

## Chapter 8. Optimal Control

An interesting attribute of an RFA network is its physically imposed performance minimum; i.e. the minimum  $J$  which is physically possible due to constraints (5.15a) and (5.15b), for a given acceleration input  $a_g$  and initial condition  $\mathbf{w}_0$ . Recall that in Chapter 4,  $J$  was optimized. However, this optimization was performed assuming a constant-damping relationship between the states in and the control force (i.e., the matrix  $\mathbf{Z}$  was assumed to be constant). The Damping-Reference controllers proposed in the previous chapter are guaranteed to perform at least as well as the constant- $\mathbf{Z}$  control law, but clearly they are sub-optimal in comparison to the absolute minimum on  $J$  imposed by the physical constraints.

The goal of this chapter is to develop a method for calculating the physical limit on  $J$ , given the Nominal System Model (i.e. the structural dynamic model, RFA network properties  $\mathbf{C}_c$  and  $\mathbf{f}_{\max}$ , and connectivity matrix  $\mathbf{N}$ ), initial condition  $\mathbf{w}_0$ , and acceleration input  $a_g$ . It should be emphasized that this discussion does not necessarily concern real-time feedback control. Rather, the problem at hand is to solve for the optimal physically-realizable  $\mathbf{u}$ , assuming that the *entire* earthquake record  $a_g(t)$  is known *a priori*.

As such, it is reasonable to ask what purpose this analysis might have. Consider a scenario where actuators are being designed for a given structural application. It is necessary in such a case to determine the number of actuators to be used, the types and ratings of machines to be used, the manner in which to distribute them about the structure, and so forth. To measure the quality of a given configuration of devices in a given structure, a typical procedure would be to design the devices, design a feedback control law relating the structural deformation  $\mathbf{w}$  to the control input  $\mathbf{u}$ , and then see how the closed-loop system performs. The problem with this approach is that it couples the assessment of the *actuators* with the assessment of the *control law*. This is inconvenient, because it is impossible to tell whether an actuation system is performing badly because the hardware is inadequate for the demands of the application, or because the control law is not using the existing hardware to its full potential. If the situation is the former, then no amount of control law redesign will ever yield the desired level of performance. If the situation is the latter, however, it may be that redesign of the control law could yield acceptable results. By evaluating the optimal  $J$  for a set of earthquake records, or obtaining its statistics for a stochastic earthquake model, conclusions can be drawn about the quality of the actuation system hardware, which precede the design of the control law. As such, an assessment of the optimal

performance is appealing because it allows for a more intelligent preliminary actuator hardware design.

The research reported in this chapter concerns the applications of fundamental optimal control theory, as applied to energy-constrained actuation systems. Although the ideas are framed in the context of RFA networks, they are readily transferable to semiactive systems. This problem has received some attention in a semiactive context, both in suspensions (Hrovat et al. 1988; Tseng and Hedrick 1994) and in earthquake engineering (Yamada and Kobori 2001).

The work in this chapter lays the foundation for this problem. However, there remain some unresolved difficulties. In particular, these stem from the fact that the optimization problem at hand is in general nonconvex. Thus, the development begun in this chapter is left open-ended and, consequently, this material should be viewed more as a work in progress than as a finished product.

## 8.1: The Optimal Control Problem

Consider a solution to Eq. (5.13), over a time interval  $t \in [0, t_f]$ , with initial condition  $\mathbf{w}(0) = \mathbf{w}_0$  and acceleration  $a_g \in C[0, t_f]$ . Then clearly there is an affine relationship between functions  $\mathbf{w}$  and  $\mathbf{u}$ ; i.e.

$$\mathbf{w}(t) = \mathbf{W}(t, \mathbf{u}(\cdot); \mathbf{w}_0, a_g) = e^{A t} \left[ \mathbf{w}_0 + \int_0^t e^{-A \tau} (\mathbf{B}_u \mathbf{u}(\tau) + \mathbf{B}_a a_g(\tau)) d\tau \right] \quad (8.1)$$

However, not all  $\mathbf{u} \in \mathfrak{R}^m \times C[0, t_f]$  satisfy  $\mathbf{u}(t) \in \mathcal{U}(\mathbf{w}(t))$  for  $t \in [0, t_f]$ , and the following definition characterizes the set of  $\mathbf{u}$  for which this condition holds.

**DEFINITION:** For the dynamic system in (5.13), define  $\mathcal{F}_u(\mathbf{w}_0, a_g) \subset \mathfrak{R}^m \times C[0, t_f]$  as

$$\mathcal{F}_u(\mathbf{w}_0, a_g) = \left\{ \mathbf{u} \in \mathfrak{R}^m \times C[0, t_f] \mid \mathbf{u} \in \mathcal{U}(\mathbf{W}(t, \mathbf{u}(\cdot); \mathbf{w}_0, a_g)) \text{ , } \forall t \in [0, t_f] \right\} \quad (8.2)$$

An input  $\mathbf{u} \in \mathcal{F}_u(\mathbf{w}_0, a_g)$  is called *Feasible*, given  $\mathbf{w}_0$  and  $a_g$ .

Thus,  $\mathcal{F}_u(\mathbf{w}_0, a_g)$  is the largest set in  $\mathfrak{R}^m \times C[0, t_f]$  of control inputs which are physically possible for the NSM.

With this terminology, the performance measure  $J$  is redefined in an equivalent form to that proposed in Chapter 4.

**DEFINITION:** The deterministic performance measure  $J: \mathcal{F}_u(\mathbf{w}_0, a_g) \mapsto \mathfrak{R}^+$  is defined as

$$J(\mathbf{u}; \mathbf{w}_0, a_g) = \int_0^{t_f} \phi(\mathbf{u}(t), \mathbf{W}(t, \mathbf{u}(\cdot); \mathbf{w}_0, a_g); a_g(t)) dt \quad (8.3)$$

where  $\phi \geq 0$  is of the form

$$\phi(\mathbf{u}(t), \mathbf{w}(t); a_g(t)) = \phi_1(\mathbf{w}(t)) + \frac{1}{2} \begin{bmatrix} \mathbf{w}^T(t) & \mathbf{u}^T(t) & a_g(t) \end{bmatrix} \begin{bmatrix} \mathbf{Q} & \mathbf{S} & \mathbf{Q}_a \\ \mathbf{S}^T & \mathbf{R} & \mathbf{S}_a \\ \mathbf{Q}_a^T & \mathbf{S}_a^T & \mathbf{R}_a \end{bmatrix} \begin{bmatrix} \mathbf{w}(t) \\ \mathbf{u}(t) \\ a_g(t) \end{bmatrix} \quad (8.4)$$

and where the following properties hold:

$$\nabla_{\mathbf{w}} \otimes \nabla_{\mathbf{w}} \phi_1 \geq 0 \quad (8.5a)$$

$$\mathbf{R} > 0 \quad (8.5b)$$

$$\mathbf{Q} - \mathbf{SRS}^T \geq 0 \quad (8.5c)$$

where  $\nabla_{\mathbf{w}}$  is the gradient with respect to  $\mathbf{w}$ , and  $\nabla_{\mathbf{w}} \otimes \nabla_{\mathbf{w}}$  is the Hessian operator. Together, these properties guarantee that  $\phi$  is convex in  $\mathbf{u}$  and semiconvex in  $\{\mathbf{w}, \mathbf{u}\}$ .

Note that it has been assumed that  $\mathbf{R}$  is positive definite. Recall that this is true if the optimization weight of each story acceleration is nonzero and  $\mathbf{N}$  is nondegenerate, or if additional weights have been added to  $\mathbf{R}$  to favor small values of  $\mathbf{u}$ .

The constraints on  $\mathbf{u}$  in Eq. (6.6) are such that  $\mathcal{F}_u(\mathbf{w}_0, a_g)$  is compact. It follows from this observation, together with the fact that  $J$  is continuous and bounded, that there must exist a set of  $\mathbf{u} \in \mathcal{F}_u(\mathbf{w}_0, a_g)$  for which  $J$  is minimal. This set will be denoted as  $U_{opt}$ ; i.e.

$$U_{opt} = \left\{ \tilde{\mathbf{u}} \in \mathcal{F}_u(\mathbf{w}_0, a_g) \mid J(\tilde{\mathbf{u}}; \mathbf{w}_0, a_g) \leq J(\mathbf{u}; \mathbf{w}_0, a_g), \forall \mathbf{u} \in \mathcal{F}_u(\mathbf{w}_0, a_g) \right\} \quad (8.6)$$

In terms of these quantities, the optimal control problem statement can be given as follows:

**OPTIMAL CONTROL PROBLEM (OCP):** Find at least one  $\mathbf{u}_{opt} \in U_{opt}$  as defined in Eq. (8.6), and the associated optimal performance  $J_{opt}$  as defined in Eq. (8.3).

In Section 8.2, necessary conditions are derived for the solution of the theoretical OCP. It is shown that  $\mathbf{u}_{opt}$  must satisfy a nonlinear two-point boundary value problem, and characteristics of  $\mathbf{u}_{opt}$  are discussed. However, satisfaction of this boundary value problem, on its own, is in general not sufficient to guarantee  $\mathbf{u}_{opt}$  yields the globally-optimal performance. Sufficiency for global performance minimization can be obtained through the derivation of the solution to the Hamilton-Jacobi-Bellman equation, discussed in Section 8.3. However, the numerical demands for the derivation of this solution are prohibitive for all but the most simple

structural systems. In Section 8.4, the OCP is numerically solved for free-vibrating SDOF systems, with one control actuator, and various performance measures. Finally, Section 8.5 summarizes the findings of this chapter, and discusses the next logical steps in the progression of this research.

## 8.2: Necessary Conditions for Local Optimality

To find necessary conditions for  $\mathbf{u} \in U_{opt}$ , the calculus of variations is used. The structure of this section is an application of the general optimal control problem with input-state constraints, which has been presented extensively in the literature (e.g. Kirk 1970; Stengel 1994). Because the proofs of the claims made in this section are somewhat lengthy, and because they are an application of a well-known body of theory, they have been relegated to an appendix.

Eq. (8.6) yields a variational statement necessary for  $\mathbf{u} \in U_{opt}$ , as

$$\mathbf{u} \in U_{opt} \Rightarrow \delta J(\mathbf{u}; \mathbf{w}_0, a_g) \geq 0 \quad \forall (\mathbf{u} + \delta \mathbf{u}) \in F_u(\mathbf{w}_0, a_g). \quad (8.7)$$

where  $\delta \mathbf{u}$  is an infinitesimal variation. Substituting (8.3) for  $J$  above gives

$$\mathbf{u} \in U_{opt} \Rightarrow \delta \int_0^{t_f} \phi(\mathbf{W}(t, \mathbf{u}(\cdot); \mathbf{w}_0, a_g), \mathbf{u}(t)) dt \geq 0 \quad \forall \text{adm. } \delta \mathbf{u} \quad (8.8)$$

where admissible variations in  $\delta \mathbf{u}$  are those notated in (8.7).

The following lemma gives the implications of the variational statement in (8.8), making use of Lagrange multipliers to enforce the constraints on  $\mathbf{u}$ .

**LEMMA 8.1:** A necessary condition for  $\mathbf{u}_{opt}$  is that there exist integrable functions  $\mathbf{p}(t) \in \mathfrak{R}^{2n}$ ,  $\boldsymbol{\lambda}(t) \in \mathfrak{R}^m$ , and  $\lambda_R(t) \in \mathfrak{R}$ , over the interval  $t \in [0, t_f]$  such that

$$\mathbf{u}_{opt}(t) = -\frac{1}{2} \mathbf{B}_u^T \mathbf{w}(t) - (\mathbf{R} + 2\lambda_R(t) \mathbf{I})^{-1} \left[ \boldsymbol{\lambda}(t) + \left( \mathbf{S}^T - \frac{1}{2} \mathbf{R} \mathbf{B}_u^T \right) \mathbf{w}(t) + \mathbf{B}_u^T \mathbf{p}(t) + \mathbf{S}_a^T a_g(t) \right] \in \mathcal{U}(\mathbf{w}(t)) \quad (8.9)$$

where  $\mathbf{p}$  satisfies the final-value problem

$$\dot{\mathbf{p}}(t) = -\nabla_{\mathbf{w}}^T \phi_1(\mathbf{w}(t)) - \mathbf{Q} \mathbf{w}(t) - (\mathbf{S} + \lambda_R(t) \mathbf{B}_u) \mathbf{u}_{opt}(t) - \mathbf{A}^T \mathbf{p}(t) - \mathbf{Q}_a a_g(t) \quad , \quad \mathbf{p}(t_f) = \mathbf{0} \quad (8.10)$$

and where  $\boldsymbol{\lambda}$  and  $\lambda_R$  must satisfy the following constraints for all  $t \in [0, t_f]$ :

$$\lambda_k(t) > 0 \Rightarrow u_k(t) = u_{k \max} \quad (8.11a)$$

$$\lambda_k(t) < 0 \Rightarrow u_k(t) = -u_{k \max} \quad (8.11b)$$

$$\lambda_R(t) \geq 0 \quad , \quad \lambda_R(t) P(\mathbf{u}_{opt}(t), \mathbf{w}(t)) = 0 \quad (8.11c)$$

*proof:* See appendix A8

The dynamic states  $\mathbf{p}$ , called the *costate* or adjoint system, constitute a set of Lagrange multipliers which constrain  $\mathbf{w}$  to obey its differential equation, enforcing the constraint between  $\mathbf{w}$  and  $\mathbf{u}$  arising from equation (8.1). The Lagrange multipliers  $\lambda_R$  and  $\boldsymbol{\lambda}$  enforce conditions (5.15a) and (5.15b) respectively. Constraint equations (8.11a-c) constitute a property called *complementary slackness*, which arises from the fact that the constraints in (5.15) are inequalities. Greater detail concerning these Lagrange multipliers is included in the proof to Lemma 8.1.

To interpret the observations of Lemma 8.1, Lemma 8.2 below makes some conclusions regarding the uniqueness of the solutions to Eqs. (8.9) and (8.11c).

**LEMMA 8.2:** For the conditions of Lemma 8.1, the following are true:

a) There exists a unique mapping  $\mathbf{U}: \mathfrak{R}^{2n} \times \mathfrak{R}^{2n} \mapsto \mathfrak{R}^m$  such that

$$\mathbf{u}_{opt}(t) = \mathbf{U}(\mathbf{w}(t), \mathbf{p}(t); a_g(t)) \quad (8.12)$$

Specifically,  $\mathbf{u}_{opt}$  is related to  $\mathbf{w}$  and  $\mathbf{p}$  by

$$\mathbf{u}_{opt}(t) = \arg \min_{\tilde{\mathbf{u}} \in \mathcal{U}(\mathbf{w}(t))} \left\{ \frac{1}{2} \tilde{\mathbf{u}}(t)^T \mathbf{R} \tilde{\mathbf{u}}(t) + \tilde{\mathbf{u}}(t)^T [\mathbf{S} \mathbf{w}(t) + \mathbf{B}_u^T \mathbf{p}(t) + \mathbf{S}_a^T a_g(t)] \right\} \quad (8.13)$$

b) There exists a mapping  $A_R: \mathfrak{R}^{2n} \times \mathfrak{R}^{2n} \mapsto \mathfrak{R}$  such that

$$\lambda_R(t) = A_R(\mathbf{w}(t), \mathbf{p}(t); a_g(t)) \quad (8.14)$$

This mapping is bounded for all  $\{\mathbf{w}(t), \mathbf{p}(t)\} \in \{\mathfrak{R}^{2n} - \mathcal{N}(B_u^T)\} \times \mathfrak{R}^{2n}$ , where  $\mathcal{N}(\cdot)$  denotes the null space of the operator. The mapping is unique for almost all  $\{\mathbf{w}(t), \mathbf{p}(t)\}$ .

c) The product  $A_R \mathbf{U}$  is bounded for all  $\{\mathbf{w}(t), \mathbf{p}(t)\} \in \mathfrak{R}^{2n} \times \mathfrak{R}^{2n}$  and is unique for almost all  $\{\mathbf{w}(t), \mathbf{p}(t)\}$ .

*proof:* See appendix A8

Note that if  $\mathbf{u}_a$  is defined as

$$\mathbf{u}_a = -\mathbf{R}^{-1} [\mathbf{S} \mathbf{w} + \mathbf{B}_u^T \mathbf{p} + \mathbf{S}_a a_g] \quad (8.15)$$

Then, Eq. (8.13) can be restated by completing the square, as

$$\mathbf{u}_{opt}(t) = \arg \min_{\tilde{\mathbf{u}} \in \mathcal{U}(\mathbf{w}(t))} \|\tilde{\mathbf{u}}(t) - \mathbf{u}_a(t)\|_{\mathbf{R}} \quad (8.16)$$

Thus, the optimal control consists of the instantaneous clipping action, described in Chapter 6, operating on a signal which is linear in  $\mathbf{w}$  and  $\mathbf{p}$ . Thus, the optimal control is a Clipped-Linear controller. Because the vector  $\mathbf{p}(t)$  depends on future values of  $a_g$ , the linear term (and therefore

the controller) is noncausal. Note that the Clipped-Linear controllers from Chapter 6 can be viewed as approximations of the optimal controller, where the quantity  $\mathbf{P}\mathbf{w}(t)$  is an approximation of the noncausal  $\mathbf{p}(t)$  term. However, it is in general difficult to quantify the error of this approximation without solving the OCP directly.

From the observations in Lemmas 8.1 and 8.2, it can be concluded that  $\mathbf{u}_{opt} = \mathbf{U}(\mathbf{w}, \mathbf{p}; a_g)$  must be a solution to the differential equation

$$\begin{aligned} \begin{bmatrix} \dot{\mathbf{w}}(t) \\ \dot{\mathbf{p}}(t) \end{bmatrix} &= \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{Q} & -\mathbf{A}^T \end{bmatrix} \begin{bmatrix} \mathbf{w}(t) \\ \mathbf{p}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ -\nabla_{\mathbf{w}}^T \phi(\mathbf{w}(t)) \end{bmatrix} + \begin{bmatrix} \mathbf{B}_a \\ -\mathbf{Q}_a \end{bmatrix} a_g(t) \\ &+ \begin{bmatrix} \mathbf{B}_u \\ -\mathbf{S} \end{bmatrix} \mathbf{U}(\mathbf{w}(t), \mathbf{p}(t); a_g(t)) + \begin{bmatrix} \mathbf{0} \\ -\mathbf{B}_u \end{bmatrix} A_R \mathbf{U}(\mathbf{w}(t), \mathbf{p}(t); a_g(t)) \end{aligned} \quad (8.17)$$

with boundary conditions

$$\mathbf{w}(0) = \mathbf{w}_0, \quad \mathbf{p}(t_f) = \mathbf{0} \quad (8.18)$$

Thus, the optimal control  $\mathbf{u}_{opt}$  must admit a solution to the nonlinear two-point Boundary Value Problem (BVP) described above.

Observations a and c of Lemma 8.2 imply that the derivatives of  $\{\mathbf{w}, \mathbf{p}\}$  are finite; i.e. that  $\mathbf{w}$  and  $\mathbf{p}$  are differentiable. However, these conditions do not imply that  $\mathbf{w}$  and  $\mathbf{p}$  are smooth. Rather, their derivatives may possess discontinuities where the values of  $\lambda_R(t)\mathbf{u}(t)$  or  $\lambda(t)$  “jump” from one value to another. Here, no claim is made as to the continuity of  $\lambda_R(t)\mathbf{u}(t)$  and  $\lambda(t)$ .

The observations in Lemmas 8.1 and 2 make it possible to present the theorem below, which is the main result of this development.

**THEOREM 8.3:** The optimal structural response for an RFA network yields solutions  $\mathbf{w} \in \mathfrak{R}^{2n} \times C[0, t_f]$  and  $\mathbf{p} \in \mathfrak{R}^{2n} \times C[0, t_f]$  to the nonlinear two-point boundary value problem stated in Eq. (8.17). Furthermore,  $\mathbf{u}_{opt}$  is uniquely determined from  $\mathbf{w}$  and  $\mathbf{p}$ .

*proof:* The existence of a solution  $\{\mathbf{w}, \mathbf{p}\}$  follows immediately from the facts that  $F_u(\mathbf{w}_0, a_g)$  is compact and  $J$  is convex in  $\{\mathbf{u}, \mathbf{W}(\mathbf{u})\}$  (thus guaranteeing that a minimal  $\mathbf{u}$  exists) and because the conditions of Lemma 8.1 are necessary for any extremal  $\mathbf{u}$ . The rest of the proof follows immediately from Lemma 8.2.

Theorem 8.3 does not state that the solution to the boundary-value problem is unique. It merely states that there exists an optimal control  $\mathbf{u}_{opt}$ , which satisfies the conditions of the BVP. There may be other solutions which are *locally* optimal in  $J$ , but which do not yield the global minimum. Because  $J$  is a convex functional, proof that the extremal control is unique (i.e. that there is exactly one  $\mathbf{u}$  satisfying the constraints of the two-point BVP) would be sufficient to

ensure that the necessary conditions discussed above yield a global minimum. However, this is in general difficult to prove, because of the role of  $\lambda_{Rk}(t)$  in the differential equation. If the BVP has multiple solutions, then the fact that  $\mathcal{F}_u(\mathbf{w}_0, a_g)$  is a nonconvex domain implies that there may be multiple local minima on its boundary, despite the convexity of  $J$ . This will be discussed in greater detail in the next section.

Because of these issues, direct application of these concepts to the derivation of the optimal performance remains an unsolved problem.

### 8.2.1: Comparisons with Related Optimal Control Problems

For systems with different types of constraints on  $\mathbf{u}$ , analogous optimal control problems can be solved. Two related problems relevant to the present study are those for semiactive and active control systems.

#### *Optimal Semiactive Control*

Analogies can be drawn to the optimal control of semiactive systems, in which the active feedback and clipping action operations also appear, although customarily presented in a different way (Karnopp 1983; Margolis 1983; Tseng and Hedrick 1994). If the same actuators used in the RFA network were operated as semiactive devices (i.e. if they were not allowed to share power with each other), then a BVP for  $\mathbf{u}_{opt}$  could be developed which is analogous to the one presented in equation (8.17). Mathematically, the difference between the OCPs for semiactive and RFA systems would be that for semiactive systems, constraint (5.15a) would be changed to

$$u_k^2(t) + \{\mathbf{B}_u^T \mathbf{w}(t)\}_k u_k \leq 0 \quad , \quad \forall k \in \{1..m\} \quad (8.19)$$

to reflect the fact that each  $u_k$  must dissipate electrical energy independently of the others.

It can then be shown that the optimal control for semiactive systems must satisfy equations analogous to Lemma 8.1, except that Eq. (8.9) would become

$$\mathbf{u}(t) = -\frac{1}{2} \mathbf{B}_u^T \mathbf{w}(t) - \left( \mathbf{R} + 2 \sum_{k=1}^m \lambda_{Rk}(t) \hat{\mathbf{e}}_k \hat{\mathbf{e}}_k^T \right)^{-1} \left[ \boldsymbol{\lambda}(t) + \left( \mathbf{S}^T - \frac{1}{2} \mathbf{R} \mathbf{B}_u^T \right) \mathbf{w}(t) + \mathbf{B}_u^T \mathbf{p}(t) + \mathbf{S}_a^T a_g(t) \right] \in \mathcal{U}(\mathbf{w}(t)) \quad (8.20)$$

and the differential equation for  $\mathbf{p}$  would be

$$\dot{\mathbf{p}}(t) = -\nabla_{\mathbf{w}}^T \phi_1(\mathbf{w}(t)) - \mathbf{Q} \mathbf{w}(t) - \left( \mathbf{S} + \sum_{k=1}^m \lambda_{Rk}(t) \hat{\mathbf{e}}_k \hat{\mathbf{e}}_k^T \mathbf{B}_u \right) \mathbf{u}(t) - \mathbf{A}^T \mathbf{p}(t) - \mathbf{Q}_a a_g(t) \quad , \quad \mathbf{p}(t_f) = \mathbf{0} \quad (8.21)$$

where  $\hat{\mathbf{e}}_k$  is the unit vector in direction  $k$ , and  $\lambda_{Rk}(t)$ ,  $k \in \{1..m\}$ , are Lagrange multipliers which enforce constraint (8.19). Similarly to Eq. (8.11c), each of these multipliers must satisfy the complementary slackness condition

$$\lambda_{Rk} \geq 0 \quad , \quad \lambda_{Rk} \left( u_k^2 + u_k \{ \mathbf{B}_u^T \mathbf{w} \}_k \right) = 0 \quad , \quad k \in \{1..m\} \quad (8.22)$$

With Eqs. (8.20) and (8.21) substituted for (8.9) and (8.10) respectively, the optimal  $\mathbf{u}$  for the semiactive system can be described as a BVP in the same way as for the RFA network.

Conclusions similar to Lemma 8.2 can also be drawn, and  $\mathbf{u}(t)$  can be expressed in the manner of Eq. (8.16), with the same  $\mathbf{u}_a(t)$ . However, for the semiactive system, the region of admissible  $\mathbf{u}(t)$  values would be a subset of  $\mathcal{U}(\mathbf{w}(t))$ , and the minimization in Eq. (8.16) would be taken over this subset.

Because of the similarity of the OCPs for semiactive and RFA systems, many of the conclusions drawn in this section are applicable to both. In fact, the majority of the theory developed in this chapter may be applied to the semiactive OCP with minimal changes.

### ***Optimal Unconstrained (i.e. Active) Control***

The results derived here for RFA and semiactive systems can also be related to the classical results for unconstrained optimal control and regulation. As the absence of constraints on  $\mathbf{u}$  implies an external power source for the control system, these results apply to ideal active control systems.

It is possible to present Eq. (8.17) in an alternate format which is more explicit in its illustration of how the Lagrange multipliers influence the BVP. It follows from Lemma 8.2 that there exists a unique  $\boldsymbol{\lambda}(t)$  vector for almost all  $\{\mathbf{w}(t), \mathbf{p}(t)\}$ . Thus, the multipliers  $\boldsymbol{\lambda}(t)$  and  $\lambda_{Rk}(t)$  may both be viewed as feedback functions of  $\mathbf{w}(t)$  and  $\mathbf{p}(t)$ . Using Eq. (8.9), the BVP may then be written as

$$\begin{bmatrix} \dot{\mathbf{w}}(t) \\ \dot{\mathbf{p}}(t) \end{bmatrix} = \mathbf{F}_{wp}(\lambda_R(t)) \begin{bmatrix} \mathbf{w}(t) \\ \mathbf{p}(t) \end{bmatrix} + \mathbf{f}_w(\mathbf{w}(t)) + \mathbf{F}_a(\lambda_R(t)) a_g(t) + \mathbf{F}_\lambda(\lambda_R(t)) \boldsymbol{\lambda}(t) \quad , \quad \mathbf{w}(0) = \mathbf{w}_0 \quad , \quad \mathbf{p}(t_f) = \mathbf{0} \quad (8.23)$$

where

$$\begin{aligned} \mathbf{F}_{wp}(\lambda_R) = & \begin{bmatrix} \mathbf{A} - \frac{1}{2} \mathbf{B}_u \mathbf{B}_u^T & \mathbf{0} \\ -\mathbf{Q} - \frac{1}{4} \mathbf{B}_u \mathbf{R} \mathbf{B}_u^T + \frac{1}{2} \mathbf{B}_u \mathbf{S}^T + \frac{1}{2} \mathbf{S} \mathbf{B}_u^T & -\mathbf{A}^T + \frac{1}{2} \mathbf{B}_u \mathbf{B}_u^T \end{bmatrix} \\ & + \begin{bmatrix} -\mathbf{B}_u \\ (\mathbf{S} - \frac{1}{2} \mathbf{B}_u \mathbf{R}) \end{bmatrix} [\mathbf{R} + 2\lambda_R \mathbf{I}]^{-1} \begin{bmatrix} (\mathbf{S} - \frac{1}{2} \mathbf{R} \mathbf{B}_u)^T & \mathbf{B}_u^T \end{bmatrix} + \frac{1}{2} \lambda_R \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{B}_u \mathbf{B}_u^T & \mathbf{0} \end{bmatrix} \end{aligned} \quad (8.24a)$$



$$\mathbf{f}_w(\mathbf{w}(t)) = \begin{bmatrix} 0 \\ -\nabla_w^T \phi_1(\mathbf{w}(t)) \end{bmatrix} \quad (8.24b)$$

$$\mathbf{F}_a(\lambda_R) = \begin{bmatrix} \mathbf{B}_a - \mathbf{B}_u(\mathbf{R} + 2\lambda_R \mathbf{I})^{-1} \mathbf{S}_a \\ -\mathbf{Q}_a + (\mathbf{S} + \lambda_R \mathbf{B}_u)(\mathbf{R} + 2\lambda_R \mathbf{I})^{-1} \mathbf{S}_a \end{bmatrix} \quad (8.24c)$$

$$\mathbf{F}_\lambda(\lambda_R) = \begin{bmatrix} -\mathbf{B}_u \\ \mathbf{S} + \lambda_R \mathbf{B}_u \end{bmatrix} [\mathbf{R} + 2\lambda_R \mathbf{I}]^{-1} \quad (8.24d)$$

This form of the BVP is useful because it shows explicitly the way in which the Lagrange multiplier  $\lambda_R(t)$  modifies the differential equation. It can be viewed as a time-varying parameter that modifies the matrices above. If constraints (5.15a) and (5.15b) were not enforced, this would be equivalent to fixing  $\lambda_R=0$  and  $\lambda=0$  in the above problem, in which circumstance the matrices above would become

$$\mathbf{F}_{wp}(0) = \begin{bmatrix} \mathbf{A} - \mathbf{B}_u \mathbf{R}^{-1} \mathbf{S}^T & -\mathbf{B}_u \mathbf{R}^{-1} \mathbf{B}_u^T \\ -\mathbf{Q} + \mathbf{S} \mathbf{R}^{-1} \mathbf{S}^T & -\mathbf{A}^T + \mathbf{S} \mathbf{R}^{-1} \mathbf{B}_u^T \end{bmatrix} \quad (8.25a)$$

$$\mathbf{F}_a(0) = \begin{bmatrix} \mathbf{B}_a - \mathbf{B}_u \mathbf{R}^{-1} \mathbf{S}_a \\ -\mathbf{Q}_a + \mathbf{S} \mathbf{R}^{-1} \mathbf{S}_a \end{bmatrix} \quad (8.25b)$$

and so in this case, the BVP becomes

$$\begin{bmatrix} \dot{\mathbf{w}}(t) \\ \dot{\mathbf{p}}(t) \end{bmatrix} = \mathbf{F}_{wp}(0) \begin{bmatrix} \mathbf{w}(t) \\ \mathbf{p}(t) \end{bmatrix} + \mathbf{f}_w(\mathbf{w}(t)) + \mathbf{F}_a(0) a_g(t) \quad , \quad \mathbf{w}(0) = \mathbf{w}_0 \quad , \quad \mathbf{p}(t_f) = \mathbf{0} \quad (8.26)$$

This BVP is a classical result of optimal active control, and it can be readily shown that there exists exactly one solution if conditions (8.5a-c) are met. In the particular circumstance that  $\mathbf{f}_w = \mathbf{0}$  (i.e. if the optimization function involves only quadratic terms in  $\mathbf{w}$ ) then Eq. (8.17) becomes a linear differential equation, and the problem reduces to the Linear Quadratic (LQ) Control problem.

### 8.2.2: Optimal Damping, Revisited

Recall Eq. (5.16), which related  $\mathbf{u}(t)$  to  $\mathbf{w}(t)$  through the matrix  $\mathbf{Z}$ . It was shown that if  $\mathbf{Z}$  satisfied the relation in Eq. (5.20), then  $\mathbf{u}(t)$  would be guaranteed to satisfy constraint (5.15a). In that analysis,  $\mathbf{Z}$  was constrained to be constant in time, but clearly any control input

$$\mathbf{u}(t) = -\mathbf{Z}(t) \mathbf{B}_u^T \mathbf{w}(t) \quad , \quad \bar{\sigma}(\mathbf{Z}(t) - \frac{1}{2} \mathbf{I}) \leq \frac{1}{2} \quad (8.27)$$

will result in satisfaction of (5.15a) over the entirety of the interval  $[0, t_f]$ .

From this perspective, the matrix  $\mathbf{Z}(t)$  becomes the control signal, and the system differential equation can be written as

$$\dot{\mathbf{w}}(t) = \mathbf{A}\mathbf{w}(t) - \mathbf{B}_u \mathbf{Z}(t) \mathbf{B}_u^T \mathbf{w}(t) + \mathbf{B}_a a_g(t) \quad (8.28)$$

This variable-structure differential equation has a form which is often called *bilinear* (Mohler 1970) because it is linear in state and control variables independently, but nonlinear in both.

By the same reasoning used in the previous chapter, for situations where constraint (5.15b) may be taken for granted (i.e. for problems where  $\mathbf{w}$  is small or where  $\mathbf{R}$  is large), it follows that  $\mathbf{u} \in \mathcal{F}_u(\mathbf{w}_0, a_g)$  if and only if constraint (5.15a) holds over  $[0, t_f]$ . In light of this fact, the following observation relates the analysis of Chapter 4 to the optimal control problem studied in this chapter.

**LEMMA 8.4:** If constraint (5.15b) is disregarded, then

$$\mathbf{u} \in \mathcal{F}_u(\mathbf{w}_0, a_g) \Leftrightarrow \exists \mathbf{Z}(t) \in \mathfrak{R}^{m \times m} \mid \mathbf{u}(t) = -\mathbf{Z}(t) \mathbf{B}_u^T \mathbf{w}(t) \text{ and } \bar{\sigma}(\mathbf{Z}(t) - \frac{1}{2} \mathbf{I}) \leq \frac{1}{2}, \quad \forall t \in [0, t_f] \quad (8.29)$$

*proof:* See appendix A8.

Another way of stating the above lemma is that for the domain of  $\mathbf{Z}(t)$  expressed by Eq. (8.27) and with  $\mathbf{w}(t)$  constrained by Eq. (8.2), the mapping from  $\mathbf{Z}(t)$  to  $\mathbf{u}(t)$  has a range space equal to  $\mathcal{F}_u(\mathbf{w}_0, a_g)$ . Thus, if constraint (5.15b) is disregarded, the OCP may be viewed as an optimization of the time-varying damping matrix  $\mathbf{Z}(t)$ , and the optimal  $\mathbf{u}$  implies at least one corresponding optimal  $\mathbf{Z}$ ; i.e.  $\mathbf{Z}_{opt}$ . If (5.15b) is disregarded, this implies that  $\lambda = \mathbf{0}$ , and the most general condition for  $\mathbf{Z}_{opt}(t)$  is

$$(\mathbf{R} + 2\lambda_R(t) \mathbf{I})^{-1} \left[ (\mathbf{S}^T - \frac{1}{2} \mathbf{R} \mathbf{B}_u^T) \mathbf{w}(t) + \mathbf{B}_u^T \mathbf{p}(t) + \mathbf{S}_a^T a_g(t) \right] = \left[ \mathbf{Z}_{opt}(t) - \frac{1}{2} \mathbf{I} \right] \mathbf{B}_u^T \mathbf{w}(t) \quad (8.30)$$

where the value of the Lagrange multiplier  $\lambda_R(t)$  can be interpreted as constraining  $\mathbf{Z}_{opt}(t) - \frac{1}{2} \mathbf{I}$  to a maximum singular value of  $\frac{1}{2}$ . Note that, as there are  $m^2$  unknowns for  $\mathbf{Z}_{opt}(t)$  and only  $m$  equations, there are in general an infinite number of damping matrices yielding optimality. This differs from semiactive control systems, where the diagonality constraint on  $\mathbf{Z}_{opt}(t)$  results in a one-to-one relationship between  $\mathbf{Z}_{opt}$  and  $\mathbf{u}_{opt} \in U_{opt}$ .

Consider  $\mathbf{Z}(t)$  and  $\mathbf{u}(t)$  related by

$$\mathbf{Z}(t) = \left( \frac{-1}{\mathbf{u}^T(t) \mathbf{B}_u^T \mathbf{w}(t)} \right) \mathbf{u}(t) \mathbf{u}^T(t) \quad (8.31)$$

Note that this relationship results in a feasible  $\mathbf{Z}(t)$  if  $\mathbf{u}(t)$  is feasible. It can be concluded that if  $\mathbf{u} \in \mathcal{F}_u(\mathbf{w}_0, a_g)$ , then there exists a feasible  $\mathbf{Z}$  which is symmetric for all  $t \in [0, t_f]$ . Resultantly, the effect of any constant asymmetric damping matrix, as studied in the previous chapter, can always be replicated by a time-varying symmetric matrix. Thus, the idea of “skew damping” is not meaningful for RFA networks, in the deterministic time-varying sense. Furthermore, it can be concluded from the above that there always exists a time-varying, symmetric optimal damping matrix.

## 8.3: Global Performance Minimization

### 8.3.1: Gradient Methods

The most common approach to the derivation of  $\mathbf{u}_{opt}$  involves asymptotic gradient-based methods (Stengel 1994). In order for this approach to be meaningful in this context, it must be shown (or assumed) that all inputs  $\mathbf{u}$  corresponding to local minima yield the same  $J$ . Otherwise, it is possible that the method will converge to a local minimum in  $\mathcal{F}_u(\mathbf{w}_0, a_g)$  which is not globally optimal. In general, these assurances are hard to make for RFA networks.

In the simplest application of a gradient-based method to this problem, the regenerative constraint (5.15a) is first converted to a penalty function. To do this, the Lagrange multiplier  $\lambda_R$  is set to zero and  $\phi$  is augmented to

$$\phi_\varepsilon(\mathbf{u}(t), \mathbf{w}(t); a_g(t)) = \phi(\mathbf{u}(t), \mathbf{w}(t); a_g(t)) + \frac{1}{\varepsilon} \max\{0, P(\mathbf{u}(t), \mathbf{w}(t))\} \quad (8.32)$$

where  $\varepsilon$  is a small positive constant. Consider that if  $\mathbf{u}(t) \in \mathcal{U}(\mathbf{w}(t))$ , then  $\phi_\varepsilon = \phi$ . However if  $\mathbf{u}(t) \notin \mathcal{U}(\mathbf{w}(t))$ , this results in  $\phi_\varepsilon > \phi$ . The integration of  $\phi_\varepsilon$  gives an augmented performance  $J_\varepsilon$ . As  $\varepsilon$  is made arbitrarily small, the addition of the penalty function to the performance measure therefore results in an arbitrarily large  $J_\varepsilon$  for  $\mathbf{u} \notin \mathcal{F}_u(\mathbf{w}_0, a_g)$ , while  $\mathbf{u} \in \mathcal{F}_u(\mathbf{w}_0, a_g)$  will result in  $J_\varepsilon = J$ .

For a given  $\varepsilon$ , the optimal control  $\mathbf{u}$  is found as follows. Starting from initial guess for  $\mathbf{u}$ , gradient methods operate by iteratively re-solving for successive, more favorable  $\mathbf{u}$  functions. For iterative cycle  $k$ , let the input function be  $\mathbf{u}_k$ . With this input,  $\mathbf{w}_k$  is solved through Eq. (8.1). Then, the corresponding costate vector  $\mathbf{p}_k$  is solved. With the penalty function above, and  $\lambda_R = 0$ , the differential equation for  $\mathbf{p}_k$  becomes

$$\dot{\mathbf{p}}_k(t) = -\nabla_{\mathbf{w}}^T \phi_1(\mathbf{w}_k(t)) - \mathbf{Q}\mathbf{w}_k(t) - \left( \mathbf{S} + \frac{\text{hvs}[P(\mathbf{u}_k(t), \mathbf{w}_k(t))]}{\varepsilon} \mathbf{B}_u \right) \mathbf{u}_k(t) - \mathbf{A}^T \mathbf{p}_k(t) - \mathbf{Q}_a a_g(t) \quad (8.33)$$

where  $\text{hvs}[\cdot]$  is the Heaviside step function. With final condition  $\mathbf{p}_k(t_f)=\mathbf{0}$ , this equation can be solved. With  $\mathbf{u}_k$ ,  $\mathbf{w}_k$ , and  $\mathbf{p}_k$  known, it is then possible to find the sensitivity of  $J_\varepsilon$  to  $\mathbf{u}_k$ . Defining the Hamiltonian as

$$\mathcal{H}_\varepsilon = \phi_\varepsilon(\mathbf{w}(t), \mathbf{u}(t); a_g(t)) + \mathbf{p}^T(t) [\mathbf{A}\mathbf{w}(t) + \mathbf{B}_u \mathbf{u}(t) + \mathbf{B}_a a_g(t)] \quad (8.34)$$

it can then be shown (see proof to Lemma 8.1) that, for an infinitesimal perturbation  $\delta\mathbf{u}$ ,

$$\delta J_\varepsilon = \int_0^{t_f} (\nabla_{\mathbf{u}} \mathcal{H}_\varepsilon \cdot \delta\mathbf{u}(t)) dt \quad (8.35)$$

It follows that the designation of  $\mathbf{u}_{k+1}$  as

$$\mathbf{u}_{k+1} = \underset{\mathbf{u}_{\max}}{\text{sat}} \{ \mathbf{u}_k - \beta \nabla_{\mathbf{u}} \mathcal{H}_\varepsilon \} \quad (8.36)$$

for  $\beta$  positive and small, should reduce the value of  $J_\varepsilon$ .

Thus, by repeating the above-described iteration, the value of  $\mathbf{u}_k$  will converge upon a local minimum as  $k \rightarrow \infty$ . By repeating this process for successively small  $\varepsilon$ , using the previous optimized  $\mathbf{u}$  as the initial guess for the next  $\varepsilon$  value, the resultant optimal  $\mathbf{u}$  will correspond to a local minimum in  $\mathcal{F}_u(\mathbf{w}_0, a_g)$  as  $\varepsilon \rightarrow 0$ .

The gradient method is an intuitive way of minimizing the functional  $J$  in the feasible input space. It also has the appealing feature that, as the size of the system becomes large, it requires small computational and data storage resources in comparison to some other methods. However, as mentioned, it works on the assumption that either there is exactly one  $\mathbf{u} \in \mathcal{F}_u(\mathbf{w}_0, a_g)$  corresponding to a minimum in  $J$ , or at least that all local minima in  $\mathcal{F}_u(\mathbf{w}_0, a_g)$  yield the same  $J$ .

### 8.3.2: Nonconvexity

The conventional wisdom concerning proofs of global optimality concerns convex analysis. It is a well-known fact that a locally optimal solution of a convex function  $J(\mathbf{u}, \mathbf{w})$ , over a convex domain  $\{\mathbf{u}, \mathbf{W}(\mathbf{u}; \mathbf{w}_0, a_g)\}$ , exists and is unique. Thus, under these circumstances, it is immediate that any  $\mathbf{u}$  yielding a local minimum also yields the global minimum.

For the problem at hand, it can be readily proven that  $J(\mathbf{u}, \mathbf{w})$ , constrained to the domain  $\{\mathbf{u}, \mathbf{W}(\mathbf{u}; \mathbf{w}_0, a_g)\}$ , is convex. This follows directly from properties (8.5a-c). Furthermore, because the system differential equation is linear, the set  $\{\mathbf{u}, \mathbf{W}(\mathbf{u}; \mathbf{w}_0, a_g)\}$  is a convex domain for  $\mathbf{u} \in \mathfrak{R}^m \times C[0, t_f]$ . Thus, if the admissible inputs  $\mathbf{u}$  were unconstrained, the optimization problem would become convex, the BVP would have a unique solution, and thus  $\mathbf{u}_{opt}$  would be unique.

However, the constraint  $\mathbf{u} \in \mathcal{F}_u(\mathbf{w}_0, a_g)$  must also be enforced, and  $\mathcal{F}_u(\mathbf{w}_0, a_g)$  is nonconvex on  $\{\mathbf{u}, \mathbf{W}(\mathbf{u}; \mathbf{w}_0, a_g)\}$ . This can be shown by observing that condition (5.15a) is equivalent to

$$\begin{bmatrix} \mathbf{u}^T(t) & \mathbf{w}^T(t) \end{bmatrix} \begin{bmatrix} \mathbf{I} & \frac{1}{2}\mathbf{B}_u^T \\ \frac{1}{2}\mathbf{B}_u & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u}(t) \\ \mathbf{w}(t) \end{bmatrix} \leq 0 \quad (8.37)$$

The boundary of the region in  $\{\mathbf{u}(t), \mathbf{w}(t)\}$  space created by this inequality is hyperbolic. This becomes clear when the above is rewritten in the equivalent form

$$\begin{bmatrix} \mathbf{u}^T(t) & \mathbf{w}^T(t) \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \frac{1}{2}\mathbf{B}_u & \frac{1}{2}\mathbf{B}_u \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \frac{1}{2}\mathbf{B}_u^T \\ \mathbf{0} & \frac{1}{2}\mathbf{B}_u^T \end{bmatrix} \begin{bmatrix} \mathbf{u}(t) \\ \mathbf{w}(t) \end{bmatrix} \leq 0 \quad (8.38)$$

To show that  $\mathcal{F}_u(\mathbf{w}_0, a_g)$  is nonconvex on  $\{\mathbf{u}, \mathbf{W}(\mathbf{u}; \mathbf{w}_0, a_g)\}$ , consider that for two feasible trajectories  $\{\mathbf{u}_1, \mathbf{w}_1\}$  and  $\{\mathbf{u}_2, \mathbf{w}_2\}$ , the linearity of the differential equation implies that a weighted average of  $\mathbf{u}_1$  and  $\mathbf{u}_2$  (i.e.  $\mathbf{u} = \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2$  with  $\alpha_{1,2} > 0$  and  $\alpha_1 + \alpha_2 = 1$ ) results in a similarly weighted trajectory (i.e.  $\mathbf{w} = \alpha_1 \mathbf{w}_1 + \alpha_2 \mathbf{w}_2$ ). If at some time  $t$  both control-input trajectories 1 and 2 lie on the boundary expressed by the inequality above, then it follows from the nonconvexity of this boundary that for some weighted average of the two trajectories, the above condition may be violated.

The nonconvex nature of the optimization problem means that there may be local minima on the boundary of  $\mathcal{F}_u(\mathbf{w}_0, a_g)$  which are not global minima. Gradient-based methods of numerical computation for the  $\mathbf{u}_{opt}$  may therefore yield erroneous results. (It is worth noting that this is also true, although seldom observed, for the semiactive optimal control problem.)

In this section, Hamilton-Jacobi-Bellman theory is discussed, which circumvents problems arising from the nonconvex nature of the optimization. However, in exchange for this favorable attribute, this method presents other more practical difficulties. It involves numerical quadrature on the system state space, and therefore requires the assembly and manipulation of arrays which grow geometrically with the state number and grid resolution. Specifically, if each state dimension is allocated  $n_w$  grid points, then the resultant grid size will be  $n_w^{2n}$ . Thus, if a reasonable resolution is given for  $n_w$  (say, 100) then the grid size grows with the number of degrees of freedom like  $10,000^n$ . It is therefore only usable for systems with very few degrees of freedom (i.e. 2 or 3).

### 8.3.3: Sufficient Conditions for Global Optimality

Even for problems which involve optimization over nonconvex domains, a sufficient condition for global optimality may be found in the Hamilton-Jacobi-Bellman equation for the optimal performance,  $J_{opt}$ . This equation starts from an intuitive reasoning. Let  $J_{opt}$  be the performance evaluated over an optimal trajectory  $\{\mathbf{w}, \mathbf{u}\}$ , for an initial condition  $\mathbf{w}_0$  and time interval  $[0, t_f]$ . Let  $t_s \in (0, t_f)$  and let  $\mathbf{w}(t_s) = \mathbf{w}_s$ . Then it follows that the trajectory  $\{\mathbf{w}, \mathbf{u}\}$  over  $[t_s, t_f]$  is the solution to the OCP over this interval, given initial condition  $\mathbf{w}_s$ .

This line of reasoning gives rise to the definition of a function  $V(\mathbf{w}_s, t_s)$ , which is equal to the optimal performance over the interval  $[t_s, t_f]$ , with initial condition  $\mathbf{w}_s$ . Theorem 8.5 below states that, because of the above reasoning,  $V$  must satisfy a partial differential equation (PDE) which is sometimes called the *recurrence relation*.

**THEOREM 8.5:** (Hamilton-Jacobi-Bellman) If the function  $V(\mathbf{w}_s, t_s)$  is a solution to the PDE

$$\frac{\partial V}{\partial t_s} = -\phi(\mathbf{u}_s(\mathbf{w}_s, t_s), \mathbf{w}_s; a_g) - \nabla_{\mathbf{w}} V [\mathbf{A}\mathbf{w}_s + \mathbf{B}_u \mathbf{u}_s(\mathbf{w}_s, t_s) + \mathbf{B}_a a_g(t_s)] \quad (8.39)$$

with the boundary condition

$$V(\mathbf{w}_s, t_f) = 0 \quad (8.40)$$

and where  $\mathbf{u}_s$  is defined as a solution to

$$\mathbf{u}_s(\mathbf{w}_s, t_s) = \arg \min_{\tilde{\mathbf{u}} \in \mathcal{U}(\mathbf{w}_s)} \left[ \phi(\tilde{\mathbf{u}}, \mathbf{w}_s; a_g) + \nabla_{\mathbf{w}} V [\mathbf{A}\mathbf{w}_s + \mathbf{B}_u \tilde{\mathbf{u}} + \mathbf{B}_a a_g(t_s)] \right] \quad (8.41)$$

then

$$\dot{\mathbf{w}}(t) = \mathbf{A}\mathbf{w}(t) + \mathbf{B}_u \mathbf{u}_s(\mathbf{w}(t), t) + \mathbf{B}_a a_g(t) \quad , \quad \mathbf{w}(0) = \mathbf{w}_0 \quad \Rightarrow \quad \mathbf{u}_s(\mathbf{w}(\cdot), \cdot) \in U_{opt} \quad (8.42)$$

and

$$J_{opt}(\mathbf{w}_0, a_g) = V(\mathbf{w}_0, 0) \quad (8.43)$$

*proof:* The proof is standard. For examples, see (Stengel 1994) or (Kirk 1970).

In this particular problem, the expression for  $\mathbf{u}_s$  is

$$\mathbf{u}_s(\mathbf{w}_s, t_s) = \arg \min_{\tilde{\mathbf{u}} \in \mathcal{U}(\mathbf{w}_s)} \left\| \tilde{\mathbf{u}} + \mathbf{R}^{-1} [\mathbf{S}^T \mathbf{w}_s + \mathbf{S}_a a_g(t_s) + \mathbf{B}_u^T \nabla_{\mathbf{w}}^T V(\mathbf{w}_s, t_s)] \right\|_{\mathbf{R}} \quad (8.44)$$

Note that this is the clipping action operation from equation (8.16), where  $\mathbf{p} = \nabla_{\mathbf{w}}^T V$ .

In general, there is no closed-form solution to Eq. (8.39), and it must be solved numerically for the optimal trajectory. In the appendix to this chapter, a simple numerical approach for this optimization is presented. In the next section, Eq. (8.39) is solved for SDOF systems in free vibration, for the infinite-horizon case (i.e.  $t_f \rightarrow \infty$ ). First, however, several corollaries are presented, concerning the characteristics of  $V(\mathbf{w}_s, t)$ .

**COROLLARY 8.6:** Let constraints (8.5a-c) hold for  $\mathbf{R}$ ,  $\mathbf{S}$ ,  $\mathbf{Q}$ , and  $\phi_1$ , and assume  $\mathbf{A}$  is stable. Then the function  $V(\mathbf{w}_s, t)$  is continuous in both arguments.

*proof:* see appendix A8.

This observation allows for  $V(\mathbf{w}_s, t)$  to be optimized for discrete points in  $\{\mathbf{w}_s, t\}$  space, with the understanding that values of  $V$  in the neighborhood of these grid points may be closely approximated through interpolation, for a sufficiently fine grid.

**COROLLARY 8.7:** Assume constraints (8.5a-c) hold for  $\mathbf{R}, \mathbf{S}, \mathbf{Q}$ , that  $\phi_1=0$ . Then if constraint (5.15b) is ignored, the solution to  $V$  is homogeneous; i.e.

$$V(\beta \mathbf{w}_s, t; \beta a_g) = \beta^2 V(\mathbf{w}_s, t; a_g) \quad (8.45)$$

*proof:* If constraint (5.15b) is disregarded then the system differential equation may be written in bilinear form. The differential equation is then linear in  $a_g$  and  $\mathbf{w}_s$ , given  $\mathbf{Z}(t)$ . If the performance measure is quadratic, then it follows that the measure of performance uniformly scales quadratically with simultaneous scaling of  $\mathbf{w}_s$  and  $a_g$ .

This corollary is useful for a number of reasons. In studies involving the free response to initial conditions (i.e. where  $\mathbf{w}_0 \neq 0$  and  $a_g = 0$ ) it leads to the conclusion is that the optimal control scales with the magnitude of the initial condition. It follows that all the cross-section contours of  $V$  in  $\mathbf{w}_0$ -space will be of similar shape.

**COROLLARY 8.8:** For the free-vibration case with  $\mathbf{A}$  stable, the value of  $V(\mathbf{w}_s, t)$  is stationary as  $t_f \rightarrow \infty$  and  $V(\mathbf{w}_s, t)$  is stable in reverse-time.

*proof:* see appendix A8

Thus, for the infinite-horizon case, if a performance function  $J(\mathbf{w}_0)$  can be found which obeys

$$0 = \min_{\mathbf{u}_s \in \mathcal{U}(\mathbf{w}_0)} \{ \phi(\mathbf{u}_s, \mathbf{w}_0) + \nabla J(\mathbf{w}_0) [\mathbf{A}\mathbf{w}_0 + \mathbf{B}_u \mathbf{u}_s] \} \quad (8.46)$$

then  $J(\mathbf{w}_0)$  is the optimal performance for initial condition  $\mathbf{w}_0$  and in general,  $V(\mathbf{w}_s, t) = J(\mathbf{w}_s)$  for all  $t$ .

As a consequence of this corollary, the optimal infinite-horizon control for the RFA network, in free-response, could be implemented exactly in real-time by a time-invariant nonlinear control law, if the above equation could be solved for  $J$ . By evaluating the gradient of  $J$ , the optimal control force could be obtained as a feedback function of  $\mathbf{w}(t)$ . Of course, this assumes that an analytical solution can be found for  $J$  above, or that it is practical to employ a numerical “table-lookup” feedback approach. It may be that either or both of these options are untrue, depending on the application.

It also follows from Corollary 8.8 that for the infinite-horizon free-vibration case, the positive-definiteness of  $\phi$  implies that  $J(\mathbf{w})$  is a Lyapunov function for the optimally-controlled system; i.e.

$$\left. \frac{d}{dt} J(\mathbf{w}(t)) \right|_{\mathbf{u}(t)=\mathbf{u}_{opt}(t)} = -\phi(\mathbf{u}_{opt}(t), \mathbf{w}(t)) \leq 0 \quad (8.47)$$

Thus, the optimal control for the free-vibration case can be viewed as a specific kind of Lyapunov-based feedback control.

## 8.4: Example: SDOF Free Vibration

The examination of optimal control for SDOF systems is convenient because the mechanical system is simple enough that data on the optimal response (i.e.  $J$ ,  $\mathbf{u}_{opt}$ ,  $\mathbf{v}$ , etc.) can be easily represented and interpreted graphically. This example considers the free vibration of a SDOF system with one actuator. Note that, because there is only one actuator, the RFA “network” reduces to a single, semiactive device.

It is straight-forward to show that any such system, with appropriate scaling, can be represented by a nondimensionalized Nominal System Model

$$\dot{\mathbf{w}} = \mathbf{A}\mathbf{w} + \mathbf{B}_u a_g \quad (8.48)$$

with state space matrices

$$\mathbf{w} = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & -2\zeta \end{bmatrix}, \quad \mathbf{B}_u = \begin{bmatrix} 0 \\ N \end{bmatrix} \quad (8.49)$$

and where the RFA network constraints are

$$u^2 + N\dot{q}u \leq 0 \quad (8.50)$$

$$|u| \leq 1 \quad (8.51)$$

Note that the actuator for this system may be viewed as a variable-damping system with a maximum viscosity  $N$ , a minimum viscosity of 0, and a force saturation.

The parameters  $\zeta$  and  $N$  are the only free parameters. Throughout this example,  $\zeta$  will be taken as 0.001. The value of  $N$  will be varied for different cases.

Recall the expression for  $\phi$  in Eq. (8.4). Because this is a free-vibration example,  $a_g=0$  and consequently, the terms  $\mathbf{Q}_a$ ,  $\mathbf{S}_a$ , and  $R_a$  may be assumed to be zero. Only quadratic performance will be considered, so  $\phi_1=0$  for this example. This gives  $\phi$  as



$$\phi = \begin{bmatrix} \mathbf{w}^T & u \end{bmatrix} \begin{bmatrix} \mathbf{Q} & \mathbf{S} \\ \mathbf{S}^T & R \end{bmatrix} \begin{bmatrix} \mathbf{w} \\ u \end{bmatrix} \quad (8.52)$$

In these examples, two performance measures will be considered; mean-square drifts and mean-square accelerations.

For the free vibration case, Corollary 8.8 dictates that  $V(\mathbf{w}_s, t)$  is stable in reverse-time. Thus, the case for  $t_f \rightarrow \infty$  can be solved by starting from the final condition  $V(t_f) = 0$  and integrating Eq. (8.39) in reverse-time until convergence is reached. The resultant equilibrium for  $V$  is equal to the optimal performance  $J_{opt}(\mathbf{w}_s)$ , for arbitrary initial condition  $\mathbf{w}_s$ .

Recalling Eq. (8.41), it follows that

$$u_{opt}(t) = \arg \min_{\tilde{u} \in \mathcal{U}(\mathbf{w}(t))} \{ \phi(\tilde{u}, \mathbf{w}(t)) + \nabla J(\mathbf{w}(t)) \cdot (\mathbf{A}\mathbf{w}(t) + \mathbf{B}_u \tilde{u}) \} \quad (8.53)$$

Thus, for free vibration, the gradient of  $J$  is solved and used to derive an optimal control law.

### 8.4.1: Displacement Optimization

Consider the case where  $\phi$ , as in Eq. (8.52), is such that

$$\mathbf{Q} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{S} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad R = 0 \quad (8.54)$$

This case corresponds to displacement regulation. For  $\mathbf{Q}$ ,  $\mathbf{S}$ , and  $R$  as above, the solution to  $J(\mathbf{w}_s)$  can be solved, and the feedback relationship in Eq. (8.53) derived. Here, this will be done first for the small-vibration case (where  $\mathbf{w}(t)$  is small enough that the force vector does not saturate) and then for the large-vibration case. For these cases, it will be assumed that  $N=1$ . Later, the effect of larger  $N$  values will be discussed.

#### *Small-Vibration Case*

Solutions for the optimal  $\{\mathbf{w}, \mathbf{u}\}$  trajectory were derived for the following initial conditions.

$$\mathbf{w} = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} -0.1 \\ 0 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 0 \\ -0.1 \end{bmatrix} \quad (8.55)$$

For these four trajectories, the hysteresis curve  $\{q(t), u(t)\}$  is shown in Figure 8.1. Note that at any given time,  $|u(t)| \in [0, |N\dot{q}(t)|]$ . As such, Figure 8.1 also shows the trajectory for  $\{q(t), -N\dot{q}(t)\}$ . From the plot, it is clear that the optimal  $u$  alternates discontinuously between 0 and  $N\dot{q}(t)$ . This is not surprising because  $R=0$  in this example. It is interesting to note that the

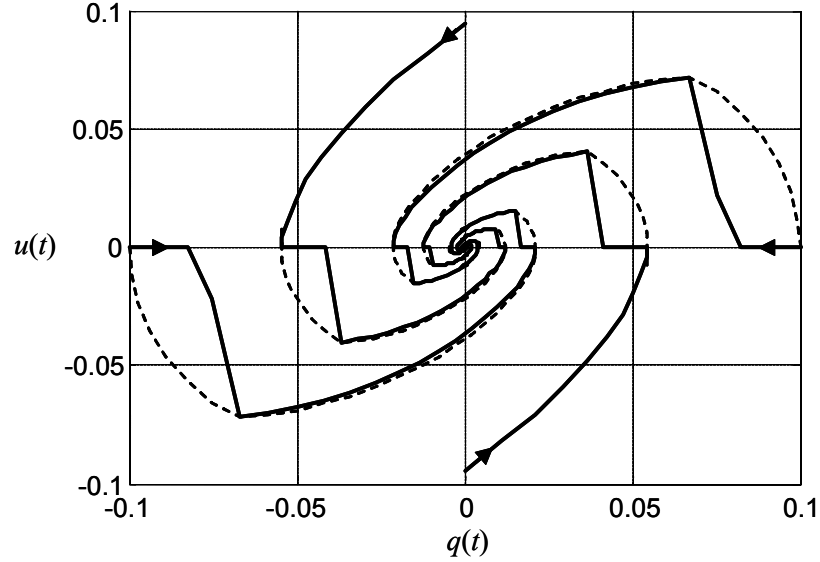


Figure 8.1:  $q$  vs.  $u$  (solid) and  $-N\dot{q}$  (dashed) for small-vibration displacement regulation

optimal hysteresis loop for optimal quadratic displacement regulation does not maximize the energy dissipation.

Consider that for the parameters in Eq. (8.54), Eq. (8.53) is equivalent to

$$u_{opt}(t) = \arg \min_{\tilde{u} \in \mathcal{V}(\mathbf{w}(t))} \{ \nabla J \cdot (\mathbf{B}_u \tilde{u}) \} \quad (8.56)$$

This minimization is

$$u_{opt}(t) = \text{sat}_1 \{ -\mathbf{B}_u \mathbf{w}(t) \} \text{hvs} \left\{ \nabla J \cdot (\mathbf{B}_u \mathbf{B}_u^T \mathbf{w}(t)) \right\} \quad (8.57)$$

where  $\text{hvs}(\cdot)$  is the Heaviside step function. For behavior near the origin, the finite value of  $N$  prohibits  $u$  from saturating at  $\pm 1$ , so the optimal control force is rather simple:

$$u_{opt}(t) = -N\dot{q}(t) \text{hvs} \left\{ \frac{\partial J}{\partial \dot{q}} \dot{q}(t) \right\} \quad (8.58)$$

The nature of this optimal control force is equivalent to an on/off damper with viscosity  $N^2$ . Because  $J$  depends on position as well as velocity, the switching of the damper on and off constitutes full-state feedback.

Figure 8.2 shows the four trajectories in  $\mathbf{w}$ -space, together with contour plots of the performance measure  $J$ . Note that the contours of  $J$  are approximately elliptical, implying that  $J$  is approximately quadratic. Because  $J$  is homogeneous, the system state space can be partitioned into sectors characterized by different control regimes. Figure 8.2 shows these sector boundaries. In the smaller regions,  $u=0$ , while in the complimentary regions,  $u=-N\dot{q}(t)$ . Because  $u$  changes

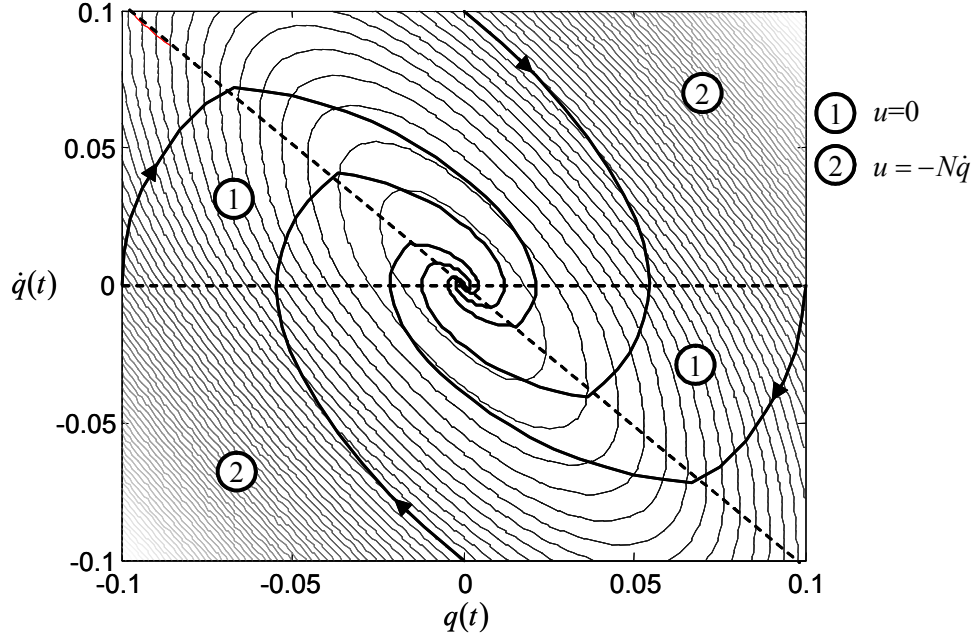


Figure 8.2: Sectors in state space separating different optimal control regimes, for small-vibration displacement regulation

discontinuously across these boundaries, they are called switching surfaces. Note that one switching surface is aligned with the  $\dot{q} = 0$  axis, while the other is coincidental with the locus  $\partial J / \partial \dot{q} = 0$ .

Thus, for optimal displacement regulation, it is not actually necessary to know  $J$  explicitly to implement the optimal control force  $u_{opt}$ , only the sector boundaries.

The derivation of the switching surfaces for this case (and any other homogeneous SDOF case corresponding to  $R=0$ ) turns out to be analytically tractable in the context of bilinear control, as originally investigated by Mohler (1973).

However, for any RFA network with  $m > 1$ , the optimal control is no longer so simple. There are not switching surfaces, but rather a continuously-varying control input on the elliptical boundary  $P(\mathbf{u}, \mathbf{w})=0$ . Thus, for  $m > 1$ , the simplicity of displacement regulation vanishes.

### ***Large-Vibration Case***

When initial condition  $\mathbf{w}_0$  is large enough such that the maximum force limit affects the optimal response, the characteristics of  $J$  change. It is no longer true that  $J$  is homogeneous in  $\mathbf{w}$ -space, and this complicates the relationship of the optimal control to the states. Consider that the expression for the optimal force, for large signals, becomes

$$u_{opt}(t) = \text{sat}_1 \{ -N\dot{q}(t) \} \text{hvs} \left\{ \frac{\partial J}{\partial \dot{q}} \dot{q}(t) \right\} \quad (8.59)$$

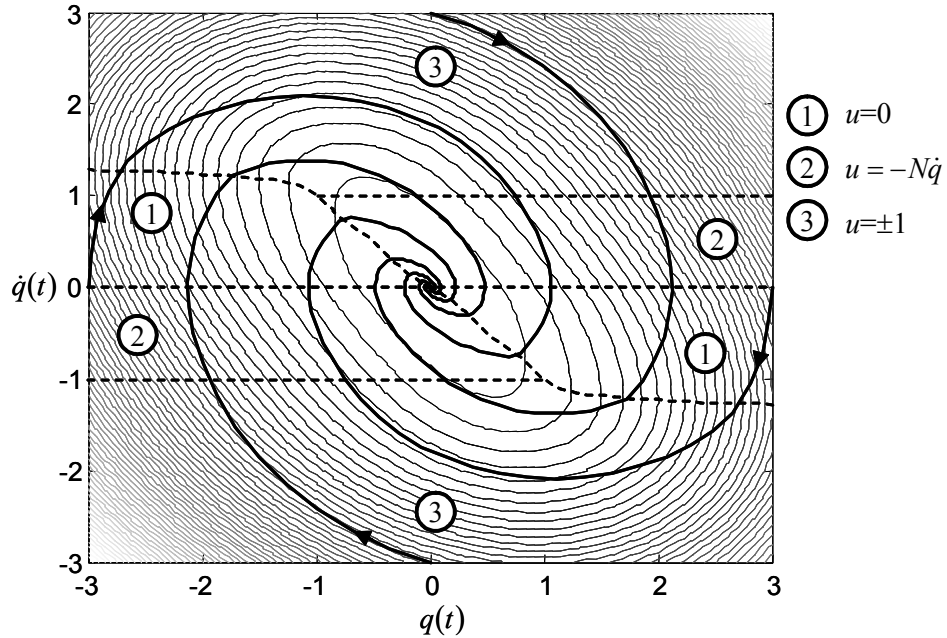


Figure 8.3: Boundaries in state space separating different optimal control regimes, for large-vibration displacement regulation with  $N=1$

As with the small-signal case, the state space can be broken up into several regions, as shown in Figure 8.3. Clearly, the switching surface boundaries for large vibrations are nonlinear. For large oscillations, the control force “switches on” earlier in each cycle, because constraint (8.51) effectively limits the maximum damping capability.

### ***Effect of Larger $N$ on Response***

As  $N$  is made larger, the maximum viscosity of the actuator becomes greater, and the region in  $\mathbf{w}$ -space corresponding to homogeneous behavior becomes smaller. Figure 8.4 shows state trajectories, switching surfaces, and performance for  $N=2$ . It is clear that for this larger  $N$ , the optimally-controlled system exhibits a sliding mode on the switching surface where  $\partial J / \partial \dot{q} = 0$ , as the trajectory decays. For intersections of the switching surface for larger  $\mathbf{w}$ , the trajectory does not slide on the surface. This is because of the limitation  $|u| \leq 1$ , which effectively works to reduce the influence of  $u$ , in comparison to the stiffness force. Note that for this example, the optimal sliding surface is nonlinear. Also note that  $J$  is clearly nonquadratic, even for small vibrations.

### 8.4.2: Acceleration Optimization

A similar analysis to the above can be conducted for acceleration optimization. For this case, the performance measure is characterized by

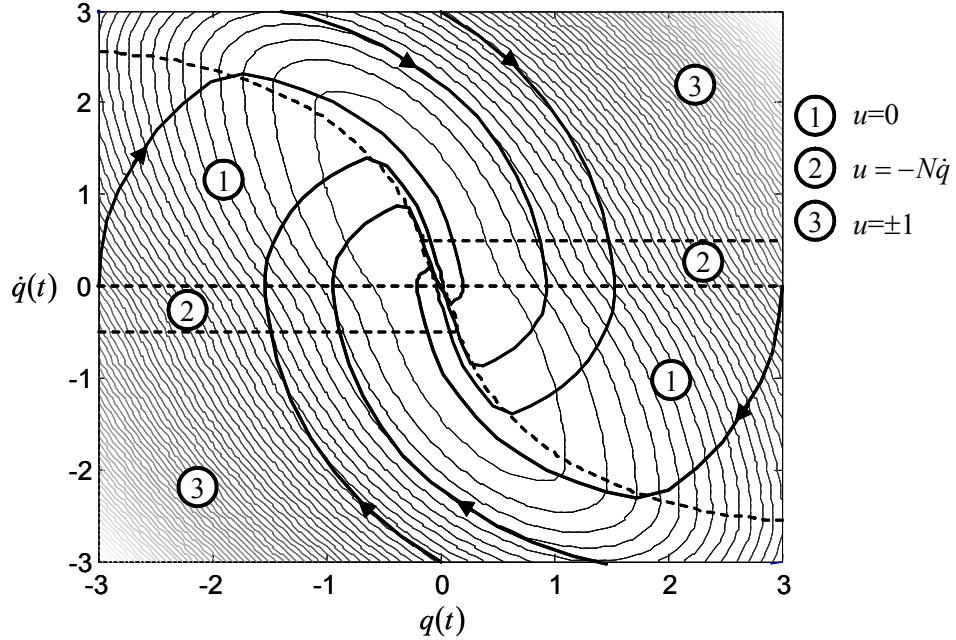


Figure 8.4: Boundaries in state space separating different optimal control regimes, for large-vibration displacement regulation with  $N=2$

$$\mathbf{Q} = \begin{bmatrix} 1 & 2\zeta \\ 2\zeta & 4\zeta^2 \end{bmatrix}, \quad \mathbf{S} = \begin{bmatrix} -N \\ -2\zeta N \end{bmatrix}, \quad R = N^2 \quad (8.60)$$

As for the previous example, the small-vibration case is analyzed first, then the case with large vibrations.

### *Small-Vibration Case*

Analogous to Figure 8.1 for the displacement optimization case, Figure 8.5 shows  $\{q, u\}$  trajectories for the acceleration-optimization case. Note that, unlike in the previous examples,  $u$  varies continuously between its maximum allowable magnitude and zero. This is not surprising, because  $R \neq 0$  for this example. It is interesting that the optimal force departs from its maximum value on the trailing edge of each half-cycle for this case, compared to the leading edge in the displacement optimization.

As with the displacement optimization, some analysis of the expression for the optimal control force sheds some light on the relationship between  $J$  and  $u$ . For acceleration optimization,  $\phi$  depends on  $u$  as well as the system states. For acceleration regulation, Eq. (8.53) is

$$u_{opt}(t) = \arg \min_{\tilde{u} \in \mathcal{U}(\mathbf{w}(t))} \left\{ R\tilde{u}^2 + (\nabla J \cdot \mathbf{B}_u + 2\mathbf{w}^T(t)\mathbf{S})\tilde{u} \right\} \quad (8.61)$$

which has the solution

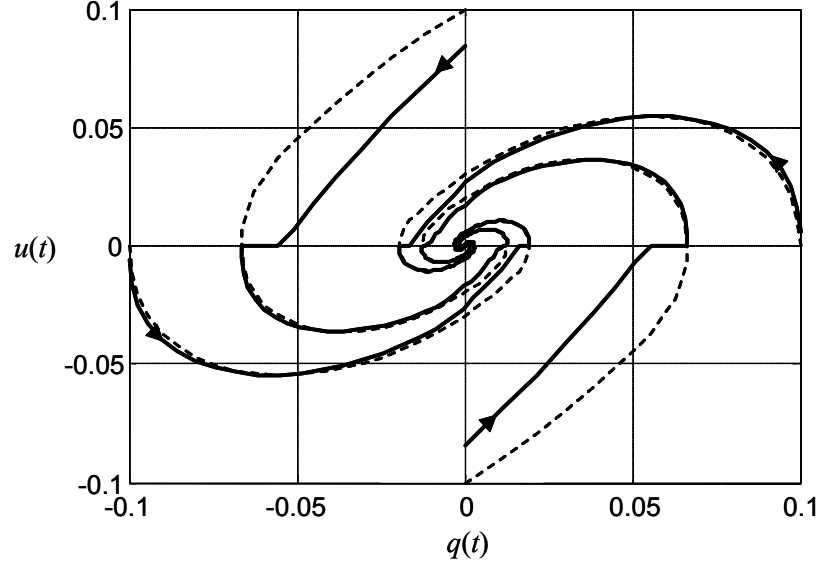


Figure 8.5:  $q$  vs.  $u$  (solid) and  $-N\dot{q}$  (dashed) for small-vibration acceleration regulation

$$u_{opt}(t) = \text{sat}_{v(w(t))} \left\{ -\frac{1}{R} \left( \frac{1}{2} \mathbf{B}_u^T \nabla J + \mathbf{S}^T \mathbf{w}(t) \right) \text{hvs} \left\{ \mathbf{w}^T(t) \mathbf{B}_u \left( \frac{1}{2} \mathbf{B}_u^T \nabla J + \mathbf{S}^T \mathbf{w}(t) \right) \right\} \right\} \quad (8.62)$$

Again, if the maximum force is ignored, this is equivalent to

$$u_{opt}(t) = -\min \left\{ \frac{1}{R} \left| \frac{1}{2} \mathbf{B}_u^T \nabla J + \mathbf{S}^T \mathbf{w}(t) \right|, \left| \mathbf{B}_u^T \mathbf{w}(t) \right| \right\} \text{hvs} \left\{ \mathbf{w}^T(t) \mathbf{B}_u \left( \frac{1}{2} \mathbf{B}_u^T \nabla J + \mathbf{S}^T \mathbf{w}(t) \right) \right\} \text{sgn} \left( \mathbf{B}_u^T \mathbf{w}(t) \right) \quad (8.63)$$

Defining

$$u_a(t) = -\frac{1}{N} \left\{ \frac{1}{2} \frac{\partial J}{\partial \dot{q}} - q(t) - 2\zeta \dot{q}(t) \right\} \quad (8.64)$$

Eq. (8.63) can be written simplified to

$$u_{opt}(t) = -\min \left\{ |u_a(t)|, |N\dot{q}(t)| \right\} \text{hvs} \left\{ -\dot{q}(t) u_a(t) \right\} \text{sgn}(\dot{q}(t)) \quad (8.65)$$

As in the displacement optimization example, the homogeneity of  $J$  yields sectors in state space inside of which different conditions hold. For the example at hand, these sectors are illustrated in Figure 8.6.

In the displacement optimization example, the division of the state space into sectors fully characterized the relationship of the optimal control force to the states, because this force was of a “bang-bang” nature. In this example, however, sectors with  $u=u_a$  still require knowledge of  $J$  to find the optimal control force. Thus, implementation of such an optimal controller would require explicit knowledge of  $J$ , or at least  $\partial J / \partial \dot{q}$ .

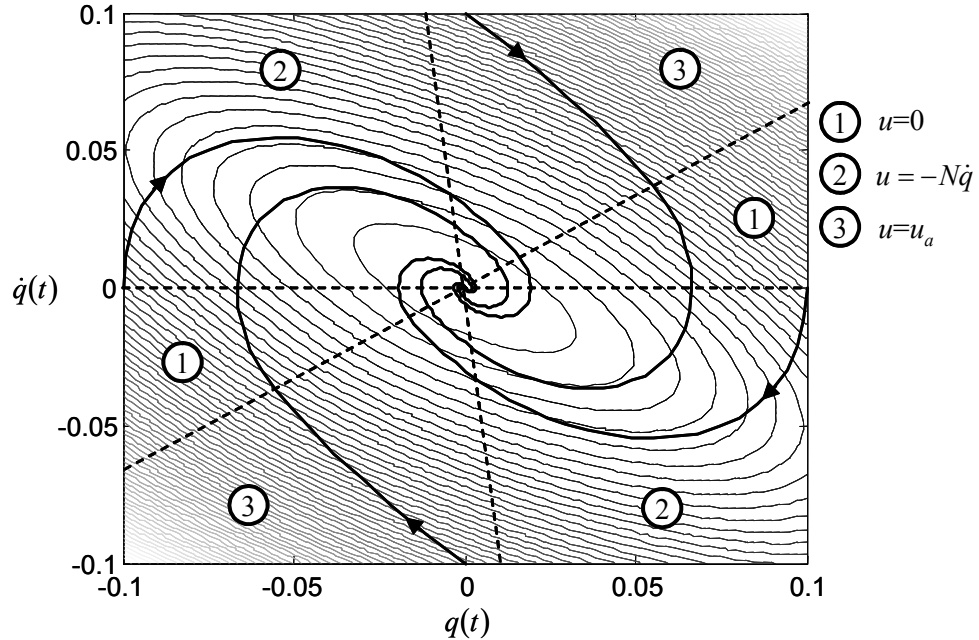


Figure 8.6: Sectors in state space separating different optimal control regimes, for small-vibration acceleration regulation

### *Large-Vibration Case*

Similarly, the optimal force for acceleration regulation, given by Eq. (8.65), can be modified to include effects of the maximum force limit. This modification gives

$$u_{opt}(t) = -\min\{1, |u_a(t)|, |N\dot{q}(t)|\} \operatorname{hvs}\{-\dot{q}(t)u_a(t)\} \operatorname{sgn}(\dot{q}(t)) \quad (8.66)$$

Graphically, this is shown in Figure 8.7. The jagged shapes of the boundaries is due to the finite spacing of the state-space grid and the influence of edge extrapolation. As in the displacement example, Figure 8.7 shows that for large vibrations, the effect of the maximum force constraint is to “bend” the switching surfaces toward the  $\dot{q}$  axis.

Figure 8.8 shows a similar plot for  $N=2$ . Unlike the displacement example, this example does not exhibit sliding modes. Rather, the value of  $u$  varies continuously as it transitions from one region to the next.

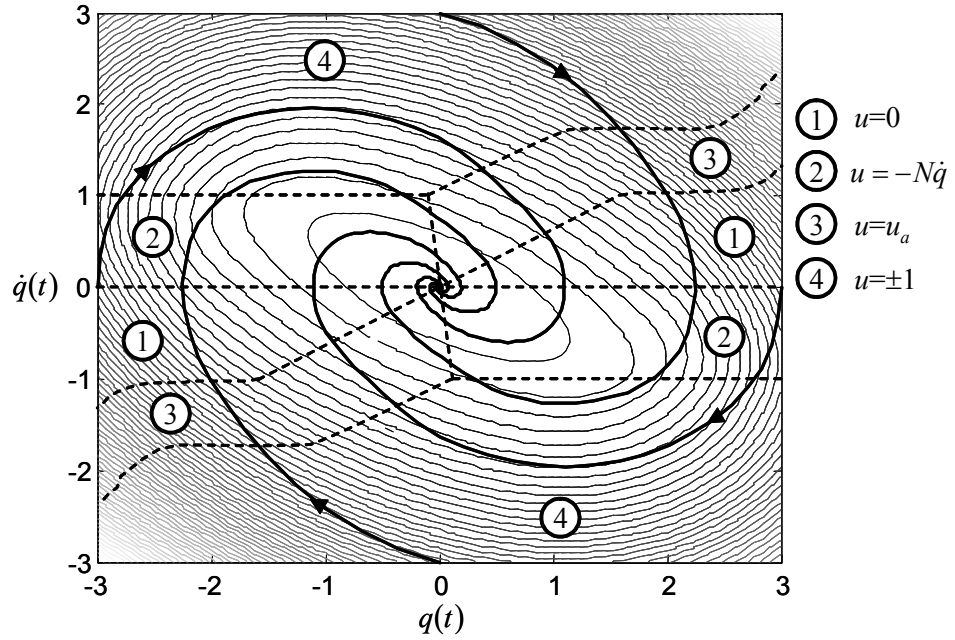


Figure 8.7: Boundaries in state space separating different optimal control regimes, for large-vibration acceleration regulation with  $N=1$

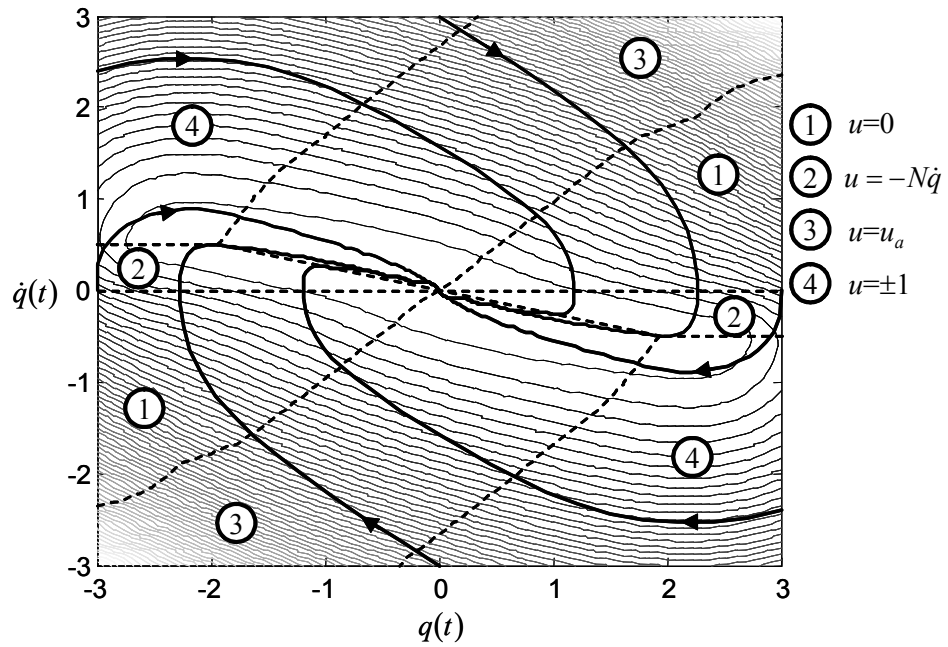


Figure 8.8: Boundaries in state space separating different optimal control regimes, for large-vibration acceleration regulation with  $N=2$



## 8.5: Some Final Comments

As mentioned in the introduction, the material presented in this chapter is somewhat inconclusive. The examples in Section 8.4 are instructive and support our intuition concerning the “best way” to damp out SDOF vibrations to achieve different performance objectives. However, these methods are too computationally costly to be applied to practical applications with dozens of degrees of freedom.

In order to find the optimal physically-attainable performance in such cases, asymptotic convergence algorithms almost surely should be used. In the literature, studies of optimal semiactive controllers for suspension systems and civil structures have invariably relied on gradient algorithms to arrive at a numerical solution for the optimal control law. However, issues concerning global optimality are absent from the literature.

## Appendix A8

**LEMMA 8.1:** A necessary condition for  $\mathbf{u}_{opt}$  is that there exist  $\mathbf{p} \in \mathfrak{R}^{2n} \times C[0, t_f]$ ,  $\boldsymbol{\lambda} \in \mathfrak{R}^m \times C[0, t_f]$ , and  $\lambda_R \in \mathfrak{R} \times C[0, t_f]$  such that

$$\mathbf{u}_{opt}(t) = -\frac{1}{2} \mathbf{B}_u^T \mathbf{w}(t) - (\mathbf{R} + 2\lambda_R(t)\mathbf{I})^{-1} \left[ \boldsymbol{\lambda}(t) + \left( \mathbf{S}^T - \frac{1}{2} \mathbf{R} \mathbf{B}_u^T \right) \mathbf{w}(t) + \mathbf{B}_u^T \mathbf{p}(t) + \mathbf{S}_a^T a_g(t) \right] \in \mathcal{V}(\mathbf{w}(t)) \quad (\text{A8.1})$$

where  $\mathbf{p}$  satisfies the final-value problem

$$\dot{\mathbf{p}}(t) = -\nabla_{\mathbf{w}}^T \phi_1(\mathbf{w}(t)) - \mathbf{Q} \mathbf{w}(t) - (\mathbf{S} + \lambda_R(t) \mathbf{B}_u) \mathbf{u}_{opt}(t) - \mathbf{A}^T \mathbf{p}(t) - \mathbf{Q}_a a_g(t) \quad , \quad \mathbf{p}(t_f) = \mathbf{0} \quad (\text{A8.2})$$

and where  $\boldsymbol{\lambda}$  and  $\lambda_R$  must satisfy the following constraints for all  $t \in [0, t_f]$ :

$$\lambda_k(t) > 0 \Rightarrow u_k(t) = u_{k \max} \quad (\text{A8.3a})$$

$$\lambda_k(t) < 0 \Rightarrow u_k(t) = -u_{k \max} \quad (\text{A8.3b})$$

$$\lambda_R(t) \geq 0 \quad , \quad \lambda_R(t) P(\mathbf{u}_{opt}(t), \mathbf{w}(t)) = 0 \quad (\text{A8.3c})$$

*Proof:*

The variational statement in Eq. (5.9) is equivalent to

$$\delta \int_0^{t_f} \left[ \phi(\mathbf{w}(t), \mathbf{u}(t)) + \mathbf{p}^T(t) (\mathbf{A} \mathbf{w}(t) + \mathbf{B}_u \mathbf{u}(t) + \mathbf{B}_a a_g(t) - \dot{\mathbf{w}}(t)) \right] dt \geq 0 \quad \forall \delta \mathbf{w}, \delta \mathbf{p} \quad , \quad \forall \text{adm. } \delta \mathbf{u} \quad (\text{A8.4})$$

where  $\mathbf{p}(t)$  is a vector of Lagrange multipliers which constrain  $\mathbf{w}(t)$  such that it equals

$\mathbf{W}(t, \mathbf{u}; \mathbf{w}_0, a_g)$ . Define the *Hamiltonian* as

$$H(\mathbf{w}(t), \mathbf{p}(t), \mathbf{u}(t), a_g(t)) = \phi(\mathbf{w}(t), \mathbf{u}(t)) + \mathbf{p}^T(t) (\mathbf{A} \mathbf{w}(t) + \mathbf{B}_u \mathbf{u}(t) + \mathbf{B}_a a_g(t)) \quad (\text{A8.5})$$

and (A8.4) is

$$\delta \int_0^{t_f} \left( H(\mathbf{w}(t), \mathbf{p}(t), \mathbf{u}(t), a_g(t)) - \mathbf{p}^T(t) \dot{\mathbf{w}}(t) \right) dt \geq 0 \quad \forall \delta \mathbf{w}, \delta \mathbf{p} \quad , \quad \forall \text{adm. } \delta \mathbf{u} \quad (\text{A8.6})$$

Taking the variation inside the integral gives

$$\mathbf{p}(t) \delta \mathbf{x}(t) \Big|_0^{t_f} + \int_0^{t_f} \left[ \left( \frac{\partial H}{\partial \mathbf{w}} + \dot{\mathbf{p}}(t) \right)^T \delta \mathbf{w}(t) + \left( \frac{\partial H}{\partial \mathbf{p}} - \dot{\mathbf{w}}(t) \right)^T \delta \mathbf{p}(t) + \left( \frac{\partial H}{\partial \mathbf{u}} \right)^T \delta \mathbf{u}(t) \right] dt \geq 0 \quad (\text{A8.7})$$

$\forall \delta \mathbf{w}, \delta \mathbf{p}, \forall \text{adm. } \delta \mathbf{u}$

Next, consider the function

$$\begin{aligned} G(\mathbf{w}(t), \mathbf{u}(t), \boldsymbol{\lambda}_U(t), \boldsymbol{\lambda}_L(t), \lambda_R(t)) \\ = \lambda_R(t) P(\mathbf{u}(t), \mathbf{w}(t)) + (\mathbf{u}(t) - \mathbf{u}_{\max})^T \boldsymbol{\lambda}_U(t) + (-\mathbf{u}(t) - \mathbf{u}_{\max})^T \boldsymbol{\lambda}_L(t) \end{aligned} \quad (\text{A8.8})$$

where  $\lambda_R(t) \in \mathfrak{R}$ . Lagrange multipliers  $\lambda_U(t), \lambda_L(t) \in \mathfrak{R}^m$  have all elements  $\geq 0$ , and are related to  $\lambda(t)$  through

$$\lambda(t) = \lambda_U(t) - \lambda_L(t) \quad (\text{A8.28})$$

Note that Eq. (A8.28), together with restrictions (A8.3a) and (A8.3b), establishes a one-to-one relationship between  $\{\lambda_U(t), \lambda_L(t)\}$  and  $\lambda(t)$ .

The variation of Eq. (A8.8) is

$$\begin{aligned} \delta G(t) = & \lambda_R(t) \delta P(\mathbf{u}(t), \mathbf{w}(t)) + (\lambda_U(t) - \lambda_L(t))^T \delta \mathbf{u}(t) \\ & + P(\mathbf{u}(t), \mathbf{w}(t)) \delta \lambda_R(t) + (\mathbf{u}(t) - \mathbf{u}_{\max})^T \delta \lambda_U(t) + (-\mathbf{u}(t) - \mathbf{u}_{\max})^T \delta \lambda_L(t) \end{aligned} \quad (\text{A8.9})$$

Consider that if  $\lambda_U(t), \lambda_L(t)$ , and  $\lambda_R(t)$  are constrained to zero as in Eqs (A8.3a-c), then (A8.9) becomes

$$\delta G(t) = \lambda_R(t) \delta P(\mathbf{u}(t), \mathbf{w}(t)) + (\lambda_U(t) - \lambda_L(t))^T \delta \mathbf{u}(t) \quad (\text{A8.10})$$

Furthermore, if the inequalities in Eqs. (A8.3a-c) hold as well, then

$$\lambda_U^T(t) \delta \mathbf{u}(t) \leq 0 \quad \forall \text{adm. } \delta \mathbf{u}(t) \quad (\text{A8.11a})$$

$$\lambda_L^T(t) \delta \mathbf{u}(t) \leq 0 \quad \forall \text{adm. } \delta \mathbf{u}(t) \quad (\text{A8.11b})$$

$$\lambda_R(t) \delta P(\mathbf{u}(t), \mathbf{w}(t)) \leq 0 \quad \forall \text{adm. } \delta \mathbf{u}(t) \quad (\text{A8.11c})$$

So, consequently,

$$(\mathbf{u} + \delta \mathbf{u}) \in \mathcal{F}_u(\mathbf{w}_0, a_g) \Rightarrow \delta G(t) \leq 0 \quad \forall t \in [0, t_f] \quad (\text{A8.12})$$

Or, equivalently,

$$(\mathbf{u} + \delta \mathbf{u}) \in \mathcal{F}_u(\mathbf{w}_0, a_g) \Rightarrow \delta \int_0^{t_f} G(t) dt \leq 0 \quad (\text{A8.13})$$

Now, consider the expression

$$\tilde{J} = J + \int_0^{t_f} G(t) dt \quad (\text{A8.14})$$

In light of the constraints in (A8.3a-c),  $G(t)=0$ , and thus  $\tilde{J} = J$ . The statement

$$\delta \tilde{J} = 0 \quad \forall \delta \mathbf{u}, \delta \mathbf{w}, \delta \mathbf{p}, \quad \forall \text{admissible } \delta \lambda_U, \delta \lambda_L, \delta \lambda_R \quad (\text{A8.15})$$

implies that  $\tilde{J}$  is extremal in  $\mathbf{u}, \mathbf{w}, \mathbf{p}$ , and the admissible values of  $\lambda_U, \lambda_L$ , and  $\lambda_R$ . But consider that the variation of  $\tilde{J}$  is

$$\begin{aligned}
\delta\tilde{J} = & \mathbf{p}(t)\delta\mathbf{w}(t)\Big|_0^{t_f} \\
& + \int_0^{t_f} \left[ \left( \frac{\partial H}{\partial \mathbf{w}} + \lambda_R(t)\mathbf{B}_u\mathbf{u}(t) + \dot{\mathbf{p}}(t) \right)^T \delta\mathbf{w}(t) + \left( \frac{\partial H}{\partial \mathbf{p}} - \dot{\mathbf{w}}(t) \right)^T \delta\mathbf{p}(t) \right. \\
& + \left. \left( \frac{\partial H}{\partial \mathbf{u}} + \lambda_U(t) - \lambda_L(t) + \lambda_R(t)[2\mathbf{u}(t) + \mathbf{B}_u^T\mathbf{w}(t)] \right)^T \delta\mathbf{u}(t) \right. \\
& \left. (\mathbf{u}(t) - \mathbf{u}_{\max})^T \delta\lambda_U(t) + (-\mathbf{u}(t) - \mathbf{u}_{\max})^T \delta\lambda_L(t) + (\mathbf{u}^T(t)\mathbf{u}(t) + \mathbf{u}^T(t)\mathbf{B}_u^T\mathbf{w}(t))\delta\lambda_R(t) \right] dt
\end{aligned} \tag{A8.16}$$

However, because variations in  $\delta\mathbf{w}$ ,  $\delta\mathbf{p}$ , and  $\delta\mathbf{u}$  are independent of each other, and because  $\delta\lambda_U$ ,  $\delta\lambda_L$ , and  $\delta\lambda_R$  are independent, so long as they are constrained to admissible variations, Eq.

(A8.16) implies that necessary conditions for a extremum in  $\tilde{J}$  are

$$\delta\tilde{J} = 0 \Leftrightarrow \begin{cases} \mathbf{p}(t_f) = \mathbf{0} & \text{(A8.17a)} \\ \frac{\partial H}{\partial \mathbf{w}} + \lambda_R(t)\mathbf{B}_u\mathbf{u}(t) + \dot{\mathbf{p}}(t) = \mathbf{0} & \text{(A8.17b)} \\ \frac{\partial H}{\partial \mathbf{p}} - \dot{\mathbf{w}}(t) = \mathbf{0} & \text{(A8.17c)} \\ \frac{\partial H}{\partial \mathbf{u}} + \lambda_U(t) - \lambda_L(t) + \lambda_R(t)[2\mathbf{u}(t) + \mathbf{B}_u^T\mathbf{w}(t)] = \mathbf{0} & \text{(A8.17d)} \\ [\mathbf{u}(t) - \mathbf{u}_{\max}]^T \delta\lambda_U(t) = 0 \quad \forall \text{adm. } \delta\lambda_U(t) & \text{(A8.17e)} \\ [-\mathbf{u}(t) - \mathbf{u}_{\max}]^T \delta\lambda_L(t) = 0 \quad \forall \text{adm. } \delta\lambda_L(t) & \text{(A8.17f)} \\ [\mathbf{u}^T(t)\mathbf{u}(t) + \mathbf{u}^T(t)\mathbf{B}_u^T\mathbf{w}(t)]\delta\lambda_R(t) = 0 \quad \forall \text{adm. } \delta\lambda_R(t) & \text{(A8.17g)} \end{cases}$$

where it has been recognized that  $\delta\mathbf{w}(0)=\mathbf{0}$ , because it is assumed that the initial condition  $\mathbf{w}_0$  is fixed.

Consider equation (A8.17c). Inspection of the Hamiltonian expression in (A8.5) verifies that this term imposes the constraint that an extremum in  $\tilde{J}$  requires that  $\mathbf{x}$  obey its differential equation.

Consider conditions (A8.17e-g), dealing with the  $\delta\lambda_U$ ,  $\delta\lambda_L$ , and  $\delta\lambda_R$  variations. For these terms to be zero for all admissible variations (which is required for  $\delta\tilde{J} = 0$ ), either their arguments must be zero, or the variations are constrained to zero. Because of the constraints on  $\lambda_U(t)$ ,  $\lambda_L(t)$ , and  $\lambda_R(t)$  in equations (A8.3a-c), it follows that

$$\left. \begin{aligned} [\mathbf{u}(t) - \mathbf{u}_{\max}]^T \delta \lambda_U(t) &= 0 \quad \forall \text{adm. } \delta \lambda_U(t), \quad \forall t \in [0, t_f] \\ [-\mathbf{u}(t) - \mathbf{u}_{\max}]^T \delta \lambda_L(t) &= 0 \quad \forall \text{adm. } \delta \lambda_L(t), \quad \forall t \in [0, t_f] \\ [\mathbf{u}^T(t) \mathbf{u}(t) + \mathbf{u}^T(t) \mathbf{B}_u^T \mathbf{w}(t)] \delta \lambda_R(t) &= 0 \quad \forall \text{adm. } \delta \lambda_R(t), \quad \forall t \in [0, t_f] \end{aligned} \right\} \Leftrightarrow \mathbf{u} \in \mathcal{F}_u(\mathbf{w}_0, a_g) \quad (\text{A8.18})$$

Thus, these three terms constrain any solution to  $\delta \tilde{J} = 0$  such that  $\mathbf{u} \in \mathcal{F}_u(\mathbf{w}_0, a_g)$ .

Furthermore, in light of Eqs. (A8.13) and (A8.14), it is true that

$$\delta \tilde{J} = 0 \quad \Rightarrow \quad \delta J \geq 0 \quad \forall (\mathbf{u} + \delta \mathbf{u}) \in \mathcal{F}_u(\mathbf{w}_0, a_g) \quad (\text{A8.19})$$

Thus, the optimization of  $\tilde{J}$  yields a solution to the variational statement, and gives a solution for the input  $\mathbf{u}$  which meets the constraint  $\mathbf{u} \in \mathcal{F}_u(\mathbf{w}_0, a_g)$ .

For  $J$  defined as in Eq. (5.4), the Hamiltonian  $H$  is

$$\begin{aligned} H &= \phi_1(\mathbf{w}(t)) + \mathbf{w}^T(t) \mathbf{Q} \mathbf{w}(t) / 2 + \mathbf{u}^T(t) \mathbf{R} \mathbf{u}(t) / 2 + \mathbf{u}^T(t) \mathbf{S} \mathbf{w}(t) + \mathbf{w}^T(t) \mathbf{Q}_a a_g(t) + \mathbf{u}^T(t) \mathbf{S}_a a_g(t) \\ &\quad + \mathbf{p}^T(t) (\mathbf{A} \mathbf{w}(t) + \mathbf{B}_u \mathbf{u}(t) + \mathbf{B}_a a_g(t)) \end{aligned} \quad (\text{A8.20})$$

With this definition of  $H$ , Eq. (A8.17a) and (A8.17b) yield the boundary condition and differential equation for  $\mathbf{p}$ , respectively, while (A8.17d) yields the expression for  $\mathbf{u}(t)$ .

◆

**LEMMA 8.2:** For the conditions of Lemma 1, the following are true:

a) There exists a unique mapping  $\mathbf{U}: \mathfrak{R}^{2n} \times \mathfrak{R}^{2n} \mapsto \mathfrak{R}^m$  such that

$$\mathbf{u}(t) = \mathbf{U}(\mathbf{w}(t), \mathbf{p}(t); a_g(t)) \quad (\text{A8.21})$$

Specifically,  $\mathbf{u}$  is related to  $\mathbf{w}$  and  $\mathbf{p}$  by

$$\mathbf{u}(t) = \arg \min_{\tilde{\mathbf{u}} \in \mathcal{U}(\mathbf{w}(t))} \left\{ \frac{1}{2} \tilde{\mathbf{u}}(t)^T \mathbf{R} \tilde{\mathbf{u}}(t) + \tilde{\mathbf{u}}(t)^T [\mathbf{S} \mathbf{w}(t) + \mathbf{B}_u^T \mathbf{p}(t) + \mathbf{S}_a^T a_g(t)] \right\} \quad (\text{A8.22})$$

b) There exists a mapping  $A_R: \mathfrak{R}^{2n} \times \mathfrak{R}^{2n} \mapsto \mathfrak{R}$  such that

$$\lambda_R(t) = A_R(\mathbf{w}(t), \mathbf{p}(t); a_g(t)) \quad (\text{A8.23})$$

which is bounded for all  $\{\mathbf{w}(t), \mathbf{p}(t)\} \in \{\mathfrak{R}^{2n} - \mathcal{N}(B_u^T)\} \times \mathfrak{R}^{2n}$ , where  $\mathcal{N}(\cdot)$  denotes the null space of the operator. This mapping is unique for almost all  $\{\mathbf{w}(t), \mathbf{p}(t)\}$ .

c) The functional product  $A_R \mathbf{U}$  is bounded for all  $\{\mathbf{w}(t), \mathbf{p}(t)\} \in \mathfrak{R}^{2n} \times \mathfrak{R}^{2n}$  and is unique for almost all  $\{\mathbf{w}(t), \mathbf{p}(t)\}$ .

*Proof:*

**part a)**

Equation (A8.17d) may be re-written as

$$\left[ \frac{\partial H}{\partial \mathbf{u}} + \frac{\partial G}{\partial \mathbf{u}} \right]^T \delta \mathbf{u}(t) = 0 \quad \forall \delta \mathbf{u}(t) \quad (\text{A8.24})$$

Together with equations (A8.13) and (A8.14), this statement is equivalent to

$$\left[ \frac{\partial H}{\partial \mathbf{u}} \right]^T \delta \mathbf{u}(t) \geq 0 \quad , \quad \forall \text{adm. } \delta \mathbf{u}(t) \quad , \quad \mathbf{u}(t) \in \mathcal{U}(\mathbf{w}(t)) \quad (\text{A8.25})$$

It is clear that the value of  $\mathbf{u}(t)$  satisfying equation (A8.25) must yield a minimum in the Hamiltonian, over the domain  $\mathcal{U}(\mathbf{w}(t))$ . This interpretation of the optimal  $\mathbf{u}(t)$  is Pontryagin's Minimum Principle, and leads directly to Eq. (A8.22). It is easy to show that  $\mathcal{U}(\mathbf{w}(t))$  is a semiconvex domain, and that  $H$  is convex in  $\mathbf{u}(t)$ . Thus, the minimum of  $H$  over  $\mathcal{U}(\mathbf{w}(t))$  is unique.

**part b)**

The proof that equation (A8.22) has a solution implies that for all  $\{\mathbf{w}(t), \mathbf{p}(t)\}$ , there exist  $\boldsymbol{\lambda}(t)$  and  $\lambda_R(t)$  which satisfy equation (A8.1). However, it does not guarantee that they are unique. Equation (A8.1) may be re-written as

$$\boldsymbol{\alpha} = - \begin{bmatrix} \mathbf{I} & \boldsymbol{\beta} \end{bmatrix} \begin{bmatrix} \boldsymbol{\lambda}(t) \\ \lambda_R(t) \end{bmatrix} \quad (\text{A8.26})$$

where

$$\boldsymbol{\alpha} = \left. \frac{\partial H}{\partial \mathbf{u}} \right|_{\mathbf{w}(t), \mathbf{p}(t), \mathcal{U}(\mathbf{w}(t), \mathbf{p}(t)), a_g(t)} \quad , \quad \boldsymbol{\beta} = 2\mathbf{U}(\mathbf{w}(t), \mathbf{p}(t)) + \mathbf{B}_u^T \mathbf{w}(t) \quad (\text{A8.27})$$

are known for given  $\mathbf{w}(t)$  and  $\mathbf{p}(t)$ , due to the uniqueness of the mapping  $\mathbf{U}$ . Taking the pseudoinverse of (A8.26), the Lagrange multipliers must satisfy

$$\begin{bmatrix} \boldsymbol{\lambda}(t) \\ \lambda_R(t) \end{bmatrix} = - \begin{bmatrix} \mathbf{I} \\ \boldsymbol{\beta}^T \end{bmatrix} \left[ \mathbf{I} + \boldsymbol{\beta} \boldsymbol{\beta}^T \right]^{-1} \boldsymbol{\alpha} + q \begin{bmatrix} -\boldsymbol{\beta} \\ 1 \end{bmatrix} \quad (\text{A8.29})$$

where  $q$  is an unknown scalar variable.

The solution to  $\mathbf{u}(t)$  will dictate that some of the Lagrange multipliers are zero, and some not. Consider that

$$\begin{aligned} \lambda_k(t) \neq 0 &\Rightarrow |u_k(t)| = u_{k \max} \\ \lambda_R(t) \neq 0 &\Rightarrow \|\boldsymbol{\beta}\|_2 = \|\mathbf{B}_u^T \mathbf{w}(t)\|_2 \end{aligned} \quad (\text{A8.30})$$

If all the above are satisfied, this implies that

$$[\pm u_{1\max}, \dots, \pm u_{m\max}] \mathbf{B}_u^T \mathbf{w}(t) = -\|\mathbf{u}_{\max}\|_2^2 \quad (\text{A8.31})$$

where the components of the row vector on the left can individually be positive or negative.

Thus, the set of  $\mathbf{w}(t)$  values which result in multiple solutions for  $\lambda(t)$  and  $\lambda_R(t)$  is confined to  $2^m$  subspaces; a set of measure zero in  $\mathfrak{R}^{2n}$ .

Note that if  $\boldsymbol{\beta}=\mathbf{0}$ , then the equations for  $\lambda_R(t)$  and  $\lambda(t)$  decouple in equation (A8.26). If  $P(\mathbf{u}(t), \mathbf{w}(t))=0$  for the case of  $\boldsymbol{\beta}=\mathbf{0}$ , then  $\lambda_R(t)$  can be any positive number. But in this circumstance,  $\mathbf{B}_u^T \mathbf{w}(t)=\mathbf{0}$ , and consequently,  $\mathcal{U}(\mathbf{w}(t))=\{\mathbf{0}\}$ . From equation (A8.1), it is then clear that in order to satisfy  $\mathbf{u}(t) \in \mathcal{U}(\mathbf{w}(t))$ ,  $\lambda_R(t)$  must be infinitely large. This proves the non-boundedness assertion in the lemma, and henceforward, attention is concentrated on the case where  $\boldsymbol{\beta} \neq \mathbf{0}$ .

Assuming at least one multiplier to be zero (i.e. assuming (A8.31) does not hold), let the vector  $\tilde{\lambda}(t)$  be a truncated vector, containing only the nonzero Lagrange multipliers for which the conditions in (A8.30) apply, and define the full-column-rank  $\mathbf{G}$  such that

$$\begin{bmatrix} \lambda(t) \\ \lambda_R(t) \end{bmatrix} = \mathbf{G} \tilde{\lambda}(t) \quad (\text{A8.32})$$

The fact that a solution exists implies that

$$\boldsymbol{\alpha} \in \mathcal{R}([\mathbf{I} \ \boldsymbol{\beta}] \mathbf{G}) \quad (\text{A8.33})$$

Noting that  $[\mathbf{I} \ \boldsymbol{\beta}] \mathbf{G}$  has full column rank for almost all  $\boldsymbol{\beta}$ , and therefore almost all  $\{\mathbf{w}(t), \mathbf{p}(t)\}$ , solutions to  $\lambda(t)$  and  $\lambda_R(t)$  are uniquely found for almost all  $\{\mathbf{w}(t), \mathbf{p}(t)\}$  as

$$\begin{bmatrix} \lambda(t) \\ \lambda_R(t) \end{bmatrix} = \mathbf{G} \left[ \mathbf{G}^T \begin{bmatrix} \mathbf{I} & \boldsymbol{\beta} \\ \boldsymbol{\beta}^T & \boldsymbol{\beta}^T \boldsymbol{\beta} \end{bmatrix} \mathbf{G} \right]^{-1} \mathbf{G}^T \begin{bmatrix} \mathbf{I} \\ \boldsymbol{\beta}^T \end{bmatrix} \boldsymbol{\alpha} \quad (\text{A8.34})$$

If  $[\mathbf{I} \ \boldsymbol{\beta}] \mathbf{G}$  is column-rank-deficient, then it has a null space and the solution is thus non-unique.

**part c)**

If  $\lambda_R(t) \neq 0$ , then  $P(\mathbf{u}(t), \mathbf{w}(t)) = 0$ . So

$$\mathbf{u}^T(t) \mathbf{B}_u^T \mathbf{w}(t) = -\mathbf{u}^T(t) \mathbf{u}(t) \quad (\text{A8.35})$$

or, with some rearranging,

$$\left\| \mathbf{u}(t) + \frac{1}{2} \mathbf{B}_u^T \mathbf{w}(t) \right\|_2^2 = \left\| \frac{1}{2} \mathbf{B}_u^T \mathbf{w}(t) \right\|_2^2 \quad (\text{A8.36})$$

Using equation (A8.1),

$$\mathbf{u}(t) + \frac{1}{2} \mathbf{B}_u^T \mathbf{w}(t) = -[\mathbf{R} + 2\mathbf{I}\lambda_R(t)]^{-1} \left[ \lambda(t) - \left( \mathbf{S} + \frac{1}{2} \mathbf{R} \mathbf{B}_u^T \right) \mathbf{w}(t) - \mathbf{B}_u^T \mathbf{p}(t) \right] \quad (\text{A8.37})$$

It can also be shown that

$$\|\mathbf{u}(t) + \frac{1}{2}\mathbf{B}_u^T \mathbf{w}(t)\|_2 \leq \|[\mathbf{R} + 2\mathbf{I}\lambda_R(t)]^{-1} [(\mathbf{S} + \frac{1}{2}\mathbf{R}\mathbf{B}_u^T) \mathbf{w}(t) + \mathbf{B}_u^T \mathbf{p}(t)]\|_2 \quad (\text{A8.38})$$

Combining (A8.36) and (A8.38) gives

$$\|\frac{1}{2}\mathbf{B}_u^T \mathbf{w}(t)\|_2 \leq \|[\mathbf{R} + 2\mathbf{I}\lambda_R(t)]^{-1} [(\mathbf{S} + \frac{1}{2}\mathbf{R}\mathbf{B}_u^T) \mathbf{w}(t) + \mathbf{B}_u^T \mathbf{p}(t)]\|_2 \quad (\text{A8.39})$$

from which it can be concluded that

$$\lambda_R(t) \leq \|(\mathbf{S} + \frac{1}{2}\mathbf{R}\mathbf{B}_u^T) \mathbf{w}(t) + \mathbf{B}_u^T \mathbf{p}(t)\|_2 / \|\mathbf{B}_u^T \mathbf{w}(t)\|_2 \quad (\text{A8.40})$$

So

$$\|\lambda_R(t)\mathbf{u}(t)\|_2 \leq \|(\mathbf{S} + \frac{1}{2}\mathbf{R}\mathbf{B}_u^T) \mathbf{w}(t) + \mathbf{B}_u^T \mathbf{p}(t)\|_2 \frac{\|\mathbf{u}(t)\|_2}{\|\mathbf{B}_u^T \mathbf{w}(t)\|_2} \quad (\text{A8.41})$$

But, from (A8.36),

$$\|\mathbf{u}(t)\|_2 \leq \|\mathbf{B}_u^T \mathbf{w}(t)\|_2 \quad (\text{A8.42})$$

Thus, equation (A8.41) implies that

$$\|\lambda_R(t)\mathbf{u}(t)\|_2 \leq \|(\mathbf{S} + \frac{1}{2}\mathbf{R}\mathbf{B}_u^T) \mathbf{w}(t) + \mathbf{B}_u^T \mathbf{p}(t)\|_2 \quad (\text{A8.43})$$

◆

**LEMMA 8.4:** If the maximum force constraint (5.15b) is disregarded, then

$$\mathbf{u} \in \mathcal{F}_u(\mathbf{w}_0, a_g) \Leftrightarrow \exists \mathbf{Z}(t) \in \mathfrak{R}^{m \times m} \mid \mathbf{u}(t) = -\mathbf{Z}(t)\mathbf{B}_u^T \mathbf{w}(t) \text{ and } \bar{\sigma}(\mathbf{Z}(t) - \frac{1}{2}\mathbf{I}) \leq \frac{1}{2}, \forall t \in [0, t_f] \quad (\text{A8.44})$$

*Proof:*

The sufficiency of this statement is evident by observing that the relation of  $\mathbf{u}$  to  $\mathbf{w}$  in (8.27) implies that (5.15a) is always satisfied. Necessity may be observed by construction. For a given  $\mathbf{u} \in \mathcal{F}_u(\mathbf{w}_0, a_g)$ , define  $\mathbf{z}$  and  $\alpha$  such that

$$\mathbf{u}(t) = -\mathbf{z}(t)\mathbf{z}^T(t)\mathbf{B}_u^T \mathbf{w}(t) \quad (\text{A8.45})$$

from which it can be ascertained that

$$\mathbf{z}(t) = \pm \left( \frac{1}{\sqrt{-\mathbf{u}^T(t)\mathbf{B}_u^T \mathbf{w}(t)}} \right) \mathbf{u}(t) \quad (\text{A8.46})$$

Constraint (5.15a) then implies that



$$\|\mathbf{B}_u^T \mathbf{w}(t)\|_2^2 \|\mathbf{z}(t)\|_2^4 \leq -[\mathbf{z}^T(t) \mathbf{B}_u^T \mathbf{w}(t)]^2 \quad (\text{A8.47})$$

The Cauchy-Schwartz inequality gives

$$|\mathbf{z}^T(t) \mathbf{B}_u^T \mathbf{w}(t)| \leq \|\mathbf{B}_u^T \mathbf{w}(t)\|_2 \|\mathbf{z}(t)\|_2 \quad (\text{A8.48})$$

Thus, from the above two equations,

$$\|\mathbf{z}(t)\|_2 \leq 1 \quad (\text{A8.49})$$

Because  $\mathbf{z}(t)\mathbf{z}^T(t)$  is symmetric,

$$\begin{aligned} \bar{\sigma}(\mathbf{Z}(t) - \frac{1}{2}\mathbf{I}) &= \bar{\sigma}(\mathbf{z}(t)\mathbf{z}^T(t) - \frac{1}{2}\mathbf{I}) \\ &= |\bar{\lambda}(\mathbf{z}(t)\mathbf{z}^T(t) - \frac{1}{2}\mathbf{I})| \\ &= |\bar{\lambda}(\mathbf{z}(t)\mathbf{z}^T(t)) - \frac{1}{2}| \\ &= |\mathbf{z}^T(t)\mathbf{z}(t) - \frac{1}{2}| \\ &\leq |1 - \frac{1}{2}| = \frac{1}{2} \end{aligned} \quad (\text{A8.50})$$

Thus, if  $\mathbf{u} \in \mathcal{F}_u(\mathbf{w}_0, a_g)$ , then the above choice for  $\mathbf{Z}$  always meets the constraints.

◆

**COROLLARY 8.6:** Let constraints (8.5a-c) hold for  $\mathbf{R}$ ,  $\mathbf{S}$ ,  $\mathbf{Q}$ , and  $\phi_1$ , and assume  $\mathbf{A}$  is stable

Then the function  $V(\mathbf{w}_s, t)$  is continuous in both arguments.

*Proof:*

For convenience in this proof, the dependency on disturbance input  $a_g$  and the final time  $t_f$  will be dropped from all expressions.

Let  $\mathbf{w}_{s1}$  be an initial condition at time  $t$ , and let  $\{\mathbf{w}_1^*(t), \mathbf{u}_1^*(t)\}$  be the optimal trajectory from this initial condition over the time interval  $[t, t_f]$ . Then  $V(\mathbf{w}_{s1}, t)$  is equal to the optimal performance,  $J(\mathbf{u}_1^*; \mathbf{w}_{s1})$ . Let  $\mathbf{w}_{s2}$  be some other initial condition and suppose that there exists a control input  $\mathbf{u}_2 \in \mathcal{F}_u(\mathbf{w}_{s2})$  such that

$$J(\mathbf{u}_2; \mathbf{w}_{s2}) \leq J(\mathbf{u}_1^*; \mathbf{w}_{s1}) + \Delta \|\mathbf{w}_{s2} - \mathbf{w}_{s1}\| \quad (\text{A8.51})$$

where  $\Delta$  is positive and finite. Likewise, let  $\{\mathbf{w}_2^*, \mathbf{u}_2^*\}$  be the optimal trajectory starting from  $\mathbf{w}_{s2}$  at time  $t$ , giving  $V(\mathbf{w}_{s2}, t)$  equal to  $J(\mathbf{u}_2^*; \mathbf{w}_{s2})$ . Suppose that there exists a control input  $\mathbf{u}_1 \in \mathcal{F}_u(\mathbf{w}_{s1})$  such that

$$J(\mathbf{u}_1; \mathbf{w}_{s1}) \leq J(\mathbf{u}_2^*; \mathbf{w}_{s2}) + \Delta \|\mathbf{w}_{s2} - \mathbf{w}_{s1}\| \quad (\text{A8.52})$$

But

$$J(\mathbf{u}_k; \mathbf{w}_{sk}) \geq J(\mathbf{u}_k^*; \mathbf{w}_{sk}) \quad , \quad k \in \{1, 2\} \quad (\text{A8.53})$$

It follows that if both Eq. (A8.53) and (A8.52) are true, then

$$|V(\mathbf{w}_{s2}, t) - V(\mathbf{w}_{s1}, t)| \leq \Delta \|\mathbf{w}_{s2} - \mathbf{w}_{s1}\| \quad (\text{A8.54})$$

For  $\|\mathbf{w}_{s2} - \mathbf{w}_{s1}\|$  arbitrarily small, the above yields the conclusion that  $V$  is continuous in  $\mathbf{w}_s$ . But from the PDE for  $V$ , continuity in  $\mathbf{w}_s$  implies continuity in  $t$ . Thus, for this proof, it is sufficient to show that for any  $\mathbf{w}_{s1}$  and  $\mathbf{w}_{s2}$ , there exists a  $\mathbf{u}_2$  such that Eq. (A8.51) holds.

For convenience, let  $\mathbf{s}$  be defined as  $\mathbf{B}_u^T \mathbf{w}$ . Then at an arbitrary time, the optimal control signal  $\mathbf{u}_1^*$  is known to be feasible; i.e.

$$\mathbf{u}_1^{*T}(t) \mathbf{u}_1(t) + \mathbf{u}_1^{*T}(t) \mathbf{s}_1(t) \leq 0 \quad (\text{A8.55})$$

$$|\mathbf{u}_1^*(t)| \leq \mathbf{u}_{\max} \quad (\text{A8.56})$$

For the  $\mathbf{w}_2$  trajectory, let the sub-optimal control input  $\mathbf{u}_2$  be defined in terms of  $\mathbf{u}_1^*$  as

$$\mathbf{u}_2(t) = [1 - \alpha(t)] \mathbf{u}_1^*(t) \quad (\text{A8.57})$$

where  $\alpha(t) \in [0, 1]$  is adjusted such that  $\mathbf{u}_2(t)$  meets the regenerative constraint; i.e.,

$$\mathbf{u}_2^T \mathbf{u}_2 + \mathbf{u}_2^T \mathbf{s}_2 \leq 0 \quad (\text{A8.58})$$

Note that the restrictions on  $\alpha(t)$  ensure that if  $\mathbf{u}_1^*(t)$  satisfies Eq. (A8.56), then  $\mathbf{u}_2(t)$  satisfies the same constraint. Inserting Eq. (A8.57) into (A8.58),  $\alpha(t)$  is required to satisfy

$$\begin{aligned} & [\alpha^2(t) - \alpha(t)] \mathbf{u}_1^{*T}(t) \mathbf{u}_1(t) + [1 - \alpha(t)] \mathbf{u}_1^{*T}(t) [\mathbf{s}_2(t) - \mathbf{s}_1^*(t)] \\ & + [1 - \alpha(t)] [\mathbf{u}_1^{*T}(t) \mathbf{u}_1^*(t) + \mathbf{u}_1^{*T}(t) \mathbf{s}_1^*(t)] \leq 0 \end{aligned} \quad (\text{A8.59})$$

But inequality (A8.55) is known to hold, so Eq. (A8.59) is conservatively satisfied by

$$[\alpha^2(t) - \alpha(t)] \mathbf{u}_1^{*T}(t) \mathbf{u}_1(t) + [1 - \alpha(t)] \mathbf{u}_1^{*T}(t) [\mathbf{s}_2(t) - \mathbf{s}_1^*(t)] \leq 0 \quad (\text{A8.60})$$

Eq. (A8.60) has two roots corresponding to values of  $\alpha(t)$  for which the equality holds exactly.

One of these is always  $\alpha(t)=1$ , corresponding to  $\mathbf{u}_2(t)=\mathbf{0}$ . The other is

$$\alpha(t) = \frac{\mathbf{u}_1^{*T}(t)}{\mathbf{u}_1^{*T}(t) \mathbf{u}_1^*(t)} [\mathbf{s}_2(t) - \mathbf{s}_1^*(t)] \quad (\text{A8.61})$$

If the value above is less than zero, the  $\alpha(t)=0$  is guaranteed to satisfy inequality (A8.60). Thus, the control input

$$\mathbf{u}_2(t) = \begin{cases} \mathbf{u}_1^*(t) - \frac{\mathbf{u}_1^*(t)\mathbf{u}_1^{*T}(t)}{\mathbf{u}_1^{*T}(t)\mathbf{u}_1^*(t)}[\mathbf{s}_2(t) - \mathbf{s}_1^*(t)] & : \mathbf{u}_1^{*T}(t)[\mathbf{s}_2(t) - \mathbf{s}_1^*(t)] \geq 0 \\ \mathbf{u}_1^*(t) & : \mathbf{u}_1^{*T}(t)[\mathbf{s}_2(t) - \mathbf{s}_1^*(t)] < 0 \end{cases} \quad (\text{A8.62})$$

is such that  $\mathbf{u}_2 \in \mathcal{F}_u(\mathbf{w}_{s2})$ . The resultant differential equation for  $\mathbf{w}_1^* - \mathbf{w}_2$  is

$$\frac{d}{dt}[\mathbf{w}_2(t) - \mathbf{w}_1^*(t)] = \left\{ \mathbf{A} - \text{hvs}\left(\mathbf{u}_1^{*T}(t)[\mathbf{s}_2(t) - \mathbf{s}_1^*(t)]\right) \mathbf{B}_u \frac{\mathbf{u}_1^*(t)\mathbf{u}_1^{*T}(t)}{\mathbf{u}_1^{*T}(t)\mathbf{u}_1^*(t)} \mathbf{B}_u^T \right\} [\mathbf{w}_2(t) - \mathbf{w}_1^*(t)] \quad (\text{A8.63})$$

where  $\text{hvs}(\cdot)$  is the Heaviside step function. The structure of  $\mathbf{A}$  is such that the second term in the brackets in Eq. (A8.63) contributes supplemental damping to the differential equation. Thus, if  $\mathbf{A}$  is asymptotically stable, then Eq. (A8.63) is as well. The resultant performance  $J(\mathbf{u}_2; \mathbf{w}_{s2})$  is

$$J(\mathbf{u}_2; \mathbf{w}_{s2}) = \int_t^{t_f} \phi(\mathbf{u}_2(t), \mathbf{w}_2(t)) dt \quad (\text{A8.64})$$

But  $\phi$  is convex and positive-definite, so

$$\begin{aligned} J(\mathbf{u}_2; \mathbf{w}_{s2}) &\leq \int_t^{t_f} \phi(\mathbf{u}_1^*(t), \mathbf{w}_1^*(t)) dt + \int_t^{t_f} \phi(\mathbf{u}_2(t) - \mathbf{u}_1^*(t), \mathbf{w}_2(t) - \mathbf{w}_1^*(t)) dt \\ &\leq J(\mathbf{u}_1^*; \mathbf{w}_{s1}) + \int_t^{t_f} \phi\left(\frac{\mathbf{u}_1^*(t)\mathbf{u}_1^{*T}(t)}{\mathbf{u}_1^{*T}(t)\mathbf{u}_1^*(t)} \mathbf{B}_u^T [\mathbf{w}_2(t) - \mathbf{w}_1^*(t)], \mathbf{w}_2(t) - \mathbf{w}_1^*(t)\right) dt \end{aligned} \quad (\text{A8.65})$$

Finally, because  $\phi$  is a bounded, polynomial function, and the difference  $\mathbf{w}_2(t) - \mathbf{w}_1^*(t)$  decays exponentially, the second integral is bounded by a norm on  $\mathbf{w}_{s2} - \mathbf{w}_{s1}$ . (The appropriate norm will depend on the polynomial powers used in  $\phi$ ). Thus, Eq. (A8.65) implies Eq. (A8.51), completing the proof.

◆

**COROLLARY 8.8:** For the free-vibration case with  $\mathbf{A}$  stable, the value of  $V(\mathbf{w}_s, t)$  is stationary as  $t_f \rightarrow \infty$  and  $V(\mathbf{w}_s, t)$  is stable in reverse-time.

*Proof:*

Let  $\mathbf{w}_s$  be an arbitrary initial condition at time  $t$ . Let the corresponding optimal trajectory over the interval  $[t, t_f]$  be  $\{\mathbf{u}^*, \mathbf{w}^*\}$ . Let the optimal trajectory over the interval  $[t, t_f + \Delta t]$  be denoted  $\{\mathbf{u}^\Delta, \mathbf{w}^\Delta\}$ .

Then

$$\begin{aligned}
V(\mathbf{w}_s, t_f + \Delta t) &= J(\mathbf{u}^\Delta[t, t_f]; \mathbf{w}_s) + J(\mathbf{u}^\Delta[t_f, t_f + \Delta t]; \mathbf{w}^\Delta(t_f)) \\
&\geq V(\mathbf{w}_s, t_f) + J(\mathbf{u}^\Delta[t_f, t_f + \Delta t]; \mathbf{w}^\Delta(t_f)) \\
&\geq V(\mathbf{w}_s, t_f)
\end{aligned} \tag{A8.66}$$

and

$$V(\mathbf{w}_s, t_f + \Delta t) \leq V(\mathbf{w}_s, t_f) + J(\mathbf{0}[t_f, t_f + \Delta t]; \mathbf{w}^*(t_f)) \tag{A8.67}$$

where  $\mathbf{0}(t)$  is shorthand for the input function  $\mathbf{u}(t)=\mathbf{0}$ . Thus,

$$0 \leq V(\mathbf{w}_s, t_f + \Delta t) - V(\mathbf{w}_s, t_f) \leq J(\mathbf{0}[t_f, t_f + \Delta t], \mathbf{w}^*(t_f)) \tag{A8.68}$$

As  $\Delta t$  goes to zero,

$$0 \leq \frac{\partial V(\mathbf{w}_s, t)}{\partial t_f} \leq \phi(\mathbf{0}, \mathbf{w}^*(t_f)) \tag{A8.69}$$

If  $\mathbf{A}$  is stable, then the optimally-controlled system is also stable, and resultantly  $\mathbf{w}^*(t_f) \rightarrow \mathbf{0}$  as

$t_f \rightarrow \infty$ .

◆

## Chapter 9. Summary and Future Work

### 9.1: Summary

It has been the intent in this research to “map out” the major issues at the heart of regenerative actuation. As such, the material in this study has taken form as a collection of connected problems.

The most basic issue regarding these systems is the question of how to build them. Setting aside the many technologically nuanced challenges, there are a few fundamental issues. In Chapters 2 and 3, a possible realization of an arbitrarily large RFA network was presented. This realization is probably the simplest design which can be applied to networks of arbitrary size and configuration. However, it may be that there are other realizations which are more practical or realistic. For instance, there may be some benefit to using a more elaborate electrical network to interface the various actuators. However, such ideas clearly would add significant complexity to the dynamic analysis.

One of the challenges concerning the electronic control of these networks is that they have to operate minimum dissipation, in order to take full advantage of their forcing capability. The electronic controller proposed in Chapter 4 represents the approach that yielded the best results, in this context. Based on sliding-mode control, its performance is fairly robust to uncertainties in system parameters when operating at near-maximum efficiency. As the development and examples in Chapter 4 illustrated, it turns out that ensuring this robustness is one of the great challenges associated with these systems.

In Chapter 5, RFA networks were placed in the context of mechanics. It was shown that they can be viewed as providing an effective damping to a mechanical system, but where the nature of this damping, and its relationship to the device configuration, is more general and abstract than in other dissipative systems. In particular, the concept of the RFA network imposing an asymmetric damping matrix on the structural differential equation is both intuitive and useful from a control point of view. In the investigation of memoryless control laws in Chapter 6, the motivation was toward controllers with guaranteed performance bounds. It was shown that, through the relation of a certain class of controllers to the damping characteristics studied in Chapter 5, the so-called “Damping-Reference” controller can be guaranteed to yield

performance in stationary and stochastic response which is at least as good as that achieved with optimal linear damping.

The simulations in Chapter 7 have several uses. First, of course, they illustrate the response of the structure-actuator system to earthquake excitation. But a fair amount of general information can be obtained concerning the coupling between the electrical and mechanical dynamics, as well as the role of the transistor switches in the response.

As a parting thought, Chapter 8 discussed the application of optimal control to these actuation systems, for which the goal is the derivation of the best possible performance over all control inputs, causal or otherwise, which are physically feasible. It is clear that this subject requires further analysis. The lingering issues concerning the nonconvex nature of the optimization, and the resultant dubiousness of asymptotic convergence methods, have impeded progress in this area. It may very well be that this optimization problem can be proved to have a unique, global minimum, thus allaying the concerns voiced in Chapter 8. However, this remains to be shown.

## 9.2: Future Work

There are essentially three main areas which constitute the next logical steps in the development of RFA networks into a viable technology.

### 9.2.1: Experimental Validation

Thus far, the development of the theoretical tools necessary for the proper use and control of RFA networks has outweighed the need for experimental validation. Because these devices are assembled from such common components, they really do not require “cutting edge technology” for implementation. This is in stark contrast to many other devices which have been proposed for civil structure control, many of which require a completely custom design and fabrication. However, now that the theoretical aspects of these systems have been adequately mapped out, it is time to build a laboratory experiment and demonstrate the concept.

In scale-model demonstrations, there is little doubt that commercial electronic, mechanical, sensory, and control hardware will be adequate for this undertaking. However, at the full-scale, there may be practical limitations to what commercial hardware can do in this context. Typical high-power commercial motor drives are not built for this kind of actuation (i.e., at such low velocities and high forces) and their designs may turn out to be rather sub-optimal. It is therefore interesting to contemplate the development of novel, custom-built motors which are developed expressly for such applications.

### 9.2.2: Actuation Configurations

In this study, only three basic structural configurations for RFA networks were considered, as shown in Fig. 1.6. All three of these were analyzed in Chapter 5 in the context of linearly-damped stationary stochastic response. However, in the remaining chapters only configuration “a” was considered. This was necessary in order to focus the analysis. However, it may be that there are many other configurations yielding interesting results.

Especially, the analysis of RFA configurations involving remote energy storage (i.e., configuration “b”) require further investigation. Presently, the most effective use of remote energy storage devices in vibration suppression is still not completely understood. Intuitively, there are clearly many benefits to storing and reusing energy in the context of forced vibration suppression. In addition to the possible improvement in drift and acceleration response which may be possible with energy reuse, there are also energy management issues which may make this kind of configuration appealing. Specifically, the storage of energy extracted from the structure, rather than the dissipation of that energy as heat, may be beneficial for practical reasons, such as thermal considerations.

There are also other configurations which have not been addressed at all in this analysis, which may yield very interesting results. For instance, an RFA network used to control vibration in two or three spacial dimensions, rather than the one-dimensional examples considered here, would be able to facilitate power sharing between actuators operating in different directions and at different locations.

### 9.2.3: Control Synthesis

In this study, a variety of methods were investigated for the synthesis of control laws for RFA networks. However, these methods were all rather simple, and may not be ideal for this application. There a number of interesting avenues for the pursuit of more sophisticated control algorithms for energy-constrained actuation systems. Two of the more promising of these are briefly discussed below.

#### ***Receding Horizon Control***

Receding Horizon control techniques (Clarke et al. 1987; Mayne et al. 2000) involve the repeated re-solving of the optimal control problem (as discussed in Chapter 5) in real-time. Assuming the dynamics of the structure are sufficiently slow, it is not unreasonable to assume that with current computing technology, it may be possible to solve the optimal control problem

at time  $t$  over a window  $[t, t+T]$ , and arrive at  $\mathbf{u}_{opt}(t; T)$ , in a negligible amount of time. If this is the case, it then becomes possible to iteratively re-solve for  $\mathbf{u}_{opt}(t; T)$  in real-time, using updated state information. For a horizon time  $T$  sufficiently large, this procedure results in a control force which closely approximates the infinite-horizon optimal control  $\mathbf{u}_{opt}(t; \infty)$ . However, because the control force is solved in real-time, the approximate solution to  $\mathbf{u}_{opt}$  is effectively “adjusted” to account for system uncertainty.

Thus, the appeal of Receding Horizon control is that it effectively makes it possible to approximate, very closely, the optimal control. If implemented, this kind of control would therefore ensure that the actuation system is yielding performance which is very close to the optimal physically-attainable performance. These controllers have attracted some attention the area of structural control (Gluck et al. 2000; Huang and Betti 1999; Inaudi et al. 1992; Mei et al. 2001; Yang and Beck 1995). However, Receding Horizon controllers are computationally expensive, and in order to be appealing, there are some issues which must first be resolved.

In Chapter 8, it was mentioned that the optimal control problem, for energy-constrained systems, is an optimization over a nonconvex domain. Although it is very likely that this issue can somehow be resolved, at the present time the solution to this problem is elusive. If a Receding Horizon controller is to iteratively solve for a solution  $\mathbf{u}_{opt}$ , this process would likely require a convex optimization framework.

Also, the fact that the future disturbance is unknown makes Receding Horizon control more difficult. Even in a stationary stochastic setting, this problem is nontrivial.

### ***Probabilistic Control Synthesis***

It turns out that typical structural control system design goals for Civil Engineering applications are somewhat ill-matched with quadratic performance measures. The ultimate performance measure for civil applications concerns failure probability, in the face of both disturbance and parameter uncertainty. With disturbances modeled as random inputs and uncertainties modeled by probability distributions, the aspects of the response statistics relating to failure probability usually concern information far in the tails of the response distributions. Thus, controllers designed to yield favorable “average” responses may not inspire a lot of confidence. Additionally, because parameter uncertainty is typically modeled by continuous distributions rather than by compact sets in parameter space, much of the existing  $\mathcal{H}_\infty$  robust control theory does not seem well-suited to the problem.

The ideal control synthesis tool for Civil Engineering applications would find the controller minimizing the probability of the first-passage of a set of failure thresholds, in a given



period of time, and given distribution functions on structural parameters. This problem is not totally new. In Civil Engineering, it has been investigated for linear active control systems with digital FIR controllers by (May and Beck 1998; Yuen and Beck 2003). There has also been some general work in the area of probabilistic robust control conducted by (Marrison and Stengel 1995; Stengel et al. 1995). However, at present there have not been any such attempts to extend this problem to energy-constrained actuation systems.

## References

- Adhikari, S. (1999). "Modal Analysis of Linear Asymmetric Nonconservative Systems." *Journal of Engineering Mechanics*, 125(12), 1372-1379.
- Akbay, Z., and Aktan, H. M. "Intelligent energy dissipation devices." *Proc., 4th U.S. National Conference on Earthquake Engineering*, 427-435.
- Akbay, Z., and Aktan, H. M. "Actively regulated friction slip devices." *Proc., 6th Canadian Conference on Earthquake Engineering*, 367-374.
- Camino, J., de Oliveira, M., and Skelton, R. E. (2003). "'Convexifying' Linear Matrix Inequality Methods for Integrating Structure and Control Design." *Journal of Structural Engineering*, 129(7), 978-988.
- Carlson, J. D., and Spencer, B. F., Jr. "Magneto-rheological fluid dampers: scalability issues for application to dynamic hazard mitigation." *Proc., Second International Workshop on Structural Control: Next Generation of Intelligent Structures*, 99-109.
- Caughey, T. K., and Ma, F. (1993). "Complex Modes and Solvability of Nonclassical Linear Systems." *Journal of Applied Mechanics*, 60(3), 26-28.
- Chang, S., and Peng, T. (1972). "Adaptive Guaranteed Cost Control of Systems with Uncertain Parameters." *IEEE Transactions on Automatic Control*, AC17(4), 474.
- Cherry, S. "Research on friction damping at the University of British Columbia." *Proc. Internatioinal Workshop on Structural Control*, 84-91.
- Clarke, D., Mohtadi, C., and Tuffs, P. (1987). "Generalized Predictive Control .1. The Basic Algorithm." *Automatica*, 23(2), 137-148.
- Croila, D. A., and Abdel-Hady, M. B. A. (1991). "Semi-Active Suspension Control for a Full Vehicle Model." *Transactions of the Society of Automobile Engineers*, 100, 1660-1666.
- Dorato, P., Chaouki, A., and Cerone, V. (1995). *Linear Quadratic Control*, Prentice Hall, Inc., Englewood Cliffs, NJ.
- Dowdell, D. J., and Cherry, S. "Semi-active friction dampers for seismic response control of structures." *Proc., Fifth U.S. National Conference on Earthquake Engineering*, 819-828.
- Dowdell, D. J., and Cherry, S. "Structural control using semiactive friction dampers." *Proc., First World Conference on Structural Control*, 59-68.
- Dyke, S. J., Spencer, B. F., Jr., Sain, M. K., and Carlson, J. D. "Seismic response reduction using magnetorheological dampers." *Proc., IFAC World Congress*, 145-150.
- Dyke, S. J., Spencer, B. F., Jr., Sain, M. K., and Carlson, J. D. "Experimental verification of

- semi-active structural control strategies using acceleration feedback." *Proc., Third International Conference on Motion and Vibration Control*, 291-296.
- Dyke, S. J., Spencer, B. F., Jr., Sain, M. K., and Carlson, J. D. (1996). "Modeling and Control of Magnetorheological Dampers for Seismic Response Reduction." *Smart Materials and Structures*, 5, 565-575.
- Edberg, D. L., and Bicos, A. S. "Design and Development of Passive and Active Damping Concepts for Adaptive Space Systems." *Symposium on Active Materials and Adaptive Structures*, Alexandria, VA.
- Ehrgott, R. C., and Masri, S. F. (1992). "Modelling the Oscillatory Dynamic Behavior of Electro-rheological Materials in Shear." *Smart Materials and Structures*, 1(4), 275-285.
- Ehrgott, R. C., and Masri, S. F. (1994). "Experimental Characterization of an Electrorheological Material Subjected to Oscillatory Shear Strains." *Journal of Vibration and Acoustics*, 116(Jan.), 53-60.
- Ehrgott, R. C., and Masri, S. F. "Structural control applications of an electrorheological device." *Proc., International Workshop on Structural Control*, 115-129.
- Feng, Q., and Shinozuka, M. "Use of variable damper for hybrid control of bridge response under earthquake." *U.S. National Workshop on Structural Control Research*.
- Feng, Q., Shinozuka, M., and Fujii, X. (1993). "Friction-Controllable Sliding Isolated Systems." *Journal of Engineering Mechanics, ASCE*, 119(9), 1845-1864.
- Fujino, Y., Soong, T. T., and Spencer, B. F., Jr. "Structural control: Basic concepts and applications." *Proc., ASCE Structural Congress, Chicago, Ill.*, 1277-87.
- Gavin, H. (2001). "Control of Seismically Excited Vibration Using Electrorheological Materials and Lyapunov Methods." *IEEE Transactions on Control Systems Technology*, 9(1), 27-36.
- Gavin, H. P., Hose, Y. D., and Hanson, R. D. "Design and control of electrorheological dampers." *Proc., First World Conference on Structural Control*, 83-92.
- Gavin, H. P., Ortiz, D. S., and Hanson, R. D. "Testing and modeling of a proto-type ER damper for seismic structural response control." *Proc., International Workshop on Structural Control*, 166-180.
- Gluck, J., Ribakov, Y., and Dancygier, A. (2000). "Predictive Active Control of MDOF Structures." *Earthquake Engineering and Structural Dynamics*, 29(1), 109-125.
- Gordaninejad, F., Rau, A., and Bindu, R. "Vibration control of structures using hybrid electrorheological-viscous fluid dampers." *Proc., First World Conference on Structural Control*, 41-49.

- Green, M., and Limebeer, D. (1995). *Linear Robust Control*, Prentice-Hall, Englewood Cliffs, NJ.
- Hagood, N. W., and von Flotow, A. (1991). "Damping of Structure Vibrations with Piezoelectric Materials and Passive Electrical Networks." *Journal of Sound and Vibration*, 146(2), 243-268.
- Hayen, J. C., and Iwan, W. D. "Response control of structural systems using active interface damping." *Proc., First World Conference on Structural Control*, 23-32.
- He, W., Agarwal, A., and Yang, J. (2003). "Novel Semiactive Friction Controller for Linear Structures Against Earthquakes." *ASCE Journal of Structural Engineering*, 129(7), 941.
- Hollkamp, J. J., and Starchville, T. F. "A Self-Tuning Piezoelectric Vibration Absorber." *Proc., 35th SDM Conference*, Hilton Head, SC.
- Housner, G. W., Bergman, L. A., Caughey, T. K., Cassiakos, A. G., Claus, R. O., Masri, S. F., Skelton, R. E., Soong, T. T., Spencer, B. F. J., and Yao, J. T. P. (1997). "Structural Control: Past, Present and Future." *Journal of Engineering Mechanics*, 123(9).
- Housner, G. W., and Jennings, P. C. (1964). "Generation of Artificial Earthquakes." *Journal of the Engineering Mechanics Division, ASCE*, 90.
- Hrovat, D., Barak, P., and Rabins, M. (1983). "Semi-active versus passive or active tuned mass dampers for structural control." *Journal of Engineering Mechanics*, 109(3), 691-705.
- Hrovat, D., Margolis, D. L., and Hubbard, M. (1988). "An Approach Toward the Optimal Semi-Active Suspension." *Journal of Dynamic Systems, Measurement, and Control*, 110(3), 288-296.
- Huang, K., and Betti, R. (1999). "Predictive Optimal Control for Seismic Analysis of Non-Linear and Hysteretic Structures." *Earthquake Engineering and Structural Dynamics*, 28(6), 585-607.
- Ikeda, Y. (2004). "Active and Semi-Active Control of Buildings in Japan." *Journal of Japan Association for Earthquake Engineering, Special Issue, August 2004*.
- Ikeda, Y., Katsuyasu, S., Sakamoto, M., and Kobori, T. (2001). "Active Mass Driver System as the First Application of Active Structure Control." *Earthquake Engineering and Structural Dynamics*, 30(7), 1575-1595.
- Inaudi, J., Lopez-Almansa, F., Kelly, J., and Rodellar, J. (1992). "Predictive Control of Base-Isolated Structures." *Earthquake Engineering and Structural Dynamics*, 21(6), 471-482.
- Inman, D. J. (1983). "Dynamics of Asymmetric Nonconservative Systems." *ASME Journal of Applied Mechanics*, 50, 199-203.
- Jolly, M. R., and Margolis, D. L. (1997). "Regenerative Systems for Vibration Control." *Journal*

- of Vibration and Acoustics*, 119(April), 208-215.
- Jolly, M. R., and Margolis, D. L. (1997). "Assessing the Potential for Energy Regeneration in Dynamic Subsystems." *Journal of Dynamic Systems, Measurement, and Control*, 119(June), 265-270.
- Kamagata, S., and Kobori, T. "Autonomous adaptive control of active variable stiffness system for seismic ground motion." *First World Conference on Structural Control*, 33-42.
- Kanai, K. (1957). "Semi-Empirical Formula for the Seismic Characteristics of the Ground." *Bulletin of the Earthquake Research Institute, University of Tokyo*, 35, 309-325.
- Kareem, A. "The next generation of tuned liquid dampers." *Proc., First World Conference on Structural Control*, 19-28.
- Karnopp, D. (1989). "Permanent Magnet Linear Motors Used as Variable Mechanical Dampers for Vehicle Suspensions." *Vehicle System Dynamics*, 18, 187-200.
- Karnopp, D., Crosby, M. M., and Harwood, R. A. (1974). "Vibration Control Using Semi-Active Force Generators." *ASME Journal of Engineering for Industry*, 96(2), 619-626.
- Karnopp, D. C. (1983). "Active Damping in Road Vehicle Suspension System." *Vehicle System Dynamics*, 12, 291-316.
- Kassakian, J. G., Schlecht, M. F., and Verghese, G. C. (1991). *Principles of Power Electronics*, Addison-Wesley Publ. Co., Inc., Reading, MA.
- Kawashima, K., Unjoh, S., and Shimizu, K. "Experiments on dynamics characteristics of variable damper." *Japan National Symposium on Structural Response Control*.
- Kim, Y., and Wang, K. (1993). "On the Sliding Mode Control of Structural Vibrations via Variable Damping." *Mechanical Systems and Signal Processing*, 7(4), 335-347.
- Kirk, D. (1970). *Optimal Control Theory*, Prentice-Hall, Inc., Englewood Cliffs, NJ.
- Kobori, T. "Past, present and future in seismic response control in civil engineering structures." *Proc., 3rd World Conference on Structural Control*, 9-14.
- Kobori, T., Takahashi, M., Nasu, T., Niwa, N., and Ogasawara, K. (1993). "Seismic response controlled structure with active variable stiffness system." *Earthquake Engineering and Structural Dynamics*, 22, 925-941.
- Krishnan, R. (2001). *Electric Motor Drives: Modeling, Analysis, and Control*, Prentice Hall, Upper Saddle River, New Jersey.
- Kurata, N., Kobori, T., Takahashi, M., Niwa, N., and Kurino, H. "Shaking table experiments of active variable damping system." *First World Conference on Structural Control*, 108-127.
- Leitmann, G. (1994). "Semiactive control for vibration attenuation." *Journal of Intelligent*

- Materials, Systems, and Structures*, 5(Sept.), 841-846.
- Leitmann, G., and Reithmeier, E. (1993). "Semi-Active Control of Vibrating System by Means of Electro-Rheological Fluids." *Dynamics and Control*, 3(1), 7-33.
- Liang, Z., Tong, M., and Lee, G. C. (1995). "Real-time structural parameter modification (RSPM): development of innervated structures." *NCEER-95-0012*, National Center for Earthquake Engineering Research, Buffalo, N.Y.
- Lou, J. Y. K., Lutes, L. D., and Li, J. J. "Active tuned liquid damper for structural control." *Proc., First World Conference on Structural Control*, 70-79.
- Makris, N., Hill, D., Burton, S., and Jordan, M. "Electrorheological fluid dampers for seismic protection of structures." *Proc., SPIE Conference on Smart Structures and Materials*, 184-194.
- Margolis, D. L. (1983). "The Response of Active and Semi-Active Suspensions to Realistic Feedback Signals." *Vehicle System Dynamics*, 12, 317-330.
- Marrison, C., and Stengel, R. (1995). "Stochastic Robustness Synthesis Applied to a Benchmark Problem." *International Journal of Robust and Nonlinear Control*, 5(1), 13-31.
- Masri, S. F., Kumar, R., and Ehrgott, R. C. (1995). "Modeling and Control of an Electrorheological Device for Structural Control Applications." *Journal of Smart Materials and Structures*, 4(Mar), 121-131.
- May, B., and Beck, J. (1998). "Probabilistic control for the active mass driver benchmark structural model." *Earthquake Engineering and Structural Dynamics*, 27(11), 1331-1346.
- Mayne, D., Rawlings, J., Rao, C., and Sokaert, P. (2000). "Constrained Model Predictive Control: Stability and Optimality." *Automatica*, 36(6), 789-814.
- McClamroch, N. H., and Gavin, H. P. "Closed loop structural control using electrorheological dampers." *Proc., American Control Conference*, 4173-4177.
- Mei, G., Kareem, A., and Kantor, J. (2001). "Real-time Model Predictive Control of Structures Under Earthquakes." *Earthquake Engineering and Structural Dynamics*, 30(7), 995-1019.
- Mirzuno, T., Kobori, T., Hirai, J., Matsunaga, Y., and Niwa, N. "Development of adjustable hydraulic dampers for seismic response control of large structure." *ASME PVP Conference*, 163-170.
- Mohler, R. R. (1970). "Natural Bilinear Control Processes." *IEEE Transactions on Systems Science and Cybernetics*, SSC6(3), 192.
- Mohler, R. R. (1973). *Bilinear Control Processes*, Academic Press, New York.
- Nagarajaiah, S., and Mate, D. "Semi-active control of continuously variable stiffness system."

- Proc., 2nd World Conference on Structural Control*, 397-405.
- Nasu, T., Kobori, T., Takahashi, M., Niwa, N., and Ogasawara, K. (2001). "Active Variable Stiffness System with Non-Resonant Control." *Earthquake Engineering and Structural Dynamics*, 30(7), 1597-1614.
- Nerves, A. C., and Krishnan, R. "A Strategy for Active Control of Tall Civil Structures Using Regenerative Electric Actuators." *Proc., Engineering Mechanics Conference*, Fort Lauderdale, FL, 503-506.
- Nishitani, A., Nitta, Y., and Ikeda, Y. (2003). "Semiactive Structural-Control Based on Variable Slip-Force Level Dampers." *ASCE Journal of Structural Engineering*, 129(7), 933-940.
- Okada, Y., Hideyuki, H., and Kohei, S. (1997). "Active and Regenerative Control of an Electrodynamical-Type Suspension." *JSME International Journal, C*, 40(2), 272-278.
- Patten, W. N., He, Q., Kuo, C. C., Liu, L., and Sack, R. L. "Seismic structural control via hydraulic semi-active vibration dampers." *1st World Conference on Structural Control*, 83-89.
- Pillay, P., and Krishnan, R. (1989). "Modeling, Simulation, and Analysis of Permanent-Magnet Motor Drives, Part II: The Brushless DC Motor Drive." *IEEE Transactions on Industrial Applications*, 25(2), 274-279.
- Rabinow, J. (1948). "The magnetic field clutch." *AIEE Transactions*, 67, 1308-1315.
- Sack, R. L., Kuo, C. C., Wu, H. C., Liu, L., and Patten, W. N. "Seismic motion control via semiactive hydraulic actuators." *U.S. Fifth National Conference on Earthquake Engineering*.
- Sack, R. L., and Patten, W. N. "Semiactive hydraulic structural control." *Proc., International Workshop on Structural Control*, University of Southern California, Los Angeles, 417-431.
- Scruggs, J. (1999). "Active, Regenerative Control of Civil Structures," M.S. Thesis, Virginia Tech, Blacksburg, VA.
- Scruggs, J., and Lindner, D. K. "Active Energy Control for Civil Structures." *SPIE 1999 North American Symposium on Smart Structures and Materials: Smart Systems for Bridges, Structures, and Highways*, Los Angeles, CA, 194-205.
- Scruggs, J. T., and Iwan, W. D. (2003). "Control of a Civil Structure Using an Electric Machine with Semiactive Capability." *ASCE Journal of Structural Engineering*, 129(7), 951-959.
- Scruggs, J. T., and Iwan, W. D. (2003). "Structural Control Using Regenerative Actuation Networks." *Journal of Structural Control and Health Monitoring (in press)*.
- Shinozuka, M., Constantinou, M. C., and Ghanem, R. "Passive and active fluid dampers in

- structural applications." *U.S./China/Japan Workshop on Structural Control*, 507-516.
- Singh, M., Matheu, E., and Suarez, L. (1997). "Active and Semi-Active Control of Structures Under Seismic Excitation." *Earthquake Engineering and Structural Dynamics*, 26(2), 193-213.
- Skelton, R. E., Hanks, B., and Smith, M. (1991). "Structure Redesign for Improved Dynamic Response." *Journal of Guidance, Control and Dynamics*, 15(5), 1272-1278.
- Soong, T. T. (1990). *Active Structural Control: Theory and Practice*, Longman Wiley, London, England.
- Spencer, B. F., Jr. "Recent trends in vibration control in the USA." *Proc., 3rd International Conference on Motion and Vibration Control*, Chiba, Japan), R1-R6.
- Spencer, B. F., Jr., Dyke, S., and Deoskar, H. (1998). "Benchmark Problems in Structural Control - Part I: Active Mass Driver System." *Earthquake Engineering and Structural Dynamics*, 27(11).
- Spencer, B. F., Jr., Dyke, S. J., and Sain, M. K. "Magneto-rheological dampers: a new approach to seismic protection of structures." *Proc., Conference on Decision and Control*, 676-681.
- Spencer, B. F., Jr., Dyke, S. J., Sain, M. K., and Carlson, J. D. (1997). "Phenomenological Model of a Magneto-rheological Damper." *Journal of Engineering Mechanics, ASCE*, 123(3), 230-238.
- Spencer, B. F., Jr., and Nagarajaiah, S. (2003). "State of the Art in Structural Control." *ASCE Journal of Structural Engineering*, 129(7), 845-856.
- Spencer, B. F., Jr., and Sain, M. K. (1997). "Controlling Buildings: A New Frontier in Feedback." *IEEE Control Systems Magazine*, 17(6), 19-35.
- Stengel, R. (1994). *Optimal Control and Estimation*, Dover Publications, Inc., New York, NY.
- Stengel, R., Ray, L., and Marrison, C. (1995). "Probabilistic Evaluation of Control-System Robustness." *International Journal of Systems Science*, 26(7), 1363-1382.
- Symans, M. D., and Constantinou, M. C. "Experimental study of seismic response of structures with semi-active damping control systems." *Structural Congress XIV*.
- Symans, M. D., and Constantinou, M. C. (1999). "Semiactive Control of Systems for Seismic Protection of Structures: A State-of-the-Art Review." *Engineering Structures*, 21(6), 469-487.
- Symans, M. D., Constantinou, M. C., Taylor, D. P., and Ganjost, K. D. "Semi-active fluid viscous dampers for seismic response control." *First World Conference on Structural Control*, 3-12.
- Tajimi, H. "A Statistical Method of Determining the Maximum Response of a Building Structure



- During an Earthquake." *2nd World Conference on Earthquake Engineering*, Tokyo and Kyoto, 781-798.
- Tseng, H. E., and Hedrick, J. K. (1994). "Semi-Active Control Laws - Optimal and Sub-Optimal." *Vehicle System Dynamics*, 23, 545-69.
- Wang, K. W., Lai, J. S., and Yu, W. K. (1996). "An Energy-Based Parametric Control Approach to Structural Vibration Suppression via Semi-Active Piezoelectric Networks." *Journal of Vibration and Acoustics*, 118(July), 505-509.
- Winslow, W. M. (1947). "Method and means for translating electrical impulses into mechanical forces." U.S. Patent Specification No. 2,417,850.
- Winslow, W. M. (1949). "Induced Fibration of Suspensions." *Journal of Applied Physics*, 20, 1137-1140.
- Yalla, S., and Kareem, A. (2003). "Semiactive Tuned Liquid Column Dampers: Experimental Study." *ASCE Journal of Structural Engineering*, 127(7), 960-972.
- Yamada, K., and Kobori, T. (2001). "Fundamental Dynamics and Control Strategies for Aseismic Structural Control." *International Journal of Solids and Structures*, 38(34-35), 6079-6121.
- Yang, C., and Beck, J. "Robust Adaptive Structural Control." *ASCE 10th Annual Engineering Mechanics Conference*, Boulder, CO.
- Yang, J. N., Wu, J. C., and Hsu, S. Y. "Parametric control of seismic-excited structures." *Proc. First World Conference on Structural Control*, 88-97.
- Yeh, H., Reed, D. A., Yu, J., and Gardarsson, S. "Performance of tuned liquid dampers under large amplitude excitation." *Proc., Second International Workshop on Structural Control*, 432-443.
- Yuen, K., and Beck, J. (2003). "Reliability-Based Robust Control for Uncertain Dynamical Systems Using Feedback of Incomplete Noisy Response Measurements." *Earthquake Engineering and Structural Dynamics*, 32(5), 751-770.
- Zhang, Y., and Iwan, W. (2002). "Active Interaction Control of Civil Structures. Part 2: MDOF Systems." *Earthquake Engineering and Structural Dynamics*, 31(1), 179-194.