Dispersive Properties of Schrödinger Equations

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Abstract

This thesis mainly concerns the dispersive property of Schrödinger equations with certain potentials, and some of their consequences.

We denote the evolution generated by the Hamiltonian $H(t) = -\Delta + V(x,t)$ as $U(t)$.

First, we consider the charge transfer models in $\mathbb{R}^n$ with $n \geq 3$. In this case, the potential $V(x,t)$ is a sum of several individual real-valued potentials, each moving with constant velocities. Our results are motivated by considering the asymptotic stability of noninteracting multisoliton states. For suitable initial data ψ_0, we obtain

$$\|U(t)\psi_0\|_{L^p} < C(n,p)t^{-n(\frac{1}{2}-\frac{1}{p})}\|\psi_0\|_{L^q}, \;\; p^{-1}+q^{-1}=1, \;\; 2 \leq p \leq +\infty. \qquad (0.0.1)$$

Second, we derive the same estimate as (0.0.1) for the derivatives of $U(t)\psi_0$ and prove the asymptotic completeness for charge transfer models in the Sobolev space $H^\kappa(\mathbb{R}^n)$.

Another kind of potential we consider is the spatially periodic case. In this case, the Hamiltonian $H(x)$ is called a Hill's operator. Generally, its spectrum is union of infinitely many intervals (bands). To get started, we focus on one class of finite band potentials, i.e., the Lamé operators. We derive a dispersive estimate with a decay rate $t^{-\frac{1}{3}}$.

Contents

Chapter 1

Background

The well-known free Schrödinger equation is of the form

$$\frac{1}{i}\partial_t\psi(t,x) = -\Delta\psi(t,x), \tag{1.0.1}$$

where the space variable $x \in \mathbb{R}^n$ and the time variable $t \in \mathbb{R}$. It is well-known that $\|\psi(t,\cdot)\|_{L^2(\mathbb{R}^n)} = \|\psi(0,\cdot)\|_{L^2(\mathbb{R}^n)}$ for any $t \in \mathbb{R}$ and the solution that satisfies the initial condition $\psi(0,x) = \psi_0(x)$ is given by the following:

$$\psi(t,x) = C_n t^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\frac{|x-y|^2}{4t}} \psi_0(y)dy. \tag{1.0.2}$$

This implies the $L^1 \to L^\infty$ estimate

$$\|\psi(t,\cdot)\|_{L^\infty(\mathbb{R}^n)} \le C_n t^{-\frac{n}{2}} \|\psi_0\|_{L^1(\mathbb{R}^n)}. \tag{1.0.3}$$

By interpolation, we derive the dispersive estimate

$$\|\psi(t,\cdot)\|_{L^p(\mathbb{R}^n)} \le C_n t^{-n(\frac{1}{2}-\frac{1}{p})} \|\psi_0\|_{L^q(\mathbb{R}^n)} \quad \frac{1}{p}+\frac{1}{q}=1 \quad 2 \le p \le +\infty. \tag{1.0.4}$$

The constant C_n depends on the dimension n and may vary from line to line. Many efforts have been made to obtain an analog of (1.0.4) for Schrödinger equations with a potential

$$\frac{1}{i}\partial_t\psi(t,x) = -\Delta\psi(t,x) + V(x)\psi(t,x), \tag{1.0.5}$$

where $V(x)$ is a time-independent and real-valued function defined on $\mathbb{R}^n$. Write $H = -\Delta + V(x)$. For $n \geq 3$ and small V, we employ a perturbative argument. When V is large, H may have bound states (eigenfunctions). It is well-known that bound states can arise for arbitrarily small potentials in dimensions $n = 1,\ 2$ (see Theorem XIII.11 in Reed and Simon [30]). Denote $P_c(H)$ as the projection onto the continuous part of the spectrum of self-adjoint operator H. The estimate should take the form

$$\|e^{itH}P_c(H)\psi_0\|_{L^p(\mathbb{R}^n)} \leq C_{n,p}t^{-n(\frac{1}{2}-\frac{1}{p})}\|\psi_0\|_{L^q(\mathbb{R}^n)} \quad \frac{1}{p}+\frac{1}{q}=1 \quad 2\leq p\leq +\infty. \tag{1.0.6}$$

1.1 Schrödinger operators in dimension $n \geq 3$

In dimension $n = 3$, Kato [24] showed that $-\Delta + V$ is unitarily equivalent to $-\Delta$, provided that $\|V\|_{Roll} < \frac{1}{4\pi}$, where

$$\|V\|^2_{Roll} = \int_{\mathbb{R}^6} \frac{V(x)V(y)}{|x-y|^2}dxdy.$$

Similar results are known for unitary equivalence for $d \geq 4$. For large V in dimension $n = 3$, Rauch [28] and Jensen and Kato [22] obtained weighted L^2 estimates,

$$\|we^{itH}wf\|_{L^2} \leq C|t|^{-\frac{3}{2}}\|f\|_{L^2}, \tag{1.1.1}$$

where $w(x) = e^{-\rho\langle x\rangle}$ with some $\rho > 0$ and V exponentially decaying (Rauch) or $w(x) = \langle x\rangle^{-\sigma}$ with some $\sigma > 0$ and V decaying at a power rate (Jensen, Kato). We denote $\langle x\rangle = (1+|x|^2)^{\frac{1}{2}}$. In addition, they assume that zero energy is neither an eigenvalue nor a resonance, i.e.,

$$\limsup_{\lambda\to 0}\|w(H-(\lambda\pm i0))^{-1}w\|_{L^2\to L^2} < \infty. \tag{1.1.2}$$

Journeé, Soffer and Sogge [23] proved (1.0.6) for dimension $n \geq 3$ under suitable power decay and regularity condition on V and under the assumption (1.1.2). Specifically, in dimension $n = 3$, they assume that $|V| < C\langle x\rangle^{-7-\epsilon}$ and $\hat{V} \in L^1(\mathbb{R}^3)$. These requirements were later relaxed by Yajima [40],[41] and [42]. Yajima also proved the L^p boundedness of the wave operators in $1 \leq p \leq \infty$. A different approach introduced by Rodnianski, Schlag [29] and by Goldberg, Schlag [16], leads to even weaker conditions on V. Finally, Goldberg [15] obtains (1.0.6) in dimension $n = 3$ under the assumption that (1.1.2) and $|V(x)| \leq C\langle x\rangle^{-2-\epsilon}$. For more details and references, see Schlag's survey [33].

For a general time-dependent potential $V(t, x)$, it is not clear how to introduce an analog of bound states and the spectral projection. In the case of small potentials, Rodnianski, Schlag treat the potential as a perturbation [29]. The case of large time-dependent potentials is still under investigation. In this thesis, we will consider one class of time-dependent potential, namely, the charge transfer model. The charge transfer model has been devised to describe the motion of a light particle in a collision between heavy ones. In this model, only the light particle is subject to quantum dynamics, while the heavy ones follow assigned classical trajectories, which are asymptotically inertial (see Graf [17]). This leads to the Hamiltonian

$$H(t) = -\frac{1}{2}\triangle + \sum_{j=1}^{m} V_j(x - \vec{v}_j t). \tag{1.1.3}$$

The charge transfer model has been extensively studied in the literature in connection with the question of asymptotic completeness, which is first established by Yajima [43]. Later, Graf [17] obtains a new proof containing a crucial argument showing that the energy associated with a solution of (1.1.4) remains bounded in time. Similar results are also obtained by Wüller [38] and Zielinski [44].

The charge transfer model also arises in the study of asymptotic stability of non-interacting multi-soliton states. This refers to solving an NLS

$$i\partial_t\psi + \frac{1}{2}\triangle\psi + \beta(|\psi|^2)\psi = 0$$

in $\mathbb{R}^n$, $n \geq 3$ with initial data $\psi_0 = \sum_{j=1}^m w_j(0,\cdot) + R_0$, where w_j are special standing wave solutions called solitons and R_0 is a small perturbation. In [32], Rodnianski, Schlag and Soffer show that if the solitons are sufficiently separated at time $t = 0$, and if R_0 is sufficiently small in a suitable norm, then the solution ψ evolves like a sum of solitons with time-dependent parameters approaching a limit, plus a radiation term that goes to zero in $L^\infty(\mathbb{R}^3)$. This leads to the problem of establishing dispersive estimates for the linear problem, which is closely related to the charge transfer model (see [31] and [32]):

$$\frac{1}{i}\partial_t \psi = -\frac{1}{2}\triangle\psi + \sum_{j=1}^m V_j(\cdot - \vec{v}_j t)\psi. \tag{1.1.4}$$

In [31] a weak version of the dispersive estimate for (1.1.4) is derived. Motivated by their approach, we will obtain a dispersive estimate for (1.1.4) in Chapter 2. Furthermore, we derive a $W^{\kappa,p'} \to W^{\kappa,p}$ estimate for the solution of (1.1.4). As an application, we will obtain the asymptotic completeness in $H^\kappa(\mathbb{R}^n)$.

1.2 Dispersive estimates for Schrödinger operators in dimension $n = 1, 2$

In dimension $n = 1, 2$, even a small potential may produce bound states, thus we cannot treat the potential as a perturbation. The dispersive estimates (1.0.6) in dimension $n = 1$ with a spatially decaying potential were first obtained by Weder [37] and Artbazar, Yajima [3]. These authors express the resolvent via the Jost solutions, namely, for $\Im z > 0$

$$(-\partial_x^2 + V - z^2)^{-1}(x,y) = \frac{f_+(x,z)f_-(y,z)}{W(z)}, \quad x > y$$

and symmetrically if $x < y$. Here $f_\pm$ are the Jost solutions defined as solutions of

$$-f''_\pm(\cdot,z) + V f_\pm(\cdot,z) = z^2 f_\pm(\cdot,z)$$

with the asymptotics

$$f_+(x,z) \sim e^{ixz} \quad \text{as} \quad x \to \infty$$
$$f_-(x,z) \sim e^{-ixz} \quad \text{as} \quad x \to -\infty,$$

and $W(z)$ is the Wronskian of $f_+(\cdot,z), f_-(\cdot,z)$. In this case, we say zero energy is a resonance if and only if $W(0) = 0$. Note that the free case $V = 0$ has a resonance at zero energy. Later, Goldberg and Schlag [16] proved (1.0.6) with a slightly less restrictive condition on V. They assumed $\langle x\rangle V(x) \in L^1(\mathbb{R})$ and the zero energy is not a resonance. Note that in term of pointwise decay, this is in agreement with the $\langle x\rangle^{-2}$ threshold. If zero is a resonance, they proved the same estimate assuming that $\langle x\rangle^2 V(x) \in L^1(\mathbb{R})$.

The dispersive estimates in dimension $n = 2$ were established by Schlag [34]. One crucial ingredient of the arguments is an asymptotic expansion of the resolvent around zero energy from Jensen and Nenciu [21]. In odd dimensions, the free resolvent $(-\Delta - z^2)^{-1}$ is analytic for all $z \neq 0$. When $\Im z > 0$,

$$(-\Delta + V - z^2)^{-1} = z^{-2}A_{-2} + z^{-1}A_{-1} + A_0 + zA_1 + O(z^2) \quad \text{as} \quad z \to 0 \qquad (1.2.1)$$

where the O-term is understood in the operator norm on a suitable weighted L^2-space. The free resolvent $(-\Delta - z^2)^{-1}$ is analytic for all $z \neq 0$. In even dimensions the Riemann surface of the free resolvent is that of the logarithm. Thus, (3.3.11) needs to include powers of $\log z$ in even dimensions.

In Chapter 3, we will study the dispersive property of Schrödinger operators with a periodic potentials on the real line. When $V(x)$ is periodic real function defined on $\mathbb{R}^1$, the spectrum of the Schrödinger operator $-(d^2/dx^2) + V(x)$ acting on $L^2(\mathbb{R}^1)$ is a union of intervals carrying a purely absolutely continuous spectrum. The absence of a point spectrum and a singular spectrum suggests the dispersion of the solution.

We will illustrate this dispersive phenomenon for a special analytic potential, whose spectrum is a union of two intervals (bands); namely, all gaps but one are degenerate. It is known that a one-gap potential $V(x)$ must be an elliptic function ([19]). With such a potential, we have

$$y''(x) - 2\wp(x+\omega_3)y(x) = -Ey(x), \tag{1.2.2}$$

where $\wp(z)$ is the Weierstrass elliptic function with periods $2\omega_1, 2\omega_3$, satisfying the following differential equations:

$$\wp'(a)^2 = 4\wp^3(a) - g_2\wp(a) - g_3, \tag{1.2.3}$$

$$\wp''(a) = 6\wp^2(a) - \frac{g_2}{2}, \tag{1.2.4}$$

where g_2, g_3 are the invariants of $\wp(z)$ defined by (3.5.1). Known as Lamé's equation, (1.2.2) arises from the theory of the potential of an ellipsoid ([39],[11]). We assume $\omega_1 = \omega > 0,\ \omega_3 = i\omega'$ and $\omega' > 0$ to guarantee that $\wp(x+\omega_3)$ is real-valued for $x \in \mathbb{R}$. If we choose the potential in (1.2.2) to be $n(n+1)\wp$, instead of $2\wp$ (n is any positive integer), then the spectrum of Lamé's equation consists of $n+1$ bands ([27]).

In Chapter 3, we give a dispersive estimate similar to (1.0.4) for the following equation:

$$\begin{aligned} \frac{1}{i}\partial_t\psi(x,t) &= -\frac{d^2}{dx^2}\psi(x,t) + 2\wp(x+\omega_3)\psi(x,t), \\ \psi(x,0) &= \psi_0(x). \end{aligned}$$

Chapter 2

Charge Transfer Models

In Section 2.1, we give our main results about the charge transfer models, and some necessary definitions. In Section 2.2, we present some lemma that are useful for proving our results. In Section 2.3, we prove the dispersive estimate for the charge transfer models. In Section 2.4, the dispersive estimates for the derivatives of the evolutions of the charge transfer models are proved by further investigating the argument in Section 2.3. In Section 2.5, we prove that the Sobolev norms of the evolution of charge transfer models remain bounded. In Section 2.6, the existence of the wave operator and the asymptotic completeness for the charge transfer models are established.

2.1 Definitions and main results

Definition 2.1.1. *By a* charge transfer model *we mean a Schrödinger equation*

$$\frac{1}{i}\partial_t\psi - \frac{1}{2}\triangle\psi + \sum_{\kappa=1}^{m} V_\kappa(x-\vec{v}_\kappa t)\psi = 0 \tag{2.1.1}$$
$$\psi|_{t=0} = \psi_0, x \in \mathbb{R}^n,$$

where $\vec{v}_\kappa$ are distinct vectors in $\mathbb{R}^n$, $n \geq 3$, and the real potentials V_κ are such that for every $1 \leq \kappa \leq m$,

1. *V_κ is time independent and has compact support (or fast decay), $V_\kappa, \nabla V_\kappa \in L^\infty$;*

2. 0 *is neither an eigenvalue nor a resonance of the operators*

$$H_\kappa = -\frac{1}{2}\triangle + V_\kappa(x).$$

Recall that ψ is a resonance if it is a distributional solution of the equation $H_\kappa\psi = 0$, which belongs to the space $L^2(\langle x\rangle^{-\sigma}dx)$ for any $\sigma > \frac{1}{2}$, but not for $\sigma = 0$.

The conditions in the above definition are always assumed when we prove and apply the dispersive estimates, i.e., Theorem 2.1.3 and Theorem 2.1.4. The conditions are not optimal, but for convenience. This definition is standard (see [17], [42]). The Schrödinger group e^{-itH_κ} is known to satisfy the decay estimates (see Journé, Soffer, Sogge [23], and Yajima [41])

$$\|e^{-itH_\kappa}P_c(H_\kappa)\psi_0\|_{L^\infty} \lesssim |t|^{-\frac{n}{2}}\|\psi_0\|_{L^1} \tag{2.1.2}$$

for $n \geq 3$ under various conditions on the potential. Here $P_c(H_\kappa)$ is the spectral projection onto the continuous spectrum of H_κ and $\lesssim$ denotes bounds involving multiplicative constants independent of ψ_0 and t. For $n = 3$, [16] proved (2.1.2) under the assumption that $|V_\kappa(x)| \leq C(1+|x|)^{-\beta}$, for some $\beta > 3$. For $n > 3$, (2.1.2) holds ([23]) under the additional assumption: $\mathcal{F}(V_\kappa) \in L^1$. Yajima [41] proved (2.1.2) with slightly weaker conditions than [23]. In this chapter we shall assume that $\mathcal{F}(V_\kappa) \in L^1$ to guarantee the estimate (2.1.2), except in Section 2.5.

To establish similar dispersive estimates for time-dependent Schrödinger equations is more involved. Heuristically, we can't project away the bounded states as they are moving in different directions. Rodnianski and Schlag [29] proved dispersive estimates for small time-dependent potentials. In this paper, we will focus on the charge transfer model.

An indispensable tool in the study of the charge transfer model is the Galilean transforms

$$\mathfrak{g}_{\vec{v},y}(t) = e^{-i\frac{|\vec{v}|^2}{2}t}e^{-ix\cdot\vec{v}}e^{i(y+t\vec{v})\cdot\vec{p}}, \tag{2.1.3}$$

cf. [17], where $\vec{p} = -i\vec{\nabla}$. Under $\mathfrak{g}_{\vec{v},y}(t)$, the Schrödinger equation transforms as follows:

$$\mathfrak{g}_{\vec{v},y}(t)e^{it\frac{\triangle}{2}} = e^{it\frac{\triangle}{2}}\,\mathfrak{g}_{\vec{v},y}(0) \tag{2.1.4}$$

and moreover, with $H = -\frac{1}{2}\triangle + V$,

$$\psi(t) := \mathfrak{g}_{\vec{v},y}(t)^{-1}e^{-itH}\,\mathfrak{g}_{\vec{v},y}(0)\phi_0,, \qquad \mathfrak{g}_{\vec{v},y}(t)^{-1} = e^{-iy\cdot\vec{v}}\,\mathfrak{g}_{-\vec{v},-y}(t) \tag{2.1.5}$$

solves

$$\begin{aligned} \tfrac{1}{i}\partial_t\psi - \tfrac{1}{2}\triangle\psi + V(\cdot - t\vec{v} - y)\psi = 0 \\ \psi|_{t=0} = \phi_0. \end{aligned} \tag{2.1.6}$$

Since in our case $y = 0$ always, we set $\mathfrak{g}_{\vec{v}}(t) := \mathfrak{g}_{\vec{v},0}(t)$. Note that the transformations $\mathfrak{g}_{\vec{v},y}(t)$ are isometries on all L^p spaces and $\mathfrak{g}_{\vec{e_1}}(t)^{-1} = \mathfrak{g}_{-\vec{e_1}}(t)$ because of (2.1.5). In the following, we shall assume that the number of potentials is $m = 2$ and that the velocities are $\vec{v}_1 = 0, \vec{v}_2 = (1, 0, \ldots, 0) = \vec{e}_1$. The arguments generalize easily to $m \geq 3$.

We now introduce the appropriate analog to project away bounded states for the problem

$$\begin{aligned} &\frac{1}{i}\partial_t\psi - \frac{1}{2}\triangle\psi + V_1\psi + V_2(\cdot - t\vec{e_1})\psi = 0 \\ &\psi|_{t=0} = \psi_0 \end{aligned} \tag{2.1.7}$$

with compactly supported potentials V_1, V_2. Let $u_1, \ldots, u_m$ and $w_1, \ldots, w_\ell$ be the normalized bound states of H_1 and H_2 corresponding to the negative eigenvalues $\lambda_1, \ldots, \lambda_m$ and $\mu_1, \ldots, \mu_\ell$, respectively (recall that we are assuming that 0 is not an eigenvalue). We denote by $P_b(H_1)$ and $P_b(H_2)$ the corresponding projections onto the bound states of H_1 and H_2, respectively, and let $P_c(H_\kappa) = Id - P_b(H_\kappa)$, $\kappa = 1, 2$.

The projections $P_b(H_{1,2})$ have the form

$$P_b(H_1) = \sum_{i=1}^{m} \langle \cdot, u_i \rangle u_i, \qquad P_b(H_2) = \sum_{j=1}^{\ell} \langle \cdot, w_j \rangle w_j.$$

We introduce the following orthogonality condition in the context of the charge transfer Hamiltonian:

Definition 2.1.2. *Let* $U(t)\psi_0 = \psi(t,x)$ *be the solutions of* (2.1.7)*. We say that* ψ_0 *(or also* $\psi(t,\cdot)$*) is* asymptotically orthogonal *to the bound states of* H_1 *and* H_2 *if*

$$\|P_b(H_1)U(t)\psi_0\|_{L^2} + \|P_b(H_2,t)U(t)\psi_0\|_{L^2} \to 0 \text{ as } t \to +\infty. \tag{2.1.8}$$

Here

$$P_b(H_2,t) := \mathfrak{g}_{-\vec{e}_1}(t) P_b(H_2)\, \mathfrak{g}_{\vec{e_1}}(t) \tag{2.1.9}$$

for all times t*.*

Remark 2.1.1. *From Corollary 2.2.3,* $\|U(t)\psi_0\|_{L^p} \leq C_t \|\psi_0\|_{L^{p'}}$*, we know that* $U(t)\psi_0 \in L^p$ *is well-defined for* $\psi_0 \in L^{p'}$*. As the bound states* u_i, w_j *are exponentially decaying at infinity, Definition 2.1.2 makes sense for any initial data* $\psi_0 \in L^{p'}$ *for* $p' \in [1,2]$*.*

Remark 2.1.2. *Clearly,* $P_b(H_2,t)$ *is again an orthogonal projection for every* t*. It gives the projection onto the bound states of* H_2 *that have been translated to the position of the potential* $V_2(\cdot - t\vec{e_1})$*. Equivalently, one can think of it as translating the solution of* (2.1.7) *from that position to the origin, projecting onto the bound states of* H_2*, and then translating back.*

Remark 2.1.3. *From Proposition 3.1 of [31], the decay rate of* (2.1.8) *is actually exponential. More precisely, the following holds:*

$$\|P_b(H_1)U(t)\psi_0\|_{L^2} + \|P_b(H_2,t)U(t)\psi_0\|_{L^2} \lesssim e^{-\alpha t}\|\psi_0\|_{L^2}, \tag{2.1.10}$$

for some $\alpha > 0$*.*

Remark 2.1.4. *It is clear that all* ψ_0 *that satisfy* (2.1.8) *form a closed subspace. This subspace coincides with the space of* scattering states *for the charge transfer problem. The latter is well-defined by Graf's asymptotic completeness result [17].*

We can only expect the dispersive estimate for (2.1.7) for the initial data satisfying Definition 2.1.2, just as we have to project away the bound states for (2.1.2). Rodnianski, Schlag, Soffer [31] established the following estimate:

$$\|U(t)\psi_0\|_{L^2+L^\infty} \lesssim \langle t\rangle^{-\frac{n}{2}}\|\psi_0\|_{L^1\cap L^2} \tag{2.1.11}$$

with initial data $\psi_0 \in L^1 \cap L^2$ satisfying (2.1.8), where $U(t)$ is the evolution of the charge transfer model and $\langle t\rangle = (1+t^2)^{\frac{1}{2}}$. By definition, $\|f\|_{L^2+L^\infty} := \inf_{f=h+g}(\|h\|_{L^2}+\|g\|_{L^\infty})$ and $\|f\|_{L^1\cap L^2} = \|f\|_{L^1} + \|f\|_{L^2}$. (2.1.11) has an important application to the asymptotic stability and asymptotic completeness for the small perturbation of non-colliding solitons for NLS ([32]).

[31] decomposes the evolution into different channels according to each potential. Every channel splits into a large velocity part and a low velocity part. For the large velocity part, Kato's smoothing estimate was employed; for the low velocity part, a propagation estimate was used. In this paper, we will combine the methods from [23] and [31] and obtain the following:

Theorem 2.1.3. *Consider the charge transfer model as in Definition 2.1.1 with two potentials, cf.* (2.1.7). *Assume* $\widehat{V_1}, \widehat{V_2} \in L^1(\mathbb{R}^n)$. *Let* $U(t)$ *denote the propagator of* (2.1.7). *Then for any initial data* $\psi_0 \in L^1$, *which is asymptotically orthogonal to the bound states of* H_1 *and* H_2 *in the sense of Definition 2.1.2, one has the decay estimates*

$$\|U(t)\psi_0\|_{L^\infty} \lesssim |t|^{-\frac{n}{2}}\|\psi_0\|_{L^1}. \tag{2.1.12}$$

An analogous statement holds for any number of potentials, i.e., with arbitrary m *in* (2.1.1).

Inspection of the argument in the following sections shows that it applies, say, to exponentially decaying potentials. But sufficiently fast power decay at infinity is also allowed. We shall prove (3.3.1) by means of a bootstrap argument. More precisely, we prove that the *bootstrap assumption*

$$\|U(t)\psi_0\|_{L^\infty} \leq C_0|t|^{-\frac{n}{2}}\|\psi_0\|_{L^1} \quad \text{for all} \quad 0 \leq t \leq T \tag{2.1.13}$$

implies that

$$\|U(t)\psi_0\|_{L^\infty} \leq \frac{C_0}{2}|t|^{-\frac{n}{2}}\|\psi_0\|_{L^1} \quad \text{for all} \quad 0 \leq t \leq T. \tag{2.1.14}$$

Here T is any given fixed large constant and (2.1.13) holds for C_0 some sufficiently large constant because of Corollary 2.2.3. C_0 may depend on T in the beginning. The above implication (2.1.13) $\Longrightarrow$ (2.1.14) holds as long as C_0 is larger than some universal constant independent of the time T. Thus iterating (2.1.13) $\Longrightarrow$ (2.1.14) then yields a constant that does not depend on T. The theorem follows by letting $T \to +\infty$.

As the L^2 norm of $U(t)\psi_0$ remains constant, by interpolation, the following holds:

$$\|U(t)\psi_0\|_{L^p} \lesssim C_p|t|^{-n(\frac{1}{2}-\frac{1}{p})}\|\psi_0\|_{L^{p'}} \quad p \geq 2, \ \frac{1}{p}+\frac{1}{p'}=1. \tag{2.1.15}$$

Our next theorem is about the decay estimates of $\partial^\alpha U(t)\psi_0$, where $\alpha = (\alpha_1, \cdots, \alpha_n)$ is an n-tuple of nonnegative integers and $\partial^\alpha = \frac{\partial^\alpha}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}$. We write $|\alpha| = \alpha_1+\cdots+\alpha_n$.

Theorem 2.1.4. *Let $U(t)$ denote the propagator of the equation* (2.1.7). *Assume* (2.1.2) *holds for H_1 and H_2. Let $V_j \in C_0^{\kappa+1}$ where κ is a positive integer and $j = 1, 2$. Moreover, assume that for $\forall|\beta| \leq \kappa$ and $j = 1, 2$, $\widehat{\partial^\beta V_j} \in L^1(\mathbb{R}^n)$. Then for any initial data $\psi_0 \in W^{\kappa,p'}$, which is asymptotically orthogonal to the bound states of H_j $(j = 1, 2)$ in the sense of Definition 2.1.2, one has the decay estimates*

$$\|U(t)\psi_0\|_{W^{\kappa,p}} \lesssim |t|^{-n(\frac{1}{2}-\frac{1}{p})}\|\psi_0\|_{W^{\kappa,p'}}, \tag{2.1.16}$$

where $2 \leq p < \infty$ and $\frac{1}{p}+\frac{1}{p'}=1$.

Remark 2.1.5. *It suffices to prove Theorem 2.1.4 for $p > \frac{2n}{n-2}$, because interpolating with Theorem 2.1.5, which holds under the assumption of Theorem 2.1.4, we derive*

Theorem 2.1.4 for any $p \in [2, +\infty]$. $p > \frac{2n}{n-2}$ *guarantees that* $\int_1^\infty |t|^{-n(\frac{1}{2}-\frac{1}{p})} < \infty$. *We need to exclude the case* $p = \infty$, *since part of our proof relies on singular integrals and we do not know whether or not* (2.1.16) *holds for* $p = \infty$.

The second part of this paper is motivated by Graf [17]. Graf proved energy boundedness for $U(t,s)$ by a geometric method, where $U(t,s)$ is the solution operator corresponding to the time-dependent Schrödinger equation

$$\begin{aligned} &\frac{1}{i}\partial_t \psi - \frac{\triangle}{2}\psi + \sum_{j=1}^{m} V_j(x - \vec{v}_j t)\psi = 0, \\ &\psi|_{t=s} = \psi_0, \end{aligned} \tag{2.1.17}$$

i.e., $\psi(t,\cdot) = U(t,s)\psi_0$. Graf proved that $\|U(t,s)\psi_0\|_{H^1}$ is bounded as $t \to \infty$ provided that the initial data $\psi_0 \in H^1(\mathbb{R}^n)$, $n \geq 1$. For the higher degree Sobolev norms, J. Bourgain [6] proved the following for the general time-dependent Hamiltonian $H(t)$:

Suppose the time-dependent potential $V(x,t)$ is bounded, real, and $\sup_{t\in\mathbb{R}} |V(x,t)|$ is compactly supported. Moreover, for any n-tuple α

$$\sup_{t\in\mathbb{R}} \|D_x^\alpha V(t)\|_\infty < C_\alpha.$$

Then for $\forall\epsilon > 0$ and $\kappa > 0$,

$$\|U(t,0)\psi_0\|_{H^\kappa} \leq C_{\epsilon,\kappa}|t|^\epsilon \|\psi_0\|_{H^\kappa} \qquad \text{for all} \qquad t. \tag{2.1.18}$$

An example ([6]) is given to show that we cannot remove the $|t|^\epsilon$ growth for general time-dependent potentials. In this paper, it is shown that (2.1.18) does hold with $\epsilon = 0$ for the case of the charge transfer Hamiltonian. More precisely, in Section 4, the time-boundedness of $\|U(t,s)\psi_0\|_{H^\kappa}$ for charge transfer models is established by the same geometric method as in [17] for any real number κ. We write $\lceil x \rceil$ as the least integer no less than x. The precise statement is as follows:

Theorem 2.1.5. *Let* $U(t,s)$ *be the evolution operator for* (2.1.17), *and let* $\kappa \in \mathbb{R}$

and the dimension $n \geq 1$. Furthermore, suppose $V_j \in C_0^{\lceil|\kappa|\rceil}(\mathbb{R}^n)$, $(j = 1, 2, \cdots, m)$, i.e., V_j has derivatives up to degree $\lceil|\kappa|\rceil$, which are all continuous and compactly supported. Then for $\forall t, s \in \mathbb{R}$

$$\|U(t,s)\psi_0\|_{H^\kappa} \leq C_\kappa \|\psi_0\|_{H^\kappa},$$

where C_κ depends on κ and the potentials V_j.

Remark 2.1.6. *By interpolation, it clearly suffices to consider the case where κ is an integer. By duality, it suffices to prove the case where κ is a positive integer. Indeed, assuming $\kappa < 0$, due to the fact that $U(t,s)$ is unitary on $L^2(\mathbb{R}^n)$, we have*

$$\begin{aligned}\|U(t,s)\psi_0\|_{H^\kappa} &= \sup_{\|\phi\|_{H^{-\kappa}}=1} \langle U(t,s)\psi_0, \phi\rangle_{L^2} \\ &= \sup_{\|\phi\|_{H^{-\kappa}}=1} \langle \psi_0, U(s,t)\phi\rangle_{L^2} \leq C_{-\kappa}\|\psi_0\|_{H^\kappa}.\end{aligned}$$

No assumption is made on the spectra of the subsystems H_j. The assumption of compact support of V_j is for convenience only and the proof works for sufficiently fast polynomial decay at infinity without essential change ([17]). Suppose all assumptions of both Theorem 2.1.4 and Theorem 2.1.5 hold; then by interpolation, the estimate (2.1.16) *holds for $2 \leq p < \infty$.*

Remark 2.1.7. *It follows from Duhamel's formula and Gronwall's inequality, that*

$$\|U(t,s)\psi_0\|_{H^\kappa} \leq C(I)\|\psi_0\|_{H^\kappa} \quad t, s \in I, \tag{2.1.19}$$

for any compact interval I. Therefore, it suffices to prove Theorem 2.1.5 when $|t|$ or $|s|$ is large.

As an important consequence, we apply Theorem 2.1.4 and Theorem 2.1.5 to obtain the following asymptotic completeness for the charge transfer model in the H^κ sense:

Theorem 2.1.6. *Let $u_1, \ldots, u_m$ and $w_1, \ldots, w_\ell$ be the eigenfunctions of $H_1 = -\frac{\triangle}{2} + V_1(x)$ and $H_2 = -\frac{\triangle}{2} + V_2(x)$, respectively, corresponding to the negative eigenvalues*

$\lambda_1, \cdots, \lambda_m$ and $\mu_1, \cdots, \mu_\ell$. Assume that $V_j \in C_0^{n+2\kappa+2}(\mathbb{R}^n)$, $(n \geq 3,\ j = 1,2)$, and that 0 is neither an eigenvalue nor a resonance of H_1, H_2, where κ is a nonnegative integer. Then for any initial data $\psi_0 \in H^\kappa$, the solution $U(t)\psi_0$ of the charge transfer problem (2.1.7) can be written in the form

$$U(t)\psi_0 = \sum_{r=1}^{m} A_r e^{-i\lambda_r t} u_r + \sum_{k=1}^{\ell} B_k e^{-i\mu_k t}\, \mathfrak{g}_{-\vec{e}_1}(t) w_k + e^{-it\frac{\triangle}{2}}\phi_0 + \mathcal{R}(t),$$

for some choice of the constants A_r, B_k and the function $\phi_0 \in H^\kappa$. The remainder term $\mathcal{R}(t)$ satisfies the estimate

$$\|\mathcal{R}(t)\|_{H^\kappa} \longrightarrow 0, \quad \text{as } t \to \infty.$$

Remark 2.1.8. *The above theorem holds for m potentials. We are not aiming to give the optimal regularity condition on the potentials. The theorem is equivalent to claiming that $H^\kappa(\mathbb{R}^n)$ is the sum of the ranges of the wave operators Ω_l^-, $(l = 0, 1, 2)$ defined in Section 6.1. [17] proved that the ranges of the wave operators are orthogonal to each other in the L^2 sense. Therefore, $H^\kappa(\mathbb{R}^n)$ again is a direct sum of $\Omega_l^-(H^\kappa)$.*

2.2 Preparation

In this section, we present some lemma that are indispensable for the proof of the above theorems. The first ingredient of our proof is the notion of cancellation. In this section, $U(t)$ will denote the evolution operator of (2.1.7) or (2.1.1). It is clear from their proofs that the following lemmas also hold for the general time-dependent Hamiltonian $H_0 + V(t)$.

Lemma 2.2.1.

$$\sup_{-\infty < t < \infty} \|e^{it\Delta} V e^{-it\Delta}\|_{p \to p} \leq \|\widehat{V}\|_1, \tag{2.2.1}$$

where $p \in [1, \infty]$ and $\|\cdot\|_{p\to p}$ means the operator norm from L^p to L^p.

For the proof of the lemma, notice that equation (2.1.4) implies $[e^{it\Delta} e^{i\zeta x} e^{-it\Delta} f](x) = \mathfrak{g}_{-\zeta}(2t) f(x) = e^{-it|\zeta|^2} e^{ix\zeta} f(x - 2\zeta t)$.

Lemma 2.2.2. *Suppose $t, s \in \mathbb{R}$, then we have the following:*

$$\sup_{r\in\mathbb{R}} \|e^{-i(t-s)H_0}V(r)U(s)\|_{1\to\infty} < |t|^{-\frac{n}{2}}CMe^{M|s|}, \tag{2.2.2}$$

where $M = \max_{r\in\mathbb{R}} \|\widehat{V(r)}\|_1 < \infty$.

Proof. Let's write $\Psi(t,s) := \sup_{r\in\mathbb{R}} \|e^{-i(t-s)H_0}V(r)U(s)\|_{1\to\infty}$. Without loss of generality, we suppose that $s > 0$. By Duhamel's formula,

$$e^{-i(t-s)H_0}V(r)U(s) = e^{-i(t-s)H_0}V(r)\{e^{-isH_0} - i\int_0^s e^{-i(s-\tau)H_0}V(\tau)U(\tau)d\tau\},$$

it follows that

$$\begin{aligned}
&\|e^{-i(t-s)H_0}V(r)U(s)\|_{1\to\infty}\\
&\leq \|e^{-i(t-s)H_0}V(r)e^{-isH_0}\|_{1\to\infty} + \int_0^s \|e^{-i(t-s)H_0}V(r)e^{-i(s-\tau)H_0}V(\tau)U(\tau)\|_{1\to\infty}d\tau\\
&\leq C\|\widehat{V}(r)\|_1|t|^{-\frac{n}{2}} + \|\widehat{V}(r)\|_1\int_0^s \|e^{-i(t-\tau)H_0}V(\tau)U(\tau)\|_{1\to\infty}d\tau\\
&\leq CM|t|^{-\frac{n}{2}} + M\int_0^s \Psi(t,\tau)d\tau.
\end{aligned}$$

Taking the supremum over r, we get $\Psi(t,s) \leq CM|t|^{-\frac{n}{2}} + M\int_0^s \Psi(t,\tau)d\tau$. By Gronwall's inequality,

$$\Psi(t,s) \leq CM|t|^{-\frac{n}{2}}e^{Ms}.$$

□

Note that the lemma still holds with other constants C and M on the right-hand side if we replace $V(r)$ with $V_j(r)$ or replace $U(s)$ with another evolution, say e^{-isH_j}. Another observation is that the lemma can be generalized to the following by the same proof:

$$\sup_{r\in\mathbb{R}} \|e^{-i(t-s)H_0}V(r)U(s)\psi_0\|_p \lesssim |t|^{-\gamma}Me^{Ms}\|\psi_0\|_{p'} \tag{2.2.3}$$

where $\gamma = n(\frac{1}{2} - \frac{1}{p})$ and $2 \le p \le \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$. This will be useful in Section 4.

Corollary 2.2.3. *Suppose $U(t)$ is the evolution operator of (2.1.7) or (2.1.1). Assume $t > 0$, then*

$$\|U(t)\|_{p' \to p} \lesssim t^{-n(\frac{1}{2}-\frac{1}{p})} e^{Mt} \quad \frac{1}{p} + \frac{1}{p'} = 1,\ 2 \le p \le \infty \tag{2.2.4}$$

Proof. By Duhamel's formula, $U(t) = e^{-itH_0} - i\int_0^t e^{-i(t-\tau)H_0} V(\tau)U(\tau)d\tau$. Write $\gamma = n(\frac{1}{2} - \frac{1}{p})$, then by Lemma 2.2.2, we have

$$\|U(t)\|_{p' \to p} \le Ct^{-\gamma} + \int_0^t \Psi(t,\tau)d\tau \le Ct^{-\gamma} + \int_0^t Ct^{-\gamma} M e^{M\tau} d\tau \le Ct^{-\gamma} e^{Mt}.$$

□

From the corollary, the bootstrap assumption (2.1.13) holds for any time T if we take $C_0 = Ce^{MT}$.

Lemma 2.2.4. *Suppose $m \ge 1$ and $\epsilon > 0$. If $u_1, u_2, \ldots, u_m$ are either all positive or all negative, satisfying $|\sum_{j=1}^m u_j| > \epsilon$, then there exists a constant $C = C(m,\epsilon)$ such that*

$$\|\prod_{j=1}^{m-1} (e^{iu_j H_0} V(s_j)) e^{iu_m H_0}\|_{1\to\infty} \le CM^{m-1} \prod_{j=1}^{m} \langle u_j \rangle^{-\frac{n}{2}} \tag{2.2.5}$$

$$\|\prod_{j=1}^{m-1} (e^{iu_j H_0} V(s_j)) U(u_m)\|_{1\to\infty} \le CM^{m-1} \prod_{j=1}^{m} \langle u_j \rangle^{-\frac{n}{2}} e^{Mu_m} \tag{2.2.6}$$

where s_j is any real number and $M = \sup_{s\in R}(\|V(s)\|_1 + \|\widehat{V}(s)\|_1)$.

Proof. The first inequality is from [23]. Assume that $u_1, u_2, \ldots, u_m$ are all positive without loss of generality. We apply the dispersive estimate for $e^{iu_j H_0}$ repeatedly and the left-hand side is dominated by $CM^{m-1}\prod_{j=1}^m u_j^{-\frac{n}{2}}$, which is dominated by the right-hand side up to a constant, provided each $u_j > \epsilon$. If some $u_j \le \epsilon$, it is inefficient to use a dispersive estimate for $e^{iu_jH_0}$. Instead, we apply the cancellation lemma 2.2.1 and obtain

$$e^{iu_jH_0}V(s_j)e^{iu_{j+1}H_0} = \int e^{iu_jH_0}e^{ix\zeta}e^{-iu_jH_0}\widehat{V(s_j)}(\zeta)\,d\zeta e^{i(u_{j+1}+u_j)H_0},$$

where $e^{iu_jH_0}e^{ix\zeta}e^{-iu_jH_0}$ is the Galilean transform $\mathfrak{g}_{-\zeta}(-u_j)$ according to (2.1.4). If again $u_j + u_{j+1} < \epsilon$, we can repeat this procedure until $u_{j-l} + \cdots + u_j + \cdots + u_{j+k} > \epsilon$ which always happens because $|\sum_{j=1}^m u_j| > \epsilon$. Then we apply the dispersive estimate to obtain the inequality.

We sketch the proof of the second equation. When $m = 1$, it is just (2.2.4) provided that $u_m > \epsilon$. When $m = 2$, if $u_1 > \frac{\epsilon}{2}$ and $u_2 > \frac{\epsilon}{2}$,

$$\begin{aligned}
&\|(e^{iu_1H_0}V(s_1))U(u_2)\|_{1\to\infty} \lesssim |u_1|^{-\frac{n}{2}}\|V(s_1)U(u_2)\|_{1\to 1}\\
&\lesssim |u_1|^{-\frac{n}{2}}\|U(u_2)\|_{1\to\infty}\\
&\lesssim |u_1|^{-\frac{n}{2}}|u_2|^{-\frac{n}{2}}e^{Mu_2} \lesssim \langle u_1\rangle^{-\frac{n}{2}}\langle u_2\rangle^{-\frac{n}{2}}e^{Mu_2}
\end{aligned}$$

If $u_1 \le \frac{\epsilon}{2}$ or $u_2 \le \frac{\epsilon}{2}$, we apply Lemma 2.2.2

$$\|e^{iu_1H_0}V(s_1)U(u_2)\|_{1\to\infty} \lesssim (|u_1| + |u_2|)^{-\frac{n}{2}}e^{Mu_2} \lesssim \langle u_1\rangle^{-\frac{n}{2}}\langle u_2\rangle^{-\frac{n}{2}}e^{Mu_2}$$

The case where $m > 2$ follows exactly as the first inequality using Lemma 2.2.1. □

Next we list a variant of Kato's $\frac{1}{2}$-smoothing estimate. It appears in [31]. We give the proof for completeness.

Lemma 2.2.5. *Let $H = -\frac{1}{2}\triangle + V$, $\|V\|_\infty < \infty$. Then for all $T, R, M \ge 1$,*

$$\sup_{B_R}\int_0^T \|F(|\vec{p}| \ge M)e^{-itH}f\|_{L^2(B_R)}\,dt \le C(n,V)\,\frac{TR}{M^{\frac{1}{2}}}\|f\|_{L^2}. \tag{2.2.7}$$

Here the supremum ranges over all balls B_R of radius $R \ge 1$ and $C(n,V)$ is a constant that depends only on $\|V\|_\infty$ and the dimension n.

Proof. We first prove the following estimate, which will then imply the lemma. Let $\psi(t) = e^{-itH}\psi_0$ with $H = -\frac{1}{2}\triangle + V$, $\|V\|_\infty < \infty$. Then for all $T > 0$ and $0 < \alpha$,

$$\sup_{x_0\in\mathbb{R}^n} \int_0^T \int_{\mathbb{R}^n} \frac{|\nabla\langle\nabla\rangle^{-\frac{1}{2}}\psi(x,t)|^2}{(1+|x-x_0|^\alpha)^{\frac{1}{\alpha}+1}}\, dxdt \le C_{\alpha,n}\, T(1+\|V\|_\infty)\|\psi_0\|_2^2. \tag{2.2.8}$$

The multiplier $\nabla\langle\nabla\rangle^{-\frac{1}{2}}$ corresponds to the symbol $\xi\langle\xi\rangle^{-\frac{1}{2}} = \xi(1+|\xi|^2)^{-\frac{1}{4}}$. It suffices to prove this with $x_0 = 0$ fixed. The proof is based on taking the commutator of H with $m := w(x)x\cdot\frac{\nabla}{\langle\nabla\rangle}$, where

$$w(x) = (1+|x|^\alpha)^{-\frac{1}{\alpha}}, \qquad \alpha > 0.$$

One has, with $\psi = \psi(t)$ for simplicity,

$$\frac{d}{dt}\langle m\psi,\psi\rangle = -i\langle[m,H]\psi,\psi\rangle$$

$$\int_0^T \langle -[m,H]\psi(t),\psi(t)\rangle\, dt = i\langle m\psi(0),\psi(0)\rangle - i\langle m\psi(T),\psi(T)\rangle$$

$$\langle -[m,H]\psi,\psi\rangle$$

$$= -\langle[m,V]\psi,\psi\rangle + \Big\langle \partial_\ell(w(x)\,x_j)\frac{\partial_j}{\langle\nabla\rangle}\psi, \partial_\ell\psi\Big\rangle + \frac{1}{2}\Big\langle \triangle(w(x)\,x_j)\frac{\partial_j}{\langle\nabla\rangle}\psi,\psi\Big\rangle \tag{2.2.9}$$

$$= \int_{\mathbb{R}^n} w|\nabla\langle\nabla\rangle^{-\frac{1}{2}}\psi|^2\, dx + \int_{\mathbb{R}^n} (\partial_\ell w)(x)x_j\partial_j\langle\nabla\rangle^{-\frac{1}{2}}\psi\,\partial_\ell\langle\nabla\rangle^{-\frac{1}{2}}\bar\psi\, dx \tag{2.2.10}$$

$$-\Big\langle [\langle\nabla\rangle^{\frac{1}{2}}, w]\frac{\nabla}{\langle\nabla\rangle}\psi, \nabla\langle\nabla\rangle^{-\frac{1}{2}}\psi\Big\rangle - \Big\langle [\langle\nabla\rangle^{\frac{1}{2}}, (\partial_\ell w)(x)x_j]\frac{\partial_j}{\langle\nabla\rangle}\psi, \partial_\ell\langle\nabla\rangle^{-\frac{1}{2}}\psi\Big\rangle \tag{2.2.11}$$

$$+\ \frac{1}{2}\Big\langle \triangle(w(x)\,x_j)\frac{\partial_j}{\langle\nabla\rangle}\psi,\psi\Big\rangle - \langle[m,V]\psi,\psi\rangle. \tag{2.2.12}$$

One now checks easily that the two terms in (2.2.10) satisfy

$$\int_{\mathbb{R}^n} w(x)|\langle\nabla\rangle^{-\frac{1}{2}}\nabla\psi(t,x)|^2\, dx + \int_{\mathbb{R}^n} (\partial_\ell w)(x)x_j\langle\nabla\rangle^{-\frac{1}{2}}\partial_j\psi(t,x)\,\langle\nabla\rangle^{-\frac{1}{2}}\partial_\ell\bar\psi(t,x)\, dx$$

$$\ge \int_{\mathbb{R}^n} \frac{w(x)}{1+|x|^\alpha}\,|\langle\nabla\rangle^{-\frac{1}{2}}\nabla\psi(t,x)|^2\, dx = \int_{\mathbb{R}^n} \frac{|\langle\nabla\rangle^{-\frac{1}{2}}\nabla\psi(t,x)|^2}{(1+|x|^\alpha)^{\frac{1}{\alpha}+1}}\, dx. \tag{2.2.13}$$

Notice that (2.2.13) is precisely the space integral in the desired lower bound from (2.2.8).

There are several ways to bound the commutators $A_1 := [\langle\nabla\rangle^{\frac{1}{2}}, w]$ and $A_2 := [\langle\nabla\rangle^{\frac{1}{2}}, (\partial_\ell w)(x)x_j]$. For example, one can use the Kato square root formula as in [23]. Alternatively, one can invoke the standard composition formula from ΨDO calculus. This gives $A_1 = T_{\{\langle\nabla\rangle^{\frac{1}{2}},w\}} + R_{a_1}$ where $\{\langle\nabla\rangle^{\frac{1}{2}}, w\}$ is the Poisson bracket of the symbols $\langle\nabla\rangle^{\frac{1}{2}}$ and w. Moreover, $T_{\{\langle\nabla\rangle^{\frac{1}{2}},w\}}$ is an associated ΨDO operator and R_{a_1} another ΨDO operator with symbol $a_1 \in S^{-\frac{3}{2}}$. A similar expression holds for A_2. One checks that $|\{\langle\nabla\rangle^{\frac{1}{2}}, w\}(\xi, x)| \lesssim |\nabla w(x)|$ for all x, ξ. Therefore, the two terms in (2.2.11) both satisfy

$$\begin{aligned} \Big|\Big\langle A_i \frac{\nabla}{\langle\nabla\rangle}\psi(t), \nabla\langle\nabla\rangle^{-\frac{1}{2}}\psi(t)\Big\rangle\Big| &\lesssim \|\psi(t)\|_2 \Big\| \frac{\langle\nabla\rangle^{-\frac{1}{2}}\nabla\psi(t)}{(1+|x|^\alpha)^{\frac{1}{\alpha}+1}} \Big\|_2 + \|\psi(t)\|_2 \|R_{a_i}\nabla\langle\nabla\rangle^{-\frac{1}{2}}\psi(t)\|_2 \\ &\leq C\|\psi(t)\|_2^2 + \frac{1}{4}\Big\| \frac{\langle\nabla\rangle^{-\frac{1}{2}}\nabla\psi(t)}{(1+|x|^\alpha)^{\frac{1}{\alpha}+1}} \Big\|_2^2. \end{aligned} \tag{2.2.14}$$

Finally, the two terms in (2.2.12) are bounded by

$$(1 + \|V\|_\infty)\|\psi(t)\|_2^2 \leq (1 + \|V\|_\infty)\|\psi_0\|_2^2. \tag{2.2.15}$$

In the above estimate we have used the boundedness of the multipliers m and $\triangle(w(x)\,x_j)\frac{\partial_j}{\langle\nabla\rangle}$ on L^2. Integrating (2.2.13), (2.2.14), and (2.2.15) in time, inserting the resulting bounds into (2.2.9), and finally using

$$|\langle m\psi(0), \psi(0)\rangle| + |\langle m\psi(T), \psi(T)\rangle| \leq 2\|\psi_0\|_2^2,$$

one obtains (2.2.8). To pass from (2.2.8) to (2.2.7), let χ_R be a smooth cutoff to the ball B_R, so that $\widehat{\chi_R}$ has compact support in a ball of size $\sim R^{-1}$. Then, by (2.2.8) with $\alpha = 1$

$$
\begin{aligned}
& \int_0^T \|F(|\vec{p}| \geq M) e^{-itH}\psi_0\|^2_{L^2(B_R)}\, dt \leq \int_0^T \|\chi_R F(|\vec{p}| \geq M) e^{-itH}\psi_0\|^2_{L^2}\, dt \\
\lesssim & \int_0^T \|F(|\vec{p}| \geq M)\chi_R e^{-itH}\psi_0\|^2_{L^2}\, dt + \int_0^T \|[\chi_R, F(|\vec{p}| \geq M)]\|^2_{2\to 2}\|e^{-itH}\psi_0\|^2_{L^2}\, dt \\
\lesssim & \; M^{-1}\int_0^T \|\nabla\langle\nabla\rangle^{-\frac{1}{2}} F(|\vec{p}| \geq M)\chi_R e^{-itH}\psi_0\|^2_{L^2}\, dt + T(MR)^{-2}\|\psi_0\|^2_{L^2} \\
\lesssim & \; M^{-1}\int_0^T \|F(|\vec{p}| \geq M)\chi_R \nabla\langle\nabla\rangle^{-\frac{1}{2}} e^{-itH}\psi_0\|^2_{L^2}\, dt \\
+ & \; M^{-1}\int_0^T \|[\nabla\langle\nabla\rangle^{-\frac{1}{2}}, \chi_R]\|^2_{2\to 2}\|e^{-itH}\psi_0\|^2_{L^2} + T(MR)^{-2}\|\psi_0\|^2_2 \\
\lesssim & \; M^{-1}R^2 \int_0^T\int_{\mathbb{R}^n} \frac{|\nabla\langle\nabla\rangle^{-\frac{1}{2}} e^{-itH}\psi_0|^2}{(1+|x|)^2} dx\, dt + TM^{-1}R^{-1}\|\psi_0\|^2_{L^2} \\
\leq & \; C(n,V)\, TM^{-1}R^2\, \|\psi_0\|^2_{L^2}.
\end{aligned}
$$

The lemma follows. □

2.3 Proof of the decay estimates

Theorem 2.1.3 will be proven in this section by a bootstrap argument. By Corollary 2.2.3, we can assume that t is large enough in Theorem 2.1.3. More precisely, t will be bigger than any constant appearing in our estimate, except the bootstrap constant C_0 in (2.3.1). By assumption, H_1, H_2 can only admit finitely many negative eigenvalues. Let $\alpha > 0$ satisfy: $-\alpha$ is bigger than any eigenvalue of H_1, H_2. For technical reasons, we will assume that the initial data ψ belong to $L^1 \cap L^2$ and employ the following bootstrap argument:

Specifically, we will show that

$$\|U(t)\psi_0\|_{L^\infty} \leq C_0|t|^{-\frac{n}{2}}(\|\psi_0\|_{L^1} + e^{-\frac{\alpha T}{2}}\|\psi_0\|_{L^2}) \;\text{ for all }\; 0 \leq t \leq T, \tag{2.3.1}$$

implies that

$$\|U(t)\psi_0\|_{L^\infty} \leq \frac{C_0}{2}|t|^{-\frac{n}{2}}(\|\psi_0\|_{L^1} + e^{-\frac{\alpha T}{2}}\|\psi_0\|_{L^2}) \;\text{ for all }\; 0 \leq t \leq T, \tag{2.3.2}$$

provided that $\frac{C_0}{2}$ remains larger than some constant that does not depend on T. The logic here is that for arbitrary but fixed T, the assumption (2.3.1) can be made to hold for some C_0 depending on T, because of Corollary 2.2.3. Iterating the implication (2.3.1) $\Longrightarrow$ (2.3.2) then yields a constant that does not depend on T. So we can let $T \to +\infty$ to eliminate $\|\psi_0\|_{L^2}$ on the right-hand side. Since $L^1 \cap L^2$ is dense in L^1 and $U(t)$ is a linear operator, we get the dispersive estimate (3.3.1) for any initial data $\psi_0 \in L^1$. To simplify the notation, we write $\|\psi_0\|_{L^1} + e^{-\frac{\alpha T}{2}}\|\psi_0\|_{L^2}$ as $|||\psi_0|||^{(T)}$ or $|||\psi_0|||$.

We proceed by expanding $U(t)$ via Duhamel's formula with respect to the free evolution H_0:

$$U(t)\phi_0 = e^{-itH_0}\phi_0 - i\int_0^t e^{-i(t-s)H_0}V(s)U(s)\psi_0\,ds \tag{2.3.3}$$

$$= e^{-itH_0}\psi_0 - i\int_0^t e^{-i(t-s)H_0}V(s)e^{-isH_0}\psi_0\,ds$$
$$- \int_0^t \int_0^s e^{-i(t-s)H_0}V(s)e^{-i(s-\tau)H_0}V(\tau)U(\tau)\psi_0\,d\tau ds. \tag{2.3.4}$$

Note that $\|e^{-itH_0}\psi_0\|_\infty \lesssim |t|^{-\frac{n}{2}}\|\psi_0\|_1$. For the second term in (2.3.4), we divide the integration interval $(0,t)$ into three pieces and handle them by means of the cancellation lemma. Firstly,

$$\|\int_0^1 e^{-i(t-s)H_0}V(s)e^{-isH_0}\psi_0\,ds\|_\infty \lesssim |t|^{-\frac{n}{2}}\sup_s \|e^{isH_0}V(s)e^{-isH_0}\|_{1\to1}\|\psi_0\|_1 \lesssim |t|^{-\frac{n}{2}}\|\psi_0\|_1.$$

Similarly, we have

$$\|\int_{t-1}^t e^{-i(t-s)H_0}V(s)e^{-isH_0}\psi_0\,ds\|_\infty \lesssim |t|^{-\frac{n}{2}}\sup_s \|e^{isH_0}V(s)e^{-isH_0}\|_{1\to1}\|\psi_0\|_1 \lesssim |t|^{-\frac{n}{2}}\|\psi_0\|_1.$$

The third piece is

$$\| \int_1^{t-1} e^{-i(t-s)H_0} V(s) e^{-isH_0} \psi_0 \, ds \|_\infty \lesssim \int_1^{t-1} |t-s|^{-\frac{n}{2}} \sup_s \|V(s)\|_1 |s|^{-\frac{n}{2}} \, ds \|\psi_0\|_1 \lesssim |t|^{-\frac{n}{2}} \|\psi_0\|_1,$$

where we observed that

$$\int_1^{t-1} |t-s|^{-\frac{n}{2}} |s|^{-\frac{n}{2}} ds \lesssim t^{-\frac{n}{2}} \quad \text{given} \quad n \geq 3. \tag{2.3.5}$$

The third term in (2.3.4) is

$$\int_0^t ds \int_0^s d\tau \, e^{-i(t-s)H_0} V(s) e^{-i(s-\tau)H_0} V(\tau) U(\tau) \psi_0. \tag{2.3.6}$$

We will decompose the domain of integration $\int_0^t ds \int_0^s d\tau$ into several pieces and treat each piece separately. We fix $A > 0$ as a large constant and $\epsilon > 0$ as a small constant. Write $\min\{s, A\} = s \wedge A$. Then Lemma 2.2.4 and (2.3.5) implies that

$$\begin{aligned}
&\| \int_0^t ds \int_0^{s \wedge A} d\tau \, e^{-i(t-s)H_0} V(s) e^{-i(s-\tau)H_0} V(\tau) U(\tau) \, d\tau ds \|_{1 \to \infty} \\
&\lesssim \int_0^t ds \int_0^{s \wedge A} d\tau \, \langle t-s \rangle^{-\frac{n}{2}} \langle s-\tau \rangle^{-\frac{n}{2}} \langle \tau \rangle^{-\frac{n}{2}} e^{AM} \\
&\lesssim t^{-\frac{n}{2}}.
\end{aligned}$$

By $\|\cdot\|_{1\to\infty}$, we mean the operator norm from L^1 to L^∞. However, when we apply the bootstrap assumption, $\|\psi_0\|_{L^1}$ has to be modified to $\|\psi_0\|_{L^1} + e^{-\frac{\alpha T}{2}} \|\psi_0\|_{L^2} := |\!|\!|\psi_0|\!|\!|$.

An application of Lemma 2.2.1 and the bootstrap assumption show that

$$\begin{aligned}
&\| \int_{t-\epsilon}^t ds \int_{t-\epsilon}^s d\tau \, e^{-i(t-s)H_0} V(s) e^{-i(s-\tau)H_0} V(\tau) U(\tau) \, d\tau ds \psi_0 \|_\infty \\
&\lesssim \int_{t-\epsilon}^t ds \int_{t-\epsilon}^s d\tau \, \|e^{-i(t-s)H_0} V(s) e^{-i(s-\tau)H_0}\|_{1\to\infty} \|V(\tau) U(\tau) \psi_0\|_1 \, d\tau ds \\
&\lesssim \int_{t-\epsilon}^t d\tau \int_\tau^t ds |t-\tau|^{-\frac{n}{2}} \max_{\tau \in (t-\epsilon, t)} \|U(\tau)\psi_0\|_\infty.
\end{aligned}$$

If $n = 3$, then the above is dominated by

$$\lesssim \int_{t-\epsilon}^{t} d\tau |t-\tau|^{-\frac{1}{2}} C_0 t^{-\frac{n}{2}} \|\psi_0\| \lesssim \sqrt{\epsilon} C_0 t^{-\frac{n}{2}} \|\psi_0\|.$$

Taking ϵ small enough, the above term can be dominated by $\frac{1}{100} C_0 t^{-\frac{n}{2}} \|\psi_0\|$. When $n > 3$, we need to expand $U(t)$ further to remove the singularity of $|t-\tau|^{-\frac{n}{2}}$ at $\tau = t$, (see [23] Section 2 for details). The following is another piece of (2.3.6):

$$\begin{aligned}
&\| \int_A^{t-A} \int_A^s e^{-i(t-s)H_0} V(s) e^{-i(s-\tau)H_0} V(\tau) U(\tau) \psi_0 \, d\tau ds \|_\infty \\
&\lesssim \int_A^{t-A} ds \int_A^s d\tau \langle t-s \rangle^{-\frac{n}{2}} \langle s-\tau \rangle^{-\frac{n}{2}} \| V(\tau) U(\tau) \psi_0 \|_1 \\
&\lesssim \int_A^{t-A} ds \int_A^s d\tau \langle t-s \rangle^{-\frac{n}{2}} \langle s-\tau \rangle^{-\frac{n}{2}} C_0 \tau^{-\frac{n}{2}} \|\psi_0\| \\
&\lesssim C_0 \|\psi_0\| \int_A^{t-A} ds \langle t-s \rangle^{-\frac{n}{2}} \langle s \rangle^{-\frac{n}{2}} \\
&\lesssim C_0 \|\psi_0\| t^{-\frac{n}{2}} \kappa_A \leq \frac{1}{100} C_0 \|\psi_0\| t^{-\frac{n}{2}},
\end{aligned}$$

where $\kappa_A < \int_A^{+\infty} ds \langle s \rangle^{-\frac{n}{2}} \to 0$ as $A \to \infty$. Lemma 2.2.4 and the bootstrap assumption are applied in turn in the above. The last line of above inequality holds provided that A is large enough. By Corollary 2.2.3, we can assume $t >> A$. Similarly, the following piece in (2.3.6) also requires that A is large:

$$\begin{aligned}
&\| \int_{t-A}^{t} \int_A^{s-A} e^{-i(t-s)H_0} V(s) e^{-i(s-\tau)H_0} V(\tau) U(\tau) \psi_0 \, d\tau ds \|_\infty \\
&\lesssim \int_{t-A}^{t} ds \int_A^{s-A} d\tau \langle t-s \rangle^{-\frac{n}{2}} \langle s-\tau \rangle^{-\frac{n}{2}} \| U(\tau) \psi_0 \|_\infty \leq \frac{1}{100} C_0 \|\psi_0\| t^{-\frac{n}{2}}.
\end{aligned}$$

So what remains in (2.3.6) is

$$\sum_{j=1}^{m} \int_{t-A}^{t} ds \int_{s-A}^{s \wedge (t-\epsilon)} d\tau \, e^{-i(t-s)H_0} V(s) e^{-i(s-\tau)H_0} V_j(\cdot - \tau \vec{v}_j) U(\tau) \psi_0. \tag{2.3.7}$$

For the term containing V_j in (2.3.7), $U(\tau)$ will be expanded with respect to H_j by Duhamel's formula. Abusing the notation, we will write $V_1(\cdot - \tau\vec{v_1})$ as $V_1(\tau)$. In the following, we only deal with the term containing V_1, which will be decomposed into two parts by $U(\tau) = P_b(H_1, \tau)U(\tau) + P_c(H_1, \tau)U(\tau)$.

2.3.1 Bound states

Proposition 2.3.1. *Let* $\psi(t, x) = (U(t)\psi_0)(x)$ *be a solution of* (2.1.7)*, which is asymptotically orthogonal to the bound states of* H_j*,* $j = 1, 2$ *in the sense of Definition 2.1.2. Provided the bootstrap assumption* (2.3.1)*, we have for any* $t \in (0, T)$

$$\|P_b(H_1, t)U(t)\psi_0\|_\infty \lesssim C_0 e^{-\frac{\alpha t}{4}} t^{-\frac{n}{2}} (\|\psi_0\|_{L^1} + e^{-\frac{\alpha T}{2}} \|\psi_0\|_{L^2}), \tag{2.3.8}$$

where C_0 *is the constant in the bootstrap assumption.*

Proof. Let $\tilde{U}(t) := \mathfrak{g}_{\vec{e_1}}(t)U(t)$ and $\phi(t) = \tilde{U}(t)\psi_0$. Then $\phi(t)$ solves

$$\frac{1}{i}\partial_t \phi - \frac{1}{2}\triangle\phi + V(\cdot + t\vec{v_1})\phi = 0, \tag{2.3.9}$$
$$\phi|_{t=0}(x) = (\mathfrak{g}_{\vec{e_1}}(0)\psi_0)(x).$$

Then $\|P_b(H_1, t)U(t)\psi_0\|_\infty = \|P_b(H_1)\tilde{U}(t)\phi_0\|_\infty$ so without loss of generality, we can assume that $\vec{v_1}$ is the zero vector. Suppose that the bound states of H_1 are $u_1, u_2, \ldots, u_l$ and we decompose

$$U(t)\psi_0 = \sum_{i=1}^{l} a_i(t)u_i + \psi_1(t, x) \tag{2.3.10}$$

with respect to H_1 so that $P_c(H_1)\psi_1 = \psi_1$ and $P_b(H_1)\psi_1 = 0$. By the asymptotic orthogonality assumption,

$$\sum_{i=1}^{l} |a_i(t)|^2 \to 0 \text{ as } t \to \infty.$$

Substituting (2.3.10) into (2.1.7) yields

$$\frac{1}{i}\partial_t\psi_1 - \frac{1}{2}\triangle\psi_1 + V_1\psi_1 + V_2(\cdot - t\vec{e_1})\psi_1 + \\ + \sum_{j=1}^{l}\left[\frac{1}{i}\dot{a}_j(t)u_j - \frac{1}{2}\triangle u_j a_j(t) + V_1\, u_j a_j(t) + V_2(\cdot - t\vec{e_1})u_j a_j(t)\right] = 0. \qquad (2.3.11)$$

Since $P_c(H_1)\psi_1 = \psi_1$, we have

$$(-\frac{1}{2}\triangle + V_1)\psi_1 = H_1\psi_1 = P_c(H_1)H_1\psi_1, \qquad \partial_t\psi_1 = P_c(H_1)\partial_t\psi_1.$$

In particular,

$$P_b(H_1)\left(\frac{1}{i}\partial_t\psi_1 - \frac{1}{2}\triangle\psi_1 + V_1\psi_1\right) = 0.$$

Thus, taking an inner product of the equation (2.3.11) with u_κ and using the fact that $\langle u_\kappa, u_j\rangle = \delta_{j\kappa}$ as well as the identity

$$-\frac{1}{2}\triangle u_j + V_1 u_j = \lambda_j u_j,$$

we obtain the ODE

$$\frac{1}{i}\dot{a}_\kappa(t) + \lambda_\kappa a_\kappa(t) + \langle V_2(\cdot - t\vec{e_1})\psi_1, u_\kappa\rangle + \sum_{j=1}^{m} a_j(t)\langle V_2(\cdot - t\vec{e_1})u_j, u_\kappa\rangle = 0$$

for each a_κ with the condition that

$$a_\kappa(t) \to 0 \text{ as } t \to +\infty.$$

Recall that u_κ is an eigenfunction of $H_1 = -\frac{1}{2}\triangle + V_1$ with eigenvalue $\lambda_\kappa < 0$. It is well-known (e.g., Agmon [1]) that such eigenfunctions are exponentially localized, i.e.,

$$\int_{\mathbb{R}^n} e^{2\alpha|x|}\,|u_\kappa(x)|^2\,dx \le C = C(V_1, n) < \infty \text{ for some positive } \alpha. \qquad (2.3.12)$$

Therefore, the assumption that V_2 has compact support implies

$$\|V_2(\cdot - t\vec{e_1})u_\kappa\|_2 \lesssim e^{-\alpha t} \quad \text{for all} \quad t \geq 0. \tag{2.3.13}$$

The implicit constant in (2.3.13) depends on the size of the support of V_2 and $\|V_2\|_{L^\infty}$.

By the bootstrap assumption, $f_\kappa(t) := \langle V_2(\cdot - t\vec{e_1})\psi_1, u_\kappa\rangle$ satisfies

$$\begin{aligned} |f_\kappa(t)| &\lesssim \|\psi_1\|_\infty \|V_2(\cdot - t\vec{e_1})u_\kappa\|_1 \lesssim e^{-\alpha t}\|\psi_1\|_\infty \\ &\lesssim e^{-\alpha t}\|(Id - P_b(H_1))U(t)\psi_0\|_\infty \\ &\lesssim e^{-\alpha t} t^{-\frac{n}{2}} C_0(\|\psi_0\|_{L^1} + e^{-\frac{\alpha T}{2}}\|\psi_0\|_{L^2}) + e^{-\alpha t}\sum_{i=1}^{l} |a_i(t)|\,\|u_i\|_\infty, \end{aligned} \tag{2.3.14}$$

where $t \in (0, T)$. Notice that (2.3.14) fails for $t > T$ because the bootstrap assumption only applies to $0 < t < T$. Instead, we have the following for $t > T$:

$$|f_\kappa(t)| < \|V_2(\cdot - t\vec{e_1})u_\kappa\|_2 \|\psi_1\|_2 \lesssim e^{-\alpha t}\|\psi_0\|_2. \tag{2.3.15}$$

In view of (2.3.1), a_κ solves the equation

$$\begin{gathered} \tfrac{1}{i}\dot{a}_\kappa(t) + \lambda_\kappa a_\kappa(t) + \textstyle\sum_{j=1}^{m} a_j(t)C_{j\kappa}(t) + f_\kappa(t) = 0, \\ a_\kappa(\infty) = 0, \end{gathered} \tag{2.3.16}$$

where $C_{j\kappa}(t) = C_{\kappa j}(t) = \langle V_2(\cdot - t\vec{e_1})u_j, u_\kappa\rangle$. By (2.3.13), $\max_{j,\kappa}|C_{j\kappa}(t)| \lesssim e^{-\alpha t}$. Solving (2.3.16) explicitly, we obtain

$$\vec{a}(t) = ie^{-i\int_0^t B(s)\,ds}\int_t^\infty e^{i\int_0^s B(\tau)\,d\tau}\vec{f}(s)\,ds,$$

where $B_{j\kappa}(t) = \lambda_j\delta_{j\kappa} + C_{j\kappa}(t)$.

By (2.3.14), (2.3.15) and the unitarity of $e^{i\int_0^s B(\tau)\,d\tau}$, we conclude that

$$|\vec{a}(t)| \leq \int_t^T + \int_T^\infty |\vec{f}(s)|\,ds$$
$$\lesssim \int_t^T e^{-\alpha s} s^{-\frac{n}{2}} C_0 ds \|\psi_0\| + \int_t^T e^{-\alpha s} \sum_{j=1}^l |a_j(s)| \|u_i\|_\infty ds + \int_T^\infty e^{-\alpha s} ds \|\psi_0\|_{L^2}.$$

Choose a large constant $t_0 > 0$ such that for all $t_1 > t_0$, the following holds:

$$\int_{t_1}^T e^{-\alpha s} \sum_{j=1}^l |a_j(s)| \|u_i\|_\infty ds \leq \frac{1}{2} \sup_{t_1<t<T} |\vec{a}(t)|, \tag{2.3.17}$$

then

$$\sup_{t_1<t<T} |\vec{a}(t)| \lesssim e^{-\alpha t_1} t_1^{-\frac{n}{2}} C_0 \|\psi_0\| + e^{-\alpha T} \|\psi_0\|_{L^2} \lesssim e^{-\frac{\alpha t_1}{4}} t_1^{-\frac{n}{2}} C_0 \|\psi_0\|.$$

□

Remark 2.3.1. *In the above proof, if we change* (2.3.14) *into the following:*

$$|f_\kappa(t)| \lesssim \|\psi_1(t)\|_p \|V_2(\cdot - t\vec{e_1}) u_\kappa\|_{p'} \lesssim e^{-\alpha t} \|\psi_1(t)\|_p$$
$$\lesssim e^{-\alpha t} \| \ (Id - P_b(H_1)) U(t) \psi_0\|_p$$
$$\lesssim e^{-\alpha t} t^{-\gamma} C_0 (\|\psi_0\|_{p'} + e^{-\frac{\alpha T}{2}} \|\psi_0\|_{L^2}) + e^{-\alpha t} \sum_{i=1}^l |a_i(t)| \, \|u_i\|_p,$$

where $\gamma = n(\frac{1}{2} - \frac{1}{p}) > 1$*, and follow the same arguments, we see that for large* t*,*

$$\|P_b(H_1, t) U(t) \psi_0\|_p \lesssim t^{-\gamma} C_0 e^{-\frac{\alpha T}{4}} (\|\psi_0\|_{p'} + e^{-\frac{\alpha T}{2}} \|\psi_0\|_{L^2}).$$

If the potential V_1 *is smooth enough, it is known (e.g., [1]) that the bound state* u_j *of* H_1 *is differentiable. Moreover, its derivatives decay exponentially at infinity. Thus,*

$$\|\partial P_b(H_1, t) U(t) \psi_0\|_p \leq \sum_{i=1}^l |a_i(t)| \|\partial u_i\|_p \lesssim t^{-\gamma} C_0 e^{-\frac{\alpha T}{4}} (\|\psi_0\|_{p'} + e^{-\frac{\alpha T}{2}} \|\psi_0\|_{L^2}). \tag{2.3.18}$$

In addition, the above claims hold with H_1 replaced by H_j, $j = 2, \cdots, m$. These results will be used to prove Theorem 2.1.4 in Section 4. $\square$

With Proposition 2.3.1, the $P_b(H_1,\tau)U(\tau)$ part of (2.3.7) can be estimated by the following:

$$\begin{aligned}
&\int_{t-A}^{t} ds \int_{s-A}^{s\wedge(t-\epsilon)} d\tau \| e^{-i(t-s)H_0} V(s) e^{-i(s-\tau)H_0} V_1(\tau) P_b(H_1,\tau) U(\tau)\psi_0\|_\infty \\
\lesssim &\int_{t-A}^{t} ds \int_{s-A}^{s\wedge(t-\epsilon)} d\tau \langle t-s\rangle^{-\frac{n}{2}} \langle s-\tau\rangle^{-\frac{n}{2}} \|V_1(\tau) P_b(H_1,\tau) U(\tau)\psi_0\|_1 \\
\lesssim &A^2 \sup_{\tau\in(t-2A,t)} \|P_b(H_1,\tau)U(\tau)\psi_0\|_\infty \\
&< \frac{C_0}{100} t^{-\frac{n}{2}} (\|\psi_0\|_{L^1} + e^{-\frac{\alpha T}{2}} \|\psi_0\|_{L^2}).
\end{aligned}$$

For the $P_c(H_1,\tau)U(\tau)$ part of (2.3.7), we need to apply Duhamel's formula again and expand (2.3.7) further with respect to H_1. We assume that $\vec{v_1} = 0$ and $m = 2$ to simplify our notation. Specifically, we plug the following

$$P_c(H_1,\tau)U(\tau) = P_c(H_1)U(\tau) = P_c(H_1)e^{-i\tau H_1} - iP_c(H_1)\int_0^\tau e^{-i(\tau-r)H_1} V_2(r)U(r)\, dr$$

into (2.3.7). For the term containing $P_c(H_1)e^{-i\tau H_1}$, we apply the dispersive decay for $P_c(H_1)e^{-i\tau H_1}$:

$$\begin{aligned}
&\int_{t-A}^{t} ds \int_{s-A}^{s\wedge(t-\epsilon)} d\tau \| e^{-i(t-s)H_0} V(s) e^{-i(s-\tau)H_0} V_1(\tau) P_c(H_1) e^{-i\tau H_1}\psi_0\|_\infty \\
\lesssim &\int_{t-A}^{t} ds \int_{s-A}^{s\wedge(t-\epsilon)} d\tau \langle t-s\rangle^{-n/2} \langle s-\tau\rangle^{-n/2} \langle \tau\rangle^{-n/2} \|\psi_0\|_1 \lesssim t^{-n/2}\|\psi_0\|_1.
\end{aligned}$$

The second term is

$$\int_{t-A}^{t} ds \int_{s-A}^{s\wedge(t-\epsilon)} d\tau \int_0^\tau dr\, e^{-i(t-s)H_0} V(s) e^{-i(s-\tau)H_0} V_1(\tau) P_c(H_1) e^{-i(\tau-r)H_1} V_2(r) U(r)\psi_0. \tag{2.3.19}$$

Now take a small constant $\delta > 0$ and a large constant $A_1 > 0$ to be specified later.

We decompose the integral $\int_0^\tau dr$ in (2.3.19) as follows:

$$\int_0^\tau dr = \int_0^\delta dr + \int_\delta^{A_1} dr + \int_{A_1}^{\tau-A_1} dr + \int_{\tau-A_1}^{\tau-\delta} dr + \int_{\tau-\delta}^{\tau} dr. \tag{2.3.20}$$

To simplify the notation, we will write A_1 as A. Our goal is to estimate each term in (2.3.20). The second term of (2.3.20) is estimated as follows:

$$\begin{aligned}
&\int_{t-A}^{t} ds \int_{s-A}^{s\wedge(t-\epsilon)} d\tau \int_{\delta}^{A} dr \| e^{-i(t-s)H_0} V(s) e^{-i(s-\tau)H_0} V_1(\tau) P_c(H_1) e^{-i(\tau-r)H_1} V_2(r) U(r)\psi_0\|_\infty \\
&\lesssim \int_{t-A}^{t} ds \int_{s-A}^{s\wedge(t-\epsilon)} d\tau \int_{\delta}^{A} dr \langle t-s\rangle^{-n/2} \langle s-\tau\rangle^{-n/2} \langle \tau-r\rangle^{-n/2} \langle r\rangle^{-n/2} e^{rM} \|\psi_0\|_1 \\
&\lesssim t^{-n/2}\|\psi_0\|_1.
\end{aligned}$$

The implicit constant above depends on A, δ.

The third term of (2.3.20) is estimated as follows:

$$\begin{aligned}
&\int_{t-A}^{t} ds \int_{s-A}^{s\wedge(t-\epsilon)} d\tau \int_{A}^{\tau-A} dr \| e^{-i(t-s)H_0} V(s) e^{-i(s-\tau)H_0} V_1(\tau) P_c(H_1) e^{-i(\tau-r)H_1} V_2(r) U(r)\psi_0\|_\infty \\
&\lesssim \int_{t-A}^{t} ds \int_{s-A}^{s\wedge(t-\epsilon)} d\tau \int_{A}^{\tau-A} dr \langle t-s\rangle^{-n/2} \langle s-\tau\rangle^{-n/2} \langle \tau-r\rangle^{-n/2} \|U(r)\psi_0\|_\infty \\
&\lesssim \int_{t-A}^{t} ds \int_{s-A}^{s\wedge(t-\epsilon)} d\tau \int_{A}^{\tau-A} dr \langle t-s\rangle^{-n/2} \langle s-\tau\rangle^{-n/2} \langle \tau-r\rangle^{-n/2} \langle r\rangle^{-n/2} C_0 |||\psi_0||| \\
&\lesssim t^{-n/2} C_0 \kappa_A |||\psi_0||| \le \frac{1}{100} C_0 t^{-n/2} |||\psi_0|||,
\end{aligned}$$

where $\kappa_A \to 0$ as $A \to \infty$. So the above inequality holds for large enough A.

For the fourth term in (2.3.20), we have

$$\begin{aligned}
&\int_{t-A}^{t} ds \int_{s-A}^{s\wedge(t-\epsilon)} d\tau \int_{\tau-\delta}^{\tau} dr \| e^{-i(t-s)H_0} V(s) e^{-i(s-\tau)H_0} V_1(\tau) e^{-i(\tau-r)H_1} P_c(H_1) V_2(r) U(r)\psi_0 \|_\infty \\
&\lesssim \int_{t-A}^{t} ds \int_{s-A}^{s\wedge(t-\epsilon)} d\tau \int_{\tau-A}^{\tau} dr \langle t-s\rangle^{-n/2} \langle s-\tau\rangle^{-n/2} \| V_1(\tau) P_c(H_1) e^{-i(\tau-r)H_1} V_2(r) U(r)\psi_0 \|_1 \\
&\lesssim \int_{t-A}^{t} ds \int_{s-A}^{s\wedge(t-\epsilon)} d\tau \int_{\tau-A}^{\tau} dr \langle t-s\rangle^{-n/2} \langle s-\tau\rangle^{-n/2} \| U(r)\psi_0 \|_\infty \\
&\lesssim \int_{t-A}^{t} ds \int_{s-A}^{s\wedge(t-\epsilon)} d\tau \int_{\tau-\delta}^{\tau} dr \langle t-s\rangle^{-n/2} \langle s-\tau\rangle^{-n/2} \langle r\rangle^{-n/2} C_0 \|\psi_0\| \\
&\lesssim t^{-n/2} C_0 \kappa_\delta \|\psi_0\| \leq \frac{1}{100} C_0 t^{-n/2} \|\psi_0\|,
\end{aligned}$$

where $\kappa_\delta \to 0$ as $\delta \to 0$. So the above inequality holds for δ small enough.

For the $\int_0^\delta dr$ part of (2.3.20), we expand

$$e^{-i(\tau-r)H_1} = e^{-i(\tau-r)H_0} - i \int_0^{\tau-r} e^{-i(\tau-r-\beta)H_1} V_1 e^{-i\beta H_0} \, d\beta.$$

Here we put H_0 after H_1 in the integral because we want H_0 to appear immediately before $U(r)$ and apply Lemma 2.2.4. Substitute this expansion into the $\int_0^\delta dr$ part of (2.3.20) and we get two terms. The first one is

$$\int_{t-A}^{t} ds \int_{s-A}^{s\wedge(t-\epsilon)} d\tau \int_0^\delta dr e^{-i(t-s)H_0} V(s) e^{-i(s-\tau)H_0} V_1(\tau) P_c(H_1) e^{-i(\tau-r)H_0} V_2(r) U(r)\psi_0. \tag{2.3.21}$$

Notice that $P_c(H_1) = Id - P_b(H_1)$, and because $\|P_b(H_1)\|_{p\to p}$ is bounded, $\|P_c(H_1)\|_{p\to p}$ is bounded as well. Therefore the L^∞ norm of (2.3.21) is estimated as follows:

$$\lesssim \int_{t-A}^{t} ds \int_{s-A}^{s\wedge(t-\epsilon)} d\tau \int_0^\delta dr \langle t-s\rangle^{-n/2} \langle s-\tau\rangle^{-n/2} \langle \tau\rangle^{-n/2} e^{Mr} \|\psi_0\|_1 \lesssim t^{-n/2} \|\psi_0\|_1.$$

The second term of the $\int_0^\delta dr$ part of (2.3.20) after substitution is

$$\int_{t-A}^{t} ds \int_{s-A}^{s\wedge(t-\epsilon)} d\tau \int_0^{\delta} dr\, e^{-i(t-s)H_0} V(s) e^{-i(s-\tau)H_0} V_1(\tau)$$
$$P_c(H_1) \int_0^{\tau-r} e^{-i(\tau-r-\beta)H_1} V_1 e^{-i\beta H_0}\, d\beta\, V_2(r) U(r)\psi_0. \tag{2.3.22}$$

Decompose $\int_0^{\tau-r} d\beta$ so we can rewrite (2.3.22)$= J_1 + J_2 + J_3$, where J_1, J_2, and J_3 correspond to $\int_0^{\delta} d\beta$, $\int_{\delta}^{\tau-r-1} d\beta$, and $\int_{\tau-r-1}^{\tau-r} d\beta$, respectively.

We proceed to estimate J_1 as follows:

$$\begin{aligned}
&\int_0^{\delta} dr \int_0^{\delta} d\beta\, \|P_c(H_1) e^{-i(\tau-r-\beta)H_1} V_1 e^{-i\beta H_0} V_2(r) U(r)\psi_0\|_\infty \\
&\lesssim \int_0^{\delta} dr \int_0^{\delta} d\beta\, \langle \tau - r - \beta\rangle^{-n/2} \|e^{-i\beta H_0} V_2(r) U(r)\psi_0\|_\infty \\
&\lesssim \langle \tau \rangle^{-n/2} \int_0^{\delta} dr \int_0^{\delta} d\beta (\beta + r)^{-n/2} e^{Mr} \|\psi_0\|_1.
\end{aligned}$$

In the above expression, when $n = 3$, $\int_0^{\delta} dr \int_0^{\delta} d\beta (\beta + r)^{-n/2} e^{Mr}$ is integrable. When $n > 3$, we need to further expand $e^{-i(\tau-r-\beta)H_1}$ to remove the singularity of $(\beta + r)^{-n/2}$ at $\beta + r = 0$. In either case, we can conclude that $\|J_1\|_\infty \lesssim t^{-n/2}\|\psi_0\|_1$.

For J_2, our estimate is the following:

$$\begin{aligned}
&\int_0^{\delta} dr \int_{\delta}^{\tau-r-1} d\beta \|P_c(H_1) e^{-i(\tau-r-\beta)H_1} V_1 e^{-i\beta H_0} V_2(r) U(r)\psi_0\|_\infty \\
&\lesssim \int_0^{\delta} dr \int_{\delta}^{\tau-r-1} d\beta \langle \tau - r - \beta\rangle^{-n/2} \|V_1 e^{-i\beta H_0} V_2(r) U(r)\psi_0\|_1 \\
&\lesssim \int_0^{\delta} dr \int_{\delta}^{\tau-r-1} d\beta \langle \tau - r - \beta\rangle^{-n/2} \|e^{-i\beta H_0} V_2(r) U(r)\psi_0\|_\infty \\
&\lesssim \int_0^{\delta} dr \int_{\delta}^{\tau-r-1} d\beta \langle \tau - r - \beta\rangle^{-n/2} (\beta + r)^{-n/2} e^{Mr} \|\psi_0\|_1 \\
&\lesssim \int_0^{\delta} dr \int_{\delta}^{\tau-r-1} d\beta \langle \tau - r - \beta\rangle^{-n/2} \langle \beta + r\rangle^{-n/2} \|\psi_0\|_1 \\
&\lesssim \tau^{-n/2} \|\psi_0\|_1.
\end{aligned}$$

The implicit constant above depends on δ and is independent of t and ψ_0. Plugging the above estimate into J_1, we derive that $\|J_2\|_\infty \lesssim t^{-n/2}\|\psi_0\|_1$.

To estimate J_3, we notice that

$$
\begin{aligned}
&\| \int_{\tau-r-1}^{\tau-r} d\beta V_1(\tau) P_c(H_1) e^{-i(\tau-r-\beta)H_1} V_1 e^{-i\beta H_0} V_2(r) U(r)\psi_0\|_1 \\
&\leq \int_{\tau-r-1}^{\tau-r} d\beta \|V_1(\tau)\|_2 \|P_c(H_1) e^{-i(\tau-r-\beta)H_1}\|_{2\to 2} \|V_1 e^{-i\beta H_0} V_2(r) U(r)\psi_0\|_2 \\
&\lesssim \int_{\tau-r-1}^{\tau-r} d\beta \|V_1\|_2 \|e^{-i\beta H_0} V_2(r) U(r)\psi_0\|_\infty \\
&\lesssim \int_{\tau-r-1}^{\tau-r} d\beta |\beta|^{-\frac{n}{2}} e^{Mr} \|\psi_0\|_1.
\end{aligned}
$$

Observe that r is small and $\beta \simeq \tau$. Plugging the above estimate into J_3, we derive that $\|J_3\|_\infty \lesssim t^{-\frac{n}{2}}\|\psi_0\|_1$. Thus, we finish the estimate of the $\int_0^\delta dr$ part of (2.3.20).

2.3.2 Low and high velocity estimates

So far we have estimated four parts of (2.3.20). This subsection is devoted to deriving the estimate of the $\int_{\tau-A}^{\tau-\delta} dr$ part of (2.3.20), which will be decomposed as follows:

$$
\begin{aligned}
&\int_{t-A}^{t} ds \int_{s-A}^{s\wedge(t-\epsilon)} d\tau \int_{\tau-A}^{\tau-\delta} dr e^{-i(t-s)H_0} V(s) e^{-i(s-\tau)H_0} V_1 P_c(H_1) e^{-i(\tau-r)H_1} V_2(r) U(r)\psi_0 \\
&= \int_{t-A}^{t} ds \int_{s-A}^{s\wedge(t-\epsilon)} d\tau \int_{\tau-A}^{\tau-\delta} dr e^{-i(t-s)H_0} V(s) e^{-i(s-\tau)H_0} \cdot \\
&\qquad V_1 P_c(H_1) e^{-i(\tau-r)H_1} (F(|\vec{p}| \geq N) + F(|\vec{p}| \leq N)) V_2(r) U(r)\psi_0 \\
&= J_{high} + J_{low}.
\end{aligned}
$$

$F(|\vec{p}| \leq N)$ and $F(|\vec{p}| \geq N)$ denote smooth projections onto the frequencies $|\vec{p}| \leq N$ and $|\vec{p}| \geq N$, respectively. For the low velocity part J_{low}, firstly, $(t-s) + (s-\tau) \geq \epsilon$ and Lemma 2.2.4 imply

$$\|e^{i(t-s)H_0}V(s)e^{i(s-\tau)H_0}\|_{1\to\infty} \lesssim \langle t-s\rangle^{-\frac{n}{2}}\langle s-\tau\rangle^{-\frac{n}{2}}. \qquad (2.3.23)$$

Secondly, we need the following proposition from [31]:

Proposition 2.3.2. *Let $\chi_1(t,x)$ be a smooth cut of the ball $B(0,\delta t)$ with respect to x, where δ is a small constant only depending on $\vec{e_1}$ and $B(0,\delta t)$ is a ball centered at 0 with radius δt. Let A, M be large positive constants and $A, M << t$. Then*

$$\sup_{|t-s|\le A} \|\chi_1(t,\cdot)e^{-i(t-s)H_1}P_c(H_1)F(|\vec{p}|\le M)V_2(\cdot - s\vec{e_1})\|_{L^2\to L^2} \le \frac{AM}{\delta t}.$$

The idea behind Proposition 2.3.2 can be explained as the following:

The support of $V_2(\cdot - s\vec{e_1})$ is contained in $B(s\vec{e_1}, R)$. Here R is the size of the support of V_2. The operator $e^{-i(t-s)H_1}P_c(H_1)F(|\vec{p}|\le M)$ can "propagate" $B(s\vec{e_1}, R)$ into $B(0, t\delta)$ only if $(t-s)M \ge \mathrm{dist}(B(s\vec{e_1}, R), B(0, t\delta))$ according to the classical picture. However if $|t-s| < A$, $tA, M \ll t$, then $(\tau - r)M \ll \mathrm{dist}(B(r\vec{e_1}, R), B(0, \tau\delta))$. This proposition appears in [31]. We give a proof here for completeness.

Proof. The proof is a commutator argument. Let $\chi_1 = \chi(t,x)$. Firstly, we claim that

$$\|[\chi_1, P_c(H_1)]\|_{L^2\to L^2} \lesssim e^{-\delta\alpha t}. \qquad (2.3.24)$$

Clearly,

$$[\chi_1, P_c(H_1)] = [\chi_1, I - P_b(H_1)] = -[\chi_1, P_b(H_1)].$$

Recall that $u_1, \dots, u_m$ are the exponentially decaying eigenvalues of H_1. Therefore,

$$\begin{aligned}[\chi_1, P_b(H_1)]f &= \sum_{i=1}^{m}(\chi_1 u_i\langle f, u_i\rangle - u_i\langle f\chi_1, u_i\rangle) \\ &= \sum_{i=1}^{m}\left((-1+\chi_1)u_i\langle f, u_i\rangle - u_i\langle f, (\chi_1 - 1)u_i\rangle\right).\end{aligned}$$

In of the support of $\chi_1 - 1$ we have that

$$\|(1-\chi_1)u_i\|_2 \lesssim e^{-\alpha\delta t}$$

and thus

$$\|[\chi_1, P_b(H_1)]f\|_{L^2} \leq e^{-\alpha\delta t}\|f\|_{L^2},$$

as desired. Secondly, we claim that

$$\|[\chi_1, e^{-i(t-s)H_1}]F(|\vec{p}| \leq M)\|_{L^2\to L^2} \lesssim \frac{AM}{\delta t}. \tag{2.3.25}$$

We write

$$[\chi_1, e^{-i(t-s)H_1}] = e^{-i(t-s)H_1}(e^{i(t-s)H_1}\chi_1 e^{-i(t-s)H_1} - \chi_1)$$

and

$$\begin{aligned} e^{i(t-s)H_1}\chi_1\, e^{-i(t-s)H_1} - \chi_1 &= i\int_0^{t-s} e^{i\tau H_1}[H_1, \chi_1]e^{-i\tau H_1}\, d\tau \\ &= i\int_0^{t-s} e^{i\tau H_1}(-\nabla\chi_1\nabla - \frac{1}{2}\triangle\chi_1)e^{-i\tau H_1}\, d\tau. \end{aligned}$$

Observe now that

$$|\nabla\chi_1(t,x)| \lesssim \frac{1}{\delta t}, \quad |\triangle\chi_1(t,x)| \lesssim \frac{1}{(\delta t)^2}.$$

Therefore,

$$\begin{aligned} &\|[\chi_1, e^{-(t-s)H_1}]F(|\vec{p}| \leq M)\|_{L^2\to L^2} \\ &\lesssim |t-s|\Big(\|\nabla\chi_1\, \nabla e^{-i\tau H_1}F(|\vec{p}| \leq M)\|_{L^2\to L^2} + \|\Delta\chi_1 e^{-i\tau H_1}F(|\vec{p}| \leq M)\|_{L^2\to L^2}\Big) \\ &\lesssim \frac{A}{\delta t}\|\nabla e^{-i\tau H_1}F(|\vec{p}| \leq M)\|_{L^2\to L^2} + \frac{A}{(\delta t)^2}\|e^{-i\tau H_1}F(|\vec{p}| \leq M)\|_{L^2\to L^2}. \end{aligned}$$

Since the potential V_1 is bounded it is standard that

$$\sup_\tau \|\nabla e^{-i\tau H_1}f\|_{L^2} \lesssim \|\nabla f\|_{L^2} + \|f\|_{L^2}. \tag{2.3.26}$$

Indeed,

$$\sup_\tau \|\triangle e^{-i\tau H_1} f\|_{L^2} \;\le\; \sup_\tau \|e^{-i\tau H_1} H_1 f\|_{L^2} + \sup_\tau \|V_1 e^{-i\tau H_1} f\|_{L^2} \lesssim \|\nabla^2 f\|_{L^2} + \|f\|_{L^2},$$

and (2.3.26) follows by interpolation with L^2. Therefore,

$$\|\nabla e^{-i\tau H_1} F(|\vec{p}| \le M)\|_{L^2 \to L^2} \le M.$$

Combining terms we obtain the bound

$$\|[\chi_1, e^{-i(t-s)H_1}]F(|\vec{p}| \le M)\|_{L^2 \to L^2} \lesssim \frac{AM}{\delta t} + \frac{A}{(\delta t)^2} \le 2\frac{AM}{\delta t}$$

since $M << t$, which is (2.3.25). Finally, we invoke one more standard fact, namely

$$\|[\chi_1, F(|\vec{p}| \le M)]\|_{L^2 \to L^2} \lesssim M^{-1} \|\nabla \chi_1\|_\infty \lesssim \frac{1}{\delta M t}. \tag{2.3.27}$$

To see this, write $F(|\vec{p}| \le M)f = [\hat{\eta}(\xi/M)\hat{f}(\xi)]^\vee$ with some smooth bump function η. Hence the kernel K of $[\chi_1, F(|\vec{p}| \le M)]$ is

$$K(x,y) = M^n \eta(M(x-y))(\chi_1(x) - \chi_1(y)),$$

and (2.3.29) follows from Schur's test. One concludes from estimates (2.3.24), (2.3.25), (2.3.29) that

$$\Big\|\chi_1(t,\cdot)e^{-i(t-s)H_1} P_c(H_1) F(|\vec{p}| \le M) V_2(\cdot - s\vec{e_1}) \\ - e^{-i(t-s)H_1} P_c(H_1) F(|\vec{p}| \le M) \chi_1(t,\cdot) V_2(\cdot - s\vec{e_1})\Big\|_{L \to L^2} \lesssim \frac{AM}{\delta t}.$$

It remains to be seen that $\chi_1(t,\cdot)V_2(\cdot - s\vec{e_1}) = 0$ since the supports of $\chi_1(t,\cdot)$ and $V_2(\cdot - s\vec{e_1})$ are disjoint. □

To apply Proposition 2.3.2 to J_{low}, note that $\chi_\tau V_1 = V_1$. Let χ_2 be a smooth cut of the support of V_2 and f be any function in $L^\infty(R^n)$. Then it follows from

Proposition 2.3.2 that

$$\begin{aligned}
&\|V_1 P_c(H_1) e^{-i(\tau-r)H_1} F(|\vec{p}| \le N) V_2(r) f\|_1 \\
&= \|V_1 \chi_\tau P_c(H_1) e^{-i(\tau-r)H_1} F(|\vec{p}| \le N) V_2(r) \chi_2(\cdot - r\vec{v}_2) f\|_1 \\
&\le \|V_1\|_2 \|\chi_\tau P_c(H_1) e^{-i(\tau-r)H_1} F(|\vec{p}| \le N) V_2(r)\|_{2\to 2} \|\chi_2\|_2 \|f\|_\infty \\
&\lesssim \frac{ANM^2}{\delta t} \|f\|_\infty.
\end{aligned}$$

Combining the above estimate with (2.3.23) and noting $A, M, N \ll t$, we conclude

$$\begin{aligned}
\|J_{low}\|_\infty &\lesssim \int_{t-A}^{t} ds \int_{s-A}^{s\wedge(t-\epsilon)} d\tau \int_{\tau-A}^{\tau-\delta} dr \langle t-s\rangle^{-n/2} \langle s-\tau\rangle^{-n/2} \frac{ANM^2}{\delta t} \|U(r)\psi_0\|_\infty \\
&\le \frac{C_0}{100} t^{-n/2} \|\psi_0\|.
\end{aligned}$$

From the above estimate for J_{low}, it is worth remarking that the purpose of the multiple expansions by Duhamel's formula is to prepare a cushion (the potentials V_1 and V_2) to apply the $L^2 \to L^2$ estimate (Prop 2.3.2) between the $L^1 \to L^\infty$ estimates.

For the high velocity part J_{high}, we shall further expand $U(r)$ with respect to H_0, followed by a commutator argument. By Duhamel's formula

$$U(r) = e^{-irH_0} - i \int_0^r e^{-i(r-\alpha)H_0} V(\alpha) U(\alpha)\, d\alpha,$$

we write $J_{high} = J_{high,1} - iJ_{high,2}$, where

$$\begin{aligned}
J_{high,1} = \int_{t-A}^{t} ds \int_{s-A}^{s\wedge(t-\epsilon)} d\tau \int_{\tau-A}^{\tau-\delta} dr\, e^{-i(t-s)H_0} V(s) e^{-i(s-\tau)H_0} V_1 P_c(H_1) e^{-i(\tau-r)H_1} \cdot \\
\cdot F(|\vec{p}| \ge N) V_2(r) e^{-irH_0} \psi_0,
\end{aligned}$$

and

$$J_{high,2} = \int_{t-A}^{t} ds \int_{s-A}^{s\wedge(t-\epsilon)} d\tau \int_{\tau-A}^{\tau-\delta} dr\, e^{-i(t-s)H_0} V(s) e^{-i(s-\tau)H_0} V_1 P_c(H_1) e^{-i(\tau-r)H_1} \cdot$$
$$\cdot F(|\vec{p}| \geq N) V_2(r) \int_0^r e^{-i(r-\alpha)H_0} V(\alpha) U(\alpha)\psi_0\, d\alpha.$$

The decay of $J_{high,1}$ will come easily from e^{-irH_0}. Indeed, we apply Lemma 2.2.4 to $e^{-i(t-s)H_0}V(s)e^{-i(s-\tau)H_0}$ as in (2.3.23) and notice that

$$\|P_c(H_1)e^{-i(\tau-r)H_1}F(|\vec{p}| \geq N)\|_{L^2\to L^2} \leq 1. \tag{2.3.28}$$

Then it is clear that $\|J_{high,1}\|_\infty$ is dominated by

$$\int_{t-A}^{t} ds \int_{s-A}^{s\wedge(t-\epsilon)} d\tau \int_{\tau-A}^{\tau-\eta} dr \langle t-s\rangle^{-n/2}\langle s-\tau\rangle^{-n/2}\langle r\rangle^{-n/2}\|\psi_0\|_1 \lesssim t^{-n/2}\|\psi_0\|_1.$$

$J_{high,2}$ will be decomposed into three parts $J^1_{high,2}$, $J^2_{high,2}$, and $J^3_{high,2}$, corresponding to $\int_0^B d\alpha$, $\int_B^{r-B} d\alpha$, and $\int_{r-B}^r d\alpha$, respectively, where $B > 0$ is a large constant to be specified.

For $J^1_{high,2}$, the decay comes from $e^{-i(r-\alpha)H_0}$. Indeed, it follows from Lemma 2.2.2 and $0 < \alpha < B$ that

$$\|e^{-i(r-\alpha)H_0}V(\alpha)U(\alpha)\|_{1\to\infty} \lesssim r^{-n/2}e^{M\alpha} \lesssim \langle r\rangle^{-n/2}.$$

Hence, it follows from (2.3.23), (2.3.28), and the above inequality that $\|J^1_{high,2}\|_\infty$ is dominated by

$$\int_{t-A}^{t} ds \int_{s-A}^{s\wedge(t-\epsilon)} d\tau \int_{\tau-A}^{\tau-\delta} dr \langle t-s\rangle^{-n/2}\langle s-\tau\rangle^{-n/2}\langle r\rangle^{-n/2}\|\psi_0\|_1 \lesssim \langle t\rangle^{-n/2}\|\psi_0\|_1.$$

$J^2_{high,2}$ will be estimated by an application of the bootstrap assumption, and the smallness comes from choosing B to be sufficiently large. Indeed, it follows from (2.3.23), (2.3.28), Lemma 2.2.4 and the bootstrap assumption that

$$\begin{aligned}\|J^2_{high,2}\|_\infty &\lesssim \int_{t-A}^{t} ds \int_{s-A}^{s\wedge(t-\epsilon)} d\tau \int_{\tau-A}^{\tau-\delta} dr \langle t-s\rangle^{-n/2}\langle s-\tau\rangle^{-n/2}\cdot \\ &\quad \cdot \int_{B}^{r-B} \langle r-\alpha\rangle^{-n/2}\langle\alpha\rangle^{-n/2} d\alpha\, C_0 |\!|\!|\psi_0|\!|\!| \\ &\lesssim \int_{t-A}^{t} ds \int_{s-A}^{s\wedge(t-\epsilon)} d\tau \int_{\tau-A}^{\tau-\delta} dr \langle t-s\rangle^{-n/2}\langle s-\tau\rangle^{-n/2}\langle r\rangle^{-n/2}\kappa_B C_0 |\!|\!|\psi_0|\!|\!| \\ &\le \frac{1}{100} C_0 t^{-n/2} |\!|\!|\psi_0|\!|\!|.\end{aligned}$$

In the above inequality, B is chosen to be sufficiently large, because $\kappa_B = \int_B^\infty \langle\alpha\rangle^{-n/2} d\alpha \to 0$ when $B \to \infty$.

The decay of $J^3_{high,2}$ can only come from $U(\alpha)$. As usual we need to generate the smallness $\frac{1}{100}$ for the bootstrap assumption. Here the smallness $\frac{1}{100}$ comes from the high velocity and a commutator argument. Write $F(|\vec{p}| \ge N)V_2(r) = [F(|\vec{p}| \ge N), V_2(r)] + V_2(r)F(|\vec{p}| \ge N)$ and correspondingly, we decompose $J^3_{high,2} = J^{3,1}_{high,2} + J^{3,2}_{high,2}$. That is to say $J^{3,1}_{high,2}$ and $J^{3,2}_{high,2}$ are just $J^3_{high,2}$ with $F(|\vec{p}| \ge N)V_2(r)$ replaced by $[F(|\vec{p}| \ge N), V_2(r)]$ and $V_2(r)F(|\vec{p}| \ge N)$.

Specifically, the smallness $\frac{1}{100}$ for $J^{3,1}_{high,2}$ comes from the following standard fact, namely

$$\|[F(|\vec{p}| \le N), V_2]\|_{L^2\to L^2} \lesssim N^{-1}\|\nabla V_2\|_\infty. \qquad (2.3.29)$$

To see this, write $F(|\vec{p}| \le N)f = [\hat{\eta}(\xi/N)\hat{f}(\xi)]^\vee$ with some smooth bump function η. Hence the kernel K of $[F(|\vec{p}| \le N), V_2]$ is

$$K(x,y) = N^n \eta(N(x-y))(V_2(y) - V_2(x)),$$

and (2.3.29) follows from Schur's test and $\sup_x \|K(x,\cdot)\|_{L^1} = \sup_y \|K(\cdot,y)\|_{L^1} \lesssim N^{-1}\|\nabla V_2\|_\infty$.

It follows from (2.3.23), $\|P_c(H_1)e^{-i(\tau-r)H_1}\|_{2\to 2} \le 1$, (2.3.29) and the bootstrap assumption that

$$\begin{aligned}
&\|J^{3,1}_{high,2}\|_\infty \\
&\lesssim \int_{t-A}^{t} ds \int_{s-A}^{s\wedge(t-\epsilon)} d\tau \int_{\tau-A}^{\tau-\delta} dr (\langle t-s\rangle\langle s-\tau\rangle)^{-n/2} \frac{\|V_1\|_2 \|\nabla V_2\|_\infty}{N} \cdot \\
&\qquad \cdot \int_{r-B}^{r} \|e^{-i(r-\alpha)H_0} V(\alpha) U(\alpha)\psi_0\|_2 d\alpha \\
&\lesssim \frac{1}{N} \sup_{t-3A-B<\alpha<t} \|U(t)\psi_0\|_\infty \lesssim \frac{C_0}{N} t^{-n/2} |\!|\!|\psi_0|\!|\!| \le \frac{C_0}{100} t^{-n/2} |\!|\!|\psi_0|\!|\!|,
\end{aligned}$$

where $\frac{1}{N}$ is chosen sufficiently small to dominate the implicit constant in " $\lesssim$ " which only depends on $n, V, \vec{v}_2$ and ϵ, δ, A, B.

The smallness for $J^{3,2}_{high,2}$ comes from the following version of Kato's $\frac{1}{2}$-smoothing estimate:

$$\| \int_{\alpha}^{\alpha+B} \chi_2(\cdot - r\vec{v}_2) F(|\vec{p}| \ge N) e^{-i(r-\alpha)H_0} \, dr \|_{2\to 2} \lesssim \frac{BR}{\sqrt{N}}, \tag{2.3.30}$$

where $\chi_2(\cdot)$ is a smooth cut around the support of V_2 and R is radius of the support of χ_2. The implicit constant only depends on n, V_2. We refer to Section 3.5 in [31] for its proof and further references.

Now observe that the region of integration $\int_{\tau-A}^{\tau-\delta} dr \int_{r}^{r-B} d\alpha$ is contained in that of $\int_{\tau-A-B}^{\tau-\delta} d\alpha \int_{\alpha}^{\alpha+B} dr$ and $\|P_c(H_1) e^{-i(\tau-r)H_1}\|_{2\to 2} \le 1$. It follows from (2.3.23), (2.3.30), and the above observation that

$$\begin{aligned}
\|J^{3,2}_{high,2}\|_\infty &\lesssim \int_{t-A}^{t} ds \int_{s-A}^{s\wedge(t-\epsilon)} d\tau \int_{\tau-A-B}^{\tau-\delta} d\alpha \langle t-s\rangle^{-n/2} \langle s-\tau\rangle^{-n/2} \frac{BR}{\sqrt{N}} \|U(\alpha)\psi_0\|_\infty \\
&\lesssim \frac{C_0}{100} t^{-n/2} |\!|\!|\psi_0|\!|\!|,
\end{aligned}$$

where $\frac{1}{\sqrt{N}}$ is chosen to be sufficiently small to dominate the implicit constant, which only depends on $n, V, \vec{v}_2$ and ϵ, δ, A, B. Therefore, we conclude that (2.3.1) implies (2.3.2), from which Theorem 2.1.3 follows.

2.4 Decay estimates of the derivatives of $U(t)$

In this section we prove Theorem 2.1.4 by induction on κ by following the same scheme as the proof of Theorem 2.1.3. The first step is to set up the cancellation lemma for $\partial U(t)\psi_0$.

Lemma 2.4.1. *Let κ be a nonnegative integer. Assume that*

$$\sup_{0\le\beta\le\kappa}\sup_{r\in R}\|\widehat{\partial^\beta V(r)}\|_{L^1} < M.$$

Let α be a nonnegative integer n-tuple with $|\alpha| = \kappa$. Suppose $U(t)$ is the evolution operator of (2.1.7) as before. Then

$$\sup_{r\in\mathbb{R}}\|e^{-i(t-s)H_0}V(r)\partial^\alpha U(s)\psi_0\|_p < |t|^{-\gamma}Me^{(\kappa+1)Ms}\|\psi_0\|_{W^{\kappa,p'}}, \tag{2.4.1}$$

where $\gamma = n(\frac{1}{2}-\frac{1}{p})$ and $2\le p<\infty$, $\frac{1}{p}+\frac{1}{p'}=1$.

Proof. Write the left-hand side of (2.4.1):= $\Psi(t,s)$. When $\kappa = 0$, (2.4.1) is just the inequality (2.2.3). Note that the inequality (2.2.3) holds with V replaced by its derivative $\partial^\beta V$, as long as $\widehat{\partial^\beta V}$ lies in $L^1(R^n)$. Assume $\kappa = 1$ and apply Duhamel's formula:

$$\begin{aligned}
&\|e^{-i(t-s)H_0}V(r)\partial U(s)\psi_0\|_p\\
&\le \|e^{-i(t-s)H_0}V(r)\partial e^{-isH_0}\psi_0\|_p + \int_0^s \|e^{-i(t-s)H_0}V(r)e^{-i(s-\tau)H_0}\partial V(\tau)U(\tau)\psi_0\|_p d\tau\\
&\le C\|\widehat{V}(r)\|_1 t^{-\gamma}\|\partial\psi_0\|_{p'} + \|\widehat{V}(r)\|_1\int_0^s \|e^{-i(t-\tau)H_0}(\partial V)(\tau)U(\tau)\psi_0\|_p d\tau\\
&\quad + \|\widehat{V}(r)\|_1\int_0^s \|e^{-i(t-\tau)H_0}V(\tau)\partial U(\tau)\psi_0\|_p d\tau\\
&\le CMt^{-\gamma}\|\psi_0\|_{W^{1,p'}} + M\int_0^s t^{-\gamma}e^{\tau M}d\tau\|\psi_0\|_{p'} + M\int_0^s \Psi(t,\tau)d\tau\\
&\le CMt^{-\gamma}e^{sM}\|\psi_0\|_{W^{1,p'}} + M\int_0^s \Psi(t,\tau)d\tau.
\end{aligned}$$

Taking supremum over r, we get $\Psi(t,s)\le CMt^{-\gamma}e^{sM}\|\psi_0\|_{W^{1,p'}} + M\int_0^s\Psi(t,\tau)d\tau$. By

Gronwall's inequality, $\Psi(t,s) \le CMt^{-\gamma}e^{2Ms}$.

For $\kappa > 1$, the above argument goes through by induction, provided that the Fourier transforms of the derivatives up to degree κ of $V(r)$ are uniformly bounded in $L^1(\mathbb{R}^n)$. □

The following is an analog of Corollary 2.2.3:

Corollary 2.4.2. *With the same notations and assumptions as in Lemma 2.4.1, we have*

$$\|U(t)\psi_0\|_{W^{\kappa,p}} \lesssim t^{-\gamma}e^{(1+\kappa)Mt}\|\psi_0\|_{W^{\kappa,p'}}. \tag{2.4.2}$$

Proof. By Duhamel's formula, Lemma 2.4.1, and the fact that ∂ commutes with e^{-itH_0}, we have the following estimate:

$$\begin{aligned}
&\|\partial^\alpha U(t)\psi_0\|_p \lesssim \|e^{-itH_0}\partial^\alpha\psi_0\|_p + \Sigma_{\beta\le\alpha}\int_0^t \|e^{-i(t-\tau)H_0}(\partial^\beta V)(\tau)\partial^{\alpha-\beta}U(\tau)\psi_0\|_p d\tau \\
&\lesssim t^{-\gamma}\|\psi_0\|_{W^{\kappa,p'}} + \Sigma_{\beta\le\alpha}\int_0^t t^{-\gamma}e^{(|\beta|+1)M\tau}d\tau\|\psi_0\|_{W^{\kappa,p'}} \\
&\le Ct^{-\gamma}e^{(\kappa+1)Mt}\|\psi_0\|_{W^{\kappa,p'}}.
\end{aligned}$$

□

Similarly, the following lemma generalizes Lemma 2.2.4:

Lemma 2.4.3. *Let α be an n-tuple with $|\alpha| = \kappa$ and $U(t)$ be the evolution operator of* (2.1.7)*. For each $m \ge 1$ and $\epsilon > 0$, $u_1, u_2, \ldots, u_m$ are all in either $\mathbb{R}_+$ or $\mathbb{R}_-$, satisfying $|\sum_{j=1}^m u_j| > \epsilon$, then there exists constant $C = C(m,\epsilon,\kappa,p)$ such that*

$$\|\prod_{j=1}^{m-1}(e^{iu_jH_0}V(s_j))\partial^\alpha U(u_m)\psi_0\|_p \le CM^{m-1}\prod_{j=1}^{m}\langle u_j\rangle^{-\gamma}e^{(\kappa+1)Mu_m}\|\psi_0\|_{W^{\kappa,p'}}, \tag{2.4.3}$$

where s_j is any real number, $\frac{2n}{n-2} < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$ and

$$M = \Sigma_{0\le\beta\le\alpha}\sup_{s\in R}(\|\partial^\beta V(s)\|_1 + \|\widehat{\partial^\beta V}(s)\|_1).$$

Using Lemma 2.4.1 and Corollary 2.4.2, the proof of Lemma 2.4.3 is exactly the same as that of Lemma 2.2.4.

We only prove Theorem 2.1.4 for the case $\kappa = 1, 2$. The case $\kappa > 2$ can be proven by induction. Specifically, we prove the following implication:

For any fixed sufficiently large time T,

$$\|U(t)\psi_0\|_{W^{\kappa,p}} \leq C_0|t|^{-\gamma}(\|\psi_0\|_{W^{\kappa,p'}} + e^{-\frac{\alpha T}{2}}\|\psi_0\|_{L^2}) \quad \text{for} \quad 0 \leq t \leq T, \kappa = 1, 2 \tag{2.4.4}$$

implies that

$$\|U(t)\psi_0\|_{W^{\kappa,p}} \leq \frac{C_0}{2}|t|^{-\gamma}(\|\psi_0\|_{W^{\kappa,p'}} + e^{-\frac{\alpha T}{2}}\|\psi_0\|_{L^2}) \quad \text{for} \quad 0 \leq t \leq T, \kappa = 1, 2 \tag{2.4.5}$$

provided that $\frac{C_0}{2}$ remains larger than some constant that does not depend on T. The assumption (2.4.4) can be made to hold for some C_0 depending on T, because of Corollary 2.4.2. Letting $T \to +\infty$ to eliminate $\|\psi_0\|_{L^2}$, Theorem 2.1.4 follows from the iteration of the above implication.

We will first prove (2.4.5) for $\kappa = 1$. For technical reasons (see (2.4.15)), we need the above bootstrap assumption (2.4.4) for $\kappa + 2$. To simplify the notation, we write $\partial^\alpha = \partial$ and

$$\|\psi_0\|_{W^{1,p'}} + e^{-\frac{\alpha T}{2}}\|\psi_0\|_{L^2} := \|\!|\psi_0|\!\|_{(1,p')}.$$

With these cancellation lemmas for $\partial U(t)\psi_0$, the proof of Theorem 2.1.4 follows the scheme of that of Theorem 2.1.3. The difference is that now we need to commute ∂_x with operators such as e^{itH_0}, V and e^{itH_1} to apply the cancellation lemma and the bootstrap assumption.

We proceed by expanding $U(t)$ with Duhamel's formula:

$$
\begin{aligned}
\partial U(t)\psi_0 &= \partial e^{-itH_0}\psi_0 - i\int_0^t \partial e^{-i(t-s)H_0}V(s)U(s)\psi_0\, ds \\
&= \partial e^{-itH_0}\psi_0 - i\int_0^t e^{-i(t-s)H_0}(\partial V)(s)U(s)\psi_0\, ds \\
&\quad - i\int_0^t e^{-i(t-s)H_0}V(s)\partial U(s)\psi_0\, ds. \qquad (2.4.6)
\end{aligned}
$$

Notice that $[\partial, V] = (\partial V)\cdot$ is a multiplication operator, which can be viewed as another potential and Theorem 2.1.3 can be applied to the second term of (2.4.6). This idea has appeared in the proof of Lemma 2.4.1. Specifically, it follows from the proof of Theorem 2.1.3 and an interpolation with the L^2 conservation of $U(t)$ that

$$
\| \int_0^t e^{-i(t-s)H_0}V(s)U(s)\psi_0\, ds\|_p \lesssim t^{-\gamma}\|\psi_0\|_{p'}.
$$

By assumption, ∂V_j satisfies the regularity and smoothness conditions for V_j in Theorem 2.1.3, and we conclude that

$$
\| \int_0^t e^{-i(t-s)H_0}(\partial V)(s)U(s)\psi_0\, ds\|_p \lesssim t^{-\gamma}\|\psi_0\|_{p'}.
$$

We expand the last term of (2.4.6) by Duhamel's formula just as in Section 2.3 and perform the same decomposition. With the cancellation lemma for $\partial U(t)$ and Remark 2.3.1, the last term (2.4.6) is reduced to the following:

$$
\sum_{j=1}^{2}\int_{t-A}^{t} ds \int_{s-A}^{s\wedge(t-\epsilon)} d\tau\, e^{-i(t-s)H_0}V(s)e^{-i(s-\tau)H_0}V_j(\cdot - \tau\vec{v}_j)\partial P_c(H_1,\tau)U(\tau)\psi_0. \quad (2.4.7)
$$

Before we proceed, we observe that our assumptions guarantee

$$
\|P_c(H_1)e^{-itH_1}\psi_0\|_{L^q} \leq C_q\, |t|^{-\gamma}\|\psi_0\|_{L^{q'}}. \qquad (2.4.8)
$$

This implies that

$$\|H_1P_c(H_1)e^{-itH_1}\psi_0\|_{L^q} = \|P_c(H_1)e^{-itH_1}H_1\psi_0\|_{L^q}$$
$$\leq C_q\,|t|^{-\gamma}\|H_1\psi_0\|_{L^{q'}} \leq C_q\,|t|^{-\gamma}\|\psi_0\|_{W^{2,q'}}.$$

As $V_1 \in L^\infty(\mathbb{R}^n)$ and double Riesz transforms are bounded on $L^q(\mathbb{R}^n)$ $1 < q < +\infty$, the above inequality in the case of $1 < q < +\infty$, implies that

$$\|P_c(H_1)e^{-itH_1}\psi_0\|_{W^{2,q}} \lesssim |t|^{-\gamma}\|\psi_0\|_{W^{2,q'}}. \tag{2.4.9}$$

Interpolating between (2.4.8) and (2.4.9) (Theorem 6.4.5 [5]), we conclude that

$$\|P_c(H_1)e^{-itH_1}\psi_0\|_{W^{1,q}} \leq C_q\,|t|^{-\gamma}\|\psi_0\|_{W^{1,q'}}, \tag{2.4.10}$$

where $2 \leq q < \infty$, $\frac{1}{q}+\frac{1}{q'} = 1$ and $\gamma = n(\frac{1}{2}-\frac{1}{q})$. Because double Riesz transforms are unbounded on $L^\infty(\mathbb{R}^n)$, we exclude $p = \infty$ in Theorem 2.1.4.

We write $P_c(H_1)U(\tau) = P_c(H_1)e^{-i\tau H_1} - iP_c(H_1)\int_0^\tau e^{-i(\tau-r)H_1}V_2(r)U(r)\,dr$ and (2.4.7) is broken into two terms.

It follows from (2.4.10), among other things, that the first term of (2.4.7), which contains $P_c(H_1)e^{-i\tau H_1}$, is dominated by $|t|^{-\gamma}\|\psi_0\|_{W^{1,p}}$.

The second term of (2.4.7) is decomposed as follows:

$$\int_0^\tau dr = \int_0^\delta dr + \int_\delta^A dr + \int_A^{\tau-A} dr + \int_{\tau-A}^{\tau-\delta} dr + \int_{\tau-\delta}^\tau dr. \tag{2.4.11}$$

We estimate each term in (2.4.11) with similar methods as those for (2.3.20). Because of (2.4.10), the terms containing $\int_\delta^A dr$ and $\int_A^{\tau-A} dr$ in (2.4.11) can be estimated exactly as there is no derivative before $P(H_1)$, and we omit the details here. Again by (2.4.10) with $q = 2$, the term containing $\int_{\tau-\delta}^\tau dr$ in (2.4.11) is estimated as follows:

$$\int_{t-A}^{t} ds \int_{s-A}^{s\wedge(t-\epsilon)} d\tau \int_{\tau-\delta}^{\tau} dr \| e^{-i(t-s)H_0} V(s) e^{-i(s-\tau)H_0} V_1(\tau) \partial e^{-i(\tau-r)H_1} P_c(H_1) V_2(r) U(r) \psi_0 \|_p$$
$$\lesssim \sup_{t-2A<\tau<t} \int_{\tau-\delta}^{\tau} dr \| \partial P_c(H_1) e^{-i(\tau-r)H_1} V_2(r) U(r) \psi_0 \|_2$$
$$\lesssim \sup_{t-2A<\tau<t} \int_{\tau-\delta}^{\tau} dr \| V_2(r) U(r) \psi_0 \|_{W_{1,2}}$$
$$\lesssim \sup_{t-2A<\tau<t} \int_{\tau-\delta}^{\tau} dr \| U(r) \psi_0 \|_{W_{1,p}}$$
$$\lesssim t^{-\gamma} C_0 \delta \| \psi_0 \|_{(1,p')} \leq \frac{C_0}{100} t^{-\gamma} \| \psi_0 \|_{(1,p')}.$$

Here $\delta > 0$ is chosen sufficiently small.

The $\int_0^\delta dr$ term in (2.4.11) is expanded by Duhamel's formula:

$$e^{-i(\tau-r)H_1} = e^{-i(\tau-r)H_0} - i \int_0^{\tau-r} e^{-i(\tau-r-\beta)H_1} V_1 e^{-i\beta H_0} \, d\beta.$$

Plugging the above expression into the $\int_0^\delta \, dr$ term, we get two terms. The first one containing $e^{-i(\tau-r)H_0}$ is

$$\int_{t-A}^{t} ds \int_{s-A}^{s\wedge(t-\epsilon)} d\tau \int_0^\delta dr e^{-i(t-s)H_0} V(s) e^{-i(s-\tau)H_0} V_1(\tau) \partial P_c(H_1) e^{-i(\tau-r)H_0} V_2(r) U(r) \psi_0. \tag{2.4.12}$$

Since $P_c(H_1) = Id - P_b(H_1)$ and $P_b(H_1)$ is a bounded operator from L^p to L^p, $P_c(H_1)$ is bounded from L^p to L^p. It follows from Lemma 2.4.1, $0 < r < \delta$, and the Leibnitz rule that

$$\| H_1 P_c(H_1) e^{-i(\tau-r)H_0} V_2(r) U(r) \psi_0 \|_p = \| P_c(H_1) H_1 e^{-i(\tau-r)H_0} V_2(r) U(r) \psi_0 \|_p$$
$$\leq C \| H_1 e^{-i(\tau-r)H_0} V_2(r) U(r) \psi_0 \|_p$$
$$\leq C \| V_1 \|_\infty \| e^{-i(\tau-r)H_0} V_2(r) U(r) \psi_0 \|_p + \| e^{-i(\tau-r)H_0} \Delta V_2(r) U(r) \psi_0 \|_p$$
$$\leq C \tau^{-\gamma} \| \psi_0 \|_{W^{2,p'}}.$$

Since $H_1 = H_0 + V_1$ and V_1 is bounded, we see that

$$\|\Delta P_c(H_1)e^{-i(\tau-r)H_0}V_2(r)U(r)\psi_0\|_p \leq C\tau^{-\gamma}\|\psi_0\|_{W^{2,p'}}.$$

Because the double Riesz transforms are bounded on $L^p(\mathbb{R}^n)$ $1 < p < \infty$, it follows that

$$\|P_c(H_1)e^{-i(\tau-r)H_0}V_2(r)U(r)\psi_0\|_{W^{2,p}} \leq C\tau^{-\gamma}\|\psi_0\|_{W^{2,p'}}.$$

Therefore, by complex interpolation, we see that

$$\|P_c(H_1)e^{-i(\tau-r)H_0}V_2(r)U(r)\psi_0\|_{W^{1,p}} \leq C\tau^{-\gamma}\|\psi_0\|_{W^{1,p'}}, \tag{2.4.13}$$

which implies that $\|(2.4.12)\|_{W^{1,p}} \lesssim t^{-\gamma}\|\psi_0\|_{W^{1,p'}}$.

For the term containing $\int_0^\delta dr \int_0^{\tau-r} d\beta$, we perform the exact same decomposition as in (2.3.22) and each step there goes through provided (2.4.10) and (2.4.13).

The term containing $\int_{\tau-A}^{\tau-\delta} dr$ in (2.4.11) is

$$\int_{t-A}^{t} ds \int_{s-A}^{s\wedge(t-\epsilon)} d\tau \int_{\tau-A}^{\tau-\delta} dr\, \|e^{-i(t-s)H_0}V(s)e^{-i(s-\tau)H_0}V_1\partial P_c(H_1)e^{-i(\tau-r)H_1}V_2(r)U(r)\psi_0\|_p. \tag{2.4.14}$$

The proof of Theorem 2.1.3 showed that $\forall\epsilon > 0$, the following holds:

$$\|V_1P_c(H_1)e^{-i(\tau-r)H_1}V_2(r)U(r)\psi_0\|_\infty < \epsilon C_0 t^{-\frac{n}{2}}\|\psi_0\|_1,$$

given t sufficiently large. Going through the proof, we see that the same argument also shows

$$\|V_1P_c(H_1)e^{-i(\tau-r)H_1}V_2(r)U(r)\psi_0\|_p < \epsilon C_0 t^{-\gamma}\|\psi_0\|_{p'}.$$

Furthermore, the above inequality holds if V_1 or V_2 is replaced by its derivative. Another observation is that, given our new cancellation lemma for $\partial U(r)\psi_0$,

$$\|V_1P_c(H_1)e^{-i(\tau-r)H_1}V_2(r)\partial^\beta U(r)\psi_0\|_p < \epsilon C_0 t^{-\gamma}|\!|\!|\psi_0|\!|\!|_{(|\beta|,p')}.$$

Indeed, to prove the above inequality, we decompose the left-hand side into a high velocity part and a low velocity part. Each part generates the small constant ϵ for the same reason as in Section 3.3. The same argument with the bootstrap assumption (2.4.4) implies

$$\begin{aligned}&\|V_1 H_1 P_c(H_1) e^{-i(\tau-r)H_1} V_2(r) U(r)\psi_0\|_p \\ &= \|V_1 P_c(H_1) e^{-i(\tau-r)H_1} H_1 V_2(r) U(r)\psi_0\|_p \lesssim \epsilon C_0 t^{-\gamma} \|\psi_0\|_{(2,p')}. \end{aligned} \tag{2.4.15}$$

It follows from the above inequality and an elementary calculation that

$$\|V_1 P_c(H_1) e^{-i(\tau-r)H_1} V_2(r) U(r)\psi_0\|_{W^{2,p}} \lesssim \epsilon C_0 t^{-\gamma} \|\psi_0\|_{(2,p')}. \tag{2.4.16}$$

Hence, by complex interpolation, for $\forall\, \epsilon > 0$,

$$\|V_1 P_c(H_1) e^{-i(\tau-r)H_1} V_2(r) U(r)\psi_0\|_{W^{1,p}} \lesssim \epsilon C_0 t^{-\gamma} \|\psi_0\|_{(1,p')}, \tag{2.4.17}$$

given t sufficiently large. This implies that $\|(2.4.14)\|_{W^{1,p}}$ can be estimated by $\frac{1}{100} C_0 t^{-\gamma} \|\psi_0\|_{1,p'}$.

Therefore, we have proven (2.4.5) for $\kappa = 1$. The same procedure also proves (2.4.5) for $\kappa = 2$. Thus, we finish the bootstrap argument and conclude that

$$\|U(t)\psi_0\|_{W^{\kappa,p}} \lesssim \|\psi_0\|_{W^{(\kappa,p')}},$$

by letting $T \to \infty$. The proof for $\kappa > 2$ is similar by induction. Thus, we have proven Theorem 2.1.4.

2.5 Boundedness of the Sobolev norm of $U(t, s)\psi_0$

The goal of this section is to prove Theorem 2.1.5 when κ is a positive integer. The intuition comes from the case $\kappa = 1$ ([17]). To bound the kinetic energy (the H^1 norm), we look at the observable $K(t) = \frac{1}{2}(p - \frac{x}{t})^2 + \sum_{l=1}^m V_l(t)$. $\langle K(t)\rangle$ will decrease if the particle is far away from any potential, since the observable $(p - \frac{x}{t})^2$ decreases like t^{-2} for the free motion (the pseudo-conformal identity). If the particle is close to the center of potential V_l, then $\frac{x}{t} \approx \vec{v_l}$ and $\langle K(t)\rangle \approx \langle \frac{1}{2}(p-\vec{v_l})^2 + V_l(x-\vec{v_l}t)\rangle$, which clearly is the total energy of this one potential stationary subsystem up to a Galilean transform. To carry this boundedness from $\langle K(t)\rangle$ to $\langle p^2\rangle$, we need to replace the vector field $\frac{x}{t}$ by $\nu(x,t)$, such that $\nu(x,t)$ is uniformly bounded and is equal to $\vec{v_l}$ in an increasingly large neighborhood of $x = \vec{v_l}t$.

Rigorously, consider a smooth, uniformly bounded vector field

$$\nu(x,t) : \mathbb{R}^n \times (-\infty, -T] \cup [T, +\infty) \to \mathbb{R}^n$$

and let

$$K_0(t) = \frac{1}{2}(p - \nu(x,t))^2 + \sum_{l=1}^{m} V_l(t),$$

where T is a large positive constant, $p = (p_1, \cdots, p_n)$, and $p_j = -i\frac{\partial}{\partial x_j}$. Note $p^2 = H_0$ and $\frac{1}{2}(p-\nu(x,t))^2$ is a well-defined self-adjoint positive operator.

In [17], Graf constructed $\nu(x,t)$ and proved $\|U(t,s)\psi_0\|_{H^1}$ is bounded as $t \to \infty$ by bounding $\frac{d}{dt}\langle K_0(t)\rangle$ from above by a time-integrable function, where $\langle K_0(t)\rangle = (U(t,s)\psi_0, K_0(t)U(t,s)\psi_0)_{L^2}$. We write (f,g) as the inner product of f, g in the $L^2(\mathbb{R})$ sense.

To prove Theorem 2.1.5, we need to define the proper analog of $K_0(t)$ suitable to the H^κ norm of $U(t,s)\psi_0$ to match the intuition given by the classical system. Fortunately the following observable works:

$$K(t) = \sum_{l=1}^{m}(\frac{1}{2}(p-\nu(x,t))^2 + V_l(t))^\kappa - (m-1)(\frac{1}{2}(p-\nu(x,t))^2)^\kappa.$$

Notice that $K(t) = K_0(t)$ if $\kappa = 1$. Because $\nu(x,t)$ and its derivatives are bounded uniformly in space-time and $V_j \in C_0^\kappa(\mathbb{R}^n)$, we have the following, writing $\langle K(t,s)\rangle = (U(t,s)\psi_0, K(t)U(t,s)\psi_0)$:

$$\|U(t,s)\psi_0\|_{H^\kappa}^2 \lesssim \langle K(t,s)\rangle + \|U(t,s)\psi_0\|_{H^{\kappa-1}}^2; \quad \langle K(t,s)\rangle \lesssim \|U(t,s)\psi_0\|_{H^\kappa}^2.$$

By induction on κ, it suffices to show $\langle K(t,s)\rangle$ is bounded uniformly in t and s.

Expand $K(t)$ as a polynomial of $(\frac{1}{2}(p-\nu(x,t))^2$. Though $(\frac{1}{2}(p-\nu(x,t))^2$ and $V_l(t)$ do not commute with each other, viewing $K(t)$ as a differential operator, the term of highest degree is $((\frac{1}{2}(p-\nu(x,t))^2)^\kappa$, which is positive, self-adjoint. The other terms in $K(t)$ are of degree no bigger than $2\kappa-2$ with bounded and smooth enough coefficients. Correspondingly, $\langle K(t,s)\rangle$ breaks into two parts. The part $(U(t,s)\psi_0, (\frac{1}{2}(p-\nu(x,t))^2)^\kappa U(t,s)\psi_0)$ is always nonnegative. The other part containing the low degree terms can be dominated by $\|U(t,s)\psi_0\|^2_{H^{\kappa-1}}$. By the induction hypothesis, it follows that $\langle K(t,s)\rangle$ is bounded from below. To bound $\langle K(t,s)\rangle$ from above, it suffices to show that for $t>T$,

$$\frac{d}{dt}\langle K(t,s)\rangle \le t^{-(1+\delta)}C(\langle K(t,s)\rangle + \|U(t,s)\psi_0\|^2_{H^{\kappa-1}}). \tag{2.5.1}$$

For $t<-T$, the opposite of the above inequality should hold:

$$\frac{d}{dt}\langle K(t,s)\rangle \ge t^{-(1+\delta)}C(\langle K(t,s)\rangle + \|U(t,s)\psi_0\|^2_{H^{\kappa-1}}). \tag{2.5.2}$$

First let's consider $t>T$, integrating (2.5.1),

$$\langle K(t_2,s)\rangle - \langle K(t_1,s)\rangle \le C\int_{t_1}^{t_2} t^{-(1+\delta)}\langle K(t,s)\rangle dt + C\sup_{t,s\in\mathbb{R}}\|U(t,s)\psi_0\|^2_{H^{\kappa-1}}.$$

Choosing $T>0$ large enough such that $C\int_T^\infty t^{-1-\delta}dt < \frac{1}{2}$, then

$$\langle K(t_2,s)\rangle \le \langle K(t_1,s)\rangle + C\sup_{t,s\in\mathbb{R}}\|U(t,s)\psi_0\|^2_{H^{\kappa-1}} + \frac{1}{2}\max_{t_1<t<t_2}\langle K(t,s)\rangle.$$

By (2.1.19), $\max_{t_1<t<t_2}\langle K(t,s)\rangle < \infty$. This implies that

$$\max_{t_1<t<t_2}\langle K(t,s)\rangle \le 2\langle K(t_1,s)\rangle + C\|\psi_0\|^2_{H^{\kappa-1}}.$$

Letting $t_2\to+\infty$ and $t_1=T$, it follows that $\max_{t>T}\langle K(t,s)\rangle < C\langle K(T,s)\rangle + C\|\psi_0\|^2_{H^{\kappa-1}}$. Hence $\langle K(t,s)\rangle \le C_T\|\psi_0\|_{H^\kappa}$ for $t>T$ and $s\in[-T,T]$. For $t<-T$, we integrate (2.5.2) to bound $\langle K(t,s)\rangle$ from above and Theorem 2.1.5 follows in this case by the same argument given that (2.5.2) holds.

Before we proceed to proving (2.5.1) and (2.5.2), let's specify some properties of the vector field $\nu(x,t)$. It is convenient to describe $\nu(x,t)$ in the rescaled coordinates $y=\frac{x}{t}$. Let $u_0 = 2\max_{1\le l\le m}|\vec{v_l}|$. When $|y|>u_0$, $\nu(x,t)=u_0\frac{y}{|y|}$. When $y\in B_l$, we specify $\nu(x,t)=\vec{v_l}$, where B_l is a fixed ball centered at $\vec{v_l}$. We suppose that B_l $(l=1,\cdots,m)$ lie in the big ball B_0 centered at the origin with radius u_0 and that they are disjoint from each other. When $y\in B_0-\cup_{l=1}^m B_l$, we specify $\nu(x,t)=y$. To make the vector field smooth, we modify and smooth the vector field in the scale of $|t|^{1-\gamma}$, where γ is a small positive number. In the rescaled coordinates y, the scale is $|t|^{-\gamma}$. Specifically, consider

$$\omega(s,\alpha)=s\,\varphi(\frac{u_0-s}{\alpha})+u_0(1-\varphi(\frac{u_0-s}{\alpha})),$$

where $\varphi\in C^\infty(\mathbb{R})$ with $\varphi'\ge 0$ and

$$\varphi(x)=0 \text{ for } x\le 0 \qquad \varphi(x)=1 \text{ for } x>1.$$

Then writing $y=\frac{x}{t}$, we define

$$\omega^{(0)}(x,t)=\omega(|y|,|t|^{-\gamma})\frac{y}{|y|} \quad \text{and} \quad \omega^{(\ell)}(x,t)=-(y-\vec{v_\ell})\varphi(2-|t|^\delta|y-\vec{v_\ell}|),$$

where $\ell=1,2,\cdots,m$. Finally, $\nu(x,t):=\sum_{\ell=0}^m\omega^{(\ell)}$

The properties of the vector field $\nu(x,t)$ that concern us are as follows:

1. ν is bounded in space time. The k-th space derivatives of ν uniformly decay as $|t|^{-k(1-\gamma)}$ as $t\to\infty$.

2. $(\nu_{i,j})_{n\times n}$ as a matrix is symmetric and positive semidefinite when $t>0$, negative semidefinite when $t<0$, where ν_i is the i-th component of vector ν and the indices following a comma stand for partial derivatives in space. As $\nu_{k,j}=\nu_{j,k}$, $p_k-\nu_k$ and $p_j-\nu_j$ commute with each other, i.e., $[p_k-\nu_k,p_j-\nu_j]=0$.

3. $\|\nu_{i,j}\nu_j + \frac{\partial \nu_i}{\partial t}\|_\infty \leq C|t|^{-(1+\delta)}$. We make the choice $1+\delta = \min\{1+\gamma, 2-2\gamma\} > 1$. Summation over double indices is understood.

These properties can be shown by a direct calculation ([17]). Now we are going to prove (2.5.1) and (2.5.2) and proceed by observing that

$$i\frac{\partial}{\partial t}U(t,s)\psi_0 = H(t)U(t,s)\psi_0 \quad \text{and} \tag{2.5.3}$$

$$-i\frac{\partial}{\partial s}U(t,s)\psi_0 = U(t,s)H(s)\psi_0. \tag{2.5.4}$$

It follows from the above that $\frac{d}{dt}\langle K(t,s)\rangle = (U(t,s)\psi_0, (i[H(t),K(t)] + \frac{\partial K}{\partial t})U(t,s)\psi_0)$. A straightforward calculation shows that

$$\frac{\partial K}{\partial t} = \sum_{l=1}^{m}\sum_{k=0}^{\kappa-1}(\frac{1}{2}(p-\nu(x,t))^2 + V_l(t))^k \frac{d}{dt}(\frac{1}{2}(p-\nu)^2 + V_l(t))(\frac{1}{2}(p-\nu)^2 + V_l(t))^{\kappa-1-k}$$
$$-(m-1)\sum_{k=0}^{\kappa-1}(\frac{1}{2}(p-\nu)^2)^k \frac{d}{dt}\frac{1}{2}(p-\nu(x,t))^2(\frac{1}{2}(p-\nu)^2)^{\kappa-1-k} := J_1 + J_2,$$

and the commutator

$$[H(t),K(t)] = [\frac{1}{2}p^2 + \sum_{l=1}^{m}V_l(t), \sum_{l=1}^{m}(\frac{1}{2}(p-\nu)^2 + V_l(t))^\kappa - (m-1)(\frac{1}{2}(p-\nu)^2)^\kappa]$$

$$= \sum_{l=1}^{m}\sum_{k=0}^{\kappa-1}(\frac{1}{2}(p-\nu)^2 + V_l(t))^k[\frac{1}{2}p^2, \frac{1}{2}(p-\nu)^2 + V_l(t)](\frac{1}{2}(p-\nu)^2 + V_l(t))^{\kappa-1-k}$$
$$-(m-1)\sum_{k=0}^{\kappa-1}(\frac{1}{2}(p-\nu)^2)^k[\frac{1}{2}p^2, \frac{1}{2}(p-\nu)^2](\frac{1}{2}(p-\nu)^2)^{\kappa-1-k}$$
$$+\sum_{l,j=1}^{m}\sum_{k=0}^{\kappa-1}(\frac{1}{2}(p-\nu)^2 + V_l(t))^k[V_j(t), \frac{1}{2}(p-\nu)^2 + V_l(t)](\frac{1}{2}(p-\nu)^2 + V_l(t))^{\kappa-1-k}$$
$$-(m-1)\sum_{j=1}^{m}\sum_{k=0}^{\kappa-1}(\frac{1}{2}(p-\nu)^2)^k[V_j(t), \frac{1}{2}(p-\nu)^2](\frac{1}{2}(p-\nu)^2)^{\kappa-1-k} := J_3 + J_4 + J_5 + J_6.$$

First, let's consider

$$J_1 + iJ_3 + iJ_5 = \sum_{l=1}^{m} \sum_{k=0}^{\kappa-1} (\frac{1}{2}(p-\nu)^2 + V_l(t))^k M_1 (\frac{1}{2}(p-\nu)^2 + V_l(t))^{\kappa-1-k}, \quad (2.5.5)$$

where $M_1 = i[\frac{1}{2}p^2 + \sum_{j=1}^m V_j(t), \frac{1}{2}(p-\nu)^2 + V_l(t)] + \frac{d}{dt}(\frac{1}{2}(p-\nu)^2 + V_l(t))$. Another elementary calculation gives the following:

$$M_1 = i[\sum_{j\neq l} V_j, \frac{1}{2}(p-\nu)^2 + V_l] + A - \frac{1}{2}p_i(\nu_{i,j}\nu_j + \frac{\partial \nu_i}{\partial t}) - \frac{1}{2}(\nu_{i,j}\nu_j + \frac{\partial \nu_i}{\partial t})p_i$$
$$+ \nu_i(\nu_{i,j}\nu_j + \frac{\partial \nu_i}{\partial t}) + \frac{1}{4}\nu_{i,ijj} + (\nu - \vec{v_l}) \cdot \nabla V_l, \quad (2.5.6)$$

where $A = -(p_i - \nu_i)\frac{\nu_{i,j}+\nu_{j,i}}{2}(p_j - \nu_j)$ is a symmetric, semidefinite negative operator when $t > 0$ and a semidefinite positive operator when $t < 0$. It follows from the properties of $\nu_{x,t}$ that

$$\|\nu_{i,j}\nu_j + \frac{\partial \nu_i}{\partial t}\|_\infty \leq C|t|^{-(1+\delta)}, \quad \|\nu_{i,ijj}\|_\infty \leq C|t|^{-1-\delta} \quad (2.5.7)$$

and that the L_∞ norm of derivatives of these terms decay even faster because each space derivative gains a factor $|t|^{\delta-1}$. Moreover, $\sum_{l=1}^m (\nu - \vec{v_l}) \cdot \nabla V_l$ vanishes as $|t| > T$ is sufficiently large, since $\nu - \vec{v_l}$ vanishes on an increasing neighborhood of $x = t\vec{v_l}$, which will eventually contain the support of ∇V_l.

Plugging the expression of M_1 into expression (2.5.5), we claim that the decaying terms listed in equation (2.5.7) only produce time integrable term. We calculate the term containing $\frac{1}{2}p_i(\nu_{i,j}\nu_j + \frac{\partial \nu_i}{\partial t})$ as an example to illustrate this point:

$$|(U(t,s)\psi_0, (\frac{1}{2}(p-\nu)^2 + V_l(t))^k \frac{1}{2}p_i(\nu_{i,j}\nu_j + \frac{\partial \nu_i}{\partial t})(\frac{1}{2}(p-\nu)^2 + V_l(t))^{\kappa-1-k}U(t,s)\psi_0)|$$
$$= |(\frac{1}{2}p_i(\frac{1}{2}(p-\nu)^2 + V_l(t))^k U(t,s)\psi_0, (\nu_{i,j}\nu_j + \frac{\partial \nu_i}{\partial t})(\frac{1}{2}(p-\nu)^2 + V_l(t))^{\kappa-1-k}U(t,s)\psi_0)|. \quad (2.5.8)$$

If $2k+1=\kappa$ or $2k+2=\kappa$, (2.5.8) can be dominated by

$$
\begin{aligned}
&C|t|^{-1-\delta}\|p_i(\frac{1}{2}(p-\nu)^2+V_l(t))^k U(t,s)\psi_0\|_{L^2}\|(\frac{1}{2}(p-\nu)^2+V_l(t))^{\kappa-1-k}U(t,s)\psi_0\|_{L^2}\\
&\le C|t|^{-1-\delta}\|U(t,s)\psi_0\|^2_{H^\kappa}\\
&\le C|t|^{-1-\delta}(\langle K(t,s)\rangle+\|U(t,s)\psi_0\|^2_{H^{\kappa-1}}).
\end{aligned}
$$

If $\kappa\neq 2k+1$ or $2k+2$, first consider $2k+2<\kappa$ and $\kappa=2d+1$, an odd integer. We need to commute $\nu_{i,j}\nu_j+\frac{\partial\nu_i}{\partial t}$ with $(\frac{1}{2}(p-\nu)^2+V_l)^{d-k}$. Specifically, we claim that

$$
(\frac{1}{2}(p-\nu)^2+V_l(t))^{d-k}(\nu_{i,j}\nu_j+\frac{\partial\nu_i}{\partial t})\frac{p_i}{2}(\frac{1}{2}(p-\nu)^2+V_l(t))^k
$$

is an differential operator of degree $2d+1$, whose coefficients are of magnitude $t^{-1-\delta}$. This is clear because $\nu_{i,j}\nu_j+\frac{\partial\nu_i}{\partial t}$ and its derivatives decay at least as $|t|^{-1-\delta}$. Hence, (2.5.8) is dominated by $C|t|^{-1-\delta}(\langle K(t,s)\rangle+\|U(t,s)\psi_0\|^2_{H^{\kappa-1}})$. In the case that $2k+2<\kappa$ and $\kappa=2d$ or $2k+1>\kappa$, (2.5.8) is dominated by $C|t|^{-1-\delta}(\langle K(t,s)\rangle+\|U(t,s)\psi_0\|^2_{H^{\kappa-1}})$ for the same reason.

Therefore, it remains to estimate the following in expression (2.5.5) :

$$
\sum_{l=1}^{m}\sum_{k=0}^{\kappa-1}(\frac{1}{2}(p-\nu)^2+V_l(t))^k(i[\sum_{j\neq l}V_j,\frac{1}{2}(p-\nu)^2+V_l]+A)(\frac{1}{2}(p-\nu)^2+V_l(t))^{\kappa-1-k}. \quad (2.5.9)
$$

Observe that for given time t, $\nu(x,t)$ is a constant vector on a ball centered at $t\vec{v_l}$ with radius growing linearly in $|t|$ approximately. So, as long as $|t|$ is large, $\nu(x,t)$ will be constant on the support of $V_l(t)$. This implies that $\nu_{j,i},\nu_{i,j}$ both vanish on the support of $V_l(t)$. Hence it follows from $A=-(p_i-\nu_i)\frac{\nu_{i,j}+\nu_{j,i}}{2}(p_j-\nu_j)$ that $A\,V_l=0$ and $V_l\,A=0$. Moreover, for $j\neq l$, $V_j(t),V_l(t)$ have disjoint supports given that t is large. So the expression (2.5.9) is reduced to the following:

$$\sum_{l=1}^{m}\sum_{k=0}^{\kappa-1}(\frac{1}{2}(p-\nu)^2)^k(i[\sum_{j\neq l}V_j,\frac{1}{2}(p-\nu)^2]+A)(\frac{1}{2}(p-\nu)^2)^{\kappa-1-k} \tag{2.5.10}$$

$$=\sum_{k=0}^{\kappa-1}(\frac{1}{2}(p-\nu)^2)^k(mA+(m-1)i[\sum_{j}V_j,\frac{1}{2}(p-\nu)^2])(\frac{1}{2}(p-\nu)^2)^{\kappa-1-k} \tag{2.5.11}$$

Secondly, we consider $J_2+iJ_4+iJ_6$, which equals

$$-(m-1)\sum_{k=0}^{\kappa-1}(\frac{1}{2}(p-\nu)^2)^k(i[\frac{1}{2}p^2+\sum_{j=1}^{m}V_j(t),\frac{1}{2}(p-\nu)^2]+\frac{d}{dt}\frac{1}{2}(p-\nu)^2)(\frac{1}{2}(p-\nu)^2)^{\kappa-1-k}. \tag{2.5.12}$$

Setting all potentials $V_l=0$ in (2.5.6), we see that

$$i[\frac{1}{2}p^2,\frac{1}{2}(p-\nu)^2]+\frac{d}{dt}\frac{1}{2}(p-\nu)^2=A+\texttt{time integrable terms},$$

where the time-integrable terms are equal to

$$A-\frac{1}{2}p_i(\nu_{i,j}\nu_j+\frac{\partial\nu_i}{\partial t})-\frac{1}{2}(\nu_{i,j}\nu_j+\frac{\partial\nu_i}{\partial t})p_i+\nu_i(\nu_{i,j}\nu_j+\frac{\partial\nu_i}{\partial t})+\frac{1}{4}\nu_{i,ijj}$$

and can be estimated exactly as those in $J_1+iJ_3+iJ_5$. We are left to estimate in $J_2+iJ_4+iJ_6$:

$$-(m-1)\sum_{k=0}^{\kappa-1}(\frac{1}{2}(p-\nu)^2)^k(A+i[\sum_{j=1}^{m}V_j(t),\frac{1}{2}(p-\nu)^2])(\frac{1}{2}(p-\nu)^2)^{\kappa-1-k} \tag{2.5.13}$$

Now adding (2.5.11) and (2.5.13) together, we see that $\langle i[H(t),K(t)]+\frac{\partial K}{\partial t}\rangle$ is simplified as some time-integrable terms plus the following:

$$\sum_{k=0}^{\kappa-1}(\frac{1}{2}(p-\nu)^2)^kA(\frac{1}{2}(p-\nu)^2)^{\kappa-1-k}, \tag{2.5.14}$$

which is a differential operator of degree 2κ.

First we observe that $[p_k - \nu_k, p_j - \nu_j] = 0$ and $(p_k - \nu_k)\frac{\nu_{i,j}+\nu_{j,i}}{2} = \frac{\nu_{i,j}+\nu_{j,i}}{2}(p_k - \nu_k) + \frac{\nu_{i,jk}+\nu_{j,ik}}{2}$. Second, $\nu_{i,jk} + \nu_{j,ik}$ and its derivatives decay at least as fast as $|t|^{-1-\delta}$ when $t \to \infty$ and thus is integrable in time. Hence if we commute A with $(p-\nu)^2$ or $p_j - \nu_j$, the commutator is time-integrable.

If $\kappa = 2d + 1$, an odd integer, then

$$(\frac{1}{2}(p-\nu)^2)^k A(\frac{1}{2}(p-\nu)^2)^{\kappa-1-k} = (\frac{1}{2}(p-\nu)^2)^d A(\frac{1}{2}(p-\nu)^2)^d + \texttt{time-integrable terms}.$$

The first summand is negative (positive) definite when $t > 0$ ($t < 0$).

If $\kappa = 2d$, an even integer, then $(\frac{1}{2}(p-\nu)^2)^k A(\frac{1}{2}(p-\nu)^2)^{\kappa-1-k} = (\frac{1}{2}(p-\nu)^2)^{h-1}\frac{1}{2}(p_j - \nu_j)A(p_j - \nu_j)(\frac{1}{2}(p-\nu)^2)^{h-1} + \texttt{time-integrable terms}$. Again the first summand is negative (positive) definite if $t > 0$ ($t < 0$).

Hence, we have written $\frac{d}{dt}\langle K(t,s)\rangle$ as a sum of a negative (positive if $t < 0$) term and other time-integrable terms. More precisely, the time-integrable terms decay at least as fast as $|t|^{-1-\delta}$. Therefore, we have proven (2.5.1) for $t > T$ and (2.5.2) for $t < -T$.

Finally, we deal with the case where $|t| < T$, $s > T$ by time reversal. Write $r = s - t$ and $\widetilde{U}(r,s) = U(s-r,s)$, $\widetilde{H}(r) = H(s-r)$. Then we have $i\partial_r \widetilde{U}(r,s) = -\widetilde{H}(r)\widetilde{U}(r,s)$. Define the corresponding observable:

$$\widetilde{K}(r) = \sum_{l=1}^{m}(\frac{1}{2}(p+\nu(x,s-r))^2 + V_l(x - s\vec{v}_l + \vec{v}_l r))^\kappa - (m-1)(\frac{1}{2}(p+\nu(x,s-r))^2)^\kappa.$$

It can be shown that $\widetilde{U}(r,s)$ is a bounded operator from H^κ to itself by the same argument with $U(t,s)$ replaced by $\widetilde{U}(r,s)$. The case of $|t| < T, s < -T$ is similar.

2.6 Asymptotic completeness in Sobolev spaces

Recall that we are considering (2.1.7). V_1 is stationary (we denote its velocity as $\vec{e}_0 = 0$) and V_2 is moving with velocity $\vec{e}_1$. There are two approaches to prove Theorem 2.1.6. Graf ([17]) proved the asymptotic completeness for the charge transfer model in the L^2 sense by proving a RAGE theorem. Our first option to prove Theorem 2.1.6 is to generalize Graf's idea. We find that this approach works, provided that each individual subsystem (i.e., $p^2 + V_l$) is asymptotically complete in the H^κ sense. However, the only direct way to prove this fact, as we know, is by the dispersive estimate. The advantage of this approach is that it requires less restrictive conditions on the potentials and the spectrum of the individual subsystem, given that nontrivial fact. Our second option to prove Theorem 2.1.6 is to apply the dispersive estimate (Theorem 2.1.4) directly. To illustrate both of these ideas, the following proof is somehow a combination of these two options. Specifically, we follow [17] to prove the existence of the wave operators and then apply Theorem 2.1.4 to prove Theorem 2.1.6.

2.6.1 Existence of wave operators

The well-known wave operators are defined as follows:

$$\Omega_0^-(s) = s - \lim_{t\to+\infty} U(s,t)e^{-i(t-s)H_0},$$

$$\Omega_1^-(s) = s - \lim_{t\to+\infty} U(s,t)e^{-i(t-s)H_1}P_b(H_1),$$

$$\Omega_2^-(s) = s - \lim_{t\to+\infty} U(s,t)\,\mathfrak{g}_{-\vec{e}_1}(t)e^{-i(t-s)H_2}P_b(H_2)\,\mathfrak{g}_{\vec{e}_1}(s).$$

Theorem 2.6.1. *Under the assumption of Theorem 2.1.5, the above wave operators exist in the space H^κ. More precisely, for $l = 0, 1, 2$ and $\forall\psi_0 \in H^\kappa$, the limits converge in the H^κ sense and $\Omega_l^-(s)\psi_0$ lies in $H^\kappa(\mathbb{R}^n)$.*

Remark 2.6.1. *The above theorem can be proven by Cook's method together with Theorem 2.1.4 and Theorem 2.1.5 if we are willing to impose more regularity on the potentials and the spectrum condition. The following proof originated in [17], which, we believe, requires the least conditions on the system.*

We present some preliminary facts before proceeding:

Lemma 2.6.2. *Let $g \in C_0^\infty(\mathbb{R}^n)$ and $\nu > 0$. Suppose*

1. *$g(p) = 0$ for $|p| \geq \nu$ and fix $\alpha > 1$. Then for $R > 0, t > 0$ and any $N > 0$,*

$$\|F(|x| > \alpha(R + \nu t))e^{-i\frac{p^2}{2}t}g(p)F(|x| < R)\psi\|_{H^\kappa} \leq C_{N,\kappa}(R + \nu t)^{-N}\|\psi\|_{L^2}.$$

2. *$g(p) = 0$ for $|p| \leq \nu$ and $\nu_0 > 0$, $0 < \alpha < 1$. Then for $t > 0$ and any $N > 0$,*

$$\|F(|x| < \alpha(\nu - \nu_0)t)e^{-i\frac{p^2}{2}t}g(p)F(|x| < \nu_0 t)\psi\|_{H^\kappa} \leq C_{N,\kappa}t^{-N}\|\psi\|_{L^2}.$$

These estimates are fairly common for $\kappa = 0$ and may be proven by the stationary phase methods (e.g., [10], Lemma (6.3)). For the case $\kappa \geq 1$, the above lemma follows from a commutator argument and the fact that the derivative on the left-hand side can be absorbed into $g(p)$ because $g \in C_0^\infty(\mathbb{R}^n)$. The next lemma represents to some extent the counterpart of Lemma 2.6.2 for $H_l = H_0 + V_l$.

Lemma 2.6.3. *Let $g \in C_0^\infty(\mathbb{R})$ and $v > 0$. Suppose $g(e) = 0$ for $e \geq v^2/2$ and fix $\alpha > 1$. Then for $l = 1, 2$, $R > 0$ and $t \geq 0$, we have*

$$\|F(|x| > \alpha(R + vt))e^{-iH_l t}g(H_l)F(|x| < R)\psi(x)\|_{H^\kappa} \leq C_{N,\kappa}(R + vt)^{-\epsilon}\|\psi\|_{L^2}. \quad (2.6.1)$$

When $\kappa = 0$, the lemma is just Lemma 4.2 of [17]. For $\kappa \geq 1$, the left-hand side of (2.6.1) is dominated up to a constant by

$$\|(H_l + M)^{\frac{\kappa}{2}}e^{-iH_l t}g(H_l)F(|x| < R)\psi(x)\|_{L^2(|x| > \alpha(R+vt))},$$

where M is chosen so large that $H_l + M$ is a positive operator. If we define $\widetilde{g}(H_l) = (H_l + M)^{\frac{\kappa}{2}}g(H_l)$, then $\widetilde{g} \in C_0^\infty(\mathbb{R})$. The above is of the form $\kappa = 0$ and the lemma follows from the case $\kappa = 0$.

Lemma 2.6.4. *1. Let $0 < v_0 < v$ and $g \in C_0^\infty(\mathbb{R}^n)$ with $g(p) = 0$ for $\{|p| < v\} \bigcup \{|p - \vec{e}_1| < v\}$. Then for any $s \in \mathbb{R}$,*

$$\lim_{t_1 \to +\infty} \sup_{t_2 > t_1} \|(U(t_2, t_1) - e^{-iH_0(t_2 - t_1)}) e^{-iH_0(t_1 - s)} g(p) \prod_{l=0}^{1} F(|x - \vec{e}_l s| < v_0(t_1 - s))\|_{L^2 \to H^\kappa} = 0.$$

2. Let $v_0, v > 0$ with $v_0 + v < |\vec{e}_1|$ and $g \in C_0^\infty(\mathbb{R})$ with $g(p) = 0$ for $p > v^2/2$. Then

$$\lim_{t_1 \to +\infty} \sup_{t_2 > t_1} \|(U(t_2, t_1) - e^{-iH_1(t_2 - t_1)}) g(H_1) F(|x| < v_0 t_1)\|_{L^2 \to H^\kappa} = 0.$$

For $\kappa = 0$, the lemma was proven in [17]. We will follow the approach there to prove the case $\kappa > 0$.

Proof. Part (1): Take $\alpha < \alpha_1 < 1$ and let $f \in C_0^\infty(R^n)$ with $f(y) = 0$ if $|y - \vec{e}_l| > \alpha(v - v_0)$ for both $l = 0, 1$. Since $\alpha t < \alpha_1(t - s)$, we have $|f(x/t)| \le |f(x/t)| \sum_{l=0}^{1} F(|x - \vec{e}_l t| < \alpha_1(v - v_0)(t - s))$ for large enough t.

$$\begin{aligned}
&\|f(\frac{x}{t}) e^{-iH_0(t-s)} g(p) \prod_{l=0}^{1} F(|x - \vec{e}_l s| < v_0(t - s))\|_{L^2 \to H^\kappa} \\
&\lesssim \sum_{|\beta| < \kappa} \sum_{l=0}^{1} \|\partial^\beta f\|_\infty \|F(|x - \vec{e}_l t| < \alpha_1(v - v_0)(t - s)) \\
&\qquad e^{-iH_0(t-s)} g^\beta(p) F(|x - \vec{e}_l s| < v_0(t - s))\|_{L^2 \to L^2} \qquad (2.6.2) \\
&\le C \sum_{|\beta| < \kappa} \sum_{l=0}^{1} \|F(|x| < \alpha_1(v - v_0)(t - s)) e^{-iH_0(t-s)} g^\beta(p + \vec{e}_l) F(|x| < v_0(t - s))\| \\
&\le C(t - s)^{-N},
\end{aligned}$$

where $g^\beta(p) = \sum_{|\beta + \gamma| = \kappa} p^\gamma g(p)$. The above inequality follows by commuting the derivative through $f(x/t)$, by applying a Galilean transform to the second expression, and by Lemma 2.6.2. By (2.6.2) and Theorem 2.1.5, it suffices to show that

$$\sup_{t_2>t_1} \big\|(U(t_2,t_1)(1-f(x/t_1)) - (1-f(x/t_2))e^{-iH_0(t_2-t_1)})e^{-iH_0(t_1-s)}g(p)\cdot$$
$$\cdot\prod_{l=0}^{1} F(|x-\vec{e}_l s| < v_0(t_1-s))\big\|_{L^2\to H^\kappa} \to 0. \qquad (2.6.3)$$

Substituting

$$(U(t_2,t_1)(1-f(x/t_1))-(1-f(x/t_2))e^{-iH_0(t_2-t_1)}) = \int_{t_1}^{t_2} \frac{d}{dt}(U(t_2,t)(1-f(x/t))e^{-iH_0(t-t_1)})dt$$

into (2.6.3), it follows from Theorem 2.1.5 that the left-hand side of (2.6.3) is dominated by

$$\int_{t_1}^{+\infty} \big\|[iH(t)(1-f(x/t)) - i(1-f(x/t))H_0 - \frac{\partial}{\partial t}f(x/t)]e^{-iH_0(t-s)}g(p)\cdot$$
$$\cdot\prod_{l=0}^{1} F(|x-\vec{e}_l s| < v_0(t_1-s))\big\|_{L^2\to H^\kappa}dt.$$

The expression within the square brackets consists of (1) – (3), which are estimated as follows:

1. Suppose t is sufficiently large, then $V_l(t)(1-f(x/t)) = 0$, because V_l is compactly supported, where we take $f(y) = 1$ for $|y-\vec{e}_l| < \alpha(v-v_0)/2$;
2. $H_0 f(x/t) - f(x/t)H_0 = -\frac{1}{2}t^{-2}(\triangle f)(x/t) - it^{-1}(\nabla f)(x/t)p$; and
3. $\frac{\partial}{\partial t}f(x/t) = -t^{-1}(x/t)(\nabla f)(x/t)$

are treated using (2.6.2).

Part (2): Choose $\alpha > 1$ and v_1 with $\alpha(v+v_0) < v_1 < |e_1|$ and let $f \in C_0^\infty(\mathbb{R}^n)$ with $f(y) = 1$ for $|y| < \alpha(v+v_0)$ and $f(y) = 0$ for $|y| > v_1$. We first claim that

$$\lim_{t_1\to+\infty}\sup_{t>t_1}\|(1-f(x/t))e^{-iH_1(t-t_1)})g(H_1)F(|x| < v_0 t_1)\|_{L^2\to H^\kappa} = 0. \qquad (2.6.4)$$

Since $1 - f(x/t)$ is supported in $|x| > \alpha(v+v_0)t > \alpha(v_0t_1 + v(t-t_1))$, it follows from Lemma 2.6.3 that

$$\|F(|x| > \alpha(v_0t_1+v(t-t_1))e^{-iH_l(t-t_1)}g(H_l)F(|x| < v_0t_1)\|_{L^2\to H^\kappa} \leq C_{N,\kappa}(v_0t_1+v(t-t_1))^{-\epsilon}. \tag{2.6.5}$$

Now by Theorem 2.1.5 and

$$(U(t_2,t_1)f(x/t_1) - f(x/t_2)e^{-iH_1(t_2-t_1)}) = \int_{t_1}^{t^2} \frac{d}{dt}(U(t_2,t)f(x/t)e^{-iH_1(t-t_1)})dt,$$

it suffices to estimate that

$$\sup_{t_2>t_1} \|(U(t_2,t_1)f(x/t_1) - f(x/t_2)e^{-iH_1(t_2-t_1)})g(H_1)F(|x| < v_0t_1)\|_{L^2\to H^\kappa}$$
$$\leq \int_{t_1}^{+\infty} dt\, \|[iH(t)f(x/t) - if(x/t)H_1 + \frac{\partial}{\partial t}f(x/t)]e^{-iH_1(t-t_1)}g(H_1)F(|x| < v_0t_1)\|_{L^2\to H^\kappa}.$$

As in Part (1), a discussion of terms (a)-(d) in the square brackets now follows:

(a) $V_1(x)f(x/t) - f(x/t)V_1(x) = 0$.

(b) $V_2(x-e_1t)f(x/t) = 0$ if t is large enough because V_2 is compactly supported and $f(y) = 0$ for $|y| > v_1$ and $|e_1| > v_1$.

(c) $[H_0, f(x/t)] = \frac{1}{2}t^{-2}\triangle f(x/t) - \frac{ip}{t}\nabla f(x/t)$. Since $V_1 \in C_0^\kappa$, we can take M large enough so that the corresponding term can be dominated by

$$\|(M+H_1)^{\frac{\kappa}{2}}(\frac{1}{2}t^{-2}\triangle f(x/t) - \frac{ip}{t}\nabla f(x/t))e^{-iH_1(t-t_1)}g(H_1)F(|x| < v_0t_1)\|_{L^2\to L^2}.$$

Commute $(M+H_1)^{\frac{\kappa}{2}}$ through $(\frac{1}{2}t^{-2}\triangle f(x/t) + \frac{1}{t}\nabla f(x/t)\nabla)$ and the commutators generated will decay at least as fast as t^{-2}, hence they are time-integrable. Note that $\|(p^2+1)^\sigma g(H_1)\|_{L^2\to L^2} < C_\sigma$. The only term that does not decay as fast as t^{-2} is

$$\|(M+H_1)^{-1}(\frac{ip}{t}\nabla f(x/t))e^{-iH_1(t-t_1)}(M+H_1)^{\frac{\kappa}{2}+1}g(H_1)F(|x|<v_0t_1)\|_{L^2\to L^2},$$

which is integrable, due to the fact that $(M+H_1)^{-1}p$ is a bounded operator from L^2 to L^2 and due to (2.6.5) (with $g(H_1)$ replaced by $(M+H_1)^{\frac{\kappa}{2}+1}g(H_1)$), and due to the support property of $\nabla f(x/t)$.

(d) $\frac{\partial}{\partial t}f(x/t)=-\frac{x}{t}f(x/t)t^{-1}$, which can be treated as part (c), using (2.6.5) with $f(x)$ replaced by $xf(x)$. □

Proof of Theorem 2.6.1. Since $U(s,t)e^{-i(t-s)H_0}$ and $U(s,t)e^{-i(t-s)H_1}$ are uniformly bounded operators from H^κ to H^κ, it suffices to prove the existence of the strong limits $\Omega_0^-(s)$ and $\Omega_1^-(s)$ on a dense set D:

$$D=\{g(p)f(x)\psi : g\in C_0^\infty(\mathbb{R}^n\backslash\{0,e_1\}), f\in C_0^\infty(\mathbb{R}^n), \psi\in L^2(\mathbb{R}^n)\}.$$

$g(p)$ satisfies the hypothesis of Lemma 2.6.4 Part (1), with a suitable $v>0$. Take $0<v_0<v$ and note that

$$\prod_{l=1}^{2}F(|x-\vec{e}_l|<v_0(t_1-s))f(x)=f(x)$$

for t_1 big enough. For $t_2>t_1$, it follows from Theorem 2.1.5 that

$$\begin{aligned}&\|(U(s,t_1)e^{-iH_0(t_1-s)}-U(s,t_2)e^{-iH_0(t_2-s)})g(p)f(x)\psi\|_{H^\kappa}\\ \lesssim&\|(U(t_2,t_1)-e^{-iH_0(t_2-t_1)})e^{-iH_0(t_1-s)}g(p)\prod_{l=0}^{1}F(|x-\vec{e}_l s|<v_0(t_1-s))\|_{L^2\to H^\kappa}\|f(x)\psi\|_{L^2}.\end{aligned}$$

Hence Lemma 2.6.4 implies that $U(s,t)e^{-iH_0(t-s)}g(p)f(x)\psi$ is Cauchy sequence in $H^\kappa(\mathbb{R}^n)$ as $t\to+\infty$, which is equivalent to the existence of $\Omega_0^-(s)$.

We will only show the existence of $\Omega_1^-(s)$. The existence of $\Omega_2^-(s)$ follows from the same argument up to a Galilean transform ([17]). Since the eigenfunctions of

H_1 span the range of $P_b(H_1)$, it suffices to prove convergence on the eigenfunctions $\psi:\ H_1\psi = E\psi$. Due to our assumptions on the potentials, the positive eigenvalues are excluded. Thus for any $v > 0$, we can find a suitable g as in Lemma 2.6.4 part (2) with $g(H_1)P(H_1) = P(H_1)$. More precisely, we take $v, v_0 > 0$ with $v + v_0 < |e_1|$. For $t_2 \geq t_1$,

$$\begin{aligned}
&\|(U(s,t_1)e^{-iH_1(t_1-s)}P(H_1) - U(s,t_2)e^{-iH_1(t_2-s)})\psi\|_{H^\kappa} \\
=&\|U(s,t_2)(U(t_2,t_1) - e^{-iH_1(t_2-t_1)})e^{-iH_1(t_1-s)}g(H_1)(F(|x| < v_0t_1) + F(|x| > v_0t_1))\psi\|_{H^\kappa} \\
\lesssim&\|(U(t_2,t_1) - e^{-iH_1(t_2-t_1)})e^{-iH_1(t_1-s)}g(H_1)F(|x| < v_0t_1)\|_{L^2\to H^\kappa}\|\psi\|_{L^2} + \|F(|x| > v_0t_1)\psi\|_{H^\kappa},
\end{aligned}$$

since $U(s,t), H_1(s)$ and $g(H_1)$ are bounded operators on $H^\kappa(\mathbb{R}^n)$ with a uniform bound.

Lemma 2.6.4 part (2) and the fact that $\|F(|x| > v_0t_1)\psi\|_{H^\kappa} \to 0$ when $t_1 \to +\infty$ imply that $U(s,t)e^{-iH_1(t-s)}P(H_1)\psi$ is a Cauchy sequence in H^κ. □

2.6.2 Asymptotic completeness

In this section we will apply theorems 2.1.4 and 2.1.5 to prove Theorem 2.1.6. For the case $\kappa = 0$, we refer the reader to [31].

Proof of Theorem 2.1.6. First let us assume that $\psi_0 \in W^{\kappa,2} \cap W^{\kappa,p'}$ for some $1 < p' < \frac{2n}{2+n}$. Decompose

$$\psi(t) := U(t)\psi_0 = P_b(H_1)U(t)\psi_0 + P_b(H_2,t)U(t)\psi_0 + R(t).$$

By construction, we clearly have

$$\begin{aligned}
P_b(H_2,t)U(t)\psi_0 + R(t) &\in \mathrm{Ran}(P_c(H_1)), \\
P_b(H_1)U(t)\psi_0 + R(t) &\in \mathrm{Ran}(P_c(H_2,t)).
\end{aligned} \tag{2.6.6}$$

We further write

$$P_b(H_1)U(t)\psi_0 = \sum_{r=1}^{m} e^{-i\lambda_r t}a_r(t)u_r(x)$$

for some choice of unknown functions $a_r(t)$. Due to the smoothness of the potentials, u_r belongs to $H^\kappa(\mathbb{R}^n)$. It follows from (3.0.3) that, similar to (2.3.1),

$$\dot{a}_r + i\,\langle V_2(\cdot - t\vec{e}_1)\psi(t), u_r\rangle = 0 \;\text{ for all }\; 1 \le r \le m.$$

The exponential localization of u_r implies that $|\langle V_2(\cdot - t\vec{e}_1)\psi(t), u_r\rangle| \lesssim e^{-\alpha t}$. Therefore, $a_r(t)$ has a limit, writing $\lim_{t\to+\infty} a_r(t) = A_r$, and

$$\Big\| P_b(H_1)U(t)\psi_0 - \sum_{r=1}^{m} A_r e^{-i\lambda_r t} u_r \Big\|_{H^\kappa} \to 0, \quad t \to +\infty. \tag{2.6.7}$$

We next define the functions $v_r = \lim_{t\to+\infty} U(t)^{-1} e^{-i\lambda_r t} u_r$. The existence of v_r and $v_r \in H^\kappa$ is guaranteed by Theorem 2.6.1. By Theorem 2.1.5, we have

$$\Big\| U(t)\big(\sum_{r=1}^{m} A_r v_r\big) - \sum_{r=1}^{m} A_r e^{-i\lambda_r t} u_r \Big\|_{H^\kappa} \to 0, \quad t \to +\infty. \tag{2.6.8}$$

We then infer from (2.6.7) that

$$\Big\| U(t)\big(\sum_{r=1}^{m} A_r v_r\big) - P_b(H_1)U(t)\psi_0 \Big\|_{H^\kappa} \to 0, \quad t \to +\infty. \tag{2.6.9}$$

The above arguments apply to $P_b(H_2,t)U(t)\psi_0$ in a similar fashion. More precisely, we write

$$U(t)\psi_0 = P_b(H_2,t)U(t)\psi_0 + \Gamma(t) = \mathfrak{g}_{-\vec{e}_1}(t) P_b(H_2)\, \mathfrak{g}_{\vec{e}_1}(t) U(t)\psi_0 + \Gamma(t).$$

Therefore,

$$\mathfrak{g}_{\vec{e}_1}(t)U(t)\psi_0 = P_b(H_2)\, \mathfrak{g}_{\vec{e}_1}(t)U(t)\psi_0 + \mathfrak{g}_{\vec{e}_1}(t)\Gamma(t). \tag{2.6.10}$$

Recall that the function $\tilde{\psi}(t) = \mathfrak{g}_{\vec{e}_1}(t)U(t)\psi_0$ is a solution of the problem

$$\frac{1}{i}\partial_t\tilde{\psi} - \frac{\triangle}{2}\tilde{\psi} + V_2(x)\tilde{\psi} + V_1(x + t\vec{e}_1)\tilde{\psi} = 0, \qquad \tilde{\psi}|_{t=0} = \mathfrak{g}_{\vec{e}_1}(0)\psi_0. \tag{2.6.11}$$

According to (2.6.10), $\tilde{\psi}(t) = P_b(H_2)\tilde{\psi}(t) + \Gamma_1(t)$, where $\Gamma_1(t) = \mathfrak{g}_{\vec{e}_1}(0)\Gamma(t)$. In particular,

$$\Gamma_1(t) \in Ran(P_c(H_2)).$$

Decompose

$$P_b(H_2)\tilde{\psi}(t) = \sum_{s=1}^{\ell} b_s(t) e^{-i\mu_s t} w_s$$

for some choice of unknown functions $b_s(t)$. Again, due to the smoothness of the potentials, $w_s \in H^{\kappa}(\mathbb{R}^n)$. After substituting the decomposition in (2.6.11) we obtain the equation

$$\dot{b}_s(t) + i\,\langle V_1(\cdot + t\vec{e}_1)\tilde{\psi}, w_s\rangle = 0 \;\text{ for all }\; 1 \le s \le \ell.$$

Using exponential localization of w_s we conclude the existence of the limit $b_s(t) \to B_s$ as $t \to +\infty$. Thus $\|P_b(H_2)\tilde{\psi}(t) - \sum_{s=1}^{\ell} B_s e^{-i\mu_s t} w_s\|_{H^\kappa} \to 0, \quad t \to \infty$. Equivalently, after applying $\mathfrak{g}_{-\vec{e}_1}(t)$, we have

$$\Big\| P_b(H_2, t)U(t)\psi_0 - \sum_{s=1}^{\ell} B_s e^{-i\mu_j t}\, \mathfrak{g}_{-\vec{e}_1}(t) w_s \Big\|_{H^\kappa} \to 0. \tag{2.6.12}$$

Now Theorem 2.6.1 allows us to define

$$\omega_s := \Omega_2^- w_s = s - \lim_{t\to+\infty} U(t)^{-1}\, \mathfrak{g}_{-\vec{e}_1}(t) e^{-itH_2} P_b(H_2) w_s \in H^\kappa.$$

Moreover,

$$\Big\| U(t)\big(\sum_{s=1}^{\ell} B_s\omega_s\big) - \sum_{s=1}^{\ell} B_s e^{-i\mu_s t}\, \mathfrak{g}_{-\vec{e}_1}(t) w_s \Big\|_{H^\kappa} \to 0, \quad t \to +\infty. \tag{2.6.13}$$

It then follows from (2.6.12) that

$$\|P_b(H_2, t)U(t)\psi_0 - U(t)\big(\sum_{s=1}^{\ell} B_s\omega_s\big)\|_{H^\kappa} \to 0, \quad t \to +\infty. \tag{2.6.14}$$

We now define the function

$$\phi := \psi_0 - \sum_{r=1}^{m} A_r v_r - \sum_{s=1}^{\ell} B_s \omega_s, \tag{2.6.15}$$

which will lead to the initial data ϕ_0 for the free channel. We have that

$$P_b(H_1)U(t)\phi = P_b(H_1)U(t)\psi_0 - P_b(H_1)U(t)\big(\sum_{r=1}^{m} A_r v_r\big) - P_b(H_1)U(t)\big(\sum_{s=1}^{\ell} B_s \omega_s\big).$$

It follows from (2.6.9) and the identity $P_b^2(H_1) = P_b(H_1)$ that

$$\Big\| P_b(H_1)U(t)\psi_0 - P_b(H_1)U(t)\big(\sum_{r=1}^{m} A_r v_r\big)\Big\|_{H^\kappa} \to 0 \qquad \text{as } t \to +\infty. \tag{2.6.16}$$

Furthermore,

$$P_b(H_1)\sum_{s=1}^{\ell} B_s e^{-i\mu_s t}\, \mathfrak{g}_{-\vec{e}_1}(t) w_j = \sum_{r=1}^{m}\sum_{s=1}^{\ell} B_s e^{-i\mu_s t} \langle\, \mathfrak{g}_{-\vec{e}_1}(t) w_j, u_r\rangle\, u_r \to 0 \tag{2.6.17}$$

in the H^κ sense as $t \to +\infty$, due to the exponential localization of the eigenfunctions u_r. We infer from (2.6.16), (2.6.13), and (2.6.17) that $\|P_b(H_1)U(t)\phi\|_{H^\kappa} \to 0$. Similarly, $\|P_b(H_2,t)U(t)\phi\|_{H^\kappa} \to 0$. Thus, $U(t)\phi$ is asymptotically orthogonal to the bound states of H_1 and H_2. $V_j \in C_0^{n+2\kappa+2}$ implies that $(1+|\xi|)^{\kappa+1+\frac{n}{2}}\widehat{V}_j(\xi) \in L^2(\mathbb{R}^n)$. So $(1+|\xi|)^\kappa \widehat{V}_j(\xi) \in L^1(\mathbb{R}^n)$. Therefore, according to Theorem 2.1.4, $U(t)\phi$ satisfies the estimate

$$\|U(t)\phi\|_{W^{\kappa,p}} \lesssim |t|^{-n(\frac{1}{2}-\frac{1}{p})}\|\phi\|_{W^{\kappa,p'}}, \tag{2.6.18}$$

where $\frac{2n}{n-2} < p < +\infty$. In order to be able to apply the estimate (2.6.18), one needs to verify that $\phi \in W^{\kappa,p'}$. By assumption, $\psi_0 \in W^{\kappa,p'}$. Thus it remains to check $v_r \in W^{\kappa,p'}$, $r = 1, \cdots, m$ and $\omega_s \in W^{\kappa,p'}$, $s = 1, \cdots, \ell$, which is guaranteed by Lemma 2.6.5 below. Assuming this lemma for the moment, we now consider the expression

$$e^{-it\frac{\triangle}{2}}U(t)\phi = \phi - i\int_0^t e^{-is\frac{\triangle}{2}}\left(V_1(x)+V_2(x-s\vec{e}_1)\right)U(s)\phi\, ds.$$

Writing $\frac{2p}{p-2}=r$, we have the following estimate:

$$\begin{aligned}
&\int_t^{+\infty} \|e^{-is\frac{\triangle}{2}}\left(V_1(x)+V_2(x-s\vec{e}_1)\right)U(s)\phi\|_{H^\kappa} ds \\
&\lesssim (\|V_1\|_{W^{\kappa,r}} + \|V_2\|_{W^{\kappa,r}})\int_t^{+\infty} \|U(s)\phi\|_{W^{\kappa,p}} ds \\
&\lesssim \int_t^{+\infty} |s|^{-n(\frac{1}{2}-\frac{1}{p})}\|\phi\|_{W^{\kappa,p'}}(\|V_1\|_{W^{\kappa,r}}+\|V_2\|_{W^{\kappa,r}}) \to 0, \quad \text{as} \quad t\to+\infty.
\end{aligned}$$

Here we note that $-n(\frac{1}{2}-\frac{1}{p})<-1$. This allows us to show the existence of the limit

$$\phi_0 := \lim_{t\to\infty} e^{it\frac{\triangle}{2}}U(t)\phi \in H^\kappa.$$

It follows that

$$\|U(t)\phi - e^{-it\frac{\triangle}{2}}\phi_0\|_{H^\kappa}\to 0, \quad t\to+\infty. \tag{2.6.19}$$

Combining (2.6.8), (2.6.13), (2.6.15), and (2.6.19) we infer that

$$\left\|U(t)\psi_0 - \sum_{r=1}^m A_r e^{-i\lambda_r t}u_r - \sum_{s=1}^{\ell} B_s e^{-i\mu_s t}\,\mathfrak{g}_{-\vec{e}_1}(t)w_s - e^{-it\frac{\triangle}{2}}\phi_0\right\|_{H^\kappa} \to 0, \quad \text{as} \quad t\to+\infty,$$

as claimed. Because $W^{\kappa,2}\cap W^{\kappa,p'}$ is dense in $W^{\kappa,2}$, for any $\psi_0\in W^{\kappa,2}$, there is a sequence $\psi_l\in W^{\kappa,2}\cap W^{\kappa,p'}$ converging to ψ_0 in the $W^{\kappa,2}$ norm. Then for each ψ_l, we have the following decomposition:

$$U(t)\psi_l = \sum_{r=1}^m A_r^l e^{-i\lambda_r t}u_r + \sum_{k=1}^{\ell} B_k^l e^{-i\mu_k t}\,\mathfrak{g}_{-\vec{e}_1}(t)w_k + e^{-it\frac{\triangle}{2}}\phi_l + \mathcal{R}_l(t).$$

It follows from Theorem 2.1.5 that $\psi_l = \sum_{r=1}^m A_r^l\Omega_1^- u_r + \sum_{k=1}^{\ell} B_k^l\Omega_2^- w_k + \Omega_0^-\phi_l$.

Since the ranges of $\Omega_{0,1,2}^-$ are orthogonal to each other in $L^2(\mathbb{R}^n)$ ([17]), the fact that ψ_l converges as $l\to+\infty$, implies that each component in the above equation converges. Hence, $\lim_{l\to+\infty} A_r^l = A_r^0$, $\lim_{l\to+\infty} B_k^l = B_k^0$. These imply that $\Omega_0^-\phi_l$ converges in H^κ, since all other terms in the above identity converge in H^κ. Write $\lim_{l\to+\infty}\Omega_0^-\phi_l = f_0 \in H^\kappa$.

By the asymptotic completeness theorem for L^2 ([17]), there are $\phi_0 \in L^2$ such that the following holds:

$$\psi_0 = \sum_{r=1}^{m} A_r^0\Omega_1^- u_r + \sum_{k=1}^{\ell} B_k^0\Omega_2^- w_k + \Omega_0^-\phi_0$$

in the L^2 sense. This implies that $f_0 = \Omega_0^-\phi_0$.

Then by the definition of the wave operator, $U(0,t)e^{-itH_0}\phi_0 - f_0 \to 0$ as $t \to +\infty$ in $L^2(\mathbb{R}^n)$. Since $U(t,0)$ and e^{itH_0} are uniformly bounded operators on $H^\kappa(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n)$, we see that $\phi_0 = \lim_{t\to+\infty} e^{itH_0}U(t,0)f_0$ in $L^2(\mathbb{R}^n)$. We claim that this implies $\phi_0 \in H^\kappa(\mathbb{R}^n)$. It suffices to prove the following:

Assume g_n is a sequence in $H^\kappa(\mathbb{R}^n)$ and $\|g_n\|_{H^\kappa} < 1$. Moreover, g_n converges to g in the L^2 norm. Then g lies in $H^\kappa(\mathbb{R}^n)$.

To see this, note that on Fourier side, $H^\kappa(\mathbb{R}^n)$ is just a weighted $L^2(\mathbb{R}^n)$ space. More precisely, $\|\hat{g_n} - \hat{g}\|_{L^2(\mathbb{R}^n)} \to 0$ implies that for the ball B_R with radius R, centered at the origin,

$$\|(1+|\xi|^2)^{\frac{\kappa}{2}}(\hat{g_n}(\xi) - \hat{g}(\xi))\|_{L^2(B_R)} \to 0 \quad \text{as} \quad n \to +\infty.$$

This implies that $\|(1+|\xi|^2)^{\frac{\kappa}{2}}\hat{g}(\xi)\|_{L^2(B_R)}$ is uniformly bounded by $\sup\|g_n\|_{H^\kappa} \le 1$. Let $R \to +\infty$, we see that $\|g\|_{H^\kappa} \le 1$.

Now it is clear that the following decomposition holds in the space H^κ for any $\psi_0 \in H^\kappa$:

$$\psi_0 = \sum_{r=1}^{m} A_r^0\Omega_1^- u_r + \sum_{k=1}^{\ell} B_k^0\Omega_2^- w_k + \Omega_0^-\phi_0.$$

To complete the proof of Theorem 2.1.6, it remains to prove the following lemma:

Lemma 2.6.5. *Assume that the potentials $V_1(x), V_2 \in C_0^{n+2\kappa+2}(\mathbb{R}^n)$. Let $U(t)$ be the evolution operator of* (2.1.7) *and $\Omega_{1,2}^-$ the wave operators corresponding to $U(t)$, as* *defined at the beginning of this section. Then for* $\forall f \in L^2(\mathbb{R}^n)$, $\Omega_{1,2}^- f$ *lies in* $W^{\kappa,p'}$, *where* $1 < p' < \frac{2n}{n+2}$.

Proof. The proof is essentially contained in [31] Section 4. For the reader's convenience, we present the details here. Without loss of generality we only consider the wave operator Ω_1^-. For an arbitrary L^2 function f

$$\Omega_1^- f = \sum_{r=1}^{m} f_r \lim_{t\to+\infty} U(t)^{-1} e^{-itH_1} u_r,$$

where $P_b(H_1)f = \sum_{r=1}^{m} f_r u_r$ for some constants f_r. It follows from Duhamel's formula that

$$\begin{aligned} U(t)^{-1} e^{-itH_1} u_r =& u_r + i \int_0^t U(s)^{-1} V_2(\cdot - s\vec{e}_1) e^{-isH_1} u_r \, ds \\ =& u_r + i \int_0^t U(s)^{-1} V_2(\cdot - s\vec{e}_1) e^{-i\lambda_r s} u_r \, ds, \end{aligned} \tag{2.6.20}$$

since u_r is an eigenfunction of H_1 corresponding to an eigenvalue λ_r. The function u_r is exponentially localized in L^2 together with its $n+2$ derivatives [1]

$$\sum_{0\le|\gamma|\le n+2} \int_{\mathbb{R}^n} e^{2\alpha|x|} |\partial_x^\gamma u_r(x)|^2 \, dx \le C$$

for some positive constant α appearing in (2.3.12). This implies that the function

$$G_r(s,x) := e^{-i\lambda_r s} V_2(x - s\vec{e}_1) u_r(x)$$

has the property that for any $k \ge 0$ and multi-index γ, $0 \le |\gamma| \le n+2$

$$\|\langle x\rangle^k \partial_x^\gamma G_r(s,\cdot)\|_{L_x^2} \le c(r,|\gamma|,k) \langle s\rangle^{-3j_0-2-\kappa}.$$

[1]The localization of higher derivatives of u_r follows from the localization of u_r stated in (2.3.12) and the equation $-\frac{\triangle}{2} u_r + V_1(x) u_r = \lambda_r u_r$ with potential $V_1(x)$, which is bounded together with all its derivatives of order $\le (n+2)$.

By Hölder's inequality, we have, writing $q = \frac{2p'}{2-p'}$,

$$\|\partial_x^\gamma U(-s)G(s,x)\|_{L^{p'}} \leq \|\langle x\rangle^{-j_0}\|_{L^q}\|\langle x\rangle^{j_0}\partial_x^\gamma U(-s)G(s,x)\|_{L^2}. \tag{2.6.21}$$

Take $j_0 = [\frac{2-p'}{2p'}n] + 1 > \frac{n}{q}$, then $\|\langle x\rangle^{-j_0}\|_{L^q} < +\infty$.

To prove the desired conclusion, it would then suffice to show that for any $|\gamma| = \kappa$, there exists a positive constant k such that for any function $g(x)$

$$\|\langle x\rangle^j \partial^\gamma U(t)g\|_{L^2} \lesssim \langle t\rangle^{3j_0+\kappa} \sum_{|\beta|\leq j_0+\kappa} \|\langle x\rangle^k \partial_x^\beta g\|_{L^2}, \qquad \forall t \geq 0. \tag{2.6.22}$$

We note that the estimates of the type (2.6.22) for problems with time-independent potentials are well-known. They have been proved in the paper by Hunziker [18]. In the time-dependent case the argument is essentially the same. More precisely, define the functions

$$\Phi_{j,|\gamma|}(t) := \sum_{j'=0}^{j} \sum_{|\gamma'|=0}^{|\gamma|} \|\langle x\rangle^{j'}\partial_x^{\gamma'} U(t)g\|_{L^2}$$

for any index $j \geq 0$ and any multi-index γ. Using equation (2.1.7) we obtain

$$\frac{d}{dt}\|\langle x\rangle^j \partial_x^\gamma U(t)g\|_{L^2}^2 = i(-1)^{|\gamma|}\langle [\frac{\triangle}{2} - V(t,x), \partial_x^\gamma \langle x\rangle^{2j}\partial_x^\gamma] U(t)g, U(t)g\rangle.$$

Computing the commutator we obtain the recurrence relation

$$\begin{aligned}\Phi_{j,|\gamma|}(t) \lesssim & \Phi_{j,|\gamma|}(0) + \langle t\rangle^2 \sum_{|\gamma'|\leq 2|\gamma|} \Big\|\frac{\langle x\rangle}{\langle t\rangle}\partial_x^{\gamma'} V\Big\|_{L^\infty_{t,x}} \sup_{0\leq\tau\leq t} \Phi_{j-1,|\gamma|+1}(\tau) \leq \\ & C(V)\Big(\sum_{k=0}^{j-1}\langle t\rangle^{2k}\Phi_{j-k,|\gamma|+k}(0) + \langle t\rangle^{2j}\Phi_{0,|\gamma|+j}(\tau)\Big),\end{aligned}$$

where $C(V)$ is a constant depending on

$$\sum_{|\gamma'|\leq 2(|\gamma|+j-1)} \Big\|\frac{\langle x\rangle}{\langle t\rangle}\partial_x^{\gamma'} V\Big\|_{L^\infty_{t,x}}. \tag{2.6.23}$$

In addition, differentiating the equation (2.1.7) $|\gamma| + j$ times with respect to x and using the standard L^2 estimate, we have

$$\Phi_{0,|\gamma|+j}(\tau) \leq C(V)(1+|\tau|^{|\gamma|+j})\Phi_{0,|\gamma|+j}(0).$$

Therefore,

$$\Phi_{j,|\gamma|}(t) \leq C(V)(1+|t|)^{3j+|\gamma|}\Phi_{j,|\gamma|+j}(0).$$

Now setting $j = j_0$ and $|\gamma| = \kappa$ we obtain the desired estimate (2.6.22) with $k = j_0$. Observe that the assumption $V_1, V_2 \in C_0^{n+2\kappa+2}(\mathbb{R}^n)$ controls the constant $C(V)$ in (2.6.23) for the potential $V(t,x) = V_1(x) + V_2(x - t\vec{e}_1)$. $\square$

Chapter 3

Schrödinger Operators with Lamé Potentials on $\mathbb{R}$

Even though the dispersive properties of Schrödinger operators have been extensively studied for potentials vanishing at infinity, they are little known in the case that the potential is periodic in space. Assuming $V(x) = V(x + 2\omega)$, $-\frac{d^2}{dx^2} + V(x)$ is called Hill's operator. The spectrum of Hill's operator is purely continuous and a union of infinitely many intervals (bands), generically. A potential $V(x)$ is called a finite band potential if the spectrum of $-\frac{d^2}{dx^2} + V(x)$ is a union of finitely many intervals, one of which is a half axis. One example of finite band potentials is the so-called Lamé potential. The corresponding $-\frac{d^2}{dx^2} + V(x)$ is called a Lamé operator. The Laḿe operator has a rich history (see [39]) and the properties of its eigenfunctions are still of interest for current research. In this chapter, we explore the dispersive property of Schrödinger operator with a Lamé potential.

We assume that $\psi_0 \in L^1(\mathbb{R})$ and denote $U(t)\psi_0$ as the solution of the following problem

$$\begin{aligned} \frac{1}{i}\partial_t \psi(x,t) &= -\frac{d^2}{dx^2}\psi(x,t) + 2\wp(x+\omega_3)\psi(x,t), \qquad (3.0.1) \\ \psi(x,0) &= \psi_0(x). \end{aligned}$$

Theorem 3.0.6. *Generically, for almost all $\omega, \omega' \in \mathbb{R}$, there exists a constant $C > 0$ such that for $t > 1$*

$$\|U(t)\psi_0\|_{L^\infty(\mathbb{R})} < C\,t^{-\frac{1}{3}}\|\psi_0\|_{L^1(\mathbb{R})}. \tag{3.0.2}$$

Moreover, for all nonzero $\omega, \omega' \in \mathbb{R}$*, there exists a constant* $C > 0$ *such that for* $t > 1$

$$\|U(t)\psi_0\|_{L^\infty(\mathbb{R})} < C\,t^{-\frac{1}{4}}\|\psi_0\|_{L^1(\mathbb{R})}. \tag{3.0.3}$$

(3.0.2) *is optimal in the sense that for any nonzero* $\omega, \omega' \in \mathbb{R}$*, there exist constants* $c > 0$ *and* $T > 0$*, depending only on* ω, ω' *such that for* $t > T$

$$\sup_{\psi_0 : \|\psi_0\|_{L^1(\mathbb{R})}=1} \|U(t)\psi_0\|_{L^\infty(\mathbb{R})} > c\,t^{-\frac{1}{3}}. \tag{3.0.4}$$

We require $t > 1$ to be large only to exclude $t \to 0$. The decay rates $t^{-\frac{1}{3}}$ and $t^{-\frac{1}{4}}$ are different from $t^{-\frac{1}{2}}$ in (1.0.4) because phase function is nonquadratic, which is a natural outcome of the periodic potential. The decay factor $t^{-\frac{1}{3}}$ as $t \to \infty$ has appeared in the analysis of the Modified KdV equation ([8]), where the nonlinear phase of the main term is cubic. In our case, the analytic phase function, roughly speaking, satisfies a cubic relation up to a change of variables. This cubic relation comes from the differential equations satisfied by the Weierstrass $\wp$ function. We denote $P(x)$ to be the real-coefficient cubic polynomial

$$2x^3 + \frac{6\zeta(\omega)}{\omega}x^2 + \frac{g_2}{2}x + g_3 - \frac{g_2\zeta(\omega)}{2\omega}. \tag{3.0.5}$$

We shall prove (3.0.2) under the assumption that

$$P(x) \text{ has no double root in } (-\infty, \wp(\omega_3)]. \tag{3.0.6}$$

If (3.0.6) does not hold, then we shall prove (3.0.3). In this case, by Lemma 3.1.2, $P(x)$ has no root of degree 3. Our proof implies that (3.0.3) is optimal in the sense stated in Theorem 3.0.6. However, we are unable to give an explicit example such that $P(x)$ does have a double root in $(-\infty, \wp(\omega_3)]$.

Finally, we prove that assumption (3.0.6) holds for almost all $\omega, \omega' \in \mathbb{R}$.

3.1 Preliminaries

Eigenfunctions of (1.2.2) are expressed in terms of the Weierstrass σ-function and ζ-function ([25], [14]) as follows:

$$f_a(x) = \frac{\sigma(x + i\omega' + a)}{\sigma(x + i\omega')} e^{-\zeta(a)x - \zeta(i\omega')a}, \tag{3.1.1}$$

where the energy

$$E = -\wp(a).$$

(3.1.1) can be verified by noticing that ([39])

$$f_a'(x) = (\zeta(x + \omega_3 + a) - \zeta(x + \omega_3) - \zeta(a)) f_a(x), \tag{3.1.2}$$

and

$$(\zeta(x + y) - \zeta(x) - \zeta(y))^2 = \wp(x + y) + \wp(x) + \wp(y).$$

Some basic properties of Weierstrass functions are listed in the appendix. f_a is periodic when a is one of the half periods $\omega_1, \omega_2 = \omega_1 + \omega_3$ or ω_3. f_{-a} and f_a are the two Floquet-type solutions of (1.2.2). We write

$$f_a(x) = m_a(x) e^{ik(a)x},$$

where

$$m_a(x) = \frac{\sigma(x + i\omega' + a)}{\sigma(x + i\omega')} e^{-a\zeta(i\omega') - a\frac{x}{\omega}\zeta(\omega)} \tag{3.1.3}$$

is periodic with period 2ω. Denote

$$\Sigma = [-\wp(\omega_1), -\wp(\omega_2)] \cup [-\wp(\omega_3), +\infty),$$

and the quasimomentum

$$k(a) = i\omega^{-1}(\omega\zeta(a) - a\zeta(\omega))$$

is real-valued for $E \in \Sigma$. f_a is bounded when $E \in \Sigma$ and is unbounded otherwise, which implies that Σ is the spectrum of (1.2.2) ([27]).

It is known that $U(t)$ is an integral operator with kernel

$$K(t, x, x') = \int_\Sigma e^{itE} P_{a.c.}(E, x, x') dE.$$

Namely,

$$\psi(x, t) = \int_\Sigma \int_\mathbb{R} e^{itE} P_{a.c.}(E, x, x') \psi_0(x') dx' dE.$$

The absolutely continuous spectral projection is

$$P_{a.c.}(E, x, x') = \frac{1}{2\pi i}[(H - (E + i0))^{-1}(x, x') - (H - (E - i0))^{-1}(x, x')],$$

and by definition

$$(H - (E \pm i0))^{-1} = \lim_{\epsilon \to 0^+} (H - (E \pm i\epsilon))^{-1},$$

which can be expressed by f_a and f_{-a}. Hence, we obtain for $x > x'$

$$\begin{aligned} K(t, x, x') &= \int_\Sigma e^{itE}(f_{-a}(x') f_a(x) + f_{-a}(x) f_a(x')) \frac{dE}{W(E)} \\ &= \int_\Sigma e^{itE}(e^{ik(a)(x-x')} m_{-a}(x') m_a(x) + e^{-ik(a)(x-x')} m_{-a}(x) m_a(x')) \frac{dE}{W(E)}, \end{aligned} \tag{3.1.4}$$

where $W(E) = W(a) = W(f_a, f_{-a}) = f_a f'_{-a} - f'_a f_{-a}$, called the Wronskian of f_a, f_{-a}, is independent of x.

Because the spectral projection $P_{a.c.}$ is self-adjoint, $P_{a.c.}(E, x, x') = \overline{P_{a.c.}(E, x', x)}$.

Therefore, when $x < x'$

$$K(t,x,x') = \int_\Sigma e^{itE}(\overline{f_{-a}}(x')\overline{f_a}(x) + \overline{f_{-a}}(x)\overline{f_a}(x'))\frac{dE}{\overline{W}(E)}. \tag{3.1.5}$$

The proof of (3.0.2) and (3.0.3) shall be reduced to proving that

$$\sup_{x,x'} |K(t,x,x')| < Ct^{-\frac{1}{3}} \text{ and } \; Ct^{-\frac{1}{4}}.$$

It follows from (3.1.2) that

$$W(f_a, f_{-a}) = f_a(x)f_{-a}(x)(\zeta(x+\omega_3-a) + 2\zeta(a) - \zeta(x+\omega_3+a)).$$

Since $W(f_a, f_{-a})$ is independent of x, we set $x = 0$ and obtain

$$W(f_a, f_{-a}) = f_a(0)f_{-a}(0)(\zeta(\omega_3-a) + 2\zeta(a) - \zeta(\omega_3+a)).$$

By the addition formula for Weierstrass functions ([2], §15)

$$\zeta(u+v) - \zeta(u-v) - 2\zeta(v) = -\frac{\wp'(v)}{\wp(u)-\wp(v)},$$

we have

$$W(E) = \frac{\sigma(i\omega'+a)\sigma(i\omega'-a)}{\sigma^2(i\omega')}\frac{\wp'(a)}{\wp(i\omega')-\wp(a)}. \tag{3.1.6}$$

Hence,

$$\frac{dE}{W(E)} = -\frac{\sigma^2(i\omega')(\wp(i\omega')-\wp(a))}{\sigma(i\omega'+a)\sigma(i\omega'-a)}da. \tag{3.1.7}$$

Remark 3.1.1. *Because $\wp'(i\omega') = 0$ and zeroes of σ are the lattice points $\{n_1 2\omega_1 + n_2 2\omega_3 : n_1, n_2 \in Z\}$, all of which are of degree 1, it follows that $\frac{\wp(i\omega')-\wp(a)}{\sigma(i\omega'+a)\sigma(i\omega'-a)}$ is bounded and smooth when $a \to i\omega'$.*

Also, it is clear that $\frac{\wp(i\omega')-\wp(a)}{\sigma(i\omega'+a)\sigma(i\omega'-a)} = O(a^{-2})$ when $a \to 0$, and that $\frac{\wp(i\omega')-\wp(a)}{\sigma(i\omega'+a)\sigma(i\omega'-a)}$

and its ∂_a-derivatives are bounded on $[\omega_1, \omega_2]$, where $[\omega_1, \omega_2]$ denotes the set

$$\{\lambda\omega_1 + (1-\lambda)\omega_2 : \lambda \in [0,1]\}.$$

By (3.1.3)*, m_a, m_{-a}, and their ∂_a-derivatives are bounded uniformly for $a \in [\omega_1, \omega_2] \cup [0, \omega_3]$, $x \in \mathbb{R}$.*

Since Σ is a union of two intervals, we shall decompose the integral of $K(t, x, x')$ into two parts. Namely,

$$K(t, x, x') = K_1(t, x, x') + K_2(t, x, x'),$$

where

$$K_1(t, x, x') = \int_{-\wp(\omega_1)}^{-\wp(\omega_2)} e^{itE} P_{a.c.}(E, x, x') dE,$$

$$K_2(t, x, x') = \int_{-\wp(\omega_3)}^{+\infty} e^{itE} P_{a.c.}(E, x, x') dE.$$

Before we proceed to analyze $K_1(t, x, x')$ and $K_2(t, x, x')$, we prove two technical lemmas.

Lemma 3.1.1. *Let $F(x)$ be a real-valued and smooth function on (a, b),*

1. *Suppose $|F'(x)| \geq \epsilon$, $|F''(x)| \leq M$ for all $x \in (a, b)$, then*

$$\Big| \int_a^b e^{-itF(x)} \psi(x) dx \Big| \leq c\,\epsilon^{-1} |t|^{-1} \Big[|\psi(b)| + \int_a^b (|\psi'(x)| + |\psi(x)|) dx \Big],$$

where c depends on M.

2. *Suppose $k \geq 2$, $k \in \mathbb{Z}$, and $|F^{(k)}(x)| \geq \epsilon$ for all $x \in (a, b)$, then*

$$\Big| \int_a^b e^{-itF(x)} \psi(x) dx \Big| \leq c\,\epsilon^{-\frac{1}{k}} |t|^{-\frac{1}{k}} \Big[|\psi(b)| + \int_a^b |\psi'(x)| dx \Big],$$

where c depends on k.

The first part of Lemma 3.1.1 follows from integration by parts. The second part is proved in [36] (p.334).

Lemma 3.1.2. *Let* $e_j = \wp(\omega_j)$, $j = 1, 2, 3$. *Then* $P(x)$ *has a unique simple root in* $[e_2, e_1]$, *and* $P(e_j)$, $j = 1, 2, 3$, *are nonzero. Also* $P(x)$ *has no root of degree* 3 *in* $\mathbb{R}$. *Moreover,* $-\frac{\zeta(\omega)}{\omega} \in (e_3, e_2)$.

Proof. $P(x) = 0$ if and only if $4x^3 - g_2 x - g_3 = (6x^2 - \frac{g_2}{2})(x + \frac{\zeta(\omega)}{\omega})$. Denote $p_1(x) = 4x^3 - g_2 x - g_3$ and $p_2(x) = (6x^2 - \frac{g_2}{2})(x + \frac{\zeta(\omega)}{\omega})$. We shall examine the roots of $p_1(x)$ and $p_2(x)$ on the real line.

It follows from (3.5.2) that $p_1(x) = 4(x - e_1)(x - e_2)(x - e_3)$, where $e_j = \wp(\omega_j)$, $j = 1, 2, 3$. Because there is no quadratic term in $p_1(x)$, $e_1 + e_2 + e_3 = 0$. Since $e_3 < e_2 < e_1$, we have $e_3 < 0 < e_1$.

Observe that $\wp''(\omega_1) > 0$, $\wp''(\omega_2) < 0$ and $\wp''(\omega_3) > 0$, and by Eq (1.2.4), we obtain

$$\wp(\omega_2)^2 < \frac{g_2}{12} < \min\{\wp(\omega_1)^2, \wp(\omega_3)^2\}.$$

Now we shall prove $\frac{\zeta(\omega)}{\omega} \in (-e_2, -e_3)$. Indeed, let $y_1(x, E)$ and $y_2(x, E)$ be the solutions of (1.2.2), which satisfy

$$y_1(0, E) = y_2'(0, E) = 1, \qquad y_1'(0, E) = y_2(0, E) = 0.$$

And we introduce the discriminant $\Delta(E) = y_1(2\omega, E) + y_2'(2\omega, E)$.

Recall a and E are related by $E = -\wp(a)$, and as $E \to +\infty$ on the real line, $a \to 0$ on the positive imaginary axis. Therefore $i\zeta(a)$ and $k(a)$ go to $+\infty$ on the real line when $E \to +\infty$.

By Lemma 2.1 of [12], $\Delta(E) = 2\cos k(a)$ and $k(E) = k(a(E))$ is the conformal map from the upper half plane to a slit quarter plane $\Omega = \{\Re z > 0, \Im z > 0\} \backslash T$, with the slit $T = \{\frac{\pi}{2\omega} + iy : 0 < y \leq h\}$, where h is some positive real number. Moreover, $k(-e_1) = 0$ and $k(-e_2) = k(-e_3) = \frac{\pi}{2\omega}$.

Denote Q_0 to be the preimage of the tip $\frac{\pi}{2\omega} + ih$ of the slit T under the map $k(E)$.

Then $-\wp(\omega_2) < Q_0 < -\wp(\omega_3)$, and $k(E)$ sends $[-\wp(\omega_2), Q_0]$ to $[\frac{\pi}{2\omega}, \frac{\pi}{2\omega} + ih]$, and $[Q_0, -\wp(\omega_3)]$ to $[\frac{\pi}{2\omega} + ih, \frac{\pi}{2\omega}]$, respectively. Thus when $E \in (-\wp(\omega_2), Q_0)$, $\frac{1}{i}\partial_E k(E) \geq 0$. We observe that

$$\partial_E k(E) = \frac{1}{-\wp'(a)}\partial_a k(a) = \frac{i}{\wp'(a)}\Big(\frac{\zeta(\omega)}{\omega} + \wp(a)\Big),$$

which implies that

$$\frac{\wp(a) + \zeta(\omega)/\omega}{\wp'(a)} \geq 0.$$

Since $\wp'(a) > 0$ when $a \in (\omega_3, \omega_2)$, we conclude that $E = -\wp(a) \leq \zeta(\omega)/\omega$ for any $E \in (-\wp(\omega_2), Q_0)$. Hence, $Q_0 \leq \zeta(\omega)/\omega$. On the other hand, $\frac{1}{i}\partial_E k(E) \leq 0$ when $E \in (Q_0, -\wp(\omega_3))$. Following a similar argument, $Q_0 \geq \zeta(\omega)/\omega$. Therefore $\zeta(\omega)/\omega = Q_0 \in (-\wp(\omega_2), -\wp(\omega_3))$.

In fact, $k(E)$ maps $E = \frac{\zeta(\omega)}{\omega}$ to the tip $\frac{\pi}{2\omega} + ih$ of the slit T and $\Delta(E)$ reaches its minimum at $E = \frac{\zeta(\omega)}{\omega}$.

The three roots of $p_2(x)$ are $\pm\sqrt{\frac{g_2}{12}}$ and $-\frac{\zeta(\omega)}{\omega}$. From the above analysis, we have that $\sqrt{\frac{g_2}{12}} \in (e_2, e_1)$ and $-\sqrt{\frac{g_2}{12}}, -\frac{\zeta(\omega)}{\omega} \in (e_3, e_2)$, which implies $p_2(e_1) > 0$ and $p_2(e_2) < 0$. Hence $P(x)$ has either one or three zeroes in (e_2, e_1) and clearly $P(e_j)$, $j = 1, 2, 3$, are nonzero.

To verify that $P(x)$ has no root of degree 3, we consider

$$P'(x) = 6x^2 + 12\frac{\zeta(\omega)}{\omega}x + \frac{g_2}{2}.$$

The minimum of $P'(x)$ is reached at $x = -\frac{\zeta(\omega)}{\omega} \in (e_3, e_2)$ and is equal to $\frac{g_2}{2} - 6(\frac{\zeta(\omega)}{\omega})^2$. Notice that $-\frac{\zeta(\omega)}{\omega} < e_2 < \sqrt{\frac{g_2}{12}}$ always holds.

If $-\frac{\zeta(\omega)}{\omega} > -\sqrt{\frac{g_2}{12}}$, then $P'(x) > 0$ holds for all $x \in \mathbb{R}$. $P(x)$ has no root of degree greater or equal to 2.

If $-\frac{\zeta(\omega)}{\omega} = -\sqrt{\frac{g_2}{12}}$, then $P'(x)$ has a double root $-\frac{\zeta(\omega)}{\omega} \in (e_3, e_2)$. Since $P(x)$ has a root in (e_2, e_1), we conclude that $P(x)$ has no root of degree 3.

If $-\frac{\zeta(\omega)}{\omega} < -\sqrt{\frac{g_2}{12}}$, then $P'(x)$ has no double root. Hence $P(x)$ has no root of degree 3 on the whole real line.

If $P(x)$ has three zeroes in (e_2, e_1), then $P'(x)$ has two roots in (e_2, e_1), which is impossible because $-\frac{\zeta(\omega)}{\omega} < e_2$. Therefore, $P(x)$ has unique simple root in (e_2, e_1). □

3.2 Analysis of $K_1(t, x, x')$

We first consider $K_1(t, x, x')$. We proceed by making the following observation:

Lemma 3.2.1. *Let $b = 2\omega_2 - a$ for $a \in [\omega_1, \omega_2]$. Write $W(a) = W(f_a, f_{-a})$. Then for $x, x' \in \mathbb{R}$*

$$\frac{f_a(x')f_{-a}(x)}{W(a)} = -\frac{f_b(x)f_{-b}(x')}{W(b)}. \tag{3.2.1}$$

Proof. It is clear that $\wp(a) = \wp(b)$ and $\wp'(a) = -\wp'(b)$. We prove (3.2.1) by direct calculation. By definition,

$$f_a(x')f_{-a}(x) = \frac{\sigma(x' + \omega_3 + a)\sigma(x + \omega_3 - a)}{\sigma(x + \omega_3)\sigma(x' + \omega_3)} e^{\zeta(a)(x - x')}.$$

By (3.5.4) and (3.5.5), this equals

$$\frac{\sigma(x' + \omega_3 - b)\sigma(x + \omega_3 + b)}{\sigma(x + \omega_3)\sigma(x' + \omega_3)} e^{\zeta(b)(x' - x)} e^{4\eta_3(\omega_3 - b)} = f_b(x)f_{-b}(x')e^{4\eta_3(\omega_3 - b)}.$$

Also by (3.1.6) and (3.5.5),

$$W(a) = \frac{\sigma(i\omega' - b)\sigma(i\omega' + b)\exp\left(4\eta_3(\omega_3 - b)\right)}{\sigma^2(i\omega')} \frac{-\wp'(b)}{\wp(\omega_3) - \wp(b)} = -W(b)e^{4\eta_3(\omega_3 - b)}.$$

Combining them, (3.2.1) follows.

□

It follows from Lemma 3.2.1 that

$$\int_{\omega_1}^{\omega_2} e^{-it\wp(a)} \frac{f_a(x')f_{-a}(x)}{W(a)} d\wp(a) = \int_{\omega_2}^{\omega_2+i\omega'} e^{-it\wp(b)} \frac{f_b(x)f_{-b}(x')}{W(b)} d\wp(b).$$

Hence we have that for $x > x'$

$$\begin{aligned} K_1(t,x,x') &= \int_{-\wp(\omega_1)}^{-\wp(\omega_2)} e^{itE}(f_{-a}(x')f_a(x) + f_{-a}(x)f_a(x'))\frac{dE}{W(E)} \\ &= \int_{\omega_1}^{\omega_1+i2\omega'} e^{-it\wp(a)} \frac{f_a(x)f_{-a}(x')}{W(a)} d(-\wp(a)) \\ &= \int_{\omega_1}^{\omega_1+i2\omega'} e^{-it\wp(a)+i(x-x')k(a)} m_a(x)m_{-a}(x')\frac{-\wp'(a)da}{W(a)}. \end{aligned} \quad (3.2.2)$$

Note $k(a)$ is real-valued and by (3.1.5), we have that for $x < x'$

$$K_1(t,x,x') = \int_{\omega_1}^{\omega_1+i2\omega'} e^{-it\wp(a)+i(x-x')k(a)} \overline{m_a}(x')\overline{m_{-a}}(x)\frac{-\wp'(a)da}{\overline{W}(a)}.$$

To simplify notation, we set $\tau = \frac{x-x'}{t} \in \mathbb{R}$ and

$$F_\tau(a) = \wp(a) - i\tau(\zeta(a) - \frac{a}{\omega}\zeta(\omega)).$$

Moreover, we write

$$K_1(t,x,x') = \int_{\omega_1}^{\omega_1+i2\omega'} e^{-itF_\tau(a)}\varphi(a,x,x')da, \quad (3.2.3)$$

where $\varphi(a,x,x') = m_a(x)m_{-a}(x')\frac{-\wp'(a)}{W(a)}$ when $x > x'$, and $\varphi(a,x,x') = \overline{\varphi(a,x',x)}$ when $x < x'$. Without losing clarity, $\varphi(a,x,x')$ will be written simply as $\varphi(a)$.

By Remark 3.1.1, $\varphi(a,x,x')$ and its ∂_a-derivatives are bounded uniformly for $a \in [\omega_1,\omega_2]$ and $x,x' \in \mathbb{R}$. To apply Lemma 3.1.1 to (3.2.3), we analyze the ∂_a-derivatives of $F_\tau(a)$. Our plan is to decompose the integral in (3.2.3) into several regions and on each region, Lemma 3.1.1 for some exponent k will be applied. We observe that

$$\partial_a F_\tau(a) = \wp'(a) + \tau i(\zeta(\omega)/\omega + \wp(a)), \tag{3.2.4}$$

$$\partial_a^2 F_\tau(a) = \wp''(a) + \tau i\wp'(a), \tag{3.2.5}$$

$$\partial_a^3 F_\tau(a) = \partial_a^3\wp(a) + \tau i\wp''(a). \tag{3.2.6}$$

Let

$$c_1 = \min\{\zeta(\omega)/\omega + \wp(a) : a \in [\omega_1, \omega_1 + 2i\omega']\}.$$

Then $c_1 = \zeta(\omega)/\omega + \wp(\omega_2)$ and by Lemma 3.1.2, $c_1 > 0$. Also we denote

$$M_1 = 1 + \max\{|\wp'(a)|, |\wp''(a)|, \zeta(\omega)/\omega + \wp(a) : a \in [\omega_1, \omega_1 + 2i\omega']\}. \tag{3.2.7}$$

When $|\tau| > \frac{2M_1}{c_1}$, we have for $a \in [\omega_1, \omega_1 + 2i\omega']$

$$|\partial_a F_\tau(a)| > |\tau|c_1 - M_1 > \frac{1}{2}|\tau|c_1,$$

and

$$|\partial_a^2 F_\tau(a)| < M_1(|\tau| + 1).$$

Integrating by parts and recalling $\varphi(a)$ and its derivatives are uniformly bounded, we obtain

$$\begin{aligned}
|K_1(t,x,x')| =& \frac{1}{t}\Big|\int_{\omega_1}^{\omega_1+i2\omega'} \frac{\varphi(a)}{\partial_a F_\tau(a)} d\, e^{-itF_\tau(a)}\Big| \\
\le& \frac{4\omega'\|\varphi(a)\|_{L^\infty[\omega_1,\omega_2]}}{t\tau c_1} + \frac{1}{t}\Big|\int_{\omega_1}^{\omega_1+i2\omega'} e^{-itF_\tau(a)}\Big(\frac{\varphi'(a)}{F_\tau'(a)} - \frac{\varphi(a)F_\tau''(a)}{(F_\tau'(a))^2}\Big)da\Big| \\
\le& Ct^{-1}.
\end{aligned} \tag{3.2.8}$$

We now estimate $K_1(t,x,x')$ when $|\tau| \leq \frac{2M_1}{c_1}$. Suppose both (3.2.4) and (3.2.5) vanish for $a = a_0 \in [\omega_1, \omega_2]$ and $\tau = \tau_0 \in [-\frac{2M_1}{c_1}, \frac{2M_1}{c_1}]$. Then

$$\wp'(a_0)^2 = \wp''(a_0)(\frac{\zeta(\omega)}{\omega} + \wp(a_0)).$$

By (1.2.3) and (1.2.4), this is equivalent to

$$2\wp(a_0)^3 + \frac{6\zeta(\omega)}{\omega}\wp(a_0)^2 + \frac{g_2}{2}\wp(a_0) + g_3 - \frac{g_2\zeta(\omega)}{2\omega} = 0.$$

Thus $\wp(a_0)$ is the simple root of $P(x)$ in $[\wp(\omega_2), \wp(\omega_1)]$ and $\wp'(a_0) \neq 0$ by Lemma 3.1.2.

Observe that (3.2.4) and (3.2.5) also vanish when $(a,\tau) = (2\omega_2 - a_0, -\tau_0)$. The analysis of $(2\omega_2 - a_0, -\tau_0)$ is the same as that of (a_0, τ_0) and we will focus on (a_0, τ_0).

Also, we observe that $\partial_a^3 F_\tau(a)$ vanishes at (a_0, τ_0) if and only if

$$\det \begin{pmatrix} \wp'(a_0) & \frac{\zeta(\omega)}{\omega} + \wp(a_0) \\ \partial_a^3\wp(a_0) & \wp''(a_0) \end{pmatrix} = 0;$$

namely,

$$\partial_a \det \begin{pmatrix} \wp'(a) & \frac{\zeta(\omega)}{\omega} + \wp(a) \\ \wp''(a) & \wp'(a) \end{pmatrix}_{a=a_0} = 0.$$

Since $\wp'(a_0) \neq 0$, that $\partial_a^3 F_{\tau_0}(a_0) = 0$ is equivalent to the fact that $\wp(a_0)$ is a double root of $P(x)$. By Lemma 3.1.2, $P(x)$ has no double root in $[\wp(\omega_2), \wp(\omega_1)]$. Hence, $\partial_a^3 F_{\tau_0}(a_0) \neq 0$ and there exists $\epsilon > 0$ such that

$$\min\{\Sigma_{j=1}^3 |\partial_a^j F_\tau(a)| : a \in [\omega_1, \omega_1 + 2i\omega'], \tau \in [-2M_1/c_1, 2M_1/c_1]\} > \epsilon > 0.$$

Let $\chi_3(a,\tau)$ be a smooth function defined on $[\omega_1, \omega_1 + 2i\omega'] \times [-\frac{2M_1}{c_1}, \frac{2M_1}{c_1}]$ such that $0 \leq \chi_3 \leq 1$, $\chi_3(a,\tau) = 1$ when $|\partial_a F_\tau(a)| + |\partial_a^2 F_\tau(a)| \leq \frac{1}{3}\epsilon$, and $\chi_3(a,\tau) = 0$ when $|\partial_a F_\tau(a)| + |\partial_a^2 F_\tau(a)| \geq \frac{2}{3}\epsilon$. Similarly, let $\chi_2(a,\tau)$ to be a smooth function defined on $[\omega_1, \omega_1 + 2i\omega'] \times [-\frac{2M_1}{c_1}, \frac{2M_1}{c_1}]$, such that $0 \leq \chi_2 \leq 1$, $\chi_2(a,\tau) = 1$ when $|\partial_a^2 F_\tau(a)| \geq \frac{1}{6}\epsilon$, and $\chi_2(a,\tau) = 0$ when $|\partial_a^2 F_\tau(a)| \le \frac{1}{9}\epsilon$.

On the support of χ_3, $|\partial_a^3 F_\tau(a)| \ge \frac{1}{3}\epsilon$. It follows from Lemma 3.1.1 that

$$\Big|\int_{\omega_1}^{\omega_1+i2\omega'} e^{-itF_\tau(a)}\chi_3(a,\tau)\varphi(a)da\Big| \le C_3(\tau)t^{-\frac{1}{3}}. \tag{3.2.9}$$

On the support of $\chi_2(1-\chi_3)$, $|\partial_a^2 F_\tau(a)| \ge \frac{1}{9}\epsilon$. And similarly

$$\Big|\int_{\omega_1}^{\omega_1+i2\omega'} e^{-itF_\tau(a)}\chi_2(a,\tau)(1-\chi_3(a,\tau))\varphi(a)da\Big| \le C_2(\tau)t^{-\frac{1}{2}}. \tag{3.2.10}$$

On the support of $(1-\chi_2)(1-\chi_3)$, $|\partial_a^2 F_\tau(a)| \le \frac{1}{6}\epsilon$ and $|\partial_a F_\tau(a)| \ge \frac{1}{6}\epsilon$. Lemma 3.1.1 yields

$$\Big|\int_{\omega_1}^{\omega_1+i2\omega'} e^{-itF_\tau(a)}(1-\chi_2(a,\tau))(1-\chi_3(a,\tau))\varphi(a)da\Big| \le C_1(\tau)t^{-1}. \tag{3.2.11}$$

Note that $C_j(\tau), j = 1,2,3$, are continuous functions of $\tau \in [-2M_1/c_1, 2M_1/c_1]$. Let

$$C = \Sigma_{j=1}^3 \max\{C_j(\tau) : \tau \in [-2M_1/c_1, 2M_1/c_1]\}.$$

Then $|K_1(t,x,x')| \le Ct^{-\frac{1}{3}}$ for large t, because

$$\chi_3 + \chi_2(1-\chi_3) + (1-\chi_2)(1-\chi_3) = 1.$$

Consequently, we have proven that for large t

$$\sup_{x,x'} \big|K_1(t,x,x')\big| < Ct^{-\frac{1}{3}}, \tag{3.2.12}$$

where C only depends on ω, ω'.

3.3 Analysis of $K_2(t,x,x')$

We now consider $K_2(t,x,x')$. Let $b = 2\omega_3 - a$ for $a \in (0,\omega_3)$, and the proof of Lemma 3.2.1 gives

$$\frac{f_a(x')f_{-a}(x)}{W(a)} = -\frac{f_b(x)f_{-b}(x')}{W(b)}.$$

Then for $x > x'$

$$\begin{aligned} K_2(t,x,x') &= \int_{-\wp(\omega_3)}^{+\infty} e^{itE}(f_{-a}(x')f_a(x) + f_{-a}(x)f_a(x'))\frac{dE}{W(E)} \\ &= \int_{-\wp(\omega_3)}^{+\infty} e^{-it\wp(a)}\frac{f_a(x)f_{-a}(x')}{W(a)}d(-\wp(a)) \\ &= \int_0^{i2\omega'} e^{-itF_\tau(a)}m_a(x)m_{-a}(x')\frac{-\wp'(a)}{W(a)}da, \end{aligned}$$

where $\tau = \frac{x-x'}{t}$. For $x < x'$, $K_2(t,x,x')$ can be written in a similar form. Therefore

$$K_2(t,x,x') = \int_0^{i2\omega'} e^{-itF_\tau(a)}\varphi(a,x,x')da,$$

where $\varphi(a,x,x')$ was defined in the previous section.

Step 1. The analysis of the nonlinear phase in $K_2(t,x,x')$ is similar to that of $K_1(t,x,x')$. However, by Remark 3.1.1, $\varphi(a,x,x')$ in $K_2(t,x,x')$ is unbounded when $a \to 0$ and $a \to 2i\omega'$, contrary to the case of $K_1(t,x,x')$. Our strategy then is to change variables to remove this singularity.

Define $\lambda^2 = \wp(\omega_3) - \wp(a)$ such that $\lambda > 0$ when $a \in (0,\omega_3)$ and $\lambda < 0$ when $a \in (\omega_3, 2\omega_3)$. Then the map $a \to \lambda$ is one-to-one, onto and analytic from $(0, 2\omega_3)$ to $\mathbb{R}$. Note that $\lambda(2i\omega' - a) = -\lambda(a)$ and the behavior of $\lambda(a)$ as $a \to 2i\omega'$ is the same as that when $a \to 0$.

We claim that $\frac{\partial a}{\partial \lambda} = \frac{2\lambda}{-\wp'(a)}$ is never zero when $a \in (0, 2\omega_3)$. In fact, the claim is obvious for $a \neq \omega_3$. When $a = \omega_3$, by L'Hôpital's rule,

$$\frac{\partial a}{\partial \lambda}(0) = \lim_{\lambda \to 0} \frac{2\lambda}{-\wp'(a)} = \lim_{\lambda \to 0} \frac{2}{-\wp''(a)\frac{\partial a}{\partial \lambda}},$$

which implies that

$$\left|\frac{\partial a}{\partial \lambda}(0)\right| = \sqrt{\frac{2}{\wp''(\omega_3)}} > 0.$$

Observe that when $\lambda \to \pm\infty$, $\lambda\varphi(\lambda) = \lambda^3 \cdot O(1)$ and $|-\wp'(a)| = |\lambda|^3 + O(\lambda^2)$. Hence, $\frac{\lambda\varphi(\lambda)}{-\wp'(a)}$ and its λ-derivatives are bounded uniformly for $x, x', \lambda \in \mathbb{R}$.

After changing the variables, we obtain

$$\int_0^{i2\omega'} e^{-itF_\tau(a)}\varphi(a)da = \int_{\mathbb{R}} e^{-itF_\tau(\lambda)}\varphi(\lambda)\frac{\partial a}{\partial \lambda}d\lambda, \tag{3.3.1}$$

where $F_\tau(\lambda) = F_\tau(a(\lambda))$ and $\varphi(\lambda) = \varphi(a(\lambda))$.

We will decompose (3.3.1) into different integral regions and estimate them separately. Define $\chi(\cdot)$ to be a smooth function supported in $(-2, 2)$ such that $\chi(x) = 1$ when $x \in [-1, 1]$, and let M be a large number to be specified.

Step 2. We claim that

$$\Big|\int_{\mathbb{R}} e^{-itF_\tau(\lambda)}\varphi(\lambda)\frac{2\lambda}{-\wp'(a)}\chi(\lambda/M)d\lambda\Big| < C_M\, t^{-\frac{1}{4}} \text{ or } C_M t^{-\frac{1}{3}}, \tag{3.3.2}$$

depending on whether $P(x)$ has a double root in $(-\infty, \wp(\omega_3)]$ or not.

The proof of (3.3.2) will follow the lines of the proof of (3.2.12). Recall that the map $\lambda \to a$ is one-to-one from $\mathbb{R}$ onto $(0, 2\omega_3)$, and satisfies $\lambda^2 = \wp(\omega_3) - \wp(a)$. Also, we observe that

$$\partial_\lambda F_\tau(\lambda) = \frac{\partial a}{\partial \lambda}(\wp'(a) + \tau i(\zeta(\omega)/\omega + \wp(a))), \tag{3.3.3}$$

$$\partial_\lambda^2 F_\tau(\lambda) = \frac{\partial^2 a}{\partial \lambda^2}(\wp'(a) + \tau i(\zeta(\omega)/\omega + \wp(a))) + \left(\frac{\partial a}{\partial \lambda}\right)^2(\wp''(a) + \tau i\wp'(a)). \tag{3.3.4}$$

By Lemma 3.1.2,

$$\inf\{|\zeta(\omega)/\omega + \wp(a)| : a \in (0, \omega_3)\} = |\zeta(\omega)/\omega + \wp(\omega_3)| = c_2 > 0.$$

Denote

$$M_2 = \max\{|\wp'(a)| + |\wp''(a)| + |\zeta(\omega)/\omega + \wp(a)| : \lambda(a) \in [-2M, 2M]\}.$$

Since $\frac{\partial a}{\partial \lambda}$ is smooth and never zero, there exist c_3 and M_3 such that $0 < c_3 < |\frac{\partial a}{\partial \lambda}| < \sqrt{M_3}$ for all $\lambda \in [-2M, 2M]$. Moreover, suppose $|\frac{\partial^2 a}{\partial \lambda^2}| < M_3$ for $\lambda \in [-2M, 2M]$. Then for $\lambda \in [-2M, 2M]$ and $|\tau| \geq 2M_2/c_2$

$$|\partial_\lambda F_\tau(\lambda)| > \frac{1}{2} c_2 c_3 |\tau|, \qquad |\partial_\lambda^2 F_\tau(\lambda)| < 2M_3 M_2 (1 + |\tau|).$$

Integrating by parts, an argument similar to (3.2.8) shows that for $|\tau| \geq 2M_2/c_2$,

$$\Big| \int_{\mathbb{R}} e^{-itF_\tau(\lambda)} \varphi(\lambda) \frac{2\lambda}{-\wp'(a)} \chi(\lambda/M) d\lambda \Big| < C_M t^{-1}.$$

To prove (3.3.2) for $|\tau| \leq 2M_2/c_2$, we first suppose that $P(x)$ has no double root in $(-\infty, \wp(\omega_3)]$. Since $\frac{\partial F_\tau(\lambda)}{\partial \lambda} = \frac{\partial a}{\partial \lambda} \frac{\partial F_\tau(a)}{\partial a}$ and $\frac{\partial a}{\partial \lambda} \neq 0$ for $a \in (0, 2\omega_3)$, it follows from (3.3.3) and (3.3.4) that $\frac{\partial F_\tau(\lambda)}{\partial \lambda}$ and $\frac{\partial^2 F_\tau(\lambda)}{\partial \lambda^2}$ vanish at (λ_0, τ_0) if and only if (3.2.4) and (3.2.5) vanish at (a_0, τ_0), where $\lambda_0^2 = \wp(\omega_3) - \wp(a_0)$. Therefore, the fact that $P(x)$ has no double root implies that there exists ϵ such that

$$\min\Big\{ \sum_{j=1}^{3} |\partial_\lambda^j F_\tau(\lambda)| : |\lambda| \leq 2\lambda|\tau| < 2M_2/c_2 \Big\} > \epsilon > 0.$$

Just as in the case of $K_1(t, x, x')$, we define $\chi_2(\lambda, \tau), \chi_3(\lambda, \tau)$ on $[-2M, 2M] \times [-2M_2/c_2, 2M_2/c_2]$. Namely, $\chi_2(\lambda, \tau) = 1$ when $|\partial_\lambda^2 F_\tau(\lambda)| > \frac{1}{6}\epsilon$, and $\chi_2(\lambda, \tau) = 0$ when $|\partial_\lambda^2 F_\tau(\lambda)| < \frac{1}{9}\epsilon$. $\chi_3(\lambda, \tau) = 1$ when $|\partial_\lambda F_\tau(\lambda)| + |\partial_\lambda^2 F_\tau(\lambda)| < \frac{1}{3}\epsilon$, and $\chi_3(\lambda, \tau) = 0$ when $|\partial_\lambda F_\tau(\lambda)| + |\partial_\lambda^2 F_\tau(\lambda)| > \frac{2}{3}\epsilon$.

Decompose the integral in (3.3.2) according to

$$1 = \chi_3 + \chi_2(1-\chi_3) + (1-\chi_2)(1-\chi_3).$$

The same arguments as those in (3.2.9), (3.2.10), and (3.2.11) yield for $|\tau| \leq 2M_2/c_2$,

$$\Big| \int_{\mathbb{R}} e^{-itF_\tau(\lambda)} \varphi(\lambda) \frac{2\lambda}{-\wp'(a)} \chi(\lambda/M) d\lambda \Big| < C_M t^{-\frac{1}{3}}.$$

In the case that $P(x)$ has a double root $\wp(a_0) \in (-\infty, \wp(\omega_3)]$, there is $\tau_0 \in \mathbb{R}$, such that $\partial_\lambda^j F_\tau(\lambda)$, $j = 1, 2, 3$, vanish at (λ_0, τ_0), where $\lambda_0^2 = \wp(\omega_3) - \wp(a_0)$. The fact that $P(x)$ has no root of degree 3 implies that $\partial_\lambda^4 F_{\tau_0}(\lambda_0) \neq 0$. Therefore, there exists ϵ such that

$$\min \Big\{ \sum_{j=1}^{4} |\partial_\lambda^j F_\tau(\lambda)| : \lambda \in [-2M, 2M], \tau \in [-2M_2/c_2, 2M_2/c_2] \Big\} > 2\epsilon > 0.$$

Define smooth function $\chi_4(\lambda, \tau) : [-2M, 2M] \times [-2M_2/c_2, 2M_2/c_2] \to [0, 1]$, such that $\chi_4 = 1$ when $\Sigma_{j=1}^3 |\partial_\lambda^j F_\tau(\lambda)| \leq \epsilon$, and $\chi_4 = 0$ when $\Sigma_{j=1}^3 |\partial_\lambda^j F_\tau(\lambda)| \geq \frac{3}{2}\epsilon$. Hence on the support of χ_4, $|\partial_\lambda^4 F_\tau(\lambda)| \geq \frac{1}{2}\epsilon$. It follows from Lemma 3.1.1 that

$$\Big| \int_{\mathbb{R}} e^{-itF_\tau(\lambda)} \varphi(\lambda) \frac{2\lambda}{-\wp'(a)} \chi(\lambda/M) \chi_4(\lambda, \tau) d\lambda \Big| < C_4(\tau) t^{-\frac{1}{4}}.$$

We decompose the integral in (3.3.2) by using

$$\chi_4 + (1-\chi_4)\chi_3 + \chi_2(1-\chi_3)(1-\chi_4) + (1-\chi_2)(1-\chi_3)(1-\chi_4) = 1.$$

The analysis of the terms containing $(1-\chi_4)\chi_3, \chi_2(1-\chi_3)(1-\chi_4)$, and $(1-\chi_2)(1-\chi_3)(1-\chi_4)$ is similar to (3.2.9), (3.2.10), and (3.2.11) respectively. Therefore, under the assumption that $P(x)$ has a double root in $(-\infty, \wp(\omega_3)]$, we have proven that

$$\Big|\int_{\mathbb{R}} e^{-itF_\tau(\lambda)}\varphi(\lambda)\frac{2\lambda}{-\wp'(a)}\chi(\lambda/M)d\lambda\Big| < C_M t^{-\frac{1}{4}}.$$

Step 3. It now remains to estimate

$$\int_{\mathbb{R}} e^{-itF_\tau(\lambda)}\varphi(\lambda)\frac{2\lambda}{-\wp'(a)}(1-\chi(\lambda/M))d\lambda,$$

which by definition equals

$$\lim_{N\to+\infty}\int_{\mathbb{R}} e^{-itF_\tau(\lambda)}\varphi(\lambda)\frac{2\lambda}{-\wp'(a)}(\chi(\lambda/N)-\chi(\lambda/M))d\lambda. \tag{3.3.5}$$

Since (3.3.5) are not integrable on the support of $1-\chi(\lambda/M)$, Lemma 3.1.1 cannot be applied to (3.3.5) directly. We shall explore the oscillation of the phase $e^{-itF_\tau(\lambda)}$ and perform integration by parts to bound (3.3.5), which requires us to exclude the zeroes of $\partial_\lambda F_\tau(\lambda)$.

By definition, $\wp(a) = a^{-2} + \frac{1}{20}g_2 a^2 + O(a^4)$, and $\wp(a) = \wp(i\omega') - \lambda^2$, hence

$$\lambda(a) = \frac{i}{a} + \alpha_1 a + O(a^3) \quad \text{as } a \to 0,\ a \in (0,\omega_3),$$

which is an meromorphic function of a. It follows that $\zeta(a) = -i\lambda + O(\lambda^{-1})$ as $a \to 0$, $a \in (0,\omega_3)$. Consequently

$$F_\tau(\lambda) = -\lambda^2 - \tau\lambda + \wp(i\omega') + O(\frac{\tau}{\lambda}), \quad \lambda \to \pm\infty; \tag{3.3.6}$$

$$\partial_\lambda F_\tau(\lambda) = -2\lambda - \tau + O(\tau\lambda^{-2}), \quad \lambda \to \pm\infty; \tag{3.3.7}$$

$$\partial_\lambda^2 F_\tau(\lambda) = -2 + O(\tau\lambda^{-3}), \quad \lambda \to \pm\infty. \tag{3.3.8}$$

We require M large enough such that

$$M > 1 + \max\Big\{|\lambda_0| : \frac{\partial F_\tau(\lambda)}{\partial\lambda}, \frac{\partial^2 F_\tau(\lambda)}{\partial\lambda^2} \text{ both vanish at } (\tau_0,\lambda_0)\Big\}.$$

Therefore, if $|\lambda| > M$, $\frac{\partial F_\tau(\lambda)}{\partial\lambda}$ and $\frac{\partial^2 F_\tau(\lambda)}{\partial\lambda^2}$ cannot vanish at the same (τ,λ).

When $|\lambda| > M$ and $|\lambda + \frac{\tau}{2}| > 1$, we claim that

$$|\partial_\lambda F_\tau(\lambda)| > |\lambda + \frac{\tau}{2}| - \frac{1}{2}. \tag{3.3.9}$$

In fact, by (3.3.7)

$$|\partial_\lambda F_\tau(\lambda)| > 2|\lambda + \frac{\tau}{2}| - O(\tau\lambda^{-2}).$$

We choose M large enough such that $O(\tau\lambda^{-2}) < \frac{|\tau|}{100|\lambda|}$. If $|\lambda + \frac{\tau}{2}| > \frac{|\tau|}{100}$, (3.3.9) clearly holds. If $|\lambda + \frac{\tau}{2}| \leq \frac{|\tau|}{100}$, then $\frac{|\tau|}{100|\lambda|} < \frac{1}{2}$ and (3.3.9) also follows.

When $(-\frac{\tau}{2} - 1, -\frac{\tau}{2} + 1)$ is not contained in $(-M, M)$, by (3.3.8), $|\partial_\lambda^2 F_\tau(\lambda)| > 1$ for $\lambda \in (-\frac{\tau}{2} - 2, -\frac{\tau}{2} + 2)$ as long as M is large enough. By Lemma 3.1.1, we have

$$\int_{\mathbb{R}} e^{-itF_\tau(\lambda)} \varphi(\lambda) \frac{2\lambda}{-\wp'(a)} \chi(\lambda + \tau/2) d\lambda < Ct^{-\frac{1}{2}},$$

where $\chi(x) = 1$ when $|x| < 1$, and $\chi(x) = 0$ when $|x| > 2$.

To estimate (3.3.5), we first consider the case when $\frac{|\tau|}{10} > M$ and estimate

$$\int_{\mathbb{R}} e^{-itF_\tau(\lambda)} \frac{2\lambda\varphi(\lambda)}{-\wp'(a)} (\chi(10\lambda/|\tau|) - \chi(\lambda/M)) d\lambda,$$

which equals

$$\frac{1}{it} \int_{\mathbb{R}} e^{-itF_\tau(\lambda)} \frac{d}{d\lambda} \Big(\frac{2\lambda\varphi(\lambda)}{-\wp'(a)} \frac{\chi(10\lambda/|\tau|) - \chi(\lambda/M)}{\partial_\lambda F(\lambda)} \Big) d\lambda. \tag{3.3.10}$$

It follows from (3.3.7) that on the support of $\chi(10\lambda/|\tau|) - \chi(\lambda/M)$

$$|\partial_\lambda F_\tau(\lambda)| > |\tau| - 2|\lambda| - O(\tau\lambda^{-2}) > |\tau|/2,$$

and

$$|\partial_\lambda^2 F_\tau(\lambda)| < |\tau|/4,$$

as long as M is large enough. Hence

$$|\partial_\lambda (\partial_\lambda F_\tau(\lambda))^{-1}| < \frac{1}{|\tau|}. \tag{3.3.11}$$

Since $(\chi(10\lambda/|\tau|) - \chi(\lambda/M))$, $\frac{2\lambda\varphi(\lambda)}{-\wp'(a)}$ and their λ-derivatives are uniformly bounded, we have

$$\Big|\frac{d}{d\lambda}\Big(\frac{2\lambda\varphi(\lambda)}{-\wp'(a)}\frac{\chi(10\lambda/|\tau|) - \chi(\lambda/M)}{\partial_\lambda F(\lambda)}\Big)\Big| < \frac{C}{|\tau|},$$

from which it follows that $|(3.3.10)| < Ct^{-1}$.

To complete the estimate on (3.3.5) when $|\tau|/10 > M$, it remains to bound

$$\int_{\mathbb{R}} e^{-itF_\tau(\lambda)}\varphi(\lambda)\frac{2\lambda}{-\wp'(a)}(\chi(\lambda/N) - \chi(10\lambda/|\tau|) - \chi(\lambda + \tau/2))d\lambda.$$

Integrating by parts, this equals

$$\frac{2}{it}\int_{\mathbb{R}} e^{-itF_\tau(\lambda)}\frac{d}{d\lambda}\Big(\frac{\lambda\varphi(\lambda)}{-\wp'(a)}\frac{(\chi(\lambda/N) - \chi(10\lambda/\tau) - \chi(\lambda + \tau/2))}{\partial_\lambda F_\tau(\lambda)}\Big)d\lambda := J_1 + J_2,$$

where

$$J_1 = \frac{2}{it}\int_{\mathbb{R}} e^{-itF_\tau(\lambda)}\frac{(\chi(\lambda/N) - \chi(10\lambda/|\tau|) - \chi(\lambda + \tau/2))}{\partial_\lambda F_\tau(\lambda)}\frac{d}{d\lambda}\frac{\lambda\varphi(\lambda)}{-\wp'(a)}d\lambda,$$

and

$$J_2 = \frac{2}{it}\int_{\mathbb{R}} e^{-itF_\tau(\lambda)}\frac{\lambda\varphi(\lambda)}{-\wp'(a)}\frac{d}{d\lambda}\frac{(\chi(\lambda/N) - \chi(10\lambda/|\tau|) - \chi(\lambda + \tau/2))}{\partial_\lambda F_\tau(\lambda)}d\lambda.$$

In J_2,

$$\partial_\lambda(\chi(\lambda/N) - \chi(10\lambda/|\tau|) - \chi(\lambda + \tau/2)) = \frac{1}{N}\chi'(\lambda/N) - \frac{10}{|\tau|}\chi'(10\lambda/|\tau|) - \chi'(\lambda + \tau/2). \tag{3.3.12}$$

On the support of (3.3.12), we have $|\lambda + \tau/2| > 1$ and $|\lambda| > M$. Thus $|(\partial_\lambda F_\tau(\lambda))^{-1}| < C$ by (3.3.9). Consequently,

$$\Big|\frac{2}{it}\int_{\mathbb{R}} e^{-itF_\tau(\lambda)}\frac{\lambda\varphi(\lambda)}{-\wp'(a)\partial_\lambda F_\tau(\lambda)}\Big(\frac{1}{N}\chi'(\lambda/N)-\frac{10}{|\tau|}\chi'(10\lambda/|\tau|)-\chi'(\lambda+\tau/2)\Big)d\lambda\Big| < Ct^{-1}.$$

On the support of $\chi(\lambda/N) - \chi(10\lambda/|\tau|) - \chi(\lambda + \tau/2)$ we have $|\lambda| > |\tau/10|$. It follows from (3.3.8) that $|\partial_\lambda^2 F_\tau(\lambda)| < 3$. Combining it with (3.3.9), we obtain

$$\partial_\lambda(\partial_\lambda F_\tau(\lambda))^{-1} < C(\lambda + \frac{\tau}{2})^{-2},$$

which is integrable. Therefore

$$\Big|\frac{2}{it}\int_{\mathbb{R}} e^{-itF_\tau(\lambda)}\frac{\lambda\varphi(\lambda)}{-\wp'(a)}\Big(\partial_\lambda\frac{1}{\partial_\lambda F_\tau(\lambda)}\Big)(\chi(\lambda/N)-\chi(10\lambda/|\tau|)-\chi(\lambda+\tau/2))d\lambda\Big| < Ct^{-1}.$$

This completes the estimate on J_2.

As for J_1, integrating by parts again, we obtain

$$J_1 = -\frac{4}{t^2}\int_{\mathbb{R}} e^{-itF_\tau(\lambda)}\frac{d}{d\lambda}\Big(\frac{(\chi(\lambda/N)-\chi(10\lambda/|\tau|)-\chi(\lambda+\tau/2))}{(\partial_\lambda F_\tau(\lambda))^2}\frac{d}{d\lambda}\frac{\lambda\varphi(\lambda)}{-\wp'(a)}\Big)d\lambda.$$

Applying Leibnitz's rule, we are left with three terms. Two terms come from $\frac{d}{d\lambda}$ hitting $\chi(\lambda/N) - \chi(10\lambda/|\tau|) - \chi(\lambda+\tau/2)$ and $(\partial_\lambda F_\tau(\lambda))^{-2}$, and the analysis is analogous to that of J_2. When $\frac{d}{d\lambda}$ hits $\frac{d}{d\lambda}\frac{\lambda\varphi(\lambda)}{-\wp'(a)}$, we obtain the third term:

$$-\frac{4}{t^2}\int_{\mathbb{R}} e^{-itF_\tau(\lambda)}\frac{\chi(\lambda/N)-\chi(10\lambda/|\tau|)-\chi(\lambda+\tau/2)}{(\partial_\lambda F_\tau(\lambda))^2}\frac{d^2}{d\lambda^2}\frac{\lambda\varphi(\lambda)}{-\wp'(a)}d\lambda.$$

Because $|(\partial_\lambda F_\tau(\lambda))^2| > |\lambda+\tau/2|^2/4$ on the support of $\chi(\lambda/N)-\chi(10\lambda/|\tau|)-\chi(\lambda+\tau/2)$ and $\frac{d^2}{d\lambda^2}\frac{\lambda\varphi(\lambda)}{-\wp'(a)}$ is uniformly bounded, the above term is dominated by Ct^{-2}, where the constant C is independent of N.

This completes the estimate of (3.3.5) when $|\tau|/10 > M$. The analysis is similar and even simpler when $|\tau|/10 \le M$. Therefore, when $P(x)$ has no double root in $(-\infty, \wp(\omega_3)]$, we have

$$\sup_{x,x'} |K_2(t,x,x')| < Ct^{-\frac{1}{3}}.$$

The decay factor $t^{-\frac{1}{3}}$ is replaced by $t^{-\frac{1}{4}}$ when $P(x)$ has a double root in $(-\infty, \wp(\omega_3)]$.
$\square$

Combining the estimates on $K_1(t,x,x')$ and $K_2(t,x,x')$, we have proven (3.0.2) under the assumption (3.0.6). We have also proven (3.0.3) for all nonzero $\omega, \omega' \in \mathbb{R}$.

It remains to prove that (3.0.6) holds for almost all $\omega, \omega' \in \mathbb{R}$.

Suppose $P(x)$ has a double root $x_0 \in (-\infty, \wp(\omega_3)]$. Then x_0 is a root of $P'(x)$. Recall that

$$P'(x) = 6x^2 + \frac{12\zeta(\omega)}{\omega}x + \frac{g_2}{2},$$

with its roots

$$r_+, r_- = -\frac{\zeta(\omega)}{\omega} \pm \sqrt{\left(\frac{\zeta(\omega)}{\omega}\right)^2 - \frac{g_2}{12}}\ .$$

That $P(x)$ has a double root in $(-\infty, \wp(\omega_3)]$ implies that $(\zeta(\omega)/\omega)^2 - g_2/12 > 0$ and $P(r_-) = 0$. By (3.5.1), g_2 and g_3 are real analytic for $\omega, \omega' \in \mathbb{R}^+$. By (3.5.3), $\zeta(\omega)$ is also real analytic for $\omega, \omega' \in \mathbb{R}^+$. Therefore, r_+, r_- are analytic when $\omega, \omega' \in \mathbb{R}^+$, with branches at $\left(\frac{\zeta(\omega)}{\omega}\right)^2 - \frac{g_2}{12} = 0$. To prove (3.0.6) for almost all $\omega, \omega' \in \mathbb{R}^+$, it suffices to show that $P(r_-)$ is nonzero at one point. This can be done by direct numerical calculation.

For example, take $\omega = 5.5$ and $\omega' = 2$. Then we have

$$g_2 = 0.507343, \qquad g_3 = -0.0695438,$$

$$r_+, r_- = 0.0628169 \pm 0.195787i,$$

$$P(r_+), P(r_-) = -0.0386656 \pm 0.0300201i.$$

This indicates that $P(r_-)$ is nonzero for almost all $\omega, \omega' \in \mathbb{R}$. Therefore, (3.0.6) holds for almost all $\omega, \omega' \in \mathbb{R}$.

3.4 Optimality of the decay factor

So far we have proven the first part of Theorem 3.0.6. To verify (3.0.4), we first reduce it to showing that there exist constants $c > 0$ and $T > 0$ such that for $t > T$

$$\|K(t, x, x')\|_{L^\infty} > c\, t^{-\frac{1}{3}}. \tag{3.4.1}$$

Accepting (3.4.1) temporarily, we obtain that for any given large t, there exist (x_0, x_0') such that $x_0 \neq x_0'$ and $|K(t, x_0, x_0')| > ct^{-\frac{1}{3}}$. Without loss of generality, suppose that

$$\Re(K(t, x_0, x_0')) > \frac{c}{2} t^{-\frac{1}{3}}.$$

As $K(t, x, x')$ is smooth away from $x = x'$, there exists $\delta > 0$ such that for any $(x, x') \in (x_0 - \delta, x_0 + \delta) \times (x_0' - \delta, x_0' + \delta)$

$$\Re(K(t, x, x')) > \frac{c}{4} t^{-\frac{1}{3}}.$$

Take the initial data $\psi_0(x') = \frac{1}{2\delta} \chi_{(x_0' - \delta, x_0' + \delta)}(x')$. Then $\|\psi_0\|_{L^1} = 1$ and for any $x \in (x_0 - \delta, x_0 + \delta)$

$$|\psi(t, x)| = \Big| \int K(t, x, x') \psi_0(x') dx' \Big| > \frac{c}{4} t^{-\frac{1}{3}}.$$

To prove (3.4.1), we need the following lemma (Prop. 3, Chap. 8 [36]):

Lemma 3.4.1. *Suppose $k \geq 2$, and*

$$\phi(x_0) = \phi'(x_0) = \cdots = \phi^{(k-1)}(x_0) = 0,$$

while $\phi^{(k)}(x_0) \neq 0$. If ψ is supported on a sufficiently small neighborhood of x_0 and $\psi(x_0) \neq 0$, then

$$\int_{\mathbb{R}} e^{i\lambda\phi(x)}\psi(x)dx = a_k\psi(x_0)(\phi^{(k)}(x_0))^{-\frac{1}{k}}\lambda^{-\frac{1}{k}} + O(\lambda^{-\frac{1}{k}-1}),$$

where $a_k \neq 0$ *only depends on* k*. The implicit constant in* $O(\lambda^{-\frac{1}{k}-1})$ *depends on only finitely many derivatives of* ϕ *and* ψ *at* x_0*.*

By Lemma 3.1.2, $P(a)$ has a unique simple root in $(\wp(\omega_2), \wp(\omega_1))$, thus we can choose $a_0 \in (\omega_1, \omega_2)$ and a corresponding τ_0 such that both $\partial_a F_\tau(a)$ and $\partial_a^2 F_\tau(a)$ vanish at (a_0, τ_0).

First, we denote $I = [0, i2\omega'] \cup [\omega_1, \omega_1 + i2\omega']$ and assume that for any $a \in I$, $a \neq a_0$, at least one of $\partial_a F_{\tau_0}(a)$ and $\partial_a^2 F_{\tau_0}(a)$ does not vanish. Then we take $\delta > 0$ small enough such that for $a \notin (a_0 - \delta, a_0 + \delta) \subset I$, $|\partial_a F_{\tau_0}(a)| + |\partial_a^2 F_{\tau_0}(a)|$ is greater than some positive constant.

Given any large t, take (x, x') such that $\frac{x-x'}{t} = \tau_0$ and

$$K(t, x, x') = \Big(\int_0^{i2\omega'} + \int_{\omega_1}^{\omega_1+i2\omega'}\Big)e^{-itF_{\tau_0}(a)}m_a(x)m_{-a}(x')\frac{-\wp'(a)da}{W(a)}.$$

The $\int_0^{i2\omega'}$-term is bounded by $C\,t^{-\frac{1}{2}}$ using an argument analogous to that in Section 4, because $|\partial_a F_{\tau_0}(a)| + |\partial_a^2 F_{\tau_0}(a)|$ is uniformly greater than some positive constant for $a \in (0, i2\omega')$.

We decompose the $\int_{\omega_1}^{\omega_1+i2\omega'}$-term as follows

$$\int_{\omega_1}^{\omega_1+i2\omega'} e^{-itF_{\tau_0}(a)}m_a(x)m_{-a}(x')\frac{-\wp'(a)da}{W(a)} := J_3 + J_4,$$

where

$$J_3 = \int_{\omega_1}^{\omega_1+i2\omega'} e^{-itF_{\tau_0}(a)}\rho(a)m_a(x)m_{-a}(x')\frac{-\wp'(a)da}{W(a)},$$

and

$$J_4 = \int_{\omega_1}^{\omega_1+i2\omega'} e^{-itF_{\tau_0}(a)}\tilde{\rho}(a)m_a(x)m_{-a}(x')\frac{-\wp'(a)da}{W(a)}.$$

Here $\rho(a)$ is a smooth cutoff function supported on $(a_0 - \delta, a_0 + \delta)$ and $\tilde{\rho}(a) = 1 - \rho(a)$.

Under our assumption, $|J_4| < C\,t^{-\frac{1}{2}}$, following the same reasoning as that in

Section 3.

Considering J_3, the phase function $F_{\tau_0}(a)$ satisfies $\partial_a F_{\tau_0}(a_0) = \partial_a^2 F_{\tau_0}(a_0) = 0$ and $\partial_a^3 F_{\tau_0}(a_0) \neq 0$. $m_a(x)$ and $m_{-a}(x')$ do not vanish when $a \in (\omega_1, \omega_2)$ by (3.1.3). $\frac{-\wp'(a)}{W(a)}$ is nonzero when $a \in (\omega_1, \omega_2)$. Therefore $m_{a_0}(x)m_{-a_0}(x')\frac{-\wp'(a_0)}{W(a_0)}$ is nonzero.

Since $\rho(a)$ is supported in a sufficiently small neighborhood of a_0, by Lemma 3.4.1, there exist $c_1 > 0$ and $T > 0$ such that for $t > T$

$$|J_3| > c_1 t^{-\frac{1}{3}},$$

where c_1 is independent of t.

Combining these estimates, we have $|K(t,x,x')| > c_1 t^{-\frac{1}{3}} - 2Ct^{-\frac{1}{2}} > c_1/2\ t^{-\frac{1}{3}}$ for any (x,x') satisfying $(x-x')/t = \tau_0$, which implies (3.4.1).

Second, suppose there are other a_1, $a_2 \in I$ such that a_1, a_2, a_0 are distinct and $\partial_a F_{\tau_0}(a)$, $\partial_a^2 F_{\tau_0}(a)$ both vanish at $a = a_1$, a_2. Then $P(x)$ vanishes at $\wp(a_j)$, $j = 0, 1, 2$.

Since $a_0 \in (\omega_1, \omega_2)$, we have that $-i\wp'(a_0) > 0$ and $\zeta(\omega)/\omega + \wp(a_0) > 0$ by Lemma 3.1.2. Thus $\tau_0 < 0$ by (3.2.4). Similar analysis shows that when $a \in (\omega_2, \omega_2 + i\omega') \cup (i\omega', 2i\omega')$, $\partial_a F_{\tau_0}(a) \neq 0$. Therefore, a_1, $a_2 \in (0, i\omega')$.

Thus $\wp(a_j)$, $j = 0, 1, 2$, are distinct and are the three roots of $P(x)$. This implies that there is no other $a \in I$ such that $\partial_a F_{\tau_0}(a) = \partial_a^2 F_{\tau_0}(a) = 0$.

We again set $\delta > 0$ small enough such that for $a \notin \bigcup_{j=0}^{3}(a_j - \delta, a_j + \delta) \subset I$, $|\partial_a F_{\tau_0}(a)| + |\partial_a^2 F_{\tau_0}(a)|$ is uniformly greater than some positive constant. Given any large t, take (x,x') such that $\frac{x-x'}{t} = \tau_0$. The earlier argument implies that

$$K(t,x,x') = \sum_{j=0}^{2} \int_I e^{-itF_{\tau_0}(a)} \rho_j(a) m_a(x) m_{-a}(x') \frac{-\wp'(a)da}{W(a)} + O(t^{-\frac{1}{2}}),$$

where $\rho_j(a) = 1$ when $|a - a_j| < \delta$ and $\rho_j(a) = 0$ when $|a - a_j| > 2\delta$.

By Lemma 3.4.1,

$$K(t,x,x') = a_3 t^{-\frac{1}{3}} \sum_{j=0}^{2} (F_{\tau_0}^{(3)}(a_i))^{-\frac{1}{3}} m_{a_j}(x) m_{-a_j}(x') \frac{-\wp'(a_j)}{W(a_j)} + O(t^{-\frac{1}{2}}).$$

Recall that x and x' are related by $(x-x')/t = \tau_0$. $m_{a_j}(x)m_{-a_j}(x')$, $j = 0, 1, 2$, are linearly independent as functions of $x \in \mathbb{R}$ and their nontrivial linear combination is a nonzero function. Therefore, there exist x_0 and x_0', satisfying $(x_0 - x_0')/t = \tau_0$ and

$$K(t, x_0, x_0') = ct^{-\frac{1}{3}} + O(t^{-\frac{1}{2}}),$$

where c is nonzero. Thus there exists some T such that for $t > T$

$$|K(t, x_0, x_0')| > \frac{c}{2} t^{-\frac{1}{3}}.$$

Finally, suppose that $a_1 = a_2$ in the second case, which is equivalent to $\wp(a_1)$ being a double root of $P(x)$ in $(-\infty, \wp(\omega_3))$. Similarly, we have for $t > T$

$$\begin{aligned} K(t,x,x') &= \sum_{j=0}^{1} \int_I e^{-itF_{\tau_0}(a)} \rho_j(a) m_a(x) m_{-a}(x') \frac{-\wp'(a)da}{W(a)} + O(t^{-\frac{1}{2}}) \\ &= a_3 t^{-\frac{1}{3}} (F_{\tau_0}^{(3)}(a_0))^{-\frac{1}{3}} m_{a_0}(x) m_{-a_0}(x') \frac{-\wp'(a_0)}{W(a_0)} + \\ &\qquad a_4 t^{-\frac{1}{4}} (F_{\tau_1}^{(4)}(a_1))^{-\frac{1}{4}} m_{a_1}(x) m_{-a_1}(x') \frac{-\wp'(a_1)}{W(a_1)} + O(t^{-\frac{1}{2}}). \end{aligned}$$

Therefore, there exists (x_0, x_0') such that $|K(t,x,x')| > ct^{-\frac{1}{4}} > ct^{-\frac{1}{3}}$. This completes the proof of (3.4.1).

Our proof also gives the optimality of (3.0.3) in the case that $P(x)$ has a double root in $(-\infty, \wp(\omega_3)]$.

By the proof of Lemma 3.1.2, we have the following corollary:

Corollary 3.4.2. *Suppose that* $(\zeta(\omega)/\omega)^2 \leq g_2/12$*. Then for* $t > 1$*,*

$$\|U(t)\psi_0\|_{L^\infty} < Ct^{-\frac{1}{3}} \|\psi_0\|_{L^1(R)}.$$

Set $\omega = 1$; it follows from (3.5.1) and (3.5.3) that $\frac{g_2}{12} - (\zeta(\omega)/\omega)^2$, as a function of $\omega' > 0$, is analytic and $\omega' = 0$ is its essential singular point. Numerical experiment indicates that $g_2/12 - (\zeta(\omega)/\omega)^2 \approx 0.966104$ when $\omega = 1$ and $\omega' > 5$. When $\omega = 1$ and $\omega' \to 0^+$, $g_2/12 - (\zeta(\omega)/\omega)^2$ assumes each real number infinitely many times.

3.5 Appendix: Elements of Weierstrass functions

Here we list some elementary properties of Weierstrass functions ([39], [7], [2], [13]). A doubly-periodic function that is meromorphic is called an elliptic function. Suppose that $2\omega_1$ and $2\omega_3$ are two periods of an elliptic function $f(z)$ and $\Im(\omega_3/\omega_1) \neq 0$. Join in succession the points 0, $2\omega_1$, $2\omega_1 + 2\omega_3$, $2\omega_3$, 0 and we obtain a parallelogram. If there is no point ω inside or on the boundary of this parallelogram (the vertices excepted) such that $f(z+\omega) = f(z)$ for all values of z, this parallelogram is called a fundamental period-parallelogram for an elliptic function with periods $2\omega_1$ and $2\omega_3$. As a set, we assume this parallelogram only includes one of four vertices and two edges adjacent to it. In this way, the z-plane can be covered with the translations of this parallelogram without any overlap. It can be shown that for any $c \in \mathbb{C}$, the number of roots (counting multiplicity) of the equation

$$f(z) = c$$

that lie in the fundamental period-parallelogram do not depend on c. This number is called the order of the elliptic function $f(z)$ and it equals the number of poles of f inside a fundamental period-parallelogram.

Given $\omega_1, \omega_3 \in \mathbb{C}$ with $\Im(\omega_3/\omega_1) \neq 0$, the Weierstrass elliptic function is defined as

$$\wp(z) = \frac{1}{z^2} + \sum_{(m,n)\neq(0,0)} \left\{(z - 2m\omega_1 - 2n\omega_3)^{-2} - (2m\omega_1 + 2n\omega_3)^{-2}\right\}.$$

The summation extends over all integer values of m and n, simultaneous zero values of m and n excepted. $\wp(z)$ is doubly-periodic, namely

$$\wp(z) = \wp(z + 2\omega_1) = \wp(z + 2\omega_3).$$

$\wp(z)$ is an elliptic function of order 2, with poles $\Omega_{m,n} = 2m\omega_1 + 2n\omega_3$. Each pole $\Omega_{m,n}$ is of degree 2. $\wp(z)$ is an even function, $\wp(z) = \wp(-z)$. The Laurent's expansion of $\wp(z)$ at $z = 0$ is written as

$$\wp(z) = z^{-2} + \frac{1}{20} g_2 z^2 + \frac{1}{28} g_3 z^4 + O(z^6),$$

where g_2, g_3 are the constants in (1.2.3) and (1.2.4). Explicitly, we have

$$g_2 = 60 \sum_{(m,n)\neq(0,0)} \Omega_{m,n}^{-4}, \qquad g_3 = 140 \sum_{(m,n)\neq(0,0)} \Omega_{m,n}^{-6}. \tag{3.5.1}$$

Here g_2 and g_3 are called the invariants of $\wp$ and they uniquely characterize $\wp$.

Since $\wp'$ is odd and elliptic of order 3, it has three zeroes in its fundamental period-parallelogram. It is clear that these zeroes are the half periods $\omega_1, \omega_2 = \omega_1 + \omega_3$, and ω_3. Denote $e_j = \wp(\omega_j), j = 1, 2, 3$. The fact that $\wp(z)$ is of order 2 implies that e_1, e_2, e_3 are distinct and that $\wp''$ does not vanish at ω_j, $j = 1, 2, 3$. Furthermore, (1.2.3) implies that e_1, e_2, e_3 are the roots of the cubic polynomial

$$4x^3 - g_2 x - g_3 = 0. \tag{3.5.2}$$

The function $\zeta(z)$ is defined by the equation

$$\frac{d}{dz}\zeta(z) = -\wp(z),$$

coupled with the condition $\lim_{z\to 0}(\zeta(z) - z^{-1}) = 0$. $\zeta(z)$ may also be represented as

$$\zeta(z) = \frac{1}{z} + \sum_{(m,n)\neq(0,0)} \left\{ \frac{1}{z - 2m\omega_1 - 2n\omega_3} + \frac{1}{2m\omega_1 + 2n\omega_3} + \frac{z}{(2m\omega_1 + 2n\omega_3)^2} \right\}. \tag{3.5.3}$$

$\zeta(z)$ is an odd meromorphic function of z over the whole complex plane except at the simple poles $\Omega_{m,n}$. The residue at each pole is 1.

Write $\zeta(\omega_1) = \eta_1$ and $\zeta(\omega_3) = \eta_3$; then

$$\eta_1\omega_3 - \eta_3\omega_1 = \frac{1}{2}\pi i.$$

$\zeta(z)$ is not doubly-periodic; however, it satisfies the following equations:

$$\zeta(z+2\omega_1)=\zeta(z)+2\eta_1, \qquad \zeta(z+2\omega_3)=\zeta(z)+2\eta_3. \tag{3.5.4}$$

Next we define $\sigma(z)$ by the equation

$$\frac{d}{dz}\log\sigma(z)=\zeta(z),$$

coupled with the condition $\lim_{z\to 0}\sigma(z)/z=1$. $\sigma(z)$ is an odd entire function with simple zeroes at $\Omega_{m,n}$. Just like $\zeta(z)$, $\sigma(z)$ satisfies

$$\sigma(z+2\omega_1)=-\sigma(z)e^{2\eta_1(z+\omega_1)}, \qquad \sigma(z+2\omega_3)=-\sigma(z)e^{2\eta_1(z+\omega_3)}. \tag{3.5.5}$$

If we assume that $\omega_1=\omega$, $\omega_3=i\omega'$ and $\omega,\,\omega'\in\mathbb{R}$, then by symmetry $\wp(z)$ is real-valued when $\Re z\in\{0,\omega_1\}$ or $\Im z\in\{0,i\omega_3\}$. $\zeta(z)$ is real-valued on the real line and is pure imaginary when $\Re z=0$. Let D to be the rectangle with vertices $0,\omega,\omega+i\omega'$, and $i\omega'$. Then $\wp(z)$ sends D to the upper half plane conformally. As z moves clockwise on the boundary of D both starting and ending at 0, $\wp(z)$ varies from $-\infty$ to ∞. This implies that $\wp(i\omega')<\wp(\omega+i\omega')<\wp(\omega)$.

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