Symmetry, Reduction and Swimming in a Perfect Fluid

Thesis by
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In Partial Fulfillment of the Requirements
for the Degree of
Doctor of Philosophy

California Institute of Technology
Pasadena, California

2003

(Defended 23 May 2003)

Acknowledgements

Many, many people have helped and encouraged me along the way. In particular I would like to thank:

- Joel Burdick, my advisor, for providing me with guidance, support and freedom, and for having the faith, patience and stamina to see me through.
- Jill Bush for being my constant companion and supporter.
- My friends and colleagues Andrew Lewis, Francesco Bullo and Scott Kelly. Each of your styles and interests have influenced me and this thesis would not have been possible without you.
- Jerry Marsden for his help, encouragement and example.
- Todd and Shelley Murphey who provided me with relentless proactive support. It meant a lot to me, and without it I might not have made it to the end.
- Emma Huang, Nikki Takarabe, Sian Romoli, Mo Salama and Greg Starbird for their friendship and for bringing smiles and fun to my life.
- Bill Caldwell for taking the time to help and teach me throughout the years, for the opportunities you have provided me, and for your friendship.
- My grad student friends Michael Scott, Mark Long, Kristo Kriechbaum and Hans Hoeg for enriching my life.
- My family for all they have done for me.
- My friend David Caldwell for providing a constant stream of interestring distractions that helped me keep my sanity.
- My friends and colleagues Patricio Vela, Kristi Morgensen, Milŏs Žefran, Bill Goodwine, Jim Ostrowski and Howie Choset for lots of ideas and discussions.

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Abstract

This thesis presents a geometric picture of a deformable body in a perfect fluid and a way to approximate its dynamics and the motion, resulting from cyclic shape deformations, of the body and, interestingly, the fluid as well. Emphasis is placed on the group structure of the configuration space of the body fluid system and the resulting symmetry in their equations of motion. Symmetry is also used to reduce a series expansion for the flow of a time dependent vector field in order to obtain a novel expansion for the path-ordered exponential. This can be used to approximate holonomy, or geometric phase, in a principal bundle when its evolution is governed by a connection on the bundle and it is subject to periodic shape inputs. Simple models for swimming in and the stirring of a perfect fluid are proposed and examined.

Contents

1 Introduction **1**
1.1 Outline of the Thesis 2

2 Background in Differential Geometry **5**
2.1 Manifolds and Tensors 5
2.1.1 A Simplified Notation 5
2.1.2 Differentiation and Integration 6
2.1.3 Volume and Tangency 7
2.2 Lie Groups and Associated Constructions 8
2.2.1 Lie Groups and Lie Algebras 8
2.2.2 Actions 9
2.2.3 Left and Right Actions and Differing Signs 9
2.2.4 Principal Bundles 9
2.2.5 Associated Bundles and Equivariance 10
2.2.6 A Section/Function Equivalence on Associated Bundles . . 10
2.2.7 The Associated Adjoint Bundle 10
2.2.8 Connections and the Horizontal Subspace 11
2.2.9 Curvature 11
2.2.10 The Covariant Derivative 12
2.2.11 Local Forms 12
2.3 Riemannian Geometry 13
2.3.1 Isometries 15
2.3.2 The Isometry Action 15
2.3.3 Transport Theorems 16
2.3.4 A Metric on Diffeomorphisms 16
2.3.5 Variations 16

3 Simple Mechanical Systems **19**
3.1 General Simple Mechanical Systems 19
3.2 Partial Symmetry 20
3.3 The Mechanical Connection 21
3.4 The Metric Lagrange-Poincaré Equations 22

4 A Deformable Body in Fluid 25

4.1 A Geometric Framework 26
4.1.1 The Diffeomorphism Group 27
4.1.2 Generalized Flows 28
4.1.3 Assumptions on the Nature of the Flow at Infinity 28
4.1.4 A Body in a Fluid 29
Invariance of the Body Fluid Metric 30
4.2 Euler's Equation 30
4.2.1 The Euler Fluid 30
4.2.2 The Rigid Body 32
4.3 The Fluid Body Group 33
4.3.1 The Momentum of a body in a Fluid 35
4.4 A Rigid Body in a Perfect Fluid 35
4.4.1 The Fluid Part 37
Fluids in a Moving Frame 38
4.4.2 The Rigid Part 39
4.4.3 The Kirchhoff Potential 41
4.4.4 Kirchhoff's Equations 41
4.5 The Geometry of the Fluid Body Group 42
4.5.1 Trivialities 43
4.5.2 A Simply Connected Body in the Plane 43
4.5.3 Construction of a Global Section 44
The Trivial Ellipse 46
4.5.4 A Multiply Connected Body in the Plane 46
4.5.5 Stirring in a Perfect Fluid 47
4.6 Body Deformations 49

5 Reduced Series Expansions 51

5.1 Motivation and Background 52
5.1.1 Local Expansion of the Group Displacement 53
5.2 The Chen Series 54
5.3 Exponential Series Expansions 56
5.3.1 The Magnus Series 56
5.3.2 Sussmann's Log Series 57
5.3.3 Sussmann's Product Series 57
5.4 Reduction 57
5.4.1 The Reduced Brackets 58
5.4.2 Holonomy 59
5.4.3 Controllability 59
5.4.4 The Reduced Series 60

6 The Fish and The Hat 63

6.1 Articulated Rigid Bodies 64
6.1.1 The Propeller Paradox? 66
6.2 Simple Planar Satellite 67

6.2.1 The Center of Added Mass 69
6.3 The Hat 69
6.3.1 Approximating a Stir 71

7 Future Work **75**
Geometric Modeling of Fluids 75
Swimming and Stirring 75
Series Expansions 76

Chapter 1

Introduction

The symmetries present in the description of a deformable body in a perfect fluid provide a useful structure that can both serve to unify the treatment of their equations of motion and aid in the analysis of swimming and stirring motions.

In particular, *I show the system to be a simple mechanical system with symmetry whose configuration space is a principal bundle.*[1] This structure codifies a natural splitting of its configuration into a shape space, assumed to be under control, and a Lie group measuring the progress of the system in its environment. In the case of swimming, cyclic undulations effect net translations and rotations of the body. In the case of stirring, motion of a paddle induces swirls of the fluid.

In both cases, the evolution of the system is governed by a connection on the principal bundle. *I provide a series expansion for the geometric phase or holonomy generated by a non-Abelian principal connection in terms of the covariant derivatives of its curvature along with geometric properties of its shape deformation.* This series is an alternative to the path ordered exponential. Though it is an extension of known results for holonomy in terms of the curvature, it will be shown to result from the reduction, made possible by the symmetry and periodic inputs, of a general series for the flow of a time dependent vector field representing the flow of the full system to a flow on the adjoint bundle and eventually to the Lie algebra.

It is well known that the Euler equations for the motion of a rigid body and for the motion of an inviscid incompressible fluid can be thought of as invariant geodesic equations. The configuration space of each is a Lie group and their corresponding kinetic energy, thought of as a metric, are invariant with respect to that group. The principle of least action implies that the system will evolve along the geodesics of the metric, and the symmetry implies that the equations can be reduced to the Lie algebra.

The equations of motion for a rigid body in an irrotational fluid are also known to have this structure. It is shown here that the irrotational restriction is not necessary. *Even with vorticity the combined body fluid system evolves along geodesics of an invariant metric on what I call the fluid body group. The configuration space*

[1] In these paragraphs my specific contributions have been italicized. A detailed literature review can be found in the beginning of each of the relevant chapters.

of the deformable body is a principal bundle over the shape of the body with this as its group.

Apart from deformations, the fluid body group itself, *the configuration of a rigid body in a fluid, is shown to be a principal bundle over the position and orientation of the body.* In the context of holonomy and motion generation through cyclic deformations, this bundle represents the configuration of the stirring problem. In two dimensions it is shown that this bundle is mathematically non trivial when the stirring rod is not simply connected. This property captures some of the well known topological features of flow in two dimensions leading to, for example, thorough mixing through braid motions. In this case the local motions captured by the series above only begin to capture the richness of ways in which the fluid can be swirled.

Deformable bodies in an inviscid incompressible fluid are governed by equations of the same form as those of an articulated satellite. Both are based on conservation of angular momentum. Whereas in space the satellite can only reorient itself, the same system in water becomes a fish that can swim. *I examine some of the features of swimmers in an ideal fluid by approximating them as mechanically coupled but hydrodynamical decoupled bodies.* These simple models can serve as the basic building blocks of low dimensional models of swimming that can be analyzed and controlled given that many features of these systems persist after adding viscosity and other complexity.

1.1 Outline of the Thesis

This thesis is presented in the language of differential geometry. Relevant concepts in manifolds, tensors, Lie groups, principal bundles and Riemannian geometry are reviewed in Chapter 2.

The equations of motion for a simple mechanical system with symmetry are derived in Chapter 3. They are the general equations for all the examples in this thesis.

Chapter 4 begins with a review of the Euler rigid body and the Euler fluid as invariant simple mechanical systems on Lie groups. A rigid body in an inviscid fluid is show to have this same structure. The geometry of its configuration space, the fluid body group, is then discussed. The group is shown to have a principal bundle structure itself. This is then discussed as the configuration space for stirring. The configuration space for deformable body in a fluid is then shown to be a principal bundle over its shape with the fluid body group as its structure group.

Chapter 5 reviews the structure of driftless control systems governed by a connection on a principal bundle and previous attempts to compute expansions for the holonomy of these systems in terms of the curvature of the connection. Then a review is presented of general series expansions for the flow of time dependent vector fields. The Lie brackets that appear in these series are shown to reduce in the presence of symmetry. The controllability conditions are then reduced, followed by the series themselves, generalizing the previously known results and giving a new expansion for the path-ordered exponential.

Chapter 6 applies some of the results of Chapter 5 to swimming and stirring examples shown to have the requisite structure in Chapter 4. Simple models for swimmers in inviscid fluids are proposed that consist of articulated rigid bodies that are approximated as mechanically coupled but hydrodynamically decoupled. A simple model of a propeller driven submarine is discussed in the context of understanding the properties added mass. Then a three link manipulator, which can only reorient itself in space is shown to be completely controllable when immersed in a fluid. The idea of controlling a fluid via stirring is then examined in an example.

Chapter 2

Background in Differential Geometry

I will be using the notation and tools of differential geometry. The reader is assumed to be familiar with the concepts of manifolds and tensors, and the standard methods of integration and differentiation on and of them. In order to make a self-contained presentation and to establish notation, this section reviews some of the mathematical concepts and tools used in this thesis. Additional information can be found in any number of books. This thesis typically uses the notation and conventions found in [Abraham, Marsden, and Ratiu, 1991]. Topics like principal bundles and Riemannian geometry not covered therein can be found in [Kolář, Slovák, and Michor, 1993, Nakahara, 1992].

2.1 Manifolds and Tensors

On a smooth manifold M, let $C^\infty(M)$ be the space of smooth functions mapping M to $\mathbb{R}$, let $\mathfrak{X}(M)$ be the space of vector fields on M. Let $\Omega^k(M, V)$ be the space of k-forms, alternating or skew-symmetric multi-linear functions of k vector fields taking values in the vector space V. The vector space V defaults to $\mathbb{R}$ if it is unspecified.

The interior product $\mathbf{i}_X\, \alpha$ of a vector field $X \in \mathfrak{X}(M)$ and a k-form $\alpha \in \Omega^k(M)$ is a $(k-1)$-form, which, when applied to vectors $Y_1, \ldots, Y_{k-1} \in \mathfrak{X}(M)$ yields

$$(\mathbf{i}_X\, \alpha)(Y_1, \ldots, Y_{k-1}) = \alpha(X, Y_1, \ldots, Y_{k-1}).$$

There is a natural pairing between a space and its dual, denoted $\langle \cdot, \cdot \rangle$, such that $\mathbf{i}_X\, \alpha = \alpha(X) = \langle X, \alpha \rangle$ for $X \in \mathfrak{X}(M)$ and $\alpha \in \Omega^1(M)$.

The wedge product of a k-form $\alpha \in \Omega^k(M)$ and an l-form $\beta \in \Omega^l(M)$ is a $(k+l)$-form $\alpha \wedge \beta$. The wedge product is associative and bi-linear. Any m-form where m is greater than the dimension of the manifold is zero.

2.1.1 A Simplified Notation

Adjacency is adopted for composition and the existence of derivatives is inferred from an object's position. Let $q\colon M \to N$, $f \in C^\infty(N)$, $v \in \mathfrak{X}(M)$ and $u \in \Omega^1(N)$. Since $v\colon M \to TM$ and $u\colon N \to T^*N$, then $uq\colon N \to T^*_q M$ and $qv\colon M \to T_q N$

make sense. In both uq and qv adjacency is taken to mean composition and in the second q is interpreted to mean its tangent map $T_xq\colon T_xM \to T_{q(x)}N$ since q itself cannot be composed before v but Tq can. The following expressions are then well defined

$$\begin{aligned} q^*f &= fq && \in C^\infty(M) \\ q_*v &= qvq^{-1} && \in \mathfrak{X}(N) \\ q^*u &= quq && \in \Omega^1(M) \\ q^*\langle q_*v, u\rangle &= \langle v, q^*u\rangle && \in C^\infty(M). \end{aligned}$$

This notation is natural for linear operators since they are their own derivatives and, with the above conventions, it becomes unambiguous even in the nonlinear case.

2.1.2 Differentiation and Integration

The exterior derivative of a k-form, $\alpha \in \Omega^k(M)$, is a $(k+1)$-form, $\mathbf{d}\,\alpha$. The operator $\mathbf{d}$ is linear and satisfies the relation $\mathbf{d}\circ\mathbf{d} = \mathbf{d}^2 = 0$.

The exterior derivative $\mathbf{d}$ and the interior product $\mathbf{i}_X$ are graded derivations. That is, if $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^l(M)$ and $X \in \mathfrak{X}(M)$ then

$$\begin{aligned} \mathbf{d}(\alpha\wedge\beta) &= \mathbf{d}\,\alpha\wedge\beta + (-1)^k\alpha\wedge\mathbf{d}\,\beta \\ \mathbf{i}_X(\alpha\wedge\beta) &= (\mathbf{i}_X\,\alpha)\wedge\beta + (-1)^k\alpha\wedge\mathbf{i}_X\,\beta. \end{aligned} \tag{2.1}$$

The Lie derivative of a vector field Y in the direction of a vector field X is another vector field defined to be

$$\mathscr{L}_X Y = \left.\frac{d}{dt}\right|_{t=0} x_t^* Y, \tag{2.2}$$

where $x_t\colon M \to M$ is the flow of X, i.e. the solution of $\dot{x}_t(x) = X(x_t(x))$ and $x_0(x) = x$ for all $x \in M$. This will often be written $X = \dot{x}_t x_t^{-1}$. The definition of the Lie derivative can be extended by replacing Y in (2.2) with any function, form or tensor.

The Lie derivative of vector fields is also called the Jacobi-Lie bracket, $[X,Y] \equiv \mathscr{L}_X Y$. When written this way, the reader should have in mind its equivalent definition as the commutator of vector fields. It is the unique vector field $[X,Y]$ such that $[X,Y]f = (XY - YX)f$ for all functions $f \in C^\infty(M)$ where a vector field acting on a function is defined by $Xf \equiv \mathscr{L}_X f$.

Theorem (Cartan's Magic Formula). *Cartan's formula relates the Lie derivative, the interior product, and the exterior derivative. For any vector field $X \in \mathfrak{X}(M)$ the following "magic" formula holds*

$$\mathscr{L}_X = \mathbf{i}_X\,\mathbf{d} + \mathbf{d}\,\mathbf{i}_X\,. \tag{2.3}$$

The exterior derivative, $\mathbf{d}$, commutes with the Lie derivative, $\mathscr{L}_X$, for all vector fields $X \in \mathfrak{X}(M)$. This can be seen by using (2.3) and $\mathbf{d}^2 = 0$,

$$\mathscr{L}_X \mathbf{d} = \mathbf{i}_X \mathbf{d}\,\mathbf{d} + \mathbf{d}\,\mathbf{i}_x \mathbf{d} = \mathbf{d}\,\mathbf{d}\,\mathbf{i}_X + \mathbf{d}\,\mathbf{i}_x \mathbf{d} = \mathbf{d} \mathscr{L}_X. \tag{2.4}$$

The following is a useful relationship between a one-form $\omega \in \Omega^1(M)$, two vector fields $X, Y \in \mathfrak{X}(M)$ and their derivatives,

$$X(\omega(Y)) - Y(\omega(X)) = \omega([X,Y]) + d\omega(X,Y). \tag{2.5}$$

2.1.3 Volume and Tangency

On an n-dimensional manifold M, n-forms are particularly important. The classical determinant is the unique, up to a scaling, alternating (skew-symmetric) multi-linear function of n vectors on an n-dimensional vector space. n-forms (alternating n-multi-linear tensors) encode orientation and volume, and differ from each other by no more than a function, i.e. a scalar at each point. The determinant (of the Jacobian) of a map, $f\colon M \to M$, is the unique function such that $f^*\omega = (\det f)\omega$ for all $\omega \in \Omega^n(M)$. f is said to be *orientation preserving* if $\det f$ is positive. These properties lead to integration being naturally defined on n-forms.

Theorem (Change of Coordinates). *Given a diffeomorphism, a smooth map whose inverse is also smooth, $f\colon N \to M$, that is orientation preserving, and an n-form $\omega \in \Omega^n(M)$, the following relationship holds:*

$$\int_M \omega = \int_N f^*\omega.$$

This is generally true for compact manifolds and, we assume, true for non-compact manifolds assuming that at least one of the integrals converges.

n-forms that are everywhere non-vanishing are called volume forms. Let μ be a volume form and let v be a vector field on a manifold M. If $\mathbf{i}_v\,\mu$ is restricted to the boundary ∂M of M then $\mathbf{i}_v\,\mu$ is a volume form on that boundary as long as v is not zero and is not tangent to the boundary. Notice that if v is tangent to the boundary, then there cannot possibly be $n-1$ independent vectors $X_1, \ldots, X_{n-1}$ parallel to the boundary such that $(\mathbf{i}_v\,\mu)(X_1, \ldots, X_{n-1}) \neq 0$, since v will be parallel to some linear combination of them. Therefore $\mathbf{i}_v\,\mu$ depends only on the component of v normal to the boundary as the tangential component will not contribute. This can be summarized as follows:

$$\mathbf{i}_v\,\mu|_{\partial M} = 0 \quad \Leftrightarrow \quad v \| \partial M.$$

Since it is linear in v, $\mathbf{i}_v\,\mu|_{\partial M}$ represents the flux of v through the boundary or the classical expression $(v \cdot n)dA$, where n is the unit normal to ∂M. The term $\mathbf{i}_v\,\mu$ arises naturally in the following two classical theorems.

Theorem (Stokes' Theorem). *Let M be an n-dimensional manifold. For any*

$(n-1)$-form $\omega \in \Omega^{n-1}(M)$,

$$\int_M \mathbf{d}\,\omega = \int_{\partial M} \omega. \tag{2.6}$$

This generalizes the standard theorems of Green, Gauss and Stokes to arbitrary dimensions. The following statement shows how the divergence theorem is also a simple consequence of Stokes' Theorem.

Theorem (Divergence Theorem). *Let $\phi \in C^\infty(M)$, $v \in \mathfrak{X}(M)$, and $\mu \in \Omega^n(M)$ a volume form. The divergence is defined (with respect to μ) by $(\mathrm{div}_\mu\, v)\mu \equiv \mathscr{L}_v \mu$. It then hold that*

$$\int_M (\mathscr{L}_v \phi)\mu = \int_{\partial M} \phi\, \mathbf{i}_v\, \mu + \int_M (\mathrm{div}_\mu\, v)\phi\mu. \tag{2.7}$$

Proof. Using first (2.3) and (2.6) and then fact that $(n+1)$-forms are zero it follows that

$$\int_M (\mathscr{L}_v \phi)\mu = \int_M (\mathbf{i}_v\, \mathbf{d}\,\phi)\mu = \int_M \mathbf{d}\,\phi \wedge \mathbf{i}_v\, \mu + \int_M \mathbf{i}_v(\mathbf{d}\,\phi \wedge \mu).$$

The result is obtained by applying the Leibniz property (2.1) of $\mathbf{i}_v$ and $\mathbf{d}$

$$= \int_M \mathbf{d}(\phi \wedge \mathbf{i}_v\, \mu) + \int_M \phi \wedge \mathbf{d}\, \mathbf{i}_v\, \mu$$

and then Stokes's Theorem (2.6). □

2.2 Lie Groups and Associated Constructions

Objects invariant under the action of a group are said to have a symmetry. Since this concept is fundamental to this thesis, Lie groups are reviewed.

2.2.1 Lie Groups and Lie Algebras

A *Lie group* is a manifold G endowed with a smooth, associative and invertible binary operation called group multiplication. A Lie group G is said to act on a manifold Q from the left if there is a smooth operation $L\colon G \times Q \to Q$, written $L_g(q) = gq$ for $g \in G$ and $q \in Q$, such that $L_g(L_h(q)) = L_{gh}(q) = ghq$ for all $g, h \in G$ and $q \in Q$. Similarly, a right action $R\colon G \times Q \to Q$, written $R_g q = qg$, satisfies $R_g(R_h(q)) = R_{hg}(q) = qhg$. The identity element is denoted by $e \in G$.

A tensor α on Q is said to be *invariant* with respect to the left (right) action of G on Q if $L_g^*\alpha = \alpha$ ($R_g^*\alpha = \alpha$) for all $g \in G$.

A *Lie algebra* is a vector space endowed with a skew symmetric bi-linear operation, called the *Lie bracket*, that satisfies the Jacobi identity. Each Lie group, G, has a natural Lie algebra, $\mathfrak{g}$, the space of left-invariant vector fields, whose Lie bracket is the Jacobi-Lie bracket. This is a vector space isomorphic to the tangent space of G at the identity element.

2.2.2 Actions

An *action* Φ of a Lie Group G on a manifold Q is a map $\Phi\colon G \times Q \to Q$ written $(g, q) \to \Phi_g(q)$. A right action $\Phi_g(q) = R_g q = qg$ must satisfy $q(gh) = (qg)h$. Similarly, a left action $\Phi_g(q) = L_g(q) = gq$ must satisfy $(gh)q = g(hq)$.

Associated to each $\xi \in \mathfrak{g}$ is a vector field ξ_Q on Q called the *infinitesimal generator* defined by

$$\xi_Q(q) \equiv \left.\frac{d}{dt}\right|_{t=0} \Phi_{e^{\xi t}}(q). \tag{2.8}$$

There is a natural left action $\mathrm{Ad}\colon G \times \mathfrak{g} \to \mathfrak{g}$ of G on its Lie algebra, called the *adjoint action* that is the derivative at the identity $\mathrm{Ad}_g \xi \equiv T_e \mathrm{I}_g$ of the conjugation map $I\colon G \times G \to G$ defined by $I_g(h) \equiv ghg^{-1}$. The infinitesimal generator of this action is the (lower case) *adjoint action* $\mathrm{ad}_\xi \equiv \xi_{\mathfrak{g}}$. The adjoint action corresponds to the Jacobi-Lie bracket of the associated left-invariant vector fields, so the same square bracket notation $\mathrm{ad}_\xi \eta = [\xi, \eta]$ will often be used for it.

2.2.3 Left and Right Actions and Differing Signs

Many expressions involving operators defined with respect to the action of a group on a manifold have signs that differ depending on whether the associated action is left, right, or a combination.

To simplify potentially cumbersome notation, this thesis uses the symbol $\cdot\!\!\vdash$ as a sign that changes based on whether the associated action is left, right or a combination. The symbol $\cdot\!\!\vdash$ should be interpreted as plus sign for the component of the action that is associated with a left action and minus for the component of the action associated with a right action. The symbol $\cdot\!\!\ulcorner$ is its negative. The obvious mnemonic is that when a plus should be plus for a left action the "L"-eft part of the plus is emphasized. Similarly, the "r"-ight part of the plus is emphasized for when the plus is a plus for a right action. For example, given the above definitions, the Lie algebra bracket and the Jacobi-Lie bracket are related by

$$[\xi, \eta]_Q = \cdot\!\!\ulcorner[\xi_Q, \eta_Q]. \tag{2.9}$$

2.2.4 Principal Bundles

Given a free and proper action Φ of a Lie group G on a manifold Q, the quotient Q/G is a manifold and the projection $\pi\colon Q \to Q/G$ is well defined. Such a manifold Q is called a *principal bundle* over the *base* space Q/G with *fibre* G. Q is locally isomorphic to $Q/G \times G$. If there exists a global isomorphism between Q and $Q/G \times G$, then the bundle is called *trivial.* A point q is said to be *over* a point $x \in Q/G$ if $\pi(q) = x$.

In what follows, expressions are given for the case of a left action. The corresponding versions for a right action often differ by a sign.

2.2.5 Associated Bundles and Equivariance

A principal bundle $\pi \colon Q \to Q/G$ and an action of G on a vector space V defines an *associated bundle* $\tilde{V} \equiv (Q \times V)/G$ with respect to that action. A typical element will be written $\overline{v} = [(q, v)] \in \tilde{V}$ where the equivalence class is defined, for a left action, by $[(q, v)] \equiv \{(gq, gv) : g \in G\}$.

Given an action of G on Q and a vector space V, a function $f \colon Q \to V$ is *equivariant* if for each $q \in Q$ and $g \in G$ it holds that $g(fq) = f(gq)$. This can be written more concisely as $gf = fg$.

Theorem (Generators on Equivariant Functions). *Given an action of G on a manifold Q and on a vector space V then*

$$\xi_Q f = \xi_V \circ f \tag{2.10}$$

holds for every equivariant ($fg = gf$) function $f \colon Q \to V$.

Proof.

$$(\xi_Q f)(q) = \left.\frac{d}{dt}\right|_{t=0} f(e^{t\xi} q) = \left.\frac{d}{dt}\right|_{t=0} e^{t\xi} f(q) = \xi_V(f(q)) = (\xi_V \circ f)(q)$$

where $q \in Q$ and $\xi \in \mathfrak{g}$. □

2.2.6 A Section/Function Equivalence on Associated Bundles

A *section* of a bundle $\pi \colon Q \to M$ is a map $s \colon M \to Q$ such that $\pi \circ s = \mathrm{Id}_M$. A vector field is a section of the tangent bundle.

The space of vector valued equivariant functions, $f \colon Q \to V$, is isomorphic to the space of sections, $s \colon Q/G \to \tilde{V}$, of the associated bundle $\tilde{V}$. A section s_f can be defined from a function f as $s_f(x) = [(q, f(q))]$. This definition makes sense for all $q \in Q$ over $x \in Q/G$ because of the equivariance of f. Conversely an equivariant function f_s can be defined in terms of the section $s(x) = [(q, v(q))]$ by $f_s = v$.

2.2.7 The Associated Adjoint Bundle

The *adjoint bundle*, $\tilde{\mathfrak{g}}$, is associated with the natural action of Ad on the vector space $\mathfrak{g}$, the Lie algebra of G. Lie algebra valued forms ψ on Q are therefore said to be Ad-*equivariant* if for all $g \in G$,

$$L_g^* \psi = \mathrm{Ad}_g \psi$$

for a left action, or

$$R_g^* \psi = \mathrm{Ad}_{g^{-1}} \psi$$

for a right action.

For a left action, if ψ is an Ad-equivariant Lie-algebra valued tensor on the full space Q, then

$$\begin{aligned}\mathscr{L}_{\xi_Q}\psi &= \left.\frac{d}{dt}\right|_{t=0} L^*_{e^{t\xi}}\psi \\ &= \left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}_{e^{t\xi}}\,\psi \\ &= \mathrm{ad}_\xi\,\psi. \end{aligned} \tag{2.11}$$

2.2.8 Connections and the Horizontal Subspace

On a principal bundle $\pi\colon Q \to Q/G$, the *vertical subspace* $VQ \subset TQ$ is defined as the kernel of the projection π or, equivalently, as the image of the infinitesimal generator, and is therefore isomorphic to $\mathfrak{g}$ at each point in Q.

A *connection* on a principal bundle Q is specified by an Ad-equivariant Lie algebra valued one-form $\omega \in \Omega^1(Q, \mathfrak{g})$ with the property that ω_Q is the identity on VQ. This implies that $\omega(\xi_Q) = \xi$ for all $\xi \in \mathfrak{g}$.

A connection ω defines a horizontal subspace HQ via the kernel of ω. This definition together with the defining properties of ω imply $TQ \equiv VQ \oplus HQ$.

There is an isomorphism between the tangent space at a point in the base and the horizontal subspace at any point in the fibre above it. The *horizontal lift* $X^h \in \mathfrak{X}(Q)$ of a base vector field $X \in \mathfrak{X}(Q/G)$ is the unique left-invariant horizontal vector field that projects to X.

The horizontal lift, $\alpha^h\colon [0,T] \to Q$, of a curve, $\alpha\colon [0,T] \to Q/G$, is the unique curve, starting at a given point in the fibre above $\alpha(0)$, which projects to α and has a horizontal derivative.

2.2.9 Curvature

The curvature $\Omega \in \Omega^2(Q, \mathfrak{g})$ of a connection $\omega \in \Omega^1(Q, \mathfrak{g})$ on a principal bundle $Q \to Q/G$ is an Ad-equivariant Lie algebra valued 2-form on Q defined to be

$$\Omega \equiv \mathrm{h}^*\,\mathbf{d}\,\omega,$$

where $\mathrm{h}\colon TQ \to HQ$ is the *horizontal projection* $\mathrm{h} \equiv \mathrm{id}_{TQ} - \omega_Q$.

Theorem (The Structure Equation). *Given a connection, ω, on a principal bundle, the curvature satisfies*

$$\Omega = d\omega - [\omega, \omega]. \tag{2.12}$$

Proof. First (2.12) is verified to hold if one of its arguments is vertical

$$i_{\xi_Q}\Omega = i_{\xi_Q}(d\omega - [\omega,\omega]) = \mathscr{L}_{\xi_Q}\omega - \mathrm{ad}_\xi\,\omega = 0$$

by using (2.11) and the Ad-equivariance of both ω and Ω. Second, If all the arguments are horizontal, then clearly $[\omega, \omega] = 0$ and $\Omega = \mathrm{h}^* \, d\omega = d\omega$. □

2.2.10 The Covariant Derivative

The *covariant derivative* of a curve $\overline{s}(t) \in \tilde{V}$ in an associated bundle is the derivative of the vector V portion of the equivalence class of $\overline{s}(t)$ as the Q part is made to follow a horizontal path. This is easily shown to be independent of the horizontal path that is chosen. Specifically let $q(t) \in Q$ be a curve over the projection $x(t) \in Q/G$ of $\overline{s}(t)$ and let $g(t) \in G$ be defined such that $q(t) = g(t)x^h(t)$, then $\overline{s}(t) = [(q(t), v(t))] = [(g(t)x^h(t), v(t))] = [(x^h(t), g^{-1}(t)v(t))]$. The covariant derivative is then

$$\begin{aligned}\nabla_{\dot{x}}\overline{s} \equiv \frac{D}{dt}\overline{s}(t) &\equiv [(x^h, \frac{d}{dt}g^{-1}v)] \\ &= [(x^h, g^{-1}\dot{v} - g^{-1}(\omega(\dot{q}))_V(v))] \\ &= [(q, \dot{v} - (\omega(\dot{q}))_V(v))]\end{aligned} \tag{2.13}$$

where

$$\omega(\dot{q}) = \omega(\dot{g}x^h + g\dot{x}^h) = \omega(\dot{g}g^{-1}q) + \mathrm{Ad}_g\,\omega(\dot{x}^h) = \dot{g}g^{-1} \tag{2.14}$$

is used. As a matter of notation, $\dot{g}g^{-1}q = (\dot{g}g^{-1})_Q(q)$.

Recall from §2.2.6 that sections of the associated bundle $(Q \times V)/G$ are in one-to-one correspondence with vector-valued equivariant functions $f\colon Q \to V$. Under this isomorphism, the covariant derivative (2.13) can be defined on vector valued equivariant functions on Q. Let $f\colon Q \to V$ be such that $fg = gf$, with $g \in G$. Then

$$\nabla_X f = Zf - (\omega(Z))_V(f), \tag{2.15}$$

for $X \in T_xM$ and some horizontal lift $Z \in T_qQ$ such that $\pi(q) = x$ and $T_q\pi Z = X$. This expression is independent of the choice of Z in the fibre above X in the same way that (2.13) is independent of the chosen representative v. To see this recall that $Z = X^h + \xi_Q$ can be decomposed for some $\xi\colon Q \to \mathfrak{g}$ so that

$$\nabla_X f = (X^h + \xi_Q)f - (\omega(X^h + \xi_Q))_V f = X^h f$$

using Lemma 2.10.

If $q(t) \in Q$ is a path above $x(t) \in Q/G$ then $\omega(\dot{q})$ can be thought of as a path in the adjoint bundle that doesn't depend on which q is chosen in

$$\overline{\omega}(t) = [(q(t), \omega(\dot{q}))] \in \tilde{\mathfrak{g}}$$

because ω is equivariant.

2.2.11 Local Forms

A principal bundle Q with projection $\pi\colon Q \to Q/G$ is locally isomorphic to $Q/G \times G$. When this holds globally, i.e. when $Q = Q/G \times G$, then the bundle is called *trivial.*

Generically there a unique point in the base $\pi(q)$ associated with each point $q \in Q$ but there is no way to globally assign a point $g(q) \in G$ in the group to each Q. A *local trivialization* of Q is defined by a local section $\sigma\colon U \subset Q/G \rightarrow Q$ such that $(\pi(q), g(q)) \in Q/G \times G$ where $g\colon Q \rightarrow G$ is defined with respect to the section by

$$g(q)\ \sigma(\pi(q)) = q \tag{2.16}$$

Henceforth assume that the coordinate function g is always defined with respect to an unspecified local section so that $q = (r, g) = Q/G \times G$. Note that g is equivariant.

The *local form* with respect to a local trivialization, σ, of an equivariant object defined on a principal bundle is its pullback by σ to the base Q/G. Due to the equivariance, the object on the full space can be locally reconstructed from it.

For example, if $f\colon Q \rightarrow V$ is an equivariant function then its local form is $\widehat{f} \equiv \sigma^* f \in C^\infty(Q/G)$. One can locally recover the function f on $Q/G \times G$ because of the equivariance using the coordinate function $g\colon Q \rightarrow G$

$$f = g\ \pi^* \widehat{f}.$$

The local form of a principal connection $\omega \in \Omega^1(()Q, \mathfrak{g})$ is $A \equiv \sigma^* \omega \in \Omega^1(Q/G, \mathfrak{g})$. By equivariance,

$$\omega = \mathrm{Ad}_g(g^{-1}\,\mathrm{d}\,g + \pi^* A)$$

wherever the section is defined. The covariant derivative of an equivariant function is again an equivariant function. Its local form is then $\widehat{\nabla}_X \widehat{f} \equiv \sigma^*(\nabla_X f)$. The local form of the covariant derivative,

$$\widehat{\nabla} \widehat{f} = \mathrm{d}\,\widehat{f} - A_V \widehat{f},$$

can be found by using (2.15). The curvature can be reconstructed from its local form, $F \equiv \sigma^* \Omega$, by

$$\Omega \approx \mathrm{Ad}_g\, \pi * F.$$

The local form of the structure equation (2.12) is

$$F = \mathrm{d}\,A - [A, A]. \tag{2.17}$$

2.3 Riemannian Geometry

Riemannian geometry is the study of smooth manifolds that are endowed with a metric. It provides the manifold with natural structures that are not found on an unadorned manifold. One is a distance measure, a second is a covariant derivative, and a third is a natural volume element.

Riemannian manifolds appear here in two contexts. First, the ambient space in which the problems of interest will be embedded will be Riemannian manifolds. An example is the Euclidean plane $\mathbb{E}^2$ which is $\mathbb{R}^2$ with its standard metric, δ. Second, the configuration space of a simple mechanical system is a Riemannian manifold whose metric is its kinetic energy. Unforced, it will evolve along geodesics.

A manifold M with a given metric m, a symmetric positive definite two tensor, is a called a Riemannian manifold and is denoted by (M, m).

Theorem (The Fundamental Theorem of Riemannian Geometry). *A Riemannian manifold (M, m) has a unique covariant derivative ∇ called the Levi-Civita connection that satisfies the following properties. For all vector fields $X, Y, Z \in \mathfrak{X}(M)$ and functions $f, g \in C^\infty(M)$*

1. $\nabla_{fX+gY} Z = f\nabla_X Z + \nabla_Y Z,$ (2.18)
2. $\nabla_X fY = f\nabla_X Y + (Xf)Y,$ (2.19)
3. $\nabla_X(m(Y, Z)) = m(\nabla_X Y, Z) + m(Y, \nabla_X Z),$ (2.20)
4. $\nabla_X Y - \nabla_Y X = \mathscr{L}_X Y.$ (2.21)

The value of the covariant derivative only depends on the value of its first argument at a particular point and not its derivatives, and it only depends on its second argument locally in the direction of the flow of the first. This allows us to define the Lagrangian derivative along a curve $x(t)$ of a vector field Y, possibly only defined along $x(t)$, as $\frac{D}{dt}Y = \nabla_{\dot{x}} Y$.

The derivative of a function, $f \in C^\infty(M)$, in the direction of a vector field, $X \in \mathfrak{X}(M)$, is another function, $Xf \in C^\infty(M)$. Both ∇_X and $\mathscr{L}_X$ can be naturally extended to operate on all tensors by specifying that they degenerate to the directional derivative when acting on functions (0-forms)

$$Xf = \nabla_X f = \mathscr{L}_X f = \mathbf{i}_X \, \mathbf{d} f = \langle X, \mathbf{d} f \rangle,$$

and that they "commute with contraction" or that they are "Leibniz". This is best illustrated by an example. Let m be a 2-tensor and let $X, Y, Z \in \mathfrak{X}(M)$ be vectors. Then $\nabla_X m$ is another tensor of the same type as m and is defined such that the following is true for all Y, Z.

$$\nabla_X(m(Y, Z)) = (\nabla_X m)(Y, Z) + m(\nabla_X Y, Z) + m(Y, \nabla_X Z)$$

Notice that (2.20) is equivalent to $\nabla_X m = 0$.

On a Riemannian manifold (M, m) the musical isomorphisms $\cdot^\flat \colon TQ \to T^*Q$ and $\cdot^\sharp \colon T^*Q \to TQ$ are defined with respect to the metric m on M by

$$\langle X, Y^\flat \rangle = m(X, Y)$$
$$\langle \alpha^\sharp, Y^\flat \rangle = \langle Y, \alpha \rangle$$

for all $X, Y \in \mathfrak{X}(M)$ and $\alpha \in \Omega^1(M)$. The *gradient* of a function $f \in C^\infty(M)$ is a vector field defined in terms of the metric and the exterior derivative $\mathbf{d}$ by

$$\operatorname{grad} f = (\mathbf{d} f)^\sharp.$$

Theorem (A Relationship Between $\mathscr{L}$, ∇ and d). *On a Riemannian manifold (M, m) the Lie derivative $\mathscr{L}$, the Levi-Civita connection ∇ and the exterior derivative* d *can be related by*

$$\mathscr{L}_v(v^\flat) = \nabla_v v^\flat + \frac{1}{2}\,\mathrm{d}\,\|v\|^2 \tag{2.22}$$

for all vector fields, $v \in \mathfrak{X}(M)$[1].

Proof. For all $X \in \mathfrak{X}(M)$,

$$\begin{aligned}
\langle X, \mathscr{L}_v(v^\flat)\rangle &= \mathscr{L}_v\langle X, v^\flat\rangle - \langle \mathscr{L}_v X, v^\flat\rangle \\
&= \nabla_v\langle X, v^\flat\rangle - \langle \mathscr{L}_v X, v^\flat\rangle \\
&= \langle X, \nabla_v v^\flat\rangle + \langle \nabla_v X, v^\flat\rangle - \langle \mathscr{L}_v X, v^\flat\rangle \\
&= \langle X, \nabla_v v^\flat\rangle + \langle \nabla_X v, v^\flat\rangle \\
&= \langle X, \nabla_v v^\flat\rangle + \frac{1}{2}\nabla_X \|v\|^2,
\end{aligned}$$

because ∇ preserves the metric (2.20) and is torsion free (2.21). □

2.3.1 Isometries

The isometry group

$$\mathcal{I}(M) = \{b\colon M \to M \mid b^* m = m\}$$

on a Riemannian manifold (M, m) is the space of maps that preserve the metric m. It is the space of rigid motions on M. The Lie algebra of $\mathcal{I}(M)$ is denoted by $\mathfrak{i}(M)$. It is the space of infinitesimal isometries, or the space of rigid velocities. For simplicity, $\mathcal{I}(M)$ is restricted to its component connected to the identity.

For Euclidean space $\mathbb{E}^n = (\mathbb{R}^n, \delta)$, that is $\mathbb{R}^n$ with the standard metric (the Kronecker delta), the isometry group $\mathcal{I}(\mathbb{R}^n) = \mathrm{SE}(n)$ is the special Euclidean group, the space of rotations and translations. The Lie algebra of the Euclidean group is, $\mathfrak{i}(\mathbb{R}^n) = \mathfrak{se}(n)$, the space of both rotational and translational rigid velocities.

2.3.2 The Isometry Action

The isometry group, $\mathcal{I}(M)$, naturally acts on its Riemannian manifold (M, m). By definition the metric is invariant with respect to this action. The infinitesimal version of this invariance is that for all $\xi \in \mathfrak{i}(M)$

$$\mathscr{L}_{\xi_M} m = 0. \tag{2.23}$$

[1]Note that $\nabla_v(v^\flat) = (\nabla_v v)^\flat$ so that leaving off the parenthesis is unambiguous.

This follows from definitions of the infinitesimal generator (2.8) and the Lie derivative (2.24).

2.3.3 Transport Theorems

Theorem (Transport Theorems). *If $x_t\colon M \to M$ is a local diffeomorphism, then it is the flow of the vector field $v_t = \dot{x}x_t^{-1}$ on M. If T_t is a tensor field defined along x_t then the following hold,*

$$\frac{d}{dt}x_t^*T_t = x_t^*(\dot{T}_t + \mathscr{L}_{v_t}T_t), \tag{2.24}$$

$$\frac{D}{dt}T_t|_{x_t} = (\dot{T}_t + \nabla_{v_t}T_t)|_{x_t}. \tag{2.25}$$

2.3.4 A Metric on Diffeomorphisms

A metric m and a volume form μ on a manifold M induce a natural metric $\mathcal{M}$ on $\mathbf{Diff}(M)$, the space of diffeomorphisms of M. Let $X, Y \in \mathfrak{X}(\mathbf{Diff}(M))$, then

$$\mathcal{M}(X,Y) \equiv \int_M m(X,Y)\mu. \tag{2.26}$$

It is not obvious that this definition makes sense since $X \notin \mathfrak{X}(M)$ is not a vector field on M, but rather is a vector field on $\mathbf{Diff}(M)$. Given a point $x \in M$ the change in a diffeomorphism $f\colon M \to M$ restricted to x is a vector at $f(x)$ or an element of $T_{f(x)}M$ so that the above means

$$\mathcal{M}_f(X_f, X_f) = \int_{x\in M} m_{f(x)}(X_f(x), X_f(x))\mu_x.$$

Notice the volume form and the metric are evaluated at different points: the metric is evaluated at $f(x)$, and the volume form is evaluated at x.

2.3.5 Variations

Our use of the calculus of variations requires a change in the order of differentiation of a path with respect to its two parameters, here called s and t.

Theorem (Exchanging s and t Differentiation). *Let $q\colon \mathbb{R}^2 \to Q$, then*

$$\nabla_{\dot{q}}\delta q - \nabla_{\delta q}\dot{q} = 0, \tag{2.27}$$

$$\frac{d}{dt}\omega(\delta q) - \frac{d}{ds}\omega(\dot{q}) = \mathbf{d}\,\omega(\dot{q}, \delta q), \tag{2.28}$$

where $\dot{q}(s,t) = T_{(s,t)}q \circ \frac{d}{dt}$, $\delta q(s,t) = T_{(s,t)}q \circ \frac{d}{ds}$, and $\omega \in \Omega^1(Q)$.

Proof. The vector fields $\dot{q}$ and δq are q-related and $\frac{d}{dt}$ and $\frac{d}{ds}$ are constant so that

$$[\dot{q}, \delta q] = Tq \circ \left[\frac{d}{dt}, \frac{d}{ds}\right] = 0$$

and since the connection ∇ is torsion-free (2.21), the first result follows. The second follows from the above and (2.5). □

Chapter 3

Simple Mechanical Systems

Simple mechanical systems are completely described by their quadratic kinetic energy which can be thought of as a metric on their configuration space. This means that the configuration space is a Riemannian manifold. The equations of motion for such a system are the geodesic equations for the configuration manifold. For background in the these system as related to control theory see [Lewis, 1995].

A coordinate invariant splitting of the geodesic equations is presented for the case when the metric has a symmetry. This work was inspired by the coordinate version found in [Bloch, Krishnaprasad, Marsden, and Murray, 1996, Ostrowski, 1998]. The key insight to the coordinate invariant formulation presented here is the use of the adjoint bundle. Though it is just the metric case of general Lagrangian reduction in the presence of symmetry found in [Cendra, Marsden, and Ratiu, 2001a], this formulation was done in parallel and is presented in a sufficiently different style to justify its presentation here. In addition all our motivating examples have a metric structure, so it is reasonable to work through this special case in detail. The results will also be used in Chapter 6.

3.1 General Simple Mechanical Systems

Because a similar technique will be used in the derivation of the Metric Lagrangian-Poincaré equations the standard formulation of the forced geodesic equations is reviewed.

Let Q be a Riemannian manifold with metric $\mathcal{M}$ and ∇ its associated Levi-Civita connection. When subject to a force $Y^\flat \in \Omega^1(Q)$, the equations of motion of the associated simple mechanical system with kinetic energy $\frac{1}{2}\|\dot{q}\|^2$ are

$$\nabla_{\dot{q}}\dot{q} = Y.$$

This can be shown by using the Lagrange d'Alembert principle

$$\delta \int_a^b L(q(s,t), \dot{q}(s,t))dt + \mathcal{M}(Y, \delta q) = 0, \tag{3.1}$$

where $\delta = \frac{d}{ds}\big|_{s=0}$, applied to a Lagrangian of the form

$$L(q, \dot{q}) = \frac{1}{2}\|\dot{q}\|^2 = \frac{1}{2}\mathcal{M}(\dot{q}, \dot{q}).$$

Recall that for scalars, like the action, $\delta = \frac{d}{ds}\big|_{s=0} = \nabla_{\delta q}$ so that (3.1) becomes

$$\int_a^b \mathcal{M}(Y, \delta q)dt = -\nabla_{\delta q} \int_a^b \frac{1}{2}\mathcal{M}(\dot{q}, \dot{q})dt$$

because ∇ respects the metric then

$$= -\int_a^b \mathcal{M}(\nabla_{\delta q}\dot{q}, \dot{q})dt$$

and because ∇ is torsion free

$$= \int_a^b \mathcal{M}(\nabla_{\dot{q}}\delta q - [\delta q, \dot{q}], \dot{q})dt$$

and here s and t derivatives commute (2.27)

$$= \int_a^b \mathcal{M}(\nabla_{\dot{q}}\dot{q}, \delta q)dt - \int_a^b \nabla_{\dot{q}}(\mathcal{M}(\delta q, \dot{q}))dt$$
$$= \int_a^b \mathcal{M}(\nabla_{\dot{q}}\dot{q}, \delta q)dt - \mathcal{M}(\delta q, \dot{q})|_a^b$$

and finally since δq vanishes at a,b.

3.2 Partial Symmetry

When the metric that describes the systems is invariant with respect to the action of a Lie group G with Lie algebra $\mathfrak{g}$, then it will have a conserved quantity called the momentum. The momentum map $J\colon TQ \to \mathfrak{g}^*$ defined by

$$\langle J(X), \xi \rangle = \mathcal{M}(X, \xi_Q) \tag{3.2}$$

for all $\xi \in \mathfrak{g}$, satisfies Noether's theorem, which states that

$$\frac{d}{dt}J(\dot{q}) = J(Y). \tag{3.3}$$

In the absence of external forcing, the momentum J is constant along a trajectory of the system. (3.3) is a simple consequence of taking the time derivative of (3.2) and using the fact that the metric is invariant.

$$\begin{aligned}\frac{d}{dt}\langle J(\dot{q}),\xi\rangle &= \nabla_{\dot{q}}(\mathcal{M}(\dot{q},\xi_Q)) \\ &= \mathcal{M}(\nabla_{\dot{q}}\dot{q},\xi_Q) + \mathcal{M}(\dot{q},\nabla_{\dot{q}}\xi_Q) \\ &= \langle J(Y),\xi\rangle\end{aligned}$$

where (2.23) and the lemma $2m(\nabla_X Y, X) = (\mathscr{L}_Y \mathcal{M})(X,X)$ were used. If the configuration space is completely invariant, $Q = G$, then (3.3) completely describes the system.

3.3 The Mechanical Connection

A simple mechanical system with symmetry described by a metric $\mathcal{M}$ on a manifold Q that is invariant with respect to the action of a Lie group G has a natural principal connection $\omega\colon TQ \to \mathfrak{g}$ called the *mechanical connection* defined on the bundle $\pi\colon Q \to Q/G$. This connection is defined for all $X \in \mathfrak{X}(Q)$ and all $\xi \in \mathfrak{g}$ by,

$$\mathcal{M}(X,\xi_Q) = \mathcal{M}(\omega_Q(X),\xi_Q). \tag{3.4}$$

The horizontal subspace is therefore normal, with respect to the metric, to the vertical subspace defined by the infinitesimal action of G. ω, is naturally Ad-equivariant because $\mathcal{M}$ is invariant. The mechanical connection diagonalizes the metric $\mathcal{M}$ on Q

$$\mathcal{M} = \pi^* m + \omega^* \mathbb{I}.$$

Here m is a metric on Q/G defined by

$$m(X,Y) = \mathcal{M}(X^h, Y^h)$$

and $\mathbb{I}$, an Ad-equivariant metric on $\mathfrak{g}$, is defined by

$$\mathbb{I}(\xi,\eta) = \mathcal{M}(\xi_Q,\eta_Q). \tag{3.5}$$

$\mathbb{I}$, because it is Ad-equivariant, can also be thought of (2.2.6) as

$$\bar{\mathbb{I}} = [(q,\mathbb{I}(q))],$$

a metric on the adjoint bundle $\tilde{\mathfrak{g}}$. The momentum map J and ω are then related by

$$J = \mathbb{I}\omega \tag{3.6}$$

3.4 The Metric Lagrange-Poincaré Equations

A connection ω, in this case the mechanical connection, splits the derivative of a time dependent path $q(t)$ on Q into a horizontal and vertical part

$$\dot{q} = \dot{r}^h + \omega_Q(\dot{q})$$

where $r(t) = \pi(q(t))$. This is a choice of connection, while convenient, is not necessary [Cendra, Marsden, and Ratiu, 2001b]. The external force can be split as follows

$$J(Y) = \frac{d}{dt}J(\dot{q}) = (\dot{r}^h + \omega_Q(\dot{q}))J(\dot{q}).$$

The covariant derivative on the dual of the adjoint bundle (2.13) can be written as $\dot{r}^h = \nabla_{\dot{r}} = \frac{D}{dt}$ so this becomes

$$\frac{D}{dt}\overline{J} - \mathrm{ad}^*_{\overline{\omega}}\overline{J} = J(Y)$$

or

$$\frac{D}{dt}(\mathbb{I}\overline{\omega}) - \mathrm{ad}^*_{\overline{\omega}}\mathbb{I}\overline{\omega} = \mathbb{I}\omega(Y)$$

using §2.2.6 where $\overline{\omega}(t) \equiv [(q(t), \omega(\dot{q}))]$ is a path in the adjoint bundle $\tilde{\mathfrak{g}}$, $\overline{\mathbb{I}}$ is defined by (3.3) and

$$\overline{J}(t) = \overline{\mathbb{I}}\overline{\omega}$$

is its momentum, a path in the dual $\tilde{\mathfrak{g}}^*$. This is the vertical part of the Lagrange-Poincaré equations [Cendra et al., 2001a] for a quadratic Lagrangian. The horizontal part can be obtained by going back to Hamilton's variational principal.

Theorem (Metric Lagrange-Poincaré Equations). *The Lagrange equations for a quadratic Lagrangian on a manifold Q defined by a metric $\mathcal{M}$ that is invariant with respect to the action of Lie group G can be reduced and split with respect to the mechanical connection ω into a vertical*

$$\frac{D}{dt}(\overline{\mathbb{I}}\overline{\omega}) - \mathrm{ad}^*_{\overline{\omega}}\overline{\mathbb{I}}\overline{\omega} = 0 \tag{3.7}$$

and horizontal

$$\nabla^?_{\dot{r}}\dot{r} - \frac{1}{2}(\nabla\overline{\mathbb{I}})(\overline{\omega}, \overline{\omega}) + \overline{\mathbb{I}}(\overline{\omega}, \mathbf{i}_{\dot{r}}\overline{\Omega}) = 0 \tag{3.8}$$

piece where $\overline{\omega}(t) \in \tilde{\mathfrak{g}}$ and $\mathcal{M} = \pi^*m + \omega^*\mathbb{I}$ as above (3.3). Note that the first ∇ is the Levi-Civita connection associated with the metric m on Q/G and the second ∇ is the covariant derivative (2.13) on the adjoint bundle $\tilde{\mathfrak{g}}$ associated with the connection ω and that $\overline{\Omega} \in \Omega^2(Q/G, \tilde{\mathfrak{g}})$ is defined by $\overline{\Omega}(X,Y) \equiv [(q, \Omega(X^h, Y^h))]$.

Proof.

$$\delta\int_a^b \frac{1}{2}\mathcal{M}(\dot{q},\dot{q})dt = \delta\int_a^b \frac{1}{2}\mathbb{I}(\omega(\dot{q}),\omega(\dot{q})) + \delta\int_a^b \frac{1}{2}m(\dot{r},\dot{r})dt$$

where $r = \pi(q)$. The second term on the right is just the usual variation (3.1) on the base space and will give $-\int_a^b m(\nabla_{\dot r}\dot r, \delta r)dt$. Continuing with the first term leads to

$$\delta \int_a^b \frac{1}{2}\mathbb{I}(\omega(\dot q), \omega(\dot q)) = \int_a^b \frac{1}{2}(\delta\mathbb{I})(\omega(\dot q), \omega(\dot q)) + \int_a^b \mathbb{I}(\delta(\omega(\dot q)), \omega(\dot q)).$$

The order of s and t differentiation can be changed using (2.28). $\xi_Q\mathbb{I} = -\operatorname{ad}^*_\xi \mathbb{I} - \mathbb{I}\operatorname{ad}_\xi$ because $\mathbb{I}$ is Ad-equivariant so $\delta\mathbb{I} = \delta q(\mathbb{I}) = \delta r^h\mathbb{I} + \delta\omega_Q\mathbb{I} = \nabla_{\delta r}\mathbb{I} - \operatorname{ad}^*_{\delta\omega}\mathbb{I} - \mathbb{I}\operatorname{ad}_{\delta\omega}$. With this the previous equation becomes

$$= \frac{1}{2}\int_a^b (\delta\mathbb{I})(\omega(\dot q), \omega(\dot q))dt + \int_a^b \mathbb{I}(\frac{d}{dt}(\omega(\delta q)) + d\omega(\delta q, \dot q), \omega(\dot q))dt$$

and after applying the structure (2.12) equation $\Omega = d\omega - [\omega, \omega]$

$$= \frac{1}{2}\int_a^b (\nabla_{\delta r}\mathbb{I} - ad^*_{\delta\omega}\mathbb{I} - \mathbb{I}\operatorname{ad}_{\delta\omega})(\omega(\dot q), \omega(\dot q))dt$$
$$+ \int_a^b \mathbb{I}(\dot q(\omega(\delta q)) + \Omega(\delta q, \dot q) + [\omega(\delta q), \omega(\dot q)], \omega(\dot q))dt$$

define $\delta\overline{\omega} \equiv [(q, \omega(\delta q))]$ and $\delta r \equiv T\pi(\delta q)$ so that

$$\int_a^b \frac{1}{2}(\nabla_{\delta r}\overline{\mathbb{I}})(\overline{\omega}, \overline{\omega}) + \mathbb{I}(\omega(\dot q), \Omega(\delta r^h, \dot r^h))dt + \int_a^b \overline{\omega}\overline{\mathbb{I}}\frac{d}{dt}(\delta\overline{\omega})dt.$$

Now apply the usual technique of the variational calculus by first doing integration by parts on the second term and then setting the integrands to zero. The m term we neglected above is returned.

$$\delta\overline{\omega}\frac{d}{dt}(\overline{\mathbb{I}}\overline{\omega}) = 0$$

$$m(\delta r, \nabla_{\dot r}\dot r) - \frac{1}{2}(\nabla_{\delta r}\overline{\mathbb{I}})(\overline{\omega}, \overline{\omega}) + \overline{\mathbb{I}}(\overline{\omega}, \overline{\Omega}(\dot r, \delta r)) = 0$$

the second of which is G-invariant. Then of course, since the variations are arbitrary, the result is obtained. □

Chapter 4

A Deformable Body in Fluid

Symmetries play a fundamental role in mathematics and physics. In physical systems, they lead to conserved quantities called momenta and to simplified evolution equations. In Chapter 3 the specific form for the momenta and the reduced equations of motion for a simple mechanical system with symmetry were presented. In this chapter the equations of motion are derived for a deformable body in a perfect fluid by showing the system fits this form. When the submersed body is rigid, the system further reduces to become an invariant one on a Lie group. This group is a combination finite/infinite dimensional one which is slightly complicated by its behavior at infinity.

I feel that the presentation here of the group structure of a deformable body in a perfect fluid is a fundamental contribution. Birkhoff [1950, p. 164], referring to the classical chapter of hydrodynamics [Lamb, 1945, Chapter VI] concerned with a rigid body in a perfect fluid accelerated from rest (a requirement dispensed here as vorticity is included), said

> ... it has great historical importance, and to this day no better model of comparable simplicity is known. I therefore feel justified in presenting a new mathematical treatment of it, ultimately in terms of the ideas of group theory, whose importance for other hydrodynamical questions has already been shown.

I proceed in the same vein and with a similar sentiment though my treatment is closer to that of Arnold and Khesin [1998].

A review is first presented of Arnold's invariant formulation of the Euler equation [Arnold and Khesin, 1998] for an incompressible inviscid fluid as a right invariant mechanical system on a Lie group, the group of volume preserving diffeomorphisms of the reference configuration. Then the analogous equations for the generalized rigid body, invariant with respect to the left action of the isometry group, are shown to yield Euler's rigid body equations.

When started from rest, the equations for a rigid body in a fluid are known to be symmetric geodesic equations [Birkhoff, 1950]. It will be shown here that the equations for the combined rigid body fluid system in the generic rotational case are also symmetric geodesic equations for what I call the *fluid-body group*. They are coupled versions of the Euler fluid and Euler rigid body which, when the fluid component of the momentum is zero, reduce to Kirchhoff's equations.

The idea of using the particle relabeling and metric symmetries to compute reduced equations for a rigid body in a perfect fluid is due to Kelly [1998]. His work is presented here in a more geometric framework and is expanded upon by relating the derived equations to the standard versions, by considering the effect of the constraints at infinity, and by investigating the geometry of the resulting fluid body group. This exposition will provide the background necessary to tackle the deformable body and will provide the necessary framework to approach the stirring problem.

I show that the deformable case is again geodesic, but this time the fluid body group is only a partial symmetry over the shape space. The equations of motion on the resulting principal bundle are then the metric Lagrange-Poincaré equations (3.7), (3.8).

I show that the fluid body group has a natural principal bundle structure over the group of rigid motions. Conservation of momentum and the conditions at infinity connect the motions of the fluid to given motions of the body. This bundle is quite complicated, in particular in two dimensions with a multiply connected body, it is mathematically non trivial. This means that it has a structure that is a generalization of a Möbius strip where global motions returning to the initial point can leave the fluid mixed in way that, without moving the body, cannot be undone. This is a model for stirring. With this in mind the curvature of the mechanical connection on fluid-body group is computed with the intention of using it in Chapter 6 as a means to approximate the holonomy or "stir" resulting from cyclic motions of a body.

While the group formulation presented here may seem obvious in retrospect, others schooled in the formalism of the reduction, symmetries, mechanical systems and fluids have had trouble seeing it. Vladimirov and Ilin [1999a,b] say that they had to extend Arnold's technique for fluid stability analysis to systems that are not Lie groups because "for the system 'body + fluid', however, the flow domain changes with time, so that the configuration space does not form a group". It does, as we shall see.

4.1 A Geometric Framework

We will be discussing the locomotion of objects in an ambient space M that is assumed to be a Riemannian manifold with metric m. This formalism is adopted, despite the fact that in the examples M will be either $\mathbb{R}^2$ or $\mathbb{R}^3$, because, among other things, it facilitates dealing with the rotational and translational components of equations in $\mathbb{R}^n$. At the same time, the formalism allows a formulation that is independent of coordinates and helps avoid the historically pervasive, but notoriously hard to generalize, cross product. Additionally, while the ambient space is usually Euclidean, its isometry group and diffeomorphism group are, in the context of the mechanics of a rigid body and a fluid, Riemannian manifolds as well and by themselves justify the formalism.

4.1.1 The Diffeomorphism Group

Let $\mathbf{Diff}(M)$ be the space of diffeomorphisms $\eta\colon M \to M$ of an n-dimensional manifold M. This space is the configuration space a compressible fluid confined to a volume modeled by M. $\mathbf{Diff}(M)$ is a Lie group with composition as the group action. Its Lie algebra is $\mathbf{diff}(M) = \{v \in \mathfrak{X}(M) \mid v \parallel \partial M\}$ the space of vector fields v on M that are parallel to the boundary ∂M on the boundary. The adjoint action $\mathrm{Ad}_\eta v = \eta v \eta^{-1} = \eta_* v$ is the push forward and the Lie algebra bracket is $\mathrm{ad}_v = -\mathscr{L}_v$ is minus the Jacobi-Lie bracket of vector fields.

The dual of $\mathbf{diff}(M)$ is $\mathbf{diff}^*(M) = \Omega^1(n)$, the space of one-forms α on M. The natural pairing between elements of $\mathbf{diff}(M)$ and $\mathbf{diff}^*(M)$ is $\langle v, \alpha\rangle = \int_M \alpha(v)\mu$ where the integral with respect to $\mu \in \Omega^n(M)$, a given volume form (the density) on M, times the standard pairing pairing between $v \in \mathbf{Diff}(M)$ and $\alpha \in \mathbf{diff}^*(M)$ at each point. The co-adjoint action, $\mathrm{Ad}^*_\eta = \eta^*$, is the pullback. The dual Lie algebra bracket, $\mathrm{ad}^*_v = \mathscr{L}_v$, is the Lie derivative.

The space of volume preserving diffeomorphisms

$$\mathbf{Diff}_{\mathrm{vol}}(M) \equiv \{\eta \in \mathbf{Diff}(M) \mid \eta^*\mu = \mu\}$$

is a sub-group of $\mathbf{Diff}(M)$. It is the configuration space of incompressible fluids. The Lie algebra of this group, the space of fluid velocities

$$\mathbf{diff}_{\mathrm{vol}}(M) = \{v \in \mathbf{diff}(M) \mid \mathrm{div}_\mu v = 0\},$$

is the space of divergence free vector fields $v \in \mathfrak{X}(M)$ (parallel to the boundary ∂M). Its dual

$$\mathbf{diff}_{\mathrm{vol}}{}^*(M) = \Omega^1(M)/\,\mathbf{d}\,\Omega^0(M)$$

is the space of one-forms modulo exact one-forms. This is the space of momenta. An element $[\alpha] \in \mathbf{diff}_{\mathrm{vol}}{}^*(M)$ is an equivalence class, the set of one-forms that differ from α only by the differential of a function. With this restriction, the pairing

$$\langle v, [\alpha]\rangle \equiv \int_M \langle v, \alpha\rangle\, \mu$$

is non-degenerate. The following theorem makes this fact precise and is fundamental to our technique for handling incompressibility.

Theorem (The Dual of Divergence Free Vectors). *[Arnold and Khesin, 1998] For an n-dimensional compact manifold M with boundary ∂M, the space dual to $\mathbf{diff}_{\mathrm{vol}}(M)$ (the space of divergence free vector fields on M tangent to ∂M) is naturally isomorphic to the quotient space $\Omega^1(M)/\,\mathbf{d}\,\Omega^0(M)$ (i.e. the space of all 1-forms on M modulo exact 1-forms on M) in the following sense*

i *If α is the differential of a function ($\alpha = \mathbf{d} f$) and $v \in \mathbf{diff}_{\mathrm{vol}}(M)$, then $\int_M \alpha(v)\mu = 0$.*

ii *If $\int_M \alpha(v)\mu = 0$ for all $\alpha = \mathbf{d}\, f$ for some f, then $v \in \mathbf{diff}_{\mathrm{vol}}(M)$ (i.e. v is a divergence-free vector field on M tangent to ∂M).*

iii *If $\int_M \alpha(v)\mu = 0$ for all $v \in \mathbf{diff}_{\mathrm{vol}}(M)$ then the 1-form α is the differential of a function.*

Proof. (i) If $\alpha = \mathbf{d}\, f$ and $v \in \mathbf{Diff}_{\mathrm{vol}}(M)$ then $\alpha(v) = \mathscr{L}_v f$. Hence by the divergence theorem (2.7),

$$\int_M \alpha(v)\mu = \int_{\partial M} f\, \mathbf{i}_v\, \mu + \int_M (\mathrm{div}_\mu\, v) f \mu = 0$$

because v is divergence free and parallel to the boundary. (ii) If

$$0 = \int_M \mathbf{i}_v\, \mathbf{d}\, f \mu = \int_M f\, \mathbf{d}\, \mathbf{i}_v\, \mu$$

for all f then by considering f as a bump function supported in the interior of M we see that $\mathbf{d}\, \mathbf{i}_v\, \mu$ must be zero in the interior, and hence v must must be divergence-free. By considering a bump function f supported on ∂M,

$$\int_M f\, \mathbf{d}\, \mathbf{i}_v\, \mu = \int_{\partial M} f\, \mathbf{i}_v\, \mu$$

the vector v must be parallel to the boundary and hence in $\mathbf{diff}_{\mathrm{vol}}(M)$. (iii) is proved in [Arnold and Khesin, 1998]. □

4.1.2 Generalized Flows

One can't proceed naively with variational principles on the group of volume preserving diffeomorphisms since the limit of sequences of diffeomorphisms is not guaranteed to exist in the space. One can construct the closure by introducing discontinuous flows.

Brenier [1989] introduced *generalized flows* where particles can cross and overlap thereby completing the space of volume preserving diffeomorphisms. With the space expanded to include these discontinuous flows, one can then proceed with the variational formalism [Arnold and Khesin, 1998]. This means that the resulting flow is possibly non smooth. We ignore these issues and proceed formally.

4.1.3 Assumptions on the Nature of the Flow at Infinity

We will often specify a geometrical object's behavior at infinity in order to be physically realistic or to ensure convergence of certain integrals that make the metric on $\mathbf{Diff}(M)$ or the momentum well defined.

For example a diffeomorphism $q \in \mathbf{Diff}(M)$ representing an inertial configuration of a volume of fluid will be required to satisfy fixed boundary conditions at infinity. We will write this constraint as $q|_\infty = \mathrm{id}_M$ by which we mean that with respect to the metric the distance between $x \in M$ and $q(x)$ approaches zero along unbounded geodesics, when they exist.

For configurations of the fluid relative to the body it makes sense to ask for $\eta \in \mathbf{Diff}(M)$ to be rigid, that is an isometry, at infinity.

In addition we will assume that for all $u \in \mathbf{diff}_{\mathrm{vol}}(M)$, the integral $\int_M m(\xi_M, u)\mu$, when it appears, converges for all elements $\xi \in \mathrm{i}(M)$ of the isometry group of M. As is well known, this requirement disallows, for example, flows with circulation in $\mathbb{R}^2$.

Generally we will be concerned with manifolds that admit rigid motions, at least in some directions. It is well known that in these directions the manifold must have constant curvature. For example a torus with the induced euclidean metric allows for rigid motions "around the outside" but not "around into the middle".

The above theorem as quoted only applies to compact manifolds, but we assume it can be extended to non-compact manifolds if we restrict our infinities to have constant curvature. It certainly holds for manifolds that are Euclidean (zero curvature) at infinity [Cantor, 1981].

4.1.4 A Body in a Fluid

This section discusses the abstract configuration space for a deformable body in a fluid. These notions will be made more tangible subsequently.

We will assume that the body $B \subset M$ is a sub-manifold of an ambient Riemannian manifold (M, m) and that the remainder of M, $F = M \setminus B$, is fluid. Here m is a given metric on M. A volume μ_ρ is given which represents the density ρ in the reference configuration of the fluid and body when compared, $\mu_\rho = \rho\mu_m$, to the induced volume form μ_m derived from m. The density μ_ρ is allowed to be discontinuous across the body's boundary ∂B and is independent of time since it is specified only for the reference configuration. When a body is present we require uniform fluid density. We will use μ instead of μ_ρ in the sequel when there is no chance for confusion.

Configurations Q of the body and fluid

$$Q = \{q\colon M_o \to M \mid q^*\mu = \mu,\ q(\partial B) = \partial B,\ q|_\infty = \mathrm{id}_M\},$$

where each map is a diffeomorphism everywhere except along ∂B. For inviscid fluids we allow slip on the boundary. Note that we will write M_o instead of M as the domain (reference configuration) of the maps q as a reminder that one cannot compose elements of Q since the body moves and possibly deforms. Doing so could lead to the nonsensical mixing of body and fluid elements. [1]

[1]This would suggest an alternative approach in which the configuration space is thought of as a groupoid [Weinstein, 1995]

Invariance of the Body Fluid Metric

The metric on Q is (2.26),

$$\mathcal{M}(X,Y) = \int_M m(X,Y)\mu \tag{4.1}$$

where $X, Y \in \mathfrak{X}(Q)$. It is clearly invariant with respect to the right action of $\mathbf{Diff}_{\text{vol}}(F)$ on Q. For all $\eta \in \mathbf{Diff}_{\text{vol}}(F)$ we have that

$$\begin{aligned}(\eta^*\mathcal{M})(X,Y) &= \mathcal{M}(\eta_* X, \eta_* Y) \\ &= \int_M m(\eta_* X, \eta_* Y)\mu \\ &= \int_{\eta(M)} m(X,Y)\eta_*\mu \\ &= \int_M m(X,Y)\mu \\ &= \mathcal{M}(X,Y)\end{aligned} \tag{4.2}$$

This is the so called particle relabeling symmetry since η acts as a change of coordinates of the reference configuration.

The metric on Q is also invariant with respect to the left action of the isometry group $\mathcal{I}(M)$ on Q. For all $b \in \mathcal{I}(M)$ we have that

$$\begin{aligned}(b^*\mathcal{M})(X,Y) &= \mathcal{M}(b * X, b_* Y) \\ &= \int_M m(b_* X, b_* Y)\mu \\ &= \int_M (b^* m)(X,Y)\mu \\ &= \int_M m(X,Y)\mu \\ &= \mathcal{M}(X,Y)\end{aligned} \tag{4.3}$$

This is the rigid symmetry of the ambient space.

4.2 Euler's Equation

It is unclear if Euler understood that his rigid body and his fluid equations could be thought of as coming from the same principle, but Poincaré [1901, 1910] did.

4.2.1 The Euler Fluid

An inviscid incompressible fluid is an example of a simple mechanical system on a Lie group. Given a metric m on our ambient space M we get an induced metric (4.1) on $\mathbf{Diff}_{\text{vol}}(M)$ along whose geodesics the fluid flows [Arnold, 1966]. The

Lagrangian for the system,

$$L(\eta, \dot{\eta}) = \frac{1}{2} \int_M \|\dot{\eta}\|^2 \mu$$

is right-invariant (4.2) with respect to the action of $\mathbf{Diff}_{\mathrm{vol}}(M)$ on itself. By changing coordinates (2.1.3) and using the fact that $\eta^* \mu = \mu$ we see that the Lagrangian depends only on

$$\ell(v) = \frac{1}{2} \int_M \|v\|^2 \mu$$

where $v = \dot{\eta}\eta^{-1} \in \mathbf{diff}_{\mathrm{vol}}(M)$. Lagrange's equations can therefore be reduced to the Euler-Poincaré equations . Since $\frac{\partial \ell}{\partial v} = [v^\flat] \in \mathbf{diff}_{\mathrm{vol}}{}^*(M)$ we have

$$\frac{d}{dt}[v^\flat] + \mathscr{L}_v[v^\flat] = 0 \tag{4.4}$$

and by using (2.22) and the fact that $\mathscr{L}$ and $\mathbf{d}$ commute we have

$$\dot{v} + \nabla_v v + \operatorname{grad} p = 0 \tag{4.5}$$

Noether's theorem (3.3) which says that the momentum, $J = \eta^*[v^\flat]$, is constant

$$\frac{d}{dt}\eta^*[v^\flat] = 0 \tag{4.6}$$

This is an equivalent formulation of the equations for a perfect fluid (4.4) which can be seen by applying (2.24) to (4.6).

Theorem (Kelvin's Circulation Theorem). *The integral of the velocity v around a closed path C moving with the fluid $C_t = \eta(C)$ is constant in time.*

$$\frac{d}{dt} \int_{C_t} v^\flat = 0$$

Proof. By Noether's Theorem (4.6) we have that $\frac{d}{dt}\eta^*[v^\flat] = 0$. Integrating this around the curve $C \subset M$ gives

$$0 = \int_C \frac{d}{dt}\left(\eta^*[v^\flat]\right) = \frac{d}{dt} \int_{\eta(C)} [v^\flat] = \frac{d}{dt} \int_{C_t} v^\flat$$

where we can remove the equivalence class since $\int_{C_t} \mathbf{d} f = 0$ for all functions f by Stokes' Theorem (2.6) as C_t has no boundary. □

Theorem (Helmholtz's Circulation Theorem). *The integral of the vorticity $\omega = \mathbf{d} v^\flat$ over the surface $S \subset M$ moving with the fluid $S_t = \eta(S)$ is constant in time.*

$$\frac{d}{dt} \int_{S_t} \omega = 0$$

Proof. Take $\mathbf{d}$ of (4.6), integrate over S, and then change coordinates. [2] □

We see from Noether's Theorem (4.6) that J is constant. If that constant is zero then

$$0 = J = \eta^*[v^\flat] \Rightarrow v^\flat = -\,\mathbf{d}\,\phi$$

where $\phi \in C^\infty(M)$ or in other words we have

$$v = -\operatorname{grad}\phi$$

potential flow. Hence a fluid that can be characterized by a potential function has zero fluid momentum and a flow of a fluid with zero momentum is characterized by a potential function.

We would like to add a body to the fluid, so we start by reviewing the body alone.

4.2.2 The Rigid Body

The equations for a rigid body in a perfect fluid are, as we shall see, a generalization of those for the rigid body alone. In the special case in which the fluid can be described by a potential function, the Kirchhoff case, they are almost identical. In order to see this we start by reviewing the rigid body in our framework.

A configuration of rigid body $B \subset M$ on a Riemannian manifold (M, m) is an element $b \in \mathcal{I}(M)$ of the isometry group of M, the space of rigid motions. The Lagrangian for the rigid body is

$$L(b, \dot{b}) = \int_B \|\dot{b}\|^2 \mu.$$

Noticing that $b^*m = m$ we see that it is left-invariant and we have

$$\ell(\xi) = \frac{1}{2}\int_B \|\xi_M\|^2 \mu$$

where $\xi = b^{-1}\dot{b} \in \mathfrak{i}(M)$ is an infinitesimal isometry or a rigid velocity (see §2.3.1). Denoting

$$\mathbb{I} = \int_B m(\cdot_M, \cdot_M)\mu$$

as the inertia of B, then we have that $\frac{\partial \ell}{\partial \xi} = \mathbb{I}\xi$. The resulting reduced equations of motion are the Euler-Poincaré equations

$$\frac{d}{dt}\frac{\partial \ell}{\partial \xi} = \operatorname{ad}^*_\xi \frac{\partial l}{\partial \xi}. \tag{4.7}$$

When $M = \mathbb{R}^3$ we have $\mathcal{I}(\mathbb{R}^3) = \mathrm{SE}(3)$ and $\mathfrak{i}(\mathbb{R}^3) = \mathrm{se}(3)$. Let $\xi = (U, \Omega)$ be the

[2] in three dimensions we have that $\omega = \mathbf{i}_{(*\omega)^\flat}\,\mu$ so that the integral is over the normal component of the vector $(*\omega)^\flat$ associated with the 2-form ω.

body linear and angular velocity respectively then the infinitesimal generator has the form $(U,\Omega)_{\mathbb{R}^3}(x) = \widehat{\Omega}x + U = \Omega \times x + U$ and the body inertia then takes the form

$$\widehat{\mathbb{I}}_{\mathbb{R}^3} = \int \begin{bmatrix} I & -\widehat{x} \\ \widehat{x} & -\widehat{x}^2 \end{bmatrix} \rho \, \mathrm{d}^3 x = \begin{bmatrix} mI & -m\widehat{c} \\ m\widehat{c} & J \end{bmatrix}$$

where m is the total mass, c the center of mass and J the rotational inertia. If we let $(T,A) = \frac{\partial \ell(U,\Omega)}{\partial(U,\Omega)} \in \mathrm{se}(3)^*$ be the respective momenta, then (4.7) becomes

$$\dot{T} = T \times \Omega + A \times U$$
$$\dot{A} = A \times \Omega$$

If we allow $\mathbb{I}$ to be an arbitrary symmetric matrix instead of having the structure above then the above are also Kirchhoff's equations for the motion of a rigid body in an irrotational perfect fluid.

4.3 The Fluid Body Group

We have seen that the dynamics of a perfect fluid and a rigid body can both be described as invariant simple mechanical systems on Lie groups. We would like to generalize this to the combined case of a rigid body in a (rotational) fluid. This generalization was first proposed by Kelly [1998]. This section expands his work, placing it in a more geometric context. The proposed symmetry group and its bundle structure are discussed in more detail, in preparation for its application to the case of a deformable body.

This is a different take from the usual presentation found in for example in [Birkhoff, 1950, Lamb, 1945] as we do not immediately apply the condition zero fluid momentum, irrotationality, to reduce to a finite dimensional system. Thus we get a complete system of equations for the rigid-body impulse [Saffman, 1992] of the fluid/body coupled to the Kirchhoff equations. These are the Euler-Poincaré equations for the rigid body in a perfect rotational fluid.

We have already described an abstract configuration space. What is the group action? There is a natural action of $\mathcal{I}(M)$ on M and hence on Q, the space of configurations of the body and fluid, by composition. However, note if $b \in \mathcal{I}(M)$ acts on $q \in Q$, then $bq \notin Q$ because the fluid particles at infinity are moved. This is a problem since we want the total system energy to be finite and that the particles at infinity remain fixed.

While it looks tempting to just combine the rigid body and the fluid into one system, it is not obvious what the group operation should be.

The solution presented here, which differs from the approach of Kelly [1998], is to combine the actions of $\mathcal{I}(M)$ and $\mathbf{Diff}_{\mathrm{vol}}(M)$ so that a pair (b,η) acts on q in such a way that the combined action does not move the fluid particles infinity. Hence we propose *the fluid-body group* defined as follows

$$\mathcal{FB}(M,B) \equiv \{(b,\eta^b) \in \mathcal{I}(M) \otimes \mathbf{Diff}_{\mathrm{vol}}(M \setminus B) : \eta^b|_\infty = b^{-1}|_\infty\} \qquad (4.8)$$

where the action of (b, η^b) on q is

$$(b, \eta^b)q = bq\eta^b. \tag{4.9}$$

For notational convenience, the combined action is written as a left action, but it is actually a left-right action as shown in (4.9). The group operation is defined as

$$(b_1, \eta_1^{b_1})(b_2, \eta_2^{b_2}) = (b_1 b_2, \eta_2^{b_2}\eta_1^{b_1}). \tag{4.10}$$

Note that $\mathcal{FB}(M, B)$ is a subgroup of $\mathcal{I}(M) \otimes \mathbf{Diff}_{\mathrm{vol}}(M \setminus B)$ that is specified by the holonomic constraint that the two individual maps must be inverses of each other at infinity. The action of $\mathcal{FB}(M, B)$ clearly leaves the Lagrangian, in this case the metric, invariant (4.2), (4.3).

The Lie algebra of the fluid-body group $\mathcal{FB}(M, B)$ is,

$$\mathfrak{fb}(M, B) = \{(\xi, v^\xi) \in \mathfrak{i}(M) \oplus \mathbf{diff}_{\mathrm{vol}}(M \setminus B) : v^\xi|_\infty = -\xi_M|_\infty\},$$

divergence free vector fields that are parallel to the body boundary and approach rigid velocity field at infinity. Its dual is

$$\mathfrak{fb}^*(M, B) = \{(\alpha, [u]) \in \mathfrak{i}^*(M) \oplus \mathbf{diff}_{\mathrm{vol}}{}^*(M \setminus B) : [u]|_\infty = 0\},$$

the space of equivalence classes of one-forms that differ by differentials of functions (see §4.1.1) that tend to zero at infinity. With these definitions the natural pairing

$$\langle(\xi, v^\xi), (\alpha, [u])\rangle = \langle\xi, \alpha\rangle + \langle v^\xi, [u]\rangle$$

is well defined and non-degenerate.

Because η^b actually contains all of the information that (b, η^b) does, the first b in the pair is redundant. Therefore *the fluid-body group* could be equivalently defined by

$$\mathcal{FB}(M, B) = \{\eta \in \mathbf{Diff}_{\mathrm{vol}}(M \setminus B) : \eta|_\infty \in \mathcal{I}(M)\} \tag{4.11}$$

where the left-right (again written as a left) action would be

$$\eta q = (\eta^{-1})|_\infty q\eta$$

and where in both of these expressions we assume that by the notation $\eta|_\infty$ we mean η's value at ∞, an isometry, extended rigidly over the whole domain. An analogous alternative definition can can be made for the Lie algebra $\mathfrak{fb}(M, B)$ as well, but not for its dual because of the lack of a relationship between the finite and infinite dimensional components.

Therefore, though these simpler equivalent definitions are certainly appealing, the pairs are chosen, with their redundancies, because they highlights the left-right nature of the action of $\mathcal{FB}(M, B)$ on Q and because the representation of Lie algebra is then symmetric with that of its dual which is necessarily a pair.

4.3.1 The Momentum of a body in a Fluid

The Lagrangian for a deformable body B in a fluid $M \backslash B$ is invariant with respect to the action of the fluid-body group $\mathcal{FB}(M,B)$. This symmetry leads to a conserved quantity, the fluid-body momentum. The infinitesimal generator of the action of $\mathcal{FB}(M,B)$ on Q the full space of configurations is

$$(\xi, v)_Q(q) = \left.\frac{d}{ds}\right|_{s=0} e^{s\xi} q e^{sv} = \xi_M(q) + qv,$$

so that the momentum defined by (3.2) is

$$J(q,\dot{q}) = (\int_M m(\dot{q}q^{-1}, \cdot_M)\mu, \quad q^*[(\dot{q}q^{-1})^\flat])$$

where $\dot{q}q^{-1}$ is the fluid velocity. The fluid part of the Euler-Poincaré equations will be the standard Euler equation, and the rigid part is just conservation of the impulse.

4.4 A Rigid Body in a Perfect Fluid

The geometric picture of fluids will be extended to include a rigid body. This description is a reformulation and extension of the work of Kelly [1998].

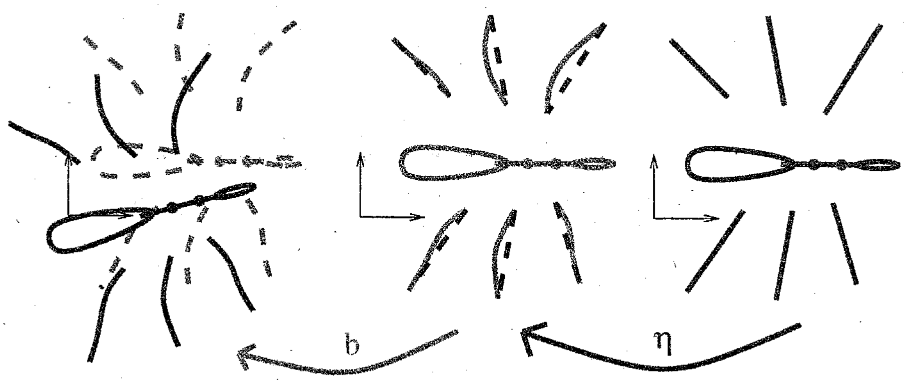

Figure 4.1: The configuration q of a rigid body in a fluid can be uniquely decomposed into a rigid part b that moves both the fluid and the body and a swirl η^b that preserves the body but cancels the rigid motion b at infinity.

The configuration space for a rigid body B in an incompressible inviscid fluid filling $F = M \setminus B$ is $\mathcal{FB}(M,B)$. If $q\colon M_o \to M$ is a map from the reference configuration of both the fluid and body particles, then it can be decomposed as $q = b \circ \eta$. The coordinates $(b,\eta) \in \mathcal{FB}(M,B)$ are unique if we extend η to all of

M by assuming that $\eta|B = \mathrm{id}_M$. The Lagrangian,

$$L(q, \dot{q}) = \int_M \frac{1}{2}\|\dot{q}\|^2 \mu,$$

is invariant and can transformed to the identity as

$$\ell((\xi, v)) = \int_M \frac{1}{2}\|v + \xi_M\|^2 \mu$$

where $(\xi, v) \in \mathfrak{fb}(M, B)$ is in the Lie algebra of $\mathcal{FB}(M, B)$.

The (conserved) spatial momentum corresponding to the symmetry is then

$$J((b, \eta), (\dot{b}, \dot{\eta})) = (\mathrm{Ad}^*_b \beta,\ \eta^*[(v + \xi_M)^\flat]) \tag{4.12}$$

where

$$\beta = \int_M m(v + \xi_M, \cdot_M)\mu \tag{4.13}$$

is the rigid component of the body momentum. The centered dot in this equation is a place holder for an element of the isometry algebra, $\mathfrak{i}(M)$. β is therefore a linear functional on $\mathfrak{i}(M)$ and hence an element of its dual $\mathfrak{i}^*(M)$.

The body inertia operator $\widehat{\mathbb{I}}\colon \mathfrak{fb} \to \mathfrak{fb}^*$ is then

$$\widehat{\mathbb{I}}((\xi, v)) = (\beta,\ [(v + \xi_M)^\flat]),$$

where β is defined in terms of (ξ, v) in (4.13). The reduced Lagrangian becomes

$$\ell((\xi, v)) = \left\langle (\xi, v), \widehat{\mathbb{I}}((\xi, v)) \right\rangle = \int_M m(v + \xi_M, v + \xi_M)\mu.$$

The equations of motion are then the Euler-Poincaré equations,

$$\dot{p} = \pm\, \mathrm{ad}^*_{\widehat{\mathbb{I}}^{-1} p}\, p, \tag{4.14}$$

for $p \in \mathfrak{fb}^*(M, B)$ where $\pm$ is plus for part of the momentum corresponding to the left part of the action and minus for that corresponding to the right part (see §2.2.3). Splitting this in to a "rigid" and "fluid" piece with $p = (\beta, [u^\flat])$ yields:

$$\dot{\beta} = \mathrm{ad}^*_\xi \beta, \tag{4.15}$$

$$\dot{[u^\flat]} = -\mathscr{L}_v [u^\flat], \tag{4.16}$$

where $u = v + \xi_M$, β is defined in (4.13). We now examine each equation in detail.

4.4.1 The Fluid Part

We can rewrite the second equation (4.16) that governs the fluid motion as either

$$\dot{v}^\flat + \nabla_v v^\flat + \mathbf{d}\, p = -\langle v, \mathbf{d}\, \xi_M^\flat \rangle - \dot{\xi}_M^\flat + \mathbf{d}\, \frac{\xi_M^2}{2} \tag{4.17}$$

or with the identification $u = v + \xi_M$ as

$$\dot{u}^\flat + \nabla_u u^\flat + \mathbf{d}\, p = \mathbf{i}_{\xi_M}\, \omega - \mathbf{d}\, \frac{u^2}{2}, \tag{4.18}$$

where $\omega = \mathbf{d}\, u^\flat$ is the vorticity corresponding to u (i.e. in the body frame). The velocity of the fluid relative to the moving body is v. It is therefore parallel to the body at its boundary and has an apparent free stream (rigid) velocity $-\xi_M$ at infinity. $u = v + \xi_M$ is the velocity as seen by an observer in a fixed frame that coincides with with the body's frame at the moment in question. It is zero at infinity and is not necessarily parallel to the boundary of the body. Neither v nor u is the Lagrangian velocity $\dot{q}$ or the body velocity $w = \dot{q}q^{-1}$, the velocity of a particle seen in the fixed reference frame.

The infinite dimensional part of the Euler-Poincaré equations for a body in a perfect fluid are just a simple change of coordinates of the Euler-Poincaré equations, the standard Euler equations (4.5), for a fluid with no body. As before let $q = b\eta$ describes the state of the fluid and body, B, relative to an inertial reference frame, where the coordinates $(b, \eta) \in \mathcal{FB}(M, B)$. The velocity $w = \dot{q}q^{-1} = (\dot{b}\eta - b\dot{\eta})\eta^{-1}b^{-1}$ is then related to those in the moving frames by

$$w = b_*(\xi_M + v) = b_* u, \tag{4.19}$$

where as usual $\xi = b^{-1}\dot{b}$ and $v = \dot{\eta}\eta^{-1}$ so that $(\xi, v) \in \mathfrak{fb}(M, B)$. Taking the time derivative and using (2.24) we see that

$$\dot{w} = b_*(\dot{u} - \mathscr{L}_{\xi_M} u).$$

Recall that the Euler equation for a fluid (without a body),

$$\dot{w} + \nabla_w w + \operatorname{grad} \tilde{p} = 0$$

using (2.22), this equation becomes

$$\dot{w}^\flat + \mathscr{L}_w(w^\flat) + \mathbf{d}(\tilde{p} - \frac{w^2}{2}).$$

Substituting (4.19) in this equation yields,

$$b_*(\dot{u}^\flat + \mathscr{L}_{(u-\xi_M)}(u^\flat) + \mathbf{d}(b^*\tilde{p} - \frac{u^2}{2})) = 0,$$

Note that the isometry b preserves the metric and that the Lie derivative commutes with pull back. After defining $p = b_* \tilde{p}$ relative to the body, the infinite dimensional component of the Euler-Poincaré equations (4.18) is obtained.

Recall the identification $u = v + \xi_M$ from (4.16) so that, as in (4.17), we can switch from (ξ, u) to (ξ, v) to get

$$\dot{v}^\flat + \dot{\xi}_M^\flat + \mathscr{L}_v (v + \xi_M)^\flat + \mathbf{d}(p - \frac{(v + \xi_M)^2}{2}) = 0.$$

Expanding the Lie derivative term using (2.22)

$$\dot{v}^\flat + \dot{\xi}_M^\flat + \nabla_v v^\flat + \mathscr{L}_v \xi_M^\flat + \mathbf{d}(p - \frac{\xi_M^2}{2} - \mathbf{i}_v \, \xi_M^\flat) = 0,$$

and using the magic formula (2.3), we get a useful alternative form of (4.18),

$$\dot{v}^\flat + \nabla_v v^\flat + \mathbf{d}\, p = -\langle v, \mathbf{d}\, \xi_M^\flat \rangle - \dot{\xi}_M^\flat + \mathbf{d}\, \frac{\xi_M^2}{2}.$$

Fluids in a Moving Frame

In Euclidean space $E^3 = (R^3, \delta)$, the terms on the right hand side of (4.17) can be written as

$$\begin{aligned} \dot{v}^\flat + \nabla_v v^\flat + \mathbf{d}\, p &= -\dot{\xi}_M^\flat - \langle v, \mathbf{d}\, \xi_M^\flat \rangle + \mathbf{d}\, \frac{\xi_M^2}{2} \\ &= -\dot{\Omega} \times x - \dot{U} - 2\Omega \times v + \Omega \times (\Omega \times x + U) \end{aligned}$$

where $\xi = (U, \Omega) \in \mathrm{se}(3) = \mathrm{i}(\mathbb{R}^3)$ is the body velocity[3]. See Batchelor [1974, p. 140] for a comparison of the rotational terms noting his Ω has the opposite sign.

The infinitesimal generator is $\xi_M(x) = \Omega \times x + U$ where $x \in \mathbb{R}^3$. In coordinates this is

$$\xi_M^\flat(x) = (\Omega_{ij} x^j + U^i)\, \mathbf{d}\, x^i.$$

Taking the exterior derivative

$$\mathbf{d}\, \xi_M^\flat(x) = \Omega_{ij}\, \mathbf{d}\, x^j \wedge \mathbf{d}\, x^i$$

and pairing with v gives

$$\begin{aligned} \langle v, \mathbf{d}\, \xi_M^\flat \rangle(x) &= \Omega_{ij} v^j\, \mathbf{d}\, x^i - \Omega_{ij} v^i\, \mathbf{d}\, x^j, \\ &= 2\Omega_{ij} v^j\, \mathbf{d}\, x^i \end{aligned}$$

[3] Ω is the rotational velocity of the body with respect to the body frame, and U is the linear velocity of the body with respect to the body frame. In the body frame these are minus the apparent free-stream velocity and rotation.

or in other words

$$\langle v, \mathbf{d}\, \xi_M^\flat \rangle = 2\Omega \times v.$$

The $\mathbf{d}\, \frac{\xi_M^2}{2}$ term can be computed from

$$\xi_M^2 = (\Omega_{ij} x^j + U^i)(\Omega_{ik} x^k + U^i)$$

by taking the exterior derivative

$$\mathbf{d}\, \xi_M^2 = 2(\Omega_{ij} x^j + U^i)\Omega_{ik}\, \mathbf{d}\, x^k.$$

Compare

$$\mathbf{d}\, \frac{\xi_M^2}{2} = \Omega \times (\Omega \times x + U)$$

with Batchelor [1974, p. 162], again with a different sign for Ω. Finally, it should be clear that

$$\dot{\xi}_M^? = \dot{\Omega} \times x + \dot{U}.$$

4.4.2 The Rigid Part

In the previous sections we looked at the infinite dimensional term in the Euler-Poincaré equations for a rigid body in a fluid. We now concentrate on the finite dimensional part (4.15) and show that it is equivalent to the pressure acting on the body. The body rigid momentum is

$$\beta = \int_M m(u, \cdot_M)\mu$$

where as before $u = v + \xi_M$ and v is parallel to the boundary ($\mathbf{i}_v\, \mu|_\partial = 0$). The implication is that $\mathbf{i}_u\, \mu|_\partial = \mathbf{i}_{\xi_M}\, \mu|_\partial$. Recall that $M = B + F$ and that v is zero on the body.

Splitting the momentum integral $\beta = \beta_B + \beta_F$ into body and fluid part by gives the standard rigid body momentum

$$\beta_B = \int_B m(\xi, \cdot_M)\mu = \widehat{\mathbb{I}}_B \xi$$

governed by the standard rigid body equations

$$\dot{\beta}_B - \mathrm{ad}_\xi^* \beta_B = -\mathcal{F},$$

forced by

$$\mathcal{F} = \int_F m(\dot{u}, \cdot_M)\mu - \int_F m(u, [\xi, \cdot]_M)\mu.$$

The infinitesimal generator is a Lie algebra anti-homomorphism (2.9), so

$$\mathcal{F} = \int_F m(\dot{u}, \cdot_M)\mu + \int_F m(u, [\xi_M, \cdot_M])\mu.$$

Continuing, insert $\dot{u}^\flat$ from our infinite dimensional partner (4.18)

$$\mathcal{F} = \int_F \mathbf{i}_{\cdot_M}(\mathscr{L}_{(u-\xi_M)}(u^\flat) + \mathbf{d}(p - \frac{u^2}{2}))\mu + \int_F m(\mathscr{L}_{\cdot_M}\xi_M, u)\mu$$

and combine the two $\mathscr{L}_{\xi_M}$ terms to get

$$\mathcal{F} = \int_F (\mathbf{i}_{\cdot_M}(\mathscr{L}_u(u^\flat) + \mathbf{d}(p - \frac{u^2}{2})) - \mathscr{L}_{\xi_M}(m(u, \cdot_M))\mu,$$

or equivalently

$$\mathcal{F} = \int (\mathbf{i}_{\cdot_M}(\mathscr{L}_u(u^\flat) + \mathbf{d}(p - \frac{u^2}{2})) - \mathbf{i}_{\xi_M}\,\mathbf{d}(m(\cdot_M, u))\mu.$$

By the divergence theorem (2.7), the last term only depends on the value of $\mathbf{i}_{\xi_M}\mu$ along the boundary, where ξ_M and u are equal. The magic formula (2.3) can be used to change the sign of the $\frac{u^2}{2}$ term

$$\mathcal{F} = \int (\mathbf{i}_{\cdot_M}(\mathbf{i}_u\,\mathbf{d}\,u^\flat + \mathbf{d}(p + \frac{u^2}{2})) - \mathbf{i}_u\,\mathbf{d}\,\mathbf{i}_{\cdot_M}u^\flat)\mu.$$

Performing more magic on the last term leads to

$$\mathcal{F} = \int (\mathbf{i}_{\cdot_M}(\mathbf{i}_u\,\mathbf{d}\,u^\flat + \mathbf{d}(p + \frac{u^2}{2})) - \mathbf{i}_u\,\mathscr{L}_{\cdot_M}u^\flat + \mathbf{i}_u\,\mathbf{i}_{\cdot_M}\,\mathbf{d}\,u^\flat)\mu.$$

Infinitesimal isometries, $\cdot_M$, preserve the metric (2.23), so $\mathbf{i}_{\cdot_M}\mathbf{d}\frac{u^2}{2} = \mathbf{i}_u\mathscr{L}_{\cdot_M}u^\flat$, which implies that everything cancels but the pressure. Using the divergence theorem (2.7), the force due to the fluid becomes

$$\mathcal{F} = \int \mathbf{i}_{\cdot_M}\,\mathbf{d}p\mu = \int_\partial p\,\mathbf{i}_{\cdot_M}\mu.$$

In other words, in the absence of any other external forces the integral of the pressure above is the only force acting on the body.

4.4.3 The Kirchhoff Potential

For inertial configurations we have assumed that we have fixed boundary condition at infinity. This couples the motion of the body to the motion of the fluid. In particular there is a unique potential flow corresponding to every body motion.

The *Kirchhoff potential* is the unique harmonic ($\Delta\phi = 0$) function $\phi \in C^\infty(M_o \setminus B, \mathrm{i}^*(M))$ on the reference configuration such that grad $\langle \phi, \xi \rangle - \xi_M$ is parallel to the boundary ∂B and $\phi|_\infty = 0$. This ϕ is a co-Lie algebra vector of potential functions corresponding to the potential (Dirichlet) flows for each rigid motion.

4.4.4 Kirchhoff's Equations

The rigid component (4.15), of the equations of motion (4.14) for a body in a fluid, are Kirchhoff's equations for the evolution of the body fluid momentum[4]. They hold even when the fluid is rotational. Generically they are coupled to Euler's equations (4.16) for the fluid, through the boundary conditions, and the fluid is coupled to the rigid equations directly through the definition of the rigid momentum (4.12).

In the absence of a fluid force, if the fluid component of the momentum is zero, it will stay zero for all time. In this case, that of potential flow, the rigid component, Kirchhoff's equations, completely describe the motion of the system.

Recall that the body velocity is $(v, \xi) \in \mathbf{diff}_{\mathrm{vol}}(M) \times \mathrm{i}(M)$ and body momentum is $\zeta = (\alpha, \beta) = ([u^\flat], \int \mathbf{i}_{\cdot_M} u^\flat \mu)$ where $u = v + \xi_M$. If the fluid component of the momentum is zero

$$[u^\flat] = 0 \Rightarrow u = \operatorname{grad} \phi$$

then, because of the boundary condition $\mathbf{i}_{u-\xi_M} \mu|_\partial = 0$, the fact that the fluid is incompressible ($0 = \delta u^\flat = -\delta \, \mathbf{d} \, \phi = -\Delta\phi$) means that u can be written as the gradient of a linear function on $\mathrm{i}(M)$,

$$u = \operatorname{grad} \phi(\xi).$$

It will stay that way for all time by the fluid part of the fluid-body equations, which means for the body momentum β that

$$\beta = \int \mathbf{i}_{\cdot_M} u^\flat \mu = \int_\partial \phi(\xi) \, \mathbf{i}_{\cdot_M} \mu$$

and since the boundary conditions in this case are linear in ξ we have $\mathbf{i}_{\operatorname{grad}\phi(\cdot) - \cdot_M} \mu|_\partial =$

[4]Here body refers to "body" coordinates, as opposed to "spatial" coordinates, not the rigid body component. Generically a "body" velocity is one that satisfies the Euler-Poincaré equations, the "spatial" velocity is the one that is conserved, and the Lagrangian velocity or just the velocity, satisfies Newton's equations, or more generically the geodesic equation

0 which means we can write the momentum as,

$$\beta = \int_{\partial} \phi(\xi)\, \mathrm{i}_{\operatorname{grad}\phi(\cdot)}\, \mu = \int m(\operatorname{grad}\phi(\xi), \operatorname{grad}\phi(\cdot))\mu = \widehat{\mathbb{I}}\xi,$$

which, given (4.15), leads to the classical Kirchhoff equations.

4.5 The Geometry of the Fluid Body Group

In addition to its group structure, the fluid body group also has the structure of a possibly non-trivial principal bundle over the space of rigid motions. This is a natural configuration space when thinking about stirring.

There is a natural right-action of $\mathbf{Diff}_{\mathrm{vol}}(M)$ on Q when restricted to diffeomorphisms that do not affect particles at infinity. The elements of $\mathcal{FB}(M, B)$ that have the identity rigid component form a subgroup of $\mathcal{FB}(M, B)$,

$$\mathcal{FB}^e(M, B) = \{(e, \eta^e) \in \mathcal{FB}(M, B) \mid \eta^e|_\infty = \mathrm{id}_M|_\infty\}. \tag{4.20}$$

$\mathcal{FB}^e(M, B)$ is a normal subgroup of $\mathcal{FB}(M, B)$ (H is a normal subgroup of G if $g^{-1}hg \in H$ for all $h \in H$ and $g \in G$, in which case G/H is a group), therefore $\mathcal{FB}(M, B)/\mathcal{FB}^e(M, B)$ is a manifold and Lie group.

Proposition 1. *Let $\mathcal{FB}(M, B)$, and $\mathcal{FB}^e(M, B)$ be defined as in* (4.8) *and* (4.20) *then we have the following isomorphism,*

$$\mathcal{FB}(M, B)/\mathcal{FB}^e(M, B) = \mathcal{I}(M). \tag{4.21}$$

Proof. Elements $[(b, \eta^b)] \in \mathcal{FB}(M, B)/\mathcal{FB}^e(M)$ are equivalence classes

$$[(b, \eta^b)] = \{(e, \eta^e)(b, \eta^b) : \forall \eta^e \in F^e(M \setminus B)\},$$

within which b is constant so the projection $[(b, \eta^b)] \mapsto b$ is well defined. For the other direction we have existence by choosing $b \mapsto [(b, \eta_1)]$ where η_1 is the solution at $t = 1$ of

$$\dot{\eta}\eta^{-1} = \operatorname{grad}\langle \phi, \log b\rangle - (\log b)_M$$

starting with $\eta_0 = \mathrm{id}_M$, where $\phi\colon M \to \mathrm{i}^*(M)$ is the Kirchhoff potential §4.4.3, so that we have $\eta_1|_\infty = b^{-1}$. $b \mapsto \eta_1$ is well defined, at least for b near the identity, since since log is. The isometry group, $\mathcal{I}(M)$, is assumed to be connected §2.3.1, a sequence of b_i's can be found, for each of which the log is defined, whose composition is b. The map $b \mapsto [(b, \eta_1)]$ make sense globally if η_1 is chosen as the composition of the respective $(\eta_1)_i$'s, since any two elements η_1^b and $\eta_1^{b'}$ constructed this way, or any other, would differ by by a diffeomorphism that is the identity at infinity and preserves the boundary. □

This proposition implies that the dynamics of a rigid body in an inviscid fluid are ripe for reduction by stages [Cendra et al., 2001b].

4.5.1 Trivialities

Is the principal bundle that describes the configuration space of a rigid body in a perfect fluid trivial? I conjecture that in three dimensions it is trivial (see [Arnold and Khesin, 1998] concerning the diameter of the diffeomorphism group in two and three dimensions), but in two dimensions the question is more subtle and the answer depends on the geometry of the body and the ambient space.

The classic example of a non-trivial bundle is the Möbius strip where the position around the strip is the base and the orientation, the direction of an arrow transversal to the path around the base, is the group. This is locally a bundle $\mathbb{S}^1 \times \{0, 1\}$. By definition it is not globally trivial because every point in the base cannot be globally associated with either 0 or 1.

For a deformable body in a fluid the shape space is the shape of the body, and the group is diffeomorphisms of a reference configuration of the body that are rigid (an isometry) at infinity. Can a group element be associated with each real configuration? If so, it would provide a map that allows comparisons diffeomorphisms of the reference configuration of the fluid to those of the fluid with the body in a different shape. So we seek for a map that for each shape gives a map from the fluid around the reference shape to fluid around the given shape. This is clearly not unique, nor does there seem to be a canonical choice of path. An example choice might be to follow the unique potential flow as a given path from the reference shape to the final shape is traversed. This depends on the path and there is no natural one to choose. Additionally it will not likely yield a global section when, like in the Möbius strip, two non reconcilable paths are taken to get to a particular point.

4.5.2 A Simply Connected Body in the Plane

Consider an ellipse, B, that moves in a plane $M = \mathbb{R}^2$ filled with an incompressible fluid $F = M \setminus B$. A configuration for this system is a map $q \colon M_o \to M$ from one configuration of the fluid and body system to another that does not move particles at infinity, is restricted to be an isometry on the body, and is allowed to slip on the boundary ∂B.

The group of diffeomorphisms of the fluid that are the identity at infinity, $G = \mathbf{Diff}(M \setminus B)^e$, act on Q by composition on the right. The base space Q/G is isomorphic to $\mathcal{I}(M)$, the isometry group of M. An element $q \in Q$ can uniquely be written as $q = b\eta^b$ where b an isometry and η^b is a diffeomorphism of the reference configuration that is b^{-1} at infinity. Because of this Q can equivalently be thought of as the space of η^b's.

A global section, $\sigma \colon \mathcal{I}(M) \to Q$, of the bundle $Q \to Q/G$ exists if and only if the bundle is trivial, i.e. $Q = G \times Q/G$. In other words, we need to smoothly assign to each location of the body a unique configuration of the fluid that moves the fluid at infinity appropriately. Consider the Kirchhoff flow f_1^b, the time one flow of the potential flow problem for $\log b$, from the proof of Proposition 1. This is only defined on the domain of the log. If only body rotations are considered, then f^{π} and $f^{-\pi}$ would be the displacement of the fluid particles resulting from spinning infinity half way around first in the positive direction and a second in the negative. For anything other than a circle, the two diffeomorphisms will not match except at infinity. A global section must therefore be generically rotational.

4.5.3 Construction of a Global Section

Consider a simply connected body and a coordinate system outside the body with a "radial" variable $r \in [r_o, \infty)$ and an "angular" variable $\varphi \in [0, 2\pi]$ such that $r = r_o$ is the boundary of the body and such that the coordinates become "polar" as r approaches infinity (Figure 4.2). We desire a flow $\eta_t(r, \varphi) = (r_t(r, \varphi), \varphi_t(r, \varphi))$

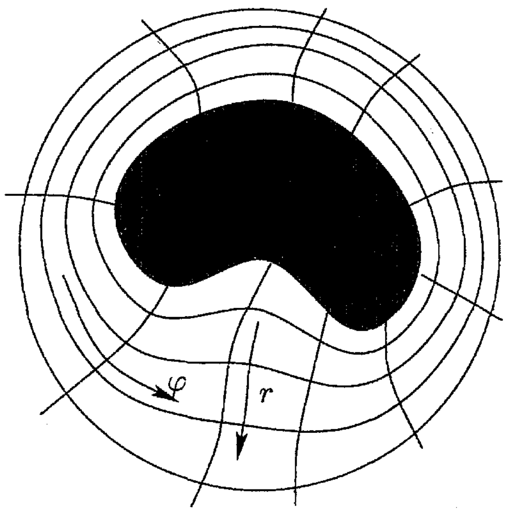

Figure 4.2: Generic simply connected body

that is

1. smooth ,
2. volume preserving ,
3. periodic i.e. $\eta_0 = \eta_{2\pi k} = \mathrm{Id}_{\mathbb{R}^2}$, and
4. uniformly traverses the circle at infinity when $r \to \infty$ for $t \in [0, 2\pi]$.

Let the volume element in these coordinates be $\mu = \mu_{\varphi r}\, \mathbf{d}\varphi \wedge \mathbf{d}r$.

Proposition 2. *The flow corresponding to the stream function,*

$$\psi(r) \equiv \frac{1}{2\pi} \int^{r} \int_{0}^{2\pi} \mu_{\varphi r}\, \mathbf{d}\varphi\, \mathbf{d}r$$

satisfies the above properties and is a global section for a simply connected rigid body that can only spin.

Proof. The velocity, defined by $\mathrm{d}\,\psi \equiv \mathrm{i}_v\,\mu$, is $v = \frac{\psi'}{\mu_{\varphi r}}\frac{\partial}{\partial\varphi}$ and the flow, $\eta_t = (r_t, \varphi_t)$, is defined by $\dot{\eta}_t(r,\varphi) = v(\eta_t(r,\varphi))$ with initial conditions $\eta_0(r,\varphi) = (r,\varphi)$. Integrating the r component we get $r_t(r,\varphi) = r$. By manipulating the expression $\dot{\varphi}_t(r,\varphi) = \frac{\psi'}{\mu_{\varphi r}}$ and by integrating the resulting expression, $\varphi_t(r,\varphi)$ is specified by

$$\frac{1}{\psi'}\int^{\varphi_t} \mu_{\varphi r}\,\mathrm{d}\,\varphi = t + C.$$

The constant of integration $C = \frac{1}{\psi'}\int^{\varphi} \mu_{\varphi r}\,\mathrm{d}\,\varphi$ is set by the initial conditions so that

$$\frac{1}{\psi'}\int_{\varphi}^{\varphi_t} \mu_{\varphi r}\,\mathrm{d}\,\varphi = t. \tag{4.22}$$

The volume form is positive so its integral is monotonic. Therefore, the above equation gives a unique smooth value of φ_t for all t. Using (4.22) and the definition of ψ,

$$\int_{\varphi}^{\varphi_{t+2\pi}} \mu_{\varphi r}\,\mathrm{d}\,\varphi = \psi'(t+2\pi) = \int_{\varphi}^{\varphi_t} \mu_{\varphi r}\,\mathrm{d}\,\varphi + \int_{\varphi_t}^{\varphi_t+2\pi} \mu_{\varphi r}\,\mathrm{d}\,\varphi,$$

we see that $\varphi_{t+2\pi} = \varphi_t + 2\pi$. In other words $\varphi_t - t$ is 2π-periodic while φ_t is 2π-periodic mod 2π. It is volume preserving, as shown by the fact that

$$\eta_t^*\mu = \mu_{\eta_t}\,\mathrm{d}\,\varphi_t \wedge \mathrm{d}\,r_t = \mu_{\varphi_t r_t}\frac{\partial\varphi_t}{\partial\varphi}\,\mathrm{d}\,\varphi\wedge\mathrm{d}\,r = \mu_{\varphi r}\,\mathrm{d}\,\varphi\wedge\mathrm{d}\,r$$

where the last step comes from taking the derivative of (4.22) with respect to φ. Finally, as r approaches infinity, the volume form becomes "polar", i.e. $\mu \to r\,\mathrm{d}\,\varphi\wedge\mathrm{d}\,r$, and therefore $\psi'(r) \to r$ so $\varphi_t(\infty,\varphi) \to \varphi + t$, which uniformly traverses the circle at infinity. Therefore, seen in the reference frame of the circle moving at infinity ($\varphi_t \to \varphi_t - t$), the body uniformly rotates the other direction and all particles move in a volume preserving periodic motion. Therefore $\eta_t(r,\varphi) = (r, \varphi_{-t}(r,\varphi) + t)$ is a global section, a global map, from orientations of the body (specified by t) to volume preserving maps of the fluid outside the reference configuration of the body to the space outside the rotated body, that leaves particles at infinity fixed. □

Corollary 1. *The configuration space for a simply connected rigid body in the plane is trivial.*

Proof. A global section can be found by composing the above map for the rotational part of the rigid motion and the unique potential flow associated with the translational part. □

Remark: Note that the global section for any simply connected shape but a circle is necessarily rotational and therefore any global canonical coordinates for the principal bundle are as well.

It can be seen from (4.22) that $\varphi_{t+s}(r,\varphi) = \varphi_t(r, \varphi_s(r,\varphi))$, so the global section, found in §4.2 for the body that can spin, is a group homomorphism. If a global section for arbitrary motions in the plane, that is also a group homomorphism, could be found, it would provide semi-direct product structure for the fluid-body bundle. I conjecture it is not possible.

The Trivial Ellipse

As an example we construct a global section for rotation of an ellipse defined by major semi-axis a and minor semi-axis b. Let (r, φ) be coordinates for the (x, y)-plane defined by

$$x = (r + f(r)) \cos\varphi \tag{4.23}$$

$$y = (r - f(r)) \sin\varphi \tag{4.24}$$

where f decays to zero as r goes to infinity, so that we recover polar coordinates, and $f(r_o = \frac{a+b}{2}) = \frac{a-b}{2}$, so that r_o is the boundary of the ellipse. For example if $c^2 = a^2 - b^2$ then, $f(r) = \frac{c^2}{4r}$ would suffice.

By (4.23), $\mu = (A + B\cos 2\varphi)\,\mathrm{d}\, r \wedge \mathrm{d}\,\varphi$ where $A = r - ff'$ and $B = rf' - f$ so that

$$\psi(r) = \frac{1}{2\pi}\int^r\int_0^{2\pi}(A + B\cos 2\varphi)\,\mathrm{d}\, r\,\mathrm{d}\,\varphi = \frac{1}{2}(r^2 - f^2).$$

Then, $\psi' = A$ and (4.22) becomes

$$\frac{1}{A}\int_{\varphi}^{\varphi_t}(A + B\cos 2\varphi)\,\mathrm{d}\,\varphi = \varphi_t - \varphi + \frac{B}{2A}(\sin 2\varphi_t - \sin 2\varphi) = t.$$

There is no explicit solution for $\varphi_t(r, \varphi, t)$, but one can easily compute it numerically by iterating

$$\varphi_t \to \varphi + t + \frac{B}{2A}(\sin 2\varphi - \sin 2\varphi_t)$$

4.5.4 A Multiply Connected Body in the Plane

Multiply connected bodies in the plane are non-trivial. To see this, consider two circles that are rigidly connected, and consider rotating the pair rigidly once around. If the fluid body group is a trivial bundle, then there is smooth path for all the particles in which, after the complete rotation, every particle has returned to its original location. In this example, a smooth path cannot be found due to the fixed boundary condition at infinity. Consider the dotted line of fluid particles in Figure 4.3 extending from one of the circles out to infinity before the rotation and compare them with the solid line representing the same line of particles after the rotation. Clearly there is no smooth way for the particles to return to their reference configuration since they are wrapped around the body. Therefore the principal bundle describing the configuration of a multiply-connected rigid body in $\mathbb{R}^2$ is non-trivial.

From this one might conjecture that, with a connected body in a perfect fluid, the same stirring effect can be achieved through only local, as opposed to global,

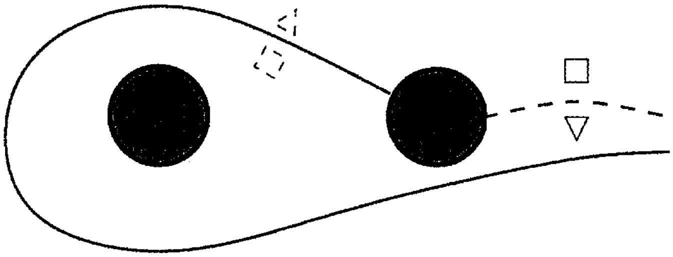

Figure 4.3: Two swizzle sticks. The fluid cannot be made to return to its initial position after a complete rotation of a multiply connected body.

motions whereas, the non-trivial swizzle sticks can expect to achieve stirring motions through global holonomy that are otherwise unattainable. This effect be seen clearly in the experiments found in Boyland, Aref, and Stremler [2000], MacKay [2001].

4.5.5 Stirring in a Perfect Fluid

Take the right principal bundle $\pi \colon \mathcal{FB}(M,B) \to \mathcal{I}(M)$ with structure group $\mathcal{FB}^e(M,B)$. The Lie algebra of $\mathcal{FB}^e(M,B)$ is $\mathfrak{fb}^0(M,B)$, the space of divergence free vector fields that are zero at infinity and tangent to the body boundary. This can be a model of stirring a perfect fluid. Moving a rigid body in a repetitive manner would be small contractible loops in the base space, the space of isometries, producing a net displacement in the fibre, which corresponds to a swirl, an element of $\mathcal{FB}^e(M,B)$. We have seen that when the body is simply connected the bundle is trivial, but for a multiply connected (rigid) body there are global stirring motions that reach fluid configurations not available by local motions.

As will be made clear in Chapter 6, the curvature of the metric orthogonal connection can be used to measure the net stirring effect resulting from periodic motions of the body, so an expression for it is presented.

The infinitesimal generator of the action of $\mathcal{FB}^e(M,B)$ on $\mathcal{FB}(M,B)$ is

$$v_Q(q) = qv$$

with $v \in \mathcal{FB}^e(M,B)$ and $q \in \mathcal{FB}(M,B)$. The connection is almost

$$\omega_q(X) = q^{-1}X \tag{4.25}$$

where $X \in T_q\mathcal{FB}(M,B)$. The connection ω, defined by (4.25), preserves the metric (3.4), is the identity on generators, and is Ad-equivariant, but it is not vertical valued because its value is not necessarily tangent to the boundary. To determine what modification are needed, we examine the defining equation for the mechanical connection (3.4).

$$\mathcal{M}(\xi_Q, \omega_Q(X)) = \mathcal{M}(\xi_Q, X),$$

in terms of the induced metric (4.1)

$$\int_M m(qv, q\omega(X))\mu = \int_M m(qv, X)\mu.$$

After a change of coordinates by q and using (4.1.1) the connection satisfies

$$[(q_*(\omega(X)))^b] = [Xq^{-1^b}].$$

In the same manner that the pressure arose from the equivalence class when deriving the Euler equations, now $\varphi \in C^\infty(M, \mathfrak{i}^*(M))$, the unique Kirchhoff potential §4.4.3 arises

$$\omega(X) = q^{-1}X + q^* \operatorname{grad} \langle \varphi, \pi X \rangle \tag{4.26}$$

to kill off the the part of X in (4.25) normal to the boundary. Because $q = b\eta^b$, this can be rewritten as

$$\omega = \eta^{b^*}(v^\xi + \xi_M + \langle \operatorname{grad} b^*\varphi, \xi \rangle)$$

where $\xi = b^{-1}\,\mathbf{d}\,b \in \Omega^1(Q, \mathfrak{i}(M))$ and $v^\xi = \mathbf{d}\,\eta^b\eta^{b^{-1}} \in \Omega^1(Q, \mathfrak{fb}(M, B))$. This looks a lot like a connection in local coordinates, but it is not since the terms inside the parenthesis are not both parallel to the boundary and zero at infinity. Their sum is, however, so the expression for the connection itself is still valid. For all the terms to be in the Lie algebra we would have to write $\eta^b = \sigma^b\eta^e$ with respect to a local section σ. It is not clear that this is worth the effort because we can just consider the local connection to be

$$A = \xi_M + \langle \operatorname{grad} \varphi, \xi \rangle$$

since, at least for computing the holonomy algebra in Chapter 6 only the curvature (2.12), will be needed and it is necessarily zero at infinity and parallel to the boundary since any rigid part, either at infinity or on the boundary, naturally satisfies Cartan's structure equation (2.12)

$$\mathbf{d}\,\xi + [\xi, \xi] = 0$$

for the Lie group $\mathcal{I}(M)$ which has no curvature. Recall that the bracket, in the expression (2.12) for the curvature, is the Lie algebra bracket of vector fields and the $\mathbf{d}$ is the exterior derivative of the rigid part of A which is linear in ξ. Therefore, when computing the local curvature (2.17) for a right action using (4.5.5), it follows that

$$F = (\mathbf{d}\,\xi)_M + \langle \operatorname{grad} b^*\varphi, \mathbf{d}\,\xi \rangle - [\xi_M + \langle \operatorname{grad} b^*\varphi, \xi \rangle\, , \xi_M + \langle \operatorname{grad} b^*\varphi, \xi \rangle].$$

Using (2.9) and (4.5.5) the curvature becomes

$$F = \langle \operatorname{grad} \xi_M b^*\varphi, \xi \rangle - \langle \operatorname{grad} b^*\varphi, [\xi, \xi] \rangle - [\langle \operatorname{grad} b^*\varphi, \xi \rangle\, , \langle \operatorname{grad} b^*\varphi, \xi \rangle]. \tag{4.27}$$

4.6 Body Deformations

We would like to construct the bundle $S = Q/\mathcal{FB}(M,B)$ where Q is the space of configurations of a deformable body in fluid and $\mathcal{FB}(M,B)$ is the fluid body group. In other words we would like to have a section of the bundle $Q \to S$, that is a smooth map taking a given shape to a placement of the shape in the ambient space M and an associated reference location of the all the fluid particles. The

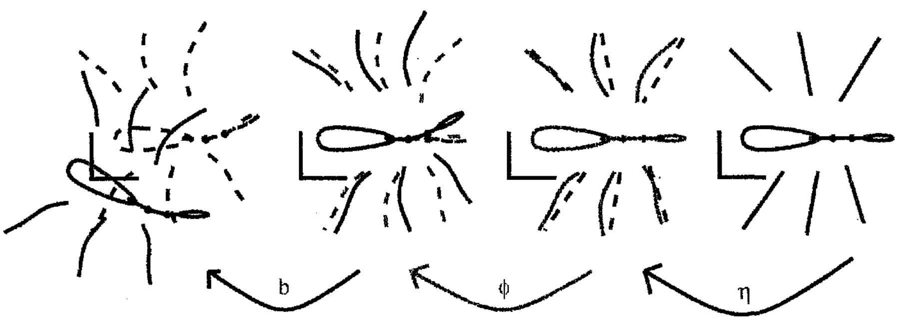

Figure 4.4: The configuration q of a deformable rigid body in a fluid can be uniquely decomposed into a rigid part b that moves both the fluid and the body, a shape which through ϕ gives a reference location for the fluid, not moving infinity, in that shape, and a swirl η^b that preserves the body but cancels the rigid motion b at infinity.

possible deformations of the body are elements of a shape space S. We assume that we are given a map $\phi\colon S \times B \to M$ such that for each shape $s \in S$, $\phi_s(B)$ describes how the body deforms in M (we will mostly write $\phi_s = \phi(s,\cdot)\colon B \to M$).

For convenience we also assume that this map can be smoothly extended to all of M (though we allow discontinuity parallel to ∂B) such that it is volume preserving (i.e. $\phi_s^*\mu = \mu$ (???) where μ is a volume element on M) off B, and for now we assume so on B as well (i.e. all of M) though this is not strictly necessary. This extension is not necessarily unique. We will discuss this later.

The body and the fluid can move by isometries (i.e. rigidly). These motions are described by elements of the isometry group $G = \mathcal{I}(M)$ (SE(3) is the isometry group for $\mathbb{R}^3$).

We want to show that $Q \approx G \times S$. Given q we can write it uniquely as

$$q = b \circ \phi_r \circ \eta$$

Where $(b,\eta)\colon M \to M \in G$, $b^*m = m$, $\eta\colon M \to M$, $\eta^*\mu = \mu$, $\eta|B = \mathrm{Id}_M$. The action of the fluid-body group on q is the same as before and so we have a principal bundle. The equations of motion are again given by conservation of momentum (3.7) since we assume we have control over the shape such that (3.8) can be ignored.

The unique horizontal path $x^h(t) \in Q$ in the configuration space over a given path $x(t) \in Q/\mathcal{FB}(M,B)$ in the shape space can be found by integrating the solutions of $\omega(\dot{x}^h) = 0$. For the mechanical connection ω(3.4) given in terms of our metric $\mathcal{M}$ have then for all $(\xi, v^\xi) \in \mathfrak{fb}(M,B)$ that

$$\mathcal{M}((\xi, v^\xi)_Q, \dot{x}^h) = 0.$$

The generator is given by $(\xi, v^\xi)_Q(q) = \xi_M(q) + qv$ so we get two equations since ξ and v are arbitrary. After changing coordinates we get

$$\int_{q_t(M_t)} m(\xi_M, \dot{x}^h x^{h^{-1}})\mu = 0$$

$$\int_{q_t(M_t)} m(x^h{}_* v, \dot{x}^h x^{h^{-1}})\mu = 0$$

The latter is equivalent to $[x^{h^*}(\dot{x}^h x^{h^{-1}})^\flat] = 0$ which implies that

$$\dot{x}^h x^{h^{-1}} = \mathrm{d}\varphi$$

the horizontal flow is potential.

Chapter 5

Reduced Series Expansions

This chapter develops a series expansion for the geometric phase or holonomy generated by a non-Abelian principal connection in terms of the covariant derivatives of its curvature along with geometric properties of its shape deformation. First, series expansions for the values of a function defined along the flow of an time dependent analytic vector field are examined. The vector field represents a non-linear control system and its time dependence represents the effects of the inputs. These systems have a rich algebraic structure and the expansions provide a connection between the Lie brackets that appear in tests for local controllability and properties of the input that are necessary to effect motion in directions corresponding to the brackets.

One of the first people to consider formal series expansions for the flow of non-commutative operators was Feynman [1951] who introduced the path ordered exponential as a way to deal with the non-commutativity. He expanded time integrals of compositions of non-commutative operators by allowing the order of composition to depend on the value of time. Magnus [1954] extend this work by generalizing the Cambell-Baker-Hausdorff formula, which dealt with the effective flow of two non-commuting linear operators, to the continuously variable case in which a single operator fails to commute with itself at different times. He found a differential equation for a time independent operator that had the same flow at a given time as the time varying one. He was then able to recursively integrate the equation to obtain a series in which the Lie algebraic structure intertwined with iterated integrations. Chen [1957] generalized these ideas to the flow of nonlinear vector fields. These series expansions have since become widely acknowledged as invaluable tools in the study of controllability of nonlinear systems [Hirschorn and Lewis, 2002, Kawski, 1998, Liu., 1997]. Agračev and Gamkrelidze [1978] modernized the treatment of these series and added convergence results. Sussmann [1986] succeeded in finding, and Kawski and Sussmann [1997] better formalized, an explicit formula for the series expansion in terms of a product of exponentials of Hall basis elements with explicit coefficients. It is still and open question, whether explicit coefficients can be found for the exponentiated sum of basis elements. The series we concentrate on has explicit coefficients, but no explicit basis. Bullo [2001] has created a series specialized to the evolution of mechanical systems. This series is cast in a similar setting to ours, but without the symmetry and the zero momentum condition. [Vela, 2003] has shown a formal correspondence between series expansions and averaging theory by showing that truncations of the single are related the standard averages. This idea was used, but not formalized, in [Leonard and Krishnaprasad, 1995].

The second half of this chapter will connect these ideas to those of the previous chapters by specializing to the case when the system in question has enough symmetry to allow these series expansions to be reduced. This reduction, while simplifying the general case, will nonetheless be an extension of the standard results for the expansions for the holonomy, or geometric phase, of systems defined by a connection on a principal bundle. The standard results cover systems defined on Abelian bundles [Berry, 1984, Koon and Marsden, 1997] where the holonomy is written as an exponential of the integral of the curvature. In the non-Abelian case we have expansions directly in the group to second [Shapere and Wilczek, 1989] or third [Rui, Kolmanovsky, McNally, and McClamroch, 1996] order. Again, the curvature and, though not explicitly stated, its derivatives playing a key role. To continue to see this structure in a higher order expansion it is necessary to move to an exponential representation like in the Abelian case. For left-invariant vector fields this technique has been used extensively [Leonard and Krishnaprasad, 1995]. We try and bridge the gap between this case and the general non-linear case by allowing the control techniques from the left-invariant case to apply to the vector field case by performing reduction.

We also review the reduced controllability tests from [Kelly and Murray, 1995] for control systems on principal bundles and then relate them to the standard notion the holonomy algebra [Kobayashi and Nomizu, 1963]. The series presented here can then be thought of as a expansion for the local holonomy in terms of its holonomy algebra.

When this work was completed it was the first appearance of the adjoint bundle that we are aware of in the non-linear control and mechanics communities. Since then however it has proven to be one of the fundamental objects in the reduction of mechanical systems with symmetry Cendra et al. [2001a].

5.1 Motivation and Background

Nonholonomic mechanical systems naturally occur when there are rolling constraints [Kelly and Murray, 1995] or Lagrangian symmetries leading to momentum constraints [Bloch et al., 1996]. Examples include kinematic wheeled vehicles, free floating satellites with appendages, and simplified models of bio-mimetic locomotion. This chapter considers the local motion planning problem for a specific class of nonholonomic systems—those whose configuration space is the total space of a principal fibre bundle and whose equations of motion are described by a connection on that bundle. The state variables of these systems naturally split into two classes. One class is the set of *base* or *shape* variables that describe the internal configuration of the system. The other variables take values in a Lie group G, and are termed *group* or *fibre* variables. They typically describe the position of the system via the displacement of a reference frame in the moving system with respect to a fixed frame. Motion in the position variables can often be realized though periodic motion of the shape variables.

The governing equations of many nonholonomic control systems locally take the form

$$\begin{aligned} \dot{g} &= -gA_i(x)u^i \\ \dot{x}^i &= u^i \end{aligned} \tag{5.1}$$

where $x \in M$ the shape manifold, $g \in G$ a Lie group with Lie algebra $\mathfrak{g}$, and $A\colon TM \to \mathfrak{g}$ is termed the *local form* of the connection. We assume that we have complete control over M. We seek to control the group variables through actuation of the shape variables. In the case of the kinematic car, the shape space consists of the wheel rolling and turning angles, while the car's position in SE(2) defines the group. The connection describes the no slip constraint between the wheels and the ground. For a more complete review of these ideas see [Kelly and Murray, 1995].

First, we construct an expansion for the system's group displacement that arises from small periodic motions in the base space. This expansion is a generalization of the work of Leonard and Krishnaprasad [Leonard and Krishnaprasad, 1995], who developed an analogous formula for case when the local connection form A is constant. Kolmanovsky et al., [Rui et al., 1996] have developed a less structured version of this formula that expands directly in the group rather than in its Lie algebra. In order to develop an intrinsic geometric understanding of these systems we next relate the terms in the expansion to the infinitesimal holonomy algebra of the bundle and to the controllability distribution. In doing so we introduce the covariant derivative on the associated adjoint bundle as a simple means to calculate the terms in the expansion and the small-time-local-controllability tests. These results represent a sharpening and intrinsic restatement of the controllability results of Kelly and Murray [Kelly and Murray, 1995]. We sum up by showing that we can write our expansion to any order as a reduction of a series for general affine control systems given by Sussmann [Sussmann, 1986].

5.1.1 Local Expansion of the Group Displacement

Since the system (5.1) is "kinematic," a path in M completely specifies the resulting path in G and the control u that will generate it. For open-loop planning it is sufficient to design paths in the base M. We can think of A_i in (5.1) as components of a Lie algebra valued one-form $A_i\colon TM \to \mathbb{R}$ and write the equivalent system as

$$\dot{g} = -gA\dot{x} \tag{5.2}$$

We would like to find a solution for this equation that will aid in designing or evaluating paths. We let $g(t) = g(0)e^{z(t)}$ and use an expansion for the Lie algebra valued function $z(t)$ given by Magnus [Magnus, 1954].

$$z = \overline{A} + \frac{1}{2}\overline{[\overline{A}, A]} + \frac{1}{3}\overline{[[\overline{A}, A], A]} + \frac{1}{12}\overline{[\overline{A}, \overline{[\overline{A}, A]}]} + \cdots \tag{5.3}$$

where $\overline{A}(t) \equiv \int_0^t A(\tau)\dot{x}(\tau)d\tau$.

To obtain useful results, examine the group displacement resulting from a periodic path $\alpha\colon [0,T] \to M$ such that $\alpha(0) = \alpha(T)$. In coordinates we have $A(x)\dot{x} = A_i(x)\dot{x}^i$. Taylor expand A_i about $\alpha(0)$ and then judiciously regroup, simplify, apply integration by parts, and use the fact that the path is cyclic. The expansion up to third order is,

$$z(\alpha) \quad = \quad -\frac{1}{2}F_{ij}\int_\alpha dx^i dx^j \; + \; \frac{1}{3}(F_{ij,k} \; - \; [A_i, F_{jk}])\int_\alpha dx^i dx^j dx^k \; + \; \cdots \tag{5.4}$$

where $F_{ij} \equiv A_{j,i} - A_{i,j} - [A_i, A_j]$ and

$$\int_\alpha dx^i dx^j dx^k \equiv \int_0^T \int_0^{t_k} \int_0^{t_j} \dot{x}^i(t_i)dt_i \; \dot{x}^j(t_j)dt_j \; \dot{x}^k(t_k)dt_k.$$

A and F are evaluated at $\alpha(0)$ so that the coefficients of the integrals are constants. The resulting motion is a geometric phase since the integrals represent parameterization independent areas and moments of the path.

Given a desired $g(T)$ that is sufficiently close to $g(0)$, the term $z(T)$ is easily computed. We then find the lowest order of the integral coefficients that can be summed to produce $z(T)$. The coefficients of the vector sum determine the desired values of the integral terms. We can now use the results of [Leonard and Krishnaprasad, 1995] to plan trajectories in M that will have the needed geometric properties. We note that for A constant our results simplify to those in [Leonard and Krishnaprasad, 1995]. While this result is sufficient to design paths for these systems, greater understanding can be gained by recasting the problem in the geometric framework found in [Kelly and Murray, 1995, Montgomery, 1993] and viewing the series as a reduction of a general series for the flow of a time dependent vector field given by Sussmann [1986]. We begin by reviewing these now.

5.2 The Chen Series

We will be dealing with a system of the following form.

$$\dot{x}(t) = X_t(x(t))$$

where for our purpose the time dependence represents the input $X_t(x(t)) = X(x(t), u(t))$ freedom of a control system. Therefore our time dependent vector field $X_t \in \mathfrak{X}(M)$ takes its values in a given input distribution D. Often we will make this restriction explicit by writing

$$\dot{x}(t) = u_a(t)X_a(x(t)) \tag{5.5}$$

where $\{X_a\}$ with $X_a \in \mathfrak{X}(M)$ is a basis for D.

We give first a review of series solutions for the flow maps of non-stationary vector fields. A non-stationary vector field X_t on a manifold M is one that depends explicitly on a real parameter t, the salient feature of which is that for two distinct times t_1 and t_2 the bracket $[X_{t_1}, X_{t_2}]$ is not necessarily 0. In the context of control, this time dependence represents the effects of the inputs on the system.

We will be constructing formal series solutions for the value of a real valued function ϕ of M along solutions of the differential equation (5.2). The first series that will be useful to us is the Chen [1957] series for $\phi(x(t))$. It can be found though repeated application of the fundamental theorem of calculus, first to the function $\phi(x(t))$,

$$\phi(x(t)) = \phi(x(0)) + \int_0^t (X_s\phi)(x(s))ds$$

and then again to the function $(X_s\phi)(x(t))$

$$(X_s\phi)(x(t)) = (X_s\phi)(x(0)) + \int_0^t (X_\tau(X_s\phi))(x(\tau))d\tau$$

and so on to get

$$\phi(x(t)) = \sum_{k\geq 0} \int_0^t \int_0^{t_k} \cdots \int_0^{t_2} (X_{t_1} \cdots X_{t_k}\phi)(x(0))dt_1 \cdots dt_k. \tag{5.6}$$

Note that in the case where the vector field is not a function of time it can be pulled outside the integral so that we obtain

$$\begin{aligned} \phi(x(t)) &= \sum_{k\geq 0} (X^k\phi)(x(0)) \int_0^t \int_0^{t_k} \cdots \int_0^{t_2} dt_1 \cdots dt_k \\ &= \sum_{k\geq 0} (X^k\phi)(x(0)) \frac{t^k}{k!} \\ &= (e^{tX}\phi)(x(0)) \end{aligned} \tag{5.7}$$

which shows that the formal exponential of tX, thought of a differential operator, is the time-t flow map for the flow of the stationary vector field X.

Therefore we define the following

$$\exp^{\int X_t} \equiv \sum_{|T|\geq 0} \int^t X_T dT, \tag{5.8}$$

where the sum is on length $|T| = k$ of the multi-variable $T = (t_1, \ldots, t_k)$. This is the time-t flow map, a partial differential operator, which when applied to the function ϕ

$$\phi(x(t)) = (\exp^{\int X_t} \phi)(x(0))$$

gives its value (5.6) along the flow $x(t)$ of the time dependent vector field X_t. The use of exp is justified because it reduces (5.7) to the standard exponential

$$\exp^{\int X} = e^{tX}$$

defined by the usual infinite series when X_t is independent of time.

5.3 Exponential Series Expansions

If we expand the time dependent vector field $X_t = u_a(t)X_a$ in a time independent basis $\{X_a\}$ for the input distribution, then we can rewrite the time t flow map (5.8) as

$$\exp^{\int X_t} = \sum_{|I|>0} X_I \int^t u_I \equiv X_{a_1} \ldots X_{a_k} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{k-1}} u_{a_1}(t_1)dt_1 \cdots u_{a_k}(t_k)dt_k$$

where there is an implied sum of over each a_i in the multi-index $I = (a_1, \ldots a_k)$. The $\int u_I$ can be thought of as the coefficients of a polynomial where the X_a are the non-commuting variables, and we can treat this partial differential operator algebraically. To do this we employ some results from the combinatorics of words [Lothaire, 1997] [1].

In particular the coefficients $\int u_I$ satisfy the necessary and sufficient condition of Ree's Theorem [Ree, 1957] such that the series $\exp^{\int X_t}$ is the formal exponential of Lie elements. Lie elements are commutators which, when thought of as vector fields, become Lie brackets.

5.3.1 The Magnus Series

Given a vector field X_t we ask if there is a vector field Z_t whose formal exponential e^{Z_t} is the time-t flow map $\exp^{\int X_t}$ for X_t. That is, can we find a vector field Z_t such that its time-1 flow with t fixed is the same at time t of the time dependent flow of X_t?

$$\exp^{\int X_t} = e^{Z_t}$$

This question was answered by Chen [1957] who gave a differential equation for the Z_t

$$\dot{Z}_t = f(\mathrm{ad}_{Z_t})X_t$$

where $f(x) = x/(1 - e^{-x}) = \beta_k x^k$ (the β_k are the Bernoulli numbers). There are a number of different forms for the solution to this equation. Magnus [1954] repeatedly integrated this equation to get

$$Z_t = \overline{X_t} + \frac{1}{2}\overline{[\overline{X_t}, X_t]} + \frac{1}{4}\overline{[\overline{[\overline{X_t}, X_t]}, X_t]} + \frac{1}{12}\overline{[\overline{X_t}, \overline{[\overline{X_t}, X_t]}]} + \cdots$$

[1] A further exposition of these ideas and their connection to control theory can be found in found in [Kawski, 1994, Kawski and Sussmann, 1997].

where $\overline{X_t} \equiv \int^t X_\tau d\tau$. Note that this has the same form as (5.3).

5.3.2 Sussmann's Log Series

Another way to find Z_t such that $\exp^{\int X_t} = e^{Z_t}$ is to take the formal logarithm of the Chen series (5.8). This was done by Sussmann [1986] who wrote down a closed form solution for Z_t by taking the formal log of the Chen series which we rewrite

$$\phi(x(t)) = \sum_{|T|\geq 0} \int^t (X_T\phi)(x(0))dT$$

using an implied sum on length $|T| = k$ of the multi-variable $T = (t_1, \ldots, t_k)$. Taking the log, $\ln X \equiv \sum_{k>0}(-1)^{k+1}\frac{(X-1)^k}{k}$, we see that

$$Z_t = \ln(\exp^{\int X_t}) = \sum_{\substack{k>0 \\ |T|>0}} \frac{(-1)^{k+1}}{k}(\int^t X_T dT)^k.$$

Since we know that Z_t is a Lie element, we can use Dynkins's theorem [Lothaire, 1997] which says $|T| \int X_T dT = \int [X_T] dT$ (there is an implied sum on the length of T even though there are three Ts) where $[X_T] \equiv [X_{t_1}, [X_{t_2}, \ldots [X_{t_{k-1}}, X_{t_k}] \ldots]]$ [2]. We then have that

$$Z_t = \ln(\exp^{\int X_t}) = \sum_{\substack{k>0 \\ |T_i|>0}} \frac{(-1)^{k+1}}{k} \frac{[(\int^t X_{T_1} dT_1) \ldots (\int^t X_{T_k} dT_k)]}{|T_1 \ldots T_k|}$$

5.3.3 Sussmann's Product Series

In addition to a single exponential, Sussmann [1986] also expanded the flow map $\exp^{\int X_t}$ of a time dependent vector field $X_t = u_a X_a$ as a product of exponentials of elements of a basis $\mathcal{B}$ for the free Lie algebra with $\{X_a\}$ as indeterminates.

$$\exp^{\int X_t} = \overrightarrow{\prod}_{B\in\mathcal{B}} e^{c_B(t)B} \tag{5.9}$$

where $c_a(t) = \int_0^t u_a dt$ and $c_{ad^m_{B_1} B_2}(t) = \int_0^t (c_{B_1}(t))^m dc_{B_2}(t)$. For a more recent take on this series, see [Kawski and Sussmann, 1997].

5.4 Reduction

The goal of this section is to show how the series we have just reviewed can be reduced in the presence of symmetry and periodic inputs. In this way we generalize (5.4) and see it both as an extension of known expansions for left-invariant systems and as a reduction of the case general non-linear systems.

[2]The brackets can nest in either direction

Given a configuration space that is a principal bundle where we assume complete control over the base space, and a non-holonomic control system that is forced by completely controllable horizontal lifts, then the the system is driftless, kinematic and takes the form

$$\dot{q} = X_t^h \tag{5.10}$$

where we are given complete control over the time dependent vector field X_t. Let $\{X_i\}$ be a basis for the tangent space of the shape such that $[X_i, X_j] = 0$, and then write, with $X_t = u^a(t)X_a$,

$$\dot{q} = u^a(t)X_a^h. \tag{5.11}$$

With respect to a local section, $X^h = X\frac{\partial}{\partial x} - gAX\frac{\partial}{\partial g}$, so (5.10) can be expressed locally as

$$\dot{g} = -gA_a(x)u^a(t)$$

where $A_a = A(X_a)$.

In order to reduce the series we first reduce the brackets that appear and show that they are left-invariant and so are defined by their values at the identity. With this we can relate the Lie algebra rank condition for controllability of the driftless systems to the infinitesimal holonomy algebra. We then show that we can reduce the exponential from a flow on the full space to one on the Lie algebra of the symmetry group.

5.4.1 The Reduced Brackets

Theorem (Bracket Symmetry). *Let ω be a connection on a principal bundle $\pi: Q \to Q/G$ and let $X, Y \in \mathfrak{X}(Q/G)$ and $f \in C^\infty(Q, \mathfrak{g})$ then*

$$[X^h, Y^h] = [X, Y]^h - \Omega_Q(X^h, Y^h) \tag{5.12}$$

$$[X^h, f_Q] = (X^h f)_Q \tag{5.13}$$

where Ω is the curvature of ω.

Proof. Because X^h and X are π-related we have $T\pi([X^h, X^h]) = [X, X]$. The first equation then follows because of the isomorphism between VQ and $\mathfrak{g}$ and

$$\begin{aligned}\omega([X^h, Y^h]) &= -d\omega(X_I^h, Y^h) + X^h(\omega(Y^h)) + Y^h(\omega(X^h)) \\ &= -(\Omega + [\omega, \omega])(X^h, Y^h) \\ &= -\Omega(X^h, Y^h)\end{aligned}$$

where we have used (2.5) and (2.12) and the fact that ω vanishes on HQ. Because vertical vectors are π-related to zero, $T\pi([X^h, f_Q]) = 0$. The second equation then follows from

$$\begin{aligned}\omega([Z^h, f_Q]) &= -d\omega(Z^h, f_Q) + Z^h(\omega(f_Q)) + f_Q(\omega(Z^h)) \\ &= (-\Omega - [\omega,\omega])(Z^h, f_Q) + Z^h(\omega(f_Q)) \\ &= Z^h f\end{aligned}$$

because Ω is zero if one of its arguments is vertical. □

5.4.2 Holonomy

For analytic systems, the *infinitesimal holonomy group*,

$$H_q = \{g \in G : \exists\ c\colon [0,T] \to M \ni c^h(0) = q, c^h(T) = gq,\ c(0) = c(T)\},$$

is the subgroup of group displacements in the fibre resulting from the horizontal lifts of small contractible loops in the base. It is shown in [Kobayashi and Nomizu, 1963] that H_q is a Lie subgroup whose Lie algebra is spanned by

$$X^h_{i_k} \circ \cdots \circ X^h_{i_2} \circ (\Omega(X^h_{i_1}, X^h_{i_0})) \tag{5.14}$$

where $X_i \in TM$.

The local curvature is defined relative to a section σ as $F \equiv \sigma^*\Omega$. It can be shown that $F = dA - [A, A]$. Clearly the coefficients of the area term in (5.4) are the local coordinates of F. As we will see, the other coefficients are also related to the pull backs by the local section σ of the expressions in (5.14).

We can now use these expressions for the Lie brackets of horizontal and vertical lifts to reduce the Lie bracket condition for small-time-local-controllability in the case of completely horizontal inputs.

5.4.3 Controllability

Theorem 1 (Reduced Controllability). *The following are equivalent.*

1. *(5.4) is small-time-locally-controllable at* $q \in Q$
2. $\mathrm{span}([X^h_{i_k}, \ldots [X^h_{i_2}, [X^h_{i_1}, X^h_{i_0}]] \ldots])(q) = T_q Q$
3. $\mathrm{span}(X^h_{i_k} \circ \cdots \circ X^h_{i_2} \circ (\Omega(X^h_{i_1}, X^h_{i_0})))(q) = \mathfrak{g}$
4. $\mathrm{span}(\nabla_{X_{i_k}} \cdots \nabla_{X_{i_2}}(\overline{\Omega}(X_{i_1}, X_{i_0})))(\pi(q)) = \tilde{\mathfrak{g}}$
5. $\mathrm{span}(\widehat{\nabla}_{X_{i_k}} \cdots \widehat{\nabla}_{X_{i_2}}(F(X_{i_1}, X_{i_0})))(\pi(q)) = \mathfrak{g}$

Proof. From Chow's theorem we know that (5.4) is small-time-locally-controllable if and only if the iterated Lie brackets of the X^h_i plus the vector fields themselves span $T_q Q$. Because of the natural splitting of TQ into HQ and VQ, we must show that the brackets span VQ since the vector fields X^h_i exactly span HQ.

The equivalence of 2. and 3. should be clear from (5.12) and their equivalence with 4. and 5. from §2.2.6 and the fact that $\Omega(X^h, Y^h)$ is Ad-equivariant, and can therefore be thought of, $\overline{\Omega}(X,Y) = [(q, \Omega(X^h, Y^h))] \in \tilde{\mathfrak{g}}$, as a section of the adjoint bundle. □

This theorem is a generalization of similar theorems for trivial bundles and the infinitesimal holonomy algebra for Abelian groups found in [Kelly and Murray, 1995]. Note that the last three expressions in 5 are only functions on M.

We have now an explicit relation between the brackets of Chow's theorem, the infinitesimal holonomy algebra from [Kobayashi and Nomizu, 1963], and the coefficients that appear (5.1). Using this framework, we next show that the series in (5.1) can generally derived as the reduction to the Lie algebra of a series given by Sussmann [Sussmann, 1986] for the general solution of an affine control system in which we replace brackets with covariant derivatives.

5.4.4 The Reduced Series

We apply the series expansion to (5.4), subject to periodic inputs, and choose $g\colon Q \to G$, defined by a local trivialization (2.16) $\sigma\colon Q/G \to Q$, as the output function. We seek to find the value of $g(q(T))$ the group displacement, after a periodic change in shape.

Notice that, because $[X_i, X_j] = 0$, the only brackets of the form $[X_I^h]$, which are horizontal, are the $\{X_i^h\}$, and they have a coefficient of $\int_0^t dx^i$. These coefficients are zero when evaluated a T because the inputs are periodic. The brackets $[X_I^h]$, with $|I| > 1$, are vertical and can be written as the infinitesimal generator of an Ad-equivariant function.

Proposition 3. *If $f\colon Q \to \mathfrak{g}$ is an* Ad-*equivariant function, then*

$$e^{f_Q} g = e^f g$$

where the exponential on the left is a formal series consisting of compositions of vector fields acting on the equivariant function $g\colon Q \to G$, defined by a local section (see §2.2.11). The exponential on the right is the Lie algebra exponential right translated by g.

Proof. Since g is equivariant, using (2.10), we have

$$f_Q g = \left.\frac{d}{dt}\right|_{t=0} L^*_{e^{tf}} g = \left.\frac{d}{dt}\right|_{t=0} e^{tf} g = fg.$$

If we apply f_Q again we see that $f_Q(f_Q g) = ffg$. This can be repeated to obtain the result. □

With respect to a local trivialization, (2.2.11) $f = \mathrm{Ad}_g \widehat{f}$ so

$$e^f = \mathrm{Ad}_g e^{\widehat{f}}.$$

Therefore, if we let

$$\widehat{\nabla} X_I \equiv \begin{cases} -F(X_i, X_j) & \text{if } I = (i,j),\ |I| = 2 \\ \widehat{\nabla}_{X_j}(\widehat{\nabla} X_J) & \text{if } I = (j, J),\ |I| > 2 \end{cases}$$

and $g(t) = g_0 e^{z(t)}$, we can write our expansion for z in this notation as

$$z(T) = \sum_{k>0, |I_i|>1} \frac{(-1)^{k+1}}{k} \frac{(\int dx^{I_1}) \dots (\int dx^{I_k})}{|(I_1, \dots, I_k)|} \widehat{\nabla} X_{(I_1, \dots, I_k)}$$

where the first few terms are

$$z(\alpha) = -\frac{1}{2} F(X_i, X_j) \int_\alpha dx^i dx^j + \frac{1}{3} \widehat{\nabla}_{X_i}(F(X_j, X_k)) \int_\alpha dx^i dx^j dx^k + \cdots \quad (5.15)$$

Though this series is limited because the curvature terms are not linearly independent (because of the Jacobi identity), we can easily apply this technique to reduce in an analogous way Sussmann's product of exponentials formula (5.9) which is based on a Philip-Hall basis.

Chapter 6

The Fish and The Hat

This chapter provides mainly two examples, one of swimming and one of stirring.

The mobility of three connected rigid bodies in a plane filled with a perfect fluid is examined first. As this is intended to be a simple model for a swimmer, complete control over the relative positions of the bodies, or more practically, the angles between the bodies is assumed. This system fits into the general framework of simple mechanical systems with symmetry introduced in Chapter 3. In addition, when started from rest, the controllability and displacement resulting from cyclic motions can be analyzed using the techniques from Chapter 5.

Since exact computations are infeasible, we approximate our deformable body in a way that can provide useful insight. In analyzing systems of this nature, many researchers, [Kelly, 1998, Mason and Burdick, 1999, Shapere and Wilczek, 1989], have computed an expansion for the velocity and pressure fields around the slightly deforming body. We take a different approach. Instead, we suggest an approximation of a deformable body in a fluid consisting of *mechanically coupled but hydrodynamically decoupled bodies*, that is, a system of rigid bodies with Kirchoff like added masses connected and articulated like they were non interacting rigid bodies.

This, of course, is an approximation that makes sense only when the bodies are well separated though still connected via links with negligible hydrodynamic effect. We assume that this technique captures the essence of the systems we study, though clearly not the exact details of their motion.

These simple low dimensional models are sufficiently rich that, together with models of lift and drag, they can capture enough of the relevant dynamics for the model to be used for path planning or controller synthesis[Morgansen, Vela, and Burdick, 2002]. A recent model of a deformable Joukowski foil that capture the effects of vorticity generation and added mass is quite complicated[Mason, 2002]. The examples concentrate on the purely inviscid case for its simplicity and to emphasize the fact that alone, i.e. without lift or drag, a stripped-down model can provide complete controllability. I feel that this fact is often overlooked and bears emphasis. These models may not be realistic or capture the most significant dynamics, but they do capture, in a simple way, effects present in the analogous real systems.

Not only can systems consisting of mechanically coupled but hydrodynamically decoupled bodies be controllable, but the resulting gaits are reasonably efficient for "bracket" motions. This stands in contrast with the classic rolling examples, like a wheelchair or car, where Lie brackets give rise to impractical parallel parking gaits.

To explain why such a simple extension of rigid body dynamics leads such different characteristic behavior, some emphasis is put on demonstrating the possibly non-intuitive characteristics of added mass. The notion of a center of mass doesn't exist generically for rigid bodies in inviscid fluids. This is demonstrated with a simple model of a propeller.

When started from rest, a fluid-body system will return to rest once the joints stop moving. The center of mass of a system of internally actuated rigid bodies in space that is at rest will stay at rest even once the bodies are actuated to move relative to each another. In a fluid, even when the center of mass exists, this is not the case. The articulated satellite in space becomes a fish when put in an inviscid fluid.

The second example presented is that of a rigid body stirring a perfect fluid. We have asked what net motions of a body in space result from performing cyclic deformations of its shape. As we have seen, the same formalism can be applied when the body is immersed in a perfect fluid. In the latter case, the "net motions" naturally include those of the fluid as well as the body. In particular, though it will be ignored in the approximations used to describe the swimmer, the series expansion Chapter 5 theoretically gives an approximation for the resulting swirl of the fluid as well the net motion of the body.

This can be highlighted by ignoring body deformations and by looking a the simpler problem of stirring with a rigid body. The formalism developed here gives a concrete way to answer questions such as: Can a Hat dropped from a dock can be moved to an arbitrary location with an arbitrary orientation. If this is true, then can any diffeomorphism of the fluid be approximated just via stirring? The latter question is not answered here but the groundwork to do so is set.

6.1 Articulated Rigid Bodies

To study deformable bodies in a fluid, we consider articulated rigid bodies. The effects of fluid around the bodies is approximated by assuming that they are rigidly coupled but hydrodynamically decoupled. This is only accurate, of course, when the bodies are sufficiently separated. This technique, however, does lead to computable and manipulable systems of equations that still qualitatively capture many of the locomotion properties of the systems of interest.

The articulated bodies will consist of collections of ellipses for simplicity. The location of each body is parameterized as follows.

The rigid displacement $b \in \mathrm{SE}(3)$ specifies the location, with respect to an inertial frame, of a frame moving with the bodies, and usually fixed in one in particular. The location of a body fixed frame located at the center of mass of each body is then specified as a rigid displacement $b_i(\theta) \in \mathrm{SE}(3)$, depending on some parameters $\theta \in S$, from to the moving frame specified by b. The inertial location of each of the bodies will then be $h_i = bb_i$. The body velocity of each of the bodies is then

$$\xi_i = h_i^{-1}\dot{h}_i = \mathrm{Ad}_{b_i^{-1}}(\xi + A_i\dot{\theta})$$

where $\xi = b^{-1}\dot{b}$ and $A_i = \mathbf{d}\, b_i b_i^{-1}$. The Lagrangian and kinetic energy of the system is then

$$\begin{aligned}\ell(\xi, \theta, \dot{\theta}) &= \sum_i \frac{1}{2}\xi_i^T \widehat{\mathbb{I}}_i \xi_i \\ &= \frac{1}{2}\xi^T \widehat{\mathbb{I}}\xi + \xi^T \widehat{\mathbb{I}A} + \frac{1}{2}\dot{\theta}^T \widehat{m}\dot{\theta}\end{aligned}$$

where $\mathbb{I}_i = \mathrm{Ad}^T_{b_i^{-1}} \widehat{\mathbb{I}}_i \mathrm{Ad}_{b_i^{-1}}$

$$\begin{aligned}\widehat{\mathbb{I}} &= \sum_i \mathbb{I}_i, \\ \widehat{\mathbb{I}A} &= \sum_i \mathbb{I}_i A_i, \\ \widehat{m} &= \sum_i \frac{1}{2} A_i^T \mathbb{I}_i A_i.\end{aligned}$$

The local versions are related to their global counterparts (3.4), (3.3), (3.5), (3.6), by

$$\begin{aligned}\omega &= \mathrm{Ad}_b(b^{-1}\,\mathbf{d}\, b + A), \\ m &= \widehat{m} - A^T \widehat{\mathbb{I}} A, \\ \mathbb{I} &= \mathrm{Ad}^T_{b^{-1}} \widehat{\mathbb{I}}\, \mathrm{Ad}_{b^{-1}}, \\ J &= \mathrm{Ad}^T_{b^{-1}} \widehat{J}.\end{aligned}$$

The momentum J is conserved but the local momentum,

$$\widehat{J} = \widehat{\mathbb{I}}(\xi + A\dot{\theta}),$$

is only conserved when the system starts at zero initial momentum, in which case we have

$$\xi = -A\dot{\theta}.$$

This equation relates shape changes to motion of the body as a whole. This is the local version of the statement which says that the velocity of the system on the full space is the horizontal lift of a path in the base.

6.1.1 The Propeller Paradox?

Elroy's Beanie, two rigid bodies connected at the their center of mass on an axis actuated with a torque relative to one another is a standard trivial example of momentum conservation giving a connection, but one without local holonomy since it is one dimensional. In a perfect fluid in $\mathbb{R}^3$, a body doesn't necessarily have a center of added mass (there is a notion of one in two dimensions however as we will in §6.2.1), since the symmetric added mass matrix without symmetry has 15 free parameters and a rigid coordinate change can only force 6 of them to zero.

[Birkhoff, 1950, p. 157] in his discussion of the Propeller Paradox for a rigid body in perfect fluid said:

> for a propeller or any other object possessing n-fold rotational symmetry about an axis ($n > 1$), *all* force components are (theoretically) zero.

By calling this a paradox it might seem he suggests somehow this notion implies that a propeller in a perfect fluid cannot act like a propeller should. Like in d'Alembert's Paradox, no force is generated by a propeller moving along its relative equilibria and it it cannot generate momentum, but when forced to spin relative to a main body, it can serve to drive them both along. This single degree of freedom system does not violate the Scallop Theorem [Purcell, 1977] as a complete rotation is not a reciprocal motion.

The propeller is our first example of dynamically coupled but hydrodynamically uncoupled bodies. In this case the propeller is modeled by three ellipses, one for the main body $b_0 \in \mathrm{SE}(3)$ and two connected rigidly for form a propeller $b_1, b_2 \in \mathrm{SE}(3)$ that can spin relative to the main body.

To avoid needless complication we assume the blades have only translational inertia $\widehat{\mathbb{I}}_1 = \widehat{\mathbb{I}}_2 = \mathrm{diag}(m_1, m_2, m_3, 0, 0, 0)$ and the body only rotational inertia about the x axis $\widehat{\mathbb{I}}_0 = \mathrm{diag}(0, 0, 0, j, 0, 0)$. Let the coordinates in each blade be such that the long semi-axis of length a is along the y axis and the second longest along the x. Let each blade have a pitch ψ so the blades are positioned relative to a coordinate frame at the hub by $b_1 = e^{\zeta}$ and $b_2 = e^{-\zeta}$ where $\zeta = (0, a, 0, 0, \phi, 0) \in$ se(3). Below we assume the pitch of the blades is $\phi = \frac{\pi}{4}$. The total inertial of the blades is then

$$
\begin{aligned}
\mathbb{I}_1 + \mathbb{I}_2 &= \mathrm{Ad}^T_{b_1^{-1}} \widehat{\mathbb{I}}_1 \mathrm{Ad}_{b_1-1} + \mathrm{Ad}^T_{b_2^{-1}} \widehat{\mathbb{I}}_2 \mathrm{Ad}_{b_2^{-1}} \\
&= \begin{bmatrix} m_1 + m_3 & 0 & 0 & (m_3 - m_1)a & 0 & 0 \\ & 2m_2 & 0 & 0 & 0 & 0 \\ & & m_1 + m_3 & 0 & 0 & (m_1 - m_3)a \\ & & & a^2(m_1 + m_3) & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & (m_1 + m_3)a^2 \end{bmatrix}.
\end{aligned}
$$

The entries that appear on the diagonal of the upper right quadrant are what give rise to screw like motion.

If the main body is oriented along the x axis with longest semi-major axis of length l and is rotated by an angle θ then its frame is given relative to the hub by $b_o(\theta) = e^{\chi}$ where $\chi = (l, 0, 0, \theta, 0, 0)$.

$$\hat{\mathbb{I}}A = \begin{bmatrix} 0 & 0 & 0 & j & 0 & 0 \end{bmatrix} \mathrm{d}\,\theta$$

The locked inertia matrix is singular but only because of the restrictive assumptions on the body and blade inertias. This poses no problem in solving for the local connection A since it is in the range of $\hat{\mathbb{I}}A$. The local form of the connection is

$$A = \frac{1}{(m_1 + m_3)j + 4m_1m_3a^2} \begin{bmatrix} (m_1 - m_3)ja \\ 0 \\ 0 \\ (m_1 + m_3)j \\ 0 \\ 0 \end{bmatrix} \mathrm{d}\,\theta.$$

By spinning the propeller with respect to the main body we get a net rotation of the body about the propellers axis and a translation along it.

Figure 6.1: A propeller propelling a body in inviscid flow. The body is only shown in the last frame.

6.2 Simple Planar Satellite

We next consider three connected bodies in the plane as shown in Figure 6.2. It is well known that in space, starting from rest and allowing only cyclic motions, this system able to reorient itself, but not to translate. This limitation on the motion is due to the structure of the inertia tensor, and as with the propeller, we will see that this satellite becomes a fish that can swim when put in a perfect fluid.

Our deformable fish is approximated as a dynamically coupled but hydrodynamically uncoupled set of three identical ellipses with only translational inertia

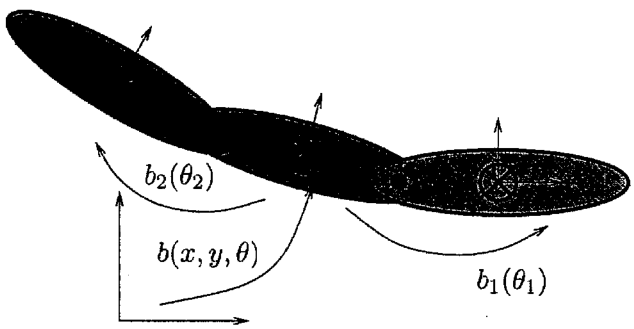

Figure 6.2: A three-link swimmer

$\widehat{\mathbb{I}}_i = \mathrm{diag}(m_1, m_2, 0)$ and major semi-axis a along the x axis. Connecting the tips of the ellipses with a revolute joint and using the notation from §6.1 we have

$$b_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$
$$b_1 = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & a(1+\cos\theta_1) \\ \sin\theta_1 & \cos\theta_1 & a\sin\theta_1 \end{bmatrix}$$
$$b_2 = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & -a(1+\cos\theta_2) \\ \sin\theta_2 & \cos\theta_2 & -a\sin\theta_2 \end{bmatrix}$$

Computing the local locked inertia gives

$$\widehat{\mathbb{I}} = \frac{1}{2}\begin{bmatrix} H_2 & \mu(S_1+S_2) & a(2m_2(s_2-s_1)+\mu(S_1-S_2)) \\ & -H_1 & a(c_1-c_2)(m_2-\mu(c_1+c_2)) \\ & & H \end{bmatrix}$$

where

$$H_i = 2(2\mu - m_1) + \mu(C_1 + C_2)$$
$$H = 2a^2(m_2(1+c_1)^2 + m_2(1+c_2)^2 + m_1(s_1^2 + s_2^2))$$

and where $\mu = m_1 - m_2$, $C_i = \cos 2\theta_i$, $c_i = \cos\theta_i$, $S_i = \sin 2\theta_i$, and $s_i = \sin\theta_i$. Using

$$\widehat{\mathbb{I}}A = am_2 \begin{bmatrix} -\sin\theta_1 \\ \cos\theta_1 \\ a(1+\cos\theta_1) \end{bmatrix} \mathbf{d}\,\theta_1 + am_2 \begin{bmatrix} \sin\theta_2 \\ -\cos\theta_2 \\ a(1+\cos\theta_2) \end{bmatrix} \mathbf{d}\,\theta_2$$

the local connection can be computed. To see that this system is locally controllable compute the local curvature and its covariant derivative evaluated in the straight configuration,

$$F = \begin{bmatrix} a*(-m1+m2))/(9*m1) \\ 0 \\ 0 \end{bmatrix}$$

$$\nabla_1 F = \begin{bmatrix} 0 \\ -(a*(m1-m2)*(2*m1+m2))/(54*m1*m2) \\ (-6+(5*m1)/m2+(4*m2)/m1)/144 \end{bmatrix}$$

$$\nabla_2 F = \begin{bmatrix} 0 \\ -(a*(m1-m2)*(2*m1+m2))/(54*m1*m2) \\ (6-(5*m1)/m2-(4*m2)/m1)/144 \end{bmatrix}.$$

Together three these span se(3) when $m_1 \neq m_2$ so the system is locally controllable in water but only orientable out of it. The forward gait can be seen in *Figure* 6.3.

6.2.1 The Center of Added Mass

One of the distinctive features of the added inertia tensor in three dimensions is that it cannot always be diagonalized by a change of frame. For a general body without symmetry the best that can be done is to find three principal, not necessarily perpendicular, screw axes about which you can sustain steady motion coupled with rotation at a given pitch[Birkhoff, 1950]. We saw an example of one, the propeller, in §6.1.1. In two dimensions however it always possible to diagonalize the added mass matrix. In two dimensional space a rigid body has no preferred orientation, while in a fluid it does. In space the center of mass is confined to the convex hull of all of the particles that make up the body. This is not the case in a fluid since we technically include the whole fluid plane as part of the body. We can get an intuitive picture of how good our ellipse approximation is by looking at Figure 6.4. The inertia tensor for the fish is diagonal when written with respect to a coordinate frame aligned with and centered on the large ellipse which drawn such that it would generate the same effective added inertia as the three links in their current configuration.

Clearly we are underestimating the inertia in the short direction and overestimating in the long direction. This makes sense, as we intuitively know that the three bodies effectively become one long streamlined body with almost the same inertia along its length as that of just one of the bodies, but our approximation doesn't notice this drafting since as far as it is concerned the bodies could be situated next to each other.

6.3 The Hat

The geometric framework and results on holonomy presented here can be applied to novel problem of stirring of an inviscid fluid. Consider the questions: If a hat is dropped off a dock, is it possible to retrieve it just by moving a sick around in the

Figure 6.3: The forward swimming gait of the three link fish.

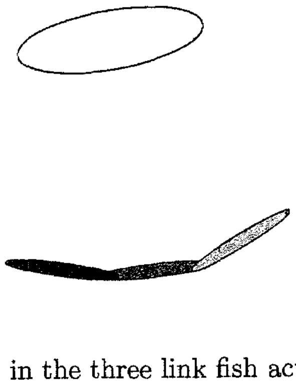

Figure 6.4: The three bodies in the three link fish act together as if they were the shown large ellipse, the center of added mass. In particular, the center of rotation of the three links is at the center of the large ellipse.

water? Can it be arbitrarily reoriented by the stirring motions of the stick? How completely can I stir a fluid with a given shaped rod using only small motions?

In §4.5.5 it was shown that stirring can be represented as holonomy of the principal bundle $\mathcal{FB}(M,B) \to \mathcal{I}(M)$ of body fluid motions over rigid body motions with with structure group $\mathcal{FB}^e(M,B)$ consisting of fluid motions that are parallel to the body boundary and do not move the fluid at infinity.

The resulting motion, when started from rest, is governed by the mechanical connection (3.4) which in this case consists of the potential flow solutions §4.4.3 corresponding to rigid motions (4.26).

The above questions can be answered by looking at the holonomy algebra from §5.4.3 consisting of the curvature of the connection and its covariant derivatives. The curvature is Lie algebra valued which in this case consists of elements of $\mathfrak{fb}^e(M,B)$ which are vector fields that are parallel to the boundary and zero at infinity. It can be decided if arbitrary motions of the "hat" are possible by looking at the span of the vector fields that make up the holonomy algebra at each point in the ambient space.

Vector fields in the Lie algebra obviously cannot be free from vorticity since irrotational flow requires boundary motion. Because the vorticity represents the local rotation of the fluid, solenoidial flows will affect the rotation of the hat, complete local control over the hat can then be achieved if there are at least three vector fields from the holonomy algebra that together span not only the two dimensional tangent space at each point but also the one dimensional space of the vorticity.

Evaluating the span of vector fields at every possible hat location in the ambient space, independently, corresponds to a more limited notion of controllability than that covered in Theorem 1. The vector fields spanning the infinite dimensional space of all vector fields in the Lie algebra would correspond, in this case, to the potential to arbitrarily stir the fluid. Because of the boundary conditions, only the span of the vorticity of the curvature its covariant derivatives needs to be considered.

6.3.1 Approximating a Stir

In this section we compute the curvature of the connection associated with moving a circle in small loops and comparing the net displacement with out calculated values of the curvature using the first term of (5.15).

The stream function for the flow around a circle centered at (r_X, r_y) is

$$\varphi(x,y) = \frac{(y-r_y)\,\mathbf{d}\,r_x + (x-r_x)\,\mathbf{d}\,r_y}{(x-r_x)^2+(y-r_y)^2}$$

so the curvature given by (4.27) evaluated at the initial point is

$$F_{xy} = \frac{4}{(x^2+y^2)^3}\begin{bmatrix}-y\\x\end{bmatrix}\mathbf{d}\,r_x \wedge \mathbf{d}\,r_y$$

The first covariant derivatives are

$$\widehat{\nabla}_x F_{xy} = \frac{1}{(x^2+y^2)^5} \begin{bmatrix} (-8xy(4+3x^2+3y^2)) \\ (4(5x^4 - y^2(1+y^2) + x^2(7+4y^2))) \end{bmatrix}$$

$$\widehat{\nabla}_y F_{xy} = \frac{1}{(x^2+y^2)^5} \begin{bmatrix} (4(x^4 - 7y^2 - 5y^4 + x^2(1-4y^2))) \\ (8xy(4+3x^2+3y^2)) \end{bmatrix}$$

Figure 6.5 shows the motion of some of the particles when the circle follows $(r_x(t), r_y(t)) = .05(1 - \cos t, \sin t)$ for two cycles. Figure 6.6 shows a blow up of the paths of four particles compared to the average displacement predicted by the curvature scaled by the appropriate area coefficients from the series (5.15). To determine if the Hat can be controlled we compute the vorticity of the curvature and that of its first covariant derivatives,

$$dF^\flat_{xy} = \frac{-16}{(x^2+y^2)^3},$$

$$d\widehat{\nabla}_x F^\flat_{xy} = \frac{-96x(x^2+y^2-2}{(x^2+y^2)^5},$$

$$d\widehat{\nabla}_y F^\flat_{xy} = \frac{-96y(x^2+y^2-2}{(x^2+y^2)^5}.$$

The determinant of the velocity field and vorticity corresponding to the first and second brackets

$$\begin{vmatrix} F_{xy} & \widehat{\nabla}_x F^\flat_{xy} & \widehat{\nabla}_y F^\flat_{xy} \\ dF^\flat_{xy} & d\widehat{\nabla}_x F^\flat_{xy} & d\widehat{\nabla}_y F^\flat_{xy} \end{vmatrix} = \frac{-256(5 + x^4 - 6y^2 + y^4 + 2x^2(-3+y^2))}{(x^2+y^2)^{11}}$$

is only zero when $r = 1$ and $r = \sqrt{5}$. The Hat is therefore controllable everywhere in the plane by doing small circles and figure eights except on the boundary of the stirring rod and on the circle of radius $\sqrt{5}$. It may still be controllable on $r = \sqrt{5}$ via higher order derivatives of the curvature.

The span of the vorticities of the curvature and of all its covariant derivatives can be checked to determine to what extent the fluid can be controlled in general, not just at one point, through stirring. Only the vorticity needs to be checked because the fixed boundary conditions preclude any irrotational component of velocity. If they span the space of functions (vorticity is a scalar) outside the circle then the fluid will be completely locally controllable. A cursory examination of a Hall basis of derivatives up to those corresponding to 5th order brackets does seem to produce linearly independent functions. Despite this, it does not seem likely that the system will turn out to be completely controllable.

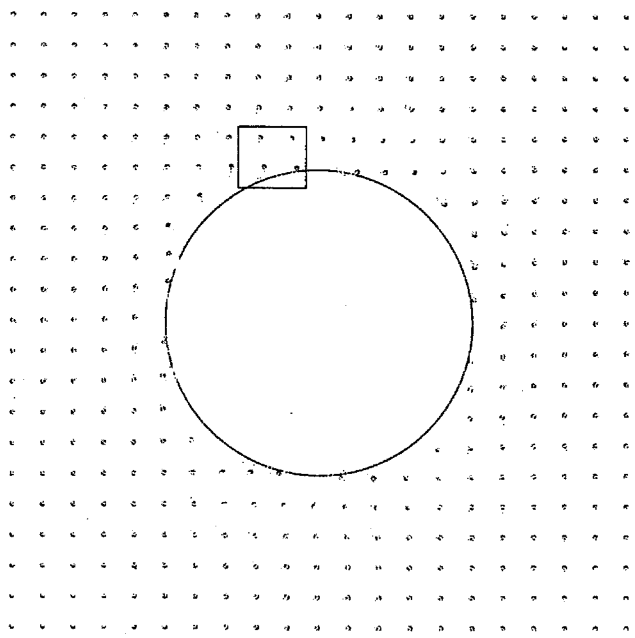

Figure 6.5: Particle paths after two stirs.

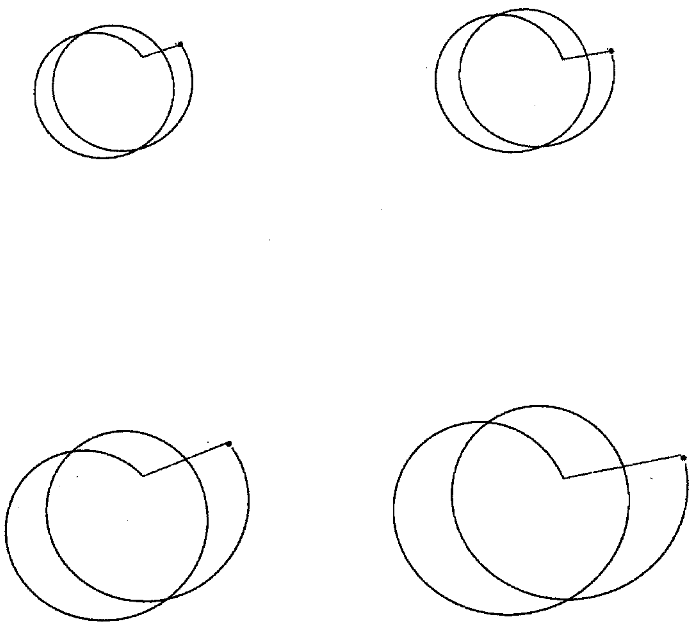

Figure 6.6: A close up of four particle paths from Figure 6.5 showing the their net displacement compared with that predicted by the curvature.

Chapter 7

Future Work

There are many avenues open for further research in the areas of geometric modeling of fluid dynamics, series expansions and the modeling of swimming and stirring.

Geometric Modeling of Fluids A formulation of fluid body problems based on groupoids, which have principal bundles as a special case, warrant further examination. I have alluded to the fact that the configuration space of a deformable body in an inviscid fluid is naturally a groupoid. The Lagrangian is also naturally right invariant with respect to the right action of composition. In addition, Lagrangian's with a parameter, for example compressible fluids, also have a natural groupoid structure.

It would be nice to use the symmetry elicited in our Lagrangian description of ideal fluid flow to develop symplectic integrators that would have well behaved properties with respect to the conserved quantities of the system. Doing this in the presence of moving boundaries would be even nicer.

No heed has been payed to time symmetries in our geometric formulation and hence energy conservation has been denied its rightful place. This glaring shortcoming might be rectified by moving to a multi-symplectic formulation.

I have been less than strict in dealing with the technicalities of convergence in infinite domains. Most of the "results" presented here are not precise mathematical statements, but more guides to precise statements that could be made. In addition assumptions have been made that explicitly forbid circulation in two dimensions for the sake of convergence.

Swimming and Stirring The simple models for swimming in terms of articulated, but hydrodynamically decoupled, rigid bodies easily capture many of the effects of a carangiform swimming. In particular they capture steering due to body bending and eel like forward gaits. However, in modeling only added mass, they cannot capture momentum generation and loss. Clearly it would be advantageous to add representations of the effects of vorticity and viscosity while keeping the model simple.

It would be interesting extend the results on stirring to determine the extent to which an arbitrary diffeomorphisms of the fluid can be achieved via stirring.

Series Expansions The series presented here was my entree into the world of series expansions, but I only touch on the structure of the generic problem. These series expansions have a clear connection to averaging, which can be studied in the generic case. The algebraic structure of these series is very beautiful and I have only hinted the bigger picture. In particular it would be nice to see an expansion for the single exponential in terms of a Hall basis with explicit coefficients.

This thesis is rooted in attempts by many people to write coordinate independent problem statements and answers. In the context of the series and in the context of non-linear control in general this has been successful with respect to the state space, it has not been so with respect to the input space. The Lie bracket between two input vectors is a coordinate free expression, but not when you consider that is relies the choice of a basis for the input space. Sussmann's good and bad bracket test is an example of this. One gets different answers based on the choice of input parameterization. This is suboptimal. This also leads to the question of nilpotentization. In some problems, certain choices of input parameterization naturally truncate the series. Determining when this is possible and what change of coordinates to use to do so is an unsolved and difficult problem. There may be a connection between tests nilpotentness and tests for flatness.

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