# ON PROBLEMS OF HEAT CONDUCTION IN A COMPRESSIBLE FLUID

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## ABSTRACT

The present work starts with a study of heat conduction in a non-viscous compressible fluid based on a linearized theory which is similar to that used in the theory of sound. Important features of exact equations of motion and their corresponding linearized equations are studied briefly. For this linear system, which preserves many of the features of the original non-linear system, the fundamental solutions are found and discussed. The additional role played by viscosity in the heat conduction problem is then investigated. The fundamental solutions for this compressible, viscous, heat-conducting flow problem are found and compared with the non-viscous case. The problem of heat conduction in a two-dimensional stationary flow of a viscous compressible fluid is further studied by finding the fundamental solutions and discussing the result in some detail. As an example proposed to show how a superposition of these fundamental solutions can be used to solve a boundary value problem, the problem of the anemometry of a heated flat plate is solved for both large and small values of the Reynolds number. The result obtained herein is discussed and compared with some existing theories and experiments. The causes of the discrepancy resulting from this linearized theory are briefly explained.

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# NOMENCLATURE

| | |
|---|---|
| $P$ | = pressure |
| $\rho$ | = density |
| $T$ | = temperature |
| $C_v, C_p,$ | = specific heats at constant volume and constant pressure respectively |
| $R$ | = gas constant = $C_p - C_v$ |
| $\gamma$ | = $\frac{C_p}{C_v}$ = ratio of specific heats |
| $t$ | = time, space coordinates: $(x, y, z)$ or $(x_1, x_2, x_3)$, $(x_i)$ |
| $\vec{Q}$ | = velocity vector |
| $\vec{F}$ | = body force vector per unit mass |
| $E$ | = internal energy per unit mass = $C_v T$ |
| $k$ | = thermal conductivity |
| $\mu$ | = coefficient of viscosity |
| $\mathcal{Q}$ | = heat addition per unit mass |
| $\vec{H}$ | = heat flux vector |
| $C_i$ | = local isothermal speed of sound = $\sqrt{RT}$ |
| $C$ | = local adiabatic speed of sound = $\sqrt{\gamma RT}$ |
| $\vec{q}$ | = perturbation velocity vector = $\vec{Q}$ (assumed small) |
| $p$ | = pressure perturbation; $P = P_0(1+p)$, subscript $_0$ denotes initial state |
| $s$ | = density perturbation (condensation) $\rho = \rho_0(1+s)$ |
| $\theta$ | = temperature perturbation $T = T_0(1+\theta)$ |
| $\vec{X}$ | = perturbation force vector = $\vec{F}$ (assumed small) |
| $\vartheta$ | = perturbation heat addition = $\mathcal{Q}$ (assumed small) |

$\kappa$ = thermometric conductivity = $\frac{k_0}{\rho_0 c_v}$

$\Omega$ = non-dimensional heat addition = $\frac{\theta}{c_v T_0}$

$\Delta$ = Laplace operator = grad div − curl curl $\left(= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right.$ in three-dimensional Cartesian coordinate system)

$\Xi$ = perturbation force potential $\vec{X} = \mathrm{grad}\, \Xi$

$\varphi$ = perturbation velocity potential $\vec{q} = \mathrm{grad}\, \varphi$

$\Gamma$ = fundamental solution for $\theta$

$\Phi$ = fundamental solution for $\varphi$ in Part I, also used as dissipation function in Parts II and III

$\overline{(\ )}$ bar denotes Laplace transform of the function

$\sigma$ = complex parameter in Laplace transformation

$\widetilde{(\ )}$ wavy bar denotes Fourier transform of the function

$\beta$ = parameter used in Fourier transformation with respect to $x$.

$z, Z, \zeta, w$ = complex variables

$\mathcal{U}$ = fundamental solution for $u$

$\lambda$ = coefficient of volume viscosity $(= -\frac{2}{3}\mu$ by Stokes' assumption)

$\nu = \frac{\mu_0}{\rho_0}$ = kinematic viscosity of the undisturbed flow

$\vec{\omega}$ = vorticity = curl $\vec{q}$

$\mathfrak{S}$ = fundamental solution for $s$.

$\mathfrak{P}$ = fundamental solution for $p$

$\vec{\Psi} = \begin{pmatrix} \mathcal{U} \\ \mathcal{V} \end{pmatrix}$ = fundamental solution for $\vec{q}$

$U$ = free stream velocity in a stationary flow

$M = \frac{U}{c}$ = Mach number of the free stream flow

$Re$ = Reynolds number = $\frac{U l \rho_0}{\mu_0}$

$Pr$ = Prandtl number evaluated at infinity = $\frac{c_p \mu_0}{k_0} = \frac{\gamma \nu}{\kappa}$.

| | | |
|---|---|---|
| | $\equiv$ | $\Gamma_\Omega$ |
| $\Gamma_2$ | $=$ | $\vec{\Gamma}_X \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ |
| $\begin{pmatrix} U_1 \\ V_1 \end{pmatrix}$ | $\equiv$ | $\vec{\Psi}_\Omega$ |
| $\begin{pmatrix} U_2 \\ V_2 \end{pmatrix}$ | | $\Psi_X \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ |
| $Q$ | | heat transfer rate |
| $T_w$ | | re temperature |
| $i$ | $=$ | rent |
| $r$ | $=$ | e tric resistance |
| $Nu$ | $=$ | Nu t number (over-all non-dimensional heat transfer coe ient) $= \dfrac{Q}{k(T_w - T_0)(length)}$ |
| $D$ | $=$ | fric al drag |
| $C_D$ | $=$ | fricti drag coefficient $= \dfrac{D}{\frac{1}{2}\rho_0 U^2 (length)}$ |
| $\gamma_E$ | $=$ | Euler's stant $= 0.577\cdots$ |

## INTRODUCTION

The conduction of heat in a compressible fluid medium is studied in this paper. An arbitrary distribution of heat sources (or sinks) and external forces is assumed and their effects on the whole flow field of this real fluid medium are investigated.

A general review of the schemes of solving problems of viscous compressible fluids is given in Ref. 1.* An outstanding method is by the use of the classical boundary layer theory. This method was first proposed by Prandtl (Ref. 2) to solve problems in a viscous incompressible fluid, and may be regarded as an asymptotic method for high Reynolds numbers. In the boundary layer theory the equations of motion are simplified by assuming that viscous effects are important only in a narrow region close to the solid wall, across which changes in physical variables are rapid compared to those in the direction along the wall. The outer flow field may then be solved to a good approximation by neglecting the viscous effects and taking the boundary conditions equal to those on the outer edge of the boundary layer. This theory has also been extended to a compressible fluid medium with heat conduction.** However, in many cases the basic assumption of neglecting the viscous and heat conducting effects outside the boundary layer is not justified, at least in regions other than the

*An extensive list of references on related topics may be found in the bibliography of Ref. 1.

**A general review and survey of references on this topic is given in Ref. 3.

boundary layer. Several of these exceptional cases are mentioned in Ref. 1, pp. 1-3. It is easy to find other examples associated with the application of the study in this paper.

For instance, if a certain amount of heat is introduced at a point of a compressible, heat conducting fluid, it can be shown that the compressibility of the fluid has the important effect of causing a pressure wave to move in the fluid. A portion of the added heat is carried away by conduction, while the rest is concentrated on the pressure wave front (cf. § 7 and § 11.1). A closely related problem is the effect of heat conduction on the propagation of sound waves in a gas. The study of this problem dates back to Stokes (Ref. 4) who solved the problem of plane waves with linearized equations of motion, neglecting only the viscosity. In both cases the total energy is divided between thermal and pressure waves. If both viscosity and heat conductivity are neglected, the pressure wave degenerates to a disturbance of discontinuities of pressure, temperature, and all other physical quantities (see the study of characteristics § 1). These discontinuities are smoothed out by the consideration of just one of them. Hence, this leads naturally to the question of the manner in which these discontinuities are smoothed out and what role they play in distributing the total energy by taking into account either viscosity or heat conduction or both. In this unsteady motion, the pressure wave propagates through the space as time goes on, so evidently viscosity and heat conduction are equally important everywhere in the medium through the time history.

Another example which may be mentioned is the problem of the disturbances produced by a stationary heat source in the supersonic flow of a compressible fluid. Tsien and Beilock (Ref. 5) solved the two-dimensional case for zero values of viscosity and heat conductivity for a line source of heat in a supersonic flow of a two-dimensional unbounded compressible fluid. Their results show that velocity and pressure perturbations behave like a Dirac $\delta$ -function along Mach lines through the heat source, while the density perturbation in addition to the $\delta$ -function behavior along Mach lines, also has a sharp wake. But again, if either viscosity or heat conductivity is considered these sharp discontinuities will be diffused out across the Mach lines. This again leads to the question of what their effects are in such a diffusion process. Apparently, viscous and heat conducting effects play an important role in these regions and can not be neglected.

All these preliminary considerations make it clear that it would be advisable to obtain solutions to some problems of a heat conducting flow without assuming a priori that the flow field is divided into regions where the heat conductivity (and/or viscosity) is considered and regions where it is not. However, to seek an exact solution seems a hopeless task, due to the complexity of the equations for viscous, heat-conducting, compressible flow (such as the Navier-Stokes equations, see §1). The principle of simplifying the problem in this paper is in accordance with a general viewpoint of trying to isolate the difficulties so that the resulting mathematical problems can be solved while at the same time retaining some of the typical features of the more exact problems. Following this principle, the program of the paper is as follows: first some exact equations of motion and heat transfer are written down and studied briefly. To make calculations possible it is next assumed that all disturbances are small so that the equations can be linearized, as in the theory of sound. For this linear system, which preserves many of the typical features of the original non-linear system, the fundamental solutions (for their definition see §4) are found and discussed in some detail. For a discussion on the justification of this linearization see Ref. 1, p. 6.

In Part I the effects of viscosity are neglected. The study of this part is for problems in which heat transfer is the main interest and in which the viscous stresses may actually be quite small. One example is the propagation of a combusting flame front in a compressible fluid. The assumption of neglecting the viscosity alone is, of course, not quite realistic since the kinetic theory of gases indicates a close relationship between the coefficients of heat conduction and viscosity (for example, see Ref. 6). However, for the problems in which the energy transport is more important than that of momentum, the simplified problem, due to the neglecting of viscosity, may throw some light on the real nature of the viscous, heat-conducting, compressible flow since the viscous effects do not change the picture qualitatively. As a matter of fact, from results obtained later, the theory based on non-viscous fluid can, in most instances (which will be made more precise later), explain not only qualitatively but also quantitatively the motion of a viscous, heat-conducting fluid.

In Part II both viscosity and heat conductivity are considered, with some simplifying assumptions (which will be mentioned later), to solve some unsteady flow problems. The fundamental solutions are found and discussed and the results are compared with the non-viscous case in Part I.

In Part III some stationary flow problems are studied considering both viscosity and heat conductivity. The fundamental solutions with the existence of both heat sources and external forces are found. An application is sought to apply these results to calculate the heat transfer rate from a heated flat plate over a range of Mach numbers to show the compressibility effect. For the problem of heat loss from a hot-wire anemometer, King (Ref. 7) obtained a solution based on the theory of an inviscid, incompressible fluid in 1914; Tchen (Ref. 8) extended King's result to an inviscid, compressible fluid in 1949. Some experimental results were given rather recently by Kovasznay (Ref. 9) (for a more detailed historic survey see Part III). Their results are compared with those in this paper. It is known in the boundary layer theory for a compressible fluid that if only the skin friction on or the heat transfer from a solid wall is concerned, their values can be calculated by the help of certain simplifying assumptions, without even solving all the equations of motion (for example, see Refs. 10, 11, 12 and 13). Several solutions for a plate thermometer problem based on boundary layer theory are known (see Refs. 14-17). As a rule, the result based on linearized theory almost always gives non-conservative values compared with that calculated from boundary layer theory and it is true in this case. However, sacrificing quantitative accuracy enables us to study the role compressibility plays in this complicated mechanism.

It should be pointed out that dissipation terms are dropped out by linearization so that the analysis can not be applied directly to any case in which this is of importance. An example is Rayleigh's problem, that is, an infinite flat plate suddenly moves parallel to itself in a non-heat-conducting compressible fluid. Then a pressure wave is sent out normal to the plate. This pressure wave is caused by dissipation and will be missed in any theory omitting it (cf. Refs. 18-21).

# PART I

## HEAT CONDUCTION IN A NON-VISCOUS COMPRESSIBLE FLUID

### § 1. Fundamental Equations and Their Characteristics

The fundamental equations used in this part of the paper may be obtained from the application of the principle of conservation of mass, momentum, and energy to the hydrodynamical continuum. As explained previously, the viscosity is neglected in Part I. It is assumed that the medium is a perfect gas, and that the pressure is confined to a moderate range so that the perfect gas law

$$P = R\rho T \tag{1.1}$$

holds. The basic equations may then be written as:

Conservation of Mass

$$\frac{\partial \rho}{\partial t} + div\,(\rho \vec{Q}) = 0 \tag{1.2}$$

Conservation of Momentum

$$\frac{\partial \vec{Q}}{\partial t} + (\vec{Q} \cdot grad)\vec{Q} = \vec{F} - \frac{1}{\rho}\, grad\, P \tag{1.3}$$

Conservation of Energy

$$\frac{\partial E}{\partial t} + (\vec{Q} \cdot grad)\, E + P\left\{\frac{\partial(\frac{1}{\rho})}{\partial t} + (\vec{Q} \cdot grad)\frac{1}{\rho}\right\} = \frac{1}{\rho}\, div\,(k\, grad\, T) + \mathcal{Q} \tag{1.4}$$

where it is assumed that Fourier's conduction law applies. The heat flux vector $\vec{H}$, which denotes the rate of heat conduction across an isothermal surface per unit area per unit time, is given by*

$$\vec{H} = -k\, grad\, T \tag{1.5}$$

The interval energy $E$ per unit mass is defined by

*This law holds in a frame of reference moving with the local center of gravity of the fluid.

$$E = C_v T \tag{1.6}$$

The notation is described in the Nomenclature. Eqs. (1.1)-(1.4) may be regarded as a non-linear system of four partial differential equations for unknowns $P$ , $\rho$ , $T$ and $\vec{Q}$ . Eq. (1.4) is of second order, with the thermal conductivity $k$ as the coefficient of second order terms. Hence, it may be noticed that if $k$ is put equal to zero, the order of (1.4) is lowered by one. If at the same time a boundary condition has to be relaxed, the perturbation problem is singular.

The mathematical characteristics of the system reveal the underlying structure of the system by showing surfaces of propagation of certain disturbances or discontinuities. According to the general theory, the characteristics are determined only by the highest order terms occurring in the equations (cf. Ref. 22, Vol. II, Chapters 5 and 6). For simplicity, we shall only consider one space dimension $x$ , together with the time $t$ as two independent variables. This, however, does not destroy the essential nature of the system. Let $\psi(x,t)$ = constant denote a characteristic, the characteristic condition is then (Ref. 1, pp. 18-20)

$$\psi_x^2 \left\{ \psi_t + (u \pm c_i)\, \psi_x \right\} = 0 \tag{1.7a}$$

where

$$c_i = \sqrt{RT} = \sqrt{(P/\rho)} \quad = \text{isothermal speed of sound} \tag{1.7b}$$

The characteristics are:

(i) $\psi_x^2 = 0$, or $t$ = constant, which occurs double (1.7c)

This set indicates propagation of certain disturbances with an infinite speed and are thus associated with the usual heat conduction.

(ii) $\psi_t + (u \pm c_i)\psi_x = 0,$ or $\left(\frac{dx}{dt}\right)_\psi = u \pm c_i$ (1.7d)

These lines indicate propagation of certain discontinuities with the isothermal speed of sound $c_i$ relative to the fluid. It may be noted that as a typical feature of non-linear equations, these characteristics are not known in advance since they depend on the unknown values of $u(x,t)$. It is not yet completely clear what the physical meaning of these characteristics is. However, from later study of the fundamental solution, one will be convinced that these characteristics at least exhibit the surfaces of discontinuities of the fundamental solutions (cf. § 6).

If we consider for a moment a fluid in which the heat conductivity $k$ also vanishes, the order and hence the number of characteristics of our system (1.1)-(1.4) is lowered by one. The characteristics then become

$$(\psi_t + u\,\psi_x)[\psi_t + (u+c)\,\psi_x][\psi_t + (u-c)\,\psi_x] = 0 \tag{1.8a}$$

where

$$c = \sqrt{\gamma RT} = \sqrt{(\gamma P/\rho)} = \text{adiabatic speed of sound} \tag{1.8b}$$

There are now only three sets of characteristics.

(i) $\psi_t + u\,\psi_x = 0,$ or $\frac{dx}{dt} = u$ (1.8c)

This indicates the streamline of the flow, across which temperature and density may now jump (because $k=0$ , $\mu=0$ ).

(ii) $\psi_t + (u \pm c)\,\psi_x = 0$, or $\dfrac{dx}{dt} = u \pm c$ (1.8d)

This indicates propagation of sound waves with the usual adiabatic speed $C$ relative to the fluid.

These characteristics, which remain undiscovered for $k \neq 0$, must play an important role in the solutions for small values of $k$. Some light will be shed on this interesting point by the solutions given explicitly later.

## § 2. Linearization of Equations and Their Properties

The necessity and justification for the linearization of equations were explained in the Introduction. Due to the difficulty of handling non-linear equations like these, the original system is not considered in detail any further. Instead, in order to look for the nature of the solutions, a set of linearized equations will be considered with which some specific problems can be solved while the important features of more exact problems are still retained. Let us consider a basic flow such that at infinity, the flow is uniform, parallel to the $x$-axis, with velocity $U$, constant pressure $P_0$, density $\rho_0$ and temperature $T_0$. The general case $U \neq 0$ may be reduced to the case $U = 0$ by a Galilean transformation (cf. § 3). The linearization may then be carried out by assuming that the flow field only differs slightly from the basic flow so that the solution may be written as

$\vec{Q} = \vec{q}$ (assumed small, vanish at infinity) (2.1a)

$P = P_0(1+p)$, $\rho = \rho_0(1+s)$, $T = T_0(1+\theta)$ (2.1b) where $s$ is usually called the condensation, $p, s, \theta \ll 1$ .* It is also assumed that

$$k = k_0(1+k'), \qquad k_0 = k_0(T_0), \qquad k' \ll 1 \tag{2.1c}$$

and

$$\vec{F} = \vec{X} \qquad \text{perturbation force vector} \tag{2.1d}$$

$$Q = \theta \qquad \text{perturbation heat addition} \tag{2.1e}$$

are small and vanish at infinity. By neglecting squared terms of small quantities, the equations then become

State $\qquad p = s + \theta \qquad$ (2.2a)

Continuity $\qquad s_t + \operatorname{div} \vec{q} = 0 \qquad$ (2.2b)

Momentum $\qquad \vec{q}_t = \vec{X} - c_i^2 \operatorname{grad} p \qquad$ (2.2c)

Energy $\qquad \theta_t - \kappa \Delta\theta = (\gamma - 1) s_t + \Omega \qquad$ (2.2d)

where

$$\kappa = \frac{k_0}{c_v \rho_0} = \text{thermometric conductivity** of the undisturbed flow} \tag{2.3a}$$

and

$$\Omega = \frac{\theta}{c_v T_0} = \text{non-dimensional heat addition per unit time***} \tag{2.3b}$$

Eqs. (2.2a)-(2.2d) are the linearized system of equations of motion. The solutions to this system can be applied to the case where

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*Almost the same linearization was used by Stokes (Ref. 4) to solve a plane oscillating wave with radiation.

**This name is due to Clerk Maxwell. It was called the thermal diffusivity by Kelvin. Its dimension is (length$^2$)/(time).

***The dimension of $\Omega$ is $[\Omega] = [t]^{-1}$.

there is a component of velocity in one direction at infinity by using a Galilean transformation. It should be remarked here that in linearizing the dissipation terms drop out in (2.2d) whether or not $\mu = 0$ . We are neglecting $\mu$ only in the momentum equations.

Before finding solutions of the system (2.2) some general properties of the equations will be discussed.

Irrotationality. First we show that the flow field described by (2.2) is irrotational if it is irrotational initially and if the external forces have a potential $\Xi$ such that

$$\vec{X} = \operatorname{grad} \Xi \tag{2.4}$$

The curl of (2.2c) then gives

$$(\operatorname{curl} \vec{q}\,)_t = 0 \tag{2.5}$$

which shows that the vorticity = curl $\vec{q}$ is independent of time. If the vorticity is initially zero it remains zero. In the case which will be considered here the irrotationality would also imply that there exists a velocity potential $\varphi$ such that

$$\vec{q} = \operatorname{grad} \varphi \tag{2.6}$$

The system (2.2) can then be reduced to a pair of equations for $\theta$ and $\varphi$

$$\Delta\varphi - \frac{1}{c_i^2}\varphi_{tt} = \theta_t - \frac{1}{c_i^2}\Xi_t \tag{2.7a}$$

$$\theta_t - \kappa\Delta\theta = \Omega - (\gamma-1)\Delta\varphi \tag{2.7b}$$

From this system it is easy to see the coupling between the thermal waves and the flow field. Eq. (2.7a) appears as a wave equation for the velocity potential with a driving force on the right-hand side provided by the temperature field in addition to the external force effect. The second equation (2.7b) is a heat diffusion equation for the temperature field but with a distribution of heat sources on the right-hand side due to the velocity field and the external heat addition. It is this interaction which will be studied shortly.

By eliminating $\varphi$ in (2.7) it can be shown that $\theta$ satisfies the following differential equation

$$\left[\left(\Delta-\frac{1}{c_i^2}\frac{\partial^2}{\partial t^2}\right)\Delta-\frac{\gamma}{\kappa}\left(\Delta-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\frac{\partial}{\partial t}\right]\theta=-\frac{1}{\kappa}\left(\Delta-\frac{1}{c_i^2}\frac{\partial^2}{\partial t^2}\right)\Omega-\frac{\gamma-1}{\kappa c_i^2}\Delta\Xi_t \qquad (2.8a)$$

similarly, $\varphi$ satisfies

$$\left[\left(\Delta-\frac{1}{c_i^2}\frac{\partial^2}{\partial t^2}\right)\Delta-\frac{\gamma}{\kappa}\left(\Delta-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\frac{\partial}{\partial t}\right]\varphi=-\frac{1}{\kappa}\Omega_t-\frac{1}{c_i^2}\left(\Delta-\frac{1}{\kappa}\frac{\partial}{\partial t}\right)\Xi_t \qquad (2.8b)$$

After disconnecting the coupling, equations for $\theta$ and $\varphi$ then have the same homogeneous part.

Several limiting cases which uncouple Eqs. (2.7a) and (2.7b) can be indicated briefly.

Incompressible Fluid. To deduce incompressible flow from the above results we may assume that the speed of sound $c, c_i$ tends to infinity. Then (2.7) becomes

$$\Delta\varphi=\theta_t-\Upsilon \qquad (2.9a)$$

$$\theta_t-\frac{\kappa}{\gamma}\Delta\theta=\frac{1}{\gamma}\Omega-\left(1-\frac{1}{\gamma}\right)\Upsilon \qquad (2.9b)$$

if we assume that some of the force potential remains. Thus the temperature can be found from (2.9b) and the velocity field follows from (2.9a). It is interesting to note that $\gamma$ still remains in this limiting process and that the result is quite different from putting $S=0$ in the beginning, so that the classical potential flow equations are obtained (for example, see Refs. 23 and 7).

Zero Heat Conductivity. As another case, we let $\kappa=0$. The resulting equations become

$$\Delta\varphi - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\varphi = \frac{1}{\gamma}\Omega - \frac{1}{c^2}\Xi_t \tag{2.10a}$$

$$\theta_t = \Omega - (\gamma-1)\Delta\varphi \tag{2.10b}$$

Now the velocity potential can be found from (2.10a) and the temperature follows from (2.10b). The isothermal speed of sound disappears in (2.10) and only the adiabatic speed of sound $c$ remains. That this is so was indicated by our previous study of the characteristics of the original system.

## § 3. Stationary Flow and Galilean Transformation

In system (2.2) it is assumed that the perturbation velocity vanishes at infinity. In many problems, particularly those involving bodies moving with constant velocity, the flow picture becomes stationary with respect to an observer who moves along with the body. Now let us consider a stationary flow relative to an observer such that the fluid at infinity moves with constant velocity $U$ parallel to the $x$ -axis while the body is at rest. The equations for this case can be deduced from (2.2) or (2.7) by means of a Galilean transformation, that is, by introducing a new system of coordinates $(\bar{x}, \bar{y}, \bar{z}, \bar{t})$ which moves with respect to the original system such that

$$\bar{x} = x + Ut, \qquad \bar{y} = y, \quad \bar{z} = z, \quad \bar{t} = t \tag{3.1a}$$

then

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \bar{t}} + U\frac{\partial}{\partial \bar{x}}, \quad \frac{\partial}{\partial x} = \frac{\partial}{\partial \bar{x}}, \quad \frac{\partial}{\partial y} = \frac{\partial}{\partial \bar{y}}, \quad \frac{\partial}{\partial z} = \frac{\partial}{\partial \bar{z}} \tag{3.1b}$$

The motion is stationary in this barred coordinate system if $\frac{\partial}{\partial \bar{t}} = 0$. Using unbarred letters again for the transformed coordinates, (2.7) becomes

$$\Delta\varphi - \frac{U^2}{c_i^2}\varphi_{xx} = U\theta_x - \frac{U}{c_i^2}\Xi_x \tag{3.2a}$$

$$U\theta_x - \kappa\Delta\theta = \Omega - (\gamma-1)\Delta\varphi \tag{3.2b}$$

In this case the existence of the force potential $\Xi$ and the velocity potential $\varphi$ can be justified as follows. The motion represented by Eq. (2.2c) becomes stationary by Galilean transformation

$$U\vec{q}_x = \vec{X} - c_i^2 \operatorname{grad} p \tag{3.3}$$

Hence, if $\Xi$ exists as given by (2.4) then the curl of (3.3) yields

$$U(\operatorname{curl}\vec{q})_x = 0 \tag{3.4}$$

Hence, if the flow is irrotational at infinity it is irrotational everywhere. This implies the existence of the velocity potential $\varphi$.

## §4. Fundamental Solutions of the Linearized System

In order to simplify the problem further, it is assumed that the force potential $\Xi$ is zero in Part I and only the effect due to heat addition $\Omega$ is considered. Eq. (2.8) then becomes

$$\left[\left(\Delta - \frac{1}{c_i^2}\frac{\partial^2}{\partial t^2}\right)\Delta - \frac{\gamma}{\kappa}\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\frac{\partial}{\partial t}\right]\theta = -\frac{1}{\kappa}\left(\Delta - \frac{1}{c_i^2}\frac{\partial^2}{\partial t^2}\right)\Omega \tag{4.1a}$$

$$\left[\left(\Delta - \frac{1}{c_i^2}\frac{\partial^2}{\partial t^2}\right)\Delta - \frac{\gamma}{\kappa}\left(\Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\frac{\partial}{\partial t}\right]\varphi = -\frac{1}{\kappa}\Omega_t \tag{4.1b}$$

Various fundamental solutions may be defined for the linear system (4.1). One definition may be given as follows. Suppose that the fluid fills an infinite space and that all perturbations are initially zero; and then suppose that a unit quantity of heat is introduced instantaneously at the origin of the coordinates. The resulting temperature and potential field given by

$$\theta = \Gamma(x_i,t), \qquad \varphi = \Phi(x_i,t) \tag{4.2}$$

is called the fundamental solution of the system due to a disturbance at the origin, or explicitly, in mathematical terms

$$\Omega = \delta(x_1)\,\delta(x_2)\,\delta(x_3)\,\delta(t) \tag{4.3}$$

where $\delta$ denotes the Dirac delta-function defined by

$$\begin{aligned} \delta(x) &= 0 \qquad x \neq 0 \\ &= \infty \qquad x = 0 \end{aligned} \tag{4.4a}$$

and

$$\int_{-\epsilon}^{\epsilon} \delta(x)\,dx = 1 \quad , \qquad \epsilon > 0 \tag{4.4b}$$

This fundamental solution (4.2) illustrates many features of the system. Furthermore, due to the linearity of the system (4.1), superposition of fundamental solutions can be used. If $\Omega$ is an arbitrary function of $(x_i, t)$ the solution, under zero initial conditions and vanishing at infinity, can be represented by

$$\theta(x_i, t) = \iiint_{-\infty}^{\infty} d\xi_i \int_0^t \Gamma(x_i - \xi_i, t-\tau)\,\Omega(\xi_i, \tau)\,d\tau \tag{4.5a}$$

$$\varphi(x_i, t) = \iiint_{-\infty}^{\infty} d\xi_i \int_0^t \Phi(x_i - \xi_i, t-\tau)\,\Omega(\xi_i, \tau)\,d\tau \tag{4.5b}$$

The fact that the fundamental solutions can be taken to depend only on the differences in coordinates follows from homogeneity of (4.1) in space and time if a Cartesian system of coordinates is used. It should be noted that this form may not be true if some other coordinate system is adopted. However, it is easy to see that the following forms always hold

$$\Gamma(X, \xi;\, t-\tau) \qquad \text{and} \qquad \Phi(X, \xi;\, t-\tau) \tag{4.6}$$

where $X = (x_i)$ and $\xi = (\xi_i)$ are position vectors. Comparing (4.5) with the original system (4.1), we see that the fundamental solution actually has its more general meaning as the kernel of an integral operator which inverts the differential operator in (4.1). Now, when dealing with a single equation of fourth order as given by (4.1), it can be expected that the fundamental solutions may not be expressible in terms of well known functions but will probably themselves define new functions. These fundamental solutions can also be used to build up the solutions of initial and boundary value problems (see § 8).

The procedure is to apply the Laplace transformation with respect to time to reduce the problem to one in space variables. The problem in $x_1$, $x_2$, $x_3$ can be solved. An integral representation of the fundamental solutions can be obtained and some asymptotic formulas for these solutions can be calculated.

The Laplace transform with respect to time of a function $f(x_i,t)$ is defined by (using barred letters to stand for its Laplace transform)

$$\bar{f}(x_i,\sigma) = \int_0^\infty e^{-\sigma t} f(x_i,t)\,dt \tag{4.7a}$$

and its inversion may be given by a complex inversion formula

$$f(x_i,t) = \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} e^{\sigma t}\,\bar{f}(x_i,\sigma)\,d\sigma \tag{4.7b}$$

where the path of integration is parallel to the imaginary axis and is to the right of all the singularities of $\bar{f}(x_i,\sigma)$ in the $\sigma$ plane (Ref. 24). Application of the transformation to (4.1), with zero initial conditions, yields

$$L(\bar{\theta}) = -\frac{1}{K}\left(\Delta - \frac{\gamma\sigma^2}{c^2}\right)\bar{\Omega} = -F(x_i)\ , \qquad say \tag{4.8a}$$

$$L(\bar{\varphi}) = -\frac{\sigma}{K}\bar{\Omega} = -H(x_i) \tag{4.8b}$$

where $L$ is the operator

$$L \equiv \Delta\Delta - \gamma\sigma\left(\frac{1}{\kappa} + \frac{\sigma}{c^2}\right)\Delta + \frac{\gamma\sigma^3}{\kappa c^2} = (\Delta - \lambda_1)(\Delta - \lambda_2) \tag{4.8c}$$

with

$$\begin{matrix}\lambda_1 \\ \lambda_2\end{matrix} = \frac{\gamma\sigma}{2c^2}\left\{\sigma + \frac{c^2}{\kappa} \pm \sqrt{(\sigma-\alpha)(\sigma-\bar{\alpha})}\right\} \tag{4.8d}$$

$$\alpha, \bar{\alpha} = \frac{c^2}{\kappa}\left\{\left(\frac{2}{\gamma} - 1\right) \pm i\,\frac{2}{\gamma}\sqrt{\gamma - 1}\right\} \tag{4.8e}$$

where $i$ stands for $\sqrt{-1}$ . Moreover, application of the transformation and convolution theorem to (4.5) gives

$$\bar{\theta}(x_i, \sigma) = \iiint_{-\infty}^{\infty} \bar{\Gamma}(x_i - \xi_i ; \sigma)\,\bar{\Omega}(\xi_i, \sigma)\, d\xi_i \tag{4.9a}$$

$$\bar{\varphi}(x_i, \sigma) = \iiint_{-\infty}^{\infty} \bar{\Phi}(x_i - \xi_i, \sigma)\,\bar{\Omega}(\xi_i, \sigma)\, d\xi_i \tag{4.9b}$$

These two equations indicate that the Laplace transforms $\bar{\Gamma}$ , $\bar{\Phi}$ of the fundamental solutions are themselves fundamental solutions to the transformed equations (4.8a) and (4.8b).

It may be remarked that $2 > \gamma > 1$ for all kinds of gases. Hence, it follows that $\alpha$ has positive real and imaginary parts. Moreover,

$$\alpha + \bar{\alpha} = 2\left(\frac{2}{\gamma} - 1\right)\frac{c^2}{\kappa}, \qquad \alpha\bar{\alpha} = \frac{c^4}{\kappa^2}\,. \tag{4.10a}$$

and

$$\lambda_1\lambda_2 = \frac{\gamma\sigma^3}{\kappa c^2}\,. \tag{4.10b}$$

The factoring of the operator (4.8c) permits us to represent the fundamental solution $G^{(2)}$ of (4.8a) and (4.8b), defined by

$$\bar{\theta}(x_i) = \iiint_{-\infty}^{\infty} G^{(2)}(x_i - \xi_i)\, F(\xi_i)\, d\xi_i \tag{4.11a}$$

$$\bar{\varphi}(x_i) = \iiint\limits_{-\infty}^{\infty} G^{(2)}(x_i - \xi_i)\, H(\xi_i)\, d\xi_i \tag{4.11b}$$

as a combination of fundamental solutions of second order equations. If $G_1^{(1)}$ and $G_2^{(1)}$ are fundamental solutions, respectively, of

$$(\Delta - \lambda_1)\,\bar{\theta} = -f_1(x_i)\,, \qquad (\Delta - \lambda_2)\,\bar{\theta} = -f_2(x_i) \tag{4.12a}$$

defined by

$$\bar{\theta}(x_i) = \iiint\limits_{-\infty}^{\infty} G_j^{(1)}(x_i - \xi_i)\, f_j(\xi_i)\, d\xi_i \qquad j = 1, 2 \tag{4.12b}$$

and if $\lambda_1 \neq \lambda_2$ (which is true in our case) then, by Theorem 1, Appendix A,

$$G^{(2)} = \frac{1}{\lambda_1 - \lambda_2}\left( G_1^{(1)} - G_2^{(1)} \right) \tag{4.13}$$

is the fundamental solution of (4.8) such that (4.11) is satisfied. Comparing definitions (4.9) and (4.11), $\bar{\Gamma}$ and $\bar{\Phi}$ can be related to $G^{(2)}$. Now $G_1^{(1)}$ and $G_2^{(1)}$, as fundamental solutions to second order equations are known functions in the case of one, two, and three space dimensions. Thus $G^{(2)}$, $\bar{\Gamma}$, and $\bar{\Phi}$ are known and the desired fundamental solutions $\Gamma$, $\Phi$ are given by the complex inversion formula (4.7b). The values of $\bar{\Gamma}$ and $\bar{\Phi}$ will be obtained for some specific cases in the next paragraph.

## § 5. Fundamental Solution in Laplace Transform for Several Special Cases

In Eq. (4.8), if the Laplace operator $\Delta$ is given explicitly in different forms for different dimensions or coordinate systems, $\bar{\Gamma}$ and $\bar{\Phi}$ can be obtained for each of these cases.

## 5.1 One-Dimensional Case

Let this one-dimensional space coordinate be denoted by $x$. The actual physical problem in this case may be visualized by thinking that the heat sources are distributed uniformly over infinite planes perpendicular to the $x$-axis. In this case the Laplace operator $\Delta$ is simply the second derivative with respect to $x$ so that (4.12a) is

$$\left(\frac{d^2}{dx^2} - \lambda_{1,2}\right) \bar{\theta} = -f \tag{5.1}$$

and its fundamental solutions defined by (4.12b) are known as (for example, Ref. 22, Vol. I, p. 325)

$$G_{1,2}^{(1)}(x) = \frac{1}{2\sqrt{\lambda_{1,2}}} e^{-\sqrt{\lambda_{1,2}}\,|x|} \tag{5.2}$$

Here the plane heat source is supposed to be located at $x = 0$. The fundamental solution $G^{(2)}$, defined by (4.11), of (4.8a) and (4.8b) is given by (4.13), for a plane heat source located at $x = \xi$, as

$$G^{(2)}(x-\xi,\sigma) = \frac{1}{2(\lambda_1-\lambda_2)} \left\{ \frac{1}{\sqrt{\lambda_1}} e^{-\sqrt{\lambda_1}\,|x-\xi|} - \frac{1}{\sqrt{\lambda_2}} e^{-\sqrt{\lambda_2}\,|x-\xi|} \right\} \tag{5.3}$$

As a remark, it can be pointed out that $G^{(2)}(x-\xi,\sigma)$, as a function of $\xi$ for fixed $x$, satisfies the following conditions:

$$\begin{aligned}
&\text{(i)} && G^{(2)},\ G^{(2)}_{\xi},\ G^{(2)}_{\xi\xi},\ G^{(2)}_{\xi\xi\xi}\ \text{etc.} \longrightarrow 0 \quad \text{as} \quad \xi \longrightarrow \pm\infty \\
&\text{(ii)} && G^{(2)}(x^+) - G^{(2)}(x^-) = 0 \\
&\text{(iii)} && G^{(2)}_{\xi}(x^+) - G^{(2)}_{\xi}(x^-) = 0 \\
&\text{(iv)} && G^{(2)}_{\xi\xi}(x^+) - G^{(2)}_{\xi\xi}(x^-) = 0 \\
&\text{(v)} && G^{(2)}_{\xi\xi\xi}(x^+) - G^{(2)}_{\xi\xi\xi}(x^-) = -1
\end{aligned} \tag{5.4}$$

Actually, these are what we need if we try to find the fundamental solution by integrating directly the fourth order equation (4.8). In order to find $\bar{\Gamma}$ and $\bar{\Phi}$, first substituting $F(\xi)$ given by (4.8a) into (4.11a), using conditions (5.4 i-iii), we have

$$\begin{aligned}\bar{\theta}(x,\sigma) &= \int_{-\infty}^{\infty} \frac{1}{\kappa} G^{(2)}(x-\xi,\sigma)\left(\frac{d^2}{d\xi^2} - \frac{\gamma\sigma^2}{c^2}\right)\bar{\Omega}(\xi)\,d\xi \\ &= \int_{-\infty}^{\infty} \frac{1}{\kappa}\left[G^{(2)}_{\xi\xi} - \frac{\gamma\sigma^2}{c^2} G^{(2)}\right]\bar{\Omega}(\xi)\,d\xi\end{aligned} \tag{5.5}$$

Comparison of (5.5) with the definition of $\bar{\Gamma}$ given by (4.9a) leads to

$$\bar{\Gamma}(x-\xi,\sigma) = \frac{1}{\kappa}\left[\frac{\partial^2}{\partial\xi^2} - \frac{\gamma\sigma^2}{c^2}\right] G^{(2)}(x-\xi,\sigma) \tag{5.6}$$

or

$$\bar{\Gamma}(x,\sigma) = \frac{1}{2(\lambda_1-\lambda_2)}\left[\left(\frac{1}{\kappa}-\frac{\lambda_2}{\sigma}\right)\sqrt{\lambda_1}\, e^{-\sqrt{\lambda_1}|x|} - \left(\frac{1}{\kappa}-\frac{\lambda_1}{\sigma}\right)\sqrt{\lambda_2}\, e^{-\sqrt{\lambda_2}|x|}\right] \tag{5.7a}$$

Similarly, we have

$$\bar{\Phi}(x,\sigma) = \frac{\sigma}{2\kappa(\lambda_1-\lambda_2)}\left[\frac{1}{\sqrt{\lambda_1}} e^{-\sqrt{\lambda_1}|x|} - \frac{1}{\sqrt{\lambda_2}} e^{-\sqrt{\lambda_2}|x|}\right] \tag{5.7b}$$

An integral representation of the desired fundamental solutions of $\Gamma$ and $\Phi$ can be obtained by applying the complex inversion formula (4.7b) to (5.7). The details of this procedure are given in §6.

### 5.2 Two-Dimensional Case

The space coordinates in this case will be denoted by $x$ and $y$. The actual physical problem may then be visualized by thinking that all heat sources are distributed uniformly along infinite straight lines perpendicular to the $x$-$y$ plane. In this case, the Laplace operator is $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ so that Eq. (4.12a) becomes

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \lambda_{1,2}\right)\bar{\theta} = -f(x,y) \tag{5.8}$$

whose fundamental solutions are known as

$$G^{(1)}_{1,2}(x,y;\lambda_{1,2}) = \frac{1}{2\pi} K_0\left(\sqrt{\lambda_{1,2}}\,\sqrt{x^2+y^2}\right) \tag{5.9}$$

where the single line heat source is supposed to be located at the origin, and $K_0$ is the modified Bessel function of the second kind. Hence, the fundamental solution $G^{(2)}$ of (4.8a) and (4.8b) is given by (4.13), for a line source located at $x = \xi$ , $y = \eta$ , as

$$G^{(2)}(x-\xi,\, y-\eta;\, \sigma) = \frac{1}{2\pi(\lambda_1-\lambda_2)}\left[K_0(\sqrt{\lambda_1}\,r) - K_0(\sqrt{\lambda_2}\,r)\right], \qquad r^2 = (x-\xi)^2+(y-\eta)^2 \quad (5.10)$$

$\overline{\Gamma}$ and $\overline{\Phi}$ can be obtained by comparing definitions of fundamental solutions (4.9) and (4.11)

$$\overline{\Gamma}(x-\xi,\, y-\eta;\, \sigma) = \frac{1}{K}\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}-\frac{\gamma\sigma^2}{c^2}\right) G^{(2)}(x-\xi,\, y-\eta;\, \sigma) \quad (5.11)$$

It can be shown that for $x \neq \xi$ , $y \neq \eta$ , the following relation is true

$$\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}\right) G^{(2)} = \frac{1}{\lambda_1-\lambda_2}\left[\lambda_1 G_1^{(1)} - \lambda_2 G_2^{(1)}\right] \quad (5.12)$$

so that by using this relation (5.11) becomes

$$\overline{\Gamma}(x,y;\sigma) = \frac{1}{2\pi(\lambda_1-\lambda_2)}\left[\left(\frac{1}{K}-\frac{\lambda_2}{\sigma}\right)\lambda_1 K_0\left(\sqrt{\lambda_1}\sqrt{x^2+y^2}\right) - \left(\frac{1}{K}-\frac{\lambda_1}{\sigma}\right)\lambda_2 K_0\left(\sqrt{\lambda_2}\sqrt{x^2+y^2}\right)\right] \quad (5.13a)$$

Similarly, we have

$$\overline{\Phi}(x,y;\sigma) = \frac{\sigma}{2\pi K(\lambda_1-\lambda_2)}\left[K_0\left(\sqrt{\lambda_1}\sqrt{x^2+y^2}\right) - K_0\left(\sqrt{\lambda_2}\sqrt{x^2+y^2}\right)\right] \quad (5.13b)$$

## 5.3 Three-Dimensional Case

In this case the operator $\Delta$ in Cartesian coordinate system $(x,y,z)$ is

$$\Delta = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2} \quad (5.14)$$

and the heat source now degenerates to a point source. Eq. (4.12a) becomes

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\lambda_{1,2}\right)\overline{\theta} = -f(x,y,z) \quad (5.15)$$

whose fundamental solution can be shown as

$$G^{(1)}_{1,2}(x,y,z;\lambda_{1,2}) = \frac{e^{-\sqrt{\lambda_{1,2}}\,r}}{4\pi r} \; ; \qquad r^2 = x^2+y^2+z^2 \tag{5.16}$$

Here the single point source is supposed to be located at the origin. Then the fundamental solution $G^{(2)}$ of (4.8) is, for a point source located at $\vec{\rho} = (\xi,\eta,\zeta)$

$$G^{(2)}(x-\xi, y-\eta, z-\zeta;\sigma) = \frac{1}{4\pi(\lambda_1-\lambda_2)|\vec{r}-\vec{\rho}|}\left[e^{-\sqrt{\lambda_1}|\vec{r}-\vec{\rho}|} - e^{-\sqrt{\lambda_2}|\vec{r}-\vec{\rho}|}\right] \tag{5.17}$$

where

$$|\vec{r}-\vec{\rho}|^2 = (x-\xi)^2+(y-\eta)^2+(z-\zeta)^2.$$

In a similar manner, as described in previous cases, we obtain

$$\overline{\Gamma}(x,y,z;\sigma) = \frac{1}{4\pi(\lambda_1-\lambda_2)r}\left[\left(\frac{1}{\kappa}-\frac{\lambda_2}{\sigma}\right)\lambda_1 e^{-\sqrt{\lambda_1}\,r} - \left(\frac{1}{\kappa}-\frac{\lambda_1}{\sigma}\right)\lambda_2 e^{-\sqrt{\lambda_2}\,r}\right] \tag{5.18a}$$

$$\overline{\Phi}(x,y,z;\sigma) = \frac{\sigma}{4\pi\kappa(\lambda_1-\lambda_2)r}\left[e^{-\sqrt{\lambda_1}\,r} - e^{-\sqrt{\lambda_2}\,r}\right] \tag{5.18b}$$

with

$$r^2 = x^2+y^2+z^2.$$

## 5.4 Two-Dimensional Stationary Flow

In this case we are dealing with a flow having constant velocity $U$ parallel to the $x$-axis at infinity past some stationary heat sources. The equation of motion can be obtained by applying a Galilean transformation (cf. § 3) to (2.8) (or (2.2)), with $\Xi = 0$. We have

$$\left[\left(\Delta - \gamma M^2\frac{\partial^2}{\partial x^2}\right)\Delta - \frac{\gamma U}{\kappa}\left(\Delta - M^2\frac{\partial^2}{\partial x^2}\right)\frac{\partial}{\partial x}\right]\theta = -\frac{1}{\kappa}\left(\Delta-\gamma M^2\frac{\partial^2}{\partial x^2}\right)\Omega \tag{5.19a}$$

$$\left[\left(\Delta - \gamma M^2\frac{\partial^2}{\partial x^2}\right)\Delta - \frac{\gamma U}{\kappa}\left(\Delta - M^2\frac{\partial^2}{\partial x^2}\right)\frac{\partial}{\partial x}\right]\varphi = -\frac{U}{\kappa}\Omega_x \tag{5.19b}$$

where the Laplace operator $\Delta$, in this two-dimensional problem, is

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{5.20}$$

and the Mach number $M$ is defined by

$$M = \frac{U}{c} \,. \tag{5.21}$$

The fundamental solutions $\Gamma$ and $\Phi$ due to heat addition for this case may be defined in a way similar to that in §4. That is, when

$$\Omega = \delta(x)\,\delta(y) \tag{5.22}$$

$$\theta = \Gamma(x,y)\,, \qquad \varphi = \Phi(x,y) \tag{5.23}$$

There are several methods available for obtaining $\Gamma$ and $\Phi$. For instance, a method of descent (see, for example, Ref. 1, p. 69) can be used to deduce $\Gamma(x,y)$ and $\Phi(x,y)$ from the two-dimensional non-stationary fundamental solutions $\Gamma(x,y,t)$, $\Phi(x,y,t)$. Another method is somewhat similar to the previous one. First, one of the independent variables has to be eliminated by means of transformation. In this case Fourier transformations are most convenient to use since the solution must be defined in the entire $(x,y)$ plane. The Fourier transform with respect to $x$ of a function $f(x,y)$ is defined by (using the wavy barred letter to stand for its Fourier transform)

$$\tilde{f}(\beta,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\beta x} f(x,y)\,dx \tag{5.24a}$$

The inverse transform may be given by

$$f(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\beta x}\, \tilde{f}(\beta,y)\,d\beta \tag{5.24b}$$

Application of this transformation to (5.19) with vanishing conditions of $\theta$, $\varphi$ and $\Omega$ at $x = \pm\infty$, assuming that the Fourier transform of $\Omega$ exists, yields

$$L(\tilde{\theta}) = -\frac{1}{\kappa}\left[\frac{\partial^2}{\partial y^2} + \beta^2(\gamma M^2 - 1)\right]\tilde{\Omega} \tag{5.25a}$$

$$L(\tilde{\varphi}) = -\frac{i\beta U}{\kappa}\tilde{\Omega} \tag{5.25b}$$

where $L$ is now the operator

$$L \equiv \tilde{\Delta}\tilde{\Delta} + (\gamma M^2\beta^2 - \frac{i\beta U\gamma}{\kappa})\tilde{\Delta} - \frac{i\gamma U M^2\beta^3}{\kappa} = (\frac{\partial^2}{\partial y^2} - \lambda_1)(\frac{\partial^2}{\partial y^2} - \lambda_2) \tag{5.25c}$$

with

$$\begin{matrix}\lambda_1 \\ \lambda_2\end{matrix} = \frac{\gamma M^2}{2}(i\beta)\left\{ i\beta\left(1 - \frac{2}{\gamma M^2}\right) + \frac{U}{\kappa M^2} \pm \sqrt{(i\beta - \alpha)(i\beta - \bar{\alpha})} \right\} \tag{5.25d}$$

$$\alpha, \bar{\alpha} = \frac{U}{\kappa M^2}\left[ \left(\frac{2}{\gamma} - 1\right) \pm i\,\frac{2}{\gamma}\sqrt{\gamma - 1} \right] \tag{5.25e}$$

In this stationary case, the value $\alpha$ is almost the same as that in the non-stationary case (4.8e) except that the parameter $c^2$ is replaced by $\frac{U}{M^2}$ .

The problem of finding $G^{(2)}$ , $\tilde{\Gamma}$ and $\tilde{\Phi}$ in this case is now almost the same as in case (5.1). The results are as follows:

$$G^{(2)}(y) = \frac{1}{2(\lambda_1 - \lambda_2)}\left[ \frac{1}{\sqrt{\lambda_1}} e^{-\sqrt{\lambda_1}|y|} - \frac{1}{\sqrt{\lambda_2}} e^{-\sqrt{\lambda_2}|y|} \right] \tag{5.26}$$

where $\lambda_{1,2}$ are given by (5.25d), and

$$\tilde{\Gamma}(\beta, y) = \frac{1}{\sqrt{2\pi}\,2\kappa(\lambda_1 - \lambda_2)}\left\{ [\lambda_1 + \beta^2(\gamma M^2 - 1)]\frac{e^{-\sqrt{\lambda_1}|y|}}{\sqrt{\lambda_1}} - [\lambda_2 + \beta^2(\gamma M^2 - 1)]\frac{e^{-\sqrt{\lambda_2}|y|}}{\sqrt{\lambda_2}} \right\} \tag{5.27a}$$

$$\tilde{\Phi}(\beta, y) = \frac{i\beta U}{\sqrt{2\pi}\,\kappa}\, G^{(2)}(y) \tag{5.27b}$$

The desired fundamental solutions $\Gamma$ and $\Phi$ can be obtained by applying the inversion formula (5.24b) to (5.27).

## § 6. Problem of Inversion in the One-Dimensional Case

Since the one-dimensional problem usually forecasts the important features of higher dimensional cases, we shall discuss the problem of inversion of these one-dimensional fundamental solutions in detail. As an example only the calculation of $\Gamma$ will be given here.

If we apply the inversion formula (4.7b) to $\bar{\Gamma}$, (5.7a), with $\Omega = \delta(x)\,\delta(t)$,

$$\theta(x,t) = \Gamma(x,t) = \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} e^{\sigma t}\,\bar{\Gamma}(x,\sigma)\,d\sigma \tag{6.1}$$

we see that we have to deal with some rather complicated integrals. The success of the method depends very much on applying conformal transformations of the complex plane to simplify these integrals, especially to facilitate obtaining approximate formulas. In order to write out $\bar{\Gamma}$ explicitly, it is convenient to split $\bar{\Gamma}$ into two parts, and by introducing non-dimensional quantities

$$s = \frac{\kappa}{c^2}\sigma, \qquad T = \frac{c^2}{\kappa}t, \qquad X = \frac{c}{\kappa}|x| \qquad (X \geqslant 0) \tag{6.2a}$$

and

$$a = \frac{\kappa}{c^2}\alpha = \left(\frac{2}{\gamma}-1\right) + i\,\frac{2}{\gamma}\sqrt{\gamma-1} = h + id, \qquad \bar{a} = h - id, \tag{6.2b}$$

we have

$$\bar{\Gamma}(x,\sigma) = \bar{\Gamma}_1(X,s) + \bar{\Gamma}_2(X,s) \tag{6.3a}$$

where

$$\bar{\Gamma}_1(X,s) = \frac{1}{4c}\sqrt{\frac{\gamma}{2}}\left(1 - \frac{s-h}{\sqrt{(s-a)(s-\bar{a})}}\right)\sqrt{\frac{s+1+\sqrt{(s-a)(s-\bar{a})}}{s}}\;\exp\left\{-\sqrt{\frac{\gamma}{2}s\left[s+1+\sqrt{(s-a)(s-\bar{a})}\right]}\,X\right\} \tag{6.3b}$$

$$\bar{\Gamma}_2(X,s) = \frac{1}{4c}\sqrt{\frac{\gamma}{2}}\left(1 + \frac{s-h}{\sqrt{(s-a)(s-\bar{a})}}\right)\sqrt{\frac{s+1-\sqrt{(s-a)(s-\bar{a})}}{s}}\;\exp\left\{-\sqrt{\frac{\gamma}{2}s\left[s+1-\sqrt{(s-a)(s-\bar{a})}\right]}\,X\right\} \tag{6.3c}$$

Then (6.1) yields

$$\theta(x,t) = \Gamma(x,t) = \Gamma_1(X,T) + \Gamma_2(X,T) \tag{6.4a}$$

with

$$\Gamma_j(X,T) = \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty}\left(\frac{c^2}{\kappa}\right)\bar{\Gamma}_j(X,s)\, e^{sT}\, ds\,, \qquad j = 1,2 \tag{6.4b}$$

In these formulas, all $\sqrt{\phantom{x}}$ are defined to have positive real parts on the path of integration (which is described in §4 after (4.7b)). If we investigate the singularities of $\bar{\Gamma}_1$ and $\bar{\Gamma}_2$ in the $s$-plane, we note that $s = 0$ is a branch point of $\bar{\Gamma}_1$, but is a regular point of $\bar{\Gamma}_2$. We also notice that $s = a$, $s = \bar{a}$ are branch points of $\bar{\Gamma}_1$ as well as $\bar{\Gamma}_2$, but they are regular points of $\bar{\Gamma} = \bar{\Gamma}_1 + \bar{\Gamma}_2$. This simple analysis gives some idea about what transformations should be introduced. As a matter of fact, which will be shown later, $\Gamma_1$ and $\Gamma_2$ given by (6.4) are two branches of the same function. Now we apply the following conformal transformation to (6.4)

$$s - h = \frac{d}{2}\left(z - \frac{1}{z}\right) \tag{6.5}$$

which splits the cut from $s = a$ to $\bar{a}$ in the $s$-plane to a circle $|z| = 1$ in the $z$-plane. (6.4) then becomes

$$\Gamma_1 = \frac{c}{2\kappa}\sqrt{\frac{\gamma-1}{\gamma}}\,\frac{1}{2\pi i}\int_{\mathcal{C}}\frac{z^{-3/2}}{\sqrt{z-\sqrt{\gamma-1}}}\exp\left\{\frac{\sqrt{\gamma-1}}{\gamma}\left(z+\frac{1}{\sqrt{\gamma-1}}\right)\sqrt{\frac{z-\sqrt{\gamma-1}}{z}}\left[\sqrt{\frac{z-\sqrt{\gamma-1}}{z}}\,T - \sqrt{\gamma}\,X\right]\right\}dz \tag{6.6a}$$

$$\Gamma_2 = \frac{c}{2\kappa}\,\frac{(\gamma-1)^{1/4}}{\sqrt{\gamma}}\,\frac{1}{2\pi i}\int_{\mathcal{C}}\frac{1}{\sqrt{z+\frac{1}{\sqrt{\gamma-1}}}}\exp\left\{\frac{\sqrt{\gamma-1}}{\gamma}\,\frac{z-\sqrt{\gamma-1}}{z}\sqrt{z+\frac{1}{\sqrt{\gamma-1}}}\left[\sqrt{z+\frac{1}{\sqrt{\gamma-1}}}\,T - \frac{\sqrt{\gamma}}{(\gamma-1)^{1/4}}X\right]\right\}dz \tag{6.6b}$$

where the contour $\mathcal{C}$, as shown in Fig. 1, starts from $-i\infty$, passes the real $z$-axis to the right of the branch point $z = \sqrt{\gamma-1}$ and ends up at $+i\infty$. It is easy to see that in the finite $z$-plane, the integrand in (6.6a) has an essential singularity and a branch point at $z = 0$, also another branch point at $z = \sqrt{\gamma-1}$; while the integrand in (6.6b) has an essential singularity at $z = 0$ and a branch point at $z = -\frac{1}{\sqrt{\gamma-1}}$.

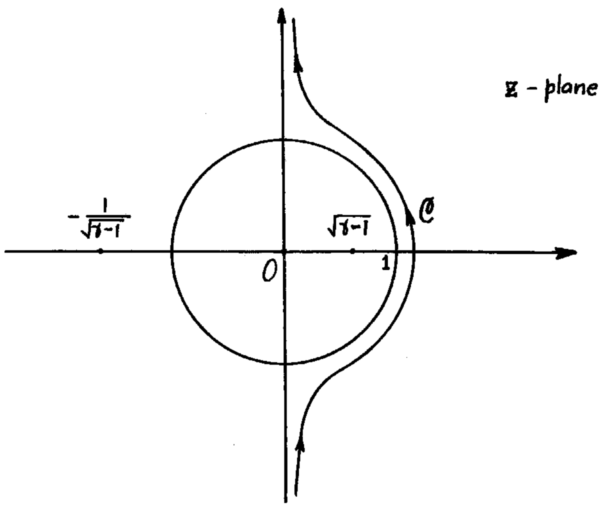


Fig. 1

Now (6.6) provides a neater expression for us to investigate the behavior and condition of convergence of the original integrals (6.4) in the neighborhood of infinity. As $|z| \to \infty$ , (6.6) gives

$$\Gamma_1 \sim \frac{c}{2\kappa}\sqrt{\frac{\gamma-1}{\gamma}}\,\frac{1}{2\pi i}\int \exp\left\{\frac{\sqrt{\gamma-1}}{\gamma}\left[T-\sqrt{\gamma}\,X\right]z\right\}\frac{dz}{z^2}$$

$$\Gamma_2 \sim \frac{c}{2\kappa}\,\frac{(\gamma-1)^{1/4}}{\sqrt{\gamma}}\,\frac{1}{2\pi i}\int \exp\left\{\frac{\sqrt{\gamma-1}}{\gamma}\left[T z-\frac{\sqrt{\gamma}\,X}{(\gamma-1)^{1/4}}\sqrt{z}\right]\right\}\frac{dz}{\sqrt{z}}$$

where the integrals are taken along parts of the path for which $|z|$ is large. From here it is natural that the following different cases must be considered to study the convergence of these integrals.

(i) $T<0$ (or $t<0$ ), the contour $\mathcal{C}$ for both $\Gamma_1$ and $\Gamma_2$ may be closed in the right half plane, so in this case $\Gamma_1 \equiv 0$ , $\Gamma_2 \equiv 0$ , which gives

$$\Gamma = \Gamma_1 + \Gamma_2 \equiv 0 \tag{6.7}$$

(ii) $\sqrt{\gamma}\,X > T > 0$ (or $|x| > c_i t > 0$ ), the contour $\mathcal{C}$ for $\Gamma_1$ may be closed in the right half plane while the contour $\mathcal{C}$ for

$\Gamma_2$ may be closed in the left half plane, then in this case only $\Gamma_1 \equiv 0$ which implies

$$\Gamma \equiv \Gamma_2 \tag{6.8}$$

(iii) $T > \sqrt{\gamma} X$ (or $|x| < c_i t$ ), the contour $\mathcal{C}$ for both $\Gamma_1$ and $\Gamma_2$ may be closed in the left half plane so that

$$\Gamma = \Gamma_1 + \Gamma_2 \tag{6.9}$$

Thus we have the important result that there is a discontinuity in the representation of the fundamental solutions across the characteristics of the system of equations indicated by the linearized form of (1.7d), namely $x = \pm c_i t$ .

If we apply a further conformal transformation

$$Z = -\frac{1}{z} \tag{6.10}$$

to (6.6a), we obtain

$$\Gamma_1(X,T) = \frac{c}{2\kappa}\frac{(\gamma-1)^{1/4}}{\sqrt{\gamma}}\frac{1}{2\pi i}\int_{\mathcal{C}'}\frac{1}{\sqrt{Z+\frac{1}{\sqrt{\gamma-1}}}}\exp\left\{\frac{\sqrt{\gamma-1}}{\gamma}\frac{Z-\sqrt{\gamma-1}}{Z}\sqrt{Z+\frac{1}{\sqrt{\gamma-1}}}\left[\sqrt{Z+\frac{1}{\sqrt{\gamma-1}}}\,T-\frac{\sqrt{\gamma}\,X}{(\gamma-1)^{1/4}}\right]\right\}dZ \tag{6.11}$$

where the contour $\mathcal{C}'$ described the path as shown in Fig. 2.

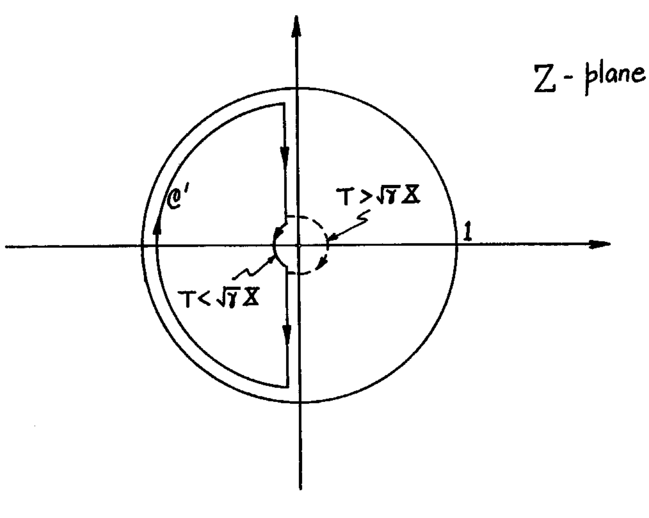


Fig. 2

According to the former analysis of the convergence of the integral, it is clear that $\mathcal{C}'$ is closed to the left of the origin for $T < \sqrt{\gamma}X$ so that in this case $\Gamma_1 \equiv 0$ since no singularity is inside the contour; but $\mathcal{C}'$ is closed to the right of the origin for $T > \sqrt{\gamma}X$ and thus encloses the essential singularity at $Z = 0$. Comparison of (6.11) with (6.6b) shows that $\Gamma_1$ and $\Gamma_2$ have the same integral representation but are taken on two different contours. This implies that $\Gamma_1$ and $\Gamma_2$ actually represent two different branches of the same function which originally bears branch points $s = a$ and $s = \bar{a}$ in the $s$-plane, as mentioned before.

Now we can combine (6.4a), (6.6b) and (6.11) into the following form:

$$\theta(x,t) = \Gamma(x,t) = \frac{c}{2\kappa}\frac{(\gamma-1)^{1/4}}{\sqrt{\gamma}}\frac{1}{2\pi i}\int_{\mathcal{C}+\mathcal{C}'} \exp\left\{\frac{\sqrt{\gamma-1}}{\gamma}\frac{z-\sqrt{\gamma-1}}{z}\sqrt{z+\frac{1}{\sqrt{\gamma-1}}}\left[\sqrt{z+\frac{1}{\sqrt{\gamma-1}}}\,T - \frac{\sqrt{\gamma}X}{(\gamma-1)^{1/4}}\right]\right\}\frac{dz}{\sqrt{z+\frac{1}{\sqrt{\gamma-1}}}} \quad (6.12)$$

where contours $\mathcal{C}$ and $\mathcal{C}'$ are described above. Now the integrand in (6.12) has an essential singularity at $z = 0$ and a branch point at $z = -\frac{1}{\sqrt{\gamma-1}}$ in the finite $z$-plane. Hence, if a branch cut is introduced from $-\infty$ along the negative $z$-axis to $z = -\frac{1}{\sqrt{\gamma-1}}$, then the integrand is an analytic function of $z$, regular in the entire cut plane except at $z = 0$ and $\infty$. It is easy to see, in accord with former analysis, that for $T < \sqrt{\gamma}X$, the integral on the contour $\mathcal{C}'$ is zero so that $\Gamma$ comes solely from the contribution on $\mathcal{C}$. For $T > \sqrt{\gamma}X$, the contribution on part of the contour $\mathcal{C}$ cancels that on the part of $\mathcal{C}'$ so that the resulting contour becomes $\mathcal{C}''$ (see Fig. 3) which starts from $-i\infty$, passes the real $z$-axis to the left of the essential singularity $z = 0$ and ends up at $+i\infty$.

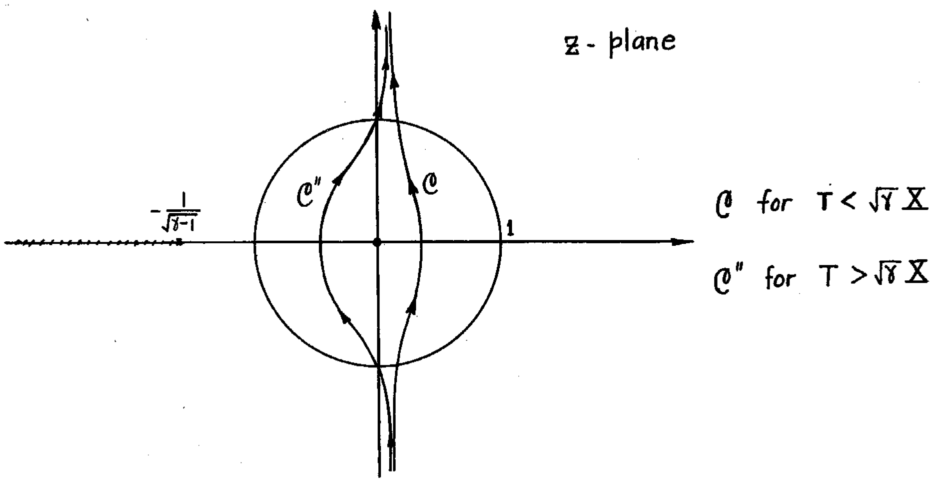


Fig. 3

The expression (6.12) may be further simplified by the following conformal transformation

$$\zeta^2 = \sqrt{\gamma-1}\left(z + \frac{1}{\sqrt{\gamma-1}}\right) \tag{6.13}$$

which removes the branch points at $z = -\frac{1}{\sqrt{\gamma-1}}$ and $\infty$ . Eq. (6.12) then reduces to a simple form

$$\theta(x,t) = \frac{c}{\kappa\sqrt{\gamma}}\,\frac{1}{2\pi i}\int_{\mathcal{C}_1,\mathcal{C}_2} \exp\left\{\frac{1}{\gamma}\,\frac{\zeta^2-\gamma}{\zeta^2-1}\,\zeta\left[\zeta T - \sqrt{\gamma}X\right]\right\} d\zeta \tag{6.14a}$$

where contours $\mathcal{C}_1$ and $\mathcal{C}_2$ are shown in Fig. 4. $\mathcal{C}_1$ is a contour

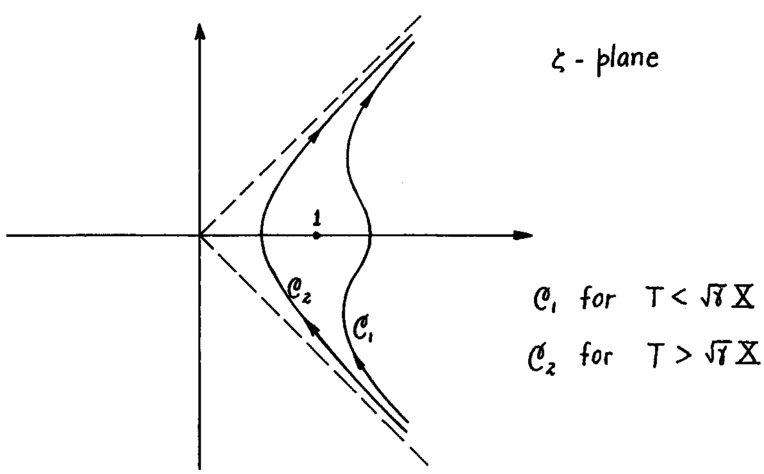


Fig. 4

starting at $\infty\, e^{-i\pi/4}$ , passing the real axis to the right of the essential singularity at $\zeta = 1$ and ending up at $\infty\, e^{+i\frac{\pi}{4}}$ . $\mathcal{C}_2$ starts and ends in the same place but passes to the left of $\zeta = 1$ .

By a similar treatment it can be shown (cf. Appendix B) that the integral representations for velocity, density, and pressure are given by

$$u(x,t) = \frac{c^2}{\kappa}\frac{\operatorname{sign} x}{\gamma}\frac{1}{2\pi i}\int_{\mathcal{C}_1,\mathcal{C}_2}\frac{\zeta}{\zeta^2-1}\exp\left\{\frac{1}{\gamma}\frac{\zeta^2-\gamma}{\zeta^2-1}\zeta\left[\zeta T-\sqrt{\gamma}X\right]\right\}d\zeta \tag{6.14b}$$

$$s(x,t) = \frac{c}{\kappa}\frac{1}{\sqrt{\gamma}}\frac{1}{2\pi i}\int_{\mathcal{C}_1,\mathcal{C}_2}\frac{1}{\zeta^2-1}\exp\left\{\frac{1}{\gamma}\frac{\zeta^2-\gamma}{\zeta^2-1}\zeta\left[\zeta T-\sqrt{\gamma}X\right]\right\}d\zeta \tag{6.14c}$$

$$p(x,t) = \frac{c}{\sqrt{\gamma}\,\kappa}\frac{1}{2\pi i}\int_{\mathcal{C}_1,\mathcal{C}_2}\frac{\zeta^2}{\zeta^2-1}\exp\left\{\frac{1}{\gamma}\frac{\zeta^2-\gamma}{\zeta^2-1}\zeta\left[\zeta T-\sqrt{\gamma}X\right]\right\}d\zeta \tag{6.14d}$$

where $\mathcal{C}_1$ , $\mathcal{C}_2$ denote the same contours as in (6.14a). It may be noted here that Eqs. (6.14) are still exact integral representations for $\theta$ , $u$ , $s$ and $p$ . No approximation has ever been made and it can be shown that they, in integral form, still satisfy the fundamental system of equations (2.2) with $\vec{X} = 0$ , $\Omega = \delta(x)\delta(t)$ . Now one can obtain asymptotic formulas suitable for either large or small values of $T$, ( $T = \frac{c^2}{\kappa}t$ ). We shall discuss these two cases separately.

## 6.1 Asymptotic Formulas Suitable for Large Values of $T$

From the expression $T = \frac{c^2}{\kappa}t$ , the large values of $T$ correspond to $c$ , $t$ both large, but $\kappa$ small, or in such a combination that $\frac{c^2 t}{\kappa}$ is large. For example, if air at standard conditions is taken as the medium, $\kappa$ = 0.28 cm.$^2$/sec., $c$ = 34 x 10$^3$ cm./sec. and $c^2t/\kappa$ = 4.1 x 10$^9$ t, thus it is clear that $t = O(1)\,sec.$ is large enough to make $T$ large. With this in mind, the required asymptotic solution in two different regions will be calculated as follows:

(i) $T > \sqrt{\gamma}\, X$ (or $|x| < c_i t$)

In this case, it is convenient to use a method related to the method of steepest descent (for example, cf. Ref. 25, Chapter VII, Ref. 26, Chapter 17, or Ref. 1). The first step is to deform the contour, if possible, until the following conditions are satisfied:

(1) The path of integration goes through a zero of $f'(\zeta)$,

$$f(\zeta) = \frac{\zeta^2-\gamma}{\zeta^2-1}\,\zeta^2$$

(2) The imaginary part of $f(\zeta)$ is constant on the path.

In addition to these, we need a third condition for our special case to aid approximation.

(3) $g(\zeta) = \dfrac{\zeta^2-\gamma}{\zeta^2-1}\,\zeta$ is purely imaginary on the path.

By keeping the coefficient of $X$ imaginary the magnitude of this term is fixed. Furthermore, it is assumed that $X$ is bounded as $T \to \infty$. Now $f'(\zeta)=0$ gives

$$\zeta\left[\zeta^2-(1+i\sqrt{\gamma-1})\right]\left[\zeta^2-(1-i\sqrt{\gamma-1})\right]=0 \tag{6.15a}$$

which has five roots

$$\zeta_0 = 0\,, \qquad \zeta_{1,2} = \sqrt{\gamma}\, e^{\pm i \frac{1}{2}\tan^{-1}\sqrt{\gamma-1}} \qquad \zeta_{3,4} = \sqrt{\gamma}\, e^{i(\pi \pm \frac{1}{2}\tan^{-1}\sqrt{\gamma-1})} \tag{6.15b}$$

The only path of steepest descent which meets all three of the above conditions is the imaginary axis which passes through one of these saddle points $\zeta_0 = 0$. Hence, if we deform the contour $C_2$ to the imaginary $\zeta$-axis, we obtain, by writing $\zeta = \xi + i\eta$

$$\theta(x,t) = \frac{c}{\sqrt{\gamma}\,\kappa}\,\frac{1}{\pi}\int_0^\infty e^{-\frac{T}{\gamma}\frac{\eta^2+\gamma}{\eta^2+1}\eta^2}\cos\left(\frac{X}{\sqrt{\gamma}}\,\frac{\eta^2+\gamma}{\eta^2+1}\,\eta\right)d\eta \tag{6.16}$$

It is shown in the theory of steepest descent that most of the contribution comes from the neighborhood of this saddle point $\eta = 0$ . Following the scheme of this method, (6.16) can be asymptotically approximated by

$$\theta(x,t) = \frac{1}{\sqrt{\gamma}} \frac{1}{\sqrt{4\pi \frac{\kappa}{\gamma} t}} e^{-\frac{x^2}{4\frac{\kappa}{\gamma} t}} + O\left(\frac{1}{ct}\right) \tag{6.17}$$

This solution should be good for $c^2 t/\kappa$ large but $x$ bounded. The detailed calculation and measure of the error term are given in Appendix C.

(ii) $T < \sqrt{\gamma}\, X$ $\quad (or \ |x| > c_i t)$

In this region both $T$ and $X$ are very large; we find that it is more convenient to use another method of approximation. Owing to the uncertainty of the value of $X$, it is desirable to choose a contour which crosses the real axis to the right of the essential singularity $\zeta = 1$ and satisfies the third condition in Case (i), that is, $g(\zeta) = \frac{\zeta^2 - \gamma}{\zeta^2 - 1}\zeta$ is purely imaginary on the path. By writing $\zeta = \xi + i\eta$ , the condition $Re\left(\zeta \frac{\zeta^2-\gamma}{\zeta^2-1}\right) = 0$ gives

$$\xi = 0 \tag{6.18a}$$

or

$$\left[ (\xi^2+\eta^2)^2 - (\gamma+1)\xi^2 + (3-\gamma)\eta^2 + \gamma \right] = 0 \tag{6.18b}$$

It is clear that $\xi = 0$ is irrelevant in this case, since $C_1$ must cross the $\xi$-axis to the right of $\xi = 1$ . Now (6.18b) gives a closed curve in the right half-plane, confined in the region $1 \leq \xi \leq \sqrt{\gamma}$ , $|\eta| \leq \frac{\gamma-1}{4}$ . It crosses the $\xi$-axis at $\xi = 1$ and $\sqrt{\gamma}$ with infinite slope and twice meets the circle $\xi^2 + \eta^2 = \frac{\gamma+1}{2}$ at $\eta = \pm \frac{\gamma-1}{4}$ with zero slope (as shown in Fig. 5).

Since we have to avoid the essential singularity $\zeta = 1$ , we

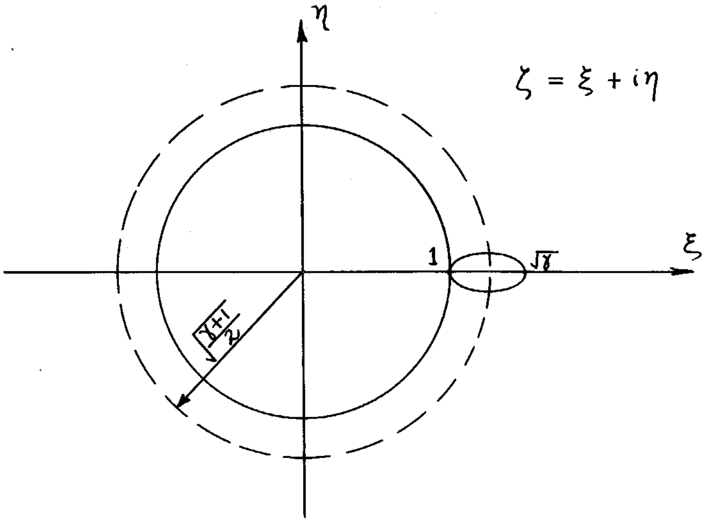


Fig. 5

choose a small portion of this curve near $(\sqrt{\gamma}, 0)$ to be part of the $\mathcal{C}_1$ -contour on which

$$\left| e^{\frac{1}{\gamma}\frac{\zeta^2-\gamma}{\zeta^2-1}\zeta^2 T} \right| = e^{-\frac{\alpha}{\gamma}\frac{\xi^2+\eta^2}{\xi^2+\eta^2-1}\eta^2 T} \tag{6.19}$$

and the rest of $\mathcal{C}_1$ may be taken as two straight lines leading from this oval curve to $\pm i\infty$ parallel to the $\eta$ -axis (Fig. 6).

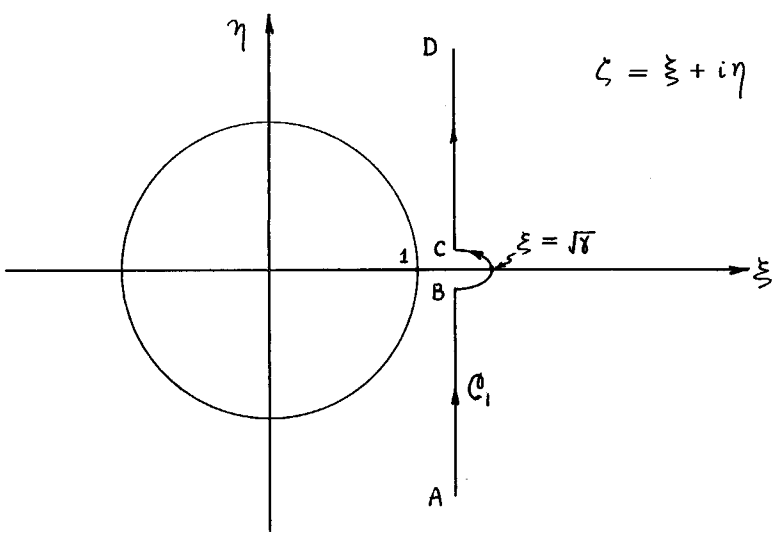


Fig. 6

From the expression (6.19) it can be shown that most of the contribution of the integral comes from the neighborhood of the point $\zeta = \sqrt{\gamma}$. This point also involves an interesting physical significance. Since the characteristics for the $\kappa = 0$ case indicate that important physical processes take place on usual lines of propagation of sound waves $x = \pm ct$ (corresponding to $X = T$ which is inside the region $T < \sqrt{\gamma} X$ under consideration). A flow with very small values of $\kappa$, which corresponds to large $T$ in our present case, should still retain this important feature. Hence, we may restrict ourselves to the region

$$|T - X| \ll T < \sqrt{\gamma} X \tag{6.20}$$

Mathematically speaking, this amounts to assuming that $|T-X|$ is bounded as $T, X \to \infty$. Then the integrand may be split into two parts

$$exp\left\{\frac{1}{\gamma}\frac{\zeta^2-\gamma}{\zeta^2-1}\zeta\left[\zeta-\sqrt{\gamma}\right]T\right\} exp\left\{\frac{1}{\sqrt{\gamma}}\frac{\zeta^2-\gamma}{\zeta^2-1}\zeta(T-X)\right\} \tag{6.21}$$

so that for $T$ large, $(T-X)$ might still be quite small; besides, $\left|exp\left\{\frac{1}{\sqrt{\gamma}}\frac{\zeta^2-\gamma}{\zeta^2-1}\zeta(T-X)\right\}\right| = 1$ on the path. This shows that the asymptotic method has to be applied only to the first exponential function of which $\zeta = \sqrt{\gamma}$ is evidently a saddle point. According to this analysis, we may foresee that our future result is important only in the neighborhood of the sound wave front $X = T$, or $(x \pm ct) = 0$.

As $Rl\,(g(\zeta))$ vanishes on the path $BC$, we have

$$g(\zeta) = 2i\,\frac{\eta(\xi^2+\eta^2)}{\xi^2+\eta^2-1} \tag{6.22}$$

Now, in order to evaluate (6.14a) on $BC$, it is advisable to represent the path parametrically by choosing a convenient parameter such that

$$\eta(\epsilon) = \epsilon \tag{6.23a}$$

then

$$\xi(\epsilon) = \left[\frac{\gamma+1}{2} - \epsilon^2 \pm \sqrt{1-16\left(\frac{\epsilon}{\gamma-1}\right)^2}\right]^{\frac{1}{2}} \tag{6.23b}$$

and

$$\zeta(\epsilon) = \xi(\epsilon) + i\epsilon \,, \qquad d\zeta = (i + \xi'(\epsilon))\,d\epsilon \tag{6.23c}$$

Consequently $\theta(x,t)$ may be asymptotically approximated by (see Appendix C)

$$\theta(x,t) \sim \frac{c}{\sqrt{\gamma}\,\kappa}\,\frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left\{-\frac{2}{\gamma-1}\epsilon^2 + i\,\frac{2\sqrt{\gamma}}{\gamma-1}(\mathrm{T}-\mathrm{X})\,\epsilon\right\} d\epsilon \tag{6.24}$$

which gives

$$\theta(x,t) = \begin{cases} \dfrac{\gamma-1}{2\gamma}\,\dfrac{1}{\sqrt{4\pi\frac{\gamma-1}{2\gamma}\kappa t}}\, e^{-\frac{(x-ct)^2}{4\frac{\gamma-1}{2\gamma}\kappa t}} + O\left(\frac{1}{ct}\right) & x > c_i t \\[2ex] \dfrac{\gamma-1}{2\gamma}\,\dfrac{1}{\sqrt{4\pi\frac{\gamma-1}{2\gamma}\kappa t}}\, e^{-\frac{(x+ct)^2}{4\frac{\gamma-1}{2\gamma}\kappa t}} + O\left(\frac{1}{ct}\right) & x < -c_i t \end{cases} \tag{6.25}$$

The velocity, density, and pressure fields may be calculated in a similar way. In summary, for $\Omega(x,t) = \delta(x)\,\delta(t)$, our results read

$$\theta(x,t) = \begin{cases} \dfrac{1}{\gamma}\,\dfrac{1}{\sqrt{4\pi\frac{\kappa}{\gamma}t}}\, e^{-\frac{x^2}{4\frac{\kappa}{\gamma}t}} + O\left(\frac{1}{ct}\right) & |x| < c_i t \\[2ex] \dfrac{\gamma-1}{2\gamma}\,\dfrac{1}{\sqrt{4\pi\frac{\gamma-1}{2\gamma}\kappa t}}\, e^{-\frac{(|x|-ct)^2}{4\frac{\gamma-1}{2\gamma}\kappa t}} + O\left(\frac{1}{ct}\right) & |x| > c_i t \end{cases} \tag{6.26a}$$

$$u(x,t) = \begin{cases} -\dfrac{1}{2\gamma}\,\dfrac{x}{\sqrt{4\pi\frac{\kappa}{\gamma}t^3}}\, e^{-\frac{x^2}{4\frac{\kappa}{\gamma}t}} + O\left(\frac{\kappa^{\frac{1}{2}}}{c\,t^{3/2}}\right) & |x| < c_i t \\[2ex] \dfrac{c}{2\gamma}\,\dfrac{\operatorname{sign} x}{\sqrt{4\pi\left(\frac{\gamma-1}{2\gamma}\right)\kappa t}}\, e^{-\frac{(|x|-ct)^2}{4\frac{\gamma-1}{2\gamma}\kappa t}} + O\left(\frac{1}{t}\right) & |x| > c_i t \end{cases} \tag{6.26b}$$

$$S(x,t)=\begin{cases} -\frac{1}{\gamma}\frac{1}{\sqrt{4\pi\frac{\kappa}{\gamma}t}}\,e^{-\frac{x^2}{4\frac{\kappa}{\gamma}t}}+O\left(\frac{1}{ct}\right) & |x|<c_i t \\ \frac{1}{2\gamma}\frac{1}{\sqrt{4\pi\frac{\gamma-1}{2\gamma}\kappa t}}\,e^{-\frac{(|x|-ct)^2}{4\frac{\gamma-1}{2\gamma}\kappa t}}+O\left(\frac{1}{ct}\right) & |x|>c_i t \end{cases} \tag{6.26c}$$

$$p(x,t)=\begin{cases} \frac{\kappa}{2\gamma c^2}\frac{1}{\sqrt{4\pi\frac{\kappa}{\gamma}t^3}}\left[1-\frac{x^2}{2\frac{\kappa}{\gamma}t}\right]e^{-\frac{x^2}{4\frac{\kappa}{\gamma}t}}+O\left(\frac{\kappa}{c^3t^2}\right) & |x|<c_i t \\ \frac{1}{2}\frac{1}{\sqrt{4\pi\frac{\gamma-1}{2\gamma}\kappa t}}\,e^{-\frac{(|x|-ct)^2}{4\frac{\gamma-1}{2\gamma}\kappa t}}+O\left(\frac{1}{ct}\right) & |x|>c_i t \end{cases} \tag{6.26d}$$

A discussion of these results is given in §7.

## 6.2 Approximated Solutions Suitable for Small Values of $T$

If we again adopt the same example given in §6.1, that is, take air at standard conditions as a medium, then the value of $T=\frac{c^2t}{\kappa}=4.1\times10^9 t$ could be regarded as "small" if $t$ is of the order of $10^{-12}$sec. or less. However, such a short duration is just about the order of magnitude of time required when gas molecules have suffered few collisions (for example, Ref. 6, p. 149) and equipartition of energy is nearly complete from the viewpoint of kinetic theory of gases. Hence, the temperature field in the immediate neighborhood of the heat source may be defined at the end of this time interval. Consequently, it is sensible to look for the behavior of solutions for such small values of time.

For our purpose, it is advisable to change those integral representations of our solutions (6.14) into a form more suitable for estimation by the following transformation

$$w=\sqrt{\frac{T}{\gamma}}\,\zeta \tag{6.27}$$

which shrinks the scale of $\zeta$ by the factor $\sqrt{T/\gamma}$ . The resulting integrals are

$$\theta(x,t) = \frac{c}{\kappa}\frac{1}{\sqrt{T}}\frac{1}{2\pi i}\int_{C_1, C_2} \exp\left\{\frac{w^2-T}{w^2-T/\gamma}\left(w^2-\frac{X}{\sqrt{T}}w\right)\right\}dw \tag{6.28a}$$

$$u(x,t) = \frac{c^2}{\gamma\kappa}\frac{\operatorname{sign} x}{2\pi i}\int_{C_1, C_2} \frac{w}{w^2-T/\gamma}\exp\left\{\frac{w^2-T}{w^2-T/\gamma}\left(w^2-\frac{X}{\sqrt{T}}w\right)\right\}dw \tag{6.28b}$$

with the corresponding formulas for $s$ and $p$, where contours $C_1$, $C_2$ still have the same starting and ending places but cross the real $w$-axis to the right and left of the essential singularity $w = \sqrt{T/\gamma}$ respectively.

One of the advantages of the transformation (6.27) is the appearance of the factor $\frac{X}{\sqrt{T}}$ in the resulting integrals (6.28). Since from our previous results (see (6.26)) it is clear that heat conduction causes the attenuation of sound waves as $\frac{1}{\sqrt{\kappa t}}$, hence for small values of $t$ the region where heat conductivity is important is of the order $x \sim \sqrt{\kappa t}$. Consequently, as $t \to 0$, the region of interest should be such that $\frac{x}{\sqrt{t}} \to O(\sqrt{\kappa})$ which corresponds in non-dimensional form to $\frac{X}{\sqrt{T}} = \frac{|x|}{\sqrt{\kappa t}} \to O(1)$ as $t \to 0$. Thus $\frac{X}{\sqrt{T}}$ is a very natural parameter for the problem of small values of $t$. Following this reasoning, we may then simplify the expressions (6.28) by letting $T \to 0$, but assuming $X/\sqrt{T} \to \alpha$ (a constant) in this limiting process. The error term may be estimated by the mean value theorem.

If the integrand in Eq. (6.28a) is written in the form

$$\exp\left\{\frac{w^2-T}{w^2-T/\gamma}\left(w^2-\frac{X}{\sqrt{T}}w\right)\right\} = g(w,T)\exp\left\{w^2-\frac{X}{\sqrt{T}}w\right\} \tag{6.29a}$$

with

$$g(w,T) = \exp\left\{-\frac{(1-\frac{1}{\gamma})T}{w^2-T/\gamma}\left(w^2-\alpha w\right)\right\}, \qquad \alpha = \frac{X}{\sqrt{T}} \tag{6.29b}$$

then for an interval of small values of $T$ excluding $T = \gamma w^2$ , such as $0 \leq T \leq T_0 < \gamma w^2$ , the mean value theorem may be applied to $g(w,T)$ . This calculation gives

$$\theta(x,t) = \frac{c}{\kappa\sqrt{T}} \frac{1}{2\pi i}\int_{\mathcal{C}} e^{\left(w^2 - \frac{X}{\sqrt{T}} w\right)} \left[1 - \left(1-\frac{1}{\gamma}\right)\frac{T}{w^2} + \frac{T^2}{2}\frac{\partial^2}{\partial T^2} g(AT)\right] dw \quad (0 < A < 1) \tag{6.30}$$

where for the first two terms in the bracket contour $\mathcal{C}$ may be taken as any path parallel to the imaginary $w$-axis in the right half-plane due to the removal of the singularity $w = \sqrt{T/\gamma}$ . For the last term, with $\frac{\partial^2}{\partial T^2} g(AT)$ , $\mathcal{C}$ must not pass through the point $\sqrt{AT/\gamma}$ . However, $\frac{\partial^2}{\partial T^2} g(AT)$ is jointly continuous in $w$ and $T$ on such a path $\mathcal{C}$ and is therefore also bounded. Carrying out this integral (6.30) with an appropriate estimation of error terms gives the following results:

$$\theta(x,t) = \frac{1}{\sqrt{4\pi\kappa t}}\, e^{-\frac{x^2}{4\kappa t}} \left[1 + O\left(\frac{c^2 t}{\kappa}\right)\right] \tag{6.31a}$$

$$u(x,t) = \frac{c^2 \operatorname{sign} x}{2\gamma\kappa}\, \operatorname{erfc}\left(\frac{|x|}{2\sqrt{\kappa t}}\right)\left[1 + O\left(\frac{c^2 t}{\kappa}\right)\right] \tag{6.31b}$$

$$S(x,t) = \frac{2}{\gamma}\frac{c^2}{\sqrt{\pi\kappa}}\frac{t^{3/2}}{x^2}\, e^{-\frac{x^2}{4\kappa t}} \left[1 + O\left(\frac{c^2 t}{\kappa}\right)\right] \tag{6.31c}$$

$$p(x,t) = \frac{1}{\sqrt{4\pi\kappa t}}\, e^{-\frac{x^2}{4\kappa t}} \left[1 + O\left(\frac{c^2 t}{\kappa}\right)\right] \tag{6.31d}$$

Now these results will be discussed in the next paragraph.

## § 7. Discussion of Results

The fundamental solutions given by (6.26) for $T$ large and (6.31) for $T$ small illustrate some remarks made in the Introduction. We shall discuss them one by one.

(i) Heat Energy Distribution

Suppose that a certain amount of heat $\Omega(x,t)$ has been put into the fluid in the time interval $0 \leqslant t < \tau$. Then if we integrate the fundamental equations (4.1a) and (4.1b) over a domain $0 \leqslant t < \tau$, $-\infty < x < \infty$ using the assumption that all perturbations and their derivatives vanish at infinity, we obtain

$$\int_{-\infty}^{\infty} \theta(x,\tau)\, dx = \int_{-\infty}^{\infty} dx \int_0^{\tau} \Omega(x,t)\, dt \tag{7.1}$$

This equation tells us that the spatial integral of the temperature perturbation $\theta$ at any instant $\tau$ is equal to the total heat input (dimensionless) during the time interval $0 \leqslant t < \tau$. For our special case, $\Omega(x,t) = \delta(x)\delta(t)$, (7.1) then becomes

$$\int_{-\infty}^{\infty} \theta(x,\tau)\, dx = 1 \qquad \text{for all} \quad \tau > 0 \tag{7.2}$$

It can be seen that this relation is also satisfied by the approximate formulas for both $t$ small and large (by using $\frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{4t}} dx = 1$ ) but their energy distributions are quite different. For $t$ small, all the total heat diffuses about the origin, with the coefficient of diffusion $\kappa$. However, for $t$ large, $\frac{1}{\gamma}$ of the total heat diffuses about the origin, with the coefficient of diffusion $\frac{\kappa}{\gamma}$; and $(1-\frac{1}{\gamma})$ of the total heat diffuses about $x = \pm ct$, the usual lines of propagation of sound waves, with the diffusion coefficient $\frac{\gamma-1}{2\gamma}\kappa$.

(ii) For very small values of $t$, pure heat conduction with unmodified heat conductivity $\kappa$ is the dominant process. This implies that the pressure waves take some time to establish themselves. Near the origin changes are mainly in temperature and pressure, while those in velocity and density are much less appreciable. From this phenomenon we might expect that the formation of pressure waves needs to be built up by density which then serves as the mechanism for sending off the pressure waves and at the same time modifies the conductivities. Nevertheless, the complete transition phenomenon still has to be sought.

(iii) For $t$ large, heat conduction near the origin with conductivity $\frac{\kappa}{\gamma}$ and along the pressure wave fronts $x = \pm ct$ with $\frac{\gamma-1}{2\gamma}\kappa$ are both important. Thus we see how the introduction of heat produces both a thermal wave and a pressure wave. Now, changes near the origin are mainly in temperature and density; while near the lines $x = \pm ct$ all changes are almost equally important. It is also interesting to see that near $x = \pm ct$, simple relations between variables exist, such as

$$p = s + \theta, \qquad u = cs, \qquad p = \gamma s \tag{7.3}$$

which are very familar in the linearized theory of isentropic sound wave propagation.

(iv) The Case $\kappa \to 0$

As $\kappa$ becomes smaller, the solutions for $T$ large are the only relevant ones. It can be seen from (6.26) that as $\kappa$ becomes smaller, all disturbances concentrate about $x = 0$ and $x = \pm ct$ and approach $\delta$-functions as $\kappa \to 0$. For example,

$$\theta(x,t) = \frac{1}{\gamma}\delta(x) + \frac{\gamma-1}{2\gamma}\delta(|x| - ct) \tag{7.4}$$

It can be shown that they approach the solutions with $\kappa = 0$.

(v) From (6.26b) it is clear that the velocity near the origin is linearly proportional to $x$ and approaches zero as $x \to 0$.

(vi) From these results it can be seen that heat conduction has an effect similar to that of viscosity, which also causes attenuation like $\frac{1}{\sqrt{\nu t}}$ . Thus, when both of them are considered in this problem the picture should be about the same as it is here.

(vii) The solution for higher dimensional waves can be worked out in a similar way and the results are very similar to the usual geometrical distortion.

## § 8. Remarks on General Initial Value and Boundary Value Problems

In previous sections it has been shown how a solution of the non-homogeneous system (2.2) is constructed if the initial conditions are zero and if the domain is unbounded ($-\infty < x < \infty$). In order to solve more general problems with non-zero initial conditions and with other boundary values, more general formulas will have to be derived. This method (for example, Ref. 27, p. 129) involves in particular an analysis of the singularity of the fundamental solution $G^{(2)}$ (defined by (4.11)) in the application of Green's formula. However, only results for several cases will be given here.

### 8.1 Representation for Formulas for the Solution in a Half-Plane $t > 0$, $-\infty < x < \infty$ with Non-Zero Initial Conditions

It is sufficient to impose the initial conditions on two of the four dependent variables (say $\theta$ and $u$), those for the other two can be deduced from the fundamental system (2.2). Having obtained $\theta$ and $u$ for the given initial value problem, $S$ and $p$ can be derived also from (2.2). An investigation of Eq. (4.1a) shows that initial values $\theta(x,0)$, $\theta_t(x,0)$, $\theta_{tt}(x,0)$ should all be given in order to fulfill the requirement for the formulation of the solution. This corresponds to any three independent initial conditions, for example, $\theta$ , $u$ and $u_t$ , being given in a practical problem. The result is

$$\begin{aligned}\theta(x,t) &= \int_{-\infty}^{\infty} d\xi \int_0^t \Gamma(x-\xi,t-\tau)\Omega(\xi,\tau)d\tau + \frac{\gamma}{\kappa C^2}\int_{-\infty}^{\infty}[G(x-\xi,t)\Omega_\tau(\xi,0)+G_t(x-\xi,t)\Omega(\xi,0)]d\xi \\ &+ \gamma\int_{-\infty}^{\infty}\left[\Gamma(x-\xi,t)+\frac{\kappa}{C^2}\Gamma_t(x-\xi,t)+\frac{\gamma-1}{\kappa C^2}G_{tt}(x-\xi,t)+\frac{\kappa}{C^2}G_{ttt}(x-\xi,t)\right]\theta(\xi,0)d\xi \\ &+ \frac{\gamma\kappa}{C^2}\int_{-\infty}^{\infty}\left[\Gamma(x-\xi,t)-\frac{1}{\kappa^2}G_t(x-\xi,t)+\frac{\gamma}{\kappa C^2}G_{tt}(x-\xi,t)\right]\theta_\tau(\xi,0)\,d\xi \\ &- \frac{\gamma}{\kappa C^2}\int_{-\infty}^{\infty} G(x-\xi,t)\,\theta_{\tau\tau}(\xi,0)\,d\xi \qquad (8.1a)\end{aligned}$$

The first term on the right hand side is the contribution due to the heat source and all the rest of the terms arise from the non-zero initial conditions, in which both $\theta(\xi,0)$ , $\theta_\tau(\xi,0)$ act like instantaneous heat sources. A similar requirement is for $u(x,t)$ which is given by

$$\begin{aligned}u(x,t) &= \int_{-\infty}^{\infty} d\xi \int_0^t \mathcal{U}(x-\xi,t-\tau)\Omega(\xi,\tau)\,d\tau + \frac{1}{\kappa}\int_{-\infty}^{\infty} G_\xi(x-\xi,t)\Omega(\xi,0)d\xi \\ &+ \gamma\int_{-\infty}^{\infty}\left[\Gamma(x-\xi,t)+\frac{\kappa}{C^2}\Gamma_t(x-\xi,t)+\frac{\gamma-1}{\kappa C^2}G_{tt}(x-\xi,t)+\frac{\kappa}{C^2}G_{ttt}(x-\xi,t)\right]u(\xi,0)d\xi \\ &+ \frac{\gamma\kappa}{C^2}\int_{-\infty}^{\infty}\left[\Gamma(x-\xi,t)-\frac{1}{\kappa^2}G_t(x-\xi,t)+\frac{\gamma}{\kappa C^2}G_{tt}(x-\xi,t)\right]u_\tau(\xi,0)\,d\xi \\ &- \frac{\gamma}{\kappa C^2}\int_{-\infty}^{\infty} G(x-\xi,t)\,u_{\tau\tau}(\xi,0)\,d\xi \qquad (8.1b)\end{aligned}$$

where $\mathcal{U} = \frac{\partial}{\partial x}\Phi(x,t)$ , $\Phi$ is given by (4.5b).

<u>Special Case (1)</u>

If all initial values are put equal to zero, then (8.1a) and (8.1b) reduce to the form (4.5a) and (4.5b) respectively.

<u>Special Case (2)</u> $\Omega \equiv 0$ (pure initial value problem)

In this case the first two integrals in (8.1a) and (8.1b) drop out.

8.2 Representation of Formulas for the Solution in a Quadrant $t > 0$, $0 \leq x < \infty$, with Non-Zero Initial Values and Non-Zero Boundary Conditions on $x = 0$

The representation of the solution in this case can be deduced from the previous results (8.1) by use of a reflection principle about $x = 0$. We also assume that

$$\begin{aligned} \Omega(x,t) &= 0 \quad \text{for} \quad x<0, \ t>0; \quad \text{and for all } x, \quad t<0 \\ &\neq 0 \quad \text{for} \quad x>0, \ t>0. \end{aligned}$$

Application of a method of reflection to (8.1) about $x = 0$ yields

$$\begin{aligned} \theta(x,t) &= \int_0^\infty d\xi \int_0^t \left[\Gamma(x-\xi,t-\tau) - \Gamma(x+\xi,t-\tau)\right]\Omega(\xi,\tau)\,d\tau + \frac{2}{\kappa}\int_0^t G_x(x,t-\tau)\Omega(0,\tau)\,d\tau \\ &+ \frac{\gamma}{\kappa C^2}\int_0^\infty \Big\{\left[G(x-\xi,t) - G(x+\xi,t)\right]\Omega_\tau(\xi,0) + \left[G_t(x-\xi,t) - G_t(x+\xi,t)\right]\Omega(\xi,0)\Big\}\,d\xi \\ &- 2\kappa\int_0^t \Gamma_x(x,t-\tau)\,\theta(0,\tau)\,d\tau + \int_0^\infty \left[\Gamma(x-\xi,t) - \Gamma(x+\xi,t)\right]\theta(\xi,0)\,d\xi \\ &- 2\int_0^t G_x(x,t-\tau)\,\theta_{\xi\xi}(0,\tau)\,d\tau + \frac{2\gamma}{\kappa}\int_0^t G_x(x,t-\tau)\,\theta_\tau(0,\tau)\,d\tau \\ &- \frac{\gamma}{\kappa C^2}\int_0^\infty \Big\{\left[G(x-\xi,t) - G(x+\xi,t)\right]\theta_{\tau\tau}(\xi,0) + \left[G_t(x-\xi,t) - G_t(x+\xi,t)\right]\theta_\tau(\xi,0)\Big\}\,d\xi \\ &+ \frac{\gamma-1}{\kappa}\int_0^\infty \left[G_{xx}(x-\xi,t) - G_{xx}(x+\xi,t)\right]\theta(\xi,0)\,d\xi \end{aligned} \tag{8.2a}$$

$$\begin{aligned} u(x,t) &= \int_0^\infty d\xi \int_0^t \left[\mathcal{U}(x-\xi,t-\tau) - \mathcal{U}(x+\xi,t-\tau)\right]\Omega(\xi,\tau)\,d\tau - \frac{1}{\kappa}\int_0^\infty \left[G_x(x-\xi,t) + G_x(x+\xi,t)\right]\Omega(\xi,0)\,d\xi \\ &- 2\kappa\int_0^t \Gamma_x(x,t-\tau)\,u(0,\tau)\,d\tau + \int_0^\infty \left[\Gamma(x-\xi,t) - \Gamma(x+\xi,t)\right]u(\xi,0)\,d\xi \\ &- 2\int_0^\infty G_x(x,t-\tau)\,u_{\xi\xi}(0,\tau)\,d\tau + \frac{2\gamma}{\kappa}\int_0^t G_x(x,t-\tau)\,u_\tau(0,\tau)\,d\tau \\ &- \frac{\gamma}{\kappa C^2}\int_0^\infty \Big\{\left[G(x-\xi,t) - G(x+\xi,t)\right]u_{\tau\tau}(\xi,0) + \left[G_t(x-\xi,t) - G_t(x-\xi,t)\right]u_\tau(\xi,0)\Big\}\,d\xi \\ &+ \frac{\gamma-1}{\kappa}\int_0^\infty \left[G_{xx}(x-\xi,t) - G_{xx}(x+\xi,t)\right]u(\xi,0)\,d\xi \end{aligned} \tag{8.2b}$$

<u>Special Case (1)</u> All initial values equal to zero

In this case (8.2) becomes

$$\theta(x,t) = \int_0^\infty d\xi \int_0^t \left[\Gamma(x-\xi,t-\tau) - \Gamma(x+\xi,t-\tau)\right]\Omega(\xi,\tau)d\tau + \frac{2}{\kappa}\int_0^t G_x(x,t-\tau)\Omega(0,\tau)d\tau$$

$$-2\kappa\int_0^t \Gamma_x(x,t-\tau)\,\theta(0,\tau)d\tau - 2\int_0^t G_x(x,t-\tau)\left[\theta_{\xi\xi}(0,\tau) - \frac{\gamma}{\kappa}\theta_\tau(0,\tau)\right]d\tau \quad (8.3a)$$

$$u(x,t) = \int_0^\infty d\xi \int_0^t \left[\mathcal{U}(x-\xi,t-\tau) - \mathcal{U}(x+\xi,t-\tau)\right]\Omega(\xi,\tau)d\tau - 2\kappa\int_0^t \Gamma_x(x,t-\tau)\,u(0,\tau)d\tau$$

$$-2\int_0^t G_x(x,t-\tau)\left[u_{\xi\xi}(0,\tau) - \frac{\gamma}{\kappa}u_\tau(0,\tau)\right]d\tau \quad (8.3b)$$

<u>Special Case (2)</u> All initial values equal to zero and $\Omega \equiv 0$

In this case we have a pure radiation problem and Eqs. (8.3a) and (8.3b) then further reduce to

$$\theta(x,t) = -2\kappa\int_0^t \Gamma_x(x,t-\tau)\,\theta(0,\tau)d\tau - 2\int_0^t G_x(x,t-\tau)\left[\theta_{\xi\xi}(0,\tau) - \frac{\gamma}{\kappa}\theta_\tau(0,\tau)\right]d\tau \quad (8.4a)$$

$$u(x,t) = -2\kappa\int_0^t \Gamma_x(x,t-\tau)\,u(0,\tau)d\tau - 2\int_0^t G_x(x,t-\tau)\left[u_{\xi\xi}(0,\tau) - \frac{\gamma}{\kappa}u_\tau(0,\tau)\right]d\tau \quad (8.4b)$$

In a pure radiation problem, boundary values $\theta(0,\tau)$ , $\theta_{\xi\xi}(0,\tau)$ and $u(0,\tau)$ , $u_{\xi\xi}(0,\tau)$ are needed.

Representation of the solution in other cases can be worked out in a similar way. With these results many practical problems can be solved.

# PART II

## HEAT CONDUCTION IN AN UNSTEADY FLOW OF A VISCOUS COMPRESSIBLE FLUID

### § 9. Fundamental Equations and Their Characteristics

The fundamental equations of a viscous, compressible, heat conducting fluid may be obtained in the following form:

The continuity equation is the same as before

$$\frac{\partial \rho}{\partial t} + div(\rho \vec{Q}) = 0 \tag{9.1}$$

Conservation of momentum now becomes

$$\rho \frac{D\vec{Q}}{Dt} = \rho \vec{F} - grad\, P + grad\,(\lambda\, div\, \vec{Q}) + div\left\{\mu\left[(\nabla\vec{Q}) + (\nabla\vec{Q})^*\right]\right\} \tag{9.2}$$

where $\lambda$ and $\mu$ are two coefficients of viscosity, which may be considered in general as functions of $\rho$ and $T$, entered naturally in the derivation of the Navier-Stokes equation. $(\nabla\vec{Q})^*$ is the transposed tensor of $(\nabla\vec{Q})$. The operator $\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \vec{Q}\cdot grad$. Conservation of energy now becomes

$$\frac{DE}{Dt} + P\frac{D(\frac{1}{\rho})}{Dt} = \frac{1}{\rho}\, div\,(k\, grad\, T) + \frac{\Phi}{\rho} + \mathcal{Q} \tag{9.3a}$$

where $\Phi$ is the dissipation function due to viscous stress.

$$\Phi = \lambda\,(div\,\vec{Q})^2 + \mu\left\{2\sum_i \left(\frac{\partial Q_i}{\partial x_i}\right)^2 + \sum_{i<j}\left(\frac{\partial Q_i}{\partial x_j} + \frac{\partial Q_j}{\partial x_i}\right)^2\right\} \tag{9.3b}$$

It is again assumed that the medium is a perfect gas, so that the equation of state is

$$P = \rho R T \tag{9.4}$$

In the following work the Stokes assumption

$$3\lambda + 2\mu = 0 \tag{9.5}$$

is taken, which is approximately true for actual gases, but does not hold for a liquid such as water (for this discussion see Ref. 28).

Eqs. (9.1)-(9.4) may be regarded as a non-linear system of four partial differential equations for four unknowns $P$, $\rho$, $T$ and $\vec{Q}$. Comparing this system with the non-viscous case (1.1)-(1.4), it can be seen that one of the differences is that the order of the momentum equation is raised from one to two, while the added viscosity has its known diffusive character. Therefore, we would expect that the limiting value of the solution of the above system of equations, as $\mu \rightarrow 0$, will not converge uniformly somewhere in the fluid to the non-viscous case.

In order to investigate this uniform convergence and the underlying construction of the solution, the characteristics of (9.1)-(9.4) for the one-dimensional case will be considered.

Let $\psi(x,t)$ = constant denote the equation of characteristics, then the characteristic condition for $\mu \neq 0$, $k \neq 0$ becomes (Ref. 1, pp. 18-20)

$$\psi_x^4 \left( \psi_t + u \psi_x \right) = 0 \tag{9.6a}$$

The characteristics are:

(i) $\psi_x^4 = 0$ or $t$ = const. which occurs quadruple (9.6b)

This set indicates the usual heat conduction.

(ii) $\psi_t + u \psi_x = 0$ or $\frac{dx}{dt} = u$ (9.6c)

This indicates the streamlines of the flow across which temperature, density and hence pressure are still continuous (because $k \neq 0$, $\mu \neq 0$).

The characteristics for different special cases when $\mu$, or $k$ or both are put equal to zero were given in §1. In Part I it was shown that the characteristic lines $\frac{dx}{dt} = u \pm c_i$ for the non-viscous, heat-conducting fluid flow indicate the boundary of different regions across which the fundamental solutions have discontinuous form. However, when the viscosity is considered in addition to heat conduction, such characteristic lines disappear. Thus it would imply that any discontinuity in the representation of the fundamental solutions is expected not to occur in the viscous, heat-conducting fluid flow.

## § 10. Linearization of Equations and Their Properties

With the same reasoning stated in §2, we apply the same linearization (2.1) to the system (9.1)-(9.4) with an additional expression for coefficients of viscosity

$$\mu = \mu_0\,(1+\mu') \qquad \mu_0 = \mu_0(T_0)\,, \qquad \mu' \ll 1 \tag{10.1}$$

If we neglect all higher order terms of small quantities, we then obtain a linearized system of equations as follows:

State $\qquad p = s + \theta \qquad$ (10.2a)

Continuity $\qquad s_t + \operatorname{div} \vec{q} = 0 \qquad$ (10.2b)

Momentum $\qquad \vec{q}_t = \vec{X} - \frac{c^2}{\gamma}\operatorname{grad} p + \frac{4}{3}\nu \operatorname{grad}\operatorname{div}\vec{q} - \nu \operatorname{curl}\operatorname{curl}\vec{q} \qquad$ (10.2c)

Energy $\qquad \theta_t - \kappa\,\Delta\theta = (\gamma-1)\,s_t + \Omega \qquad$ (10.2d)

Here the dissipation terms are dropped out by linearization due to the fact that the whole expression of the dissipation function $\Phi$ contains only squared terms of small quantities. Thus the linearized energy equation has the same form whether the fluid is viscous or not.

If we consider for a moment the property of rotationality of the flow, we can see that the scalar velocity potential in general does not exist. Because if we define the vorticity $\vec{\omega}$ by

$$\vec{\omega} = \operatorname{curl} \vec{q} \tag{10.3}$$

we can show, by taking curl of (10.2c), that $\vec{\omega}$ satisfies

$$\vec{\omega}_t = \nu \Delta \vec{\omega} - \operatorname{curl} \vec{X} \tag{10.4}$$

in our linearized theory so that $\vec{\omega}$ diffuses like heat with diffusivity $\nu = \frac{\mu_0}{\rho_0}$ . If $\vec{\omega}$ has some singularity initially, or if $\operatorname{curl} \vec{X} \neq 0$ in the whole domain, then the flow is rotational everywhere for $t > 0$ . This is a remarkable difference from the inviscid fluid. Consequently, we have to deal with the velocity itself instead of its potential as in the non-viscous case. It may be remarked here that in most cases the splitting of the velocity field into a longitudinal wave (whose curl is equal to zero) and a transversal wave (whose div is equal to zero) (for their definitions see Ref. 1, p. 27) is possible (see for example, Part III).

## § 11. Fundamental Solution in the One-Dimensional Case

We shall denote this one-dimensional space coordinate by $x$ and the velocity by $u$ . In this section we again assume that the external force $\vec{X}$ is zero and consider only the effect due to heat addition $\Omega$ . The actual physical problem in this case may also be visualized as that which is described in §5.1. The linearized fundamental system of equations in this case then becomes:

$$p = S + \theta \tag{11.1a}$$

$$S_t + u_x = 0 \tag{11.1b}$$

$$u_t - \frac{4}{3} \nu u_{xx} = -\frac{c^2}{\gamma} p_x \tag{11.1c}$$

$$\theta_t - \kappa \theta_{xx} = \Omega + (\gamma - 1) S_t \tag{11.1d}$$

In this system all dependent variables are coupled in a complicated manner. It can be seen that $u$ satisfies a diffusion equation with $p_x$ acting as a forcing function; while $\theta$ satisfies another diffusion equation with heat addition $\Omega$ and $S_t$ as heat sources. This system can then be reduced to a pair of equations for $\theta$ and $u$

$$\theta_t - \kappa\,\theta_{xx} = \Omega - (\gamma-1)\,u_x \tag{11.2a}$$

$$\frac{4\nu}{3c^2}u_{xxt} + u_{xx} - \frac{1}{c^2}u_{tt} = \frac{1}{\gamma}(\kappa\,\theta_{xxx} + \Omega_x) \tag{11.2b}$$

Here $\theta$ and $u$ are still coupled. Furthermore, for $\nu \neq 0$ , $u$ satisfies a parabolic equation instead of a hyperbolic one in the non-viscous case, so it is impossible in this case to have any discontinuities propagated with finite speed.

The system (11.1) can also be reduced further to a single fifth order equation for each of the dependent variables

$$L(\theta) = -\frac{4\gamma\nu}{3\kappa c^2}\Omega_{xxt} - \frac{1}{\kappa}\left(\frac{\partial^2}{\partial x^2} - \frac{\gamma}{c^2}\frac{\partial^2}{\partial t^2}\right)\Omega \tag{11.3a}$$

$$L(u) = -\frac{1}{\kappa}\Omega_{xt} \tag{11.3b}$$

$$L(S) = \frac{1}{\kappa}\Omega_{xx} \tag{11.3c}$$

$$L(p) = -\frac{4\gamma\nu}{3\kappa c^2}\Omega_{xxt} + \frac{\gamma}{\kappa c^2}\Omega_{tt} \tag{11.3d}$$

where $L$ is the operator

$$L \equiv \left\{\frac{4\gamma\nu}{3c^2}\frac{\partial^5}{\partial t\,\partial x^4} + \left[\frac{\partial^2}{\partial x^2} - \left(1+\frac{4\nu}{3\kappa}\right)\frac{\gamma}{c^2}\frac{\partial^2}{\partial t^2}\right]\frac{\partial^2}{\partial x^2} - \frac{\gamma}{\kappa}\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\frac{\partial}{\partial t}\right\} \tag{11.3e}$$

It may be noted that due to $\nu$ being different from zero, the order of these differential equations is raised by one compared with the non-viscous case (see (2.8)). Since both viscosity and heat conduction are considered here, it is clear that the Prandtl number

$$Pr = \frac{c_p \mu_0}{k_0} = \frac{\gamma \nu}{\kappa} \tag{11.4}$$

is a very important parameter in this problem. It is of interest to note that when $Pr = \frac{3}{4}$ , a remarkable simplification of these equations can be carried out. Since this value is very close to the observed value for air*, we shall take advantage of the simplification offered and limit our attention to this value. When

$$Pr = \frac{3}{4} \qquad \text{so that} \qquad \frac{\kappa}{\nu} = \frac{4}{3}\gamma \tag{11.5}$$

the operator $L$ given by (11.3e) can be written in terms of two factors

$$L \equiv \left[\left(1 + \frac{\kappa}{c^2}\frac{\partial}{\partial t}\right)\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\left[\frac{\partial^2}{\partial x^2} - \frac{\gamma}{\kappa}\frac{\partial}{\partial t}\right] \tag{11.6}$$

The operator in the first bracket is exactly the one discussed in Ref. 1, Appendix A; while the operator in the second bracket is a diffusion operator with diffusivity $\kappa/\gamma$ . With this particular value of the Prandtl number, the system (11.3) may be written as follows:

$$L(\theta) = -\frac{1}{\kappa}\left[\left(1 + \frac{\kappa}{c^2}\frac{\partial}{\partial t}\right)\frac{\partial^2}{\partial x^2} - \frac{\gamma}{c^2}\frac{\partial^2}{\partial t^2}\right]\Omega \tag{11.7a}$$

$$L(u) = -\frac{1}{\kappa}\,\Omega_{xt} \tag{11.7b}$$

$$L(s) = \frac{1}{\kappa}\,\Omega_{xx} \tag{11.7c}$$

$$\left[\left(1 + \frac{\kappa}{c^2}\frac{\partial}{\partial t}\right)\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right] p = -\frac{1}{c^2}\,\Omega_t \tag{11.7d}$$

with $L$ given by (11.6). It is interesting to note that $p$ satisfies

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*For air, $Pr$ = 0.72 to 0.76 over a wide range of temperature (Ref. 13). Another example that simplification can be achieved when $Pr = \frac{3}{4}$ is the shock wave theory. See, for example, Puckett and Stewart, Quart. Appl. Math., Vol. 7, p. 457, 1950; von Mises, Journ. Aero. Sci. Vol. 9, 1950.

a partial differential equation of the third order in $x$ and $t$, instead of the fifth order as the original one. This has some physical interpretation from the result obtained later.

The fundamental solutions of (11.7) are defined in the same way as stated in §4, so that for $\Omega = \delta(x)\delta(t)$, the solutions to (11.7) are given by

$$\theta = \Gamma(x,t), \qquad u = \mathfrak{U}(x,t), \qquad s = \mathfrak{S}(x,t), \qquad p = \mathfrak{P}(x,t) \tag{11.8}$$

Here the procedure is very similar to that in Part I. If we apply the Laplace transformation (4.7) to the system (11.7) with all zero initial conditions, we obtain

$$\left(\frac{d^2}{dx^2} - \lambda_1\right)\left(\frac{d^2}{dx^2} - \lambda_2\right)\bar{\theta} = -\frac{1}{\kappa}\left(\frac{d^2}{dx^2} - \gamma\lambda_1\right)\bar{\Omega} \tag{11.9a}$$

$$\left(\frac{d^2}{dx^2} - \lambda_1\right)\left(\frac{d^2}{dx^2} - \lambda_2\right)\bar{u} = -\frac{\sigma}{\kappa}\left(1 + \frac{\kappa\sigma}{c^2}\right)^{-1}\bar{\Omega}_x \tag{11.9b}$$

$$\left(\frac{d^2}{dx^2} - \lambda_1\right)\left(\frac{d^2}{dx^2} - \lambda_2\right)\bar{s} = \frac{1}{\kappa}\left(1 + \frac{\kappa\sigma}{c^2}\right)^{-1}\bar{\Omega}_{xx} \tag{11.9c}$$

$$\left(\frac{d^2}{dx^2} - \lambda_1\right)\bar{p} = -\frac{\lambda_1}{\sigma}\bar{\Omega} \tag{11.9d}$$

where

$$\lambda_1 = \left(1 + \frac{\kappa\sigma}{c^2}\right)^{-1}\frac{\sigma^2}{c^2}, \qquad \lambda_2 = \frac{\gamma\sigma}{\kappa}. \tag{11.9e}$$

The fundamental solution for the pressure $\bar{p}$ is easily obtained (see (5.2))

$$\bar{p}(x,\sigma) = \bar{\mathfrak{P}}(x,\sigma) = \frac{\sqrt{\lambda_1}}{2\sigma}\, e^{-\sqrt{\lambda_1}\,|x|}. \tag{11.10a}$$

while for the rest of the fundamental solutions the factoring of the operator again permits us to apply Theorem 1 in Appendix A. Following the same procedure described in §5, we obtain

$$\bar{\theta}(x,\sigma) = \bar{\Gamma}(x,\sigma) = \frac{1}{2(\lambda_1-\lambda_2)}\left[\left(\frac{1}{\kappa}-\frac{\lambda_2}{\sigma}\right)\sqrt{\lambda_1}\, e^{-\sqrt{\lambda_1}|x|} - \left(\frac{1}{\kappa}-\frac{\lambda_1}{\sigma}\right)\sqrt{\lambda_2}\, e^{-\sqrt{\lambda_2}|x|}\right] \tag{11.10b}$$

$$\bar{u}(x,\sigma) = \bar{\mathcal{U}}(x,\sigma) = -\frac{\frac{\sigma}{\kappa}}{1+\frac{\kappa\sigma}{c^2}}\,\frac{\operatorname{sign} x}{2(\lambda_1-\lambda_2)}\left[e^{-\sqrt{\lambda_1}|x|} - e^{-\sqrt{\lambda_2}|x|}\right] \tag{11.10c}$$

$$\bar{S}(x,\sigma) = \bar{\mathcal{S}}(x,\sigma) = -\frac{\frac{1}{\kappa}}{1+\frac{\kappa\sigma}{c^2}}\,\frac{1}{2(\lambda_1-\lambda_2)}\left[\sqrt{\lambda_1}\, e^{-\sqrt{\lambda_1}|x|} - \sqrt{\lambda_2}\, e^{-\sqrt{\lambda_2}|x|}\right] \tag{11.10d}$$

**Applying the inversion formula (4.7b) to the system (11.10), and introducing**

$$s = \frac{\kappa}{c^2}\sigma, \qquad T = \frac{c^2 t}{\kappa}, \qquad X = \frac{c}{\kappa}|x| \qquad (X \geq 0) \tag{11.11}$$

**we obtain:**

$$\theta(x,t) = \frac{c}{2\kappa}\,\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} e^{sT}\left\{\frac{e^{-\sqrt{\gamma s}\,X}}{\sqrt{\gamma s}\left(1+\frac{\gamma-1}{\gamma}s\right)} + \frac{\gamma-1}{\gamma}\,\frac{\sqrt{1+s}}{1+\frac{\gamma-1}{\gamma}s}\, e^{-\frac{s}{\sqrt{1+s}}X}\right\} ds \tag{11.12a}$$

$$u(x,t) = -\frac{c^2}{2\gamma\kappa}\operatorname{sign} x\,\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} \frac{e^{sT}}{1+\frac{\gamma-1}{\gamma}s}\left\{e^{-\sqrt{\gamma s}\,X} - e^{-\frac{s}{\sqrt{1+s}}X}\right\} ds \tag{11.12b}$$

$$S(x,t) = -\frac{c}{2\gamma\kappa}\,\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} e^{sT}\left\{\frac{\sqrt{\gamma}\, e^{-\sqrt{\gamma s}\,X}}{\sqrt{s}\left(1+\frac{\gamma-1}{\gamma}s\right)} - \frac{1}{\sqrt{1+s}\left(1+\frac{\gamma-1}{\gamma}s\right)}\, e^{-\frac{s}{\sqrt{1+s}}X}\right\} ds \tag{11.12c}$$

$$p(x,t) = \frac{c}{2\kappa}\,\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} \frac{1}{\sqrt{1+s}}\, e^{sT - \frac{s}{\sqrt{1+s}}X}\, ds \tag{11.12d}$$

**where the path of integration is parallel to the imaginary axis and to the right of all the singularities of the integrand in the $s$-plane. If we consider for a moment the integrands in (11.12) we see that $e^{-\sqrt{\gamma s}\,X}$ has a branch point at the origin $s = 0$, and $e^{-\frac{s}{\sqrt{1+s}}X}$ has a branch point and an essential singularity at $s = -1$ so that we must in general restrict $|\arg s| \leq \pi$ in the former case and $|\arg(1+s)| \leq \pi$ in the latter. Another singularity in the integrand of $\theta$, $u$, and**

S is a simple pole at $s = -\frac{\gamma}{\gamma - 1}$. By investigating the behavior of these integrands for large values of $|s|$, it can be seen that the original contour may always be closed to the right half-plane for $T < 0$ so that $\theta, u, S, p \equiv 0$; for $T > 0$ the contour may be closed to the left half-plane with some proper branch cuts. This fact is foreseen in our study of the characteristics of the system and is quite different from the non-viscous case where we have a different choice of contours for different regions of $T$ ( $T > \sqrt{\gamma} X$ and $0 < T < \sqrt{\gamma} X$ ).

The integral

$$\mathcal{J}(x,t) = \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} e^{sT - \frac{s}{\sqrt{1+s}} X} F(s)\, ds$$

is calculated and discussed in Ref. 1, Appendix A by contour integration in the $s$-plane. However, it is found that the application of some appropriate conformal transformations of the complex plane can simplify the calculation. If we apply in (11.12) the transformation

$$s = \zeta^2 \tag{11.13}$$

to terms with $e^{-\sqrt{\gamma s} X}$ and another transformation

$$s = \zeta^2 - 1 \tag{11.14}$$

to terms with $e^{-\frac{s}{\sqrt{1+s}} X}$, then all the original branch points are removed, the resulting integrals are

$$\theta(x,t) = \frac{c}{\kappa} \frac{1}{2\pi i} \left\{ \frac{\sqrt{\gamma}}{\gamma - 1} \int_{C_1} \frac{1}{\zeta^2 + \frac{\gamma}{\gamma-1}} e^{\zeta^2 T - \sqrt{\gamma} X \zeta} d\zeta + \int_{C_2} \frac{\zeta^2}{\zeta^2 + \frac{1}{\gamma-1}} e^{(\zeta^2 - 1)T - \frac{\zeta^2 - 1}{\zeta} X} d\zeta \right\} \tag{11.15a}$$

$$u(x,t) = -\frac{c^2 \operatorname{sign} x}{(\gamma - 1)\kappa} \frac{1}{2\pi i} \left\{ \int_{C_1} \frac{\zeta}{\zeta^2 + \frac{\gamma}{\gamma-1}} e^{\zeta^2 T - \sqrt{\gamma} X \zeta} d\zeta - \int_{C_2} \frac{\zeta}{\zeta^2 + \frac{1}{\gamma-1}} e^{(\zeta^2 - 1)T - \frac{\zeta^2 - 1}{\zeta} X} d\zeta \right\} \tag{11.15b}$$

$$S(x,t) = -\frac{c}{(\gamma-1)\kappa}\frac{1}{2\pi i}\left\{\sqrt{\gamma}\int_{\mathcal{C}_1}\frac{1}{\zeta^2+\frac{\gamma}{\gamma-1}}\, e^{\zeta^2 T-\sqrt{\gamma}X\zeta}\, d\zeta - \int_{\mathcal{C}_2}\frac{1}{\zeta^2+\frac{1}{\gamma-1}}\, e^{(\zeta^2-1)T-\frac{\zeta^2-1}{\zeta}X}\, d\zeta\right\} \quad (11.15c)$$

$$p(x,t) = \frac{c}{\kappa}\frac{1}{2\pi i}\int_{\mathcal{C}_2} e^{(\zeta^2-1)T-\frac{\zeta^2-1}{\zeta}X}\, d\zeta \quad (11.15d)$$

where $\mathcal{C}_1$ and $\mathcal{C}_2$ are the transformed contours of the original path by the transformations (11.13) and (11.14) respectively. They may be

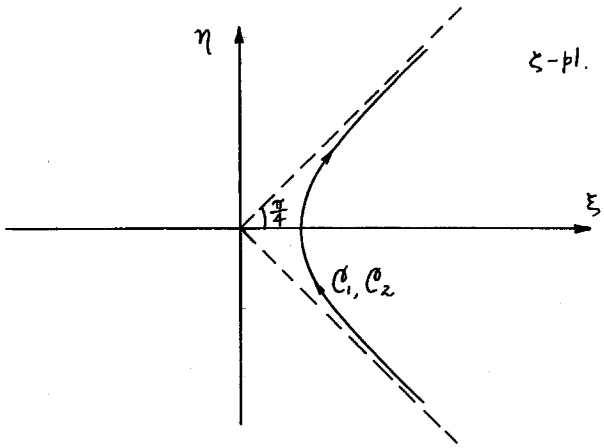



Fig. 7

shown as in Fig. 7. In the integrand of these integrals, $e^{-\frac{\zeta^2-1}{\zeta}X}$ has an essential singularity at $\zeta = 0$ ; $\left(\zeta^2+\frac{1}{\gamma-1}\right)^{-1}$ has two simple poles on the imaginary axis at $\zeta = \pm i\sqrt{\frac{1}{\gamma-1}}$ ; and $\left(\zeta^2+\frac{\gamma}{\gamma-1}\right)^{-1}$ has two simple poles at $\zeta = \pm i\sqrt{\frac{\gamma}{\gamma-1}}$ . An investigation of the behavior of the integrands for large values of $|\zeta|$ shows that contours $\mathcal{C}_1$ , $\mathcal{C}_2$ may be deformed only in such a way that they start from $\infty e^{-i\delta}$ and end up at $\infty e^{i\delta}$ where $\frac{3\pi}{4} > \delta > \frac{\pi}{4}$ . It may be pointed out here that Eqs. (11.15) are still exact integral representations for $\theta$ , $u$ , $S$ and $p$ and that they still satisfy the fundamental system of equations (11.1). From these expressions one can obtain the asymptotic formulas suitable for $T$ large or small.

## 11.1 Asymptotic Formulas Suitable for Large Values of $T$

In Eqs. (11.15) there are two types of integrals, namely

$$\mathcal{J}_1(x,t) = \frac{1}{2\pi i}\int_{\mathcal{C}_1} f(\zeta)\, e^{\zeta^2 T - \sqrt{\gamma} X \zeta}\, d\zeta \tag{11.16a}$$

$$\mathcal{J}_2(x,t) = \frac{1}{2\pi i}\int_{\mathcal{C}_2} g(\zeta)\, e^{(\zeta^2-1)T - \frac{\zeta^2-1}{\zeta} X}\, d\zeta \tag{11.16b}$$

It is desired here to find asymptotic expressions for $\mathcal{J}_1$ and $\mathcal{J}_2$ for $T$ large. Now for $\mathcal{J}_1$, it is convenient to apply a modification of the method of steepest descent as described in §6.1. It is easy to see that $\zeta = 0$ is the only saddle point through which the path $\mathcal{C}_1$ must go. This is possible because $f(\zeta) = \zeta^a\left(\zeta^2 + \frac{\gamma}{\gamma-1}\right)^{-1}$ ( $a = 0$, or $1$ ) is regular in the neighborhood of $\zeta = 0$. The next step is to deform the contour $\mathcal{C}_1$ such that both (i) $\mathcal{J}m\,(\zeta^2) =$ constant and (ii) $R\ell(\zeta) = 0$ on the path are satisfied. The requirement of the second condition is explained in §6.1. These two conditions lead to a conclusion that the imaginary axis is the only path of the steepest descent. Hence, we deform $\mathcal{C}_1$ into the whole imaginary axis, with two indentations at two simple poles of $f(\zeta)$, $\zeta = \pm i\sqrt{\frac{\gamma}{\gamma-1}}$, to the right half-plane. Then, with the exception of the contributions from these two indentations, the most of the contribution of the integral comes from the neighborhood of the saddle point $\zeta = 0$. Hence, if we write

$$f(\zeta) = \zeta^a f_1(\zeta) \qquad (a = 0,\ \text{or}\ 1\ ); \qquad f_1(\zeta) = \left(\zeta^2 + \frac{\gamma}{\gamma-1}\right)^{-1} \tag{11.17}$$

then near $\zeta = 0$, we may approximate $f_1$ as

$$f_1(\zeta) = f_1(0) + O(\zeta^2) = \frac{\gamma-1}{\gamma} + O(\zeta^2) \tag{11.18}$$

Hence the integral $\mathcal{J}_1$ can be asymptotically approximated by

$$\mathcal{J}_1(x,t) \sim \frac{f_1(0)}{2\pi i}\int_{\mathcal{C}_1} \zeta^a e^{\zeta^2 T - \sqrt{\gamma} X \zeta} d\zeta + \frac{1}{2}\left(\Sigma \text{ residues at } \pm i\sqrt{\tfrac{\gamma}{\gamma-1}}\right) \tag{11.19}$$

The value of this integral is given in Appendix D. Hence, we have

$$\mathcal{J}_1(x,t) = \frac{\gamma-1}{2\gamma}\frac{1}{\sqrt{\pi T}}\, e^{-\frac{\gamma X^2}{4T}} + O\left(\frac{1}{T}\right) \qquad \text{for} \qquad a=0 \tag{11.20a}$$

$$= \frac{\gamma-1}{4}\frac{X}{\sqrt{\gamma\pi T^3}}\, e^{-\frac{\gamma X^2}{4T}} + O\left(\frac{1}{T^2}\right) \qquad \text{for} \qquad a=1 \tag{11.20b}$$

The estimation of error term is similar to that given in Appendix C, and the contribution due to residues is included in the error term. These formulas are important in the region where $X^2/T$ is small.

For the integral $\mathcal{J}_2$ it is convenient to use a method of approximation similar to that described in §6.1, Case (ii). First we choose a contour which crosses the real axis to the right of the essential singularity $\zeta = 0$ in such a way that the quantity $\frac{\zeta^2-1}{\zeta}$ is purely imaginary on the path. By writing $\zeta = \xi + i\eta$, the condition $Re\left(\frac{\zeta^2-1}{\zeta}\right) = 0$ gives

$$\xi\,(\xi^2+\eta^2-1) = 0 \tag{11.21a}$$

so that

$$\xi = 0 \qquad or \qquad \xi^2+\eta^2 = 1 \tag{11.21b}$$

Since the line $\xi = 0$ passes through the essential singularity $\zeta = 0$, we have to use both conditions in (11.21b) by combining them in a manner as shown in Fig. 8. The contour $\mathcal{C}_2$ goes from $-i\infty$ along the line $\xi = 0$ to $-i$, with an indentation at the simple pole $\zeta = -i\sqrt{\frac{1}{\gamma-1}}$ to the right half-plane, and then goes on the semi-circle $BCD$ of unit radius in the right half-plane to $+i$, from $+i$ proceeds to $+i\infty$

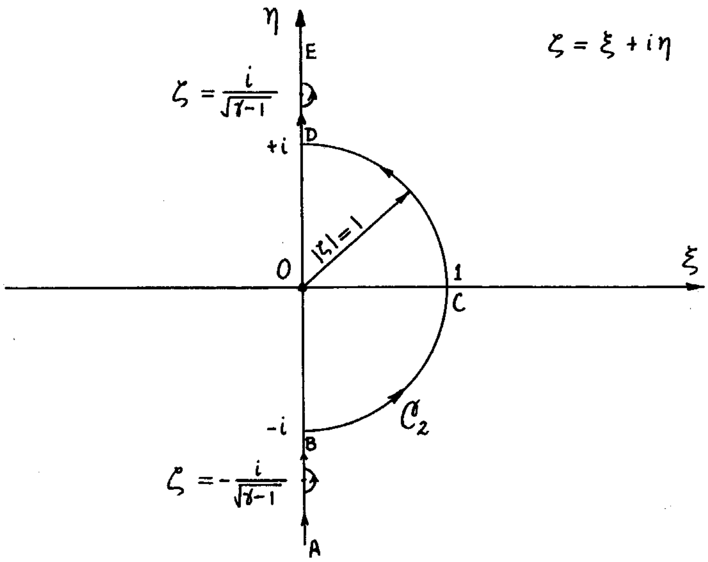



Fig. 8

along $\xi = 0$ , again with another indentation at $\zeta = +i\sqrt{\frac{1}{\gamma-1}}$ .

If the contribution of the integral on $AB$ and $DE$ is denoted by $\mathcal{J}_{AB}$ and $\mathcal{J}_{DE}$ respectively, it can be shown that

$$|\mathcal{J}_{AB}, \mathcal{J}_{DE}| \leqslant \frac{1}{2\pi}\int_1^\infty |g(i\eta)| e^{-(\eta^2+1)T} d\eta + \frac{1}{2}\left(\Sigma \text{ residue at } \pm i\sqrt{\frac{1}{\gamma-1}}\right) \tag{11.22}$$

Now we have

$$g(\zeta) = \zeta^b \left(\zeta^2 + \frac{1}{\gamma-1}\right)^{-1} \qquad (b = 0, 1 \text{ or } 2 \text{ for } s, u \text{ or } \theta) \tag{11.23a}$$

$$= 1 \qquad (\text{for } p) \tag{11.23b}$$

After some detailed manipulation it can be shown that, for all cases

$$|\mathcal{J}_{AB}, \mathcal{J}_{DE}| < \frac{A}{T} e^{-2T} \tag{11.24}$$

$A$ being a constant. On the semi-circle $BCD$ , it can also be shown that most of the contribution of the integral comes from the neighborhood of the point $\zeta = 1$ . This point involves some physical significance similar to that mentioned in §6.1.

If we denote the contribution of the integral on $BCD$ by $\mathcal{J}_{BD}$ and choose a convenient parameter such that

$$\eta(\epsilon) = \epsilon \tag{11.25a}$$

and

$$\xi(\epsilon) = \sqrt{1-\epsilon^2} \qquad \text{(positive branch being chosen)} \tag{11.25b}$$

then $\mathcal{J}_{BD}$ may be expressed parametrically by

$$\mathcal{J}_{BD} = \frac{1}{2\pi i}\int_{-1}^{+1} g(-2\epsilon^2 + 2i\epsilon\,\xi(\epsilon))\, e^{-[2\epsilon^2 + 2i\epsilon\,\xi(\epsilon)]T - 2i\epsilon X} (\xi' + i)\, d\epsilon \tag{11.26a}$$

which can be asymptotically approximated as

$$\mathcal{J}_{BD} \sim \frac{g(\epsilon=0)}{2\pi}\int_{-\infty}^{\infty} e^{-2\epsilon^2 T + 2i(T-X)\epsilon}\, d\epsilon$$

or

$$\mathcal{J}_{BD} = \frac{g(\zeta=1)}{2\sqrt{2\pi T}}\, e^{-\frac{(X-T)^2}{2T}} + O\left(\frac{1}{T}\right) \tag{11.26b}$$

This formula is important only in the region where $(X-T)^2/T$ is small. The estimation of the error term may be obtained by a calculation similar to that described in Appendix C. Therefore, we have the asymptotic expressions of the fundamental solutions for $\frac{c^2 t}{\kappa}$ large, with $P_r = 3/4$, as follows:

$$\theta(x,t) = \frac{1}{\gamma}\frac{1}{\sqrt{4\pi\frac{\kappa}{\gamma}t}}\, e^{-\frac{\gamma x^2}{4\kappa t}} + \frac{\gamma-1}{2\gamma}\frac{1}{\sqrt{2\pi\kappa t}}\, e^{-\frac{(|x|-ct)^2}{2\kappa t}} + O\left(\frac{1}{ct}\right) \tag{11.27a}$$

$$u(x,t) = -\frac{x}{2\gamma\sqrt{4\pi\frac{\kappa}{\gamma}t^3}}\, e^{-\frac{\gamma x^2}{4\kappa t}} + \frac{c}{2\gamma}\frac{\operatorname{sign} x}{\sqrt{2\pi\kappa t}}\, e^{-\frac{(|x|-ct)^2}{2\kappa t}} + O\left(\frac{1}{t}\right) \tag{11.27b}$$

$$S(x,t) = -\frac{1}{\gamma}\frac{1}{\sqrt{4\pi\frac{\kappa}{\gamma}t}}\, e^{-\frac{\gamma x^2}{4\kappa t}} + \frac{1}{2\gamma}\frac{1}{\sqrt{2\pi\kappa t}}\, e^{-\frac{(|x|-ct)^2}{2\kappa t}} + O\left(\frac{1}{ct}\right) \tag{11.27c}$$

$$p(x,t) = \frac{1}{2}\frac{1}{\sqrt{2\pi\kappa t}}\, e^{-\frac{(|x|-ct)^2}{2\kappa t}} + O\left(\frac{1}{ct}\right) \tag{11.27d}$$

Discussion of Results

Comparing these formulas with those of the non-viscous case (6.26) several interesting points may be noted as follows:

(i) For $\frac{c^2 t}{\kappa}$ large and $P_\lambda = 3/4$, the role of the additional viscosity gives exactly the same asymptotic expressions for $\theta$, $u$, and $s$ near the origin as the non-viscous case, but it rules out completely the pressure field $p$ near $x=0$ where the heat is introduced. For the non-viscous case $p$ is shown to be of negligible value there.

(ii) For $P_\lambda = 3/4$, the effective coefficient of diffusion about the origin is $\kappa/\gamma$, which is the same as the non-viscous case. The effective diffusivity about the lines $x = \pm ct$ is $\kappa/2$, which is independent of $\gamma$ in the first place, and secondly, is greater than that of the non-viscous case, $\frac{\gamma-1}{2\gamma}\kappa$ ($= \frac{1}{7}\kappa$ for air). This would imply that the viscosity has its property to increase the rate of heat diffusion about the usual lines of propagation of sound waves.

(iii) As far as the heat energy distribution is concerned, the integration of the linearized energy equation (11.2a) throughout space and time again gives the relation

$$\int_{-\infty}^{\infty} \theta(x,\tau)\,dx = 1 \qquad \text{for} \quad \Omega(x,t) = \delta(x)\,\delta(t), \ \text{and} \ \tau > 0, \tag{11.28}$$

Substitution of (11.27) into (11.28) shows that the heat energy distribution is exactly the same as the non-viscous case, that is, $\frac{1}{\gamma}$ of the total heat diffuses about the origin and $(1 - \frac{1}{\gamma})$ of the total heat diffuses about the lines $x = \pm ct$ .

(iv) The relative magnitude between $\theta$ , $u$ , $S$ and $p$ along $x = \pm ct$ is the same as the non-viscous case (cf. Eq. (7.3)).

(v) As $k \to 0$ , $\mu \to 0$ , but keeping $Pr = 3/4$ , these disturbances approach the same limiting forms as the non-viscous case to their first order terms.

(vi) For $Pr = 3/4$ , so that $\kappa = \frac{4}{3}\gamma\nu$ , we may replace $\kappa$ by $\frac{4}{3}\gamma\nu$ in solutions (11.27), but the forms in terms of $\kappa$ are more convenient for comparison.

It may be remarked here that all of these results verify the statement made in §7 (vi).

11.2 Approximated Solutions Suitable for Small Values of $T$

Following the scheme described in §6.2, we first apply the transformation

$$z = \sqrt{T}\,\zeta \tag{11.29}$$

to the integral representation of solutions (11.15). We then have

$$\theta(x,t) = \frac{c}{\kappa}\frac{1}{2\pi i}\left\{\frac{\sqrt{\gamma}}{\gamma-1}\int_{C_1}\frac{\sqrt{T}}{z^2+\frac{\gamma T}{\gamma-1}}e^{z^2-\sqrt{\gamma}\frac{X}{\sqrt{T}}z}\,dz + \frac{e^{-T}}{\sqrt{T}}\int_{C_2}\frac{z^2 e^{z^2-\frac{X}{\sqrt{T}}\frac{z^2-T}{z}}}{z^2+\frac{T}{\gamma-1}}\,dz\right\} \tag{11.30a}$$

$$u(x,t) = -\frac{c^2\,\mathrm{sign}\,x}{(\gamma-1)\kappa}\frac{1}{2\pi i}\left\{\int_{C_1}\frac{z\exp\left[z^2-\sqrt{\gamma}\frac{X}{\sqrt{T}}z\right]}{z^2+\frac{\gamma T}{\gamma-1}}\,dz - e^{-T}\int_{C_2}\frac{z\exp\left[z^2-\frac{X}{\sqrt{T}}\frac{z^2-T}{z}\right]}{z^2+\frac{T}{\gamma-1}}\,dz\right\} \tag{11.30b}$$

with the corresponding formulas for $S$ and $p$ . We can now approximate these integrals for $T$ very small. For terms like $(z^2+AT)^{-1}$ , $A$ being a constant, we may expand it into a convergent series in $T$ for sufficiently small $T$ so that

$$(z^2+AT)^{-1} = z^{-2}\left[1+O\left(\frac{T}{z^2}\right)\right], \quad \text{if } \frac{T}{|z^2|}<1 \quad \text{for all } z \text{ on } C_1, C_2. \tag{11.31}$$

And for the term $exp\left[-\frac{X}{\sqrt{T}}\frac{z^2-T}{z}\right]$, we may write

$$exp\left[-\frac{X}{\sqrt{T}}\frac{z^2-T}{z}\right] = exp\left[-\frac{X}{\sqrt{T}}z\right] g(z,T)\,; \qquad g(z,T) = exp\left(\frac{X\sqrt{T}}{z}\right) \tag{11.32}$$

Again, for the same reason concerning the rate of attenuation as stated in §6.2, $X = O(\sqrt{T})$ for $T$ small, so we may assume

$$\frac{X}{\sqrt{T}} \longrightarrow \alpha \quad (\text{a constant}) \qquad \text{as} \quad T \longrightarrow 0 \tag{11.33}$$

Then (11.32) takes the value

$$exp\left[-\frac{X}{\sqrt{T}}\frac{z^2-T}{z}\right] = e^{-\alpha z}\left(1+O\left(\frac{T}{z}\right)\right) \qquad \text{as} \quad T \to 0 \tag{11.34}$$

if $\frac{T}{|z|} < 1$ for all $z$ on $\mathcal{C}_1$, $\mathcal{C}_2$. From the imposed conditions required in (11.31) and (11.34) we may deform $\mathcal{C}_1$, $\mathcal{C}_2$ into a path parallel to the imaginary axis, to the right of the origin such that on the path $\min(|z|, |z^2|) \gg T$ so that instead of $\left[1+O\left(\frac{T}{|z|}\right)\right]$ and $\left[1+O\left(\frac{T}{|z^2|}\right)\right]$, we can write $\left[1+O(T)\right]$ on the path. Therefore, for very small values of $T$, we have

$$\theta(x,t) = \frac{c}{\kappa}\frac{1}{2\pi i}\left\{\frac{\sqrt{\gamma T}}{\gamma-1}\int_{\mathcal{C}}\frac{1}{z^2}e^{z^2-\sqrt{\gamma}\alpha z}dz + \frac{1}{\sqrt{T}}\int_{\mathcal{C}}e^{z^2-\alpha z}dz\right\}\left[1+O(T)\right] \tag{11.35a}$$

$$u(x,t) = -\frac{c^2 \operatorname{sign} x}{(\gamma-1)\kappa}\frac{1}{2\pi i}\left\{\int_{\mathcal{C}}\frac{1}{z}e^{z^2-\sqrt{\gamma}\alpha z}dz - \int_{\mathcal{C}}\frac{1}{z}e^{z^2-\alpha z}dz\right\}\left[1+O(T)\right] \tag{11.35b}$$

again with corresponding formulas for $S$ and $p$, which give (cf. Appendic D)

$$\theta(x,t) = \frac{1}{\sqrt{4\pi\kappa t}}\,e^{-\frac{x^2}{4\kappa t}}\left[1+O\left(\frac{c^2 t}{\kappa}\right)\right] \tag{11.36a}$$

$$u(x,t) = \frac{c^2 \operatorname{sign} x}{2(\gamma-1)\kappa}\left[\operatorname{erfc}\frac{|x|}{2\sqrt{\kappa t}} - \operatorname{erfc}\frac{|x|}{2\sqrt{\frac{\kappa}{\gamma}t}}\right]\left[1+O\left(\frac{c^2 t}{\kappa}\right)\right] \tag{11.36b}$$

$$S(x,t) = \frac{c^2}{2(\gamma-1)\kappa^2}\left\{2\sqrt{\frac{\kappa t}{\pi}}\left[e^{-\frac{x^2}{4\kappa t}} - e^{-\frac{\gamma x^2}{4\kappa t}}\right] + |x|\left[\gamma\operatorname{erfc}\frac{|x|}{2\sqrt{\frac{\kappa}{\gamma}t}} - \operatorname{erfc}\frac{|x|}{2\sqrt{\kappa t}}\right]\right\}\left[1+O\left(\frac{c^2 t}{\kappa}\right)\right] \tag{11.36c}$$

$$p(x,t) = \frac{1}{\sqrt{4\pi\kappa t}}\, e^{-\frac{x^2}{4\kappa t}} \left[1 + O\left(\frac{c^2 t}{\kappa}\right)\right] \tag{11.36d}$$

Discussion of Results

Comparing these solutions (11.36) with the corresponding non-viscous solutions (6.31), it can be seen that $\theta$ and $p$ have exactly the same expression to the first order of approximation; both diffuse about the origin with conductivity $\kappa$. However, there is a slight difference for $u$ and $s$; in the non-viscous case we have only the unmodified conductivity $\kappa$, while in the viscous case there are two apparent conductivities $\kappa$ and $\kappa/\gamma$. Terms with $\kappa/\gamma$ have a counter effect against those with $\kappa$. However, the qualitative behavior is the same in both cases so that for $T$ small, heat conduction is the dominant process and changes near the origin are mainly in temperature and pressure. The important point is that the viscosity does not change the temperature and pressure, which are the two major ones, to their first order of magnitude. This seems to imply that they are independent of Prandtl number.

These approximated formulas can be shown to satisfy the fundamental equations (11.1) asymptotically; and $\theta$ to satisfy the energy relation (11.28).

As a remark, a perturbation may be applied to these equations with respect to $P_r$ when $P_r$ is very close to $3/4$.

# PART III

# HEAT CONDUCTION IN A TWO-DIMENSIONAL STATIONARY FLOW OF A VISCOUS COMPRESSIBLE FLUID AND THE APPLICATION TO ANEMOMETRY OF A HEATED FLAT PLATE

## § 12. Linearized Equations and Their Characteristics

The significant equations governing the motion of a two-dimensional stationary flow with the presence of heat addition and external forces can be obtained by a method mentioned in §5.4. This system of equations is then linearized in a manner similar to previous cases. For a flow whose velocity at upstream infinity is $U$ parallel to the $x$-axis, the resulting linearized equations read:

State $\qquad p = s + \theta \qquad$ (12.1a)

Continuity $\qquad U s_x + \operatorname{div} \vec{q} = 0 \qquad$ (12.1b)

Momentum $\qquad U \vec{q}_x = -\frac{c^2}{\gamma} \operatorname{grad} p + \frac{4}{3} \nu \operatorname{grad} \operatorname{div} \vec{q} - \nu \operatorname{curl} \operatorname{curl} \vec{q} + \vec{X} \qquad$ (12.1c)

Energy $\qquad U \theta_x - \kappa \Delta \theta = U(\gamma - 1) s_x + \Omega \qquad$ (12.1d)

where

$$\operatorname{grad} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}, \quad \operatorname{grad} \operatorname{div} = T_1 = \begin{pmatrix} \frac{\partial^2}{\partial x^2} & \frac{\partial^2}{\partial x \partial y} \\ \frac{\partial^2}{\partial x \partial y} & \frac{\partial^2}{\partial y^2} \end{pmatrix}, \quad \operatorname{curl} \operatorname{curl} = T_2 = \begin{pmatrix} -\frac{\partial^2}{\partial y^2} & \frac{\partial^2}{\partial x \partial y} \\ \frac{\partial^2}{\partial x \partial y} & -\frac{\partial^2}{\partial x^2} \end{pmatrix}$$

and

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \qquad \vec{q} = \begin{pmatrix} u \\ v \end{pmatrix}$$

In Eqs. (12.1), $\Omega$ and $\vec{X}$ are assumed to be independent of time.

If the characteristics of this system are considered, it can be shown that for $\kappa \neq 0$, $\nu \neq 0$, no real characteristics exist. This means that any discontinuity of physical quantities is smoothed out instantaneously by the viscous and heat-conducting effects. If we consider for a moment a special case $\mu = 0$ , $k = 0$ , the order of the momentum and the energy equations is each lowered by one, and the characteristic condition then becomes

$$\psi_x^2 \left[ \psi_y^2 - (M^2 - 1)\psi_x^2 \right] = 0 \quad ; \qquad M = \frac{U}{c} \tag{12.2}$$

where $\psi(x,y)$ = constant denotes a characteristic. Hence the characteristics in this case are:

(i) $\psi_x^2 = 0$ or $y = \text{const.}$ which occurs double (12.3a)

This set indicates possible discontinuities across the linearized streamlines $y$ = constant.

(ii) $\psi_y^2 - (M^2-1)\psi_x^2 = 0,$ or $\left(\frac{dy}{dx}\right)_\psi = \pm \frac{1}{\sqrt{M^2-1}}$. (12.3b)

This set is imaginary for subsonic flow $M < 1$ ; but becomes real for supersonic flow $M > 1$ , which then represents the familiar Mach lines. Across these lines temperature, density and even velocity may jump (because $\mu = 0$ , $k = 0$).

These characteristics (i) and (ii) must play an important role in the solutions for small values of $\mu$ and $k$ . In other words, a two-dimensional disturbance introduced at a point inside a fluid of small viscosity and heat conductivity tends to propagate downstream more concentrated along the streamline and Mach lines (if $M > 1$ ) through that point, and at the same time diffuses about these lines (due to the non-vanishing $\mu$ and $k$ even though they are very small). This interesting point can be seen more clearly from the solutions given explicitly later.

§ 13. Representation of Fundamental Solutions in Fourier Transform

For the definition of fundamental solutions in this problem reference is made to §4 and §5.4. As the effects due to $\Omega$ and $\vec{X}$ are independent of each other, we may treat them separately. If we denote the fundamental solutions of the temperature and velocity field by $\Gamma_\Omega$, $\vec{\Psi}_\Omega$ due to $\Omega$ and $\vec{\Gamma}_X$ and $\Psi_X$ due to $\vec{X}$ (with corresponding notations for density and pressure), then when $\Omega$ and $\vec{X}$ are arbitrary functions of $(x,y)$, the solutions to (12.1) can be represented by:

$$\theta(x,y) = \iint_{-\infty}^{\infty} \left\{ \Gamma_\Omega(x-\xi, y-\eta)\,\Omega(\xi,\eta) + \vec{\Gamma}_X(x-\xi, y-\eta)\,\vec{X}(\xi,\eta) \right\} d\xi\, d\eta \tag{13.1a}$$

$$\vec{q}(x,y) = \iint_{-\infty}^{\infty} \left\{ \vec{\Psi}_\Omega(x-\xi, y-\eta)\,\Omega(\xi,\eta) + \Psi_X(x-\xi, y-\eta)\,\vec{X}(\xi,\eta) \right\} d\xi\, d\eta \tag{13.1b}$$

$$s(x,y) = \iint_{-\infty}^{\infty} \left\{ \Sigma_\Omega(x-\xi, y-\eta)\,\Omega(\xi,\eta) + \vec{\Sigma}_X(x-\xi, y-\eta)\,\vec{X}(\xi,\eta) \right\} d\xi\, d\eta \tag{13.1c}$$

$$p(x,y) = \iint_{-\infty}^{\infty} \left\{ \Pi_\Omega(x-\xi, y-\eta)\,\Omega(\xi,\eta) + \vec{\Pi}_X(x-\xi, y-\eta)\,\vec{X}(\xi,\eta) \right\} d\xi\, d\eta. \tag{13.1d}$$

where $\Gamma_\Omega$, $\Sigma_\Omega$, $\Pi_\Omega$ are scalars; $\vec{\Gamma}_X$, $\vec{\Psi}_\Omega$, $\vec{\Sigma}_X$, $\vec{\Pi}_X$ are vectors and $\Psi_X$ is a two-by-two tensor. These fundamental solutions can also be used to build up the solutions of boundary value problems as will be shown later in the application to the anemometry problem.

The procedure adopted here is similar to that in §5.4, that is, first to eliminate the independent variable $x$ by applying Fourier transformation to the system (12.1) and thus reduce the system to ordinary differential equations in $y$ only. After this the method of integration in the complex plane is used to obtain an integral representation of the fundamental solutions from which asymptotic formulas can then be derived. The Fourier transform $\tilde{f}$ of a function $f(x,y)$ with respect to $x$ and its inverse transform is given by (5.24).

Application of Fourier transformation to system (12.1) with the condition that all perturbations vanish as $x \longrightarrow \pm\infty$, gives

$$\tilde{p} = \tilde{S} + \tilde{\theta} \tag{13.2a}$$

$$i\beta U\,\tilde{S} + \widetilde{div}\,\tilde{\vec{q}} = 0 \tag{13.2b}$$

$$\frac{c^2}{\gamma}\,\widetilde{grad}\;\tilde{\theta} = \left[\left(\frac{4}{3}\nu + \frac{c^2}{\gamma}\frac{1}{i\beta U}\right)\tilde{T}_1 - \nu\,\tilde{T}_2 - i\beta U\right]\tilde{\vec{q}} + \tilde{\vec{X}} \tag{13.2c}$$

$$(-i\beta U + \kappa\,\tilde{\Delta})\,\tilde{\theta} = (\gamma-1)\,\widetilde{div}\,\tilde{\vec{q}} - \tilde{\Omega} \tag{13.2d}$$

where all wavy barred operators are those obtained by replacing $\frac{\partial}{\partial x}$ by $i\beta$ in their original forms.

Transforming (13.1) and using the convolution theorem, we obtain

$$\tilde{\theta}(\beta,y) = \sqrt{2\pi}\int_{-\infty}^{\infty}\left\{\tilde{\Gamma}_\Omega(\beta,y-\eta)\,\tilde{\Omega}(\beta,\eta) + \tilde{\vec{\Gamma}}_X(\beta,y-\eta)\,\tilde{\vec{X}}(\beta,\eta)\right\}d\eta \tag{13.3a}$$

$$\tilde{\vec{q}}(\beta,y) = \sqrt{2\pi}\int_{-\infty}^{\infty}\left\{\tilde{\vec{\Psi}}_\Omega(\beta,y-\eta)\,\tilde{\Omega}(\beta,\eta) + \tilde{\Psi}_X(\beta,y-\eta)\,\tilde{\vec{X}}(\beta,\eta)\right\}d\eta \tag{13.3b}$$

and corresponding formulas for $\tilde{S}(\beta,y)$ and $\tilde{p}(\beta,y)$. Eqs. (13.3) tell us that the Fourier transform $\tilde{\Gamma}_\Omega$, $\tilde{\vec{\Gamma}}_X$ etc. are themselves fundamantal solutions to the transformed system (13.2) divided by $\sqrt{2\pi}$. In order to find these fundamental solutions in Fourier transform the system (13.2) is reduced to a single higher order equation for each of $\tilde{\theta}$, $\tilde{\vec{q}}$, $\tilde{S}$ and $\tilde{p}$. The resulting equations can be greatly simplified by choosing the Prandtl number $Pr = 3/4$. Under this assumption, our equations then become as follows:

$$L_2(\tilde{\theta}) = -\frac{1}{\kappa}\left\{\frac{\partial^2}{\partial y^2} - \beta^2\frac{(1-\gamma M^2)+i\alpha\beta}{1+i\alpha\beta}\right\}\tilde{\Omega} - \frac{\gamma(\gamma-1)\,i\alpha\beta}{\kappa^2(1+i\alpha\beta)}\,\widetilde{div}\,\tilde{\vec{X}} = -F_\theta(y) \tag{13.4}$$

where $L_2$ is the operator

$$L_2 \equiv \left(\frac{\partial^2}{\partial y^2} - \lambda_1\right)\left(\frac{\partial^2}{\partial y^2} - \lambda_2\right) \tag{13.5a}$$

$$\lambda_1 = \beta^2 \frac{1-M^2+ia\beta}{1+ia\beta}, \qquad \lambda_2 = \beta^2 + \frac{i\gamma U\beta}{\kappa}, \qquad a = \frac{M^2\kappa}{U}. \tag{13.5b}$$

The equation for the velocity is slightly more complicated, since an equation of order higher than four is involved. However, it can be expressed most simply by splitting $\vec{q}$ into four parts

$$\vec{q} = \vec{q}_\Omega + \vec{q}_X = \vec{q}_\Omega + \vec{q}_1^{\,*} + \vec{q}_2^{\,*} + \vec{q}_3^{\,*} \tag{13.6a}$$

in such a way that their transforms satisfy

$$L_2(\tilde{\vec{q}}_\Omega) = -\frac{i\frac{U}{\kappa}\beta}{1+ia\beta}\, \widetilde{grad}\, \tilde{\Omega} \tag{13.6b}$$

$$L_2(\tilde{\vec{q}}_1^{\,*}) = \frac{\gamma}{\kappa}\frac{ia\beta}{1+ia\beta}\left(\beta^2 + \frac{iU\beta}{\kappa} - \frac{\partial^2}{\partial y^2}\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\tilde{\vec{X}} \tag{13.6c}$$

$$L_2(\tilde{\vec{q}}_2^{\,*}) = \frac{i\beta}{\nu}\frac{1+\frac{1}{4}ia\beta}{1+ia\beta}\begin{pmatrix} -\frac{1}{i\beta}\frac{\partial^2}{\partial y^2} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & -i\beta \end{pmatrix}\tilde{\vec{X}} \tag{13.6d}$$

and

$$L_3(\tilde{\vec{q}}_3^{\,*}) = \frac{U}{4\nu^2}(i\beta)^2\frac{1+(1-\frac{3}{4\gamma})ia\beta}{1+ia\beta}\begin{pmatrix} -\frac{1}{i\beta}\frac{\partial^2}{\partial y^2} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & -i\beta \end{pmatrix}\tilde{\vec{X}} = -F_3(y) \tag{13.6e}$$

where

$$L_3 \equiv \left(\frac{\partial^2}{\partial y^2} - \lambda_1\right)\left(\frac{\partial^2}{\partial y^2} - \lambda_2\right)\left(\frac{\partial^2}{\partial y^2} - \lambda_3\right), \qquad \lambda_3 = \beta^2 + \frac{iU\beta}{\nu} \tag{13.6f}$$

Furthermore, the equations for density and pressure perturbations are

$$L_2(\tilde{s}) = \frac{1}{\kappa(1+ia\beta)}\left(\frac{\partial^2}{\partial y^2} - \beta^2\right)\tilde{\Omega} + \frac{\frac{\gamma M^2}{U^2}}{1+ia\beta}\left[\frac{\partial^2}{\partial y^2} - \left(\beta^2 + \frac{iU\beta}{\kappa}\right)\right]\widetilde{div}\,\tilde{\vec{X}} \tag{13.7}$$

$$L_1(\tilde{p}) = -\frac{ia\beta}{\kappa(1+ia\beta)}\tilde{\Omega} + \frac{\gamma M^2}{U^2}\frac{1}{1+ia\beta}\widetilde{div}\,\tilde{\vec{X}} = -F_p(y) \tag{13.8a}$$

where

$$L_1 \equiv \left(\frac{\partial^2}{\partial y^2} - \lambda_1\right) \tag{13.8b}$$

The factoring of these operators $L_2$ and $L_3$ again permits us to represent these fundamental solutions as a combination of fundamental solutions of second order equations as indicated in Appendix A. Applying the theorems in Appendix A we have the following statement:

The fundamental solution $G^{(1)}$ of (13.8a) defined as the solution vanishing at infinity when $F_p(y) = \delta(y)$ is known as

$$G^{(1)}(y) = \frac{1}{2\sqrt{\lambda_1}} e^{-\sqrt{\lambda_1}|y|} \tag{13.9}$$

The fundamental solution $G^{(2)}$ of (13.4) (as well as of (13.6b)-(13.6d) etc.) defined as the solution when $F_\theta(y) = \delta(y)$ is given by

$$G^{(2)}(y) = \frac{1}{\lambda_1 - \lambda_2}\left[\frac{1}{2\sqrt{\lambda_1}} e^{-\sqrt{\lambda_1}|y|} - \frac{1}{2\sqrt{\lambda_2}} e^{-\sqrt{\lambda_2}|y|}\right] \tag{13.10}$$

while the fundamental solution $G^{(3)}$ of (13.6e) defined by

$$\tilde{\vec{q}}_3^{\,*}(y) = \int_{-\infty}^{\infty} G^{(3)}(y-\eta)\; F_3(\eta)\, d\eta$$

is again a linear combination of $G^{(1)}$

$$G^{(3)}(y) = \frac{\frac{1}{2\sqrt{\lambda_1}} e^{-\sqrt{\lambda_1}|y|}}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} + \frac{\frac{1}{2\sqrt{\lambda_2}} e^{-\sqrt{\lambda_2}|y|}}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)} + \frac{\frac{1}{2\sqrt{\lambda_3}} e^{-\sqrt{\lambda_3}|y|}}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)} . \tag{13.11}$$

Following the definition of $G^{(2)}$ in (13.10) the solution to (13.4) may be written as

$$\tilde{\theta}(\beta,y) = \int_{-\infty}^{\infty} G^{(2)}(\beta, y-\eta)\left\{\frac{1}{K}\left[\frac{\partial^2}{\partial\eta^2} - \beta^2\frac{1-\gamma M^2 + i\alpha\beta}{1+i\alpha\beta}\right]\tilde{\Omega}(\beta,\eta) + \frac{\gamma(\gamma-1)i\alpha\beta}{K^2(1+i\alpha\beta)}\, \tilde{div}_{(\eta)}\, \tilde{\vec{X}}(\beta,\eta)\right\} d\eta$$

$$= \int_{-\infty}^{\infty}\left\{\tilde{\Omega}(\beta,\eta)\frac{1}{K}\left[\frac{\partial^2}{\partial y^2} - \beta^2\frac{1-\gamma M^2+i\alpha\beta}{1+i\alpha\beta}\right] G^{(2)}(\beta, y-\eta) + \frac{\gamma(\gamma-1)i\alpha\beta}{K^2(1+i\alpha\beta)}\, \tilde{grad}_{(y)} G^{(2)}(y-\eta)\cdot\tilde{\vec{X}}(\beta,\eta)\right\} d\eta \tag{13.12a}$$

where

$$\tilde{grad}_{(y)} = \begin{pmatrix} i\beta \\ \frac{\partial}{\partial y}\end{pmatrix}, \qquad \tilde{div}_{(\eta)}\,\tilde{\vec{X}} = \left(i\beta\,\tilde{X}_1 + \frac{\partial}{\partial\eta}\tilde{X}_2\right), \qquad \vec{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \tag{13.12b}$$

Comparing (13.12a) with the definition of $\tilde{\Gamma}_\Omega$ and $\tilde{\vec{\Gamma}}_X$ given by (13.3a) and using the form of $G^{(2)}$ given by (13.10), we obtain

$$\tilde{\Gamma}_{\Omega}(\beta, y) = \frac{1}{2\kappa\sqrt{2\pi}} \frac{1}{1+\frac{\gamma-1}{\gamma}ia\beta} \left\{ \frac{\frac{\gamma-1}{\gamma}ia\beta}{\sqrt{\lambda_1}} e^{-\sqrt{\lambda_1}|y|} + \frac{e^{-\sqrt{\lambda_2}|y|}}{\sqrt{\lambda_2}} \right\} \tag{13.13a}$$

and

$$\tilde{\vec{\Gamma}}_X(\beta, y) = -\frac{1}{\sqrt{2\pi}} \frac{\gamma-1}{2} \frac{M^2}{U^2} \frac{1}{1+\frac{\gamma-1}{\gamma}ia\beta} \begin{pmatrix} i\beta \\ \frac{\partial}{\partial y} \end{pmatrix} \left\{ \frac{e^{-\sqrt{\lambda_1}|y|}}{\sqrt{\lambda_1}} - \frac{e^{-\sqrt{\lambda_2}|y|}}{\sqrt{\lambda_2}} \right\} \tag{13.13b}$$

For the velocity field, if we follow the similar procedure to treat each component separately, we have

$$\tilde{\vec{\Psi}}_{\Omega}(\beta, y) = -\frac{1}{2\gamma\sqrt{2\pi}} \frac{1}{1+\frac{\gamma-1}{\gamma}ia\beta} \begin{pmatrix} i\beta \\ \frac{\partial}{\partial y} \end{pmatrix} \left\{ \frac{e^{-\sqrt{\lambda_1}|y|}}{\sqrt{\lambda_1}} - \frac{e^{-\sqrt{\lambda_2}|y|}}{\sqrt{\lambda_2}} \right\} = \frac{1}{\gamma(\gamma-1)} \frac{U^2}{M^2} \tilde{\vec{\Gamma}}_X \tag{13.14a}$$

$$\tilde{\Psi}_1^*(\beta, y) = \frac{1}{2\kappa\sqrt{2\pi}} \frac{ia\beta}{1+\frac{\gamma-1}{\gamma}ia\beta} \left\{ \frac{1}{1+ia\beta} \frac{e^{-\sqrt{\lambda_1}|y|}}{\sqrt{\lambda_1}} + (\gamma-1) \frac{e^{-\sqrt{\lambda_2}|y|}}{\sqrt{\lambda_2}} \right\} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{13.14b}$$

$$\begin{aligned} \tilde{\Psi}_2^*(\beta, y) = \frac{2}{3U} \frac{1}{\sqrt{2\pi}} \frac{1+\frac{1}{4}ia\beta}{1+\frac{\gamma-1}{\gamma}ia\beta} \Bigg\{ & \begin{pmatrix} i\beta & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & -i\beta \end{pmatrix} \left[ \frac{e^{-\sqrt{\lambda_1}|y|}}{\sqrt{\lambda_1}} - \frac{e^{-\sqrt{\lambda_2}|y|}}{\sqrt{\lambda_2}} \right] \\ & + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \left[ -\frac{i\beta M^2}{1+ia\beta} \frac{e^{-\sqrt{\lambda_1}|y|}}{\sqrt{\lambda_1}} + \frac{\gamma U}{\kappa} \frac{e^{-\sqrt{\lambda_2}|y|}}{\sqrt{\lambda_2}} \right] \Bigg\} \end{aligned} \tag{13.14c}$$

where the following relations have been used

$$-\frac{1}{i\beta} \frac{\partial^2}{\partial y^2} e^{-\sqrt{\lambda_1}|y|} = \left( i\beta - \frac{i\beta M^2}{1+ia\beta} \right) e^{-\sqrt{\lambda_1}|y|}$$

$$-\frac{1}{i\beta} \frac{\partial^2}{\partial y^2} e^{-\sqrt{\lambda_2}|y|} = \left( i\beta - \frac{\gamma U}{\kappa} \right) e^{-\sqrt{\lambda_2}|y|}$$

Similarly, we have

$$\begin{aligned} \tilde{\Psi}_3^*(\beta, y) = -\frac{1}{6U\sqrt{2\pi}} \frac{1}{1+\frac{\gamma-1}{\gamma}ia\beta} \Bigg\{ & \begin{pmatrix} i\beta & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & -i\beta \end{pmatrix} \left[ (1+ia\beta) \frac{e^{-\sqrt{\lambda_1}|y|}}{\sqrt{\lambda_1}} - 4\left(1+\left(1-\frac{3}{4\gamma}\right)ia\beta\right) \frac{e^{-\sqrt{\lambda_2}|y|}}{\sqrt{\lambda_2}} + 3\left(1+\frac{\gamma-1}{\gamma}ia\beta\right) \frac{e^{-\sqrt{\lambda_3}|y|}}{\sqrt{\lambda_3}} \right] \\ & - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \left[ i\beta M^2 \frac{e^{-\sqrt{\lambda_1}|y|}}{\sqrt{\lambda_1}} - \frac{4\gamma U}{\kappa}\left(1+\left(1-\frac{3}{4\gamma}\right)ia\beta\right) \frac{e^{-\sqrt{\lambda_2}|y|}}{\sqrt{\lambda_2}} + \frac{3U}{\nu}\left(1+\frac{\gamma-1}{\gamma}ia\beta\right) \frac{e^{-\sqrt{\lambda_3}|y|}}{\sqrt{\lambda_3}} \right] \Bigg\} \end{aligned} \tag{13.14d}$$

Combining Eqs. (13.14b)-(13.14d), $\tilde{\Psi}_X$ is then obtained

$$\tilde{\Psi}_x(\beta,y) = \tilde{\Psi}_1^* + \tilde{\Psi}_2^* + \tilde{\Psi}_3^*$$

$$= \frac{1}{2U\sqrt{2\pi}} \begin{pmatrix} i\beta & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & -i\beta \end{pmatrix} \left\{ \frac{1}{1+\frac{\gamma-1}{\gamma} i\alpha\beta} \left[ \frac{e^{-\sqrt{\lambda_1}|y|}}{\sqrt{\lambda_1}} + \frac{\gamma-1}{\gamma} i\alpha\beta \frac{e^{-\sqrt{\lambda_2}|y|}}{\sqrt{\lambda_2}} \right] - \frac{e^{-\sqrt{\lambda_3}|y|}}{\sqrt{\lambda_3}} \right\}$$

$$+ \frac{1}{2\nu\sqrt{2\pi}} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \frac{e^{-\sqrt{\lambda_3}|y|}}{\sqrt{\lambda_3}}$$

$$+ \frac{1}{2\kappa\sqrt{2\pi}} \frac{i\alpha\beta}{1+\frac{\gamma-1}{\gamma} i\alpha\beta} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \left[ \frac{1}{1+i\alpha\beta} \frac{e^{-\sqrt{\lambda_1}|y|}}{\sqrt{\lambda_1}} + (\gamma-1) \frac{e^{-\sqrt{\lambda_2}|y|}}{\sqrt{\lambda_2}} \right] \qquad (13.14e)$$

From these expressions it can be seen that $\vec{\Psi}_\Omega$ is a longitudinal wave (since it is a grad of a certain function) while $\Psi_2^*$, $\Psi_3^*$ represent transversal waves (since each is a curl of a certain function), but the splitting of $\Psi_1^*$ into longitudinal and transversal waves is not readily seen.

By a calculation similar to the above, we obtain, for the density and pressure fields

$$\tilde{\mathfrak{S}}_\Omega(\beta,y) = \frac{1}{2\kappa\sqrt{2\pi}} \frac{1}{1+\frac{\gamma-1}{\gamma} i\alpha\beta} \left[ \frac{1}{\gamma} \frac{i\alpha\beta}{1+i\alpha\beta} \frac{e^{-\sqrt{\lambda_1}|y|}}{\sqrt{\lambda_1}} - \frac{e^{-\sqrt{\lambda_2}|y|}}{\sqrt{\lambda_2}} \right] \qquad (13.15a)$$

$$\tilde{\vec{\mathfrak{S}}}_x(\beta,y) = -\frac{M^2}{2U^2\sqrt{2\pi}} \frac{1}{1+\frac{\gamma-1}{\gamma} i\alpha\beta} \begin{pmatrix} i\beta \\ \frac{\partial}{\partial y} \end{pmatrix} \left[ \frac{1}{1+i\alpha\beta} \frac{e^{-\sqrt{\lambda_1}|y|}}{\sqrt{\lambda_1}} + (\gamma-1) \frac{e^{-\sqrt{\lambda_2}|y|}}{\sqrt{\lambda_2}} \right] \qquad (13.15b)$$

$$\tilde{\mathfrak{P}}_\Omega(\beta,y) = \frac{1}{2\kappa\sqrt{2\pi}} \frac{i\alpha\beta}{1+i\alpha\beta} \frac{e^{-\sqrt{\lambda_1}|y|}}{\sqrt{\lambda_1}} \qquad (13.16a)$$

$$\tilde{\vec{\mathfrak{P}}}_x(\beta,y) = -\frac{\gamma M^2}{2U^2\sqrt{2\pi}} \frac{1}{1+i\alpha\beta} \begin{pmatrix} i\beta \\ \frac{\partial}{\partial y} \end{pmatrix} \frac{e^{-\sqrt{\lambda_1}|y|}}{\sqrt{\lambda_1}}. \qquad (13.16b)$$

The desired fundamental solutions can be obtained by applying the inversion formula (5.24b) to (13.12)-(13.16). The details of this procedure are given in the following section.

## § 14. Problem of Inversion of the Transformation

Application of the inversion formula (5.24b) to (13.12)-(13.16) gives

$$\Gamma_\Omega(x,y) = \frac{1}{4\pi\kappa}\int_{-\infty}^{\infty} \frac{e^{i\beta x}}{1+\frac{\gamma-1}{\gamma}ia\beta}\left[\frac{\gamma-1}{\gamma}ia\beta\,\frac{e^{-\sqrt{\lambda_1}|y|}}{\sqrt{\lambda_1}} + \frac{e^{-\sqrt{\lambda_2}|y|}}{\sqrt{\lambda_2}}\right]d\beta \tag{14.1a}$$

$$\vec{\Gamma}_X(x,y) = -\frac{\gamma-1}{4\pi}\frac{M^2}{U^2}\begin{pmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\end{pmatrix}\int_{-\infty}^{\infty}\frac{e^{i\beta x}}{1+\frac{\gamma-1}{\gamma}ia\beta}\left[\frac{e^{-\sqrt{\lambda_1}|y|}}{\sqrt{\lambda_1}} - \frac{e^{-\sqrt{\lambda_2}|y|}}{\sqrt{\lambda_2}}\right]d\beta \tag{14.1b}$$

$$\vec{\Psi}_\Omega(x,y) = \frac{1}{\gamma(\gamma-1)}\frac{U^2}{M^2}\vec{\Gamma}_X \tag{14.1c}$$

$$\Psi_X(x,y) = \frac{1}{4\pi U}\begin{pmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y}\\ \frac{\partial}{\partial y} & -\frac{\partial}{\partial x}\end{pmatrix}\int_{-\infty}^{\infty}\left\{\frac{1}{1+\frac{\gamma-1}{\gamma}ia\beta}\left[\frac{e^{-\sqrt{\lambda_1}|y|}}{\sqrt{\lambda_1}} + \frac{\gamma-1}{\gamma}ia\beta\,\frac{e^{-\sqrt{\lambda_2}|y|}}{\sqrt{\lambda_2}}\right] - \frac{e^{-\sqrt{\lambda_3}|y|}}{\sqrt{\lambda_3}}\right\}e^{i\beta x}d\beta$$

$$+\frac{1}{4\pi\nu}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\int_{-\infty}^{\infty}\frac{e^{i\beta x-\sqrt{\lambda_3}|y|}}{\sqrt{\lambda_3}}d\beta + \frac{1}{4\pi\kappa}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\int_{-\infty}^{\infty}\frac{ia\beta\, e^{i\beta x}}{1+\frac{\gamma-1}{\gamma}ia\beta}\left[\frac{e^{-\sqrt{\lambda_1}|y|}}{(1+ia\beta)\sqrt{\lambda_1}} + \frac{(\gamma-1)\,e^{-\sqrt{\lambda_2}|y|}}{\sqrt{\lambda_2}}\right]d\beta \tag{14.1d}$$

$$\mathfrak{S}_\Omega(x,y) = \frac{1}{4\pi\kappa}\int_{-\infty}^{\infty}\frac{e^{i\beta x}}{1+\frac{\gamma-1}{\gamma}ia\beta}\left[\frac{\frac{1}{\gamma}ia\beta}{1+ia\beta}\,\frac{e^{-\sqrt{\lambda_1}|y|}}{\sqrt{\lambda_1}} - \frac{e^{-\sqrt{\lambda_2}|y|}}{\sqrt{\lambda_2}}\right]d\beta \tag{14.1e}$$

$$\vec{\mathfrak{S}}_X(x,y) = -\frac{M^2}{4\pi U^2}\begin{pmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\end{pmatrix}\int_{-\infty}^{\infty}\frac{e^{i\beta x}}{1+\frac{\gamma-1}{\gamma}ia\beta}\left[\frac{1}{1+ia\beta}\,\frac{e^{-\sqrt{\lambda_1}|y|}}{\sqrt{\lambda_1}} + (\gamma-1)\,\frac{e^{-\sqrt{\lambda_2}|y|}}{\sqrt{\lambda_2}}\right]d\beta \tag{14.1f}$$

$$\mathfrak{P}_\Omega(x,y) = \frac{1}{4\pi\kappa}\int_{-\infty}^{\infty}\frac{ia\beta}{1+ia\beta}\,\frac{e^{i\beta x-\sqrt{\lambda_1}|y|}}{\sqrt{\lambda_1}}\,d\beta \tag{14.1g}$$

$$\vec{\mathfrak{P}}_X(x,y) = -\frac{\gamma M^2}{4\pi U^2}\begin{pmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\end{pmatrix}\int_{-\infty}^{\infty}\frac{1}{1+ia\beta}\,\frac{e^{i\beta x-\sqrt{\lambda_1}|y|}}{\sqrt{\lambda_1}}\,d\beta \tag{14.1h}$$

All of these integrals may be evaluated in the complex $\beta$-plane, the path is taken from $-\infty$ to $+\infty$ along the real $\beta$-axis, on which all $\sqrt{\lambda_1}$ , $\sqrt{\lambda_2}$ , $\sqrt{\lambda_3}$ are defined to take their branches having positive real parts.

Let us consider for a moment the case

$$\Omega = 0 \qquad \vec{X} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \delta(x)\,\delta(y) \tag{14.2}$$

The temperature and velocity due to this force are formally given by

$$\theta(x,y) = -\frac{\gamma-1}{4\pi}\frac{M^2}{U^2}\frac{\partial}{\partial x}\int_{-\infty}^{\infty}\frac{e^{i\beta x}}{1+\frac{\gamma-1}{\gamma}ia\beta}\left[\frac{e^{-\sqrt{\lambda_1}|y|}}{\sqrt{\lambda_1}} - \frac{e^{-\sqrt{\lambda_2}|y|}}{\sqrt{\lambda_2}}\right]d\beta \tag{14.3a}$$

$$\vec{q}(x,y) = \begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{4\pi U}\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}\int_{-\infty}^{\infty}\left\{\frac{1}{1+\frac{\gamma-1}{\gamma}ia\beta}\left[\frac{e^{-\sqrt{\lambda_1}|y|}}{\sqrt{\lambda_1}} + \frac{\gamma-1}{\gamma}ia\beta\,\frac{e^{-\sqrt{\lambda_2}|y|}}{\sqrt{\lambda_2}}\right] - \frac{e^{-\sqrt{\lambda_3}|y|}}{\sqrt{\lambda_3}}\right\}e^{i\beta x}d\beta$$

$$+\frac{1}{4\pi\nu}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\int_{-\infty}^{\infty}\frac{e^{i\beta x-\sqrt{\lambda_3}|y|}}{\sqrt{\lambda_3}}\,d\beta \tag{14.3b}$$

It is of interest to note that the $v$-component of the velocity vanishes at $y=0$, $v(x,0)=0$. Furthermore, the heat transfer in this case is proportional to $\frac{\partial}{\partial y}\theta(x,0)$, which also vanishes. Similarly, when

$$\Omega = 0 \qquad \vec{X} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \delta(x)\,\delta(y) \tag{14.4}$$

it can also be shown that $u(x,y)$ has the same expression as that of $v(x,y)$ in the case (14.2), so that it also vanishes where the force is applied. The heat transfer in this case depends on $\frac{\partial}{\partial x}\theta(x,y)$ evaluated at the singular force, which is also zero. The case (14.2) (or (14.4)) may be visualized as a very short flat plate, of zero thickness, being placed in the flow, parallel (or perpendicular in case (14.4)) to the free stream, with its length tending to zero as a limit. Then the above results may be interpreted that such a singular flat plate does not create any hot or cold spots (that is, singular heat source) in the flow, nor generate on itself a velocity perpendicular to the plate.

On the other hand, if we consider the case

$$\Omega = \delta(x)\delta(y), \qquad \vec{X} = 0 \tag{14.5}$$

then $\vec{q}$ is given as a grad of a certain quantity (longitudinal wave) and is irrotational so that there is no net force on this singular heat source. Tsien and Beilock obtained the same result in treating a single heat source with $\mu = 0$ , $k = 0$ (Ref. 5).

All the above argument tells us the fact that $\Omega$ and $\vec{X}$ are linearly independent of each other. This follows from our linearization. Physically interpreted, $\Omega$ and $\vec{X}$ lose their coupling because the dissipation term drops out from the energy equation in linearization. In actual cases, $\vec{X}$ should affect $\Omega$ through the dissipation mechanism.

To proceed with our inversion problem, we see that only two terms in (14.1d) with $e^{-\sqrt{\lambda_3}|y|}$ can be evaluated exactly as

$$\int_{-\infty}^{\infty} \frac{e^{i\beta x - \sqrt{\lambda_3}|y|}}{\sqrt{\lambda_3}} d\beta = \int_{-\infty}^{\infty} \frac{e^{i\beta x - |y|\sqrt{\beta^2 + \frac{iU}{\nu}\beta}}}{\sqrt{\beta^2 + \frac{iU}{\nu}\beta}} d\beta = 2\, e^{\frac{Ux}{2\nu}} K_0\left(\frac{U\lambda}{2\nu}\right) \tag{14.6}$$

where $\lambda = \sqrt{x^2 + y^2}$ (Ref. 29). These two terms, which do not depend on $M$ , result solely from the viscous effect. They remain unaltered if heat conduction and compressibility are neglected (for example, see the incompressible, non-heat conducting Oseen solution given in Ref. 1, Appendix Da, Eq. (98)). The rest of these integrals can be approximated very easily for some special values of $M$ , for instance $M = 0$ and $\infty$ . For general values of $M$ , no closed forms could be achieved successfully for these integrals, but one can find their asymptotic formulas for either large or small values of $\frac{U|x|}{\kappa M^2}$ and $\frac{U|y|}{\kappa M^2}$ .*

*These two terms are actually proportional to the local Reynolds number divided by $M^2$ , usually assumed to be large in boundary layer theory.

## 14.1 Asymptotic Solutions Suitable for Large Values of $\frac{U|y|}{\kappa M^2}$.

Here we have two types of integrals to treat

$$\mathcal{J}_1 = \int_{-\infty}^{\infty} \frac{1}{1+Ai\alpha\beta} \frac{e^{i\beta x - \sqrt{\lambda_1}|y|}}{\sqrt{\lambda_1}} d\beta \qquad \left( A = 1 \ \text{ or } \ \frac{\gamma-1}{\gamma} \right) \tag{14.7a}$$

$$\mathcal{J}_2 = \int_{-\infty}^{\infty} \frac{1}{1+Ai\alpha\beta} \frac{e^{i\beta x - \sqrt{\lambda_2}|y|}}{\sqrt{\lambda_2}} d\beta \tag{14.7b}$$

while other integrals can be derived from these two, for example

$$\mathcal{J}_3 = \int_{-\infty}^{\infty} \frac{i\beta}{1+Ai\alpha\beta} \frac{e^{i\beta x - \sqrt{\lambda_1}|y|}}{\sqrt{\lambda_1}} d\beta = \frac{\partial}{\partial x} \mathcal{J}_1 \tag{14.7c}$$

$$\mathcal{J}_4 = \int_{-\infty}^{\infty} \frac{i\beta}{1+Ai\alpha\beta} \frac{e^{i\beta x - \sqrt{\lambda_2}|y|}}{\sqrt{\lambda_2}} d\beta = \frac{\partial}{\partial x} \mathcal{J}_2 \tag{14.7d}$$

By introducing non-dimensional quantities

$$\lambda = a\beta, \qquad X = \frac{x}{a} = \frac{Ux}{\kappa M^2}, \qquad Y = \frac{|y|}{a} = \frac{U|y|}{\kappa M^2} \quad (Y \geq 0) \tag{14.8}$$

Eqs. (14.7a) and (14.7b) become

$$\mathcal{J}_1 = \int_{-\infty}^{\infty} \frac{g(\lambda)}{\sqrt{\lambda^2 \frac{\lambda - i(1-M^2)}{\lambda - i}}} e^{i\lambda X - \sqrt{\lambda^2 \frac{\lambda - i(1-M^2)}{\lambda - i}}\, Y} d\lambda \tag{14.9a}$$

$$\mathcal{J}_2 = \int_{-\infty}^{\infty} \frac{g(\lambda)}{\sqrt{\lambda^2 + i\gamma M^2 \lambda}} e^{i\lambda X - \sqrt{\lambda^2 + i\gamma M^2\lambda}\, Y} d\lambda \tag{14.9b}$$

where $g(\lambda) = \frac{1}{1+Ai\lambda}$ , independent of $a$. Now our problem is to find the asymptotic formulas of $\mathcal{J}_1$ and $\mathcal{J}_2$ for $Y$ large ( $Y \geq 0$ ).

To approximate the integral $\mathcal{J}_1$ , we have to consider the subsonic and supersonic cases separately, because the branches of $\sqrt{\lambda_1}$ have a different behavior for $M < 1$ and $M > 1$ .

(i) Subsonic Case, $M < 1$

It is easy to obtain a rather crude approximation in the subsonic case by considering contributions from small $\lambda$ only as an asymptotic formula for large $Y$. The error term can be estimated in the same manner as described in Appendix C. Here only the first term will be given

$$\mathcal{J}_1 \sim \int_{-\infty}^{\infty} \frac{g(0)}{\sqrt{1-M^2}} \frac{1}{|\lambda|} e^{i\lambda X - |\lambda|\sqrt{1-M^2}\, Y} d\lambda = -\frac{2}{\sqrt{1-M^2}} \log \sqrt{X^2+(1-M^2)Y^2}$$

which is the principal value of the above integral. Hence

$$\mathcal{J}_1 = -\frac{2}{\sqrt{1-M^2}} \log\left(\frac{U}{\kappa M^2}\left[x^2+(1-M^2)y^2\right]^{\frac{1}{2}}\right) + O\left(\frac{\kappa M^2}{U|y|}\right) \tag{14.10a}$$

and from (14.7c) we have

$$\mathcal{J}_3 = -\frac{2}{\sqrt{1-M^2}} \frac{x}{x^2+(1-M^2)y^2} + O\left(\left(\frac{\kappa M^2}{U|y|}\right)^2\right) \tag{14.10b}$$

$\mathcal{J}_1$ behaves like a subsonic source, while $\mathcal{J}_3$ behaves like a doublet.

(ii) Supersonic Case, $M > 1$

Here we denote $m^2 = M^2 - 1$, so that $m^2 > 0$. Now special attention is required to choose the proper branch in order to assure that $Rl \sqrt{\lambda^2 \frac{\lambda+im^2}{\lambda-i}}$ is always greater than zero on the path of integration. If we write the integral $\mathcal{J}_1$ in the following form

$$\mathcal{J}_1 = \int_{-\infty}^{\infty} \frac{g(\lambda)}{\lambda\sqrt{\frac{\lambda+im^2}{\lambda-i}}} e^{i\lambda X - \lambda\sqrt{\frac{\lambda+im^2}{\lambda-i}}\, Y} d\lambda \tag{14.11a}$$

we then require that

$$Rl \sqrt{\frac{\lambda+im^2}{\lambda-i}} > 0 \quad \text{for} \quad \lambda > 0\ ; \quad \text{and} \quad Rl \sqrt{\frac{\lambda+im^2}{\lambda-i}} < 0 \quad \text{for} \quad \lambda < 0 \tag{14.11b}$$

In the following a method is described to obtain the required approximation by applying a conformal transformation of the complex plane to simplify the integral. Another method is given in Ref. 1, pp. 190-192.

If we apply the following transformation

$$i\zeta = \sqrt{\frac{\lambda+im^2}{\lambda-i}} \tag{14.12}$$

to (14.11), we then obtain

$$\mathcal{J}_1 = 2iM^2 \int_{\mathcal{C}} \frac{g(\lambda(i\zeta))}{(\zeta^2+1)(\zeta^2-m^2)} \, e^{-\frac{\zeta^2-m^2}{\zeta^2+1}(X-\zeta Y)} d\zeta \qquad (14.13)$$

where the contour $\mathcal{C}$, which is determined by (14.12) and condition (14.11b), starts from $\zeta = -i$, passes through points $\sqrt{m}\, e^{-i\pi/4}$, $m$, $\sqrt{m}\, e^{+i\pi/4}$ and ends up at $+i$, as shown in Fig. 9. In the integral (14.13), $\zeta = \pm i$ (corresponding to $\lambda = \pm\infty$) are essential singularities of the integrand. For a similar reason explained in §6.1, it is

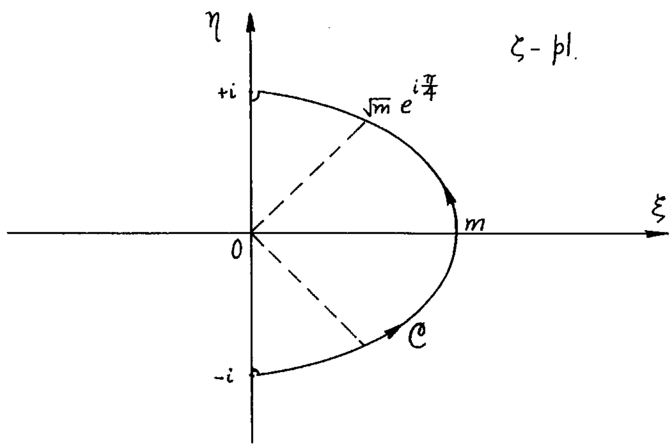



Fig. 9

desirable to deform the path $\mathcal{C}$, if necessary, such that the quantity $\frac{\zeta^2-m^2}{\zeta^2+1}$ is purely imaginary on the path, to aid the approximation. If we write $\zeta = \xi + i\eta$, the condition $Rl\left(\frac{\zeta^2-m^2}{\zeta^2+1}\right) = 0$ gives

$$(\xi^2+\eta^2) + (1-m^2)(\xi^2-\eta^2) - m^2 = 0$$

which passes the points $(0,-1)$, $\left(\sqrt{\frac{m}{2}}, -\sqrt{\frac{m}{2}}\right)$, $(m,0)$, $\left(\sqrt{\frac{m}{2}}, \sqrt{\frac{m}{2}}\right)$, $(0,+1)$ and is actually the transformed contour $\mathcal{C}$ of our original path, real $\lambda$-axis. The fact that this is so may be seen from the point that $i\lambda X$ is purely imaginary on the real $\lambda$-axis. On this contour it can be shown that most of the contribution of the integral comes from the neighborhood of the point $\zeta = m$ . This point also has some underlying physical significance similar to what is mentioned in §6.1 (ii) so that our solution is important only in regions close to the non-viscous Mach lines $x \pm my = 0,\ x > 0$ .

If we choose a parameter $\epsilon$ on the path such that

$$\eta(\epsilon) = \epsilon \tag{14.14a}$$

$$\xi(\epsilon) = \left[\frac{m^2-1}{2} - \epsilon^2 + \frac{m^2+1}{2}\sqrt{1 - 8\frac{m^2-1}{(m^2+1)^2}\epsilon^2}\,\right]^{\frac{1}{2}} \tag{14.14b}$$

and noting that most of the contribution comes from the neighborhood of $\epsilon = 0$ , then the integral $\mathcal{J}_1$ can be approximated as

$$\mathcal{J}_1 \sim i\,\frac{g(0)}{m}\int_{-\infty}^{\infty} \exp\left\{-\frac{2mY}{M^2}\epsilon^2 - \frac{2im}{M^2}(X - mY)\,\epsilon\right\}\frac{d\epsilon}{\epsilon}$$

$$\sim -\frac{\pi}{m}\,\mathrm{erfc}\,\frac{X - mY}{\sqrt{\frac{2M^2}{m}Y}} \qquad (\text{cf. Appendix D})$$

The estimation of the error term is the same as stated in Appendix C. Hence we have

$$\mathcal{J}_1(x,y) = -\frac{\pi}{\sqrt{M^2-1}}\,\mathrm{erfc}\,\frac{x - \sqrt{M^2-1}\,|y|}{\sqrt{\frac{2M^4}{\sqrt{M^2-1}}\frac{\kappa|y|}{U}}} + O\left(\frac{\kappa M^2}{U|y|}\right) \qquad (M>1) \tag{14.15a}$$

and by (14.8c)

$$\mathcal{J}_3(x,y) = \frac{2\pi}{\sqrt{M^2-1}}\frac{1}{\sqrt{\frac{2\pi M^4}{\sqrt{M^2-1}}\frac{\kappa|y|}{U}}}\exp\left\{-\frac{\left[x - \sqrt{M^2-1}\,|y|\right]^2}{\frac{2M^4}{\sqrt{M^2-1}}\frac{\kappa|y|}{U}}\right\} + O\left(\frac{\kappa M^2}{U|y|}\right) \qquad (M>1) \tag{14.15b}$$

A rather crude approximation of $\mathcal{J}_2$ for $Y$ large can be obtained by approximating only $g(\lambda)$ for $\lambda$ small, then we have

$$\mathcal{J}_2 \sim g(0)\int_{-\infty}^{\infty}\frac{e^{i\lambda X - \sqrt{\lambda^2 + i\gamma M^2\lambda}\,Y}}{\sqrt{\lambda^2 + i\gamma M^2\lambda}}\,d\lambda = 2\,e^{\frac{\gamma M^2}{2}X}\,K_0\left(\frac{\gamma M^2}{2}\sqrt{X^2+Y^2}\right)$$

or

$$\mathcal{I}_2(x,y) = 2\sqrt{\frac{\pi\kappa}{\gamma U r}}\, e^{\frac{\gamma U}{2\kappa}(x-r)} + O\left(\frac{\kappa M^2}{U|y|}\right) \tag{14.16a}$$

The error term can be estimated as before. From (14.7d)

$$\mathcal{I}_4(x,y) = \sqrt{\frac{\pi\gamma U}{\kappa r}}\left(1-\frac{x}{r}\right) e^{\frac{\gamma U}{2\kappa}(x-r)} + O\left(\frac{\kappa M^2}{U|y|}\right) \tag{14.16b}$$

The Mach number effect disappears in these first order terms.

With these approximations the asymptotic expressions of the fundamental solutions for $\frac{U|y|}{\kappa M^2}$ large, with $P_r = 3/4$, then become,

$$\Gamma_\Omega(x,y) = \frac{1}{4\pi\kappa}\left\{\frac{\gamma-1}{\gamma}\frac{\kappa M^2}{U}\mathcal{I}_3 + \mathcal{I}_2\right\} \tag{14.17a}$$

$$\vec{\Gamma}_X(x,y) = -\frac{\gamma-1}{4\pi}\frac{M^2}{U^2}\,\mathrm{grad}\left\{\mathcal{I}_1 - \mathcal{I}_2\right\} \tag{14.17b}$$

$$\vec{\Psi}_\Omega(x,y) = -\frac{1}{4\pi\gamma}\,\mathrm{grad}\left\{\mathcal{I}_1 - \mathcal{I}_2\right\} \tag{14.17c}$$

$$\Psi_X(x,y) = \frac{1}{4\pi U}\begin{pmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y}\\ \frac{\partial}{\partial y} & -\frac{\partial}{\partial x}\end{pmatrix}\left\{\mathcal{I}_1 + \frac{\gamma-1}{\gamma}\frac{\kappa M^2}{U}\mathcal{I}_3 - 2e^{\frac{Ux}{2\nu}}K_0\left(\frac{Ur}{2\nu}\right)\right\} + \frac{1}{2\pi\nu}\begin{pmatrix}1&0\\0&0\end{pmatrix}e^{\frac{Ux}{2\nu}}K_0\left(\frac{Ur}{2\nu}\right)$$

$$+\frac{M^2}{4\pi U}\begin{pmatrix}0&0\\0&1\end{pmatrix}\left\{\mathcal{I}_3 + (\gamma-1)\mathcal{I}_4\right\} \tag{14.17d}$$

$$\mathfrak{S}_\Omega(x,y) = \frac{1}{4\pi\kappa}\left\{\frac{1}{\gamma}\frac{\kappa M^2}{U}\mathcal{I}_3 - \mathcal{I}_2\right\} \tag{14.17e}$$

$$\vec{\mathfrak{S}}_X(x,y) = -\frac{M^2}{4\pi U^2}\,\mathrm{grad}\left\{\mathcal{I}_1 + (\gamma-1)\mathcal{I}_2\right\} \tag{14.17f}$$

$$\mathfrak{P}_\Omega(x,y) = \frac{M^2}{4\pi U}\mathcal{I}_3 \tag{14.17g}$$

$$\vec{\mathfrak{P}}_X(x,y) = -\frac{\gamma M^2}{4\pi U^2}\,\mathrm{grad}\,\mathcal{I}_1 \tag{14.17h}$$

where $\mathcal{I}_2$ and $\mathcal{I}_4$ are given by (14.16) and $\mathcal{I}_1$ and $\mathcal{I}_3$ by (14.10) for $M < 1$; and by (14.15) for $M > 1$.

These formulas, valid if the free stream Mach number $M$ is not close to $1$ or $\infty$, show a diffusion term representing a heated wake (coming from the asymptotic expression of $\mathcal{J}_2$ and the term $e^{\frac{Ux}{2\nu}} K_0\left(\frac{Ur}{2\nu}\right)$ ) and a dynamic term representing the temperature-pressure field (from $\mathcal{J}_1$ and $\mathcal{J}_3$ ). The wake diffusion damps out exponentially upstream but decreases as $\frac{1}{\sqrt{r}}$ downstream for all Mach numbers. The dynamic term, which dominates the outer flow field, has a different behavior for $M < 1$ and $M > 1$ . For $M < 1$ the temperature-pressure field behaves like that of flow around a cylinder, but for $M > 1$ the disturbance is concentrated along the non-viscous Mach lines $x \pm \sqrt{M^2-1}\, y = 0$ . A careful study of $\vec{\Gamma}_x$ , $\vec{\Psi}_r$ and $\vec{\Xi}_x$ shows, however, that there is no wake for these quantities. It is also of interest to note that pressure satisfies a differential equation of lower order and its solution shows that it does not have a diffusive wake. Hence, due to heat addition alone the wake consists of changes in temperature and density; while due to shearing force the wake has only the change in velocity. Now, when both $\kappa$ and $\nu$ tend to zero we note that

$$\left(\frac{2\pi M^4}{m}\frac{\kappa|y|}{U}\right)^{-\frac{1}{2}} exp\left[-\frac{(x-m|y|)^2}{\frac{2M^4}{\sqrt{M^2-1}}\frac{\kappa|y|}{U}}\right] \longrightarrow \delta(x-m|y|) \qquad as\ \kappa\to 0 \qquad (14.18a)$$

and

$$\frac{1}{\kappa}e^{\frac{A}{\kappa}x} K_0\left(\frac{A}{\kappa}\sqrt{x^2+y^2}\right) \longrightarrow \frac{\pi}{A}\delta(y)\,\mathbb{1}(x) \qquad as\ \kappa\to 0 \qquad (14.18b)$$

where $A$ is a constant, and $\mathbb{1}(x)$ is a Heaviside unit step function.

$$\begin{aligned}\mathbb{1}(x) &= 1 \qquad x>0\\ &= 0 \qquad x<0\end{aligned} \qquad (14.18c)$$

In Ref. 5 Tsien and Beilock solved the problem $\vec{X}=0$ , $\Omega = \delta(x)\,\delta(y)$ with the assumption $\kappa = \nu = 0$ . It is easy to verify that our results reduce to theirs as $\kappa\to 0$ .

For most aerodynamic problems $U/\kappa$ is large so that these asymptotic solutions are good approximations everywhere except close to the singularities. The fields in the neighborhood of the singularity can also be obtained by approximating the $\mathcal{J}$'s for $\frac{U\lambda}{\kappa}$ small.

14.2 Approximated Solutions Suitable for Small Values of $\frac{U\lambda}{\kappa}$

In addition to the condition that local Reynolds number $Re = \frac{4\gamma}{3}\frac{U\lambda}{\kappa}$ should be small in this case, we shall also assume that $M^2/Re$ tends to zero with $Re$ so that the neglecting of the dissipation is still justified. It is plausible to see that these two assumptions are in general correct for very slow flow problems. It can be shown that for both $Re$ and $M^2/Re$ small, $\mathcal{J}_1$, $\mathcal{J}_2$ etc. can be approximated as

$$\mathcal{J}_1 = -2 \log\left(\frac{U}{\kappa}\sqrt{x^2+y^2}\right) + O\left(\frac{M^2}{Re}\right) \tag{14.19a}$$

$$\mathcal{J}_2 = -2\left[\log\left(\frac{\gamma U}{4\kappa}\sqrt{x^2+y^2}\right) + \gamma_E\right]\left(1+\frac{\gamma U}{2\kappa}x\right) + O(Re^2) \tag{14.19b}$$

$$\mathcal{J}_3 = -\frac{2x}{x^2+y^2} + O\left(\frac{M^2}{Re}\right) \tag{14.19c}$$

$$\mathcal{J}_4 = -\frac{2x}{x^2+y^2}\left(1+\frac{\gamma U}{2\kappa}x\right) - \frac{\gamma U}{\kappa}\left[\log\left(\frac{\gamma U}{4\kappa}\sqrt{x^2+y^2}\right)+\gamma_E\right] + O(Re^2) \tag{14.19d}$$

where $\gamma_E = 0.577\cdots$, Euler's constant. With these formulas, fundamental solutions then read, for both $Re$ and $\frac{M^2}{Re}$ small, and $P_\lambda = 3/4$, as follows:

$$\Gamma_\Omega = -\frac{1}{2\pi\kappa}\left[\log\sqrt{x^2+y^2} + \log\frac{\gamma U}{4\kappa} + \gamma_E\right] + O(Re) \tag{14.20a}$$

$$\vec{\Gamma}_X = O(Re) \tag{14.20b}$$

$$\vec{\Psi}_\Omega = O(Re) \tag{14.20c}$$

$$\Psi_X = -\frac{1}{4\pi\nu}\begin{pmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & -\frac{\partial}{\partial x}\end{pmatrix}\left[ x \log\sqrt{x^2+y^2} + x\left( \log\frac{U}{4\nu} + \gamma_E\right)\right]$$
$$- \frac{1}{2\pi\nu}\begin{pmatrix}1 & 0 \\ 0 & 0\end{pmatrix}\left[ \log\left(\frac{U}{4\nu}\sqrt{x^2+y^2}\right) + \gamma_E \right] + O(Re) \tag{14.20d}$$

$$\Xi_\Omega = -\Gamma_\Omega \tag{14.20e}$$

$$\vec{\Xi}_X = -\frac{\gamma-1}{2\pi c^2}\,\mathrm{grad}\left[ \log\left(\frac{\gamma U}{4\kappa}\sqrt{x^2+y^2}\right) + \gamma_E \right]\left(1+\frac{\gamma U}{2\kappa}x\right) + O(Re) \tag{14.20f}$$

$$\Phi_\Omega = O(Re) \tag{14.20g}$$

$$\vec{\Phi}_X = \vec{\Xi}_X \tag{14.20h}$$

The above results show that pure diffusion is the dominant process in the neighborhood of the heat source and shearing force.

In the following paragraph an example will be given to show how a superposition of these fundamental solutions can be used to solve a boundary value problem.

## § 15. A Boundary Value Problem - Anemometry of a Heated Flat Plate

As a simple example, we shall consider a two-dimensional steady flow of a compressible, viscous, heat-conducting fluid past a flat plate of zero thickness and length $l$ at zero angle of attack. The plate is assumed to be heated, in the usual application by the passage of an electric current. The difference between the surface temperature $T_w$ of the plate and that of free stream $T_0$ is supposed to be only a fraction of $T_0$ so that our linearized theory may be applied and the radiation may also be neglected. On the other hand, $(T_w - T_0)$ is also supposed not to be too small (for example, insulated condition) such that the neglecting of the dissipation is justified. The exact problem probably should be viewed more realistically by considering the plate to be very thin (but not of zero thickness) so the boundary condition of the surface temperature for the fluid flow should be determined together with the heat generation and conduction rate inside the solid plate. Our assumptions used to simplify this problem are as follows: (1) The heat conductivity of the plate material is supposed to be so large and the plate so thin that it remains at a uniform constant temperature. (2) The over-all Reynolds number based on the plate length $Re = \frac{U\ell}{\nu}$ is assumed to be large (say, $> 10$). (3) $Pr = \frac{3}{4}$. This method is analogous to the linearized boundary layer theory, and our result may be regarded as an asymptotic solution for large $Re$. If we choose the coordinate system such that the plate is located at $0 \leq x \leq \ell$, $y = 0$, then our simplified boundary conditions are, on the plate

$$\begin{aligned} &\text{(i)} \quad \theta = \theta_0 \quad \text{(constant)} \\ &\text{(ii)} \quad u = u_0 \quad (u_0 = -U \text{ if no slip}) \\ &\text{(iii)} \quad v = 0 \end{aligned} \tag{15.1}$$

while at infinity all disturbances are required to vanish.

$$\begin{aligned} &\text{(i)} \quad \theta = 0 \\ &\text{(ii)} \quad \vec{q}_\infty = 0 \end{aligned} \tag{15.2}$$

In order to solve this problem the plate is represented by a distribution of heat sources and shearing forces, each with unknown strength which can be determined by applying the boundary conditions. Hence, if we write

$$\Omega(x,y) = f(x)\,\delta(y), \qquad \vec{X}(x,y) = \binom{1}{0} g(x)\,\delta(y) \quad \text{for} \quad 0 < x < \ell \tag{15.3a}$$

and

$$\Omega = 0 \qquad \vec{X} = 0 \qquad \text{elsewhere} \tag{15.3b}$$

where $f(x)$ and $g(x)$ are respectively the unknown strength of the heat source and shearing force to be determined. Then the solutions to this problem satisfying the boundary condition at infinity (15.2) may be given formally as

$$\theta(x,y) = \int_0^\ell \Gamma_1(x-\xi,y)\, f(\xi)\,d\xi + \int_0^\ell \Gamma_2(x-\xi,y)\, g(\xi)\,d\xi \tag{15.4a}$$

$$u(x,y) = \int_0^\ell \mathcal{U}_1(x-\xi,y)\, f(\xi)\,d\xi + \int_0^\ell \mathcal{U}_2(x-\xi,y)\, g(\xi)\,d\xi \tag{15.4b}$$

$$v(x,y) = \int_0^\ell \mathcal{V}_1(x-\xi,y)\, f(\xi)\,d\xi + \int_0^\ell \mathcal{V}_2(x-\xi,y)\, g(\xi)\,d\xi \tag{15.4c}$$

where

$$\Gamma_1 \equiv \Gamma_\Omega, \quad \Gamma_2 = \vec{\Gamma}_X \binom{1}{0}, \quad \binom{\mathcal{U}_1}{\mathcal{V}_1} \equiv \vec{\Psi}_\Omega, \quad \binom{\mathcal{U}_2}{\mathcal{V}_2} = \Psi_X \binom{1}{0}$$

As investigated previously (cf. Eqs. (14.1c) and (14.3b)), $\mathcal{V}_1(x,0) = \mathcal{V}_2(x,0) = 0$, so that the boundary condition (15.1 (iii)) is always satisfied by the solution (15.4c) for any value of $f$ and $g$. Application of the other two boundary conditions leads to a system of two simultaneous integral equations in $f$ and $g$

$$\theta_0 = \int_0^\ell \Gamma_1(x-\xi,0)\, f(\xi)\,d\xi + \int_0^\ell \Gamma_2(x-\xi,0)\, g(\xi)\,d\xi \qquad (0<x<\ell) \tag{15.5a}$$

$$u_0 = \int_0^\ell \mathcal{U}_1(x-\xi,0)\, f(\xi)\,d\xi + \int_0^\ell \mathcal{U}_2(x-\xi,0)\, g(\xi)\,d\xi \qquad (0<x<\ell) \tag{15.5b}$$

Following our assumptions, we shall regard $\kappa$ to be small, then (15.5a,b) can be solved approximately by a method of estimating the order of terms with respect to $\sqrt{\kappa}$ in the $\Gamma$'s and $\mathcal{U}$'s and expanding $f$ and $g$ similarly. It can be shown (cf. Appendix E) that in general

$$\Gamma_1(x,0) = \frac{1}{\sqrt{\kappa}}\, \Gamma_1^{(0)}(x) + \Gamma_1^{(1)}(x) + O(\sqrt{\kappa}) \tag{15.6a}$$

$$\Gamma_2(x,0) = \Gamma_2^{(1)}(x) + O(\sqrt{\kappa}) \tag{15.6b}$$

$$\mathcal{U}_1(x,0) = \mathcal{U}_1^{(1)}(x) + O(\sqrt{\kappa}) \tag{15.6c}$$

$$\mathcal{U}_2(x,0) = \frac{1}{\sqrt{\kappa}}\,\mathcal{U}_2^{(0)}(x) + \mathcal{U}_2^{(1)}(x) + O(\sqrt{\kappa}) \tag{15.6d}$$

These functions may be approximated crudely as follows:

$$\Gamma_1^{(0)}(x) = \frac{a_1}{\sqrt{x}}\,\mathbb{1}(x), \qquad \mathcal{U}_2^{(0)}(x) = \frac{a_2}{\sqrt{x}}\,\mathbb{1}(x) \qquad \text{for all } M \tag{15.7a}$$

$$\begin{aligned}\Gamma_1^{(1)}(x) &= -\frac{a_3}{x} + a_4\,\delta(x) && M<1\\ &= [a_5 + a_4]\,\delta(x) && M>1\end{aligned} \tag{15.7b}$$

$$\begin{aligned}\Gamma_2^{(1)}(x) &= \frac{\gamma}{U}\,\frac{a_3}{x} && M<1\\ &= -\frac{\gamma}{U}\,a_5\,\delta(x) && M>1\end{aligned} \tag{15.7c}$$

$$\mathcal{U}_1^{(1)}(x) = \frac{U^2}{\gamma(\gamma-1)M^2}\,\Gamma_2^{(1)} \tag{15.7d}$$

$$\mathcal{U}_2^{(1)}(x) = -\frac{U}{(\gamma-1)M^2}\,\Gamma_2^{(1)} \tag{15.7e}$$

where $\mathbb{1}(x)$ denotes the Heaviside step function defined by (14.18c), and

$$\begin{aligned} a_1 &= \frac{1}{2}\sqrt{\frac{1}{\gamma\pi U}}, \qquad a_2 = \sqrt{\frac{\gamma}{3\pi U}}, \qquad a_3 = \frac{1}{2\pi U}\,\frac{\gamma-1}{\gamma}\,\frac{M^2}{\sqrt{1-M^2}}.\\ a_4 &= \frac{1}{\gamma\pi U}\,\frac{1}{1+(\gamma-1)M^2}, \qquad a_5 = \frac{1}{2U}\,\frac{\gamma-1}{\gamma}\,\frac{M^2}{\sqrt{M^2-1}}.\end{aligned} \tag{15.7f}$$

As these $\Gamma$'s and $\mathcal{U}$'s only serve here as kernels, these forms may be regarded as good approximations as long as they give accurate weighting factors in the integrals.

If a similar asymptotic expression for $f$ and $g$ is assumed,

$$f(x) = \sqrt{\kappa}\,f_0(x) + \kappa\,f_1(x) + O(\kappa^{3/2}) \tag{15.8a}$$

$$g(x) = \sqrt{\kappa}\,g_0(x) + \kappa\,g_1(x) + O(\kappa^{3/2}) \tag{15.8b}$$

where $f_0$ and $g_0$ are the main terms in the local heat transfer and skin friction respectively, and $f_1$ and $g_1$ are their correction terms, all independent of $\kappa$. Then by substituting (15.6) and (15.8) into (15.5), and collecting terms of order $1$, $\sqrt{\kappa}$, $\kappa$, etc. a set of integral equations is obtained

Order $1$ :

$$\theta_0 = \int_0^\ell \Gamma_1^{(0)}(x-\xi)\, f_0(\xi)\, d\xi \tag{15.9a}$$

$$u_0 = \int_0^\ell \mathcal{U}_2^{(0)}(x-\xi)\, g_0(\xi)\, d\xi \tag{15.9b}$$

Order $\sqrt{\kappa}$ :

$$\int_0^\ell \Gamma_1^{(0)}(x-\xi)\, f_1(\xi)\, d\xi = -\int_0^\ell \left[\Gamma_1^{(1)}(x-\xi)\, f_0(\xi) + \Gamma_2^{(1)}(x-\xi)\, g_0(\xi)\right] d\xi \tag{15.10a}$$

$$\int_0^\ell \mathcal{U}_2^{(0)}(x-\xi)\, g_1(\xi)\, d\xi = -\int_0^\ell \left[\mathcal{U}_1^{(1)}(x-\xi)\, f_0(\xi) + \mathcal{U}_2^{(1)}(x-\xi)\, g_0(\xi)\right] d\xi \tag{15.10b}$$

These integral equations can be solved in succession. With the values of these kernels as given in (15.7), (15.9a) and (15.9b) reduce to the well-known Abel equation

$$\theta_0 = a_1 \int_0^x \frac{f_0(\xi)}{\sqrt{x-\xi}}\, d\xi\,, \qquad u_0 = a_2 \int_0^x \frac{g_0(\xi)}{\sqrt{x-\xi}}\, d\xi \qquad (\text{for all } M) \tag{15.11}$$

which have solutions (cf. Ref. 30, p. 141)

$$f_0(x) = \frac{\theta_0}{\pi a_1 \sqrt{x}} = 2\theta_0 \sqrt{\frac{\gamma U}{\pi x}} \tag{15.12a}$$

$$g_0(x) = \frac{u_0}{\pi a_2 \sqrt{x}} = u_0 \sqrt{\frac{3U}{\gamma \pi x}} \tag{15.12b}$$

Having obtained $f_0$ and $g_0$, the right-hand sides of (15.10a) and (15.10b) are then known functions of $x$. The integral equations in $f_1$ and $g_1$ are again of Abel's type, and their solutions may be formally given as

$$f_1(x) = -\frac{\theta_0 4\gamma U}{\pi}\frac{d}{dx}\int_0^x \frac{d\xi}{\sqrt{x-\xi}}\int_0^\ell \left\{\Gamma_1^{(1)}(\xi-\sigma) + \frac{\sqrt{3}\,u_0}{2\gamma\theta_0}\Gamma_2^{(1)}(\xi-\sigma)\right\}\frac{d\sigma}{\sqrt{\sigma}} \tag{15.13a}$$

$$g_1(x) = -\frac{2\sqrt{3}\theta_0 U}{\pi}\frac{d}{dx}\int_0^x \frac{d\xi}{\sqrt{x-\xi}}\int_0^\ell \left\{\mathcal{U}_1^{(1)}(\xi-\sigma) + \frac{\sqrt{3}\,u_0}{2\gamma\theta_0}\mathcal{U}_2^{(1)}(\xi-\sigma)\right\}\frac{d\sigma}{\sqrt{\sigma}} \tag{15.13b}$$

which should be treated separately for subsonic and supersonic cases.

Furthermore, what we are interested in is the heat transfer rate from the plate which is given by

$$Q = \rho_0 C_v T_0 \int_0^\ell f(x)\,dx \tag{15.14}$$

With the asymptotic expansion of $f(x)$ given by (15.8a), $Q$ can be consequently expanded in a similar form

$$Q = \sqrt{\kappa}\,Q^{(0)} + \kappa\,Q^{(1)} + O(\kappa^{3/2}) \tag{15.15a}$$

where

$$Q^{(0)} = C_v \rho_0 T_0 \int_0^\ell f_0(x)\,dx = 4\,C_v \rho_0 T_0\,\theta_0\sqrt{\frac{\gamma U \ell}{\pi}} \tag{15.15b}$$

$$Q^{(1)} = C_v \rho_0 T_0 \int_0^\ell f_1(x)\,dx \tag{15.15c}$$

By substituting (15.13a) into (15.15c), we have

$$Q^{(1)} = -C_v \rho_0 T_0 \theta_0 U \frac{4\gamma}{\pi}\int_0^\ell \frac{d\xi}{\sqrt{\ell-\xi}}\int_0^\ell \left[\Gamma_1^{(1)}(\xi-\sigma) + \frac{\sqrt{3}\,u_0}{2\gamma\theta_0}\Gamma_2^{(1)}(\xi-\sigma)\right]\frac{d\sigma}{\sqrt{\sigma}} \tag{15.16}$$

This integral will be calculated for $M<1$ and $M>1$ separately.

(i) Subsonic Case, $M<1$

Using the explicit formulas of $\Gamma_1^{(1)}$ and $\Gamma_2^{(1)}$ for $M<1$, we have

$$\int_0^\ell \frac{d\xi}{\sqrt{\ell-\xi}}\int_0^\ell \Gamma_1^{(1)}(\xi-\sigma)\frac{d\sigma}{\sqrt{\sigma}} = -a_3\,\delta\,\alpha_1 + a_4\pi \tag{15.17a}$$

where

$$\alpha_1 = -\int_0^1 \frac{\log x}{1+x^2}\,dx = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2} \doteq 0.917 \tag{15.17b}$$

$$\int_0^l \frac{d\xi}{\sqrt{l-\xi}} \int_0^l \frac{\Gamma_2^{(1)}(\xi-\sigma)}{\sqrt{\sigma}}\, d\sigma = \frac{\gamma a_3}{U}\, \delta\alpha_1 \tag{15.17c}$$

Therefore, we obtain

$$Q^{(1)} = 2 C_v \rho_0 T_0 \theta_0 \left\{ \frac{\delta\alpha_1}{\pi^2}\left(1 - \frac{\sqrt{3}\, u_0}{2U\theta_0}\right)(\gamma-1)\frac{M^2}{\sqrt{1-M^2}} - \frac{2}{\pi}\, \frac{1}{1+(\gamma-1)M^2} \right\} \qquad (M<1) \tag{15.18}$$

(ii) Supersonic Case, $M > 1$

In this case we have

$$\int_0^l \frac{d\xi}{\sqrt{l-\xi}} \int_0^l \Gamma_1^{(1)}(\xi-\sigma)\, \frac{d\sigma}{\sqrt{\sigma}} = (a_5 + a_4)\,\pi \tag{15.19a}$$

$$\int_0^l \frac{d\xi}{\sqrt{l-\xi}} \int_0^l \Gamma_2^{(1)}(\xi-\sigma)\, \frac{d\sigma}{\sqrt{\sigma}} = -\frac{\gamma a_5}{U}\,\pi \tag{15.19b}$$

so that

$$Q^{(1)} = -2 C_v \rho_0 T_0 \theta_0 \left\{ \left(1 - \frac{\sqrt{3}\, u_0}{2U\theta_0}\right)(\gamma-1)\frac{M^2}{\sqrt{M^2-1}} + \frac{2}{\pi}\, \frac{1}{1+(\gamma-1)M^2} \right\} \qquad (M>1) \tag{15.20}$$

In the usual application of a hot-wire anemometer, the heat delivered from the anemometer to the fluid is

$$Q = i^2 r \tag{15.21}$$

where $i$ is the electric current passing through the hot-wire and $r$ is its resistance. In our notation the temperature of the wire is

$$T_w = T_0(1+\theta_0) \tag{15.22}$$

so that

$$T_0\theta_0 = T_w - T_0$$

and

$$\frac{Q}{T_0\theta_0} = \frac{i^2 r}{T_w - T_0} \tag{15.23}$$

is the variable usually used in the hot-wire measurement.

In summary, the result based on our linearized theory gives

$$\frac{Q}{T_w - T_0} = 2 C_v \rho_0 \left\{ 2\sqrt{\frac{\kappa U l}{\pi}} + \kappa\left[ \frac{\delta\alpha_1}{\pi^2}\left(1 - \frac{\sqrt{3}\, u_0}{2U\theta_0}\right)\frac{(\gamma-1)M^2}{\sqrt{1-M^2}} - \frac{2}{\pi}\,\frac{1}{1+(\gamma-1)M^2}\right] + O(\kappa^{3/2}) \right\}, \qquad (M<1) \tag{15.24a}$$

$$\frac{Q}{T_w - T_0} = 2 c_v \rho_0 \left\{ 2\sqrt{\frac{\gamma \kappa U \ell}{\pi}} - \kappa \left[ \left(1 - \frac{\sqrt{3}\, u_0}{2 U \theta_0}\right) \frac{(\gamma-1) M^2}{\sqrt{M^2-1}} + \frac{2}{\pi} \frac{1}{1+(\gamma-1)M^2} \right] + O(\kappa^{3/2}) \right\}, \quad (M>1) \qquad (15.24b)$$

where $\alpha_1$ is given by (15.17b).

The above result may be put in a non-dimensional form by defining a non-dimensional heat transfer coefficient (usually called the Nusselt number), counting both sides of the plate

$$Nu = \frac{Q}{2 k (T_w - T_0)} \qquad \left( = \frac{Q}{2 \kappa c_v \rho_0 (T_w - T_0)} \quad \text{in our notation} \right) \qquad (15.25a)$$

and a non-dimensional parameter depending on hydrodynamical behavior of the flow, say, Reynolds number based on the plate length $\ell$

$$Re = \frac{U \ell}{\nu} = \frac{4\gamma}{3} \frac{U \ell}{\kappa} \qquad (15.25b)$$

Then for no-slip condition, $u_0 = -U$ , (15.24) becomes

$$Nu = \sqrt{\frac{3}{\pi}} \sqrt{Re} + \left[ \frac{8\alpha_1}{\pi^2} \left(1 + \frac{\sqrt{3}}{2\theta_0}\right) \frac{(\gamma-1) M^2}{\sqrt{1-M^2}} - \frac{2}{\pi} \frac{1}{1+(\gamma-1)M^2} \right] + O\left(\frac{1}{\sqrt{Re}}\right), \quad (M<1) \qquad (15.26a)$$

$$Nu = \sqrt{\frac{3}{\pi}} \sqrt{Re} - \left[ \left(1 + \frac{\sqrt{3}}{2\theta_0}\right) \frac{(\gamma-1) M^2}{\sqrt{M^2-1}} + \frac{2}{\pi} \frac{1}{1+(\gamma-1)M^2} \right] + O\left(\frac{1}{\sqrt{Re}}\right). \quad (M>1) \qquad (15.26b)$$

Another interesting part of this problem is to consider the skin friction drag on this heated flat plate

$$D = -\rho_0 \int_0^\ell g(x)\, dx \qquad (15.27a)$$

The drag coefficient is given by

$$C_D = \frac{D}{\frac{1}{2} \rho_0 U^2 \ell} = -\frac{2}{U^2 \ell} \int_0^\ell g(x)\, dx \qquad (15.27b)$$

By a calculation similar to that for $Q$ we obtain

$$C_D = \frac{8}{\sqrt{\pi}} \frac{1}{\sqrt{Re}} + \frac{64\alpha_1}{\pi^2 \sqrt{3}} \theta_0 \left(1 + \frac{\sqrt{3}}{2\theta_0}\right) \frac{1}{\sqrt{1-M^2}} \frac{1}{Re} + O\left(Re^{-3/2}\right). \quad (M<1) \qquad (15.28a)$$

$$C_D = \frac{8}{\sqrt{\pi}} \frac{1}{\sqrt{Re}} - \frac{8}{\sqrt{3}} \theta_0 \left(1 + \frac{\sqrt{3}}{2\theta_0}\right) \frac{1}{\sqrt{M^2-1}} \frac{1}{Re} + O\left(Re^{-3/2}\right), \quad (M>1) \qquad (15.28b)$$

These results for heat transfer (Eqs. (15.24) and (15.26)) and frictional drag (15.28) are good only for large values of $Re$ (say, greater than 10, which is the usual operational range of anemometry). However, some useful formulas good for small values of $Re$ can also be obtained by solving the boundary value problem, using the approximated fundamental solutions for both $Re$ and $\frac{M^2}{Re}$ small. Since in the present case, for a flow creeping very slowly over the heated flat plate, the assumption that both $Re$ and $\frac{M^2}{Re}$ are small would be rather realistic. In the calculation of this problem it is convenient to shift the origin to the center of the plate so that the leading and trailing edges are $(-\frac{\ell}{2},0)$ $(\frac{\ell}{2},0)$ respectively. It may be noted that now the boundary conditions can be imposed only at the plate, which are the same as (15.1); the boundary conditions at infinity have to be released. Here again the boundary condition (15.1 iii) for the $v$-component is automatically satisfied while conditions (i) and (ii) give

$$\theta_0 = \int_{-\ell/2}^{\ell/2} \Gamma_1(x-\xi,0)\, f(\xi)\, d\xi + \int_{-\ell/2}^{\ell/2} \Gamma_2(x-\xi,0)\, g(\xi)\, d\xi \qquad (-\ell/2 < x < \ell/2) \tag{15.29a}$$

$$u_0 = \int_{-\ell/2}^{\ell/2} \mathcal{U}_1(x-\xi,0)\, f(\xi)\, d\xi + \int_{-\ell/2}^{\ell/2} \mathcal{U}_2(x-\xi,0)\, g(\xi)\, d\xi \qquad (-\ell/2 < x < \ell/2) \tag{15.29b}$$

When both $Re$ and $\frac{M^2}{Re}$ are small, the $\Gamma$'s and $\mathcal{U}$'s can be deduced from (14.20) as

$$\Gamma_1(x-\xi,0) = -\frac{1}{2\pi\kappa}\left[\log|x-\xi| + \log\frac{\gamma U}{4\kappa} + \gamma_E\right](1+O(Re)) \tag{15.30a}$$

$$\Gamma_2(x-\xi,0) = O(Re) \tag{15.30b}$$

$$\mathcal{U}_1(x-\xi,0) = O(Re) \tag{15.30c}$$

$$\mathcal{U}_2(x-\xi,0) = -\frac{1}{4\pi\nu}\left[\log|x-\xi| + \log\frac{U}{4\nu} + \gamma_E - 1\right](1+O(Re)) \tag{15.30d}$$

It is clear that if we want to find the first order term of $f(x)$ and $g(x)$, then terms with $\Gamma_2$ and $\mathcal{U}_1$ in (15.29) may be neglected and thus the equations are uncoupled. Physically, this means that the flow rate is so small that there is practically no interference between the heat transfer and shearing force. By using the formula (cf. Ref. 30, p. 143)

$$1 = -\frac{1}{\pi \log \frac{2}{a}} \int_{-a}^{a} \frac{\log|x-\xi|}{\sqrt{a^2-\xi^2}}\, d\xi \tag{15.31}$$

the integral equations (15.29), neglecting $\Gamma_2$ and $\mathcal{U}_1$ have the solution

$$f(x) = \frac{2\kappa\theta_0}{\log \frac{64}{3Re} - \gamma_E} \frac{1}{\sqrt{\left(\frac{l}{2}\right)^2 - x^2}} \tag{15.32}$$

and

$$g(x) = \frac{4\nu u_0}{\log \frac{16}{Re} - \gamma_E + 1} \frac{1}{\sqrt{\left(\frac{l}{2}\right)^2 - x^2}} \tag{15.33}$$

The heat transfer rate from the plate is given by (15.14) as

$$Q = \rho_0 C_v T_0 \theta_0 \frac{2\pi\kappa}{\log \frac{64}{3Re} - \gamma_E} \tag{15.34}$$

and in non-dimensional form

$$Nu = \frac{\pi}{\log \frac{64}{3Re} - \gamma_E}. \tag{15.35}$$

The drag coefficient can be obtained from (15.27b), for the no-slip case

$$C_D = \frac{8\pi}{Re} \frac{1}{\log \frac{16}{Re} - \gamma_E + 1} \qquad \text{per unit span} \tag{15.36}$$

These solutions should be good for $Re \leq 4$.

## §16. Discussion and Comparison of Results with Existing Theories and Experiments

The formulas listed above show what type of behavior to expect from a hot-wire anemometer in two different ranges of $Re$ . For $Re$ large (15.24) and (15.26) exhibit the behavior of the heat transfer rate as $Re$ and $M$ vary. The first order term in the expression of $Nu$ is proportional to $\sqrt{Re}$ (in dimensional form, $Q \sim \rho_0 \sqrt{U}$ ), but is independent of $M$ . The second order term, which is independent of $Re$ , shows clearly the different behavior of the heat loss for $M < 1$ and $M > 1$ . This implies that $M$ in its moderate range has a rather slight effect on the value of the heat transfer, although for $M > 1$ the increasing $M$ tends to decrease $Nu$ and the converse is true for $M < 1$ . This is qualitatively in agreement with experiments (for example, cf. Refs. 9, 31 and 32). For $Re$ small, (15.35) and (15.36) show that $Nu$ and $C_D$ depend on $Re$ only, the Mach number effect drops out. Now, as far as the local transfer rate is concerned, we see that for $Re$ large, the first order term $f_0$ is asymmetric, proportional to $\frac{1}{\sqrt{x}}$ which has a square root singularity at the leading edge. For $Re$ small, Eq. (15.32) shows that the distribution of heat loss is symmetric with respect to the center of the plate and has a square root singularity at both the leading and trailing edges.

In order to compare our results with some existing theories and experiments, a brief historical survey is given in the following. The experimental study of thermal losses from heated bodies under various conditions dates back to the classical researches of Dulong and Petit in 1817 (Ref. 33). The first experimenter to have shown that the forced convection loss in a current of air is proportional to the temperature difference and to the square root of velocity seems to have been Ser (Ref. 34) in 1888. In 1901 the theoretical study of this problem was taken by Boussinesq (Ref. 23). Following Boussinesq, King (Ref. 7) in 1914 considered the circular cylinder immersed in an incompressible, non-viscous fluid by choosing a surface heat flux distribution which involves discontinuities of temperature between the solid surface and the outer fluid to give results in agreement with his experimental data. King's result for a circular cylinder of radius $a$, in our notation, is

$$\frac{Q}{T_0\theta_0} = 2C_v\rho\left\{\sqrt{2\pi\kappa Ua} + \frac{\kappa}{2}\right\} \qquad (\text{large } U) \tag{16.1a}$$

$$\frac{Q}{T_0\theta_0} = \frac{2\pi C_v\rho\kappa}{\log\frac{\kappa}{Ua} - \gamma_E + 1}\,. \qquad (\text{small } U) \tag{16.1b}$$

This theory is not very satisfactory since it is clear that viscous effects are of great importance. Some improvement on King's assumptions and method was made by Piercy and Winny (Ref. 35) in 1933. A little later Piercy and Schmidt et.al. (cf. Refs. 36-38) extended King's theory to viscous incompressible fluid by using Oseen's velocity profile. Their result for a flat plate is, in the present notation

$$\frac{Q}{T_0\theta_0} = 1.616\,\rho\,C_v\,\kappa\,(\gamma\,P_\lambda)^{-\frac{1}{3}}\,Re^{\frac{1}{2}} \qquad \text{for } Re \text{ large} \tag{16.2}$$

In 1949 Tchen (Ref. 8) generalized King's theory to compressible but still non-viscous fluid. His result, in our notation, becomes

$$\frac{Q}{T_w - T_a} = \lambda^{-1} + \pi C_v\rho_0\,\lambda^{-\frac{\gamma}{2(\gamma-1)}}\sqrt{\frac{\gamma U\kappa l}{\pi}} + \kappa\,\frac{3\sqrt{3}}{32}\,\pi\,C_v\rho_0\,\frac{1+\gamma M^2}{1-M^2}\,\frac{\theta_0}{T_a} + O(\kappa^{3/2}), \qquad (M<1) \tag{16.3a}$$

$$\frac{Q}{T_w - T_a} = \lambda^{-1} + \pi C_v\rho_0\,\lambda^{-\frac{\gamma}{2(\gamma-1)}}\sqrt{\frac{\gamma U\kappa l}{\pi}} - \kappa\,\frac{C_v\rho_0}{\gamma}\lambda^{-1}(1+\gamma M^2)\sqrt{M^2-1}\,\log\left[\frac{\sqrt{2}-1}{4}\,\lambda^{-\frac{2-\gamma}{2(\gamma-1)}}\sqrt{\frac{\gamma U l}{\kappa}}\right], \qquad (M>1) \tag{16.3b}$$

where $\lambda = 1 + \frac{\gamma-1}{2} M^2$ and $T_a$ is the isentropic stagnation temperature.

Another quite different category of attacking this problem is the use of boundary layer theory. In 1921 Pohlhausen (Ref. 15) calculated the heat transfer from a flat plate of length $\ell$ immersed in an incompressible viscous fluid. His result is

$$Nu \doteq 0.644\,(Pr)^{\frac{1}{3}}(Re)^{\frac{1}{2}}, \qquad C_f = 1.33\,(Re)^{-\frac{1}{2}} \tag{16.4}$$

Another solution to the same problem has been obtained by Fage and Falkner (Ref. 17) as

$$Nu = 0.623\sqrt{Re} \tag{16.5a}$$

(by putting $Pr = \frac{3}{4}$ in their formula). Their experimental measurement gives the following empirical formula

$$Nu_{average} = 0.75\sqrt{Re} \tag{16.5b}$$

In 1948 Tsien and Finston (Ref. 39) considered the heat loss of a flat ribbon hot-wire of length $\ell$ immersed in a compressible viscous flow. Their result may be repeated here as

$$Q = 1.328\, k_w \frac{\mu_0}{\mu_w} T_0 \sqrt{Re}\left[\left(\frac{T_w}{T_0}-1\right) - \frac{\gamma-1}{2}M^2\right] \tag{16.6a}$$

or

$$Nu = 0.664\sqrt{Re}\, f(M) \tag{16.6b}$$

where $f(M)$ is a function of $M$ given below

(16.6c)

| $M$ | 0 | 0.5 | 1.0 | 1.5 | 2.0 | 2.5 | 3.0 | 5.0 | $\infty$ |
|---|---|---|---|---|---|---|---|---|---|
| $f(M)$ | 1.00 | .98 | .93 | .87 | .80 | .73 | .67 | .50 | 0 |

Rather recently (1950) an empirical formula was given by Kovasznay (Refs. 9 and 31) based on his experiments of circular hot-wires at moderate supersonic speeds (between M = 1.15 and 2.03) using platinum and tungsten wires of diameter $d$ from 0.00015 inch to 0.0003 inch.

His empirical formula is

$$Nu = 0.58\sqrt{\frac{U\rho_0 d}{\mu_w}} - 0.795 \tag{16.7}$$

where a special combination is chosen to form $Re$ such that the dependence on $M$ disappears. It is claimed that this formula is good for low temperature loadings and at large values of $Re$. Experimental results show that the dependence on $M$ is not appreciable at all (the standard deviation of the data from Eq. (16.7) is less than 1%). However, Lowell's experiments (Ref. 32) show that the effects due to changes in $M$ and $\theta_0$ are both quite appreciable.

Now we can compare these results with those obtained in this paper. It may be pointed out that at small values of $Re$, our heat transfer rate agrees with Piercy and Winny's formula (Ref. 35) for incompressible fluid, and our result of $C_D$ (Eq. (15.36)) is exactly that given by Bairstow (Refs. 1 and 40) for an incompressible, non-heat conducting fluid. Therefore, when both $Re$ and $M^2/Re$ are small, our compressible fluid theory reduces to a special case of incompressible flow, even though the heat transfer may still take place. For $Re$ large the first order term of the heat transfer in our result gives the right type of dependence on $Re$, but its numerical coefficient has some discrepancy from other results. For example, when $M = 0$ our result is about 30% too high compared with Fage's experiment, while Fage's theory gives a value about 18% too low and Tsien's result is about 12% too low (cf. Fig. 10). In this case, our linearized theory gives over-estimation. However, the dependence on $M$ should be correct as shown by our formulas. The over-estimation of the exact heat transfer rate which resulted from this theory may be explained as mainly being due to the linearization procedure. Because during linearization the momentum transport rate $uu_x + vu_y$ due to $u$ (where $\binom{u}{v}$ denotes the total velocity here) is approximated by $Uu_x$ which over-estimates the exact picture. This can be seen from the fact that $u$ is considerably smaller than $U$ near the body and on the body surface, where $u = 0$, the approximation is further impaired. This over-estimation is the same as that which occurs in using Rayleigh's formula for skin friction instead of Blasius' result. Furthermore, the neglecting of dissipation adds another cause of over-estimation. The order of magnitude of the total dissipation in this problem for $Re$ large can be estimated as follows: It can be shown that most of the contribution of the dissipation comes from the second integral in (14.1d). Hence, the estimation of the dissipation due to this term only should give a rough idea about its order of magnitude. The velocity field generated by this term due to the frictional force $g_0(x)$ (15.12b) can be obtained. For instance, the $u$-component is

$$u(x,y) \sim -\frac{U}{\pi\sqrt{l}} \frac{1}{(x^2+y^2)^{1/4}} e^{\frac{U}{2\nu}\left[x-\sqrt{x^2+y^2}\right]} \tag{16.8}$$

Then the total dissipation in this velocity field can be approximated as*

$$\Phi \doteq \mu \iint_{-\infty}^{\infty} u_y^2(x,y)\,dx\,dy < \frac{1}{2\pi\sqrt{\pi}}\,\mu U^2\sqrt{Re} \qquad \text{for } Re \text{ large} \tag{16.9}$$

It is this value of $\Phi$ that should be compared with the total heat

*Incidentally, this relationship may also be seen from the viewpoint of the work done by the drag, because if there were no heat added, we should have $\Phi = D\cdot U = \frac{1}{2}\rho U^3 l\, C_D \propto \mu U^2\sqrt{Re}$ by using (15.28).

loss $Q = i^2 r$ in the operation of a hot-wire

$$\frac{\Phi}{Q} = \frac{\text{total dissipation}}{\text{total heat loss}} < \frac{0.098\,\mu U^2 \sqrt{Re}}{i^2 r} \tag{16.10}$$

In order to give a rough idea about the value of this ratio, we may pick up a typical operating condition (for example, cf. Refs. 9 and 31) and take the air at standard conditions as an example, so that

$$\begin{array}{lll} \mu = 1.8 \times 10^{-4} \ \text{c.g.s.} & U = 3 \times 10^{4} \ \text{cm/sec.} & \ell = 3 \times 10^{-4} \ \text{cm} \\ \nu = 0.15 \ \text{cm}^2/\text{sec.} & i = 0.03 \ \text{amp} & r = 1000 \ \text{ohms} \end{array} \tag{16.11}$$

Then $Re = 60$ , which is in the range of our consideration, and

$$\frac{\Phi}{Q} < 1.5\,\% \tag{16.12}$$

This shows that the dissipation is usually very small compared to the total heat loss if we keep the wire temperature sufficiently high. However, the magnitude of dissipation might be appreciably larger for less favorable cases so that by omitting it a further over-estimated heat transfer rate results.

# APPENDIX A

## SOME THEOREMS ABOUT FUNDAMENTAL SOLUTIONS

In this appendix we shall state and prove some simple theorems about fundamental solutions of some special linear differential equations.

**Theorem 1**

If $G_i^{(1)}$ is the fundamental solution of

$$(M-\lambda_i)\,u = -f(x) \qquad i = 1,2 \tag{A.1}$$

defined by

$$u(x) = \iint^{\infty} G_i^{(1)}(x,\xi)\, f(\xi)\, d\xi \tag{A.2}$$

where $M$ stands for any linear differential operator in one, two, or three dimensional space, $\lambda_i$ are constants with the condition $\lambda_1 \neq \lambda_2$ and $\iint^{\infty} d\xi$ means to integrate over all components of the position vector $\xi$. Then the fundamental solution of

$$(M-\lambda_1)(M-\lambda_2)\,u = -f(x) \tag{A.3}$$

over the same region of the space is given by

$$G^{(2)}(x,\xi) = \frac{1}{\lambda_1-\lambda_2}\left(G_1^{(1)} - G_2^{(1)}\right) \tag{A.4}$$

such that

$$u(x) = \iint^{\infty} G^{(2)}(x,\xi)\, f(\xi)\, d\xi \tag{A.5}$$

**Proof**

Since $M$ is a linear operator and $\lambda_1$, $\lambda_2$ are constants, ($\lambda_1 \neq \lambda_2$), the operators $(M-\lambda_1)$ and $(M-\lambda_2)$ are commutable. Thus (A.3) may also be written as

$$(M-\lambda_2)(M-\lambda_1)\,u = -f(x) \tag{A.6}$$

Considering $(M-\lambda_2)u$ as an unknown function in (A.3) and applying the definition of fundamental solutions given by (A.2), we have

$$(M-\lambda_2)u = \iint^{\infty} G_1^{(1)}(x,\xi)\, f(\xi)\, d\xi \tag{A.7}$$

Similarly, from (A.6) we obtain

$$(M-\lambda_1)u = \iint^{\infty} G_2^{(1)}(x,\xi)\, f(\xi)\, d\xi \tag{A.8}$$

Substracting (A.8) from (A.7) yields

$$(\lambda_1-\lambda_2)u = \iint^{\infty} (G_1^{(1)}-G_2^{(1)})\, f(\xi)\, d\xi \tag{A.9}$$

Comparing (A.9) with (A.5) we obtain, for $\lambda_1 \neq \lambda_2$

$$G^{(2)}(x,\xi) = \frac{1}{\lambda_1-\lambda_2}(G_1^{(1)}-G_2^{(1)})$$

This proves the theorem.

<u>Theorem 2</u>

Use the same notations and definitions as in Theorem 1, for $i = 1,2,3$, $\lambda_1 \neq \lambda_2 \neq \lambda_3$. The fundamental solution of

$$(M-\lambda_1)(M-\lambda_2)(M-\lambda_3)\,u = -f(x) \tag{A.10}$$

is given by

$$G^{(3)}(x,\xi) = \frac{G_1^{(1)}}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} + \frac{G_2^{(1)}}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)} + \frac{G_3^{(1)}}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)} \tag{A.11}$$

such that

$$u(x) = \iint^{\infty} G^{(3)}(x,\xi)\, f(\xi)\, d\xi$$

<u>Proof</u>

Here operators $(M-\lambda_1)$, $(M-\lambda_2)$ and $(M-\lambda_3)$ are again commutable, so by applying the definitions of $G_i^{(1)}$ given by (A.2) to Eq. (A.10) we have

$$(M-\lambda_2)(M-\lambda_3)\,u = \left[M^2-(\lambda_2+\lambda_3)M+\lambda_2\lambda_3\right]u = \iint G_1^{(1)}(x,\xi)\, f(\xi)\, d\xi \tag{A.12}$$

$$(M-\lambda_3)(M-\lambda_1)u = \left[M^2-(\lambda_3+\lambda_1)M+\lambda_3\lambda_1\right]u = \iint^{\infty} G_2^{(1)}(x,\xi)\,f(\xi)\,d\xi \qquad (A.13)$$

$$(M-\lambda_1)(M-\lambda_2)u = \left[M^2-(\lambda_1+\lambda_2)M+\lambda_1\lambda_2\right]u = \iint^{\infty} G_3^{(1)}(x,\xi)\,f(\xi)\,d\xi \qquad (A.14)$$

For the case $\lambda_1 \neq \lambda_2 \neq \lambda_3$ , the following identities are easily proven

$$\frac{1}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}+\frac{1}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)}+\frac{1}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)} = 0 \qquad (A.15)$$

$$\frac{\lambda_2+\lambda_3}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}+\frac{\lambda_3+\lambda_1}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)}+\frac{\lambda_1+\lambda_2}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)} = 0 \qquad (A.16)$$

and

$$\frac{\lambda_2\lambda_3}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}+\frac{\lambda_3\lambda_1}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)}+\frac{\lambda_1\lambda_2}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)} = 1 \qquad (A.17)$$

Dividing (A.12) by $(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)$ , (A.13) by $(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)$ and (A.14) by $(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)$ and adding, using the relations (A.15) - (A.17), we then obtain

$$u(x) = \iint^{\infty}\left[\frac{G_1^{(1)}}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}+\frac{G_2^{(1)}}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)}+\frac{G_3^{(1)}}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}\right]f(\xi)\,d\xi \qquad (A.18)$$

Therefore the result (A.11) follows.

In Theorem 3 we shall consider a special ordinary linear differential equation of any even order whose operator can be factorized into second order operators.

<u>Theorem 3</u>

If $M = \frac{d^2}{dy^2}$ and constants $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \cdots \neq \lambda_n \neq 0$ (again the same notations and definitions as in previous cases are used), then the fundamental solution of

$$\left(\frac{d^2}{dy^2}-\lambda_1\right)\left(\frac{d^2}{dy^2}-\lambda_2\right)\cdots\cdots\left(\frac{d^2}{dy^2}-\lambda_n\right)u = -f(y) \qquad \begin{pmatrix} -\infty < y < \infty \\ n = 1,2,\cdots \end{pmatrix} \qquad (A.19)$$

is given by

$$G^{(n)}(y-\eta) = \frac{G_1^{(1)}}{(\lambda_1-\lambda_2)\cdots\cdots(\lambda_1-\lambda_n)} + \frac{G_2^{(1)}}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)\cdots(\lambda_2-\lambda_n)} + \cdots + \frac{G_n^{(1)}}{(\lambda_n-\lambda_1)\cdots\cdots(\lambda_n-\lambda_{n-1})} \tag{A.20}$$

such that

$$u(y) = \int_{-\infty}^{\infty} G^{(n)}(y-\eta)\, f(\eta)\, d\eta \tag{A.21}$$

**Proof**

We shall prove this theorem by mathematical induction.

This theorem is certainly true for $n=1$, since then $G^{(1)} = G_1^{(1)} = \frac{1}{2\sqrt{\lambda_1}} e^{-\sqrt{\lambda_1}|y-\eta|}$ where the positive branch of the $\sqrt{\ }$ is chosen. The theorem also holds for $n = 2, 3$ because they are special cases of Theorem 1 and Theorem 2 respectively. Now suppose that (A.20) holds up to $n = k$. That is, the equation

$$\left(\frac{d^2}{dy^2}-\lambda_1\right)\left(\frac{d^2}{dy^2}-\lambda_2\right)\cdots\cdots\left(\frac{d^2}{dy^2}-\lambda_k\right) u = -f(y) \tag{A.22}$$

has the fundamental solution

$$G^{(k)}(\lambda_1,\cdots\lambda_k;\, y-\eta) = \frac{\frac{1}{2\sqrt{\lambda_1}} e^{-\sqrt{\lambda_1}|y-\eta|}}{(\lambda_1-\lambda_2)\cdots(\lambda_1-\lambda_k)} + \frac{\frac{1}{2\sqrt{\lambda_2}} e^{-\sqrt{\lambda_2}|y-\eta|}}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)\cdots(\lambda_2-\lambda_k)} + \cdots\cdots + \frac{\frac{1}{2\sqrt{\lambda_k}} e^{-\sqrt{\lambda_k}|y-\eta|}}{(\lambda_k-\lambda_1)\cdots\cdots(\lambda_k-\lambda_{k-1})} \tag{A.23}$$

which satisfies Eq. (A.21). Then in

$$\left(\frac{d^2}{dy^2}-\lambda_1\right)\left(\frac{d^2}{dy^2}-\lambda_2\right)\cdots\cdots\left(\frac{d^2}{dy^2}-\lambda_k\right)\left(\frac{d^2}{dy^2}-\lambda_{k+1}\right) u = -f(y) \tag{A.24}$$

the quantity $\left(\frac{d^2}{dy^2}-\lambda_{k+1}\right)u$ may be regarded as a new variable $v$. Hence by the definition of the fundamental solution $G^{(k)}$, we have

$$v = \left(\frac{d^2}{dy^2}-\lambda_{k+1}\right)u = \int_{-\infty}^{\infty} f(\eta)\, G^{(k)}(\lambda_1,\cdots\lambda_k;\, y-\eta)\, d\eta = -H(y), \quad \text{say} \tag{A.25}$$

However, the fundamental solution of (A.25) is $\frac{1}{2\sqrt{\lambda_{k+1}}} e^{-\sqrt{\lambda_{k+1}}|y-\eta|}$.

Hence

$$u = \int_{-\infty}^{\infty} \frac{1}{2\sqrt{\lambda_{k+1}}} e^{-\sqrt{\lambda_{k+1}}\,|y-\zeta|} H(\zeta)\, d\zeta$$

$$= -\int_{-\infty}^{\infty} f(\eta)\,d\eta \, \frac{1}{4} \int_{-\infty}^{\infty} \left\{ \frac{\frac{1}{\sqrt{\lambda_1 \lambda_{k+1}}} e^{-\sqrt{\lambda_1}|\zeta-\eta| - \sqrt{\lambda_{k+1}}|y-\zeta|}}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\cdots(\lambda_1-\lambda_k)} + \cdots\cdots + \frac{\frac{1}{\sqrt{\lambda_k \lambda_{k+1}}} e^{-\sqrt{\lambda_k}|\zeta-\eta| - \sqrt{\lambda_{k+1}}|y-\zeta|}}{(\lambda_k-\lambda_1)\cdots\cdots(\lambda_k-\lambda_{k-1})} \right\} d\zeta \quad \textbf{(A.26)}$$

It is easy to show that the interchange of the order of integration is justified in the present case. Furthermore, it can be shown that

$$\frac{1}{2}\int_{-\infty}^{\infty} e^{-a|y-\zeta| - b|\zeta-\eta|}\, d\zeta = \frac{1}{a^2-b^2}\left[a\, e^{-b|y-\eta|} - b\, e^{-a|y-\eta|}\right] \qquad (a>0,\ b>0) \quad \textbf{(A.27)}$$

Hence Eq. (A.26) becomes

$$u(y) = \int_{-\infty}^{\infty} f(\eta)\,d\eta \, \frac{1}{2} \left\{ \frac{1}{(\lambda_1-\lambda_2)\cdots\cdots(\lambda_1-\lambda_k)(\lambda_1-\lambda_{k+1})}\left[\frac{e^{-\sqrt{\lambda_1}|y-\eta|}}{\sqrt{\lambda_1}} - \frac{e^{-\sqrt{\lambda_{k+1}}|y-\eta|}}{\sqrt{\lambda_{k+1}}}\right] \right.$$

$$+ \frac{1}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)\cdots(\lambda_2-\lambda_k)(\lambda_2-\lambda_{k+1})}\left[\frac{e^{-\sqrt{\lambda_2}|y-\eta|}}{\sqrt{\lambda_2}} - \frac{e^{-\sqrt{\lambda_{k+1}}|y-\eta|}}{\sqrt{\lambda_{k+1}}}\right]$$

$$+ \text{-------------}$$

$$\left. + \frac{1}{(\lambda_k-\lambda_1)\cdots\cdots(\lambda_k-\lambda_{k-1})(\lambda_k-\lambda_{k+1})}\left[\frac{e^{-\sqrt{\lambda_k}|y-\eta|}}{\sqrt{\lambda_k}} - \frac{e^{-\sqrt{\lambda_{k+1}}|y-\eta|}}{\sqrt{\lambda_{k+1}}}\right]\right\}$$

$$= \int_{-\infty}^{\infty} f(\eta)\,d\eta \left[ \frac{\frac{1}{2\sqrt{\lambda_1}} e^{-\sqrt{\lambda_1}|y-\eta|}}{(\lambda_1-\lambda_2)\cdots(\lambda_1-\lambda_k)(\lambda_1-\lambda_{k+1})} + \frac{\frac{1}{2\sqrt{\lambda_2}} e^{-\sqrt{\lambda_2}|y-\eta|}}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)\cdots(\lambda_2-\lambda_{k+1})} + \cdots\cdots\cdots \right.$$

$$\left. + \frac{\frac{1}{2\sqrt{\lambda_k}} e^{-\sqrt{\lambda_k}|y-\eta|}}{(\lambda_k-\lambda_1)\cdots(\lambda_k-\lambda_{k-1})(\lambda_k-\lambda_{k+1})} + \frac{\frac{1}{2\sqrt{\lambda_{k+1}}} e^{-\sqrt{\lambda_{k+1}}|y-\eta|}}{(\lambda_{k+1}-\lambda_1)\cdots\cdots(\lambda_{k+1}-\lambda_k)} \right] \quad \textbf{(A.28)}$$

where in the last step, the following identity is used:

$$\frac{1}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)\cdots(\lambda_1-\lambda_{k+1})} + \cdots\cdots + \frac{1}{(\lambda_k-\lambda_1)\cdots(\lambda_k-\lambda_{k-1})(\lambda_k-\lambda_{k+1})} = \frac{-1}{(\lambda_{k+1}-\lambda_1)\cdots(\lambda_{k+1}-\lambda_k)} \quad \textbf{(A.29)}$$

Eq. (A.28) shows that the relation (A.20) also holds for $n = k+1$.

Therefore, the proposition follows.

# APPENDIX B

## INTEGRAL REPRESENTATIONS OF VELOCITY, DENSITY AND PRESSURE FIELDS IN UNSTEADY ONE-DIMENSIONAL FLOW OF A NON-VISCOUS FLUID

If we denote fundamental solutions of velocity, density and pressure by $\mathfrak{U}$ , $\mathfrak{S}$ , and $\mathfrak{P}$ respectively, which may be defined by similar equations like (4.5)

$$u(x,t) = \int_{-\infty}^{\infty} d\xi \int_0^t \mathfrak{U}(x-\xi,\, t-\tau)\,\Omega(\xi,\tau)\,d\tau \tag{B.1}$$

etc. and if we take

$$\Omega = \delta(x)\,\delta(t) \tag{B.2}$$

then we have

$$u(x,t) = \mathfrak{U}(x,t), \qquad s(x,t) = \mathfrak{S}(x,t), \qquad p(x,t) = \mathfrak{P}(x,t) \tag{B.3}$$

### B.1 Velocity Field

The velocity of this one-dimensional flow may be deduced from $\varphi$ according to (2.6), using their Laplace transforms

$$\bar{\mathfrak{U}}(x;\sigma) = \frac{\partial}{\partial x}\bar{\Phi}(x;\sigma) = -\frac{\sigma \operatorname{sign} x}{2\kappa(\lambda_1-\lambda_2)}\left[e^{-\sqrt{\lambda_1}|x|} - e^{-\sqrt{\lambda_2}|x|}\right] \tag{B.4}$$

where the values of $\lambda_1$ , $\lambda_2$ are given in (4.8d). The desired velocity is then given by the inverse transform of (B.4)

$$u(x,t) = \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} e^{\sigma t}\left(-\frac{\sigma \operatorname{sign} x}{2\kappa(\lambda_1-\lambda_2)}\right)\left[e^{-\sqrt{\lambda_1}|x|} - e^{-\sqrt{\lambda_2}|x|}\right] d\sigma \tag{B.5}$$

If now the following conformal transformations are applied successively to (B.5)

$$T = \frac{c^2 t}{\kappa}, \qquad X = \frac{c|x|}{\kappa}, \tag{B.6a}$$

$$\sigma = \frac{c^2}{\kappa}\left[\frac{2}{\gamma} - 1 + \frac{\sqrt{\gamma-1}}{\gamma}\left(z_{1,2} - \frac{1}{z_{1,2}}\right)\right] \tag{B.6b}$$

where $z_1$ applies to the term with $e^{-\sqrt{\lambda_1}|x|}$ and $z_2$ to that with $e^{-\sqrt{\lambda_2}|x|}$ and

$$z_2 = -\frac{1}{z_1} = \frac{1}{\sqrt{\gamma-1}}(\zeta^2-1) \tag{B.6c}$$

we then obtain

$$u(x,t) = \frac{c^2}{\gamma\kappa}\,\operatorname{sign} x\,\frac{1}{2\pi i}\int_{C_1,C_2} \frac{\zeta}{\zeta^2-1}\, e^{\frac{1}{\gamma}\frac{\zeta^2-\gamma}{\zeta^2-1}\zeta\,[\zeta T-\sqrt{\gamma}X]}\,d\zeta \tag{B.7}$$

where the paths $C_1$, $C_2$ are described in §6.

B.2 Density Field

We may deduce the density $s(x,t)$ from the basic system of Eqs. (2.2b) or (2.2d) by using the integral representation of $u$ or $\theta$. However, for the sake of later checking, we use an alternative method. The Laplace transform of the fundamental system of equations (2.2) is

$$\bar{p} = \bar{s} + \bar{\theta} \tag{B.8a}$$

$$\sigma\bar{s} + \operatorname{div}\bar{\vec{q}} = 0 \tag{B.8b}$$

$$\sigma\bar{\vec{q}} = -\frac{c^2}{\gamma}\operatorname{grad}\bar{p} \tag{B.8c}$$

$$\sigma\bar{\theta} - \kappa\Delta\bar{\theta} = \sigma(\gamma-1)\,\bar{s} + \bar{\Omega} \tag{B.8d}$$

Eliminating $\bar{p}$, $\bar{\theta}$, $\bar{\vec{q}}$, it can be shown that $\bar{s}$ satisfies

$$L(\bar{s}) = \frac{1}{\kappa}\Delta\bar{\Omega} \tag{B.9}$$

where $L$ is the same operator as (4.8c). In one-dimensional case, with $\Omega$ given as (B.2), the solution to (B.9) is then

$$\bar{s}(x;\sigma) = -\frac{1}{2\kappa(\lambda_1-\lambda_2)}\left[\sqrt{\lambda_1}\,e^{-\sqrt{\lambda_1}|x|} - \sqrt{\lambda_2}\,e^{-\sqrt{\lambda_2}|x|}\right] \tag{B.10}$$

such that

$$s(x,t) = -\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} \frac{e^{\sigma t}}{2\kappa(\lambda_1-\lambda_2)}\left[\sqrt{\lambda_1}\,e^{-\sqrt{\lambda_1}|x|} - \sqrt{\lambda_2}\,e^{-\sqrt{\lambda_2}|x|}\right]d\sigma \tag{B.11}$$

Applications of the same transformations (B.6) to (B.11) yield

$$S(x,t) = \frac{c}{\kappa\sqrt{\gamma}}\,\frac{1}{2\pi i}\int_{\mathcal{C}_1,\mathcal{C}_2} \frac{1}{\zeta^2-1}\, \exp\left\{\frac{1}{\gamma}\,\frac{\zeta^2-\gamma}{\zeta^2-1}\,\zeta\left[\zeta T-\sqrt{\gamma}\,X\right]\right\} d\zeta \tag{B.12}$$

## B.3 Pressure Field

The shortest way to obtain $p$ at this stage is by using (2.2a), so by adding up (6.14a) and (B.12) we have

$$p(x,t) = S+\theta = \frac{c}{\kappa\sqrt{\gamma}}\,\frac{1}{2\pi i}\int_{\mathcal{C}_1,\mathcal{C}_2} \frac{\zeta^2}{\zeta^2-1}\, \exp\left\{\frac{1}{\gamma}\,\frac{\zeta^2-\gamma}{\zeta^2-1}\,\zeta\left[\zeta T-\sqrt{\gamma}\,X\right]\right\} d\zeta \tag{B.13}$$

The same result is obtained if we solve the equation

$$\mathsf{L}(\bar{p}) = \frac{\gamma\sigma^2}{\kappa c^2}\,\bar{\Omega} \tag{B.14}$$

with $\mathsf{L}$ given by (4.8c) and $\bar{\Omega} = \delta(x)$.

# APPENDIX C

## ESTIMATION OF ERROR TERM IN THE APPROXIMATION OF A CERTAIN INTEGRAL

In §6.1 and §6.2 a general outline of the scheme to approximate integrals in Eq. (6.14) for large or small values of the time is described. However, many of the detailed calculations are not given there. This appendix may be regarded as the supplementary note of §6.1 and §6.2, by showing the systematic procedure of how the error term is estimated in approximating (6.14a)

$$\theta(x,t) = \frac{c}{\kappa\sqrt{\gamma}}\,\frac{1}{2\pi i}\int_{\mathcal{C}_1,\mathcal{C}_2} \exp\left\{\frac{1}{\gamma}\,\frac{\zeta^2-\gamma}{\zeta^2-1}\,\zeta\left[\zeta T-\sqrt{\gamma}\,X\right]\right\}d\zeta \tag{C.1}$$

where $\mathcal{C}_1$ , $\mathcal{C}_2$ are shown in Fig. 4. The rest of the integrals in (6.14) may be calculated in a way similar to this.

When $T$ is large ( $T = c^2t/\kappa$ ) it is shown in §6 that the above integral should take different contours for $T > \sqrt{\gamma}\,X$ and $T < \sqrt{\gamma}\,X$ .

(i) $T > \sqrt{\gamma}\,X$ ( or $|x| < c_i t$ )

We found in §6.1 that $\zeta = 0$ is a saddle point and that the imaginary axis is the path of steepest descent. If we write $\zeta = \xi + i\eta$ then on this path we have $\zeta = i\eta$ so that

$$\begin{aligned}\theta(x,t) &= \frac{c}{\kappa\sqrt{\gamma}}\,\frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left\{-\frac{1}{\gamma}\,\frac{\eta^2+\gamma}{\eta^2+1}\left[\eta^2 T + i\eta\sqrt{\gamma}\,X\right]\right\}d\eta \\ &= \frac{c}{\kappa\sqrt{\gamma}}\,\frac{1}{2\pi}\left[\mathcal{J}_1 + 2\,\mathcal{J}_2\right]\end{aligned} \tag{C.2}$$

where

$$\mathcal{J}_1 = \int_{-1}^{1} \exp\left\{-\frac{1}{\gamma}\,\frac{\eta^2+\gamma}{\eta^2+1}\left[T\eta^2 + i\sqrt{\gamma}\,X\eta\right]\right\}d\eta \tag{C.3}$$

$$\mathcal{J}_2 = \int_{1}^{\infty} \exp\left\{-\frac{T}{\gamma}\,\frac{\eta^2+\gamma}{\eta^2+1}\,\eta^2\right\}\cos\left(\frac{X}{\sqrt{\gamma}}\,\frac{\eta^2+\gamma}{\eta^2+1}\,\eta\right)d\eta \tag{C.4}$$

Now

$$|\mathcal{I}_2| \leqslant \int_1^\infty \exp\left\{-\frac{T}{\gamma}\frac{\eta^2+\gamma}{\eta^2+1}\eta^2\right\} d\eta < \int_1^\infty e^{-\frac{T}{\gamma}\eta^2} d\eta < \frac{\gamma}{2T} e^{-\frac{T}{\gamma}} \tag{C.5}$$

In $\mathcal{I}_1$ we write

$$-\frac{1}{\gamma}\frac{\eta^2+\gamma}{\eta^2+1} \equiv -1 + \frac{(\gamma-1)\eta^2}{\gamma(\eta^2+1)}$$

then we obtain

$$\mathcal{I}_1 = \int_{-1}^{1} e^{-\eta^2 T - i\eta\sqrt{\gamma}X} \, e^{\frac{\gamma-1}{\gamma}\frac{\eta^2}{\eta^2+1}[\eta^2 T + i\eta\sqrt{\gamma}X]} \, d\eta$$

The mean value theorem may be used to approximate

$$f(\eta) = \exp\left\{\frac{\gamma-1}{\gamma}\frac{\eta^2}{\eta^2+1}\left[\eta^2 T + i\eta\sqrt{\gamma}X\right]\right\}$$

$$= 1 + \eta f'(A_1\eta) \qquad (|\eta| \leqslant 1, \quad 0 < A_1 < 1)$$

Hence

$$\mathcal{I}_1 = \int_{-1}^{1} e^{-\eta^2 T - i\eta\sqrt{\gamma}X} d\eta + \int_{-1}^{1} \eta f'(A_1\eta) \exp\left\{-[\eta^2 T + i\sqrt{\gamma}X\eta]\right\} d\eta$$

$$= 2\int_0^\infty e^{-\eta^2 T} \cos\sqrt{\gamma}X\eta \, d\eta + \mathcal{I}_{11} + \mathcal{I}_{12} = \sqrt{\frac{\pi}{T}} e^{-\frac{\gamma X^2}{4T}} + \mathcal{I}_{11} + \mathcal{I}_{12} \tag{C.6}$$

where

$$\mathcal{I}_{11} = -2\int_1^\infty e^{-\eta^2 T} \cos\sqrt{\gamma}X\eta \, d\eta$$

$$\mathcal{I}_{12} = \int_{-1}^{1} \eta f'(A_1\eta) \exp\left\{-\eta^2 T - i\sqrt{\gamma}X\eta\right\} d\eta$$

It is easy to see that

$$|\mathcal{I}_{11}| < \frac{1}{T} e^{-T} \tag{C.7}$$

and a careful estimation of $\mathcal{I}_{12}$ gives

$$|\mathcal{I}_{12}| < \frac{\text{const.}}{T} \tag{C.8}$$

Therefore, by summing up these error terms we obtain

$$\theta(x,t) = \frac{c}{\kappa\sqrt{\gamma}} \frac{1}{2\pi} \left[ \sqrt{\frac{\pi}{T}}\, e^{-\frac{\gamma X^2}{4T}} + O\left(\frac{1}{T}\right) \right]$$

which is Eq. (6.17).

(ii) $T < \sqrt{\gamma}X$ $\quad (or \quad |x| > c_i t)$

It is shown that the right contour in this case is the one drawn in Fig. 6, on which we may break the integral into three parts

$$\theta(x,t) = \frac{c}{\kappa\sqrt{\gamma}} \frac{1}{2\pi i} \left( \int_A^B + \int_B^C + \int_C^D \right) \exp\left\{ \frac{1}{\gamma} \frac{\zeta^2-\gamma}{\zeta^2-1} \zeta\, (\zeta T - \sqrt{\gamma} X) \right\} d\zeta \qquad (C.9)$$

Denote these three integrals by $\mathcal{J}_3$ , $\mathcal{J}_\gamma$ , $\mathcal{J}_4$ , respectively. For $\mathcal{J}_\gamma$ we may take a parametric representation of the path $BC$ as given by (6.23)

$$\begin{aligned} \zeta(\epsilon) &= \xi(\epsilon) + i\,\eta(\epsilon) \\ \eta(\epsilon) &= \epsilon \\ \xi(\epsilon) &= \left[ \frac{\gamma+1}{2} - \epsilon^2 + \frac{\gamma-1}{2}\sqrt{1-16\left(\frac{\epsilon}{\gamma-1}\right)^2} \right]^{\frac{1}{2}} \end{aligned} \qquad (C.10)$$

If $\epsilon = \pm \frac{\gamma-1}{4}$ , then

$$\xi = \frac{1}{4}\sqrt{10\gamma+7-\gamma^2} \equiv \xi_0 , \qquad \eta = \pm\frac{\gamma-1}{4} \equiv \pm\eta_0 \qquad (C.11)$$

It is convenient to choose our end points to be $B(\xi_0, -\eta_0)$, $C(\xi_0, \eta_0)$ then on paths $AB$ and $CD$ we have $\zeta = \xi_0 + i\eta$ and $|\eta| \geqslant \eta_0$ , so

$$|\mathcal{J}_3 + \mathcal{J}_4| = 2\exp\left\{ \frac{1}{\gamma}\left[ \xi_0^2 T - \sqrt{\gamma} X \xi_0 - (\gamma-1)T \right] \right\} \int_{\eta_0}^{\infty} \exp\left\{ -\frac{T}{\gamma}\eta^2 - \frac{\gamma-1}{\gamma} \frac{(\xi_0^2-\eta^2-1)T - \sqrt{\gamma} X \xi_0 (\xi_0^2+\eta^2-1)}{(\xi_0^2-\eta^2-1)^2 + 4\xi_0^2\eta^2} \right\} d\eta$$

$$< 2K_1 \exp\left\{ \frac{1}{\gamma}\left[ \xi_0^2 T - \sqrt{\gamma} X \xi_0 - (\gamma-1)T \right] \right\} \int_{\eta_0}^{\infty} e^{-\frac{T}{\gamma}\eta^2} d\eta$$

$$< \frac{\gamma}{\eta_0 T} K_1 \exp\left\{ \frac{1}{\gamma}\left[ -(\eta_0^2 - \xi_0^2)T - \sqrt{\gamma} X \xi_0 - (\gamma-1)T \right] \right\}$$

where

$$K_1 = \sup \left| \exp\left\{ -\frac{\gamma-1}{\gamma} \frac{(\xi_0^2-\eta^2-1)T - \sqrt{\gamma}X\xi_0(\xi_0^2+\eta^2-1)}{(\xi_0^2-\eta^2-1)^2 + 4\xi_0^2\eta^2} \right\} \right| = \exp\left\{ \frac{1}{\gamma}\left[ \frac{\gamma-5}{4} T + \sqrt{\gamma}X\xi_0 \right] \right\} \quad \text{in } \eta \geq \eta_0$$

For these special end points chosen we have

$$|\mathcal{I}_3 + \mathcal{I}_4| < \frac{\gamma}{\eta_0 T} e^{-\frac{\gamma^2-1}{8\gamma}T} \tag{C.12}$$

Now if we apply the parametric representation (C.10) to $\mathcal{I}_\gamma$ , we have

$$\mathcal{I}_\gamma = \int_{-\eta_0}^{\eta_0} \exp\left\{ \frac{2i}{\gamma}\epsilon \frac{\xi^2+\epsilon^2}{\xi^2+\epsilon^2-1} \left[ (\xi(\epsilon)+i\epsilon)T - \sqrt{\gamma}X \right] \right\} (i + \xi'(\epsilon))\,d\epsilon \tag{C.13}$$

and write

$$\frac{\xi^2+\epsilon^2}{\xi^2+\epsilon^2-1} = 1 + g(\epsilon) \ , \qquad g(\epsilon) = \frac{1}{\xi^2+\epsilon^2-1} \tag{C.14}$$

then on its path $BC$ , which is outside of the circle $\xi^2+\epsilon^2 = 1$ , we may apply the mean value theorem to $g(\epsilon)$ , which leads to

$$\mathcal{I}_\gamma = -\int_{-\eta_0}^{\eta_0} e^{-\frac{2}{\gamma-1}\left[T\epsilon^2 - i\epsilon\sqrt{\gamma}(T-X)\right]} h(\epsilon) \left[ i + \epsilon\,\xi''(A_2\epsilon) \right] d\epsilon \tag{C.15a}$$

where

$$h(\epsilon) = \exp\left\{ \frac{2i}{\gamma-1} \left[ \frac{\gamma-1}{\sqrt{\gamma}} g''(A_3\epsilon)(T-X) + \xi''(A_4\epsilon)T + O(\epsilon) \right] \frac{\epsilon^3}{2} \right\} \tag{C.15b}$$

and

$$0 < A_2, A_3, A_4 < 1 \tag{C.15c}$$

The mean value theorem can be again applied to $h(\epsilon)$ for $|\epsilon| \leq \eta_0$ . This process yields:

$$\begin{aligned} \mathcal{I}_\gamma &= i\int_{-\eta_0}^{\eta_0} e^{-\frac{2}{\gamma-1}\left[T\epsilon^2 - i\epsilon\sqrt{\gamma}(T-X)\right]} d\epsilon + \mathcal{I}_{\gamma 1} \\ &= 2i\int_0^\infty e^{-\frac{2T}{\gamma-1}\epsilon^2} \cos \frac{2\sqrt{\gamma}}{\gamma-1}(T-X)\epsilon\,d\epsilon + \mathcal{I}_{\gamma 1} + \mathcal{I}_{\gamma 2} \end{aligned} \tag{C.16a}$$

where

$$\mathcal{J}_{\gamma 1} = \int_{-\eta_0}^{\eta_0} e^{-\frac{2T}{\gamma-1}\epsilon^2 + i\epsilon\frac{2\sqrt{\gamma}}{\gamma-1}(T-X)} \left\{ \left[ \xi''(A_5\epsilon) + i h'(A_6\epsilon) \right]\epsilon + O(\epsilon^2) \right\} d\epsilon, \quad \left(0 < \begin{matrix} A_5 \\ A_6 \end{matrix} < 1\right) \tag{C.16b}$$

$$\mathcal{J}_{\gamma 2} = -2i \int_{\eta_0}^{\infty} e^{-\frac{2T}{\gamma-1}\epsilon^2} \cos \frac{2\sqrt{\gamma}}{\gamma-1}(T-X)\epsilon \, d\epsilon \tag{C.16c}$$

It can be shown that

$$|\mathcal{J}_{\gamma 1}| < \frac{K_2}{T}\left(1 - e^{-\frac{2T}{\gamma-1}\eta_0^2}\right) \tag{C.17a}$$

where $K_2$ = constant $(\eta_0)$ , independent of $T$ , and

$$|\mathcal{J}_{\gamma 2}| < \frac{\gamma-1}{4\eta_0}\frac{1}{T} e^{-\frac{2T}{\gamma-1}\eta_0} \tag{C.17b}$$

Therefore, by combining these order terms we obtain

$$\theta(x,t) = \frac{c}{2\kappa}\left[\sqrt{\frac{\gamma-1}{2\gamma\pi T}} \exp\left\{-\frac{\gamma(T-X)^2}{2(\gamma-1)T}\right\} + O\left(\frac{1}{T}\right)\right] \tag{C.18}$$

which is Eq. (6.25).

# APPENDIX D

## EVALUATION OF SOME CONTOUR INTEGRALS

In the following some contour integrals which are used in this paper are calculated. The contour $\mathcal{C}$ denotes a path parallel to the imaginary axis, to the right of the origin of the complex plane $z$ .

$$\text{(i)} \quad \mathcal{J} = \frac{1}{2\pi i}\int_{\mathcal{C}} e^{az^2-bz}\,dz = \frac{1}{2\sqrt{\pi a}}\, e^{-\frac{b^2}{4a}} \qquad (a>0) \qquad \text{(D.1)}$$

This result is immediately seen by deforming $\mathcal{C}$ into the imaginary axis.

$$\text{(ii)} \quad \mathcal{J} = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{1}{z}\, e^{az^2-bz}\,dz = \frac{1}{2}\,\mathrm{erfc}\left(\frac{b}{2\sqrt{a}}\right) \qquad (a>0) \qquad \text{(D.2)}$$

One way to prove this is by deforming $\mathcal{C}$ into the imaginary axis except with an indentation at the origin to the right half-plane

$$\therefore \int_{\mathcal{C}} \frac{1}{z}\, e^{az^2}\,dz = \lim_{\epsilon\to 0}\int_{\supset} \frac{1}{z}\, e^{az^2}\,dz + \lim_{\epsilon\to 0}\left\{\int_{i\epsilon}^{i\infty} + \int_{-i\infty}^{-i\epsilon} \frac{1}{z}\, e^{az^2}\,dz\right\} = \pi i$$

where $\int_{\supset} dz$ denotes the integration taken on the indentation of radius $\epsilon$ . Hence

$$\mathcal{J} - \frac{1}{2} = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{e^{az^2}}{z}\left[e^{-bz}-1\right]dz = -\frac{1}{2\pi i}\int_{\mathcal{C}} e^{az^2}\int_0^b e^{-\xi z}\,d\xi\,dz$$

and by justifying the change of order of integration and using (D.1) we have

$$\mathcal{J} = \frac{1}{2} - \frac{1}{2\sqrt{\pi a}}\int_0^b e^{-\frac{\xi^2}{4a}}\,d\xi = \frac{1}{2}\,\mathrm{erfc}\left(\frac{b}{2\sqrt{a}}\right) \qquad \text{for } a>0$$

$$\text{(iii)} \quad \mathcal{J} = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{1}{z^2}\, e^{az^2-bz}\,dz = \sqrt{\frac{a}{\pi}}\, e^{-\frac{b^2}{4a}} - \frac{b}{2}\,\mathrm{erfc}\left(\frac{b}{2\sqrt{a}}\right), \quad (a>0) \qquad \text{(D.3)}$$

It can be justified that we may carry out the differentiation of $\mathcal{J}$ with respect to either $a$ or $b$ under the integral sign,

$$\frac{\partial \mathcal{J}}{\partial a} = \frac{1}{2\pi i}\int_{\mathcal{C}} e^{az^2-bz}\,dz = \frac{1}{2\sqrt{\pi a}}\, e^{-\frac{b^2}{4a}}$$

by (D.1). Hence

$$\mathcal{J} = \sqrt{\frac{a}{\pi}}\, e^{-\frac{b^2}{4a}} - \frac{b}{2}\,\mathrm{erfc}\left(\frac{b}{2\sqrt{a}}\right) + C(b)$$

where $C(b)$ is a function of $b$ only, which can be determined by comparing with

$$\mathcal{J}(0,b) = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{1}{z^2} e^{-bz}\,dz = \begin{cases} 0 & b>0 \\ -b & b<0 \end{cases}$$

This gives the value of $C(b) = 0$ for all $b$ . Therefore the result (D.3) follows

(iv) $\mathcal{J} = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{1}{z^3} e^{az^2-bz}\,dz = \frac{1}{2}\left[\left(a+\frac{b^2}{2}\right)\mathrm{erfc}\,\frac{b}{2\sqrt{a}} - b\sqrt{\frac{a}{\pi}}\, e^{-\frac{b^2}{4a}}\right],\quad (a>0)$ (D.4)

(v) $\mathcal{J} = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{1}{z^4} e^{az^2-bz}\,dz = \frac{1}{6}\sqrt{\frac{a}{\pi}}(4a+b^2)e^{-\frac{b^2}{4a}} - \frac{b}{2}\left(a+\frac{b^2}{6}\right)\mathrm{erfc}\left(\frac{b}{2\sqrt{a}}\right),\quad (a>0)$ (D.5)

(D.4) and (D.5) can be proven by a method similar to (iii)

(vi) $\mathcal{J} = \frac{1}{2\pi i}\int_{\mathcal{C}} z\, e^{az^2-bz}\,dz = \frac{b}{4\sqrt{\pi a^3}}\, e^{-\frac{b^2}{4a}},\quad (a>0)$ (D.6)

This result can be obtained, with justification, by differentiating (D.1) with respect to $b$ .

(vii) $\mathcal{J} = \frac{1}{2\pi i}\int_{\mathcal{C}} z^2 e^{az^2-bz}\,dz = \frac{1}{4\sqrt{\pi a^3}}\left(\frac{b^2}{2a}-1\right) e^{-\frac{b^2}{4a}},\quad (a>0)$ (D.7)

This result can be obtained by differentiating (D.1) with respect to $a$ .

# APPENDIX E

## ASYMPTOTIC EXPANSION OF FUNDAMENTAL SOLUTIONS IN PART III IN ORDER OF TERMS WITH RESPECT TO $\sqrt{\kappa}$

The fundamental solutions $\Gamma_1$ , $\Gamma_2$ , $\mathcal{U}_1$ and $\mathcal{U}_2$ defined by Eqs. (15.4) can be expressed in their original integral forms (cf. Eq. (14.1)) as follows:

$$4\pi\kappa\,\Gamma_1(x,0) = \Gamma_{11} + \Gamma_{12} = \int_{-\infty}^{\infty} \frac{\frac{\gamma-1}{\gamma} ia\beta}{1+\frac{\gamma-1}{\gamma} ia\beta} \frac{e^{i\beta x}}{\sqrt{\lambda_1}}\, d\beta + \int_{-\infty}^{\infty} \frac{1}{1+\frac{\gamma-1}{\gamma} ia\beta} \frac{e^{i\beta x}}{\sqrt{\beta^2 + i\frac{\gamma U}{\kappa}\beta}}\, d\beta \tag{E.1a}$$

$$4\pi\kappa\,\Gamma_2(x,0) = \Gamma_{21} + \Gamma_{22} = -\frac{\gamma}{U}\Gamma_{11} + \frac{\gamma-1}{U} a \frac{\partial}{\partial x}\Gamma_{12} \tag{E.1b}$$

$$4\pi\kappa\,\mathcal{U}_1(x,0) = \mathcal{U}_{11} + \mathcal{U}_{12} = \frac{U^2}{\gamma(\gamma-1)M^2}\,(4\pi\kappa\,\Gamma_2) \tag{E.1c}$$

$$4\pi\kappa\,\mathcal{U}_2(x,0) = \mathcal{U}_{21} + \mathcal{U}_{22} + \mathcal{U}_{23} = \frac{\gamma}{\gamma-1}\frac{\Gamma_{11}}{M^2} + \frac{\gamma-1}{\gamma}\frac{M^2\kappa}{U^3}\frac{\partial^2}{\partial x^2}\Gamma_{12} + \frac{\kappa}{\nu}\int_{-\infty}^{\infty} \frac{(1-\frac{\nu}{U} i\beta)e^{i\beta x}}{\sqrt{\beta^2 + i\frac{U}{\nu}\beta}}\, d\beta \tag{E.1d}$$

where each term is indicated according to its order, and all notations are the same as given in Part III. Note that $x = 0$ is a singular point of some of these integrals, so in our approximation an attempt is made to avoid this difficulty either by keeping away from $x = 0$ or by finding its principal value, if it exists.

As an example, we shall approximate $\Gamma_{12}$ first. $\Gamma_{12}$ represents a diffusive wake and thus behaves differently upstream and downstream. Hence, we first convert them into real integrals of Laplace type for $x > 0$ and $x < 0$ separately. The resulting integrals are then approximated for $\kappa$ small. Now

$$\Gamma_{12} = \int_{-\infty}^{\infty} \frac{1}{1+\frac{\gamma-1}{\gamma} ia\beta} \frac{e^{i\beta x}}{\sqrt{\beta(\beta + \frac{i\gamma U}{\kappa})}}\, d\beta = \Gamma_a + \Gamma_b \tag{E.2a}$$

with

$$\Gamma_a = -\int_0^{\infty} \frac{\frac{\gamma}{\gamma-1} i\alpha}{\beta - \frac{\gamma}{\gamma-1} i\alpha} \frac{e^{i\beta x}}{\sqrt{\beta(\beta+ib)}} d\beta , \qquad \Gamma_b = -\int_{-\infty}^{0} \frac{\frac{\gamma}{\gamma-1} i\alpha}{\beta - \frac{\gamma}{\gamma-1} i\alpha} \frac{e^{i\beta x}}{\sqrt{\beta(\beta+ib)}} d\beta \tag{E.2b}$$

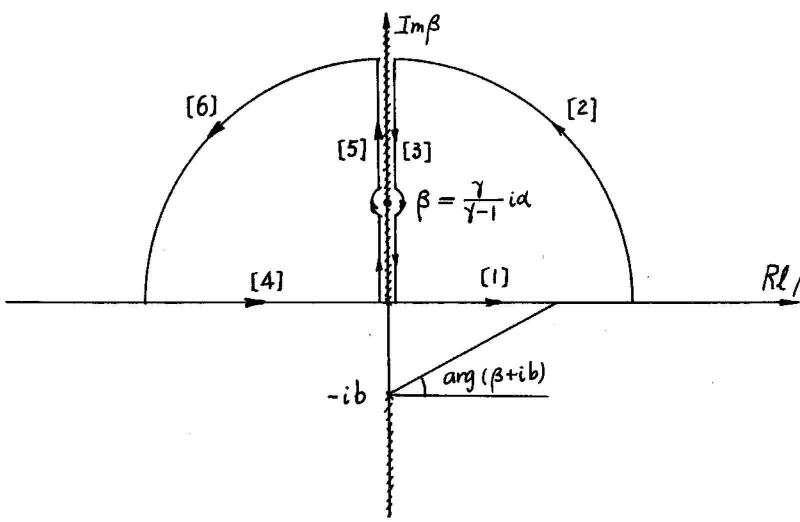
where

$$\alpha = \frac{1}{a} = \frac{U}{\kappa M^2} , \qquad b = \frac{\gamma U}{\kappa} \tag{E.2c}$$

and

$\sqrt{\beta(\beta+ib)}$ is defined to take the branch having a positive real part on the path of integration. (E.2d)

The integrand in (E.2) has two branch points at $\beta = 0$ and $-ib$ and a simple pole at $\beta = \frac{\gamma}{\gamma-1} i\alpha$ . A proper branch cut may be introduced as shown in Fig. E-1 so that the condition (E.2d) is satisfied on the whole real axis.




For $x > 0$ , we may close the paths for $\Gamma_a$ and $\Gamma_b$ both in the upper half of the cut plane. It can be shown that the contributions on [2] and [6] tend to zero as $|\beta| \to \infty$ . Then as $|\beta| \to \infty$ on [2] and [6], we have

$$\int_{[1]} + \int_{[3]} = \pi i \times \text{(Residue at } \beta = \tfrac{\gamma}{\gamma-1} i\alpha \text{ evaluated on [3])} \qquad \text{(E.3a)}$$

$$\int_{[4]} + \int_{[5]} = \pi i \times \text{(Residue at } \beta = \tfrac{\gamma}{\gamma-1} i\alpha \text{ evaluated on [5])} \qquad \text{(E.3b)}$$

Now on [3], $\beta = t e^{i\frac{\pi}{2}}, \quad \beta + ib = (t+b) e^{i\frac{\pi}{2}},$

$$\therefore \int_{[3]} = \int_0^\infty \frac{\frac{\gamma\alpha}{\gamma-1}}{t - \frac{\gamma\alpha}{\gamma-1}} \frac{e^{-xt}}{\sqrt{t(t+b)}} dt$$

and Residue on $[3] = -\frac{1}{\sqrt{1+(\gamma-1)M^2}} e^{-\frac{\gamma\alpha}{\gamma-1}x}$

while on [5] $\beta = t e^{-i\frac{3\pi}{2}}, \quad \beta + ib = (t+b) e^{i\frac{\pi}{2}}, \quad \therefore \int_{[5]} = \int_{[3]}$

and Residue on $[5] = \frac{1}{\sqrt{1+(\gamma-1)M^2}} e^{-\frac{\gamma\alpha}{\gamma-1}x}$

Therefore

$$\Gamma_{12} = \Gamma_{a_{[1]}} + \Gamma_{b_{[4]}} = -\int_{[3]} - \int_{[5]} = -2\int_0^\infty \frac{\frac{\gamma\alpha}{\gamma-1}}{t - \frac{\gamma\alpha}{\gamma-1}} \frac{e^{-xt}}{\sqrt{t(t+b)}} dt, \quad (x>0) \qquad \text{(E.4)}$$

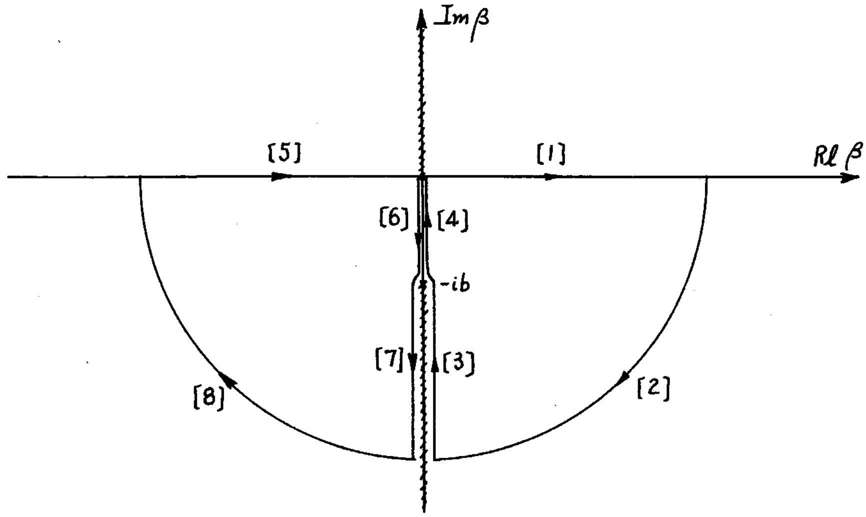
For $x < 0$ , close the contours in the lower half-plane (as shown in Fig. E-2). Here again as $|\beta| \to \infty$ on [2] and [8], we have

$$\int_{[1]} + \int_{[3]} + \int_{[4]} = 0 \qquad \text{(E.5a)}$$

$$\int_{[5]} + \int_{[6]} + \int_{[7]} = 0 \qquad \text{(E.5b)}$$

Now on [3], $\beta = t e^{-i\frac{\pi}{2}}, \quad \beta + ib = (t-b) e^{-i\frac{\pi}{2}}$

$$\therefore \int_{[3]} = -\int_b^\infty \frac{\frac{\gamma\alpha}{\gamma-1}}{t + \frac{\gamma\alpha}{\gamma-1}} \frac{e^{xt}}{\sqrt{t(t-b)}} dt$$




on [4], $\beta = t\,e^{-i\frac{\pi}{2}}$, $\beta + ib = (b-t)\,e^{i\frac{\pi}{2}}$

$$\therefore \int_{[4]} = i\int_0^b \frac{\frac{\gamma\alpha}{\gamma-1}}{t+\frac{\gamma\alpha}{\gamma-1}}\,\frac{e^{xt}}{\sqrt{t(b-t)}}\,dt$$

while on [6], $\beta = t\,e^{-i\frac{\pi}{2}}$, $\beta + ib = (b-t)\,e^{i\frac{\pi}{2}}$ $\therefore \int_{[6]} = -\int_{[4]}$

and on [7], $\beta = t\,e^{-i\frac{\pi}{2}}$, $\beta + ib = (t-b)\,e^{\frac{3\pi i}{2}}$, $\therefore \int_{[7]} = \int_{[3]}$

Therefore

$$\Gamma_{12} = -2\int_{[3]} = 2\int_b^\infty \frac{\frac{\gamma\alpha}{\gamma-1}}{t+\frac{\gamma\alpha}{\gamma-1}}\,\frac{e^{xt}}{\sqrt{t(t-b)}}\,dt \qquad (x<0) \qquad \text{(E.6)}$$

In summary, $\Gamma_{12}$ has different representations in terms of real integrals for $x \gtrless 0$.

$$\Gamma_{12}(x,0) = -2\int_0^\infty \frac{\frac{\gamma\alpha}{\gamma-1}}{t-\frac{\gamma\alpha}{\gamma-1}}\,\frac{1}{\sqrt{t(t+b)}}\,e^{-xt}\,dt \qquad (x>0) \qquad \text{(E.7a)}$$

$$= 2\,e^{\frac{\gamma U}{\kappa}x}\int_0^\infty \frac{\frac{\gamma\alpha}{\gamma-1}}{t+b\left[1+\frac{1}{(\gamma-1)M^2}\right]}\,\frac{e^{xt}}{\sqrt{t(t+b)}}\,dt \qquad (x<0) \qquad \text{(E.7b)}$$

It can be seen that $\Gamma_{12}$ decays exponentially for $x<0$, so the important contribution comes from the neighborhood of $x = 0^-$. If we integrate $\Gamma_{12}$ from $-\epsilon$ to $0$, where $\epsilon$ is a positive small quantity, we have

$$\int_{-\epsilon}^{0} \Gamma_{12}(x,0)\,dx = \frac{2}{b(\gamma-1)M^2}\int_0^\infty \left[\lambda + \frac{1+(\gamma-1)M^2}{(\gamma-1)M^2}\right]^{-1} \lambda^{-\frac{1}{2}}(\lambda+1)^{-\frac{1}{2}}\,\frac{1-e^{-\epsilon(\lambda+1)b}}{\lambda+1}\,d\lambda$$

This integral converges uniformly with respect to $\epsilon$ in any sub-region of $\epsilon \geqslant 0$, so we may obtain the limit of the integral as $\kappa \to 0$, or $b \to \infty$ by taking the limit of the integrand as $b \to \infty$.

$$\lim_{\kappa\to 0} \int_{-\epsilon}^{0} \Gamma_{12}(x,0)\,dx = \frac{4\kappa}{\gamma U}\left[1 - \frac{(\gamma-1)M^2}{\sqrt{1+(\gamma-1)M^2}}\, \coth^{-1}\sqrt{1+(\gamma-1)M^2}\right]$$

which is independent of $\epsilon$. This implies, in an approximated form, that

$$\Gamma_{12}(x,0) = \kappa\,\frac{4}{\gamma U}\,\frac{1}{1+(\gamma-1)M^2}\,\delta(x) + O\left(\left(\frac{\kappa}{U}\right)^{3/2}\right) \qquad \text{as } \kappa\to 0, \quad (x<0) \qquad \text{(E.8)}$$

(E.7a) may be approximated by noting that most of the contribution comes from the neighborhood of $\lambda = 0$ (cf. Watson's lemma, Ref. 41). The procedure is to expand

$$\frac{1}{1-(\gamma-1)M^2\lambda}\,\frac{1}{\sqrt{\lambda(\lambda+1)}} = \frac{1}{\sqrt{\lambda}}\left[1 + O(\lambda)\right]$$

and integrate termwise to obtain its asymptotic expression

$$\Gamma_{12}(x,0) = 2\sqrt{\frac{\pi\kappa}{\gamma U x}} + O\left(\left(\frac{\kappa}{U}\right)^{3/2}\right) \qquad (x>0) \qquad \text{(E.9)}$$

By a similar method applied to $\Gamma_{11}$, we have

$$\Gamma_{11}(x,0) = -\frac{2\kappa}{U}\,\frac{\gamma-1}{\gamma}\,\frac{M^2}{\sqrt{1-M^2}}\,\frac{1}{x} + O\left(\left(\frac{\kappa}{U}\right)^{3/2}\right), \qquad M<1 \qquad \text{(E.10a)}$$

$$= \frac{2\pi\kappa}{U}\,\frac{\gamma-1}{\gamma}\,\frac{M^2}{\sqrt{M^2-1}}\,\delta(x) + O\left(\left(\frac{\kappa}{U}\right)^{3/2}\right), \qquad M>1 \qquad \text{(E.10b)}$$

Now, the integral $\mathcal{U}_{23}$ given in (E.1d) may be written as

$$\mathcal{U}_{23} = -\frac{i\kappa}{U}\int_{-\infty}^{\infty}\sqrt{\frac{\beta+iU/\nu}{\beta}}\, e^{i\beta x}\, d\beta \tag{E.11}$$

If this is converted into real integrals by contour integration by a method similar to the previous one, the following result is obtained

$$\mathcal{U}_{23}(x) = \frac{8\gamma}{3}\int_0^{\infty}\sqrt{\frac{\lambda+1}{\lambda}}\, e^{-\frac{Ux}{\nu}\lambda}\, d\lambda \qquad (x>0) \tag{E.12a}$$

$$= -\frac{8\gamma}{3}\, e^{\frac{Ux}{\nu}}\int_0^{\infty}\sqrt{\frac{\lambda}{\lambda+1}}\, e^{\frac{Ux}{\nu}\lambda}\, d\lambda \qquad (x<0) \tag{E.12b}$$

For $x>0$ , Watson's lemma may be applied, which yields

$$\mathcal{U}_{23}(x) = 4\sqrt{\frac{\gamma\pi\kappa}{3Ux}} + O\left(\left(\frac{\kappa}{U}\right)^{3/2}\right) \qquad (x>0) \tag{E.13}$$

However, it needs extra attention to approximate $\mathcal{U}_{23}$ for $x<0$ . If we apply Watson's lemma to (E.12b), the asymptotic expansion starts with the term $\left(\frac{\kappa}{U|x|}\right)^{3/2} e^{-U|x|/\nu}$ ( $x<0$ ) which is not integrable at $x=0$ . To make sure that $\mathcal{U}_{23}(x)$ does not give any contribution for $x<0$ up to the term of order $\kappa$ , we integrate $\mathcal{U}_{23}$ from $-\epsilon$ to $\epsilon$ with respect to $x$ whose principal value exists.

$$\int_{-\epsilon}^{\epsilon}\mathcal{U}_{23}(x)\,dx = \frac{2\kappa}{U}\left\{\int_1^{\infty}\frac{1-e^{-\frac{U\epsilon}{\nu}\lambda}}{\lambda^{3/2}}\left[\sqrt{\lambda+1}-\sqrt{\lambda-1}\right]d\lambda + \int_0^1\frac{1-e^{-\frac{U\epsilon}{\nu}\lambda}}{\lambda^{3/2}}\sqrt{\lambda+1}\,d\lambda\right.$$

$$= \frac{2\kappa}{U}\left\{\mathcal{I}_a + \mathcal{I}_b\right\} \tag{E.14}$$

For $P_r = 3/4$ , $\nu$ tends to zero with $\kappa$ . It is clear that $\mathcal{I}_a$ converges and tends to a limit as $\nu\to 0$ .

$$\lim_{\nu\to 0}\mathcal{I}_a = \int_0^1\frac{\sqrt{1+x}-\sqrt{1-x}}{x}\,dx = 2\sqrt{2} - 2\log(\sqrt{2}+1) \tag{E.15}$$

For $\mathcal{I}_b$ , we have to integrate it by parts first in order to obtain convergent integrals as $\nu\to 0$ .

$$\mathcal{J}_b = -2\sqrt{2}\,(1-e^{-\frac{U\epsilon}{\nu}}) + \int_0^1 \frac{1-e^{-\frac{U\epsilon}{\nu}\lambda}}{\sqrt{\lambda(\lambda+1)}}\,d\lambda + 2\,\frac{U\epsilon}{\nu}\int_0^1 \sqrt{\frac{\lambda+1}{\lambda}}\, e^{-\frac{U\epsilon}{\nu}\lambda} d\lambda$$

$$= \mathcal{J}_{b1} + \mathcal{J}_{b2} + \mathcal{J}_{b3} \tag{E.16}$$

Now each of these three terms tends to a limit as $\nu \to 0$, namely

$$\lim_{\nu\to 0} \mathcal{J}_{b1} = -2\sqrt{2} \tag{E.17a}$$

$$\lim_{\nu\to 0} \mathcal{J}_{b2} = 2\log(\sqrt{2}+1) \tag{E.17b}$$

$$\frac{1}{\sqrt{\nu}} \lim_{\nu\to 0} \sqrt{\nu}\, \mathcal{J}_{b3} = 2\sqrt{\frac{\pi U\epsilon}{\nu}}\left[1 + O\left(\frac{\nu}{U}\right)\right] \tag{E.17c}$$

so that $\mathcal{J}_a + \mathcal{J}_{b1} + \mathcal{J}_{b2} \to 0$ as $\nu \to 0$ and

$$\int_{-\epsilon}^{\epsilon} \mathcal{U}_{23}(x)\,dx = \frac{4\kappa}{U}\sqrt{\frac{\pi U\epsilon}{\nu}}\left[1 + O\left(\frac{\nu}{U}\right)\right] \qquad as\ \kappa,\nu \to 0 \tag{E.18}$$

Comparing (E.13) and (E.18) we may draw the conclusion that

$$\mathcal{U}_{23}(x) = 4\sqrt{\frac{\pi\gamma\kappa}{3Ux}}\,\mathbb{1}(x) + O\left(\left(\frac{\kappa}{U}\right)^{3/2}\right) \tag{E.19}$$

where $\mathbb{1}(x)$ is the Heaviside step function defined by (14.18c).

In summary, we have:

(i) $M < 1$

$$4\pi\kappa\,\Gamma_1(x,0) = \sqrt{\kappa}\left(\frac{2\sqrt{\pi}}{\sqrt{\gamma U x}}\right) + \kappa\left(-\frac{2}{U}\,\frac{\gamma-1}{\gamma}\,\frac{M^2}{\sqrt{1-M^2}}\,\frac{1}{x}\right) + O(\kappa^{3/2}), \qquad x>0 \tag{E.20a}$$

$$= \kappa\left(-\frac{2}{U}\,\frac{\gamma-1}{\gamma}\,\frac{M^2}{\sqrt{1-M^2}}\,\frac{1}{x} + \frac{4}{\gamma U}\,\frac{\delta(x)}{1+(\gamma-1)M^2}\right) + O(\kappa^{3/2}), \quad x<0$$

$$4\pi\kappa\,\Gamma_2(x,0) = \kappa\left(2\,\frac{\gamma-1}{U^2}\,\frac{M^2}{\sqrt{1-M^2}}\,\frac{1}{x}\right) + O(\kappa^{3/2}) \tag{E.20b}$$

$$4\pi\kappa\,\mathcal{U}_1(x,0) = \kappa\left(\frac{2}{\gamma}\,\frac{1}{\sqrt{1-M^2}}\,\frac{1}{x}\right) + O(\kappa^{3/2}) \tag{E.20c}$$

$$4\pi\kappa\,\mathcal{U}_2(x,0) = \sqrt{\kappa}\left(4\sqrt{\frac{\pi}{3U}}\right)\frac{\mathbb{1}(x)}{\sqrt{x}} + \kappa\left(-\frac{2}{U\sqrt{1-M^2}}\,\frac{1}{x}\right) + O(\kappa^{3/2}) \tag{E.20d}$$

**(ii)** $M > 1$

$$4\pi\kappa\,\Gamma_1(x,0) = \sqrt{\kappa}\left(\frac{2\sqrt{\pi}}{\sqrt{\gamma U x}}\,\mathbb{1}(x)\right) + \kappa\left(\frac{2\pi}{U}\,\frac{\gamma-1}{\gamma}\,\frac{M^2}{\sqrt{M^2-1}} + \frac{4}{\gamma U}\,\frac{1}{1+(\gamma-1)M^2}\right)\delta(x) + O(\kappa^{3/2}) \quad \text{(E.20e)}$$

$$4\pi\kappa\,\Gamma_2(x,0) = \kappa\left(-\frac{2\pi}{U^2}\,\frac{(\gamma-1)\,M^2}{\sqrt{M^2-1}}\,\delta(x)\right) + O(\kappa^{3/2}) \quad \text{(E.20f)}$$

$$4\pi\kappa\,\mathcal{U}_1(x,0) = \kappa\left(-\frac{2\pi}{\gamma}\,\frac{1}{\sqrt{M^2-1}}\,\delta(x)\right) + O(\kappa^{3/2}) \quad \text{(E.20g)}$$

$$4\pi\kappa\,\mathcal{U}_2(x,0) = \sqrt{\kappa}\left(4\sqrt{\frac{\gamma\pi}{3U}}\,\frac{\mathbb{1}(x)}{\sqrt{x}}\right) + \kappa\left(\frac{2\pi}{U}\,\frac{1}{\sqrt{M^2-1}}\,\delta(x)\right) + O(\kappa^{3/2}) \quad \text{(E.20h)}$$

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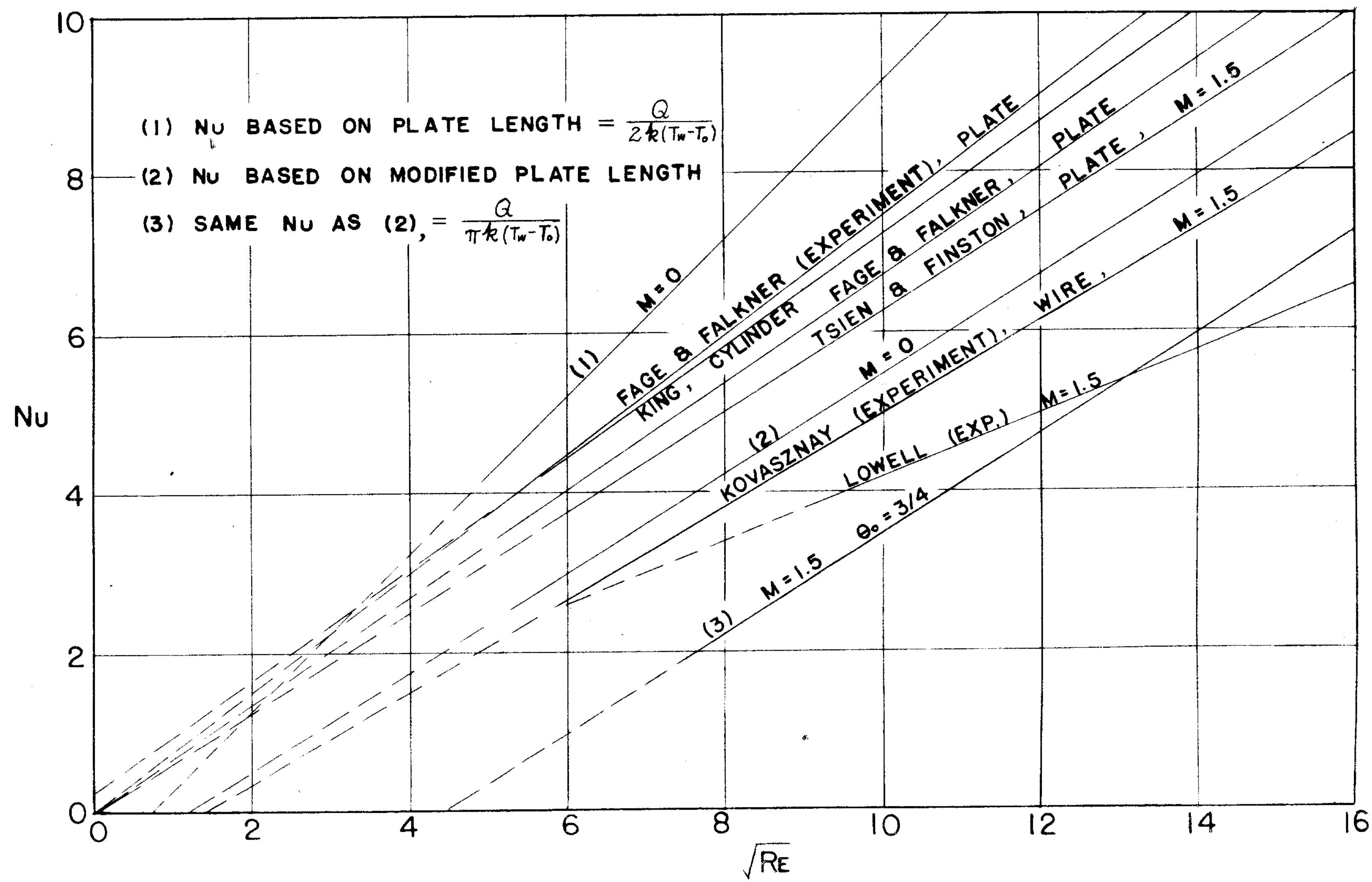


FIG.10. HEAT LOSS FROM HEATED FLAT PLATE OR CIRCULAR HOT-WIRE