On quantum interacting embedded geometrical objects of various dimensions

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Dedicated to the victims of scientific progress.

Acknowledgments

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Abstract

Modern string theory naturally gives rise to an assortment of dynamical geometrical objects of various dimensions (collectively referred to as “branes”) embedded into space-time. The aim of this thesis is to present a series of results (of varying novelty and rigor) pertinent to dynamics of the low-dimensional geometrical objects of this kind. The processes considered are the D0-brane recoil and annihilation, “local recoil” of D1-branes (which is a peculiar effect manifested by one-dimensional topological defects in response to an impact, and closely related to soliton recoil), and D- and F-string loop mixing. Apart from the practical relevance within the formalism of string theory, such considerations are worthwhile in that the quantum dynamics of the geometrical objects involved is complex enough to be interesting, yet simple enough to be tractable. Furthermore, some of the results derived here within the string theory formalism may give valuable insights into the dynamics of low-dimensional field-theoretical topological defects.

Contents

Introduction

String theory is both praised and denounced for its mathematical complexity, yet the fact remains that it provides surprisingly simple solutions to *some* physically non-trivial problems. Thus, the question of graviton scattering, which lies at the very heart of the subject, becomes eventually reduced to the dynamics of *free* quantum fields in two dimensions.

Among such remarkable mathematical simplifications is the seminal discovery that certain topological defects in space-time (D-branes) can be described by introducing Dirichlet boundary conditions for strings [1]. The question of scattering in the presence of such topological defects is thereby reduced to calculating correlation functions in a free two-dimensional field theory ("worldsheet CFT") with the Dirichlet boundary conditions.

As attractive as the description of the D-branes in terms of Dirichlet boundary conditions may appear, it fails for the low-dimensional cases, due to the presence of infrared divergences. The simplest example of this phenomenon is the case of D-instantons, which had received a fairly complete treatment in [2]. It is one of the primary motivations for this thesis to develop an adequate formalism for the cases of D0- and D1-branes, which, in analogy to the D-instantons, suffer from an infrared failure of the worldsheet CFT description.

For the case of D0-branes, the presence of the infrared divergences in the worldsheet CFT is intimately related to the motion of the D0-brane center-of-mass. The strategy we adopt in handling this situation is then very similar to the case of D-instantons. Instead of integrating over the positions of the D-instanton, as it has been done in [2], we shall integrate over the trajectories of the D0-brane. Such formalism was first introduced in [4], and it will be described below in further detail.

Once the translational motion of the D0-brane is explicitly introduced, and the corresponding difficulties of the conventional CFT description are eliminated, there are two processes (related by crossing symmetry) that naturally call for consideration: the D0-brane recoil and the D0-brane annihilation. These subjects dominate the considerations in the first three chapters of the present thesis.

The remaining chapters are devoted to the studies of the D1-branes. It appears to be fairly little appreciated that the worldsheet CFT description of the D1-branes breaks down in a manner very similar to the case of D0-branes. One should therefore expect that the D1-brane exhibits response phenomena analogous to the D0-brane recoil. This subject will be discussed in chapter 4, where such a response phenomenon exhibited by the D1-branes (termed the *local recoil*) will be analyzed in detail.

Finally, in chapter 5, I'll present some preliminary considerations pertinent to the question of dynamical mixing of loops made of the D1-branes and fundamental strings, and its potential role in the $SL(2, Z)$ duality of the type IIB string in light of graviton uniqueness.

Mathematically, all the problems considered here deal with low-dimensional interacting quantum geometrical objects embedded in space-time, as reflected in the title of this thesis. The first three chapters focus on the interactions of particles and strings, whereas the latter two focus on the interactions of two different types of strings. These quantum geometries are certainly quite simple compared, say, to the (conjectural) quantum geometry of the space-time itself. Nevertheless, working with quantum geometrical structures is certainly quite pleasing aesthetically.

All the considerations of this thesis are carried out for the case of bosonic string. It is certainly true that, by itself, this theory suffers from the well-known tachyon problems, yet the degrees of freedom relevant to the considerations of this thesis are the ones corresponding to the space-time motion of the embedded geometrical objects whose dynamics is being investigated, and thus these degrees of freedom should be present in a universal form in any realistic string theory.

I have refrained from giving a detailed account of the basic dynamics of D-branes. This material is widely known by now and presented in a number of textbooks. Another due organizational remark concerns the order of chapters. The results and considerations presented in this thesis are interrelated in a number of ways, and it would be in any case impossible to do justice to these interrelations while maintaining the linear exposition structure dictated by the framework of a thesis. I have therefore chosen to present the various issues in order in which they were investigated. It is for the lack of opportunity to do full justice to the logic of the subject that one has to resort to the logic of autobiography.

Chapter 1

Worldline approach to D0-brane annihilation

The physics of topological defects in relativistic field systems is a rather accomplished area of study [3], yet fairly little remains known about how such localized objects are pair-produced in collisions of elementary particles, and how they annihilate. It is of very little use that the scattering of the fundamental field quanta off a topological defect (a process related to annihilation by crossing symmetry) has been analyzed in quite some detail. The masses of the solitons are typically inversely proportional to the coupling constant, forcing the annihilation products into the high-energy regions of the phase space. Transition between the scattering and annihilation kinematic regions would then require an analytic continuation *non-perturbatively* far in the phase space, and the *perturbative* expansion of the amplitude in the scattering region will be essentially unrelated to the values of the amplitude in the annihilation region.

Even though there must exist a classical solution of the field equations describing the annihilation process, it does not appear to be accessible to any degree of analytic control. Indeed, there is apparently no straightforward way to prove that such a solution encapsulates even the most fundamental features to be expected from the annihilation process, e.g. the unitarity restrictions on the exclusive annihilation/pair-production amplitudes that we'll discuss below.

One may expect that the situation becomes even more involved for the solitonic objects of string theory, the D-branes, since, in that case, we would have to deal with all the complexities of quantum gravity in addition to the problems that plague the treatment of the field-theoretical process. It has been known since the seminal work by Polchinski [1] that the solitons of string theory admit a particularly simple description in terms of the Dirichlet boundary conditions for open strings. Indeed, the analysis of elementary quanta scattering in the presence of D-branes is substantially simpler than the corresponding problem in the conventional relativistic field theory, as it merely amounts to computing the expectation values of certain operators in a free two-dimensional field system. One may therefore hope that worldsheet techniques would provide a valuable tool for understanding the D-brane annihilation, a tool unavailable to the corresponding field theory considerations.

Here, we attempt to construct a computational scheme for the D0-brane annihilation amplitude inspired by the simplicity of the CFT description of the extended D-branes. Since the center-of-mass motion of the D0-branes is of a crucial importance for the processes we intend to consider, it is only natural that we introduce an explicit functional integration over the D0-brane worldlines, to which the string worldsheets are attached via the Dirichlet

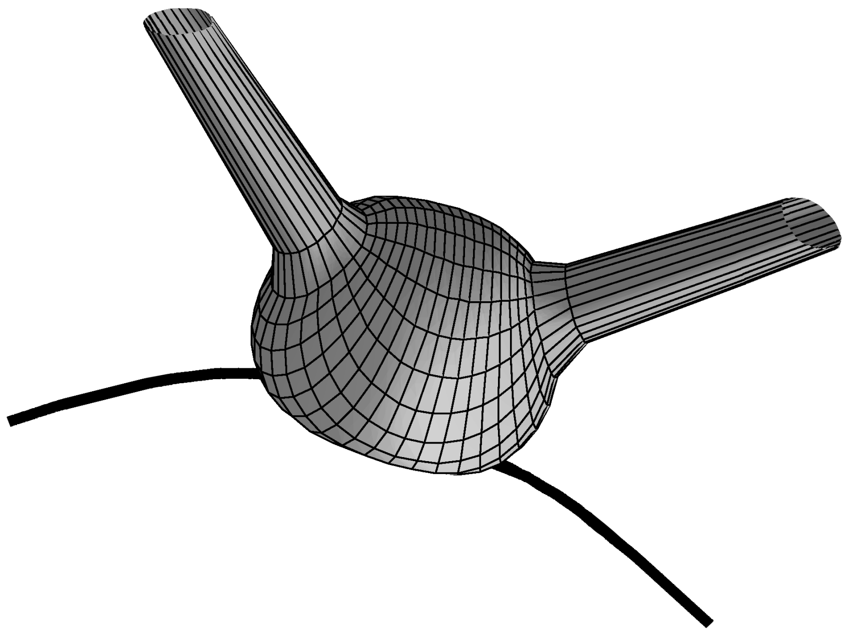

Figure 1.1: A possible D0-brane annihilation diagram. The two incoming D0-branes are converted into the two outgoing closed strings.

boundary conditions. A similar method has been employed by Hirano and Kazama in their treatment of D0-brane recoil [4]. It has been shown that, for low-energy gravitational quanta, the scattering amplitudes obtained via this method can be matched with the standard no-recoil computation and respect the consistency requirements, such as BRST-invariance. The goal of this chapter is to address the question of how instructive this kind of approach can be for essentially high-energy processes, in particular, the D0-brane annihilation. Similarly to Hirano and Kazama's paper, we restrict ourselves to bosonic D0-branes, keeping in mind that the generalization to the potentially physically interesting case of superstring theory is likely to be conceptually straightforward, if only technically more challenging.

It is worth a notice that the attitude taken here is in many ways complementary to the attempts to address the same physical problem within the tachyon condensation paradigm of string field theory [5]. Indeed, the tachyon condensation approach typically deals with the initial state of coincident D-branes, whereas our aim is to analyze the genuine scattering[1]. On the other hand, should there be a need to generalize the approach advocated here to the case of higher dimensional D-branes, those will have to be wrapped around cycles in the compactification manifold, a requirement that does not appear within the string field theory considerations.

Speaking of the possible phenomenological applications, one should certainly mention the kinetics of primordial topological defects. Even though the inflationary scenario suggests that the density of primordial topological defects is dramatically diluted during the period of exponential expansion [7], they could play an important role at the earlier stages of the life of the Universe[2]. The problem we're considering is also in a (somewhat indirect) relation to the microscopic black hole pair-production, a subject of much theoretical controversy in the recent past [9].

Determining the dependence of the annihilation amplitude on the coupling constant is the central theme of this present investigation. It is a requirement imposed by unitarity [10, 11] that the *exclusive* annihilation amplitudes for the topological defects must be non-perturbatively suppressed, i.e., the amplitude should vanish identically when expanded in powers of the coupling constant[3]. The general intuition about how such suppression could

[1] A scattering-like process in a tachyon condensation setting has been recently considered in [6].

[2] In the cosmological context, the medium in which the topological defects move may exert substantial influence upon the annihilation process [8].

[3] Note that the *total* annihilation cross-section does not need to be non-perturbatively suppressed, and it

arise dynamically appeals to the well-known property that the inverse mass of the topological defects is typically much smaller (by powers of the coupling constant) than their size. Therefore, the typical wavelength of the annihilation products will be much smaller than the size of the objects they are produced by, resulting in exponential damping.

The problem with this kind of explanations is that the path integral includes arbitrarily singular configurations, and those could easily upset the exponential damping, which is supposed to underlie the non-perturbative suppression of the annihilation amplitude. (The precise meaning of this statement will become more apparent once we proceed with constructing the actual formalism.) The saddle point estimate of the amplitude we obtain in the following sections can be seen as a proof that the measure of those singular configurations in the path integral is too small to alter the coupling constant dependence necessary to maintain unitarity.

It is a fairly general principle that the processes involving topological defects upset the decreasing significance of the final states with large multiplicity familiar from the perturbative Feynman calculus. Producing a large number of quanta may often turn out to be advantageous compared to the low-multiplicity final states. A possibly more familiar manifestation of this kind of behavior is the emission of a large number of gluons in the instanton-induced processes of gauge theories [12]. In our context, the bias for producing $O(1/g_{st})$ soft quanta is even more dramatic, as we'll be able to see after completing the saddle point evaluation

is expected to be comparable to the square of the size of the topological defect for the case of gauge theory. This is one of the indications that the final state multiplicity distribution for topological defect annihilation is typically rather non-trivial.

of the amplitude.

1.1 Quantization of D0-brane worldlines

When dealing with extended D-branes, one does not have to worry much about the center-of-mass motion: the infinite D-brane mass makes recoil impossible, at least when a localized object scatters off the extended D-brane. The situation is different, however, for D0-branes or compactified D-branes: for the low-energy scattering experiments, the recoil is considerably suppressed by the mass of D0-brane (inversely proportional to the coupling constant), but the center-of-mass motion becomes absolutely crucial in any essentially relativistic process, such as annihilation.

In the context of field theory, this problem has been addressed in [13]. One introduces the translational (center-of-mass) modes for the soliton explicitly, and performs quantization with the center-of-mass coordinate treated as a canonical variable. It has been a subject of some controversy how the center-of-mass degrees of freedom should be introduced for D0-branes [14, 15]. Hirano and Kazama's recipe [4] constitutes a specific proposal to this end. (The various approaches will be further contrasted and compared in chapter 3.)

To account for the motion of the D0-brane as a whole, one introduces its coordinates explicitly and integrates over all the possible worldlines $f^\mu(t)$, with t being the proper time. The boundaries of the string worldsheet are restricted to the D0-brane worldline, and the emission of closed strings is described by insertions of the closed string vertex operators in the interior of the worldsheet. Our main object of interest is the amplitude for two D0-branes starting off at the positions x_1^μ and x_2^μ to annihilate into m closed strings carrying momenta k_1 to k_m:

$$\begin{aligned} G(x_1, x_2|\, k_1, \cdots, k_m) = \sum \frac{(g_{st})^\chi}{V_\chi} \int [\mathcal{D}f]_{\text{diff}} \mathcal{D}t\, \mathcal{D}X\, \delta\left(X_\mu(\theta) - f_\mu(t(\theta))\right) \\ \times \exp\left[-S_D(f) - S_{st}(X)\right] \prod_{a=1}^{m} \left\{g_{st}\mathcal{V}_a(k_a)\right\} \end{aligned} \tag{1.1}$$

where S_D is the action for the D0-brane to be discussed below, S_{st} is the standard conformal gauge action

$$S_{st} = \frac{1}{4\pi\alpha'} \int d^2\sigma \nabla X_\mu \nabla X^\mu$$

the integration with respect to f_μ extends over all the inequivalent (unrelated by diffeomorphisms) curves starting at x_1 and ending at x_2, the boundary of the worldsheet is parametrized by θ, and $t(\theta)$ describes how this boundary is mapped onto the D0-brane worldline. The sum is over all the topologies of the worldsheets (not necessarily connected, but without any disconnected vacuum parts) and χ is the Euler number. V_χ is the conformal Killing volume (the negative regularized value of [16] should be used for the disk). The fully integrated form of the vertex operators is implied. We work in the Euclidean space-time, keeping in mind a subsequent analytic continuation to the Minkowski signature. The integration over moduli of the worldsheet is suppressed for the rest of this paper, as it does not affect the qualitative results. The annihilation amplitude can be deduced from (1.1) by means of the standard reduction formula:

$$\begin{aligned} &\langle k_1, \cdots, k_m | p_1, p_2 \rangle \\ &= \lim_{p_1^2, p_2^2 \to -M^2} \left(p_1^2 + M^2\right)\left(p_2^2 + M^2\right) \int dx_1 dx_2 e^{ip_1 x_1} e^{ip_2 x_2} G(x_1, x_2|\, k_1, \cdots, k_m) \end{aligned} \tag{1.2}$$

where M is the D0-brane mass.

Several general questions have to be addressed regarding the expression (1.1). Firstly, the issue with the Weyl invariance appears to be rather subtle. On physical grounds, one would believe that making the D-branes fully dynamical reinforces the consistency of the amplitudes, much in the same way as respecting the supergravity equations of motion makes the non-linear σ-models consistent. We shall examine in chapter 3 how this works explicitly to the lowest order of the recoil perturbation theory. The case of annihilation/pair-production is considerably less computationally straightforward, and assessing the issues of Weyl invariance there is likely to require some new ideas.

Of course, whether or not the integration over the D0-brane worldlines reinforces the consistency of the string amplitudes depends crucially on the choice of the D0-brane worldline action. It appears to be a fairly general principle [17] that the value of the effective action for a background which couples to strings is given by (minus) the sum of all connected vacuum string graphs evaluated in this background. Thus, very much in the spirit of [2]:

$$S_D[f] = \sum_{\text{connected}} \frac{(g_{st})^\chi}{-V_\chi} \int \mathcal{D}t\, \mathcal{D}X\, \delta\left(X_\mu(\theta) - f_\mu(t(\theta))\right) \exp\left[-S_{st}(X)\right] \tag{1.3}$$

Again, the negative regularized value of the conformal Killing volume should be used for the disk [16]. The exponentiation of the action in the path integral can be seen as a result of summing up the disconnected graphs containing vacuum parts [2]. It can be shown[4] that, for nearly straight worldlines, the above action is reduced to the naïve point-particle result $M \int dt$. For curved worldlines, (1.3) would take into account the back-reaction from the

[4]Essentially, M is a constant produced by the integration over the non-zero modes of $t(\theta)$, and $\int dt$ comes from the integration over the zero mode of $t(\theta)$. Similar considerations for the action functionals constructed from string amplitudes will be quite important in chapter 5.

space-time fields excited by the accelerating D0-brane. Some properties of this action will become more apparent as we proceed with the computation of the amplitude.

Given the close relation between the integrand in (1.1) and the worldline action (1.3), it is convenient to rewrite (1.1) in the following form:

$$\begin{aligned} G(x_1, x_2|\, k_1, \cdots, k_m) = \sum_{\text{all}} \frac{\mathcal{C}\,(g_{st})^{\chi}}{V_{\chi}} \int [\mathcal{D}f]_{\text{diff}} \mathcal{D}t\, \mathcal{D}X\, \delta\left(X_\mu(\theta) - f_\mu(t(\theta))\right) \\ \times \exp\left[-S_{st}(X)\right] \prod_{a=1}^{m} \left\{g_{st} \mathcal{V}_a(k_a)\right\} \end{aligned} \tag{1.4}$$

Here, the sum extends over all the string diagrams including arbitrary disconnected vacuum parts. There is no explicit action for the D0-brane worldline, but it arises after resumming the contributions from all the disconnected vacuum parts, which exponentiates to restore the S_D of (1.3). $\mathcal{C}$ is a combinatorial factor that can be deduced from (1.3).

Using the transformation properties of the vertex operators under the target space translations, it is easy to see that

$$G(x_1, x_2|\, k_i) = \exp\left[\frac{i}{2}(x_1^\mu + x_2^\mu) \sum k_i\right] G\left(\frac{x_1 - x_2}{2}, -\frac{x_1 - x_2}{2}\right|\, k_i\Big)$$

The first term here merely provides for the momentum conservation δ-function in the Fourier transform, and (1.2) can be rewritten as

$$\begin{aligned} \langle k_1, \cdots, k_m | p_1, p_2\rangle = (2\pi)^{26} \delta(p_1 + p_2 + \textstyle\sum k_i) \\ \times \lim_{p_1^2, p_2^2 \to -M^2} \left(p_1^2 + M^2\right)\left(p_2^2 + M^2\right) \int dx \exp\left[\frac{i}{2}(p_1 - p_2)x\right] G\left(\frac{x}{2}, -\frac{x}{2}\right|\, k_i\Big) \end{aligned} \tag{1.5}$$

We shall work with this representation in our subsequent calculation of the amplitude.

1.2 The Gaussian integration

By a direct inspection of (1.4), it is easy to see that the integration over X is Gaussian and can be performed exactly. We will thus be able to recast the formalism into a (0+1)-dimensional form. The Gaussian integration we have to perform is closely related to the derivations in [17] and can be implemented by applying the formula

$$
\begin{aligned}
\int \mathcal{D}X\, \delta\left(X(\theta) - \xi(\theta)\right) \exp\left[\int d^2\sigma \left(-\frac{1}{4\pi\alpha'}\nabla X \nabla X + iJX\right)\right] \\
= \exp\Big[-\pi\alpha' \int J(\sigma) D(\sigma,\sigma') J(\sigma') d^2\sigma d^2\sigma' - i\int \xi(\theta)\partial_n D(\theta,\sigma') J(\sigma') d\theta d^2\sigma' \\
-\frac{1}{4\pi\alpha'}\int \xi(\theta)\partial_n \partial_{n'} D(\theta,\theta')\xi(\theta') d\theta d\theta'\Big]
\end{aligned}
\tag{1.6}
$$

(The derivation of this basic yet important formula together with a few underlying subtleties is described in Appendices A and B.) Here, D is the Dirichlet Green function of the Laplace operator $\Delta D(\sigma,\sigma') = -\delta(\sigma-\sigma')$, and ∂_n denotes the normal derivative evaluated at the boundary (which is parametrized by θ). It is convenient to consider

$$
\begin{aligned}
G\left(\frac{x}{2}, -\frac{x}{2}\middle| J\right) = \sum_{\text{all}} \frac{\mathcal{C}\,(g_{st})^{\chi}}{V_\chi} \int [\mathcal{D}f]_{\text{diff}} \mathcal{D}t\, \mathcal{D}X\, \delta\left(X_\mu(\theta) - f_\mu(t(\theta))\right) \\
\times \exp\left[-S_{st}(X) + i\int d^2\sigma J_\mu X^\mu\right]
\end{aligned}
\tag{1.7}
$$

instead of (1.4). Indeed, differentiating with respect to the source J and setting it to $\sum k_i \delta(\sigma - \sigma_i)$ allows us to reproduce the amplitude for an arbitrary vertex operator insertion.

Performing the integration in (1.7) by means of (1.6) yields:

$$G\left(\frac{x}{2},-\frac{x}{2}\middle| J\right) = \sum_{\text{all}} \frac{\mathcal{C}\,(g_{st})^{\chi}}{V_{\chi}} \exp\left[-\pi\alpha' \int J^{\mu}(\sigma) D(\sigma,\sigma') J_{\mu}(\sigma') d^2\sigma d^2\sigma'\right]$$
$$\int [\mathcal{D}f]_{\text{diff}} \mathcal{D}t \, \exp\left[-i \int f^{\mu}(t(\theta)) \partial_n D(\theta,\sigma') J_{\mu}(\sigma') d\theta d^2\sigma'\right] \tag{1.8}$$
$$\exp\left[-\frac{1}{4\pi\alpha'} \int f^{\mu}(t(\theta)) \partial_n \partial_{n'} D(\theta,\theta') f_{\mu}(t(\theta')) d\theta d\theta'\right]$$

It should be noted that $D(\sigma,\sigma')$, χ, $\mathcal{C}$ and V depend on the topology of the diagram corresponding to each particular term in the sum. For diagrams with disconnected parts, $D(\sigma,\sigma')$ is block diagonal in the sense that it vanishes whenever the two arguments belong to different disconnected components.

The path integral in (1.8) may appear rather cumbersome, as one of the functions to be integrated over appears in the argument of the other one. Nevertheless, the integration over $f^{\mu}(t)$ can be performed exactly by means of a technique very similar to the treatment of the free point-like particle in [18].

We first rewrite the measure on reparametrization equivalence classes of $f^{\mu}(t)$ as

$$[\mathcal{D}f]_{\text{diff}} = \mathcal{D}f\,\delta\left[\dot{f}^2-1\right] = \mathcal{D}f \int \mathcal{D}z \exp\left[-\int_0^T z(\dot{f}^2-1)dt\right]$$

where $\delta[\dot{f}^2-1]$ is a product of δ-functions at every point (reinforcing t to be the proper time), and, for each t, the integration over $z(t)$ is along a contour going from $c-i\infty$ to $c+i\infty$ in the complex z plane, with c being an arbitrary (positive) constant. This contour can of course be deformed, an opportunity implicit in our subsequent application of the saddle point method.

If we now introduce

$$\mathcal{N}(t,t') = \int d\theta d\theta' \partial_n \partial_{n'} D(\theta,\theta')\, \delta\left(t - t(\theta)\right) \delta\left(t' - t(\theta')\right), \quad d(t,\sigma) = \int d\theta \partial_n D(\theta,\sigma)\, \delta\left(t - t(\theta)\right)$$

the f-integration in (1.8) can be recast into a manifestly Gaussian form:

$$G\left(\frac{x}{2}, -\frac{x}{2}\middle| J\right) = \sum_{\text{all}} \frac{\mathcal{C}\left(g_{st}\right)^{\chi}}{V_\chi} \exp\left[-\pi\alpha' \int J^\mu(\sigma) D(\sigma,\sigma') J_\mu(\sigma') d^2\sigma d^2\sigma'\right]$$

$$\int \mathcal{D}t\, \mathcal{D}z\, \mathcal{D}f \exp\left[-\int z(\dot{f}^2 - 1)dt\right] \exp\left[-i\int f^\mu(t) d(t,\sigma) J_\mu(\sigma) dt d^2\sigma'\right]$$

$$\exp\left[-\frac{1}{4\pi\alpha'}\int f^\mu(t)\mathcal{N}(t,t') f_\mu(t') dt dt'\right]$$

It is convenient to pass on to the integration over $\mathcal{D}\dot{f}$ by means of the relations

$$\mathcal{D}f = dT \mathcal{D}\dot{f}\, \delta\left(\int_0^T \dot{f}^\mu dt + x^\mu\right) \qquad f^\mu(t) = \frac{x}{2} + \int_0^t \dot{f}^\mu dt$$

We should also keep in mind that

$$\int_0^T dt \mathcal{N}(t,t') = 0 \qquad \int_0^T dt\, d(t,\sigma) = -1$$

as

$$\int d\theta\, \partial_n \partial_{n'} D(\theta,\theta') = 0 \qquad \int d\theta\, \partial_n D(\theta,\sigma') = -1$$

If we also perform the Fourier transform of (1.5), we arrive at the following representation

$$G(p_1,p_2|J) = \int dx \exp\left[\frac{i}{2}(p_1 - p_2)x\right] G\left(\frac{x}{2}, -\frac{x}{2}\middle| J\right)$$

$$= \sum_{\text{all}} \frac{\mathcal{C}\left(g_{st}\right)^{\chi}}{V_\chi} \exp\left[-\pi\alpha' \int J^\mu(\sigma) D(\sigma,\sigma') J_\mu(\sigma') d^2\sigma d^2\sigma'\right]$$

$$\int dT \mathcal{D}t\, \mathcal{D}z\; e^{\int z\, dt} \int \mathcal{D}\dot{f}\; \exp\left[i\int \dot{f}_\mu(t)\left(p_1^\mu - \int_0^t d(\tilde{t},\sigma) J^\mu(\sigma) d\tilde{t} d^2\sigma\right) dt\right]$$

$$\exp\left[-\int \dot{f}^\mu(t) \mathcal{B}(t,t') \dot{f}_\mu(t') dt dt'\right]$$

where we've introduced

$$\mathcal{B}(t,t') = z(t)\,\delta(t-t') + \frac{1}{4\pi\alpha'}\int\limits_0^t d\tilde{t}\int\limits_0^{t'} d\tilde{t}'\,\mathcal{N}(\tilde{t},\tilde{t}')$$

At this point, the Gaussian integration becomes completely straightforward and yields

$$G(p_1,p_2|J) = \sum_{\text{all}} \frac{\mathcal{C}\,(g_{st})^{\chi}}{V_\chi}\exp\left[-\pi\alpha'\int J^\mu(\sigma)D(\sigma,\sigma')J_\mu(\sigma')d^2\sigma d^2\sigma'\right]$$
$$\int dT\mathcal{D}t(\theta)\,\mathcal{D}z(t)\ \det[\mathcal{B}]^{-1/2}\exp\left[\int z\,dt\right]$$
$$\exp\left[-\frac{1}{4}\int\Big(p_1^\mu - \int\limits_0^t d(\tilde{t},\sigma)J^\mu(\sigma)d\tilde{t}d^2\sigma\Big)\mathcal{B}^{-1}(t,t')\Big(p_{1\mu} - \int\limits_0^{t'} d(\tilde{t}',\sigma')J_\mu(\sigma')d\tilde{t}'d^2\sigma'\Big)dtdt'\right] \tag{1.9}$$

From this representation, it is apparent that the endpoints of the z integration contour can be moved towards $-\infty$. The contour itself cannot shrink to $-\infty$, however, on account of the singularities of $(\det\mathcal{B})^{-1/2}$. Should there be a discontinuity in $t(\theta)$, these singularities move towards $-\infty$ allowing the contour to be deformed arbitrarily far to the left in the complex z plane. Then the integrand will vanish due to the last factor in the second line of (1.9). This is as it should be, since discontinuous worldsheets do not give any contribution to the original path integral (1.1).

The fact that we have been able to perform the integration over the D0-brane worldlines exactly is rather remarkable, since, for any *given* worldline, the action S_D featured in (1.1) cannot be computed. This action however is itself constructed from string amplitudes and can be completely eliminated from the path integral at the cost of including the disconnected string worldsheets, as it has been done in (1.4). This peculiar simplification plays an important role in making possible the saddle point evaluation of the annihilation/pair-production amplitudes.

1.3 The saddle point method

The presence of the p_1 and J in the exponential of the last line of (1.9) alludes to the relevance of the saddle point techniques, as those quantities become non-perturbatively large in the annihilation kinematic region. This can be made more apparent by introducing $\wp_i = p_i/M$ and $\mathcal{J} = J/M$ (M being the D0-brane mass) and rewriting (1.9) as

$$G(p_1, p_2|J) = \sum_{\text{all}} \frac{\mathcal{C}\,(g_{st})^{\chi}}{V_\chi} \exp\left[-\pi\alpha' M^2 \int \mathcal{J}^\mu(\sigma) D(\sigma,\sigma')\mathcal{J}_\mu(\sigma') d^2\sigma d^2\sigma'\right]$$

$$\int dT \mathcal{D}t(\theta)\, \mathcal{D}z(t)\, \det[\mathcal{B}]^{-1/2} \exp\left[\int z\, dt\right]$$

$$\exp\left[-\frac{M^2}{4}\int\Big(\wp_1^\mu - \int_0^t d(\tilde{t},\sigma)\mathcal{J}^\mu(\sigma) d\tilde{t} d^2\sigma\Big)\mathcal{B}^{-1}(t,t')\Big(\wp_{1\mu} - \int_0^{t'} d(\tilde{t}',\sigma')\mathcal{J}_\mu(\sigma') d\tilde{t}' d^2\sigma'\Big) dt dt'\right]$$

The latter representation is still somewhat inconvenient, since the position of the saddle point depends on the saddle point parameter M. This dependence is very simple, however, and can be completely eliminated by rescaling[5] $z(t) \to M z(t/4\pi M\alpha')$, $t(\theta) \to 4\pi M\alpha' t(\theta)$,

[5] It should be kept in mind that, from now on, the geometrical parameters of the worldsheet in physical space are related to those represented by $t(\theta)$ through a factor of $O(1/g_{st})$.

$T \to 4\pi M\alpha' T$, which brings the above integral to the form

$$G(p_1, p_2|J) = \sum_{\text{all}} \frac{\mathcal{C}\,(g_{st})^\chi}{V_\chi} \exp\left[-\pi\alpha' M^2 \int \mathcal{J}^\mu(\sigma) D(\sigma,\sigma') \mathcal{J}_\mu(\sigma') d^2\sigma d^2\sigma'\right]$$
$$\int dT \mathcal{D}t(\theta)\, \mathcal{D}z(t)\, \det[\mathcal{B}]^{-1/2} \exp\left[4\pi\alpha' M^2 \int z\, dt\right]$$
$$\exp\left[-\pi\alpha' M^2 \int \Big(\wp_1^\mu - \int\limits_0^t d\cdot \mathcal{J}^\mu d\tilde{t} d^2\sigma\Big) \mathcal{A}^{-1}(t,t') \Big(\wp_{1\mu} - \int\limits_0^{t'} d\cdot \mathcal{J}_\mu d\tilde{t}' d^2\sigma'\Big) dt dt'\right] \tag{1.10}$$

where

$$\mathcal{A}(t,t') = z(t)\,\delta(t-t') + \int\limits_0^t d\tilde{t} \int\limits_0^{t'} d\tilde{t}'\, \mathcal{N}(\tilde{t},\tilde{t}') \tag{1.11}$$

In view of the subsequent application of the saddle point method, it is necessary to specify how the integrand is analytically continued to the complex $t(\theta)$. It may seem worrisome that the definitions of $\mathcal{N}(t,t')$ and $d(t,\sigma)$ involve δ-functions, but these δ-functions appear in integral convolution, and the result can be made manifestly analytic by expressing them through their Fourier series on the interval $[0,T]$. Such a representation defines $\mathcal{N}(t,t')$ and $d(t,\sigma)$ as periodic with respect to shifting t or t' by multiples of T. It is worth a notice that analogous analytic continuation is not possible at the level of (1.8) due to the lack of smoothness[6] in $f^\mu(t)$. In this respect, the summation over all the worldlines performed in the previous section is a crucial prerequisite for the saddle point techniques to be applicable.

With these specifications, the saddle point equations can be readily derived by requiring the sum of the two exponents in (1.10) to be stationary with respect to variations of $t(\theta)$

[6]It is a familiar fact (related to the central limit theorem) that the measure on random paths is dominated by curves of fractal dimension 2: see, for example, [19].

and $z(t)$. In this manner, one obtains

$$\mathcal{P}^\mu(t(\theta))\left(\int d\theta'\,\partial_n\partial_{n'}D(\theta,\theta')\int\limits_0^{t(\theta')}\mathcal{P}_\mu(t')dt'+\int\partial_n D(\theta,\sigma)\mathcal{J}_\mu(\sigma)d^2\sigma\right)=0 \tag{1.12}$$

$$\mathcal{P}^\mu\mathcal{P}_\mu=-4 \tag{1.13}$$

where we've introduced

$$\mathcal{P}^\mu(t)=\int\limits_0^T dt'\mathcal{A}^{-1}(t,t')\left(\wp_1^\mu-\int\limits_0^{t'}d(\tilde{t}',\sigma')\mathcal{J}^\mu(\sigma')d\tilde{t}'d^2\sigma'\right)$$

Multiplying (1.12) by $\delta(t-t(\theta))$ and integrating with respect to θ, we get

$$\dot{z}\,\mathcal{P}^2+\frac{1}{2}\,z\left(\mathcal{P}^2\right)^{\cdot}=0$$

which, coupled with (1.13), implies $z=$ const. The relevant value of the constant can be determined from the following argument. Because of the subsequent application of the reduction formula, the value of the amplitude is determined by the divergent piece of $G(p_1,p_2|J)$ as the momenta go on-shell. This divergence can only come from large values of T in the integral (which, of course, corresponds to the long distance propagation of the on-shell particles). For large values of T, there will always be parts of the worldline arbitrarily remote from the location of the worldsheet. There, (1.13) will fix $z=\pm i\sqrt{\wp^2}/2$ (with $\wp^2=\wp_1^2=\wp_2^2$). Since the saddle point value of z does not depend on t, we conclude that, for the purposes of evaluating the amplitude, z can be set to $\pm i\sqrt{\wp^2}/2$ everywhere.

Thus, we have two complex conjugate saddle points for z. After substituting the corresponding values to (1.12), we should obtain two complex conjugate solutions for $t(\theta)$. The equation (1.12) is clearly beyond the reach of analytic methods, but the saddle point configuration of the disconnected vacuum components of the worldsheet can be easily identified as $t(\theta) = \text{const}$. Indeed, for θ belonging to a disconnected vacuum component, the second term in the brackets of (1.12) vanishes. Enforcing $t(\theta) = \text{const}$ makes the first term in the brackets vanish as well, the equation being thereby satisfied.

Moreover, the values of the second functional derivative of the saddle point functional with respect to $t(\theta)$ can be calculated for the parts of the worldsheet which do not emit any final state particles. Indeed, if θ and θ' both belong to a disconnected vacuum component with $t(\theta) = t_0$,

$$\frac{\delta^2}{\delta t(\theta)\delta t(\theta')}\left[\int \Big(\wp_1^\mu + \int_0^t d\cdot \mathcal{J}^\mu d\tilde{t} d^2\sigma\Big)\mathcal{A}^{-1}(t,t')\Big(\wp_{1\mu} + \int_0^{t'} d\cdot \mathcal{J}_\mu d\tilde{t}' d^2\sigma'\Big)\right]$$

$$= -2\mathcal{P}^2(t_0)\partial_n\partial_{n'}D(\theta,\theta') = 8\,\partial_n\partial_{n'}D(\theta,\theta')$$

Furthermore, if θ belongs to a disconnected vacuum component and θ' belongs to any other component of the worldsheet, the corresponding second derivative vanishes.

The above specifications in fact allow to perform a complete resummation (to the leading order of the saddle point approximation) of the contributions to (1.10) coming from the disconnected vacuum components. Indeed, for each disconnected component in a given diagram, we'll obtain a factor

$$\int \mathcal{D}t(\theta)\exp\left[-8\pi\alpha' M^2\int t(\theta)\partial_n\partial_{n'}D(\theta,\theta')t(\theta')d\theta d\theta'\right] \tag{1.14}$$

The value of this integral is a constant[7] (depending on the topology of each particular disconnected vacuum component) times a factor of T, which comes from the integration

[7]One may argue that, for higher genus worldsheets, the modular integrations (that we do not write explicitly) will exhibit a tachyonic divergence. The same problem arises if one tries to compute the higher genus corrections to the D0-brane mass. Both problems will disappear in the superstring case, which we

over the constant mode of $t(\theta)$. If we now recall that the combinatorial coefficients $\mathcal{C}$ in (1.10) originated from expanding the exponential in (1.1), it is easy to see that, to the leading order of the saddle point approximation, the effect of all the disconnected vacuum components amounts to a factor of $\exp[\mu T]$, where μ is a constant, which can be evaluated by computing the Gaussian integral (1.14). In fact, it is easier to notice that $G(p_1, p_2|J = 0)$ is merely the D0-brane propagator, and, identifying its pole with the D0-brane mass M, conclude that $\mu = -4\pi\alpha' M^2$.

For the parts of the worldsheet which do emit final state strings, it is not possible to solve the equation (1.12) explicitly. However, valuable information can be extracted from the mere assumption that a saddle point exists (this can be checked in the low-energy scattering case). It is an important circumstance that, for the worldsheets located not too close to the endpoints of the worldline, in the limit $T \to \infty$, the value of the saddle point function does not change under the shifts of $t(\theta)$ by a constant (we assume $\wp_1^2 = \wp_2^2 = \wp^2$, as it is on-shell). It is not hard to take into account this quasi-zero mode though, as the subsequent application of the reduction formula discards everything but the term growing most rapidly as T goes to ∞. Since extending the interval to which t and t' belong to $[-\infty, \infty]$ does not introduce any singularities to (1.12), we should expect that, as T goes to ∞, the solutions to (1.12) located not too close to the endpoints of the worldline approach a fixed shape. The saddle point evaluation of the $\mathcal{D}t$ and $\mathcal{D}z$ integrals in (1.10) can be therefore written (up to

see as a conceptual rationale behind the formal identification of the value of the integral (1.14) with the D0-brane mass, which we're about to make.

the pre-exponential factors) as

$$e^{-4i\pi\alpha' M^2 T\sqrt{\wp^2}}\left(Te^{-\alpha' M^2\mathcal{F}[\mathcal{J}]}+o(T)\right) \tag{1.15}$$

where $\mathcal{F}$ does not depend on T and can be determined from the solution to the equation (1.12) with t and t' running from $-\infty$ to ∞. The above expression is written for the saddle point $z = i\sqrt{\wp^2}/2$, the contribution from $z = -i\sqrt{\wp^2}/2$ is the complex conjugate of this. We shall omit the $o(T)$ term in the following, as it vanishes upon the application of the reduction formula.

Assembling everything together, up to the pre-exponential factors, the saddle point estimate of (1.10) is

$$\begin{aligned}\Big[G(p_1,p_2|J)\Big]_{\text{saddle}} \approx \sum \frac{(g_{st})^\chi}{V_\chi}\exp\left[-\pi\alpha' M^2\int \mathcal{J}^\mu(\sigma)D(\sigma,\sigma')\mathcal{J}_\mu(\sigma')d^2\sigma d^2\sigma'\right] \\ \times e^{-\alpha' M^2\mathcal{F}[\mathcal{J}]}\int_0^\infty T\exp\left[-4\pi\alpha' M^2\left(1+i\sqrt{\wp^2}\right)T\right]dT + \text{c.c.}\end{aligned}$$

where the summation is once again performed only over those worldsheets which do not contain any disconnected vacuum parts. Performing the T integration and applying the reduction formula, we finally obtain

$$\begin{aligned}\langle p_1,p_2|J\rangle_{\text{saddle}} \approx \sum \frac{(g_{st})^\chi}{V_\chi}\exp\left[-\pi\alpha'\int J^\mu(\sigma)D(\sigma,\sigma')J_\mu(\sigma')d^2\sigma d^2\sigma'\right] \\ \times\left(\exp\left\{-\alpha' M^2\mathcal{F}[J/M]\right\}+\text{c.c.}\right)\end{aligned} \tag{1.16}$$

To compute the actual amplitude, one has to combine this expression and its functional derivatives with respect to J so as to imitate the vertex operator insertions, as well as to integrate over the worldsheet moduli and the positions of the vertex operators (these will be saddle point integrations themselves, in analogy to [20]). All these tasks are likely to require numerical computations, as is the evaluation of the $\mathcal{F}$-functional.

An essential piece of information contained in the expression (1.16) is, however, that it identifies the dependence of the annihilation amplitude on the coupling constant. We can see that, for a final state with the number of quanta fixed as $g_{st} \to 0$ and momenta growing proportionally to the mass of the D0-brane, the amplitude diminishes as $\exp(-O(1/g_{st}^2))$. Indeed, the anticipated non-perturbative suppression has become manifest[8]. It is worth noting that the character of this coupling dependence is not at all obvious at the intermediate stages of our computation. Only upon the application of the saddle point method can we see that the qualitative arguments outlined in the introduction do, in fact, result in a non-perturbative suppression of the amplitude.

It may appear rather surprising that the estimate of the amplitude obtained above is exactly the same as if we had used the simplistic free particle action $M \int dt$ in place of the expression (1.3), which takes into account the emission and re-absorption of virtual string states by the accelerating D0-branes. The qualitative explanation here is that, being objects of effective size $\sqrt{\alpha'}$, the D0-branes are not likely to emit *virtual* string states of energy much higher than $1/\sqrt{\alpha'}$. These states will produce worldline curvatures of order g_{st} and will not therefore affect the leading order of the saddle point approximation. One may have suspected

[8]There doesn't appear to be any straightforward way to prove that $\mathcal{F}$ is greater than zero in the entire annihilation kinematic region (after the analytic continuation to Minkowski signature). Nevertheless, (1.16) suggests that the amplitude is either non-perturbatively suppressed or non-perturbatively enhanced, and the inherent absurdity of the latter option is quite apparent.

that the worldline curvature could become large in the region where the *final* state strings are emitted. Yet, the saddle point equation (1.12) implies that the amplitude is dominated by worldsheets of physical size $\sim 1/g_{st}$ (which corresponds to $t(\theta)$ being of order 1). Therefore, the emission occurs from a segment of the worldline whose size is of order $1/g_{st}$, and the relevant curvatures are of order g_{st} again.

1.4 Emission of a large number of strings

One may be somewhat worried by the fact that the annihilation amplitudes computed via the saddle point method are suppressed much stronger in the limit $g_{st} \to 0$ than $\exp(-O(1/g_{st}))$, which is generally expected to be the order of the non-perturbative corrections in string theory [23]. The resolution of this apparent paradox is that the actual annihilation is most likely to proceed into a large ($\sim 1/g_{st}$) number of quanta, whereas the number of strings in the final state has been kept fixed as $g_{st} \to 0$ in our preceding considerations. Indeed, as we could see in the above, the annihilation process was dominated by worldsheets of size $\sim 1/g_{st}$, and it would hardly be natural for such extended configurations to decay into a small number of strings, a circumstance responsible for the anomalously strong suppression of the amplitude (1.16).

We shall now obtain some estimates of the amplitude for the D0-branes to annihilate into a large number of quanta, and show that those contributions are in fact likely to saturate the bound of [23]. The analysis of the previous sections still applies, but the position of the saddle point (and hence the functional $\mathcal{F}$) now depends on g_{st}, and therefore extra care needs to be taken in making statements about the coupling constant dependence of the amplitude.

It can be seen by inspecting the expression (1.16) that, unless all but an infinitesimal fraction of the final state strings are emitted from disconnected worldsheet components of disk topology with exactly one vertex operator on each of them, the g_{st}^{χ} factor itself will provide suppression of order $\exp(O(\ln(g_{st})/g_{st}))$. The extent to which the dynamics (accounted for by the saddle point estimates) enhances this suppression depends on the topology of each particular diagram. When most of the final state strings are emitted from disks with a single vertex operator, the non-perturbative suppression can only arise from the value of the saddle point exponential. We shall now estimate the contribution from those diagrams.

When the worldsheet consists of many disconnected components each of which carries away only a small fraction of the momentum, the independent displacements of those disconnected components cause only a small change of the saddle point function. We therefore encounter quasi-zero modes very similar to the ones described above (1.15). One way to account for these quasi-zero modes is to introduce the integration over them explicitly into the path integral by inserting $\delta(\int_i t(\theta)d\theta - \bar{t}_i \int_i d\theta)$ for each of the disconnected components of the worldsheet (the θ integral runs over the ith component of the worldsheet, $\bar{t}_i$ thereby being the average position of its boundary). The variational problem of (1.12–1.13) has then to be solved subject to the constraints specified by the δ-functions. (This would introduce the corresponding Lagrange multipliers into the right-hand side of (1.12), but they are inessential to our subsequent considerations.)

In order to obtain constraints on the value of the amplitude, let us expand the $\mathcal{P}(t)$ in

(1.12) in Taylor series around $\bar{t}_i$ (for each i). It is easy to see that unless more than $O(1/g_{st}^{1-\varepsilon})$ (with ε arbitrarily small and positive) of the $\bar{t}_i$'s come into an interval of size $O(g_{st})$, the solution to (1.12) for θ belonging to the ith disconnected component is given by[9]

$$t(\theta) = \bar{t}_i + \frac{1}{4}\int \left[\partial_n \partial_{n'} D(\theta, \theta')\right]^{-1} \partial_{n'} D(\theta', \sigma') \mathcal{P}^\mu(\bar{t}_i) \mathcal{J}_\mu(\sigma') d^2\sigma' d\theta' + O(g_{st}^{1+\varepsilon})$$

(The second term here is closely related to the saddle point which dominates the low-energy closed string scattering off a D0-brane to the lowest order in g_{st}.) For the value of the saddle point function (the exponential in the last line of (1.10)), we obtain

$$-\pi\alpha' M^2 \left(\int_0^T dt\, z(t)\mathcal{P}^2(t) + \int d\theta d\theta' \partial_n \partial_{n'} D(\theta, \theta') \int_0^{t(\theta)} \mathcal{P}(t) dt \int_0^{t(\theta')} \mathcal{P}(t')dt' \right)$$

$$= -2\pi i\, \alpha' M^2 \sqrt{\wp^2}\, T$$

$$+\frac{\pi\alpha'}{4} \sum_i \int d\theta d\theta' d^2\sigma d^2\sigma' \mathcal{P}^\mu(\bar{t}_i) J_\mu(\sigma) \partial_n D(\theta, \sigma) \left[\partial_n \partial_{n'} D(\theta, \theta')\right]^{-1} \partial_{n'} D(\theta', \sigma') \mathcal{P}^\nu(\bar{t}_i) J_\nu(\sigma')$$

$$+O(1/g_{st}^{1-\varepsilon})$$

The second line here merely provides for the correct pole structure (and disappears after the application of the reduction formula). Each term in the sum from the third line is of order $O(g_{st}^0)$ and negative ($\mathcal{P}$ is imaginary to the lowest order in g_{st}). However, since there are $O(1/g_{st})$ terms, we should expect a non-perturbative suppression of order $\exp(-O(1/g_{st}))$.

We still have to deal with the excluded region of integration over the $\bar{t}_i$'s, namely, the configurations with more than $O(1/g_{st}^{1-\varepsilon})$ of the $\bar{t}_i$'s in an interval of size $O(g_{st})$. The volume

[9] We are essentially demanding the variation of $\mathcal{P}$ to be small over the extent of a single disconnected component of the worldsheet, and then using the expression for the saddle point value of $t(\theta)$ in the background of a straight worldline.

of this region is proportional to $\exp(O(\ln(g_{st})/g_{st}^{1-\varepsilon}))$, however, and this is the amount of non-perturbative suppression provided regardless of the value of the saddle point exponential. Barring any unfathomable cancellations, the sum of the contributions from the two regions would be of order $\exp(-O(1/g_{st}))$, in accordance with the general estimates of the non-perturbative effects in string theory.

1.5 Bosonic D0-brane decay

It is a notable feature of the worldline formalism that most of the results obtained in the previous sections are immediately generalized to the case of bosonic D0-brane decay that has received a large amount of attention within the tachyon condensation considerations. Indeed, the natural proposal for the D0-brane decay amplitude is

$$\langle k_1, \cdots, k_m | p \rangle = \lim_{p^2 \to -M^2} \int dx\, dx'\, e^{ipx}\, G(x, x' |\, k_1, \cdots, k_m) \qquad (1.17)$$

where $G(x, x')$ is as it has been defined in (1.1). The meaning of this formula is that the endpoint of the D0-brane worldline is unconstrained in the path integral, and the reduction formula is applied only to the starting point. The path integral thus describes a "disappearing" D0-brane.

Given the formal similarities between the expressions (1.2) and (1.17), it is not surprising that the derivations presented above for the annihilation/pair-production case carry over to the decay process with a formal substitution $p_1 = p$, $p_2 = 0$. One important difference is that, for $p_1^2 \neq p_2^2$ the quasi-zero mode described in the paragraph above (1.15) will be absent. The saddle point configuration of the worldsheet will be localized near the endpoint of the

D0-brane worldline, as one would expect on general grounds. For that reason, the factor of T featured in (1.15) will not make an appearance for the case of D0-brane decay. This is reassuring, since it precisely replaces the double pole necessary for the application of the reduction formula (1.2) by a single pole necessary for the application of the reduction formula (1.17). The conclusions regarding the coupling constant dependence of the amplitude to the leading order of the saddle point approximation will remain intact, i.e., the exclusive decay amplitudes will be non-perturbatively suppressed[10] as $\exp[-O(1/g_{st}^2]$.

It would be interesting to see to what extent the results obtained here are compatible with the investigations of the closed string emission by a decaying D0-brane within the tachyon condensation approach [5]. Unfortunately, the (non-self-consistent) neglect of the closed string back-reaction that underlies the derivations of [5] makes it hard to judge which features of the emission spectrum described there should be trusted.

Conclusions and speculations

As the calculations of this chapter reveal, the problem of D0-brane annihilation appears in many ways much more tractable in our present context than the field-theoretical soliton-anti-soliton annihilation or, more generally, any other topological defect annihilation process that comes to mind. Indeed, the saddle point method provides rigorous results regarding the coupling constant dependence for the annihilation with a fixed number of final state quanta,

[10]For the case of D0-brane decay, additional non-perturbative suppression is likely to arise from the presence of an endpoint of the worldline, as it will be advocated in chapter 5.

and certain qualitative estimates can be devised for final states whose multiplicity increases as g_{st} goes to zero. For a fixed number of final state closed strings (and for D0-brane pair production), the amplitudes turn out to be suppressed as $\exp[-O(1/g_{st}^2)]$, much stronger than the naïve expectation $\exp[-O(1/g_{st})]$.

It is a somewhat dissatisfying feature of our present formalism that only *exclusive* amplitudes appear to be computationally accessible. Should one have had actual D0-branes at one's disposal, exploring the refined properties of their annihilation (such as final state multiplicity distribution) would certainly provide a valuable insight into their nature. Exclusive annihilation amplitudes would be *the* relevant theoretical prediction to guide this sort of experimentation. However, in most phenomenological applications (such as cosmology), the most important quantity is the simplest characteristic of the annihilation process, namely, the total cross-section.

It is the non-trivial nature of the final state multiplicity distribution, i.e. the dominant role of the final states with a huge number of quanta, that makes the reconstruction of the total cross-section from the exclusive annihilation amplitudes so hard. If one could overcome this difficulty, it would be possible, for example, to examine in our setting the stretched D-string interconnection, a process that has so far only admitted a semi-quantitative treatment [24]. The generalization of our present formalism to the case of D-strings is not likely to pose much difficulty. The problem is that, in the cosmological context, one is interested in the *total* interconnection probability.

For the case of D0-brane pair production, we have established a *very strong* non-perturbative suppression (of order $\exp(-1/g_{st}^2)$). It would be interesting to contemplate whether it may have anything to do with the exponential suppression of the microscopic black hole pair-production amplitudes argued by one of the sides in the dispute [9].

Another amusing feature is that there does not appear to be a unitarity relation that constrains the absolute normalization of the D0-brane annihilation amplitude. It is hard to judge at this point whether or not this represents an actual non-perturbative ambiguity in string theory.

Since many qualitative features of the topological defect annihilation, such as the non-perturbative suppression of the exclusive amplitudes and the high mean multiplicity of the final state, appear to be rather general, it is natural to ask whether our computation can shed any light on the field-theoretical soliton annihilation, a process that has thus far defied an analytic treatment. The Christ and Lee formalism of [13] provides a rather conspicuous link to our approach, as it introduces explicitly the center-of-mass position of the soliton, a worldline to be integrated over. It is certainly true that performing the path integral over the fluctuations of the fields around the given trajectory of the soliton is a much more difficult task than the worldsheet integrations pertinent to the D0-brane case. This circumstance is also likely to undermine the direct applicability of the saddle point method, since it seems to hinge upon the summation over the worldlines. Yet, one could try to develop some intuition about the kinds of worldlines that contribute most, and establish some bounds, say, on the coupling constant dependence. If such a program were to succeed, it would effectively provide a path-integral derivation of certain non-trivial properties of the classical annihilation solution. Mathematically and physically, such a derivation would present a fair amount of elegance.

Chapter 2

The Fischler-Susskind mechanism

String theories generically compute the S-matrix elements for the various scattering processes as sums over the topologies of the worldsheet. It is important to realize that, under many circumstances, the separate terms in these sums may be divergent. It is then imperative (as far as the computation of physical observables is concerned) to implement the cancellation of such divergences between the different terms in an explicit form. The cancellation of divergences between the worldsheets of different topologies has been generally referred to as the Fischler-Susskind mechanism.

The most familiar example of how such cancellations work comes from the analysis of the divergences in the modular integration of the torus amplitude in closed string theory [21, 22]. This divergence is related to the non-vanishing torus tadpole and signifies an inconsistency of the closed bosonic string theory in the flat space-time background. The resolution is to introduce a small correction to the geometry of the background space-time in such a way that it induces a divergence for the string theory amplitude on the sphere, which, in turn, cancels the original divergence from the torus.

The situation for the divergences in the presence of the D-branes is somewhat analogous, even though the physical interpretation is different. To the lowest non-trivial order in the string coupling, the relevant divergence comes from the modular integration on the annulus. This divergence is related to the propagation of virtual massless open string states, and it is directly analogous to the infrared divergences appearing when the classical background of a topological defect is introduced in field theories. This correspondence is helpful in developing intuition about the nature of the infrared divergences. In particular, the divergence in the presence of D0-brane is a manifestation of the D0-brane recoil, which is the string theory analog of soliton recoil.

The purpose of this chapter is to demonstrate how the Fischler-Susskind mechanism is implemented in the worldline representation of the D0-brane scattering amplitudes developed in the previous chapter. We'll first go through the general derivation of the annular divergence in the background of a general bosonic D-brane (keeping in mind the case of D1-branes, which will be the primary focus of chapter 4). The consistency of the theory will then require that this divergence should be matched against the divergence on the disk arising in the Hirano-Kazama formalism at the next-to-leading order of the string coupling. Note that the Hirano-Kazama formalism differs from the conventional Dirichlet CFT by the presence of curved D0-brane worldlines in the path integral (1.1). It is precisely those worldlines that induce the divergence on the disk. If not for them, the disk amplitude would have been finite and there would have been nothing that the modular divergence on the annulus could be cancelled against.

2.1 D-branes and the annular divergence

The non-perturbatively large tension of the D-branes ($\sim 1/g_{st}$) makes any background modification unnecessary to the lowest order in string coupling, when the D-brane is hit by closed strings (whose masses are fixed as g_{st} goes to zero). The infrared divergences (and the corresponding need for the background modification) may arise however if one tries to compute the next-to-leading order corrections to the scattering amplitude. In string theory, these next-to-leading order corrections come from string worldsheets of annular topology.

In field theory, it is a familiar fact that the relevant infrared divergences in the loops come from the large distance propagation of the zero modes corresponding to shifting the entire topological defect. In string theory, such large distance propagation corresponds to the small values of the modular parameter of the annulus, i.e., to the annulus degenerating into a thin strip.

Arguably, the most intuitive and powerful method to analyze divergences from degenerating Riemann surfaces is the familiar Polchinski's plumbing fixture construction [22], which relates the divergences to the amplitudes evaluated on a lower genus Riemann surface. In particular, the annulus amplitude with an insertion of the operators $V^{(1)}, \cdots, V^{(n)}$ (in the interior) can be expressed through the disk amplitudes with additional operator insertions at the boundary as follows:

$$\left\langle V^{(1)} \cdots V^{(n)} \right\rangle_{\text{annulus}} = \sum_{\alpha} \int \frac{dq}{q}\, q^{h_\alpha - 1} \int d\theta d\theta' \left\langle V_\alpha(\theta) V_\alpha(\theta') V^{(1)} \cdots V^{(n)} \right\rangle_{D_2} \tag{2.1}$$

where the summation extends over a compete set of local operators $V_\alpha(\theta)$ with conformal weights h_α, and q is the gluing parameter that can be related to the annular modulus. (θ

parametrizes the boundary of the disk.) The divergence in the integral over q coming from the region $q \approx 0$ (i.e., from an annulus degenerating into a thin strip) will be dominated by the terms with the smallest possible h_α.

Let us consider the annular divergence in a somewhat more general setting, when closed strings scatter off a static Dp-brane with d non-compact Neumann directions, $p+1-d$ compact Neumann directions and $25-p$ (non-compact) Dirichlet directions. Neglecting the tachyon divergence, which is a pathology peculiar to the case of the bosonic string, we identify (in close relation to the investigations of [14]) the following set of relevant operators with conformal weights $h = 1 + \alpha'\kappa^2/4 + \pi^2 \sum n_\alpha^2/L_\alpha^2$:

$$V^i(\theta, \kappa^i, n^\alpha) =: \partial_n X^i(\theta) \exp\left[i\kappa^i X^i(\theta)\right] \exp\left[\frac{2\pi i n^\alpha X^\alpha(\theta)}{L_\alpha}\right] :$$

These operators correspond to the massless open string states carrying the momentum κ^i in the non-compact Neumann directions and n^α units of Kaluza-Klein momentum in the α'th compact Neumann direction. L_α are the compactification radii.

It is easy to see that only the operators with $n_\alpha = 0$ will contribute to the leading divergence in (2.1). Furthermore, for small values of q (which is the region we're interested in), only small values of κ^i will contribute into the integral. With these specifications, we can transform the annular divergence as follows:

$$
\begin{aligned}
&\left\langle V^{(1)} \cdots V^{(n)} \right\rangle_{\text{annulus}} \\
&\quad \sim \int_0^\infty dq \int d\kappa^i q^{-1+\alpha'\kappa^2/4} \int d\theta d\theta' \left\langle V^i(\theta,\kappa^i,0) V^i(\theta',\kappa^i,0) V^{(1)} \cdots V^{(n)} \right\rangle_{D_2} \\
&\quad \sim \int_0^\infty dq \int d\kappa^i q^{-1+\alpha'\kappa^2/4} \int d\theta d\theta' \left\langle V^i(\theta,0,0) V^i(\theta',0,0) V^{(1)} \cdots V^{(n)} \right\rangle_{D_2} \\
&\quad \sim P^2 \left\langle V^{(1)} \cdots V^{(n)} \right\rangle_{D_2} \int_0^\infty dq \int d\kappa^i q^{-1+\alpha'\kappa^2/4}
\end{aligned}
\tag{2.2}
$$

where we've taken into account that the operator $\int \partial_n X(\theta) d\theta$ merely shifts the position of the D-brane and inserting it into any amplitude amounts to multiplication by the total (Dirichlet) momentum P transferred by the closed strings to the D-brane during scattering. We further notice that

$$
\int_0^\infty dq \int d\kappa^i q^{-1+\alpha'\kappa^2/4} \sim \int_0^\infty dq \int_0^\infty d\kappa \kappa^{d-1} q^{-1+\alpha'\kappa^2/4} \sim \int_0^\infty \frac{dq}{q\,(\log q)^{d/2}} \tag{2.3}
$$

where, once again, d is the number of non-compact Neumann directions[1].

The divergences from the region $q \sim 0$ in the above integral reveal the peculiarity of the low-dimensional cases.

For $d = 0$, i.e., an instanton-like topological defect, introducing a cut-off ε on the lower bound of the integral (2.3) reveals a $\log \varepsilon$ divergence first described in [2].

[1] Please note that the vertex operators corresponding to the D-brane gauge field do not contribute to the divergence in the modular integration. Mathematically, this comes from the fact that the operator $\int d\theta : \partial_\theta X^i(\theta) \exp\left[i\kappa^i X^i(\theta)\right] :$ vanishes for small κ^i. Physically, it conforms to the notion that the infrared divergences are related to recoil, i.e., an excitation of the scalar fields corresponding to the displacement of the D-brane, rather than the gauge field.

The divergence for $d = 1$, i.e., a particle-like topological defect, is $\sqrt{|\log\varepsilon|}$. It is indicative of recoil and will be our main focus in this and the subsequent chapter.

For $d = 2$, we observe a $\log|\log\varepsilon|$ divergence that is a manifestation of the *local recoil*, a peculiar phenomenon that will be further described and analyzed in chapter 4.

There are no annular divergences for $d > 2$. Once again, we'll postpone the discussion of the underlying physics until chapter 4 of this thesis.

2.2 The general structure of recoil perturbation theory

Let us now return to the setting of the previous chapter and examine how the Hirano-Kazama description of the dynamical D0-branes works in the recoil regime, i.e., for closed strings scattering off a D0-brane. Since the mass of the D0-brane diverges in the limit $g_{st} \to 0$, it can be treated as static to the lowest order in g_{st} (if we keep the momenta of the incident closed strings fixed as $g_{st} \to 0$), and the corrections due to the motion of the D0-brane's center-of-mass (i.e. recoil) will appear as a perturbative expansion in powers of g_{st}. This is the familiar recoil perturbation theory.

The formalism of the previous section is, of course, perfectly applicable in this case. Since the momenta of the closed string states (and hence J) are kept fixed as $g_{st} \to 0$ (and hence $M \to \infty$), $\mathcal{J} \equiv J/M$ can be treated as perturbation. The saddle point equation (1.12) implies then that the saddle point configuration $t(\theta) = \text{const} + O(1/M)$, i.e., the path integral (1.10) is dominated by nearly constant $t(\theta)$ (the fluctuations of $t(\theta)$ are of order $1/M$ generically, and so is the saddle point value, as we've just remarked). One could look for solutions of the saddle point equation as an expansion in $\mathcal{J}$. However, it is much more efficient to generate the perturbation theory directly from the path integral (1.10) by constructing a suitable expansion of the "effective action" functional of (1.10):

$$S_{eff}[z(t), t(\theta)] = -4\pi\alpha' M^2 \int z\, dt$$
$$+\pi\alpha' M^2 \int \Big(\wp_1^\mu - \int\limits_0^t d\cdot \mathcal{J}^\mu d\tilde{t} d^2\sigma\Big) \mathcal{A}^{-1}(t,t') \Big(\wp_{1\mu} - \int\limits_0^{t'} d\cdot \mathcal{J}_\mu d\tilde{t}' d^2\sigma'\Big) dt dt'$$

There is one difficulty one encounters in implementing such a program. It is a most straightforward approach to try to construct a Taylor-like expansion of S_{eff} in powers of $t(\theta)$ around $t(\theta) = \text{const}$:

$$S_{eff}(\text{const} + t(\theta))$$
$$= S_{eff}(\text{const}) + \int d\theta \left.\frac{\delta S_{eff}}{\delta t(\theta)}\right|_{\text{const}} t(\theta) + \frac{1}{2}\int d\theta d\theta' \left.\frac{\delta^2 S_{eff}}{\delta t(\theta)\delta t(\theta')}\right|_{\text{const}} t(\theta)t(\theta') + \cdots$$

This strategy comes to mind in immediate relation to the computational techniques most commonly used in σ-models, and it has been employed in [4] for the purposes we're presently pursuing here. Unfortunately, such an expansion does not exist. In [4], various ζ-function prescriptions have been devised (in the lowest order) to deal with the infinities arising when one tries to brute-force the Taylor-like expansion, but the situation quickly becomes hopeless, if one tries to envisage the general structure of the recoil perturbation theory in such a framework.

The origin of the above complication can be traced back to the non-analytic properties of the worldlines in the path integral (1.8). Indeed, for non-analytic $f^\mu(t)$ (most worldlines are fractal and therefore non-differentiable [19]) the "effective action" in (1.8) does not admit a Taylor-like expansion in $t(\theta)$ around *any* configuration of $t(\theta)$. The integration over $f^\mu(t)$

improves the situation considerably: the resulting effective action can be expanded around any $t(\theta) \neq \text{const}$, but the non-analyticity still survives for the worldsheets whose boundary shrinks to a single point. (One may become worried about whether such non-analyticity could undermine the application of the saddle point method from the previous section. Whereas any discussions of explicit contour deformation in multi-dimensional cases are necessarily rather subtle, there are still good chances that the deformation implicit in the saddle point evaluation of the integral can be attained, since S_{eff} *is* analytic for any $t(\theta) \neq \text{const}$.)

Luckily, the Taylor-like expansion is not the only way to generate a sensible perturbation theory. Appearing as insertions in a Gaussian path integral, the exponentials of $t(\theta)$ are just as tractable as powers of $t(\theta)$. We shall therefore resort to a combination of a Taylor-like and a Fourier-like expansion. We first introduce

$$A(t,t') = i\frac{\sqrt{\wp^2}}{2}\delta(t-t') \qquad B(t,t') = \delta z(t)\,\delta(t-t') + \int\limits_0^t d\tilde{t}\int\limits_0^{t'} d\tilde{t}'\,\mathcal{N}(\tilde{t},\tilde{t}')$$

such that (cf. (1.11)) $\mathcal{A}(t,t') = A(t,t') + B(t,t')$, and expand formally

$$\mathcal{A}^{-1} = \frac{1}{A} - \frac{1}{A}B\frac{1}{A} + \frac{1}{A}B\frac{1}{A}B\frac{1}{A} + \cdots \tag{2.4}$$

(we work with one of the two saddle points in $z(t)$, the other one will give a complex conjugate contribution, as in the previous section). Upon inserting this expression in the S_{eff} we can isolate the following terms that need to be retained in the exponent (everything else is small and may be treated perturbatively, as it will become apparent after we exhibit the powers of the expansion parameter $1/M$):

$$S^{(1)}_{eff} = 4\pi i\alpha' M^2\sqrt{\wp^2}T$$

$$S^{(2)}_{eff} = -4\pi\alpha' M^2\int dt dt'\int_0^t d\tilde{t}\int_0^{t'} d\tilde{t}'\,\mathcal{N}(\tilde{t},\tilde{t}') \equiv -4\pi\alpha' M^2\int t(\theta)\partial_n\partial_{n'}D(\theta,\theta')t(\theta')d\theta d\theta'$$

$$S^{(3)}_{eff} = -\frac{8\pi i\alpha' M^2}{\sqrt{\wp^2}}\int \delta z^2 dt$$

$$S^{(4)}_{eff} = \frac{4\pi i\alpha' M^2}{\sqrt{\wp^2}}\wp_1^\mu\int dt\int_0^t d\cdot\mathcal{J}_\mu d\tilde{t}d^2\sigma \equiv \frac{4\pi i\alpha' M}{\sqrt{\wp^2}}\wp_1^\mu\int t(\theta)\partial_n D(\theta,\sigma)J(\sigma)_\mu d\theta d^2\sigma \tag{2.5}$$

The first term here merely provides for the correct pole structure. It will disappear after integration over T and application of the reduction formula. The remaining three terms define a Gaussian integral with respect to $z(t)$ and $t(\theta)$.

A general term in the expansion (2.4) can be constructed as follows. We should take a certain number of factors

$$\int_0^{t_i} d\tilde{t}\int_0^{t_{i+1}} d\tilde{t}'\,\mathcal{N}(\tilde{t},\tilde{t}') \tag{2.6}$$

and, for each i add a factor

$$\frac{(\delta z(t_i))^{n_i}}{(i\sqrt{\wp^2}/2)^{-n_i+1}}$$

where n_i are non-negative integers. Then, for the first and last t_i we either add a factor of $\wp_1^\mu$ or a factor of

$$\int_0^{t_i} d(\tilde{t},\sigma)\mathcal{J}^\mu(\sigma)d\tilde{t}d^2\sigma \tag{2.7}$$

We then integrate over all t_i's and multiply the result by $\pi\alpha' M^2$.

We now proceed with the Gaussian integral over $\delta z(t)$. As it can be seen from the expression for $S^{(3)}_{eff}$, each factor of $\delta z(t)$ will produce a factor of $1/M$ in the result of the Gaussian integration. Where several δz's occur at the same t_i, divergences from the singularity of the δz propagator (proportional to the δ-function) will be present. Just like in the case of free particle considered in [18], these divergences will merely renormalize the saddle point (expectation) value of $z(t)$.

We are left with a Gaussian integral over $t(\theta)$. As we've remarked before, the structures of the type (2.6) and (2.7) cannot be expanded in powers of $t(\theta)$ around $t(\theta) = \text{const}$. Instead, we'll Fourier-trasform in t_i's:

$$\int_0^{t_i} d\tilde{t} \int_0^{t_{i+1}} d\tilde{t}' \mathcal{N}(\tilde{t}, \tilde{t}') \to \frac{1}{\kappa_i \kappa_{i+1}} \int e^{i\kappa_i t(\theta)} \partial_n \partial_n' D(\theta, \theta') e^{i\kappa_{i+1} t(\theta')} d\theta d\theta'$$

$$\int_0^{t_i} d(t, \sigma) \mathcal{J}^\mu(\sigma) d\tilde{t} d^2\sigma \to \frac{1}{\kappa_i} \int e^{i\kappa_i t(\theta)} \partial_n D(\theta, \sigma) \mathcal{J}^\mu(\sigma) d\theta d^2\sigma$$

At this point, the integration over $t(\theta)$ can be explicitly performed. The substitution $t(\theta) \to t(\theta)/M$, $\kappa_i \to M\kappa_i$ reveals the additional powers of $1/M$ that each term will receive upon the integration over $t(\theta)$.

2.3 Divergence cancellation in the Hirano-Kazama formalism

Having sketched the general structure of the recoil perturbation theory, we shall now turn to the important issue of how the consistency of the string amplitudes is respected to the lowest non-trivial order. As we've remarked early on in the course of our present investigation, the inclusion of curved D0-brane worldlines into the path integral (1.1) will generally induce a Weyl anomaly thereby threatening the decoupling of the negative norm states, which is a crucial requirement for the consistency of the string theory S-matrix. The educated hope is that (a suitable modification of) the familiar Fischler-Susskind mechanism [21, 22] will come to rescue the formalism that has been constructed so far. Indeed, it is a well known fact that there are singularities in the modular integration for the higher genus worldsheets coming from the corners of the moduli space. Should we have chosen the D0-brane worldline action correctly, the Weyl anomalies induced by those singularities would precisely cancel the Weyl anomalies coming from the coupling to an accelerating D0-brane. Let us see how this works out to the first non-trivial order of the recoil perturbation theory.

The Fischler-Susskind mechanism in our set-up implies a cancellation between an infrared divergence on a higher genus worldsheet and an ultraviolet divergence on a lower genus worldsheet. To the lowest order, this means that we should examine the ultraviolet divergences on a disk to the order $1/M$ and compare them with the modular integration divergences from an annulus coupled to a straight D0-brane worldline (the curved worldlines will only contribute to the higher orders in $1/M$).

When we expand S_{eff} (for a disk) according to (2.4) and treat all the terms except for the ones indicated in (2.5) as a perturbation, we can identify the following insertions that can in principle contribute to the order $1/M$ in the resulting recoil perturbation theory Gaussian integral:

$$F_1 = -\frac{2\pi i\alpha' M^2}{\sqrt{\wp^2}} \int dt \left(\int\limits_0^t d(\tilde{t},\sigma)\mathcal{J}^\mu(\sigma) d\tilde{t} d^2\sigma \right)^2$$

$$F_2 = \frac{8\pi\alpha' M^2}{\wp^2} \wp^\mu \int dt dt' \int\limits_0^t d\tilde{t} \int\limits_0^{t'} d\tilde{t}' \, \mathcal{N}(\tilde{t},\tilde{t}') \int\limits_0^{t'} d\tilde{t}'' d^2\sigma d(\tilde{t}'',\sigma)\mathcal{J}^\mu(\sigma)$$

$$F_3 = \frac{1}{2}\left(\frac{8\pi\alpha' M^2}{\wp^2} \wp^\mu \int dt \delta z(t) \int\limits_0^t d(\tilde{t},\sigma)\mathcal{J}^\mu(\sigma) d\tilde{t} d^2\sigma \right)^2$$

After we integrate over $\delta z(t)$ and Fourier-transform, these three terms become:

$$F_1 \sim \alpha' \int \frac{d\kappa}{(\kappa + i0)^2} \left| \int e^{i\kappa t(\theta)} \partial_n D(\theta,\sigma) J_\mu(\sigma) d\theta d^2\sigma \right|^2$$

$$F_2 \sim \alpha' M \wp_1^\mu \int \frac{d\kappa}{\kappa + i0)^2} \int d\theta d\theta' t(\theta) \partial_n \partial_n' D(\theta,\theta') e^{i\kappa t(\theta')} \int e^{-i\kappa t(\theta'')} \partial_n D(\theta'',\sigma) J_\mu(\sigma) d\theta'' d^2\sigma$$

$$F_3 \sim \alpha' \int \frac{d\kappa}{\kappa + i0)^2} \left| \wp_1^\mu \int e^{i\kappa t(\theta)} \partial_n D(\theta,\sigma) J_\mu(\sigma) d\theta d^2\sigma \right|^2 \tag{2.8}$$

We now have to compute the path integral

$$\int \mathcal{D}t(\theta) \; (F_1 + F_2 + F_3) \exp\left[-S_{eff}^{(2)} - S_{eff}^{(4)} \right]$$

The relevant structure that is featured in all the three terms is

$$\begin{aligned} &\int \frac{d\kappa}{\kappa^2} \int \mathcal{D}t(\tilde{\theta}) \, e^{i\kappa(t(\theta) - t(\theta'))} \\ &\times \exp\left[-4\pi\alpha' M^2 \int t(\tilde{\theta}) \partial_n \partial_{n'} D(\tilde{\theta},\tilde{\theta}') t(\tilde{\theta}') d\tilde{\theta} d\tilde{\theta}' + \frac{4\pi i \alpha' M}{\sqrt{\wp^2}} \wp_1^\mu \int t(\tilde{\theta}) \partial_n D(\tilde{\theta},\sigma) J_\mu(\sigma) d\tilde{\theta} d^2\sigma \right] \\ &\quad \sim \frac{1}{\alpha' M} \exp\left[-\pi\alpha' \wp_1^\mu \wp_1^\nu \int J_\mu(\sigma) \partial_n D(\tilde{\theta},\sigma) [\partial_n \partial_{n'} D]^{-1}(\tilde{\theta},\tilde{\theta}') \partial_{n'} D(\tilde{\theta}',\sigma') J_\nu(\sigma') d\tilde{\theta} d\tilde{\theta}' d^2\sigma d^2\sigma' \right] \\ &\qquad \times \int \frac{d\kappa}{\kappa + i0)^2} \exp\left[-2\kappa^2 \log\varepsilon + \kappa h_1(\theta,\theta') + \kappa^2 h_2(\theta,\theta') \right] \end{aligned}$$

Where $\log\varepsilon \equiv \partial_n \partial_{n'} D(\theta,\theta)$ is the regularized value of the singularity of the boundary-to-boundary propagator (ε being the worldsheet cut-off), and $h_{1,2}(\theta,\theta')$ are certain (cut-off independent) functions. If we now substitute $\tilde{\kappa} = \kappa/\sqrt{\log \varepsilon}$, we observe that the functions $h_{1,2}$ do not contribute to the divergent piece of the path integral, which turns out to be (θ, θ')-independent and proportional to

$$\frac{\sqrt{\log \varepsilon}}{\alpha' M}\, e^{-\pi \alpha' \wp_1^\mu \wp_1^\nu \int J_\mu(\sigma) \partial_n D(\tilde{\theta},\sigma)[\partial_n \partial_{n'} D]^{-1}(\tilde{\theta},\tilde{\theta}') \partial_{n'} D(\tilde{\theta}',\sigma') J_\nu(\sigma') d\tilde{\theta} d\tilde{\theta}' d^2\sigma d^2\sigma'} \tag{2.9}$$

Now, in the path integral of F_2, the expression (2.9) will appear in convolution with $\partial_n \partial_{n'} D(\theta, \theta')$, and, since the latter is orthogonal to constant modes, the path integral of F_2 will not contain any UV-divergent piece. Both path integrals of F_1 and F_3 will simplify due to the relation

$$\int \partial_n D(\theta, \sigma) J^\mu(\sigma) d\theta d^2\sigma = -\int J^\mu d^2\sigma = p_1^\mu - p_2^\mu$$

and the UV divergent part of the path integral of F_3 will turn out to be of order $1/M^2$, not $1/M$ due to the relation $2\wp_1(\wp_1 - \wp_2) = (\wp_1 - \wp_2)^2$. If we also notice that the exponential in (2.9) is merely the value of the Gaussian integral

$$\int \mathcal{D}t(\theta)\, \exp\left[-S_{eff}^{(2)} - S_{eff}^{(4)}\right]$$

(without any insertions), we conclude that the UV divergent part of the disk amplitude to the order $1/M$ is proportional to

$$\frac{(p_1 - p_2)^2}{\alpha' M} A_{D_2}^0 \sqrt{\log \varepsilon} \tag{2.10}$$

where $A_{D_2}^0$ is the zeroth order value of the disk amplitude, i.e., the amplitude computed in the background of a straight D0-brane worldline, in neglect of recoil.

We now have to compare this result with the divergence in the modular integration of the annulus amplitude to the lowest order in $1/M$, i.e., the annulus amplitude in the background of a straight D0-brane worldline. This is precisely the computation that has been performed in section 2.1, and, for the case of D0-brane, the divergence was found to be proportional to

$$g_{st}\,(p_1-p_2)^2\left\langle V^{(1)}\cdots V^{(n)}\right\rangle_{D_2}\sqrt{\log\varepsilon}\equiv g_{st}\,(p_1-p_2)^2\,A^0_{D_2}\sqrt{\log\varepsilon}$$

To the order $1/M$ (i.e. to the order g_{st}), we recognize precisely the same structure, as in (2.10). We shall not work out here the actual coefficients that depend on a number of conventions, but it is clear that the non-trivial dependencies on the momenta and the cut-off do in fact match, as it is necessary for the successful implementation of the Fischler-Susskind mechanism.

One cannot help but notice that the version of the Fischler-Susskind mechanism that we've just described bears a strong resemblance to the investigations of [15]. Indeed, limiting themselves to the lowest order of the string coupling expansion, the authors of [15] have shown that, to that particular order, the consistency of the string S-matrix should be restored if the background of a straight D0-brane worldline is augmented by an inclusion of the operator

$$V_{TF}=\frac{p_1^\mu-p_2^\mu}{M}\int\partial_n X_\mu(\theta)X^0(\theta)\,\Theta(X^0(\theta))d\theta$$

(where $\Theta(X^0)$ is a step function). Heuristically, such an operator corresponds to the D0-brane abruptly starting to move with the appropriate recoil velocity at the moment $X^0=0$. (Once again, the operators $\int\partial_n X^i(\theta)d\theta$ with i being one of the Dirichlet directions shift the entire trajectory of the D0-brane.) The UV divergence induced by this operator is precisely what we've found in (2.10). Moreover, the divergent integral

$$\int\frac{d\kappa}{\kappa^2}e^{-\kappa^2\log\varepsilon}$$

makes an actual appearance in the derivations of [15]. These parallels should not be too surprising since, for example, the structure

$$\int_0^t d(\tilde{t},\sigma)J^\mu(\sigma)d\tilde{t}d^2\sigma$$

(entering the expression for the saddle point value of the D0-brane trajectory $f^\mu(t)$) develops a discontinuity in its first derivative as the boundary of the worldsheet shrinks to a single point.

It will be one of the primary objectives of the subsequent chapter to compare and contrast the propositions of the Periwal-Tafjord paper with our present treatment. We'll see that, despite of the notable algebraic similarities, the two approaches are separated by essential differences when it comes to their physical effect and implications. It will be explained, in particular, why the Periwal-Tafjord approach in its straightforward form cannot provide an adequate treatment of the D0-brane recoil.

Is it possible to reach a deeper understanding of how the Fischler-Susskind mechanism works in our present formalism beyond the lowest order of the recoil perturbation theory? Extending our above analysis to the higher orders in g_{st} is likely to require a more systematic picture of both the general structure of the recoil perturbation theory and the general structure of the divergences in the modular integration. Yet, conceptually, there doesn't seem to be anything puzzling about how the cancellations could actually occur.

Of course, the situation is much more intricate for the annihilation/pair-production case. In that regime, the first non-trivial UV divergence will arise if we try to compute the pre-exponential factor in the saddle point estimate (1.16). There doesn't appear to exist any meaningful expansion of this pre-exponential factor in any small parameter. It is merely a determinant of some integral operator, whose divergent part needs to be matched against the modular divergences from the higher genus worldsheets. Any cancellations in this setting will necessarily be rather subtle. In particular, they may require certain relations between the values of the saddle point exponentials, and therefore certain relations between the solutions to the saddle point equation (1.12) for worldsheets of different genera. We shall not pursue this line of thought any further in this present investigation. Even though the considerations related to the Fischler-Susskind mechanism are of a crucial importance for establishing the consistency of the worldline formalism, they do not affect the estimates of the saddle point exponential and, in particular, its coupling constant dependence, which is the main objective of our present inverstigation of the annihilation process.

Chapter 3

D0-brane recoil

The recoil of D0-branes has been essentially the primary concern of the previous chapter. And, by analyzing the cancellation of divergences between the different worldsheet topologies, we have established the Hirano-Kazama formalism as a viable approach to describing the physical phenomenon of recoil.

The primary concern of this present chapter is to make contact between the implementation of recoil we have already developed and the work on this subject that has been previously described in the literature[1]. To this end, we shall look at the same problem from a different perspective. Namely, we'll start with the worldsheet CFT description of the D0-brane, observe its inconsistency (manifested by the infrared divergences in the modular integration of the higher genus worldsheets), conclude that these divergences are indicative of recoil, and, furthermore, ask the question of how the worldsheet CFT should be modified, to the

[1]This and the subsequent chapter rely crucially upon an extensive collaboration with Ben Craps and Shin Nakamura.

lowest non-trivial order in string coupling, in order to accommodate recoil. The results of the previous chapter will provide guidance for this sort of undertaking.

The question of how the translational mode of the D0-brane is treated in the standard CFT and its deformations relevant to the phenomenon of recoil will be completely central to our investigations. To this end, it will be prudent to start this chapter with a brief reminder on how the translational mode of solitons is treated in field theory. Such preliminary exercise is instructive since, in field theory, the various versions of recoil perturbation theory (whether infrared-divergent or properly accommodating the translational motion of the soliton) can be *derived* from the fundamental field-theoretical Lagrangian. In string theory, this deeper level of description is lacking, and one must be content with operating at the level of the various forms of the worldsheet perturbative expansion themselves.

3.1 A few remarks on soliton recoil in field theory

It is a familiar fact that the solitonic quasi-particles in field theory arise as quantum descendants of the static stable finite energy solutions of the classical equations of motion. For convenience, let us generically denote such solutions as $\Phi^a_{(0)}(x)$ where Φ^a is a collective label for all the fields taking non-trivial values in the solitonic configuration.

To investigate the impact such classical solutions have on the structure of the quantum field theory, one examines the fluctuations of the fields linearized in the vicinity of the classical solutions. One might be looking for a mode expansion with the terms of the form

$$\phi^a(x,t) = e^{iEt}\phi^a(x)$$

The typical structure of such a mode expansion is as follows:

- There will be a few modes with $E = 0$. In particular, the modes corresponding to the displacement of the original solution ($\phi_i^a(x) = \partial\Phi_{(0)}^a(x)/\partial x^i$) are always present.

- There may be a few *normalizable* modes with E taking values in a discrete spectrum. These modes correspond to "internal" quantum excitations of the soliton.

- There will be a continuous spectrum of modes approaching plane waves far away from the soliton. These modes correspond to the quanta of the fundamental fields.

One therefore arrives at the picture of a solitonic quasi-particle with translational degrees of freedom (and possibly some internal degrees of freedom) capable of interacting with the quanta of the fundamental fields.

The presence of the $E = 0$ modes corresponding to the displacement of the solitonic solutions may appear very natural and completely innocuous. Nevertheless, they introduce a number of formal complications into the Feynman diagram representation of the scattering amplitudes, since they make the propagators for the fields from which the solitonic solution is constructed formally divergent.

The most straightforward attempt to deal with the complications from the translational modes has become known as "naïve semiclassical quantization". If one simply discards the contribution of the translational mode from the functional determinants, then the lowest order contributions to the scattering amplitudes are finite. Such scattering amplitudes would ignore the recoil of the soliton (which is legitimate at the lowest order in coupling, due to the large mass of the soliton).

If one pursues this program to higher orders, one encounters infrared divergences coming from the fact that the linearized wave operator in the background of the soliton is non-invertible (on account of the zero-mode). One could try to regularize the Gaussian integrations and work with the infrared-divergent theory. Performing the integral over the translational mode as if it were a Gaussian integration would effectively treat the free solitonic particle as if it were confined in a harmonic oscillator potential with a frequency tending to zero. As we'll see below, this picture is very similar to how the translational mode of the D0-brane is treated in the worldsheet CFT.

One could try, however, to develop a more intelligent treatment of the translational mode. This has been accomplished by Christ and Lee [13]. Instead of writing the naïve linearized mode expansions in the background of the soliton, one introduces the following ansatz:

$$\Phi^a(x,t) = \Phi^a_{(0)}(x - X(t)) + {\sum_k}' \eta_k \, \phi^a_k(x - X(t), t)$$

where the prime in the sum stands for the omission of the translational mode. Substituting this expansion in the original Lagrangian of the field theory, one ends up with a description in which the position of the center-of-mass of the soliton appears as a canonical variable whose state can be explicitly specified. This is reminiscent of the Hirano-Kazama formalism. We'll keep in mind the analogies between the different ways to treat the translational mode in field theory and string theory as we proceed with the analysis of the D0-brane recoil.

3.2 Worldsheet CFT and the translational mode of D0-brane

It is a conventional statement in relation to D-branes that their dynamics is described by the massless scalar vibrational states of the open strings attached to them. For the purposes of evaluating the S-matrix, such massless scalar vibrational states should be represented by their vertex operators:

$$\int d\theta : \partial_n X^i(\theta) \exp\left[ik_\mu X^\mu(\theta)\right] :$$

(where the index i runs over the Dirichlet directions and the index μ — over the Neumann directions).

Indeed, for the higher-dimensional D-branes, such massless states of open strings attached to the D-brane can be identified with one-particle states of the worldvolume fields corresponding to the deformations of the D-brane (plus the gauge field). It is also quite intuitive: the open strings move at the speed of light in the Neumann directions, and so do the excitations of the worldvolume. Furthermore, should one be willing to specify the initial and final vibrational states of a higher-dimensional brane in a scattering process involving, say, closed strings, this can be immediately accomplished by including the appropriate open string states into the initial and final states of the string theory S-matrix.

One must, however, realize that a significant subtlety is encountered if one attempts to extend this picture to the case of D0-branes. Indeed, the only dynamical degree of freedom of the D0-brane is the translational mode (i.e. the spatial co-ordinate of the D0-brane). The quantum mechanical spectrum of such mode is well known to be the spectrum of momentum eigenstates. This spectrum is continuous and it cannot be straightforwardly matched to the discreet spectrum of open strings attached to the D0-brane. To fully appreciate the difficulty it may be illuminating to contemplate the question of how one would use open strings attached to the D0-brane to specify the quantum state in which the D0-brane moves at 1 mile-per-hour.

One would sometimes hear that the translational motion of the D0-brane should correspond to some "condensate of the open strings". While such statements are not necessarily completely meaningless, their practical value is limited, unless one is capable to specify a recipe for how the open strings should be used to describe the moving D0-branes.

It is an important (even if trivial) observation that the energy of the "massless scalar" open string state attached to the D0-brane, namely,

$$\int d\theta : \partial_n X^i(\theta) \exp\left[ik_0 X^0(\theta)\right] :$$

is exactly 0. Adding more "massless scalar" open strings will not change energy either. One ends up with a (formally) discrete spectrum of evenly infinitesimally spaced energy levels. This is the spectrum of a harmonic oscillator with a frequency tending to 0 — a hardly surprising finding, given that the translational degree of freedom of the D0-brane is freely moving. Using open strings to describe the translational mode of the D0-brane would thus be the same as using the frequency-goes-to-0 limit of the harmonic oscillator to describe a free particle.

One cannot help but notice the analogy between the situation I've just described and the naïve quantization of a soliton that does not go through the trouble of carefully treating the zero-mode. The resulting description features, in particular, a harmonic oscillator of zero frequency, which, in turn, introduces a number of infrared pathologies. What would be the string theory analog of the Christ-Lee method that has been described in the previous section?

Well, it is precisely the Hirano-Kazama formalism. One explicitly *forbids* open strings attached to the D0-brane in the initial and final states. Instead, one introduces the translational mode as an explicit dynamical variable, much in the same way as the Christ-Lee method does for the case of solitons. The resulting amplitudes are free of the infrared pathologies.

One must realize that there is a direct analogy between this approach and the standard treatment of D-instantons [2]. Namely, in the latter case, one does not attempt to describe the translational zero-modes of the D-instantons in terms of open strings. Instead, one introduces the collective coordinate (i.e. the position of the D-instanton) explicitly and integrates over it. The resulting amplitudes are free of infrared pathologies. The Hirano-Kazama formalism is a direct analog of this procedure for the case of D0-branes.

Even though it may be rather cumbersome to specify the final state of the D0-brane in terms of open strings in the worldsheet CFT, one may still ask the question what is the initial and final state of the D0-brane *implied* in the standard Dirichlet CFT formalism when one computes the matrix elements for closed strings to scatter off it, *without* any open strings. This question is relevant inasmuch as we're trying to implement recoil within the CFT, in order to make contact with the existing literature on the subject. As we'll see below, the answer to this question is not completely trivial, and, in order to deduce the answer, we'll make an appeal to the well-known low-energy description of the D-branes, the DBI action.

3.3 The DBI picture of D0-brane recoil

Since the state of the translational mode of the D0-brane is specified in a rather implicit way in the worldsheet CFT computations, one needs to resort to some external framework in order to identify which initial and final states of the translational mode correspond to, say, a worldsheet scattering amplitude that does not contain any open strings.

To this end, we'll appeal to the DBI action that reproduces the low-energy limit of the string amplitudes and introduces the translational mode very explicitly. The question we'll ask is what initial and final states for the translational mode of the D0-brane one needs to specify (in the DBI language) in order to make the resulting DBI amplitude mimic the low-energy limit of the worldsheet CFT amplitude without open strings. As I've already mentioned, the answer will not be completely trivial; for example, the naïve guesses that the state of the D0-brane in the worldsheet CFT is the momentum eigenstate with a zero momentum, or a coordinate eigenstate with the D0-brane located at the origin are *incorrect*.

I'll start with the DBI action (Polchinski, v.1 p.270):

$$S_{DBI} = -\tau \int d^{p+1}\xi \, e^{-\Phi(X(\xi))} \left\{ \det \left[\frac{\partial X^\mu}{\partial \xi^a} \frac{\partial X^\nu}{\partial \xi^b} \left(G_{\mu\nu}(X(\xi)) + B_{\mu\nu}(X(\xi)) \right) + 2\pi\alpha' F_{ab} \right] \right\}^{1/2}$$

and restrict myself, for the sake of convenience, to scattering of one dilaton off a (non-relativistic) D0-brane[2]. In this case, the relevant part of the above action is

$$M \int dt \left(\frac{1}{2} \left(\dot{X} \right)^2 - \Phi \left(t, X^i(t) \right) + \frac{1}{2} \left(\Phi(t, X^i(t)) \right)^2 \right) \tag{3.1}$$

The interactions of the dilaton and the translational mode involving time derivatives are omitted, as they do not contribute to infrared divergences. We further Taylor expand:

$$\Phi \left(t, X^i(t) \right) = \exp \left[X^i(t) \frac{\partial}{\partial x^i} \right] \Phi(t, 0) \tag{3.2}$$

When embedded into Feynman diagrams, $\partial / \partial x^i$ will turn into the momentum flowing from the closed strings into each particular vertex.

Let me consider the contribution to the dilaton scattering from the last term in (3.1). In the operator language (I'll work in the interaction picture), it is just

$$\langle f, k_2 | \frac{M}{2} \int dt \exp \left[X^i(t) \frac{\partial}{\partial x^i} \right] (\Phi(t, 0))^2 | i, k_1 \rangle \tag{3.3}$$

where $\langle f|$ and $|i\rangle$ describe the initial and final state of the D0-brane, and $\langle k_2|$ and $|k_1\rangle$ describe the outgoing dilaton of momentum k_2 and incoming dilaton of momentum k_1. Furthermore,

$$\langle k_2 | \left(\Phi(t, x^i) \right)^2 | k_1 \rangle \sim \exp \left[i(k_2^0 - k_1^0) t \right] \exp \left[-i(k_2^i - k_1^i) x^i \right]$$

(the overall coefficient of the amplitude will not interest us here) and (3.3) can be rewritten as

$$\langle f | \int dt \exp \left[-i(k_2^i - k_1^i) X^i(t) \right] \exp \left[i(k_2^0 - k_1^0) t \right] | i \rangle \tag{3.4}$$

[2]Note that even though there are higher derivative corrections to the DBI action, they would not have a major bearing upon any infrared issues, such as recoil. Indeed, adding derivative interactions to any given process will soften the infrared behavior, and hence will only be able to introduce subleading contributions (compared to the ones coming from the DBI action).

I will simply specify the initial and final states of the D0-brane to be spatially broad Gaussian wave packets (of width d at $t = 0$) with momentum centered around 0 and P respectively. We'll see that the inverse width of the wave packets should be identified in a particular way with the infrared cut-off imposed in the worldsheet theory.

It is most convenient to compute the matrix element in (3.4) in the "Schrödinger" picture[3] and in the momentum representation, since the expressions for the Schrödinger (momentum) wavefunctions of the initial and final states we've chosen are well known:

$$\begin{aligned} \Psi_f(p,t) &\sim d^{D/2} \exp\left[-\frac{(p-P_2)^2 d^2}{2}\right] \exp\left[\frac{ip^2 t}{2M}\right] \\ \Psi_i(p,t) &\sim d^{D/2} \exp\left[-\frac{(p-P_1)^2 d^2}{2}\right] \exp\left[\frac{ip^2 t}{2M}\right] \end{aligned} \tag{3.5}$$

where D is the number of spatial dimensions. Moreover, in the momentum representation, the operator $\exp[i(k_1 - k_2)X]$ is nothing other than a shift of the wavefunction by $k_1 - k_2$. Hence, (3.4) becomes

$$\begin{aligned} &\langle f| \int dt \exp\left[-i(k_2^i - k_1^i)X^i(t)\right] \exp\left[i(k_2^0 - k_1^0)t\right] |i\rangle \\ &= \int dt \exp\left[i(k_2^0 - k_1^0)t\right] \int dp\, \Psi_f^*(p,t)\Psi_i(p + k_2 - k_1, t) \\ &\sim \exp\left[-\frac{(P_2 + k_2 - P_1 - k_1)^2 d^2}{2}\right] \\ &\times \int dt \exp\left[i\left(k_2^0 - k_1^0 + \frac{P_2^2}{2M} - \frac{P_1^2}{2M}\right)t\right] \exp\left[-t^2\,\zeta\left(P_i, k_i\right)/d^2\right] \end{aligned}$$

where ζ is some function of the momenta whose precise form will drop out of the final result. The important point is that, as d goes to infinity, the integral over dt becomes just

[3] It's in quotation marks because it's the Shrödinger picture of the free particle, not of the particle described by (3.1).

the energy conserving δ-function, and, as one should have expected, the DBI expression for the particular amplitude we're considering (scattering of one dilaton off the D0 due to the contact interaction term in the DBI action) becomes proportional to

$$\delta\left(k_2^0 - k_1^0 + \frac{P_2^2}{2M} - \frac{P_1^2}{2M}\right) \exp\left[-\frac{(P_2 + k_2 - P_1 - k_1)^2 d^2}{2}\right] \tag{3.6}$$

(Note that, because we've been considering the normalized wavepacket states rather then the momentum states, the momentum conservation δ-function does not appear in the amplitude in the $d \to \infty$ limit: rather, the amplitude is finite (in this limit) if the momentum conservation is satisfied, and vanishes otherwise.)

The question is now how this amplitude should be expanded to match the structure of the IR-divergent CFT-based worldsheet perturbation theory. One important circumstance to realize is that the CFT amplitude is recovered in the limit $d \to \infty$ and there is an identification between the worldsheet cut-off ε and d (which specifies which particular amplitude on the DBI side we're considering) of the type

$$d^2 \sim g_{st}\alpha' \sqrt{\log \varepsilon} \tag{3.7}$$

Note that this relation is not obvious a priori and the identification needs to imposed precisely because it makes the structure of the IR-divergent worldsheet perturbative expansion visible at the level of the DBI action. This IR-divergent structure is extremely unnatural from the standpoint of the DBI description, but it is forced upon us by the worldsheet CFT, where it is implemented by construction.

With the above identification, one can readily expand the amplitude (3.6) in powers of g_{st} (or $1/M$) and observe the emergence of the IR-divergent structure reminiscent of the

CFT. The lowest-order term is just

$$\delta\left(k_2^0 - k_1^0\right)$$

which is obviously just one of the terms in the low-energy expansion of the worldsheet disk amplitude (please keep in mind that we've restricted our analysis to the dilaton contact interaction term in the DBI Lagrangian). There are two corrections to the above δ-function arising at the first order of g_{st}:

$$\left(\frac{P_2^2}{2M} - \frac{P_1^2}{2M}\right)\delta'\left(k_2^0 - k_1^0\right) \tag{3.8}$$

which arises from expanding the energy conservation δ-function and

$$d^2(P_2 + k_2 - P_1 - k_1)^2\delta\left(k_2^0 - k_1^0\right) \tag{3.9}$$

which arises from expanding the exponential (please keep in mind that $d^2 \sim g_{st}$ if one is match the DBI and CFT descriptions).

If $P_1 = P_2 = 0$, which is precisely the case of the standard Dirichlet CFT, (3.8) vanishes, whereas (3.9) reproduces the IR-divergence in the modular integration of the annulus. The conclusion (reached here through an appeal to the DBI formalism) is then that the standard Dirichlet CFT amplitudes without open strings describe a D0 brane whose initial and final states are Gaussian wavepackets centered around $P = 0$, $X = 0$, and the width should be identified with the CFT infrared cut-off as in (3.7) and taken to infinity.

3.4 Critique of the previously advocated approaches to recoil

Let me now summarize the points of dissatisfaction that I believe should arise upon a scrutinizing examination of the recipes aimed at incorporating the phenomenon of recoil into the worldsheet CFT description of the string amplitudes that have been previously advocated in the literature. Three publications are usually mentioned in relation to this subject, with their authors being Periwal and Tafjord [15]; Fischler, Paban and Rozali [14]; and Hirano and Kazama [4].

It must be evident that, of the three aforementioned treatments of the subject, the paper by Hirano and Kazama bears the closest similarity to the pursuits undertaken in this thesis. I shall nevertheless emphasize that the presentation given in that paper cannot be satisfactory for someone with a specific interest for the problem of recoil. Indeed, as it has been repeatedly highlighted throughout the exposition of this thesis, the cancellations between the divergences coming from worldsheets of different topologies are extremely essential to the implementation of recoil in string theory. The authors of [4] do not recover the disk divergence in a form that would make it possible to see how it cancels the divergence in the modular integration of the annulus. The second chapter of this thesis is specifically dedicated to making up for this shortcoming.

The paper by Periwal and Tafjord is also quite essential to our present considerations in that it gives particular attention to the role of the Fischler-Susskind mechanism for the formal implementation of recoil in string theory. It has been already mentioned chapter 2 that the version of the Fischler-Susskind mechanism that naturally arises from the Hirano-Kazama formalism bears a strong resemblance to the considerations of the Periwal-Tafjord paper. We will eventually see that an algebraically minor (even though essential) modification of the Periwal-Tafjord approach reconciles it completely with the considerations building upon the Hirano-Kazama formalism. However, it is important to understand that, as it is presented in [15], the Periwal-Tafjord approach does *not* provide an adequate treatment of recoil. Let me describe below some of its essential failures.

In order to cancel the modular integration divergence coming from the annulus, Periwal and Tafjord suggest to introduce a background correction of the form

$$V_{PT} \sim \int d\theta \, \partial_n X^i(\theta) F^i(X^0(\theta)) \tag{3.10}$$

where $F(t)$ is some function and the index i, once again, labels the Dirichlet directions. Periwal and Tafjord make a particular choice

$$F^i(t) = \frac{p_2^i - p_1^i}{M} \, t \, \Theta(t)$$

(where p_1, p_2 and M are the initial and final momentum and the mass of the D0-brane respectively, and $\Theta(t)$ is the step function). However, as we'll see below the divergence cancellation only depends on the asymptotic behavior of $F(t)$ in the infinite past and future. Note the absence of normal ordering in (3.10).

The physical interpretation of this background correction ("the recoil operator") is that the D0-brane starts to move along the trajectory $F(t)$. In particular, the choice of $F(t)$ made in [15] corresponds to the D0-brane abruptly starting to move at the moment $t = 0$. The authors then express a minor dissatisfaction with the abruptness of recoil which is manifest in their implementation.

As we're about to show, the divergence cancellation does not depend on the detailed shape of the D0-brane trajectory $F(t)$ and, therefore, the abruptness of the particular trajectory chosen in [15] can hardly be a problem. However, the operator (3.10) indeed exhibits unattractive features, even at the purely heuristic level:

- in an actual recoil process, the D0-brane does *not* move along a classical trajectory;
- the operator (3.10) explicitly breaks the time translation invariance.

Let us now show that these pathologies indeed lead to unacceptable behavior of the physical amplitudes.

If one starts with $F(t)$ satisfying the asymptotic conditions

$$F^i(t \to -\infty) \sim 0 \qquad F^i(t \to +\infty) \sim v^i t + x_*^i$$

the Fourier transform of $F(t)$ can be written in the form

$$F^i(\omega) = \frac{v^i}{(\omega + i0)^2} + \frac{x_*^i}{\omega + i0} + \varphi^i(\omega) \tag{3.11}$$

where $\varphi(\omega)$ is analytic at $\omega = 0$. Correspondingly, the operator (3.10) can be rewritten in the form

$$V_{PT} \sim \int d\theta \, d\omega \, F(\omega) \, \partial_n X^i(\theta) \exp\left[-i\omega X_0(\theta)\right] \tag{3.12}$$

Let us now insert this operator into the disk amplitude containing the closed string vertex operators $V_1(\sigma_1),\cdots,V_n(\sigma_n)$ carrying momenta $k_1,\cdots,k_n$ with $P = \sum k_i$:

$$\langle V_{PT} V_1(\sigma_1) \cdots V_n(\sigma_n) \rangle_{D_2}$$

Performing first the path integral over X^0, we obtain

$$\int d\theta\, F^i(P^0)\,\partial_n X^i(\theta)\exp\left[(P^0)^2\log\varepsilon + P^0 h(\theta,\sigma_i,k_i)\right]\langle V_1(\sigma_1)\cdots V_n(\sigma_n)\rangle_{D_2}^{\int X^0} \tag{3.13}$$

Here, the $(P^0)^2\log\varepsilon$ term in the exponent comes from the self-contraction in the recoil operator, all the contractions among the various vertex operators are symbolically assembled as $\langle V_1(\sigma_1)\cdots V_n(\sigma_n)\rangle_{D_2}^{\int X^0}$, and the $P^0 h(\theta,\sigma_i,k_i)$ term comes from the contractions between the recoil operator and the vertex operators. We shall not need the explicit form of the function $h(\theta,\sigma_i,k_i)$. Let us only note that it is non-singular as long as σ_i's stay away from the boundary of the worldsheet. I shall briefly comment below on the important subject of integration over the positions of the vertex operators. Note also that the integration over ω present in (3.12) has disappeared in (3.13) due to the energy conservation δ-function.

We shall now examine the $\varepsilon\to 0$ limit of the expression (3.13). In this limit, it vanishes unless $P^0 = 0$ and should therefore be thought of a distribution (made of the δ-function and its derivatives) rather than an ordinary function. Hence, to analyze the $\varepsilon\to 0$ limit, we'll examine the convolution of (3.13) with an arbitrary smooth function $\mathcal{G}(P^0)$ admitting a Taylor expansion

$$\mathcal{G}(P^0) = \mathcal{G}(0) + P^0\mathcal{G}'(0) + \cdots$$

It is important to keep in mind the expression (3.11) for the function F. Only a few terms of the Taylor expansion will contribute to the final answer, since

$$\int dP^0 \left(P^0\right)^n \exp\left[(P^0)^2\log\varepsilon\right]$$

goes to 0 in the limit $\varepsilon\to 0$ if $n\geq 0$. Some other relevant formulas for the integral expressions we'll have to use are

$$\int dP^0 \, \frac{\exp\left[(P^0)^2 \log\varepsilon\right]}{\left(P^0+i0\right)^2} \sim \sqrt{|\log\varepsilon|} \qquad \int dP^0 \, \frac{\exp\left[(P^0)^2 \log\varepsilon\right]}{P^0+i0} \sim 1$$

If we now contract (3.13) with $\mathcal{G}(P^0)$:

$$\int d\theta \, dP^0 \, \mathcal{G}(P^0) \, F^i(P^0) \, \partial_n X^i(\theta) \exp\Big[(P^0)^2 \log\varepsilon + P^0 h(\theta,\sigma_i,k_i)\Big] \, \langle V_1(\sigma_1)\cdots V_n(\sigma_n)\rangle_{D_2}^{\int X^0}$$

and use the above integral formulas, we obtain a few distinct terms:

1) The only term that diverges in the limit $\varepsilon \to 0$ is proportional to

$$\sqrt{|\log\varepsilon|} \, v^i \, \mathcal{G}(0) \int d\theta \, \partial_n X^i(\theta) \, \langle V_1(\sigma_1)\cdots V_n(\sigma_n)\rangle_{D_2}^{\int X^0}$$

After integration over X^i, this becomes

$$\sqrt{|\log\varepsilon|} \, \mathcal{G}(0) \, v^i P^i \, \langle V_1(\sigma_1)\cdots V_n(\sigma_n)\rangle_{D_2}$$

The corresponding term in the expression (3.13) itself (rather than its convolution with $\mathcal{G}(P^0)$) is

$$\sqrt{|\log\varepsilon|} \, \delta(P^0) \, v^i P^i \, \langle V_1(\sigma_1)\cdots V_n(\sigma_n)\rangle_{D_2}$$

This is precisely the structure that, in [15], has been claimed to be responsible for cancelling the annular divergence after the final recoil velocity v is adjusted to the value P/M. Note, however, that, by itself, this argument does not even determine the velocity v, but only its component along P. Already at this stage, there are discrepancies between the intuitive notion of recoil and the mathematical structure provided by the Periwal-Tafjord operator.

2) Another contribution comes from the $P^0\mathcal{G}'(0)$ term in the Taylor expansion for $\mathcal{G}(P^0)$ and is proportional to

$$\delta'(P^0) \, v^i P^i \, \langle V_1(\sigma_1)\cdots V_n(\sigma_n)\rangle_{D_2}$$

This term should correct the energy conservation in such a way that the final energy of the D0-brane (proportional to $v^i P^i$) is taken into account. Note the similarities with the corresponding term found in the DBI formalism in the previous section.

3) The last contribution we shall consider here come from the second term in (3.11). It is proportional to

$$\delta(P^0)\, x_*^i P^i \,\langle V_1(\sigma_1)\cdots V_n(\sigma_n)\rangle_{D_2}$$

The essential feature that needs to be highlighted here is that the above expression depends on x_*, the parameter that shifts the D0-brane trajectory. Different values of x_* would produce different contributions to the finite part of the closed string scattering amplitude.

However, there is clearly no preferred physical choice for x_*. The D0-brane is a quantum particle that does not have a definite position in space as long as its final velocity v is specified. An attempt to implement the Periwal-Tafjord approach to recoil thus leads to an apparent absurdity: an infinite-fold ambiguity in the values of the physical amplitudes governed by the value of the fictitious position of the quantum D0-brane.

Further examination is likely to reveal additional pathologies introduced by shifting the classical trajectory of the D0-brane in a manner inconsistent with time translation invariance. In particular, the above consideration did not pay the due attention to the integration over the positions of the vertex operators. The limit of this integration when the closed string vertex operators approach the boundary is likely to produce specific singularities due to the presence of the recoil operator. It is to be anticipated that the study of such singularities will exhibit additional inconsistencies of the Periwal-Tafjord approach. I shall not pursue this line of thought further in this present exposition.

Finally, let me briefly comment on the paper [14] by Fischler, Paban and Rozali. In its spirit and general attitude to implementing the Fischler-Susskind mechanism, it is very similar to the Periwal-Tafjord paper. However, many of the mathematical details there have been reportedly found obscure by a number of readers. In particular, the authors of [14] claim that it is necessary to introduce simultaneously a few different recoil operators, one of which is bilocal (and resembles, but is not quite identical to the one we're about to introduce in relation to the Hirano-Kazama formalism), the other one is a shift of the classical D0-brane trajectory analogous to the Periwal-Tafjord approach. The rationale behind such algebraic excesses is not too apparent. There are further computational details that are hard to agree with. For example, it is claimed that the annular divergence is proportional to $\log \varepsilon/T$ (where T is designated as a "large time cut-off"). As it has been shown in the previous chapter, the correct form of the annular divergence is $\sqrt{|\log \varepsilon|}$.

3.5 Hirano-Kazama formalism and the worldsheet CFT

As it has been already remarked, the most natural way to deal with the translational mode of the D0-brane is to introduce the corresponding dynamical degree of freedom explicitly in the path integral. However, it may be convenient to derive a certain recipe that would allow to compute the relevant amplitudes by a suitable deformation of the conventional worldsheet CFT. This would also bring us in contact with the previous work on the subject.

This line of pursuit should be encouraged by the similarity between the version of the

Fischler-Susskind mechanism that arises in the Hirano-Kazama formalism and has been described in chapter 3 on one hand and the considerations of [15] on the other hand. (Note, however, that these similarities do not amount to identity.) To complete the picture, it suffices to realize that the structure of divergence cancellation found in chapter 3 can be mimicked by deforming the worldsheet CFT with the following *bilocal* operator:

$$V_{bilocal} = \frac{1}{M} \int d\theta d\theta' G_{(0+1)} \left(X^0(\theta) - X^0(\theta') \right) \partial_n X^i(\theta) \partial_{n'} X^i(\theta') \tag{3.14}$$

Here, $G_{(0+1)}$ is the retarded Green function of the free particle:

$$G_{(0+1)}(t) = t\,\Theta(t)$$

If one inserts the operator (3.14) into the CFT path integral for a closed string scattering amplitude, performs the integration over the values of $X^0(\sigma)$ in the *interior* of the worldsheet, and then identifies $X^0(\theta) \to t(\theta)$ and $p_1 \to (M, 0)$, one concludes that $V_{bilocal}$ is the same as the operator F_1 of (2.8), which was found to be responsible for the divergence cancellation within the Hirano-Kazama formalism.

Heuristically, the operator (3.14) can be understood as follows. $\partial_{n'} X^i(\theta')$ is the operator of momentum density flowing out of the point θ' of the boundary of the worldsheet (integrated over θ', it gives the total transferred Dirichlet momentum P). Then,

$$\int d\theta' G_{(0+1)} \left(X^0(\theta) - X^0(\theta') \right) \partial_{n'} X^i(\theta') \tag{3.15}$$

evaluates the displacement (in the Dirichlet directions) of the point on the D1-brane world-surface with the co-ordinates $X^0(\theta)$ and $X^1(\theta)$ due to the momentum influx from the closed strings. Finally, the remaining $\partial_n X^i(\theta)$ actually displaces the point on the D0-brane worldline with the co-ordinate $X^0(\theta)$ by the amount (3.15). Overall, the operator (3.14) deforms the D0-brane worldline precisely in the way that intuitively corresponds to its response to the impact by the closed strings.

Of course, the above description merely serves to relate the background modification we're proposing to the intuitive notion of recoil, and its precise justification comes from the corresponding cancellation of divergences.

It is not hard to check that the operator (3.14) does in fact induce the appropriate divergence on the disk:

$$g_{st} P^2 \left\langle V^{(1)} \cdots V^{(n)} \right\rangle_{D_2} \sqrt{\log \varepsilon} \tag{3.16}$$

(we shall not concern ourselves with the coefficients in this treatment). The algebraic considerations leading to this result are essentially identical to what has been done for the Periwal-Tafjord recoil operator in the previous section. Note, however, that the bilocal operator itself is by no means identical to the Periwal-Tafjord recoil operator. In particular, it is manifestly time-translation invariant and does *not* correspond to a classical deformation of the worldsheet trajectory.

An important question to be asked here is what is the physical meaning of the closed string scattering amplitudes that the (divergence-free) CFT deformed with the bilocal recoil operator computes, or, more precisely, what are the initial and final quantum states of the D0-brane that correspond to such amplitudes.

This question is closely related to the considerations of the section 3.3, and, again, there is a (minor) subtlety. One could naïvely guess that, after the recoil operator is introduced as a deformation of the worldsheet CFT, the resulting closed string scattering amplitudes should imply D0-brane at rest as the initial state, and D0-brane moving with the appropriate velocity (developed in the course of recoil) in the final state. In other words, one could think that the initial and final states of the D0-brane are momentum eigenstates.

This naïve guess cannot be correct, however, since the amplitude to transition between the zero momentum state and a state carrying the amount of momentum transferred by the incident closed strings must be infinite on account of the momentum conservation δ-function, whereas the amplitude computed by the worldsheet CFT deformed with the recoil operator is finite by construction. (Note that the presence of the momentum conservation δ-function is closely related to the non-normalizability of the momentum eigenstates.)

The resolution here is that (in close parallel to the considerations of the section 3.3) the CFT deformed with the recoil operator really describes the transition between *normalized* wavepacket states (in the limit of spatial extent of the wavepacket going to infinity), rather than the momentum eigenstates. The difference between these amplitudes is precisely that the limit of the wavepacket amplitude does *not* contain the momentum conservation δ-function (yet it does contain the energy conservation δ-function). This is precisely what makes such amplitudes into an adequate interpretation for the closed string scattering computations within the worldsheet CFT deformed by the bilocal operator.

Summary

Worldsheet CFT with the D0-brane Dirichlet boundary conditions suffers from loop divergences indicative of recoil. The most natural way to deal with this complication is to introduce explicitly the collective co-ordinate corresponding to the translational motion of the D0-brane. This leads to the Hirano-Kazama formalism.

Nevertheless, one could try to come up with a recipe that, to the lowest non-trivial order in the string coupling, would reproduce the physical amplitudes involving a recoiling D0-brane by means of deforming the CFT with an appropriate "recoil operator".

The ansätze for such recoil operators that have been previously put forward in the literature suffer, upon a closer examination, from pathologies that make them unacceptable for the role for which they've been proposed. Building on the considerations within the Hirano-Kazama formalism presented in the previous chapter, it is possible to deduce the appropriate form of the recoil operator, which, in particular, does not lead to the aforementioned pathologies.

Chapter 4

Local recoil of extended D-branes

The phenomenon of soliton recoil has been familiar for a few decades by now [3]. The solitons in quantum field theory are quantum descendants of the topologically non-trivial solutions of the classical field equations. In attempting to examine the scattering of fundamental field quanta in the background of these classical solutions, one discovers infrared divergences at higher orders of perturbation theory. These divergences signify a need for a background modification. This is not surprising, since the soliton has a finite mass and necessarily starts moving as a result of the impact by the incident particles. The original classical solution does not properly accommodate this detail of scattering dynamics. Hence the need for the background modification.

The string theory analog of the above phenomenon is the D0-brane recoil under the impact of the incident closed strings. Indeed, if one starts with a background of a D0-brane at rest, the modular integrations on the higher genus worldsheets exhibit divergences, thus signalling the need for a background modification. This background modification precisely corresponds to the D0-brane recoil, which has been a primary subject of the previous two chapters.

Extended topological defects cannot recoil due to their infinite mass. Nevertheless, by energy-momentum conservation, during the impact, the incident particles necessarily transfer to the extended topological defect a certain amount of momentum in the transverse directions. This momentum influx will, of course, induce a wave-like perturbation propagating along the worldvolume. The fate of this perturbation depends dramatically on the dimensionality of the topological defect under consideration.

For topological defects with more than one spatial dimension, the worldvolume perturbation will decay into spherical waves leaving no trace in the final state of the system. However, spherical waves do not exist when there is only one spatial dimension available. Therefore, for one-dimensional topological defects (vortex lines), the perturbation produced by the impact of the incident particles will not decay in the asymptotic future, and the classical background of a stationary vortex line will then have to be modified, much in the same way as for the case of soliton recoil. If one tries to perform the computation in the stationary classical background, the loop diagrams will exhibit infrared divergences.

A more formal statement about the behavior of the perturbation of the vortex line produced by the incident particles can be derived by inspecting the Green function of the (1+1)-dimensional wave operator describing the propagation of small transverse perturbations of the vortex line:

$$G_{(1+1)}(t,x) \sim \theta(|x|-t) \tag{4.1}$$

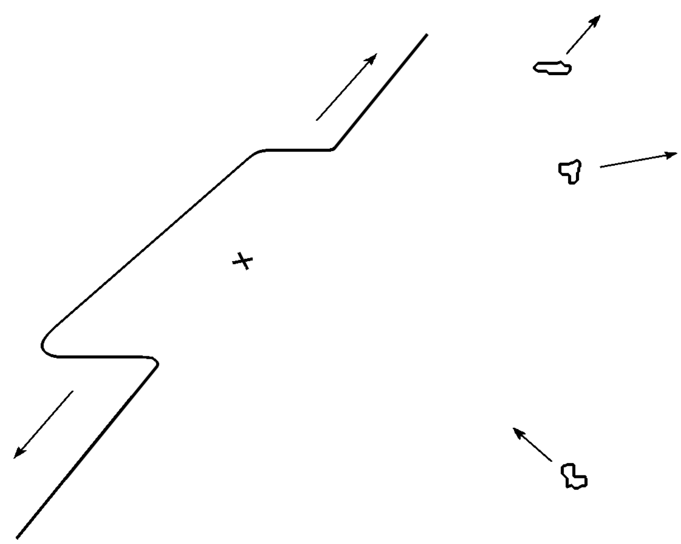

Figure 4.1: A pictorial representation of the local recoil. The closed strings scattering off a D1-brane induce two waves propagating away from the point of impact (denoted by x).

where $\theta(x)$ is the step function. The Green function (4.1) describes the response of the vortex line to a kick at $t = 0$ and $x = 0$. Quite obviously, it does not decay for large times. Rather, the system exhibits two kink-like waves propagating towards infinity and shifting the vortex line by a finite distance off its initial position as they pass along. The pictorial representation of this process is given in Fig. 4.1. We shall term it "local recoil" in the subsequent discussions.

One may wonder why the local recoil wave does not decay into radiation. (As it happens, for example, to the excitations of macroscopic strings in a cosmological context.) One intuitive explanation invokes the non-oscillating character of the local recoil wave. Moreover, by momentum conservation, the momentum transferred to the topological defect by the incident particles is purely space-like. There is no assortment of outgoing particles into which an excitation carrying such momentum can decay.

It is precisely the purely kinematic nature of the local recoil phenomenon that makes it so universal. One should expect that a generic vortex line configuration will undergo a local recoil whenever particles scatter off it, to a great extent irrespectively of the dynamical contents of the theory. In particular, local recoil should occur in a string theory setting when closed strings scatter off a D1-brane.

In the remainder of this chapter, I shall examine the local recoil of a (bosonic) D1-brane in some detail. It will be shown how the infrared divergences signalling the need for a background modification arise, and how they can be resummed in a way that enforces momentum conservation.

4.1 The limited role of the Fischler-Susskind mechanism

The general structure of the divergences in the annular modular integration in the presence of D-branes has been presented in section 2.1. In particular, we could see that the divergences were absent for Dp-branes with $p \geq 2$, in accord with the general kinematic considerations given in the introduction to this chapter.

As it has been emphasized throughout the exposition in this thesis, for the case of the D0-branes, the annulus divergence is indicative of the failure of the worldsheet CFT to describe the translational motion of the D0-brane, and it is most properly dealt with by introducing the corresponding collective co-ordinate explicitly, as per construction of the

Hirano-Kazama formalism. Equivalently, to the lowest order in string coupling, one could deform the worldsheet CFT by an appropriate recoil operator, so that the annular divergence is cancelled *and* the final state of the D0-brane is changed in such a way that the momentum is conserved.

Does this strategy translate in a meaningful way to the case of D1-brane? The most practical answer is no, and the reason is twofold. First, unlike for the case of the D0-brane, for a D1-brane one can explicitly specify its vibrational state by including the corresponding incoming and outgoing open strings into the scattering amplitude. Second, unlike for the case of the D0-brane, there is no *unique* momentum-conserving final state of the D1-brane. Hence, even if some sort of recoil operator is introduced, one would still need to specify the final vibrational state of the D1-brane in terms of open strings. Deforming the CFT with a recoil operator will merely re-shuffle the assignment of the various open sting states to the various vibrational modes of the D1-brane.

What is then the appropriate way to deal with the double-log annular divergence present in the background of a D1-brane? One could simply construct (to the lowest order in g_{st}) the various assortments of open strings that carry the appropriate amount of Dirichlet momentum and show that the amplitude to scatter into such states is manifestly finite.

In fact, an even clearer picture of the inner workings of the divergence cancellation emerges, if such computations are performed at the level of the DBI action (with the momenta of the incident closed string restricted to values much smaller than the string scale). Such considerations are similar to what has been done in the section 3.3 for the case of D0-branes. Indeed, the DBI formalism allows to work with the infrared divergences to all orders in the string coupling, and, as a result, it becomes apparent that the infrared divergences in the worldsheet computations arise from the attempt expand in a Taylor series the non-analytic dependences appearing in the scattering amplitudes. Such non-analytic momentum dependences are a direct consequence of momentum conservation (and a direct analog of the momentum conservation δ-function familiar from the case of the D0-brane). The presence of the infrared divergences in the CFT formalism is thereby related to the physical underpinnings of the local recoil phenomenon.

4.2 The final state of D1-brane, effective field theory and divergence cancellation.

The effective field theory description of the D-branes is a very powerful framework to study the infrared divergences associated with recoil, because the resummed non-polynomial form of the low energy effective action (the DBI action) allows to analyze the structure of the infrared divergences to all orders in coupling. A drawback inherent to this kind of approach is that the momenta of the incident closed strings must be restricted to values much smaller than the string scale. Considering this kinematic region is nevertheless sufficient to corroborate the qualitative description of the local recoil outlined in the introduction.

We shall start with the familiar DBI action:

$$S_{DBI} = -\tau \int d^{p+1}\xi \, e^{-\Phi(X(\xi))} \left\{ \det \left[\frac{\partial X^\mu}{\partial \xi^a} \frac{\partial X^\nu}{\partial \xi^b} \left(G_{\mu\nu}(X(\xi)) + B_{\mu\nu}(X(\xi)) \right) + 2\pi\alpha' F_{ab} \right] \right\}^{1/2}$$

and restrict ourself, for the sake of convenience, to scattering of one dilaton off a (non-relativistic) D1-brane. In this case, the G and B fields can be set to their (Minkowski) background values. Since the energy of the incident closed strings is small, the D1-brane will be perturbed off its stationary configuration only slightly:

$$X^0(x,t) = t \qquad X^1(x,t) = x \qquad X^i(x,t) = Y^i(x,t)$$

(we're considering a D1-brane stretched in the 1st spatial direction, and the Dirichlet directions are labelled by i). As it was remarked in section 2.1, the D-brane gauge fields do not contribute to the infrared divergences. At the level of effective field theory, this originates from the fact that the gauge fields only interact with the closed string fields through terms in the Lagrangian that contain the derivatives in the Neumann directions. Such derivatives will soften the infrared behavior in the loops. We shall therefore omit the gauge fields from the Lagrangian for the purposes of our analysis. If we further limit ourselves, for the purposes of demonstration, to one dilaton scattering off the D1-brane, the relevant part of the Lagrangian is

$$\tau \int dt\, dx \left\{ \frac{1}{2} (\partial_t Y)^2 - \frac{1}{2} (\partial_x Y)^2 - \Phi \left(t, x, Y^i(t,x) \right) + \frac{1}{2} \left(\Phi(t,x,Y^i(t,x)) \right)^2 \right\} \tag{4.2}$$

where τ is the tension of the D1-brane. The interactions of the Y-scalars involving derivatives in the Neumann directions have been omitted as they do not contribute to the infrared divergences (in analogy to the gauge fields). We further formally Taylor expand:

$$\Phi \left(t, x, Y^i(t) \right) = \exp \left[Y^i(t) \frac{\partial}{\partial y^i} \right] \Phi(t,x,y^i) \bigg|_{y^i=0}$$

$$\Phi^2 \left(t, x, Y^i(t) \right) = \exp \left[Y^i(t) \frac{\partial}{\partial y^i} \right] \Phi^2(t,x,y^i) \bigg|_{y^i=0}$$

To analyze the pattern of infrared divergence resummation, let us examine the contribution to the dilaton scattering amplitude from the last term in (4.2). In the operator language (we'll work in the interaction picture), it is just

$$\langle f, k_2| \frac{\tau}{2} \int dt dx \exp\left[Y^i(t,x)\frac{\partial}{\partial y^i}\right] (\Phi(t,x,0))^2 |i, k_1\rangle \tag{4.3}$$

where $\langle f|$ and $|i\rangle$ describe the initial and final state of the D1-brane, and $\langle k_2|$ and $|k_1\rangle$ describe the outgoing dilaton of momentum k_2 and incoming dilaton of momentum k_1. Furthermore,

$$\langle k_2| \left(\Phi(t,x,y^i)\right)^2 |k_1\rangle \sim \exp\left[i(k_2^0 - k_1^0)t\right] \exp\left[-i(k_2^1 - k_1^1)x\right] \exp\left[-i(k_2^i - k_1^i)y^i\right]$$

(the overall coefficient of the amplitude will not interest us here) and (4.3) can be rewritten as

$$\langle f| \int dx dt \exp\left[-i(k_2^i - k_1^i)Y^i(t,x)\right] \exp\left[i(k_2^0 - k_1^0)t\right] \exp\left[-i(k_2^1 - k_1^1)x\right] |i\rangle \tag{4.4}$$

Let us choose $|i\rangle$ to be the vacuum state $|0\rangle$ of the stretched D1-brane, and let $|f\rangle$ be a coherent state corresponding to the complex amplitude $v^i(\kappa)$ for the D1-brane oscillation of wavelength κ:

$$|v(\kappa)\rangle = \exp\left[-\frac{1}{2}\int d\kappa |v(\kappa)|^2\right] \exp\left[\int d\kappa\, v^i(\kappa) a^{i\dagger}(\kappa)\right] |0\rangle$$

where $a^{i\dagger}$ are the creation operators corresponding to the excitations of the D1-brane:

$$Y^i(t,x) = \int \frac{d\kappa}{\sqrt{2\tau|\kappa|}} \left(a^i(\kappa) e^{i(\kappa x - |\kappa| t)} + a^{i\dagger}(\kappa) e^{-i(\kappa x - |\kappa| t)}\right)$$

If we now take into account that an exponential of the field creates a coherent state, we can rewrite (4.4) as

$$\int dx dt \exp\left[i(k_2^0 - k_1^0)t\right] \exp\left[-i(k_2^1 - k_1^1)x\right] \left\langle v(\kappa) \middle| -\frac{i(k_2^i - k_1^i)}{\sqrt{2\tau|\kappa|}} e^{-i(\kappa x - |\kappa| t)} \right\rangle \tag{4.5}$$

where

$$\left| -\frac{i(k_2^i - k_1^i)}{\sqrt{2\tau|\kappa|}} e^{-i(\kappa x - |\kappa| t)} \right\rangle$$

is the coherent state in which (for every κ) the oscillation of the string with the wave vector κ is excited with the complex amplitude

$$-\frac{i(k_2^i - k_1^i)}{\sqrt{2\tau|\kappa|}} e^{-i(\kappa x - |\kappa| t)}$$

The inner product of the two coherent states in (4.5) can be evaluated using the standard formula. Once this is done, (4.5) becomes

$$\begin{aligned} &\int dx dt \exp\left[i(k_2^0 - k_1^0)t\right] \exp\left[-i(k_2^1 - k_1^1)x\right] \\ &\times \exp\left[-\frac{1}{2}\int d\kappa \left(\left|v^i(\kappa) - \frac{i(k_2^i - k_1^i)}{\sqrt{2\tau|\kappa|}} e^{-i(\kappa x - |\kappa| t)}\right|^2 + i\,\mathrm{Re}\left(\frac{v(\kappa)(k_2^i - k_1^i)}{\sqrt{2\tau|\kappa|}} e^{-i(\kappa x - |\kappa| t)}\right)\right)\right] \end{aligned}$$

The apparent feature of the above amplitude that is important for our purposes is that the integral in the exponential of the second line will diverge at small κ for a generic $v(\kappa)$. This behavior is not surprising, since, for a generic $v(\kappa)$, the final state of the D1-brane does not satisfy momentum conservation, and the corresponding amplitude should vanish.

To investigate further into this matter, let us write

$$v^i(\kappa) = \frac{A^i}{\sqrt{\kappa}} + \tilde{v}^i(\kappa)$$

where A is a constant[1] and $\tilde{v}(\kappa)$ is less singular than $1/\sqrt{\kappa}$ as κ goes to 0. The amplitude

[1]Note that, for $A \neq 0$, the state $|v(\kappa)\rangle$ (as constructed above) formally vanishes on account of the divergence in the normalization factor. This is directly analogous to the situation with the infinitely broad wave packet states considered in section 3.3, and is merely an artifact of our attempt to normalize a state

(4.5) can then be rewritten as

$$\int dxdt \exp\left[i(k_2^0 - k_1^0)t\right] \exp\left[-i(k_2^1 - k_1^1)x\right]$$
$$\times \exp\left[-\frac{1}{2}\left|A^i - \frac{i(k_2^i - k_1^i)}{\sqrt{2\tau}}\right|^2 \int \frac{d\kappa}{|\kappa|} + (\text{finite terms})\right] \tag{4.6}$$

the integral of $d\kappa/|\kappa|$ is obviously divergent, so the amplitude will vanish unless

$$A^i = \frac{i(k_2^i - k_1^i)}{\sqrt{2\tau}} \tag{4.7}$$

What's the meaning of this condition?

Consider the Dirichlet momentum carried by the D1-brane:

$$P^i = \tau \int dx\, \partial_t Y^i(x,t) = -i \lim_{\kappa \to 0} \sqrt{\frac{\tau|\kappa|}{2}} (a^i(\kappa) - a^{i\dagger}(\kappa)) \tag{4.8}$$

If we now compute the expectation value of P for our final state $|v(\kappa)\rangle$, we get

$$\langle v(\kappa)|P^i|v(\kappa)\rangle = -i\sqrt{2\tau}A^i$$

Therefore, (4.7) is nothing other than the momentum conservation condition:

$$P + k_2 - k_1 = 0$$

In other words, we have proved that the scattering amplitude (4.3) vanishes unless the momentum conservation condition is satisfied. (Please note that our argument has technically been restricted to the contribution of the last term in the action (4.2) into the dilaton scattering amplitude. However, similar arguments can be devised for the other contributions.)

from a continuous spectrum. Note, however, that it is the physical contents of the state $|v(\kappa)\rangle$ that interests us, and not the normalization (which can, after all, be adjusted at the end of the computation whichever way one pleases).

Can we relate the property that the amplitude vanishes unless the momentum conservation condition is satisfied to the annular divergences in the string diagrams found in the previous section? To answer this question, we should see how the (IR-divergent) perturbative expansion in $1/\tau$ (equivalent to a perturbative expansion in g_{st}) is implemented in our present effective field theory setting.

Let us turn back to the action (4.2). Since the kinetic term for the Y-field is accompanied by the D1-brane tension τ, the powers of Y entering the various matrix elements contributing to the amplitude will be translated into the powers of $1/\tau$. Let us concentrate for a moment on the case when the initial and final states of the D1-brane are vacuum, since this is the amplitude we've been studying in the string theory context. Then, the lowest order contribution (in $1/\tau$) to the amplitude (4.3) is

$$\langle 0, k_2| \frac{\tau}{4} \int dtdx Y^i(t,x) Y^j(t,x) \frac{\partial^2}{\partial y^i \partial y^j} \left(\Phi(t,x,0)\right)^2 |0, k_1\rangle \tag{4.9}$$

This is a divergent expression proportional to

$$(k_1 - k_2)^2 \int \frac{d\kappa}{|\kappa|} \tag{4.10}$$

We immediately recognize the same dependence[2] on the transferred Dirichlet momentum $(k_1 - k_2)^2$ as we found for the annular divergence (2.2). In fact, (4.9) is nothing other than the contribution from the particular process we're considering to the string annular diagram.

[2]The analog of $\left\langle V^{(1)} \cdots V^{(n)} \right\rangle_{D_2}$ of (2.2) in our present computation is

$$\langle 0, k_2| \frac{1}{2} \int dtdx \left(\Phi(t,x,0)\right)^2 |0, k_1\rangle$$

which is momentum-independent.

The annular diagram is divergent, and so is our present contribution. (Please note that one needs to be careful in relating the cut-off parameter used to regularize the divergent integral in (4.10) to the world-sheet cut-off employed in string theory).

What happens to this divergence (in the context of the effective field theory) when the higher order corrections are included? In fact, we have already derived the full resummed expression (4.6). If all the powers of $1/\tau$ are kept in the expression of the amplitude (4.3), the result vanishes if the initial and final states of the D1-brane are vacuum. This is in accord with momentum conservation.

Moreover, if we include more general final states of the D1-brane, we notice that the amplitude is only non-vanishing for the final states of the D1-brane that carry the amount of Dirichlet momentum dictated by the momentum conservation. This is the local recoil phenomenon we've described in the introduction.

4.3 Relation to the kinematic peculiarities of low-dimensional field theories

It has been repeatedly emphasised throughout the above presentation that the infrared divergences of the string perturbative expansion in the presence of a topological defect are related to the momentum conservation. This statement may seem paradoxical, since the momentum is conserved for all possible D-branes, yet the divergences are present only for Dp-branes with $p < 2$.

One can furthermore examine, at the level of the DBI action, the higher-dimensional analog of the expression (4.6) derived in the previous section for the case of the D1-brane and observe, for the case of higher dimensional branes, that the transition amplitude is non-vanishing, for example, when both initial and final state of the D-brane are chosen to be vacuum. Does it mean that the momentum conservation is compromised?

The answer to the above question is most certainly no. However, the resolution of the paradox is instructive and somewhat subtle. One just needs to be conscious about what states of the D-branes appear in the CFT amplitudes. And this is, in turn, related to the kinematic properties of the free massless scalar field theories in various dimensions.

Let us start with the intuitively straightforward case of the D0-brane. In this case, the dynamics of the topological defect is described by a free massless (0+1)-dimensional scalar field $X^i(t)$, which is simply the position of the D-brane. The momentum conservation is due to the translational symmetry of $X^i(t)$, and the scattering states of the D0-brane are momentum eigenstates. The scattering amplitudes contain the momentum conservation δ-function, and, as it was described in the previous chapter, the worldsheet CFT "attempts" to expand this δ-function in a Taylor series thereby producing the infrared divergences.

Imagine now we are to examine closed string scattering off a Dp-brane with $p \geq 2$. Again, the deformations of its worldvolume are described by a free massless (p+1)-dimensional scalar field $X^i(\xi, t)$, and the (Dirichlet) momentum conservation is due to the field translation symmetry $X^i \to X^i + a^i$. However, this symmetry is *spontaneously broken*! The ground state of a Dp-brane with $p \geq 2$ is localized in the Dirichlet directions and does not have a definite value of the Dirichlet momentum. This explains why, even though momentum is conserved irrespectively of which D-brane one works with, for the higher dimensional D-branes, the amplitude for the D-brane to remain in its vacuum state (as the closed strings scatter off it) is non-vanishing, the general scattering amplitude does not contain the momentum conservation δ-function, and the infrared divergences in the worldsheet CFT are absent.

Why doesn't the case of the D1-brane fall into the same category as the higher-dimensional branes? The deformations of a D1-brane worldsheet are once again described by a free massless (1+1)-dimensional scalar field $X^i(\xi, t)$, and the (Dirichlet) momentum conservation is due to the field translation symmetry $X^i \to X^i + a^i$. However this symmetry is *not* spontaneously broken, since, according to the well-known result by Coleman [25], spontaneous breaking of continuous symmetries is *impossible* in two dimensions. The ground state of a D1-brane is *not* localized in the Dirichlet directions, and it *does* have a definite value of the Dirichlet momentum. Hence the discontinuous momentum dependence of the scattering amplitudes and the associated infrared divergences in the worldsheet CFT. Mathematically, these statements about scattering of closed strings off a D1-brane are intimately related to the much-discussed kinematic peculiarities of free massless scalar fields in two dimensions [26, 27].

Summary

Contrarily to what one might have naïvely expected, the behavior manifested by the extended D-branes in response to the impact by the incident closed string is vastly different for the D1-branes and the higher-dimensional branes. This difference is purely kinematic in nature and, therefore, it pertains to the case of non-string-theory topological defects as well.

In response to impact by the closed strings, the D1-branes develop specific wave patterns propagating away from the point of incidence. These wave patterns do not decay in the asymptotic future and cause infrared divergences, if one tries to compute scattering amplitudes in the background of a static D1-brane. Such situation is directly analogous to the (considerably) more familiar case of D0-brane recoil. We have therefore termed it the *local recoil*, and described how it should be accommodated within the formalism of string theory.

Chapter 5

String loop mixing

It is a much-exercised conjecture that the type IIB superstring theory admits an $SL(2,Z)$ self-duality group, which, in particular, acts on the complexified dilaton as

$$\tau \to \frac{a\tau + b}{c\tau + d}$$

with some relatively prime integers a, b, c and d satisfying $ad - bc = 1$. This transformation law implies, in particular, that the weak coupling regime can be mapped to a finite or even strong coupling.

Another important aspect of the conjectured duality action is the transformation it induces upon the various branes. Type IIB theory is known to possess a rich spectrum of 1-dimensional geometrical entities known as (p,q)-branes, which can be thought of as bound states of p fundamental strings and q D1-branes with relatively prime p and q. It is conjectured that the $SL(2,Z)$ self-duality should permute such (p,q)-branes according to

$$\begin{pmatrix} p \\ q \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}$$

This brings us to a subtlety (perhaps even with a touch of paradox) which will be the primary concern of this chapter. Indeed, it is well known that, in the perturbative string expansion, the graviton appears as a particular vibrational state of a loop made of the fundamental string (F-string). It should then be natural to expect that, at strong coupling, the graviton should be a vibrational state of a loop made of the D1-brane (D-string). How would the transition between the two pictures work?

Since we're still lacking a non-perturbative extension of string theory, the answer to this question can hardly be obvious. Nor does it need to be unique. Nevertheless, one could hope that the transition occurs smoothly, without an abrupt switching, say, between the F-string graviton and the D-string graviton. If that is the case, the physical graviton should be a mixture of the various (p,q)-loop states, which, at small string couplings, will be almost entirely an F-string loop; at strong string couplings, will be almost entirely a D-string loop; and, at finite rational string couplings p/q, will be a loop made of the (p,q)-string[1].

In the rest of this chapter, I will present some preliminary considerations intended to corroborate the picture of dynamical mixing between string loops. Once again, the setting will be the bosonic string theory rather then the superstring. In this case, there is no $SL(2,Z)$ duality. Nevertheless, the F- and D-string are still present, and one could pose the question of their dynamical mixing. The issue is then whether the relevant mixing patterns can be identified in a way that allows to make contact with how the string loop mixing should work

[1] I owe this qualitative picture to an incidental conversation with Steven Gubser. It does not appear to have been extensively discussed in the literature, and should perhaps be seen as a concept of folklore extraction.

in the context of the type IIB superstring.

5.1 The fully dynamical worldsheets

In describing the vibrational states of a stretched D-string in the previous chapter, we have successfully employed the open fundamental strings. It is hard to see how such program would work for a D-string loop. One should therefore be compelled to resort to a description in which the spatial configuration of the D-string loop appears explicitly. This is a direct generalization of the Hirano-Kazama formalism of chapters 1 and 2 to the case of D1-branes.

Of a central importance to our present investigation is the D-string-F-string mixing amplitude, which describes a transition between a state containing one D-string loop and a state containing one F-string loop. The formal expression for such an amplitude can be constructed along the lines of (1.1), with the dynamical degrees of freedom of the D1-brane replacing those of the D0-brane:

$$\int \mathcal{D}\Xi\, \mathcal{D}X\, \mathcal{D}s_1\, \mathcal{D}s_2\; \delta\left[X^\mu(\theta) - \Xi^\mu(s_1(\theta), s_2(\theta))\right] \exp\left[-S_D(\Xi) - S_F(X)\right] \tag{5.1}$$

In this formula, the embedding of the F-string worldsheet into space-time is described by the function $X^\mu(\sigma)$, and the embedding of the D-string worldsheet into space-time is described by the function $\Xi^\mu(s)$. The δ-function enforces the Dirichlet boundary conditions on the F-string: each boundary point θ is attached to the D-string worldsheet, and the coordinates of the junction (on the D-string worldsheet) are given by $(s_1(\theta), s_2(\theta))$. The topology of the F-string-D-string transition diagram is presented on Fig. 5.1a. Note that suitable boundary conditions should be specified on the open ends of the both cylindrical worldsheets.

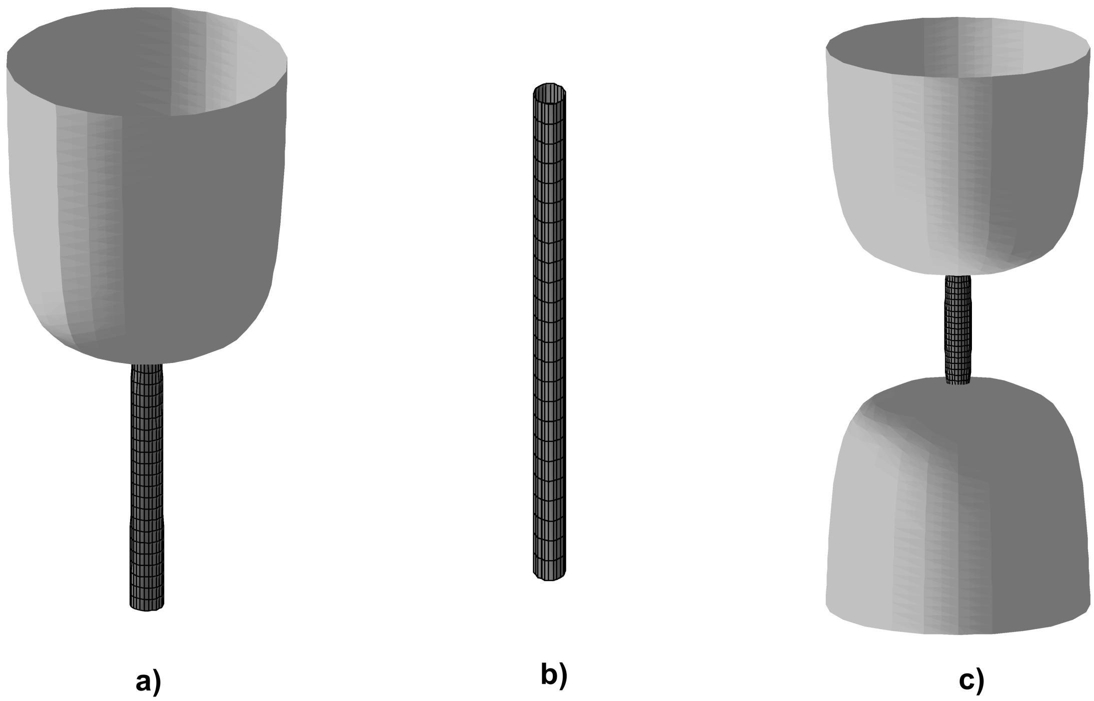

Figure 5.1: Examples of worldsheet diagrams relevant to D-string-F-string loop mixing: a) D-string-F-string loop transition; b) D-string propagation; c) mass correction for the F-string loop.

Once again the action S_D should be constructed from the string amplitudes themselves, as it was done in (1.3)

$$S_D[\Xi] = \sum_{\text{connected}} \frac{(g_{st})^\chi}{-V_\chi} \int \mathcal{D}X\, \mathcal{D}s_1\, \mathcal{D}s_2\; \delta\left[X^\mu(\theta) - \Xi^\mu(s_1(\theta), s_2(\theta))\right]\, \exp\left[-S_F(X)\right] \quad (5.2)$$

This choice of action will, in particular, provide for the non-perturbative suppression of the mixing amplitude. (Note the close analogy with how, in [2], this choice of action played a central role in making the D-instanton contribution into string scattering non-perturbatively small.)

To understand what physical effects are induced by the mixing amplitude (5.1), it is instructive to recall momentarily how the dynamical mixing of particles occurs in the familiar setting of quantum field theory. One can start with a Lagrangian containing two species of particles with masses m and M, and then turn on an interaction which induces mixing between these two species. In this case, the relevant quantity is the one-particle irreducible two-point function $\mu(k^2)$ corresponding to the transition between the two species of particles occuring at the value of the 4-momentum k. The exact propagator will be given by a familiar sum of the "geometrical progression" terms of the form

$$\cdots \frac{1}{k^2 - m^2 - i0}\,\mu(k^2)\,\frac{1}{k^2 - M^2 - i0}\,\mu(k^2)\,\frac{1}{k^2 - m^2 - i0}\cdots$$

The series can, as usually, be resummed, and, if the mm-, mM- and MM-propagators are arranged in a matrix:

$$\begin{pmatrix} D_{mm} & D_{mM} \\ D_{mM} & D_{MM} \end{pmatrix}$$

The resummed expression can be conveniently written as

$$\frac{1}{k^2 - \mathcal{M}^2 - i0}$$

where

$$\mathcal{M}^2 = \begin{pmatrix} m^2 & \mu(k^2) \\ \mu(k^2) & M^2 \end{pmatrix}$$

Since the masses of the physical states are determined by the positions of the poles of the propagator, the spectral analysis for the case of mixing between two particle species is reduced to diagonalizing the matrix $\mathcal{M}^2$, which would yield the eigenvalues $\lambda_{1,2}(k^2)$. With these eigenvalues, the masses of the physical particle states can be determined by solving the two equations

$$m_{1,2}^2 = \lambda_{1,2}(m_{1,2}^2)$$

If we now turn to the case of F-string-D-string mixing bearing in mind the above field-theoretical consideration, we must realize that, besides the mixing matrix element (5.1), we shall need to know the spectra of the D-string loop, which can be extracted from the cylindrical diagram Fig. 5.1b. Furthermore, the interactions with the D-string loop will introduce corrections to the graviton two-point function, Fig. 5.1c. These diagrams should be used to extract the entries of the F-string-D-string mixing matrix:

$$\begin{pmatrix} m^2 & \mu \\ \mu & M^2 \end{pmatrix}$$

where m^2 and M^2 are the masses of the F-string and D-string loop states (in neglect of the mixing), and μ is the mixing.

The condition of graviton uniqueness can now be formulated in terms of the mixing matrix. After the diagonalization procedure, one of the two eigenstates must be massless, and the other one should have a non-perturbatively large mass. The first condition requires that

$$m^2 M^2 = \mu^2 \tag{5.3}$$

The second condition requires that M should be non-perturbatively large. Furthermore, the massless eigenstate (the physical graviton) should be predominantly made of an F-string loop, with a non-perturbatively small admixture of the D-string loop (as long as we stay in the weak coupling regime). This requres that μ should be non-perturbatively suppressed.

5.2 The mixing matrix

Unfortunately, no rigorous derivations can be given at this point on the basis of the path integral (5.1). One complication (in comparison to the D0-brane case analyzed in chapter 1) is that the diffeomorphism-invariant measure for the D1-brane worldsheets is harder to construct explicitly than the diffeomorphism-invariant measure for the D0-brane worldlines. One can nevertheless contemplate at the qualitative level how the correct mixing matrix structure described in the previous section could emerge from the path integral representation for the mixing amplitudes.

Let us start with the non-perturbatively large value of the D-loop mass M. At the very first sight, it may appear rather counterintuitive given the definition of the action (5.2). Indeed, for a nearly flat worldsheet, if we first fix the zero modes of the functions s_1 and s_2, the integration over the non-zero modes will produce a constant (independent on the values of the zero-modes). The integration over the zero modes will then give the area of the worldsheet A. Thus, for nearly flat worldsheets, one should expect

$$S_D = \tau A \tag{5.4}$$

where $\tau \sim 1/g_{st}$. This is the standard area action, and if it were the whole story, one would end up with *massless* excitations, as one does for the fundamental string. Note, however, that the tension τ is non-perturbatively large. Hence, the typical spatial extent of the D-string loop will be very small compared to the string scale, and the D-string worldsheet will be highly curved. Therefore, the approximations made in deriving (5.4) would be invalid.

The non-perturbatively large tension of the D-string suggests that the dynamics of the

D-string loop will be dominated by tube-like worldsheets with the diameter of the tube much smaller then the string scale. For a "probe" string appearing in the path integral (5.2), such thin tube D-string worldsheets will be indistinguishable from a D0-brane worldline (which would be produced if the diameter of the tube were actually reduced to 0). One should therefore expect that a sensible estimate for the action of the thin-tube worldsheet will be

$$S_D = M_0 T + S_A$$

where M_0 is the D0-brane mass, T is the length of the tube, and S_A vanishes as the area of the worldsheet goes to 0. Note that all the dependence on the vibrational state of the D-string loop (and hence all the information on the vibrational excitation spectrum) is contained in S_A. However, regardless of the exact vibrational state of the D-string loop, its mass will receive additional contribution M_0, on account of the difference between the action (5.2) and the naïve area action (5.4). Since M_0 is non-perturbatively large, this explains qualitatively why the mass of a D-string loop state M should be large.

We shall now turn to the question of mixing. As it has been remarked above, for the F-strings (whose typical worldsheets are of the size $\sqrt{\alpha'}$), the thin-tube worldsheet of the D-string appears as though it were a 1-dimensional worldline. The estimate $S_D = M_0 T$ of the action (which, for a worldline, is given by the expression (1.3)) comes from looking at the integral over the constant mode of the function $t(\theta)$. One must realize, however, that, for worldsheets located near the endpoints of the worldline, the integral over the *non-constant* modes of $t(\theta)$ will be different from the worldsheets attached to the interior of the worldline.

For that reason, the endpoints will give a contribution to the action:

$$S_D = M_0 T + S_{endpoint}$$

where $S_{endpoint} \sim 1/g_{st}$ does not depend on T. Hence, each endpoint will contribute a factor of $\exp[-S_{endpoint}] = \exp[-O(1/g_{st})]$ to the amplitude. (Note the close similarity of this supression with the non-perturbative suppression of the hard D-instanton amplitudes suggested in [2].)

The above consideration explains why the mixing μ deduced from the diagram in Fig. 5.1a is non-perturbatively suppressed. It also suggests that the non-perturbative suppression (by the factors of $\exp[-S_{endpoint}]$) is co-ordinated between the diagram on Fig. 5.1a and the one on Fig. 5.1c precisely in the way suggested by (5.3). Indeed, the thin-tube D-string worldsheet has one "endpoint" in Fig. 5.1a and two "endpoints" in Fig. 5.1c.

One can therefore conclude that the general kind of coupling constant dependences that arise in the worldsheet representation for the D-string-F-string mixing problem appear to be compatible with the educated physical expectations. Of course, one would still need a more thorough analysis of how the relation (5.3) is fulfilled. More importantly, the various statements about the asymptotic behavior of the action S_D made above would certainly benefit from strengthening the rigor of the derivations. This would require a more explicit understanding of the properties of the relevant geometrical path integrals.

Concluding remarks

The exposition of the previous five chapters has traversed a wide range of topics, and it is only appropriate to present here a concise summary of the central ideas and results described in this thesis:

1. Attempts to describe the dynamics of low-dimensional D-branes encounter subtleties due to the infrared divergences which plague the conventional worldsheet approach.

2. For D0-branes, these infrared divergences signify the failure of the Dirichlet CFT to describe the D0-brane translational motion.

3. As a powerful remedy against this complication, one can introduce an explicit dynamical degree of freedom corresponding to the translational motion of the D0-brane. The resulting system is described by a geometrical path integral involving dynamical worldlines and dynamical worldsheets coupled via the Dirichlet boundary conditions.

4. In application to the process of D0-brane annihilation, such formalism permits to identify the coupling constant dependence of the amplitudes. In particular, one can exhibit the saddle point structure of the relevant path integration and establish the non-perturbative suppression of the form $\exp[-O(1/g_{st}^2)]$, which is required by unitarity.

5. In application to the process of D0-brane recoil, this formalism allows to clarify how the Dirichlet CFT should be deformed to accommodate the translational motion of the D0-brane to the lowest order of g_{st}. The appropriate deformation deduced in this

thesis is different from the ones previously described in the literature. This is all for the better, since the deformations previously described in the literature lead to pathological features of the physical amplitudes, as advocated in chapter 3.

6. For the case of D1-branes, the infrared problems of the Dirichlet CFT description signify that the response to the D1-brane to the impact by closed strings is of such nature that the background of a static D1-brane is inadequate when it is bombarded by the closed strings. This kind of dynamical behavior is closely related to the D0-brane recoil, and we called it the local recoil. The nature of this phenomenon is purely kinematical and, therefore, it should be manifest for other 1-dimensional topological defects, such as the field theory vortex lines.

7. Finally, some preliminary considerations have been given for the question of mixing between the D1-brane and F-string loops, and how such mixing can be relevant to the issue of graviton uniqueness in the context of the $SL(2, Z)$ duality of the type IIB string.

Appendix A

Identities for the Green functions of Laplace equation on a unit disk

In evaluating path integrals over the open string worldsheets in flat spacetime, it proves useful to employ certain identities relating the Neumann and Dirichlet Green's functions of the Laplace equation describing the dynamics of the string. The general reason for such identities to exist is that the difference between Neumann and Dirichlet Green's functions satisfies the equation $\triangle_\sigma(N(\sigma,\sigma') - D(\sigma,\sigma')) = \text{const}$ (where the constant accounts for the constant mode part) and therefore can be expressed through the boundary values of the Neumann Green function and the normal derivative of the Dirichlet Green function. We thus arrive at the representation

$$N(\sigma,\sigma') - D(\sigma,\sigma') = \int d\tilde{\theta}\, \partial_{\tilde{n}} D(\sigma,\tilde{\theta}) N(\tilde{\theta},\sigma') - \frac{1}{\pi}\int D(\sigma,\tilde{\sigma})\, d^2\tilde{\sigma} \tag{A.1}$$

where θ parametrizes the boundary (for definiteness, we work with a unit disk in these notes: $\sigma = (r,\theta)$, $0 \le r \le 1$, $0 \le \theta < 2\pi$; $\triangle_\sigma N(\sigma,\sigma') = \delta(\sigma-\sigma') - 1/\pi$, $\triangle_\sigma D(\sigma,\sigma') = \delta(\sigma-\sigma')$).

In order to resolve this equation with respect to D, we have to prove that $\partial_n N(\theta,\theta') = -\delta(\theta-\theta')$. This relation may appear to contradict the Neumann boundary conditions for $N(\sigma,\sigma')$, but, in fact, it doesn't, because of the subtleties of how the on-boundary limits are taken. Indeed,

$$N(\sigma,\theta') \equiv \lim_{r'\to 1} N(\sigma,\sigma') \qquad \partial_n N(\theta,\theta') \equiv \lim_{r\to 1} \partial_r N(\sigma,\theta') = \lim_{\substack{r,r'\to 1\\ r<r'}} \partial_r N(\sigma,\sigma')$$

Now, by virtue of the Neumann boundary conditions,

$$\lim_{\substack{r,r'\to 1\\ r>r'}} \partial_r N(\sigma,\sigma') = 0$$

(note the different order in which the on-boundary limits are taken!).

Let's integrate the Laplacian of the Neumann Green function $N(\sigma,\sigma')$ with respect to σ over the region $M = (r_0 < r < 1,\, \theta_1 < \theta < \theta_2)$, $r_0 < r'$:

$$\int_{\partial M} (d\vec{n}\cdot\vec{\nabla}_\sigma) N(\sigma,\sigma') = \int_M \triangle_\sigma N(\sigma,\sigma') d^2\sigma = \int_M \left(\delta(\sigma-\sigma') - \frac{1}{\pi}\right) d^2\sigma = \int_{\theta_1}^{\theta_2} \delta(\theta-\theta') d\theta - \frac{\mathrm{Area}(M)}{\pi}$$

Since N satisfies Neumann boundary conditions and is non-singular away from $\sigma = \sigma'$, in the limit $r_0, r' \to 1, r_0 < r'$, this reduces to

$$-\int_{\theta_1}^{\theta_2} \lim_{\substack{r,r'\to 1\\ r<r'}} \partial_r N(\sigma,\sigma')\, d\theta = \int_{\theta_1}^{\theta_2} \delta(\theta-\theta')\, d\theta$$

Because θ_1 and θ_2 are arbitrary, we are forced to conclude that

$$\partial_n N(\theta,\theta') = -\delta(\theta-\theta') \tag{A.2}$$

It is now a matter of rather straightforward manipulations to resolve (A.1) with respect to D and obtain further useful identities. We first take σ' to the boundary, differentiate with respect to r and take σ to the boundary. This yields

$$-\delta(\theta-\theta') = \int \partial_n \partial_{\tilde{n}} D(\theta,\tilde{\theta}) N(\tilde{\theta},\theta')\, d\tilde{\theta} - \frac{1}{\pi}\int \partial_n D(\theta,\tilde{\sigma})\, d^2\tilde{\sigma}$$

The second term should not depend on θ by the symmetries of the disc and can therefore be evaluated by first integrating over θ and then dividing by 2π. This gives $1/2\pi$, i.e.,

$$\int \partial_n \partial_{\tilde{n}} D(\theta,\tilde{\theta}) N(\tilde{\theta},\theta')\, d\tilde{\theta} = -\left(\delta(\theta-\theta') - \frac{1}{2\pi}\right)$$

It can be proved via exactly the same method as we used for evaluating $\partial_n N(\theta,\theta')$ that $\partial_n \partial_{\tilde{n}} D(\theta,\tilde{\theta})$ is orthogonal to the constant modes (in θ and $\tilde{\theta}$). It is also known [17] that $N(\theta,\theta')$ on a disk is orthogonal to constant modes and invertible in the sense of integral convolution. We therefore have to conclude that

$$\partial_n \partial_{n'} D(\theta,\theta') = -N^{-1}(\theta,\theta') \qquad \int d\tilde{\theta} N(\theta,\tilde{\theta}) N^{-1}(\tilde{\theta},\theta') \equiv \delta(\theta-\theta') - \frac{1}{2\pi} \tag{A.3}$$

We now differentiate (A.1) with respect to σ and take it to the boundary, which yields

$$\partial_n D(\theta,\sigma') = \int d\tilde{\theta} N^{-1}(\theta,\tilde{\theta}) N(\tilde{\theta},\sigma') + \frac{1}{2\pi} \tag{A.4}$$

Substituting this back into (A.1) gives the expression for D:

$$D(\sigma,\sigma') = N(\sigma,\sigma') - \int d\tilde{\theta} d\tilde{\theta}' N(\sigma,\tilde{\theta}) N^{-1}(\tilde{\theta},\tilde{\theta}') N(\tilde{\theta}',\sigma') - \frac{1}{2\pi}\int N(\tilde{\theta},\sigma')\, d\tilde{\theta} + \frac{1}{\pi}\int D(\sigma,\tilde{\sigma})\, d^2\tilde{\sigma}$$

$\int D(\sigma,\tilde{\sigma})\, d^2\tilde{\sigma}$ equals in fact $-\int N(\sigma,\tilde{\theta})\, d\tilde{\theta}/2$, since the difference of the two is zero at the boundary and satisfies Laplace equation in the interior (it should also be evident from analyzing the symmetries of the above equation with respect to permuting σ and σ'). With this in mind, we arrive finally at a representation for the Dirichlet Green's function completely in terms of the Neumann Green's function:

$$D(\sigma,\sigma') = N(\sigma,\sigma') - \int d\tilde{\theta} d\tilde{\theta}' N(\sigma,\tilde{\theta}) N^{-1}(\tilde{\theta},\tilde{\theta}') N(\tilde{\theta}',\sigma') - \frac{1}{2\pi} \int \left(N(\sigma,\tilde{\theta}) + N(\tilde{\theta},\sigma') \right) d\tilde{\theta} \tag{A.5}$$

This relation and the ones derived above occur ubiquitously in evaluating path integrals over the open string worldsheets.

Appendix B

Path integrals over the interior of the open string worldsheet

Path integrals over the string worldsheets can certainly be handled in a variety of ways. When the interactions/non-linearities are restricted to the worldsheet boundary (systems with accelerating D-branes or open string backgrounds, e.g.), it is most efficient to perform first the integral over the interior of the worldsheet (which can be done exactly). Such an integral can be evaluated once and for all, regardless of what kind of dynamics takes place at the boundary. We will thus be interested in

$$I[\xi, J] = \int \mathcal{D}X \delta(X|_{boundary} - \xi) \exp\left[-\int d^2\sigma \left(\nabla X \nabla X + iJX\right)\right] \tag{B.1}$$

i.e. we'll be considering the string action with sources, integrated over all the worldsheets with a given boundary. We parametrize the worldsheet by mapping it to a unit disk. The boundary is then a unit circle with the coordinate $0 \leq \theta < 2\pi$.

We first revisit the subtleties with how the path integral (B.1) is defined. The fact that the action does not depend on derivatives higher than the first suggests that we have to integrate over all the continuous functions regardless of whether or not their first derivative has breaks ("fault lines"). Indeed, the most physical way to define the path integral, explicitly preserving locality and implementing the finite resolution of a realistic experiment, is provided by latticization. Then, the path integral is just a multiple integral over the values of X specified at each lattice cite independently. This allows to approximate *any* function $X(\sigma)$. However, as the lattice spacing is taken to 0, the action will diverge unless $X(\sigma)$ is continuous. Therefore, only configurations with continuous $X(\sigma)$ (regardlessly of whether or not its first derivative has breaks) will contribute to the path integral.

Whereas latticization provides a rather operational definition of the path integral, it is not particularly convenient for mathematical manipulations. We are therefore forced to look for a different, more handy representation. We may try to expand $X(\sigma)$ in terms of the eigenfunctions of Laplace operator on the unit disk satisfying Neumann boundary conditions:

$$X(\sigma) = \sum_I x_I X_I(\sigma) \qquad \triangle X_I = -\omega_I^2 X_I \qquad \partial_n X_I|_{boundary} = 0 \qquad \int X_I X_J d^2\sigma = \delta_{IJ} \tag{B.2}$$

Analogously to the standard Fourier expansion on a unit interval, a wide class of functions will be amenable to this kind of decomposition. For continuous $X(\sigma)$, the convergence will be uniform, which will allow for termwise differentiation. Furthermore, if the first derivative of $X(\sigma)$ does not have singularities stronger than breaks, we'll be able to rewrite the action as

$$\int \nabla X \nabla X d^2\sigma = \sum_{IJ} x_I x_J \int \nabla X_I \nabla X_J d^2\sigma = \sum_I \omega_I^2 x_I^2$$

We finally notice that *all* the continuous $X(\sigma)$ will be taken an account of if we integrate over x_I's, and the above series for the action will *only* converge to a finite value for continuous configuration, integration over x_I's will *exactly* amount to doing the path integral over configurations suggested by the lattice considerations. We therefore rewrite the integral of interest as

$$I[\xi, J] = \int dx_I \delta(X|_{boundary} - \xi) \exp\left[-\sum_I \omega_I^2 x_I^2 - i \int JX d^2\sigma\right] \tag{B.3}$$

Two side remarks may be in order here. *First*, the choice of Neumann eigenfunctions is singled out be the fact that any function appearing in the path integral can be expanded in that basis with a uniform convergence of the series. If we tried to use, for example, the Dirichlet eigenfunctions, for any $X(\sigma)$ not satisfying the Dirichlet boundary conditions, convergence of the resulting series would not be uniform near the boundary, and the action for the series representation would diverge instead of converging to the action of the original configuration $X(\sigma)$. *Second*, the choice of Neumann eigenfunctions does not imply *at all* that we impose Neumann boundary conditions in the path integral. In fact, since we have allowed for breaks in the first derivative of $X(\sigma)$, it is impossible to impose Neumann boundary conditions in the path integral: a break approaching the boundary will change $\partial_n X|_{boundary} = 0$ to any other value we like. The Neumann boundary conditions are *not imposed*, but *emerge* as an operator identity in this approach—as we would expect—since they could be obtained from the variational principle in the classical theory.

The path integral (B.3) has been evaluated in [17], although in a less general setting. We first substitute the integral representation for the on-boundary δ-function $\delta(\mu) =$

$\int \mathcal{D}\nu \exp[-i\int \mu\nu d\theta]$. After performing the Gaussian integration over x_I, we get:

$$I[\xi, J] = \int \mathcal{D}\nu\, \delta\left(\int J(\sigma)d^2\sigma + \int \nu(\theta)d\theta\right) \exp\left[\frac{1}{4}\int J(\sigma)N(\sigma,\sigma')J(\sigma')d^2\sigma d^2\sigma'\right.$$
$$\left. +\frac{1}{2}\int \nu(\theta)N(\theta,\sigma')J(\sigma')d\theta d^2\sigma' + \frac{1}{4}\int \nu(\theta)N(\theta,\theta')\nu(\theta')d\theta d\theta' + i\int \nu(\theta)\xi(\theta)d\theta\right] \tag{B.4}$$

where N is the Neumann Green function:

$$\triangle_\sigma N(\sigma,\sigma') = \delta(\sigma-\sigma') - \frac{1}{\pi} \qquad N(\sigma,\sigma') = -\sum_{I\neq 0}\frac{X_I(\sigma)X_I(\sigma')}{\omega_I^2}$$

We first integrate over the constant mode of ν taking into account that $N(\theta,\theta')$ does not have a constant mode on the boundary of the unit disk:

$$I[\xi, J] = \int \mathcal{D}\tilde{\nu} \exp\left[\frac{1}{4}\int J(\sigma)N(\sigma,\sigma')J(\sigma')d^2\sigma d^2\sigma' + \frac{1}{2}\int \tilde{\nu}(\theta)N(\theta,\sigma')J(\sigma')d\theta d^2\sigma'\right.$$
$$+\frac{1}{4}\int \tilde{\nu}(\theta)N(\theta,\theta')\tilde{\nu}(\theta')d\theta d\theta' + i\int \tilde{\nu}(\theta)\xi(\theta)d\theta \tag{B.5}$$
$$\left. -\frac{1}{4\pi}\int J(\sigma)d^2\sigma\int d\theta\left(2i\xi(\theta) + \int N(\theta,\sigma')J(\sigma')d^2\sigma'\right)\right]$$

where $\tilde{\nu}$ does not contain the constant mode. Upon the final integration,

$$I[\xi, J] = \exp\left[\frac{1}{4}\int J(\sigma)N(\sigma,\sigma')J(\sigma')d^2\sigma d^2\sigma'\right.$$
$$-\frac{1}{4}\int\left(2i\xi(\theta) + \int N(\theta,\sigma)J(\sigma)d^2\sigma\right)N^{-1}(\theta,\theta')\left(2i\xi(\theta') + \int N(\theta',\sigma')J(\sigma')d^2\sigma'\right)d\theta d\theta' \tag{B.6}$$
$$\left. -\frac{1}{4\pi}\int J(\sigma)d^2\sigma\int d\theta\left(2i\xi(\theta) + \int N(\theta,\sigma')J(s')d^2\sigma'\right)\right]$$

Or, by virtue of the identities from Appendix A:

$$I[\xi, J] = \exp\left[\frac{1}{4}\int J(\sigma)D(\sigma,\sigma')J(\sigma')d^2\sigma d^2\sigma' - i\int \xi(\theta)\partial_n D(\theta,\sigma')J(\sigma')d\theta d^2\sigma'\right.$$
$$\left. + \int \xi(\theta)N^{-1}(\theta,\theta')\xi(\theta')d\theta d\theta'\right] \tag{B.7}$$

We can readily check that these expressions allow to reproduce the conventional results for the Neumann and Dirichlet partition functions:

$$Z_{Neumann}[J] \equiv \int \mathcal{D}\xi \, I[\xi, J] \qquad\qquad Z_{Dirichlet}[J] \equiv I[0, J]$$

The appearance of the Dirichlet function in the final result is not, in fact, surprising, since an alternative way to do the computation would be to shift the integration variable in (B.1) by a solution to the Laplace equation, so that the Dirichlet boundary condition $\delta(X|_{boundary})$ is enforced for the new $X(\sigma)$. Then we would be justified to expand in the set of Dirichlet eigenfunctions. The Gaussian integration will give the first term in the exponent of (B.7), and the last two terms would be surface contributions arising from the shift of the integration variable.

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