Holomorphic Anomaly Equations in Topological String Theory

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Abstract

In this thesis we discuss various aspects of topological string theories. In particular we provide a derivation of the holomorphic anomaly equation for open strings and study aspects of the Ooguri, Strominger, and Vafa conjecture.

Topological string theory is a computable theory. The amplitudes of the closed topological string satisfy a holomorphic anomaly equation, which is a recursive differential equation. Recently this equation has been extended to the open topological string. We discuss the derivation of this open holomorphic anomaly equation. We find that open topological string amplitudes have new anomalies that spoil the recursive structure of the equation and introduce dependence on wrong moduli (such as complex structure moduli in the A-model), unless the disk one-point functions vanish. We also show that a general solution to the extended holomorphic anomaly equation for the open topological string on D-branes in a Calabi-Yau manifold, is obtained from the general solution to the holomorphic anomaly equations for the closed topological string on the same manifold, by shifting the closed string moduli by amounts proportional to the 't Hooft coupling.

An important application of closed topological string theory is the Ooguri, Strominger, and Vafa conjecture, which states that a certain black hole partition function is a product of topological and anti-topological string partition functions. However when the black hole has finite size, the relation becomes complicated. In a specific example, we find a new factorization rule in terms of a pair of functions which we interpret as the "non-perturbative" completion of the topological string partition functions.

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Chapter 1 Introduction

String theory is a candidate theory that unifies all four forces of nature. As a fundamental theory, it has great beauty. This theory has constituents that are tiny 1-dimensional objects called strings. Their typical scale is thought to be about 10^{-35}m which in terms of energy is about 10^{19}GeV, the Planck scale. It is therefore hard to do any direct observation of strings, since the highest energy scale experimentally accessible today is 14 TeV at the Large Hadron Collider (LHC). We can perhaps obtain cosmological evidence, but even the Big Bang does not seem to provide high-enough energies. Another way to test string theory is to investigate its low energy behavior and compare with experiments. String theory requires spacetime to have a critical dimension. Superstring theory gives a spacetime dimension of 10. This raises questions about how to compactify this theory on a particular manifold, so that when the size of this manifold is very small it gives us 4-dimensional physics as we observe it. Unfortunately, compactification is not so simple and causes many problems.

In particular we care about a 4-dimensional low-energy effective theory of superstring theory compactified on certain manifolds. The manifold we consider throughout this thesis is a 3-complex or 6-real dimensional Calabi-Yau manifold, termed a Calabi-Yau 3-fold, which reduces supersymmetries to a quarter in $d = 4$.

Computations in physical string theory are very difficult. To calculate string amplitudes—the scattering of one string with another—we need quantum gravity on the string worldsheet. The string states are in one-to-one correspondence with vertex operators. Since there are many gauge symmetries on the worldsheet, we have to perform a path integral on gauge-inequivalent slices of worldsheets, which is very complicated. Bosonic string theory seems to be a little simpler, as the vacuum of every string loop has a specific form associated to the topology of the worldsheet. However bosonic string theory does not contain fermionic fields, so it is less useful physically. A useful string theory containing both bosons and fermions is topological string theory. Furthermore the topological string theory in a Calabi-Yau 3-fold has the same structure as a bosonic string theory. This theory became more important when its physical application was discovered.

Closed topological string theory is well-understood [1, 2]. Topological string theory with D-branes, i.e., open strings, is also interesting, because these carry gauge degrees of freedom. In particular we hope it will help us to understand open-closed dualities in string theory. These dualities originate in the so called holographic principle—gauge/geometry correspondence. It was found that Chern-Simons gauge theory on a 3-dimensional manifold M can be viewed as a topological string theory on T^*M [3]. The argument is that the perturbative expansion of this string theory coincides with Chern-Simons perturbation theory. Later, it was found that the large N limit of $SU(N)$ Chern-Simons theory on S^3 is the same as an $\mathcal{N} = 2$ topological closed string on the S^2 blowup of the conifold geometry in T^*S^3. There also exists an open topological string description of the duality where N 3-branes are wrapped on an S^3 inside the conifold T^*S^3 [4]. Furthermore, [5, 6] gave a worldsheet explanation of the duality between open and closed topological strings.

In the closed topological string theory, [1] developed a systematic method—the so called holomorphic anomaly equation—for calculating closed topological string amplitudes, and one could ask if there is a similar way to calculate the ones for open strings. Recently Walcher conjectured an open holomorphic anomaly equation [7], analagous to the closed one, under two assumptions, the absence of open string moduli and tadpole cancellation. The first assumption makes sense when the open string ends on a D-brane and the disk amplitude gives rise to a superpotential on the D-brane worldvolume, for then open string moduli only correspond to flat directions of this superpotential. Since the disk amplitude can not depend on them, other amplitudes should not develop any dependence on them through recursion relation. Meanwhile, we found that the second assumption is related to the absence of a new class of anomalies [8]. When the new anomalies are present, not only is the nice recursive structure broken, but the string amplitudes also depend on wrong moduli; that is, the A-model topological string depends on complex structure moduli and the

B-model depends on Kähler moduli.

We also derived a generating function for amplitudes of the open topological string on D-branes in a Calabi-Yau manifold. Interestingly, it is related to the generating function of the closed topological string on the same manifold [1], by shifting the closed string moduli by amounts proportional to the 't Hooft coupling [9].

One important application of the closed topological string theory is the Ooguri, Strominger, and Vafa (OSV) conjecture. It says that, in the large black hole limit where the black hole was obtained by type II superstring compactified on a Calabi-Yau 3-fold, the black hole partition function is the absolute value of square of a topological string partition function. For small black holes, there are non-perturbative corrections, for example in the factorization of two baby universes, there will be contributions from $2n$ ($n > 2$) baby universes [10]. We attempted to build a new factorization and tried to interpret it as a "non-perturbative" completion of the topological string partition function. In order to prove that, it is necessary to check if it satisfies holomorphic anomaly equation. For topological string theory, holomorphicity and modularity can be shown to be traded with each other. The partition functions of baby universes are already written in a holomorphic form but they are not modular. We thus want to understand how we can restore the modular property and then obtain the holomorphic anomaly equation.

In the first part of this thesis, we investigate a holomorphic anomaly equation for the open topological string amplitudes. Chapter 2 is a brief review of closed topological string theory. Chapter 3 contains our work on the derivation of the holomorphic anomaly equation of open topological string theory; we also discuss in detail the new class of anomalies for the open topological string. Chapter 4 is about our work on the relation between closed and open holomorphic anomaly equations. Finally, in Chapter 5 we review the OSV conjecture and discuss some of our work on studying factorization, holomorphic and modular properties, and Gromov-Witten invariants of closed topological string theory. Chapter 6 summarizes the thesis and discusses open questions.

Chapter 2 Topological String Theory

2.1 Introduction

There are various formulations of superstring theory. The worldsheet, for example, can be described by an $\mathcal{N} = 1$ supersymmetric Ramond-Neveu-Schwarz (RNS) formalism or a spacetime Green-Schwarz (GS) formalism. However calculating string scattering, the most important physical process in a quantum theory of strings, is very difficult in both formalisms. The physical string theory must have excitations that correspond to every kind of elementary particle. Bosonic string theory does not provide a description of fermions, but fermion fields can be added to the action by considering superstring theory. It was found when string worldsheet supersymmetry is extended to $\mathcal{N} = 2$, some nice geometrical feature appears which makes this theory computable. The resulting theory is shown to be a topological field theory. The most important feature for this theory is that it is not only a toy model, but it has some important physical implications as well.

A $(2, 2)$ sigma model contains bosonic fields which are maps from a Riemann surface to a Kähler manifold. Supersymmetry is manifest when the Riemann surface is flat. If the surface becomes curved, in general we can not find a covariant constant spinor as a supersymmetry transformation parameter, and therefore supersymmetry is broken. However we can modify the theory so that some fermionic symmetry remains. This procedure is called a topological twist. After the twist, we obtain a scalar supercharge which is preserved on any Riemann surface. Therefore, we can modify the flat sigma model to a curved one by changing the flat metric to a curved one and the partial derivative to the covariant one. Since this scalar supercharge is nilpotent in any Riemann surface, we can define an associated cohomology. The physical spectrum of the topological theory is given by this cohomology. The theory is called a topological sigma model. The word “topological” means that in this theory the correlation functions are independent of the metric on the Riemann surface.

A topological sigma model on a Calabi-Yau 3-fold only has a few non-trivial correlators. We can generalize the theory by coupling it to gravity on the Riemann surface. The theory is then called a topological string theory. The Riemann surface is also called a worldsheet as in string theory.

2.2 $\mathcal{N} = (2,2)$ supersymmetry

The dimensional reduction of the $\mathcal{N} = 1$, 4-dimensional supersymmetry algebra gives the $\mathcal{N} = (2,2)$ algebra in 2 dimensions. The Lagrangian for a $(2,2)$ theory of a chiral superfield on a Riemann surface is [11],

$$\mathcal{L} = \int d\theta^+ d\theta^- d\bar{\theta}^- d\bar{\theta}^+ K(\Phi, \bar{\Phi}) + \left(\int d\theta^+ d\theta^- W(\Phi) + c.c. \right), \tag{2.1}$$

where Φ is a chiral superfield, K is the Kähler potential, W is the superpotential, and the θs are fermionic coordinates of superspace. In terms of component fields, the superfield Φ is written as

$$\begin{aligned} \Phi \;=\; & \phi - i\theta^+\bar{\theta}^+\partial_+\phi - i\theta^-\bar{\theta}^-\partial_-\phi - \theta^+\theta^-\bar{\theta}^-\bar{\theta}^+\partial_+\partial_-\phi \\ & +\theta^+\psi_+ - i\theta^+\theta^-\bar{\theta}^-\partial_-\psi_+ + \theta^-\psi_- - i\theta^-\theta^+\bar{\theta}^+\partial_+\psi_- + \theta^+\theta^- F, \end{aligned} \tag{2.2}$$

where ϕ is a scalar field, ψ is a spinor field, and F is an auxiliary field. The Kähler potential can be written as

$$K = g_{I\bar{J}}\Phi^I\bar{\Phi}^{\bar{J}} + \cdots, \qquad \text{where} \qquad g_{I\bar{J}} = \partial_I\partial_{\bar{J}}K. \tag{2.3}$$

The superpotential has a Taylor expansion,

$$W = \partial_I W \Phi^I + \frac{1}{2}\partial_I\partial_J W \Phi^I \Phi^J + \cdots. \tag{2.4}$$

There are four supercharges: Q^+, $\overline{Q}^+$ for left-movers, Q^- and $\overline{Q}^-$ for right-movers. The supersymmetry transformation of an operator is defined as

$$\delta_{susy}\mathcal{O} = [\delta_Q, \mathcal{O}], \tag{2.5}$$

where

$$\delta_Q = i\epsilon_+ Q^- - i\epsilon_- Q^+ - i\bar{\epsilon}_+ \overline{Q}^- + i\bar{\epsilon}_- \overline{Q}^+. \tag{2.6}$$

There are R-symmetries for $\mathcal{N} = 2$. We denote them $U(1)_V$ and $U(1)_A$, where V and A refer to the vector and axial rotation. Under these symmetries, the superfields transform as either

$$\begin{aligned} &(V) \qquad \Phi^i(x, \theta^\pm, \bar{\theta}^\pm) \mapsto \Phi^i(x, e^{-i\alpha}\theta^\pm, e^{i\alpha}\bar{\theta}^\pm), \qquad \text{or} \\ &(A) \qquad \Phi^i(x, \theta^\pm, \bar{\theta}^\pm) \mapsto \Phi^i(x, e^{\mp i\beta}\theta^\pm, e^{\pm i\beta}\bar{\theta}^\pm). \end{aligned} \tag{2.7}$$

Correspondingly, the component fields transform as

$$\begin{aligned} & \qquad \phi \mapsto \phi, \\ (V) & \qquad \psi_\pm \mapsto e^{-i\alpha}\psi_\pm, \\ (A) & \qquad \psi_\pm \mapsto e^{\mp i\beta}\psi_\pm. \end{aligned} \tag{2.8}$$

The supercharges transform in the same way as the fermionic fields,

$$\begin{aligned} (V) & \qquad Q_\pm \mapsto e^{-i\alpha}Q_\pm, \\ (A) & \qquad Q_\pm \mapsto e^{\mp i\beta}Q_\pm. \end{aligned} \tag{2.9}$$

The Nöther charges associated to the $U(1)_V$ and $U(1)_A$ are F_V and F_A. Supercharges, Nöther charges, and conserved charges from the Poincaré symmetry give rise to a closed supersymmetry algebra.

2.3 Topological sigma model

A sigma model is a field theory in which the bosonic field ϕ is a map from a Riemann surface Σ to a target space M,

$$\phi : \Sigma \to M. \tag{2.10}$$

The 2-dimensional Riemann surface with genus g is also refered to as the string worldsheet with genus g, the difference being that in a sigma model we are not allowing variation of the worldsheet metric at this stage. When there is $(2,2)$ supersymmetry on the Riemann surface, it is called a $(2,2)$ sigma model. We can use Kähler geometry to write down the kinematic term of the Lagrangian of an $(2,2)$ non-linear sigma model,

$$\mathcal{L}_{kin} = -g_{i\bar{j}}\partial^\mu\phi^i\partial_\mu\phi^{\bar{j}} + ig_{i\bar{j}}\bar{\psi}_-^{\bar{j}}(D_0+D_1)\psi_-^i + ig_{i\bar{j}}\bar{\psi}_+^{\bar{j}}(D_0-D_1)\psi_+^i + R_{i\bar{j}k\bar{l}}\psi_+^i\psi_-^k\bar{\psi}_-^{\bar{j}}\bar{\psi}_+^{\bar{l}}, \tag{2.11}$$

where fermions are spinors with values in the pull-back of the tangent bundle, $\psi_\pm \in \Gamma(\Sigma, \phi^*TM^{(1,0)} \otimes S_\pm)$, $g_{i\bar{j}}$ is the metric of M, and $R_{i\bar{j}k\bar{l}}$ is the Riemann tensor of M. We can apply a Wick rotation to the time direction to obtain a Euclidean worldsheet with $SO(2) \cong U(1)_E$ "Lorentz" symmetry. Now we consider two types of twists in which the $U(1)_E$ is replaced by $U(1)_E \otimes U(1)_R$. After the twist, the newly defined scalar supercharge is $Q = Q_A$ or $Q = Q_B$, with

$$\begin{aligned} (A) \qquad & Q_A = Q_- + \overline{Q}_+, \\ (B) \qquad & Q_B = \overline{Q}_- + \overline{Q}_+. \end{aligned} \tag{2.12}$$

The supersymmetry transformations with respect to the scalar supercharges are

$$\begin{aligned} (A) \qquad & \delta_{Q_A} = i\epsilon Q_A, \\ (B) \qquad & \delta_{Q_B} = i\epsilon Q_B. \end{aligned} \tag{2.13}$$

The charges of the fermionic fields with respect to various $U(1)$ symmetries are listed in the following table: Performing the A-twist (B-twist), we find that two fermionic

				A-twist	B-twist
	$U(1)_V$	$U(1)_A$	$U(1)_E$	$U(1)'_E = U(1)_E \otimes U(1)_V$	$U(1)'_E = U(1)_E \otimes U(1)_A$
ψ_-	-1	1	1	0	2
$\bar{\psi}_+$	1	1	-1	0	0
$\bar{\psi}_-$	1	-1	1	2	0
ψ_+	-1	-1	-1	-2	-2

Table 2.1: $U(1)$ charges of fermionic fields under A- and B-twist

fields ψ_- and $\bar{\psi}_+$ ($\bar{\psi}_-$ and $\bar{\psi}_+$) turn into scalar fields.

The physical observables are then restricted to be in the Q-cohomology (from now we suppress the subscript), meaning,

$$[Q, \mathcal{O}] = 0, \qquad \text{and} \qquad \mathcal{O} \sim \mathcal{O} + [Q, \Lambda]. \tag{2.14}$$

Now we will consider a Calabi-Yau 3-fold X as the target space. For the A-twist, the scalar fermionic fields have the structure of de Rham cohomology; and for the B-twist, Dolbeault cohomology, where z^i and $\bar{z}^{\bar{i}}$ are the coordinates of the Calabi-Yau

	Conformal dimensions		Forms in X	
	A-twist	B-twist	A-twist	B-twist
ψ^i_-	0	1	dz^i	
$\bar{\psi}^{\bar{i}}_+$	0	0	$d\bar{z}^i$	$\frac{1}{2}(-d\bar{z}^{\bar{i}} + *d\bar{z}^{\bar{i}}*)$
$\bar{\psi}^{\bar{i}}_-$	1	0		$\frac{1}{2}(-d\bar{z}^{\bar{i}} - *d\bar{z}^{\bar{i}}*)$
ψ^i_+	1	1		

Table 2.2: Cohomological structures under A- and B-twist

3-fold and $*$ is the Hodge star with respect to the metric $g_{i\bar{j}}$.

For each twist, we can define a chiral ring of Q-closed operators. By convention, we say that an operator $\mathcal{O}$ belongs to the (c, c) ring if

$$[Q_-, \mathcal{O}] = 0, \quad [\overline{Q}_+, \mathcal{O}] = 0, \tag{2.15}$$

and the (a, c) ring if

$$[\overline{Q}_-, \mathcal{O}] = 0 \quad [\overline{Q}_+, \mathcal{O}] = 0. \tag{2.16}$$

Operators in the (c, c) and (c, a) chiral rings can then be built up as

$$\phi = k_{i_1 \cdots i_p \bar{j}_1 \cdots \bar{j}_q} \psi_-^{i_1} \cdots \psi_-^{i_p} \psi_+^{\bar{j}_1} \cdots \psi_+^{\bar{j}_q}, \tag{2.17}$$

$$\varphi = V_{\bar{i}_1 \cdots \bar{i}_p}{}^{j_1 \cdots j_q} \eta^{\bar{i}_1} \cdots \eta^{\bar{i}_p} \theta_{j_1} \cdots \theta_{j_q}, \tag{2.18}$$

where $\eta^{\bar{i}} = \bar{\psi}_+^{\bar{i}} + \bar{\psi}_-^{\bar{i}}$ and $\theta_i = g_{i\bar{j}}(\bar{\psi}_+^{\bar{i}} - \bar{\psi}_-^{\bar{j}})$. Define the tangent vector as

$$\frac{\partial}{\partial z^i} \wedge = g_{i\bar{j}} * d\bar{z}^{\bar{j}} * . \tag{2.19}$$

The chiral ring then corresponds to cohomology $H^{p,q}(X)$ and $H^p_{\bar{\partial}}(X, \wedge^q T^{(1,0)} X)$, respectively. According to the operator-state correspondence, the fields in the chiral rings also correspond to the supersymmetric ground states.

The twisted $(2, 2)$ sigma model is topological, because the variation of the action with respect to the worldsheet metric is Q-exact,

$$\frac{\delta}{\delta g_{\mu\nu}} S = T^{\mu\nu} = [Q, \Lambda^{\mu\nu}], \tag{2.20}$$

and so it should not change the correlation functions. Unfortunately, the non-trivial correlation functions are limited to be the 3-point functions on a sphere and the partition function on a torus. If we want to obtain information for higher genus, we must couple the sigma model to gravity on the worldsheet, that is, we must allow for variation of the worldsheet metric. After doing that, the theory extends to a topological string theory.

2.3.1 $U(1)_R$ anomaly

The reason that almost all correlation functions vanish in the topological sigma model is because of a $U(1)_R$ anomaly. As we know, a physical theory should not be anomalous. For comparison, consider the chiral anomaly in a gauge theory. When we calculate the generating function, the measure of the fermionic fields is not invariant under chiral transformations. Therefore, the $U(1)$ current associated to chiral symmetry is not a conserved current anymore. The divergence of the current is proportional to the difference between numbers of zero modes of chiral fermions. In mathematics, this difference is calculated by an index of the Dirac operator.

Since the correlation functions in the topological sigma model are independent of the metric of the Riemann surfaces, we can deform the metric by a scaling factor. When this factor goes to infinity, the action should be a minimum. This process is called localization. The path integral then picks up contributions only from the loci where the Q-variation of fermions vanishes [11]. In A-twisted sigma models, the Q-fixed point restricts the map from a worldsheet to target space X to be a holomorphic map; $\bar{\partial}_{\bar{z}}\phi = 0$. The anomaly is related to the index of the Dirac operator on the worldsheet,

$$k = (2-2g)\dim_{\mathbb{C}} X + c_1(X)\cdot\beta, \tag{2.21}$$

where β is the homology class of the map, $\beta = \phi_*[\Sigma] \in H_2(X,\mathbb{Z})$, $c_1(X)$ is the first Chern class of X, and g is the genus of the worldsheet. When X is a Calabi-Yau manifold, $c_1(X) = 0$, k is simplified to be $(2-2g)\dim_{\mathbb{C}} X$.

In B-twisted sigma models, the Q-fixed point restricts the map to be a constant map, $\partial_\mu\phi = 0$, where ∂_μ is a worldsheet derivative. The index is

$$k = (2-2g)\dim_{\mathbb{C}} X. \tag{2.22}$$

2.4 Closed topological string theory

Most correlation functions vanish in a sigma model because of the $U(1)$ R-symmetry anomaly. One way to solve this problem is to couple the sigma model to worldsheet gravity. We then obtain a quantum gravity theory on the worldsheet, which is similar to a string theory. A twisted $(2,2)$ sigma model on a Calabi-Yau 3-fold coupled to worldsheet gravity is called a topological string theory. Depending on A- (B-) twist, it is also called A- (B-) model topological string theory.

Actually it was shown that a topological string on Calabi-Yau 3-fold has the same structure as a bosonic string theory in 26-dim. For example, for the correspondence of A-model topological string, see Table 2.3.

Bosonic string	A-model topological string
Q_{BRST}	Q_A
b ghost	G^+
energy-stress tensor $T = \{Q_{BRST}, b\}$,	$T = \{Q_A, G^+\}$
ghost number anomaly	$U(1)_R$ anomaly

Table 2.3: Bosonic string vs A-model topological string

2.4.1 Bosonic string theory

The path integral for the bosonic string is over worldsheet inequivalent metrics of the gauge group,

$$\text{diff} \times \text{Weyl}/\text{CKG}, \tag{2.23}$$

where "diff" is the diffeomorphism group, "Weyl" is the Weyl group (scaling transformation of metric), and "CKG" is the conformal Killing group. We can use the Faddeev-Popov method to fix the gauge, which gives rise to an action for the b, c ghosts [12],

$$S_{gh} = \int d^2 z b \bar{\partial} c + c.c.\ . \tag{2.24}$$

Similar to the chiral anomaly in a gauge theory, there is a ghost number anomaly from the difference between numbers of zero modes of b and c ghost. This number is equal to the dimension of the conformal Killing group κ, minus the dimension of the modular group μ of the worldsheet. Using the Grothendieck-Riemann-Roch formula,

$$\kappa - \mu = 3\chi, \tag{2.25}$$

where $\chi = 2 - 2g$ is the Euler number of the worldsheet. Therefore the anomaly is the same as that of the topological sigma model on a Calabi-Yau 3-fold (2.22). The moduli space of Riemann surfaces for $g > 1$, denoted as $\mathcal{M}_{g,h}$, has complex dimension $3g-3$. For $g=1$ it has dimension 1, and for $g=0$ it has dimension 0. The conformal Killing group for $g>1$ has dimension 0; for $g=1$ it has dimension 1, and for $g=0$ it has dimension 3. Therefore, for $g>1$, the path integral is just over the moduli space of the worldsheet. For $g=1$ and 0, the topological string is the same as the topological sigma model. For $g=1$, there is one modulus and one conformal Killing vector, so one operator insertion is needed in the path integral to fix this isometry. For $g=0$, the non-zero physical quantities are the 3-point functions, with no moduli space left to integrate over. For $g>1$ ($g=1$), gauge fixing not only adds a ghost action to the original one, but also inserts μ (1) operators which are Beltrami differentials folded with 2-form supercurrents in the integrand. A Beltrami differential parametrizes a deformation of complex structure on the worldsheet. It is defined as

$$(\mu_a)_\nu{}^\mu = \frac{1}{2} g^{\mu\rho}\partial_a g_{\nu\rho}, \tag{2.26}$$

where $a=1,\cdots,3g-3$ labels the deformations on the worldsheet and $g_{\mu\nu}$ is the metric. In a complex coordinate system, it can be written as $\mu_{a\bar{z}}{}^z d\bar{z}\partial_z$, with $\mu_a \in H^1_{\bar{\partial}}(\Sigma, T^{(1,0)}\Sigma)$.

2.4.2 Closed topological string amplitudes

The correspondence between bosonic string theory and topological string theory allows us to write down the topological string amplitudes. Now the $U(1)_R$ anomaly comes from the difference between numbers of zero modes of 1-form and scalar fields, the fields which are fermionic fields on the worldsheet before the twist. Like the bosonic string, there are $3(2g-2)$ operators with $U(1)$ charge -1 inserted in the integrand; therefore, we are able to define topological string amplitudes [1] as

$$\begin{aligned} F_g &= \int_{\mathcal{M}_{g,h}} [dm] \left\langle \prod_{a=1}^{3g-3} \int \mu_a G^- \int \bar{\mu}_a \overline{G}^- \right\rangle_{\Sigma_{g,h}}, \qquad (g>1) && (2.27) \\ \partial_i F_1 &= \int_{\mathbb{F}} \frac{d\tau d\bar{\tau}}{\mathrm{Im}\tau} \left\langle \int \mu G^- \int \bar{\mu} \overline{G}^- \mathcal{O}_i^{(1,1)} \right\rangle_{T^2}, && (2.28) \\ F_1 &= \int_{\mathbb{F}} \frac{d\tau d\bar{\tau}}{\mathrm{Im}\tau} Tr(-1)^F F_L F_R q^{H_L} \bar{q}^{H_R}, && (2.29) \\ \partial_i \partial_j \partial_k F_0 &= \left\langle \mathcal{O}_i^{(1,1)} \mathcal{O}_j^{(1,1)} \mathcal{O}_k^{(1,1)} \right\rangle_{S^2}, && (2.30) \end{aligned}$$

where $\mathcal{O}_i^{(1,1)}$ have $U(1)_R$ charge $(1,1)$ ((left,right)).

2.4.3 Relation between closed topological strings and physical strings

Type II superstring theories have $\mathcal{N}=2$ supersymmetry in 10 dimensions. After compactification on a Calabi-Yau 3-fold only a quarter of the supersymmetry is preserved, corresponding to $\mathcal{N}=2$ supersymmetry in 4 dimensions. The 4-dimensional theory contains one $\mathcal{N}=2$ supergravity multiplet, $h^{1,1}+1$ ($h^{2,1}+1$) $\mathcal{N}=2$ vector multiplets, and $h^{2,1}+1$ ($h^{1,1}+1$) hypermultiplets for A- (B-) type superstring theory, where $h^{1,1}$ and $h^{2,1}$ are the Hodge numbers of the Calabi-Yau 3-fold. The lowest components of the vector multiplets correspond to Kähler (complex structure) moduli of the Calabi-Yau 3-fold. Remarkably, the F-term of the low-energy effective theory is calculated by the closed topological string theory. This term is

$$\sum_g \int d^4x \int d^4\theta F_g(t^i) W^{2g} = \int d^4x F_g(t^i) R_+^2 F_+^{2g-2}, \tag{2.31}$$

where W is the $\mathcal{N}=2$ supergravity multiplet, R_+ is the curvature, F_+ is the field strength of the $U(1)$ vector field component of W, t^i are Kähler (complex structure) moduli of the Calabi-Yau 3-fold, and F_g is exactly the closed topological string amplitudes defined above.

Chapter 3 The Open Holomorphic Anomaly Equation

3.1 Introduction

The closed holomorphic anomaly equation gives a recursion relation for the partition function F_g with respect to the genus g of the string worldsheet [1]. The equation has proven to be useful in evaluating topological string amplitudes. In fact, for *compact* Calabi-Yau manifolds, it is the only known method for computing these amplitudes systematically for higher g. This method has seen remarkable progress in recent years. The Feynman diagram method developed in [1] has been made more efficient by [13]. This, combined with the knowledge on the behavior of F_g at the boundaries of the Calabi-Yau moduli space, has made it possible to integrate the holomorphic anomaly equation to very high values of g [14].

Recently, Walcher generalized the holomorphic anomaly equation to the case of topological string theory in the presence of D-branes [7]. Attempts to derive such an equation had been made before, for example in [1]. The new ingredients in [7] are two assumptions: that open string moduli do not contribute to factorizations in open string channels and that disk one-point functions vanish. A Feynman diagram method for integrating the holomorphic anomaly equation in the presence of D-branes has subsequently been proven [9] and enhanced [15, 16], as well as considered in the context of background independence [17]. Furthermore, initial attempts have been made to understand the situation where open string moduli may contribute [18].

In this chapter, we will focus on the assumption of vanishing disk one-point functions. We find that, for a *compact* Calabi-Yau manifold, disk one-point functions generate new terms in the holomorphic anomaly equation and spoil its recursive structure. Moreover, with non-zero disk one-point functions, F_g can develop dependence on "wrong" moduli, that is complex structure moduli in the A-model and Kähler moduli in the B-model.

That disk one-point functions themselves depend on wrong moduli has been known for a long time. In [19], it was shown that D-branes in the A-model are associated to Lagrangian 3-cycles and that their disk one-point functions depend on B-model moduli. Conversely, D-branes in the B-model are associated to holomorphic even-cycles and their disk one-points functions depend on A-model moduli. One might then imagine that the disk one-point functions could introduce wrong moduli dependence into higher genus partition functions, and indeed we will find this effect explicitly, as a new type of anomalies in compact Calabi-Yau manifolds.

The cancellation of overall D-brane charge provides a means to remove the contribution of the new anomalies. Indeed, such a cancellation appears to be required for the successful counting of the number of BPS-states in M-theory using the topological string partition function. In [4], it was conjectured that the partition function of the closed topological string can be interpreted as counting BPS states in M-theory compactified to five dimensions on a Calabi-Yau 3-fold. This conjecture was extended to cases with D-branes in [5, 6]. Recently, Walcher [20] applied the formulae of [5] to examples of *compact* Calabi-Yau manifolds and found that the integrality of BPS state counting can be assured only when the topological charges of the D-branes were cancelled by introducing orientifold planes [20], such that the disk one-point functions vanish. Our result gives a microscopic explanation of this observation.

Furthermore, the absence of these new anomalies appears to be a prerequisite for large N duality between open and closed topological string theories. Specifically, this duality implies that topological string amplitudes in both theories should obey the same equations, notably the holomorphic anomaly equation, and should not depend on the wrong moduli. Indeed, the holomorphic anomaly equations for the open string derived in [7] under the assumption of vanishing disk one-point functions are compatible with large N duality [21]. Also, [9] pointed out a similarity between the holomorphic anomaly equations of the closed [1] and open [7] strings, requiring shifts of closed string moduli by amounts proportional to the 't Hooft coupling. Conversely, the presence of the new anomalies is correlated with the breakdown of large N duality. For compact Calabi-Yau manifolds, the conifold transition requires homology relations among vanishing cycles [22, 23]. For example, if a single 3-cycle of non-trivial homology shrinks and the singularity is blown up, the resulting manifold cannot be Kähler. Thus, the presence of D-branes with nontrivial topological charge implies a topological string theory without closed string dual; simultaneously the disk one-point functions do not vanish, so the new anomalies are present.

3.2 The open topological string theory

In this section, we will discuss some properties of the open topological string. We will write down the holomorphic anomaly equation and then prove it in Section 3.3.

3.2.1 Boundary condition

In order to preserve the scalar supercharge, there must be some boundary conditions for the supercurrents. We denote the supercurrents $G^{\pm}$ and $\overline{G}^{\pm}$, where barred quantities are right-moving, with conventions such that for both models the BRST operator is written as

$$Q_{BRST} = \oint G_z^+ dz + \oint \overline{G}_{\bar{z}}^+ d\bar{z}. \qquad (3.1)$$

The appropriate worldsheet boundary conditions for the supercurrents are then

$$(G_z^+ dz + \overline{G}_{\bar{z}}^+ d\bar{z})|_{\partial\Sigma} = 0, \qquad \text{and} \qquad (G_{zz}^- \chi^z dz + \overline{G}_{\bar{z}\bar{z}}^- \bar{\chi}^{\bar{z}} d\bar{z})|_{\partial\Sigma} = 0, \qquad (3.2)$$

where χ is a holomorphic vector along the boundary direction.

3.2.2 Some aspects of the moduli spaces of Riemann surfaces

An open string worldsheet is a Riemann surface with genus g, boundary number h, n marked points in the interior, and m marked points on the boundary; we denote it as $\Sigma_{g,h,n,m}$.

- The Euler number for this manifold is $\chi = -(2g - 2 + h + n + m/2)$.

- Since every handle is associated with 3 complex moduli, every boundary is associated with 3 real moduli, every marked point in the interior is associated with 2 real moduli, and every marked point on the boundary is associated with 1 real modulus. The dimension of the moduli space, is thus,

$$\dim \mathcal{M}_{g,h,n,m} = 6g - 6 + 3h + 2n + m \ . \tag{3.3}$$

- The boundary of the moduli space of Riemann surfaces corresponds to various degenerations of the surface (marked points are ignored). We use a set of cartoons in Figures 3.1–3.6.

- In the closed topological string theory, we need to insert a certain number of supercurrents folded with Beltrami differentials into the path integral on a worldsheet. These Beltrami differentials describe the complex deformations of the worldsheet. For a worldsheet with genus g and h boundaries, the dimension of the moduli space is $6g-6+3h$, and this is the number of possible independent Beltrami differentials. To study the Beltrami differentials, we will double the Riemann surface $\Sigma_{g,h}$ to be $\hat{\Sigma}_{2g+h-1,0}$, such that boundaries of $\Sigma_{g,h}$ are fixed points of a $\mathbb{Z}_2$ involution of $\hat{\Sigma}_{2g+h-1,0}$; in other words,

$$\Sigma_{g,h} = \hat{\Sigma}_{2g+h-1}/\mathbb{Z}_2. \tag{3.4}$$

  By this trick, Riemann surfaces with boundaries can generally be related to orientable Riemann surfaces without boundaries.

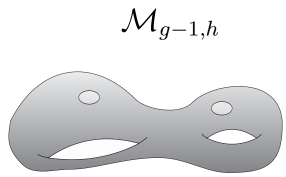

Figure 3.1: A handle pinches off, leaving $\Sigma_{g-1,h}$ plus a degenerating thin tube.

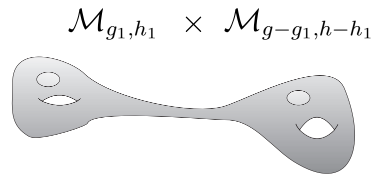

Figure 3.2: An equator pinches off, splitting the Riemann surface into two non-trivial daughter surfaces Σ_{g_1,h_2} and $\Sigma_{g-g_1,h-h_1}$, joined by a degenerating thin tube.

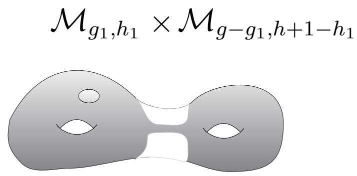

Figure 3.3: A path from a boundary to the same boundary, around an equator, degenerates to leave two surfaces Σ_{g_1,h_1} and $\Sigma_{g-g_1,h+1-h_1}$, with the two daughter surfaces joined by a degenerating thin strip.

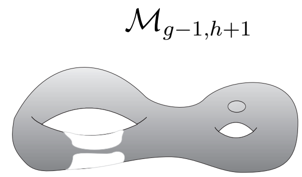

Figure 3.4: A path from a boundary to the same boundary, around a handle, degenerates to leave $\Sigma_{g-1,h+1}$, with the two child boundaries joined by a degenerating thin strip.

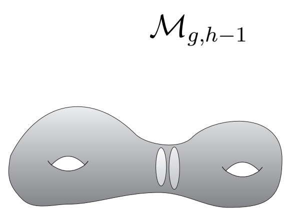

Figure 3.5: A path between two different boundaries degenerates, leaving $\Sigma_{g,h-1}$, with a degenerating thin strip across the newly-joined boundary.

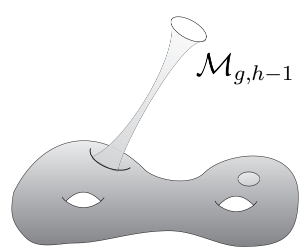

Figure 3.6: A cycle around a boundary shrinks, that is the boundary closes off. Conformally this is a boundary on the end of degenerating thin tube attached to the remaining surface $\Sigma_{g,h-1}$.

3.2.3 Open holomorphic anomaly equation

Based on the previous discussion, the topological string partition function at a given genus g and boundary number h is written as

$$F^{(g,h)} = \int_{\mathcal{M}_{g,h}} \left\langle \prod_{a=1}^{3g-3+h} \int \mu_a G^- \int \bar{\mu}_a \overline{G}^- \prod_{b=1}^{h} \int \lambda_b (G^- + \overline{G}^-) \right\rangle, \tag{3.5}$$

where μ_a $(a = 1, \cdots, 3g-3)$ are the Beltrami differentials associated with the moduli of the bulk, μ_a $(a = 3g-3+1, \cdots, 3g-3+h)$ are those associated with moduli of the positions of the boundaries, and λ_b $(b = 1, \cdots, h)$ are those associated with moduli of the length of the boundaries.

Our analysis below completes the derivation of [7], namely that under the assumption that open string moduli do not contribute to open string factorizations, for $2g-2+h > 0$, we have a set of open holomorphic anomaly equations,

$$\frac{\partial}{\partial \bar{t}^{\bar{i}}} F^{(g,h)} = \frac{1}{2} \overline{C}_{\bar{i}\bar{j}\bar{k}} e^{2K} G^{j\bar{j}} G^{k\bar{k}} \left(\sum_{g_1=0}^{g} \sum_{h_1=0}^{h} D_j F^{(g_1,h_1)} D_k F^{(g-g_1,h-h_1)} + D_j D_k F^{(g-1,h)} \right)$$
$$- e^K G^{j\bar{j}} \Delta_{\bar{i}\bar{j}} D_j F^{(g,h-1)}, \tag{3.6}$$

$$\frac{\partial}{\partial y^p} F^{(g,h)} = \frac{\partial}{\partial \bar{y}^{\bar{p}}} F^{(g,h)} = 0, \tag{3.7}$$

if and only if

$$\overline{C}_{\bar{p}} = \langle \bar{\omega}_{\bar{p}} | B \rangle = 0. \tag{3.8}$$

Here B is the boundary and $\overline{C}_{\bar{p}}$ is a disk one-point function, with $\bar{\omega}_{\bar{p}}$ an (a, a) chiral primary state with charges $q_{\bar{a}} + \bar{q}_{\bar{a}} = -3$. If the disk one-point functions do not vanish, then all three of these equations receive anomalous contributions, which cannot be written in terms of lower-genus amplitudes. Note that in the presence of orientifolds, $\overline{C}_{\bar{p}}$ is the sum of the disk and crosscap one-point functions [20].

3.3 New anomalies in topological string theory

In this section we will show how the new anomalies can enter (3.6) and (3.7) when the disk one-point functions do not vanish.

3.3.1 Physical meaning of the new anomalies

In quantum field theory, the vacuum expectation value of a field ϕ at tree level is ϕ_{cl}. The field is then expanded as,

$$\phi = \phi_{cl} + \eta, \tag{3.9}$$

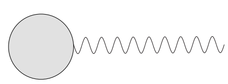

Figure 3.7: Tadpole

however the expectation value of ϕ may get quantum corrections through higher loops, i.e., $\langle \eta \rangle \neq 0$. This is called a tadpole contribution. In order for the effective action method to work, the effective action $\Gamma[\phi_{cl}]$ should not depend on the external current J. The effective action is defined as,

$$\Gamma[\phi_{cl}] = \int d^4x \mathcal{L}_{ren}[\phi_{cl}] + \frac{i}{2} \log \det \left[-\frac{\delta^2 \mathcal{L}_{ren}}{\delta\phi\delta\phi} \right] - i(\text{connect diagrams}) + \int d^4x \delta\mathcal{L}[\phi_{cl}], \tag{3.10}$$

where $\mathcal{L}_{ren}$ is the renormalized Lagrangian. It was shown that a counterterm coming from the current δJ will cancel the tadpole contribution. δJ is the difference between J, the one which satisfies the equation of motion, and J_{ren}, the one which satisfies the tree level equation of motion. Therefore we obtain a tadpole cancellation condition.

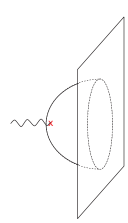

Figure 3.8: Disk one-point function

In open topological string theory, from target space point of view, the boundary conditions that preserve the scalar supercharge require open strings ending on D-branes wrapping on a Lagrangian 3-cycle or holomorphic even-cycle of a Calabi-Yau 3-fold. Here Lagrangian means that the Kähler form ω restricted to the 3-cycle is zero. [19] has proved that the disk one-point functions depend on wrong moduli.

3.3.2 Anomalous worldsheet degenerations

In this section we consider the dependence of the genus g and boundary number h amplitude $F^{(g,h)}$ on both the anti-holomorphic moduli $\bar{t}^{\bar{i}}$ and the "wrong" moduli, labelled y^p and $\bar{y}^{\bar{p}}$. Our results are independent of choosing the A- or B-model; wrong moduli are complex structure moduli for the A-model and Kähler structure moduli for the B-model. To derive the extended holomorphic anomaly equation we follow the approach of [1], with the addition of some important details. Taking the $\bar{t}^{\bar{i}}$ derivative of $F^{(g,h)}$ is equivalent to inserting the operator

$$\int_\Sigma \{G^+, [\overline{G}^+, \bar{\phi}_{\bar{i}}]\} \tag{3.11}$$

into the amplitudes with the integral going over the worldsheet Σ. $\bar{\phi}_{\bar{i}}$ is a state in the (a,a) chiral ring with left- and right-moving $U(1)_R$ charge $(-1,-1)$, which satisfies $[G^-, \bar{\phi}_{\bar{i}}] = 0$ and $[\overline{G}^-, \bar{\phi}_{\bar{i}}] = 0$. Here $[\overline{G}^+, \bar{\phi}_{\bar{i}}]$ means $\oint_{C_z} dw\ \overline{G}^+(w)\bar{\phi}_{\bar{i}}(z)$, with C_z a small contour surrounding z. In general the integrals of G^+ and $\overline{G}^+$ independently do not annihilate the boundary, so to derive the holomorphic anomaly we rewrite the insertion as

$$-\frac{1}{2}\int_\Sigma \{G^+ + \overline{G}^+, [G^+ - \overline{G}^+, \bar{\phi}_{\bar{i}}]\}, \tag{3.12}$$

allowing at least one contour to be deformed freely around the worldsheet.

For the other case, namely a dependence on wrong moduli, taking the y^p derivative of $F^{(g,h)}$ is equivalent to inserting

$$\int_\Sigma \{\overline{G}^+, [G^-, \varphi_p]\} + 2\int_{\partial\Sigma} \varphi_p, \tag{3.13}$$

where φ_p is a charge $(1,-1)$ marginal operator from the (c,a) ring, which satisfies $[G^+, \varphi_p] = 0$ and $[\overline{G}^-, \varphi_p] = 0$. The second term is a boundary term required to resolve the so-called Warner problem [24]: we require the deformation to be Q-exact, but the Q variation of the first term alone is a boundary term, as can be seen using $\{G^+, G^-\} = 2T$ and converting T to a total derivative. Since $\int_\Sigma \{G^+, [G^-, \varphi_p]\} =$

$2\int_{\partial\Sigma}\varphi_p$, we can rewrite (3.13) as

$$\int_\Sigma \{G^+ + \overline{G}^+, [G^-, \varphi_p]\}, \tag{3.14}$$

where again the first contour can be deformed past the boundaries on the worldsheet.

Thus, for both the $\bar{t}^{\bar{i}}$ and y^p derivatives, the combination $(G^+ + \overline{G}^+)$ can be moved around the Riemann surface, and will produce terms corresponding to all possible degenerations of the Riemann surface, as listed below. For each degeneration, there also remain the insertions

$$\bar{\phi}^{(1)}_{\bar{i}} \equiv -\frac{1}{2}\int_\Sigma [G^+ - \overline{G}^+, \bar{\phi}_{\bar{i}}], \tag{3.15}$$

$$\varphi^{(1)}_p \equiv \int_\Sigma [G^-, \varphi_p], \tag{3.16}$$

for $\bar{t}^{\bar{i}}$ and y^p dependence, respectively.

Moving the contour of the supercurrent $G^+ + \overline{G}^+$ around the Riemann surface, we pick up contributions from the commutation relations,

$$[G^+, G^-] = 2T, \qquad [\overline{G}^+, \overline{G}^-] = 2\overline{T}. \tag{3.17}$$

For the $\bar{t}^{\bar{i}}$ derivative we then get

$$\begin{aligned} \bar{\partial}_{\bar{t}^{\bar{i}}} F^{(g,h)} = - \int_{\mathcal{M}_{g,h}} [dm][dl] \Bigg[\sum_{c=1}^{3g-3+h} \Big\langle \bar{\phi}^{(1)}_{\bar{i}} \Big(2\int \mu_c T \int \bar{\mu}_c \overline{G}^- - 2\int \mu_c G^- \int \bar{\mu}_c \overline{T} \Big) \\ \times \prod_{a\neq c} \int \mu_a G^- \int \bar{\mu}_a \overline{G}^- \prod_{b=1}^{h} \int (\lambda_b G^- + \bar{\lambda}_b \overline{G}^-) \Big\rangle \\ + \sum_{c=1}^{h} \Big\langle \bar{\phi}^{(1)}_{\bar{i}} \int 2\left(\lambda_c T - \bar{\lambda}_c \overline{T}\right) \\ \times \prod_{a=1}^{3g-3+h} \int \mu_a G^- \int \bar{\mu}_a \overline{G}^- \prod_{b\neq c} \int (\lambda_b G^- + \bar{\lambda}_b \overline{G}^-) \Big\rangle \Bigg]. \end{aligned} \tag{3.18}$$

For the y^p derivative we just replace $\bar{\phi}^{(1)}_{\bar{i}}$ with $\varphi^{(1)}_p$. The insertions of the $\int \mu_a T$ and $\int \bar{\mu}_a \overline{T}$ can be converted into derivatives with respect to the moduli m^a and $\overline{m}^a$. By Cauchy's theorem, this reduces the integral over moduli space to contributions coming from the boundary of the moduli space, where we need to consider boundaries corresponding to degenerations of both the complex and real moduli, that is, both closed and open string degenerations. Equation (3.18) will then just sum the contributions from all the boundary components of moduli space.

To classify all boundary components of the moduli space of a Riemann surface with boundaries, a useful technique is to consider the degeneration, in turn, of all closed 1-cycles and open 1-paths with endpoints on (possibly distinct) boundaries. The various cases resulting from degenerations of closed 1-cycles were shown in Figures 3.1, 3.2, and 3.6, and of open 1-cycles in Figures 3.3, 3.4, and 3.5.

The first class of degenerations are *closed string factorizations*, corresponding to a closed 1-cycle degenerating, and either removing a handle (Figure 3.1), or splitting the Riemann surface in two (Figure 3.2). The remaining modulus of the long, thin tube created is represented by an integrated $(G^- - \overline{G}^-)$ insertion, folded with Beltrami differential. This insertion annihilates the ground states propagating on the long tube, so for a non-zero result the remaining insertion (3.15) or (3.16) must also be on the tube. Now the absence of boundaries on the tube makes the results of BCOV [1] directly applicable. The long tubes project to two sets of ground states associated to two end points $\sum_{j\bar{j}} |j\rangle g^{j\bar{j}} \langle \bar{j}|$ and $\sum_{k\bar{k}} |k\rangle g^{k\bar{k}} \langle \bar{k}|$, where $|j\rangle$ is the topological twist state, $|\bar{j}\rangle$ is the anti-topological twist state, and $g_{j\bar{j}} = \langle j|\bar{j}\rangle$ is the tt^* metric. By R-charge argument, the non-trivial contribution is from states in the (c, c) chiral ring with $U(1)_R$ charge $(1, 1)$. The y^p derivative contributions vanish, and for the $\bar{t}^{\bar{i}}$ derivative we get two terms. The first term corresponding to Figure 3.1 is

$$\frac{1}{2}\overline{C}_{\bar{i}\bar{j}\bar{k}} e^{2K} G^{j\bar{j}} G^{k\bar{k}} \int_{\mathcal{M}_{g-1,h}} [dm'] \Bigg\langle \int_{\Sigma_{g-1,h}} \{G^-, [\overline{G}^-, \phi_j]\} \int_{\Sigma_{g-1,h}} \{G^-, [\overline{G}^-, \phi_k]\} \\ \times \prod_{a=1}^{3g-6+h} \int \mu_a G^- \int \bar{\mu}_a \overline{G}^- \prod_{b=1}^{h} \int (\lambda_b G^- + \bar{\lambda}_b \overline{G}^-) \Bigg\rangle, \qquad (3.19)$$

where the set of moduli m' correspond to the remaining Riemann surfaces $\Sigma_{g-1,h}$. The overall factor 1/2 results from the $\mathbb{Z}_2$ symmetry $j \leftrightarrow k$, C_{ijk} is a three-point function, namely the Yukawa coupling, and $G_{i\bar{i}}$ is the Zamolodchikov metric. This expression can be further simplified. If we define

$$\phi_j^{(2)} = \int_{\Sigma_{g-1,h}} \{G^-, [\overline{G}^-, \phi_j]\}, \tag{3.20}$$

then the insertions $\phi_j^{(2)}$ and $\phi_k^{(2)}$ can be replaced by covariant derivatives D_j and D_k of the amplitude without insertions, namely of $F_{g-1,h}$. Therefore we get

$$\frac{1}{2}\overline{C}_{\bar{i}\bar{j}\bar{k}} e^{2K} G^{\bar{j}j} G^{\bar{k}k} D_j D_k F^{(g-1,h)}. \tag{3.21}$$

The second term corresponding to Figure 3.2 is

$$\begin{aligned}
&\frac{1}{2}\overline{C}_{\bar{i}\bar{j}\bar{k}} e^{2K} G^{j\bar{j}} G^{k\bar{k}} \sum_{r=0}^{g}\sum_{s=0}^{h} \int_{\mathcal{M}_{r,s}} [dm'] \left\langle \phi_j^{(2)} \prod_{a=1}^{3r-3+s} \left| \int \mu_a G^- \right|^2 \prod_{b=1}^{s} \int (\lambda_b G^- + \bar{\lambda}_b \overline{G}^-) \right\rangle \\
&\qquad \times \int_{\mathcal{M}_{g-r,h-s}} [dm''] \left\langle \phi_k^{(2)} \prod_{a=1}^{3(g-r)-3+h-s} \left| \int \mu_a G^- \right|^2 \prod_{b=1}^{h-s} \int (\lambda_b G^- + \bar{\lambda}_b \overline{G}^-) \right\rangle \\
&= \frac{1}{2}\overline{C}_{\bar{i}\bar{j}\bar{k}} e^{2K} G^{j\bar{j}} G^{k\bar{k}} \sum_{r=0}^{g}\sum_{s=0}^{h} D_j F^{(r,s)} D_k F^{(g-r,h-s)},
\end{aligned} \tag{3.22}$$

where the sets of moduli m' and m'' correspond to the remaining moduli on each of the daughter surfaces. The overall factor 1/2 results from the $\mathbb{Z}_2$ symmetry of the sum generated by simultaneously taking $r \to (g-r)$, $s \to (h-s)$, and $j \leftrightarrow k$. It is worth recalling that $D_j F^{(g,h)} = 0$ for $3g + h < 2$, so there is no contribution for sufficiently trivial daughter surfaces.

Next are *open string factorizations*, where a boundary expands and meets itself (Figure 3.3); a handle is removed (Figure 3.4); or two boundaries collide (Figure 3.5). The degeneration produces a thin strip, with each end encircled by a G^- or $\overline{G}^-$ folded with a Beltrami differential, associated with the position of the attachment of the strip to the boundary. The strip can be replaced by a complete set of open string states. However, our assumption that open string moduli do not contribute removes all but charge 0 and 3 states, and these are annihilated by the G^- or $\overline{G}^-$ integrated around the attachment point, regardless of the location of the insertion (3.15) and (3.16). Thus the open string factorizations give gives no contribution.

The last interesting case is that of a *boundary shrinking*, or equivalently moving far from the rest of the Riemann surface (Figure 3.6). That such a degeneration is part of the boundary of moduli space can be seen by doubling the Riemann surface $\Sigma_{g,h}$ to form the closed surface $\Sigma'_{2g+h-1,0}$, as described in Section 3.2.2. The pinching off of a $\Sigma'_{2g+h-1,0}$ handle which crosses the $\mathbb{Z}_2$ fixed plane is equivalent to a shrinking boundary in $\Sigma_{g,h}$.

This boundary is associated with three real moduli insertions, specifying the location of the boundary and its length τ. The boundary degeneration is thus equivalent to a boundary at the end of a long tube, with the Beltrami differentials associated with the two remaining moduli localized to the attachment point of the tube to the rest of the Riemann surface. The absence of additional moduli on the tube distinguishes this class from the closed string factorization class above, and furthermore allows the remaining insertion (3.15) and (3.16) to be anywhere on the worldsheet.

Firstly, insertions (3.15) and (3.16) may be on the tube. The degeneration $\tau \to \infty$ projects the intermediate states on both sides of the insertion to ground states, since excited states decay as $e^{-h\tau}$ where $h > 0$ is the total (left+right) conformal weight. Now, however, $G^\pm$ and $\overline{G}^\pm$ annihilate the ground states, so this case is zero.

Secondly, the insertions may be near the shrinking boundary. The tube pinching off in the middle gives rise to a disk $\Sigma_{0,1}$ and the remaining Riemann surface $\Sigma_{g,h-1}$. This will project to one set of ground states $\sum_{j\bar{j}} |j\rangle g^{j\bar{j}} \langle \bar{j}|$ on the pinching off point. For the $\bar{t}^{\bar{i}}$ derivative, the disk part gives us

$$-\frac{1}{2} e^K G^{j\bar{j}} \langle \bar{j}| \int_{\Sigma_{0,1}} [G^+ - \overline{G}^+, \bar{\phi}_{\bar{i}}] |B\rangle. \qquad (3.23)$$

Define the anti-topological disk two-point function as

$$\Delta_{\bar{i}\bar{j}} = \frac{1}{2}\langle\bar{j}|\int_{\Sigma_{0,1}}[G^+ - \overline{G}^+, \bar{\phi}_{\bar{i}}]|B\rangle. \tag{3.24}$$

The remaining Riemann surface $\Sigma_{g,h-1}$ then has a $\phi_j^{(2)}$ insertion, which produces $D_j F^{(g,h-1)}$. Multiplying the two contributions, we obtain

$$-e^K G^{j\bar{j}} \Delta_{\bar{i}\bar{j}} D_j F^{(g,h-1)}. \tag{3.25}$$

For the y^p derivative, the near-boundary region is shown in Figure 3.9. We can replace the tube with a complete set of closed-string ground states, $\sum_{I,\bar{J}} |I\rangle g^{I\bar{J}} \langle\bar{J}|$, where $g^{I\bar{J}}$ is the tt^* metric, and I, $\bar{J}$ run over all (c,c) and (a,a) chiral primary states, respectively. Standard considerations of global consistency on the Riemann surface force $|I\rangle = |i\rangle$ to be a charge $(1,1)$ (i.e., marginal) state, and $\langle\bar{J}| = \langle\bar{j}|$ to be a charge $(-1,-1)$ state from the (a,a) chiral ring. Near the boundary the theory is anti-topologically twisted, making G^- and $\overline{G}^-$ of dimension 1 as supercurrents, and so allowing contour deformation. Using the properties of the chiral rings, (3.16) can be written as $\int_\Sigma [G^- + \overline{G}^-, \varphi_p]$. The contour of $(G^- + \overline{G}^-)$ can be deformed off the disk, annihilating both $\langle\bar{j}|$ and the boundary, so this case is zero.

Lastly, (3.15) and (3.16) may be inserted somewhere else on the Riemann surface, as shown in Figures 3.10 and 3.11 for the y^p and $\bar{t}^{\bar{i}}$ derivatives, respectively. The tube is again replaced with a complete set of ground states $\sum_{I,\bar{J}} |I\rangle g^{I\bar{J}} \langle\bar{J}|$. To avoid annihilation by G^- and $\overline{G}^-$ localized to the tube attachment point, $|I\rangle$ must be in the (c,c) chiral ring and have $q_I, \bar{q}_I \neq 0$. Furthermore, both of the insertions (3.15) and (3.16) are (linear combinations of) states with $(0,-1)$ or $(-1,0)$ left- and right-moving $U(1)_R$ charge, and the tube end-point moduli contribute charge $(-1,-1)$, so $|I\rangle$ is required to be a (linear combination of) charge $(1,2)$ or $(2,1)$ states. We denote these states ω_p, with index p running over charge $(1,2)$ and $(2,1)$ chiral primaries. Note that the ω_p are not associated with marginal deformations of the topological string in question, but with deformations of its mirror. In the A-model, they correspond to

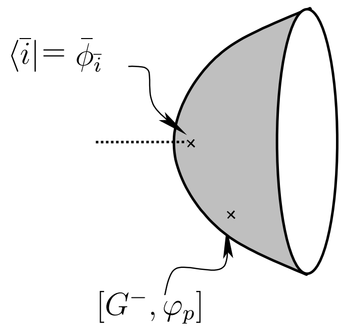

Figure 3.9: The near-boundary region of the shrinking boundary degeneration for y^p derivative, with insertion (3.16) near the boundary. This amplitude vanishes, as described in the text.

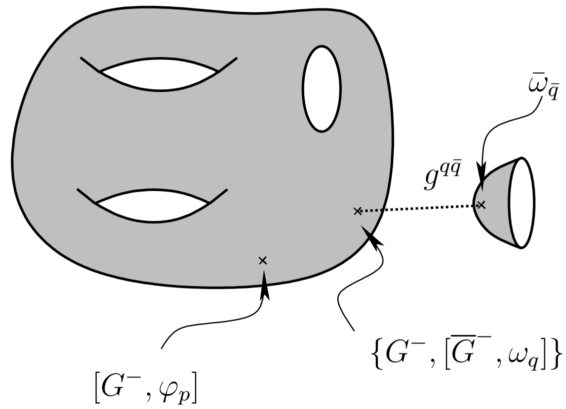

Figure 3.10: Amplitude for the shrinking boundary degeneration for y^p derivative, with insertion (3.16) elsewhere on the Riemann surface. This is non-zero unless the disk one-point function vanishes.

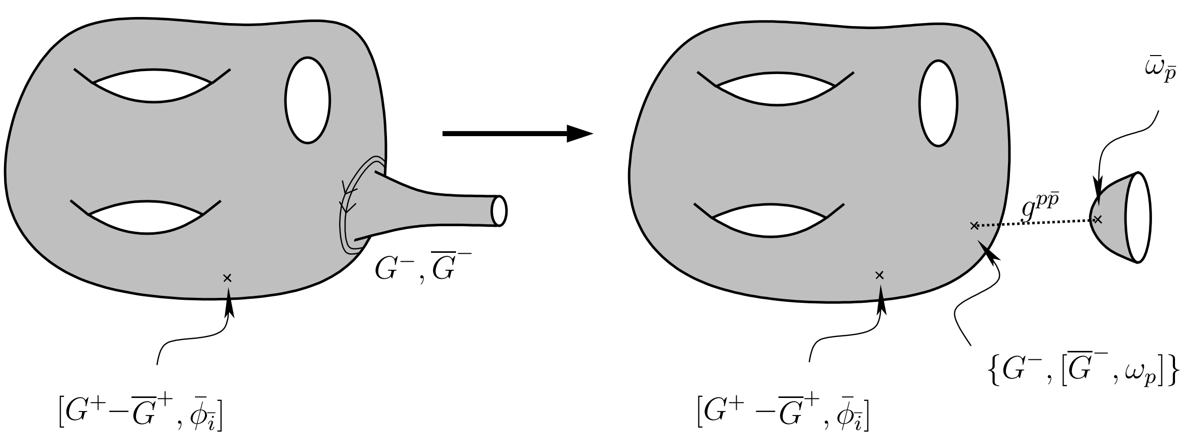

Figure 3.11: Amplitude for the shrinking boundary degenerating for $\bar{t}^{\bar{i}}$ derivative, with the insertion (3.15) located away from the shrinking boundary. On the right we have replaced the tube with a sum over states ω_a of charge $(1,2)$ and $(2,1)$, rendering the near-boundary region a disk one-point function.

target space 3-forms, and hence to complex structure variation, and in the B-model they are $(1,1)$ forms, and so correspond to Kähler deformations. Near the shrinking boundary the resulting amplitude is the disk one-point function,

$$\overline{C}_{\bar{p}} = \langle \bar{\omega}_{\bar{p}} | B \rangle. \tag{3.26}$$

This amplitude is in general not zero, and indeed our boundary conditions are such that the disk one-point function is only non-zero when the closed string state is from the wrong model [19]. This case thus contributes the following new terms to (3.6) and (3.7): for the derivative with respect to $\bar{t}^{\bar{i}}$,

$$g^{\bar{p}q}\overline{C}_{\bar{p}} \int_{\mathcal{M}_{g,h-1}} [dm'] \left\langle \int_{\Sigma_{g,h-1}} \{G^-, [\overline{G}^-, \omega_q]\} \int_{\Sigma_{g,h-1}} [G^+ - \overline{G}^+, \bar{\phi}_{\bar{i}}] \right\rangle_{\Sigma_{g,h-1}}, \tag{3.27}$$

and for the derivative with respect to y^p,

$$g^{\bar{p}q}\overline{C}_{\bar{p}} \int_{\mathcal{M}_{g,h-1}} [dm'] \left\langle \int_{\Sigma_{g,h-1}} \{G^-, [\overline{G}^-, \omega_q]\} \int_{\Sigma_{g,h-1}} [G^-, \varphi_p] \right\rangle_{\Sigma_{g,h-1}}, \tag{3.28}$$

where the m's are the moduli of the Riemann surface $\Sigma_{g,h-1}$—the corresponding insertions of G^- and $\overline{G}^-$ folded with Beltrami differentials have been suppressed. Note that the G^- and $\overline{G}^-$ contours around ω_q and φ_p cannot be deformed as they are dimension 2 as supercurrents and that the $(G^+ - \overline{G}^+)$ contour around $\bar{\phi}_{\bar{i}}$ cannot be deformed as it does not annihilate any additional boundaries that may be present.

3.4 Small number of moduli

We have so far discussed the holomorphic anomaly equation for $2g - 2 + h > 0$. For low-genus and low-boundary cases, that is $2g - 2 + h \leq 0$, $F^{(1,0)}$ does not have new anomalies [1], while $F^{(0,2)}$ has new anomalies similar to those discussed above.

$F^{(0,0)}$ and $F^{(0,1)}$ are sphere and disk amplitudes, but since neither the sphere nor the disk has a moduli space, the previous discussion does not apply to them.

3.4.1 Cylinder $\Sigma_{0,2}$

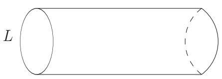

Figure 3.12: Cylinder

The open string 1-loop partition function is denoted as $F^{(0,2)}$. Quantum mechanically, if an open string state $|\alpha\rangle$ evolves to a state $|\beta\rangle = e^{-iHt}|\alpha\rangle$, the amplitude from $|\alpha\rangle$ to $|\beta\rangle$ is $\langle\beta|\alpha\rangle$. After applying a wick rotation along the time direction, we obtain a partition function $Tre^{-HL}|\alpha\rangle\langle\alpha|$. In terms of a sigma-model on a Calabi-Yau 3-fold, we can compute this partition function. Since there is only one real modulus and one isometry associated to the circular direction, we can write down the amplitude

$$F^{(0,2)} = \int \frac{dL}{L} Tr[(-1)^F F e^{-LH}], \tag{3.29}$$

where F is the $U(1)_R$ current, L is the modulus of the cylinder and H is the Hamiltonian for the string. The derivative with respect to a modulus of the Calabi-Yau is

$$\frac{\partial}{\partial t^j} F^{(0,2)} = \int \frac{dL}{L} \left\langle \int_{\Sigma_{0,2}} \{G^-, [\overline{G}^-, \phi_j]\} F \right\rangle. \tag{3.30}$$

The holomorphic anomaly equation is

$$\frac{\partial}{\partial \bar{t}^{\bar{i}}} \frac{\partial}{\partial t^j} F^{(0,2)} = \int_0^1 \frac{dL}{L} \left\langle \int_{\Sigma_{0,2}} \{G^+, [\overline{G}^+, \bar{\phi}_{\bar{i}}]\} \int_{\Sigma_{0,2}} \left\{ (G^- + \overline{G}^-), \phi_j^{(1)} \right\} F \right\rangle, \tag{3.31}$$

where $\phi_j^{(1)}$ is a 1-form and $\bar{\phi}_{\bar{i}}$ is a $(1,1)$ form on the cylinder. We will then use commutation relations,

$$[F, G^-] = G^-, \qquad [F, \overline{G}^-] = \overline{G}^-, \qquad [F, \phi_j] = 0, \tag{3.32}$$

to get that the relevant insertions on the degenerate Riemann surfaces are $\bar{\phi}_{\bar{i}}^{[1]}$ and $\phi_j^{(1)}$, where $\bar{\phi}_{\bar{i}}^{[1]}$ is a $U(1)_R$ charge -1 operator and $\phi_j^{(1)}$ is a charge 1 operator. Let's consider three types of degenerations.

i) Two inserted operators colliding (Figure 3.13). The operator product expansion of these two operators is zero, so this degeneration has zero contribution.

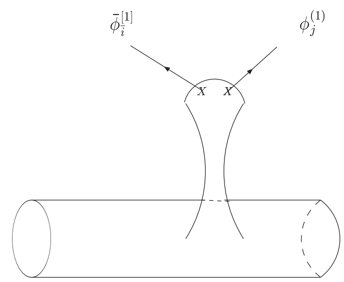

Figure 3.13: Blow up of the colliding of two operators

ii) Since a cylinder is conformally equivalent to an annulus, there is another kind of degeneration corresponding to two boundaries of the annulus colliding (Figure 3.14) and producing a long and narrow strip. The only non-trivial degeneration is when $\bar{\phi}_{\bar{i}}^{[1]}$ and $\phi_j^{(1)}$ are both away from the strip, and we project the strip to open string ground states. By the charge argument and also the assumption of a non-trivial open string ground state, we can just insert open string ground states with charges 0 and 3. It was discovered [1] that

$$\left\langle \mathcal{O}_\alpha \int_C \phi_j^{(1)} \int_{\Sigma_{0,1}} \bar{\phi}_{\bar{i}}^{[1]} \mathcal{O}_\beta \right\rangle = -R_{i\bar{j}\alpha\beta}. \tag{3.33}$$

where $\mathcal{O}$ denotes a $U(1)$ charge 0 open string ground state, and D is the degeneracy of the states. Since $\phi_j^{(1)}$ is a one-form on the disk, it must be integrated along a path C.

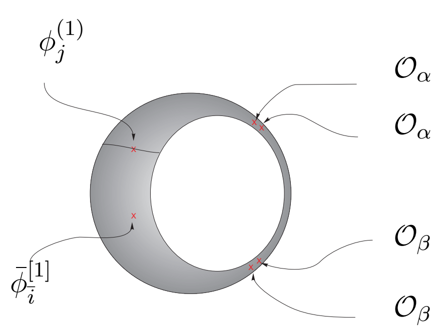

Figure 3.14: Boundary colliding

iii) Cylinder splitting. If $\bar{\phi}_{\bar{i}}^{[1]}$ and $\phi_j^{(1)}$ are inserted at opposite ends of this degenerating cylinder (Figure 3.15), by the charge argument, we have to project to ground

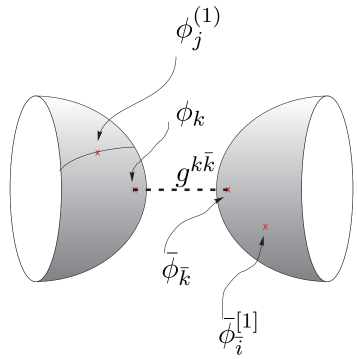

Figure 3.15: Disk two-point functions

states which stay in the (c,c) or (a,a) rings with charge $(1,1)$ or $(-1,-1)$. Recall the definition of disk two-point function (3.24), we get

$$\left\langle \int_C \phi_j^{(1)} \phi_k \right\rangle g^{k\bar{k}} \left\langle \int_{\Sigma_{0,1}} \bar{\phi}_{\bar{i}}^{[1]} \bar{\phi}_{\bar{k}} \right\rangle = \Delta_{jk} \Delta_{\bar{i}\bar{k}} e^K G^{k\bar{k}}. \tag{3.34}$$

If $\bar{\phi}_{\bar{i}}^{[1]}$ and $\phi_j^{(1)}$ are inserted on the same side (Figure 3.16), then by the charge argument, we have to project to ground states ω_p which stay in the (c,c) ring of charge $(1,2)$ and $(2,1)$. The three-point function on the disk is

$$\left\langle \int_C \phi_j^{(1)} \int_{\Sigma_{0,1}} \bar{\phi}_{\bar{i}}^{[1]} \omega_p \right\rangle . \tag{3.35}$$

If the disk one-point function does not vanish, we obtain a wrong moduli dependent term

$$g^{\bar{p}p} \overline{C}_{\bar{p}} \left\langle \int_C \phi_j^{(1)} \int_{\Sigma_{0,1}} \bar{\phi}_{\bar{i}}^{[1]} \omega_p \right\rangle , \tag{3.36}$$

where $\overline{C}_{\bar{p}}$ is defined as (3.26). When the disk one-point functions vanish, we get the holomorphic anomaly equation for a cylinder,

$$\frac{\partial}{\partial \bar{t}^{\bar{i}}} \frac{\partial}{\partial t^j} F_{0,2} = e^K G^{i\bar{j}} \Delta_{\bar{i}\bar{j}} \Delta_{ij} + \frac{D}{2} G_{\bar{i}j}. \tag{3.37}$$

Since $\phi_i^{(1)}$, $\varphi_p^{(1)}$; $\bar{\phi}_{\bar{i}}^{(1)}$, $\bar{\varphi}_{\bar{p}}^{(1)}$ have charges 1 and -1, respectively, the requirement of charge 0 will give us 3 cases for which disk-one point functions may contribute:

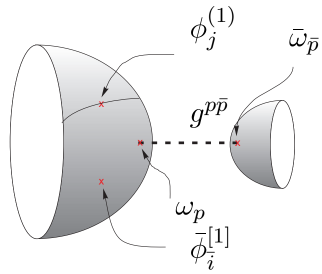

Figure 3.16: Disk one-point function

$\partial_{\bar{i}}\partial_p F^{(0,2)}$, $\partial_i\partial_{\bar{p}} F^{(0,2)}$, and $\partial_{\bar{p}}\partial_q F^{(0,2)}$. Correspondingly, the insertions are $\bar{\phi}_{\bar{i}}^{[1]}\varphi_p^{(1)}$, $\phi_i^{(1)}\bar{\varphi}_{\bar{p}}^{[1]}$, and $\bar{\varphi}_{\bar{p}}^{[1]}\varphi_q^{(1)}$. We can repeat the same discussion and find that the only contribution comes from the disk one-point functions, and therefore the vanishing of the disk one-point function makes them all zero.

3.4.2 Some discussion

The derivation above may not seem to distinguish between compact and non-compact Calabi-Yau target spaces. In fact, the anomalies can only appear in the compact case. Beforehand, note that this agrees with our expectations: D-branes wrapped on cycles in compact Calabi-Yau manifolds and filling spacetime (or perhaps even two directions in spacetime [20]) give an inconsistent setup unless there are sinks for the topological D-brane charges. Simultaneously, these sinks cancel the disk one-point functions, and so the appearance of the new anomalies is correlated with an invalid spacetime construction.

Furthermore, the standard results of Chern-Simons gauge theory and matrix models as open topological string theories are not affected by the new anomalies. For example, N D-branes wrapping the S^3 of the space T^*S^3, gives $\overline{C}_{\bar{p}} \neq 0$. The total space of T^*S^3 is Calabi-Yau and non-compact, with the S^3 radius as the complex structure modulus. It is well-known that open topological string theory on this space is the $U(N)$ Chern-Simons theory, which is topological and should be independent of the S^3 radius. To resolve this apparent contradiction, consider embedding T^*S^3 in a compact space containing a second 3-cycle in the same homology class as the base S^3, wrapped by N anti-D-branes. The boundary states of the two stacks combine to give $\overline{C}_{\bar{p}} = 0$, and the new anomalies do not appear. Now take the limit where the second 3-cycle moves infinitely far away from the base S^3 to recover an anomaly-free local Calabi-Yau construction. The point is that in non-compact Calabi-Yau manifolds, the new anomalies can be removed by an appropriate choice of boundary conditions at infinity.

3.5 Feynman rules to solve the holomorphic anomaly equation

The series of open holomorphic anomaly equations without new anomalies are

$$\begin{aligned}\frac{\partial}{\partial \bar{t}^{\bar{i}}} F^{(g,h)} &= \frac{1}{2}\overline{C}_{\bar{i}\bar{j}\bar{k}} e^{2K} G^{j\bar{j}} G^{k\bar{k}} \left(\sum_{r=0}^{g} \sum_{s=0}^{h} D_j F^{(r,s)} D_k F^{(g-r,h-s)} + D_j D_k F^{(g-1,h)} \right) \\ &\quad - e^K G^{j\bar{j}} \Delta_{\bar{i}\bar{j}} D_j F^{(g,h-1)}, \end{aligned} \tag{3.38}$$

$$\frac{\partial}{\partial t^i}\frac{\partial}{\partial \bar{t}^{\bar{j}}} F^{(1,0)} = \frac{1}{2} C_{ik\ell} \overline{C}_{\bar{j}\bar{k}\bar{\ell}} e^{2K} G^{k\bar{k}} G^{\ell\bar{\ell}} - (\frac{\chi}{24} - 1) G_{i\bar{j}}, \tag{3.39}$$

$$\frac{\partial}{\partial \bar{t}^{\bar{i}}}\frac{\partial}{\partial t^j} F^{(0,2)} = e^K G^{i\bar{j}} \Delta_{\bar{i}\bar{j}} \Delta_{ij} + \frac{N}{2} G_{\bar{i}j}. \tag{3.40}$$

Now we discuss the solution to these holomorphic anomaly equations. The moduli space of a Calabi-Yau 3-fold enjoys Kähler geometry. There exists a line bundle $\mathcal{L}$ over it corresponding to rescalings of the Kähler potential K. The $F^{(g,h)}$'s are sections of this line bundle $\mathcal{L}^{2-2g-h}$, so the covariant derivatives on this line bundle are defined as

$$D_i = \partial_i - (2 - 2g - h)\partial_i K. \tag{3.41}$$

In order to solve the above holomorphic anomaly equations, we will use Feynman rules. Firstly, we define the propagators [1] $-S$, $-S^j$, and $-S^{jk}$, where

$$S, \qquad \text{such that} \quad \overline{C}_{\bar{i}\bar{j}\bar{k}} = e^{-2K} D_{\bar{i}} D_{\bar{j}} \bar{\partial}_{\bar{k}} S, \tag{3.42}$$

$$S^j = G^{j\bar{j}} S_{\bar{j}}, \qquad \text{where} \quad S_{\bar{j}} = \bar{\partial}_{\bar{j}} S, \tag{3.43}$$

$$S^{jk} = G^{j\bar{j}} S^k_{\bar{j}}, \qquad \text{where} \quad S^k_{\bar{j}} = \bar{\partial}_{\bar{j}} S^k, \tag{3.44}$$

and the terminators [7]

$$\Delta, \qquad \text{such that} \quad \Delta_{\bar{i}\bar{j}} = e^{-K} D_{\bar{i}} \partial_{\bar{j}} \Delta, \tag{3.45}$$

$$\Delta^j = G^{j\bar{j}} \partial_{\bar{j}} \Delta. \tag{3.46}$$

Feynman diagrams for those propagators and terminators are given in Figure 3.17.

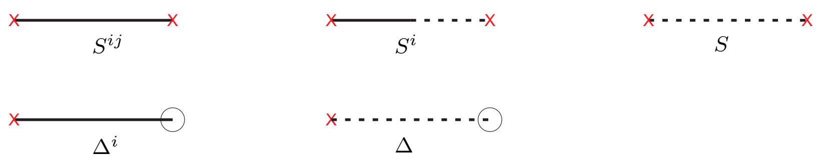

Figure 3.17: The propagators for Feynman diagrams of topological string amplitudes

The low-genus and boundary cases, $F^{(2,0)}$, $F^{(3,0)}$, $F^{(1,1)}$, and $F^{(0,3)}$, were studied in [1] and [7]. For example, for genus 2, the amplitudes is determined up to a holomorphic ambiguity,

$$\begin{aligned} F^{(2,0)} \quad = \quad & \frac{1}{2} S^{ij} C^{(1)}_{ij} + \frac{1}{2} C^{(1)}_i S^{ij} C^{(1)}_j - \frac{1}{8} S^{jk} S^{mn} C_{jkmn} - \frac{1}{2} S^{ij} C_{ijm} S^{mn} C^{(1)}_n \\ + \quad & \frac{1}{8} S^{ij} C_{ijp} S^{pq} C_{qmn} S^{mn} + \frac{\chi}{24} S^i C^{(1)}_i + \frac{1}{12} S^{ij} S^{pq} S^{mn} C_{ipm} C_{jqn} - \frac{\chi}{48} S^i C_{ijk} S^{jk} \\ & + \frac{\chi}{24}\left(\frac{\chi}{24} - 1\right) S + \text{hol. amb.}, \end{aligned} \tag{3.47}$$

where

$$C^{(g)}_{i_1 \cdots i_n} \equiv D_{i_1} \cdots D_{i_n} F^{(g,0)}, \tag{3.48}$$

$$C^{(1)}_{\varphi} \equiv \frac{\chi}{24} - 1, \tag{3.49}$$

and the Yukawa coupling C_{ijk} can be interpreted as the building blocks of the Feynman diagrams (Figure 3.18).

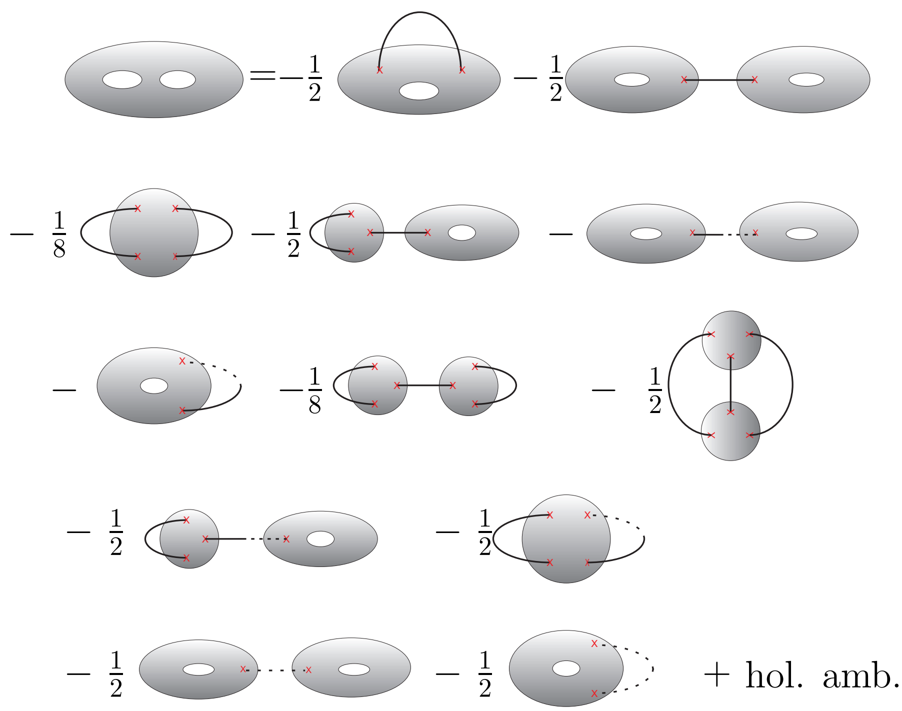

Figure 3.18: Feynman diagrams for $F^{(2,0)}$

For genus 1 with 1 boundary, the amplitude is

$$F^{(1,1)} = \frac{1}{2}S^{jk}\Delta jk - C_j^{(1)}\Delta^j + \frac{1}{2}C_{jkl}S^{kl}\Delta^j - (\frac{\chi}{24} - 1)\Delta + \text{hol. amb.}. \tag{3.50}$$

We can draw Feynman diagrams as Figure 3.19.

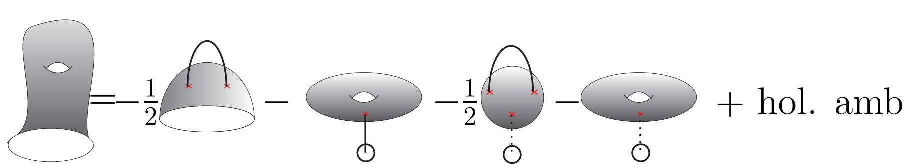

Figure 3.19: Feynman diagrams for $F^{(1,1)}$

Chapter 4 The Relation between the Open and Closed Topological String

In this chapter we show that a general solution to the extended holomorphic anomaly equations for the open topological string on D-branes in a Calabi-Yau manifold, recently written down by Walcher [7], is obtained from the BCOV solution to the holomorphic anomaly equations for the closed topological string on the same manifold [1], by shifting the closed string moduli by amounts proportional to the 't Hooft coupling [9].

4.1 Generating function

In Chapter 3, we derived the open holomorphic anomaly equations. The extended equation for the string amplitudes with closed-string operator insertions [7] is

$$\begin{aligned}\bar{\partial}_{\bar{i}} F^{(g,h)}_{i_1,\cdots,i_n} &= \frac{1}{2}\sum_{\substack{g_1+g_2=g\\ h_1+h_2=h}} \overline{C}_{\bar{i}}^{jk} \sum_{s,\sigma} \frac{1}{s!(n-s)!} F^{(g_1,h_1)}_{ji_{\sigma(1)},\cdots,i_{\sigma(s)}} F^{(g_2,h_2)}_{ki_{\sigma(s+1)},\cdots,i_{\sigma(n)}} + \frac{1}{2}\overline{C}_{\bar{i}}^{jk} F^{(g-1,h)}_{jki_1,\cdots,i_n} \\ &\quad -\Delta_{\bar{i}}^{j} F^{(g,h-1)}_{ji_1,\cdots,i_n} - (2g-2+h+n-1)\sum_{s=1}^{n} G_{i_s\bar{i}} F^{(g,h)}_{i_1,\cdots,i_{s-1},i_{s+1},\cdots,i_n}. \end{aligned} \tag{4.1}$$

The last term comes from the collision of two closed-string marginal operators [1], since

$$G_{i\bar{j}} = \langle \phi_i^{(2)} \bar{\phi}_{\bar{j}}^{(2)} \rangle_{\Sigma_{0,0}}, \tag{4.2}$$

where

$$\phi_j^{(2)} = \int_{\Sigma_{g,h}} \{G^-, [\overline{G}^-, \phi_j]\}. \tag{4.3}$$

This equation is valid for $2g-2+h+n>0$, except for $F_i^{(1,0)}$ and $F_i^{(0,2)}$ for which there are additional terms in the equation, which we will take into account below.

The ingredients $F^{(g,h)}_{i_1,\ldots,i_n}$ are topological string amplitudes with worldsheet genus g, h boundaries, and n insertions of closed-string marginal operators indexed by $i_1, \cdots, i_n$; also $\overline{C}_{\bar{i}}^{jk} = \overline{C}_{\bar{i}\bar{j}\bar{k}} e^{2K} G^{j\bar{j}} G^{k\bar{k}}$, where $\overline{C}_{\bar{i}\bar{j}\bar{k}}$ is the Yukawa coupling, indices are raised and lowered using the Zamolodchikov metric $G_{i\bar{j}} = \partial_i \partial_{\bar{j}} K$, and $\Delta_{\bar{i}}^{j} = e^K G^{j\bar{k}} \Delta_{\bar{i}\bar{k}}$, where $\Delta_{\bar{i}\bar{k}}$ is the disk two-point function. Note that these are different from the Δ (with or without indices) that appear in BCOV, which we will denote as $\hat{\Delta}$ below.

Following BCOV, we define the generating function for open topological string amplitudes,

$$W(x,\varphi;t,\bar{t}) = \sum_{g,h,n} \frac{1}{n!} g_s^{2g-2} \lambda^h F^{(g,h)}_{i_1,\cdots,i_n} x^{i_1} \cdots x^{i_n} \left(\frac{1}{1-\varphi}\right)^{2g-2+h+n} + \left(\frac{\chi}{24} - 1 - \frac{D}{2} g_s^{-2} \lambda^2\right) \log\left(\frac{1}{1-\varphi}\right), \tag{4.4}$$

where the sum is over $g, h, n \geq 0$ such that $(2g - 2 + h + n) > 0$, g_s is the topological string coupling constant, and λ is the 't Hooft coupling constant, namely g_s times the topological string Chan-Paton factor. In the last term on the right, χ is the Euler characteristic of the Calabi-Yau manifold and D is the number of open-string ground states with zero charge. This term contributes to the holomorphic anomaly equations for $F_i^{(1,0)}$ and $F_i^{(0,2)}$, reproducing [2]

$$\frac{\partial}{\partial t^i} \frac{\partial}{\partial \bar{t}^{\bar{j}}} F^{(1,0)} = \frac{1}{2} C_{ikl} \overline{C}_{\bar{j}\bar{k}\bar{l}} e^{2K} G^{k\bar{k}} G^{l\bar{l}} - \left(\frac{\chi}{24} - 1\right) G_{i\bar{j}} \tag{4.5}$$

and [7]

$$\frac{\partial}{\partial t^i} \frac{\partial}{\partial \bar{t}^{\bar{j}}} F^{(1,0)} = \frac{1}{2} \Delta_{ik} \Delta_{\bar{j}\bar{k}} e^K G^{k\bar{k}} - \frac{D}{2} G_{i\bar{j}}. \tag{4.6}$$

We will show this in the Appendix A.1. The generating function W satisfies an extension of Equation (6.11) in BCOV by a λ-dependent term, namely

$$\frac{\partial}{\partial \bar{t}^{\bar{i}}} e^{W(x,\varphi;t,\bar{t})} = \left(\frac{g_s^2}{2} \overline{C}_{\bar{i}}^{jk} \frac{\partial^2}{\partial x^j \partial x^k} - G_{\bar{i}j} x^j \frac{\partial}{\partial \varphi} - \lambda \Delta_{\bar{i}}^{j} \frac{\partial}{\partial x^j}\right) e^{W(x,\varphi;t,\bar{t})}, \tag{4.7}$$

which reproduces the open topological string holomorphic anomaly Equation (4.1) for each genus and boundary number.

Our key result is that Equation (4.7) can be rewritten in the same form as the closed topological string analogue by simply shifting

$$x^i \to x^i + \lambda\Delta^i, \quad \varphi \to \varphi + \lambda\Delta, \tag{4.8}$$

where Δ^i and Δ are defined implicitly, modulo holomorphic ambiguities, by $\Delta_{\bar{i}\bar{j}} = e^{-K}G_{\bar{j}k}\partial_{\bar{i}}\Delta^k = e^{-K}D_{\bar{i}}D_{\bar{j}}\Delta$. After this shift Equation (4.7) becomes,

$$\frac{\partial}{\partial \bar{t}^{\bar{i}}} e^{W(x+\lambda\Delta,\varphi+\lambda\Delta;t,\bar{t})} = \left(\frac{g_s^2}{2}\overline{C}_{\bar{i}}^{jk}\frac{\partial^2}{\partial x^j \partial x^k} - G_{\bar{i}j}x^j\frac{\partial}{\partial\varphi}\right)e^{W(x+\lambda\Delta,\varphi+\lambda\Delta;t,\bar{t})}. \tag{4.9}$$

This is exactly the same as the original BCOV equation for the closed topological string generating function, with the μ-dependent term absorbed by means of the shift (4.8).

Our result follows from a straightforward application of the chain rule: noting that $\bar{\partial}_{\bar{i}}\Delta^j = \Delta_{\bar{i}}^j$, the variable shift produces two new terms on the left,

$$\left(\lambda\Delta_{\bar{i}}^j\frac{\partial}{\partial x^j} + \lambda\Delta_{\bar{i}}\frac{\partial}{\partial\varphi}\right)e^W. \tag{4.10}$$

The first is the additional μ-dependent term on the right of (4.7). Using $G_{\bar{i}j}\Delta^j = \Delta_{\bar{i}}$, the second term combines with the second term on the right of (4.9) to give $-G_{\bar{i}j}(x^j + \lambda\Delta^j)\frac{\partial}{\partial\varphi}e^W$, which is required for matching powers of $x + \lambda\Delta$ in the expansion of the generating function. Thus we have reproduced the open topological string holomorphic anomaly equations from the closed topological string holomorphic anomaly equations, simply by a shift of variables.

An immediate consequence of this is a general proof of the Feynman rule method of solving the open topological string anomaly equations appearing in Section 2.10 of Walcher. Since our shifted W satisfies the closed-string differential Equation (4.9), the proof of the closed-string Feynman rules presented in Section 6.2 of BCOV applies immediately. The shift has, in fact, an elegant interpretation in terms of the Feynman rules. Equation (6.12) in BCOV defines the function

$$Y(x,\varphi;t,\bar{t}) = -\frac{1}{2g_s^2}(\hat{\Delta}_{ij}x^i x^j + 2\hat{\Delta}_{i\varphi}x^i\varphi + \hat{\Delta}_{\varphi\varphi}\varphi^2) + \frac{1}{2}\log\left(\frac{\det\hat{\Delta}}{g_s^2}\right), \tag{4.11}$$

where the $\hat{\Delta}_{ij}$ are the inverses of the corresponding propagators S^{ij}. Expanding $Z = \int dx d\varphi \exp(Y + W)$ in powers of g_s then produces the full Feynman diagram expansion of the closed topological string amplitudes. The shift (4.8) produces the additional terms appearing in the open string Feynman diagrams, shown in Section 2.10 of Walcher. Therefore the Feynman rules method in Section 3.5 is generalized to solve the open holomorphic anomaly equations. In field theory language, the shift effectively generates the vacuum expectation values $\langle x^i\rangle = \Delta^i$ and $\langle\varphi\rangle = \Delta$, and so terms containing Δ^i and Δ correspond to diagrams with tadpoles.

4.2 The closed string moduli and coupling

The shift we use above is, strictly speaking, a shift of the variables x and φ, rather than the closed-string moduli t and g_s. However, the two sets of variables are simply related. The generating function for the closed string is

$$\widehat{W}(g_s,x;t,\bar{t}) = \sum_{g,n}\frac{1}{n!}g_s^{2g-2}C^{(g)}_{i_1\cdots i_n}x^{i_1}\cdots x^{i_n} + \left(\frac{\chi}{24}-1\right)\log g_s, \tag{4.12}$$

where the correlation functions are defined as,

$$\begin{aligned} D_i C^{(g)}_{i_1\cdots i_n} &= C^{(g)}_{ii_1\cdots i_n}, && (g\geq 1), && (4.13)\\ C_{i_1\cdots i_n} &= D_{i_1}\cdots D_{i_{n-3}}C_{i_{n-2}i_{n-1}i_n}, && (g=0;\ n\geq 3)\ . && (4.14)\end{aligned}$$

φ dependence was introduced into the closed string generating function $\widehat{W}$ as

$$\widetilde{W}(g_s,x,\varphi;t,\bar{t}) = \widehat{W}\left(\frac{g_s}{1-\varphi},\frac{x}{1-\varphi};t,\bar{t}\right) - \left(\frac{\chi}{24}-1\right)\log g_s. \tag{4.15}$$

Thus a shift of φ is identified with a re-scaling of g_s and x. The generating function $\widehat{W}$ satisfies the equation

$$\left[\frac{\partial}{\partial t^i} + \Gamma^k_{ij} x^j \frac{\partial}{\partial x^k} + \frac{\partial K}{\partial t^i}\left(\frac{\chi}{24} - 1 - g_s \frac{\partial}{\partial g_s}\right)\right] e^{\widehat{W}} = \left(\frac{\partial}{\partial x^i} - \frac{\partial F_1}{\partial t^i} - \frac{1}{2g_s^2} C_{ijk} x^j x^k\right) e^{\widehat{W}}. \tag{4.16}$$

To prove this, we can extract the coefficients of different powers of g_s on both sides and apply (4.13) and (4.14) to find equivalence for high genus and high n, i.e., $2g-2+n > 1$. For low genus and low n ($2g-2+n=1$), there is a cancellation,

$$\frac{\partial}{\partial x^i}\left(\frac{1}{3!} g_s^{-2} C_{i_1 jk} x^{i_1} x^j x^k\right) - \frac{1}{2g_s^2} C_{ijk} x^j x^k = 0, \tag{4.17}$$

and another one,

$$\frac{\partial}{\partial x^i}(C^{(1)}_{i_1} x^{i_1}) + \partial_i F_1 = 0. \tag{4.18}$$

We can now adopt Kähler normal coordinates. As explained in Section 2.6 of BCOV, we can choose coordinates of the closed string moduli space and a section of the vacuum line bundle so that, at a given point $(t_0, \bar{t}_0)$,

$$\partial_{i_1} \cdots \partial_{i_n} \Gamma^k_{ij} = 0, \qquad \partial_{i_1} \cdots \partial_{i_n} K = 0. \tag{4.19}$$

This removes all but the first term on the left of Equation (4.16). On the right of Equation (4.16) the second and third terms contribute at low genus only, and can be absorbed by redefining the sum in Equation (4.4) to have only the restrictions $g \geq 1$ and $n \geq 0$, that is,

$$\widehat{W}' = \widehat{W} + F_1 - g_s^{-2} \sum_{n=3} \frac{1}{n!} C_{i_1 \cdots i_n} x^{i_1} \cdots x^{i_n}. \tag{4.20}$$

With these choices,

$$\frac{\partial}{\partial t^i} \widehat{W}' = \frac{\partial}{\partial x^i} \widehat{W}', \tag{4.21}$$

that is, $\widehat{W} = \widehat{W}(t+x; \bar{t})$.

The simple reformulation of the open string anomaly in terms of the closed string anomaly should make it possible to apply Yamaguchi and Yau's [13] reformulation of the closed string amplitude diagram expansion to the open string case, which would give a computationally more tractable formulation than the Feynman diagram rules used here. Related work was published in [17] soon after this material was published. It considered a different shift, which is convenient to show background independence, however the Feynman diagram description is obscure.

This open-closed relationship is reminiscent of large N duality, where the background is shifted by an amount proportional to the 't Hooft coupling. It would be interesting to explore the implications of this for the Gromov-Witten and Gopakumar-Vafa invariants.

Chapter 5 Applying the Ooguri, Strominger, and Vafa Conjecture

5.1 Introduction

Topological string theory has a very rich mathematical content, but it also has physical applications. Our focus will be the application to counting 4-dimensional black hole entropy which was discovered by Ooguri, Strominger, and Vafa [25].

A black hole is a solution to Einstein's equations. Classically it does not have many physical quantities for us to study; however quantum-mechanically, it is a thermal dynamical system. An analogy between the laws of black hole dynamics and the laws of thermodynamics was discovered more than 30 years ago. In particular, Bekenstein and Hawking showed that a black hole's entropy is one-quarter the area of its event horizon in gravitational units. In 1995 Strominger and Vafa gave a microscopic description of black hole entropy [26] in terms of D-brane bound states, where D-branes are the sources of the black hole. Later on, it was found that the partition function of a 4-dimensional black hole is related to the partition function of a closed topological string. In the limit of a large black hole, that is, small curvature of the event horizon, this beautiful relation is known as the Ooguri, Strominger, and Vafa (OSV) conjecture [25]. On one side, the black hole is a solution to Einstein's equations in 4 dimensions which is obtained by type II string compactification on Calabi-Yau 3-folds; on the other side, the closed topological string is evaluated at the attractor point of moduli space of the same Calabi-Yau 3-fold associated to the black hole charge.

Type IIA or IIB string theory compactified on a Calabi-Yau 3-fold gives rise to $\mathcal{N} = 2$ supersymmetry in 4-dimensional Minkowski spacetime. The lowest components of the vector multiplets and hypermultiplets are the Kähler and complex structure moduli of the Calabi-Yau manifold. D-branes which wrap 3-cycles or even cycles in the Calabi-Yau produce black hole solutions to Einstein's equations in 4 dimensions. Simultaneously, the integration of D-brane form fluxes on corresponding non-trivial cycles of the Calabi-Yau gives rise to the moduli of the Calabi-Yau. Since the black hole entropy is proportional to the area of the horizon, which is a function of electric and magnetic charges, the moduli must be fixed by those charges on the horizon. Since the vector multiplets are driven to the attractor values on the black hole horizon, while the hypermultiplets depend on values at infinity and are not fixed on the horizon, the black hole entropy depends on either Kähler or complex structure moduli. This is analogous to the closed topological string amplitudes having dependence on either Kähler or complex structure moduli.

In the large black hole case, the black hole partition function was shown to be a product of the topological and anti-topological string partition functions. There has also been progress for small black holes, where there are some corrections to the square law, which appear as an expansion in terms of baby universes. We will review the OSV conjecture in Section 5.2. In Section 5.3, we review the result of [10] for small black hole partition function. In Section 5.4, we will discuss our attempts at factorization of the black hole partition function. We also study some properties of Gromov-Witten invariants and modular vs holomorphic properties in Section 5.5 and 5.6.

5.2 OSV conjecture

Bekenstein-Hawking entropy for a black hole is related to the area of its horizon as

$$S_{BH} = \frac{A}{4}. \tag{5.1}$$

At the attractor point, the real part of the moduli of the Calabi-Yau are fixed by the black hole magnetic and electric charges,

$$p^\Lambda = Re\ CX^\Lambda, \qquad q_\Lambda = Re\ CF_\Lambda, \tag{5.2}$$

where X^Λ $(\Lambda = 0, 1, \cdots, n_V)$ are the lowest components of $\mathcal{N} = 2$ vector multiplets in 4 dimensions, C is the scale factor of the vector multiplets, CX^Λ are the moduli of the Calabi-Yau. F_Λ are not independent moduli of the Calabi-Yau, actually they are related to X^Λ in the way that

$$F_\Lambda = \frac{\partial F}{\partial X^\Lambda}, \tag{5.3}$$

where F is the prepotential F-term of the $\mathcal{N} = 2$ theory.

In the leading order of the action (Einstein-Hilbert action), the BPS black hole entropy is

$$S_{BH} = \frac{\pi}{4} C\overline{C} e^{-K}, \tag{5.4}$$

where K is the Kähler potential,

$$K = -\ln i(\bar{X}^\Lambda F_\Lambda - X^\Lambda \bar{F}_\Lambda). \tag{5.5}$$

The higher-order corrections to the action come from the higher derivative (Wald) terms beyond the Einstein-Hilbert action. Then the entropy function is modified to be

$$S = \frac{\pi i}{2}(q_\Lambda \bar{C}\bar{X}^\Lambda - p^\Lambda \bar{C}\bar{F}_\Lambda) + \frac{\pi}{2} Im[C^3 \partial_C F]. \tag{5.6}$$

The prepotential F-term can be calculated by the topological string loop computation,

$$F(X^\Lambda, W^2) = \sum_{g=0} F_g(X^\Lambda) W^{2g}, \tag{5.7}$$

where F_g is the genus g topological string amplitude [27, 1]. Since the black hole entropy is a Legendre transformation of the free energy $\mathcal{F}(\phi, p)$, it satisfies

$$S_{BH}(q,p) = \mathcal{F}(\phi,p) - \phi^\Lambda \frac{\partial}{\partial \phi^\Lambda}\mathcal{F}(\phi,p), \tag{5.8}$$

where ϕ^Λ is the chemical potential of the electric charge q_Λ,

$$q_\Lambda = -\frac{\partial}{\partial \phi^\Lambda}\mathcal{F}(\phi,p). \tag{5.9}$$

Thus the free energy is related to the F-term by

$$\begin{aligned} \mathcal{F}(\phi^\Lambda, p^\Lambda) &= -\pi Im[C^2 F(X^\Lambda, \frac{256}{C^2})] = -\pi Im[F(CX^\Lambda, 256)] \\ &= -\pi Im[F(p^\Lambda + \frac{i}{\pi}\phi^\Lambda, 256)] \end{aligned} \tag{5.10}$$

in the gauge $C^2 W^2 = 256$.

From (5.8), the partition function (more precisely, elliptic genus) of the black hole mixed ensemble (it is a canonical ensemble of magnetic charges and grand canonical ensemble of electric charges) is

$$Z_{BH}(\phi^\Lambda, p^\Lambda) = \sum \Omega(q_\Lambda, p^\Lambda) e^{-\phi^\Lambda q_\Lambda}. \tag{5.11}$$

From (5.10) we get

$$\ln Z_{BH} = \mathcal{F} = -\pi Im F(p^\Lambda + \frac{i}{\pi}\phi, 256). \tag{5.12}$$

The F-term in the large volume limit (supergravity) [28] has the form

$$F(CX^\Lambda, 256) = C^2 D_{ABC}\frac{X^A X^B X^C}{X^0} - \frac{1}{6}c_{2A}\frac{X^A}{X^0} + \cdots, \tag{5.13}$$

where $A = 1, \cdots, n_V$, D_{ABC} is an intersection number, and c_{2A} is the second Chern class of the Calabi-Yau 3-fold.

Until now we just discussed some properties from the black hole side. The partition function and the free energy of a topological string are also related by

$$Z_{top}(t^A, g_{top}) = e^{F_{top}(t^A, g_{top})}, \tag{5.14}$$

and the topological string amplitude in the same limit [2, 1] is

$$F_{top} = -\frac{(2\pi)^3 i}{g_{top}^2} D_{ABC} t^A t^B t^C - \frac{\pi i}{12} c_{2A} t^A + \cdots . \tag{5.15}$$

We can show that, under the identification

$$g_{top} = \frac{4\pi i}{X^0}, \qquad t^A = 2\pi i \frac{X^A}{X^0}, \tag{5.16}$$

and with the attractor equations

$$CX^0 = p^0 + i\frac{\phi^0}{\pi}, \qquad CX^A = p^A + i\frac{\phi^A}{\pi}, \tag{5.17}$$

there is a correspondence,

$$F(CX^\Lambda, 256) = -\frac{2i}{\pi} F_{top}(t^A, g_{top}). \tag{5.18}$$

Therefore, from Equations (5.12) and (5.18), we get,

$$\ln Z_{BH}(p, \phi) = F_{top}(t^A, g_{top}) + \overline{F}_{top}(\bar{t}^A, g_{top}). \tag{5.19}$$

It results in a square rule,

$$Z_{BH} = |Z_{top}|^2. \tag{5.20}$$

5.3 Small black hole

In [10], the type IIA superstring is compactified on a non-compact Calabi-Yau 3-fold which is a sum of two line bundles on a torus, $\mathcal{O}(m) \oplus \mathcal{O}(-m) \rightarrow T^2$. D-branes wrapping on even cycles give rise to a 4-dimensional BPS black hole solution of $\mathcal{N} = 2$

supergravity. We will take $m = 1$ for simplicity. The D-branes are the sources of 4-dimensional black hole electric and magnetic charges. From the attractor equations, we have

$$CX^1 = N + i\frac{\theta}{g_s}, \tag{5.21}$$

$$CX^0 = \frac{1}{g_s}. \tag{5.22}$$

The Kähler modulus of the Calabi-Yau manifold is

$$t = \frac{X^1}{X^0} = g_s N + i\theta. \tag{5.23}$$

The D-brane configuration can also be identified with a Yang-Mills theory with gauge group $U(N)$ on T^2, the coupling of Yang-Mills theory being related to the coupling of string theory by $g_{YM}^2 = g_s$. The number of D4-branes corresponds to the rank N of the gauge group and the chemical potentials of D2- and D0-branes are identified with some combination of the theta angle and the gauge coupling of the Yang-Mills theory. In a pure gauge theory on an arbitrary orientable Riemann surface Σ_G of genus G and unit area, the partition function is,

$$Z_M = \int [DA^\mu] e^{-\frac{1}{g_{YM}^2}\int_{\Sigma_G} d^2x\sqrt{g} Tr F^{\mu\nu}F_{\mu\nu}}. \tag{5.24}$$

For $SU(N)$, the partition function is [29],

$$Z_M = \sum_R (\dim R)^{2-2G} e^{-\frac{g_{YM}^2}{2}C_2(R)}, \tag{5.25}$$

where R is the representation of $SU(N)$ and $C_2(R)$ is the second Casimir of R. For a pure gauge theory with gauge group $U(N)$ on the torus, there is a simplification to (5.25). The black hole partition function can be expressed as a sum over all irreducible representations R of $U(N)$,

$$Z_{U(N)} = \sum_R \exp\left\{-\frac{1}{2}g_{YM}^2 C_2(R) + i\theta C_1(R)\right\}, \tag{5.26}$$

where $C_1(R)$ is the first Casimir of R, e.g., it corresponds to the number of boxes in the Young diagram of $U(N)$.

It was found that 2-dimensional $U(N)$ Yang-Mills theory on T^2 has a reformulation in terms of a system of N non-relativistic free fermions moving on a circle [30]. The anti-periodic boundary condition for fermion wave functions makes fermion momenta half integers. A fermion configuration with momenta $p_i = n_i - \frac{1}{2}$, where $n_1 > n_2 > \cdots > n_N \geq 0$ $(i = 1, \cdots, N)$, can be represented by a representation R of $U(N)$. The excitation number for each fermion can be expressed as the length of each row in the Young diagram, $n'_1 = n_1-(N-1), n'_2 = n_2-(N-2), \cdots, n'_k = n_k-(N-k), \cdots, n'_N = n_N \geq 0$. For example, the ground state is the state with all n' zero, and $n'_1 = 1$ corresponds to a state with the highest fermion excited by one unit. Figure 5.1 is a cartoon of the correspondence. The energy for the fermion system in the center of

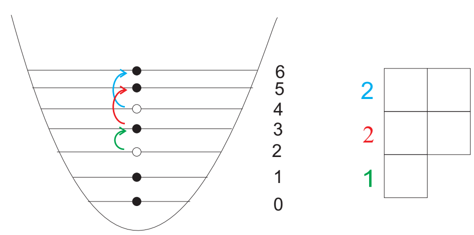

Figure 5.1: Fermion system vs Young diagram

momentum frame is the second Casimir of the representation R of $SU(N)$,

$$\begin{aligned} C_2(R) &= \frac{1}{2}\sum_i \left[(n'_i + N - i + \frac{1}{2})^2 - (N - i + \frac{1}{2})^2\right] - \frac{1}{2N}\left[\left(\sum_i n'_i + \frac{N^2}{2}\right)^2 - \left(\frac{N^2}{2}\right)^2\right] \\ &= \frac{1}{2}\left[\sum_i n'_i(n'_i + 1 - 2i) + N\sum_i n'_i - \frac{(\sum_i n'_i)^2}{N}\right]. \end{aligned} \tag{5.27}$$

In this frame, there are fermions which have positive momentum as well as negative momentum. Fermion excitations can occur at both sides. In gauge theory Figure 5.2

denotes the two types of fermion excitation. In the large N limit, there is a chiral factorization of the partition function of 2-dimensional $U(N)$ Yang-Mills,

$$Z_{U(N)} = \sum_{\ell=-\infty}^{+\infty} Z_+^{YM}(t + \ell g_{YM}^2) Z_-^{YM}(\bar{t} - \ell g_{YM}^2), \tag{5.28}$$

where $Z_\pm^{YM}$ are the chiral blocks of the $U(N)$ theory. The chiral block Z_+ is similar

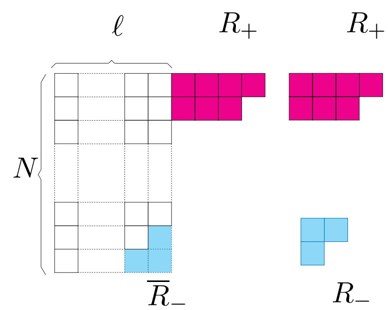

Figure 5.2: Chiral factorization of the partition function of large N $U(N)$ Yang-Mills theory

to the expressions of topological strings on this Calabi-Yau 3-fold [31],

$$Z_+^{YM} = \psi_{top}. \tag{5.29}$$

The OSV conjecture for large N then gives rise to

$$Z^{YM} = \psi_{top}\overline{\psi}_{top}. \tag{5.30}$$

Now we will show this in terms of the free fermion system. We first write down the partition function in terms of N free fermions explicitly. According to the Fermi-Dirac distribution, the free fermion partition function for the N-fermion canonical ensemble is

$$Z_N^{YM} = \oint \frac{dx}{x} x^N \sum_{p=-\infty}^{+\infty} (1 + x^{-1} e^{ip\theta} q^{p^2/2}), \tag{5.31}$$

where $q = e^{-g_s}$, $x = e^{\mu N}$, and μ is the chemical potential for the number of fermions N. The partition function of the topological string, however, corresponds to a fermi sea with arbitrary number of excitations above the fermi surface and the same number of holes in the sea. Therefore the partition function for the string in the Calabi-Yau manifold $\mathcal{O}(1) \oplus \mathcal{O}(-1) \to T^2$ with the Kähler modulus $t = g_s N + i\theta$ is

$$\psi_N = e^{-\frac{N^2}{2}t} e^{g_s(\frac{N^3}{3}+\frac{N}{24})} \oint \frac{dx}{x} \prod_{p>0}(1 + x e^{-tp} q^{p^2/2}) \prod_{p'>0}(1 + x^{-1} e^{-tp'} q^{-(p')^2/2}). \tag{5.32}$$

In terms of free fermions, this proves that for large N,

$$Z_N^{YM} = \sum_{N_+ + N_- = N} \psi_{top}(N_+)\bar{\psi}_{top}(N_-). \tag{5.33}$$

In Figure 5.3, we use the fermi sea of free fermions to show the factorization. For $N \to \infty$, the excitations only happen on the fermi sea surfaces, since the excitations inside of the sea are suppressed by e^{-N}.

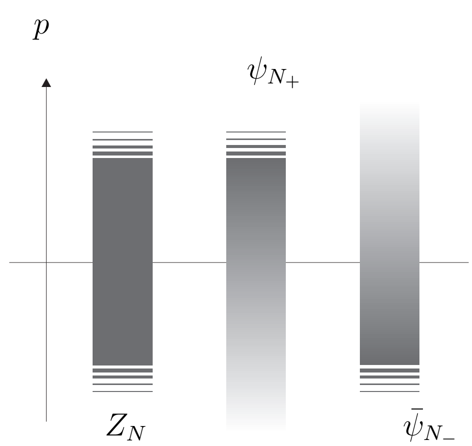

Figure 5.3: Free fermion realized large N factorization

When we consider finite N, the previous factorization is not correct since there are over countings—the double counting of an excitation of fermions to the upper and lower surfaces of the fermi sea. The modification of the factorization can be expressed by a recursive equation [10]

$$Z_N(g_s) = \sum_{k=0}^{N} \psi_k(g_s)\bar{\psi}_{N-k}(g_s) - \sum_{n>0} Z_{N+n}(g_s) Z_n(-g_s). \tag{5.34}$$

5.4 Second approach to handling non-perturbative corrections

From the previous section, we have the partition function for Yang-Mill theory (5.31), and the partition function for the topological string theory (5.32). We developed a resummation distinct from (5.34). In our resummation, the partition function of the Yang-Mills theory is factorized as

$$Z_N = \sum_{N_+ + N_- = N} \Psi_{N_+}(t)\bar{\Psi}_{N_-}(\bar{t}), \tag{5.35}$$

where

$$t = g_s N_+ + i\theta, \qquad \bar{t} = g_s N_- - i\theta. \tag{5.36}$$

The modified topological string partition function is

$$\Psi_N = e^{-tN^2 + \frac{g_s}{2}N^3} \oint \frac{dx}{x} x^N \prod_{p>-N} \left(1 + xe^{-tp}e^{-g_s p^2/2}\right) . \tag{5.37}$$

This result is exact; the non-perturbative corrections are captured in Ψ. Figure 5.4 describes how the new factorization works. In Appendix A.2, we prove that the

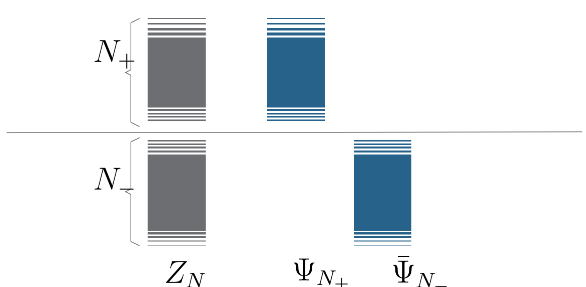

Figure 5.4: New resummation laws

newly-defined function have a recursive relation. Simply speaking,

$$\psi_N = \sum_{k=0} \Psi_{N+k}\Psi_{-k}, \tag{5.38}$$

we can show that $\Psi_0 = 1$ and that for large N, $\Psi_N = \psi_N$. Thus the newly-defined function satisfies

$$\Psi_N = \psi_N - \sum_{k=1}^{\infty} \Psi_{N+k}\Psi_{-k}. \tag{5.39}$$

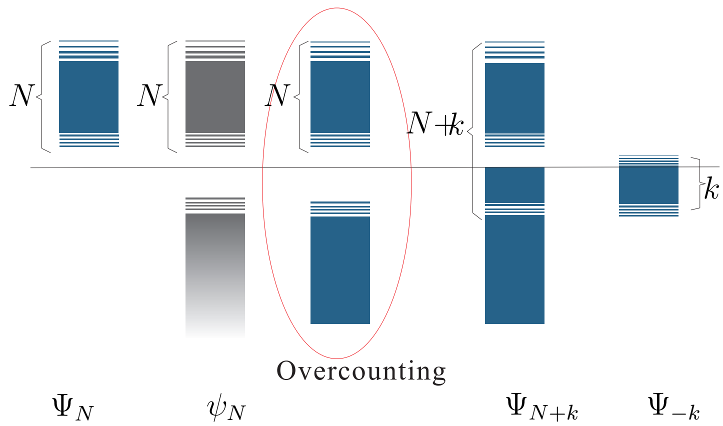

Figure 5.5: New partition function

5.5 Gromov-Witten invariants

We will discuss Gromov-Witten invariants in topological string theory and give an example for topological strings on the Calabi-Yau 3-fold $\mathcal{O}(1) \oplus \mathcal{O}(-1) \to T^2$.

We denote the moduli space of holomorphic maps $\phi : \Sigma \to X$ of degree β as,

$$\mathcal{M}_\Sigma(X, \beta) = \{\phi \quad | \quad \phi_*[\Sigma] = \beta\}, \tag{5.40}$$

where Σ is a Riemann surface and X is a Calabi-Yau manifold. The Gromov-Witten invariants $n_{\beta,D_1,\cdots,D_k}$ are defined to be the intersection numbers of k divisors of X, where those divisors are evolution maps from Σ to X; that is,

$$n_{\beta,D_1,\cdots,D_k} = \#\{\phi \in \mathcal{M}_\Sigma(X,\beta) | \phi(p_i) \in D_i, i = 1, \cdots, k\}, \tag{5.41}$$

where D_i are the divisors of X and p_i are points in X.

Now we want to extract the Gromov-Witten invariants of the modified topological string partition function Ψ. It has the nice property that the topological string free energy at each genus is convergent. From Section 5.3,

$$\begin{aligned}
\Psi_N &= e^{-\frac{t}{2}N^2+g_s(\frac{N^3}{3}+\frac{N}{24})}\sum_n H_n(g_s)e^{-tn}, & (5.42)\\
\text{where} \qquad n &= \sum_{\substack{i \\ p_i\in\mathbb{Z}_+ +\frac{1}{2} \\ p_i'\in[\frac{1}{2},N-\frac{1}{2}]}} (p_i+p_i'), & (5.43)\\
H_n(g_s) &= \exp\Big\{-\frac{g_s}{2}\sum_{\substack{i \\ p_i\in\mathbb{Z}_+ +\frac{1}{2} \\ p_i'\in[\frac{1}{2},N-\frac{1}{2}]}} \big[{p_i}^2-(p_i')^2\big]\Big\}. & (5.44)
\end{aligned}$$

There is a relation among N, t and g_s,

$$N = \frac{t-i\theta}{g_s}, \tag{5.45}$$

where θ is the imaginary part of t. Rewriting (5.42), we get

$$\Psi_N = \exp\left(-\frac{t^3}{6g_s^2}-\frac{t}{2g_s^2}\theta^2+\frac{t}{24}+i\frac{\theta^3}{3g_s^2}-i\frac{\theta}{24}\right)\sum_n H_n(g_s)e^{-tn}. \tag{5.46}$$

For simplicity we let $\theta=0$, and take the logarithm of Ψ_N,

$$\ln\Psi_N = -\frac{t^3}{6g_s^2}+\frac{t}{24}+\ln\sum_{n=0} H_n(g_s)e^{-tn}. \tag{5.47}$$

We can extract H_n from the definition (5.44) as

$$\begin{aligned} H_0 &= 1 \\ H_1 &= 1 \\ H_2 &= e^{-g_s} + e^{g_s} \\ H_3 &= 1 + e^{-3g_s} + e^{3g_s} \\ H_4 &= 1 + e^{-2g_s} + e^{2g_s} + e^{-6g_s} + e^{6g_s} \\ &\ldots \end{aligned}$$

Note that $H_n(g_s = 0) = P[n]$, where $P[n]$ is the number of partitions of n. For $n > N = \frac{t}{g_s}$ fixed, the symmetry of those terms with positive and negative power in H_n is broken, and some terms are truncated. In the following, we will consider N large enough to neglect the symmetry breaking. According to the identity

$$\ln(1 + \sum_{n=1}^{\infty} H_n e^{-tn}) = \sum_m (-1)^m \frac{1}{m} (\sum_{n=1}^{\infty} H_n e^{-tn})^m, \tag{5.48}$$

the coefficients of e^{-nt} are

$$\begin{aligned} e^0 &: 0 \\ e^{-t} &: H_1 \\ e^{-2t} &: H_2 - \frac{1}{2} H_1^2 \\ e^{-3t} &: H_3 - \frac{1}{2}(2H_1H_2) + \frac{1}{3} H_1^3 \\ e^{-4t} &: H_4 - \frac{1}{2}(2H_1H_3 + H_2^2) + \frac{1}{3}(3H_1^2H_2) - \frac{1}{4} H_1^4 \\ \ldots &\quad \ldots \\ e^{-nt} &: H_n + \cdots + (-1)^{k-1} \frac{C\{m_\ell\}}{k} \prod_{j=1}^{k} H_{i_j} + \cdots + (-1)^{n-1} \frac{1}{n} H_1^n. \end{aligned}$$

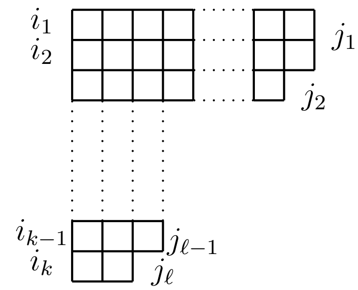

Figure 5.6: Partition

A partition of n into k integers is

$$n = \sum_{j=1}^{k} i_j = \sum_{i=1}^{\ell} m_i j_i, \qquad \sum_i m_i = k, \tag{5.49}$$

where m_i counts the multiplication of integer j_i. We can draw a diagram of the partition as in Figure 5.6. $C\{m_\ell\}$ is the weight for a partition of n into k integers,

$$C\{m_\ell\} = \frac{k!}{m_1! \cdots m_\ell!}. \tag{5.50}$$

In the weak coupling limit $g_s \to 0$, the coefficient for e^{-nt} is

$$\sum_{\{m_\ell\}} (-1)^{k-1} \frac{C\{m_\ell\}}{k} \prod_{\sum_i^\ell m_i j_i = n} (P[j_i])^{m_i}, \qquad \text{where} \qquad k = \sum_i^\ell m_i. \tag{5.51}$$

Substituting the expression from (5.50), the coefficient for e^{-nt} is

$$\sum_{\{m_\ell\}} (-1)^{(\sum_i m_i - 1)} (\sum_i^\ell m_i - 1)! \prod_{\sum_i^\ell m_i j_i = n} \frac{(P[j_i])^{m_i}}{m_i!}. \tag{5.52}$$

We want to get the Gromov-Witten invariants for

$$\Psi = \exp\left\{ \sum_{g=0} g_s^{2g-2} F_g(t) \right\} = \exp\left\{ \sum_{g=0} g_s^{2g-2} \left(N_{g,d} e^{-dt} + \text{Poly}(t) \right) \right\}. \tag{5.53}$$

The polynomials for leading and subleading terms F_0 and F_1 are

$$F_0^{(0)}(t) = -\frac{t^3}{6}, \qquad F_1^{(0)}(t) = \frac{t}{24}. \tag{5.54}$$

For $d \geq 1$, non-zero coefficients are

$$\begin{aligned}
e^{-t} &: \sum_{g=0} g_s^{2g-2} N_{g,1} = H_1; \qquad N_{1,1} = 1 \\
e^{-2t} &: \sum_{g=0} g_s^{2g-2} N_{g,2} = H_2 - \frac{1}{2}H_1^2; \qquad N_{g,2} = \oint dg_s (H_2 - \frac{1}{2}H_1^2) g_s^{1-2g} \\
e^{-3t} &: N_{g,3} = \oint dg_s (H_3 - H_1 H_2 + \frac{1}{3}H_1^3) g_s^{1-2g} \\
\cdots & \quad \cdots \\
e^{-nt} &: N_{g,n} = \oint dg_s \left[\sum_{\{m_\ell\}} (-1)^{(\sum_i m_i - 1)} (\sum_{i=1}^{\ell} m_i - 1)! \prod_{\sum_i m_i j_i = n} \frac{H_{j_i}^{m_i}}{m_i!} \right] g_s^{1-2g}.
\end{aligned}$$

The non-zero Gromov-Witten invariants are therefore

$$\begin{aligned}
N_{1,1} &= 1 \\
N_{g,2} &= \oint dg_s (e^{-g_s} + e^{g_s} - \frac{1}{2}) g_s^{1-2g} \\
& \quad N_{1,2} = \frac{3}{2}, \quad N_{2,2} = \frac{2}{2!}, \quad N_{3,2} = \frac{2}{4!}, \quad N_{p,2} = \frac{2}{(2p-2)!} (p > 1) \\
N_{g,3} &= \oint dg_s \left[1 + e^{-3g_s} + e^{3g_s} - (e^{-g_s} + e^{g_s}) + \frac{1}{3} \right] g_s^{1-2g} \\
& \quad N_{1,3} = \frac{4}{3}, \quad N_{2,3} = 2\frac{3^2}{2!} - 2\frac{1^2}{2!}, \quad N_{3,3} = 2\frac{3^4}{4!} - 2\frac{1^4}{4!}, \quad N_{p,3} = \frac{2(3^{2p-2} - 1)}{(2p-2)!} (p > 1) \\
N_{g,4} &= \oint dg_s \left[H_4 - \frac{1}{2}(2H_1 H_3 + H_2^2) + H_1^2 H_2 - \frac{1}{4} H_1^4 \right] g_s^{1-2g} \\
\cdots & \quad \cdots
\end{aligned}$$

There is not an explicit form for the Gromov-Witten invariants; however, Paul Cook wrote a mathematica code [32] to calculate these numbers. Only for $N_{1,d}$ is there an explicit form,

$$
\begin{aligned}
N_{1,d} &= \sum_{\{m_\ell\}} (-1)^{(\sum_i m_i - 1)} (\sum_i^{\ell} m_i - 1)! \prod_{\sum_i^{\ell} m_i j_i = d} \frac{(P[j_i])^{m_i}}{m_i!} \\
&= \sum_{\sum m_i i = d} (-1)^{(\sum_i m_i - 1)} (\sum_i^{\infty} m_i - 1)! \prod_{i=1}^{\infty} \frac{(P[i])^{m_i}}{m_i!}.
\end{aligned}
$$

We can also define the generating function

$$
W(q) = \sum_q q^d N_{1,d} = \sum_{\sum m_i i = d} (-1)^{(\sum_i m_i - 1)} (\sum_i^{\infty} m_i - 1)! \prod_{i=1}^{\infty} \frac{(P[i] q^i)^{m_i}}{m_i!}. \tag{5.55}
$$

We can use an identity

$$
\prod_{n=1}^{\infty} \frac{1}{(1-q^n)} = \sum_{i=0}^{\infty} P[i] q^i \tag{5.56}
$$

to simplify the generating function,

$$
W(q) = -\int_0^{\infty} \frac{dz}{z} \left[\exp\left\{ -z \prod_n \frac{1}{1-q^n} \right\} - 1 \right]. \tag{5.57}
$$

Note that

$$
\int_0^{\infty} \frac{dz}{z} \exp\left\{ -z \prod_{n=1} \frac{1}{1-q^n} \right\} \sim -\ln \prod_{n=1} \frac{1}{1-q^n} = \sum_{n=1} \ln(1-q^n). \tag{5.58}
$$

Rudd's paper [33] has calculated F_g up to $g = 8$ using another method. We can extract the Gromov-Witten invariants from that paper and find that they successfully match up with our results.

5.6 Modularity and holomorphicity

On a target space M which has a modular group, topological string amplitudes are (almost) modular forms. For example, let us consider the B-model topological string theory on a Calabi-Yau 3-fold X where the periods have a modular group Γ. The partition function of this theory is a wave function of a certain quantum mechanical system where $H^3(X)$ acts as the phase space. There are two useful choices of basis of $H^3(X)$ which are called polarizations. For the real polarization, the string amplitudes F_g will be holomorphic but quasi-modular; for the holomorphic one, F_g is modular but not quite holomorphic. The transition between these two polarizations was studied in [34]. Because of mirror symmetry, the A-model must have similar choices also.

In general, when the modular group is $SL(2,\mathbb{Z})$, the modular form for F_g can be rewritten as

$$\hat{F}_g = \sum_{k=0}^{3g-3} \sum_{2\ell+3m=3g-3-k} c_{k\ell} \hat{E}_2^k E_4^\ell E_6^m, \tag{5.59}$$

where $\hat{E}_2$, E_4 and E_6 are modular forms of weight 2, 4, and 6, respectively. They are called Eisenstein series and are defined in the Appendix A.3. In order to satisfy the holomorphic anomaly equation, we have to use a modular form for each genus topological string amplitude. Recall that the holomorphic anomaly equation is

$$\partial_{\bar{t}^{\bar{i}}} \hat{F}_g = \frac{1}{2} \overline{C}_{\bar{i}\bar{j}\bar{k}} e^{2K} G^{j\bar{j}} G^{k\bar{k}} \left[D_j D_k \hat{F}_{g-1} + \sum_{r=1}^{g-1} D_j \hat{F}_r D_k \hat{F}_{g-r} \right]. \tag{5.60}$$

In the case of a single modulus τ, we define a new variable $t = -2\pi i \tau$. The modular transformation is then

$$\tau \to -\frac{1}{\tau}, \quad \text{i.e.,} \quad t \to \frac{4\pi^2}{t}. \tag{5.61}$$

For genus 1, we have

$$\hat{F}_1 = -\ln \left\{ \left[\frac{(t+\bar{t})}{-2\pi i} \right]^{1/2} \eta(t)\eta(\bar{t}) \right\}, \tag{5.62}$$

and

$$\partial_t \hat{F}_1(t) = \frac{1}{24} \hat{E}_2(t), \tag{5.63}$$

where $\hat{F}_1$ is a modular form of weight 0.

The prepotential in this case is given by

$$F_0 = -\frac{1}{6}t^3. \tag{5.64}$$

Further, the Yukawa coupling, Kähler potential, and Zamolodchikov metric can be calculated as

$$C_{\tau\tau\tau} = \partial_\tau\partial_\tau\partial_\tau F_0 = -(2\pi)^3 i, \quad e^{-K} = \frac{(t+\bar{t})^3}{3}, \quad G_{t\bar{t}} = \frac{12\pi^2}{(t+\bar{t})^2}. \tag{5.65}$$

The covariant derivative is

$$D_\tau = \partial_\tau - \frac{n}{2\pi i(\tau - \bar{\tau})}, \quad \text{i.e.,} \quad D_t = \partial_t + \frac{n}{t+\bar{t}}, \tag{5.66}$$

where n depends on the weight of a modular form. It can be shown that the covariant derivative increases the modular weight by 2.

We now summarize some further properties of Einsenstein series and the F_g's:

1. Covariant derivatives are given in Table 5.1.

Partial derivative	Covariant derivative
$\partial_t \hat{E}_2 = \frac{1}{12}(E_4 - E_2^2) + \frac{12}{(t+\bar{t})^2}$	$D_t \hat{E}_2 = \frac{1}{12}(E_4 - \hat{E}_2^2)$
$\partial_t E_4 = E_4' = \frac{1}{3}(E_6 - E_2 E_4)$	$D_t E_4 = \frac{1}{3}(E_6 - \hat{E}_2 E_4)$
$\partial_t E_6 = E_6' = \frac{1}{2}(E_4^2 - E_2 E_6)$	$D_t E_6 = \frac{1}{2}(E_4^2 - \hat{E}_2 E_6)$

Table 5.1: Partial derivative vs covariant derivative of modular forms

2. $\hat{F}_g$ is a modular form of weight 6g-6, so the covariant derivative with respect to t turns out to be

$$D_t \hat{F}_g = \left(\partial_t + \frac{6g-6}{t+\bar{t}}\right)\hat{F}_g. \tag{5.67}$$

The Equation (5.60) for genus 2 is

$$\bar{\partial}_{\bar{t}} \hat{F}_2 = -\frac{1}{2(t+\bar{t})^2}[D_t \partial_t \hat{F}_1 + (\partial_t \hat{F}_1)^2]. \tag{5.68}$$

We can show that $\hat{F}_2$ is a modular form of weight 6, so it contains generally 3 terms,

$$\hat{F}_2 = C_1^{(2)} \hat{E}_2^3 + C_2^{(2)} \hat{E}_2 E_4 + C_3^{(2)} E_6. \tag{5.69}$$

Solving the holomorphic anomaly equation we find that

$$C_1^{(2)} = \frac{1}{2^9 \times 3^4}, \quad C_2^{(2)} = -\frac{1}{2^8 \times 3^3}. \tag{5.70}$$

3. The derivative with respect to the modulus can be changed to the one with respect to $\hat{E}_2$ by the chain rule,

$$\bar{\partial}_{\bar{t}} \hat{F}_g = \frac{12}{(t+\bar{t})^2} \frac{\partial \hat{F}_g}{\partial \hat{E}_2}. \tag{5.71}$$

Combining these facts, we get

$$D_t \hat{F}_g = \frac{\partial \hat{F}_g}{\partial \hat{E}_2} \partial_t \hat{E}_2 + \frac{\partial \hat{F}_g}{\partial E_4} \partial_t E_4 + \frac{\partial \hat{F}_g}{\partial E_6} \partial_t E_6 + \frac{6g-6}{t+\bar{t}} \hat{F}_g. \tag{5.72}$$

This can be simplified to

$$D_t \hat{F}_g = \frac{\partial \hat{F}_g}{\partial \hat{E}_2} D_t \hat{E}_2 + \frac{\partial \hat{F}_g}{\partial E_4} D_t E_4 + \frac{\partial \hat{F}_g}{\partial E_6} D_t E_6. \tag{5.73}$$

Similarly,

$$D_t(D_t \hat{F}_g) = \frac{\partial (D_t \hat{F}_g)}{\partial \hat{E}_2} D_t \hat{E}_2 + \frac{\partial (D_t \hat{F}_g)}{\partial E_4} D_t E_4 + \frac{\partial (D_t \hat{F}_g)}{\partial E_6} D_t E_6. \tag{5.74}$$

The holomorphic anomaly equation is therefore turned into an equation with variable $\hat{E}_2$. This method is useful when we know the form of F_g. Once we know the equation, we can use Feynman rules as discussed in Section 3.5 to solve it.

Now let us come back to the example of the topological strings on $\mathcal{O}(1) \oplus \mathcal{O}(-1) \to T^2$. A chiral fermion on a circle is identified with bosonic string theory on a torus. Bosonic string theory on the torus has been studied by [35]. The string amplitudes for genus up to 8 were derived in [33]. Those amplitudes are almost modular forms. We considered the transition from a holomorphic form to a modular form and studied the holomorphic anomaly equation for it. By solving the closed holomorphic anomaly equation, we find the solution is different from [33]. We have an argument for why this is so.

The Yukawa coupling for this theory is

$$C_{\tau\tau\tau} = \partial\partial\partial F_0 = \partial\partial\partial(-\frac{t^3}{6}) = \partial\partial\partial(-\frac{(-2\pi i\tau)^3}{6}) = (2\pi i)^3. \tag{5.75}$$

From special Kähler geometry,

$$e^{-K} = \int_{CY_3} \omega \wedge \omega \wedge \omega. \tag{5.76}$$

Suppose we choose a polarization,

$$a = \int_{\mathcal{O}(-1)} \omega = 1, \qquad a_D = \int_{C_4} \omega \wedge \omega = \frac{1}{2}(\tau - \bar{\tau}), \tag{5.77}$$

where $C_4 = \mathcal{O}(1) \to T^2$. Then (5.76) can be simplified to

$$e^{-K} = \int_{T^2} dz \wedge d\bar{z} = i(a\overline{a}_D - \overline{a}a_D) = i(\tau - \bar{\tau}). \tag{5.78}$$

The Weil-Petersson metric is

$$G_{\tau\bar{\tau}} = \partial_\tau \bar{\partial}_{\bar{\tau}} K = -\frac{1}{(\tau - \bar{\tau})^2}, \tag{5.79}$$

and its Christoffel symbol can be computed as

$$\Gamma^{\tau}{}_{\tau\tau} = G^{\tau\bar{\tau}} \partial_\tau G_{\tau\bar{\tau}} = -\frac{2}{\tau - \bar{\tau}}. \tag{5.80}$$

We can show that it satisfies the special Kähler condition. To show this, we first write down the curvature tensor,

$$R_{i\bar{j}k}{}^{\ell} = G_{i\bar{j}}\delta_k^\ell + G_{k\bar{j}}\delta_i^\ell - C_{ikm}\overline{C}_{\bar{j}}^{\ell m}. \tag{5.81}$$

Contracting the last two indices, we get

$$R_{i\bar{j}} = (n+1)G_{i\bar{j}} - C_{ikm}\overline{C}_{\bar{j}}^{km} = -\bar{\partial}_j \Gamma^k{}_{ik} = -\partial_i \bar{\partial}_{\bar{j}} \log G, \tag{5.82}$$

where

$$\bar{C}_{\bar{j}}^{km} = e^{2K}\overline{C}_{\bar{j}\bar{k}\bar{m}} G^{k\bar{k}} G^{m\bar{m}}. \tag{5.83}$$

In the 1-modulus case,

$$2G_{\tau\bar{\tau}} - C_{\tau\tau\tau} e^{2K} \bar{C}_{\bar{\tau}\bar{\tau}\bar{\tau}} G^{\tau\bar{\tau}} G^{\tau\bar{\tau}} = -\partial\bar{\partial}\log G. \tag{5.84}$$

Now we want to use the formula in [1] to calculate the propagators $S^{\tau\tau}$, S^{τ}, and S. For $S^{\tau\tau}$ we have

$$S^{\tau\tau} C_{\tau\tau\tau} = 2\partial_\tau K + \Gamma^\tau{}_{\tau\tau} + f^\tau{}_{\tau\tau}, \tag{5.85}$$

where $f^\tau{}_{\tau\tau}$ is some meromorphic object which was added to make the left hand side covariant. We can choose

$$f^\tau{}_{\tau\tau} = 2\partial_\tau \log f + v^{-1}\partial v + \tilde{f}^\tau{}_{\tau\tau}, \tag{5.86}$$

where f is a meromorphic section of the line bundle $\mathcal{L}$ of the Calabi-Yau, v is the meromorphic tangent vector, and $\tilde{f}^\tau{}_{\tau\tau}$ is a meromorphic section of $T \times \mathrm{Sym}^2 T^*$. In the case of 1 modulus, it can be set to be zero. The propagator $S^{\tau\tau}$ is

$$S^{\tau\tau} = \frac{1}{(-2\pi i)^3}\left[2\partial\log(e^K|f|^2) + (G_{\tau\bar{\tau}}v)^{-1}\partial(G_{\tau\bar{\tau}}v)\right]. \tag{5.87}$$

Since the target space is a torus which has modular group $SL(2,\mathbb{Z})$, we require $S^{\tau\tau}$ to be a modular form. We have to choose $f = \eta^{-4}(\tau)$, $v = 0$,

$$S^{\tau\tau} = -\frac{4}{(-2\pi i)^3}\partial\log(\tau-\bar{\tau})|\eta^2(\tau)|^2. \tag{5.88}$$

In terms of an Eisenstein series,

$$S^{\tau\tau} = \frac{1}{3(2\pi i)^2}\hat{E}_2. \tag{5.89}$$

Likewise, the propagator S^τ is defined as

$$\begin{aligned}\bar\partial S^\tau &= G_{\tau\bar\tau}S^{\tau\tau} = \frac{1}{C_{\tau\tau\tau}}\left[2G_{\tau\bar\tau}\partial\log(e^K|f|^2) + v^{-1}\partial(G_{\tau\bar\tau}v)\right] \\ &= \frac{1}{C_{\tau\tau\tau}}\bar\partial\left[(\partial\log e^K|f|^2)^2 + v^{-1}\partial(v\partial K)\right].\end{aligned} \tag{5.90}$$

After integration we get

$$S^\tau = \frac{1}{C_{\tau\tau\tau}}\left[(\partial\log e^K|f|^2)^2 + v^{-1}\partial(v\partial\log e^K)\right] + \text{hol. ambiguity}. \tag{5.91}$$

We find that S^τ is a modular form of weight 4,

$$S^\tau = -\frac{1}{2\pi i}\frac{1}{72}(\hat{E}_2^2 + E_4). \tag{5.92}$$

The last propagator S is defined as

$$\bar\partial S = G_{\tau\bar\tau}S^\tau, \tag{5.93}$$

and is a modular form of weight 6,

$$S = \frac{1}{36\times 72}(\hat{E}_2^3 + 3\hat{E}_2E_4). \tag{5.94}$$

According to [2],

$$\partial_i\bar\partial_{\bar j}F_1 = \frac{1}{2}C_{ik\ell}\bar C_{\bar j\bar k\bar\ell}e^{2K}G^{k\bar k}G^{\ell\bar\ell} - (\frac{\chi}{24} - 1)G_{i\bar j}, \tag{5.95}$$

which here becomes

$$\begin{aligned}\partial\bar{\partial}F_1 &= -\frac{1}{2}(2\pi)^6\frac{36}{(2\pi)^6(\tau-\bar{\tau})^6}\frac{(\tau-\bar{\tau})^2}{3}\frac{(\tau-\bar{\tau})^2}{3}+(\frac{\chi}{24}-1)\frac{3}{(\tau-\bar{\tau})^2} \\ &= (\frac{3\chi}{24}-5)\frac{1}{(\tau-\bar{\tau})^2}. \end{aligned} \tag{5.96}$$

The general solution for F_1 is

$$F_1 = (\frac{\chi}{8}-5)\log(\tau-\bar{\tau})|\eta^2|^2, \tag{5.97}$$

and the correlation functions are

$$\begin{aligned} C^{(1)}_{\tau} &= \partial_\tau F_1 = (\frac{\chi}{8}-5)\frac{2\pi i}{12}\hat{E}_2, \\ C^{(1)}_{\tau\tau} &= D_\tau\partial_\tau F_1 = (\frac{\chi}{8}-5)\left(\frac{2\pi i}{12}\right)^2(\hat{E}_2^2-E_4). \end{aligned}$$

Rudd's result [33] for genus 1 and 2 are

$$F_1 = -\frac{1}{2}\log(\tau-\bar{\tau})|\eta^2|^2, \tag{5.98}$$

and

$$F_2 = \frac{1}{2^7\cdot 3^4}(\hat{E}_2^3-\frac{3}{5}\hat{E}_2E_4-\frac{2}{5}E_6). \tag{5.99}$$

We can match our results for F_1 with (5.98) when $\chi = 36$. The correlation functions are then

$$C^{(1)}_{\tau} = \partial_\tau F_1 = -\frac{2\pi i}{24}\hat{E}_2, \tag{5.100}$$

$$C^{(1)}_{\tau\tau} = D_\tau\partial_\tau F_1 = -\frac{(2\pi i)^2}{24\times 12}(\hat{E}_2^2-E_4). \tag{5.101}$$

However, when we use Feynman rule to calculate the genus 2 string amplitude

$$\begin{aligned} F_2 &= \frac{1}{2}S^{ij}C^{(1)}_{ij} + \frac{1}{2}C^{(1)}_i S^{ij} C^{(1)}_j - \frac{1}{8}S^{jk}S^{mn}C_{jkmn} \\ &- \frac{1}{2}S^{ij}C_{ijm}S^{mn}C^{(1)}_n + \frac{\chi}{24}S^i C^{(1)}_i + \frac{1}{8}S^{ij}C_{ijp}S^{pq}C_{qmn}S^{mn} \\ &+ \frac{1}{12}S^{ij}S^{pq}S^{mn}C_{ipm}C_{jqn} - \frac{\chi}{48}S^i C_{ijk}S^{jk} + \frac{\chi}{24}(\frac{\chi}{24}-1)S + \text{hol. amb} \end{aligned} \tag{5.102}$$

we obtain

$$F_2 = \frac{1}{2^7 \cdot 3^4}(\hat{E}_2^3 - 6\hat{E}_2 E_4). \tag{5.103}$$

we find only the first term agrees with Rudd's result (5.99). In order to get Rudd's F_2, one has to let χ equal some irrational number, and that cannot be true. Another discrepancy happens when we compare our result with the result from [33] for high genus amplitudes. The possible reason is that we have to choose the holomorphic polarization to apply the holomorphic anomaly equation, and then change the polarization to a real polarization by converting $E_2(\tau)$ to $\hat{E}_2(\tau,\bar{\tau})$ to get a modular form. Here the periods we compute are on some non-compact cycles, there is some ambiguity for the result which means the polarization we have chosen may not be the holomorphic one. Therefore we might solve this problem by choosing an appropriate polarization. This remains to be clarified.

Chapter 6 Summary and Open Questions

In this thesis we have presented various aspects and applications of open and closed topological string theories. Let us now summarize our results and mention open questions in relation to them.

In the first part of the thesis we gave a new derivation of the open string holomorphic anomaly equation. These are equations that determine the genus g string partition functions, with h holes and possibly some marked points. One might ask how these partition functions could be used in the low-energy effective action of superstrings. Let us recall that in the closed topological string, the genus g amplitude computes the superpotential F-term of the 4-dimensional $\mathcal{N} = 2$ supergravity multiplet. More precisely, compactifying the type II superstrings on a Calabi-Yau 3-fold yields in the low-energy theory an F-term

$$F = \int d^2\theta d^2\tilde{\theta} \sum_g F_g(t^i)W^{2g}, \tag{6.1}$$

where F_g is the genus g topological string amplitude which captures the scattering amplitude of $2g-2$ graviphotons and two gravitons. One can now also consider open strings on a Calabi-Yau. This gives rise to a 4-dimensional low-energy effective theory of $\mathcal{N} = 1$ super Yang-Mills coupled to supergravity. For genus 0, the topological string amplitude $F^{(0,h)}$ describes the scattering amplitude of $h-2$ gaugini and two gravitons [1]. What is the physical meaning of higher genus amplitudes? One would expect much more complicated scattering amplitudes of graviphotons, gaugini, and gravitons.

The mathematical structure of the open topological string theory is also very rich. The moduli space of Riemann surfaces with holes is much more complicated. The stable maps of Riemann surfaces with marked points, but without holes, have been well studied by mathematicians. The stable maps of Riemann surfaces with holes are less well understood. One way to approach this problem is to study the open Gromov-Witten invariants. In this way, our open holomorphic anomaly equation may have some interesting mathematical implications.

In Chapter 3, we derived the holomorphic anomaly equation for the open topological string under two assumptions. We have not discussed the first assumption, that is, when open string moduli are not present. Although there have been some attempts [18] to generalize to a theory with open string moduli, the topological structure is not yet known.

Another interesting direction for generalization is to include fluxes in topological string theory. This might be also phenomenologically important. For example, in geometric transitions for superstrings, the deformed conifold with D-branes wrapping on S^3 is dual to the resolved conifold with flux on the blown up S^2. One may wonder if there will be a geometric transition in the topological string theory as well. There are some studies on generalizing topological sigma models in the presence of non-trivial H-flux [36], and recently there were some studies on the integrality condition when fluxes are included [37]. It would be very interesting to understand these issues further.

In the second part of the thesis we discussed the connection of topological string theories with black holes, in the context of the OSV conjecture. The microscopic origin of black hole entropy is from BPS state degeneracy of D-branes wrapping on cycles of a Calabi-Yau. The OSV conjecture relates the black hole entropy with the topological string partition function. We tried to understand how the factorization will work for a small black hole. In order to do that we defined a modified partition function for the closed topological string, and we studied its Gromov-Witten invariants. We have not been able to study its modular and holomorphic properties, due to some technical difficulties in the topological string for particular models. It would be nice to solve these, so we can study the properties of the modified function.

It would be an interesting project to study an open OSV conjecture. A first step would be to determine which ensemble the open topological string corresponds to, physically.

In summary, we have discussed a wide variety of applications of topological string theories, which undoubtedly have both a rich mathematical structure and also many interesting applications in physics.

Appendix A Explicit Extrapolation Formulae

We gather together some formulae discussed in Chapters 4 and 5 in this appendix. We also present the definition of modular forms at the end.

A.1 Low genus/boundary topological string holomorphic equation

The generating function for open topological string amplitudes is

$$\begin{aligned} W(\lambda, x, \varphi; t, \bar{t}) &= \sum_{g,h=0}^{\infty} \sum_{n}^{\infty} \frac{1}{n!} \lambda^{2g-2} \mu^h \mathcal{F}^{(g,h)}_{i_1,\cdots,i_n} x^{i_1} \cdots x^{i_n} \left(\frac{1}{1-\varphi}\right)^{2g-2+n+h} \\ &\quad + A \log\left(\frac{1}{1-\varphi}\right) + B\lambda^{-2}\mu^2 \log\left(\frac{1}{1-\varphi}\right), \end{aligned} \tag{A.1}$$

where A and B are coefficients to be determined later. We can substitute it into the master equation,

$$\bar{\partial}_{\bar{i}} e^{W(\lambda,x,\varphi;t,\bar{t})} = \left(\frac{\lambda^2}{2} \bar{C}^{jk}_{\bar{i}} \frac{\partial^2}{\partial x^j \partial x^k} - G_{\bar{i}j} x^j \frac{\partial}{\partial \varphi} - \mu \Delta^j_{\bar{i}} \frac{\partial}{\partial x^j}\right) e^{W(\lambda,x,\varphi;t,\bar{t})}, \tag{A.2}$$

and extract the coefficients of $\frac{1}{1-\varphi}$. For the left hand side, it's quite straightforward. For the right hand side, $\frac{\partial^2}{\partial x^j \partial x^k}$ contains two terms, with either both derivatives acting on the same term in W or on different terms in W. We need to consider two cases.

a) If $2g-2+h+n=1$, $2g+h+n=3$, the corresponding coefficients of $\frac{1}{1-\varphi}$ in W are

(g,h,n)	terms	(g,h,n)	terms
(0,0,3)	$\frac{1}{3!}\lambda^{-2}\mathcal{F}^{(0,0)}_{i_1 i_2 i_3} x^{i_1} x^{i_2} x^{i_3}$	(0,2,1)	$\lambda^{-2}\mu^2 \mathcal{F}^{(0,2)}_{i_1} x^{i_1}$
(0,3,0)	$\lambda^{-2}\mu^3 \mathcal{F}^{(0,3)}$	(1,1,0)	$\mu \mathcal{F}^{(1,1)}$
(0,1,2)	$\frac{1}{2!}\lambda^{-2}\mu \mathcal{F}^{(0,1)}_{i_1 i_2} x^{i_1} x^{i_2}$	(1,0,1)	$\mathcal{F}^{(1,0)}_{i_1} x^{i_1}$.

b) In the second case, we need to consider pairs of terms with $2g_1 - 2 - h_1 + n_1 + 2g_2 - 2 - h_2 + n_2 = 1$. The corresponding coefficients of $\frac{1}{1-\varphi}$ are

(g_1, h_1, n_1)	(g_2, h_2, n_2)	terms
(0,1,1)	(0,2,1)	$\lambda^{-2}\mu\mathcal{F}^{(0,1)}_{i_1}x^{i_1} \times \lambda\mu^2\mathcal{F}^{(0,2)}_{i_2}x^{i_2}$
(0,1,1)	(0,1,2)	$\lambda^{-2}\mu\mathcal{F}^{(0,1)}_{i_1}x^{i_1} \times \frac{1}{2!}\lambda^{-2}\mu\mathcal{F}^{(0,1)}_{i_2i_3}x^{i_2}x^{i_3}$
(0,1,1)	(0,0,3)	$\lambda^{-2}\mu\mathcal{F}^{(0,1)}_{i_1}x^{i_1} \times \frac{1}{3!}\lambda^{-2}\mathcal{F}^{(0,0)}_{i_2i_3i_4}x^{i_2}x^{i_3}x^{i_4}$
(0,1,3)	(0,0,1)	$\frac{1}{3!}\lambda^{-2}\mu\mathcal{F}^{(0,1)}_{i_1i_2i_3}x^{i_1}x^{i_2}x^{i_3} \times \lambda^{-2}\mathcal{F}^{(0,0)}_{i_4}x^{i_4}$
(0,1,2)	(0,0,2)	$\frac{1}{2!}\lambda^{-2}\mu\mathcal{F}^{(0,1)}_{i_1i_2}x^{i_1}x^{i_2} \times \frac{1}{2!}\lambda^{-2}\mathcal{F}^{(0,0)}_{i_3i_4}x^{i_3}x^{i_4}$
(1,0,1)	(0,1,1)	$\mathcal{F}^{(1,0)}_{i_1}x^{i_1} \times \lambda^{-2}\mu\mathcal{F}^{(0,1)}_{i_2}x^{i_2}$
(1,1,1)	(0,0,1)	$\mu\mathcal{F}^{(1,1)}_{i_1}x^{i_1} \times \lambda^{-2}\mathcal{F}^{(0,0)}_{i_2}x^{i_2}$
(1,0,1)	(0,0,2)	$\mathcal{F}^{(1,0)}_{i_1}x^{i_1} \times \frac{1}{2!}\lambda^{-2}\mathcal{F}^{(0,0)}_{i_2i_3}x^{i_2}x^{i_3}$
(1,0,2)	(0,0,1)	$\frac{1}{2!}\mathcal{F}^{(1,0)}_{i_1i_2}x^{i_1}x^{i_2} \times \lambda^{-2}\mathcal{F}^{(0,0)}_{i_3}x^{i_3}$.

The coefficient of $\frac{1}{1-\varphi}e^W$ in the LHS of (A.2) is

$$\lambda^{-2}\mu^3\bar{\partial}_{\bar{i}}\mathcal{F}^{(0,3)} + \lambda^{-2}\mu^2\bar{\partial}_{\bar{i}}\mathcal{F}^{(0,2)}_{i_1}x^{i_1} + \frac{1}{2}\lambda^{-2}\mu\bar{\partial}_{\bar{i}}\mathcal{F}^{(0,1)}_{i_1i_2}x^{i_1}x^{i_2} + \mu\bar{\partial}_{\bar{i}}\mathcal{F}^{(1,1)} + \bar{\partial}_{\bar{i}}\mathcal{F}^{(1,0)}_{i_1}x^{i_1}. \quad \text{(A.3)}$$

From the above, we find that the coefficient of $\frac{1}{1-\varphi}e^W$ in the RHS of (A.2) is

$$\begin{aligned} & \lambda^{-2}\mu^3 \times \left(\frac{1}{2}\bar{C}^{jk}_{\bar{i}}\mathcal{F}^{(0,1)}_j\mathcal{F}^{(0,2)}_k - \Delta^j_{\bar{i}}\mathcal{F}^{(0,2)}_j\right) \\ + \quad & \lambda^{-2}\mu^2 \times \left(\frac{1}{2}\bar{C}^{jk}_{\bar{i}}\mathcal{F}^{(0,1)}_j\mathcal{F}^{(0,1)}_{ki_1}x^{i_1} - \Delta^j_{\bar{i}}\mathcal{F}^{(0,1)}_{ji_1}x^{i_1} - BG_{\bar{i}i_1}x^{i_1}\right) \\ + \quad & \lambda^{-2}\mu \times \Bigg(\frac{1}{4}\bar{C}^{jk}_{\bar{i}}\mathcal{F}^{(0,1)}_j\mathcal{F}^{(0,0)}_{ki_1i_2}x^{i_1}x^{i_2} + \frac{1}{4}\bar{C}^{jk}_{\bar{i}}\mathcal{F}^{(0,1)}_{ji_1i_2}x^{i_1}x^{i_2}\mathcal{F}^{(0,0)}_k + \frac{1}{2}\bar{C}^{jk}_{\bar{i}}\mathcal{F}^{(0,1)}_{ji_1}x^{i_1}\mathcal{F}^{(0,0)}_{ki_2}x^{i_2} \\ & \qquad + \frac{1}{2}\bar{C}^{jk}_{\bar{i}}\mathcal{F}^{(0,0)}_{ji_1}x^{i_1}\mathcal{F}^{(0,1)}_{ki_2}x^{i_2} - \frac{1}{2}\Delta^j_{\bar{i}}\mathcal{F}^{(0,0)}_{ji_1i_2}x^{i_1}x^{i_2}\Bigg) \\ + \quad & \lambda^0\mu \times \left(\frac{1}{2}\bar{C}^{jk}_{\bar{i}}\mathcal{F}^{(0,1)}_{jk} + \frac{1}{2}\bar{C}^{jk}_{\bar{i}}\mathcal{F}^{(1,0)}_j\mathcal{F}^{(0,1)}_k + \frac{1}{2}\bar{C}^{jk}_{\bar{i}}\mathcal{F}^{(1,1)}_j\mathcal{F}^{(0,0)}_k - \Delta^j_{\bar{i}}\mathcal{F}^{(1,0)}_j\right) \\ + \quad & \lambda^0\mu^0 \times \left(\frac{1}{2}\bar{C}^{jk}_{\bar{i}}\mathcal{F}^{(0,0)}_{jki_1}x^{i_1} - AG_{\bar{i}i}x^i\right). \end{aligned} \quad \text{(A.4)}$$

Comparing the two sides we get

$$\bar{\partial}_{\bar{i}}\mathcal{F}^{(0,3)} = \frac{1}{2}\bar{C}_{\bar{i}}^{jk}\mathcal{F}_j^{(0,1)}\mathcal{F}_k^{(0,2)} - \Delta_{\bar{i}}^j\mathcal{F}_j^{(0,2)} \tag{A.5}$$

$$\bar{\partial}_{\bar{i}}\mathcal{F}_{i_1}^{(0,2)} = \frac{1}{2}\bar{C}_{\bar{i}}^{jk}\mathcal{F}_j^{(0,1)}\mathcal{F}_{ki_1}^{(0,1)} + \frac{1}{2}\bar{C}_{\bar{i}}^{jk}\mathcal{F}_{ji_1}^{(0,1)}\mathcal{F}_k^{(0,1)} - \Delta_{\bar{i}}^j\mathcal{F}_{ji_1}^{(0,1)} - BG_{\bar{i}i_1} \tag{A.6}$$

$$\begin{aligned}\bar{\partial}_{\bar{i}}\mathcal{F}_{i_1i_2}^{(0,1)} = {}& \frac{1}{4}\sum_\sigma \bar{C}_{\bar{i}}^{jk}\mathcal{F}_j^{(0,1)}\mathcal{F}_{ki_{\sigma1}i_{\sigma2}}^{(0,0)} + \frac{1}{4}\sum_\sigma \bar{C}_{\bar{i}}^{jk}\mathcal{F}_{ji_{\sigma1}i_{\sigma2}}^{(0,1)}\mathcal{F}_k^{(0,0)} + \Bigg(\frac{1}{4}\sum_\sigma \bar{C}_{\bar{i}}^{jk}\mathcal{F}_{ji_{\sigma1}i_{\sigma2}}^{(0,1)}\mathcal{F}_k^{(0,0)} \\ & + \frac{1}{2}\sum_\sigma \bar{C}_{\bar{i}}^{jk}\mathcal{F}_{ji_{\sigma1}}^{(0,1)}\mathcal{F}_{ki_{\sigma2}}^{(0,0)} + \frac{1}{2}\bar{C}_{\bar{i}}^{jk}\mathcal{F}_{ji_{\sigma_1}}^{(0,0)}\mathcal{F}_{ki_{\sigma_2}}^{(0,1)}\Bigg) - \Delta_{\bar{i}}^j\mathcal{F}_{ji_1i_2}^{(0,0)}\end{aligned} \tag{A.7}$$

$$\bar{\partial}_{\bar{i}}\mathcal{F}^{(1,1)} = \frac{1}{2}\bar{C}_{\bar{i}}^{jk}\mathcal{F}_{jk}^{(0,1)} + \frac{1}{2}\bar{C}_{\bar{i}}^{jk}\mathcal{F}_j^{(1,0)}\mathcal{F}_k^{(0,1)} + \frac{1}{2}\bar{C}_{\bar{i}}^{jk}\mathcal{F}_j^{(1,1)}\mathcal{F}_k^{(0,0)} - \Delta_{\bar{i}}^j\mathcal{F}_j^{(1,0)}, \tag{A.8}$$

and for $\lambda^0\mu^0$ term,

$$\bar{\partial}_{\bar{i}}\mathcal{F}_\ell^{(1,0)} = \frac{1}{2}\bar{C}_{\bar{i}}^{jk}\mathcal{F}_{jk\ell}^{(0,0)} - AG_{\bar{i}\ell}. \tag{A.9}$$

From the closed topological string holomorphic anomaly equation for genus one with one insertion, we can determine

$$A = \frac{\chi}{24} - 1. \tag{A.10}$$

When $h = 0$, $\mathcal{F}_{i_1,\cdots,i_n}^{(g,0)} = 0$ for $2g - 2 + n \leq 0$, so we get

$$\bar{\partial}_{\bar{i}}\mathcal{F}^{(0,3)} = \frac{1}{2}\bar{C}_{\bar{i}}^{jk}\mathcal{F}_j^{(0,1)}\mathcal{F}_k^{(0,2)} - \Delta_{\bar{i}}^j\mathcal{F}_j^{(0,2)} \tag{A.11}$$

$$\bar{\partial}_{\bar{i}}\mathcal{F}_{i_1}^{(0,2)} = \frac{1}{2}\bar{C}_{\bar{i}}^{jk}\mathcal{F}_j^{(0,1)}\mathcal{F}_{ki_1}^{(0,1)} + \frac{1}{2}\bar{C}_{\bar{i}}^{jk}\mathcal{F}_{ji_1}^{(0,1)}\mathcal{F}_k^{(0,1)} - \Delta_{\bar{i}}^j\mathcal{F}_{ji_1}^{(0,1)} - BG_{\bar{i}i_1} \tag{A.12}$$

$$\bar{\partial}_{\bar{i}}\mathcal{F}_{i_1i_2}^{(0,1)} = \frac{1}{2}\bar{C}_{\bar{i}}^{jk}\mathcal{F}_j^{(0,1)}\mathcal{F}_{ki_1i_2}^{(0,0)} + \frac{1}{2}\bar{C}_{\bar{i}}^{jk}\mathcal{F}_{ji_1i_2}^{(0,1)}\mathcal{F}_k^{(0,0)} - \Delta_{\bar{i}}^j\mathcal{F}_{ji_1i_2}^{(0,0)} \tag{A.13}$$

$$\bar{\partial}_{\bar{i}}\mathcal{F}^{(1,1)} = \frac{1}{2}\bar{C}_{\bar{i}}^{jk}\mathcal{F}_{jk}^{(0,1)} + \frac{1}{2}\bar{C}_{\bar{i}}^{jk}\mathcal{F}_j^{(1,0)}\mathcal{F}_k^{(0,1)} - \Delta_{\bar{i}}^j\mathcal{F}_j^{(1,0)}. \tag{A.14}$$

These give rise to Equations (2.87), (2.106), and (2.107) of [7]:

$$
\begin{aligned}
&(2.87) \qquad \bar{\partial}_{\bar{i}}\mathcal{F}^{(1,1)} = \frac{1}{2}\bar{C}_{\bar{i}}^{jk}\Delta_{jk} - \mathcal{F}_j^{(1,0)}\Delta_{\bar{i}}^{j} \\
&(2.106) \qquad \bar{\partial}_{\bar{i}}\mathcal{F}_{i_1}^{(0,2)} = \Delta_{i_1 k}\Delta_{\bar{i}}^{k} + \frac{N}{2}G_{\bar{i}i_1} \\
&(2.107) \qquad \bar{\partial}_{\bar{i}}\mathcal{F}^{(0,3)} = -(\bar{\partial}_{\bar{i}}\mathcal{F}_j^{(0,2)})\Delta^j - \mathcal{F}_j^{(0,2)}\bar{\partial}_{\bar{i}}\Delta^j + \frac{N}{2}\bar{\partial}_{\bar{i}}\Delta^j - \frac{1}{2}(\bar{\partial}_{\bar{i}}\Delta_{jk})\Delta^j\Delta^k \\
&\qquad\qquad -\Delta_{jk}(\bar{\partial}_{\bar{i}}\Delta^j)\Delta^k - \frac{1}{2}C_{jk\ell}(\bar{\partial}_{\bar{i}}\Delta^j)\Delta^k\Delta^\ell.
\end{aligned}
$$

Since $\bar{\partial}_{\bar{i}}\Delta_{jk} = -C_{jk\ell}\Delta_{\bar{i}}^{\ell}$, we also have

$$\bar{\partial}_{\bar{i}}\mathcal{F}^{(0,3)} = -\mathcal{F}_j^{(0,2)}\Delta_{\bar{i}}^{j}. \tag{A.15}$$

To match up Walcher's equations with these three equations, we require

$$B = -\frac{D}{2}, \qquad \text{where } D = \dim Ext^0(B,B), \tag{A.16}$$

and

$$\mathcal{F}_{jk}^{(0,1)} + \mathcal{F}_j^{(1,0)}\mathcal{F}_k^{(0,1)} = \Delta_{jk} \tag{A.17}$$

$$\bar{C}_{\bar{i}}^{jk}\mathcal{F}_k^{(0,1)}\mathcal{F}_{ji_1}^{(0,1)} = -\Delta_{\bar{i}}^{j}\mathcal{F}_j^{(1,0)}\mathcal{F}_{i_1}^{(0,1)} \tag{A.18}$$

$$\bar{C}_{\bar{i}}^{jk}\mathcal{F}_j^{(0,1)}\mathcal{F}_k^{(0,2)} = 0. \tag{A.19}$$

These equations require

$$\mathcal{F}_i^{(0,1)} = 0, \quad \mathcal{F}_{jk}^{(0,1)} = \Delta_{jk}. \tag{A.20}$$

(A.13) is Walcher's (2.99), with the definition $C_{ijk} = \mathcal{F}_{ijk}^{(0,0)}$. We then conclude $\mathcal{F}_{i_1,\cdots,i_n}^{(g,h)} = 0$, for $2g - 2 + h + n \leq 0$.

A.2 New approach to factorize Yang-Mills partition function

A.2.1 Topological string side

The partition function for topological string theory with $t = g_s N_+ + i\theta$ is

$$\psi_{N_+} = F(N_+) \oint \frac{dx}{x} \prod_{p>0} (1 + x e^{-tp} q^{p^2/2}) \prod_{p'>0} (1 + x^{-1} e^{-tp'} q^{-(p')^2/2}). \tag{A.21}$$

For the bottom fermi surface with $\bar{t} = g_s N_- - i\theta$, we have similarly

$$\bar{\psi}_{N_-} = F(N_-) \oint \frac{dx}{x} \prod_{p>0} (1 + x e^{-\bar{t}p} q^{p^2/2}) \prod_{p'>0} (1 + x^{-1} e^{-\bar{t}p'} q^{-(p')^2/2}). \tag{A.22}$$

The prefactor is defined as

$$F(N) = e^{-\frac{N^2}{2}t} e^{g_s(\frac{N^3}{3} + \frac{N}{24})}. \tag{A.23}$$

We can expand ϕ_{N_+} and ϕ_{N_-} as

$$\psi_{N_+} = e^{-\frac{N_+^2}{2}t} e^{g_s(\frac{N_+^3}{3} + \frac{N_+}{24})} \sum_n f_n(q) e^{-tn}, \tag{A.24}$$

$$n = 0, \sum_i (p_i + p_i'), \quad (p_i, p_i' = \frac{1}{2}, \frac{3}{2}, \cdots); \tag{A.25}$$

$$\bar{\psi}_{N_-} = e^{-\frac{N_-^2}{2}\bar{t}} e^{g_s(\frac{N_-^3}{3} + \frac{N_-}{24})} \sum_n f_m(q) e^{-\bar{t}m}, \tag{A.26}$$

$$m = 0, \sum_i (p_i + p_i'), \quad (p_i, p_i' = \frac{1}{2}, \frac{3}{2}, \cdots). \tag{A.27}$$

For fixed n and m, f_n and f_m have finite terms of q expansion,

$$f_n(q) = q^{\sum_{p_i} p_i^2/2 - \sum_{p_i'} (p_i')^2/2}. \tag{A.28}$$

The product of the chiral and anti-chiral topological string partition function is,

$$\sum_{N_++N_-=N} \psi_{N_+}\bar{\psi}_{N_-} = \sum_{N_++N_-=N} G(N_+,N_-) \sum_{n=0,m=0} f_n(q) f_m(q) e^{-tn-\bar{t}m}, \qquad \text{(A.29)}$$

where

$$G(N_+,N_-) := e^{-\frac{1}{2}(N_+^2 t + N_-^2\bar{t})} e^{\frac{g_s}{3}(N_+^3+N_-^3)+\frac{g_s(N_++N_-)}{24}}. \qquad \text{(A.30)}$$

A.2.2 Yang-Mills side

The partition function for 2-dimensional Yang-Mills theory is

$$\begin{aligned} Z_N &= \prod_{p=-\infty}^{p=+\infty} (1+x^{-1}e^{-i\theta p}e^{g_s p^2/2})\Big|_{x^N} \\ &= \sum_{N_++N_-=N} \prod_{p>0} (1+xe^{-tp}e^{-g_s p^2/2+g_s N_+ p})\Big|_{x^{N_+}} \prod_{p>0} (1+xe^{-\bar{t}p}e^{-g_s p^2/2+g_s N_- p})\Big|_{x^{N_-}} \\ &= \sum_{N_++N_-=N} \left[e^{-tN_+^2+\frac{g_s}{2}N_+^3} \prod_{p>-N_+} (1+xe^{-tp}e^{-g_s p^2/2})\Big|_{x^{N_+}} \right] \\ &\qquad \times \left[e^{-\bar{t}N_-^2+\frac{g_s}{2}N_-^3} \prod_{p>-N_-} (1+xe^{-\bar{t}p}e^{-g_s p^2/2})\Big|_{x^{N_-}} \right]. \end{aligned}$$

We get

$$Z_N = \sum_{N_++N_-=N} \Psi_{N_+}(t)\Psi_{N_-}(\bar{t}), \qquad \text{(A.31)}$$

where we define the modified topological string partition function as

$$\Psi_N = e^{-tN^2+\frac{g_s}{2}N^3} \prod_{p>-N} (1+xe^{-tp}e^{-g_s p^2/2})\Big|_{x^N}. \qquad \text{(A.32)}$$

Now we want to prove the recursive relation between our newly-defined function Ψ and the topological string partition function ψ,

$$\Psi_N = \psi_N - \sum_{k>0} \Psi_{N+k}\Psi_{-k}. \qquad \text{(A.33)}$$

We write down the product explicitly,

$$\Psi_{N+k}\Psi_{-k} = e^{-t_{(N+k)}(N+k)^2+\frac{g}{2}(N+k)^3} \prod_{p>-(N+k)} (1+xe^{-t_{(N+k)}p}e^{-gp^2/2})\Big|_{x^{N+k}} \times e^{-t_{(-k)}k^2-\frac{g}{2}k^3} \prod_{p>-k} (1+x^{-1}e^{-t_{(-k)}p}e^{gp^2/2})\Big|_{x^{-k}}. \quad \text{(A.34)}$$

We can use the following formula,

$$\prod_{p>-(N+k)} (1+xe^{-t_{(N+k)}p}e^{-gp^2/2})\Big|_{x^{N+k}} \prod_{p>-k} (1+x^{-1}e^{-t_{(-k)}p}e^{gp^2/2})\Big|_{x^{-k}} = \prod_{p>-(N+k)} (1+xe^{-tp}e^{-\frac{g}{2}(p+k)^2+\frac{g}{2}k^2})\Big|_{x^{N+k}} \prod_{p>-k} (1+x^{-1}e^{-tp}e^{\frac{g}{2}[p+(N+k)]^2-\frac{g}{2}(N+k)^2})\Big|_{x^{-k}} \quad \text{(A.35)}$$

Shifting the variable in the first product by $p \to p-k$ and the one in the second product by $p \to p-(N+k)$, this becomes

$$\prod_{p>-N} (1+xe^{-t(p-k)}e^{-\frac{g}{2}p^2+\frac{g}{2}k^2})\Big|_{x^{N+k}} \prod_{p>N} (1+x^{-1}e^{-t[p-(N+k)]}e^{\frac{g}{2}p^2-\frac{g}{2}(N+k)^2})\Big|_{x^{-k}} = e^{tk(N+k)+\frac{g}{2}k^2(N+k)} \prod_{p>-N} (1+xe^{-tp}e^{-\frac{g}{2}p^2})\Big|_{x^{N+k}} \times e^{t(N+k)k-\frac{g}{2}(N+k)^2k} \prod_{p>N} (1+x^{-1}e^{-tp}e^{\frac{g}{2}p^2})\Big|_{x^{-k}}. \quad \text{(A.36)}$$

Using the same method as before,

$$\prod_{p>-N} (1+xe^{-tp}e^{-\frac{g}{2}p^2})\Big|_{x^{N+k}} \prod_{p>N} (1+x^{-1}e^{-tp}e^{\frac{g}{2}p^2})\Big|_{x^{-k}} = \prod_{p=-N+1/2}^{-1/2} e^{-tp}e^{-\frac{g}{2}p^2} \prod_{p>0}(1+xe^{-tp}e^{-\frac{g}{2}p^2}) \prod_{p=1/2}^{N-1/2} (1+x^{-1}e^{-tp}e^{\frac{g}{2}p^2})\Big|_{x^k} \times \prod_{p>N} (1+x^{-1}e^{-tp}e^{\frac{g}{2}p^2})\Big|_{x^{-k}} = e^{\frac{N^2}{2}t}e^{-\frac{g}{2}(\frac{N^3}{3}-\frac{N}{12})} \prod_{p>0}(1+xe^{-tp}e^{-\frac{g}{2}p^2}) \prod_{p=1/2}^{N-1/2} (1+x^{-1}e^{-tp}e^{\frac{g}{2}p^2})\Big|_{x^k} \times \prod_{p>N} (1+x^{-1}e^{-tp}e^{\frac{g}{2}p^2})\Big|_{x^{-k}}. \quad \text{(A.37)}$$

All of the prefactors in (A.34), (A.36), and (A.37) give rise to a prefactor

$$e^{-t_{(N+k)}(N+k)^2+\frac{g}{2}(N+k)^3}e^{-t_{(-k)}k^2-\frac{g}{2}k^3}e^{tk(N+k)+\frac{g}{2}k^2(N+k)}e^{t(N+k)k-\frac{g}{2}(N+k)^2k}e^{\frac{N^2}{2}t}e^{-\frac{g}{2}(\frac{N^3}{3}-\frac{N}{12})}$$
$$= e^{-\frac{N^2}{2}t}e^{\frac{g}{3}N^3+\frac{g}{24}N}, \tag{A.38}$$

so finally we get

$$\begin{aligned}
&\sum_{k=0}\Psi_{N+k}\Psi_{-k} \\
&\quad = e^{-\frac{N^2}{2}t}e^{\frac{g}{3}N^3+\frac{g}{24}N}\sum_{k=0}\prod_{p>0}(1+xe^{-tp}e^{-\frac{g}{2}p^2})\prod_{p=1/2}^{N-1/2}(1+x^{-1}e^{-tp}e^{\frac{g}{2}p^2})\Big|_{x^k} \\
&\qquad \times\prod_{p>N}(1+x^{-1}e^{-tp}e^{\frac{g}{2}p^2})\Big|_{x^{-k}} \\
&\quad = e^{-\frac{N^2}{2}t}e^{\frac{g}{3}N^3+\frac{g}{24}N}\sum_{k=0}\prod_{p>0}(1+xe^{-tp}e^{-\frac{g}{2}p^2})\Big|_{x^k}\prod_{p>0}(1+x^{-1}e^{-tp}e^{\frac{g}{2}p^2})\Big|_{x^{-k}} \\
&\quad = \psi_{_N}.
\end{aligned} \tag{A.39}$$

The power expansion of the partition function is

$$Z_N = \sum_{N_++N_-=N} e^{-tN_+^2-\bar{t}N_-^2+\frac{g_s}{2}N_+^3+\frac{g_s}{2}N_-^3}\sum_{n=-\frac{N_+^2}{2}}^{\infty}h_n(q)e^{-tn}\sum_{m=-\frac{N_-^2}{2}}^{\infty}h_m(q)e^{-\bar{t}m}. \tag{A.40}$$

Shifting n and m by $-\frac{N_+^2}{2}$, and defining

$$H_n(q) := h_{n-\frac{N_+^2}{2}}(q), \tag{A.41}$$

we get the partition function of Yang-Mills theory,

$$Z_N = \sum_{N_++N_-=N} G(N_+,N_-)\sum_{n=0,m=0}H_n(q)H_m(q)e^{-tn}e^{-\bar{t}m}, \tag{A.42}$$

where

$$n = \sum_{p_i>0} p_i - \sum_{p'_i \in [-N_+ + 1/2, -1/2]} p'_i = \sum_{p_i>0} p_i + \sum_{p'_i \in [1/2, N_+ - 1/2]} p'_i, \qquad \text{(A.43)}$$

and

$$H_n(q) := h_{n - \frac{N_+^2}{2}}(q) = q^{\sum_{p_i>0} p_i^2/2 - \sum_{p'_i \in [1/2, N_+ - 1/2]} (p'_i)^2/2}. \qquad \text{(A.44)}$$

A.3 (Almost) modular forms

Eisenstein series are particular modular forms of the modular group $SL(2, \mathbb{Z})$. For $k \geq 2$, the Eisenstein series is defined as

$$G_{2k}(\tau) = \sum_{(m,n) \neq (0,0)} \frac{1}{(m + n\tau)^{2k}}. \qquad \text{(A.45)}$$

Therefore it is a modular form of weight $2k$. This series is absolutely convergent to a holomorphic function of τ in the upper half-plane. Under $SL(2, \mathbb{Z})$ transformation,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \tau \mapsto \tau, \qquad a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1, \qquad \text{(A.46)}$$

the Eisenstein series transforms as

$$G_{2k}\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^{2k} G_{2k}(\tau). \qquad \text{(A.47)}$$

An alternative definition of the Eisenstein series is

$$E_{2k}(\tau) = \frac{G_{2k}(\tau)}{2\zeta(2k)}, \qquad \text{(A.48)}$$

where $\zeta(z)$ is the Riemann Zeta function. The series contains 3 basic forms and their derivatives,

$$E_2(\tau) = 1 - 24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}, \tag{A.49}$$

$$E_4(\tau) = 1 + 240\sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^n}, \tag{A.50}$$

$$E_6(\tau) = 1 - 504\sum_{n=1}^{\infty}\frac{n^5q^n}{1-q^n}, \tag{A.51}$$

where $q = e^{2\pi i\tau}$. The recursive equations for derivatives are

$$q\frac{dE_2}{dq} = \frac{E_2^2 - E_4}{12}, \tag{A.52}$$

$$q\frac{dE_4}{dq} = \frac{E_2E_4 - E_6}{3}, \tag{A.53}$$

$$q\frac{dE_6}{dq} = \frac{E_2E_6 - E_4^2}{2}. \tag{A.54}$$

There is a recursion structure to relate high-weight modular forms to low-weight ones, and eventually to E_4 and E_6:

$$E_{k+2} = E_kE_2 + \frac{12}{k}E_k'. \tag{A.55}$$

E_2 is not quite a modular form, but it is a holomorphic function. We can convert it to a modular form, however we will lose holomorphicity. We recall that the Dedekind eta function,

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n), \tag{A.56}$$

is a modular form of weight 1/2. Therefore we can define a modular form of weight 2,

$$\hat{E}_2(\tau,\bar{\tau}) = \frac{12}{2\pi i}\partial\log(\tau-\bar{\tau})|\eta^2(\tau)|^2 = \frac{12}{2\pi i}\left[\frac{1}{(\tau-\bar{\tau})} + 2\frac{\eta'(\tau)}{\eta(\tau)}\right], \tag{A.57}$$

such that $\hat{E}_2(\tau,\bar{\tau})$ is related to $E_2(\tau)$ by

$$\hat{E}_2(\tau,\bar{\tau}) = E_2(\tau) + \frac{12}{2\pi i}\frac{1}{(\tau-\bar{\tau})}. \tag{A.58}$$

The holomorphicity is restored only in the limit $\mathrm{Im}\tau \to \infty$.

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