, the wave function is very sharply peaked about some values of the vacuum angles and in a way that is not very sensitive to initial conditions. (Just as in a system with a mostly shallow potential and a deep well, one would expect most states to have a maximum amplitude in the well.) Let us try to compute the dependence of the partition function on the couplings. First, integrate out all the matter degrees of freedom up to some low energy scale. We are left with an effective Lagrangian for gravity alone. This will look like the Einstein action up to terms suppressed by the cutoff (Planck) scale with the cosmological constant and gravitational constant being some complicated functions of the bare coupling constants and the vacuum angles induced by wormholes. Now we try to extremize this low energy action. For $\Lambda > 0$, this is extremized by large four spheres, with action $\sim G^{-2}\Lambda^{-1}$. Coleman [6] actually shows that what appears is the exponential of the exponential of this quantity so that the amplitude is very sharply peaked at $\Lambda = 0, G \rightarrow 0$. One can argue, following Wise, Grinstein and Preskill [7], that if one tries to regulate this very divergent function, that one would find $\Lambda = 0$ and G at the minimum possible value. Thus the most likely values of the vacuum angles can in principle be computed, by computing how they affect the two gravitional couplings. In practice, these are very sensitive to high energy effects, and only about the low energy parameter $\theta_{Q.C.D}$ do we have a concrete prediction[13]. Knowledge of the high-energy theory is not superfluous after all, it is needed even here to make a prediction.

The problem of the conformal factor may invalidate these ideas. No solution of the Euclidean Einstein equations is a minimum of the action. There are always conformal deformations that lower the action. The leading quantum corrections around a classical solution now involve the computation of the determinant of non-positive definite operator. Depending on how this object is defined, there is a sign ambiguity that may turn the supposed singular peak at $\Lambda = 0$ into a well-behaved function [8].

The following two chapters summarize some investigations in this field. The material appearing in them was published in the papers [9]. Each chapter is more or less self-contained, and the reader is referred to them for further details.

It is customary for the theoretical physicist, when frustrated with his problems, to retreat to a simpler universe (usually two dimensional) where his problems are more tractable. Actually, theories in two dimensions with a dynamical metric are interesting in their own right. They are a non-trivial version of quantum gravity. (Also, we do have a candidate for a quantum theory of gravity, namely, superstrings. The problem is that we do not know how to do non-perturbative calculations in the theory. Viewing the superstring as a random surface theory may lead us to a method of computing.) Versions of the Polyakov string may be exactly solvable. In recent days it has been shown [10], using random matrix methods to sum the discretized partition functions, that models of conformally invariant matter living on world sheets can be rigorously formulated and exactly solved, at least for the conformal anomaly of the matter $d \leq 1$. Continuum limits can be explicitly taken. Non-perturbative information is available about these theories. The final chapter of this thesis[11] deals with computations on the $d \leq 1$ Polyakov string within an ansatz suggested by David, Distler and Kawai[12], that purports to be the continuum version of these theories.

References

1. For example, E. Alvarez, Rev. Mod. Phys.**61** (1989) 561

2. G. W. Gibbons, S. W. Hawking, M. J. Perry, Nucl. Phys.**B138** (1978) 141

3. J. J. Halliwell, Phys. Rev **D39** (1989) 2206

4. S. W. Hawking, Phys. Lett. **B195** (1987) 337, Phys. Rev **D37** (1988) 904

   G. V. Lavrelashvili, V. A. Rubakov, P.G. Tinyakov, JETP Lett **46** (1987) 167

   S. Giddings, A. Strominger, Nucl. Phys. **B306** (1988) 890

5. S. Coleman, Nucl. Phys. **B307** (1988) 867

6. S. Coleman, Nucl. Phys. **B310** (1988) 643

7. B. Grinstein, M. B. Wise, Phys. Lett. **212B** (1988) 407;

   J. Preskill, Nucl. Phys. **B323** (1989) 141

8. J. Polchinski, Phys. Lett. **B219** (1989) 251

9. A. Gupta, M. B. Wise, Phys. Lett. **B218** (1989) 21;

   A. Gupta, J. Hughes, J. Preskill, M. B. Wise, Nucl. Phys. **B333** (1990) 195;

10. For example, M. Douglas, S. Shenker, Rutgers preprint RU-89-34 (1989)

11. A. Gupta, S. Trivedi, M. B. Wise, appear in Nucl. Phys.

12. J. Distler, H. Kawai, Nucl. Phys. **B321** (1989) 509;

    F. David, Mod. Phys. Lett. **A3** (1988) 1651

13. J. Preskill, S. Trivedi, M. B. Wise, Phys. Lett.**B223**(1989) 26

    K. Choi, R. Holman, Phys. Rev. Lett. **62** (1989) 2575

CHAPTER 1

I. Introduction

1. Wormhole Correlations

Recently it has been shown that topology changing field configurations (wormholes) in the functional integral can have important effects on physics below the wormhole scale [1,2,3]. Using the dilute instanton gas approximation the result of summing over wormhole configurations is to produce an effective field theory, for physics below the wormhole scale, that has an action* of the following form [3]

$$S_{\text{eff}} = \int d^4x\, \sqrt{g(x)}\, \mathcal{L}_0(x) \,+\, \hat{\alpha} \int d^4x\, \sqrt{g(x)}\, K(x) \quad . \tag{1}$$

Here $K(x)$ is a linear combination of gauge invariant local operators O made from fields with masses less than the wormhole scale. Possible operators that can occur in $K(x)$ are: 1, $R(x)$ (the curvature scalar), $\bar{e}_R(x)\not{D}e_R(x)$ (the gauge invariant kinetic term for right–handed electron field), etc. Nonrenormalizable operators in K are suppressed, presumably, by powers of the wormhole scale.

The parameter $\hat{\alpha}$, which occurs in eq. (1), is given by

$$\hat{\alpha} = \frac{1}{[S_W]^{p/2}} e^{-S_W} \alpha \quad , \tag{2}$$

where S_W is the action for the wormhole field configuration and α is a dimensionless quantity that characterizes the possible vacuum states of the theory much in the way θ_{QCD} determines the possible vacuum states in the strong interactions.

⋆ S_{eff} is *not* an effective action that includes all loop effects and is to be used at tree level. We shall use Γ to denote such an effective action. Rather S_{eff} is the action for a quantum field theory which describes physics below the wormhole scale.

We have assumed in eqs. (1) and (2) that there is only one type of wormhole. If there are many then there is an α parameter for each type of wormhole and eq. (1) contains a sum over $\hat{\alpha}$'s for each different type of wormhole. Since the $K(x)$'s corresponding to different types of wormholes will (in general) contain a different linear combination of operators, the coupling constants in the low energy effective field theory become different linear functions† of the α's. In eq. (2) the factor of $[S_W]^{-p/2}$ arises from p zero modes. Translational invariance implies four zero modes and so $p \geq 4$.

In eq. (1) $\mathcal{L}_0(x)$ is the Lagrangian density appropriate to the effective field theory obtained without summing over wormhole configurations. It is the piece proportional to $\hat{\alpha}$ which takes into account the effects of summing over wormholes. Note that in the dilute instanton gas approximation summing over wormholes produces a local effective field theory even though the probability for (two) wormholes to connect space–time points separated by a large distance is not suppressed.

Neglecting quantum fluctuations, in the dilute instanton gas approximation the wormholes are treated as uncorrelated: the probability density $P_n(x_1, \ldots, x_n)$ for wormholes to be located at Euclidean space–time points $x_1, \ldots, x_n$ is taken to be

$$P_n(x_1, \ldots, x_n) = [P_1]^n \quad , \tag{3}$$

where $P_1(x) = P_1$ is independent of x due to translational invariance. In general we expect there to be some correlations between the wormholes, and these are characterized by correlation functions $\xi_n(x_1, \ldots, x_n)$ defined by

$$1 + \xi_n(x_1, \ldots, x_n) = \frac{P_n(x_1, \ldots, x_n)}{[P_1]^n} \quad . \tag{4}$$

The interactions between wormholes can lead to nonlocal terms in the effective action S_{eff} for physics below the wormhole scale. For example, a two–point correlation may

† If one redefines the fields, so that their kinetic terms have the usual normalization, then the α dependence of the coupling constants is not linear. We are imagining here that the coefficients of the kinetic terms are also treated as coupling constants.

give rise to terms in $S_{\rm eff}$ of the form

$$\hat{\alpha}^2 \int d^4x_1 \sqrt{g(x_1)} \int d^4x_2 \sqrt{g(x_2)}\, O(x_1)\, f(x_1,x_2)\, O(x_2) \quad . \tag{5}$$

If the function $f(x_1,x_2)$ falls off fast enough with separation $|x_1 - x_2|$ then even for $\hat{\alpha}$ of order unity the term in the effective field theory below the wormhole scale, given by eq. (5), will be approximately local. For example consider the effects of this term in the effective action when $O(x) = \phi(x)$, where ϕ is a gauge singlet scalar field. If the function $f(x,y)$ falls off as $|x-y|^{-p}$ for large separations, then using

$$\phi(x)\phi(y) \;=\; \phi^2(x) + (x-y)^\mu \phi(x)\partial_\mu\phi(x) + \frac{1}{2}(x-y)^\mu(x-y)^\nu\phi(x)\partial_\mu\partial_\nu\phi(x) + \cdots \tag{6}$$

we see that the coefficients of $\phi^2(x)$ and $\phi(x)\Box\phi(x)$ in the effective Lagrangian density for physics below the wormhole scale are finite if $p > 6$. When $p > 6$ the first divergence encountered, in the expansion of the Lagrangian density in local operators, occurs in a nonrenormalizable operator and so nonlocalities induced by the long range two–point wormhole correlation are suppressed by powers of the wormhole scale and hence (even for $\hat{\alpha}$ of order unity) only have a small effect on low energy physics.

Wormhole solutions have been found for a U(1) Goldstone boson coupled gravitationally [4,5]. The Euclidean action for this case is (a surface term involving the extrinsic curvature also contributes)

$$S = \frac{-M_{PL}^{(0)\,2}}{16\pi} \int d^4x \sqrt{g}R + \frac{1}{2} \int d^4x \sqrt{g}\; g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \quad , \tag{7}$$

where ϕ is the Goldstone boson field. In eq. (7) $M_{PL}^{(0)}$ denotes the "bare" Planck mass, which is the coefficient of R in the Lagrangian density before integrating out the wormholes. Wormhole solutions appropriate to tunnelling between states of definite charge [5] are found by solving the field equations:

$$M_{PL}^{(0)\,2} G_{\mu\nu} = 16\pi \left[\frac{1}{2} g_{\mu\nu}\, g^{\lambda\sigma} \partial_\lambda \phi \partial_\sigma \phi - \partial_\mu \phi \partial_\nu \phi \right] \quad , \tag{8a}$$

$$\frac{1}{\sqrt{g}} \partial_\mu \sqrt{g} g^{\mu\nu} \partial_\nu \phi = 0 \quad , \tag{8b}$$

where $G_{\mu\nu}$ is the Einstein tensor. Adopting spherical symmetry about the wormhole one introduces spherical coordinates

$$ds^2 = dr^2 + a^2(r) d\Omega^2 \quad , \tag{9}$$

where $d\Omega^2$ is the line element on the three–sphere. The Goldstone boson field for a wormhole solution of charge n is determined by

$$\dot{\phi}(r) = n / f 2\pi^2 a^3(r) \quad , \tag{10}$$

and the metric satisfies the differential equation

$$\frac{3 M_{PL}^{(0)\,2}}{16\pi} \left(\frac{\dot{a}^2(r)}{a^2(r)} - \frac{1}{a^2(r)} \right) = \frac{-n^2}{8\pi^4 a^6(r) f^2} \quad , \tag{11}$$

where a dot denotes a derivative with respect to r and f is the "decay constant" of the Goldstone boson. For a wormhole solution $\dot{a}(0) = 0$ and so

$$a(0) = \left(\frac{2n^2}{3\pi^3 M_{PL}^{(0)\,2} f^2} \right)^{1/4} \quad , \tag{12}$$

is the wormhole length scale. Far from the wormhole "location" $(r = 0)$ the metric is approximately flat (i.e., $a(r) = r$). For large r

$$a(r) \simeq r \left[1 + \frac{1}{6} \left(\frac{a(0)}{r} \right)^4 \right] \quad . \tag{13}$$

Using this wormhole solution we can estimate the large distance behavior of the two–point correlation for wormholes carrying charges n_1 and n_2 of opposite sign.

The two–point correlation is given, according to eq. (4), by

$$1 + \xi_2(x_1, x_2) \simeq \exp(-S_W[1,2] + S_W[1] + S_W[2]) \quad , \tag{14}$$

where $S_W[1,2]$ is the minimal action for a field configuration consisting of a wormhole carrying charge n_1 located at x_1 and a wormhole carrying charge n_2 located at x_2, $S_W[1]$ is the action of a wormhole carrying charge n_1 and $S_W[2]$ is the action for a wormhole carrying charge n_2. Here we estimate the contribution to $\xi_2(x_1, x_2)$ coming from the part of the two wormhole field configuration that is far from either wormhole. We consider separations $|x_1 - x_2|$ that are much larger than the wormhole length scales. For an approximate two wormhole field configuration we take (in cartesian coordinates)

$$g_{\mu\nu}[1,2] = \delta_{\mu\nu} + h_{\mu\nu}[1,2] \tag{15a}$$

$$\partial_\mu \phi[1,2] = \partial_\mu \phi[1] + \partial_\mu \phi[2] \quad . \tag{15b}$$

In eq. (15a) $h_{\mu\nu}$ is chosen to satisfy the linearized Einstein equations with the field configuration (15b) as the source for the gravitational field. We first estimate the contribution to $-S_W[1,2] + S_W[1] + S_W[2]$ that comes from the Goldstone boson part of the action. Eqs. (10) and (13) imply that the region near the wormholes contributes an amount of order $1/|x_1 - x_2|^3$ to this difference of actions. It is the region a distance of order $|x_1 - x_2|$ away from either wormhole that dominates. Thus we have that the Goldstone boson part of the action contributes an amount (using cartesian coordinates)

$$\simeq \frac{n_1 n_2}{f^2 4\pi^4} \int d^4x \frac{x_\mu (x_\mu - (x_2 - x_1)_\mu)}{|x|^4 |x - (x_2 - x_1)|^4}$$
$$= \frac{n_1 n_2}{f^2 4\pi^2 |x_1 - x_2|^2}$$

to $S_W[1,2] - S_W[1] - S_W[2]$. Next we consider the gravitational part of the action. It is easy to see that in the region far from either of the wormholes the extrinsic curvative surface term cancels the order $1/|x_1 - x_2|^2$ contribution from the gravitational part of the action in eq. (7). So we have the following estimate of the two-point correlation

$$1 + \xi_2(x_1, x_2) \approx \exp\left(\frac{-n_1 n_2}{f^2 4\pi^2 |x_1 - x_2|^2}\right) \quad . \tag{16}$$

The two wormhole contribution from wormholes of charge n_1 and n_2 to the vacuum–vacuum amplitude is thus estimated to be

$$\int d^4x \sqrt{g(x)} \int d^4y \sqrt{g(y)}\ \alpha_{n_1} \alpha_{n_2} c_1 c_2 e^{-Sw[1]}\, e^{-Sw[2]}\Big(1 + \xi_2(x, y)\Big) \quad . \tag{17a}$$

On the other hand, since the effective field theory (1) contains terms (here α_{n_1} and α_{n_2} are complex since we are not in a CPT eigenstate basis)

$$\int d^4x \sqrt{g(x)} \Big\{ \alpha_{n_1}\ c_1\ e^{in_1\phi(x)/f}\ e^{-Sw[1]}\ +\ \alpha_{n_2}\ c_2\ e^{in_2\phi(x)/f}\ e^{-Sw[2]} \Big\}$$

it gives a contribution to the vacuum–vacuum amplitude that is

$$\int d^4x \sqrt{g(x)} \int d^4y \sqrt{g(y)}\ \alpha_{n_1}\ \alpha_{n_2}\ c_1\ c_2\ e^{-Sw[1]}\ e^{-Sw[2]} \Big\langle e^{in_1\phi(x)/f}\ e^{in_2\phi(y)/f} \Big\rangle \quad . \tag{17b}$$

But:

$$\Big\langle e^{in_1\phi(x)/f}\ e^{in_2\phi(y)/f} \Big\rangle \ = \ \exp\left(\frac{-n_1 n_2}{4\pi^2 f^2 |x_1 - x_2|^2}\right)$$

so the two-point correlation in eq. (16), which arose due to the contribution of the scalar field far away from either wormhole location to the action, does not give rise to a nonlocal term in the effective field theory below the wormhole scale. Similarly, correlations arising from the long range field of one wormhole perturbing the solution for the other wormhole, near its location, will be reproduced by adding local terms to the effective field theory below the wormhole scale that involve derivatives of ϕ and have coefficients linear in $\hat{\alpha}$. It thus seems likely that f in eq. (5) and the consequent nonlinear $\hat{\alpha}$ dependence will arise only from interactions where the two wormholes are very near each other and hence will not destroy the locality of the effective field theory below the wormhole scale.

Dimensional analysis suggests that at small separations the correlations of large wormholes are enormous. For large wormholes coupling constants, λ, may have a power series expansion in $\hat{\alpha}$ of the form

$$\begin{aligned}\lambda(\hat{\alpha}) &= \lambda_0 + \lambda_1\hat{\alpha} + \lambda_2 X\hat{\alpha}^2 + \lambda_3 X^2\hat{\alpha}^3 + \cdots \\ &= \lambda_0 + \frac{1}{X}F(\hat{\alpha}X) \quad .\end{aligned} \tag{18}$$

Here X, which arises due to the wormhole correlations, is enormous (e.g., of order $\exp(M_{PL}^{(0)^2} a^2(0))$). If F is a bounded function then it is possible that for all $\hat{\alpha}$ the effects of large wormholes are suppressed by $1/X$. For example, this occurs if $F(y) = (y + y^2)e^{-y^2}$. However, this possibility seems somewhat unnatural since typically a bounded function $\lambda(\hat{\alpha})$, with a power series expansion where the terms in this expansion have the order of magnitude given by eq. (18), will not have wormhole effects suppressed by $1/X$. Of course, since the action S_W for a wormhole of large size is large, if α is of order unity then eq. (2) implies that $\hat{\alpha}$ itself is very small.

Recently it has been argued that the probability distribution for the α's is [6]

$$dP = N\prod_i d\alpha_i Z(\alpha)f(\alpha) \quad , \tag{19}$$

where a subscript i denotes the wormhole type and dP is the probability of finding the α's in an interval between α_i and $\alpha_i + d\alpha_i$. In eq. (19) $f(\alpha)$ is a function determined by the boundary conditions used to define the wavefunction of the Universe. With Hartle–Hawking boundary conditions [7] $f(\alpha) = \prod_i e^{-\alpha_i^2/2}$. N is a normalization factor independent of α. The function $Z(\alpha)$ is given by

$$Z(\alpha) = \exp\left(\sum_{\text{topologies}} e^{-\Gamma_\alpha}\right) \quad , \tag{20}$$

where Γ_α is equal to the background gravitational field effective action $\Gamma_\alpha(g)$ evaluated at the stationary point in g of minimum action. For large smooth manifolds the effective action $\Gamma_a(g)$ can be expanded in the curvature tensor

$$\Gamma_\alpha(g) = \int d^4x\sqrt{g}\left[\Lambda - \frac{M_{PL}^2}{16\pi}R\right] \quad . \tag{21}$$

The α dependence of $\Gamma_\alpha(g)$ resides in the cosmological constant Λ and in the Planck mass M_{PL}. For positive Λ the stationary point of minimum action is a four sphere. Evaluating the above effective action at this stationary point gives

$$\Gamma_\alpha = -3M_{PL}^4/8\Lambda \quad . \tag{22}$$

The probability distribution for the α's is infinitely strongly peaked on the subspace of α's for which the cosmological constant vanishes. A volume regulation of the probability distribution seems to show that it is also infinitely strongly peaked at the value of α, within the subspace for which $\Lambda = 0$, that maximizes M_{PL} [8]. It is important that M_{PL} be a bounded function of the α's since in the limit $M_{PL} \to \infty$ gravity becomes a free theory. Assuming M_{PL} is a bounded function of the α's, then typically α will be chosen to maximize M_{PL}^2 when $\hat{\alpha}$ is of order unity, even if this corresponds to a value of α that is enormous: naively, wormholes of large size are important, even though their action is large [9].

Wormholes break global symmetries, and in spontaneously broken global symmetries, the Goldstone boson acquires a mass. We are then hard pressed to save the invisible axion as a solution to the strong CP puzzle [10,11].We have to ensure that the wormhole generated mass is sufficiently small. Naively, wormholes of length scale $a(0)$ and charge n give (in the dilute instanton gas approximation) a contribution to the axion mass, m_a, of order

$$m_a^2 \sim \frac{n^2\hat{\alpha}}{f_a^2 a(0)^4} \sim M_{PL}^{(0)\,2}\,\hat{\alpha} \quad . \tag{23}$$

For the axion to be able to solve the strong CP puzzle this contribution to its mass–squared must be much less than $(m_u m_d/(m_u + m_d))(\langle \bar{u}u\rangle / f_a^2)$. But if f_a is much smaller than $M_{PL}^{(0)}$, then, since n is quantized, all wormholes that contribute to m_a are large in size compared with $1/M_{PL}^{(0)}$ (see eq. (12)). It is then crucial to suppress large wormholes.

Actually for n of order unity and $f_a \ll M_{PL}^{(0)}$ the size of the wormhole is small compared to $1/f_a$. We should include the field that gets a vacuum expectation value. We assume that this field has a small mass in Planck units, and that the mass term can be neglected near the neck of the wormhole. Then it turns out that f_a, the v.e.v. of this field, is always small, and the wormhole is always large (in Planck units)[12].

References

1. S. W. Hawking, Phys. Lett. **195B** (1987) 337.

   S. W. Hawking, Phys. Rev. **D37** (1988) 904.

2. G. Lavrelashvili, V. A. Rubakov, and P. G. Tinyakov, JETP Lett. **46** (1987) 167.

3. S. Coleman, Nucl. Phys. **B307** (1988) 867; S. B. Giddings and A. Strominger, Nucl. Phys. **B307** (1988) 854.

4. S. B. Giddings and A. Strominger, Nucl. Phys. **B306** (1988) 890.

5. K. Lee, Phys. Rev. Lett. **61** (1988) 263.

6. S. Coleman, HUTP 88/A022 (1988) unpublished.

7. J. B. Hartle and S. W. Hawking, Phys. Rev. **D28** (1983) 2960.

8. B. Grinstein and M. B. Wise, Phys. Lett. **212B** 407 (1988). The sensitive dependence of conclusions, on the choice of regulator, was stressed to one of us (M. B. W.) by S. Weinberg (private communication).

9. The potential problem with large instantons was first noted by V. Kaplunovsky and communicated to us by L. Susskind (private communication). For a discussion on this issue see: I. Klebanov, L. Susskind, and T. Banks, SLAC–PUB–4705 (1988) unpublished.

10. R. D. Peccei and H. R. Quinn, Phys. Rev. Lett. **38** (1977) 1440.

    S. Weinberg, Phys. Rev. Lett. **40** (1978) 223.

    F. Wilczek, Phys. Rev. Lett. **40** (1978) 279.

11. J. E. Kim, Phys. Rev. Lett. **43** (1979) 103; M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, Nucl. Phys. **B166** (1980) 493; M. Dine, W. Fishler, and M. Srednicki, Phys. Lett. **104B** (1981) 199.

12. L. F. Abbott and M. B. Wise, Nucl. Phys. **B325** (1989) 687. vfill

Chapter Two

1. Introduction

We have seen so far some of the effects that wormholes might have on macroscopic physics. It will in general be true that given a wormhole solution involving some particular fields, we will be able to construct solutions that swallow the appropriate combinations of quanta of other fields that are Lorentz scalars , gauge invariant, etc. (For example see the discussion of baryon number violation in [1].) The cost in action is higher for the more complicated wormholes. The apparent unboundedness of the coupling constants can compensate for the suppression by the exponential of the action. Thus large wormholes may be as relevant to the semiclassical limit as small ones. We must suppose that there is some resolution to this problem in order to motivate the search for other wormhole solutions.We also would like to see whether wormholes require exotic matter or whether they would appear in theories that have as matter content the fields that we know to exist in nature. The interesting thing about the solutions described below is that they could occur in the Standard Model of particle interactions.

The geometry of the solution that we look for is as follows. The manifold has the topology of $R^1 \otimes S^3$. We call this a wormhole. Let us parametrize the R^1 manifold by a "time" coordinate, t, $t \in (-\infty, \infty)$. The slices of constant t in this geometry are assumed to be three spheres, with the radius of the three sphere at a minimum at t=0, and approaching infinity as $t \to \pm\infty$. The size of the wormhole this minimum radius. This geometry requires negative stress-energy, as may be seen by computing the Einstein tensor. Alternatively, one may note that the throat of the wormhole acts as a defocussing gravitational lens; hence negative stress-energy is required to support the throat. In Euclidean space, it is not so hard to find matter that gives rise to negative stress-energy. For example, the Euclidean Lagrange density of an electromagnetic field is

$$\mathcal{L}_E = \frac{1}{2}\left(\vec{E}^2 + \vec{B}^2\right) \quad , \tag{1.1}$$

and the corresponding Euclidean "energy density" is

$$(T_{00})_E = \frac{1}{2}\left(\vec{E}^2 - \vec{B}^2\right) \quad . \tag{1.2}$$

Therefore, a magnetic field acts as a source of negative "energy" in the Euclidean Einstein equation, and is capable of supporting the throat of a wormhole.

This observation was used by Hosoya and Ogura [2], who described wormhole solutions for gravity coupled to gauge fields in both 3 and 4 Euclidean dimensions. Our main objective here will be to elucidate the physical implications of these solutions.

We first review the properties of the 3 dimensional electromagnetic wormholes and note that by choosing an Abelian subgroup of the gauge group, we can generalize these to Yang-Mills theory. The electrodynamic wormholes may be arbitrarily large, but Yang-Mills wormholes have a maximum size above which they are unstable.

In three dimensions the magnetic wormholes violate the topological conservation laws of the gauge theory. We argue that wormholes convert ordinary noncompact electrodynamics to compact electrodynamics, and drive electric confinement. Matter that tranforms faithfully under the center of the gauge group makes the wormhole solutions forbidden. The low energy physics induced by magnetic wormholes cannot be distinguished from the low energy physics of a ordinary non-Abelian gauge theory (without wormholes) that undergoes the Higgs mechanism at a large mass scale.

We then describe a class of wormhole solutions of four dimensional Yang-Mills gravity system. However, we argue that these are unstable with respect to small deformations of the gauge field, and so do not correspond to semiclassical tunnelling processes.

2. Magnetic Wormholes in 2+1 Dimensions

Consider first the case of a U(1) gauge field coupled to gravity in 3 Euclidean dimensions. The action is

$$S = \int d^3x \sqrt{g} \left[-\frac{1}{16\pi G} R + \frac{1}{4e^2} g^{ij} g^{k\ell} F_{ik} F_{j\ell} \right] . \tag{2.1}$$

Here G is Newton's constant, R is the scalar curvature, e is the electromagnetic coupling. In three dimensions, both G^{-1} and e^2 have the dimensions of mass (with $\hbar = c = 1$). $F_{ij} = \partial_i A_j - \partial_j A_i$ is the electromagnetic field strength. We choose to describe F_{ij} as a three component magnetic field that is the curl of the gauge field A.

The field equations of this theory have a spherically symmetric solution in which the metric takes the form

$$ds^2 = dt^2 + a(t)^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2 \right) . \tag{2.2}$$

The slice at constant "Euclidean time" is a two-sphere of radius $a(t)$, as advertised above. The magnetic field on this slice points in the $\hat{t}$ direction and carries magnetic flux $\Phi = 2\pi n$; thus the field strength is

$$F = \frac{n}{2a^2} , \tag{2.3}$$

and the electromagnetic Lagrange density on the two-sphere of radius a is

$$\frac{1}{2e^2} F^2 = \frac{n^2}{8e^2 a^4} . \tag{2.4}$$

The magnetic flux is quantized in anticipation of including charged matter fields in the theory. The scale factor a satisfies the Einstein equation

$$\dot{a}^2 = 1 - \frac{\pi G n^2}{e^2 a^2} , \tag{2.5}$$

which is solved by

$$a(t)^2 = b_n^2 + t^2 , \quad b_n^2 = \frac{\pi G n^2}{e^2} . \tag{2.6}$$

Here b_n is the minimal value of the scale factor, or the thickness of the throat of the wormhole; b_1 is roughly the geometric mean of the two length scales G and e^{-2} in the problem. Semiclassically, we may interpret b as the radius of a baby universe that spontaneously nucleates. It is, in fact, the maximum radius of a $(2+1)$-dimensional Friedmann-Robertson-Walker cosmology that is supported by the magnetic flux $\Phi = 2\pi n$. This radius is large in Planck units as long as the gauge coupling e^2 is small in those units. The Euclidean action of a "semiwormhole" (comprised of one of the asymptotically flat regions and half of the throat) is

$$\begin{aligned} S_{E,n} &= 2 \int d^3\, x\sqrt{g}\, \frac{1}{2e^2}\, F^2 = \int_0^\infty dt\; 4\pi a^2\, \frac{n^2}{4e^2 a^4} \\ &= 2\pi^2 \left(\frac{n}{2e}\right)^2 \frac{1}{b_n} = \frac{\pi^{3/2}}{2}\; \frac{n}{e\sqrt{G}} \quad . \end{aligned} \tag{2.7}$$

(Half of $S_{E,n}$ comes from the scalar curvature term in S, and half from the magnetic field strength term.) Roughly, this is the "self-energy" of a monopole with magnetic charge $g = ng_0$, and core size b_n.

For a Yang-Mills gauge field, we merely imbed this solution in an Abelian subgroup of the gauge group. Since the Yang-Mills fields themselves are charge carrying fields, the magnetic flux that the wormhole carries is automatically quantized. Secondly, while we can construct solutions that carry an arbitrary integer value of flux, most of these solutions are unstable [3]. If slightly perturbed, the field strength decays, with the excess energy being radiated to infinity. There is thus a maximum size to the stable wormholes. The stable solutions are characterized by the value of the conserved magnetic flux on a two-sphere, which takes values in the center of the gauge group G. The magnetic field strength is

$$F = \frac{1}{2a^2}\, Q \quad , \tag{2.8}$$

where Q is an element of the Lie algebra of G such that $\exp(2\pi i Q)$ is a nontrivial element of the center of G.

For example, if $G = SU(N)$, with center Z_N there are N-1 distinct wormhole solutions, one for each nontrivial element of Z_N. The values of Q that satisfy the

Brandt-Neri-Coleman [3] stability criterion are, in a particular gauge,

$$Q_n \;=\; \mathrm{diag}\left(\underbrace{\frac{n}{N},\dots\frac{n}{N}}_{N-n \text{ times}} \;,\; \underbrace{\frac{n-N}{N},\dots\frac{n-N}{N}}_{n \text{ times}} \right) \;,\; n=1,2,\dots,N-1 \;. \quad (2.9)$$

The Yang-Mills Lagrange density on the two-sphere of radius a is

$$\frac{1}{e^2}\,\mathrm{tr}\,F^2 \;=\; \frac{1}{4e^2a^4}\,\frac{n(N-n)}{N} \quad , \qquad (2.10)$$

and an analysis identical to that described above shows that the thickness of the wormhole throat is

$$b_n^2 \;=\; \frac{2\pi G}{e^2}\,\frac{n(N-n)}{N} \qquad (2.11)$$

and the action of a semiwormhole is

$$S_{E,n} \;=\; \sqrt{\frac{\pi^3}{2}}\,\frac{1}{e\sqrt{G}}\,\sqrt{\frac{n(N-n)}{N}} \quad . \qquad (2.12)$$

There are also solutions if the gauge fields are coupled to Higgs fields, and the gauge symmetry is realized in a Higgs phase. For example, if the Higgs mass scale is μ, and μ^{-1} is large compared to the size of the wormhole $a(0)$, then a solution exists that closely resembles that described above. But at larger length scales, the magnetic flux finds it energetically favorable to collapse to a vortex with thickness of the order μ^{-1} rather than to be uniformly distributed. Now, if n vortices come in from far away to vanish down the wormhole, the action of the configuration diverges like

$$n\,m_v \int dt \qquad (2.13)$$

where m_v is precisely the mass of the stable vortex particle that appears in the spectrum of the theory in the Higgs phase.

The condition that the Dirac magnetic flux string be invisible to the Yang-Mills field forces it to be quantized. This quantization, however means that it cannot change through continuous deformations, and in particular, through time evolution, which is continuous. The magnetic flux is characterized by the "Aharanov-Bohm" phase associated with circumnavigating it. This phase must be constant, and we have a topological conservation law for the magnetic flux.

An observer concerned with physics on scales of length large compared to the wormhole would percieve them to be magnetic monopoles. Only on closer examination would he discern the wormhole threaded with magnetic flux.

3. Intrinsic Breaking of Topological Symmetry, and its Consequences

To consider the effects of the wormholes described above on physics at low energy we integrate out the wormholes, incorporating their effect in a effective local field theory with a cutoff scale on the order of the wormhole size. The local operators in the effective action induced by the wormholes must behave as the source of the magnetic flux that the wormholes carry.

We follow [4] to define a local operator $\phi(x)$ that creates or destroys a unit of magnetic flux. The operator can be described as follows. In the canonical formalism, impose the gauge condition $A_0 = 0$. Construct a Hilbert space H spanned by the eigenstates of the operators $A_i(\vec{x})$. The physical states form a subspace H_P of H: these are the states that are invariant under time-independent gauge transformations. The operator $\phi(x)$ acts on a basis for H as follows:

$$\phi(\vec{x})\,|A_i(\vec{y})\rangle \;=\; \left|A_i^{\Omega(\vec{x})}(\vec{y})\right\rangle \quad . \tag{3.1}$$

$\Omega(\vec{x})$ is a time independent gauge transformation that has a minimal winding number around the point $\vec{x}$, and so has a singularity at $\vec{x}$, but is smooth everywhere else. (A cutoff procedure or renormalization must be performed to avoid infinite self-energy at the singularity.) For example, if the gauge group is $G = \mathrm{SU(N)}$, then $\Omega \in \mathrm{SU(N)}$

has a discontinuity

$$\Omega_{\text{above cut}} \; = \; e^{2\pi i/N} \; \Omega_{\text{below cut}} \tag{3.2}$$

across a cut in the plane that terminates at the point $\vec{x}$. Since the gauge field actually transforms as a representation of $SU(N)/Z_N$, the discontinuity cannot be detected by the gauge field. On a state in the subspace H_P, at every point away from $\vec{x}$ the action of $\phi(\vec{x})$ is merely a time-independent gauge transformation, under which the state is invariant. So the action of $\phi(\vec{x})$ on physical states is completely defined by the point $\vec{x}$ and the winding number of the gauge transformation Ω. It is easy to see that in H_P, $\phi(\vec{x})$ is a gauge-invariant operator. It commutes at spacelike separation with all gauge-invariant smeared polynomials in the gauge fields. Its action on H_P cannot be duplicated by polynomials in the gauge fields. It must therefore be included in the local field algebra. A charged field acquires a Aharanov-Bohm phase corresponding to the winding number of the gauge transformation Ω when carried once in a curve around the point $\vec{x}$. We see therefore that $\phi(\vec{x})$ creates a Dirac string of magnetic flux at the point $\vec{x}$. The states of H_P divide into superselection sectors that are characterized by the value of the topologically conserved magnetic flux; this quantity takes values in Z for G =U(1) and in Z_N for G =SU(N). While the smeared gauge-invariant polynomials in the gauge fields preserve these superselection sectors, the operator $\phi(\vec{x})$ interpolates between sectors by creating or destroying a unit of magnetic flux.

In the Euclidean path integral language, a correlation function with an insertion of the operator $\phi(x)$ would be computed by summing over all gauge field configurations that have a Dirac magnetic monopole singularity located at the point x.

We now have the operator needed to complete the sum over wormholes. By integrating out wormholes that carry n units of quantized magnetic flux, we generate a term in the effective action that in the leading semiclassical approximation is

$$\mathcal{L}_{E,n} \; = \; C_n \; e^{-S_{E,n}} \; (a_n^\dagger \; + \; a_{-n})\phi^n(x) \; + \; \text{h.c.} \quad . \tag{3.3}$$

Here ϕ is the local operator defined above, $a_n^\dagger$ is an operator that creates a baby universe that carries n units of magnetic flux, and a_{-n} annihilates a baby universe that carries $-n$ units of magnetic flux; $S_{E,n}$ is the Euclidean action of the corresponding semiwormhole solution, eq. (2.7). (More precisely, $S_{E,n}$ is the part of action coming from a region near the wormhole throat; this distinction is important in the case where the gauge symmetry is spontaneously broken and there are stable vortices, since then the total action of the semiwormhole diverges in the infrared.) The constants C_n can in principle be determined by matching the Green functions of our effective field theory to Green functions computed on the wormhole background, in the leading semiclassical approximation. (The required formalism has been outlined in [5,6].) To interpret eq. (3.3), one observes that the Hilbert space of baby universes is spanned by eigenstates of the operators $a_n^\dagger + a_{-n}$, with eigenvalues α_n. Thus, the low-energy physics described by our effective field theory divides into superselection sectors [7,8]. The distinct sectors are labeled by $\{\alpha_n\}$, and each sector has distinct physics, for the coefficient of the local operator $\mathcal{L}_{E,n}$ in the effective action depends on the value of α_n. For the purpose of discussing the qualitative effects of wormholes on low-energy physics, it suffices to consider the physical effects of the interactions $\mathcal{L}_{E,n}$.

In the absence of wormholes, for both Abelian and non-Abelian gauge theories the magnetic flux is conserved for topological reasons. The Aharanov-Bohm phase associated with circumnavigation of the Dirac string of magnetic flux must be invisible to the charged fields and so must lie in the center of the group, which is Z for the $U(1)$ case and Z_N for $SU(N)$. No smooth deformation, and in particular continuous time evolution, can change the value of this element of the center, because the center is a discrete group. In the case of $G = SU(N)$, multiples of N times the flux quantum can be deformed away, because these correspond to the identity element in the center.

Since the action of ϕ on a state changes the value of the topologically conserved flux, we may define a unitary operator U that "counts" this flux. It commutes with the Hamiltonian, and obeys

$$\mathrm{U}(\alpha)\ \phi(x)\ \mathrm{U}^{-1}(\alpha)\ =\ e^{i\alpha}\ \phi(x)\ ,\ \alpha \in [0, 2\pi] \tag{3.4}$$

in the case $G = \mathrm{U}(1)$, or

$$\mathrm{U}_k\ \phi(x)\ \mathrm{U}_k^{-1}\ =\ e^{2\pi i k/N}\ \phi(x)\quad ,\quad k \in \{0, 1, 2, \ldots\ N-1\} \tag{3.5}$$

in the case G =SU(N). Thus we may say that the gauge theory respects a "topological symmetry." This is a continuous global U(1) symmetry if the gauge group is U(1), and a discrete Z_N symmetry if the gauge group is SU(N). The terms induced by wormholes in the action explicitly break this symmetry.

As stressed by Polykov[9] and t'Hooft [4], the realization of the topological symmetry characterises the phase structure of gauge theories, i.e., whether the gauge theory is in a Higgs phase, Coloumb phase, or confining phase. Wormholes carry topological charge and so their impact on the phases of the gauge theory must be assessed.

In 2+1 dimensions the Hilbert space of a $U(1)$ gauge theory with an infrared cutoff divides into sectors labelled by the magnetic flux $n \in Z$. In ordinary electrodynamics with a massless photon, the energy of the ground state $|n\rangle$ of sector n tends to zero as the infrared cutoff is removed, and the vacuum state of the theory becomes infinitely degenerate. The local operator ϕ is not diagonal in the basis $\{|n\rangle\}$ for vacuum states:

$$\langle m|\ \phi(x)\,|n\rangle\ =\ v\delta_{m,n-1}\quad . \tag{3.6}$$

Correlation functions evaluated in this basis do not satisfy cluster decomposition. It is more convenient to use the basis

$$|\theta\rangle\ =\ \frac{1}{\sqrt{2\pi}}\sum_{n=-\infty}^{\infty} e^{in\theta}\,|n\rangle \tag{3.7}$$

such that

$$\langle\theta'|\theta\rangle \;=\; \delta(\theta'-\theta) \text{ and } \langle\theta'|\phi(x)|\theta\rangle \;=\; ve^{i\theta}\delta(\theta-\theta') \quad . \tag{3.8}$$

The action of the topological U(1) symmetry on the vacuum states is

$$\mathrm{U}(\alpha)\,|n\rangle \;=\; e^{-in\alpha}\,|n\rangle \tag{3.9}$$

or

$$\mathrm{U}(\alpha)\,|\theta\rangle \;=\; |\theta-\alpha\rangle \quad . \tag{3.10}$$

We see that the topological $U(1)$ is spontaneously broken. The Goldstone boson can be identified with the photon. (In two spatial dimensions, massless particles have no helicity, and the representations of the Poincare group do not distinguish between the massless photon and a scalar.)

We can also couple a Higgs scalar to electrodynamics, and have the $U(1)$ gauge symmetry spontaneously broken (the Higgs mode). Then there is a massive stable vortex particle in the spectrum of the theory, created by the action of ϕ on the vacuum. It carries a unit of magnetic flux, and so the true ground state is now in the $n = 0$ sector. The topological symmetry is now manifest.

Let us now include finite action monopole configurations in the Euclidean path integral of the 2+1 dimensional $U(1)$ theory. Wormholes, for example, do this for us. The monopoles terminate lines of magnetic flux, and the topological symmetry is explicitly broken. In the Coloumb phase, the photon which was massless by virtue of being a Goldstone boson, now acquires a mass, suppressed by the factor e^{-S_M}, where S_M is the Euclidean action of the magnetic monopole. Another way of saying this is that the plasma of magnetic monopoles and antimonopoles has induced Debye screening, correlation functions decay exponentially, and the theory has a mass gap.

The dilute plasma of magnetic monopoles and antimonopoles is a source of weakly correlated magnetic fluctuations in the vacuum [9]. (The long range magnetic field has no long range correlations because of Debye screening.) The Wilson loop operator defined on the closed oriented loop C

$$W(C) \;=\; \exp\,(i \oint_C \, A_\mu \, dx^\mu) \quad . \tag{3.11}$$

is a measure of the magnetic flux enclosed by the loop. The weakly correlated magnetic fluctuations cause $\langle 0|\, W(C)\, |0\rangle$ to behave like a product of uncorrelated factors, with the number of factors growing like the area of the loop, and the Wilson loop has area-law behavior

$$\langle 0|\, W(C)\, |0\rangle \;\sim\; \exp\,[-\kappa(\text{ Area})] \quad . \tag{3.12}$$

We have electric confinement by Wilson's criteria, with κ the electric string tension.

In the Higgs phase, ordinarily the magnetic vortex acquires a mass. Vortex particles are stable, the vacuum is not magnetically disordered and electric flux is not confined. Monopoles and antimonopoles act as sources that create or destroy the vortex particles, but since they are joined by massive magnetic flux tubes they are strongly correlated and magnetic disorder is not generated. Wormholes do break intrinsically the topological $U(1)$ symmetry so that vortex number is not precisely conserved, but they do not forbid the Higgs phase.

We now turn to the 2+1 dimensional $SU(N)$ Yang-Mills wormholes. The phases the gauge theory may exist in are described by the realization of the Z_N symmetry [4]. If the Z_N symmetry is unbroken, vortex particles are stable, there is no magnetic disorder and the theory is in a Higgs phase. If the Z_N symmetry is spontaneously broken, then the vacuum, which has an indefinite number of vortices, is magnetically disordered, and the theory has electric confinement. The electric flux tubes are boundaries (domain walls) between vacua with different values of the order parameter ϕ. Now in a $SU(N)$ theory N units of magnetic flux enclosed in a compact region can be smoothly deformed away into a trivial field configuration, without affecting the long range fields. This is just a statement about the Z_N symmetry. This means that there are always gauge field histories with weakly correlated magnetic fluctuations at long distances [13], and magnetic disorder implies electric confinement. Thus a (2+1) dimensional Yang-Mills theory which is not in a Higgs phase is in a confining phase.

Now to the effects of wormholes. We first note that wormholes carry Z_N magnetic flux and behave as Z_N monopoles. If we introduce matter that transforms faithfully under the center of the group, (e.g., the fundamental representation) it cannot propagate consistently on a wormhole background because it can detect the Dirac string of a Z_N monopole. Then we cannot have any wormhole configurations in which a non-trivial amount of Z_N magnetic flux vanishes down the wormhole. So we have eliminated the possibility of stable magnetic wormholes altogether.

Let us therefore assume that the matter fields in the theory are invariant under at least some subgroup of the center of the gauge group. Then some wormholes are allowed, and they induce an intrinsic breakdown of the topological Z_N symmetry. However, this is not novel to a theory with wormholes. In a theory with a hierarchy of gauge symmetry breakdown, we may generate nonsingular monopoles of finite action in the first stage of symmetry breakdown and vortices in the second stage. These vortices can terminate on the heavy monopoles generated at the first stage [12,13] and the topological symmetry is intrinsically broken. Thus the physics that arises due to wormholes can be duplicated by enlarging the gauge group, and causing it to undergo the Higgs mechanism at a large mass scale. A low energy observer will not be able to differentiate between the two cases.

However, we point out that a magnetic wormhole with a large action, and large exponential suppression may still have significant effects at low energies, compared to a conventional monopole at the same mass scale because of the arbitrariness in coupling constants induced by the sum over wormholes. The really significant contributions of these wormholes may lie in their contribution to the suppression of the cosmological constant.

4. Magnetic Wormholes in 3+1 Dimensions

First, we very briefly indicate the construction of wormhole solutions to the Yang-Mills-Einstein system in 3+1 dimensions.

One way to arrive at the solutions is to realize that classical Yang-Mills theory is conformally invariant. This means that a flat space solution to the Yang-Mills equations remains a solution on spaces with conformally flat metrics. Then it is just a question of satisfying the Einstein equation, and verifying that the geometry of the solution is that appropriate to a wormhole.

We follow Hosoya and Ogura (HO) [2,14]. We start with the action of a $SU(2)$ gauge field coupled to gravity.

$$S = \int d^4x\sqrt{g} \left[\Lambda - \frac{1}{16\pi G} R + \frac{1}{4e^2} g^{\mu\nu} g^{\sigma\rho} F^a{}_{\mu\sigma} F^a{}_{\nu\rho}\right] \quad . \tag{4.1}$$

The equations of motion derived from the action are the Einstein equation

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G(T_{\mu\nu} - \Lambda g_{\mu\nu}) \quad . \tag{4.2a}$$

$$T_{\mu\nu} = \frac{1}{e^2}\left[F^a{}_{\rho\mu}F^{a\rho}{}_{\nu} - \frac{1}{4}g_{\mu\nu}F^a{}_{\rho\sigma}F^{a\rho\sigma}\right] \quad . \tag{4.2}$$

and the Yang-Mills equation

$$D^\mu F^a{}_{\mu\nu} = 0 \quad . \tag{4.2c}$$

We use a generalization of the HO ansatz that is most conveniently described in their language. We define the one-forms σ^a that satisfy the Cartan-Mourer structure equation

$$d\sigma^a + \epsilon^{abc}\sigma^b \wedge \sigma^c = 0. \tag{4.3}$$

The σ may be obtained from

$$\sigma = \frac{1}{2}\Omega^{-1}d\Omega \tag{4.4}$$

where Ω is the standard identity map form the three-sphere to the group $SU(2)$.

The ansatz for the metric and gauge field are

$$ds^2 = dr^2 + a^2(r)\sigma^a \otimes \sigma^a \tag{4.4a}$$

$$A^a = A^a_\mu dx^\mu = h(r)\sigma^a. \tag{4.4b}$$

(HO consider the case $h(r) = 1$). The identification of the color and space indices and the condition Eq.(4.3) make the solution be simple. After some work we find the Einstein equation to be

$$a'^2 = 1 - \frac{r_0^2}{a^2}h^2(2-h)^2 + r_0^2 h'^2 - H^2 a^2 \quad , \tag{4.5}$$

where $H^2 = 8\pi G\Lambda/3$ and $r_0 = \sqrt{4\pi G/e^2}$. The Yang-Mills equation is

$$2a(ah')' - h(h-1)(2-h) = 0 \quad . \tag{4.6}$$

Here $'$ denotes differentiation with respect to r.

Important in the sequel is the "charge" Q evaluated on each concentric three sphere

$$Q = \int_{S^3} \omega_{cs} \quad . \tag{4.7}$$

Here ω_{cs} is the Chern-Simons from

$$\omega_{cs} = \frac{1}{8\pi^2} \text{ tr } (AdA + \frac{2}{3} A^3) \quad , \tag{4.8}$$

which satisfies

$$d\omega_{cs} = \frac{1}{8\pi^2} \text{ tr } (F^2) \quad . \tag{4.9}$$

The quantity on the right-hand side of Eq. (4.9) is the Pontriagin topological charge density, or instanton density. The Pontriagin charge turns out to be

$$Q = \frac{3}{4}(h^2 - \frac{h^3}{3}) \quad . \tag{4.10}$$

.

The total instanton number in the four-volume bounded by two nested closed three-surfaces is the difference between the values of the charge Q on the two surfaces. The instanton in flat spacetime (with $Q = 1$) corresponds to $h = 2, a(r) = r$. The HO solution with $h = 1, a(r) = \sqrt{(r_0^2 + r^2)}$ has $Q = 1/2$. For each value of $Q \in (0,1)$ wormhole solutions can be constructed.

If we ignore the Einstein equation, then the Yang-Mills field with $h = 1$ is the flat space meron solution of de Alfaro, Fubini and Furlan [14]. This solution has a field strength that varies as a function of the distance r from its center like r^{-2}. The components of the field strength are

$$F^a_{ri} = 0 \quad , \quad F^a_{ij} = \frac{\epsilon_{aij}}{r^2} \quad . \tag{4.11}$$

If we identify r with Euclidean "time" then the "electric" field strength vanishes and the solution is purely "magnetic." Since the meron field strength behaves like r^{-2}, its Euclidean action diverges logarithmically at both $r = 0$ and $r = \infty$. The two singularities are identical in form and by a conformal transformation we may convert the spherically symmetric solution to a "meron pair" configuration that is nonsingular at $r = \infty$ and has two singular points that are separated by a finite distance R. If the cores of the meron pair are smeared out we get a configuration that has a finite action that increases logarithmically with the separation R.

When Yang-Mills theory is coupled to gravity the singularity at $r = 0$ opens out a wormhole with a throat of size r_0. The wormhole swallows the Pontriagin charge Q. The Euclidean action of this wormhole diverges logarithmically for large distances because of the r^{-2} decay of the field strength. This remains true for $Q \in (0,1)$. These solutions however have non-vanishing electric field strengths. The solutions may be constructed so that the electric field vanishes at the neck of the wormhole. The neck of the wormhole then can be matched on to real time evolution, and the wormhole describes quantum mechanical tunnelling between universes.

What role do these solutions play in the low energy behavior of Yang-Mills theories? We suspect, none at all. The flat space meron, or meron pair, is unstable.

A widely separated meron pair, for example, has a Euclidean action that increases logarithmically with the separation R. It is clear that by dressing the meron with one-half unit of instanton number the action can be reduced [15]. Now the pair has the long range field of an instanton and is non-interacting. Gravity does not prevail over this long range instability of the Yang Mills field and the wormhole is also unstable. This instability afflicts all the solutions discussed above. It is indicative of another solution of lower action, (rather than a configuration of lower action, via say, a conformal transformation of the metric). This instability makes it difficult to make sense of the sum over the small fluctuations about the wormhole solution. These wormholes would contribute only if we force the existence of more than two disconnected components of the bounding three geometries by some choice of boundary conditions.

We do note that the presence of wormholes makes it possible to have large gauge transformations that change the Pontriagin number of the vacuum without instantons. (An instanton is necessary to smooth out the singular core of a large gauge transformation. The wormhole neck serves this purpose just as well.) However these would be suppressed by the wormhole action rather than an instanton action, and secondly, would occur at too high an energy scale to be relevant to the determination of the low energy value of $\theta_{Q.C.D}$.

The fact that the standard model-gravity system has wormhole solutions may have no significant consequences for low energy physics.

References

1. G. Gilbert, Nucl. Phys. **B328** (1989) 159
2. A. Hosoya and W. Ogura, "Wormhole Instanton Solution in the Einstein-Yang-Mills System," University of Tokyo preprint INS-Rep.-773 (1989).
3. R. Brandt and F. Neri, Nucl. Phys. **B161** (1979) 253.
4. G. 't Hooft, Nucl. Phys. **B138** (1978) 1.
5. S. Coleman and K. Lee, "Wormholes Made Without Massless Matter Fields," Harvard preprint HUTP-89/A022 (1989).
6. L. F. Abbott and M. B. Wise, "Wormholes and Global Symmetries," Caltech preprint CALT-68-1523 (1989).
7. S. Coleman, Nucl. Phys. **B307** (1988) 867.
8. S. B. Giddings and A. Strominger, Nucl. Phys. **B307** (1988) 854.
9. A. M. Polyakov, Phys. Lett. **59B** (1975) 82R BACKWA; Nucl. Phys. **B120** (1978) 477.
10. J. Preskill, Ann. Rev. Nucl. Part. Sci. **34** (1984) 461; in *Architecture of Fundamental Interactions at Short Distances*, ed. P. Ramond and R. Stora (North-Holland, Amsterdam, 1987).
11. S. Coleman, in *The Unity of the Fundamental Interactions,* ed. A. Zichichi (Plenum, London, 1983).
12. R. P. Feynman, Nucl. Phys. **B188** (1981) 479.
13. S.-J. Rey, "Spacetime Wormholes with Yang-Mills Fields,"
14. C. G. Callan, R. Dashen, and D. J. Gross, Phys. Rev. **D17** (1978) 2717.
15. V. de Alfaro, S. Fubini, and G. Furlan, Phys. Lett. **B65** (1976) 163.

Chapter Three

I. Introduction

Classical gravity in two dimensions is trivial. This is because of the very simple geometry of two-manifolds. However quantum gravity in two dimensions has recently attracted a lot of attention. It is interesting from the viewpoint of the statistical mechanics of random surfaces, strings in non-critical dimensions. In many cases, it is exactly solvable, by discretizing and taking a continuum limit, or the methods of topological field theory. The field is progressing very rapidly at present and the final words have yet to be written. The work presented here was performed in the infancy of the field, when many of the new results and techniques were not yet developed.

Following the work of Polyakov[1] and subsequently Knizhik, Polyakov and Zamolodchikov [2] there has been progress in the conformal gauge by David, Distler and Kawai (DDK) [3,4]. Anomalous dimensions derived from these agree with the exact solutions using random matrix methods. However it would be interesting to compute other physical quantities and comparing the answers using all available approaches, if for no other reason than verifying the correspondence and consistency between the various methods. (We note that there has been some work in conformal gauge on random surfaces with the topology of the disc using operator methods [7].)

Here we restrict ourselves to the conformal gauge and calculate in the semiclassical limit [8] and exactly, following the DDK ansatz, partition functions and some correlation functions. We find that exact results on genus zero surfaces in the DDK approach are possible for some "magic" values of the central extension (for which values the operator algebra is solvable in the work of [7]).

We consider conformally invariant matter fields X that live on a two-dimensional worldsheet. When we study the dynamics of these fields, we find that the metric of the worldsheet enters the dynamics via the conformal anomaly. Indeed, only when the anomaly has a critical value does the metric decouple, and only then can we construct the theory of critical strings. In two-dimensions the form of the metric is very simple. On any coordinate patch it is possible to make a change of coordinates in which the metric is conformal to a fiducial metric, i.e.,

$$g_{\mu\nu} = e^{\phi}\hat{g}_{\mu\nu}.$$

What enters the dynamics then, is the conformal factor ϕ. Apart from this, there are additional parameters, the modular parameters τ which parametrize the globally inequivalent ways of attaining conformal gauge. (These are absent on genus zero surfaces.)

For manifolds with a fixed topology the partition function is

$$Z = \frac{1}{\mathrm{V_{CKV}}} \int (d\tau)[D_g\phi][D_gX]Je^{-[S_M(X,g)+S_{\mathrm{grav}}]} \quad . \tag{1.1}$$

Here $(\mathrm{V_{CKV}})$ is the volume generated by the conformal killing vectors, and $J/(\mathrm{V_{CKV}})$ is a determinant that arises from gauge fixing. $S_{\mathrm{grav}}[g]$, the Einstein–Hilbert action in two dimensions, is given by

$$S_{\mathrm{grav}}[g] = \int d^2\xi\sqrt{g}\left(\frac{\ell n\lambda}{8\pi}R + \tilde{\mu}\right) \tag{1.2}$$

where λ and $\tilde{\mu}$ are the "string coupling constant" and the cosmological constant, respectively. In two dimensions

$$\frac{1}{8\pi}\int d^2\xi\sqrt{g}R = (1-h) \tag{1.3}$$

where h is the genus of the surface. So

$$S_{\mathrm{grav}}[g] = \ell n\lambda\,(1-h) + \int d^2\xi\sqrt{g}\,\tilde{\mu} \quad . \tag{1.4}$$

The conformal invariance of the matter action implies that if

$$g = e^{\phi}\hat{g} \quad , \tag{1.5}$$

where $\hat{g}$ is an arbitrary reference metric, then

$$S_M[X,g] = S_M[X,\hat{g}] \quad . \tag{1.6}$$

However, since the path–integral in eq. (1.1) must be regulated in a generally covariant fashion the measure D_gX depends on ϕ. Since, as far as X is concerned,

ϕ can be treated as a background field it is straightforward to explicitly display the dependence of the measure on ϕ

$$\int [D_g X]\ e^{-S_M[X,g]} \ = \ \int \ [D_{\hat{g}} X]\ e^{-S_M[X,\hat{g}]}\ e^{+\frac{d}{48\pi} S_L[\phi,\hat{g}]} \tag{1.7}$$

where d is the central charge of the matter theory and $S_L[\phi, \hat{g}]$ is the Liouville action

$$S_L[\phi,\hat{g}] \ = \ \int \ d^2\xi\sqrt{\hat{g}}\ \left[\frac{1}{2}\hat{g}^{ab}\partial_a\phi\partial_b\phi + \hat{R}\phi + \bar{\mu}e^{\phi}\right] \quad . \tag{1.8}$$

Similarly, the dependence of the determinant J on the conformal factor ϕ can be determined

$$Z \ = \ \frac{1}{(\mathrm{V_{CKV}})} \ \int \ (d\tau)[D_g\phi][D_{\hat{g}}X]\hat{J}\ e^{-S_M[X,\hat{g}]}\ e^{+\left(\frac{d-26}{48\pi}\right)\ S_L[\phi,\hat{g}]}\ e^{-S_{\mathrm{grav}}[g]} \quad . \tag{1.9}$$

In the measure for the Liouville field ϕ, the metric cannot be treated as a background field. This makes it more difficult to deduce the Jacobian involved in going from $[D_g\phi]$ to $[D_{\hat{g}}\phi]$.

DDK assume that it can be expressed as a sum over local terms involving the ϕ field and that these terms are of the same form as those originally appearing in the Liouville action. After appropriate rescalings this ansatz gives

$$Z \ = \ \frac{\lambda^{(h-1)}}{(\mathrm{V_{CKV}})} \ \int \ (d\tau)[D_{\hat{g}}\phi][D_{\hat{g}}X]\hat{J}\ e^{\frac{1}{8\pi}\int d^2\xi\sqrt{\hat{g}}[\phi\ \hat{\Box}\ \phi + Q\hat{R}\phi + \mu e^{\alpha\phi}]}\ e^{-S_M[X,\hat{g}]} \quad . \tag{1.10}$$

Treating μ perturbatively, they show that the choice of the background metric $\hat{g}$ is irrelevant if

$$Q = \sqrt{\frac{25-d}{3}} \tag{1.11a}$$

$$\alpha = \frac{-1}{2\sqrt{3}}\left[\sqrt{25-d} \pm \sqrt{1-d}\right] \quad . \tag{1.11b}$$

Comparison with results of the semiclassical approximation reveals that the minus sign should be chosen in eq. (1.11b).

Similarly, if an insertion of the average over the two–dimensional space of a primary scalar field, O_M, of the matter theory is desired then

$$\int d^2\xi\sqrt{g(\xi)}\; O_M(\xi) \tag{1.12a}$$

is replaced by

$$\int d^2\xi\sqrt{\hat{g}}\; O_M(\xi)e^{\beta\phi} \tag{1.12b}$$

Dependence on the choice of background metric drops out of correlations functions if the operator $O_M(\xi)e^{\beta\phi}$ is a (1,1) operator. This condition determines (again perturbatively in μ) that

$$\beta \;=\; \frac{-1}{2\sqrt{3}}\left[\sqrt{25-d}\pm\sqrt{1-d+24\Delta}\right] \tag{1.13}$$

where Δ is the scaling dimension of O_M. Comparison with semiclassical results determines that the minus sign is appropriate in eq. (1.13). The partition function in eq. (1.10) doesn't exist for genus zero and one surfaces since the action (for a positive cosmological constant) is minimized as $\phi \to -\infty$; implying that the partition function is dominated by surfaces with very small area. A quantity which is well–defined (and one we will study) is the partition function for surfaces with fixed area A. With zero cosmological constant it is

$$\begin{aligned} Z[A] = \frac{\lambda^{(h-1)}}{(\mathrm{V_{CKV}})} &\int [D_{\hat{g}}\phi][D_{\hat{g}}X]\hat{J}\; e^{\frac{1}{8\pi}\int d^2\xi\sqrt{\hat{g}}[\phi\Box\phi+Q\hat{R}\phi]}\; e^{-S_M[X,\hat{g}]} \\ &\delta\left(\int d^2\xi\sqrt{\hat{g}}\; e^{\alpha\phi}-A\right) \quad . \end{aligned} \tag{1.14}$$

Typically, the delta function couples together the various modes of ϕ, making this theory non–trivial.

Partition Function on genus one surfaces

The method of computation follows the original string theory calculations of Polchinski [11]. Eq. (1.10) factors into contributions from the X integrations, the contributions from the ghosts (i.e., $\hat{J}$) and the contribution from the Liouville field. The latter is

$$Z_L[A,\tau] \;=\; \int [D_{\hat{g}}\phi]\; e^{\frac{1}{8\pi}\int d^2\xi\sqrt{\hat{g}}(\phi\hat{\Box}\phi + Q\hat{R}\phi)}\;\delta\left(\int d^2\xi\sqrt{\hat{g}}\;e^{\alpha\phi} - A\right) \quad . \qquad (2.1)$$

Choose $\hat{g}$ to be a constant curvature metric ($\hat{g}_{11} = 1$, $\hat{g}_{12} = \tau_1$, $\hat{g}_{22} = \tau_1^2 + \tau_2^2$). Here $\tau = \tau_1 + i\tau_2$ is the modular parameter of the torus. In this metric $\hat{R} = 0$.

Expand the Liouville field ϕ in eigenfunctions of the Laplacian $\hat{\Box}$:

$$\phi = \frac{C_{0,0}}{\sqrt{\tau_2}} + \sum_{n_1,n_2} C_{n_1,n_2}\;\phi_{n_1,n_2} \qquad (2.2)$$

The normalization is

$$\int d^2\xi\sqrt{\hat{g}}\phi^*_{n_1,n_2}\;\phi_{n_1',n_2'} \;=\; \delta_{n_1,n_1'}\delta_{n_2,n_2'} \qquad (2.3)$$

The measure is

$$[D_{\hat{g}}\phi] \;=\; \prod_{n_1,n_2}\left(\frac{dC_{n_1,n_2}}{\sqrt{2\pi}}\right) \quad . \qquad (2.4)$$

The vanishing of $\hat{R}$ means that the zero mode part of ϕ does not appear in the action, and so the delta function integral can be performed without introducing interactions in the theory. We find then that

$$Z_L[A,\tau] = \frac{1}{\sqrt{2\pi}}\cdot\frac{\sqrt{\tau_2}}{\alpha A}\left[\det{}'\left(-\frac{\hat{\Box}}{4\pi}\right)\right]^{-1/2} \qquad (2.5)$$

$$= \left(\frac{1}{\sqrt{8\pi}}\cdot\frac{\sqrt{\tau_2}}{\alpha A}\right)\left[\tau_2^2\;e^{-\pi\tau_2/3}\left|f(e^{2\pi i\tau})\right|^4\right]^{-1/2} \qquad (2.6)$$

with

$$f(e^{2\pi i\tau}) \;=\; \prod_n (1 - e^{2\pi i n\tau}) \qquad (2.7)$$

.

The τ–dependence of the Liouville field's contribution to the partition function is the same as that of a free scalar field. The contribution from the ghosts is the same as in string theory

$$\hat{J} = \frac{1}{2\tau_2^2}\left[\tau_2 \, e^{-\pi\tau_2/3}|f|^4\right] \tag{2.8}$$

while the matter contribution $Z_M(\tau)$ depends on the particular conformal field theory chosen.

Putting these results together gives

$$Z(A) = \left(\frac{1}{2\sqrt{8}\pi}\,\frac{1}{\alpha A}\right)\int \frac{d^2\tau}{\tau_2^2}\left(\tau_2 e^{-\pi\tau_2/3}|f|^4\right)^{1/2} Z_M(\tau) \tag{2.9}$$

where the τ integral is over the usual fundamental region.

The modular dependence of the partition function arising from integrating over the Liouville field is independent of the matter sector and modular invariant on its own. Since we expect for unitary matter that its contribution to the modular dependence for large τ_2 is

$$Z_M(\tau) \rightarrow e^{(\pi\tau_2 \; d/6)} \tag{2.10}$$

we expect the complete integral to have the factor

$$Z_M(\tau) \rightarrow e^{(\pi\tau_2 \; (1-d)/6)} \tag{2.11}$$

For $d > 1$ the surfaces tend to degenerate to those with infinite values of τ_2. This is in accordance with the view that the surface becomes a branched polymer, and is one more manifestation of the well-known change in character of the theory for $d > 1$. The string theory view would be that there is a tachyon in the spectrum, though of course we have no space-time interpretation for $d \leq 1$.

III. Partition Function on the Sphere

Choose the background metric $\hat{g}$ to correspond to the sphere of unit radius. Proceed as in the case of the torus: expand the Liouville field in eigenfunctions of the Laplacian on the sphere, and use the delta function to do the integral over the constant mode. This yields

$$Z_L[A] = \frac{\sqrt{2}}{\alpha A} \int [D_{\hat{g}}\phi'] \exp\left(\frac{1}{8\pi} \int d^2\xi \sqrt{\hat{g}}(\phi' \hat{\Box} \phi')\right) \left[\frac{\int d^2\xi \sqrt{\hat{g}}\, e^{\alpha\phi'}}{A}\right]^{-Q/\alpha} . \tag{3.1}$$

where ϕ' denotes the nonzero modes of ϕ. The calculation can proceed without approximation when the central extension d is such that $-Q/\alpha$ is a non-negative integer n. The path integral then corresponds to a free field theory with n vertex operator insertions. It is convenient to map the sphere to the complex plane by stereographic projection and we find that

$$Z_L[A] = \frac{\left(e^{-(\frac{n^2\alpha^2}{2})} \cdot \sqrt{2}\right)}{\alpha} \cdot \frac{1}{A^{n+1}} \cdot (\epsilon^2)^{\frac{-n\alpha^2}{2}} \cdot \left[\det{}'\left(-\frac{\hat{\Box}}{4\pi}\right)\right]^{-1/2} \cdot \int d^2\, z_1 \ldots \int d^2\, z_n \prod_{i<j} \frac{1}{|z_i - z_j|^{2\alpha^2}} \tag{3.2}$$

where ϵ^2 is an invariant short distance cutoff on the unit sphere. There are cutoff dependent pieces that arise form the determinant in eq.(3.2) and from the matter and ghost integrals as well. These can all be absorbed in the bare string coupling. The renormalized partition function is then defined in terms of a dimensionful renormalized string coupling.

The partition function $Z_L(A)$ has a $SL(2, C)$ symmetry which enables us to cancel the volume of conformal Killing vectors (see eq.(1.14)) against three of the integrals in eq.(3.2). Naively, when $n < 3$, the partition function on the sphere vanishes. (This argument may be too naive. See ref [13].) The relationship between the central extension and n is, using eq.(1.11),

$$d \;=\; -\frac{6n^2-25n+25}{n-1} \quad . \tag{3.3}$$

For $d=1$ the number of integrals is only two and the partition function vanishes. This may be related to the logarithmic contribution to scaling seen in the random matrix approach [6].

For consistency, it is necessary that we show that the partition function that we compute is independent of the choice of background metric $\hat{g}$. We show that this is indeed the case for genus zero surfaces, when$-Q/\alpha = n$, a non-negative integer.

We start with a fiducial metric $\bar{\hat{g}} = e^{\sigma}\hat{g}$, where $\hat{g}$ is the (constant curvature) metric on the unit sphere and show that the partition function is independent of σ. $Z_L[A]$ is given by

$$Z_L[A] = \int [D_{\bar{\hat{g}}}\phi] \exp\left[\frac{1}{8\pi}\int d^2\xi\sqrt{\bar{\hat{g}}}\left[\phi\bar{\hat{\Box}}\phi + Q\bar{\hat{R}}\phi\right]\right]\cdot\delta\Big(\int d^2\xi\sqrt{\bar{\hat{g}}}\, e^{\alpha\phi} - A\Big) \quad . \tag{3.4}$$

Since σ can be treated as a background field

$$[D_{\bar{\hat{g}}}\phi] \;=\; [D_{\hat{g}}\phi]\; e^{\frac{1}{48\pi}S_L[\sigma]} \quad . \tag{3.5}$$

Expanding ϕ in eigenfunctions of $\hat{\Box}$, as before, gives

$$Z_L[A] = \exp\left(\frac{1}{48\pi}S_L[\sigma]\right)\frac{\sqrt{2}}{\alpha A}\int [D_{\hat{g}}\phi']\; e^{\frac{1}{8\pi}\int d^2\xi\sqrt{\hat{g}}[\phi'\hat{\Box}\phi' - Q\hat{\Box}\sigma\phi']} \cdot\left[\frac{\int d^2\xi\sqrt{\hat{g}}\; e^{\alpha\phi'}\; e^{\sigma}}{A}\right]^n , \tag{3.6}$$

where ϕ' denotes the nonzero mode part of ϕ. Shifting the integration variable ϕ' to $\phi' - Q\sigma'/2$ (the measure is invariant under this) gives

$$Z_L[A] = \exp\left(\frac{1}{48\pi} S_L[\sigma]\right) \exp\left[\frac{1}{48\pi} 3Q^2 \int d^2\xi \sqrt{\hat{g}} \sigma \hat{\Box} \sigma\right]$$
$$\cdot \frac{\sqrt{2}}{\alpha A} \int [D_{\hat{g}}\phi'] e^{\frac{1}{8\pi}\int d^2\xi \sqrt{\hat{g}} \phi' \hat{\Box} \phi'} \left[\frac{1}{A} \int d^2\xi \sqrt{\hat{g}} e^{\sigma}\ e^{\alpha(\phi' + Q\frac{\sigma'}{2})}\ e^{\alpha\phi'}\right]^n \tag{3.7}$$

σ' denotes the nonzero mode part of σ. Performing the functional integral, regulating the short distance singularities with a cutoff that is invariant with respect to the full metric $\bar{\hat{g}}$, gives

$$Z_L[A] = \left\{\exp \frac{1}{48\pi}(1+3Q^2) S_L[\sigma]\right\} \frac{1}{\alpha A} \left\{\prod_{i=1}^{n} \int d^2\xi_i \sqrt{\hat{g}(\xi_i)}\right.$$
$$\left. e^{\sigma(\xi_i)} e^{\left(\frac{\alpha^2}{2} + \frac{\alpha Q}{2}\right)\ \sigma(\xi_i)}\right\} \times (\text{terms independent of } \sigma) \quad . \tag{3.8}$$

In the derivation of eq. (3.8) we redistributed some of the dependence on the zero mode of σ between the two terms in brace brackets. Since $\frac{1}{2}\alpha(\alpha + Q) = -1$ the σ–dependence drops out of the second term in brace brackets. Furthermore, in $Z[A]$, the matter and ghost contributions dependence on σ cancels that of $Z_L[A]$ since

$$1 + 3Q^2 + (d - 26) \ = \ 0 \quad . \tag{3.9}$$

IV. Correlation Functions

We outline here the procedure that enables us to compute some integrated correlation functions for certain "magic " values of the central extension. For these values these functions reduce to free field theory with some number of vertex operator insertions, as in the computation of the partition function.

Consider an operator in the matter sector $O_M(\xi)$ with dimensions (Δ, Δ). This operator is replaced by the operator $O(\xi) = e^{\beta\phi(\xi)} O_M(\xi)$, where the dressing factor $e^{\beta\phi}$ is determined by demanding that $O(\xi)$ be a $(1,1)$ operator. Quantities independent of the choice of background metric are formed by integrating the operator over the surface. The correlation functions of conformally coupled matter factorize into a product of Liouville and matter sector functions. The Liouville sector correlation function is given by

$$\int [D_{\hat{g}}\phi] e^{\frac{1}{8\pi}\int d^2\xi\sqrt{\hat{g}}[\phi\hat{\Box}\phi+Q\hat{R}\phi]}\, \delta\left(\int d^2\xi\sqrt{\hat{g}}\; e^{\alpha\phi} - A\right)\cdot e^{\beta\phi(\xi_1)}\ldots e^{\beta\phi(\xi_N)} \tag{4.1}$$

Performing the integral over the constant mode of ϕ gives that the above is equal to

$$\frac{\sqrt{2}}{\alpha A}\int [D_{\hat{g}}\phi']\; e^{\frac{1}{8\pi}\int d^2\xi\sqrt{\hat{g}}[\phi'\hat{\Box}\phi']}\; e^{\beta\phi'(\xi_1)}\ldots e^{\beta\phi'(\xi_N)}\left[\frac{\int d^2\xi\sqrt{\hat{g}}e^{\alpha\phi'}}{A}\right]^{-\frac{Q}{\alpha}-\frac{N\beta}{\alpha}} \tag{4.2}$$

where ϕ' is the non–constant part of the Liouville field. As before, if

$$-\left(\frac{Q}{\alpha}+\frac{N\beta}{\alpha}\right) = n \quad , \tag{4.3}$$

where n is a non–negative integer, the above path integral is straightforward to do. The restriction in eq. (4.3) means that only correlation functions involving specific numbers of operators can be done with this method. Once the Liouville correlation function is known, it can be combined with the matter correlation, deduced from conformal field theory [17], and then integrations over the surface can be performed to get a gauge invariant quantity.

In this procedure there will be short distance divergences that have to be regulated. One procedure is to introduce a world sheet cutoff, and to subtract all cutoff dependent pieces. When logarithmic divergences occur, they cannot be subtracted without introducing another scale into the theory. In string theory, these divergences are interpreted as poles in the S-matrix. Here, the interpretation is not clear. We believe that subtraction of power law divergences is correct. It is straightforward to show that the answers obtained in these cases is independent of the choice of background metric. Many such examples may be constructed using the $d < 1$ minimal models of conformal field theory.

Correlations that are not computable by the above methods may be estimated using the semiclassical approximation, which improves as $d \to -\infty$. We do not perform the DDK rescaling (eq. 1.1) but merely integrate small fluctuations about a stationary point of the action. For example, here we compute all the connected correlations of the operator R^2 integrated over the surface. We write the generating functional

$$Z(\alpha) = \frac{1}{(\mathrm{V_{CKV}})} \int [D_g \phi] \, e^{-\left(\frac{26-d}{48\pi}\right) S_L} \delta\left(\int d^2\xi \sqrt{g} - A \right) \, e^{-\alpha \int d^2\xi \sqrt{g} R^2} \quad . \quad (4.5)$$

The connected correlation functions are generated by differentiation with respect to α. Actually our generating function is singular at $\alpha = 0$, so what we mean is the piece left over after subtracting away this singularity. We choose the fiducial metric $\hat{g}_a b(\xi)$ to be the metric on a unit sphere, and the Liouville mode to be given by

$$g_{ab}(\xi) = e^{\phi(\xi)} \hat{g}_{ab}(\xi) \quad . \quad (4.7)$$

We expand ϕ in properly normalized spherical harmonics and perform the path integral, after expanding to terms of quadratic order in the coefficients of the expansion. A point to note is that the integration of the $l = 1$ modes is proportional to the volume of conformal killing vectors, in this approximation, and so they drop out of the integral. With

$$\phi(\xi) = \sum_{\ell,m} C_{\ell,m} \, Y_{\ell,m} \, (\xi) \sqrt{\frac{4\pi}{A}} \quad , \quad (4.8)$$

we obtain

$$Z(\alpha) = \frac{1}{(\mathrm{V_{CKV}})} \, e^{\frac{(8\pi)^2 \alpha}{A}} \int \prod_{\substack{\ell,m \\ \ell \geq 1}} \frac{dC_{\ell m}}{\sqrt{2\pi}} \, \exp\Bigg\{ - \left(\frac{-d}{12A}\right) \sum_{\ell,m} \frac{1}{2} |C_{\ell m}|^2$$

$$\left(\Big[\ell(\ell+1) - 2\Big] + \frac{384\pi^2 \alpha}{(-d)A} \Big[\ell(\ell+1) - 2\Big]^2 \right) \Bigg\} \quad . \quad (4.10)$$

The integration over the $C_{l,m}$ with $l \geq 2$ produces a determinant which is regulated using zeta function techniques. We obtain

$$\ell n Z(\alpha) = \left(\frac{(8\pi)^2\alpha}{A}\right) - \frac{1}{2}\,\zeta'(0) + \frac{1}{2}\,\ell n\left[\left(\frac{-d}{12A}\right)\right]\zeta(0) + \ldots \qquad (4.11)$$

where

$$\zeta(s) = \sum_{\ell=2}^{\infty} (2\ell+1)\left\{\left[\ell(\ell+1)-2\right] + \frac{384\pi^2\alpha}{(-d)A}\left[\ell(\ell+1)-2\right]^2\right\}^{-s} \quad . \qquad (4.12)$$

Now we expand in α:

$$\zeta(s) = \sum_{k=1}^{\infty}\left(\frac{-384\pi^2\alpha}{(-d)A}\right)^k \frac{s(s+1)\ldots[s+k-1]}{k!}\sum_{\ell=2}^{\infty}(2\ell+1)\left[\ell(\ell+1)-2\right]^{-s+k} + \ldots \qquad (4.13)$$

where the ellipses denote the term independent of α. The sum over ℓ can be put in a convenient form for our purposes using the methods of Weisberger [19]

$$\sum_{\ell=2}^{\infty}(2\ell+1)[\ell(\ell+1)-2]^{-s+k} = \sum_{n=0}^{\infty}\frac{(-1)^n 3^n \Gamma(s-k+n-1)}{n!\Gamma(s-k)}\,(2s-2k+n-2)\zeta_R(2s-2k+n-1)\,. \qquad (4.14)$$

Combining equations (4.13) and (4.14) gives that $\zeta(0)$ is independent of α and that

$$\zeta'(0) = \sum_{k=1}^{\infty}\Gamma(k)\left[\frac{-384\pi^2\alpha}{(-d)A}\right]^k \sum_{m=0}^{[(k+1)/2]}\Gamma(2m+1)\,\frac{3^{2m}}{\Gamma(k+2-2m)}\,B_{2k+2-2m}, \qquad (4.15)$$

where B_m denotes the m'th Bernoulli number and $[(k+1)/2]$ denotes the largest natural number less than or equal to $(k+1)/2$. Because $\zeta(0)$ is independent of

α, semiclassically, the operator $\int d^2\xi\sqrt{g(\xi)}R^2(\xi)$ has no anomalous scaling while eq. (4.15) implies that its connected k–point correlations are given by

$$\left\langle\left(\int d^2\xi\sqrt{g(\xi)}R^2(\xi)\right)^k\right\rangle_c = \left(\frac{(8\pi)^2}{A}\right)\delta_{k,1} - \left[\frac{384\pi^2}{(-d)A}\right]^k \Gamma(k) \sum_{m=0}^{[(k+1)/2]} \frac{3^{2m}}{\Gamma(2m+1)} \frac{B_{2k+2-2m}}{2\Gamma(k+2-2m)} \quad . \tag{4.16}$$

So, for example, using $B_2 = 1/6$ and $B_4 = -1/30$ gives

$$\left\langle\int d^2\xi\sqrt{g(\xi)}R^2(\xi)\right\rangle = \frac{(8\pi)^2}{A} - \frac{192\pi^2}{(-d)A}\left[\frac{11}{15}\right] \quad . \tag{4.17}$$

For large negative d fluctuations decrease the average value of $\int d^2\xi\sqrt{g(\xi)}R^2(\xi)$. Similarly, correlations of higher powers of the curvature can be calculated and we find no anomalous scaling in the semiclassical approximation for these operators.

V. Conclusions

In short, we have seen how to compute certain functions in the DDK formalism. In principle some of them can be computed exactly. The weakness is that the overall normalization of the DDK partition function is not known, or derivable. The divergences that arise can be handled as long as they are not logarithmic. The interpretation of these divergences is problematic; perhaps some correlations are indeed infinite.

The partition function on the torus shows signs of the $d = 1$ transition which is believed to be caused by the domination of manifolds with large τ_2 which are like long thin tubes.

We would also like to remark that the extension of this formalism to open surfaces has not yet been made. One issue that arises is the choice of boundary conditions for the Liouville field. Our investigations in the semiclassical limit indicate that among simple choices of boundary conditions, only Neumann boundary conditions give results consistent with the general coordinate invariance of the underlying theory.

References

1. A. M. Polyakov, Mod. Phys. Lett. **A2** (1987) 893.
2. V. G. Knizhnik, A. M. Polyakov and A. B. Zamolodchikov, Mod. Phys. Lett. **A3** (1988) 819.
3. F. David, Mod. Phys. Lett. **A3** (1988) 1651.
4. J. Distler and H. Kawai, Nucl. Phys. **B321** (1989) 509.
5. F. David, Nucl. Phys. **B257** (1985) 45; V. A. Kazakov, Phys. Lett. **B156** (1985) 282; J. Ambjorn, B. Durhuus and J. Fröhlich, Nucl. Phys. **B257** (1985) 433; V. A. Kazakov, I. K. Kostov and A. A. Migdal, Phys. Lett. **B157** (1985) 295; V. A. Kazakov, Niels Bohr Inst. preprint NBI-HE-89-25 (1985).
6. V. A. Kazakov, A. A. Migdal, Nucl. Phys. **B311** (1988) 171.
7. J. L. Gervais and A. Neveu, Nucl. Phys. **B224**, 329 (1983); Nucl. Phys. **B238**, 125 (1984); **B238**, 396 (1984).
8. A. B. Zamolodchikov, Phys. Lett. **117B** (1982) 87; S. Chaudhuri, H. Kawai and S. H. Tye, Phys. Rev. **D36** (1987) 1184.
9. A. M. Polyakov, Phys. Lett. **103B** (1981) 202.
10. D. Friedan, Introduction to Polyakov's string theory, in *Proceedings of 1982 Les Houches Summer School*, ed. J. B. Zuber and R. Stora (Elsevier) (1984) 839; O. Alvarez, Nucl. Phys. **B216** (1983) 125.
11. J. Polchinski, Commun. Math. Phys. **104** (1986) 36.
12. H. W. J. Bloete, J. L. Cardy, H. P. Nightingale, Phys. Rev. Lett. **56** (1986) 742; I. Affleck, Phys. Rev. Lett. **56** (1986) 746.
13. M. Cates, Europhys. Lett. **8** (1988) 719; F. David and E. Guitter, Europhys. Lett. **3** (1987) 1169.
14. M. B. Green, J. M. Schwarz and E. Witten, *Superstring Theory*, (Cambridge University Press, Cambridge) (1987).

15. J. Liu and J. Polchinski, Phys. Lett. **B203** (1988) 39.

16. B. Grinstein and M. B. Wise, Phys. Rev. **D35** (1987) 655.

17. A. A. Belavin, A. M. Polyakov and A. B. Zamolodchikov, Nucl. Phys. **B241** (1984) 333.

18. J. Polchinski, Nucl. Phys. **B307** (1988) 61; M. Green and N. Seiberg, Nucl. Phys. **B299** (1988) 559.

19. W. L. Weisberger, Nucl. Phys. **B284** (1987) 171.

Epilogue

To quickly recount the substance of previous chapters: Topology change in gravity theories lead to a new indeterminacy of the fundamental coupling constants at low energies. This conclusion is likely to survive the ultimate correct theory of gravity. Everything else is on much shakier ground. That the low energy cosmological constant vanishes, and that its vanishing (along with the minimization of the gravitational coupling) can determine the low energy parameters for us may hold. The semiclassical sum which is an uncorrelated sum of wormhole insertions has much of the behavior we want: no non-localities are introduced, quantum coherence is maintained. Correlations between wormhole pairs do not make a difference to this. On the other hand, there is no infrared cutoff, or equivalently, a principle which bounds the range of values within which the coupling constants may lie. The suppression of large wormholes seems to require knowledge of the high energy physics. This theme of the mixing of scales that occurs again and again in the literature on the subject may be a message to us of something fundamental, or more likely, that we are simply wrong.

Topology changes can occur in models with the standard matter content. However, apart from an influence via large wormholes, and providing a mechanism for Λ to vanish, they do not seem to play a role in the real world.

Two-dimensional gravity may reward us some day by indirectly solving for us via an improved understanding of string theory the problem of gravity. Recently there has been great advances in the exact formulation and solutions of models of conformally invariant matter and dynamical metric in two dimensions. These proceed through solving a discretized version of the model and proceeding to the continuum limit. On the other hand, we may try to directly build a continuum model (say by the DDK approach). The question is, of course, is our continuum theory describing the same physics? As far as the $d = 1$ string is concerned, it appears that there are extra logarithms of the cutoff scale that appear when taking the continuum limit of the discretized version, that do not appear in the DDK approach. It may be possible to reinterpret DDK to give the correct results, but this is an ongoing story.