THE STRUCTURE OF STRONG INTERACTION SYMMETRIES

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ABSTRACT

All higher symmetry schemes involving from three to ten baryons with reasonable isotopic assignments are found. The case of eight baryons is done explicitly; the results in the other cases are stated. The predictions of the schemes regarding mesons and electromagnetic form factors are found and briefly compared with experiment. A study is made of the connections between different schemes.

The methods developed to treat the above problems are used to make a systematic study of chiral symmetries (symmetries involving γ_5). It is shown that only two special cases, called free chiral symmetry and bound chiral symmetry, are of interest. These are discussed in detail.

Several recently proposed theories are treated as special cases.

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I. INTRODUCTION

"What we do here is nothing to what we dream of doing."
-- D. A. F. de Sade: Justine

Perhaps the strong interactions can be divided into two kinds, one of which is much more highly symmetric than the observed strong interactions, and the other of which breaks the symmetry. This is a very old idea (2, 3), at least in the units in which particle physicists measure time. When it was first proposed the hope was that the symmetry-breaking interactions might be considerably weaker than the symmetric interactions, so that (at least for certain processes) reasonable results could be obtained by neglecting the symmetry-breaking term, or perhaps treating it as a small perturbation (2). More recently, it has been proposed that even if the symmetry-violating interactions are relatively strong, such a split might be useful if the higher symmetries generate the currents of the weak interactions (4).

The greatest encouragement for the first viewpoint comes from the observed pattern of baryon masses, which all deviate less than 20 per cent from a common mean mass. This is large compared to the splitting within the isotopic multiplets, but small compared to baryon-meson or baryon-lepton mass differences. It is attractive to believe that the baryons form a completely degenerate supermultiplet under the symmetric interactions.

Much the same can be said about the recently discovered resonances in multi-meson systems. There are three multiplets all believed to be vector and all of whose masses lie close together. The ρ is an isotriplet at 750 mev. (5). The ω is an isosinglet at 790 mev. (6). The K^* is an isodoublet with hypercharge 1 at 880 mev. (7). (The spin of the K^* is only conjectured to be 1.) All of these masses differ less than 10 percent from their common mean mass. This is all the more surprising considering the vastly different strong decay modes of these particles. Certainly any theory that purports to explain the baryons as members of a broken supermultiplet should do the same for these resonances.

In our analysis we begin with the baryons, since it is always possible to build the mesons out of baryon-antibaryon resonances, but it is impossible to do the reverse. The only way that we can insure that the eight baryons will form a completely degenerate supermultiplet under the highly symmetric interactions is by making them the basis for an irreducible representation of the symmetry group. (Cases where energy levels are degenerate without this happening--accidental degeneracy--are known in atomic problems, but they are uncommon, and, more important, we do not know how to insure their occurrence.)

To introduce some terminology which will be used later, we want to find a group G that possesses an irreducible representation by 8 x 8 matrices, D(g). Further, we certainly want the highly symmetric part of the interaction to be invariant under those transformations under which all the strong interactions are invariant, so we want G to contain a subgroup isomorphic to the usual baryon number-hypercharge-isospin rotation group, such that under this subgroup the basis of D(g) decomposes into a hypercharge zero isotriplet (Σ), a hypercharge zero isosinglet (Λ), and two hypercharge plus and minus one isodoublets (N and Ξ), all with baryon number one. Also, we want the matrices in D(g) to preserve the symmetry of the free Lagrangian; thus they must be unitary and the group G compact.

We call the scheme described in the preceding paragraph "a higher symmetry scheme", or, sometimes, just " a symmetry" or "a scheme". The principal aim of the first half of this thesis will be to find <u>all</u> possible higher-symmetry schemes (not only for eight baryons but for interesting systems with other numbers of particles) and to extract some of their more elementary consequences. By elementary, we mean those consequences that are independent of our assumptions about the nature of the symmetry-breaking interactions.

To step out from behind the first person plural for a moment, I like to think of this part of the work as a calculation--not strikingly different in spirit from a calculation of a radiative correction.* It is true that the first half of this thesis does not look like the usual physics calculation, but that is only because the computational tools are algebraic rather than analytic.

The second viewpoint (that the symmetries are the generators of the weak-interaction currents) has a close analog in atomic physics. There we make great use of the fact that the generator of the symmetries of the free Hamiltonian (the linear momentum) is also, in a good approximation, the electromagnetic current. Useful sum rules are derived from this, independent of the strength of the poten-

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* Prof. R. Christy has informed me that the same calculation has been done independently by Speiser and Tarski (8). Their paper only states results, so it is impossible to determine if their methods are the same as those used here. They have missed some higher symmetry schemes, due to insufficient ambition in the case of disconnected groups.

tial. If we know how the potential transforms under the symmetry, we can obtain even more powerful rules. To my knowledge, the first attempt to apply similar principles to elementary particle physics was the derivation of the Goldberger-Treiman relation by Gell-Mann and Levy (9). This derivation has since been superseded (10); however, its starting point--the assumption that the divergence of the axial vector current is the pion field--still remains a good example of the sort of argument we have in mind. More recently Gell-Mann (4) has derived an approximate relation connecting the ratio of pion-muon decay to kaon-muon decay to the pion-kaon mass ratio, using a model of a symmetric interaction with a symmetry-breaking term that transforms in a specified way.

The currents of the weak interactions are axial vector as well as vector; therefore the transformations that generate them do not merely shuffle the baryon fields among each other, as do the transformations of the higher symmetry schemes discussed earlier, but multiply some of the baryon fields by γ_5. We call such transformations "chiral transformations".

The second half of this thesis is a study of chiral symmetries. We will not develop this investigation as far as the investigation of ordinary higher symmetry schemes.

This is not because we can not perform the same calculations, but because there is no point to them. There is always the possibility that an ordinary higher symmetry scheme might be a good approximation if the baryon mass differences were neglected. A chiral symmetry can only hold if the baryon masses are neglected altogether; it is automatically a bad approximation. There is no purpose in finding those chiral symmetry schemes for which the eight baryons form an irreducible representation, determining the predictions of chiral symmetries about electromagnetic form factors, etc.

In everything that follows, we phrase our arguments in terms of quantized fields and invariant interaction Lagrangians. We emphasize that this is done for convenience's sake alone. Our conclusions, like all conclusions drawn from symmetry considerations alone, are independent of detailed dynamical assumptions. The discussion of invariant Yukawa coupling in Part IV could equally well be phrased in terms of three-point functions, or spin zero resonances in baryon-antibaryon scattering, or mesons identified with the divergences of baryon currents.

We now summarize the development of the remainder of the thesis.

In Part II we give some facts about compact continuous groups and develop some techniques for working with their low-lying irreducible representations. All of this material is in the mathematical literature in some form; we hope its presentation here is more suitable for physicists. A calculus is developed for manipulating low-lying representations. It is designed to imitate the familiar techniques of tensor and spinor analysis. Our methods are most useful when applied to representations of low dimension; they become increasingly awkward and inconvenient when extended to representations of higher dimension. Fortunately it is rarely necessary in our investigations to deal with supermultiplets of more than fifteen members.

In Part III we find all possible higher symmetry schemes for eight baryons. There are ten such; they are listed in Table I. In some of these schemes the underlying symmetry group is connected; we will be able to deal with these cases in a fairly straightforward manner. In other schemes, slightly more difficult to treat, the underlying symmetry group has several components. If we only consider the connected part, the eight baryons no longer form an irreducible representation of the group, but decompose into two quadruplets. (This is exactly like the decomposition of the Dirac bispinors into two two-component spinors if we consider the connected Lorentz group only.)

In Part IV we extend our formalism slightly and use it to predict the number of Bose particles that may be coupled bilinearly to the baryon field in an invariant manner. We formulate this discussion in terms of interaction Lagrangians and invariant Yukawa couplings, but, as explained earlier, all of our conclusions are consequences of symmetry only, independent of detailed dynamical assumptions.

In Part V we determine which higher symmetry schemes contain which other higher symmetry schemes. We obtain the important result that every eight-baryon scheme is either a generalization of unitary symmetry (11) or of minimal global symmetry (12). If, in any range of applications, the predictions of these two schemes are inaccurate, then the prediction of no other eight-baryon scheme can be better.

All of these calculations can be done as well for schemes involving other than eight baryons. We have performed them for from three to ten baryons. The arguments are so similar to those in the eight-baryon case that we have not written them out here, but the answers are presented in tabular form. The tables will be found at the end of the thesis.

Seven baryons (Table II) is of interest because the Λ

may be considered apart from the other baryons. Five baryons (Table IV) is of interest because of speculations (13) that the Σ may be a bound state of the Λ and the π. Three baryons (Table VI) is of interest because all the baryons can be constructed in principle from nucleons and Λ (or from Λ and Ξ). This is the principle of the Sakata model. The schemes examined here are generalized Sakata models (14, 15). To my knowledge, no one has seriously proposed higher symmetries involving six or four baryons (Tables III and V) but they are included here for the sake of completeness. (Of course, the doublet approximation (16) is a sort of four-baryon higher symmetry and in such guise it appears in Table VI.)

We also consider theories with nine or ten baryons (Tables VII and IX); the hope here is that some of the nine or ten baryon symmetry schemes will make predictions concerning the eight observed baryons so attractive that we will be willing to search for extra baryons to prove them true, or perhaps identify some spin $\frac{1}{2}$ baryon-meson resonances with the new particles. (A case where something like this has happened is the π^0 meson, predicted by charge independence from the existence of the charged π's.) Out of proper order, we might as well state here that the results of the investigation show the hope is vain.

Part VI uses the results in the tables to briefly compare some of the more interesting higher symmetry schemes with experiment.

In Part VII we begin the study of chiral transformations, using the methods developed in the first half of the thesis. We distinguish two important special cases. In one case, the group of chiral transformations factors into the product of two groups, one of which acts on the left-handed baryons alone, and the other of which acts on the right-handed baryons alone. We call this "free chiral symmetry". In the other case, the chiral transformations act on the left- and right-handed baryons together. The left-handed baryons form a basis for a representation of the chiral symmetry group which is either equivalent to the representation with the right-handed baryons for its basis, or equivalent to its conjugate. (This actually is what happens under Lorentz transformations.) We call this "bound chiral symmetry". These definitions are somewhat vague; we will be able to give sharper ones in Part VII, after we have developed more machinery.

Parts VIII and IX treat free and bound chiral symmetries, respectively. Construction of an invariant Yukawa interaction requires the introduction of both scalar and pseudoscalar mesons. Free chiral symmetries require an embarassingly large number of mesons. (The minimum number of mesons is the square of the number of baryons.) Bound chiral symmetries are much more attractive in that they require far fewer extra mesons. However, these considerations are not as compelling as they would be for ordinary higher symmetry schemes. Chiral symmetries are bound to be bad approximations; the symmetry-violating part of the interaction, which creates the baryon masses, might well also make the extra scalar mesons extremely unstable. Part X briefly discusses some of the more interesting chiral symmetries.

The appendices deal with points auxiliary to the main part of the text. Appendix I reviews briefly the connection between a Lie group, its algebra, and the conserved currents of a theory whose Lagrangian is invariant under the group. Appendix II is a directory of all representations of simple Lie groups with dimension less than sixteen. Appendix III proves a difficult theorem that is essential to the analysis of chiral symmetries given in Part VII. Appendix IV discusses some chiral symmetries that are neither free nor bound, and explains why we believe they will not be of much use in particle physics. Appendix V is on notation.

II. SOME FACTS ABOUT LIE GROUPS

Most of the material in this section comes from the standard texts (17, 18, 19) on Lie groups, but the presentation and some of the techniques have been suited to our special purposes.

In all that follows, we exploit a fundamental theorem on Lie groups: if G_o is a compact connected Lie group, it is locally isomorphic to a direct product of simple Lie groups. A simple Lie group is a continuous group with no continuous normal subgroup. All the simple Lie groups are known; there are three infinite families and six exceptional cases.

The three infinite families are:

(1) SU(n), $n \geq 2$, the group of all n x n unitary matrices with determinant one. SU(n) is called the special unitary group. The dimension of SU(n) is n^2-1.

(2) Sp(n), $n \geq 2$, the group of all 2n x 2n unitary matrices with determinant one that satisfy the equation

$$U^T \sigma U = \sigma , \tag{2-1}$$

where σ is the matrix,

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} , \tag{2-2}$$

and I is the n x n identity matrix. Sp(n) is called the symplectic group. It is isomorphic to the group of all unitary transformations on an n-dimensional quaternionic vector space. The dimension of Sp(n) is $2n^2 + 1$.

(3) SO(n), $n \geq 7$, the group of all n x n real orthogonal matrices with determinant one. SO(n) is called the rotation group. The dimension of SO(n) is n(n-1)/2. The sequence begins with SO(7) because SO(3) is locally isomorphic to SU(2), SO(4) is not simple (it is locally isomorphic to the direct product SU(2) $\times$ SU(2).), SO(5) is locally isomorphic to Sp(2), and SO(6) is locally isomorphic to SU(4).

We will only need two of the exceptional Lie groups in our investigation:

(1) U(1), the group of all complex numbers with modulus one, and

(2) G2, a subgroup of SO(7) that leaves invariant a certain antisymmetric trilinear form. It is possible to choose coordinates in the seven-dimensional vector space such that this form, which we will call E_{ijk}, equals +1 for (ijk) equal to (123), (145), (167), (246), (275), (365), and (374). The dimension of G2 is fourteen.

The four other exceptional Lie groups are too large to interest us. Their dimensions are 52, 78, 133, and 248.

Sometimes we will use the notation U(n) for the group of all n x n unitary matrices. U(n) is isomorphic to U(1) X SU(n).

If G_o is locally isomorphic to a direct product of simple Lie groups, any irreducible representation of G_o must be equivalent to a direct product of irreducible representations of its factors. Thus we are interested in methods of finding the irreducible representations of simple Lie groups. The method we will describe is a straightforward generalization of the construction of the representations of SO(3) as irreducible tensors. (E.g., spin zero is a scalar; spin one is a vector; spin two is a traceless symmetric tensor; etc.) In principle, it is capable of yielding all the representations of any simple Lie group.* It is by far the simplest and most direct method of extracting information about the representations of low dimension. There exist methods of greater complexity that simplify the treatment of representations of high dimension, but they require considerably more labor to develop, and are not necessary for our purposes.

We begin with SU(n). We already introduced an n-dimensional representation of SU(n) when we defined the group. We call the basis vectors for this representation

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* The exceptions are the two-valued "spinor" representations of SO(n). These will be treated separately.

simply "vectors". We indicate the components of a vector by a roman letter with a superscript,

$$x^i ,$$

with i understood as running from 1 to n. The complex conjugate vectors, $(x^i)^*$ form the basis for the conjugate representation. We use a notation that mimics that of ordinary tensor analysis, and indicate the components of a conjugate vector by a roman letter with a subscript,

$$x_i .$$

The notation is in accord with that of tensor analysis in that $x^i y_i$ (sum on i understood) is an invariant form. The corresponding form constructed from two vectors, or from two conjugate vectors, is not invariant. We may form tensors with arbitrary numbers of upper and lower indices by taking direct products of vectors and conjugate vectors. Just as in ordinary tensor analysis, we may impose symmetry conditions among the upper indices and among the lower indices, but not between upper and lower indices. Likewise, we may invariantly sum an upper and a lower index, but not two upper indices nor two lower indices.

The basis vectors for the irreducible representations of SU(n) are formed by the irreducible tensorial sets:

The representation of lowest dimension is the scalar representation, of dimension one.

The next representation has the vectors for its basis; it is of dimension n. The conjugate representation has the conjugate vectors for its basis.

The next representation has the antisymmetric tensors with two upper indices for its basis. It is of dimension $n(n-1)/2$. The conjugate representation has the antisymmetric tensors with two lower indices for its basis.

Likewise, the symmetric tensors with two upper indices form the basis for a representation of dimension n^2-1, which is equivalent to its conjugate.

The higher representations correspond to tensors with more indices and more complicated symmetry properties, but we will not need to use any of these, except for the representations of SU(2), which should be familiar to the reader.

We designate the representations of SU(n) as scalar, vector, symmetric tensor, etc. Sometimes we will make an exception for SU(2) and, yielding to tradition, designate the representations by their spin, as (0), ($\frac{1}{2}$), (1), etc.

SO(n) may be considered as a subgroup of SU(n) that leaves invariant the symmetric tensor δ_{ij}. We can use this tensor to lower all indices and also to sum over lower indices. We can obtain the tensorial representations of SO(n) in the same manner we obtained those of SU(n). The ones of lowest dimension are scalar (dimension one), vector (dimension n), antisymmetric tensor (dimension n(n-1)/2), and symmetric traceless tensor (dimension $(n^2+n-2)/2$). All of these are equivalent to their complex conjugates.

G2 is a subgroup of SO(7). However, it possesses an invariant antisymmetric tensor, E_{ijk}. We can use E_{ijk} to split the general antisymmetric tensor, b_{ij}, into two parts,

$$b_{ij} = E_{ijk}b_k + \hat{b}_{ij} , \qquad (2\text{-}3)$$

where $\hat{b}_{ij}$ is an antisymmetric tensor that obeys the equation,

$$E_{ijk}\hat{b}_{jk} = 0 . \qquad (2\text{-}4)$$

The tensors of the form b_{jk} form the basis of a representation of G2 of dimension 14. The other low representations of G2 can not be reduced with the aid of E_{ijk}; they are the same as those of SO(7).

The situation for Sp(n) is the same as that for SO(n), except that instead of an invariant symmetric tensor δ_{ij}, there is an invariant antisymmetric tensor σ_{ij} (the matrix σ introduced above). We can use σ, just as we used δ, to lower all upper indices and to sum over lower indices. The lowest irreducible representations of Sp(n) are scalar (dimension one), vector (dimension 2n), antisymmetric "traceless"*

* In this case, of course, when we say b_{ij} is "traceless", we mean $\sigma_{ij}b_{ij} = 0$.

tensor (dimension $2n^2 - n-1$), and symmetric tensor (dimension $n(2n+1)$).

The two-valued representations of SO(n) can not be constructed by our tensorial methods. We construct them by a method which is closely analogous to that used by Dirac to construct the spinor representations of the Lorentz group.

We define the Clifford algebra of order $(2\nu+1)$ as a set of $(2\nu+1)$ matrices Γ_i that obey the anticommutation rules,

$$\{ \Gamma_i , \Gamma_j \} = 2\delta_{ij} \qquad (2\text{-}5)$$

The Clifford algebra of order three is the Pauli spin matrices. The Clifford algebra of order five consists of the Dirac matrices α_x, X α_y, α_z, β, and γ_5. It is easy to see that the matrices in the Clifford algebra must be at least $2^\nu \times 2^\nu$. It is also easy to construct the Clifford algebra out of Pauli spin matrices. Consider direct products of Pauli matrices of the form

$$(I)^r \times \sigma_x \times (\sigma_z)^{\nu-r-1} ,$$

where I is the 2 x 2 identity matrix, and the exponentiation symbolizes repeated direct products. There are ν such matrices. Likewise there are ν matrices of the same form, but with σ_y replacing σ_x. If we adjoin to these the direct product of ν σ_z's, we have a set of $(2\nu+1)$ Hermitian $2^\nu \times 2^\nu$ matrices; it is trivial to verify that they obey the anticommutation rules.

We can use the Clifford matrices to construct the lowest two-valued representation of SO(n). If ψ is a 2^ν component vector ("a spinor"), we represent the infinitesimal transformation

$$x_i \longrightarrow x_i + \epsilon_{ij} x_j \tag{2-6}$$

in $SO(2\nu+1)$ by the transformation

$$\psi \longrightarrow \psi + \tfrac{1}{4}\, \epsilon_{ij}\, \sigma_{ij}\, \psi \tag{2-7}$$

in spinor space, where

$$\sigma_{ij} = \tfrac{1}{2}\left(\Gamma_i \Gamma_j - \Gamma_j \Gamma_i\right). \tag{2-8}$$

It is easy to verify that this generates a representation of $SO(2\nu+1)$ which is two-valued. This is the lowest two-valued. representation. For $\nu \geq 3$ (the only cases we need consider), the other two-valued representations are of too high a dimension to interest us.

We can construct two-valued representations of $SO(2\nu+2)$ in the same spinor space, simply by following the prescription above and defining

$$\sigma_{i,\, 2\nu+2} = - \sigma_{2\nu+2,i} = \pm i \Gamma_i \quad . \qquad (2\text{-}9)$$

The $\pm$ sign lead to **inequivalent** representations; they are complex conjugates.

G2 does not have any two-valued representations.

Appendix II is the **fruit** of this section. It lists all representations of simple Lie groups with dimension less than sixteen.

III. THE EIGHT BARYON PROBLEM

We now want to use the methods of the previous section to solve the problem formulated in the introduction, and find all higher symmetry schemes for eight baryons. We need some notation: Let G be the group of symmetries and D(g) its irreducible eight-dimensional representation. There is no reason why G should be connected; it may have several components. Let G_o be the component of the identity, the subgroup that consists of all group elements that can be connected to the identity by a continuous curve. The restriction of D(g) to G_o forms a representation of G_o, but it may not be irreducible.

Such a situation exists in the case of Lorentz invariance. The Dirac bispinors form the basis for an irreducible four-dimensional representation of the full Lorentz group. However, if we just consider the component of the identity--those Lorentz transformations with determinant one that do not reverse the direction of time--the four-dimensional representation reduces to the sum of two two-dimensional ones.

Let H be the eight-dimensional vector space that forms the basis for D(g) and let us suppose it decomposes into several invariant subspaces.

$$H = H_1 + H_2 + \dots H_n , \qquad (3\text{-}1)$$

where, if g_o is in G_o, then

$$D(g_o)\, H_i = H_i \; . \qquad (3\text{-}2)$$

Let g' be an element of G. Then $g'^{-1}\, g_o\, g'$ is in G_o, and

$$D(g'^{-1}\, g_o\, g')\, H_i = H_i \; ,$$

or

$$D(g')^{-1}\, D(g_o)\, D(g')\, H_i = H_i \; ;$$

thus,

$$D(g_o)\, D(g')\, H_i = D(g')\, H_i \; .$$

This implies that

$$D(g')\, H_i = H_j \qquad (3\text{-}3)$$

where i may or may not equal j. However, given any i and j, we must be able to find a g' such that the above equation is true; otherwise $D(g)$ would not be irreducible. Equation 3-3 and the unitarity of $D(g')$ imply

$$\dim H_i = \dim H_j \; . \qquad (3\text{-}4)$$

Thus H may decompose into two four-dimensional subspaces, or eight one-dimensional subspaces. However, we know the exact symmetries of the strong interactions are in G_o, and these have a three-dimensional invariant subspace (Σ). Thus if the representation of G_o is not irreducible, the eight baryons must decompose into two quadruplets, (of which contains Λ and the Σ's, the other, nucleo1 and Ξ's.

We are now ready to decompose G_o into a a1rect product of simple Lie groups. For simplicity, in everything that follows, we neglect factors of U(1). We know G_o must contain at least one factor of U(1), corresponding to transformations that multiply all eight baryons by the same phase factor. Such transformations lead to baryon conservation. If the representation of G_o is not irreducible, there may be two such factors, each of which leads to baryon conservation within each quadruplet. Likewise, when we speak of the exact symmetries in the sequel, we will mean only hypercharge and isospin, not baryon number.

At this point, reading dimensions from Appendix II, we have the following possibilities:

I. The representation of G_o is irreducible.

a. G_o is simple. G_o can be SU(2), SO(7), SO(8) (in either its vector or its spinor representation), SU(8), SU(3), or Sp(4).

b. G_o is a direct product. G_o can be SU(2) x SU(2), SU(2) x Sp(2), SU(2) x SU(4), or SU(2) x SU(2).

II. The representation of G_o is the sum of two four-dimensional representations.

a. G_o is simple. G_o can be SU(2), Sp(2), or SU(4).

b. G_o is a direct product.

1. Every factor of G_o is represented non-trivially in each of the four dimensional representations. G_o is SU(2) x SU(2).

2. Some factor of G_o is represented trivially in one of the irreducible representations. Then an isomorphic factor must be represented trivially in the other one. G_o can be SU(2) x SU(2) x SU(2), Sp(2) x Sp(2), SU(4) x SU(4), or SU(2) x SU(2) x SU(2) x SU(2).

We will now examine these cases, and apply to each of them the one criterion we have not yet used: is it possible to imbed the exact symmetries of the strong interactions in the representation?

Ia. SU(2) is a three parameter group, and the exact symmetries form a four parameter group, so they can not be imbedded in SU(2).

The exact symmetries are all defined in terms of unitary matrices, so they are all contained in the vector representation of SU(8).

Likewise, if we use the fields, Λ, Σ, $p + \Xi^-$, $i\ (p - \Xi^-)$, $n + \Xi^0$, and $i\ (n - \Xi^0)$, the exact symmetries are all defined in terms of real orthogonal matrices, and are thus contained in the vector representation of SO(8).

If SU(3) contains the exact symmetries in such a way that the vector representation contains a hypercharge zero isodoublet and a hypercharge one isosinglet, then the space of all mixed tensors (nine objects) will contain hypercharge plus and minus one isodoublets, two hypercharge zero isosinglets, and a hypercharge zero isotriplet. The space of all traceless mixed tensors has one of the isosinglets missing, giving just the right isospin and hypercharge distribution. This is the unitary symmetry scheme of Gell-Mann (11).

SO(8) contains SO(3) x SO(5), which is locally isomorphic to SU(2) x Sp(2). The spinor representation of SO(8) must be double-valued both under rotations in SO(3) and under those in SO(5); thus it must become the product of the vector representation of SU(2) and the vector representation of Sp(2). We will show below that these contain the exact symmetries.

SO(7), in the same manner, contains SO(3) x SO(4), which is locally isomorphic to SU(2) x SU(2) x SU(2). The spinor representation of SO(7) must be double-valued both under rotations in SO(3) and under rotations in SO(4); therefore, as above, it must become the product 3-spinor x 4-spinor, or, in terms of SU(2), (vector x vector x scalar) x (vector x scalar x vector), We will show below that this contains the exact symmetries. Lagrangians possessing the symmetry of SO(7) have been suggested by Tiomno (20) and Dallaporta (21).

We will now show that the exact symmetries can not be imbedded in Sp(2). Sp(2) is defined as the group of all 4 x 4 unitary matrices satisfying the equation

$$U^T \sigma U = \sigma . \qquad (2-1)$$

Let us choose a set of basis vectors such that the exact symmetries of the strong interactions are all represented by real matrices. For these symmetries,

$$\sigma U = U \sigma . \qquad (3-5)$$

If ψ_Λ is the Λ field, this implies

$$\sigma\ \psi_\Lambda = \lambda\ \psi_\Lambda \quad , \tag{3-6}$$

where λ is a complex number of modulus one. Now, in this set of basis vectors, σ may not be represented by the same matrix as in equation 2-2, but it must still be antisymmetric, and it must still have determinant one. Antisymmetry implies

$$\lambda = 0 \quad . \tag{3-7}$$

Hence $$\det \sigma = 0 \quad , \tag{3-8}$$

a contradiction.

Ib. Assume G_o is SU(2) x SU(2), where the first factor has the four-dimensional representation. Consider the generator of isotopic rotations, $\vec{T}\cdot\vec{T}$ must be expressible as a sum of elements of the algebras of the two factors. If the proper commutation rules are to be obeyed, then $\vec{T}$ must be $\vec{T}_1$, $\vec{T}_2$, or $\vec{T}_1 + \vec{T}_2$, where $\vec{T}_1$ is the vector formed from the basis elements of the algebra of the first factor, and $\vec{T}_2$ that formed from those of the second factor. But if $\vec{T}$ is $\vec{T}_1$, all the isotopic multiplets occur in pairs; if it is $\vec{T}_2$, they all occur in quadruplets; and if it is $\vec{T}_1 + \vec{T}_2$, the eight baryons break up into an isotopic quintuplet and an isotopic triplet.

If G_o is SU(2) x SU(2) x SU(2), exactly parallel arguments to those above lead to the conclusion that either all the isotopic multiplets occur in pairs, or there exist particles with isospin 3/2.

We may write the eight baryons as a product of a quadruplet and a doublet, where the doublet is an isospinor with hypercharge 0, and the quadruplet is an isospinor with hypercharge 0 and two isosinglets with hypercharge $\pm$ 1. The exact symmetries acting on the quadruplet are symplectic matrices, so the exact symmetries may be imbedded in SU(2) x Sp(2). This is contained in SU(2) x SU(4), so the exact symmetries may also be imbedded in this group. This is the symmetry of the original globally-symmetric pion-nucleon coupling (2).

II a. SU(2), Sp(2) and SU(4) each have only one four-dimensional representation. So the two invariant subspaces must each break up into the same pattern of isotopic multiplets -- this is not what happens.

II b 1. If G_o is SU(2) x SU(2), we can not use the four-dimensional representation of SU(2), because this leads to isotopic quadruplets. So each factor must be represented by spinors, and the same arguments apply as in the preceding paragraph.

II b 2. One of the invariant subspaces must contain Λ and the Σ's. Argument parallel to those at the end of I a. show that the exact symmetries of these particles can not be imbedded in a representation of Sp(2), so G_o can not be Sp(2) x Sp(2).

On the other hand, the exact symmetries are certainly all unitary, so G_o can be SU(4) x SU(4).

If G_o is SU(2) x SU(2) x SU(2) x SU(2), let us choose the ordering so an element not in the identity exchanges the first and second factors, and also the third and fourth. If we consider Λ and the Σ's as the product of two hypercharge 0 isospinors (first and third factors), and nucleons and Ξ's as the product of a pair of hypercharge ± 1 isosinglets (second factor) and a hypercharge 0 isospinor (fourth factor), we have imbedded the exact symmetries in this group.

Let us write the general element of G_o above as (g_1, g_2, g_3, g_4), where g_i is an element of SU(2). If we consider the subgroup of all elements of the form (g_1, g_2, g_3, g_3), it is isomorphic to SU(2) x SU(2) x SU(2), and still contains the exact symmetries. This is the "minimal global symmetry group" of Yang and Lee (12).

The first three columns of Table I summarize the results of this part. I would like to emphasize in just what sense we have found "all possible symmetries" of the eight baryons. We have found all possible components of the identity, G_o. (With the neglect of factors of U(1); see the remarks earlier in this section.) We have not investigated the possibility of adjoining additional discrete symmetries. Such possibilities do exist; for example, in the symmetry scheme based on SU(3) (unitary symmetry), there is a discrete symmetry, called R by Gell-Mann (11), and hypercharge reflection by Sakurai (22), that may or may not be adjoined to the group.

For those symmetry schemes in which the representation of G_o is reducible, we know that there must be at least one discrete element in G that has the effect of exchanging two of the isomorphic components of G_o, but beyond this, we know little of its nature, and nothing of the possible existence of other discrete elements.

The third column lists the dimension of G_o (the number of linearly independent elements in its algebra). Since every infinitesimal transformation yields a conserved current (see Appendix I), this is also the number of conserved currents in the theory.

IV. YUKAWA COUPLINGS AND FORM FACTORS

We would like to consider the possible ways in which we may couple mesons (scalar, pseudoscalar, vector, or axial vector) in the Yukawa manner, to our eight baryons, without destroying our symmetry. For simplicity we consider the scalar source

$$\overline{\psi}_i \ \psi_j \qquad (i,\ j = 1,\ \cdots 8.);$$

the other three sources transform in the same manner. This source forms a basis for the 64 dimensional product representation $\overline{D}(g) \times D(g)$. In general, this representation is not irreducible; we may break it up into its irreducible parts,

$$\overline{D}(g) \times D(g) = D^{(1)}(g) + D^{(2)}(g) + \cdots .$$

If we introduce a set of meson fields that form a basis for one of the representations $D^{(i)}(g)$, we may couple these fields symmetrically to the Yukawa source. The number of fields required is dim $D^{(i)}$. If $D^{(i)}$ can be written in real form, this is also the number of mesons needed. If $D^{(i)}$ can not be written in real form, the meson fields must be complex, and 2 dim $D^{(i)}$ mesons are needed, the real and the imaginary parts of the fields. In the table we list the minimum number of mesons required, after one meson. (We can always couple a single meson to $\overline{\psi}_i \psi_i$.) This is the smallest dim $D^{(i)}$. It turns out, in our cases, that the $D^{(i)}$ of smallest dimension can always be written in real form.*

Of course, all of this is quite independent of our formulation of the problem in terms of Yukawa coupling. Instead of speaking of fields and Lagrangians, we could just as well have talked of nucleon-antinucleon resonances and three-point functions and have obtained the same results. Conclusions drawn from symmetry conditions alone are independent of the detailed dynamics.

This point is not as widely appreciated as it should be. Gursey has invented a theory which requires the introduction of an extra scalar meson+, in addition to the known pseudoscalar ones (23). He then tries to remove the extra meson by writing its field as a function of the fields of the pseudoscalar mesons in a way that preserves the symmetry group. Thus he removes the unwanted meson at the cost of

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* It is possible to prove that this is always the case. The proof is both tedious and unenlightening, and will not be given here.

\+ This is a chiral symmetry; elements of the symmetry group exchange scalar and pseudoscalar mesons. It will be discussed in Part IX.

introducing a nonlinear coupling. This is clearly a fallacy: if the theory is symmetric under Gursey's group, and if there are resonances in the nucleon-antinucleon system corresponding to the known pseudoscalar mesons, then there must be a resonance corresponding to the extra meson. (This is under the assumption that the symmetry is not too badly broken. If it is badly broken, then the extra meson might not be observable, even if it had been introduced as a dynamically independent field. In either case, there is no reason to introduce non-linear coupling.)

The actual calculation of the decomposition by our tensor methods is very straightforward. Rather than explaining the method generally, we give two examples.

(1) The eight-baryon theory based on SU(8). The direct product is the space of all mixed tensors. It decomposes into the space of all traceless mixed tensors (dimension 63) and the space of all scalars (dimension 1).

(2) The eight-baryon theory based on SU(3). The baryons transform like traceless mixed tensors ψ^i_j. The Yukawa source is $\overline{\psi}^k_\ell \psi^i_j$. If we have a set of eight mesons ϕ^i_j, which also transform like traceless mixed tensors, then we may couple the mesons to the baryons by

$$L_I = g \; \overline{\psi}^k_j \; \psi^i_k \; \phi^j_i \, . \tag{4-1}$$

In this case we can also couple the same eight meson to the baryons in a completely independent way, through

$$L_I = g \quad \overline{\psi}^{\,i}_{\,k} \; \psi^{\,k}_{\,j} \; \phi^{\,j}_{\,i} \quad . \qquad (4\text{-}2)$$

Note that it is not necessary to completely decompose the product, merely to find which of the low-lying representations occur in it.

Also note that it is not necessary to trace through the Yukawa coupling in order to determine the isotopic content of the meson fields. This is determined simply by the representation and the way in which the exact symmetries are imbedded in G_o, properties which were found in Part III. Thus in the unitary symmetry scheme we know that the eight mesons must decompose into an isoscalar, two isospinors, and an isotriplet, just because they belong to the mixed tensor representation of SU(3).

The original eight-fold way involved the pseudoscalar equivalent of this coupling, with the mesons identified as the π's, the K's and a hypothetical meson called χ^o. It has been suggested that the recently discovered resonance in the 3π system at 550 mev (<u>24</u>) might be χ^o.

We should remark that the mesonic Yukawa interaction may often have a higher symmetry than the original group. Thus if we start off with a theory of eight baryons which are a basis for a representation of SU(2) x Sp(2), and couple three mesons in a symmetric manner, we find that the resultant Lagrangian has the symmetry of SU(2) x SU(4). We must be careful when we make a prediction on the basis of some symmetry scheme, to be sure the prediction is made on the basis of the symmetry scheme alone, and not on some particular Lagrangian, which may possess higher symmetries than those we know. (In the case above, we may remove the "unwanted" symmetry by coupling, let us say, ten vector mesons to the Yukawa source.)

The decomposition of products of spinor representations does not follow from tensor methods; however, it can be obtained directly by analogy with the familiar properties of the Dirac wave functions. As usual, we must treat SO(2ν + 1) and SO(2ν + 2) separately. We begin with SO(2ν + 1). If we call the fundamental spinor representation ψ , we can decompose the direct product (spinor) x (spinor) into the forms

$$\psi^+\psi, \quad \psi^+\Gamma_i\psi, \quad \psi^+[\Gamma_i,\Gamma_j]\psi,$$

$$\psi^+\{\Gamma_i,[\Gamma_j,\Gamma_k]\}\psi, \qquad \text{etc.}$$

The same arguments involving the generators of the rotation group that are used in Dirac theory show here that these transform, respectively, like scalar, vector, antisymmetric tensor with two indices, antisymmetric tensor with three indices, etc.

As we demonstrated in Part II, we can use the same spinors and Cayley matrices to form a representation of $SO(2\nu + 2)$. Here we have two inequivalent cases, (spinor) x (spinor) and (spinor) x (conjugate spinor). In either case we can make the decomposition shown above. However, for (spinor) x (conjugate spinor), the forms after the scalar have to be grouped together pairwise to make scalar, antisymmetric tensor with two indices, etc. Likewise, for (spinor) x (spinor), the forms have to be grouped pairwise, beginning with the first two, to form vector, antisymmetric tensor with three indices, etc.

Thus, in the eight-baryon theory based on the spinor representation of SO(7), we can couple seven mesons to the Yukawa source, while in that based on the spinor representation of SO(8) we need 28 mesons.

Electromagnetic Form Factors

We can use techniques very similar to the ones above to calculate the electromagnetic form factors. To lowest order in electromagnetism, the electromagnetic form factor is linearly related to the matrix element of the electric current between single nucleon states, calculated in the absence of electromagnetic interactions.

Charge conservation is one of the symmetries we have taken care to include in our group, so the electric current is associated with some element of the algebra of G_o.

Now if a is an element of the algebra of G_o, and g is an element of G, then $g^{-1} a g$ is also an element of the algebra. That is to say, the elements of the algebra (and hence the associated currents) form the basis for a representation of G, called the adjoint representation. If G_o is simple, the adjoint representation is irreducible; otherwise, it may or may not be reducible.

If G_o is SU(n), the adjoint representation is the mixed traceless tensor representation. If G_o is SO(n), it is the antisymmetric tensor representation. If G_o is Sp(n), it is the symmetric tensor representation. If G_o is U(1), it is the trivial representation. If G_o is G2, it is the representation of dimension 14.

To find the possible electromagnetic form factors, we must:

(1) Find the adjoint representation;

(2) Decompose it into irreducible parts;

(3) Discard those irreducible parts that make no contribution to the electric current; and

(4) Decompose the representation of the current $\bar{D}(g) \times D(g)$ into its irreducible parts.

Then, we may couple each irreducible part of the adjoint representation to the corresponding irreducible part of the product representation. Each such possible coupling is a possible contribution to the form factor. (Actually, a possible contribution to either the charge or moment form factors.)

We list the number of independent form factors in the last column of our tables, and do three cases explicitly here, to demonstrate the technique. All of these are eight-baryon theories.

(1) Theory based on SO(7). The adjoint representation is the antisymmetric tensor representation. H occurs once in the direct product: there is one independent form factor. That is to say, all form factors are proportional to the charge.

(2) Theory based on SU(2) x SU(2) x SU(2) (minimal global symmetry). We label the representations of G_o (j_1, j_2, j_3), where the j's are the "angular momenta". The baryons transform according to $(\frac{1}{2}, \frac{1}{2}, 0) \times (\frac{1}{2}, 0, \frac{1}{2})$.

(Contrary to appearances, this is an irreducible representation of the full group, because the full group contains a discrete element that exchanges the second and third factors of G_o.)

The adjoint representation is

$$(1,0,0) \; + (0,1,0) + (0,0,1) \quad .$$

This is the sum of two irreducible parts (indicated by the brackets). Each part contributes to the electric current. The product representation is

$$(1,0,0) + (1,0,0) + (0,1,0) + (0,0,1) + \cdots ,$$

where we only write out the part of the sum that is of interest. This contains one of our irreducible representations twice and the other one once; there are thus three independent form factors.

Since there are three independent form factors, but nine experimentally measurable (in principle!) form factors (the eight baryons and the $\Sigma^o \Lambda$ cross term), we can find six equations between these factors. All we have to do is write out the representations explicitly in terms of the component baryons. We express the results in terms of magnetic moments, although they are true for all form factors at all energies.

$$\mu(\quad) = 0,$$

$$\mu(\quad^{o}) = 0,$$

$$\mu(\quad^{+}) + \mu(\quad^{-}) = 0,$$

$$\mu(\quad^{o}) + \mu(n) = 0, \qquad (4\text{-}4)$$

$$\mu(\quad^{-}) + \mu(p) = 0,$$

$$\mu_m = \mu(\quad^{+}) - (\mu(p) + \mu(n)) \quad ,$$

where μ_m is the mixed $\quad^{o}$ moment, responsible for decay.

(3) Theory based on SU(3) (unitary symmetry). The adjoint representation is the mixed traceless tensor representation. The direct product contains this twice; once as $\overline{\Psi}^{i}_{k} \Psi^{k}_{j}$ and once as $\overline{\Psi}^{k}_{j} \Psi^{i}_{k}$. If Q^{i}_{j} is the matrix that generates the electric current, then the most general form for the moment form factor is

$$a \, \overline{\Psi}^{i}_{k} \Psi^{k}_{j} Q^{j}_{i} + b \, \overline{\Psi}^{k}_{j} \Psi^{i}_{k} Q^{j}_{i} \quad ,$$

where we have omitted gamma matrices. This is the expression found by Coleman and Glashow (25).* All of the form factors may be found in terms of those of the neutron and proton.

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* The techniques used in this section are an attempt to generalize the methods of this paper.

$$\mu(\quad{}^{+}) = \mu(p),$$

$$\mu(\quad) = \tfrac{1}{2}\mu(n),$$

$$\mu(\quad{}^{0}) = \mu(n),$$

$$\mu(\quad{}^{-}) = \mu(\quad{}^{-}) = -\quad \mu(p) + \mu(n) \quad , \tag{4-5}$$

$$\mu(\quad{}^{0}) = \tfrac{1}{2}\mu(n),$$

$$\mu_m = \tfrac{1}{2}\quad 3\mu(n).$$

There is one final remark to be made about the calculation of moments. In doing the eight-baryon problem, we factored out of G_o a factor of U(1) that led to nucleon number conservation. We have ignored this factor in the discussion of moments. This is unobjectionable in the eight-baryon case, because the total charge of the eight baryons is zero, and nucleon number makes no contribution to the charge. This is not the case for some of the other symmetry schemes listed in the tables; for these we must include the factor of U(1), which leads to one more independent form factor.

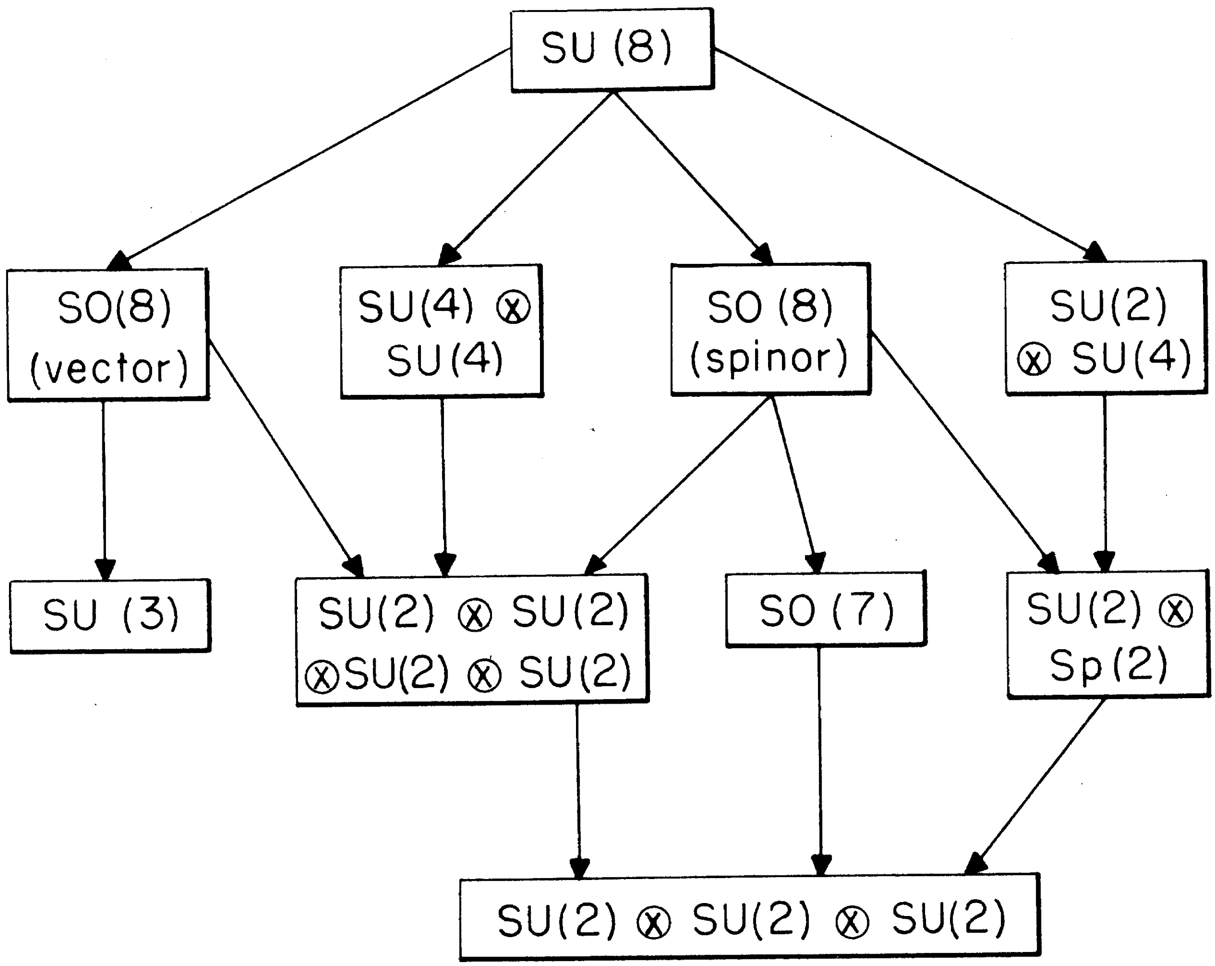

Figure 1 Hierarchy of the eight-baryon higher symmetry schemes. Note that every scheme is a generalization of either unitary symmetry or minimal global symmetry.

V. CONNECTIONS

We say one symmetry scheme contains another if the group of the first scheme contains the group of the second as a subgroup in such a way that the representation of the second group is obtained by restricting the representation of the first group to the subgroup. A more physical way of stating the relation is to say that the first symmetry scheme is a more symmetric generalization of the second, but, of course, the transition from the more symmetric scheme to the less is easier to prove.

Figure I shows the complete set of containment relations among the eight-baryon symmetry schemes. The most significant fact shown in the figure is that every eight-baryon symmetry scheme is a generalization either of the one based on SU(3) (unitary symmetry) or of the one based on SU(2) x SU(2) x SU(2) (minimal global symmetry). We will now prove the relations shown in the figure. In the sequel, "SU(8)" is used as an abbreviation for "the eight-baryon symmetry scheme for which G_o is SU(8)", etc.

It is clear that SU(8) contains all symmetries. Likewise SO(8) (vector) contains all symmetries that can be written in real form.

In Part III we showed that SO(8) (spinor) contained SU(2) x Sp(2). We can argue similarly that it contains SO(4) x SO(4), which is locally isomorphic to SU(2) x SU(2) x SU(2) x SU(2). Since the representation of SO(8) must be two-valued under rotations of either factor of SO(4), it must become (scalar x vector x scalar x vector) + (vector x scalar x vector x scalar), any other combination leading either to too high a dimension or to a loss of the symmetry between the two factors of SO(4).

Likewise, SO(8) contains SO(7), and the spinor representation of SO(8) must become a two-valued representation of SO(7). The only possibility is the lowest spinor representation.

We have shown in Part III that SO(7) contains SU(2) x SU(2) x SU(2); also that SU(2) x SU(4) contains SU(2) x Sp(2).

SU(4) contains SO(4) in such a way that the vector representation of SU(4) becomes the vector representation of SO(4), or the vector x vector representation of SU(2) x SU(2), locally isomorphic to SO(4). Thus SU(4) x SU(4) contains SU(2) x SU(2) x SU(2) x SU(2).

Sp(2) contains Sp(1) x Sp(1), which is isomorphic to SU(2) x SU(2). The vector representation of Sp(2) becomes the (vector x scalar) + (scalar x vector) representation of

SU(2) x SU(2). Thus SU(2) x Sp(2) contains the minimal global symmetry group.

We have shown in Part III that SU(2) x SU(2) x SU(2) x SU(2) contains SU(2) x SU(2) x SU(2).

We have now explained all the arrows that appear in Figure I; we still have to explain the arrows that do not occur there, by showing that those symmetry schemes not joined by arrows do not contain each other. We can take care of most of the work with three trivial observations: a theory with less currents can not contain a theory with more; a theory with lower meson number can not contain a theory with greater; a theory in which G_o is represented reducibly can not contain a theory in which G_o is represented irreducibly.

For the rest:

SO(8) (vector) can not contain any theory that can not be written in real form.

SO(8) (spinor) is a scheme in which it is possible to couple 28 meson symmetrically to the Yukawa source. These mesons include two isotopic quadruplets. Thus this scheme can not contain SU(2) x SU(4), in which the lowest irreducible component of the product which contains isotopic quadruplets is of dimension 45.

Likewise, SO(7) (spinor) is a scheme in which it is possible to couple seven mesons symmetrically to the Yukawa source. These are an isotriplet and two isodoublets. Thus this can not contain SU(2) x SU(2) x SU(2) x SU(2), in which isotriplets always come in pairs.

VI. DISCUSSION OF HIGHER SYMMETRY SCHEMES

We will begin by considering the eight-baryon schemes, and then discuss some of the other possibilities in the tables. Since we have just shown that every eight-baryon scheme contains either minimal global symmetry or unitary symmetry, it suffices to examine the consequences of these theories and their simpler generalizations.

Unitary symmetry requires a minimum of eight pseudoscalar mesons to construct an invariant Yukawa interaction. These eight have the right isospin and hypercharge assignments to be the pions, the kaons, and the recently observed three-pion resonance (24). If this assignment is correct, unitary symmetry is clearly a very bad approximation (at least at low energies (26), because of the very large difference between K and π coupling constants and masses. The eight observed (possible) vector mesons mentioned in the introduction fit nicely into this scheme; they may be coupled to the baryons in precisely the same manner as the pseudoscalar mesons.

Global symmetry, on the other hand, has the possibility of being a reasonable low energy approximation, since the pions, the most strongly interacting particles, form a supermultiplet by themselves. The scheme has been applied to the analysis of several experiments (27, 28) and does not yield results inconsistent with the data. If we wish to couple the eight vector mesons in a symmetrical way, we may choose the symmetry scheme based on SU(2) x Sp(2). We can then couple ten mesons that transform according to the symmetric tensor representation of Sp(2) to the factor of Sp(2). Eight of these have the desired isospin-hypercharge assignments; the other two are singlets with hypercharge plus and minus two. Let us call these hypothetical particles G^+ and G^-. G^+ could be produced in the scattering of K^+ off nuclei, that is to say, in the reaction

$$K^+ + n \longrightarrow G^+ + \Lambda .$$

G^+ could also be produced in the reaction

$$\pi^o + p \longrightarrow G^+ + \Xi^o ,$$

which could occur virtually in high energy nucleon-nucleon scattering. If G were lighter than two K's, it would be stable under the strong interactions and easily detectable by its weak decay into π and K. If it were heavier than two K's, it could decay strongly into two K's in an $I = 0$, $J = 1$ state. Presumably this could be detected by the same methods used to detect the K*(7). (The group Sp(2) was first introduced by Salam and Ward (29), in a somewhat different context.)

As for electromagnetic processes, the Λ moment is currently being measured by Cool et. al. at Brookhaven. Global symmetry predicts that this should be zero; unitary symmetry that it should be one half the neutron moment. Preliminary results (30) indicate that it is on the order of minus one, in agreement with the predictions of unitary symmetry. Unfortunately, there is no reason why this particular prediction of unitary symmetry should be taken seriously. If we examine, in terms of diagrams, the contributions to the Λ moment in unitary symmetry, we find that the main contribution comes from two K intermediate states. The large mass of the K, relative to the π, and the small K coupling constant, should reduce these terms considerably.

We can also predict the electromagnetic mass splittings within the baryon multiplets, in the neglect of the symmetry-breaking interactions. We have not developed systematic methods here. Unitary symmetry predicts only the Ξ mass splitting in terms of the other splittings (25). The result is in agreement with (none too accurate) experiment. Minimal global symmetry, predicts, among other things, that the charged Σ mass difference should vanish. This is easy to see; the minimal global symmetry group contains a transformation that changes the fundamental fields as follows:

$$\Sigma^+ \longleftrightarrow \Sigma^-$$

$$\Sigma^0 \longleftrightarrow \Sigma^0$$

$$\Lambda \longleftrightarrow \Lambda$$

$$n \longleftrightarrow \Xi^0$$

$$p \longleftrightarrow \Xi^-$$

This completely reverses the sign of the electromagnetic current, so the electromagnetic mass shifts, which are even in e, should not be changed.

The remarks made about the accuracy of the magnetic moment predictions of unitary symmetry also apply to the mass shifts. We thus have a peculiar picture: unitary symmetry, which can not be taken seriously as a good approximation, predicts the electromagnetic properties of the baryons fairly well, while global symmetry, which might be a reasonable approximation, fails completely. Probably the accuracy of unitary symmetry is merely coincidental.

No improvement is obtained by studying symmetry schemes involving more baryons. The only nine-baryon scheme that does not have an exorbitantly large number of mesons is that based on SU(3) x SU(3), which contains unitary symmetry. The only ten-baryon schemes with reasonable meson numbers are that based on SU(2) x SU(5), which contains global symmetry, that based on SO(5) x SO(5), which also contains global symmetry, and that based on Sp(2). The same mesons can be coupled to this as were coupled to the factor of Sp(2) in the global symmetry variant studied above. Here there are two additional hypercharge two pseudoscalar mesons as well as two additional vector mesons. This scheme predicts all baryon moments to be proportional to the charge.

Among schemes involving lesser numbers of baryons, the only ones of interest are the seven-baryon scheme based on G2, the five-baryon scheme based on Sp(2), and the three-baryon scheme based on SU(3). G2 has been suggested by Behrends and Sirlin (31). It is contained in the eight-baryon scheme based on the spinor representation of SO(7). It predicts all baryon moments to be proportional to the charge. The five-baryon scheme based on Sp(2) has the same mesonic possibilities as the two other theories we have discussed using Sp(2). It also predicts all moments to be proportional to the charge. The three-baryon scheme based on SU(3) is the generalized Sakata model (14, 15). There are the same mesonic possibilities as in the eight-baryon scheme based on SU(3). This scheme predicts equality of the Λ and neutron moments. Since all of these schemes involve placing the pions and kaons in the same supermultiplet, their predictions about moments are as untrustworthy as those of the eightfold way.

There are two possible checks on a symmetry scheme which we have neglected. One is the correlation of baryon-meson resonances. Glashow and Sakurai (32) have studied such resonances using the coupling constants obtained from unitary symmetry but the correct particle masses. Their results indicate that, in this case at least, the particle mass differences seriously distort the results obtained from the symmetric interaction alone. The other is the prediction of high-energy scattering amplitudes. Gell-Mann and Zachariasen (26) have shown that, if the interaction is truly a Yukawa interaction mediated by elementary pseudoscalar fields, and if the symmetry is violated by mass differences only, then the broken symmetry will become visible at sufficiently high energies. There is, of course, no data available on high-energy hyperon-hyperon scattering. We should also remark that predictions of this kind are not merely a consequence of symmetry, but strongly dependent on the detailed dynamics.

VII. THE STRUCTURE OF CHIRAL SYMMETRIES

As explained in the introduction, this point marks a natural break in our investigation. Until now, we have been studying symmetries that shuffle the baryon fields among themselves. We shall refer to such symmetries as "ordinary symmetries" in what follows. Now we widen our field of interest to include also symmetries that multiply the baryon fields by γ_5 and which interchange scalar and pseudoscalar mesons. We call these "chiral symmetries".

A chiral transformation has the general form:

$$\psi \rightarrow (\mathcal{A} + \mathcal{B}\gamma_5)\,\psi\,, \qquad \phi \rightarrow C\,\phi\,. \tag{7-1}$$

in which $\mathcal{A}$, $\mathcal{B}$ are n x n matrices acting upon the n fermion fields ψ. C acts upon the mesons and may mix scalar with pseudoscalar mesons and vector with pseudovector mesons. The fermion part may be rewritten:

$$\psi \rightarrow (Aa + B\bar{a})\,\psi\,. \tag{7-2}$$

where a and $\bar{a}$ are the chiral projection operators

$$a = \frac{1}{2}(1 + i\gamma_5)$$

$$\bar{a} = \frac{1}{2}(1 - i\gamma_5)$$

If two chiral transformations are performed sequentially,

$$(A_1 a + B_1 \bar{a})(A_2 a + B_2 \bar{a})\,\psi = (A_1 A_2 a + B_1 B_2 \bar{a})\,\psi \;. \quad (7\text{-}3)$$

Thus the chiral transformation corresponding to the product of (A_1, B_1) with (A_2, B_2) is given by the matrix pair (A_1A_2, B_1B_2). For this reason, most of our subsequent results are expressed in terms of chiral eigenstates rather than parity eigenstates.

In order to leave invariant the kinematic part of the Lagrangian, and hence the commutation relations, A, B, and C must be unitary matrices. We shall be concerned principally with the behavior of the fermions under chiral symmetries -- each such transformation is characterized by a pair of unitary matrices, (A, B), which act, respectively, upon right- and left-handed fermions. When A = B, the transformation is a conventional non-chiral one. Otherwise, the transformation treats the two chiral eigenstates differently, and cannot be an invariance of the entire Lagrangian unless the fermions are massless.

It is convenient to consider the matrix pair (A,B) as a pair of representations ($D_R(g)$, $D_L(g)$) of an abstract continuous (Lie) group G.

$$\psi \rightarrow D_R(g) a \; \psi \; + D_L(g) \bar{a} \; \psi \; .$$

Likewise,

$$\phi \longrightarrow D_{\phi}(g) \; \phi \qquad . \qquad (7\text{-}4)$$

We will assume G is connected. This is true of all the chiral symmetry groups studied in the literature, as well as of the examples we will present later, and for good reason: we introduce chiral symmetries to obtain partially conserved currents. But only the connected part of the group generates currents -- the addition of discrete elements to a connected Lie group would place additional restrictions on the theory with no corresponding gain in currents.

Since A and B are unitary matrices, the group G must be compact. Just as before, we can decompose G into a direct product

$$G = G_1 \times G_2 \times \cdots G_n \; , \qquad (7\text{-}5)$$

where the G_i are simple Lie groups. Also, just as before, we neglect factors of U(1).

We are interested in theories conserving parity. Parity is an operation of the form

$$\psi(x,t) \longrightarrow P\,\gamma_o\,\psi(-x,t) \qquad (7\text{-}6)$$

where P is an n x n unitary matrix acting upon the n fermion fields. We assume that all the baryons have real relative parity, hence $P^2 = 1$. Thus we may write

$$P = P^+ - P^- \quad , \qquad (7\text{-}7)$$

where $P^\pm$ are projection operators whose sum is unity.

In we now introduce new fields ψ' defined by

$$\psi' = (P^+ + \gamma_5 P^-)\,\psi \quad , \qquad (7\text{-}8)$$

then, under parity,

$$\psi'(x,t) \longrightarrow \gamma_o\,\psi'(-x,t) \quad . \qquad (7\text{-}9)$$

We shall always use the ψ' as the basic fermion fields, but henceforth, we omit the prime.

This change of basis fields may transform a group of chiral transformations into a group of conventional nonchiral ones. By the same token, the inverse transformation may be used to obtain a theory with differing fermion parities, which admits a group of chiral symmetries, from an ordinary non-chiral higher symmetry scheme, in which all of the fermion parities are the same.

Thus we see how a given symmetry group may accomodate a multiplet of baryons not all of the same parity. For example, unitary symmetry may be adopted to describe the eight baryons with Σ parity opposite to that of N, Λ, and Ξ. Four of the currents are vector; they are the isospin and strangeness currents. The remaining four currents are pseudovedtor, and are only conserved when all baryon masses are neglected.

Under parity, 7-2 becomes

$$(A\bar{a} + Ba) \quad .$$

Conservation of parity implies that if the matrix pair (A,B) occurs in the chiral symmetry group, then the pair (B,A) must also occur.

We can assume with no loss of generality that the representations $D_R(g)$ and $D_L(g)$ are irreducible, since we can always construct any representation as a direct sum of irreducible representations. Then $D_R(g)$ and $D_L(g)$ must each be equivalent to a direct product of irreducible representations of the simple factors of G. We may choose the basis fields so that $D_R(g)$ is actually a direct product,

$$D_R(g) = D_1(g_1) \times D_2(g_2) \times \cdots\cdots D_m(g_m), \qquad (7\text{-}10)$$

where $D_1(g_1)$ is an irreducible representation of G_1, etc.

Parity conservation guarantees that $D_L(g)$ is also such a direct product, but possibly with its factors reordered. Hence

$$D_L(g) = R \quad D_1'(g_1) \times D_2'(g_2) \times \cdots\cdots D_m'(g_m) \quad R^+ \quad , \qquad (7\text{-}11)$$

where $D_1'(g_1)$ is an irreducible representation of G_1, etc., and R is a unitary operator which may interchange the factors of $D_L(g)$.

The parity operation exchanges the matrices in $D_R(g)$ and $D_L(g)$. At the same time, it transforms the abstract group G. Clearly parity must turn the factors of G into the factors of G. Since the square of the parity transformation is unity, either a factor of G must remain fixed or it must change places with another, isomorphic, factor. We will examine these two cases separately:

a) A factor of G remains fixed.

$$G = G_1 \times \cdots\cdots\cdots$$

$$D_R(g) = D_1(g_1) \times \cdots\cdots\cdots$$

$$D_L(g) = D_1'(g_1) \times \cdots\cdots\cdots \qquad (7\text{-}12)$$

The parity transformation exchanges the matrices in D_1 with those in D_1'. Since the theory conserves parity, every matrix that occurs in $D_1(g_1)$ must also occur in $D_1'(g_1)$. This does not mean that $D_1(g_1)$ is necessarily the same representation as $D_1'(g_1)$. It may be some equivalent representation,

$$D_1'(g_1) = S\, D_1(g_1)\, S^+ \quad ,$$

or it may be the complex conjugate representation,

$$D_1'(g_1) = \bar{D}_1(g_1) \quad ,$$

or it may even be some representation equivalent to the complex conjugate representation,

$$D_1'(g_1) = S\, \bar{D}_1(g_1)\, S^+ \quad .$$

In both instances, it follows from parity conservation that

$$S^2 = 1 \quad . \tag{7-13}$$

We denote these possibilities collectively by

$$D_1'(g) = \tilde{D}_1(g) = \begin{cases} S\, D_1(g)\, S^+ & \text{or} \\ \\ S\, \bar{D}_1(g)\, S^+ & . \end{cases} \tag{7-14}$$

In Appendix III we show that these are the only possibilities. We call a chiral symmetry of this type a bound chiral symmetry (b.c.s.).

b) Two isomorphic factors of **G** are interchanged

$$G = G_1 \times G_2 \times \cdots\cdots$$

$$D_R(g) = D_1(g_1) \times D_2(g_2)$$

$$D_L(g) = D_2'(g_2) \times D_1'(g_1) \quad . \tag{7-15}$$

Although G_1 and G_2 are isomorphic, there are in general many possible isomorphisms between them. Simply to be definite, let us define an isomorphism between them in the following standard manner: D_1 and D_2 cannot be both trivial. Let D_1 be non-trivial. Since G_1 is simple, D_1 must be faithful; that is to say, $D_1(g_1)$ is isomorphic to G_1. Likewise $D_2'(g_2)$ is isomorphic to G_2. Parity reversal exchanges the matrices in D_1 with those in D_2', so they must contain the same matrices. Thus there is an isomorphism between D_1 and D_2' defined by matrix equality. The standard isomorphism between G_1 and G_2 is defined as the product of these three isomorphisms. With this definition, it follows that

$$D_1(g) = D_2'(g) \quad . \tag{7-16}$$

The parity transformation also exchanges the matrices in $D_2(g)$ with those in $D_1'(g)$. By the same arguments used in case (a), we find,

$$D_1'(g) = \tilde{D}_2(g) \qquad . \tag{7-17}$$

To summarize, any chiral symmetry group may be written as the direct product of simple Lie groups. These groups either occur as singlets,

$$G = G_1 \times \ldots\ldots$$

with the corresponding representations,

$$\begin{aligned} D_R(g) &= D_1(g_1) \times \ldots\ldots \\ D_L(g) &= \tilde{D}_1(g_1) \times \ldots\ldots \end{aligned} \tag{7-18}$$

or as isomorphic pairs

$$G = G_1 \times G_2 \times \ldots\ldots \quad , \; G_1 \cong G_2 \quad ,$$

with the corresponding representations

$$\begin{aligned} D_R(g) &= D_1(g_1) \times D_2(g_2) \times \ldots\ldots \quad , \\ D_L(g) &= D_1(g_2) \times \tilde{D}_2(g_1) \times \ldots\ldots \quad . \end{aligned} \tag{7-19}$$

There are two important special cases of equation 7-19 . If

$$D_1(g_1) = \overline{D}_2(g_1) \quad , \qquad (7\text{-}20)$$

the resulting chiral symmetry resembles very closely a product of two b.c.s., which are interlinked by the parity operation. We call this a doubly bound chiral symmetry; it will be discussed along with b.c.s. in Part IX.

Alternatively, it may be that

$$D_1(g_1) = 1 \quad , \qquad (7\text{-}21)$$

which implies

$$\tilde{D}_1(g_1) = 1 \quad . \qquad (7\text{-}22)$$

The left- and right-handed fermions transform quite independently. We call this situation free chiral symmetry (f.c.s.).

In Appendix IV we discuss briefly chiral symmetries that are neither bound nor free, and explain why they are not of much interest.

VIII. FREE CHIRAL SYMMETRY

For the discusiion of f.c.s., we will adopt a simplified notation for the fundamental transformation,

$$\psi \rightarrow D(g_1)a\,\psi + D(g_2)\bar{a}\,\psi \quad . \tag{8-1}$$

The conserved currents occur in pairs: vector currents are associated with transformations for which $g_1 = g_2$, and pseudovector currents with transformations for which $g_1 = g_2^{-1}$. They are identical in structure, differing, in their fermion parts, only by the presence or absence of γ_5.

The Yukawa sources

$$\bar{\psi}_i\,\bar{a}\,\psi_j \quad ,$$

transform under 4-1 in the following manner:

$$\bar{\psi}_i\,\bar{a}\,\psi_j \longrightarrow \bar{D}_{ik}(g_1)\,D_{j\ell}(g_2)\,\bar{\psi}_k\,\bar{a}\,\psi_\ell \,. \tag{8-2}$$

Thus they form a basis for the representation $\bar{D}(g_1) \times D(g_2)$ of the chiral symmetry group $G_1 \times G_2$. If $D(g_1)$ is an irreducible representation of G_1 of degree n, this is an irreducible representation of $G_1 \times G_2$ of degree n^2. Likewise, the conjugate sources

$$\overline{\psi}_i \, a \, \psi_j ,$$

form a basis for the conjugate representation $\overline{D}(g_2) \times D(g_1)$.

In order to introduce Yukawa interactions of spinless mesons to these sources in a symmetric way, we must couple to the n fermions n^2 meson fields ϕ_{ij}, which also form a basis for $\overline{D}(g_1) \times D(g_2)$. The invariant interaction is

$$L_n = g \; \overline{\psi}_i \, a \, \psi_j \, \phi_{ij} + \text{H.c.} = g \, \overline{\psi}_i \, a \, \psi_j \; \phi_{ij}$$

$$+ \, g \, \overline{\psi}_i \, \overline{a} \, \psi_j \, \phi^*_{ji} \qquad . \qquad (8\text{-}3)$$

If we define "Hermitian" and "anti-Hermitian" mesons by

$$\sigma_{ij} = \tfrac{1}{2} \, (\phi_{ij} + \phi^*_{ji})$$

$$\pi_{ij} = -i/2 \, (\phi_{ij} - \phi^*_{ji}) \quad , \qquad (8\text{-}4)$$

then,

$$L_n = g \, \overline{\psi}_i \; \psi_j \; \sigma_{ij} + g \; \overline{\psi}_i \, \gamma_5 \, \psi_j \, \pi_{ij} \quad . \; (8\text{-}5)$$

The σ_{ij} are scalar and the π_{ij} are pseudoscalar

These mesons are complex. If $\overline{D}(g)$ is not equivalent to $D(g)$, $\overline{D}(g_1) \times D(g_2)$ cannot be equivalent to $D(g_1) \times \overline{D}(g_2)$, and $\overline{D}(g_1) \times D(g_2)$ cannot be expressed in real form. The n^2

complex mesons decompose into $2n^2$ real mesons, n^2 scalar ones and n^2 pseudoscalar ones, all of which mingle irreducibly under the action of the chiral symmetry group.

Note that in this case the form of the Yukawa interaction is quite independent of the nature of the original chiral symmetry group. It possesses the symmetry of $U(n) \times U(n)$ -- the largest possible chiral symmetry group on n fermions. If the Yukawa coupling is the only interaction present, this is the symmetry of the Lagrangian.

If $\bar{D}$ is equivalent to D, there exists a matrix C such that $\bar{\psi}C$ transforms in the same way as ψ. We can use C to decompose L_n into two invariant parts, each of which involves n^2 real mesons:

$$L_n = 1/2\ (L_{ns} + L_{np}) \quad , \tag{8-6}$$

where

$$L_{ns} = g\left[(\bar{\psi}C)_i\ \psi_j + (\bar{\psi}C)_j\ \psi_i\right]\sigma_{ij}$$

$$+ g\left[(\bar{\psi}C)_i\ \gamma_5\ \psi_j - (\bar{\psi}C)_j\ \gamma_5\ \psi_i\right]\pi_{ij} \quad , \tag{8-6a}$$

and

$$L_{np} = g \left[(\overline{\psi} C)_i \psi_j - (\overline{\psi} C)_j \psi_i \right] \sigma_{ij}$$

$$+ g \left[(\overline{\psi} C)_i \gamma_5 \psi_j + (\overline{\psi} C)_j \gamma_5 \psi_i \right] \pi_{ij}. \quad (8\text{-}6b)$$

L_{ns} describes the interactions of ½ n(n + 1) scalar mesons and ½ n(n - 1) pseudoscalar meson; L_{np} describes the interactions of ½ n(n + 1) pseudoscalar mesons and ½ n(n - 1) scalar mesons.

Because of the appearance of C in equations 8-6, it seems that L_{ns} and L_{np} may describe many theories whose natures depend upon the structure of the f.c.s., in contrast to the situation for L_n. In actuality, this is not the case. It may be shown (33) that C satisfies the equation

$$C \overline{C} = \pm 1 \quad . \qquad (8\text{-}7)$$

If $C \overline{C} = +1$, it is possible to perform a unitary transformation such that

$$C = 1 \quad . \qquad (8\text{-}8)$$

If $C \overline{C} = -1$ (this can only happen if n is even), it is possible to perform a unitary transformation such that

$$C = \sigma \quad , \qquad (8\text{-}9)$$

where σ is the matrix defined in equation 2-2. We denote these two cases by a <u>superscript</u> $\pm$.

Just as for L_n, the $L_{ns}^{\pm}$ and $L_{np}^{\pm}$ have symmetries of large f.c.s. groups. L_{ns}^{+} and L_{np}^{+} have the symmetry of SO(n) x SO(n); if these are the only interactions, there are $n^2 - n$ conserved currents. L_{ns}^{-} and L_{np}^{-} have the symmetry of Sp(n/2) x Sp(n/2); if these are the only interactions there are $n^2 + n$ conserved currents.

Thus there are only five kinds of f.c.s. groups describing (irreducibly) the interactions of n fermions with spinless mesons. They are: L_n (which involves n^2 scalar mesons and n^2 pseudoscalar mesons), $L_{np}^{\pm}$ and $L_{ns}^{\pm}$ (which each involves n^2 mesons of mixed parities). If n is odd, the theories L_{np}^{-} and L_{ns}^{-} do not exist, and there are only three possibilities.

We explicitly display the possible interaction Lagrangian for n = 1, 2, and 3. For n = 1, we call the fermion Λ. The only Lie group with a one-dimensional representation is U(1), and D(g) is clearly not equivalent to $\overline{D}(g)$. The corresponding theory involves two isosinglet mesons, one scalar and one pseudoscalar:

$$L_1 = g \ \overline{\Lambda} \ \Lambda \ \sigma \ + g \ \overline{\Lambda} \ \gamma_5 \ \Lambda \ \pi \quad . \tag{8-10}$$

For n = 2, we call the two fermions nucleons. The only Lie groups with two-dimensional representations are SU(2) and U(2). SU(2) leads to interactions of the form L_2^-,

$$L_{2s}^- = g\,\bar{N}\,\vec{\tau}\cdot\vec{\sigma}\,N + g\,\bar{N}\,\gamma_5\,N\,\pi$$

$$L_{2p}^- = g\,\bar{N}\,\gamma_5\,\vec{\tau}\cdot\vec{\pi}\,N + g\,\bar{N}\,N\,\sigma \quad . \qquad (8\text{-}11)$$

L_{2p}^- is invariant under the free chiral generalization of isospin -- it is the σ-model, first investigated by Schwinger (34).

U(2) leads to the interaction L_2,

$$L_2 = \tfrac{1}{2}\,(L_{2s}^- + L_{2p}^-) \quad . \qquad (8\text{-}12)$$

This model is invariant both under free chiral isotopic transformations and under free chiral phase transformations of both fermions: both isotopic spin and baryon number are free chiral symmetries.

For n = 3, we may have interactions of the form L_3^+ and L_3. In L_3^+ we must identify the fermions as a triplet if isospin is to be a subgroup of the f.c.s. group.

$$L_{3s}^+ = g\,\vec{\pi}\cdot(\vec{\bar{\Sigma}}\times\gamma_5\,\vec{\Sigma}) + g\,\vec{\bar{\Sigma}}\cdot\overleftrightarrow{\sigma}\cdot\vec{\Sigma} \qquad (8\text{-}13)$$

where $\overleftrightarrow{\sigma}$ is a symmetric isotensor -- it includes doubly charged particles. Likewise

$$L_{3p}^{+} = g\,\vec{\pi}' \cdot (\vec{\Sigma} \times \vec{\Sigma}) + g\,\bar{\vec{\Sigma}} \cdot \overleftrightarrow{\sigma}' \cdot \gamma_5 \overleftrightarrow{\Sigma} \,. \quad (8\text{-}14)$$

There are two ways we can embed the baryons in L_3. One is as the Σ 's

$$L_3 = L_{3s}^{+} + L_{3p}^{+} \quad . \qquad (8\text{-}15)$$

The other is as Λ and nucleons. In this case there are also eighteen ($2n^2$) spinless mesons. We may identify seven of the pseudoscalar ones as the known spinless mesons. The two other pseudoscalar ones are isosinglets, which we call χ and σ. We write out only the ps-ps interaction (the s-s terms are identical in form but lack γ_5's):

$$L_3 = g \Big[\sqrt{\tfrac{1}{6}}\, \chi (\bar{N} \gamma_5 N - 2\, \bar{\Lambda} \gamma_5 \Lambda)$$
$$+ \sqrt{\tfrac{1}{3}}\, \sigma (\bar{N} \gamma_5 N + \bar{\Lambda} \gamma_5 \Lambda)$$
$$+ \sqrt{\tfrac{1}{2}}\, \pi \cdot (\bar{N} \gamma_5 \vec{\tau} N) + \bar{K}\, \bar{\Lambda} \gamma_5 N + K \bar{N} \gamma_5 \Lambda \Big]$$
$$+ \text{scalar terms} \,. \qquad (8\text{-}16)$$

This is the first Lagrangian we have displayed that contains partially conserved strangeness-changing currents. It may be thought of as the generalization to f.c.s. of the Sakata model (4).

IX. BOUND CHIRAL SYMMETRIES

The general structure is

$$D_R(g) = D(g)$$

$$D_L(g) = D(g) \quad , \qquad (9\text{-}1)$$

where D(g) is defined by equation 7-14.

We want our symmetry group to include the known exact symmetries of the strong interactions; if g is one of these, then

$$D_R(g) = D_L(g) \quad . \qquad (9\text{-}2)$$

If

$$D_L(g) = SD(g)\, S^+ \; , \qquad (9\text{-}3)$$

then equation 9-2 implies

$$\left[I, S\right] = \left[Y, S\right] = 0 \quad , \qquad (9\text{-}4)$$

where I is isotopic spin and Y is hypercharge. Since the baryon multiplets are uniquely determined by their isotopic spin and hypercharge, S must simply multiply each multiplet by a constant. S^2 is 1, so this constant must be ± 1.

We may write

$$S = P^{+} - P^{-} \quad , \tag{9-5}$$

where P^{+} and P^{-} are projection operators whose sum is 1. We will show that every theory of this type may be obtained from an ordinary symmetry group.

Consider a Lagrangian $L(\psi_o)$ which admits a group G of ordinary symmetries. The elements of G transform both chiral eigenstates of ψ_o identically; they are described by pairs of matrices (D(g), D(g)). Since the D are irreducible, the n fermions possess the same relative parities. Now define the new fields

$$\psi = (P^{+} + \gamma_5 P^{-}) \psi_o \quad . \tag{9-6}$$

If L is expressed in terms of ψ, it is still invariant under a group of transformations isomorphic to g. However, the symmetries are bound chiral symmetries described by the pairs of matrices $(D(g), SD(g)S^{+})$. In the resulting theory, the fermions $P^{+}\psi$ must possess relative parities opposite to the $P^{-}\psi$.

This is in apparent contradiction with the convention extablished in equation 7-9, which defined our fundamental fields in such a way that all baryons had the same parity. The explanation is that two different definitions of parity are being used. If we only look at the highly symmetric part of the Lagrangian, there are many conservation laws among the baryons. Each of these conservation laws leads to an ambiguity in the definition of relative baryon parity, just like the ambiguity in the definition of relative baryon parity caused by strangeness conservation in the real world. We can only tell which parity is the real parity by looking at the symmetry-breaking part of the interaction.

The preceding analysis has shown that we will not obtain any surprising new theories by considering bound chiral symmetries satisfying equation 9-3. However, the variety of theories which utilize the alternative

$$D(g) = S\bar{D}(g)S^{+} \tag{9-7}$$

is much richer. We give three examples of such theories.

Our first model is based on SU(3). This is the Lie group of lowest dimension that possesses representations that are not equivalent to their conjugates. We label the representations by their dimension; in this notation the lowest representations of SU(3) are

$$3,\ \bar{3},\ 6,\ \bar{6},\ 8,\qquad \ldots\ldots$$

In the general nomenclature of Part II, these are vector, conjugate vector, symmetric tensor, conjugate symmetric tensor, and mixed traceless tensor. We use these abbreviations to avoid handling these awkward phrases.

(D_R, D_L) is $(3, \bar{3})$. The real 3 x 3 orthogonal transformations are the same for both chiral eigenstates; we may take them as a representation of isospin. This is a theory of the Σ's. The Yukawa source,

$$\bar{\Sigma}_i \; a \; \Sigma_j \qquad ,$$

forms a basis for the representation

$$D_L(g) \times D_R(g) = 3 \times 3$$

$$= 6 + \bar{3} \qquad (9\text{-}8)$$

We may couple three spinless mesons to this source; and their conjugates to the conjugate source. These mesons are complex; just as in Section IV, they decompose into scalar and a pseudoscalar isovectors:

$$L_{int} = g \; \vec{\bar{\Sigma}} \, \gamma_5 \times \vec{\Sigma} \cdot \vec{\pi} + g \; \vec{\bar{\Sigma}} \times \vec{\Sigma} \cdot \vec{\sigma}. \qquad (9\text{-}9)$$

The conserved currents form a vector triplet and a pseudovector quintuplet.

The second model (4) has SU(3) x SU(3) for its chiral symmetry group. If we indicate the elements of this group by (g_1, g_2), then,

$$D_R(g) = \bar{3}\,(g_1) \times 3\,(g_2)$$

$$D_L(g) = \bar{3}\,(g_2) \times 3\,(g_1) \quad . \tag{9-10}$$

This is an example of a doubly bound chiral symmetry. The group contains not only isospin, but the eightfold unitary symmetry group. If we consider the subgroup of all elements of the form (g_1, g_1), it is represented by

$$D_R(g_1) = D_L(g_1) = \bar{3}(g_1) \times 3(g_1) = 8(g_1) + 1(g_1) \quad . \tag{9-11}$$

The model contains unitary symmetry as an ordinary non-chiral symmetry. In addition to the eight known baryons, it contains a ninth baryon, with I = 0, Y = 0.

As above, the Yukawa source transforms equivalently to

$$\bar{D}_L(g) \times D_R(g) = \bar{3}(g_1) \times \bar{3}(g_1) \times 3(g_2) \times 3(g_2)$$

$$= (3(g_1) \times \bar{3}(g_2)) + (3(g_1) \times 6(g_2))$$

$$+ (\bar{3}(g_1) \times \bar{6}(g_2))$$

$$+ (6(g_1) \times \bar{6}(g_2)) \quad . \qquad (9\text{-}12)$$

We couple nine spinless mesons that form a basis for the representation $(3(g_1) \times \bar{3}(g_2))$ to this source, and couple their conjugates to the conjugate source. $3(g_1) \times \bar{3}(g_2)$ is not equivalent to its conjugates; the nine meson fields must be complex, describing eighteen real mesons. These eighteen mesons have the same quantum numbers as those appearing in equation 8-16.

We can write the right-handed baryons as a tensor ψ^{i}_{α}, where the Latin index runs from 1 to 3 and indicates transformation properties under the first factor of SU(3), and the Greek index also runs from 1 to 3 and indicates transformation properties under the second factor of SU(3). Then the invariant Yukawa coupling is

$$L_{int} = \overline{\psi}^{\,i}{}_{a\,\alpha}\, \psi^{j}_{\beta}\, \phi^{k}{}_{\gamma}\, \epsilon^{\alpha\beta\gamma}\, \epsilon_{ijk} + \text{herm. conj.} \qquad (9\text{-}13)$$

where ϕ is the meson field.

Were it not for the introduction of many unobserved particles, this would be an extremely satisfactory theory -- it contains all the (partially) conserved currents needed to describe both weak and electromagnetic interactions. There are eight vector currents (the eight currents of unitary symmetry), the four associated with isospin and hypercharge, plus a pair of $Y = \pm 1$, $I = \frac{1}{2}$, currents. The eight pseudo-vector currents are of the same isotopic character.

Besides SU(n) and the exceptional group E6, the only simple Lie groups that possess representations not equivalent to their conjugates are the family $SO(2\nu + 2)$. Our third model (or, more properly, family of models) is based on this group. We take $D_R(g)$ to be the lowest spinor representation, of dimension 2^ν (as defined in Part II), and $D_L(g)$ to be the conjugate spinor representation. If we define Γ_o to be i, then the bilinear form

$$\bar{\psi}\, a \Gamma_i\, \psi \qquad\qquad 0 \leq i \leq 2\nu + 1$$

transform like a $2\nu + 2$ component vector under the chiral symmetry group, and we may invariantly couple $2\nu + 2$ real spinless mesons to it by Yukawa couplings. The resulting interaction is hermitian only if the $2\nu + 2$ mesons are also coupled to the conjugate sources, $\bar{\psi}\ \bar{a}\ \Gamma_j\, \psi$.

Depending upon the relative sign of this term, the interaction describes $2\nu + 1$ pseudoscalar (or, scalar) mesons and one scalar (or, pseudoscalar) meson.

In terms of $SO(2\nu + 1)$ the "pseudoscalar" interaction is

$$L_- = g(\bar{\psi}\, \gamma_5\, \Gamma_i\, \psi\, \phi_i - \bar{\psi}\psi\,\sigma) \tag{9-14}$$

$$1 \leq i \leq 2\nu + 1 \quad .$$

If $\nu = 1$, this is Schwinger's -model again, obtained by a quite different process than that of Section IV.

If $\nu = 3$, this is the generalization of the seven-dimensional model of Tiomno and Dallaporta first discovered by Gursey (23). There are two I = 0, Y = 0 currents, and one isotopic multiplet for each of the following (I,Y) values: (0, 2), (0, -2), (1,0), (3/2, 1), (3/2, -1), (1/2, 1), (1/2, -1). The pseudovector currents have the same isotopic assignments as the pseudoscalar mesons: (1, 0), (1/2, 1), and (1/2, -1).

This model has many attractive features. It contains more than enough currents to describe weak interactions and requires the introduction of only one new particle. The presence of I = 3/2 strangeness changing currents is particularly interesting in the light of recent experimental evidence (35) indicating that these play a role in leptonic decay modes of neutral K-mesons.

X. DISCUSSION OF CHIRAL SYMMETRIES

Our discussion of chiral symmetries will be even more fragmentary than the discussion of ordinary higher symmetry schemes in Part VI. There is a reason for this: most interesting conclusions (4, 9) drawn from chiral symmetries require some detailed hypothesis about the nature of the symmetry-violating part of the interaction, and we do not want to begin here an investigation of such possibilities. Nevertheless, we will be able to make some observations.

Gell-Mann (4) has emphasized that one can always associate a Lie group with the weak interactions in the following manner: One takes the total isopin, the total hypercharge, and the spatial integrals of the fourth components of the beta-decay currents ("the total weak charges"). One then forms all possible sums and iterated commutators of these operators. These then form the generators of a Lie group. If one follows conventional field theory and assumes that all these currents are bilinear forms in baryon fields obeying canonical commutation rules, this group is a chiral symmetry group. (This is also true under less restrictive conditions; for example, it is sufficient to assume that the charge operators introduced above turn one-baryon states into one-baryon states, with coefficients independent of the momentum.) We will call this group the full weak group.

Of course, chiral symmetries may be of use even if the chiral symmetry group is not the full weak group. We could obtain useful results even if the generators of the chiral symmetry group only contain part of the weak current -- say, the strangeness conserving part.

If we assume all eight baryons are fundamental fields, then the existence of leptonic Λ and Σ decays (37) shows that the full weak group must mix these baryons with the nucleons. If similar decays were observed for the Ξ , it would show that the eight baryons must form a basis for an irreducible representation of the full weak group. (Note that this group must be connected, in contrast to the groups of ordinary higher symmetry schemes.)

Now, for all we know, the full weak group may be as large as U(8) x U(8), the largest free chiral symmetry on eight baryons. However, if we optimistically speculate that the baryon-meson interactions have something to do with this group, we can exclude this case, which requires 128 mesons. The same argument excludes other varieties of f.c.s.

As for b.c.s., we can exclude immediately the variety discussed at the beginning of Part IX. Such a group implies that the nucleon beta-decay current must be purely vector, in contradiction to experiment. Any b.c.s. we use to construct the full weak group must involve complex conjugation. It is now a simple matter to find interesting chiral symmetry schemes from the tables. We only want to look at connected groups whose representations are not equivalent to their conjugates. We also want hypercharge and isospin to be represented by the same transformations on the left- and right-handed baryons. For eight baryons, the only possibilities that do not involve excessive numbers of mesons are the b.c.s. based on the spinor representation of SO(8), discussed on page 76, and the chiral symmetry based on the product group SU(4) x SU(2). In this theory we may introduce the chiral transformations either through the factor of SU(4), or through that of SU(2), or through both. We will discuss the case in which both factors involve chiral transformations. It is most convenient to consider SU(4) in its guise of SO(6); the model can then be considered as a member of the family discussed on page 76.

G is SO(6) x SU(2) x SU(2). D_R is (spinor) x ($\frac{1}{2}$) x (0). D_L (is conjugate spinor) x (0) x ($\frac{1}{2}$). This is the product of a b.c.s. and an f.c.s. We can couple three pseudoscalar mesons (π's) and one scalar meson to the factor of SU(2) x SU(2). Likewise, we can couple five pseudoscalar mesons and one scalar meson to the factor of SO(6). The pseudoscalar mesons are a hypercharge zero isosinglet and two hypercharge plus and minus one isodoublets. We may identify them with the K's and with χ^{o}. The scalar meson is, of course, a hypercharge zero isosinglet. The vector currents have (I, Y) assignments of (½, +1), (½, -1), (0, 0), (0, +1), (0, -1), (1, 0) and (1, 0). The pseudovector currents have (I, Y) assignments of (½, +1), (½, -1), (0, 0), and (1, 0). The chiral symmetry group contains the global symmetry variant based on Sp(2) x SU(2) as an ordinary higher symmetry. This scheme was discussed in detail in Part VI.

The only interesting schemes involving more than eight baryons are the ten-baryon scheme based on SU(5) x SU(2), which contains the model discussed above, the nine-baryon scheme based on SU(3) x SU(3), which was discussed on page 74, and the ten-baryon b.c.s. based on SU(5). Here D_R is the antisymmetric tensor representation and D_L its conjugate. We can couple ten mesons to the Yukawa source. Five are scalar and five pseudoscalar. Each quintuplet has (I, Y) distribution (½, +1), (½, -1) and (0, 0). This contains the ordinary higher symmetry scheme based on SO(5).

The only interesting schemes involving less than eight baryons are the five-baryon b.c.s. based on SU(5), which requires twenty mesons, and the f.c.s. based on SU(3). This is the generalization of the Sakata model discussed on page .

We would like to stress that we have obtained such a small number of interesting models only because we have defined to be uninteresting all groups that lead to very large numbers of mesons. If the full weak group has nothing to do with baryon-meson interactions, this is without justification.

TABLE I. POSSIBLE SYMMETRIES OF THE EIGHT BARYONS

G_o	Representation	Currents	Meson Number*	Form Factors
SU(8)	vector	63	63	1
SO(8)	vector	28	28	1
SO(8)	spinor	28	28	1
SO(7)	spinor	21	7	1
SU(3)	traceless tensor	8	8	2
SU(2) x SU(4)	(½) x vector†	18	3	2
SU(2) x Sp(2)	(½) x vector	13	3	2
SU(4) x SU(4)	(vector x scalar) + + (scalar x vector)	30	15	1
SU(2) x SU(2) x SU(2)	((½) x (0) x (½)) + + ((0) x (½) x (0))	9	3	3
SU(2) x SU(2) x SU(2) x SU(2)	((½) x (0) x (½) x (0)) + ((0) x (½) x (0) x (½))	12	4	2

* This is the smallest number of mesons that can be coupled to the Yukawa source, after one meson coupled trivially.

† We list representations of SU(2) by listing their "angular momentum", thus: (j).

TABLE II. POSSIBLE SYMMETRIES OF SEVEN BARYONS*

G_o	Representation	Currents	Meson Number	Form Factors
SU(7)	vector	48	48	1
SO(7)	vector	21	21	1
G2	vector	14	7	1

TABLE III. POSSIBLE SYMMETRIES OF SIX BARYONS†

G_o	Representation	Currents	Meson Number	Form Factors
SU(6)	vector	35	35	2
SU(2) x SU(3)	(½) x vector	11	3	3
SU(3) x SU(3)	(vector x scalar) + (scalar x vector)	16	8	2

---

* We take these to be nucleons, Σ's and Ξ's.

† We take these to be either Λ, Σ, and Ξ, or Λ, Σ's, and nucleons.

TABLE IV. POSSIBLE SYMMETRIES OF FIVE BARYONS*

G_o	Representation	Currents	Meson Number	Form Factors
SU(5)	vector	24	24	1
SO(5)	vector	10	10	1

TABLE V. POSSIBLE SYMMETRIES OF FOUR BARYONS

G_o	Representation	Currents	Meson Number	Form Factors
SU(4)	vector	15	15	1
Sp(2)[a]	vector	10	5	1
SU(2) x SU(2)	$(\frac{1}{2}) \times (\frac{1}{2})$	6	3	2†, 3**
SU(2) x SU(2)[a]	$(\frac{1}{2}) \times (0) + (0) \times (\frac{1}{2})$	6	4	1

* We take these to be nucleons, Ξ's, and Λ .

† If the baryons are nucleons and Ξ .

** If the baryons are Λ and Σ .

TABLE VI. POSSIBLE SYMMETRIES OF THREE BARYONS*

G_o	Representation	Currents	Meson Number	Form Factors
SU(2)†	(1)	3	3	2
SU(3)	vector	8	8	2

* We take these to be either Σ , nucleon and Λ , or Λ and Ξ .

† Only if the baryons are Σ . This is just isospin invariance.

TABLE VII. POSSIBLE SYMMETRIES OF NINE BARYONS*

G_o	Representation	Currents	Meson Number	Form Factors
SU(9)	vector	80	80	1
SO(9)	vector	36	36	1
SU(3) x SU(3)	vector x vector	16	8	2
SU(3) x SU(3) x SU(3)	(vector x scalar x scalar) + (scalar x vector x scalar) + (scalar x scalar x vector)	24	24	1

* We take these to be the usual eight, plus an isosinglet.

TABLE VIII. POSSIBLE SYMMETRIES OF TEN BARYONS

G_o	Representation	Currents	Meson Number	Form Factors
SU(10)	vector	99	99	1
SO(10)*	vector	45	45	1
SU(5)†	antisymmetric tensor	24	24	1
SU(4)†	symmetric tensor	15	15	1
Sp(2)†	symmetric tensor	10	5	1
SU(2) x SU(5)**	(½) x vector	27	3	2
SU(5) x SU(5)*	(vector x scalar) + (scalar x vector)	48	24	1
SO(5) x SO(5)*	(vector x scalar) + (scalar x vector)††	20	10	1

* Only if the two extra baryons are two singlets of opposite hypercharge.

† Only if the two extra baryons are two singlets of hypercharge ± 2.

** Only if the two extra baryons are a doublet.

†† This is the same as the antisymmetric tensor representation of Sp(2).

APPENDIX I. LIE ALGEBRAS AND CONSERVED CURRENTS

We want to review here the relation between the algebra of a symmetry group of a theory and the conserved currents of that theory.

The connection between a Lie group and its algebra is this: if g is an element of the group "infinitesimally close to the identity", then g may be written as $1 + \lambda A$, where A is an element of the algebra. Every representation of the group defines a representation of the algebra; we define T(A) by

$$D(1 + \lambda A) = 1 + \lambda T(A) \quad . \tag{A1-1}$$

Since D(g) is unitary, T(A) is anti-Hermitian. Note that $D(g_1 g_2) = D(g_1)D(g_2)$ implies $T(A + B) = T(A) + T(B)$.

We will show that the elements of the Lie algebra generate the conserved currents. If the Lagrangian is invariant under transformations of the form,

$$\begin{aligned} \psi &\rightarrow D_R(g)\, a\, \psi + D_L(g)\, \bar{a}\, \psi \quad , \\ \phi &\rightarrow D_\phi(g)\, \phi \quad , \end{aligned} \tag{A1-2}$$

the most general form we consider here, then it is also invariant under the infinitesimal transformations

$$\psi \rightarrow [1 + \lambda(T_R(A)a + T_L(A)\bar{a})]\,\psi\,,$$

$$\phi \rightarrow [1 + \lambda\, T_\phi(A)]\quad \phi \quad . \tag{A1-3}$$

Now, consider transformations of the same form as A1-3, but with λ a function of space-time. Hamilton's principle asserts that

$$\delta I = \int \frac{\delta L}{\delta \lambda}\, \delta\lambda d^4x = 0 \tag{A1-4}$$

for any variation, which means that

$$\frac{\delta L}{\delta \lambda} = 0 \quad . \tag{A1-5}$$

But if L does not involve higher than first derivatives,

$$\frac{\delta L}{\delta \lambda} = \frac{\partial L}{\partial \lambda} - \frac{\partial}{\partial x_\mu}\,\frac{\partial L}{\partial(\partial_\mu \lambda)} \quad . \tag{A1-6}$$

We have assumed $\partial L/\partial\lambda$ is zero. Thus

$$\frac{\partial j_\mu}{\partial x_\mu} = 0 \quad , \tag{A1-7}$$

where

$$j_\mu = \frac{\partial L}{\partial(\partial_\mu \lambda)} \quad . \tag{A1-8}$$

If L involves derivatives only through the kinematic term,

$$i\,\overline{\psi}\cdot\not{\partial}\,\psi \quad + \tfrac{1}{2}\,\partial_\mu\,\emptyset\cdot\partial_\mu\,\emptyset \quad ,$$

then

$$j_\mu = -i\ \overline{\psi}\cdot\gamma_\mu\left[T_R(A)a + T_L(A)\overline{a}\right]\psi$$
$$+\ \partial_\mu\,\emptyset\cdot T_\emptyset(A)\emptyset \quad . \qquad \text{(A1-9)}$$

For arbitrary A, j_μ is not a parity eigencurrent. However, consider its parity transform $P^+ j_\mu P$. The parity invariance of the theory guarantees that this must also be a current generated by some element of the algebra, which we will call B.

$$P^+ j_\mu P = -i\ \overline{\psi}\cdot\gamma_\mu\left[T_R(B)a + T_L(B)\overline{a}\right]\psi$$
$$+\ \partial_\mu\,\emptyset\ T_\emptyset(B)\ \emptyset \quad . \qquad \text{(A1-10)}$$

Taking the sum and difference of these equations, we find that (A + B) generates a conserved vector current, and that (A - B) generates a conserved pseudovector current.

In practice, we will take care to choose the basis elements of our algebra A_i so that they generate only vector and pseudovector currents. If all the baryons have the same intrinsic parity, the condition for this to hold is that

$$T_R(A_i) = \pm T_L(A_i) \quad , \qquad (A1\text{-}11)$$

where the upper sign holds for vector, the lower for pseudo-vector, currents.

For any realistic theory, the chiral symmetries are merely partial symmetries. All of the pseudovector currents and some of the vector currents are only partially conserved; A1-7 is replaced by

$$\frac{\partial j_\mu}{\partial x_\mu} = \frac{\partial L'}{\partial \lambda} \quad , \qquad (A1\text{-}12)$$

where L' is the symmetry-breaking part of the Lagrangian.

APPENDIX II: A DIRECTORY OF REPRESENTATIONS

In this appendix we list all representations of simple Lie groups with dimension n less than sixteen. An asterisk (*) indicates a pair of inequivalent complex conjugate representations.

Any n: The representation of SU(2) with spin (n-1)/2. The vector representation of SU(n).* The vector representation of SO(n).

Any even n: The vector representation of Sp(n/2).

n = 1: The scalar representation of any group.

n = 5: The antisymmetric tensor representation of Sp(2).

n = 6: The symmetric tensor representation of SU(3).* The antisymmetric tensor representation of SU(4).*

n = 7: The vector representation of G2.

n = 8: The mixed, traceless tensor representation of SU(3). The spinor representation of SO(7). The spinor representation of SO(8).*

n = 10: The symmetric tensor representation of SU(4).* The antisymmetric tensor representation of SU(5).* The symmetric tensor representation of Sp(2).

n = 14: The antisymmetric tensor representation of G2. The antisymmetric tensor representation of Sp(3).

n = 15: The mixed, traceless tensor representation of SU(4). The symmetric tensor representation of SU(5).* The antisymmetric tensor representation of SU(6)*.

APPENDIX III: PROOF OF A THEOREM

In this appendix we prove the theorem we need in Part VII. We show that if D(g) and D'(g) are two irreducible representations of the same simple Lie group that involve the same matrices, then either D is equivalent to D', or D is equivalent to $\overline{D}'$. Actually, we will be able to prove this theorem for every simple Lie group except SO(8); for this group we will construct a special argument showing that we may perform an automorphism of the group such that D(g) and D'(g) are transformed into two other representations that <u>are</u> connected in the manner we desire. This latter is all that is needed for our purposes.

Let D(g) and D'(g) be two irreducible representations of the simple Lie group G, such that D and D' involve the same matrices. If D is trivial, the theorem is immediate. If D is not trivial, since G is simple, it must be faithful. Thus we can always solve

$$D'(g') = D(g)$$

for g' as a function of g. From the definition this function is an automorphism of G. If it is an inner automorphism, D(g) and D'(g) are equivalent representations.

The structure of A, the group of automorphisms of a simple Lie group, has been extensively studied (36). In general, A decomposes into several components. The unit component is the group of inner automorphisms. Each of the other components is the product of the group of inner automorphisms and a particular outer automorphism.

Now, in order for our assertion above to be true, A must have either one component (for those groups for which all representations are equivalent to their conjugates) or two components (for those groups for which there exist representations inequivalent to their conjugates). Indeed, for all of the simple Lie groups except SU(n) ($n = 3$), SO(2ν) ($\nu = 3$), and the exceptional group E6, A has only one component, and for these it has two. These groups possess representations inequivalent to their conjugates.

The sole exception to the above is SO(8). For this group A has six components. We must sharpen our arguments here: we are not searching for a general automorphism, but one associated with parity. Since the square of the parity operation is one, the automorphism must be an involution. Four of the six components of A contain involutions. Let g' be an outer automorphism that is also an involution. Then it can be shown that there exists an outer automorphism $g^{\dagger}$, such that $(g^{\dagger})' = (g^*)^{\dagger}$, where g^* is in the same component of A as the outer automorphism that leads to complex conjugation. Let (D_R, D_L) be $D(g), D(g')$. Define $D'(g)$ to be $D(g^{\dagger})$ and define h by $g = h^{\dagger}$. Then (D_R, D_L) is $D'(h), D'(h^*)$, and the proof proceeds as for all other simple Lie groups.

APPENDIX IV: OTHER CHIRAL SYMMETRIES

Let us return to the decomposition of a general chiral symmetry group according to equation 7-5. Suppose a pair of isomorphic factors G_1 and G_2 is interchanged under the parity operation, and that both groups act non-trivially upon the chiral eigenstates,

$$G = G_1 \times G_2 \times \cdots\cdots$$

$$D_R(g) = D_1(g_1) \times D_2(g_2) \times \cdots\cdots$$

$$D_L(g) = D_1(g_2) \times D_2(g_1) \times \cdots\cdots \quad ,$$

where neither D_1 nor D_2 is trivial, and where D_2 is equivalent neither to D_1 nor to its conjugate. This kind of component of the general chiral symmetry group is neither a b.c.s. nor a f.c.s. We shall show, however, that its structure is too complicated to provide a model of the strong interactions.

Charge independence is presumably an exact symmetry of the strong interactions -- thus isotopic rotations must form a subgroup of any chiral symmetry proposed for the baryons. It is sufficient for us to examine the behavior of right-handed baryons; however, they behave under isotopic rotations, the left-handed baryons must follow suit.

In general, the irreducible representations D_1 and D_2 of the full chiral symmetry group provide reducible representations of the isotopic spin subgroup. Possibly, isotopic spin is described entirely by factors of G other than G_1 and G_2. Then it is described only trivially by D_1 and D_2. In this case, every isotopic multiplet of baryons must occur at least $n_1 \times n_2$ times (where n_i is the dimension of D_i) -- an evidently unsatisfactory alternative.

On the other hand, if D_1 contains a non-trivial representation of isotopic spin, so must D_2, since parity exchanges G_1 and G_2. That is to say, the generators of isotopic rotations are

$$\vec{I} = \vec{I}_1 + \vec{I}_2 \qquad ,$$

where $\vec{I}_1$ are three elements of the Lie algebra of G_1 and $\vec{I}_2$ are the three isomorphic elements in the Lie algebra of G_2. If doubly-charged baryons ($I \leq 3/2$) are not to occur, $\vec{I}_1$ and $\vec{I}_2$ must be reducible into isotopic singlets and doublets only.

The simplest possibility occurs for $G_1 \cong G_2 \cong SU(2)$. We may take D_1 to be the spin 1/2 representation -- then D_2 must be any inequivalent (and hence higher isospin) representation. Thus there must occur baryons of $I \geq 1$ and hence of charge greater than one.

If $G_1 \cong G_2 \cong SU(3)$, the vector representation must split into a singlet and a doublet. But any inequivalent representation (except the conjugate) must involve isospins of one or greater and again the theory necessarily involves baryons of higher charge.

If G is isomorphic to any other simple Lie group, the smallest inequivalent representations are four- and five-dimensional at least. Such a theory must involve at least twenty baryons.

APPENDIX V. NOTATION

We adhere to the standard notation of the literature in the discussion of fields and particles, except that we use an ordinary L for the Lagrange density, rather than a script L.

On page 3 we use the word "compact". This is a topological term. For our purposes it suffices to know that a group of matrices is compact if the set of matrix elements is bounded. Thus any group of unitary matrices (for which every matrix element is bounded by one) is compact, while the group of all linear transformations is not compact.

In the discussion of group theory, we define all terms that are not defined in the text of Wigner (33) and adhere to Wigner's notation, with the following exceptions:

We use an ordinary multiplication sign (x) for the direct product. Likewise, we use an ordinary addition sign (+) for the direct sum.

We indicate the complex conjugate representation by an overbar ($\bar{D}(g)$), instead of by an asterisk ($D(g)^*$).

In the case of Lie groups, for the sake of brevity, we sometimes speak of "isomorphisms" and "faithful representations" when we only mean "local isomorphisms" and "locally faithful representations". We never do this when the distinction is important.

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