NON-CONSERVATION OF BARYON NUMBER AT HIGH ENERGY IN THE STANDARD MODEL

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ABSTRACT

We calculate the coupling constant and energy dependence of the scattering amplitudes for baryon- and lepton-number violating processes in the context of the standard model, in the semiclassical approximation. It is found that, to leading order in this expansion, the spin-averaged total cross sections for these processes grow as a power of the CM-energy and thus violate the bound imposed by unitarity. This result has a twofold implication: first, perturbation theory in the instanton sector of the electroweak theory must break down at high energies and, second, it strongly suggests that baryon and lepton number non-conservation might be observed experimentally at energies accesible in the near future.

TABLE OF CONTENTS

Chapter I. Introduction

It has been known for some time that baryon and lepton numbers are not exactly conserved in the standard model. Due to the chiral nature of the fermionic representations the baryonic and leptonic currents suffer from $[SU(2)]^2$ and $[U(1)]^2$ anomalies [1].

However, any nonvanishing scattering amplitude that violates baryon and lepton number conservation must involve (in Euclidean space) topologically nontrivial large gauge field configurations with Euclidean actions greater or equal than $8\pi^2/g^2$, where g is the $SU(2)$ coupling constant. These instanton configurations [2] describe tunneling events of the gauge field under the potential barrier that separates topologically distinct vacua. Thus the rate for any process mediated by tunneling of the gauge field is (for g small) suppressed by the exponentially small factor $\exp(-16\pi^2/g^2)$.

Taking this potential barrier picture seriously, several authors [3–5] suggested that these anomalous processes can be unsuppressed (enhanced considerably) if high temperatures or high energies are involved: instead of tunneling through the barrier the gauge (and Higgs) field can pass over the barrier.

The discovery [5,6] of a static and unstable solution of the classical equations of motion for the $SU(2)$ Yang-Mills-Higgs field theory which interpolates between topologically distinct vacua and therefore corresponds to the configuration sitting at the top of the barrier (the so-called sphaleron), made possible a semi-quantitative estimate of the magnitude of the rate of transitions over the barrier at finite temperatures [3,7]. These authors further computed the corresponding dissipation rate of the baryonic asymmetry of the universe and concluded that the effect is big enough to wipe out an initial baryonic asymmetry generated at an earlier epoch (such as GUT baryogenesis). The connection between transitions over the barrier and baryon and lepton production is rather *ad-hoc* in their calculations, however. (In Ref. [8] it is claimed that although fluctuations in $F\tilde{F}$ can be unsuppressed at high temperatures, the net rate of baryon and lepton number generation is negligible due to the absence of occupation number factors in the rates.)

In an attempt to reconcile their sphaleron-based calculation of the baryonic asymmetry dissipation with a more conventional one based on individual anomalous scattering events, Arnold and McLerran suggest in Ref. [9] that although the scattering amplitude for a purely fermionic process like $q + q \rightarrow (3n_f - 2)\overline{q} + 3n_f\overline{l}$ (n_f is the number of families) is exponentially suppressed, an anomalous scattering event containing many gauge bosons and/or Higgs particles in the final state might not be suppressed. The motivation to consider the latter kind of processes comes from the suggestion that a classical object like the sphaleron with energy $E_{sp} \sim m_W/\alpha_W$ and size $\sim m_W^{-1}$ has typical Fourier components $k \sim m_W$ and thus is likely to decay into $\sim 1/\alpha_W$ W's with typical momenta m_W. They claim that a Green function of the type $\langle (qqql)^{n_f} A^n \rangle$ although proportional to the small factor $\exp(-8\pi^2/g^2)$ is also proportional to $(1/g)^n$, the $1/g$'s coming from the instanton configuration. So naively, for large n the latter factor can compensate for the smallness of the former, avoiding the exponential suppression. This observation cannot be taken too seriously however, since perturbation theory around the instanton breaks down for n too large.

There is a peculiar point in the previous reasoning however: for a given small value of the coupling constant g it seems that the scattering amplitude should be largest for largest n. This is not what one would expect for the emission of weakly coupled gauge bosons. Furthermore, it would be hard to understand how the unitarity equation for a process like $3n_f q + n_f l \rightarrow 3n_f\overline{q} + n_f\overline{l}$ that can produce real intermediate states containing W-bosons could work, for in the equation

$$A_{fi} - A_{if}^* = \sum_m i(2\pi)^4 \delta^{(4)}(p_i - p_m) A_{fm} A_{in}^*, \tag{1.1}$$

where $i = 3n_f q + n_f l$ and $f = 3n_f\overline{q} + n_f\overline{l}$, the LHS carries no factors of $1/g$ (apart from the usual $(g^{-8})^2$ coming from the collective coordinate Jacobian) whereas a term in the RHS containing an intermediate state with n gauge bosons would carry a factor of $(1/g)^{2n}$.

The purpose of this thesis is to study the coupling constant and energy dependence of the baryon and lepton number violating amplitudes involving in general also some number of gauge and Higgs bosons, in leading order in the semiclassical expansion ($g^2 \to 0$, keeping m_H and m_W fixed). It is essential for the feasibility of the calculation to introduce a Higgs field that spontaneously breaks the $SU(2)$ gauge symmetry completely, since otherwise the S-matrix is not defined due to the severe infrared divergences of the theory (reflected in an instanton calculation in the divergent integration over the instanton size ρ). We also investigate the effects of small Yukawa couplings (i.e., fermion masses) on the anomalous scattering amplitudes.

The main results of our leading order calculation are the following: *i*) for fixed gauge and Higgs boson masses, the amplitude for a scattering process involving n gauge bosons and m Higgs bosons is proportional to $g^{n+m}e^{-8\pi^2/g^2}$. Thus, the emission of an extra gauge or Higgs boson is *suppressed* by an extra power of g. *ii*) surprisingly enough, amplitudes grow with external momenta as $p^{N_f/2}q^n$, where N_f, n are respectively the number of fermions and gauge bosons participating in the scattering process, and p, q represent respectively typical fermion and gauge boson momenta. We find thus that the unitarity bound is not respected by the leading term in the semiclassical expansion.

Work substantially overlapping with the work presented in this thesis has been published in a recent paper [16]. The author of Ref. [16] has performed a calculation very similar to ours, reaching basically the same qualitative conclusions as we do. However, he neglects to include the effect of the global isospin orientation of the constrained instanton (see chapter 3), which *does* induce a nontrivial dependence of the anomalous scattering amplitudes on momentum transfers. This nontrivial dependence of the amplitudes on external momenta makes the corresponding cross sections grow faster with energy, which would make the anomalous processes experimentally detectable at fairly lower energies than estimated in Ref. [16] (provided, of course, higher order corrections, which should be responsible for the unitarization the cross sections at high energies, do not make the amplitudes start falling off when they are still too small.)

The rest of this thesis is organized as follows: after setting up the problem in chapter 2, our main calculational tool, the method of constrained instantons developed by Affleck [10], is described in some length in chapter 3, and used to obtain a simple expression for the relevant momentum space Green functions. In chapter 4 we study the fermionic zero modes in the field of the constrained instanton for nonzero Yukawa couplings. The result for the total spin-averaged cross section for the process $\overline{q}+\overline{q} \to (3n_f - 2)q + 3n_f l + nW + nH$ and its high energy behavior, to leading order in the semiclassical approximation, are presented in chapter 5. Our conclusions are discussed at the end of that chapter. Technical details are relegated to three appendices.

Chapter II.

Baryon and lepton number non-conserving Green functions

2.1 Anomalous currents in a simplified standard model

Our aim is to compute a semiclassical approximation to the baryon- and lepton-number-nonconserving scattering amplitudes in the standard model with n_f families of quarks and leptons. Right from the outset we make several simplifications:

a) We neglect interfamily mixings, that is, we set the (generalized) Kobayashi-Maskawa matrix equal to the identity.

b) We neglect strong interaction corrections, that is, we set $g_s = 0$.

c) We set the $U(1)$ coupling constant g' equal to zero. We think this is a reasonable approximation since on the one hand the Weinberg angle is small ($\sin^2\theta_W \simeq 0.22$) and on the other, although the anomalous divergence equation for the baryon and lepton currents involves g' in addition to the $SU(2)$ coupling constant g, $(F\tilde{F})_{U(1)}$ fluctuates around zero in the semiclassical expansion.

The model reduces therefore to a $SU(2)$ chiral gauge theory with the following matter content: one scalar Higgs doublet H and n_f identical families of two-component fermions each containing four left-handed doublets l_L and $q_{L\alpha}$ and seven right-handed singlets e_R and $u_{R\alpha}, d_{R\alpha}$, where $\alpha = 1, 2, 3$, is the $SU(3)$ color index.

As we shall soon see, the multiplicity of families as well as of quark colors is trivially taken into account. Thus, we only need to consider the role of one lepton field l_L, e_R and one quark field q_L, u_R, d_R in the scattering process.

Componentwise,

$$q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}, \qquad l_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}, \qquad H = \begin{pmatrix} H_1 \\ H_2 \end{pmatrix}.$$

The Lagrangian (in Minkowski space) is:

$$\mathcal{L} = \mathcal{L}_{YM} + \mathcal{L}_H + \sum_{(fam)}^{n_f} (\mathcal{L}_l + \sum_{\alpha=1}^{3} \mathcal{L}_{q\alpha}), \tag{2.1}$$

where

$$\mathcal{L}_{YM} = -\frac{1}{2}\mathrm{tr}\, F_{\mu\nu}F^{\mu\nu}, \tag{2.2a}$$

$$\mathcal{L}_H = (D_\mu H)^\dagger (D^\mu H) - \frac{\lambda}{4}(H^\dagger H - v^2)^2, \tag{2.2b}$$

$$\mathcal{L}_l = i l_L^\dagger \sigma^\mu D_\mu l_L + i e_R^\dagger \overline{\sigma}^\mu \partial_\mu e_R - (g_l l_L^\dagger H e_R + h.c.), \tag{2.2c}$$

$$\mathcal{L}_q = i q_L^\dagger \sigma^\mu D_\mu q_L + i u_R^\dagger \overline{\sigma}^\mu \partial_\mu u_R + i d_R^\dagger \overline{\sigma}^\mu \partial_\mu d_R - (g_u q_L^\dagger \epsilon H^* u_R + g_d q_L^\dagger H d_R + h.c.), \tag{2.2d}$$

and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu], \tag{2.3}$$

$$D_\mu = \partial_\mu - igA_\mu^a \frac{\sigma^a}{2}. \tag{2.4}$$

The hypercharge assignments are those of the standard model. In $(2c, d)$ $\sigma^\mu = (1, \vec{\sigma})$, $\overline{\sigma}^\mu = (1, -\vec{\sigma})$, $\epsilon = i\tau^2 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and the phases of the fermionic fields have been adjusted so that g_l, g_u and g_d are positive.

Although lepton and baryon numbers are global symmetries of $\mathcal{L}$, at the quantum level the corresponding currents are not conserved [1]:

$$\partial_\mu J^\mu_L = \partial_\mu J^\mu_B = \frac{g^2}{16\pi^2}\,\mathrm{tr}\,F_{\mu\nu}\tilde{F}^{\mu\nu}, \tag{2.5}$$

where

$$J^\mu_L = l^\dagger_L \sigma^\mu l_L + e^\dagger_R \overline{\sigma}^\mu e_R, \tag{2.6a}$$

$$J^\mu_B = \frac{1}{3}(q^\dagger_L \sigma^\mu q_L + u^\dagger_R \overline{\sigma}^\mu u_R + d^\dagger_R \overline{\sigma}^\mu d_R), \tag{2.6b}$$

and

$$\tilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}. \tag{2.7}$$

We shall calculate the "anomalous" scattering amplitudes by the LSZ procedure: we first compute the corresponding multiparticle Green function in Euclidean space and after continuing it back to Minkowski space we amputate the propagators for the external particles and put the latter on mass shell.

One must exercise some care when writing down the fermionic piece of the Euclidean space Lagrangian if one wants to obtain an $SO(4)$ invariant object. The subtlety here is that complex conjugation does not interchange the two spinor representations of $SO(4)$. Defining A- and B-type spinors in $SO(4)$ (the analogs of R- and L-spinors of the Lorentz group, see appendix A), the correct transition is made as follows: if ψ_R and ψ_L are two-component Weyl spinors appearing in (Minkowski space) $\mathcal{L}$, then one has to substitute

$$\begin{aligned} \psi_R &\to \psi_A, & \psi_L &\to \psi_B, \\ \psi^\dagger_R &\to \psi^\dagger_B, & \psi^\dagger_L &\to \psi^\dagger_A, \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} \sigma^\mu &\to \sigma_\mu = (\vec{\sigma}, i), \\ \overline{\sigma}^\mu &\to -\overline{\sigma}_\mu = (-\vec{\sigma}, i). \end{aligned} \tag{2.9}$$

In this way the Euclidean counterparts of eqns. (2.2a–d) are:

$$\tilde{\mathcal{L}}_{YM} = \frac{1}{2}\mathrm{tr}\, F_{\mu\nu}F_{\mu\nu}, \tag{2.10a}$$

$$\tilde{\mathcal{L}}_{H} = (D_\mu H)^\dagger (D_\mu H) + \frac{\lambda}{4}(H^\dagger H - v^2)^2, \tag{2.10b}$$

$$\tilde{\mathcal{L}}_{l} = i l_A^\dagger \sigma_\mu D_\mu l_B - i e_B^\dagger \overline{\sigma}_\mu \partial_\mu e_A - g_l l_A^\dagger H e_A - g_l e_B^\dagger H^\dagger l_B, \tag{2.10c}$$

$$\begin{aligned}\tilde{\mathcal{L}}_{q} = {} & i q_A^\dagger \sigma_\mu D_\mu q_B - i u_B^\dagger \overline{\sigma}_\mu \partial_\mu u_A + i d_B^\dagger \overline{\sigma}_\mu \partial_\mu d_A - g_u q_A^\dagger \epsilon H^* u_A \\ & - g_d q_A^\dagger H d_A + g_u u_B^\dagger H^T \epsilon q_B - g_d d_B^\dagger H^\dagger q_B.\end{aligned} \tag{2.10d}$$

Defining

$$Q = \begin{pmatrix} q_B \\ u_A \\ d_A \end{pmatrix}, \qquad Q^\dagger = (q_A^\dagger \quad u_B^\dagger \quad d_B^\dagger), \tag{2.11}$$

$$L = \begin{pmatrix} l_B \\ e_A \end{pmatrix}, \qquad L^\dagger = (l_A^\dagger \quad e_B^\dagger), \tag{2.12}$$

and

$$\hat{D}_Q = \begin{pmatrix} i\sigma_\mu D_\mu & -g_u \epsilon H^* & -g_d H \\ g_u H^T \epsilon & -i\overline{\sigma}_\mu \partial_\mu & 0 \\ -g_d^* H^\dagger & 0 & -i\overline{\sigma}_\mu \partial_\mu \end{pmatrix}, \tag{2.13}$$

$$\hat{D}_L = \begin{pmatrix} i\sigma_\mu D_\mu & -g_l H \\ -g_l H^\dagger & -i\overline{\sigma}_\mu \partial_\mu \end{pmatrix}, \tag{2.14}$$

we can write

$$\tilde{\mathcal{L}}_q = Q^\dagger \hat{D}_Q Q \tag{2.15a}$$

and

$$\tilde{\mathcal{L}}_l = L^\dagger \hat{D}_L L. \tag{2.15b}$$

2.2 Anomalous Green functions in Euclidean space

We are interested in the Fourier transform into momentum space of Green functions of the type

$$\langle 0|\psi(x_1)\cdots\psi(x_p)A(y_1)\cdots A(y_n)\eta(z_1)\cdots\eta(z_m)|0\rangle =$$

$$= \frac{\int \mathcal{D}\mu(\psi,A,H)\, e^{-S_E}\, \psi(x_1)\cdots\psi(x_p)A(y_1)\cdots A(y_n)\eta(z_1)\cdots\eta(z_m)}{\int \mathcal{D}\mu(\psi,A,H)\, e^{-S_E}}, \qquad (2.16)$$

where we have suppressed all (Euclidean) spacetime and isospin indices and ψ, A, η denote any fermionic field, the gauge field, and the (shifted) Higgs field respectively. $\mathcal{D}\mu(\psi, A, H)$ denotes the path integral measure including the appropriate gauge fixing and ghost terms.

The fermionic piece of the path integrals in (2.16) can be computed exactly in terms of the backgroud bosonic fields A, H since the fermionic fields appear quadratically in S_E. More generally, we consider the generating functional

$$Z[\zeta,\overline{\zeta}] = \int \mathcal{D}\Psi\, \mathcal{D}\Psi^\dagger \exp\{-\textstyle\int d^4x(\Psi^\dagger\hat{D}\Psi - \overline{\zeta}\Psi - \Psi^\dagger\zeta)\}, \qquad (2.17)$$

where $\Psi(x)$ and $\Psi^\dagger(x)$ are, in general, independent multi-component objects, their entries being two-component Grassmann spinors. $\zeta(x)$ and $\overline{\zeta}(x)$ are the external sources, and $\hat{D}$ is, in general, a nonhermitian operator (with respect to the standard inner product $(\psi,\chi) = \int d^4x\, \psi^\dagger\chi$).

The result for Z is:

$$Z[\zeta,\overline{\zeta}] = \prod_{i=1}^{N} \overline{v}_i(\overline{\zeta}) \prod_{j=1}^{N'} u_j(\zeta) \left[Det'(\hat{D}^\dagger\hat{D})\right]^{1/2} \exp \textstyle\int d^4x d^4y\, \overline{\zeta}(x) S'(x,y)\zeta(y), \qquad (2.18)$$

where

$$u_i(\zeta) = \int d^4x\, \varphi_{0i}^\dagger(x)\zeta(x), \qquad i = 1,\ldots,N', \tag{2.19a}$$

$$\overline{v}_i(\overline{\zeta}) = \int d^4x\, \overline{\zeta}(x)\psi_{0i}(x), \qquad i = 1,\ldots,N, \tag{2.19b}$$

N, N' being the number of normalizable zero modes ψ_{0i}, φ_{0i} of $\hat{D}$ and $\hat{D}^\dagger$ respectively:

$$\hat{D}\psi_{0i} = 0, \qquad i = 1,\ldots,N, \tag{2.20a}$$
$$\hat{D}^\dagger\varphi_{0i} = 0, \qquad i = 1,\ldots,N', \tag{2.20b}$$

chosen such that

$$(\psi_{0i}, \psi_{0j}) = (\varphi_{0i}, \varphi_{0j}) = \delta_{ij}. \tag{2.20c}$$

The meaning of the primed objects in (2.18) is explained in appendix B, where a rough derivation of formula (2.18) is given as well.

Green functions are obtained by differentiating $Z[\zeta,\overline{\zeta}]$ with respect to ζ and/or $\overline{\zeta}$ and then putting $\zeta = \overline{\zeta} = 0$. It is clear that from the sector with no zero modes ($N = N' = 0$, and hence no u_i or $\overline{v}_i$ factors) the only nonvanishing Green functions are those containing an equal number of Ψ and $\Psi^\dagger$ fields which therefore conserve any fermionic number.

Only the sector of the sum over histories over the bosonic fields A, H that leads to the existence of normalizable zero modes of either of the operators $\hat{D}, \hat{D}^\dagger$ contributes to an anomalous Green function.

It is possible to prove (see Section 3.3) that the index of the operators $\hat{D}_Q$ and $\hat{D}_L$ (recall that $\operatorname{index} A = \dim\ker A - \dim\ker A^\dagger$) is independent of the Higgs field configuration and depends only on the Pontryagin index of the $SU(2)$ gauge field as in the pure Yang-Mills case. (One might argue that this is a trivial result since the index of an operator is unchanged by continuous deformations of the operator. This is not so, however, because finiteness of the Euclidean action implies a nontrivial topology for the Higgs field as well [11]. Moreover, the configuration $H \equiv 0$ has infinite action and hence cannot be taken as the "undeformed" configuration.)

The result above is a reflection of the fact that the anomalous divergence equation (2.5) is independent of the scalar Higgs field [12].

We shall consider the simplest case: $\hat{D}$ has exactly one (normalizable) zero mode and $\hat{D}^\dagger$ none. Then the simplest nonvanishing Green function is

$$\int \mathcal{D}\Psi\, \mathcal{D}\Psi^\dagger \Psi(x) \exp\{-\textstyle\int d^4x(\Psi^\dagger \hat{D}\Psi)\} = \frac{\delta}{\delta\overline{\zeta}(x)} Z[\zeta,\overline{\zeta}]\Big|_{\zeta=\overline{\zeta}=0}$$

$$= \psi_0(x) Det'(\hat{D}^\dagger\hat{D}). \qquad (2.21)$$

In our theory (2.1) we have $4n_f$ independent fermionic fields to integrate over. So the simplest anomalous Green function must contain $p = 4n_f$ different fermionic fields and violates L-number and R-number by n_f units. It is proportional to

$$\int \mathcal{D}\mu(A,H)_{\nu=1}\, e^{-S'_E} \prod_{i=1}^{n_f} \Big\{ \big[Det'(\hat{D}_L^\dagger \hat{D}_L)\big]^{1/2} \big[Det'(\hat{D}_Q^\dagger \hat{D}_Q)\big]^{3/2}$$

$$L_0(x_1^{(i)}) Q_0(x_2^{(i)}) Q_0(x_3^{(i)}) Q_0(x_4^{(i)}) \Big\} A(y_1)\cdots A(y_n)\eta(z_1)\cdots\eta(z_m), \qquad (2.22)$$

where

$$S'_E = \int d^4x\, (\mathcal{L}_{YM} + \mathcal{L}_H), \qquad (2.23)$$

and the subscript $\nu = 1$ indicates that we are to integrate over the winding number one bosonic configurations. To very good approximation, widely separated instanton–anti-instanton configurations will not contribute to the integration since they imply too many zero modes.

In the following chapter we shall perform a semiclassical evaluation of (2.22).

2.3 Index theorem in the presence of Yukawa couplings

We want to compute the index of the operators $\hat{D}_Q$ and $\hat{D}_L$ defined in (2.13) and (2.14). Using the equality of the nonzero eigenvalues of the operators $\hat{D}^\dagger \hat{D}$ and $\hat{D}\hat{D}^\dagger$ (see (B.1–3)), the index of the operator $\hat{D}$,

$$\mathrm{index}\,\hat{D} = \dim\ker\hat{D} - \dim\ker\hat{D}, \tag{2.24}$$

can be written as

$$\mathrm{index}\,\hat{D} = \mathrm{Tr}\, e^{-t\hat{D}^\dagger\hat{D}} - \mathrm{Tr}\, e^{-t\hat{D}\hat{D}^\dagger}, \qquad (t>0). \tag{2.25}$$

To get a manageable form for (2.25) we write

$$\begin{aligned}
\mathrm{Tr}\, e^{-t\hat{D}^\dagger\hat{D}} &= \sum_{\ell=n,0i} \langle\psi_\ell|e^{-t\hat{D}^\dagger\hat{D}}|\psi_\ell\rangle \\
&= \sum_\ell \int d^4x\, \psi_\ell^\dagger(x) e^{-t\hat{D}^\dagger\hat{D}} \psi_\ell(x) \\
&= \int d^4x\, d^4x' \sum_\ell \psi^*_{\ell\alpha a}(x)\psi_{\ell\beta b}(x') \left[e^{-t\hat{D}^\dagger\hat{D}}\delta^{(4)}(x-x')\right]_{\alpha a,\beta b},
\end{aligned}$$

where α, β are spin indices and a, b are isospin indices. Using the completeness of the eigenfunctions of the hermitian operator $\hat{D}^\dagger\hat{D}$ and a momentum representation for the delta function inside brackets in the last formula, we can write

$$\mathrm{Tr}\, e^{-t\hat{D}^\dagger\hat{D}} = \int d^4x \int \frac{d^4k}{(2\pi)^4} \mathrm{tr}\,(e^{-ik\cdot x} e^{-t\hat{D}^\dagger\hat{D}} e^{ik\cdot x}), \tag{2.26}$$

and thus the index of the operator $\hat{D}$ can be computed as

$$\mathrm{index}\,\hat{D} = \int d^4x \int \frac{d^4k}{(2\pi)^4} \mathrm{tr}\,(e^{-ik\cdot x}[e^{-t\hat{D}^\dagger\hat{D}} - e^{-t\hat{D}\hat{D}^\dagger}] e^{ik\cdot x}), \tag{2.27}$$

where the trace is both over spin and isospin indices.

We will first consider

$$\hat{D}_L = \begin{pmatrix} i\sigma_\mu D_\mu & H \\ H^\dagger & -i\overline{\sigma}_\mu \partial_\mu \end{pmatrix}, \tag{2.28}$$

where we have absorbed the coupling constants g and g_l into the definition of the fields. Using formulas (C.6) one gets

$$\hat{D}_L^\dagger \hat{D}_L = \begin{pmatrix} -D^2 - \sigma_{\mu\nu}F_{\mu\nu} + HH^\dagger & i\overline{\sigma}_\mu D_\mu H \\ -i\sigma_\mu (D_\mu H)^\dagger & -\partial^2 + H^\dagger H \end{pmatrix}, \tag{2.29}$$

$$\hat{D}_L \hat{D}_L^\dagger = \begin{pmatrix} -D^2 - \overline{\sigma}_{\mu\nu}F_{\mu\nu} + HH^\dagger & i\sigma_\mu D_\mu H \\ -i\overline{\sigma}_\mu (D_\mu H)^\dagger & -\partial^2 + H^\dagger H \end{pmatrix}. \tag{2.30}$$

It is seen that $\hat{D}_L^\dagger \hat{D}_L$ and $\hat{D}_L \hat{D}_L^\dagger$ can be obtained from each other by interchanging $\sigma_{\mu\nu} \leftrightarrow \overline{\sigma}_{\mu\nu}$ and $\sigma_\mu \leftrightarrow \overline{\sigma}_\mu$.

We will compute (2.27) in the limit $t \to 0$, where several simplifications occur. First we split $-t\hat{D}_L^\dagger \hat{D}_L = tA + tB$ and $-t\hat{D}_L \hat{D}_L^\dagger = tA + tB'$, where

$$A = \begin{pmatrix} D^2 & 0 \\ 0 & \partial^2 \end{pmatrix}, \qquad B = \begin{pmatrix} \sigma_{\mu\nu}F_{\mu\nu} - HH^\dagger & -i\overline{\sigma}_\mu D_\mu H \\ i\sigma_\mu (D_\mu H)^\dagger & -H^\dagger H \end{pmatrix}, \tag{2.31}$$

and B' can be obtained from B replacing $\sigma_{\mu\nu} \to \overline{\sigma}_{\mu\nu}$, and interchanging $\sigma_\mu \leftrightarrow \overline{\sigma}_\mu$. Using then the operator identity

$$\lim_{t \to 0} e^{t(A+B)} = \lim_{t \to 0} e^{tA} e^{tB} = \lim_{t \to 0} e^{tB} e^{tA}, \tag{2.32}$$

we can write

$$\lim_{t \to 0} [e^{-t\hat{D}^\dagger \hat{D}} - e^{-t\hat{D}\hat{D}^\dagger}] e^{ik\cdot x} = \lim_{t \to 0} (e^{tB} - e^{tB'}) e^{tA} e^{ik\cdot x}. \tag{2.33}$$

Since the index of the Dirac operator $\hat{D}_L$ is a gauge invariant quantity, without loss of generality we can require the gauge field to satisfy the gauge condition $\partial_\mu A_\mu = 0$, so that the following simple formula holds:

$$e^{tD^2} e^{ik\cdot x} = e^{ik\cdot x} e^{-t(k_\mu - A_\mu)^2}. \tag{2.34}$$

Hence, from (2.27, 2.33, 2.34) we have

$$\operatorname{index} \hat{D}_L = \int d^4x \lim_{t\to 0} \operatorname{tr} \left[(e^{tB} - e^{tB'}) \int \frac{d^4k}{(2\pi)^4} \begin{pmatrix} e^{-t(k_\mu - A_\mu)^2} & 0 \\ 0 & e^{-tk^2} \end{pmatrix} \right]. \tag{2.35}$$

Rescaling $k_\mu = k'_\mu/\sqrt{t}$, the k-integral becomes

$$\frac{1}{t^2} \int \frac{d^4k'}{(2\pi)^4} \begin{pmatrix} e^{-(k'_\mu - \sqrt{t}A_\mu)^2} & 0 \\ 0 & e^{-k'^2}, \end{pmatrix}$$

which in the limit $t \to 0$ has the value $(16\pi^2 t^2)^{-1}$ and is proportional to the identity matrix in isospin space. Thus, e^{tB} and $e^{tB'}$ in (2.35) need to be expanded up to order t^2 only:

$$\operatorname{tr}(e^{tB} - e^{tB'}) = t \operatorname{tr}(B - B') + \frac{t^2}{2} \operatorname{tr}(B^2 - B'^2) + \mathcal{O}(t^3).$$

$B - B'$ is easily seen to have vanishing trace in isospin space, whereas for the term of order t^2 one finds

$$\begin{aligned} \operatorname{tr}(B^2 - B'^2) = {} & \operatorname{tr}(\sigma_{\mu\nu}\sigma_{\alpha\beta} - \overline{\sigma}_{\mu\nu}\overline{\sigma}_{\alpha\beta}) \operatorname{tr} F_{\mu\nu}F_{\alpha\beta} \\ & + \operatorname{tr}(\overline{\sigma}_\mu \sigma_\nu - \sigma_\mu \overline{\sigma}_\nu) \operatorname{tr}(D_\mu H)(D_\nu H)^\dagger \\ & + 2\operatorname{tr}(\sigma_\mu \overline{\sigma}_\nu - \overline{\sigma}_\mu \sigma_\nu)(D_\mu H)^\dagger (D_\nu H) \\ = {} & \varepsilon_{\mu\nu\alpha\beta} \operatorname{tr} F_{\mu\nu} F_{\alpha\beta}, \end{aligned} \tag{2.36}$$

where we have used equations (C.6) and (C.7), and the tracelessness of the matrices $\sigma_{\mu\nu}$ and $\overline{\sigma}_{\mu\nu}$.

We have proven therefore that

$$\operatorname{index} \hat{D}_L = \frac{1}{16\pi^2} \int d^4x \operatorname{tr} F_{\mu\nu} \tilde{F}_{\mu\nu}, \tag{2.37}$$

where

$$\tilde{F}_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\alpha\beta} F_{\alpha\beta}, \tag{2.38}$$

completely independent of the Higgs field H. An analogous calculation for $\hat{D}_Q$ shows that

$$\operatorname{index} \hat{D}_Q = \operatorname{index} \hat{D}_L. \tag{2.39}$$

Chapter III.

Saddle point expansion of the bosonic functional integral

3.1 Constrained instantons

Our strategy is to perform a semiclassical evaluation of the integral (2.22). From now on we will drop the prime in S_E' so that S_E will stand for the right-hand-side of equation (2.23). It is convenient to work in terms of the rescaled fields

$$A_\mu \to \frac{1}{g} A_\mu, \qquad H \to \frac{1}{\sqrt{\lambda}} H, \tag{3.1}$$

so that

$$S_E(A,H) = \frac{1}{g^2} \int d^4x \left\{ \frac{1}{2} \mathrm{tr}\, F^2 + \kappa \left[(D_\mu H)^\dagger (D_\mu H) + \frac{1}{4} (H^\dagger H - \langle\phi\rangle^2)^2 \right] \right\}, \tag{3.2}$$

where we have defined

$$\kappa = g^2/\lambda \tag{3.3}$$

and

$$\langle\phi\rangle^2 = \lambda v^2, \tag{3.4}$$

and now

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu], \tag{3.5}$$

$$D_\mu = \partial_\mu - i A_\mu. \tag{3.6}$$

From (3.2) it is seen that the semiclassical approximation, $\hbar \to 0$, corresponds to taking $g^2 \to 0$, keeping κ and $\langle\phi\rangle$ fixed. (This is, $m_H = \sqrt{\lambda} v = \langle\phi\rangle$ and $m_W = gv/\sqrt{2} = \sqrt{\kappa/2}\langle\phi\rangle$ should stay fixed as $g^2 \to 0$.)

The following analysis has been given by Affleck [10]. We want to expand S_E around some stationary field configuration and approximate (for g small) the full integral by its Gaussian approximation. When $\langle\phi\rangle \neq 0$, S_E has no nontrivial stationary points, however. This is seen from a simple scaling argument: for a given finite action configuration $A_\mu(x)$, $H(x)$ we consider the rescaling $A_\mu(x) \to aA_\mu(ax)$, $H(x) \to H(ax)$ (which preserves the finite action boundary conditions), under which S_E scales as

$$S_E \longrightarrow \frac{1}{g^2}\int d^4x \left\{\frac{1}{2}\mathrm{tr}\, F^2 + \kappa\big[a^{-2}(D_\mu H)^\dagger(D_\mu H) + a^{-4}\frac{1}{4}(H^\dagger H - \langle\phi\rangle^2)^2\big]\right\}.$$

Thus, given any field configuration we can always rescale it to get a smaller action (except in the trivial case $A_\mu = 0$, $H = constant$).

From the point of view of the Euler-Lagrange equations of motion,

$$\begin{aligned}(D_\mu F_{\mu\nu})^a - i\kappa H^\dagger \frac{\tau^a}{2}\overleftrightarrow{\partial}_\nu H - \frac{\kappa}{2}A^a_\nu H^\dagger H = 0,\\ (D^2H)_a - \frac{1}{2}(H^\dagger H - \langle\phi\rangle^2)H_a = 0,\end{aligned} \tag{3.7}$$

the situation is peculiar, however. We know that for $\langle\phi\rangle = 0$ a finite action solution with unit winding number of (3.7) exists, namely the instanton

$$A_\mu = \frac{2x_\nu}{x^2+\rho^2}\tau_{\mu\nu}, \qquad\qquad H = 0. \tag{3.8}$$

(For the definition and properties of the matrices $\tau_{\mu\nu}$ and $\overline{\tau}_{\mu\nu}$, see Appendix C.)

For $\langle\phi\rangle$ small enough ($\langle\phi\rangle \ll 1/\rho$) we should expect to find a solution of (3.7) in perturbation theory in $\rho\langle\phi\rangle$ that reduces to (3.8) (actually, a gauge transformation of it, as explained below) when $\langle\phi\rangle \to 0$.

The finite action boundary conditions ($F_{\mu\nu} \to 0$, $D_\mu H \to 0$, $H^\dagger H \to \langle\phi\rangle^2$ faster than $1/x^2$ as $x \to \infty$) take different forms in different gauges. In the particularly convenient gauge where $H \to \langle\phi\rangle \underline{h}$ at infinity, where $\underline{h}$ is a constant isospinor with $\underline{h}^\dagger \underline{h} = 1$, finiteness of the action requires $A_\mu \to 0$ faster than $1/x^2$ at infinity. The solution (3.8) does not have this asymptotic behavior; it actually approaches a pure gauge at infinity:

$$A_\mu \to i\Omega\partial_\mu\Omega^{-1}, \tag{3.9}$$

where

$$\Omega = i\frac{x_\mu \overline{\tau}_\mu}{(x^2)^{1/2}}. \tag{3.10}$$

To obtain the desired asymptotic behavior of the instanton, we have to recast it in the so-called "singular gauge" (obtained from (3.8) by the gauge transformation singular at the origin Ω^{-1}):

$$A_\mu = \frac{2\rho^2}{x^2(x^2+\rho^2)} x_\nu \Lambda_{\mu\nu}, \tag{3.11}$$

where $\Lambda_{\mu\nu} = \overline{\tau}_{\mu\nu}$ for the instanton ($\tau_{\mu\nu}$ and $\overline{\tau}_{\mu\nu}$ get interchanged when one goes from the regular gauge to the singular gauge and vice versa) but in general $\Lambda_{\mu\nu}$ will stand for $U\tau_{\mu\nu}U^\dagger$ or $U\overline{\tau}_{\mu\nu}U^\dagger$, where the $SU(2)$ matrix U represents the orientation of the instanton. $\Lambda_{\mu\nu}$ satisfies the commutation relations of the $so(4)$ Lie algebra.

If we neglect terms of order $(\rho\langle\phi\rangle)^2$ then equations (3.7) are approximately solved by (3.11) and

$$H = \left(\frac{x^2}{x^2+\rho^2}\right)^{1/2} \langle\phi\rangle \underline{h}. \tag{3.12}$$

(In fact, (3.11,3.12) satisfy $D_\mu F_{\mu\nu} = 0$, $D^2 H = 0$ and render the source terms negligible.)

A perturbative solution of (3.7) around the zeroth-order solution (3.11,3.12) does not exist because the appropiate finite action boundary conditions for the higher order terms cannot be enforced. The operators that act on higher order terms possess zero modes that determine *a priori* the behavior of these terms at infinity (which is not the required one).

A way of getting around this difficulty is to extremize S_E subject to a constraint. We follow Affleck [10] who chooses to constrain the spacetime integral of the sum of two gauge invariant local operators of dimension greater than four and vanishing rapidly enough at infinity, one depending only on the gauge field and the other only on the Higgs field. If we insert the identity

$$1 = \int d\rho\, \Delta(\rho) \delta\big(\textstyle\int d^4x\, [O_A(A) + \frac{\sigma_H}{\sigma_A} O_H(H)] - \int d^4x\, [O_A(\bar{A}) + \frac{\sigma_H}{\sigma_A} O_H(\bar{H})]\big) \tag{3.13}$$

in the functional integral (2.22), then the relevant stationary configuration will be a solution of the constrained Euler-Lagrange equations

$$\begin{aligned} \frac{\delta S_E}{\delta A(x)} + \sigma_A \frac{\delta}{\delta A(x)} \int d^4x\, O_A(A) = 0, \\ \frac{\delta S_E}{\delta H(x)} + \sigma_H \frac{\delta}{\delta H(x)} \int d^4x\, O_H(H) = 0. \end{aligned} \tag{3.14}$$

Given O_A and O_H, the Lagrange multipliers σ_A and σ_H are to be determined order by order in perturbation theory in $\rho\langle\phi\rangle$ in order to enforce the correct boundary conditions for the higher order terms. The "constrained instanton" $\bar{A}$, $\bar{H}$ is the unique solution of (3.14) obtained by this procedure that reduces to (3.11,3.12) when $\rho\langle\phi\rangle \to 0$.

It is easy to see how the constrained instanton behaves at large distances ($x \gg \rho$). Since the extra terms added to (3.7) decay more rapidly than the linear mass terms as $H \to \langle\phi\rangle \underline{h}$ and $A_\mu \to 0$, at infinity $\bar{A}$, $\bar{H} - \langle\phi\rangle \underline{h}$ become proportional to the solutions of

$$\begin{aligned} \partial_\mu(\partial_\mu A_\nu - \partial_\nu A_\mu) - \frac{\kappa}{2} \langle\phi\rangle^2 A_\nu = 0, \\ \partial_\mu \partial_\mu \tilde{H} - \langle\phi\rangle^2 \tilde{H} = 0. \end{aligned} \tag{3.15}$$

In terms of the function

$$G_m(x) = m^2 \frac{K_1(mx)}{mx}, \tag{3.16}$$

(where K_1 is a modified Bessel function) which satisfies

$$(\partial^2 - m^2)G_m(x) = -4\pi^2\delta^{(4)}(x), \tag{3.17}$$

the solutions to (3.15) can be written as

$$\begin{aligned} A_\mu &\propto \Lambda_{\mu\nu}\partial_\nu G_{m_W}(x), \\ H - \langle\phi\rangle\underline{h} &\propto G_{m_H}(x), \end{aligned} \tag{3.18}$$

where $m_W = \sqrt{\kappa/2}\langle\phi\rangle = gv/\sqrt{2}$ and $m_H = \langle\phi\rangle = \sqrt{\lambda}v$ are the gauge boson and Higgs masses respectively.

$G_m(x)$ decays exponentially at infinity and has the series expansion

$$\begin{aligned} G_m(x) &= \frac{1}{x^2} + \frac{1}{2}m^2 \ln mx + (const.)m^2 + \cdots \\ &= \frac{1}{x^2} + \frac{1}{2}m^2 \ln m\rho + m^2(const. + \frac{1}{2}\ln x/\rho) + \cdots . \end{aligned} \tag{3.19}$$

From (3.19) one can determine the constants of proportionality in (3.18) and the asymptotic behavior of the terms in the correct perturbative expansion of the constrained instanton (which contains logarithms of m). One finds

$$\begin{aligned} \bar{A}_\mu(x) &= A_{0\mu}(x;\rho) + (\rho\langle\phi\rangle)^2 A_{1\mu}(x;\rho) + \cdots, \\ \bar{H}(x) &= H_0(x;\rho) + (\rho\langle\phi\rangle)^3 \ln\rho\langle\phi\rangle H_1(x;\rho) + \cdots, \end{aligned} \tag{3.20}$$

where $A_{0\mu}(x;\rho)$ and $H_0(x;\rho)$ are given by (3.11,3.12). The higher order terms in these expansions generally diverge at infinity and therefore (3.20) is useful only at small distances ($x \not\gg \rho$).

For large distances ($x \gg \langle\phi\rangle^{-1}$) one finds expansions of the type

$$\begin{aligned}\bar{A}_\mu(x) &= \rho^2\left[A^0_\mu(x;\langle\phi\rangle) + (\rho\langle\phi\rangle)^2 A^1_\mu(x;\langle\phi\rangle) + \cdots\right],\\ \bar{H}(x) &= \rho^2\left[H^0(x;\langle\phi\rangle) + (\rho\langle\phi\rangle)^2 H^1(x;\langle\phi\rangle) + \cdots\right],\end{aligned} \tag{3.21}$$

where

$$\begin{aligned}A^0_\mu(x;\langle\phi\rangle) &= -2x_\nu\Lambda_{\mu\nu}\frac{d}{dx^2}G_{mw}(x),\\ H^0(x;\langle\phi\rangle) &= -\frac{1}{2}\langle\phi\rangle G_{m_H}(x)\underline{h}.\end{aligned} \tag{3.22}$$

All the terms in these expansions vanish exponentially at infinity but blow up as $x \to 0$.

It turns out that the ansatz

$$\begin{aligned}A_\mu(x) &= x_\nu\Lambda_{\mu\nu}\mathcal{A}(x^2),\\ H(x) &= \phi(x^2)\underline{h},\end{aligned} \tag{3.23}$$

where $\underline{h}$ is a constant isospinor and $\mathcal{A}$, ϕ are real functions of x^2, reduces the constrained Euler-Lagrange equations (3.14) to two coupled real nonlinear equations for $\mathcal{A}$ and ϕ (at least for the particular choice $O_A(A) = i\,\mathrm{tr}\,F^3 = i\,\mathrm{tr}\,F_{\mu\nu}F_{\nu\lambda}F_{\lambda\mu}$ and $O_H(H) = (H^\dagger H - \langle\phi\rangle^2)^3$, but this should be true in general).

The perturbative expansions for $\mathcal{A}$ and ϕ are analogous to (3.20) and (3.21) with

$$\mathcal{A}_0 = \frac{2\rho^2}{x^2(x^2+\rho^2)}, \qquad \phi_0 = \left(\frac{x^2}{x^2+\rho^2}\right)^{1/2}\langle\phi\rangle, \tag{3.24}$$

$$\mathcal{A}^0 = \rho^2 m_W^2\frac{K_2(m_W x)}{x^2}, \qquad \phi^0 = \langle\phi\rangle - \frac{1}{2}\rho^2 m_H\langle\phi\rangle\frac{K_1(m_H x)}{x}. \tag{3.25}$$

The constrained instanton resembles the instanton (3.11,3.12) at short distances but decays exponentially at large distances (recall that $K_n(z) \to e^{-z}/\sqrt{2\pi z}$ as $z \to \infty$).

The action of the constrained instanton can be calculated using expansions (3.20) and (3.21). One finds

$$\int d^4x \, \mathrm{tr}\, F^2(\bar{A}) = 16\pi^2 + \mathcal{O}\big[(\rho\langle\phi\rangle)^4\big], \tag{3.26a}$$

$$\int d^4x \, |\bar{D}_\mu \bar{H}|^2 = 2\pi^2\rho^2\langle\phi\rangle^2 + \mathcal{O}\big[(\rho\langle\phi\rangle)^4 \ln \rho\langle\phi\rangle\big], \tag{3.26b}$$

$$\int d^4x \, (\bar{H}^\dagger \bar{H} - \langle\phi\rangle^2)^2 = \mathcal{O}\big[(\rho\langle\phi\rangle)^4 \ln \rho\langle\phi\rangle\big], \tag{3.26c}$$

so that

$$S_E(\bar{A}, \bar{H}) = \frac{8\pi^2}{g^2} + \frac{2\pi^2\rho^2\langle\phi\rangle^2}{\lambda} + \mathcal{O}\left[\frac{(\rho\langle\phi\rangle)^4 \ln \rho\langle\phi\rangle}{\lambda}\right]. \tag{3.27}$$

For λ (and g) small, only those values of ρ such that $\rho\langle\phi\rangle \lesssim \sqrt{\lambda} \ll 1$ will contribute appreciably to the ρ-integral present in (2.22) after we use (3.13). In what follows we shall work to lowest order in $\rho\langle\phi\rangle$. To this order the operators O_A and O_H play a role in the integration over the constrained bosonic fluctuations (see (3.34)), but otherwise they drop out of the calculation altogether.

3.2 Collective coordinates

It has been noted that $\mathcal{A}$ and ϕ satisfy a set of equations that is independent of the matrix U appearing in $\Lambda_{\mu\nu}$ and of the isospinor $\underline{h}$. This is because the action $S_E(A_\mu = x_\nu \Lambda_{\mu\nu}\mathcal{A}(x^2),\ H = \phi(x^2)\underline{h})$ is invariant under *independent* global rotations of A_μ and H. In fact, the possibly noninvariant Higgs kinetic term takes the form

$$|D_\mu H|^2 = 4x^2\phi'^2 + \frac{3}{4}x^2(\phi\mathcal{A})^2,$$

independently of U and $\underline{h}$, where the prime denotes d/dx^2.

Thus we see that in the constrained instanton the gauge field and Higgs field orientations are not correlated. The latter, however, is completely fixed once we choose our spontaneously-broken vacuum state. Conventionally we take $\underline{h} = \binom{0}{1}$.

The orientation of A_μ remains a symmetry of the constrained instanton action and therefore has to be treated as a collective coordinate along with the position x_0 of the constrained instanton (of course we are assuming that the gauge choice is invariant under independent global rotations of the gauge field).

The size of the instanton is no longer a collective coordinate, but it acts like one since it parametrizes a family of local minima of the action when the constraint is imposed.

3.3 Momentum space Green functions in the Gaussian approximation

Schematically we have

$$\langle\psi(x_1)\cdots\eta(z_m)\rangle \sim \int d^4x_0 \int dU \int d\rho\,\Delta(\rho,\langle\phi\rangle)J(\rho,\langle\phi\rangle) \int \mathcal{D}\mu^{\perp}(A,H)\delta(\mathcal{C}(A,H))$$
$$\left[Det'\hat{D}^{\dagger}\hat{D}(A,H)\right]^{2n_f} e^{-S_E(A,H)}\psi_0(x_1-x_0)\cdots\eta(z_m-x_0), \tag{3.28}$$

where $J(\rho,\langle\phi\rangle)$ is the Jacobian factor coming from the transition to the collective coordinates x_0, U, the measure $\mathcal{D}\mu^{\perp}$ does not include fluctuations along the zero modes associated with the collective coordinates, and $\mathcal{C}(A,H)$ denotes the constraint in (3.13).

The Gaussian approximation gives

$$\langle\psi(x_1)\cdots\eta(z_m)\rangle = \int d^4x_0 \int dU \int d\rho\,\Delta(\rho,\langle\phi\rangle)J(\rho,\langle\phi\rangle)e^{-S_E(\bar{A},\bar{H})}$$
$$F(\rho,\langle\phi\rangle;\mu)\psi_0(x_1-x_0)\cdots\eta(z_m-x_0), \tag{3.29}$$

where $F(\rho\langle\phi\rangle;\mu)$ includes the (regularized) contributions coming from the fermionic determinants and the quadratic fluctuations about the constrained instanton, μ being the renormalization point.

When we go to momentum space, the integration over the position of the constrained instanton x_0 (which ensures translational invariance) produces a total momentum conserving delta function. Defining

$$(2\pi)^4\delta^4(p_1+\cdots+q_1+\cdots+k_1+\cdots)\tilde{G}(\{p\},\{q\},\{k\}) =$$
$$\int d^4x_1\cdots d^4y_1\cdots d^4z_1\cdots e^{ip_1\cdot x_1}\cdots e^{iq_1\cdot y_1}\cdots e^{ik_1\cdot z_1}\cdots\langle\psi(x_1)\cdots A(y_1)\cdots\eta(z_1)\cdots\rangle,$$

we can write for $\tilde{G}$:

$$\tilde{G}(\{p\},\{q\},\{k\}) = \int dU \int d\rho\, \Delta(\rho,\langle\phi\rangle)J(\rho,\langle\phi\rangle)e^{-S_E(\bar{A},\bar{H})} \\ F(\rho,\langle\phi\rangle;\mu)\psi_0(p_1)\cdots A(q_1)\cdots\eta(k_1)\cdots, \tag{3.30}$$

where $\psi_0(p)$, $A_\mu(q)$ and $\eta(k)$ are the Fourier transforms of the zero mode and the constrained instanton respectively:

$$\psi_0(p) = \int d^4x\, e^{ip\cdot x}\psi_0(x), \tag{3.31}$$

etc.

Dimensional analysis requires

$$\Delta(\rho,\langle\phi\rangle)J(\rho,\langle\phi\rangle)F(\rho,\langle\phi\rangle;\mu) \sim \rho^{2n_f-5}f(\rho\langle\phi\rangle,\rho\mu) \\ \sim \rho^{2n_f-5}\left[f_0(\rho\mu)+(\rho\langle\phi\rangle)^2 f_1(\rho\mu)+\cdots\right], \tag{3.32}$$

where f, f_0, f_1, etc., are dimensionless functions of the indicated variables.

The coupling constant dependence of the factors in (3.30) is obtained as follows: *i*) from the Jacobian factor J we pick up a factor g^{-7} [13], g^{-4} coming from the zero modes associated with translational invariance, and g^{-3} coming from the zero modes associated with global isospin invariance. *ii*) $\Delta(\rho,\langle\phi\rangle)$ defined in (3.13) is given by

$$\Delta(\rho,\langle\phi\rangle) = \left|\frac{\partial}{\partial\rho}\int d^4x\,\left[O_A(\bar{A}) + \frac{\sigma_H}{\sigma_A}O_H(\bar{H})\right]\right|. \qquad (3.33)$$

Thus Δ carries no factors of g; *iii*) The integration over constrained bosonic fluctuations in (3.28) is of a form analogous to the finite dimensional integral

$$\int d^n x\,\delta(b^T x)e^{-x^T A x} = \pi^{(n-1)/2}\left[(det\,A)b^T A^{-1} b\right]^{-1/2}, \qquad (3.34)$$

(only linear fluctuations need to be kept inside the delta function in the Gaussian approximation) where A is proportional to g^{-2}. The powers of g^2 coming from the determinant get cancelled out against powers of g present in the (rescaled) measure, but a factor g^{-1} coming from A^{-1} survives (this factor corresponds to the one coming from the collective coordinate ρ in the case $\langle\phi\rangle = 0$); *iv*) finally, we get n factors g^{-1} and m factors $\lambda^{-1/2} = \sqrt{\kappa}g^{-1}$ from the external gauge and Higgs fields. In fact, in (3.30) $A(q)$ and $\eta(k)$ are the Fourier transform of the gauge and Higgs components of the constrained instanton respectively. The rescaled fields appearing in (3.2) do not have the correct normalization to create relativistically normalized one particle states. To get these fields we have to undo the rescaling (3.1), which has the effect of multiplying the expressions for $\mathcal{A}$ and ϕ in (3.24), (3.25) by the factors g^{-1} and $\lambda^{-1/2}$ respectively.

Putting everything together we can write, using (3.27) and (3.32)

$$\tilde{G}(\{p\},\{q\},\{k\}) = \kappa^{m/2}\frac{e^{-8\pi^2/g^2}}{g^{8+n+m}}\int dU\int d\rho\, e^{-2\pi^2\kappa\rho^2\langle\phi\rangle^2/g^2}\rho^{2n_f-5}f_0(\rho\mu)$$
$$\psi_0(p_1)\cdots A(q_1)\cdots\eta(k_1)\cdots, \qquad (3.35)$$

where now g and κ are understood to be running coupling constants at the scale μ.

The function $f_0(\rho\mu)$ was computed by 't Hooft [14]:

$$f_0(\rho\mu) = c\,(\rho\mu)^{\frac{43-8n_f}{6}}, \tag{3.36}$$

ensuring the RG-invariance of the scattering amplitudes (to lowest order). The constant c is given by

$$c \simeq 861.94\, e^{-0.997n_f} 2^{10+6n_f} \pi^{6+4n_f}, \tag{3.37}$$

($c \simeq 1.03 \times 10^{19}$ for $n_f = 3$). In the following chapter we give the explicit forms of the zero modes of $\hat{D}_Q$ and $\hat{D}_L$ in the field of the constrained instanton. Their Fourier transforms, as well as the Fourier transform of the constrained instanton, are given in chapter 5.

Chapter IV.

Fermionic zero mode in the constrained instanton background

4.1 The fermionic zero mode

We shall consider the slightly more general case of $\hat{D}_Q$ first (the result for $\hat{D}_L$ can then be obtained by letting $g_d \to g_l$ and $g_u \to 0$). We want to solve the following system of simultaneous differential equations:

$$i\sigma_\mu D_\mu q_B - \tilde{g}_u \epsilon H^* u_A - \tilde{g}_d H d_A = 0, \tag{4.1a}$$

$$\tilde{g}_u H^T \epsilon q_B - i\overline{\sigma}_\mu \partial_\mu u_A = 0, \tag{4.1b}$$

$$-\tilde{g}_d H^\dagger q_B - i\overline{\sigma}_\mu \partial_\mu d_A = 0, \tag{4.1c}$$

in perturbation theory in $\rho\langle\phi\rangle$. Here $\tilde{g}_{u,d} = \lambda^{-1/2} g_{u,d}$ and A_μ, H are given by (3.20). We assume that $g_{u,d}$ are small (at least of order $\lambda^{1/2}$) so that $\tilde{g}_{u,d}$ are of order one. The *ansatz*

$$q_B(x) = x_\alpha \overline{\sigma}_\alpha \varphi(x^2), \qquad u_A = u_A(x^2), \qquad d_A = d_A(x^2), \tag{4.2}$$

where φ carries spin and isospin degrees of freedom, reduces equations (4.1) to the simpler

$$\hat{\mathcal{D}}_U \varphi(u) + i\tilde{g}_u \phi(u) u_A(u) \epsilon \underline{h}^* + i\tilde{g}_d \phi(u) d_A(u) \underline{h} = 0, \tag{4.3a}$$

$$\tilde{g}_u \phi(u) \underline{h}^T \epsilon \varphi(u) - 2iu_A'(u) = 0, \tag{4.3b}$$

$$-\tilde{g}_d \phi(u) \underline{h}^\dagger \varphi(u) - 2id_A'(u) = 0, \tag{4.3c}$$

where

$$\hat{\mathcal{D}}_U \varphi(u) = 4\varphi(u) + 2u\varphi'(u) + \frac{1}{2} u \mathcal{A}(u) U(\vec{\sigma}\cdot\vec{\tau}) U^\dagger \varphi(u), \tag{4.4}$$

$u = x^2 = x_\mu x_\mu$ and the prime denotes d/du. Upon substituting $\mathcal{A} = \mathcal{A}_0$ and $\phi = \phi_0$

(see (3.24)) it is seen that to lowest order in $\rho\langle\phi\rangle$ (4.3a–c) are solved by

$$\varphi = \varphi_0, \tag{4.5}$$

$$u_A = -\frac{i}{2}\tilde{g}_u \int^u du\, \phi_0 \underline{h}^T \epsilon \varphi_0, \tag{4.6a}$$

$$d_A = \frac{i}{2}\tilde{g}_d \int^u du\, \phi_0 \underline{h}^\dagger \varphi_0, \tag{4.6b}$$

Here $x_\mu \overline{\sigma}_\mu \varphi_0$ is the (normalized) zero mode of the Dirac operator $\hat{D} = \sigma_\mu D_\mu$ in the field of the pure Yang-Mills instanton (in the singular gauge) $A_\mu = \mathcal{A}_0 x_\nu U \overline{\tau}_{\mu\nu} U^\dagger$. The explicit form of φ_0 is

$$\varphi_0(x) = \frac{\sqrt{2}}{\pi} \frac{\rho}{(x^2)^{1/2}(x^2+\rho^2)^{3/2}} U \varphi_s, \tag{4.7}$$

where φ_s is the singlet in the coupled spin-isospin space, satisfying $(\vec{\sigma}\cdot\vec{\tau})\varphi_s = -3\varphi_s$ and $\varphi_s^\dagger \varphi_s = 1$, and the matrix U acts on isospin indices. From (4.6a,b) we find

$$u_A(x) = \frac{i}{2\pi}\tilde{g}_u \rho\langle\phi\rangle \frac{1}{x^2+\rho^2} U^\dagger \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{4.8a}$$

$$d_A(x) = -\frac{i}{2\pi}\tilde{g}_d \rho\langle\phi\rangle \frac{1}{x^2+\rho^2} U^\dagger \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{4.8b}$$

where $U^\dagger$ is now acting on the two-component spinors indicated.

As with the constrained instanton, the correct forms of the perturbative series for φ, u_A and d_A are obtained analyzing the behavior of the solutions of (4.3a–c) at infinity. In this limit $u\mathcal{A} \to 0$ and $\phi \to \langle\phi\rangle$ so that (4.3a–c) reduce to

$$4\varphi + 2u\varphi' + i\tilde{g}_u\langle\phi\rangle u_A \epsilon \underline{h}^* + i\tilde{g}_d\langle\phi\rangle d_A \underline{h} = 0, \tag{4.9a}$$

$$\tilde{g}_u\langle\phi\rangle \underline{h}^T \epsilon\varphi - 2iu_A' = 0, \tag{4.9b}$$

$$-\tilde{g}_d\langle\phi\rangle \underline{h}^\dagger \varphi - 2id_A' = 0. \tag{4.9c}$$

From (4.9a) we can solve for u_A and d_A (at infinity):

$$u_A = -\frac{i}{\tilde{g}_u\langle\phi\rangle}\underline{h}^T\epsilon(4\varphi + 2u\varphi'), \tag{4.10a}$$

$$d_A = \frac{i}{\tilde{g}_d\langle\phi\rangle}\underline{h}^\dagger(4\varphi + 2u\varphi'). \tag{4.10b}$$

Substituting these expressions in (4.9b, c) respectively, we find equations for the behavior of $\underline{h}^T\epsilon\varphi$ and $\underline{h}^\dagger\varphi$ at infinity:

$$\begin{aligned} u(\underline{h}^T\epsilon\varphi)'' + 3(\underline{h}^T\epsilon\varphi)' - \frac{1}{4}m_u^2(\underline{h}^T\epsilon\varphi) &= 0, \\ u(\underline{h}^\dagger\varphi)'' + 3(\underline{h}^\dagger\varphi)' - \frac{1}{4}m_d^2(\underline{h}^\dagger\varphi) &= 0, \end{aligned} \tag{4.11}$$

where we have identified $m_{u,d} = \tilde{g}_{u,d}\langle\phi\rangle$. The solutions of these equations that vanish at infinity and match the leading $x \gg \rho$ term of $\underline{h}^T\epsilon\varphi_0$ and $\underline{h}^\dagger\varphi_0$ respectively, when $\langle\phi\rangle \to 0$ can be written as

$$\underline{h}^T\epsilon\varphi = \frac{1}{2\pi}\rho m_u^2\frac{K_2(m_u x)}{x^2}U^\dagger\begin{pmatrix}0\\1\end{pmatrix}, \tag{4.12a}$$

and

$$\underline{h}^\dagger\varphi = \frac{1}{2\pi}\rho m_d^2\frac{K_2(m_d x)}{x^2}U^\dagger\begin{pmatrix}1\\0\end{pmatrix}, \tag{4.12b}$$

and consequently

$$u_A = \frac{i}{2\pi}\rho m_u^2\frac{K_1(m_u x)}{x}U^\dagger\begin{pmatrix}0\\1\end{pmatrix}, \tag{4.13a}$$

and

$$d_A = -\frac{i}{2\pi}\rho m_d^2\frac{K_1(m_d x)}{x}U^\dagger\begin{pmatrix}1\\0\end{pmatrix}. \tag{4.13b}$$

We note that u_A, d_A are correctly suppressed by m_u, m_d respectively.

We have shown, therefore, that the zero mode of $\hat{D}_Q$ in the field of the constrained instanton, as in the case of the latter, can be expressed perturbatively in $\rho\langle\phi\rangle$ by means of two series expansions analogous to (3.20) and (3.21). The respective leading terms in these series are given by (4.8a) and (4.13a) for u_A, and (4.8b) and (4.13b) for d_A. For $u_B = -\underline{h}^T\epsilon q_B$ and $d_B = \underline{h}^\dagger q_B$ we have, putting together (4.2),(4.7) and (4.12a,b):

$$\begin{aligned} u_B(x) &= -\frac{1}{\pi}\frac{\rho}{(x^2)^{1/2}(x^2+\rho^2)^{3/2}}x_\mu\overline{\sigma}_\mu U^\dagger\begin{pmatrix}0\\1\end{pmatrix}+\cdots, \\ &= -\frac{1}{2\pi}\rho m_u^2\frac{K_2(m_u x)}{x^2}x_\mu\overline{\sigma}_\mu U^\dagger\begin{pmatrix}0\\1\end{pmatrix}+\cdots, \qquad x \gg \langle\phi\rangle^{-1}, \end{aligned} \tag{4.14a}$$

$$\begin{aligned} d_B(x) &= \frac{1}{\pi}\frac{\rho}{(x^2)^{1/2}(x^2+\rho^2)^{3/2}}x_\mu\overline{\sigma}_\mu U^\dagger\begin{pmatrix}1\\0\end{pmatrix}+\cdots, \\ &= \frac{1}{2\pi}\rho m_d^2\frac{K_2(m_d x)}{x^2}x_\mu\overline{\sigma}_\mu U^\dagger\begin{pmatrix}1\\0\end{pmatrix}+\cdots, \qquad x \gg \langle\phi\rangle^{-1}. \end{aligned} \tag{4.14b}$$

The expressions for the components e_B, e_A, ν_B of the zero mode of $\hat{D}_L$ can readily be obtained from (4.14b), (4.8b,4.13b) and (4.9a) respectively. For the massless neutrino the large distance expansion becomes:

$$\nu_B(x) = -\frac{1}{\pi}\frac{\rho}{x^4}x_\mu\overline{\sigma}_\mu U^\dagger\begin{pmatrix}0\\1\end{pmatrix}+\cdots, \qquad x \gg \langle\phi\rangle^{-1}, \tag{4.14c}$$

with a power law falloff.

Chapter V.

Cross section for anomalous processes

5.1 Fourier transforms of the fermionic zero mode and the constrained instanton

The Fourier transforms we need to compute are of the generic forms

$$\int d^4x\, e^{ip\cdot x} f(x^2) = \frac{4\pi^2}{p} \int_0^\infty dr\, J_1(pr) r^2 f(r^2), \tag{5.1a}$$

$$\int d^4x\, e^{ip\cdot x} x_\mu f(x^2) = 4\pi^2 i \frac{p_\mu}{p^2} \int_0^\infty dr\, J_2(pr) r^3 f(r^2), \tag{5.1b}$$

where $p = (p_\mu p_\mu)^{1/2}$. For high energies, $|p_\mu| \gg \langle\phi\rangle$, we expect the integrals (5.1) to be sensitive to the form of f for small values of r. As we shall see, however, the pole term of these integrals turns out to depend only on the long range tail of f.

The functions f are integrable at the origin and decay exponentially at infinity. We cannot use the perturbative expansion

$$f(x^2) = f_0(x^2;\rho) + (\rho\langle\phi\rangle)^2 \ln \rho\langle\phi\rangle f_1(x^2;\rho) + (\rho\langle\phi\rangle)^2 f_2(x^2;\rho) + \cdots \tag{5.2}$$

in the whole integration range since the functions f_1, $f_2, \ldots$ do not decay to zero at infinity (in fact, all but a finite number of them diverge there). The correct thing to do is to arbitrarily split the integration range into the two intervals $(0, R)$ and (R, ∞) and use (5.2) in the first, but the large distance expansion of f in the second. It is clear that the first integral will have no singularities in the finite complex p-plane. The only singularities in p may come from the integration over the second interval. It is easy to see that if f decays at infinity as $r^\alpha e^{-Mr}$, then the second integral will exist for all p inside the circle of radius M in the complex p-plane but will diverge for $p = \pm iM$, since $J_n(\pm iz) = (\pm i)^n I_n(z)$, n integer, and $I_n(z)$ grows as $e^z/\sqrt{z}$ as $z \to \infty$. This singularity is actually a pole (as expected): in general, in

$$\int_R^\infty dr\, r^{n+1} J_n(pr) f(r),$$

$(n = 1, 2)$, to lowest order in $\rho\langle\phi\rangle$, f will have the form

$$f(r) = f^0(r) \sim \frac{K_n(Mr)}{r^n},$$

(see (3.25,4.13,4.14)) and the relevant integral will be

$$\int_R^\infty dr\, r J_n(pr) K_n(Mr) = \int_0^\infty dr\, r J_n(pr) K_n(Mr) - \int_0^R dr\, r J_n(pr) K_n(Mr),$$

the second integral being again an entire function of p^2 ($K_n(z) \sim z^{-n}$, $J_n(z) \sim z^n$ as $z \to 0$). Ref. [15] gives us

$$\int_0^\infty dr\, r J_n(pr) K_n(Mr) = \frac{p^n}{M^n(p^2 + M^2)}. \tag{5.3}$$

After amputating the propagators for the external particles and putting the latter on mass shell, only the residue of the pole piece of the Fourier transforms survive. These are given by:

$$\left[A_\mu(p)\right]_{amp} = 4\pi^2 i\, \rho^2 p_\nu U \overline{\tau}_{\mu\nu} U^\dagger, \tag{5.4a}$$

$$\left[\eta(p)\right]_{amp} = -2\pi^2\, \rho^2 \langle\phi\rangle, \tag{5.4b}$$

$$\left[\psi_A(p)\right]_{amp} = -2\pi i\, \rho m_f U^\dagger \chi, \tag{5.4c}$$

$$\left[\psi_B(p)\right]_{amp} = 2\pi i\, \rho p_\mu \overline{\sigma}_\mu U^\dagger \chi, \tag{5.4d}$$

where $[f(p)]_{amp} \equiv \lim_{p^2 \to -m^2} (p^2 + m^2) f(p)$ and $\chi = \binom{0}{-1}$ for $\psi = u, \nu$, $\chi = \binom{1}{0}$ for $\psi = d, e$.

5.2 The scattering amplitude for anomalous processes

The ρ-integral appearing in (3.35) can now be explicitly calculated. Collecting all the factors of ρ we have:

$$\int_0^\infty d\rho\, e^{-2\pi^2\kappa\rho^2\langle\phi\rangle^2/g^2}\rho^{6n_f-5+2n+2m+\frac{43-8n_f}{6}} = \frac{1}{2}\left(\frac{g}{2\pi\langle\phi\rangle}\right)^{2t}(\kappa/2)^{-t}\Gamma(t), \qquad (5.5)$$

where

$$t = \frac{19}{12} + n + m + \frac{7}{3}n_f.$$

A note on the applicability of equation (5.5) is in order. Although (5.5) is true for any values of the particle multiplicities n, m, the approximation of neglecting the terms of higher order in $\rho\langle\phi\rangle$ in the action of the constrained instanton (3.27) will be justified only if the dominant contribution to the ρ-integral in (5.5) comes from the region where $\rho\langle\phi\rangle \ll 1$. Now, the function

$$e^{-a(\rho\langle\phi\rangle)^2}(\rho\langle\phi\rangle)^N,$$

($a = 2\pi^2\kappa/g^2 = 2\pi^2/\lambda$, $N = \frac{13}{6}+2n+2m+\frac{14}{3}n_f$) appearing in the ρ-integration (5.5) is peaked at $\rho\langle\phi\rangle = \left(\frac{N}{2a}\right)^{1/2}$. Hence, if we want to treat the constrained instanton in perturbation theory in $\rho\langle\phi\rangle$, we must require that

$$N = \frac{13}{6} + 2n + 2m + \frac{14}{3}n_f \ll \frac{4\pi^2\kappa}{g^2}. \qquad (5.6)$$

The amplitude for a scattering process that violates lepton and baryon numbers by n_f units, involving the least possible number of fermions ($4n_f$) in addition to n gauge bosons and m Higgs particles (n, m satisfying (5.6)), will therefore be equal to

$$T_{n_f,n,m} = i^n(-1)^m c' \left(\frac{\kappa\mu}{2}\right)^{-(\frac{19}{12}+n+\frac{m}{2}+\frac{7}{3}n_f)} \left(\frac{\mu}{\langle\phi\rangle}\right)^{\frac{43}{6}-\frac{4}{3}n_f}$$
$$e^{-8\pi^2/g_\mu^2} g_\mu^{n+m+\frac{14}{3}n_f-\frac{29}{6}} \frac{1}{\langle\phi\rangle^{2n+m+6n_f-4}} \int dU\, h(U), \qquad (5.7)$$

where

$$c' = 1.26 \times 10^6 e^{6.516 n_f} \Gamma(\frac{19}{12} + n + m + \frac{7}{3} n_f), \qquad (5.8)$$

and all the information about spins, polarizations and momenta is encoded in the function h:

$$h(U) = \prod_{i=1}^{4n_f} \frac{\chi_i'^\dagger (p_{i\mu}\overline{\sigma}_\mu - m_i) U^\dagger \chi_i}{\sqrt{2(E_i + m_i)}} \prod_{j=1}^{n} \epsilon_\mu^{(j)}(q_j) q_{j\nu} tr(U \overline{\tau}_{\mu\nu} U^\dagger \mathcal{P}_j). \qquad (5.9)$$

Here χ_i' is a two-component spinor representing the spin of the i-th fermion, the χ_i's were defined in (5.4), and $\mathcal{P}_j$ is the 2×2 matrix that projects out the charge state of the j-th gauge boson (with polarization $\epsilon^{(j)}$).

5.3 High energy behavior of anomalous cross sections and violation of unitarity to leading order

After analytically continuing back to Minkowski space by means of $p_4 = -ip_0$, $p_0 = E > 0$, and performing the U-integration (which enforces charge conservation), one can perform the average and summation over initial and final spins and polarizations respectively of the quantity $\left|\int dU\, h(U)\right|^2$ to obtain

$$|\overline{h}|^2 = \langle \left| \int dU\, h(U) \right|^2 \rangle_{unpol} = \sum c_i \pi_i, \qquad (5.10)$$

where the c_i's are numerical constants and each π_i is a relativistically invariant function of the momenta $p_1, \ldots, p_{4n_f}, q_1, \ldots, q_n$, consisting in a product of factors $k_i \cdot k_j$

and/or $\varepsilon_{\mu\nu\alpha\beta}k_i^\mu k_j^\nu k_l^\alpha k_m^\beta$ ($k = p$ or q), such that each fermion momentum appears once and each gauge boson momentum appears twice.

For a scattering process of the type

$$\overline{q} + \overline{q} \rightarrow n_f l + (3n_f - 2)q + nW + mH, \tag{5.11}$$

the phase space integral for the total unpolarized cross section will be

$$D = \frac{1}{S}\int \prod_{i=1}^{4n_f-2} \frac{d^3p_i}{(2\pi)^3 2E_i} \prod_{j=1}^{n} \frac{d^3q_j}{(2\pi)^3 2E_j} \prod_{l=1}^{m} \frac{d^3k_l}{(2\pi)^3 2E_l} (2\pi)^4 \delta^{(4)}(p_1 + p_2 - P_{out})|\overline{h}|^2, \tag{5.12}$$

where S is the symmetry factor for identical particles in the final state. We can extract the dependence of D on the total energy in the extreme relativistic case ($E = |\vec{p}|$). In terms of $s = (p_1 + p_2)^2$, we can rescale $\vec{p}_i = \sqrt{s}\vec{x}_i$, etc., to obtain

$$D = \tilde{D}(n_f, n, m)\, s^{6n_f+2n+m-4}, \tag{5.13}$$

where

$$\tilde{D}(n_f, n, m) = \frac{1}{S}\int \prod_{i=1}^{4n_f+n+m-2} \frac{d^3\vec{x}_i}{16\pi^3|\vec{x}_i|} \delta(1 - \textstyle\sum|\vec{x}_i|)\delta^{(3)}(\sum\vec{x}_i)|\overline{h}(\vec{x}_1, \ldots, \vec{x}_{4n_f+n-2})|^2. \tag{5.14}$$

To calculate the total cross section for the process (5.11), we need to give a value to the renormalization point μ. Normally this is chosen in order to minimize the error introduced by neglecting higher order corrections of the type $g_\mu^2 \ln s/\mu^2$, which vanish if we set $\mu = \sqrt{s}$. In the case at hand, however, we have two energy scales, $\sqrt{s}$ and $\langle\phi\rangle$, and we expect to get possibly large additional logarithmic corrections of the type $g_\mu^2 \ln\langle\phi\rangle^2/\mu^2$. We choose to set $\mu = \langle\phi\rangle = m_H$. Therefore, large logarithms will be unimportant only up to moderately large s, i.e., if $g_{m_H}^2 \ln s/m_H^2$ stays less than some number of order one.

Our result for the total unpolarized cross section for the process (5.11) in the extreme relativistic case is then

$$\sigma = \frac{c''(n_f,n,m)}{m_H^2}\left(\frac{m_H^2}{m_W^2}\right)^{\frac{19}{6}+2n+m+\frac{14}{3}n_f} e^{-16\pi^2/g_{m_H}^2}(g_{m_H}^2)^{n+m+\frac{14}{3}n_f-\frac{29}{6}}\left(\frac{s}{m_H^2}\right)^{2n+m+6n_f-5}, \tag{5.15}$$

where

$$c''(n_f,n,m) = 7.94\times 10^{11} e^{13.0 n_f}\left[\Gamma(\frac{19}{12}+n+m+\frac{7}{3}n_f)\right]^2 \tilde{D}(n_f,n,m), \tag{5.16}$$

and we have rewritten $\langle\phi\rangle$ as m_H and substituted $\kappa_{m_H} = g_{m_H}^2/\lambda_{m_H} = 2m_W^2/m_H^2$.

From (5.15) we see that the emission of an extra gauge boson or Higgs particle in an anomalous scattering process costs an extra power of g^2 in the cross section, and not an extra factor of g^{-2} as conjectured in Ref. [9]. Therefore, as far as the semiclassical expansion is concerned, the unitarity equation (1.1) for anomalous processes has a chance to work.

However, it is apparent from (5.15) that for s high enough σ can be big even to the point of violating the boundedness requirement imposed by unitarity. Can this unphysical result be attributed to the breakdown of our approximations when s is very large? Only a detailed analysis of higher order corrections can tell us the answer. These should include corrections to the stationary point approximation of the path integral (2.22) in the one instanton sector, as well as contributions from multi-instanton–anti-instanton configurations, treated beyond the dilute gas approximation.

5.4 Conclusions

We have calculated in the semiclassical approximation the energy and coupling constant dependence of cross sections for baryon and lepton number violating processes involving the least possible number of fermions and an arbitrary number of gauge and Higgs bosons.

The calculated cross section displays the unphysical behavior of growing like a power of the total CM-energy squared s, in contradiction with the unitarity-bound.

We must conclude therefore that perturbation theory in the instanton sector breaks down for large energies, although we have no idea as to what the mechanism responsible for this might be. (That perturbation theory may be completely unreliable when high energies are involved was suggested in Ref. [6], in an analogy with the quantum pendulum. If this were so, the most we could expect is that our results are valid only when $\sqrt{s}$ is much less than the height of the potential barrier between topologically distinct vacua, that is, the sphaleron energy $E_{sp} \simeq (1.5\text{–}2.7)4\pi\sqrt{2}v/g = (1.5-2.7)8\pi m_W/g^2$). This breakdown of perturbation theory has nothing to do with the high multiplicity of particles that would be expected in a high energy $(B+L)$-non-conserving process: the unitarity bound is also violated in a purely fermionic anomalous process $(n, m = 0)$.

Our approach, however, is limited by the condition (5.6) to small multiplicities of gauge and Higgs bosons, and cannot be expected to apply to processes involving $\sim 1/\alpha_W$ W's or H's, claimed to be the relevant ones at high energy [9], although it strongly suggests that these processes may be, in fact, unsuppressed in this regime. A different method to calculate amplitudes at high energy and high multiplicity seems to be needed if one wants to draw any reasonable conclusions about the many particle processes that possibly play a role in baryon number violation at high energies.

We would also like to point out, disagreeing with Ref. [8], that our calculation shows that the fermions involved in an anomalous scattering event can be in arbitrary momentum states (consistent with four-momentum conservation), and therefore should carry the corresponding occupation factors that arise due to the Pauli exclusion principle in a given physical situation.

APPENDIX A

For a rotation R in $SO(4)$ we define $SU(2)$ matrices $A(R)$ and $B(R)$ such that a four-vector in R^4, V_μ, transforms as

$$V_\mu \sigma_\mu \quad \xrightarrow{R} \quad V_\mu^R \sigma_\mu = A(R) V_\mu \sigma_\mu B(R)^\dagger, \tag{A.1}$$

where σ_μ is defined in (2.9). ((A.1) is an explicit realization of the group homomorphism $SO(4) \simeq SU(2) \times SU(2)$.) A-type and B-type spinors transform then as

$$\begin{aligned} \psi_A \quad &\xrightarrow{R} \quad A(R)\psi_A, \\ \psi_B \quad &\xrightarrow{R} \quad B(R)\psi_B. \end{aligned} \tag{A.2}$$

One can verify that the fermion bilinears $\psi_A^\dagger \psi_A$, $\psi_B^\dagger \psi_B$ are $SO(4)$ invariants, whereas $\psi_B^\dagger \overline{\sigma}_\mu \psi_A$, $\psi_A^\dagger \sigma_\mu \psi_B$ transform as $SO(4)$-vectors.

APPENDIX B

We want to sketch the derivation of formula (2.18) for

$$Z[\zeta,\overline{\zeta}] = \int \mathcal{D}\Psi\, \mathcal{D}\Psi^\dagger \exp\{-\textstyle\int d^4x(\Psi^\dagger\hat{D}\Psi - \overline{\zeta}\Psi - \Psi^\dagger\zeta)\}. \qquad (2.17)$$

In general, Ψ is a "multi-spinor" like Q or L in (2.11, 2.12), containing both A-type and B-type spinors. The operators $\hat{D}$, $\hat{D}^\dagger$ do not have well-defined eigenvalue problems since they take A-spinors into B-spinors and vice versa. The hermitian operators $\hat{D}^\dagger\hat{D}$ and $\hat{D}\hat{D}^\dagger$, on the other hand, preserve the spinor type and thus have the well-defined eigenvalue problems,

$$\hat{D}^\dagger\hat{D}\psi_n = \lambda_n^2\psi_n, \qquad (B.1)$$

$$\hat{D}\hat{D}^\dagger\varphi_n = \lambda_n^2\varphi_n, \qquad (B.2)$$

their eigenfunctions forming complete orthonormal basis sets for the objects Ψ and $(\Psi^\dagger)^\dagger$ respectively. One can adjust the relative phase of ψ_n and φ_n such that for $\lambda_n \neq 0$ the latter is given by

$$\varphi_n = \frac{1}{\lambda_n}\hat{D}\psi_n. \qquad (B.3)$$

The zero modes of $\hat{D}$ and $\hat{D}^\dagger\hat{D}$ are the same (and therefore this also holds for $\hat{D}^\dagger$ and $\hat{D}\hat{D}^\dagger$):

$$\hat{D}^\dagger\hat{D}\psi_{0i} = 0 \quad \Leftrightarrow \quad \hat{D}\psi_{0i} = 0, \qquad i = 1,\ldots,N, \qquad (\mathrm{B.4})$$

$$\hat{D}\hat{D}^\dagger\varphi_{0i} = 0 \quad \Leftrightarrow \quad \hat{D}^\dagger\varphi_{0i} = 0, \qquad i = 1,\ldots,N'. \qquad (\mathrm{B.5})$$

We expand

$$\Psi(x) = \sum_{i=1}^{N} a_i\psi_{0i}(x) + \sum_{\lambda_n\neq 0} b_n\psi_n(x), \qquad (B.6)$$

$$\Psi^\dagger(x) = \sum_{i=1}^{N'} \overline{a}_i \varphi_{0i}^\dagger(x) + \sum_{\lambda_n \neq 0} \overline{b}_n \varphi_n^\dagger(x), \qquad (B.7)$$

and define the fermionic measure as

$$\mathcal{D}\Psi\mathcal{D}\Psi^\dagger = \prod_{i=1}^{N} da_i \prod_{j=1}^{N'} d\overline{a}_j \prod_n db_n d\overline{b}_n. \qquad (B.8)$$

Upon substituting (B.6,7) into (2.17), the integration over nonzero modes results in

$$\left[Det'(\hat{D}^\dagger \hat{D})\right]^{1/2} \exp \int d^4x d^4y \, \overline{\zeta}(x) S'(x,y) \zeta(y), \qquad (B.9)$$

where

$$Det'(\hat{D}^\dagger \hat{D}) = \prod_{\lambda_n \neq 0} \lambda_n^2, \qquad (B.10)$$

and

$$S'(x,y) = \sum_{\lambda_n \neq 0} \frac{\psi_n(x) \varphi_n^\dagger(y)}{\lambda_n} \qquad (B.11)$$

satisfies

$$\hat{D}_x S'(x,y) = \delta^{(4)}(x-y)\mathbf{1} - P_0(x,y), \qquad (B.12)$$

$$P_0(x,y) = \sum_{i=1}^{N'} \varphi_{0i}(x) \varphi_{0i}^\dagger(y) \qquad (B.13)$$

being the projector onto the subspace spanned by the zero modes of $\hat{D}^\dagger$.

The integrations over the zero modes are

$$\int \prod_{i=1}^{N} da_i \exp\left(\sum_{i=1}^{N} \overline{v}_i(\overline{\zeta}) a_i\right) = \prod_{i=1}^{N} \overline{v}_i(\overline{\zeta}), \qquad (B.14)$$

$$\int \prod_{i=1}^{N'} d\overline{a}_i \exp\left(\sum_{i=1}^{N'} \overline{a}_i u_i(\zeta)\right) = \prod_{i=1}^{N'} u_i(\zeta), \tag{B.15}$$

where

$$u_i(\zeta) = \int d^4x\, \varphi^{\dagger}_{0i}(x)\zeta(x) \qquad i = 1, \ldots, N', \tag{B.16}$$

$$\overline{v}_i(\overline{\zeta}) = \int d^4x\, \overline{\zeta}(x)\psi_{0i}(x) \qquad i = 1, \ldots, N. \tag{B.17}$$

Equation (2.18) is now obtained putting together results (B.9,14,15).

APPENDIX C

Throughout this thesis the 2x2 matrices denoted by the letters σ and τ represent the same numerical matrices, but they act on spin and isospin spaces respectively.

We define

$$\sigma_{\mu\nu} = \frac{1}{4i}\,(\,\overline{\sigma}_\mu\sigma_\nu - \overline{\sigma}_\nu\sigma_\mu\,), \qquad (C.1)$$

$$\overline{\sigma}_{\mu\nu} = \frac{1}{4i}\,(\,\sigma_\mu\overline{\sigma}_\nu - \sigma_\nu\overline{\sigma}_\mu\,), \qquad (C.2)$$

where

$$\begin{aligned} \sigma_\mu &= (\vec{\sigma}, i), \\ \overline{\sigma}_\mu &= (\vec{\sigma}, -i) \equiv \sigma_\mu^\dagger. \end{aligned} \qquad (C.3)$$

Here $\vec{\sigma}$ are the Pauli matrices and the indices μ, ν run from 1 to 4.

$\sigma_{\mu\nu}$ and $\overline{\sigma}_{\mu\nu}$ are antisymmetric in μ, ν. Explicitly they are:

$$\begin{aligned} \sigma_{ij} = \overline{\sigma}_{ij} &= \frac{1}{2}\varepsilon_{ijk}\sigma_k, \\ \sigma_{i4} = -\overline{\sigma}_{i4} &= \frac{1}{2}\sigma_i, \end{aligned} \qquad (C.4)$$

for $i, j = 1, 2, 3$, which also shows that they are traceless. $\sigma_{\mu\nu}$ is selfdual and $\overline{\sigma}_{\mu\nu}$ antiselfdual:

$$\begin{aligned} \sigma_{\mu\nu} &= \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}\sigma_{\alpha\beta}, \\ \overline{\sigma}_{\mu\nu} &= -\frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}\overline{\sigma}_{\alpha\beta}. \end{aligned} \qquad (C.5)$$

There are a number of useful properties:

$$\begin{aligned} \overline{\sigma}_\mu\sigma_\nu &= \delta_{\mu\nu} + 2i\sigma_{\mu\nu}, \\ \sigma_\mu\overline{\sigma}_\nu &= \delta_{\mu\nu} + 2i\,\overline{\sigma}_{\mu\nu}, \end{aligned} \qquad (C.6)$$

$$\begin{aligned}\mathrm{tr}\,(\sigma_{\mu\nu}\sigma_{\alpha\beta}) &= \frac{1}{2}(\delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha}) + \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}, \\ \mathrm{tr}\,(\overline{\sigma}_{\mu\nu}\overline{\sigma}_{\alpha\beta}) &= \frac{1}{2}(\delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha}) - \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}.\end{aligned} \tag{C.7}$$

The matrices $\sigma_{\mu\nu}$ and $\overline{\sigma}_{\mu\nu}$ satisfy the commutation relations of the $so(4)$ Lie algebra, e.g.,

$$[\sigma_{\mu\nu}, \sigma_{\alpha\beta}] = i(\delta_{\mu\alpha}\sigma_{\nu\beta} + \delta_{\nu\beta}\sigma_{\mu\alpha} - \delta_{\mu\beta}\sigma_{\nu\alpha} - \delta_{\nu\alpha}\sigma_{\mu\beta}). \tag{C.8}$$

The 't Hooft symbols $\eta_{a\mu\nu}$ [14] relate the SO(4) generators $\sigma_{\mu\nu}$, $\overline{\sigma}_{\mu\nu}$ to the SU(2) generators $\sigma^a/2$:

$$\begin{aligned}\sigma_{\mu\nu} &= \eta_{a\mu\nu}\frac{\sigma^a}{2}, \\ \overline{\sigma}_{\mu\nu} &= \overline{\eta}_{a\mu\nu}\frac{\sigma^a}{2}.\end{aligned} \tag{C.9}$$

(Recall that SO(4)=SU(2)×SU(2).)

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