D-Brane Actions and $N = 2$ Supergravity Solutions

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Abstract

Among the most remarkable recent developments in string theory are the AdS/CFT duality, as proposed by Maldacena, and the emergence of noncommutative geometry. It has been known for some time that for a system of almost coincident D-branes the transverse displacements that represent the collective coordinates of the system become matrix-valued transforming in the adjoint representation of $U(N)$. From a geometrical point of view this is rather surprising but, as we will see in Chapter 2, it is closely related to the noncommutative descriptions of D-branes.

A consequence of the collective coordinates becoming matrix-valued is the appearance of a "dielectric" effect in which D-branes can become polarized into higher-dimensional fuzzy D-branes. This last aspect has inspired Polchinski and Strassler to find a nonsingular string dual of a confining four-dimensional gauge theory. The nonsingular geometry is sourced by an extended brane arising from Myers' "dielectric" effect. Following the spirit of the Polchinski-Strassler paper, we find $N = 2$ supergravity solutions with polarized branes and a field-theory dual. In our case we are able to present exact supergravity solutions by using M-theory reductions to type IIA supergravity.

Contents

Chapter 1

Introduction

1.1 A very brief history of string theory

String theory emerged in the late 1960's as an attempt to explain the strong nuclear force. The basic idea was that the fundamental excitations are one-dimensional loops rather than pointlike objects as in standard quantum field theory. In superstring theories the world-sheet actions used to describe the string are supersymmetric versions of the Polyakov action. The latter are derived from the Nambu-Goto action, which has the interpretation of the area swept out by the string as it propagates in space-time. In the 1970's QCD was recognized as the correct theory of the nuclear forces, while string theory embarked on a more ambitious role of unifying all interactions. This followed the remarkable work of Schwarz and Scherk [1] in which it was realized that the theory includes general relativity. There are two established equivalent formulations of the superstring. The first one has a manifest world-sheet supersymmetry, and is the result of work by Ramond [2], Neveu and Schwarz [3]. The second formulation, developed by Green and Schwarz in [4] makes the ten-dimensional target space supersymmetry manifest. In the 1980's it has been discovered that there exist 5 distinct string theories having space-time supersymmetry, type I, type IIA, type IIB, SO(32) heterotic and $E_8 \times E_8$ heterotic, all of them living in a critical space-time dimension of 10. Phenomenology would then require to have 6 of the 10 dimensions compactified on a space of a size comparable to the string scale and Planck length. One of the 5 superstring theories, the $E_8 \times E_8$ heterotic, upon compactification on a Calabi-Yau space, gives a low-energy theory that can come close to the standard model physics. There are very many choices in choosing the compactification space, thus it became clear that more work was needed in understanding the nonperturbative properties of the theory. Significant developments occurred in the mid 90's with the discovery of dualities which showed that the five superstring theories are essentially equivalent to one another, and they are all related to a fundamental eleven-dimensional theory, called M-theory [57].

To study perturbation theory for strings one has to deal with a dimensionless coupling constant, g_s, given by the expectation of a dilaton field ϕ, i.e., $g_s =< e^{\phi} >$. The S duality, for example, relates theories having reciprocal coupling constants, $g_s \rightarrow \frac{1}{g_s}$. Another example is T duality which in the simplest case relates a theory compactified on a small circle of radius R, to another one compactified on a large circle of radius $R' = \frac{\alpha'}{R}$.

A key role in understanding these dualities is played by extended p-dimensional objects, p-branes, that have a nonperturbative origin. Among these excitations a particular class, the D-branes, has played an outstanding role. We are going to analyze certain aspects of D-brane actions in the first part of this work. D-branes also played an important role in the proposal of Maldacena [27], regarding a duality between 4-dimensional $\mathcal{N} = 4$ $U(N)$ super-Yang-mills theory associated with N D3-branes and type IIB string theory on $AdS_5 \times S^5$.

1.2 Basic review of D-branes

Dp branes can be viewed as dynamical $(p+1)$-dimensional hyperplanes on which open strings can end. Alternatively they can be thought of as non-perturbative objects with tension that depends on the string coupling constant as $\frac{1}{g_s}$, and carry closed-string Ramond-Ramond (RR) charges. In type IIA/IIB string theories a Dp brane is a BPS saturated object preserving 16 supercharges for p taking even/odd values. A single D-brane world volume theory admits a $U(1)$ gauge field. In type II superstring theories when we consider the case of N almost coincident D-branes the $U(1)$ gauge is enhanced to $U(N)$ gauge theory [62]. The $U(N)$ gauge theory has N^2 gauge bosons, say A^{ij} with $i = 1, ..N$, and their supersymmetric partners. The off-diagonal degrees of freedom corresponding to $i \neq j$, represent the ground state of a string connecting 2 different branes. In the low-energy limit one can integrate out the massive open string modes, to write a D-brane action that includes only massless modes. The first term of the action for a single Dp-brane takes a Born-Infeld form:

$$S_{\rm BI} = -T_p \int d^{p+1} \left(e^{-\phi} \sqrt{-\det(P[G+B]_{ab} + 2\pi\alpha' F_{ab})} \right) \tag{1.1}$$

where T_p is the Dp-brane tension and F_{ab} is the $U(1)$ field strength along the brane. The other fields appearing in the above action represent massless Neveu-Schwarz (NS) fields of the closed string theory: the dilaton ϕ, the metric G, and the NS two-form B. For constant field strength, the Born-Infeld action includes the entire α' corrections. The interaction with the massless RR fields are given by the Chern-Simons term of the D-brane actions:

$$S_{\rm CS} = \mu_p \int P[\sum C^{(n)} e^B] e^{2\pi\alpha' F}, \tag{1.2}$$

where the $C^{(n)}$ denotes an n-form RR potential. Notice that n takes only even values for type IIB theory and only odd values in type IIA. The generalizations to multiple (almost) coincident branes is challenging, but significant progress has been done in [19; 17] using the idea that the action has to be consistent under T duality transformations. Still, some questions remained open, like whether the resulting non-abelian action has the gauge symmetries required by the full string theory. This question is addressed in Section 2.1 and in reference [59]. An outstanding characteristic of non-abelian D-brane actions is that the transverse scalars that play the role of collective coordinates becomes matrices transforming in the adjoint representation of $U(N)$. This fact allows for the existence of a "dielectric" effect, for which a group of N D-branes can have a ground state corresponding to a higher dimensional "fuzzy" D-brane. This possibility can be anticipated from the fact that terms containing commutators of the transverse scalars appear in the expression of the N D-brane potential in certain background fields. The "fuzziness" in localizing the multiple Dp

branes in the transverse directions occurs when the $(9-p)$ transverse scalars cannot be diagonalized simultaneously. This effect is related to noncommutative descriptions of D-branes, a connection that we will study in Section 2.2. In a noncommutative description the coordinates of the D-brane satisfy the following commutation relations:

$$[x^i, x^j] = i\theta^{ij}, \tag{1.3}$$

where θ is an antisymmetric constant matrix. On such a noncommutative manifold the ordinary product of fields is replaced by a star product as follows:

$$f(x) * g(x) = e^{\frac{i}{2}\theta^{ij}\partial_i\partial'_j} f(x)g(x')|_{x'=x}. \tag{1.4}$$

A noncommutative gauge theory along the D-branes has been shown to exist in the presence of an antisymmetric 2 form NS field B in the bulk, as was studied in [20].

1.3 Review of AdS/CFT duality and generalizations

An outstanding development in string theory occurred in 1997, when Maldacena [27] made the conjecture that $\mathcal{N}=4$ four-dimensional $SU(N)$ Super Yang-Mills theory is dual to string theory on $AdS_5 \times S^5$. The conjecture was inspired by considering a system of N parallel D3 branes and then taking a limit such that the field theory along the D3 branes decouples from the bulk theory. The duality reveals that when the Yang Mills theory is weakly coupled, i.e. $g_{YM}^2 N << 1$, the string theory in the bulk is strongly coupled, and vice versa. One can notice that the supergravity approximation is valid if the string length l_s is much less than the radius of the AdS_5 and S^5, i.e:

$$R = (4\pi g_s N)^{1/4} l_s >> l_s. \tag{1.5}$$

Thus when the four-dimensional field theory is strongly coupled one can in principle use the duality to perform calculations on the supergravity side. It has been shown in [27] that for a local operator O_i of dimension Δ_i in the conformal field theory, there correspond two solutions of the linearized supergravity equations. One of the solutions is non-normalizable and is related to the coefficient of the operator O_i. The other solution is normalizable and is related to the vacuum expectation value of O_i.

By considering a system of parallel Dp branes one can extend this type of duality to other cases, as long as there exists a decoupling limit. In [50], it has been shown that this limit exists for $p < 6$. For generic p, the $(p+1)$-dimensional field theory in general is not conformal, and related to this, the supergravity dual is valid only for a certain range of energies. If we denote by r the transverse location of the parallel brane system, we can break the $U(N)$ gauge symmetry to $U(N-1)$, i.e. $U(N) \rightarrow U(N-1) \times U(1)$, by displacing one brane by r_0. This corresponds to giving a expectation value to some of the fields, $U = r_0/\alpha'$, which also represents the ground state energy of a string connecting the displaced brane to the rest. If we want to use the supergravity picture to describe the boundary field theory both the curvature of the bulk space and the string coupling constant have to be small. Since both quantities depend in general on the radial coordinate r, which in turn is related to the energy scale of the field theory, we conclude that in general a reliable dual supergravity description exists only for certain energy scales.

The AdS/CFT duality, as proposed in its initial form by Maldacena, applies to conformal $\mathcal{N} = 4$ gauge theories. In order to obtain a confining field theory one has to perturb the duality by adding mass terms that preserve less supersymmetry. Polchinski and Strassler in [38] have obtained the first example of a 4-dimensional confining gauge theory having a dual supergravity description without a naked singularity. Their motivation was provided by Myers' observation in [17] that multiple D-branes in transverse background fields can "blow up" to form higher-dimensional D-branes. Using rather elaborate perturbative calculations, Polchinski and Strassler showed that a system of N D3 branes placed in certain transverse perturbative fields polarizes into a D5 (or NS5) brane.

1.4 Outline

The rest of the thesis is organized as follows. In Chapter two we address the question whether the non-abelian Chern-Simons term of the D-brane action is invariant under gauge transformations of RR fields of the form $C_p \to C_p + d\Lambda_{p-1}$. This is based on work done in [59]. Detailed proofs are given in appendices A and B. In the same chapter we also obtain conceptually clear derivations of arbitrary noncommutative descriptions of D-brane actions starting from multiple lower dimensional D-branes, based on [60]. Chapter 3 of this thesis is based on [61]. We are employing perturbative techniques to show that N D4 branes can be "polarized" into NS5 branes. Remarkably, in our case, we can go beyond perturbative analysis and find the exact supergravity solution by studying the corresponding 11-dimensional supergravity picture and reducing it to 10 dimensional type IIA supergravity.

Chapter 2

Aspects of D-Brane Actions

2.1 Gauge invariance of the Chern-Simons term

Using the principle of consistency under T-duality transformation, the authors of [17; 5] extended the world-volume action for a single D-brane to the case of N coincident D-branes. Naively generalizing the single D-brane action to multiple coincident D-branes, by considering non-abelian fields and including a trace over the gauge group $U(N)$ does not lead to the correct result. This naive method fails to account, for example, for known potential terms involving commutators of the transverse scalars. The extended Chern-Simons action contains extra terms that, in general, give a non-trivial coupling between the N D-branes and a higher rank RR form. As mentioned in [6], it is not obvious whether the extended action is still invariant under gauge transformations of the type, $C_p \to C_p + d\Lambda_{p-1}$. It is the purpose of this sectionto investigate this question. It was not clear, a priori, whether to expect this to work. The fact that it does seems quite remarkable.

The world-volume action for the D$(p-1)$ branes will be written in the static gauge: one can use space-time diffeomorphisms to define the fiducial world-volume to $x^i = 0$, $i = p, \ldots, 9$, and world-volume diffeomorphisms to match the coordinates of the branes with the remaining space-time coordinates, i.e. $\sigma^a = x^a$, $a = 0, \ldots, p-1$. The transverse displacements of the branes are $\Delta x^i = (2\pi\alpha')\phi^i \equiv \lambda\phi^i$, where ϕ^i is an $N \times N$ matrix. For both the Born-Infeld part and the Chern-Simons part of the non-abelian action, the background fields are considered to be functionals of the nonabelian scalars ϕ's, as suggested in [7], while the pull-backs are defined in terms of covariant derivatives, $D_a\phi^i$, as in [8]. Furthermore the action includes a symmetrized trace prescription: we have to take a symmetrized average over all orderings of ϕ^i, $D_a\phi^i$, F_{ab}, and pairs of $\underbrace{\phi^{2k'}\phi^{2k'-1}}$ from the inner product. This prescription is in agreement with results obtained in [9] from matrix theory considerations. However, it should be noted that the symmetrized trace (STr) prescription requires corrections at order six and higher in the world-volume field strength [12].

The Chern-Simons term for N coincident D$(p-1)$ branes is given by [17],

$$S_{\rm CS} = \mu_{p-1} \int \mathrm{STr}\left(P\left[e^{i\lambda i_\phi i_\phi}\left(\Sigma C^{(n)} e^{B}\right)\right] e^{\lambda F}\right), \tag{2.1}$$

$P(\ldots)$ represents the pullback from the 10 dimensional target space to the p-dimensional D-brane world-volume, $i_\phi i_\phi$ defines an inner product, e.g., $i_\phi i_\phi C^{(2)} = \frac{1}{2}\left[\phi^j, \phi^i\right] C^{(2)}_{ij}$, F_{ab} is the gauge field strength living on the D-brane, and σ's are the coordinates parallel to the directions of the branes. We should emphasize that because of the existence of the previously defined inner product, multiple D-branes can couple to higher dimensional RR potentials, unlike for the case of a the single D-brane.

For simplicity, the gauge field living on the brane (F_{ab}) and the background NS-NS field B, are initially taken to vanish. Even for this simplified case, the demonstration of gauge invariance is rather long and subtle. We have tried to make it as clear and simple as possible. The proof involves writing the coupling as a sum of terms and then integrating these terms by parts; one uses various symmetries and identities to recombine the resulting terms into total derivatives and RR field strength terms. We refer the reader to appendices A and B for the complete analysis of gauge invariance for the $F = 0$ and $F \neq 0$ cases, where F is the $U(N)$ field strength along the branes. Here we just quote and discuss the final results.

The total coupling between N D$(p-1)$ branes and a C_{p+2k} potential can be expressed in a gauge invariant way as

$$\mu_{p-1}\sum_{r,l}\frac{\lambda^{k+1+2r+l}i^{k+r}p!}{2^r r!(k+r)!l!(p-2r-l)!}\mathrm{STr}(\overline{F}^{(2k+p+1)}_{r,l}(\phi)_{i_1 i'_1 i'_2\ldots i'_{2(k+r)-1} i'_{2(k+r)} j_1\ldots j_l a_{l+1}\ldots a_{p-2r}}\phi^{i_1}$$
$$D_{a_1}\phi^{j_1}\ldots D_{a_l}\phi^{j_l}\underbrace{\phi^{i'_{2(k+r)}}\phi^{i'_{2(k+r)-1}}}\ldots\underbrace{\phi^{i'_2}\phi^{i'_1}}F_{a_{p-2r+1}a_{p-2r+2}}\ldots F_{a_{p-1}a_p}), \tag{2.2}$$

where we defined

$$\overline{F}^{(2k+p+1)}_{r,l}(\phi)=\sum_{n\geq 0}\frac{\lambda^n}{(n)!(n+l+2k+2r+1)}\phi^{i_1}\ldots\phi^{i_n}$$
$$\partial_{x^{i_1}}\ldots\partial_{x^{i_n}}F^{0,(2k+p+1)}(\sigma,x^i)|_{x^i=0} \tag{2.3}$$

and $F^{0,(2k+p+1)}\equiv dC^{0,(2k+p)}$. Since the above coupling was derived assuming $k>0$, for $k\leq 0$ there is an additional monopole coupling term given by

$$\mu_{p-1}\frac{\lambda^{|k|}p!}{2^{|k|}(|k|)!(p-2|k|)!}C^0_{[a_1\ldots a_{p-2|k|}}F\ldots F_{a_{p-1}a_p]}. \tag{2.4}$$

In conclusion, we have obtained a manifestly gauge invariant expression for the Chern-Simons coupling between N D$(p-1)$ branes and a RR potential C_{p+2k}. In the presence of a 2-form B field, the gauge transformations of the RR fields become

$$\sum_n C^{(n)}e^B\rightarrow\sum_n C^{(n)}e^B+d\sum_p\Lambda^{(p)}. \tag{2.5}$$

The presence of the B field does not affect the generality of the previous proof since, from the point of view of the gauge transformations, we can absorb B into the definition of the RR fields. However, the proof applies only for finite N. For $N\rightarrow\infty$ we can no longer use the property of cyclicity of the trace, and we expect monopole couplings even to higher rank RR fields. As in the matrix model, one can construct a higher dimensional brane out of an infinite number of lower dimensional ones, hence in (2.1) we should have source terms for higher dimensional D-brane charges.

2.2 Noncommutative descriptions of D-brane actions

2.2.1 Introduction

The purpose of this part is to derive noncommutative descriptions of D-branes from nonabelian D-brane actions. The noncommutative descriptions are characterized by an antisymmetric 2 form Φ that appears in the expression for the noncommutative Born-Infeld determinant. This point will be explained in further detail later on. It is a known result of [21] that from the non-abelian Born-Infeld action of infinitely many D(-1) instantons one can construct the background-independent, $\Phi = -B$, description of noncommutative D-branes. Similarly, in [13; 14; 15], the same type of equivalence was shown for the Chern-Simons terms. In [16], it has been remarked that by placing D(-1) instantons in a constant B-field one can construct noncommutative D-branes with arbitrary noncommutativity. We clarify this point by starting from the action of N coincident D(-1) instantons in a constant B-field as given by [17; 18; 19]. We show that such actions lead us to construct D-brane actions in an arbitrary noncommutative description. The map relating the Born-Infeld terms is seen to be consistent with the map relating the Chern-Simons terms.

We will now review some relevant results of [17; 18] and [20]. For concreteness, we will assume Euclidean space-time and maximal rank constant B-field along the directions of a Dp-brane. We use the convention $2\pi\alpha' = 1$. Then the world-volume Dp-brane action can be described in noncommutative variables, i.e. $[x^i, x^j] = i\theta^{ij}$, as

$$\hat{S}_{\mathrm{BI}} = \frac{(2\pi)^{\frac{1-p}{2}}}{G_s} \int d^{p+1}x \sqrt{\det(G + \hat{F} + \Phi)}, \tag{2.6}$$

where the noncommutative star product is implicit in the above equation. For abelian and constant F, the Seiberg-Witten transformations relating F to $\hat{F}$ are given by

$$F = \hat{F}\frac{1}{1 - \theta\hat{F}}, \qquad \hat{F} = \frac{1}{1 + F\theta}F. \tag{2.7}$$

For every closed string background characterized by the NS-NS 2-form B, the closed string metric g, and the closed string coupling constant g_s, there is a continuum of descriptions given by a choice of Φ. The open string metric G, the open string coupling constant G_s and the noncommutativity parameter θ can be expressed in terms of closed string variables as follows:

$$\frac{1}{G+\Phi}+\theta=\frac{1}{g+B}, \tag{2.8}$$
$$G_s=g_s\left(\frac{\det(G+\Phi)}{\det(g+B)}\right)^{\frac{1}{2}}.$$

Finally, let us review the main results of [17; 18]. The remarks we have made in the previous section are very useful for understanding these results.

The non-abelian Born-Infeld action describing N (Euclidean) coincident Dp-branes in a closed string background defined by ϕ, B' and g is

$$S_{\mathrm{BI}}=\frac{(2\pi)^{\frac{1-p}{2}}}{g_s}\int d^{p+1}\sigma\mathrm{STr}\left(e^{-\phi}\sqrt{\det(P[E_{ab}+E_{ai}(M^{-1}-\delta)^{ij}E_{jb}]+F_{ab})\det(M^i_j)}\right), \tag{2.9}$$

where $E\equiv g+B'$ and ϕ is the bulk dilaton(not to be confused with the Φ introduced earlier). Furthermore, i,j are indices for the transverse coordinates, a,b are indices for the coordinates parallel to the D-brane. We also define[1]

$$M^i_j\equiv\delta^i_j-i[X^i,X^k]E_{kj} \tag{2.10}$$

where the X's are $N\times N$ matrices representing the transverse displacements expressed in the static gauge.

For the non-abelian Chern-Simons action, we have

$$S_{\mathrm{CS}}=\mu_p\int\mathrm{STr}\left(P[e^{-i(i_Xi_X)}(\sum C^{(n)}e^{B'})]e^F\right), \tag{2.11}$$

where μ_p is the RR charge of a Dp-brane. In the aforementioned actions, the bulk

[1] Unlike in [17], we used the convention $F_{ab}=\partial_aA_b-\partial_bA_a-i[A_a,A_b]$ in order to be consistent with the definition of $\hat{F}$ in [20].

fields should be considered functionals of the $N \times N$ matrices X, and the trace should be symmetrized between all expressions of the form $F_{ab}, D_a X^i, [X^i, X^j]$, and X^k. However, since we are only going to consider D(-1) instantons in constant background fields, these details are irrelevant for our purposes.

More precisely, in the next two section we consider an infinite number of D(-1) instantons with $\phi = 0$ and where g and B' are constants. The presence of the B' field will allow us to construct D-brane actions in an arbitrary noncommutative description. In Section 2.2.2, we show that the Born-Infeld action of D(-1) instantons in a constant B' field naturally leads to NC Born-Infeld action, where the B field is identified as $B = B' + \theta^{-1}$ for arbitrary noncommutativity parameter θ. Having shown this, the nonabelian generalization of the Chern-Simons action for an infinite number of D(-1) instantons should correspond to the NC Chern-Simons action in the same noncommutative description as the BI action. This fact is confirmed in Section 2.2.3.

2.2.2 Noncommutative Born-Infeld action

In this Section, we follow the line of thought in [21] and derive the equivalence of the nonabelian BI action of an infinite number of D(-1) instantons and the BI action of a noncommutative Dp-brane in a general noncommutative description. First consider the nonabelian BI action of N D(-1) branes ($N \to \infty$) in a constant B'-field:

$$S_{\rm BI} = \frac{2\pi}{g_s} \mathrm{STr} \sqrt{\det{}_{\rm ij} \left(\delta_i{}^j - i(g + B')_{ik}[X^k, X^j] \right)}. \tag{2.12}$$

We are interested in a particular classical configuration given by

$$[x^i, x^j] = i\theta'^{ij}. \tag{2.13}$$

Notice that the θ' is a measure of noncommutativity of the transverse coordinates for multiple D-branes, while the θ we introduced in the previous sectioncharacterizes the noncommutativity along the coordinates of a D-brane. The degrees of freedom on the noncommutative Dp-brane arise by expanding the matrix variable X^i around this classical configuration as follows:

$$X^i = x^i + \theta'^{ij}\hat{A}'_j. \tag{2.14}$$

Then, we have

$$i[X^i, X^j] = (\theta'\hat{F}'\theta' - \theta')^{ij}, \tag{2.15}$$

where

$$\hat{F}'_{ij} = -i\theta'^{-1}_{ik}[x^k, \hat{A}'_j] + i\theta'^{-1}_{jk}[x^k, \hat{A}'_i] - i[\hat{A}'_i, \hat{A}'_j]. \tag{2.16}$$

We can reexpress Tr over the Hilbert space as an integral over the volume of noncommutative space by replacing

$$\mathrm{Tr} \to \frac{1}{(2\pi)^{\frac{(p+1)}{2}}\mathrm{Pf}\theta'}\int d^{p+1}x, \tag{2.17}$$

where $\mathrm{Pf}\theta'$ is the Pfaffian of θ'. We write the action in terms of new variables,

$$\begin{aligned} S_{\mathrm{BI}} &= \frac{(2\pi)^{\frac{1-p}{2}}}{g_s}\int \frac{d^{p+1}x}{\mathrm{Pf}\theta'}\sqrt{\det\left[1-(g+B')(\theta'\hat{F}'\theta'-\theta')\right]} && (2.18)\\ &= \frac{(2\pi)^{\frac{1-p}{2}}}{g_s}\int d^{p+1}x\sqrt{\det\left[\theta'^{-1}-(g+B')(\theta'\hat{F}'-\mathbf{1})\right]} && (2.19)\\ &= \frac{(2\pi)^{\frac{1-p}{2}}}{g_s}\int d^{p+1}x\sqrt{\det\left[g+B'+\theta'^{-1}-(g+B')\theta'\hat{F}'\right]}. && (2.20)\end{aligned}$$

We would like to compare this with the BI action of a noncommutative Dp-brane in a description with the same noncommutativity parameter θ which appears in the above action. The NC BI action for a Dp-brane is

$$S_{\mathrm{NCBI}} = \frac{(2\pi)^{\frac{1-p}{2}}}{G_s}\int d^{p+1}x\sqrt{\det\left(G+\hat{F}+\Phi\right)}. \tag{2.21}$$

Reexpressing it in terms of closed string variables by using the relations (2.8) gives us

$$S_{\rm NCBI} = \frac{(2\pi)^{\frac{1-p}{2}}}{g_s} \frac{\sqrt{\det(g+B)}}{\sqrt{\det(G+\Phi)}} \int d^{p+1}x \sqrt{\det\left(G+\Phi+\hat{F}\right)} \tag{2.22}$$

$$= \frac{(2\pi)^{\frac{1-p}{2}}}{g_s} \int d^{p+1}x \sqrt{\det\left(g+B+(g+B)\frac{1}{G+\Phi}\hat{F}\right)} \tag{2.23}$$

$$= \frac{(2\pi)^{\frac{1-p}{2}}}{g_s} \int d^{p+1}x \sqrt{\det\left(g+B+(1-(g+B)\theta)\hat{F}\right)}. \tag{2.24}$$

We observe that (2.12) agrees with (2.21) once we make the following identifications:

$$\theta = \theta', \;\; \hat{F} = \hat{F}', \;\; B = B' + \theta'^{-1}. \tag{2.25}$$

Notice that here θ is a free parameter, not fixed to be B^{-1} as in [21]. By identifying B' in the nonabelian action for N D(-1) instantons ($N \to \infty$) with $B - \theta^{-1}$, we can go to the noncommutative description of Dp-brane with arbitrary noncommutativity parameter θ. It is interesting to note that Φ takes the following form in matrix-model-like variables:

$$\Phi = -\theta^{-1}\left(1 + (g+B')_A^{-1}\theta^{-1}\right), \tag{2.26}$$

where A denotes the antisymmetric part.

2.2.3 Noncommutative Chern-Simons action

If the nonabelian BI action for an infinite number of D(-1) instantons in a constant B' field gives rise to the NC BI action with $B = B' + \theta^{-1}$ and noncommutativity parameter θ, then we should expect the same identification to relate the Chern-Simons term of the nonabelian action with that of the NC theory. This is precisely what occurs, and the Chern-Simons action for a Dp-brane with a constant B field and noncommutativity θ can be expressed as the nonabelian CS action for an infinite number of D(−1) branes in a constant B' field given by [17]

$$S_{\rm CS}=\frac{2\pi}{g_s}{\rm STr}\left[e^{-i({\rm i}_X{\rm i}_X)}\sum_n C^{(n)}e^{B'}\right],\quad B'=B-\theta^{-1}. \tag{2.27}$$

Here i_X acts on an n-form $\omega^{(n)}$ as

$$\mathrm{i}_X\omega^{(n)}=\frac{1}{(n-1)!}X^{\nu_1}\omega^{(n)}_{\nu_1\nu_2...\nu_n}dx^{\nu_2}...dx^{\nu_n}. \tag{2.28}$$

This provides a natural explanation of the rather surprising result recently derived by [14], where they express an arbitrary NC CS action in terms of matrix-model like variables, which turns out to be identical to (2.27). For simplicity, we follow the proof of [22] to show that the nonabelian action gives rise to the NC action for D9-branes, where we can ignore transverse scalar fields. In that case, the NC CS action is given by [14; 22]

$$S_{\rm NCCS}=\mu_9\int_x\sqrt{\det(1-\theta\hat{F})}\sum_n C^{(n)}e^{B+\hat{F}(1-\theta\hat{F})^{-1}}, \tag{2.29}$$

where $\mu_9=(2\pi)^{-4}/g_s$ is the RR charge of a BPS D9-brane. In terms of $Q=-\theta+\theta\hat{F}\theta$, (2.29) can be expressed as

$$S_{\rm NCCS}=\mu_9\int_x\sqrt{\det(1-\theta\hat{F})}\sum_n C^{(n)}e^{B'}e^{-Q^{-1}}. \tag{2.30}$$

The nonabelian CS action for an infinite number of D(−1) instantons (2.27) naturally leads to the NC CS action for Dp-branes (2.29). Expanding the action (2.27) and using the fact that $i[X,X]=Q$ give terms of the form

$$\frac{2\pi}{g_s}\frac{(10-2r)!}{2^{5-r}(s-r)!(5-r)!2^{s-r}(10-2s)!}\times \\ {\rm Tr}\left[Q^{i_{2r+1}i_{2r+2}}...Q^{i_9i_{10}}B'_{[i_{2r+1}i_{2r+2}}...B'_{i_{2s-1}i_{2s}}C^{(10-2s)}_{i_{2s+1}...i_{10}]}\right], \tag{2.31}$$

where [...] denotes antisymmetrization and $5 \geq s > r \geq 0$. Employing the identity (2.17), one gets

$$\mu_9 \int d^{10}x \frac{(10-2r)!}{2^{5-r}(5-r)!(s-r)!2^{s-r}(10-2s)!\mathrm{Pf}\theta} Q^{i_{2r+1}i_{2r+2}}...Q^{i_9 i_{10}} \times B'_{[i_{2r+1}i_{2r+2}}...B'_{i_{2s-1}i_{2s}} C^{(10-2s)}_{i_{2s+1}...i_{10}]}. \tag{2.32}$$

Finally, the above expression can be simplified to

$$\mu_9 \int d^{10}x \frac{\mathrm{Pf}Q(-1)^r}{\mathrm{Pf}\theta 2^s r!(s-r)!(10-2s)!} \epsilon^{i_1...i_{10}} Q^{-1}_{i_1 i_2}...Q^{-1}_{i_{2r-1}i_{2r}} B'_{[i_{2r+1}i_{2r+2}}...B'_{i_{2s-1}i_{2s}} C^{(10-2s)}_{i_{2s+1}...i_{10}]}. \tag{2.33}$$

One can immediately see that (2.33) are the terms coming from the expansion of (2.29). We have shown that our claim holds for the special case $p = 9$. The general case has been already considered in [14].

Up to now, we have restricted the Ramond-Ramond fields to be constants, but we can generalize our procedure to the case where the Ramond-Ramond fields are varying by writing the fields as fourier transforms[2] such that

$$S_{\mathrm{CS}} = \frac{2\pi}{g_s} \int d^{10}q \mathrm{STr} \left[e^{-i(\mathrm{i}_X \mathrm{i}_X)} \sum_n C^{(n)}(q) e^{B'} e^{iq\cdot X} \right], \quad B' = B - \theta^{-1}. \tag{2.34}$$

To conclude, motivated by the identification relating the nonabelian BI action of D(-1) instantons to the BI action of Dp-branes in the last section, we have proposed and verified that the NC CS action of a Dp-brane with arbitrary noncommutativity and varying Ramond-Ramond fields can be derived from considering the nonabelian CS action for an infinite number of D(-1) branes after identifying $B' = B - \theta^{-1}$.

Finally, let's remark that since $B = B' + \theta^{-1}$, the freedom of description of NC Dp-branes translates in the matrix-model-like variables into how one separates the B-field into the external part B' and the internal part θ^{-1}. The internal part, θ^{-1}, is generated by the configuration of D(-1) instantons and B' corresponds to the external field imposed on them.

[2]See [23] for how to relate the currents expressed in matrix model language to those in noncommutative gauge theory.

Chapter 3

$N = 2$ Supergravity Solutions with Polarized Branes

3.1 Introduction

In this chapter we construct several classes of exact supersymmetric supergravity solutions describing D4 branes polarized into NS5 branes and F-strings polarized into D2 branes. These examples belong to the same universality class as the perturbative solutions used by Polchinski and Strassler to describe the string dual of $\mathcal{N} = 1^*$ theories. The D4-NS5 setup can be interpreted as a string dual to a confining $4+1$ dimensional theory with 8 supercharges, whose properties we discuss. By T-duality, our solutions give Type IIB supersymmetric backgrounds with polarized branes.

Ever since the remarkable discovery of the AdS-CFT duality [27] there has been a lot of interest in finding supergravity duals to four-dimensional field theories with reduced supersymmetry, and to use these duals to understand real-world phenomena like confinement and the generation of a mass gap.

In several cases the supergravity dual of the field theory is pure geometry [28; 29; 30; 31; 32; 33; 34], and the exact supergravity solution, although challenging, was found. In other cases, like the $\mathcal{N} = 1^*$ theory, the string/supergravity dual (found by Polchinski and Strassler in [38]) contains D3 branes polarized into 5 branes, and the exact geometry is still not known.

We attempt to make one step in that direction. We find exact supergravity solutions with polarized branes and with 8 supercharges. These solutions describe D4 branes polarized into NS5 branes, and F1 strings polarized into D2 branes. They are very similar to the Polchinski-Strassler (PS) case, both because polarization takes place in the near horizon geometry of the branes, and because the fields inducing it are tensor harmonics on the transverse space.

In fact, by T and S duality, these solutions give type IIB exact solutions containing D3 branes smeared along one direction, which polarize into cylindrical NS5 or D5 branes. These solutions are dual to a limit of the Coulomb phase of the $\mathcal{N} = 4$ Super Yang Mills, which can have screening or confining vacua when the $\mathcal{N} = 4$ supersymmetry is broken to $\mathcal{N} = 2$. As we will see, the radius and orientation of the cylinders parametrize a moduli space of vacua, for each type of (p, q) 5-brane.

We will first perform a perturbative investigation of the polarization of D4 branes into NS5 branes, along the lines of [38]. As explained in [50], supergravity in the near-horizon geometry of D branes describes a certain strongly coupled regime of the field theory living on these branes. Both sides of this duality can be perturbed. Introducing an operator in the Lagrangian of the field theory side is dual to turning on a non-normalizable mode of the corresponding supergravity field in the bulk [51].

In the Polchinski-Strassler case, the 3+1 dimensional $\mathcal{N} = 4$ Super Yang Mills theory was perturbed to the $\mathcal{N} = 1^*$ theory by giving mass to the 3 chiral multiplets. This was dual to perturbing the $AdS_5 \times S^5$ geometry with RR and NS 3 forms along the space transverse to the branes. These forms were responsible for polarizing the D3 branes in (p, q) 5 branes. The resulting setups were dual to the different phases of the $\mathcal{N} = 1^*$ theory, and made visible many features of this theory.

In Section 3.2, we similarly perturb the near horizon background of a large number of D4 branes with the operator corresponding to a mass term for the chiral multiplet in the 4+1 dimensional $\mathcal{N} = 1$ theory on the branes. This operator preserves 8 of the original 16 supercharges, and transforms in the **10** of the $SO(5)$ R symmetry group. It corresponds in the supergravity dual to a non-normalizable mode of the RR 2-form and NS 3-form field strengths on the 5-dimensional space transverse to the branes.

We will find that N D4 branes can polarize into k NS5 branes only for a very specific value of transverse field perturbation: $F_2 \sim \frac{k}{N g_s \sqrt{\alpha'}}$. For all other values no polarization happens. Moreover, our analysis shows that the polarization radius is a modulus. An identical phenomenon happens when F1 strings polarize into D2 branes [36].

Since the radius is a modulus, it is natural to suspect that these configurations could descend from a Coulomb branch configuration of M5/M2 branes in M-theory. Moreover, all the fields present could descend from the fields of the M5/M2 brane supergravity solution by a twisted Melvin reduction. It is therefore not hard to see what the full picture is.

If we have, for example, N M5 branes uniformly spaced on a circle, the angle between two of them is $\Delta\phi = 2\pi/N$. If one compactifies with a twist of $2\pi/N$, the upper end of an M5 brane is joined with the lower end of its neighboring M5 brane (figure 3.1). Thus, the whole Coulomb branch descends into a configuration of N D4 branes polarized into **one** NS5 brane. If one increases the twist k times, the upper end of an M5 brane is joined with the lower end of its k th neighbor, and this gives k chains of M5 branes, which descend into N D4 branes polarized into k NS5 branes.

For all values of the twist that do not match an M5 brane end with another, the descending configuration has no type IIA brane interpretation (it would be like N D4 branes polarized into a configuration with a noninteger NS5 brane charge). Therefore, compactifications with twists that do not match the brane ends only give consistent type IIA solutions when all the 5 branes are coincident.

Since the RR 2-form and NS 3-form field strengths acting on the 5 dimensional space transverse to the D4 branes are proportional to the twist, we can see that the above picture matches perfectly the one obtained via the Polchinski-Strassler analysis. The discrete set of values of the fields for which the D4 branes polarize corresponds to the discrete set of twists compatible with the M5 branes being on the Coulomb branch. Moreover, the Killing vectors of the M-theory solution do not depend on the radius. Hence, a twist by $2k\pi/N$ will match the brane ends at any radius. This implies that the descending configuration will be a solution at any radius, and therefore the polarization radius is a modulus, exactly as the field theory analysis implies. As an aside we note that, if the Killing vectors had different radial dependence, the twist would match the ends of neighboring branes only at certain values of the radius. In this situation the radius is no longer a modulus.

The immediate bonus of the above picture is its application for finding exact Polchinski-Strassler(PS)-like IIA solutions with polarized branes by simply reducing with a twist M-theory supergravity solutions with branes spread on a circle. In Section 3.3 we will find these solution, and show that they reduce to the first order solution obtained in Section 3.2. We will also link the boundary theory fermion mass parameters to the M-theory twists and show that the supergravity solution preserves 8-supercharges, just as expected from the gauge/gravity analysis.

One can also give an identical description to the polarization of F1 strings into D2 branes described in [36]. In that case the M2 branes on the Coulomb branch are compactified with a twist which matches their ends. This gives a geometry with F1 strings polarized into D2 branes. The radius is again a modulus, and this is consistent with the Killing vectors for x_{11} and ϕ having no radial dependence. The compactification twist preserves 8 supercharges, and can be again related to the masses of the fermion bilinears turned on in the boundary theory to induce polarization. This exact solution is discussed in Section 3.4.

In fact, both the twisted M2 and M5 supergravity backgrounds (without the branes being polarized) have recently been obtained by Figueroa O'Farill and Simon [39]. These solutions are basically superpositions of the supersymmetric flux 5 brane with D4 branes and F1 strings respectively. The new feature of our supergravity solutions is that for certain values of the fluxes, the D4 branes/F1 strings can polarize into NS5/D2 branes, and that moreover, the polarization radius is a modulus. Thus, the most general $\mathcal{N} = 2$ exact solution we can write contains several D4-NS5 (or F1-D2) concentric circles of different radii, and different orientations. One can also generate F1-D2 solutions with $\mathcal{N} = 1$ supersymmetry, which can have 2 different kinds of F1-D2 solutions, at various radii and orientations.

Using our methods it is also possible to obtain nonsupersymmetric exact solutions with polarized branes [1]. Indeed, as long as the twist along the circle where the branes are placed matches their ends, one can twist along other directions by arbitrary amounts, and still obtain a good solution. Supersymmetry was necessary in PS-like setups to control the backreaction of the various fields on the metric. However, here we have the **exact** metric, with the polarized branes, and we know that our setup is a solution simply because it is the compactification of an M-theory solution along a Killing vector direction.

In Section 3.5 we use T-duality to obtain exact Type IIB supergravity backgrounds containing D3 branes polarized into cylindrical (p, q) 5-branes. The origin of these solutions suggests that they are dual to the Coulomb branch of the $\mathcal{N} = 2^*$ theory in the limit when the number of D3 branes becomes infinite and the distance between them is kept fixed.

However, these solutions are not asymptotically AdS. The dual field theory cannot therefore be interpreted as a UV-finite deformation of the $\mathcal{N} = 4$ Super Yang Mills. In a way this theory is similar to the one that is dual to the Klebanov-Strassler flow [28], in that the rank of the gauge group grows as one goes to higher and higher energies. This theory has confining, screening, and oblique vacua, much like the one studied by Polchinski and Strassler. In fact, when one of the $\mathcal{N} = 1^*$ masses becomes much smaller than the others, the D3 branes polarize into a very elongated ellipsoid [38]. In the limit that this mass goes to zero while the thickness of the ellipsoid is kept fixed, the ellipsoid degenerates into a cylinder. As we will discuss in Section 3.2, the background with D4 branes polarized into NS5 branes is dual to a 4+1 dimensional theory with 8 supercharges. Since, when the branes are polarized, supergravity is valid everywhere, the corresponding phases of the 4+1 dimensional theory have no weakly coupled field theory description. Thus, they can only be described by their supergravity dual, much like the (2,0) and little string theories. In Section 3.6 we investigate the phase structure and the objects of this theory. We will find phases in which electric quarks are confined and "magnetic little strings" are screened. The exact supergravity dual allows us to find the tension of the confining flux tubes and the

[1] Such nonsupersymmetric solutions have been obtained in the past via Melvin reductions [42].

masses of the baryons. The theories dual to the nonsupersymmetric exact solutions can also be investigated, and exhibit similar phenomena.

3.2 Polarizing D4 branes into NS5 branes - the gauge theory/supergravity picture

In type IIA supergravity a D4-brane is a BPS object that has world-volume symmetry group $SO(1,4)$ and transverse symmetry group $SO(5)$. We will consider a system of parallel D4-branes. As explained in [50], 10-dimensional type IIA supergravity has a solution that describes the near-horizon geometry of a large number N of D4 branes. In the string frame the solution is

$$\begin{aligned} ds^2 &= Z^{-1/2} dx_{\parallel}^2 + Z^{1/2} dx_{\perp}^2 \\ e^{\Phi} &= g_s Z^{-1/4} \\ C_{01234} &= \frac{1}{g_s Z}. \end{aligned} \tag{3.1}$$

The D4 branes are aligned along the 0-4 dimensions, and $dx_{\parallel}^2 = \eta_{\mu\nu} dx^{\mu} dx^{\nu}$, where $\eta_{\mu\nu}$ is the Minkovski metric along the D4 branes directions, while $dx_{\perp}^2 = (dx^i)^2$, $i = \overline{5,9}$. When the branes are coincident(located at $r^2 \equiv x_{\perp}^2 = 0$), the harmonic function Z is given by

$$Z = \frac{\pi N g_s \alpha'^{3/2}}{r^3} \equiv \frac{R^3}{r^3}.$$

The same function Z appears both in the expression for the metric and for the electrically coupled RR potential $C^{(5)}$, as a consequence of supersymmetry. Notice that Z satisfies $\partial_{x_{\perp}}^2 Z = 0$. However Z fails to be harmonic at the origin, $r = 0$, where the horizon of the D4-brane geometry is located.

In the dual picture this supergravity solution describes a certain strongly coupled regime of the field theory living on these branes. Both sides of this duality can be perturbed. We can introduce a hypermultiplet mass in the Lagrangian of the field theory; this corresponds in the bulk to turning on a supergravity non-normalizable mode of the RR 2-form and NS 3-form field strengths on the directions transverse to the branes [51]. Indeed, the boundary fermions transform in the **4** of the $SO(5)$ R symmetry group, and therefore the fermion mass in the **10** has the same representation as a 2 or 3 form on the 5-dimensional space transverse to the branes.

Our plan is to we perturb the background (3.1) by a transverse RR 2 form F_2 and a NS 3 form H_3, and then find the supergravity solution to first order in the perturbation parameters. This solution is the dimensional reduction of the one used in [37] to explore the polarization of M5 branes into Kaluza-Klein monopoles, so many of the equations will be similar.

By expanding the IIA supergravity equations of motion:

$$\begin{aligned} d * F_2 &= *F_4 \wedge H_3 \\ 2d(e^{-2\Phi} * H_3) &= F_4 \wedge F_4 - 2d(*F_4 \wedge C_1) \end{aligned} \tag{3.2}$$

about the background (3.1), we find that the first-order perturbation fields satisfy

$$\begin{aligned} d\left(\frac{1}{Z}(*_5 H_3 + g_s F_2)\right) &= 0 \\ d\left(\frac{1}{Z}(g_s *_5 F_2 + H_3)\right) &= 0 \\ dF_2 = 0 &= dH_3, \end{aligned} \tag{3.3}$$

where $*_5$ is the flat Hodge operator on the transverse 5-dimensional space[2]. The metric, dilaton, and 6-form field strength (or its Hodge dual F_4) only receive 2nd order corrections coming from the backreaction of F_2 and H_3.

The derivation of (3.3) is rather straightforward if one uses the fact that both F_2 and H_3 act along the 5-dimensional transverse space and makes use of the following

[2]These equations are very similar to the ones satisfied by the perturbation in [38] (Eqns. 25,27).

relations:

$$
\begin{aligned}
*H_3 &= \frac{1}{Z^{3/2}}(*_5 H_3) \wedge dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 \\
*F_2 &= \frac{1}{Z}(*_5 F_2) \wedge dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4. \qquad (3.4)
\end{aligned}
$$

We should notice that the 2 form $\frac{1}{Z}(*_5 H_3 + g_s F_2)$, is harmonic, and thus it is given by its value at infinity. In particular, if one changes Z, the form of F_2 and H_3 might change, but the combination $\frac{1}{Z}(*_5 H_3 + g_s F_2)$ does not. Also, since $F_4 \wedge F_4 = 0$, eq. (3.2) implies that the NS 6-form potential B_6 satisfies

$$
dB_6 = e^{-2\phi} * H_3 + *F_4 \wedge C_1. \qquad (3.5)
$$

We must now relate the precise form of the supergravity perturbations with the fermion bilinears that are turned on, by analyzing their R-symmetry properties. Luckily, this work has already been done in [37]. The theory along the D4-branes has a field content consisting of a 5-dimensional hypermultiplet and a vector multiplet. The five real scalars correspond to the transverse directions and transform in the vector representation of the R-symmetry group. By pairing the 4 world-volume fermions and 4 of the transverse space coordinates into complex combinations

$$
Z_1 = x^5 + ix^6 \qquad Z_2 = x^8 + ix^9 \qquad (3.6)
$$

$$
\Lambda_1 = \lambda_1 + i\lambda_3 \qquad \Lambda_2 = \lambda_2 + i\lambda_4, \qquad (3.7)
$$

we can see that under an $SO(5)$ rotation $Z_i \to e^{i\phi^i} Z_i$, $i = 1, 2$ the fermions transform as

$$
\Lambda_1 \to e^{i(\phi^1 - \phi^2)/2} \Lambda_1 \qquad (3.8)
$$

$$
\Lambda_2 \to e^{i(\phi^1 + \phi^2)/2} \Lambda_2. \qquad (3.9)
$$

The factor of 1/2 in the exponents appears since the fermions transforms as spinors under $SO(5)$. We consider a diagonal mass term of the form:

$$m\text{Re}[\Lambda_1^2] + m'\text{Re}[\Lambda_2^2]. \tag{3.10}$$

Thus, a fermion mass term behaves in the same way under SO(5) rotations as

$$T_2 = \text{Re}[m dZ_1 \wedge d\bar{Z}_2 + m' dZ_1 \wedge dZ_2] \equiv \frac{1}{2} T_{ij} dx^i \wedge dx^j. \tag{3.11}$$

We are interested in giving mass to half of the world-volume fermions (together with their corresponding scalars). This preserves $\mathcal{N} = 1$ supersymmetry in 4+1 dimension (8 supercharges), and corresponds to $m' = 0$. For future reference, we should note that in this case the perturbation breaks the $SO(5)$ R symmetry to $U(1)$. Besides T_2 there exists another 2-tensor with exactly the same $SO(5)$ transformation properties:

$$V_2 = \frac{1}{2}(\frac{x^q x^i}{r^2} T_{qj} + \frac{x^q x^j}{r^2} T_{iq}) dx^i \wedge dx^j. \tag{3.12}$$

Thus, a general 2 form corresponding to the fermion mass will be a linear combination of T_2 and V_2, with r-dependent coefficients. Similarly, the 3 form will be a combination of the duals of these tensors [3]. In order to find the 1-form potentials that give the aforementioned 2-form field strength it is also useful to introduce the 1 form:

$$S_1 = T_{mn} x^m dx^n \tag{3.13}$$

satisfying

$$d(S_1) = 2T_2, \qquad d(r^p S_1) = r^p(2T_2 + pV_2). \tag{3.14}$$

In order to obtain the first-order perturbation corresponding to the fermion mass (3.11) one has to find the form that solves (3.3) and can be written as a combination of T_2 and V_2. The equations are identical to the ones in [37]. They have four solutions, given in eq. (2.22) of [37]. These solutions are the normalizable and non-normalizable modes dual to a fermion mass and to another irrelevant operator.

[3]Several useful identities involving these tensors are given in Appendix C.

One can see both from the M-theory picture [37] or by direct analysis that the non-normalizable mode dual to a fermion mass operator is

$$\begin{aligned} g_s F_2 &= Z(2T_2 - 3V_2) = d(ZS_1) \\ *_5 H_3 &= 3ZV_2. \end{aligned} \tag{3.15}$$

This Z is the factor appearing in the equation (3.1), not to be confused with Z_1 and Z_2 defined in (3.6). Note that the actual boundary fermion mass term is not the parameter m appearing in this supergravity solution through T_2 (3.11), but is proportional to it [38; 48; 40]. One can use these fields to compute the value of the 6 form NS field which couples electrically to NS5 branes:

$$\begin{aligned} d(B_6 - C_5 \wedge C_1) &= e^{-2\Phi} * H_3 + C_5 \wedge F_2 \\ &= \frac{1}{g_s^2 Z}(*_5 H_3 + g_s F_2) \wedge dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 \\ &= 2g_s^{-2} T_2 \wedge dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4. \end{aligned} \tag{3.16}$$

Since the expression $B_6 - C_5 \wedge C_1$ only depends on the harmonic combination $\frac{1}{Z}(*_5 H_3 + g_s F_2)$, its value is given by the boundary conditions only and does not change when Z changes.

To determine whether the solution (3.1), (3.15) allows the D4 branes to be polarized into NS5 branes, one must first find the potential of a probe NS5 brane with large D4 charge n (such that $n \ll N$) in the geometry created by the N D4 branes. One can thereafter find the potential for **all** the N D4 branes to be polarized into several NS5 brane shells by treating each shell as a probe in the geometry created by the others.

The action of type IIA NS5 branes is not an easy one to handle, and was found rather recently [43] by reducing the action of the M-theory M5 brane [45; 44]. Fortunately, the components responsible for the D4 charge have a rather simple form. If all the brane and bulk 3 form fields are turned off, the action becomes

$$S_{\rm BI} = \tau_5 \int d^6\xi e^{-2\Phi}\sqrt{-\det(g_{ij} - e^{2\phi}\mathcal{F}_i\mathcal{F}_j)} \tag{3.17}$$

$$S_{\rm WZ} = \tau_5 \int B_6 - C_5 \wedge C_1 + C_5 \wedge \mathcal{F}_1, \tag{3.18}$$

where $\mathcal{F}_1 \equiv F_1 + C_1$, and $F_1 = da$ is the field strength of the scalar living on the NS5 world-volume. This scalar descends from the M5 brane scalar describing its position on the M-theory circle. Thus, it is no wonder that a nontrivial value of F_1 corresponds to a nonzero D4 charge. Moreover, we can see from (3.18) that to give a circular NS5 brane the D4 charge n, one needs to turn on an F_1 such that[4]:

$$\int_0^{2\pi} F_\phi d\phi = \frac{n\tau_4}{\tau_5}, \tag{3.19}$$

which, assuming F_ϕ constant, implies $F_\phi = \frac{n\tau_4}{2\pi\tau_5} = n\alpha'^{1/2}$, where τ_4 and τ_5 are the D4 and NS5 brane tensions respectively. We assume that the NS5 brane probe has D4 charge n, and geometry $S^1 \times R^5$, where the S^1 lies in the ij plane, i and j being two of the transverse directions. The action per unit 4+1 volume in the geometry (3.1,3.15) has the Born-Infeld part:

$$\begin{aligned} V_{\rm BI} &= 2\pi Z^{1/2}\tau_5 g_s^{-2}\sqrt{g_\parallel}\sqrt{g_{\phi\phi} + e^{2\Phi}\mathcal{F}_\phi\mathcal{F}_\phi} \\ &= 2\pi Z^{-3/4}\tau_5 g_s^{-2}\sqrt{Z^{1/2}r^2 + g_s^2 Z^{-1/2}\left(\frac{n\tau_4}{2\pi\tau_5} + C_\phi\right)^2}. \end{aligned} \tag{3.20}$$

As one can see, the first and the second terms under the square root represent, respectively, the NS5 and the D4 contribution to the mass of the probe. We are interested in the limit where the D4 contribution dominates the NS5 contribution. In this limit the Born-Infeld action can be Taylor expanded as

$$V_{\rm BI} \approx Z^{-1}g_s^{-1}(n\tau_4 + 2\pi\tau_5 C_\phi) + \frac{2\pi\tau_5 r^2}{2ng_s^3\alpha'^{1/2}} + ... \tag{3.21}$$

[4] The argument for F^1 being quantized (as opposed to $\mathcal{F}^1$) is similar to the one put forth in [35] for the D-brane world-volume 2 form.

The first term represents the gravitational attraction between the N D4 branes sourcing the geometry and the n D4 branes in the probe. The second is the "left over" mass from the NS5 brane.

The Wess-Zumino action (3.18) similarly contains two terms, one representing the RR 3-form mediated repulsion between the D4 branes, and the second coming from the integral of B_6 (3.16) over the world volume:

$$V_{\rm WZ} = -\frac{Z^{-1}(n\tau_4 + 2\pi\tau_5 C_\phi)}{g_s} - \frac{2m\pi\tau_5 r^2}{g_s^2}. \tag{3.22}$$

As expected, due to supersymmetry, the leading contributions in the WZ and BI actions coming from interactions between parallel D4 branes cancel each other. Thus, the probe action seems to be given by the two remaining terms in (3.21,3.22):

$$V_{\rm naive} = \frac{2\pi\tau_5 r^2}{2ng_s^3\alpha'^{1/2}} - \frac{2m\pi\tau_5 r^2}{g_s^2}. \tag{3.23}$$

Nevertheless, there exists another term in the action which comes from the interaction of the n D4 branes with the backreaction of the first-order fields (3.15) on the metric and dilaton. In the next section, we will find the exact form of the metric, which allows one to determine this term exactly. However, we can also determine this term using the fact that our setup is supersymmetric, and thus the effective potential for the probe comes from a superpotential. As we will see, the two procedures give the same result, which confirms the validity of our approach. To obtain the superpotential, it is helpful to express the potential in terms of complex variables. We can also consider a more generic probe, by allowing the transverse circle to deform into an ellipse. If Z_1 and Z_2 (defined as in eq. (3.6)) give the length and orientation of the two semiaxes of the ellipse, then $V_{\rm naive}$ becomes

$$V_{\rm naive} = \frac{\pi\tau_5}{2ng_s^3\alpha'^{1/2}}\left(|Z_1|^2 + |Z_2|^2 - 4mng_s\alpha'^{1/2}{\rm Re}(Z_1\bar{Z}_2)\right), \tag{3.24}$$

and it is not hard to see that it contains two of the three terms coming from the superpotential:

$$W \sim Z_1 Z_2 - \frac{1}{2} m n g_s \alpha'^{1/2} (Z_1^2 + Z_2^2). \tag{3.25}$$

The full potential of the probe is then

$$V_n = \frac{\pi \tau_5}{2 n g_s^3 \alpha'^{1/2}} \left(|Z_1 - Z_2 m n g_s \alpha'^{1/2}|^2 + |Z_2 - Z_1 m n g_s \alpha'^{1/2}|^2 \right), \tag{3.26}$$

and its minima are at

$$Z_1 = m n g_s \alpha'^{1/2} Z_2, \qquad Z_2 = m n g_s \alpha'^{1/2} Z_1 \tag{3.27}$$

Evidently the only nontrivial solutions are obtained for

$$m = \pm \frac{1}{n g_s \alpha'^{1/2}}. \tag{3.28}$$

This implies that for some special values of the parameter m, the radius and orientation of the polarization configuration combine to form a complex modulus. For all other values, the only solution is $Z^1 = Z^2 = 0$, so there is no polarization. This rather surprising result has subsequentely been reconfirmed in a Dijkgraaf-Vafa type analysis in [58].

One should furthermore notice that the polarization potential does not depend on the specific form of the harmonic function Z. If the metric is of the form (3.1), the perturbation (3.15) is weaker then the background, and the energy of the probe comes predominantly from D4 branes, then Z does not enter the first term of the potential. Moreover, Eq. (3.16) implies that Z does not influence $B_6 - C_5 \wedge C_1$, which gives the second term of the potential. Since the third term is related to the first two by supersymmetry, it likewise has no Z dependence. Thus, the probe potential is independent of the positions of the N D4 branes that source the geometry. Therefore, we can find the full potential of the N D4 branes polarized into several rings of NS5 branes by treating each ring as a probe in the geometry created by the others. The potential is just

$$V_{\text{full}} = \sum_i V_{n_i}, \tag{3.29}$$

where n_i is the D4 brane charge of the i th tube. For a given m, only the tubes with $n_i = \frac{1}{g_s \alpha'^{1/2} m}$ can have a nonzero radius. It is also possible to superpose several of these tubes, and obtain tubes with $k \times n_i$ D4 branes polarized into k NS5 branes. The energy of such a tube is k times the energy of a simple tube. One can also extrapolate this formula to find that the potential for all N D4 branes to be polarized into one NS5 brane is given by simply replacing n by N in (3.26).

Notice that the orientation of the NS5 in the x^5x^6 and x^8x^9 planes is given by the phases of the complex-valued Z_1 and Z_2. When polarization occurs these phases can be arbitrary as long as they are equal to each other, as can be seen from (3.27). We have found a very interesting phenomenon. For certain values of the polarizing field strength the generic configuration consists of several rings of D4 branes polarized into NS5 branes at generic radii and generic orientations in the x^5x^6 and x^8x^9 planes. For other values, no solution with polarized branes exists. In the next chapter we will see how this phenomenon beautifully emerges from M-theory.

3.3 The exact supergravity solution describing the D4 $\rightarrow$ NS5 polarization

In this section we will find the M-theory description of the polarized D4 brane configuration found in the previous chapter. This enables us to find the exact type IIA supergravity solution containing these polarized branes. Moreover, this description provides an intuitive geometric explanation of the moduli space of polarization vacua we found perturbatively. The strong coupling limit of type IIA string theory is 11-dimensional supergravity. The bosonic sector of the 11-dimensional theory contains only the metric and the three-form gauge potential that couples electrically to M2-branes and magnetically to M5-branes. The M5-brane is a BPS object with world-volume symmetry group $SO(1,5)$ and transverse symmetry group $SO(5)$.

Ten-dimensional type IIA supergravity solutions can be obtained starting with a 11-dimensional supergravity background and then performing a Kaluza-Klein dimensional reduction along a spacelike direction. However, to be able to perform the reduction we need the existence of a symmetry of the 11-dimensional background, more exactly we need a Killing vector field along which the three-form gauge potential is constant. To dimensionally reduce the background we first perform an identification along the direction of the Killing vector and use the symmetry to remove the dependence of the background on the coordinate along the Killing vector orbit.

We will consider the solution describing a system of parallel M5-branes. This supersymmetric solution admits 16 supercharges. An interesting question is how much of the initial supersymmetry remains after a dimensional reduction. The answer is given by the number of 11-dimensional Killing spinors that remain invariant along the orbits of the Killing vectors used for the reduction [39; 41]. Let us start by consider the near horizon 11-dimensional supergravity background of N parallel M5 branes:

$$\begin{aligned} ds^2 &= Z^{-1/3}dx_{\parallel}^2 + Z^{2/3}dx_{\perp}^2 \\ \hat{F}^7 &\equiv *\hat{F}_4 = d(Z^{-1}) \wedge dx^0 \wedge ...dx^4 \wedge dx^{11}, \end{aligned} \tag{3.30}$$

where the branes are aligned along the $0, 1, 2, 3, 4,$ and 11 directions, and Z is a harmonic function on the transverse space. When the branes are coincident

$$Z = Z_0 = \frac{R_{M5}^3}{r^3}, \quad r^2 = x^i x^i, \quad R_{M5}^3 = N\pi l_p^3, \tag{3.31}$$

where i runs over the 5 transverse directions. For non-coincident branes, Z is a superposition of the harmonic functions sourced by the individual branes. If the M5 branes are smeared on a circle of radius r_0 in the $\rho - \phi$ plane, Z is given by:

$$Z = \frac{R_{M5}^3}{2\pi} \int_0^{2\pi} \frac{d\phi}{(\mathbf{x}^2 + \rho^2 + r_0^2 - 2r_0\rho\cos\phi)^{3/2}}, \tag{3.32}$$

where $\mathbf{x}$ denotes the other 3 transverse directions.

As we explained in the Introduction to this chapter, the polarized state from the previous section can be obtained by uniformly distributing the N M5 branes on a transverse circle of radius r_0 and performing a dimensional reduction along $\partial_{\tilde{11}} = \partial_{11} + B\partial_\phi$. This is the type of reduction that gives the usual Melvin background.

However, in our case only those twists that identify the upper end of one brane with the lower end of the other are consistent with the setup. These are twists by multiples of $\frac{2\pi}{N}$. The smallest twist joins neighboring M5 branes; the N M5 branes join to form one "slinky-like" object, which when reduced to type IIA becomes a circular NS5 brane with D4 charge N. Larger twists join branes which are further apart, and thus give several slinkies. In general, if

$$B = \frac{1}{2\pi R^{11}} \left(\frac{2k\pi}{N} \right), \tag{3.33}$$

we obtain N D4 branes polarized into k NS5 branes.

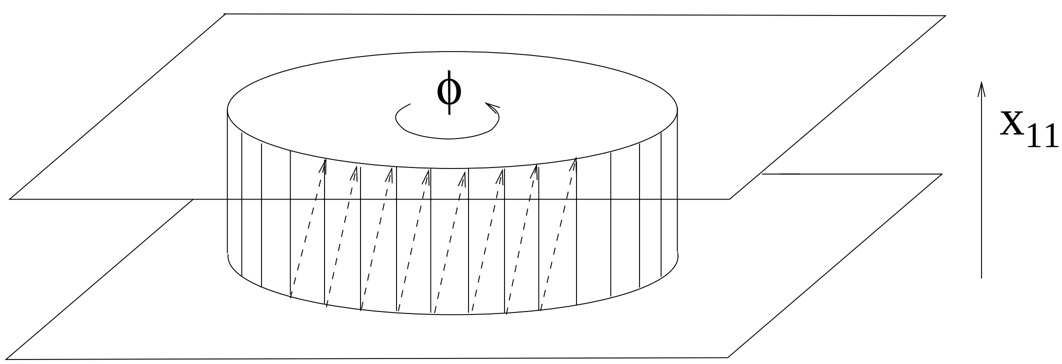

Figure 3.1: The twisted compactification of the M5 branes.

To obtain a Type IIA background, one needs to smear the M5 branes along the circle. Naively, this seems to allow a twist by an arbitrary B. Nevertheless this would give a configuration with a non-integer NS5 brane charge, which is non-physical. The condition that locally the NS5 charge be quantized is equivalent to the constraint (3.33) on the possible values of the shifts.

For large enough N, the discretely arrayed branes are seen in supergravity as smeared. Indeed, if the distance between two M5 branes on the slinky $(\frac{2\pi r_0}{N})$ is smaller than the radius where the curvature created by one brane becomes larger than the string length, supergravity is only valid away from the slinky. Therefore, the branes appear as effectively smeared.

The Killing vectors of interest in the 11-dimensional geometry sourced by the smeared branes (3.31), (3.32) are ∂_{11},∂_{ϕ_1}, and ∂_{ϕ_2}, where ϕ_1 is the angular coordinate in the plane of smearing, and ϕ_2 is the angle in an orthogonal plane. It is possible to obtain a polarized configuration by simply reducing with a twist along ϕ_1. Such a configuration would not be supersymmetric. To preserve some supersymmetry, we need two twists of equal magnitude. In the absence of M5 branes, such a reduction would give the supersymmetric flux 5-brane found in [41]. Adding the M5 branes does not spoil the supersymmetry [39]. For consistency with the previous chapter, let us choose the smearing plane to be $x^5 - x^8$, and call ρ_1 and ϕ_1 the polar coordinates in this plane. We can also denote by ρ_2 and ϕ_2 the polar coordinates in the orthogonal $x^6 - x^9$ plane. Since the M5 branes are smeared at $\rho_1 = r_0$ in the $x^5 - x^8$ plane, the harmonic function will only depend on ρ_1,ρ_2 and x^7. Reducing along the Killing vector $l = \partial_{11} + B_1\partial_{\phi_1} + B_2\partial_{\phi_2}$ is consistent with performing the identifications:

$$\begin{aligned} x^{11} &\sim x^{11} + 2\pi R^{11} n_1 \\ \phi_1 &\sim \phi_1 + 2\pi n_2 + 2\pi n_1 R^{11} B_1 \\ \phi_2 &\sim \phi_2 + 2\pi n_3 + 2\pi n_1 R^{11} B_2. \end{aligned} \tag{3.34}$$

Supersymmetry requires the ϕ_1 and ϕ_2 twists (B_1 and B_2) to be equal in magnitude [39; 41]. The type IIA coordinates descend from 11-dimensional coordinates with standard periodicity, which are constant along orbits of the Killing vector l:

$$\tilde{\phi}_1 = \phi_1 - B_1 x^{11}, \qquad \tilde{\phi}_2 = \phi_2 - B_2 x^{11} \ . \tag{3.35}$$

Using the relation[5] between the M-theory metric and the string frame metric, the

[5] We use Type IIA conventions in which the dilaton is e^{ϕ}.

dilaton field and the RR 1-form potential:

$$ds_{11}^2 = (g_s e^{-\phi})^{2/3} ds_{10}^2 + (g_s e^{-\phi})^{-4/3} (dx^{11} + g_s C_\mu dx^\mu)^2, \tag{3.36}$$

we can determine

$$g_s^{-4/3} e^{4\phi/3} = (Z)^{-1/3} + (Z)^{2/3} (\rho_1^2 B_1^2 + \rho_2^2 B_2^2) \equiv \Lambda \tag{3.37}$$

$$g_s C_{\tilde{\phi}_1} = \Lambda^{-1} \rho_1^2 B_1 Z^{2/3} \tag{3.38}$$

$$g_s C_{\tilde{\phi}_2} = \Lambda^{-1} \rho_2^2 B_2 Z^{2/3}$$

$$\begin{aligned} ds_{10}^2 &= \Lambda^{1/2} (Z^{-1/3} dx_{\parallel}^2 + Z^{2/3} dx_{\perp}^2) \\ &- \Lambda^{-1/2} Z^{4/3} (\rho_1^2 B_1 d\tilde{\phi}_1 + \rho_2^2 B_2 d\tilde{\phi}_2)^2. \end{aligned} \tag{3.39}$$

The fields H_3 and F_4 descend from the 11 dimensional 4-form $\hat{F}_4$:

$$\hat{F}_4 = g_s F_4 + dx^{11} \wedge H_3, \tag{3.40}$$

and are given by

$$g_s F_4 = *_5 dZ \tag{3.41}$$

$$*_5 H_3 = (B_1 \rho_1^2 d\tilde{\phi}_1 + B_2 \rho_2^2 d\tilde{\phi}_2) \wedge (-dZ), \tag{3.42}$$

where $*_5$ is the flat Hodge dual on the 5-dimensional space transverse to the branes. It is not hard to obtain from (3.39) and (3.42) the first order perturbations found in the previous section (3.15). The tensors S_1 and V_2 can be expressed in polar coordinates as

$$S_1 = m(\rho_1^2 d\tilde{\phi}_1 + \rho_2^2 d\tilde{\phi}_2) \tag{3.43}$$

$$V_2 = m(\rho_1^2 d\tilde{\phi}_1 + \rho_2^2 d\tilde{\phi}_2) \wedge (dr/r), \tag{3.44}$$

where $r^2 = \rho_1^2 + \rho_2^2 + x_7^2$. Identifying $m \equiv B_1 = B_2$, we can see that to first order in B the exact solution found in this chapter reproduces the one given in (3.1, 3.15). Also, the discrete values of m which allow polarization (3.28) are the same as the values of B which match the brane ends (3.33). Furthermore, we can see from the equations (3.38), (3.39) that the dilaton and the metric receive only second and higher order corrections in the perturbative field F_2 and H_3. Keeping only the first order terms in $B's$, (3.39) gives

$$g_s C^{(1)} = \Lambda^{-1} Z^{2/3} S_1 \approx Z S_1,$$

in agreement with (3.15). Also from (3.42), using $dZ = -3(dr/r)Z$ and (3.44) one recovers the first order solution for H_3 given in (3.15).

3.4 The exact supergravity solution describing the F1 $\rightarrow$ D2 polarization

In this section we find the M-theory description of the supersymmetric polarization of N F1 strings into D2 branes. The perturbative analysis of this polarization was performed in [36]. In that paper it was shown that a large number N of parallel fundamental strings can polarize into cylindrical D2 branes in the presence of transverse RR 2-form and 6-form field strengths [6]:

$$\begin{aligned} g_s F_2 &= Z(2T_2 - 6V_2) \\ g_s(*_8 F_6) &= -Z(6V_2), \end{aligned} \tag{3.45}$$

where T_2 and V_2 are again antisymmetric tensors on the 8-dimensional space transverse to the strings. By grouping the 8 transverse coordinates into 4 complex coordinates:

$$z^1 = x^2 + ix^3, \quad z^2 = x^4 + ix^5, \quad z^3 = x^6 + ix^7, \quad z^4 = x^8 + ix^9, \tag{3.46}$$

[6] We use for convenience the conventions of [36], $F_6 \equiv *\tilde{F}_4 = *(F_4 - C_1 \wedge H_3)$, and $*_8$ is the flat Hodge dual on the transverse space.

and by using the $SO(8)$ R-symmetry transformation properties of the fields, it was argued that a perturbation with

$$T_2 = m\mathrm{Re}(dz^2 d\bar{z}^4) \tag{3.47}$$

preserves 4 supercharges.

The M-theory picture of this polarization is very similar to the one found in the previous chapter. The only change comes from replacing the M5 branes with M2 branes.

Let us consider the 11-dimensional supergravity background describing the near horizon of a large number N of coincident M2 branes:

$$\begin{aligned} ds_{11}^2 &= Z_2^{-2/3} dx_{\|}^2 + Z_2^{1/3} dx_{\perp}^2 \\ \hat{F}_4 &= d(Z^{-1}) \wedge dx^0 \wedge dx^1 \wedge dx^{11} \\ Z_0 = \frac{R_{M2}^6}{r^6}, &\qquad r^2 = x^i x^i, \qquad R_{M2}^6 = 32\pi^2 N l_p^6, \end{aligned} \tag{3.48}$$

where the branes are aligned along $0, 1, 11$, and $i = 2, 3, ..., 9$.

When the M2 branes are smeared on a circle of radius r_0, the only change in the metric above is the harmonic function:

$$\begin{aligned} Z &= \frac{R_{M2}^6}{2\pi} \int_0^{2\pi} \frac{d\phi}{(\mathbf{x}^2 + \rho^2 + r_0^2 - 2r_0\rho\cos\phi)^{6/2}} \\ &= R_{M2}^6 \frac{(\mathbf{x}^2 + \rho^2 + r_0^2)^2 + 2r_0^2\rho^2}{(\mathbf{x}^2 + (\rho - r_0)^2)^{5/2}(\mathbf{x}^2 + (\rho + r_0)^2)^{5/2}}. \end{aligned} \tag{3.49}$$

where $\mathbf{x}$ denotes the 6 transverse directions perpendicular to the smearing plane.

To obtain the polarized state we again distribute the M2 branes on a circle, and compactify with a twist, as in (3.34). Local D2 charge quantization implies that only certain values of the twist (given by eq.(3.33)) give consistent backgrounds. Alternatively, one can see that only twists by multiples of $\frac{2\pi}{N}$ link an end of an M2 brane with the end of another, like in Figure 3.1.

We can assume without loss of generality that the M2 branes are distributed in the x^4-x^8 plane, and introduce polar coordinates (ρ_1,ϕ_1) for the x^4-x^8 plane, and (ρ_2,ϕ_2) for the x^5-x^9 plane.

If the number of M2 branes is large, supergravity sees them as effectively smeared. We can therefore dimensionally reduce the background (3.48, 3.49) along the Killing vector $l=\partial_{11}+B_1\partial_{\phi_1}+B_2\partial_{\phi_2}$ as in Section 3. If $|B_1|=|B_2|$ the resulting background preserves 8 supercharges.

For completeness, we should note that one can consider a more general reduction, involving twists in the x^2-x^6 and x^3-x^7 planes as well. For certain values of the twists these reductions can also give supersymmetric backgrounds with polarized branes. The comprehensive analysis done by [39] for coincident branes applies here without change.

Using the reduction formula (3.36), we can determine

$$\begin{aligned}
g_s^{-4/3}e^{4\phi/3} &= (Z)^{-2/3}+(Z)^{1/3}(\rho_1^2B_1^2+\rho_2^2B_2^2)\equiv\Lambda\\
g_sC_{\tilde\phi_1} &= \Lambda^{-1}\rho_1^2B_1Z^{1/3}\\
g_sC_{\tilde\phi_2} &= \Lambda^{-1}\rho_2^2B_2Z^{1/3}\\
ds_{10}^2 &= \Lambda^{1/2}(Z^{-2/3}dx_\parallel^2+Z^{1/3}dx_\perp^2)-\Lambda^{-1/2}Z^{2/3}(\rho_1^2B_1d\tilde\phi_1+\rho_2^2B_2d\tilde\phi_2)^2.
\end{aligned}\tag{3.50}$$

Also using (3.40) we obtain

$$\begin{aligned}
F_4 &= 0=\tilde F_4+C_1\wedge H_3\\
H_3 &= d(Z^{-1})\wedge dx^0\wedge dx^1.
\end{aligned}\tag{3.51}$$

Identifying $m\equiv B_1=B_2$, we will verify that to first order in B this exact solution reproduces the perturbative one (3.45). Notice that (3.50) implies

$$g_sC^{(1)}\approx ZS_1.$$

Thus(using Appendix C),

$$g_s F_2 \approx d(ZS_1) = Z(2T_2 - 6V_2). \tag{3.52}$$

Also, from (3.51) we have

$$\tilde{F}_4 = (-ZS_1) \wedge d(Z^{-1}) \wedge dx^0 \wedge dx^1. \tag{3.53}$$

Using $F_6 = *\tilde{F}_4$ and (3.44) we can easily recover the second equation of the perturbative solution given in (3.45).

3.5 Towards the full Polchinski-Strassler solution

It is possible to obtain the exact Type IIB solution describing smeared D3 branes polarized into a cylindrical NS5 branes by simply T-dualizing the background (3.39) along one of the directions parallel to the D4 branes. Indeed, the D4 branes become D3 branes smeared along the T-duality direction, while the NS5 branes remain the same. By an $SL(2,Z)$ transformation this configuration can give configurations with D3 branes polarized into (p,q) 5 branes.

These configurations have the same types of fields as in the PS solution. Nevertheless, they have $\mathcal{N}=2$ supersymmetry and have a different topology from the case discussed in [38]. The NS5 branes we obtain have topology $S^1 \times R^5$, while the ones in [38] have topology $S^2 \times R^4$.

To our knowledge there seem to be two major difficulties in obtaining the full PS solution. The first one is finding the exact $\mathcal{N}=1^*$ supergravity background without the polarized branes, and the second one is finding the modification of this background when the branes are polarized. Our solutions are insensitive to the exact form of Z, and seem to suggest that the second step only involves changing the harmonic function Z. It would be interesting to see if by applying this intuition to the solution obtained by lifting the 5-dimensional $\mathcal{N}=1^*$ supergravity flow one could find the full PS solution[7].

If we choose the T-duality direction y to be x^4, the exact solution (3.39) becomes

$$
\begin{aligned}
e^{\phi} &= g_s\tilde{\Lambda}^{1/2} && (3.54)\\
g_s C_2 &= \tilde{\Lambda}^{-1}Z(S_1\wedge dy) && (3.55)\\
*_6 H_3 &= (S_1\wedge dy)\wedge dZ && (3.56)\\
d^2 s_{10} &= \tilde{\Lambda}^{1/2}\left(Z^{-1/2}dx_{\parallel}^2 + Z^{1/2}(dx_{\perp}^2+dy^2)\right) && \\
&- \tilde{\Lambda}^{-1/2}Z^{1/2}\left(Z(S_1)^2+(\tilde{\Lambda}-1)dy^2\right) && (3.57)\\
g_s F_5 &= *_6 dZ\ , && (3.58)
\end{aligned}
$$

with

$$\tilde{\Lambda} = 1 + Z(B_1^2\rho_1^2 + B_2^2\rho_2^2) \quad \text{and} \quad S_1 = B_1\rho_1^2 d\tilde{\phi}_1 + B_2\rho_2^2 d\tilde{\phi}_2, \tag{3.59}$$

where the parallel directions are 0123, Z is given by equation (3.32), and the Hodge dual $*_6$ on the space transverse to the branes has flat indices.

As a side note we should note that this solution exists even for $B_1 \neq B_2$, when there is no supersymmetry. The exact type IIB solution for a circular D5 brane with large D3 brane charge can be easily obtained using S-duality.

3.6 More about the theory on the D4 branes

As we explained in the previous sections, the strongly coupled theory dual to the supergravity background with polarized branes is related to the 4+1 Super Yang Mills theory living on the D4 branes. As is well known, this theory is not renormalizable and becomes strongly coupled in the UV. In that regime it can be described by string theory on the background (3.39), which can be thought of as the dual of the UV completion of this theory.

By turning on the supergravity modes corresponding to fermion masses, the UV completion is modified and can in some cases include polarized branes. In these

[7] A related problem which might be easier to approach would be using an $\mathcal{N}=2$ AdS_4 flow [46] to find the full solution corresponding to M2 branes polarized into M5 branes [47].

cases, the supergravity solution is valid everywhere, and thus there is no regime where the boundary theory is weakly coupled. When there are no polarized branes, the supergravity background again becomes singular, and the IR limit of the field theory becomes weakly coupled.

For fermion masses allowing brane polarization ($m \sim \frac{k}{N g_s \alpha'^{1/2}}$) one can pass from a phase where the theory has a weakly coupled field theory description (as a mass-deformed $\mathcal{N} = 1$ Super Yang Mills theory in 4+1 dimensions) to a phase where there is no weakly coupled field theory description, by simply changing the polarization radius.

The purpose of this section is to learn as much as possible about these theories by studying their supergravity duals. The first thing to notice is that these theories have 8 supercharges. One can see this both directly (a mass for a chiral superfield in 4+1 dimensions preserves $\mathcal{N} = 1$ supersymmetry) or by noticing that the exact supergravity dual of these theories has 8 supercharges [39].

As in the case of D3 branes, the ends of objects ending on the branes can be interpreted as "states" in the boundary theory. As both F1 strings and D2 branes can end on a D4 brane, this theory will have both "quarks" and "little strings." Thus, an infinite F1 string ending on a D4 brane can be interpreted as a quark. In the confining phase, the energy of the flux tube between two such quarks is given by the energy of an F-string with its ends on the boundary, lowered into the bulk [53].

The potential of an quark-antiquark pair is given in the gauge theory by the expectation value of the Wilson loop operator $< W(C) >$. Since correlations of local operators are given by the supergravity action for fields with sources on the boundary, a natural proposal for the expectation of the Wilson loop is the action of a string ending on the loop C at the boundary [53]. In the leading order approximation this corresponds to the minimum area, and due to the curvature of the metric the string surface will go deep into the interior of the space. Thus if we move the quarks apart on the boundary, the variation of their potential is given by the string tension calculated with the string frame metric next to the D-brane sources. Thus for the initial configuration of D4-branes, since the metric components along the branes vanish as we approach the D4-branes, it costs no energy to move the quarks apart. For the NS5 branes the metric components along its world-volume do not vanish, therefore the string tension will have a non-zero value, indicating that the quarks are confined.

One can also see that the D2 brane "little strings," are screened. The generalized Wilson surface that describes the properties of these strings is given by the energy of a D2 brane lowered in the bulk [53; 37]. Since this D2 brane can attach itself to the NS5 brane, there is no energy cost to move the two "little strings" apart. Therefore the little strings are screened. Since we are in a phase where the quarks are confined, it is appropriate to call the little strings "magnetic little strings."

It is quite easy to find the tension of the confining flux tube. When the quarks are far apart, the bulk string joining them is basically composed of two vertical segments, and one segment sitting near the polarized branes. The energy of the two vertical segments is essentially constant, and therefore the flux tube tension is given by the tension of an F1 string sitting near the NS5-D4 shell.

It is possible to extract the components of the near-shell geometry from the exact solution. At $\rho_1 = r_0 + \epsilon$, the harmonic form (3.32) becomes

$$Z_{\text{near shell}} = \frac{R_{M5}^3}{\pi r_0 \epsilon^2}, \tag{3.60}$$

and therefore $g_{\|} = r_0 B + O(\epsilon)$, and $e^{-\phi} \sim \epsilon$.

Thus, the flux tube tension is

$$T_{\text{flux tube}} = \sqrt{-g_{00} g_{11}}\ |_{\text{near shell}} = r_0 B, \tag{3.61}$$

independent of the 't Hooft coupling of the boundary theory. Note that as $r_0 \to 0$ the weakly coupled infrared region is recovered, there is no confinement, and the string tension becomes zero as expected. One can also see that the magnetic little strings are screened, by estimating the energy of a D2 brane in the near shell limit:

$$V_{\text{D2}} = e^{-\Phi} \sqrt{-g_{00} g_{11} g_{22}}\ |_{\text{near shell}} \sim \epsilon \to 0 \tag{3.62}$$

Since the weakly coupled theory has an $SU(N)$ gauge symmetry, one expects that in the confining phase a baryon made of N quarks is a free object. One can see that the bulk dual of a baryon in the unperturbed field theory is a D4 brane wrapping the warped 4-sphere transverse to the D4 branes. Nevertheless, unlike its 3+1 dimensional "cousin" [54], this baryon is not stable because of the lack of conformal invariance. It tends to slide off towards the infrared and self-annihilate.

Nevertheless, when the D4 branes are polarized, the D4 brane baryon sliding towards the infrared crosses the polarized configuration at a finite radius. Via the Hanany-Witten effect [52], the resulting baryon is a D2 brane ending on the NS5-D4 shell, and filling the 2-ball whose boundary is the polarization circle. There are then N fundamental strings that can end on the junction between the D2 brane and the NS5-D4 shell.

Indeed, by investigating the NS5 brane action [43] (formulas 54,55), we can see that the D2 brane ends source a nonzero NS5 world-volume 3-form db_2, and the dissolved D4 branes create a nonzero world-volume 1-form $\mathcal{F}_1$. The anomaly given by the term

$$db_2 \wedge \mathcal{F} \wedge B_2 \tag{3.63}$$

under the gauge transformation $\delta B_2 = d\chi_1$ is proportional to the number of dissolved D4 branes (N), and can only cancel if N F1 strings end on the NS5-D2 junction. Therefore, the D2 brane filling the 2-ball inside the polarization circle is indeed the baryon of this theory.

One can also estimate the dependence of the mass of this baryon on the parameters of the theory. Assuming the order of magnitude of Z to be R^3/r_0^3, we find the mass of the baryon to be

$$M_{\text{baryon}} = \tau_{D2} \int_0^{r_0} d\rho_1 d\phi_1 e^{-\Phi} \sqrt{-g_{00} g_{\rho_1\rho_1} g_{\phi_1\phi_1}} \approx \sqrt{N^3 r_0 g_s}. \tag{3.64}$$

Chapter 4

Summary

Two of the major recent developments in string theory have been the discovery of gauge theory - string theory duality and the emergence of noncommutative field theory. In this thesis we have studied aspects of both of these directions. A central role in understanding these developments is played by D-branes.

We have dedicated Chapter 2 of this thesis to study some properties of D-branes actions. We analyzed the gauge invariance of the Chern-Simons term under gauge transformations of RR fields, and in the process we rewrote the action in a form that makes the symmetry manifest. In the same chapter we also gave a simple derivation of noncommutative D-brane actions starting from the action of a large number of lower dimensional D-branes.

In Chapter 3 we have investigated the polarization of D4 branes into NS5 branes both by perturbing their near-horizon geometry and performing a Polchinski-Strassler type analysis, and by investigating the M-theory origin of this polarization. This enabled us to obtain the exact supergravity solutions describing this polarization, which to the best of our knowledge is the first exact solution that contains polarized branes and has a field theory dual. We also obtained the exact supergravity solution describing the polarization of F1 strings into D2 branes. We then used T-duality to obtain type IIB solutions with 8 supercharges describing smeared D3 branes polarized into concentric cylindrical (p, q) 5 branes.

In the last section of Chapter 3 we investigated some of the properties of the supersymmetric 4+1 dimensional theory dual to the D4-NS5 exact background, and gave string theory descriptions of some of the objects this theory contains. The solutions found in Chapter 3 belong to the same universality class as the exact Polchinski-Strassler solution. We hope that the ideas presented here will be useful steps towards finding this solution.

Appendix A: Gauge invariance for the $F = 0$ case

In the following, we will show that the coupling between N D$(p-1)$branes and a C_{p+2k} RR form is invariant under the transformation $C \to C + d\Lambda$. A particular case of this problem was proved in [6], in a matrix theory context, working in the momentum basis. Here, we generalize, considering D$(p-1)$branes instead of D0 branes, with non-trivial pull-back and F_{ab} terms. (Nonzero F will be considered in the next appendix.)

Specializing (2.1) to the case $F = B = 0$, the coupling between a C_{p+2k} RR form and N D$(p-1)$branes is given by

$$\mu_{p-1} \int \mathrm{STr}\left(P\left[\frac{(i\lambda i_\phi i_\phi)^k}{k!} C_{p+2k} \right] \right). \tag{A-1}$$

Each of the RR fields C_{p+2k} are functionals of the transverse coordinates ϕ :

$$C(\sigma, \phi) = e^{\lambda \phi^i \partial_{x^i}} C^0(\sigma, x^i)|_{x^i=0} = \sum_{n, i_n} \frac{\lambda^n}{n!} \phi^{i_1} \dots \phi^{i_n} \partial_{x^{i_1}} \dots \partial_{x^{i_n}} C^0(\sigma, x^i)|_{x^i=0}, \tag{A-2}$$

where $C^0(\sigma, x^i)$ is the background RR field. If $\lambda\phi^i$ are the transverse displacements of the branes, the pullback of a p form, Ω_p, in the static gauge is

$$[P(\Omega_p)]_{a_1 \dots a_p} = \Omega_{\mu_1 \dots \mu_p} \left(\delta^{\mu_1}_{a_1} I_N + \lambda \frac{\partial \phi^{\mu_1}}{\partial \sigma^{a_1}} \right) \dots \left(\delta^{\mu_p}_{a_p} I_N + \lambda \frac{\partial \phi^{\mu_p}}{\partial \sigma^{a_p}} \right), \tag{A-3}$$

where I_N is an $N \times N$ unit matrix, and Ω_p should be considered a functional of the ϕ's. The indices μ's run over all coordinates, so we will take $\phi^\mu = 0$ for the μ's parallel to the direction of the branes. As defined in the previous equation the pullback of an antisymmetric form is not necessarily an antisymmetric form since, as $N \times N$ matrices, $\partial_{a_i} \phi^i$ do not commute in general. However, as part of the symmetrized trace prescription we should take a symmetrized average over all orderings of $\partial_{a_i} \phi^i$, thus enforcing antisymmetry on the a's.

With antisymmetry enforced on the a's, (A-3) becomes

$$[P(\Omega_p)]_{[a_1\ldots a_p]} = \Omega_{a_1\ldots a_p} + \lambda p \Omega_{i_1[a_2\ldots a_p}\partial_{a_1]}\phi^{i_1} + \ldots +$$
$$\lambda^l \frac{p!}{l!(p-l)!}\Omega_{i_1\ldots i_l[a_{l+1}\ldots a_p}\partial_{a_1}\phi^{i_1}\ldots\partial_{a_l]}\phi^{i_l} + \ldots + \lambda^p \Omega_{i_1\ldots i_p}\partial_{[a_1}\phi^{i_1}\ldots\partial_{a_p]}\phi^{i_p}. \quad \text{(A-4)}$$

We are going to use this equation for $\Omega_p \equiv \frac{\left(i\lambda i_\phi i_\phi\right)^k}{k!}C_{p+2k}$. Combining equation (A-2) and (A-4), one gets the C_{p+2k} coupling of N D$(p-1)$branes (for $F_{ab}=0$) as

$$\sum_{l,n}\frac{\mu_{p-1}\lambda^{k+n+l}i^k p!}{k!n!l!(p-l)!}\partial_{x^{i_1}}\ldots\partial_{x^{i_n}}C^0_{i'_1\ldots i'_{2k}j_1\ldots j_l[a_{l+1}\ldots a_p}$$
$$\times \mathrm{STr}\left(\partial_{a_1}\phi^{j_1}\ldots\partial_{a_l]}\phi^{j_l}\phi^{i_1}\ldots\phi^{i_n}\phi^{i'_{2k}}\phi^{i'_{2k-1}}\ldots\right), \quad \text{(A-5)}$$

where $0 \le l \le p$.

Notice that the $\mathrm{STr}(\ldots)$ expression involves symmetrizing over all the $\partial_{a_s}\phi^{j_s}$, for $s = 1,2,\ldots,l$, also over all the ϕ^{i_q}, for $q = 1,2,\ldots n$, and all the pairs $\phi^{i'_{2j}}\phi^{i'_{2j-1}}$, for $j = 1,2\ldots k$. We can rewrite this term as $\mu_{p-1}\sum_{l,n}\lambda^{k+n+l}b^n_l$, where

$$b^n_l = \frac{i^k p!}{k!n!l!(p-l)!}\left(\partial_{x^{i_1}}\ldots\partial_{x^{i_n}}\right)C^0_{i'_1\ldots i'_{2k}j_1\ldots j_l[a_{l+1}\ldots a_p} \quad \text{(A-6)}$$
$$\times \mathrm{STr}\left(\partial_{a_1}\phi^{j_1}\ldots\partial_{a_l]}\phi^{j_l}\phi^{i_1}\ldots\phi^{i_n}\phi^{i'_{2k}}\phi^{i'_{2k-1}}\ldots\right). \quad \text{(A-7)}$$

In the previous equation we antisymmetrized over all the a's, and this will be implicit in the rest of this proof.

In order to show that the coupling is invariant, up to a total derivative, under gauge transformations $C \to C + d\Lambda$, we will try to write $\sum_{l,n} b^n_l$ as a sum of total derivatives and gauge invariant terms that depend on the field strength of the RR field. Integrating $b^n_{l>0}$ by parts with respect to σ^{a_l}, and dropping the resulting total derivatives and field strength terms, we can express $b^n_{l>0}$ as a sum of two types of terms. (we will keep track of the field strength terms and will present them later.) The first type of term for b^n_l will cancel against the second type of term in the expansion for b^{n+1}_{l-1}. In this way all the terms cancel, except for the first term in $b^n_{l=1}$ and $b^n_{l=0}$. (The second term of b_l^n will turn out to be 0 for $l = l_{max} = p$, or for $n = n_{min} = 0$.)

When integrating b_l^n by parts with respect to σ^{a_l}, we will get terms in which ∂_{a_l} acts either outside the trace on C^0, or inside on ϕ's. For the part inside the trace, for simplicity of notation, we will only write down the ϕ terms that have changed after integration by parts. Note that due to the antisymmetry in the a's, $\partial_{a_l}\partial_{a_s}\phi^j \to 0$. Let's denote by U_l^n the factor outside the trace,

$$U_l^n = \frac{i^k p!}{k!n!l!(p-l)!} \left(\partial_{x^{i_1}} \ldots \partial_{x^{i_n}}\right) C^0_{i'_1 \ldots i'_{2k} j_1 \ldots j_l [a_{l+1} \ldots a_p}.$$

With these conventions, dropping the total derivative part,

$$\begin{aligned} b_l^n &= (U_l^n)\mathrm{STr}\left(\ldots \partial_{a_l}\phi^{j_l} \ldots\right) = (-\partial_{a_l})(U_l^n)\mathrm{STr}\left(\ldots \phi^{j_l} \ldots\right) \\ &- k\left[(U_l^n)\mathrm{STr}\left(\ldots \phi^{j_l} \ldots \partial_{a_l}\phi^{i'_{2k}}\phi^{i'_{2k-1}} \ldots\right) + (U_l^n)\mathrm{STr}\left(\ldots \phi^{j_l} \ldots \phi^{i'_{2k}}\partial_{a_l}\phi^{i'_{2k-1}} \ldots\right)\right] \\ &- n(U_l^n)\mathrm{STr}\left(\ldots \phi^{j_l} \ldots \partial_{a_l}\phi^{i_n} \ldots\right). \end{aligned} \tag{A-8}$$

The factor of k comes from the k pairs of $\phi^{j'}\phi^{j'-1}$ of the inner product and n from the n ϕ's of the Taylor series expansion of the RR form. Let

$$\begin{aligned} A_1 &= (U_l^n)\mathrm{STr}\left(\ldots \phi^{j_l} \ldots \partial_{a_l}\phi^{i'_{2k}}\phi^{i'_{2k-1}} \ldots\right) \\ A_2 &= (U_l^n)\mathrm{STr}\left(\ldots \phi^{j_l} \ldots \phi^{i'_{2k}}\partial_{a_l}\phi^{i'_{2k-1}} \ldots\right) \\ D_l^n &= (U_l^n)\mathrm{STr}\left(\ldots \phi^{j_l} \ldots \partial_{a_l}\phi^{i_n} \ldots\right). \end{aligned}$$

$$b_l^n = (-\partial_{a_l})(U_l^n)\mathrm{STr}\left(\ldots \phi^{j_l} \ldots\right) - k(A_1 + A_2) - nD_l^n. \tag{A-9}$$

Writing

$\partial_{x^{i_n}} C^0_{i'_1 \ldots i'_{2k} j_1 \ldots j_l a_{l+1} \ldots a_p} = (p+2k+1)\partial_{[x^{i_n}} C^0_{i'_1 \ldots i'_{2k} j_1 \ldots j_l a_{l+1} \ldots a_p]} + \partial_{x^{i'_1}} C^0_{i_n i'_2 \ldots a_p} +$
$[(p+2k-1)$ more terms obtained interchanging i_n with all the other indices],

we can rewrite D_l^n as:

$$D_l^n = (\text{gauge invariant term}) + D_l^n|_{i_n \leftrightarrow i'_1} + \ldots + D_l^n|_{i_n \leftrightarrow i'_{2k}} + D_l^n|_{i_n \leftrightarrow j_1} + \ldots + D_l^n|_{i_n \leftrightarrow j_l}$$

$$+ D_l^n|_{i_n \leftrightarrow a_{l+1}} + \ldots + D_l^n|_{i_n \leftrightarrow a_p}. \tag{A-10}$$

(notation: $D^n_l|_{i_n\leftrightarrow i_s}$ means that in the term outside the symmetrized trace, i.e. U^n_l, we interchange i_n with i_s, and if these are dummy indices, this is equivalent to keeping the outside term U^n_l the same while interchanging i_n with i_s inside the trace.) Note that every time we interchange two indices from the set of indices i' and j inside the trace we get a minus sign since these indices are contracted with indices of the RR form in the term outside the trace.

Since,

$$D^n_l = \frac{i^k p!}{k!n!l!(p-l)!}\left(\partial_{x^{i_1}}\ldots\partial_{x^{i_n}}\right) C^0_{i'_1\ldots i'_{2k}j_1\ldots j_l[a_{l+1}\ldots a_p}\mathrm{STr}\left(\ldots\phi^{j_l}\ldots\partial_{a_l]}\phi^{i_n}\ldots\right), \tag{A-11}$$

we have the following relations:

$$D^n_l|_{i_n\leftrightarrow j_l} = b^n_l, \tag{A-12}$$

the original term.

$$\begin{aligned} D^n_l|_{i_n\leftrightarrow j_s} &= (U^n_l)\mathrm{STr}\left(\ldots\phi^{j_l}\ldots\partial_{a_l}\phi^{j_s}\ldots\partial_{a_s}\phi^{i_n}\ldots\right) \\ &= -(U^n_l)\mathrm{STr}\left(\ldots\phi^{j_l}\ldots\partial_{a_s}\phi^{j_s}\ldots\partial_{a_l}\phi^{i_n}\ldots\right) = -D^n_l, \end{aligned} \tag{A-13}$$

for any $s = 1,\ldots l-1$, from the antisymmetry in the a's.

$$D^n_l|_{i_n\leftrightarrow i'_{2k}} + D^n_l|_{i_n\leftrightarrow i'_{2k-1}} = D^n_l|_{i_n\leftrightarrow i'_{2j}} + D^n_l|_{i_n\leftrightarrow i'_{2j-1}} \tag{A-14}$$

Using the above relations, (A-10) becomes

$$\begin{aligned} D^n_l = &\text{ (gauge invariant term) } + k(D^n_l|_{i_n\leftrightarrow i'_{2k}} + D^n_l|_{i_n\leftrightarrow i'_{2k-1}}) - (l-1)D^n_l + b^n_l \\ &+ D^n_l|_{i_n\leftrightarrow a_{l+1}} + \ldots + D^n_l|_{i_n\leftrightarrow a_p}. \end{aligned} \tag{A-15}$$

Dropping the gauge invariant term,

$$l(D^n_l) = k(B_1 + B_2) + b^n_l + D^n_l|_{i_n\leftrightarrow a_{l+1}} + \ldots + D^n_l|_{i_n\leftrightarrow a_p}, \tag{A-16}$$

where we have defined:

$$B_1 = D_l^n|_{i_n \leftrightarrow i'_{2k}} = (U_l^n)\mathrm{STr}\left(\ldots\phi^{j_l}\ldots\underbrace{\phi^{i_n}\phi^{i'_{2k-1}}}\ldots\partial_{a_l}\phi^{i'_{2k}}\ldots\right)$$
$$B_2 = D_l^n|_{i_n \leftrightarrow i'_{2k-1}} = (U_l^n)\mathrm{STr}\left(\ldots\phi^{j_l}\ldots\underbrace{\phi^{i'_{2k}}\phi^{i_n}}\ldots\partial_{a_l}\phi^{i'_{2k-1}}\ldots\right). \qquad \text{(A-17)}$$

The notation $\underbrace{\phi^i\phi^j}$ means that ϕ^i, ϕ^j show up together, as one entry, in the symmetrized trace prescription. In this way, the prescription, after interchanging some of the indices inside the trace, is consistent with the initial one.

Using the last equation to replace the D_l^n term in (A-9), we find:

$$b_l^n = (-\partial_{a_l})(U_l^n)\mathrm{STr}\left(\ldots\phi^{j_l}\ldots\right) - k(A_1 + A_2) - \frac{n}{l}[k(B_1 + B_2) + b_l^n$$
$$+D_l^n|_{i_n \leftrightarrow a_{l+1}} + \ldots + D_l^n|_{i_n \leftrightarrow a_p}]. \qquad \text{(A-18)}$$

$$b_l^n(n+l) = l(-\partial_{a_l})(U_l^n)\mathrm{STr}\left(\ldots\phi^{j_l}\ldots\right) - k(l(A_1 + A_2) + n(B_1 + B_2))$$
$$-n(D_l^n|_{i_n \leftrightarrow a_{l+1}} + \ldots + D_l^n|_{i_n \leftrightarrow a_p}). \qquad \text{(A-19)}$$

Note that $(-\partial_{a_l})(U_l^n)\mathrm{STr}\left(\ldots\phi^{j_l}\ldots\right) = (-\partial_{a_s})(U_l^n)\mathrm{STr}\left(\ldots\phi^{j_s}\ldots\right)$, for any $s = 1, \ldots l-1$; we get a minus sign from $a_l \leftrightarrow a_s$, and another minus sign from $j_s \leftrightarrow j_l$.

Let's evaluate, $l(A_1 + A_2) + n(B_1 + B_2) = (lA_1 + nB_1) + (lA_2 + nB_2)$,

$$\begin{aligned} lA_1 + nB_1 &= lU_l^n\mathrm{STr}\left(\ldots\phi^{j_l}\ldots\underbrace{\partial_{a_l}\phi^{i'_{2k}}\phi^{i'_{2k-1}}}\ldots\right) \\ &+ nU_l^n\mathrm{STr}\left(\ldots\phi^{j_l}\ldots\underbrace{\phi^{i_n}\phi^{i'_{2k-1}}}\ldots\partial_{a_l}\phi^{i'_{2k}}\ldots\right), \end{aligned} \qquad \text{(A-20)}$$

$$\begin{aligned} lA_1 + nB_1 &= lU_l^n\mathrm{STr}\left(\ldots\phi^{i'_{2k-1}}\ldots\underbrace{\partial_{a_l}\phi^{j_l}\phi^{i'_{2k}}}\ldots\right) \\ &+ nU_l^n\mathrm{STr}\left(\ldots\phi^{i'_{2k-1}}\ldots\underbrace{\phi^{i_n}\phi^{i'_{2k}}}\ldots\partial_{a_l}\phi^{j_l}\ldots\right). \end{aligned} \qquad \text{(A-21)}$$

All the STr (...) terms are multiplied by

$$U_l^n = \frac{i^k p!}{k!n!l!(p-l)!} \left(\partial_{x^{i_1}} \dots \partial_{x^{i_n}}\right) C^0_{i'_1 \dots i'_{2k} j_1 \dots j_l [a_{l+1} \dots a_p}.$$

Using the antisymmetry in the a's , and the symmetries in the dummy indices of the factor outside the trace, we have,

$$\begin{aligned} lA_1 + nB_1 &= (U_l^n) \sum_{q=1}^{l} \mathrm{STr}\left(\dots \phi^{i'_{2k-1}} \dots \underbrace{\partial_{a_q} \phi^{j_q} \phi^{i'_{2k}}} \dots\right) \\ &+ (U_l^n) \sum_{s=1}^{n} \mathrm{STr}\left(\dots \phi^{i'_{2k-1}} \dots \underbrace{\phi^{i_s} \phi^{i'_{2k}}} \dots\right). \end{aligned} \tag{A-22}$$

Let's denote by $\mathrm{STr}\left(\dots \underbrace{\phi^{i'_{2k-1}}} \dots \underbrace{\phi^{i'_{2k}}} \dots\right)$, the expression in which $\phi^{i'_{2k-1}}$ and $\phi^{i'_{2k}}$ are distinct entries in the symmetrized trace prescription, without any constraint on the "left neighbour" of $\phi^{i'_{2k}}$. However, the "left neighbour" can only be one of the following: $\partial_{a_q} \phi^{j_q}|_{q=1,\dots,l}$, $\phi^{i_s}|_{s=1,\dots,n}$, $(\phi^{i'_{2j}} \phi^{i'_{2j-1}})|_{j=1,\dots k-1}$, or $\phi^{i'_{2k-1}}$.

$$\begin{aligned} lA_1 + nB_1 = (n+l+k) U_l^n \mathrm{STr}\left(\dots \underbrace{\phi^{i'_{2k-1}}} \dots \underbrace{\phi^{i'_{2k}}} \dots\right) - \\ - \sum_j U_l^n \mathrm{STr}\left(\dots \phi^{i'_{2k-1}} \dots \underbrace{\phi^{i'_{2j}} \phi^{i'_{2j-1}} \phi^{i'_{2k}}} \dots\right) - U_l^n \mathrm{STr}\left(\dots \underbrace{\phi^{i'_{2k-1}} \phi^{i'_{2k}}} \dots\right) \end{aligned} \tag{A-23}$$

One can notice that the last term is b_l^n, while the first term is 0, since U_l^n is antisymmetric in i'_{2k-1} and i'_{2k}.

$$lA_1 + nB_1 = b_l^n - (k-1) U_l^n \mathrm{STr}\left(\dots \phi^{i'_{2k-1}} \dots \underbrace{\phi^{i'_{2j}} \phi^{i'_{2j-1}} \phi^{i'_{2k}}} \dots\right). \tag{A-24}$$

We can ilustrate the type of identity that we used with a concrete example:

$$\mathrm{STr}\left(XY \underbrace{ZT}\right) = 3\mathrm{STr}\left(XYZT\right) - \mathrm{STr}\left(XZ \underbrace{YT}\right) - \mathrm{STr}\left(YZ \underbrace{XT}\right), \tag{A-25}$$

where X, Y, Z, T are some $N \times N$ matrices.

Similarly for

$$\begin{aligned} nB_2 + lA_2 &= nU_l^n \mathrm{STr}\left(\ldots \phi^{i'_{2k}} \ldots \underbrace{\phi^{i'_{2k-1}}\phi^{i_n}} \ldots \partial_{a_l}\phi^{j_l} \ldots\right) \\ &+ lU_l^n \mathrm{STr}\left(\ldots \phi^{i'_{2k}} \ldots \underbrace{\phi^{i'_{2k-1}}\partial_{a_l}\phi^{j_l}} \ldots\right) \end{aligned} \tag{A-26}$$

$$nB_2 + lA_2 = (n+l+k)U_l^n \mathrm{STr}\left(\ldots \underbrace{\phi^{i'_{2k}}} \ldots \underbrace{\phi^{i'_{2k-1}}} \ldots\right) -$$
$$-(k-1)U_l^n \mathrm{STr}\left(\ldots \underbrace{\phi^{i'_{2k-1}}\phi^{i'_{2j}}\phi^{i'_{2j-1}}} \ldots\right) - U_l^n \mathrm{STr}\left(\ldots \underbrace{\phi^{i'_{2k-1}}\phi^{i'_{2k}}} \ldots\right). \tag{A-27}$$

$$nB_2 + lA_2 = b_l^n - (k-1)U_l^n \mathrm{STr}\left(\ldots \phi^{i'_{2k}} \ldots \underbrace{\phi^{i'_{2j}}\phi^{i'_{2j-1}}\phi^{i'_{2k-1}}} \ldots\right). \tag{A-28}$$

From (A-24) and (A-28), we get

$$l(A_1 + A_2) + n(B_1 + B_2) = 2b_l^n \tag{A-29}$$

Then equation (A-19) gives

$$b_l^n = \frac{l}{n+l+2k}(-\partial_{a_l})(U_l^n)\mathrm{STr}\left(\ldots \phi^{j_l} \ldots\right) - \frac{n}{n+l+2k}\left[D_l^n|_{i_n \leftrightarrow a_{l+1}} + \ldots + D_l^n|_{i_n \leftrightarrow a_p}\right]. \tag{A-30}$$

Let's remind ourselves what these terms really are

$$\begin{aligned} (-\partial_{a_l})(U_l^n)\mathrm{STr}\left(\ldots \phi^{j_l} \ldots\right) &= \frac{-i^k p!}{k!n!l!(p-l)!}\left(\partial_{x^{i_1}} \ldots \partial_{x^{i_n}}\partial_{a_l}\right) C^0_{i'_1 \ldots i'_{2k} j_1 \ldots j_l a_{l+1} \ldots a_p} \\ &\times \mathrm{STr}\left(\partial_{a_1}\phi^{j_1} \ldots \partial_{a_{l-1}}\phi^{j_{l-1}}\underline{\phi}^{j_l}\phi^{i_1} \ldots \phi^{i_n}\underbrace{\phi^{i'_{2k}}\phi^{i'_{2k-1}}} \ldots\right). \end{aligned} \tag{A-31}$$

$$\begin{aligned} D_l^n|_{i_n \leftrightarrow a_{l+1}} &= \frac{i^k p!}{k!n!l!(p-l)!}\left(\partial_{x^{i_1}} \ldots \partial_{x^{i_{n-1}}}\partial_{a_{l+1}}\right) C^0_{i'_1 \ldots i'_{2k} j_1 \ldots j_l i_n a_{l+2} \ldots a_p} \\ &\times \mathrm{STr}\left(\partial_{a_1}\phi^{j_1} \ldots \partial_{a_{l-1}}\phi^{j_{l-1}}\underline{\phi}^{j_l}\phi^{i_1} \ldots \phi^{i_{n-1}}\partial_{a_l}\phi^{i_n}\underbrace{\phi^{i'_{2k}}\phi^{i'_{2k-1}}} \ldots\right). \end{aligned} \tag{A-32}$$

If we use the same expansion for b_{l+1}^{n-1}, we get

$$\begin{aligned} b_{l+1}^{n-1} &= \frac{l+1}{n+l+2k}(-\partial_{a_{l+1}})(U_{l+1}^{n-1})\mathrm{STr}\left(\ldots\phi^{i_{n+l+1}}\ldots\right) \\ &- \frac{n-1}{n+l+2k}\left[D_{l+1}^{n-1}|_{i_n\leftrightarrow a_{l+2}}+\ldots+D_{l+1}^{n-1}|_{i_n\leftrightarrow a_p}\right]. \end{aligned} \tag{A-33}$$

We can see that the second term in the expression for b_l^n is the same as the first in the expression for b_{l+1}^{n-1}. All the $D_l^n|_{(\ldots)}$ terms from (A-30) are equal to each other, due to the antisymmetry in the a's, and as it turns out, they come with the right sign to cancel the first term from (A-33). Let's check the numerical coefficients. The second term of b_l^n is

$$\frac{p!}{k!n!l!(p-l)!}\frac{n}{n+l+2k}(p-l)$$

Note that this is 0, for $l = l_{max} = p$, or for $n = n_{min} = 0$. The first term of b_{l+1}^{n-1} is

$$\frac{p!}{k!(n-1)!(l+1)!(p-l-1)!}\frac{l+1}{n+l+2k}$$

The numerical factors are the same so all the terms cancel against each other, except for the b_0^n, and the first term of b_1^n given by (A-30). The $l = 0$ case has to be analysed separately, since we cannot integrate by parts in this case.

$$b_0^n = \frac{1}{k!n!}(\partial_{x^{i_1}}\ldots\partial_{x^{i_n}})C^0_{i'_1\ldots i'_{2k}a_1\ldots a_p}\mathrm{STr}\left(\phi^{i_1}\ldots\phi^{i_n}\ldots\underbrace{\phi^{i'_{2k}}\phi^{i'_{2k-1}}}\ldots\right). \tag{A-34}$$

As in equation (A-23), we can write

$$0 = (n+k)U_0^n\mathrm{STr}\left(\ldots\underbrace{\phi^{i'_{2k}}}\ldots\underbrace{\phi^{i'_{2k-1}}}\ldots\right) = nU_0^n\mathrm{STr}\left(\ldots\underbrace{\phi^{i_n}\phi^{i'_{2k-1}}}\ldots\right)+$$

$$+(k-1)U_0^n\mathrm{STr}\left(\ldots\underbrace{\phi^{i'_{2j}}\phi^{i'_{2j-1}}\phi^{i'_{2k-1}}}\ldots\right)+U_0^n\mathrm{STr}\left(\ldots\underbrace{\phi^{i'_{2k}}\phi^{i'_{2k-1}}}\ldots\right). \tag{A-35}$$

Similarly,

$$0 = nU_0^n \mathrm{STr}\left(\ldots \underbrace{\phi^{i'_{2k}}\phi^{i_n}}\ldots\right) + (k-1)U_0^n \mathrm{STr}\left(\ldots \underbrace{\phi^{i'_{2k}}\phi^{i'_{2j}}\phi^{i'_{2j-1}}}\ldots\right)$$

$$+U_0^n \mathrm{STr}\left(\ldots \underbrace{\phi^{i'_{2k}}\phi^{i'_{2k-1}}}\ldots\right). \qquad \text{(A-36)}$$

From (A-35) and (A-36) we have

$$\begin{aligned} 2U_0^n \mathrm{STr}\left(\ldots \underbrace{\phi^{i'_{2k}}\phi^{i'_{2k-1}}}\ldots\right) &= -nU_0^n \mathrm{STr}\left(\ldots \underbrace{\phi^{i_n}\phi^{i'_{2k-1}}}\ldots\right) - \\ -nU_0^n \mathrm{STr}\left(\ldots \underbrace{\phi^{i'_{2k}}\phi^{i_n}}\ldots\right) &= -n(U_0^n|_{i_n\leftrightarrow i'_{2k}} + U_0^n|_{i_n\leftrightarrow i'_{2k-1}})\mathrm{STr}\left(\ldots \underbrace{\phi^{i'_{2k}}\phi^{i'_{2k-1}}}\ldots\right). \end{aligned} \qquad \text{(A-37)}$$

Repeating this k times, we end up with

$$U_0^n \mathrm{STr}\left(\ldots \underbrace{\phi^{i'_{2k}}\phi^{i'_{2k-1}}}\ldots\right) = \frac{n}{n+2k}(U_0^n - U_0^n|_{i_n\leftrightarrow i'_1} - \ldots - U_0^n|_{i_n\leftrightarrow i'_{2k}})$$
$$\times \mathrm{STr}\left(\ldots \underbrace{\phi^{i'_{2k}}\phi^{i'_{2k-1}}}\ldots\right). \qquad \text{(A-38)}$$

From (A-30) we find that the first term of b_1^{n-1} is (rename $j_1 \to i_n$)

$$-\frac{i^k p}{k!(n-1)!(n+2k)}\partial_{x^{i_1}}\ldots\partial_{x^{i_{n-1}}}\partial_{a_1}C^0_{i'_1\ldots i'_{2k}i_n a_2\ldots a_p}\mathrm{STr}\left(\phi^{i_1}\ldots\phi^{i_n}\ldots\underbrace{\phi^{i'_{2k}}\phi^{i'_{2k-1}}}\ldots\right). \qquad \text{(A-39)}$$

Now we can sum the b_0^n term and the first term of b_1^{n-1} to get a gauge invariant term equal to

$$\frac{i^k}{k!(n-1)!(n+2k)}\partial_{x^{i_1}}\ldots\partial_{x^{i_{n-1}}}F^{0,(2k+p+1)}_{i_1 i'_1\ldots i'_{2k}a_1\ldots a_p}\mathrm{STr}\left(\phi^{i_1}\ldots\phi^{i_n}\underbrace{\phi^{i'_{2k}}\phi^{i'_{2k-1}}}\ldots\right), \qquad \text{(A-40)}$$

where $F^{0,(2k+p+1)} \equiv dC^{0,(2k+p)}$. The monopole coupling doesn't show up in the previous expression, since in deriving (A-38) we assumed $k > 0$. Keeping track of the gauge invariant terms dropped in (A-16), we can express the total coupling, for $F = 0$, as

$$\mu_{p-1} \sum_{l,n>0} \frac{\lambda^{(k+n+l)} i^k p!}{k!(n-1)!l!(p-l)!(2k+n+l)}$$
$$\times \quad \partial_{x^{i_1}} \ldots \partial_{x^{i_{n-1}}} F^{0,(2k+p+1)}_{i_1 i'_1 \ldots i'_{2k} j_1 \ldots j_l a_{l+1} \ldots a_p} \mathrm{STr}\left(\partial_{a_1}\phi^{j_1} \ldots \partial_{a_l}\phi^{j_l}\phi^{i_1} \ldots \phi^{i_n}\phi^{i'_{2k}}\phi^{i'_{2k-1}} \ldots\right). \tag{A-41}$$

For $k = 0$ we need to add the usual monopole coupling given by $\mu_{p-1} C^0_{a_1 \ldots a_p}$ from (A-34).

Appendix B: Gauge invariance for the $F \neq 0$ case

For non-zero F along the brane, the pull-back is defined using covariant derivatives. There are a few useful relations involving covariant derivatives that allow us to use the previous proof in the case when F is nonzero. If Y, Y_1, Y_2 are $N \times N$ matrices transforming in the adjoint representation of the gauge group ($D_a Y = \partial_a Y + i[A_a, Y]$), and f is a scalar function, then:

(a) $\mathrm{Tr}[D_a(Y_1 Y_2)] = \mathrm{Tr}[D_a(Y_1)Y_2] + \mathrm{Tr}[Y_1(D_a Y_2)]$

(b) $D_a(fY) = (\partial_a f)Y + f D_a Y$

(c) $[D_1, D_2]Y = i[F_{12}, Y]$, where $F_{12} = \partial_1 A_2 - \partial_2 A_1 + i[A_1, A_2]$

(d) $D_{[a}F_{bc]} = 0$, by the Bianchi identity

In this case, the equivalent of (A-5), which gives the coupling between N D$(p-1)$ branes and C_{p+2k} is

$$\sum_{l,n,r} \frac{\mu_{p-1}\lambda^{k+n+l+2r} i^{k+r} p!}{2^r r!(k+r)!(n)!l!(p-l-2r)!} \partial_{x^{i_1}} \dots \partial_{x^{i_n}} C^0_{i'_1 \dots i'_{2(k+r)} j_1 \dots j_l [a_{l+1} \dots a_{p-2r}} \\ \times \mathrm{STr}\left(D_{a_1}\phi^{j_1} \dots D_{a_l}\phi^{j_l} \phi^{i_1} \dots \phi^{i_n} \underbrace{\phi^{i'_{2(k+r)}} \phi^{i'_{2(k+r)-1}}} \dots \underbrace{\phi^{i'_2}\phi^{i'_1}} \dots F_{a_p a_{p-1}]} \right), \qquad \text{(B-1)}$$

where $0 \le l \le p - 2r$, and r is the number of F's appearing inside the STr part. As in the proof for $F = 0$, we will write the sum in (B-1) as $\mu_{p-1}\sum_{l,n,r}\lambda^{k+n+l+2r} b^n_{l,r}$, and denote by $U^n_{l,r}$ the term outside the trace corresponding to $b^n_{l,r}$. When integrating $b^n_{l,r}$ by parts, now we will have extra terms containing $\mathrm{STr}\left(\dots D_{[a_1}D_{a_2]}\phi \dots\right)$. Since $D_{[a_1}D_{a_2]}\phi = \frac{1}{2}[D_{a_1}, D_{a_2}]\phi = \frac{i}{2}[F_{a_1 a_2}, \phi]$ these extra terms will cancel against other terms in the expansion for $b^n_{l-2,r+1}$. Given these facts, the right-hand side of (A-8) has an additional term equal to:

$$(l-1)(-U^n_{l,r})\mathrm{STr}\left(\phi^{j_1} \dots D_{a_1} D_{a_l}\phi^{j_l} \dots\right), \qquad \text{(B-2)}$$

while (A-24) changes to:

$$lA_1 + nB_1 = b^n_{l,r} - (k+r-1)U^n_{l,r}\mathrm{STr}\left(\ldots\phi^{i'_{2(k+r)-1}}\ldots\underbrace{\phi^{i'_{2j}}\phi^{i'_{2j-1}}\phi^{i'_{2(k+r)}}}\ldots\right) - $$
$$-rU^n_{l,r}\mathrm{STr}\left(\ldots\underbrace{F\phi^{i'_{2(k+r)}}}\ldots\right). \quad \text{(B-3)}$$

Now, we can see that the generalizations of equations (A-29) and (A-30) are

$$l(A_1 + A_2) + n(B_1 + B_2) = 2b^n_{l,r} - rU^n_{l,r}\mathrm{STr}\left(\ldots\underbrace{[F,\phi^{i'_{2(k+r)}}]}\ldots\right) \quad \text{(B-4)}$$

$$\begin{aligned} b^n_{l,r} &= \frac{l}{n+l+2(k+r)}(-\partial_{a_l})(U^n_{l,r})\mathrm{STr}\left(\ldots\phi^{j_l}\ldots\right) \\ &- \frac{l(l-1)}{n+l+2(k+r)}(U^n_{l,r})\mathrm{STr}\left(\phi^{j_1}\ldots D_{a_1}D_{a_l}\phi^{j_l}\ldots\right) \\ &- \frac{n}{n+l+2(k+r)}\left[D^n_{l,r}|i_n \leftrightarrow a_{l+1} + \ldots + D^n_{l,r}|i_n \leftrightarrow a_{p-2r}\right] \\ &+ \frac{r(k+r)}{n+l+2(k+r)}(U^n_{l,r})\mathrm{STr}\left(\ldots\underbrace{[F,\phi^{i'_{2(k+r)}}]}\ldots\right). \end{aligned} \quad \text{(B-5)}$$

The second term in the expansion for $b^n_{l,r}$ has the same structure as the fourth term in the expansion for $b^n_{l-2,r+1}$. While it is easy to see that these extra terms have the required form to produce the (partial) cancelation between $b^n_{l,r}$, and $b^n_{l-2,r+1}$. The numerical pre-factor from $b^n_{l,r}$ is

$$\frac{i^{k+r}p!}{2^r(k+r)!r!(n)!l!(p-l-2r)!}l(l-1)\frac{i}{2}, \quad \text{(B-6)}$$

while from $b^n_{l-2,r+1}$, the pre=factor is

$$\frac{i^{k+r+1}p!}{2^{r+1}(k+r+1)!(r+1)!n!(l-2)!(p-l-2r)!}(r+1)(k+r+1). \quad \text{(B-7)}$$

Since the numerical factors are the same, when we are summing the $b^n_{l,r}$'s over r, all the extra terms that we get in the case of a non-zero F will cancel against each other, except for the second term of $b^n_{l=2,r}$. At the limits, when $r = r_{min}$ the fourth term in the expression for $b^n_{l,r}$ is 0, since $r_{min}(r_{min}+k)=0$. If $k>0$, $r_{min}=0$, otherwise $r_{min}=-k$. When $r=r_{max}$, $l<2$ so the second term in the expansion for $b^n_{l,r}$ is 0. After summing over l and r we are left with

$$\sum_r (b^n_{l=0,r}) + \sum_r \frac{1}{n+2(k+r)}(-\partial_{a_1})(U^{n-1}_{l=1,r})\mathrm{STr}\left(\ldots\phi^{j_1}\ldots\right) + \sum_{r>r_{min}} \frac{-i}{n+2(k+r)} U^n_{2,r-1}\mathrm{STr}\left(\ldots\phi^{j_1}\ldots\underbrace{[F_{a_1a_l},\phi^{j_l}]}\ldots\right). \tag{B-8}$$

For $b_{l=0,r}$ we are using a transformation as in (A-38):

$$U^n_{0,r}\mathrm{STr}\left(\ldots\underbrace{\phi^{i'_{2(k+r)}}\phi^{i'_{2(k+r)-1}}}\ldots\right) =$$

$$\frac{n}{n+2(k+r)}(U^n_{0,r} - U^n_{0,r}|_{i_n\leftrightarrow i'_1} - \ldots - U^n_{0,r}|_{i_n\leftrightarrow i'_{2(k+r)}})\mathrm{STr}\left(\ldots\underbrace{\phi^{i'_{2(k+r)}}\phi^{i'_{2(k+r)-1}}}\ldots\right) - \frac{(k+r)r}{n+2(k+r)}U^n_{0,r}\mathrm{STr}\left(\ldots\underbrace{[F,\phi^{i'_{2(k+r)-1}}]}\ldots\right) \tag{B-9}$$

Using (B-9) to replace $b^n_{l=0,r}$ in (B-8), we get

$$\sum_r \frac{n}{n+2(k+r)}(U^n_{0,r} - U^n_{0,r}|_{i_n\leftrightarrow i'_1} - \ldots - U^n_{0,r}|_{i_n\leftrightarrow i'_{2(k+r)}}) \times\mathrm{STr}\left(\ldots\underbrace{\phi^{i'_{2(k+r)}}\phi^{i'_{2(k+r)-1}}}\ldots\right) + \sum_r \frac{1}{n+2(k+r)}(-\partial_{a_1})(U^{n-1}_{l=1,r})\mathrm{STr}\left(\ldots\phi^{j_1}\ldots\right). \tag{B-10}$$

The above expression is gauge invariant since we can write it as a field strength term noticing that after renaming $j_1\rightarrow i_n$, the STr parts are identical and

$$(\partial_{a_1})(U^{n-1}_{l=1,r}) = n(p-2r)U^n_{0,r}|_{i_n\leftrightarrow a_1}. \tag{B-11}$$

Taking into account the corresponding gauge invariant terms dropped in equation (A-16) the total coupling between N D$(p-1)$ branes and a C_{p+2k} potential can be expressed in a gauge invariant way as

$$\mu_{p-1}\sum_{r,l}\frac{\lambda^{k+1+2r+l}i^{k+r}p!}{2^r r!(k+r)!l!(p-2r-l)!}\mathrm{STr}(\overline{F}^{(2k+p+1)}_{r,l}(\phi)_{i_1 i'_1 i'_2 \ldots i'_{2(k+r)-1} i'_{2(k+r)} j_1 \ldots j_l a_{l+1} \ldots a_{p-2r}}\phi^{i_1}$$
$$D_{a_1}\phi^{j_1}\ldots D_{a_l}\phi^{j_l}\underbrace{\phi^{i'_{2(k+r)}}\phi^{i'_{2(k+r)-1}}}\ldots\underbrace{\phi^{i'_2}\phi^{i'_1}}F_{a_{p-2r+1}a_{p-2r+2}}\ldots F_{a_{p-1}a_p}), \tag{B-12}$$

where we defined,

$$\overline{F}^{(2k+p+1)}_{r,l}(\phi)=\sum_{n\geq 0}\frac{\lambda^n}{(n)!(n+l+2k+2r+1)}\phi^{i_1}\ldots\phi^{i_n}\partial_{x^{i_1}}\ldots\partial_{x^{i_n}}F^{0,(2k+p+1)}(\sigma,x^i)|_{x^i=0}$$

Since equation (B-9) was derived assuming $k>0$, for $k\leq 0$ there is an additional monopole coupling term given by:

$$\mu_{p-1}\frac{\lambda^{|k|}p!}{2^{|k|}(|k|)!(p-2|k|)!}C^0_{[a_1\ldots a_{p-2|k|}}F\ldots F_{a_{p-1}a_p]}. \tag{B-13}$$

Appendix C: Tensor spherical harmonics

We give several useful relations between the transverse space tensors used in this paper. If the transverse space is 5-dimensional, and we are interested in describing antisymmetric 2-form and 3-form harmonics, they depend on the tensors

$$T_2 = \frac{1}{2!} T_{mn} dx^m \wedge dx^n \tag{C-1}$$
$$V_2 = \frac{1}{2!} \left(\frac{x^q x^i}{r^2} T_{qj} + \frac{x^q x^j}{r^2} T_{iq} \right) dx^i \wedge dx^j \tag{C-2}$$
$$T_3 = *_5 T_2 \tag{C-3}$$
$$V_3 = \frac{1}{3!} \left(\frac{x^q x^m}{r^2} T_{qnp} + 2 \text{ more} \right) dx^m \wedge dx^n \wedge dx^p, \tag{C-4}$$

which satisfy:

$$T_2 - V_2 = *_5 V_3 \tag{C-5}$$
$$d(\ln r) \wedge V_3 = 0 \tag{C-6}$$
$$d(\ln r) \wedge *_5 V_3 = d(\ln r) \wedge *_5 T_3 \tag{C-7}$$
$$d(V_3) = -3 d(\ln r) \wedge T_3 \tag{C-8}$$
$$d(*_5 V_3) = 2 \ln r \wedge *_5 T_3. \tag{C-9}$$

In order to express the 1-form potential it is also useful to introduce the transverse space 1-form

$$S_1 = T_{mn} x^m dx^n \tag{C-10}$$

satisfying

$$d(S_1) = 2 T_2 \tag{C-11}$$
$$d(r^p S_1) = r^p (2 T_2 + p V_2). \tag{C-12}$$

If the transverse space is 8 dimensional one can similarly introduce 2-form and 6-form tensors [1]. We give all the fields in terms of T_2, V_2 and S_1, and equations (C-1,C-2,C-10,C-12) are the only ones needed.

[1]For the precise formulas see the Appendix in [36].

Appendix D: Consistency checks

As explained in [50], in the case of a large number N of D4 branes there exists a decoupling limit, $\alpha' \to 0$, keeping $g_s(\alpha')^{1/2}$ fixed, where the field theory on the branes decouples from the theory in the bulk.

The type II A supergravity solution can be trusted [50] in the region: $\frac{\alpha'}{N} \ll r \ll N^{1/3}\alpha'$. For smaller r the curvature becomes too large, and the weakly coupled description of the physics is provided by the 4+1 dimensional Super Yang Mills theory. For larger r the dilaton becomes too large, and a weakly coupled description of the physics is provided by 11 dimensional supergravity.

The condition for the validity of the perturbative calculations done in Section 3.2 is: $\frac{|F_2|^2}{|F_6|^2} \ll 1$:

$$\begin{aligned} |F_2|^2 &= F_{ij}F_{ij}g^{ii}g^{jj} \sim g_s^{-2}m^2 Z \sim \frac{m^2 N g_s^{-1}\alpha'^{3/2}}{r^3} \\ |F_6|^2 &= F_{01234r}F_{01234r}g^{00}...g^{44}g^{rr} \sim \frac{1}{r^2}. \end{aligned} \tag{D-1}$$

Thus the perturbation is small if $m^2 N g_s^{-1}\alpha'^{3/2} \ll r$. For the smallest mass which allows for a moduli space this is equivalent (in the decoupling limit) to $\frac{\alpha'}{g_s^2 N} \ll r$, which is trivially satisfied. For the other masses that allow polarization the perturbative calculations are valid for $\frac{k^2\alpha'}{g_s^2 N} \ll r$.

Finally, let us consider the regime where the M5 branes become effectively smeared. The curvature near a single M5 brane is large in string units for distances of order α'. Therefore the smearing approximation is justified for $\frac{2\pi r_0}{N} \ll \alpha'$. This constraint is satisfied in the energy region of interest.

The regime where the M5 branes are seen as smeared is the same as the regime where the D4 contribution to the energy of the polarized configuration is dominant (3.21). Outside this regime, both the supergravity perturbative approach in Chapter 2 and the exact solution in Chapter 3 stop being valid.

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