THE DECAY OF BOUND MUONS

Thesis by

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The typing was done by Lucille Lozoya and the diagrams are the work of Adelheid von Hohenlohe.

Abstract

The decay rate of bound muons is calculated and numerical results are presented for a number of nuclei up to lead. The decay rate is found to be a monotonically decreasing function of the atomic number, in contrast with recent experimental results.

The muon is represented by a relativistic wave function for a point nucleus with parameters adjusted for finite-nuclear-size effects. The outcoming electron is represented by a Sommerfeld-Maue wave function. The errors involved in these approximations are discussed. An estimate of the error in the electron wave function is obtained by comparing with an exact calculation for the lowest electron angular momentum state.

The spectrum and angular distribution of the electrons are also presented.

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I. INTRODUCTION

The decay of negative muons in matter is complicated by the formation of mu-atoms. However, in a typical situation almost all the decays are from the ground state since both the time to slow down an incident beam and the time of descent to the ground state are negligibly small compared with the muon lifetime.*

The decay of bound muons differs from the free muon decay rate for several reasons: The available energy is less, the initial muon has a different momentum distribution, and there is a strong Coulomb interaction between the outcoming electron and the nucleus. For large nuclei the finite nuclear size is significant.

With the neglect of radiative corrections, which presumably are less than one percent, one should represent the initial muon and the final electron by solutions of the Dirac equation with the electric potential of a finite nucleus. This is rather difficult and we shall find it necessary to approximate both wave functions.

The main results are presented in Table I and in Figures 5-8. The bound muon decay rate is a monotonically decreasing function of

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*For example, in solid matter the muons from a 100 Mev beam descend to the ground state in less than 10^{-9} sec. See Reference 1.

atomic number.* This is in conflict with experiments (2) but in agreement with the result obtained by Überall (3), who used a Born approximation electron wave function to find the lowest order Coulomb correction.

The angular distribution of the electrons is also presented. These results, together with observed asymmetries (4), can be used to determine the degree of polarization of bound muons.

The calculation is outlined in Section II, the muon wave function and the Sommerfeld-Maue (S-M) electron wave function are discussed in Section III, and the body of the calculation and the results are presented in Section IV. In order to obtain an estimate of the error involved in using the Sommerfeld-Maue wave function, we also calculate the partial decay rate to the lowest angular momentum state of the electron and this is described in Section V. Section VI is devoted to estimating the error involved in dropping higher order terms in the S-M calculation. The result for the total decay rate is compared with experiments and the results of other calculations in Sections VII and VIII.

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*These results were presented by J. Mathews at the December 1959 meeting of the American Physical Society at the California Institute of Technology.

II. OUTLINE OF THE CALCULATION

A. General Result

In this section we shall outline the calculation for the spectrum and the angular distribution of the electrons in the process

$$\mu_{bound} \rightarrow e + \nu + \bar{\nu}$$

The matrix element T for bound muon decay is

$$T = \sqrt{8}\, G \int d^3x \;\bar{\psi}_e(\vec{x})\, \gamma_\sigma a \psi_\mu(\vec{x})\;\bar{\psi}_\nu(\vec{x})\, \gamma_\sigma a\, \psi_{\bar{\nu}}(\vec{x})\; 2\pi\delta(Q_4 - p_4 - k - s - t) \tag{1}$$

where the momenta are defined in Figure 1.*

Let us define M by

$$T = \sqrt{8}\; G\; M\; 2\pi\,\delta\,(Q_4 - p_4 - k - s - t) \tag{2}$$

The the total decay rate for polarized muons is

$$R = 8G^2 \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3s}{(2\pi)^3} \int \frac{d^3t}{(2\pi)^3} \sum_{\substack{\text{final}\\ \text{spins}}} |M|^2 (2\pi)^4\, \delta^4(Q-p-k-s-t) \tag{3}$$

We can insert suitable projection operators and take traces to obtain

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*I use the conventions $\hbar = c = 1$ and $x \cdot x = x_4^2 - x_1^2 - x_2^2 - x_3^2$, and the following representation for the γ-matrices:

$$\gamma_4 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \text{ and } \gamma_a = \begin{bmatrix} 0 & \sigma_a \\ -\sigma_a & 0 \end{bmatrix} \qquad \text{for } a = 1, 2, 3.$$

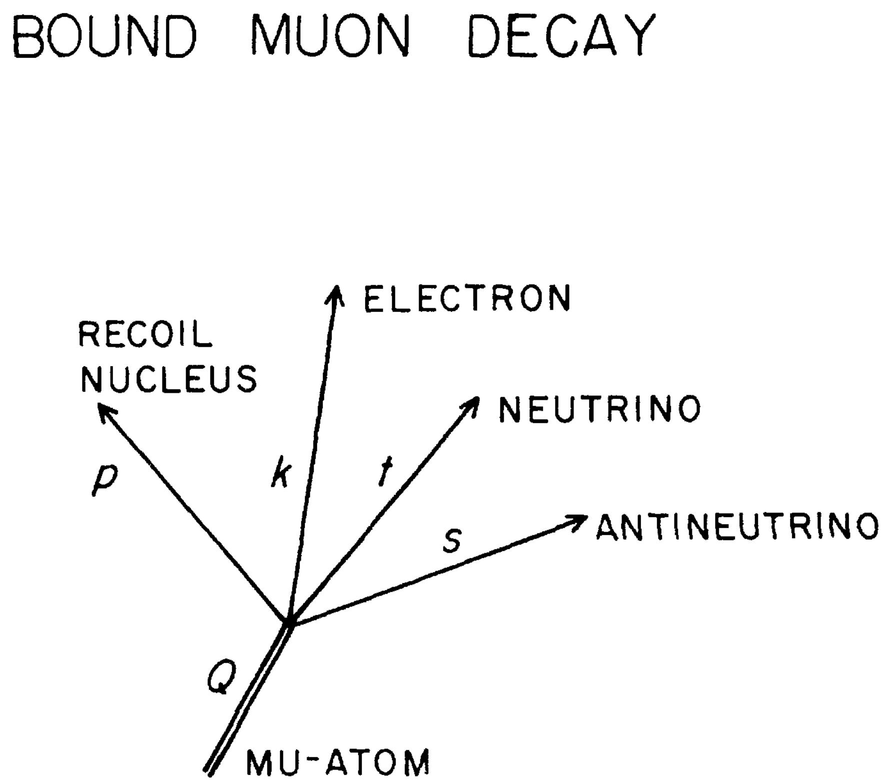
BOUND MUON DECAY
ELECTRON
RECOIL
NUCLEUS
NEUTRINO
p
k
t
s
ANTINEUTRINO
Q
MU-ATOM

Fig. 1

a simpler result.**

The neutrinos are unobserved, so it is convenient to eliminate them from the calculation at an early stage. The neutrino wave functions are

$$\psi_\nu(\vec{x}) = e^{i\vec{t}\cdot\vec{x}}\, u_{\vec{t}} \qquad \psi_\nu(\vec{x}) = e^{-i\vec{s}\cdot\vec{x}}\, v_{\vec{s}} \tag{4}$$

Therefore, using the momentum delta function, we have

$$M = \int d^3x\ \bar{\psi}_e(\vec{x})\, \gamma_\sigma a\, \psi_\mu(\vec{x})\, e^{i(\vec{p}+\vec{k})\cdot\vec{x}}\, (\bar{u}_{\vec{t}}\, \gamma_\sigma a\, v_{\vec{s}}) \tag{5}$$

For convenience, let us also define a quantity N_σ by $M = N_\sigma(\bar{u}_{\vec{t}}\, \gamma_\sigma a\, v_{\vec{s}})$. Then

$$\int \frac{d^3s}{(2\pi)^3} \int \frac{d^3t}{(2\pi)^3} \sum_{\substack{\text{neutrino}\\ \text{spins}}} |M|^2\, (2\pi)^4\ \delta^4(Q-p-k-s-t)$$

$$= N_\sigma \bar{N}_\lambda \int \frac{d^3s}{(2\pi)^3} \int \frac{d^3t}{(2\pi)^3}\ \frac{1}{2s\ 2t}\ \text{Trace}\ \not{t}\, \gamma_\sigma a\, \not{s}\, \gamma_\lambda a (2\pi)^4 \delta^4(Q-p-k-s-t)$$

$$= N_\sigma \bar{N}_\lambda \frac{1}{12\pi}\ (G_\sigma G_\lambda - G^2 \delta_{\sigma\lambda}) \tag{6}$$

The spin projection operator is $a = \frac{1}{2}(1 + i\gamma_5)$ where $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$. The spinor u_o will denote either $\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$ or $\begin{bmatrix}0\\1\\0\\0\end{bmatrix}$. All spinors will where $G = Q - p - k$. We have made use of the following identity (see Appendix A):

$$\int \frac{d^3 s}{(2\pi)^3} \int \frac{d^3 t}{(2\pi)^3} \frac{s \cdot a}{2s} \frac{t \cdot b}{2t} (2\pi)^4 \delta^4(G - s - t)$$
$$= \frac{1}{96\pi} \left(2\, G \cdot a\, G \cdot b + G^2\, a \cdot b\right) \qquad (7)$$

Let us denote the available energy*** $Q_4 - p_4$ by W. Then $G = (W - k, -\vec{p} - \vec{k})$.

The total decay rate for the decay of polarized muons can now be written

$$R = 8 G^2 \int \frac{d^3 k}{(2\pi)^3} \int \frac{d^3 p}{(2\pi)^3} \sum_{\substack{\text{electron} \\ \text{spins}}} N_\sigma \bar{N}_\lambda \frac{1}{12\pi} \left(G_\sigma G_\lambda - G^2 \delta_{\sigma\lambda}\right) \qquad (8)$$

This is the general result into which we shall insert successively better muon and electron wave functions.

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normalized by $u^+_{\vec{k}} u_{\vec{k}} = 1$. I also take $m_\mu = 1$ and let Z denote $Z/137.0$ where Z_e is the atomic number. G is the coupling constant of the weak interaction.

**The neutrinos are left-handed, so the integration over momenta immediately takes care of the sum over spins. The mass of the electron will be neglected so the same thing applies to the electron.

***We shall neglect the motion of the residual nucleus so that it can take up momentum but no kinetic energy. Hence W depends only on the masses and the binding energy.

B. Simple Example

It is instructive to carry through the calculation with simple wave functions to see how things go and especially to see which regions contribute most to the integrals. The simplest wave functions we can choose are

$$\psi_e(\vec{x}) = e^{i\vec{k}\cdot\vec{x}}\, u_{\vec{k}} \qquad \psi_\mu(\vec{x}) = \frac{\mu^{3/2}}{\pi^{1/2}}\, e^{-\mu r}\, u_o \tag{9}$$

where $\mu = Z_m/137.0$ and Z_m is the charge seen by the muon. It is useful to keep Z and μ distinct.

We immediately have

$$N_\sigma = 8\pi^{1/2} \frac{\mu^{5/2}}{(p^2+\mu^2)^2} \left(\bar{u}_k \gamma_\sigma a\, u_o\right) \tag{10}$$

and after the sum over spins we have, for oriented muons,*

$$R = \frac{G^2}{192\pi^3} \frac{256\mu^5}{\pi} \frac{1}{2} \int_{-1}^{+1} d\cos\alpha \int_0^W dp \int_{-1}^{+1} dy \int_0^{k_{max}} dk \frac{p^2 k^2}{(p^2+\mu^2)^4} \times$$

$$\times \Big\{ \left[3W^2 - 4Wk + 2py(W-2k) - p^2\right] \tag{11}$$

$$+ \left[W^2 - 4Wk - 2py(W+2k) - p^2 - 2p^2y^2\right] \cos\alpha \Big\}$$

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*The projection operator for spin up muons is $\frac{1}{2}(1+\gamma_4)\frac{1}{2}(1+i\gamma_1\gamma_2)$.

where $\cos\alpha = \hat{k}\cdot\hat{z}$, $y = \hat{p}\cdot\hat{k}$, and

$$k_{max} = \frac{W^2 - p^2}{2(W + py)} \tag{12}$$

The muon spin is along $\hat{z}$ so the term dependent $\cos\alpha$ gives the angular distribution of electrons. The trivial integrations have been done and the shape of the remaining region in phase space is shown in Figure 2. The region of integration depends, of course, only on the kinematics and will be the same for all subsequent integrations. The kinematics are treated in Appendix B. The traces which arise in this problem are given in Appendix F. It is simpler first to contract $N_\sigma \bar{N}_\lambda (G_\sigma G_\lambda - G^2 \delta_{\sigma\lambda})$ and then to perform the traces.

For small μ the p-integral peaks sharply near $p = \mu/\sqrt{3}$ so to a first approximation we can simply take $p = \mu/\sqrt{3}$. When we do this in the radial integral of N_σ,

$$4\pi \int dr\, r^2\, e^{-\mu r}\, \frac{\sin pr}{pr}$$

we find that about .7 of the integral comes from the region $1 < r\mu < 3$, that is, from between one and three muon Bohr radii.

In general, when expanding various results for small μ, we must remember that p is of the same order as μ.

The remaining integrals in Equation 11 can be performed analytically for this simple case and the result is

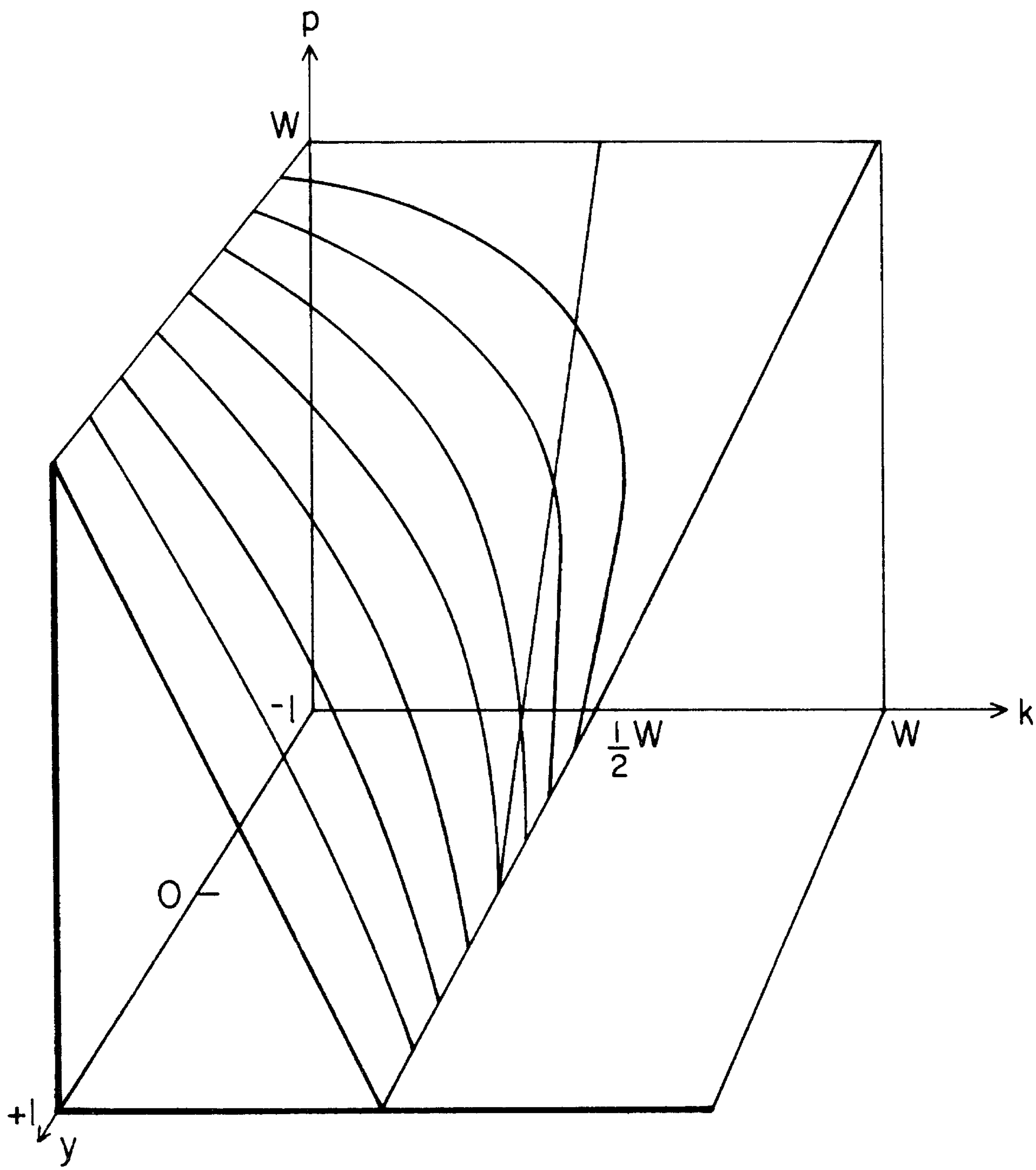

Fig. 2

The region of integration in (k, p, y) space. It is bounded on the right by the surface $k_m = \frac{w^2 - p^2}{(2(w+py)}$.

$$R = \frac{G^2}{192\pi^3} W^5 \frac{2}{3\pi}\left[\frac{\lambda}{1+\lambda^2} (3 - 4\lambda^2 - 15\lambda^4) + 3(1 - 2\lambda^2 + 5\lambda^4)\tan^{-1}\frac{1}{\lambda} \right] \tag{13}$$

where $\lambda = \frac{\mu}{W}$. For small μ we have

$$R \approx \frac{G^2}{192\pi^3} W^5 (1 - 2\mu^2) \tag{14}$$

and, of course, $W \approx 1 - \frac{1}{2}\mu^2$. This result is well known (5). The first factor, $\frac{G^2}{192\pi^3}$, is just the decay rate for free muons.

III. WAVE FUNCTIONS

Now we shall obtain more realistic wave functions for the muon and for the electron. In both cases, it is still necessary to use approximate wave functions.

A. Muon Wave Function

The muons decay from their ground state: the state of $j = 1/2$ and positive parity. Any $1/2^+$ state can be written in the form

$$\psi_\mu(\vec{x}) = \frac{1}{(4\pi)^{1/2}} \left[g(r) + f(r)\, i\, \vec{\alpha}\cdot\hat{x} \right] u_o \tag{15}$$

For the case of a point nucleus we have*

*See Reference 6, but note that equation 10.13 is incorrect and there is an error in sign in the subsequent work on the Dirac equation with a central potential.

$$g(r) = (1+\gamma)^{1/2} \frac{2^{\gamma}}{\left[\Gamma(2\gamma+1)\right]^{1/2}} \mu^{3/2} e^{-\mu r} (\mu r)^{\gamma-1}$$

$$f(r) = \left(\frac{1-\gamma}{1+\gamma}\right)^{1/2} g(r) \tag{16}$$

with $\gamma = (1-\mu^2)^{1/2}$.

Even for the lightest elements the singularity at the origin is not realistic and for heavier elements the finite size of the nucleus becomes important. For example, in iron the nuclear radius is about one-half the muon Bohr radius, so that the effect of a finite nucleus must be considered.

For the purposes of this calculation it was found convenient to take $g(r)$ and $f(r)$ of the form

$$g(r) = \frac{2}{(1+\Lambda^2)^{1/2}} \mu^{3/2} e^{-\mu r}, \quad f(r) = \Lambda\, g(r) \tag{17}$$

It is necessary to have a relatively simple analytic expression for the radial wave functions so that the integrals remain tractable. The constants μ and Λ are then adjusted to obtain the best fit to exact solutions of the Dirac equation with the potential of a uniformly charged finite nucleus.* This approximation is discussed in Section VI-E.

The radial Dirac equations for $G(r) = rg(r)$ and $F(r) = r f(r)$ are

*The nuclear radius is obtained from $R = R_o A^{1/3}$ with $R_o = 1.2 \times 10^{-13}$ cm. In our units, $R_o = 0.6425$.

$$\left[\frac{d}{dr} - \frac{1}{r} \right] G(r) + \left[W+1 - V(r) \right] F(r) = 0$$

$$\left[W-1 - V(r) \right] G(r) - \left[\frac{d}{dr} + \frac{1}{r} \right] F(r) = 0 \tag{18}$$

with

$$V(r) = \begin{cases} -\frac{Z}{R} \left[\frac{3}{2} - \frac{1}{2} \left(\frac{r}{R}\right)^2 \right] & r \leq R \\ -\frac{Z}{r} & r > R \end{cases}$$

These equations were solved numerically on the Caltech Burroughs 205 and 220 computers.

From our empirical μ we can define an effective charge $Z_m = 137.0\ \mu$. For instance, at $Z_e = 26$, $Z_m = 24.2$. Other values are listed in Table I on page 23.

One can obtain a slightly better approximate wave function by using a different effective charge for the small component, since in a finite nucleus the small component starts off with one higher power of r than the large component, and so the peak is slightly displaced. However, the effect of the small component on the total rate is only about 3% at $Z_e = 26$, so this improvement is not necessary.

The eigenvalues used here can be compared with values obtained from other muon calculations. Ford and Wills (7) calculate the 1S energy level using a complicated potential which has been adjusted to give a good fit to electron scattering data. For Z = 26 and Z = 82, we have (using m_μ = 105.68 Mev)

Z	1S level (Ford and Wills)	1S level (Table I)
26	1.705 Mev	1.783 Mev
82	10.594 Mev	10.483 Mev

The difference is clearly a small fraction of the total energy and can be disregarded.

Figures 3 and 4 show exact and empirical muon radial wave functions for the cases Z = 26 and Z = 82.

B. The Electron Wave Function

The Dirac equation with a Coulomb potential cannot be solved in closed form for scattering states. Since the decay of bound muons proceeds through many angular momentum states of the outcoming electron, each of which is not especially easy to handle, it would be very nice to have a fairly simple approximate solution in closed form. Sommerfeld and Maue have worked out very useful wave functions (8, 9). They are most easily arrived at in the following way.

Let us start with the Dirac for a massless electron in the field of a central potential V(r),

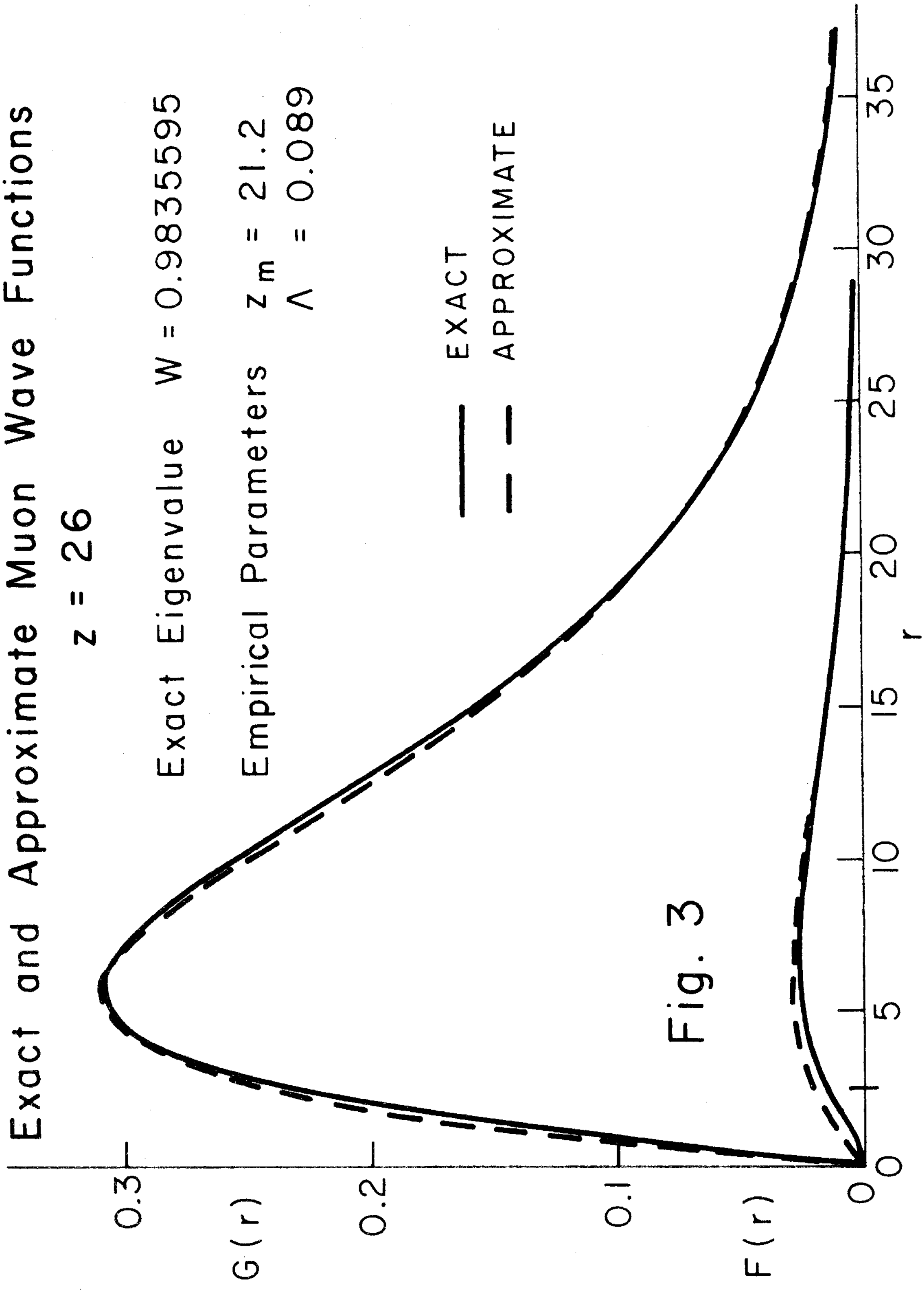
Exact and Approximate Muon Wave Functions
z = 26
Exact Eigenvalue W = 0.9835595
Empirical Parameters z m = 21.2
Λ = 0.089
EXACT
APPROXIMATE
G(r)
F(r)
0.3
0.2
0.1
0
0
5
10
15
20
25
30
35
r

Fig. 3

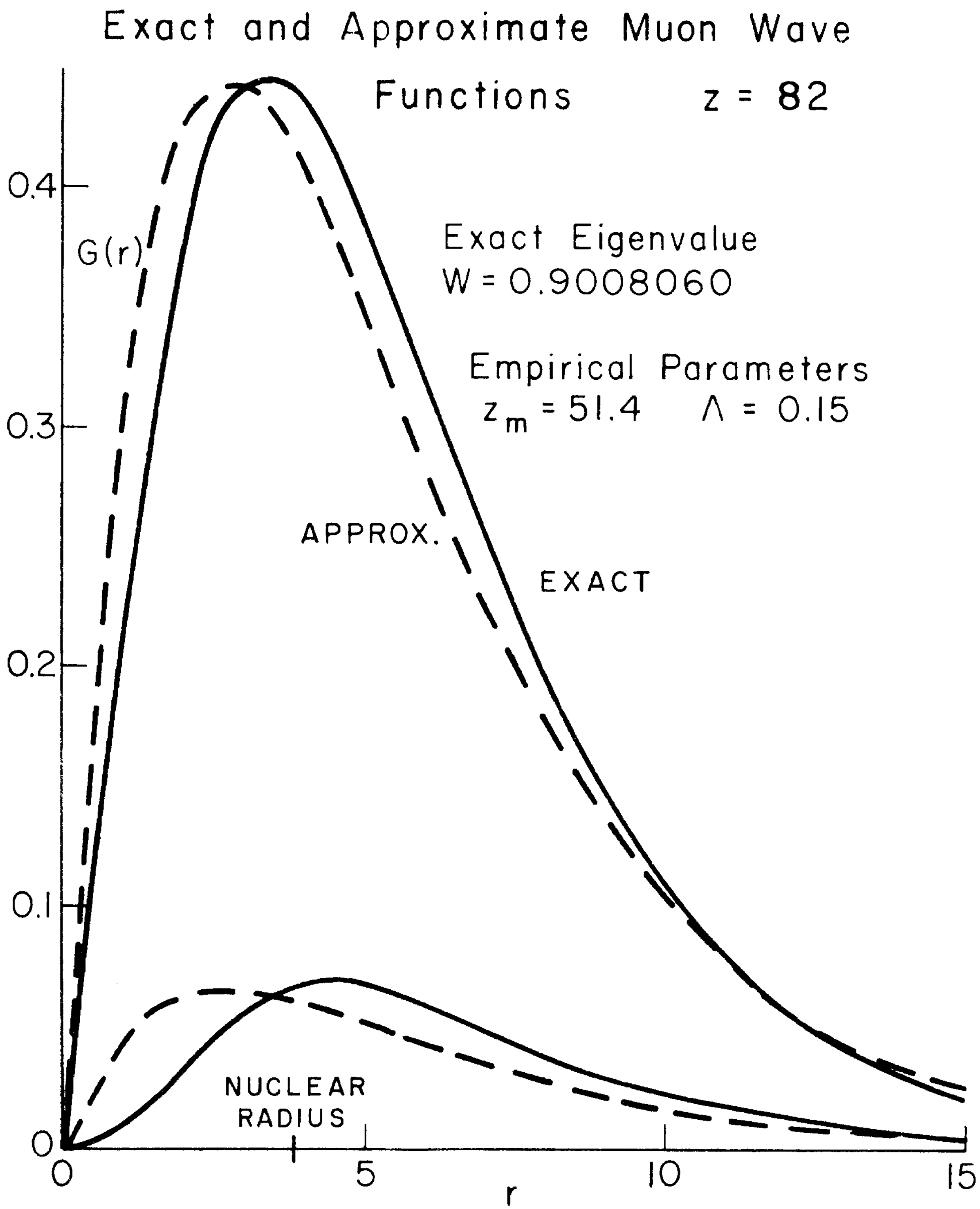
Exact and Approximate Muon Wave
Functions z = 82
0.4
0.3
0.2
0.1
0
G(r)
Exact Eigenvalue
W = 0.9008060
Empirical Parameters
z_m = 51.4 Λ = 0.15
APPROX.
EXACT
NUCLEAR
RADIUS
0
5
10
15
r

Fig. 4

$$(k - V + i\vec{\alpha}\cdot\vec{\nabla})\ \psi_e(\vec{x}) = 0 \tag{18}$$

Now let

$$\psi_e(\vec{x}) = \frac{1}{2k}(k - V - i\vec{\alpha}\cdot\vec{\nabla})\,\phi(\vec{x})$$

Then $\phi(\vec{x})$ satisfies

$$\left[(k - V)^2 + \nabla^2 - i\vec{\alpha}\cdot(\vec{\nabla}V)\right]\ \phi(\vec{x}) = 0 \tag{19}$$

The Coulomb potential is $-Z/r$, so we can write

$$\left[\nabla^2 + k^2 + 2k\frac{Z}{r}\right]\phi = -\left[i\,\vec{\alpha}\cdot(\vec{\nabla}\frac{Z}{r}) + \frac{Z^2}{r^2}\right]\phi \tag{20}$$

Consider this equation with the right-hand side set equal to zero. A solution of this truncated equation, with incoming scattered waves, is

$$\phi_o(\vec{x}) = e^{i\vec{k}\cdot\vec{x}}\,F\left(-iZ\,;\,1\,;\,-i(kr + \vec{k}\cdot\vec{x})\right) \tag{21}$$

The function $F(a;b;x)$ is the confluent hypergeometric function.*

It is now easy to show that the function

$$\chi(\vec{x}) = e^{i\vec{k}\cdot\vec{x}}\left(1 + \frac{1}{2k}\,i\vec{\alpha}\cdot\vec{\nabla}\right)F\,u_{\vec{k}} \tag{22}$$

is a solution of Equation 20, with the neglect of the terms

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*See Appendix D for a discussion of these functions.

$$-e^{i\vec{k}\cdot\vec{x}} \left\{ \frac{Z^2}{r^2} + \left[i\vec{\alpha}\cdot\vec{\nabla}\left(\frac{Z}{r}\right) + \frac{Z^2}{r^2} \right] \frac{1}{2k}\, i\vec{\alpha}\cdot\vec{\nabla} \right\} F\, u_{\vec{k}} \quad . \tag{23}$$

The arguments of F are the same as in Equation 21. The approximate solution to the Dirac equation is now

$$\psi(\vec{x}) = e^{i\vec{k}\cdot\vec{x}} \left(1 - \frac{1}{2k}\, i\vec{\alpha}\cdot\vec{\nabla}\right) F\, u_{\vec{k}} \tag{24}$$

$$+ \; e^{i\vec{k}\cdot\vec{x}}\, \frac{Z}{r}\, \frac{1}{4k^2}\, i\vec{\alpha}\cdot\vec{\nabla} F\, u_{\vec{k}}$$

The first part, aside from a normalizing factor, is the celebrated Sommerfeld-Maue wave function. This is the wave function that we shall use. Correctly normalized, it is

$$\psi_{SM}(\vec{x}) = e^{\frac{\pi}{2} Z}\, \Gamma(1 + iZ)\, e^{i\vec{k}\cdot\vec{x}} \left(1 - \frac{1}{2k}\, i\vec{\alpha}\cdot\vec{\nabla}\right) F\, u_{\vec{k}} \tag{25}$$

It is easy to show that $F(-iZ; 1; i(kr + \vec{k}\cdot\vec{x}))$ is $O(Z/r)$ asymptotically and $O(Zk)$ at the origin. In bound muon decay, the spectrum of the outcoming electrons has a maximum at $k \approx \frac{1}{3}$, and the integrand in the matrix element has a maximum at $r \approx \frac{2}{\mu}$. Then, roughly speaking, we can say that the significant kr is about $kr = .7/\mu$. Since even for lead, $Z_e = 82$, the empirical μ is less than 0.7, we can treat ∇F as being $O(Z/r)$ for the purposes of this problem.

The terms neglected in Equation 20, i.e., line 23, are then

$O(Z^2/(kr)^2)$ compared with the leading term. Since, at least for small Z, the spatial region of importance is in the neighborhood of two muon Bohr radii, $O(Z^2/(kr)^2)$ is roughly equivalent to $O(Z^4/k^2)$ for the purpose of bound muon decay.

In choosing the Sommerfeld-Maue wave function, we have also dropped the last term in Equation 24. However, in the region of interest, it is also $O(Z^2/(kr)^2)$.

The normalization of the wave function depends only on the leading term because all the other terms vanish in the limit of large distances.

The wave function $\psi_{SM}(\vec{x})$ was obtained by approximating the Dirac equation for an electron in the field of a point nucleus but fortunately it turns out to be a slightly better approximation for an electron in the field of a finite nucleus. Finite size effects will be estimated later by expanding $\psi_{SM}(\vec{x})$ in angular momentum states and comparing with the exact solution.

We have been careful to select an electron state with incoming scattered waves. However, for calculating the decay rates it is also permissible to use a state with outgoing spherical waves.

$$\psi_a(\vec{x}) = \Gamma(1 - iZ)\, e^{\frac{\pi}{2} Z}\, e^{i\vec{k}\cdot\vec{x}} \left(1 - \frac{1}{2k} i\vec{\alpha}\cdot\vec{\nabla}\right) F\left(iZ; 1; i(kr - \vec{k}\cdot\vec{x})\right) u_{\vec{k}} \tag{26}$$

Now

$$N_\sigma = \int d^3x\ \bar{\psi}_e(\vec{x})\ \gamma_\sigma\ a\ \psi_\mu(\vec{x})\ e^{i(\vec{p}+\vec{k})\cdot\vec{x}}$$

$$= \Gamma(1-iZ)\ e^{\frac{\pi}{2}Z} \int d^3x\ \bar{u}_{\vec{k}} \left(1 - \frac{1}{2k} i\vec{\alpha}\cdot\vec{\nabla}\right) F\left(iZ; 1; i(kr + \vec{k}\cdot\vec{x})\right) \times$$

$$\times\ \gamma_\sigma\ a\ \frac{\mu^{3/2}}{\pi^{1/2}}\ e^{-\mu r}\ (1 + \Lambda\ i\vec{\alpha}\cdot\hat{x})\ u_o\ e^{i\vec{p}\cdot\vec{x}} \tag{27}$$

If we put $\psi_a(\vec{x})$ for $\psi_e(\vec{x})$ in $\bar{N}_\lambda$ we have

$$\bar{N}_\lambda = \int d^3x\ \bar{\psi}_\mu(\vec{x})\ \gamma_\lambda\ a\ \psi_a(\vec{x})\ e^{-i(\vec{p}+\vec{k})\cdot\vec{x}}$$

$$= \Gamma(1-iZ)\ e^{\frac{\pi}{2}Z} \int d^3x\ e^{-i\vec{p}\cdot\vec{x}}\ \frac{\mu^{3/2}}{\pi^{1/2}}\ e^{-\mu r}\ \bar{u}_o(1+\Lambda\ i\vec{\alpha}\cdot\hat{x})\ \gamma_\lambda\ a \times$$

$$\times \left(1 - \frac{1}{2k} i\vec{\alpha}\cdot\vec{\nabla}\right) F\left(iZ; 1; i(kr + \vec{k}\cdot\vec{x})\right) u_{\vec{k}} \tag{28}$$

Now put $\vec{x} \rightarrow -\vec{x}$ in $\bar{N}_\lambda$. The integral is now, except for a factor,

$$\int d^3x\ e^{-\mu r}\ \bar{u}_o(1-\Lambda\ i\vec{\alpha}\cdot\hat{x})\ \gamma_\lambda\ a\left(1 + \frac{1}{2k}\ i\vec{\alpha}\cdot\vec{\nabla}\right) F\ \ u_{\vec{k}} \tag{29}$$

where F has the same arguments as in Equation 27. Now let's transpose the spinors in line 29:

$$\int d^3x\ e^{-\mu r}\ u_{\vec{k}}^T \left(1 + \frac{1}{2k} i\ \vec{\alpha}^T\cdot\vec{\nabla}\right) F \times$$

$$\times\ (\gamma_\lambda\ a)^T (1 - \Lambda\ i\ \vec{\alpha}^T\cdot\hat{x})\ u_o \tag{29.1}$$

But $\gamma_\mu^{\ T} = \sigma_y \gamma_\mu \sigma_y$, so we have

$\vec{\alpha}^{\,T} = - \sigma_y \vec{\alpha} \sigma_y$ and $(\gamma_\lambda a)^T = - \sigma_y \gamma_\lambda a \sigma_y$

and finally we get

$$\int d^3x \quad e^{-\mu r} \ \bar{w}_{\vec{k}} (1 - \frac{1}{2k} i \vec{\alpha} \cdot \vec{\nabla}) F \gamma_\lambda a (1 + \Lambda i \vec{\alpha} \cdot \hat{x}) w_o \quad . \tag{29.2}$$

where the w spinors have spin down when the u spinors are spin up. After we sum over spins, this makes no difference and, since we can also interchange σ and λ , we find that we get the same result with $\psi_a(\vec{x})$.

IV. CALCULATION AND RESULTS

To obtain the rate of decay, it now remains to do the space integrals in N_σ , to sum over electron spins, and to do the momentum integrals.

The space integrals can be performed anayltically, and for muons with spin along the z-axis, after summing over electron spins, we are left with an expression of the form

$$R = \frac{1}{2} \int d\cos\alpha \int dk \int dp \int dy \quad \times$$

$$\times \left[R(k,p,y) + S(k,p,y) \cos\alpha \right] \tag{30}$$

To obtain the spectrum, angular distribution, and total rate, the three non-trivial integrations over k, p, and y were done numerically. A method of working out the integrals is presented in Appendix E and the numerical work is discussed in Appendix C.

We dropped a number of terms and to make this clear let us make the following definitions. Let

$$\psi_G(\vec{x}) = e^{\frac{\pi}{2} Z} \Gamma(1+iZ)\, e^{i\vec{k}\cdot\vec{x}} F(-iZ;\, 1;\, i(kr+\vec{k}\cdot\vec{x}))\, u_{\vec{k}} \tag{31}$$

where the subscript G stands for Gordon, who first used this wave function. If we also let $\psi_o(\vec{x})$ denote the non-relativistic muon wave function times a suitable spinor, u_o, we can write

$$\begin{aligned} \psi_{SM} &= \psi_G + \Delta\,\psi_G \\ \psi_\mu &= \psi_o + \Delta\,\psi_o \end{aligned} \tag{32}$$

We now have

$$\begin{aligned} N_\sigma = \int d^3x\, \Big[\, &\bar{\psi}_G\, \gamma_\sigma\, a\, \psi_o + \Delta\,\bar{\psi}_G\, \gamma_\sigma\, a\, \psi_o \\ &+ \bar{\psi}_G\, \gamma_\sigma\, a\, \Delta\,\psi_o + \Delta\,\bar{\psi}_G\, \gamma_\sigma\, a\, \Delta\,\psi_o \Big]\, e^{-(\vec{p}+\vec{k})\cdot\vec{x}} \end{aligned} \tag{33}$$

$$N_\sigma \equiv M_1 + M_2 + M_3 + M_4$$

In the results presented in Table I and the figures, M_4, the last piece in N_σ, is neglected, although it has been calculated to lowest order in Z^{-1}(See Appendix G). In the square of the matrix element we also neglected all terms which involved the product of two small terms. That is, we kept the square of the leading term and cross-terms with the leading term. The validity of the approximation will be discussed later.

Let $R_1(k,p,y)$ and $S_1(k,p,y)$ refer to the leading term in the rate and angular distribution, i.e., the term arising from M_1 alone.

$$R_1(k,p,y) + S_1(k,p,y)\cos\alpha =$$

$$\frac{G^2}{192\pi^3}\ \frac{256\mu^5}{\pi}\ \frac{\pi Z}{\sinh \pi Z}\ \frac{k^2p^2}{(p^2+\mu^2)^4}\ \frac{e^{2Z\tan^{-1}\frac{Q^2}{2k\mu}}}{Q^4+(2k\mu)^2} \times$$

$$\times \left\{ \left[Q^2 + \frac{Z}{\mu}k(p^2-\mu^2)\right]^2 + \left[2\mu k + Z\ 2\ pky\right]^2 \right\} \times \tag{34}$$

$$\times \left\{ \left[3W^2 - 4Wk + 2\ py\ (W-2k) - p^2\right] \right.$$

$$\left. + \left[W^2 - 4Wk - 2py\ (W+2k) - p^2 - 2p^2y^2 \right] \cos\alpha \right\}$$

where $Q^2 = 2pky + p^2 + \mu^2$.

Now we add the corrections arising from the cross terms of the leading term in the matrix element with M_2, the piece involving $\Delta\psi_G$ and ψ_o.

$$\frac{G^2}{192\pi^3}\ \frac{256\mu^5}{\pi}\ \frac{\pi Z}{\sinh \pi Z}\ \frac{k^2p^2}{(p^2+\mu^2)^4}\ \frac{e^{2Z\tan^{-1}\frac{Q^2}{2k\mu}}}{Q^4+(2k\mu)^2}\ \times$$

$$\times \Big[\ [2\mu k + Z\,2pky]\ Z\,(p^2+\mu^2)\ \times$$

$$\times \Big\{ [3W^2 - 4Wk + 2py(W-2k) - p^2]$$

$$+ [W^2 - 4Wk - 2py(W+2k) - p^2 - 2p^2y^2]\cos\alpha \Big\}$$

$$+ \Big[Q^2 + \frac{Z}{\mu}k(p^2-\mu^2)\Big]\ \frac{Z}{\mu}(p^2+\mu^2)\ \times \tag{35}$$

$$\times \Big\{ [3W^2 - 4Wk + 2py(W-2k) - p^2]\,py$$

$$+ 2(W-k)\,p^2(1-y^2)$$

$$+ [W^2 - 4Wk - 2py(W+2k) - p^2 - 2p^2y^2]\cos\alpha$$

$$- 2p^2(k+py)(1-y^2)\cos\alpha \Big\}\ \Big]$$

The following are the additional corrections to $R_1(k,p,y)$ and $S_1(k,p,y)$ arising from the cross-terms of the leading term with M_3, the term involving ψ_G and $\Delta\psi_o$.

$$\frac{G^2}{192\pi^3}\ \frac{256\mu^5}{\pi}\ \frac{\pi Z}{\sinh \pi Z}\ \frac{k^2p^2}{(p^2+\mu^2)^4}\ \frac{e^{2Z\tan^{-1}\frac{Q^2}{2k\mu}}}{Q^4+(2k\mu)^2}\ \times$$

$$\times\Big[\,2\Lambda\frac{p}{\mu}\Big\{\Big[Q^2+\frac{Z}{\mu}k(p^2-\mu^2)\Big]\Big[Q^2-Z\,2\mu k\Big]$$

$$+\Big[2\mu k+Z\,2pky\Big]^2\Big\}\ \times$$

$$\times\Big\{\Big[(W^2-4Wk)y-2p(W+2ky^2)-3p^2y\Big] \tag{36}$$

$$+\Big[3W^2-4Wk+2py(W-2k)-p^2\Big]y\cos\alpha\Big\}$$

$$-\ 2\Lambda\frac{k}{\mu}\Big[2\mu k+Z\,2pky\Big]Z\,(p^2+\mu^2)\ \times$$

$$\times\Big\{\Big[W^2-4Wk-2py(W+2k)-p^2-2p^2y^2\Big]$$

$$+\Big[3W^2-4Wk+2py(W-2k)-p^2\Big]\cos\alpha\Big\}\Big]$$

The small component of ψ_μ has been neglected in the normalization. This introduces an error of less than one percent at $Z_e = 26$ and about two percent at $Z_e = 82$. Recall that the parameters in the muon wave function are empirical and, for instance, Λ for $Z_e = 82$ is 0.15. In fact, this improves the result. This will be discussed later.

The final results are presented in Table I and Figures 5-8. The rates R_G and R_{SM} are obtained by using ψ_G and ψ_{SM}, respectively, for the electron wave function. Note that the difference between using

Table I.

Bound Muon Decay Rate

The decay rates R_G and R_{SM} were obtained by using ψ_G and ψ_{SM}, respectively, for the electron wave function, and are given in units of the free muon decay rate. The total available energy W is units of the muon mass. Z_m and Λ are the empirical parameters of the muon wave function.

Z_e	W	Z_m	Λ	R_G	R_{SM}
0	1.0000	0	0	1.000	1.000
16	.9934	15.6	.057	-	.991
26	.9836	24.2	.089	.889	.962
35	.9723	30.8	.10	.821	.928
50	.9507	39.6	.14	.703	.853
82	.9008	51.4	.15	.493	.684

FIGURE CAPTIONS

Fig. 5 The decay rate of bound muons in units of free decay rate. The rates R_G and R_{SM} are obtained by using ψ_G and ψ_{SM}, respectively, for the electron wave function.

Fig. 6 The spectrum of decay electron from bound muons in iron ($Z_e = 26$). The electron is represented by an S-M wave function. The free muon decay spectrum is shown for comparison. The energy k is in units of the muon mass.

Fig. 7 The integrated asymmetry parameter of decay electrons from bound muons. The asymmetry parameter S is defined by $R(\alpha) = R + S \cos \alpha$. For free muon decay S is $-\frac{1}{3}$. The subscripts G and SM have the same meaning as in Fig. 5.

Fig. 8 The asymmetry parameter of decay electrons from bound muons in iron ($Z_e = 26$). The electron is represented by an S-M wave function. The free muon asymmetry parameter is shown for comparison. The scale is the same as in Fig. 6.

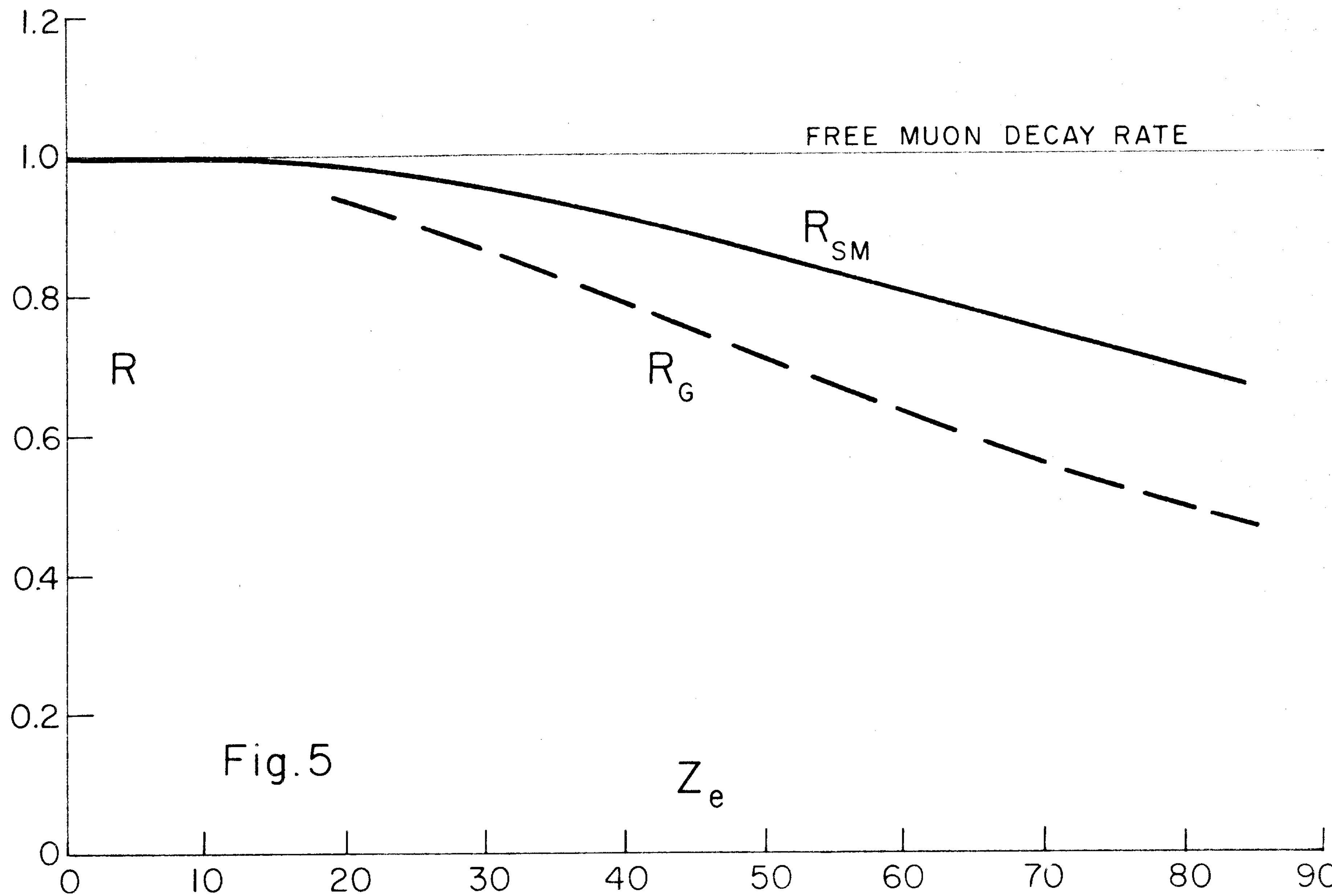

FREE MUON DECAY RATE
R
R_SM
R_G
Z_e
1.2
1.0
0.8
0.6
0.4
0.2
0
10
20
30
40
50
60
70
80
90

Fig. 5

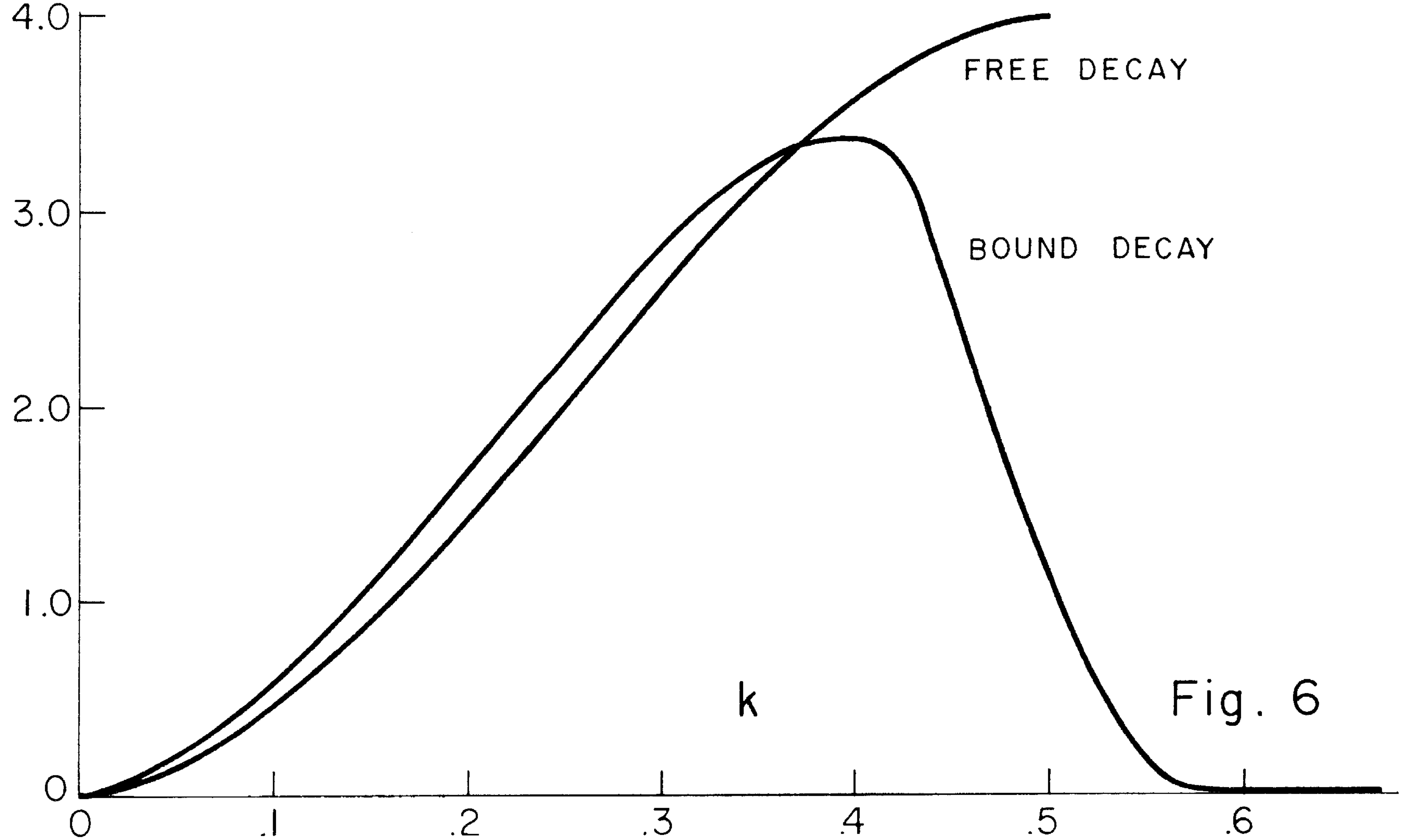


Fig. 6

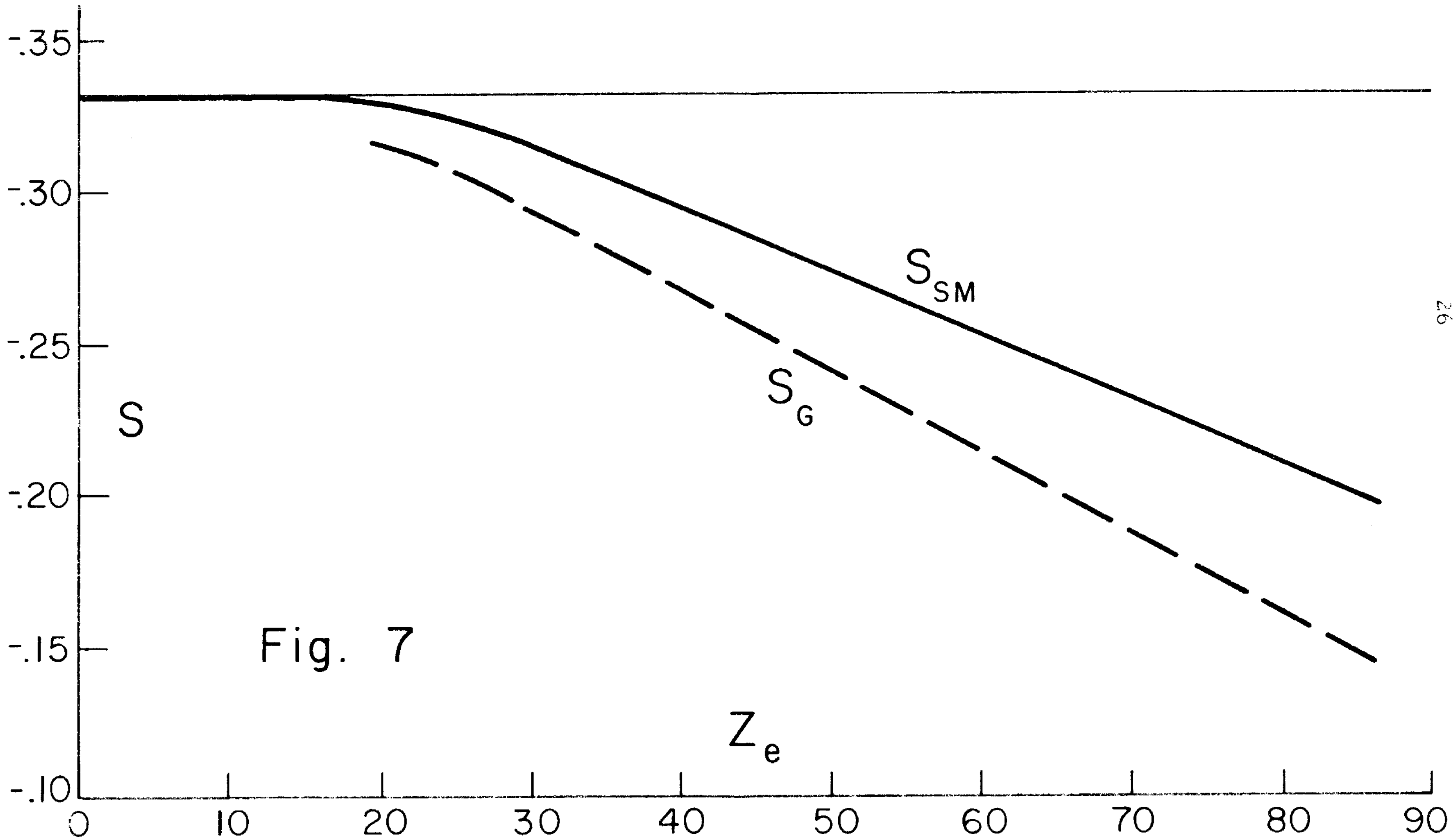


Fig. 7

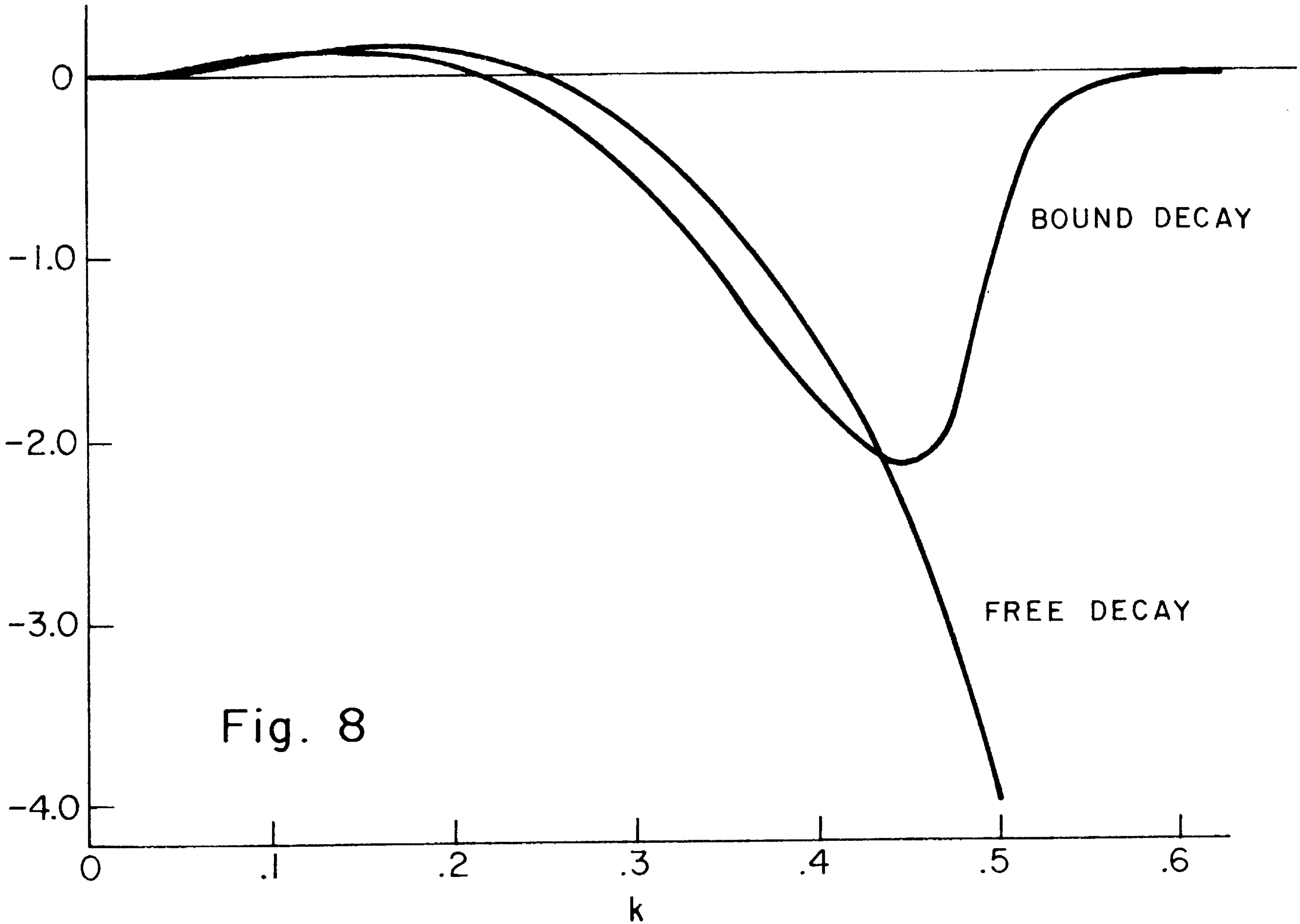


Fig. 8

ψ_G and ψ_{SM}, or the effect of $\Delta\psi_G$, is about 7% in the rate at $Z_e = 26$. The relativistic correction $\Delta\psi_o$ to the muon wave function produced about a 3% change in the rate at $Z_e = 26$.

In the square of the matrix element, we have kept only

$$|M_1|^2 + 2\,\mathrm{Re}\, M_1 M_2 + 2\,\mathrm{Re}\, M_1 M_3 .$$

The other terms will be discussed later. Another source of error is our approximate electron wave function. The next section is devoted to providing a reliable estimate of this error.

V. PARTIAL RATE

Since the Dirac equation separates in spherical coordinates, the muon decay problem can be solved exactly for a given angular momentum state of the electron. We can also pick out the corresponding piece of the S-M wave function, perform the same calculation, and then compare the results.

A. Expansion of ψ_{SM}

Let us restrict our attention to the positive parity $j = \frac{1}{2}$ state, since the error will be greatest in the lowest angular momentum state. Any such state can be written in the form

$$\psi_{\frac{1}{2}}+(\vec{x}) = \frac{1}{(4\pi)^{1/2}} \left[a(r) - b(r)\ \vec{\alpha}\cdot\hat{x} \right] u_o \tag{37}$$

where the radial functions remain to be determined.

The expansion of ψ_{SM} is simplified by the use of the well-known identity proved by Gordon (10):

$$e^{\frac{\pi}{2} Z}\,\Gamma(1 + iZ)\, e^{ikz}\, F\,(-iZ; 1; -i(kr + \vec{k}\cdot\vec{x}))$$

$$= (4\pi)^{1/2} \sum_{\ell} (2\ell + 1)^{1/2}\, i^{\ell}\, L_{\ell}(r)\, Y_{\ell o}(\hat{x}) \quad , \tag{38}$$

where

$$L_{\ell}(r) = e^{\frac{\pi}{2} Z} \frac{(\ell+1+iZ)}{(2\ell+1)!} (2kr)^{\ell}\, e^{-ikr}\, F(\ell+1+iZ; 2\ell+2; 2ikr)$$

This immediately gives the expansion of ψ_G. The second part, $\Delta\,\psi_G$, is rewritten as

$$-\frac{1}{2k}\,(\vec{\alpha}\cdot\vec{k} + i\,\vec{\alpha}\cdot\vec{\nabla})\, e^{\frac{\pi}{2} Z}\,\Gamma(1+iZ)\, e^{i\vec{k}\cdot\vec{x}}\, F(-iZ; 1; -i(kr+\vec{k}\cdot\vec{x}))\, u_{\vec{k}} \tag{39}$$

We then use the Gordon identity, apply the gradient in spherical form, and then make use of some of the identities connecting confluent hypergeometric functions listed in Appendix D. The result is

$$a_{SM} = (2\pi)^{1/2}\ e^{\frac{\pi}{2}Z}\ \Gamma\frac{(1+iZ)}{2}\ \times$$

$$\times\ \left[(1+iZ)\ e^{-ikr}\ F(2+iZ;\ 3;\ 2ikr) + e^{ikr}\ F(2-iZ;3;-2ikr) \right]$$

$$b_{SM} = -(2\pi)^{1/2}\ e^{\frac{\pi}{2}Z}\ \Gamma\frac{(1+iZ)}{2}\ \times \tag{40}$$

$$\times\ \left[(1+iZ)\ e^{-ikr}\ F(2+iZ;\ 3;\ 2ikr) - e^{ikr}\ F(2-iZ;3;-2ikr) \right]$$

It is interesting to compare these functions with the radial functions of the exact solution of the Dirac equation for the case of a point nucleus. The exact solution with incoming spherical waves is

$$a_E = (2\pi)^{1/2}\ e^{\frac{\pi}{2}Z}\ \frac{\Gamma(\gamma+iZ)}{\Gamma(2\gamma+1)}\ e^{i\frac{\pi}{2}(\gamma-1)}\ (2\,kr)^{\gamma-1}\ \times$$

$$\times\ \left[(\gamma+iZ)\ e^{-ikr}\ F(\gamma+1+iZ;\ 2\gamma+1;\ 2ikr) + e^{ikr}\ F(\gamma+1-iZ;2\gamma+1;-2ikr) \right]$$

$$b_E = -(2\pi)^{1/2}\ e^{\frac{\pi}{2}Z}\ \frac{\Gamma(\gamma+iZ)}{\Gamma(2\gamma+1)}\ e^{i\frac{\pi}{2}(\gamma-1)}\ (2kr)^{\gamma-1}\ \times \tag{41}$$

$$\times\ \left[(\gamma+iZ)\ e^{-ikr}\ F(\gamma+1+iZ;2\gamma+1;2ikr) - e^{ikr}\ F(\gamma+1-iZ;2\gamma+1;-2ikr) \right]$$

where $\gamma = (1 - Z^2)^{1/2}$.

The S-M radial wave functions for the $\frac{1+}{2}$ state are obtained by replacing γ by 1 in the exact solution. In general, for higher angular momentum states, the prescription for obtaining the S-M radial function from the exact radial function is: replace $\gamma = (\kappa^2 - Z^2)^{1/2}$ by $|\kappa|$ where κ is the eigenvalue of the operator $-\gamma_4(1 + \vec{L}\cdot\vec{S})$. That is, we neglect the Z^2 inside the square root defining γ, but nowhere else. Of course, to get the radial functions of a plane wave we simply set Z equal to zero everywhere.

We can obtain a state of $j = \frac{1}{2}$ and negative parity by multiplying $\psi_{\frac{1}{2}+}$ by γ_5. This holds only for zero mass particles, but we have used this approximation throughout. The two states of opposite parity contribute equally to the rate. In each of these states, we make an error of order $\frac{1}{2}\, Z^2$ in the electron wave function. However, for higher angular momentum states, the error is of order $\frac{1}{2}\, Z^2/\kappa^2$, so it drops off rapidly with increasing $|\kappa|$.

B. Rate to the $\frac{1}{2}^+$ Electron State

Using the $j = \frac{1}{2}$ part of the S-M wave function, we obtain the partial rate to that state. The rates $R_{SM}(\frac{1}{2}^+)$ for several atomic numbers are listed in Table II along with the rates $R_p(\frac{1}{2}^+)$ obtained when the electron is represented by plane wave. In calculating these rates, the small component of the muon wave function was neglected. We are no longer dealing with oriented muons.

The analytical expression for $R_{SM}(\frac{1}{2}^+)$ is

Table II.

Bound Muon Decay Rate to the $\frac{1+}{2}$ Electron State

The small components of the muon wave function have been neglected here. The rates R_p and R_{SM} are obtained by using plane waves and S-M wave functions, respectively. The last column gives the fraction of the total number of decays which proceed through the $\frac{1+}{2}$ electron state. The same fraction proceeds through the $\frac{1-}{2}$ state.

Z_e	$R_p(\frac{1+}{2})$	$R_{SM}(\frac{1+}{2})$	$\frac{R_{SM}(\frac{1+}{2})}{R_{SM}}$
0	0	0	0
16	.094	.101	.102
26	.151	.176	.181
35	.179	.223	.240
50	.191	.266	.312
82	.159	.274	.400

$$R_{SM}\left(\frac{1+}{2}\right) = \frac{G^2}{192\pi^3}\ \frac{32\mu^3}{\pi}\ \frac{2\pi Z}{1-e^{-2\pi Z}} \int dk \int dp \int dy\ k^2 p^2 \times$$

$$\times \left\{ \left[K^2 + L^2 + M^2 + N^2 \right] \left[3(W-k)^2 - q^2 \right] - \left[KM + LN \right] 4q\,(W-k) \right\} \tag{42}$$

where

$$K + iL = \frac{1}{8k^2 q}\ \frac{e^{Z(\alpha+\beta-\pi)}}{1+Z^2} \times$$

$$\times \left\{ \left[ZC + (2+Z^2)D - 2\sin\phi \right] + i\left[-Z^2 C - ZD + 2Z\sin\phi \right] \right\}$$

$$M + iN = \frac{1}{8k^2 q}\ \frac{e^{Z(\alpha+\beta-\pi)}}{1+Z^2} \times$$

$$\times \left\{ -\left[ZE + (2+Z^2)F + \frac{k}{q}\ \frac{2}{Z}\left(Z + 2\frac{\mu}{k} + Z^2\frac{\mu}{k}\right)\sin\phi \right] + i\left[Z^2 E + ZF + 2\frac{k}{q}\left(Z + \frac{\mu}{k}\right)\sin\phi \right] \right\}$$

and

$$q^2 = k^2 + 2\,kpy + p^2$$

$$\alpha = \tan^{-1}\frac{\mu}{k+q} \qquad \beta = \tan^{-1}\frac{\mu}{k-q} \qquad 0 < \beta < \pi$$

$$A = \frac{1}{2k}\left[(k+q)^2 + \mu^2\right]^{1/2} \qquad B = \frac{1}{2k}\left[(k-q)^2 + \mu^2\right]^{1/2}$$

$$\phi = Z \log \frac{B}{A}$$

$$C = \frac{1}{B}\cos(\phi+\beta) - \frac{1}{A}\cos(\phi-\alpha)$$

$$D = \frac{1}{B}\sin(\phi+\beta) + \frac{1}{A}\sin(\phi-\alpha)$$

$$E = \frac{1}{B}\cos(\phi+\beta) + \frac{1}{A}\cos(\phi-\alpha)$$

$$F = \frac{1}{B}\sin(\phi+\beta) - \frac{1}{A}\sin(\phi-\alpha)$$

In the limit of $Z \to 0$, we have

$$R_p(\tfrac{1}{2}^+) = \frac{G^2}{192\pi^3}\,\frac{32\mu^3}{\pi}\int dk \int dp \int dy\; k^2 p^2 \times$$

$$\left\{\left[K^2 + M^2\right]\left[3(W-k)^2 - q^2\right] - KM\,4q\,(W-k)\right\} \qquad (43)$$

where now

$$K = \frac{2\mu}{(q^2-k^2+\mu^2)^2 + 4k^2\mu^2}$$

$$M = \frac{\mu}{kq}\left[-\frac{q^2+k^2+\mu^2}{(q^2-k^2+\mu^2)^2+4k^2\mu^2} + \frac{1}{4kq}\log\frac{(q+k)^2+\mu^2}{(q-k)^2+\mu^2}\right]$$

C. Error Estimate for $\frac{1+}{2}$ State

We need to compare these results with the rates obtained with exact electron wave functions. In using S-M wave functions, we make two types of errors: we neglect part of the Coulomb effect and we ignore the finite size of the nucleus. Fortunately, these errors cancel to some extent, but it is still interesting to separate them. Ultimately, we want to know how well the S-M wave function approximates the exact solution for a finite nucleus.

Making this comparison presents a difficulty. The exact solutions for a point nucleus are known analytically, but they are difficult to deal with, and the exact solutions for a uniformly charged finite nucleus are not known in closed form. However, it is easy to obtain exact numerical solutions for any central potential and then to perform the radial integrals. Unfortunately, machine calculation of these integrals takes too long for it to be practical to perform the integrals over k, p, and y, but it is practical to compare the integrands at a number of points (k, p, y) in the neighborhood of the maximum. This has been done for $Z_e = 26$ and $Z_e = 82$. At $Z_e = 26$, using a generous estimate of the error, the rate obtained with the S-M wave function for the $\frac{1+}{2}$ state is about 3% low when compared with the exact solution for a point nucleus and 2% low when compared with the exact solution for a finite nucleus. The finite nuclear size has a rather small effect.

At large Z_e, say 82, the S-M approximation is also not bad because the neglected Coulomb effects and the finite size effects cancel each other to some extent, and the result is that the total decay rate is low by about 10%. One can make a firm statement here about the total rate from looking at the $j = \frac{1}{2}$ state because about 80% of the electrons decay to this state. We do not reap the full benefits of the accuracy because we have not kept the higher cross-terms in the numerical integration.

The error in states of $j = 3/2$ will be about one-quarter of the error in the $j = 1/2$ states, so, roughly speaking, the total percent error is approximately the percent error in the $j = 1/2$ states times the fraction of the rate which proceeds through these states plus one-quarter of the percent error in the $j = 1/2$ states times the fraction of the rate which proceeds through $j = 3/2$ states and higher. This is a generous estimate of the error.

It is of some interest to see the effect of a potential on the continuum electron wave functions. Figures 9 and 10 show radial wave functions for the $\frac{1+}{2}$ state for $Z_e = 82$.

FIGURE CAPTIONS

Fig. 9 The radial function $G(r) = r\,g(r)$ for a massless electron of energy $k = .3\ m\mu$ in a $\frac{1+}{2}$ state in the field of a nucleus of $Z_e = 82$. The electron wave function is $\psi_e(\vec{x}) = (4\pi)^{-1/2}\,[g(r) + i\,f(r)\,\vec{\alpha}\cdot\hat{x}]\,u_o$. We show two cases: (a) a point nucleus, and (b) a uniformly charged finite nucleus. The corresponding radial function of a plane wave, i.e. $r\,j_o(kr)$, is shown for comparison. The distance is in units of $m\mu^{-1}$.

Fig. 10 The radial function $F(r) = r\,f(r)$ for a massless electron for the same cases as in Fig. 8. The corresponding radial function of a plane wave, i.e., $r\,j_1(kr)$, is shown for comparison.

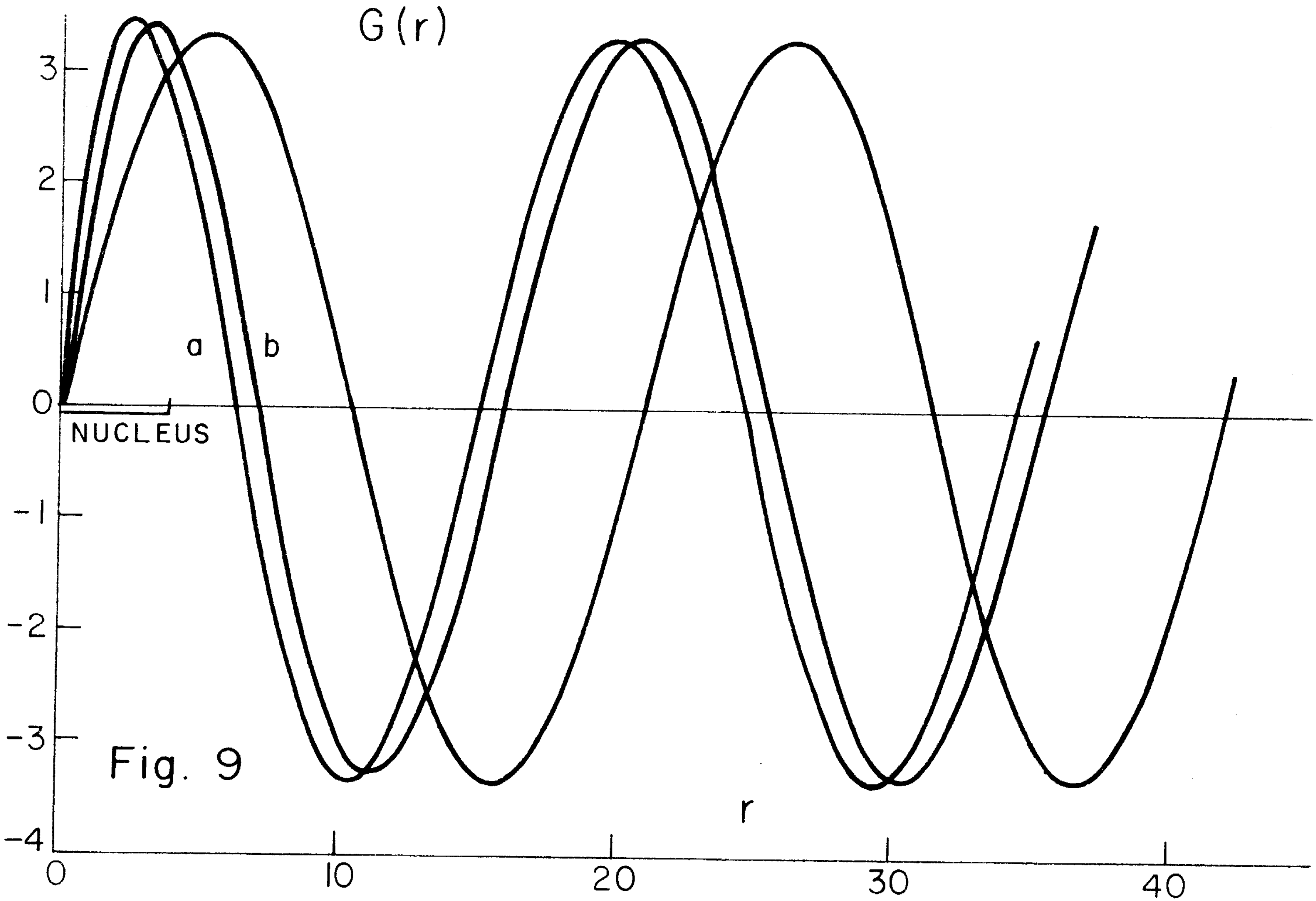


Fig. 9

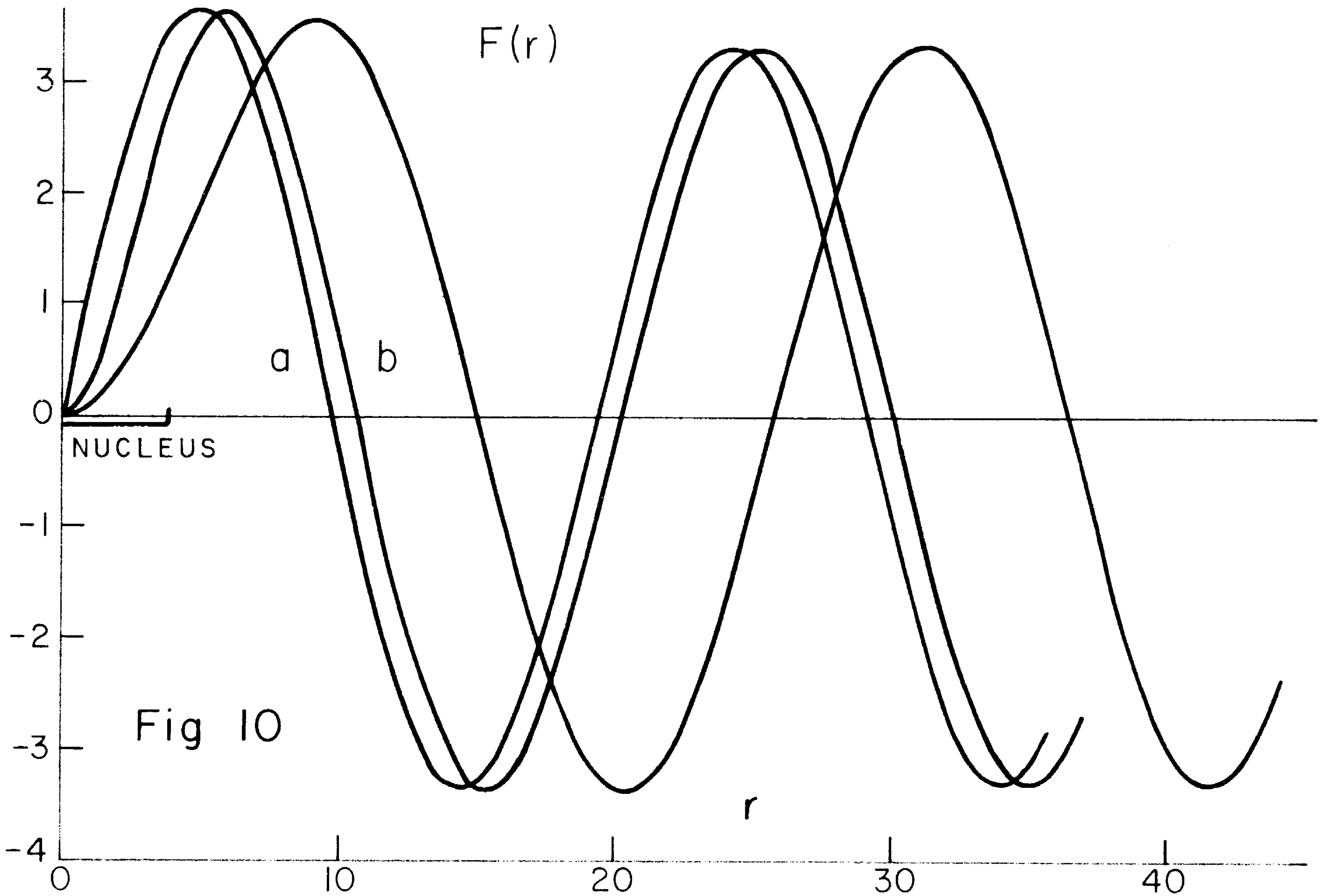


Fig 10

VI. THE NEGLECTED TERMS

A. Tiny Piece M_4

The tiny piece of the matrix element - that is, the piece which involves $\Delta\psi_G$ and $\Delta\psi_O$ - is difficult to evaluate exactly because the method which worked in the first three cases fails. However, it can be worked out to first order in Z by a procedure which is straightforward, though tedious. The method is outlined in Appendix E.

I have worked out the integral to lowest order in Z and then computed the contribution to the rate from the cross-term with the leading term, again the first order in Z. The expression is very long (it can be found in Appendix G) but the result is that the contribution to the rate is about 0.002 at Z = 26, so we can safely forget about it.

It is worth noting that this correction to the rate contains no pieces larger than $O(Z\mu^2)$. This agrees with what one would intuitively expect, since Z/r is effectively $O(Z\mu)$ and $\Lambda \approx \mu/2$.

B. Other Terms $O(Z\mu^2)$

We have now accounted for all terms of order $Z\mu^2$ which arise from the large component, but terms of the same order could arise from the product of M_2 and M_3. Strictly speaking, these would be of order $Z\mu\Lambda$. However, it can be easily shown by direct calculation that these terms vanish to this order and contribute only in higher order. Details appear in Appendix H.

C. M_3^2

We have neglected the square of the term

$$M_3 = \int d^3x \; \bar{\psi}_G \, \gamma_\sigma \; a \, \Delta \psi_o \; e^{i(\vec{p}+\vec{k})\cdot\vec{x}} \quad ,$$

but it is easy to show that to lowest order it exactly cancels the neglected term in the normalization.* Notice that the square of this term and the cross-term with the leading term are of the same order, $O(\mu^2)$. This is because the leading piece in the larger term, which is $O(\mu)$ in the spectrum, vanishes in the total rate.

There are no terms of order $Z\Lambda^2$.

D. Higher Order Terms

The main source of error is probably the square of the term

$$M_2 = \int d^3x \; \overline{\Delta \psi_G} \, \gamma_\sigma \; a \, \psi_o \; e^{i(\vec{p}+\vec{k})\cdot\vec{x}}$$

We can expect that the contribution to the rate is $O(Z^2\mu^2)$, which would make it something less than a percent at $Z_e = 26$. The other higher order terms are certainly quite small.

If one could calculate the entire matrix element exactly,

---

*See Reference 3.

there would be this advantage in keeping all terms, even though they are smaller than the error involved in using S-M wave functions: One could then be certain that the comparison in a given angular momentum state of the S-M rate and the exact rate gave a true estimate of the error. At present, there is still some play.

E. Error in Muon Wave Function

The parameters in the muon wave function are those that minimize the squared differences between the exact and empirical wave functions.* These are not necessarily the best parameters for this problem. To estimate the error in the rate arising from the error in the muon wave function, I made the following comparison.

At a number of points (k, p, y), I compared the partial rates computed with exact and empirical wave functions. (For the purposes of this check, I used plane wave electrons.) The rate computed with exact wave functions was slightly higher, about 0.6% at $Z_e = 26$, 3.0% at $Z_e = 50$, and 5.6% at $Z_e = 82$. The effect is very smooth.

*Actually, the comparison was made for r^2 times the wave function.

VII. COMPARISON WITH EXPERIMENTS

The results presented here are in marked disagreement with experimental results (2) both in the region $20 < Z_e < 30$ and for large Z_e. In the first region we do not obtain a sharp peak in the rate, and for large Z_e - say near lead - we obtain rates considerably higher than the experimental rates. Both of these disagreements are far beyond our estimates of the error in the calculation.

The disagreement at large Z_e is perhaps less serious because the experiments are very difficult to perform and the calculation is less reliable. However, at $Z_e = 26$, the error in the calculation is less than 2 percent. The only possible conclusions are: (a) there is some property of muons which has not been taken into account, or (b) the experiments are wrong. In any case, the experiment should be repeated.

Experimental Procedure

A beam of muons is stopped in a target and the outgoing electrons are counted. If M muons are stopped at $t = 0$, then the rate at which electrons come out is

$$\frac{dN}{dt} = MR\, e^{-R_t t} \tag{44}$$

where R is the decay rate and R_t is the total disappearance rate which includes the capture rate. The total rate R_t is easy to obtain from the decrease in the counting rate with time, and if we know N/M we can find R.

In practice, N/M is hard to measure and the quantity measured is $y = n/M = \Omega N/M$, where Ω is a factor which depends on the counter efficiency and the geometry. Since Ω is almost the same for different targets Z and Z', we have

$$\frac{y(Z)}{y(Z')} \approx \frac{N(Z)}{N(Z')} \, \frac{M(Z')}{M(Z)} \tag{45}$$

and

$$\frac{R(Z)}{R(Z')} \approx \frac{y(Z)}{y(Z')} \, \frac{R_t(Z)}{R_t(Z')} \tag{46}$$

In the calibrated efficiency method, electron and positron yields from identical negative and positive muon beams are compared with the same target. In the sandwich method, a stack is made up of different materials Z and Z', and the electron rate is fitted to

$$\frac{dn}{dt} = A\, e^{-R_t(Z)\, t} + A'\, e^{-R_t(Z')\, t} + B \tag{47}$$

since

$$\frac{dN}{dt} = R(Z)\,M(Z)\;e^{-R_t(Z)\,t} + R(Z')\,M(Z')\;e^{-R_t(Z')\,t} \tag{48}$$

B represents the background. The quantities $R_t(Z)$, $R_t(Z')$, $M(Z)$, and $M(Z')$ are known from other experiments.

VIII. COMPARISON WITH BORN APPROXIMATION

Überall (3) has used the Born approximation to calculate the coefficient of the leading term in the Coulomb correction to the rate. His result is that

$$R \simeq \frac{G^2}{192\,\pi^3}\;W^5\,(1 - 2\mu^2 - \mu^2 + 5\,Z\mu) \tag{49}$$

The first two terms are familiar from Equation 14. The next term is the lowest order contribution of the small component of the muon, which is easily obtained. The last term is the leading piece of the Coulomb correction.

It might at first be thought that there is a linear term, $O(Z)$, which could produce an initial increase in the rate as a function of Z. It can, however, be easily shown that there cannot be such a term or any other term in Z alone. To see this, imagine that we let $\mu \rightarrow 0$. Then the muon is spread uniformly over all of space, and the probability that it will decay within range of the Coulomb field of the nucleus (say, screened) is zero. Hence, as $\mu \to 0$, the decay rate must tend to the free decay rate, and so there cannot be any terms in Z alone. This remark simplifies the calculation in Born approximation.

The above result, Equation 49, is also obtained from our expression for the rate (Equations 34-36). It is instructive to examine this in detail.

In the S-M result (Equations 34-36), the term linear in Z is

$$
\begin{aligned}
R = Z \frac{G^2}{192\pi^3} \frac{256\mu^5}{\pi} \int dp \int dy \int dk \; \frac{k^2 p^2}{(p^2+\mu^2)^4} \; \times \\
\times \Bigg[\Big[2 \tan^{-1} \frac{Q^2}{2k\mu} + 2 \frac{k}{\mu} \frac{(Q^2-2\mu^2)(p^2+\mu^2)}{Q^4+(2k\mu)^2} \Big] \times \\
\times \left[3W^2 - 4Wk + 2py\,(W-2k) - p^2 \right] \\
+ \Big\{ \frac{py}{\mu} \frac{(Q^2-2\mu^2)(p^2+\mu^2)}{Q^4+(2k\mu)^2} \left[3W^2 - 4Wk + 2py\,(W-2k) - p^2 \right] \\
+ \frac{1}{\mu} \frac{Q^2(p^2+\mu^2)}{Q^4+(2k\mu)^2} \, 2\,(W-2k)\, p^2(1-y^2) \Big\} \\
+ 2\Lambda \Big\{ \Big(2 \tan^{-1} \frac{Q^2}{2k\mu} + \frac{k}{\mu} \frac{Q^2(p^2-3\mu^2)+8\mu^2 kpy}{Q^4+(2k\mu)^2} \Big) \times \\
\times \frac{p}{\mu} \left[(W^2 - 4Wk)\, y - 2p(W+2ky^2) - 3p^2 y \right] \\
- \frac{2k^2(p^2+\mu^2)}{Q^4+(2k\mu)^2} \left[W^2 - 4Wk - 2py\,(W+2k) - p^2 - 2p^2y^2 \right] \Big\} \Bigg]
\end{aligned}
\tag{50}
$$

The terms have been separated in the same way as Equations 34-36. To expand in μ, recall that p is $O(\mu)$. Then

$$\tan^{-1}\frac{Q^2}{2k\mu} \approx \tan^{-1}\frac{py}{\mu} + \frac{\mu}{2k}\,\frac{p^2+\mu^2}{p^2y^2+\mu^2} \qquad (51)$$

and the other terms are expanded in a straightforward way, keeping only $O(Z\mu)$. At this point, the integrals over k, y, and p can be done analytically (and most easily in that order). The result is that the first term in Equation 50 contributes $\frac{5}{3}\,Z\mu$, the second term contributes $\frac{10}{3}\,Z\mu$, and the last term contains no corrections to the rate of $O(Z\mu)$.

The integral over the last two pairs of brackets in Equation 50 has the limits

$$\int_0^1 dp \int_{-1}^{+1} dy \int_0^{1/2} dk$$

In the integral over the first term, we must use

$$\int_0^1 dp \int_{-1}^{+1} dy \int_0^{1/2\,(1-py)} dk$$

and we find that no matter how small μ is, there is a finite contribution, $-2Z\mu$, to the rate from the region $2k + p > 1$, which becomes infinitely small as $\mu \rightarrow 0$. This can be interpreted in the following way. As $\mu \to 0$, the entire spectrum tends to the free spectrum except at $k = W/2$. This one point approaches the free spectrum only when $Z \to 0$. Of course, one point cannot affect the rate, and $R \to R_o$ as $\mu \to 0$. It is interesting, however, that the Coulomb effect tends to increase the rate for low momentum electrons, but it depresses the rate for electrons with momentum $k \approx W/2$.

Born Approximation

For completeness, we shall outline here the direct calculation of the Born approximation result as done by Überall (3). This is equivalent to using, in Equation 5, an electron wave function which is obtained by iterating the Dirac equation.

From the Dirac equation,

$$(i \vec{\alpha} \cdot \vec{\nabla} + k) \ \psi_e(\vec{x}) = V(\vec{x}) \ \psi_e(\vec{x}) \tag{52}$$

we get

$$\psi_e(\vec{x}) = \psi_o(\vec{x}) + \int d^3z \ G(\vec{x} - \vec{z}) \ V(\vec{z}) \ \psi_o(\vec{z}) + \cdots \tag{53}$$

where ψ_o is a plane wave and G is the appropriate Green's function. We shall only consider the first Born approximation.

Explicitly, for $V(\vec{x}) = -Z \frac{e^{-\lambda r}}{r}$, we have

$$\psi_e(\vec{x}) = e^{i\vec{k}\cdot\vec{x}} u_{\vec{k}} + 4\pi Z \int \frac{d^3 s}{(2\pi)^3} \frac{e^{i\vec{s}\cdot\vec{x}}}{s^2 - k^2 + i\epsilon} \frac{\vec{\alpha}\cdot\vec{s} + k}{(s-k)^2 + \lambda^2} u_{\vec{k}} \qquad (54)$$

Now we just have to work out N_σ and then compute the rate from Equation 8. It is necessary to evaluate a number of integrals, all of which, however, can be expressed as derivatives of the integral

$$I = \int d^3 s \frac{1}{s^2 - k^2 - i\epsilon} \frac{1}{(\vec{s} - \vec{k})^2 + \lambda^2} \frac{1}{(\vec{s} - \vec{q})^2 + \mu^2} \qquad (55)$$

In fact, we need only the real part, and this is given by Überall (3).

$$\mathrm{Re}\, I = \frac{1}{k(p^2 + \mu^2)} \tan^{-1} \frac{Q^2}{2k\mu} \qquad (\lambda = 0) \qquad (56)$$

To order $Z\mu$, the result obtained by performing the indicated operations is, of course, identical with that obtained by expanding Equations 34, 35, and 36.

APPENDICES

APPENDIX A

The Sum over Neutrino States

We use a method due to Feynman to evaluate the following integral which arises in the sum over neutrino states:

$$I(a,b) = \int \frac{d^3 s}{(2\pi)^3} \int \frac{d^3 t}{(2\pi)^3} \frac{s\cdot a}{2s} \frac{t\cdot b}{2t} (2\pi)^4 \delta^4 (G-s-t) \tag{A1}$$

Now

$$\int \frac{d^3 s}{2s} = \int d^4 s \;\delta(s^2)$$

so

$$I(a,b) = \int \frac{d^4 s}{(2\pi)^4} 2\pi\delta(s^2) \int \frac{d^4 t}{(2\pi)^4} 2\pi\delta(t^2)\; s\cdot a\; t\cdot b\; (2\pi)^4 \delta^4 (G-s-t) \tag{A2}$$

$$= \int \frac{d^4 s}{(2\pi)^4} 2\pi\,\delta(s^2)\, 2\pi\delta((G-s)^2)\;\; s\cdot a\; (G-s)\cdot b$$

It is convenient to orient the coordinate system so that G has only a time-like component. Then

$$I(a,b) = \frac{1}{(2\pi)^2} \int d^4 s\;\; \delta(s^2)\, \delta(G_o(G_o-2s))(s_o a_o - \vec{s}\cdot\vec{a})((G_o-s_o) b_o + \vec{s}\cdot\vec{b}) \tag{A3}$$

But $\int d^4 s = \int ds_o \int \vec{s}^{\,2}\, ds \int d\Omega_s$, and

$$\int d\Omega_s = 4\pi \qquad \int d\Omega_s \ \vec{s}\cdot\vec{a} = 0$$

$$\int d\Omega_s \ \vec{s}\cdot\vec{a} \ \vec{s}\cdot\vec{b} = \frac{4\pi}{3} \ \vec{s}^{\,2} \ \vec{a}\cdot\vec{b} \tag{A4}$$

so

$$I(a,b) = \frac{1}{\pi} \int ds_o \int \vec{s}^{\,2} \, ds \ \frac{1}{2s} \ \delta(s-s_o) \ \frac{1}{2G_o} \ \delta(s_o - \tfrac{1}{2} \ G_o) \times$$

$$\times \left[s_o \ a_o \ (G_o - s_o) \ b_o - \frac{1}{3} \ \vec{s}^{\,2} \ \vec{a}\cdot\vec{b} \ \right] \tag{A5}$$

$$= \frac{1}{32\pi} \left[G_o \ a_o \ G_o \ b_o - \frac{1}{3} \ G_o^{\,2} \ \vec{a}\cdot\vec{b} \ \right]$$

Clearly, the result in an arbitrary coordinate system is

$$I(a,b) = \frac{1}{96\pi} \left[2 \ G\cdot a \ \ G\cdot b + G^2 \ a\cdot b \right] \tag{A6}$$

The sum over the neutrinos is

$$J_{\sigma\lambda} = \frac{1}{2} \int \frac{d^3 s}{(2\pi)^3} \int \frac{d^3 t}{(2\pi)^3} \ \frac{1}{2s\,2t} \ \ 4(s_\sigma t_\lambda + t_\sigma s_\lambda - s\cdot t \ \delta_{\sigma\lambda})(2\pi)^4 \delta^4(G-s-t) \tag{A7}$$

Since

$$J_{\sigma\lambda} = \frac{1}{2} \left[8 \ I(\sigma,\lambda) - 4 \ I(a,b) \ \delta_{ab} \ \delta_{\sigma\lambda} \right] \tag{A8}$$

we have the final result

$$J_{\sigma\lambda} = \frac{1}{12\pi} \left(G_\sigma G_\lambda - G^2 \delta_{\sigma\lambda} \right) \tag{A9}$$

The antisymmetric part of the trace which comes from the γ_5 is easily seen to vanish upon integration.

APPENDIX B

Kinematics and Integration over Momenta

The rate R involves an integration over all momenta of the interacting particles. The neutrino momenta are eliminated early in the calculation but the integrations over the recoil momentum p and the electron momentum k remain until the last step. We have

$$R = \int d^3p \int d^3k \quad \left[R(\vec{p},\vec{k}) + S(\vec{p},\vec{k}) \right] \tag{B1}$$

where $R(\vec{p},\vec{k})$ gives the rate and $S(\vec{p},\vec{k})$ the angular distribution.

Actually, $R(\vec{p},\vec{k})$ depends only on p, k, and $\hat{p}.\hat{k}$, while $S(\vec{p},\vec{k})$ also depends on $\hat{k}\cdot\hat{z}$ and $\hat{p}\cdot\hat{z}$. For the moment let us forget about S and consider the integration of $R(p,k,y)$. It is clear from rotational invariance that we can write

$$\begin{aligned} \int d^3p \int d^3k &= 4\pi \int k^2 dk \;\; 2\pi \int dy \int p^2 dp \\ &= 8\pi^2 \int dk \int dy \int dp\, k^2\, p^2 \end{aligned} \tag{B2}$$

The limits of the integrals come from an application of energy and momentum conservation. We can safely disregard the kinetic energy of the recoiling atom. For example, for iron $\frac{p^2}{2M} \approx 10^{-4}\ m\mu$. The electron mass is neglected so that the available energy in the decay simply goes into electron and neutrino momenta. Let $\vec{q}$ be the sum of the two neutrino momenta. Clearly, for a given k and y, p will have a maximum, or minimum, when the two neutrino momenta are in the same direction. See Figure 11.

$$\begin{aligned} q + k &= W \\ p_m \cos\theta + k &= q \cos\psi \\ p_m \sin\theta &= q \sin\psi \end{aligned} \qquad \text{(B4)}$$

and so

$$p_m = -ky \pm (W^2 - 2kW + k^2 y^2)^{1/2} \quad , \; p_m > 0 \; . \qquad \text{(B5)}$$

The region of integration has a rather complicated shape and it is best understood by drawing a picture of the three-dimensional region. The situation is quite different for $k < \frac{W}{2}$ and $k > \frac{W}{2}$. Let us consider k and y as fixed, and then find the limits on p. (See Figure 13 and also Figure 2.)

Since we shall be interested in the electron spectrum, we must leave the k-integration until the end. The limits on the y-integration are trivial for $k < \frac{W}{2}$: $-1 < y < +1$. For $k > \frac{W}{2}$, we need to find the equation of the line in the (k, y) plane which is the projection of the line of turning points of the parabolas in the (p, k) plane. At a turning point

$$(k^2 y^2_{max} + W^2 - 2kW)^{1/2} = 0$$

or

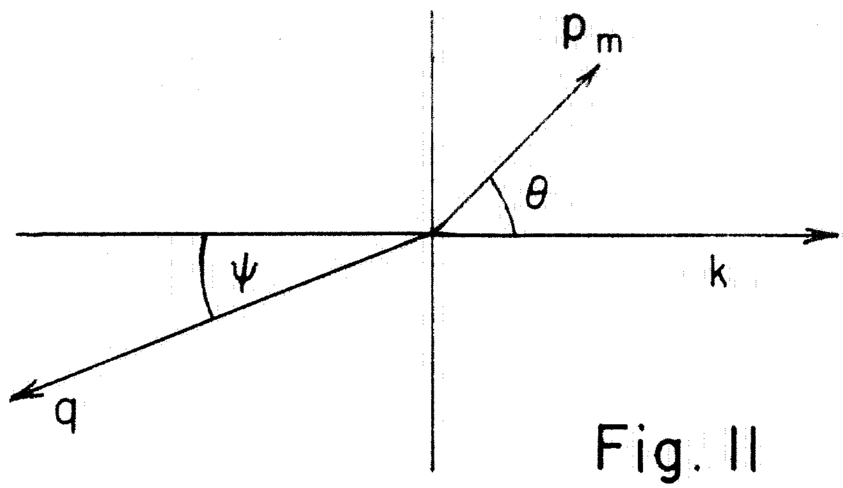
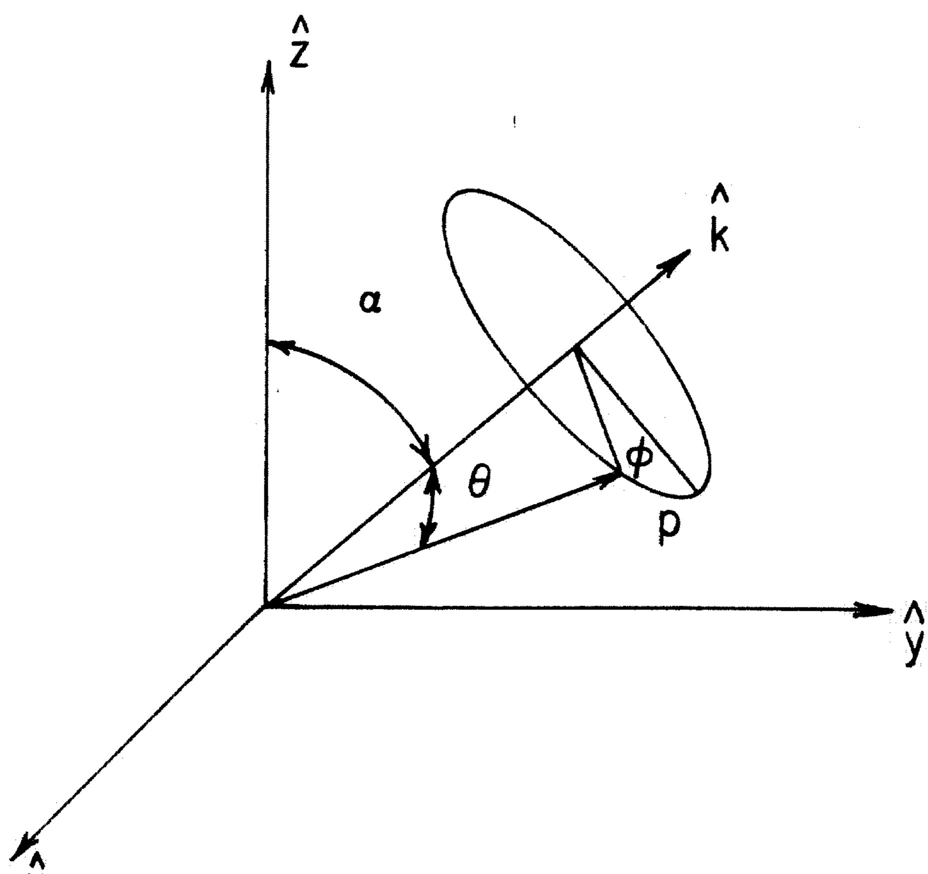


Fig. 11

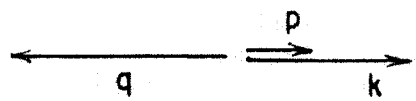
Definition of p_m



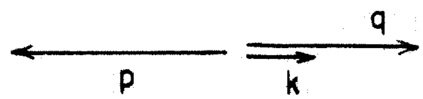
Coordinates for Angular Distribution

Fig. 12

Case 1. $k < \frac{W}{2}$

$y = 1$ 

$p_{max} = W - 2k$

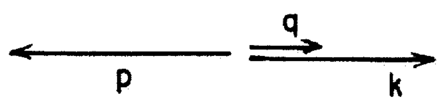
$y = -1$ 

$p_{max} = W$

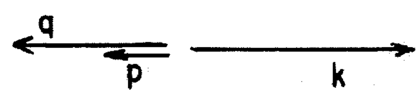
y arbitr. $p_{max} = -ky + \sqrt{k^2 y^2 + W^2 - 2Wk}$

Case 2. $k > \frac{W}{2}$

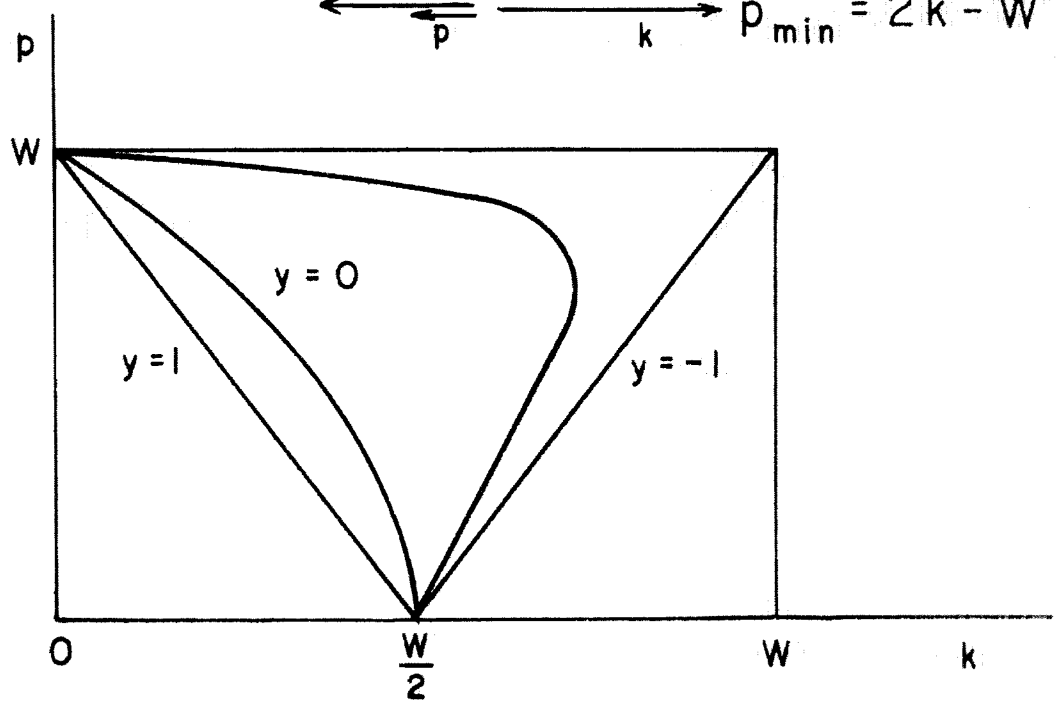
$y > 0$ no solutions

$y = -1$ 

$p_{max} = W$



$p_{min} = 2k - W$



KINEMATICS

Fig. 13

$$y_{max} = \left(\frac{2kW - W^2}{k^2} \right)^{1/2} \tag{B5}$$

If we needed only the total rate, it would be simpler to perform the k-integration first, for then the p and y limits are trivial.

Now consider the integration over $S(\vec{p}, \vec{k})$. Let $\hat{k} \cdot \hat{z} = \cos\alpha$ We replace Equation B2 by

$$\int d^3p \int d^3k = 2\pi \int d\cos\alpha \int k^2 dk \int d\phi \int dy \int p^2 dp \tag{B6}$$

where ϕ is the angle indicated in Figure 12. The dependence of $S(\vec{p}, \vec{k})$ on $\hat{z}$ is linear. But

$$\hat{p} \cdot \hat{z} = \cos\theta \cos\alpha + \sin\theta \sin\alpha \cos\phi \tag{B7}$$

so that the ϕ-integration can be performed easily. The result is that we can simply replace $\hat{p} \cdot \hat{z}$ by $\hat{p} \cdot \hat{k}\ \hat{k} \cdot \hat{z}$. With this in mind,

$$\int d^3p \int d^3k = 8\pi^2 \frac{1}{2} \int d\cos\alpha \int dk \int dy \int dp\, k^2 p^2 \quad . \tag{B8}$$

APPENDIX C

Numerical Methods and Computation

1. Machine Solution of the Dirac Equation.

If we write the solution to the Dirac equation with $j = \frac{1}{2}$, and positive parity in the form

$$\psi(\vec{x}) = \frac{1}{(4\pi)^{1/2}} \left[g(r) + f(r)\, i\, \vec{\alpha} \cdot \hat{x} \right] u_o \tag{C1}$$

then the solutions for $G(r) = r\,g(r)$ and $F(r) = r\,f(r)$, written in dimensionless form, satisfy

$$\left[\frac{d}{dx} - \frac{1}{x} \right] G(x) + \left[W + M + V(x) \right] F(x) = 0 \tag{C2}$$

$$\left[W - M + V(x) \right] G(x) - \left[\frac{d}{dx} + \frac{1}{x} \right] F(x) = 0$$

For the muon $M = 1$, and for a massless electron, $M = 0$. We use the potential

$$V(x) = \begin{cases} \dfrac{Z}{x_o} \left[\dfrac{3-t}{2} + \dfrac{t-1}{2} \dfrac{x^2}{x_o^2} \right] & x < x_o \\[2ex] \dfrac{Z}{x} & x > x_o \end{cases} \tag{C3}$$

where $x_o = R_o A^{1/3}$ and $R_o = 0.6425$. By varying t between 0 and

1, we can go from a surface charge distribution to a uniform charge distribution.

The solutions are started by means of power series. Near the origin

$$G(x) \approx G_1 x + G_3 x^3 + G_5 x^5 + G_7 x^7$$
$$F(x) \approx F_2 x^2 + F_4 x^4 + F_6 x^6 + F_8 x^8 \qquad (C4)$$
$$V(x) = V_o + V_2 x^2$$

Let

$$A = W + M + V_o \qquad B = W - M + V_o \quad , \qquad (C5)$$

then we have

$$G_1 \quad \text{arbitrary}$$
$$F_2 = \frac{1}{3} B G_1$$
$$G_3 = -\frac{1}{2} A F_2$$
$$F_4 = \frac{1}{5}[B G_3 + V_2 G_1] \qquad (C6)$$
$$G_5 = -\frac{1}{4}[A F_4 + V_2 F_2]$$
$$F_6 = \frac{1}{7}[B G_5 + V_2 G_3]$$
$$G_7 = -\frac{1}{6}[A F_6 + V_2 F_4]$$
$$F_8 = \frac{1}{9}[B G_7 + V_2 G_5]$$
$$G_9 = -\frac{1}{8}[A F_8 + V_2 F_6]$$

The derivatives $G'(x)$ and $F'(x)$ are computed in a similar way. The solution is then continued by using the extrapolation formula

$$G(m+1) = G(m) + \frac{h}{24}\left[55\, G'(m) - 59\, G'(m-1) + 37\, G'(m-2) - 9\, G'(m-3) \right]$$

(C7)

$$F(m+1) = F(m) + \frac{h}{24}\left[55\, F'(m) - 59\, F'(m-1) + 37\, F'(m-2) - 9F'(m-3) \right]$$

where h is the interval. The derivatives at $x = m + 1$ are then computed from the differential equations, and so on.

To obtain bound solutions, $W < M$, it is necessary to vary W to find the eigenvalue for which the solution doesn't blow up.* Continuum solutions for all values of $W > M$ and some of these are shown in Figures 9 and 10 for a massless electron.

All the radial wave functions corresponding to higher angular momentum states can be obtained in a similar way.

The question of the error involved in using this approximation scheme is best settled in any given case by decreasing the interval until the desired accuracy is obtained. For the calculations in work the interval $h = 0.10$ was found satisfactory.

2. Numerical Integration over Phase Space

The numerical integration over the region shown in Figure 2 was performed with the following intervals.

---

*Actually, the values of W listed in Table I were obtained previously by Prof. Mathews using a different program.

$$\Delta y = \frac{1}{8} \quad , \quad \Delta p = \frac{1}{32}\,W \text{ to } \frac{1}{64}\,W \, ,$$

and $\Delta k = \frac{1}{16}\,W$. The accuracy was checked by doubling the number of points for a sample case. In general, the error is about 1/2 percent.

APPENDIX D

Confluent Hypergeometric Functions (11)

These functions are discussed in many references. We will merely list some properties that are useful in this calculation.

The confluent hypergeometric function $F(a;b;x)$ is defined by the series

$$F(a;b;x) = 1 + \frac{a}{b}\,\frac{x}{1!} + \frac{a(a+1)}{b(b+1)}\,\frac{x^2}{2!} + \cdots \qquad \text{(D1)}$$

where $b \neq 0, -1, -2$. The function satisfies the differential equation

$$x\ u'' + (b - x)\ u' - a\ u = 0 \qquad \text{(D2)}$$

Direct substitution shows that this equation is satisfied by

$$U(x) = c \int dt\quad e^{tx}\ t^{a-1}\ (t-1)^{b-a-1}$$

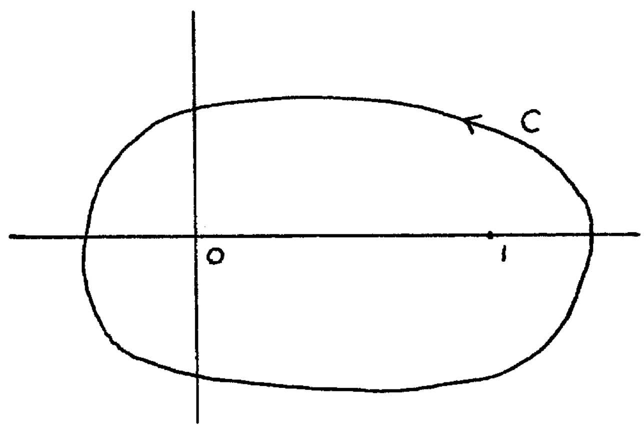
if the path of integration in the complex t plane is such that the function

$$V(t) = e^{tx}\ t^{a}\ (t-1)^{b-a}$$

returns to its initial value. This integral is then a representation of the confluent hypergeometric function if we fix the constant c so that $U(0) = 1$.

We have occasion to use the integral representation for $F(iZ;\, 1;\, i(kr + \vec{k}\cdot\vec{x}))$. For integer b the function $V(t)$ returns to its original value along the contour C.

t-plane



The integral representation is then

$$F(iZ;\, 1;\, i(kr+\vec{k}\cdot\vec{x})) = \frac{1}{2\pi i} \oint_C dt\; e^{i(kr+\vec{k}\cdot\vec{x})t}\; t^{iZ-1}\, (t-1)^{-iZ} \tag{D3}$$

In general, for integer b, we have

$$F(a;b;x) = \frac{1}{2\pi i}\, (-1)^{b-1}\, \frac{\Gamma(1-a)\Gamma(b)}{\Gamma(b-a)} \oint_C dt\; e^{xt}\, t^{a-1}(t-1)^{b-a-1} \tag{D4}$$

Another representation, which is suitable when b is not an integer, is

$$F(a;b;x) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \int_0^1 dt\; e^{xt}\; t^{a-1}\,(t-1)^{b-a-1} \tag{D5}$$

The integral can be taken over any contour connecting 0 and 1.

APPENDIX E

Integrals

The matrix-element for the total decay rate contains the four integrals:

$$H = \int d^3x \; e^{-\mu r} \; e^{i \vec{p}\cdot\vec{x}} \; F(iZ;1;i\,(kr + \vec{k}\cdot\vec{x}))$$

$$I_a = \frac{1}{2ik} \int d^3x \; e^{-\mu r} \; e^{i \vec{p}\cdot\vec{x}} \; \hat{a}\cdot\vec{\nabla} \; F(iZ;1;i(kr+\vec{k}\cdot\vec{x}))$$

$$J_b = \int d^3x \; e^{-\mu r} \; e^{i \vec{p}\cdot\vec{x}} \; F(iZ;1;i(kr + \vec{k}\cdot\vec{x}) \; \hat{b}\cdot\hat{x} \tag{E1}$$

$$K_{ab} = \frac{1}{2ik} \int d^3x \; e^{-\mu r} \; e^{i \vec{p}\cdot\vec{x}} \; \hat{a}\cdot\vec{\nabla} \; F(iZ;1;i(kr+ k\cdot x)) \; \hat{b}\cdot\hat{x}$$

For the moment, let us consider the first three integrals because they are much easier to deal with than K_{ab}. Notice that all three integrals can be obtained from

$$I = \int d^3x \; \frac{e^{-\mu r}}{r} \; e^{i \vec{p}\cdot\vec{x}} \; F(iZ;1;\, i(kr+ \vec{k}\cdot\vec{x})) \tag{E2}$$

since

$$H = -\frac{\partial}{\partial \mu} I \,, \quad I_a = \frac{1}{2i} \; \hat{a}\cdot\vec{\nabla}_k I \,, \quad J_b = \hat{b}\cdot\vec{\nabla}_p I \tag{E3}$$

where the subscript on the gradient indicates the variable it operates on.

To perform the integral I use the integral representation of

$F(iZ\,;\,1\,;\,i(kr+\vec{k}\cdot\vec{x}))$:

$$F(iZ;1;i(kr+\vec{k}\cdot\vec{x})) = \frac{1}{2\pi i}\int_C dt\; t^{iZ-1}(t-1)^{-iZ}\; e^{it(kr+\vec{k}\cdot\vec{x})}$$

The closed contour C encircles the points $t = 0$ and $t = 1$ once counter-clockwise. We can now write

$$I = \frac{1}{2\pi i}\int dt\; t^{iZ-1}(t-1)^{-iZ}\int dr\; r\; e^{-\mu r}\; e^{iktr}\int d\Omega\; e^{i\vec{p}\cdot\vec{x}}\; e^{it\vec{k}\cdot\vec{x}} \tag{E4}$$

Now

$$\int d\Omega\quad e^{i(\vec{p}+\vec{k}t)\cdot\vec{x}} = 4\pi\; j_o(Qr) \tag{E5}$$

where $\vec{Q} = \vec{p} + \vec{k}t$, and, with $\alpha = \mu - ikt$,

$$4\pi\int dr\; r\; e^{-\alpha r}\; j_o(Qr) = 4\pi\,\frac{1}{\alpha^2+Q^2} \tag{E6}$$

so the integral I can be written

$$I = 4\pi\,\frac{1}{2\pi i}\int dt\; t^{iZ-1}(t-1)^{-iZ}\,\frac{1}{2(pky-i\mu k)\,t + p^2+\mu^2} \tag{E7}$$

We can now expand the contour to infinity and it is clear that the only contribution comes from a simple pole at

$$t_o = -\,\frac{p^2+\mu^2}{2(pky-i\mu k)} \tag{E8}$$

In terms of t_o, we have

$$I = - \frac{4\pi}{2(pky - i\mu k)} \quad t_o^{\,iZ-1} \quad (t_o - 1)^{-iZ}$$

and finally

$$I = 4\pi \; (p^2 + \mu^2)^{iZ-1} \; (p^2 + \mu^2 + 2\,pky - i\mu k)^{-iZ} \qquad \text{(E9)}$$

It remains to show that the pole is always outside the original contour. Consider E-4 and imagine that we do the radial integral first. This will converge only if

$$\mathrm{Re}\,[\,\mu - ikt\,(1+u)\,] > 0 \quad ,$$

where $u = \hat{k} \cdot \hat{x}$, so we must have

$$\mathrm{Im}\; t > -\frac{\mu}{(1+u)k}$$

Since this must hold for all u, $-1 \leq u \leq 1$, we have finally

$$\mathrm{Im}\; t > -\frac{\mu}{2k}$$

But from E8 we see that

$$\mathrm{Im}\; t_o = -\frac{p^2 + \mu^2}{(p^2 y^2 + \mu^2)} \quad \frac{\mu}{2k} \quad \leq \quad \frac{\mu}{2k}$$

so the pole is always outside the initial contour.

I have been unable to evaluate K_{ab} exactly. The following shows how K_{ab} may be evaluated to first order in Z.

First notice that

$$K_{ab} = \frac{1}{2i} \quad \hat{a} \cdot \vec{\nabla}_k \quad \hat{b} \cdot \vec{\nabla}_p \quad K(\mu) \tag{E10}$$

where

$$K(\mu) = \int d^3x \; \frac{e^{-\mu r}}{r^2} \quad e^{i \vec{p} \cdot \vec{x}} \; F(iZ; 1; i(kr + \vec{k} \cdot \vec{x})) \tag{E11}$$

The method used to evaluate I fails here; the last integral over t cannot be done by the method of residues. But notice $I = - \frac{d}{d\mu} K$, or, if we put $I = I(\mu)$ and $K = K(\mu)$,

$$K(\mu) = - \int_{\infty}^{\mu} ds \quad I(s) \tag{E12}$$

From E9, we have

$$\begin{aligned} \hat{a} \cdot \vec{\nabla}_k \; I(s) &= 4\pi (p^2 + s^2)^{iZ-1} \; iZ \; \frac{\hat{a} \cdot (2\vec{p} - 2 i s \hat{k})}{(p^2 + s^2 + 2 pky - i s k)^{iZ+1}} \\ &= 8\pi i Z \; \frac{\hat{a} \cdot (\vec{p} - i s \hat{k})}{(p^2 + s^2)\left[(s - ik)^2 + (\vec{p} + \vec{k})^2\right]} \quad + \; O(Z^2) \end{aligned} \tag{E13}$$

Put $\vec{q} = \vec{p} + \vec{k}$. Then to first order in Z

$$\hat{a} \cdot \vec{\nabla}_k \, I(s) = 8\pi Z \; \frac{\hat{a} \cdot (s \hat{k} - i\vec{p})}{(s - ip)(s + ip)\,[\, s - i(k+q)\,]\,[\, s - i(k-q)\,]} \tag{E14}$$

The integer over s can now be done easily using the method of partial fractions and then the final gradient can be taken to yield K_{ab} to lowest order in Z.

The matrix element for the rate to the $\frac{1+}{2}$ electron state involves integrals which can be obtained by differentiation from

$$L = \int d^3x \; \frac{e^{-\mu r}}{r} \; e^{i(\vec{p}+\vec{k})\cdot\vec{x}} \; e^{ikr} \; F(2-iZ;3;-2ikr) \tag{E15}$$

We again write the confluent hypergeometric function in terms of the integral representation D4 and we have

$$F(2-iZ; 1; -2ikr) = -\frac{1}{2\pi i} \; \frac{2}{iZ(1-iZ)} \int dt \; t^{1-iZ} \, (t-1)^{iZ} \, e^{-2ikrt} \tag{E16}$$

and

$$L = \frac{1}{2\pi i} \; \frac{2}{iZ(1-iZ)} \int dt \; t^{1-iZ} \, (t-1)^{iZ} \int dr \; r \; e^{-\mu r} \; e^{-ikr(2t-1)} \int d\Omega \; e^{i(\vec{p}+\vec{k})\cdot\vec{x}} \tag{E17}$$

$$= \frac{1}{2\pi i} \; \frac{2}{iZ(1-iZ)} \int dt \; t^{1-iZ} \, (t-1)^{iZ} \; \frac{4\pi}{4k^2} \; \frac{1}{(t-t_a)(t-t_b)}$$

where

$$t_a = \frac{1}{2k}\,(k+q+i\,\mu) \qquad\qquad t_b = \frac{1}{2k}\,(k-q+i\,\mu)$$

Finally,

$$L = -\frac{2}{iZ(1-iZ)} \; \frac{\pi}{k^2} \; \frac{t_a^{\,1-iZ}(t_a-1)^{iZ} - t_b^{\,1-iZ}\,(t_b-1)^{iZ}}{t_a-t_b} \tag{E18}$$

To evaluate the matrix element when $Z = 0$, that is, for a plane wave electron, we use the representation D5 and do the integrals directly.

Of course, we can simply take the limit $Z \to 0$ of the integrals for finite Z.

Prof. Mathews has worked out another method for doing integrals of the type obtainable from L(12). The method follows from the remark that the Laplace transform of a product of two functions can be expressed as a convolution integral over the individual Laplace transforms of the two functions.

APPENDIX F

Traces

Since all the traces in this problem contract with $G_\sigma G_\lambda - G^2 \delta_{\sigma\lambda}$ it is simpler first to contract and then to take the trace of the contracted quantity.

The projection operators $\frac{k\!\!\!/}{2k}$ and $\frac{1+\gamma_4}{2}$ project out electron and muon states. The operator $\frac{1+i\gamma_1\gamma_2}{2}$ projects out spin-up muons, and the other operator, $\frac{1+i\gamma_5}{2}$, comes from the coupling.

The traces we need are

$$
\begin{aligned}
\ell_{1,\sigma\lambda} &= \text{Trace } k\!\!\!/ \,\gamma_\sigma (1+\gamma_4)(1+i\gamma_1\gamma_2)\gamma_\lambda (1+i\gamma_5) \\
\ell_{2,\sigma\lambda} &= \text{Trace } k\!\!\!/ \,\gamma_\sigma \gamma_p (1+\gamma_4)(1+i\gamma_1\gamma_2)\gamma_p \gamma_\lambda (1+i\gamma_5) \\
\ell_{3,\sigma\lambda} &= \text{Trace } k\!\!\!/ \,\gamma_\sigma \gamma_k (1+\gamma_4)(1+i\gamma_1\gamma_2)\gamma_k \gamma_\lambda (1+i\gamma_5) \\
\ell_{4,\sigma\lambda} &= \text{Trace } k\!\!\!/ \,\gamma_\sigma (1+\gamma_4)(1+i\gamma_1\gamma_2)\gamma_p \gamma_\lambda (1+i\gamma_5) \\
\ell_{5,\sigma\lambda} &= \text{Trace } k\!\!\!/ \,\gamma_\sigma (1+\gamma_4)(1+i\gamma_1\gamma_2)\gamma_k \gamma_\lambda (1+i\gamma_5) \qquad \text{(F1)} \\
\ell_{6,\sigma\lambda} &= \text{Trace } k\!\!\!/ \,\gamma_\sigma \gamma_p (1+\gamma_4)(1+i\gamma_1\gamma_2)\gamma_k \gamma_\lambda (1+i\gamma_5) \\
\ell_{7,\sigma\lambda} &= \text{Trace } k\!\!\!/ \,\gamma_\sigma (1+\gamma_4)(1+i\gamma_1\gamma_2)\gamma_\lambda \gamma_k \gamma_4 (1+i\gamma_5) \\
\ell_{8,\sigma\lambda} &= \text{Trace } k\!\!\!/ \,\gamma_\sigma (1+\gamma_4)(1+i\gamma_1\gamma_2)\gamma_\lambda \gamma_p \gamma_4 (1+i\gamma_5)
\end{aligned}
$$

The matrices γ_p and γ_k are spacelike.

$$\gamma_p = \vec{\gamma}\cdot\hat{p} \qquad\qquad \gamma_k = \vec{\gamma}\cdot\hat{k}$$

We can eliminate many terms in the traces because the trace of an odd number of γ-matrices vanishes and all pseudo-tensors will vanish after contraction with the symmetric quantity $G_\sigma G_\lambda - G^2\delta_{\sigma\lambda}$ and integration over momenta.

Let

$$L_i = \frac{1}{4k}\, \ell_{i,\sigma\lambda}\, (G_\sigma G_\lambda - G^2 \delta_{\sigma\lambda}) \tag{F2}$$

The explicit expressions for the L_i's are:

$$\begin{aligned}
L_1 &= [\,3W^2 - 4Wk + 2py\,(W-2k) - p^2\,] \\
&\quad + [\,W^2 - 4Wk - 2py\,(W+2k) - p^2 - 2p^2y^2\,]\cos\alpha \\
L_2 &= [\,3W^2 - 4Wk + 2py\,(W-2k) - p^2\,] \\
&\quad - [\,W^2 - 4Wk - 2py\,(W-2k) - p^2 - 2p^2y^2\,]\cos\alpha
\end{aligned} \tag{F3}$$

$$L_3 = L_1$$

$$\begin{aligned}
L_4 &= [\,(W^2 - 4Wk)y - 2p\,(W + 2ky^2) - 3p^2y\,] \\
&\quad + [\,3W^2 - 4Wk + 2py\,(W-2k) - p^2\,]\,y\cos\alpha \\
L_5 &= [\,W^2 - 4Wk - 2py\,(W-2k) - p^2 - 2p^2y^2\,] \\
&\quad + [\,3W^2 - 4Wk + 2py\,(W-2k) - p^2\,]\cos\alpha
\end{aligned}$$

$$L_6 = \left[3W^2 - 4Wk + 2py\,(W-2k) - p^2 \right]$$

$$+\left[-(W^2 - 4Wk)y - 2p\,(W+2ky^2) - 3p^2 y \right] \cos\alpha$$

$$L_7 = -L_1$$

$$L_8 = -y\,L_1 - 2p\,(1-y^2)(W-k) - 2p\,(1-y^2)\,(k+py)\cos\alpha$$

In writing out the L_i's $\hat{p}\cdot\hat{z}$ was replaced everywhere by $\hat{p}\cdot\hat{k}\;\hat{k}\cdot\hat{z}$. This is explained in Appendix G.

APPENDIX G

The Cross Term $M_1 M_4$

To first order in Z the contribution to the rate from the cross term $M_1\, M_4$ is given by

$$\Delta R = \frac{G^2}{192\pi^3} \; \frac{256\,\mu^5}{\pi} \int dp \int dy \int dk \; \frac{p^2 k^2}{(p^2+\mu^2)^2} \;\times$$

$$\times \; \frac{Z\Lambda}{2\mu} \sum_{i=0}^{4} 2K_i\, t_i$$

where $2K_i = J_i(k) + J_i(-k)$

and the t_i's and J_i's are defined below.

$$t_o = 3W^2 - 4Wk + 2py\,(W-2k) - p^2$$

$$t_1 = W^2 - 4Wk - 2py\,(W-2k) - p^2 - 2p^2y^2$$

$$+ 2\,(Wk-p^2)(1-y^2)$$

$$t_2 = (W^2 - 4Wk)y - 2p\,(W+2ky^2) - 3p^2y$$

$$+ 2p\,(W-k)\,(1-y^2)$$

$$t_3 = (W^2 - 4Wk)y - 2p\,(W+2ky^2) - 3p^2y$$

$$t_4 = W^2 - 4Wk - 2py\,(W+2k) - p^2 - 2p^2y^2$$

$$J_o = \frac{1}{p}\,\frac{1}{(p-k)^2-q^2}\,\tan^{-1}\frac{p}{\mu} - \frac{1}{q}\,\frac{1}{p^2-(q+k)^2}\,\tan^{-1}\frac{q+k}{\mu}$$

$$J_1 = -\frac{1}{p}\,\frac{1}{(p-k)^2-q^2}\,\tan^{-1}\frac{p}{\mu} + 2k\,\frac{1}{\left[(p-k)^2-q^2\right]^2}\,\tan^{-1}\frac{p}{\mu}$$

$$+\,\mu\,\frac{1}{(p-k)^2-q^2}\,\frac{1}{p^2+\mu^2} + \frac{p^2}{q^3}\,\frac{1}{p^2-(q+k)^2}\,\tan^{-1}\frac{q+k}{\mu}$$

$$-\,\frac{2kp^2}{q^2}\,\frac{1}{\left[p^2-(q+k)^2\right]^2}\,\tan^{-1}\frac{q+k}{\mu} - \frac{\mu p^2}{q^2}\,\frac{1}{p^2-(q+k)^2}\,\frac{1}{(q+k)^2+\mu^2}$$

$$J_2 = 2k\,\frac{1}{\left[(p-k)^2-q^2\right]^2}\,\tan^{-1}\frac{p}{\mu} + \frac{kp}{q^3}\,\frac{1}{p^2-(q+k)^2}\,\tan^{-1}\frac{q+k}{\mu}$$

$$-\,\frac{2p(q+k)}{q}\,\frac{1}{\left[p^2-(q+k)^2\right]^2}\,\tan^{-1}\frac{q+k}{\mu} - \frac{\mu pk}{q^2}\,\frac{1}{p^2-(q+k)^2}\,\frac{1}{(q+k)^2+\mu^2}$$

$$J_3 = 2k\,\frac{1}{\left[(p-k)^2-q^2\right]^2}\,\tan^{-1}\frac{p}{\mu} + \mu\,\frac{1}{(p-k)^2-q^2}\,\frac{1}{p^2+\mu^2}$$

$$-\,2\,\frac{kp(q+k)}{q^2}\,\frac{1}{\left[p^2-(q+k)^2\right]^2}\,\tan^{-1}\frac{q+k}{\mu}$$

$$-\,\frac{\mu p(q+k)}{q^2}\,\frac{1}{p^2-(q+k)^2}\,\frac{1}{(q+k)^2+\mu^2}$$

$$+\,\frac{pk}{q^3}\,\frac{1}{p^2-(q+k)^2}\,\tan^{-1}\frac{q+k}{\mu}$$

$$
\begin{aligned}
J_4 = \; & 2k \frac{1}{\left[(p-k)^2 - q^2\right]^2} \tan^{-1} \frac{p}{\mu} \\
& - \frac{2k(q+k)^2}{q^2} \frac{1}{\left[p^2 - (q+k)^2\right]^2} \tan^{-1} \frac{q+k}{\mu} \\
& - \frac{\mu k(q+k)}{q^2} \frac{1}{p^2 - (q+k)^2} \frac{1}{(q+k)^2 + \mu^2} \\
& + \frac{k^2}{q^3} \frac{1}{p^2 - (q+k)^2} \tan^{-1} \frac{q+k}{\mu}
\end{aligned}
$$

APPENDIX H

The Cross-Term M_2M_3

Aside from a common overall factor, the matrix element $(N_1 + N_2 + N_3)_\sigma$ is

$$-\left[1-iZ - \left(1+\frac{Z}{\mu}k\right) t_o\right] \quad (\bar{u}_{\vec{k}}\, \gamma_\sigma\, a\, u_o)$$

$$+\left[\frac{1}{2}\, iZ\, t_o\, (\bar{u}_{\vec{k}}\, \gamma_4 \gamma_k \gamma_\sigma\, a\, u_o) - \frac{1}{2}\, Z\, t_o\, \frac{p}{\mu}\, (\bar{u}_{\vec{k}}\, \gamma_4 \gamma_p\, \gamma_\sigma\, a\, u_o)\right]$$

$$+\left[-\Lambda\, \frac{p}{\mu}(1-iZ-t_o)\, (\bar{u}_{\vec{k}}\, \gamma_\sigma\, a\, \gamma_p u_o) \right. \tag{H1}$$

$$\left. + iZ\Lambda\, \frac{t_o}{\mu}\, (\bar{u}_{\vec{k}}\, \gamma_\sigma\, a\, \gamma_k\, u_o)\right]$$

where

$$t_o = -\frac{1}{2}\, \frac{p^2+\mu^2}{(\vec{p}\cdot\vec{k}-i\mu k)} = a + ib \tag{H2}$$

We need $2\,\mathrm{Re}\ M_2M_3$ to lowest order in $\frac{Z}{137}$. Now

$$2\,\mathrm{Re}\, N_{2\sigma} N_{3\lambda} \approx 2\,\mathrm{Re}\left[\frac{1}{2}\, iZ\, t_o\, (\bar{u}_{\vec{k}}\gamma_4\gamma_k\,\gamma_\sigma\, a u_o)\right.$$

$$\left. -\frac{1}{2}\, Z\, t_o\, \frac{p}{\mu}\, (\bar{u}_{\vec{k}}\, \gamma_4\, \gamma_p\, \gamma_\sigma\, a\, u_o)\right] (-1)\Lambda\, \frac{p}{\mu}(\bar{u}_o\, \gamma_p\, \gamma_\lambda\, a\, u_{\vec{k}})$$

$$= -\Lambda\frac{p}{\mu}(-Zb)\, (\bar{u}_{\vec{k}}\, \gamma_4\gamma_k\gamma_\sigma\, a\, u_o)\, (\bar{u}_o\gamma_p\,\gamma_\lambda\, a\, u_{\vec{k}}) \tag{H3}$$

$$-\Lambda\frac{p}{\mu}\left(-Za\,\frac{p}{\mu}\right)(\bar{u}_{\vec{k}}\gamma_4\,\gamma_p\gamma_\sigma\, a\, u_o)\,(\bar{u}_o\gamma_p\,\gamma_\lambda\, a\, u_{\vec{k}})$$

If we now contract with $G_\sigma G_\lambda - G^2 \delta_{\sigma\lambda}$ and sum over spins, the spinor terms are replaced by traces and we get for the cross-term $M_2 M_3$ to lowest order

$$\frac{G^2}{192\pi^3} \frac{256\mu^5}{\pi} \int dp \int dy \int dk \frac{p^2 k^2}{(p^2+\mu^2)^4} \times$$

$$\times \left[Z\Lambda b \frac{p}{\mu} (-1)(W^2 - 4Wk)y \right. \tag{H4}$$

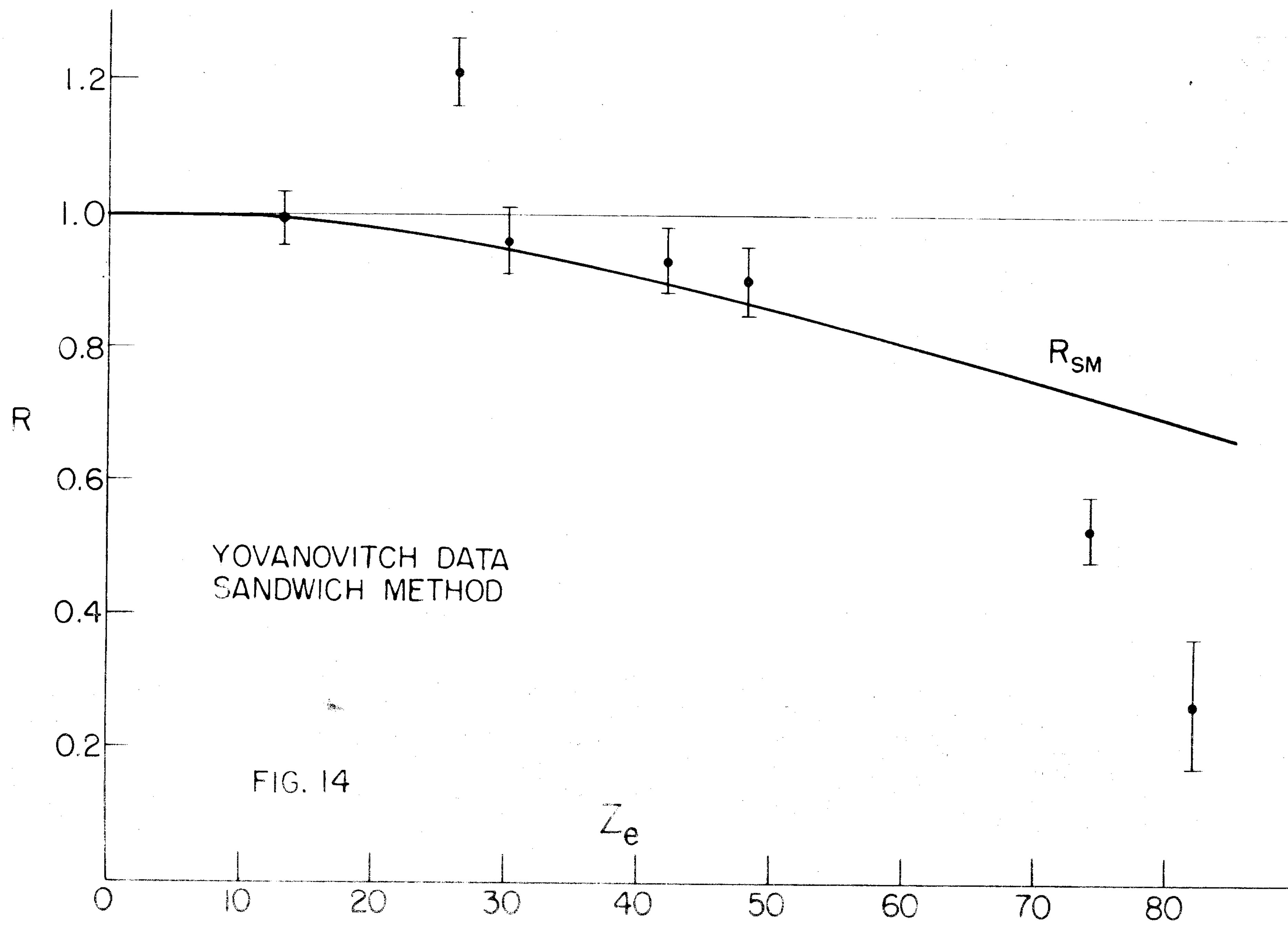
$$\left. + Z\Lambda a \frac{p^2}{\mu^2} (W^2 - 4Wk) \right]$$

Since a and b are $O(\mu)$, this term would be $O(Z\Lambda\mu)$, but upon integration it vanishes to this order since it is odd in y (a is odd and b is even).

APPENDIX I

Experimental Data

Figures 14 and 15 show the experimental points obtained by Yovanovitch (2). For comparison we also show R_{SM}. Since we estimate that the curve R_{SM} is a little lower than the true R (about 15% low at Z = 82) the disagreement with experiment at high Z is even worse than it appears.



1.2
1.0
0.8
0.6
0.4
0.2
R
R_SM
YOVANOVITCH DATA
SANDWICH METHOD
Z_e
0
10
20
30
40
50
60
70
80

FIG. 14

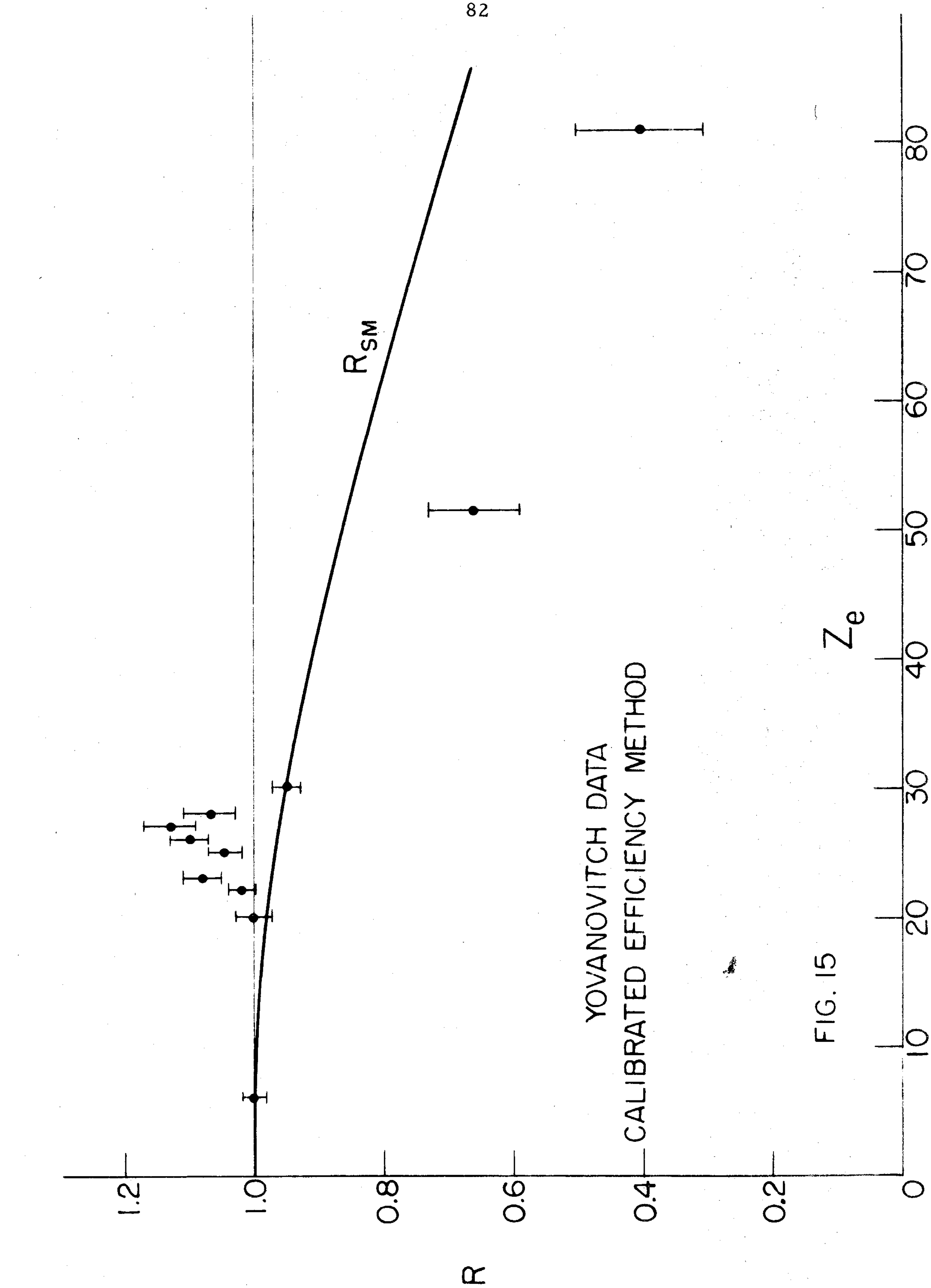

R
1.2
1.0
0.8
0.6
0.4
0.2
0
10
20
30
40
50
60
70
80
Z_e
R_{SM}
YOVANOVITCH DATA
CALIBRATED EFFICIENCY METHOD

FIG. 15

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