THE REARRANGEMENT OF FUNCTIONS AND
MAXIMIZATION OF A CONVOLUTION INTEGRAL

Thesis by
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In Partial Fulfillment of the Requirements
For the Degree of
Doctor of Philosophy

California Institute of Technology
Pasadena, California
1958
ACKNOWLEDGEMENTS

I am indebted to Professor Bohnenblust for the valuable help that he has given me as my advisor. He suggested this problem to me and has given his time and advice very generously during the past three years.

I wish to thank the California Institute of Technology for the financial assistance given me during the past four years. Special thanks go to my wife for her patience and encouragement.
ABSTRACT

For the additive group of the real numbers, the value, at the origin, of the convolution of three non-negative integrable functions is less than or equal to the corresponding value obtained from the symmetric rearrangements of those functions. That is

$$f * g * h(0) \leq f^* \ast g^* \ast h^*(0).$$

This problem is investigated on general groups and it is shown that under a wide interpretation of the notion of symmetric rearrangement the validity of the inequality implies serious restrictions on the group.

Both new and known results in $\mathbb{E}^n$ are developed using methods which in modified form yield the answer to the similar problem on the sphere.
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INTRODUCTION. Let $G$ be a locally compact topological group whose unit element is denoted by '$e$'. It is known that such a group has a left invariant measure which is unique up to a multiplicative constant.\(^1\) Left invariance means that $\mu(A) = \mu(sA)$ for every element $s$ of $G$. The uniqueness of this measure implies the existence of a function $\Delta(s)$ such that $\mu(As) = \Delta(s) \mu(A)$. If this function is identically equal to one, the group $G$ is called unimodular.

The convolution, $f \ast g$, of two integrable functions $f, g$ is defined by

$$f \ast g(t) = \int f(x) g(x^{-1} t) \, dx = \int f(tx) g(x^{-1}) \, dx.$$  

If one function is bounded the convolution function is defined everywhere and is continuous.\(^2\)

We shall be concerned with bounded, integrable, non-negative functions. For the additive group of the real numbers, it is known that for any three such functions

$$f \ast g \ast h(e) \leq f \ast g \ast h^*(e)$$

where $f^*$ is the symmetric rearrangement of $f$, and similarly for $g$ and $h$.\(^3\)

The main objective of this study is to investigate this problem on general groups. It is shown that under a wide interpretation of

\(1\) Ref. 1.

\(2\) Ref. 2, p. 50.

\(3\) cf. Ref. 3.
'The symmetric rearrangement of a function' the validity of the above inequality implies serious restrictions on the group $G$. These restrictions are discussed in the first section and a result for the group of rotations in two dimensions is obtained. The second section deals with the group of translations in two dimensions. Although the main result was known, the methods used are more constructive and more direct. Furthermore, properly modified, they are applied in section 3 to a study of functions defined on a sphere. The Appendix contains the proofs of some measure theoretic results which are needed in the main part of the text.

**First Section.** The family of non-negative integrable functions is designated $L^1^+$. 

**Def.** Two functions $f$, $g$ will be said to be a rearrangement of one another if for every $a$, $\mu\{x|f(x) > a\} = \mu\{x|g(x) > a\}$.

This notion is obviously an equivalence relation and will be denoted by $f \sim g$.

On the additive group of real numbers, in each equivalence class of functions there exists one symmetrically decreasing function. This member of a class is called the symmetric rearrangement of any function in the class. On a general group it is not evident whether or not a natural notion of symmetric rearrangement exists. To investigate this question the following two hypotheses are introduced.
Hypothesis A. There exists three mappings from $L^{1+}$ into itself taking $f$ into $f^1$, $i = 1, 2, 3$, such that $f \sim f^1$, $f \sim g$ implies $f^1 = g^1$ and for all $f, g, h \in L^{1+}$

$$f \ast g \ast h(e) \leq f^1 \ast g^2 \ast h^3(e).$$

Hypothesis B. Hypothesis A is satisfied by three mappings which are identical.

The additive group of real numbers satisfies the stronger hypothesis $B$. The additive group of integers is an example where $A$ but not $B$ is satisfied provided the domain of the third mapping is restricted to those functions in $L^{1+}$ which assume their maximum an odd number of times and other values an even number of times.

If a function is in the range of the $i$-th mapping it will be called symmetric of type $i$, and if the function is also the characteristic function of a set the corresponding set will also be called symmetric of type $i$.

THEOREM 1. If $G$ satisfies Hypothesis $A$ then $G$ is unimodular.

Proof: We first prove that if $\Phi_A$, $\Phi_B$, $\Phi_C$ are the characteristic functions of sets of positive measure $A$, $B$, $C$ and $G$ is not unimodular then

$$\max_{A' \sim A, B' \sim B, C' \sim C} \Phi_A' \ast \Phi_B' \ast \Phi_C'(e) = \mu(A) / \mu(B).$$

4) For $f, g, h$ non-negative and measurable, the convolution is defined everywhere provided $+\infty$ is allowed as a value. Thus $f \ast g \ast h(e)$ exists.

5) Cf. Ref. 4 and Ref. 5.
Secondly we prove that this can not always be achieved by symmetric sets.

**First Part:** Since \( \Phi_{c'}(x) \leq 1 \) for all \( x \), \( \Phi_{a'} \ast \Phi_{b'} \ast \Phi_{c'}(e) \leq \mu(A) \mu(B) \).

On the other hand the upper bound is obtained as follows. By the corollary to Theorem B (Appendix) there exist sets \( A_1, B_1 \) with compact closure such that \( \mu(A_1) = \mu(A), \mu(B_1) = \mu(B) \). Define

\[
D = \left\{ z \mid \mu(A_1 \cap z^{-1} B_1^{-1}) > 0 \right\}.
\]

Then \( D \subseteq B_1^{-1} A_1^{-1} \) and the compactness of the closure of this last set implies \( \mu(D) < \infty \). By considering \( \Delta(z^n) = [\Delta(z)]^n \) where \( n \) is any positive or negative integer, it is clear that \( t \) can be chosen so that \( \Delta(t) \) is arbitrarily close to zero. In particular there exists \( t \) such that \( \mu(Dt) = \Delta(t) \mu(D) < \mu(C) \) (since \( \mu(C) > 0 \)). Again by Theorem B there exists \( C' \) such that \( Dt \subseteq C' \) and \( \mu(C') = \mu(C) \).

Now consider \( \Phi_{t^{-1}A_1}, \Phi_{B_1}, \Phi_{C'} \). We have

\[
\Phi_{t^{-1}A_1} \ast \Phi_{B_1} \ast \Phi_{C'}(e) \geq \Phi_{t^{-1}A_1} \ast \Phi_{B_1} \ast \Phi_{Dt}(e) \\
= \Phi_{A_1} \ast \Phi_{B_1} \ast \Phi_{D}(e).
\]

Using the definition of \( D \) this becomes

\[
\int \int \Phi_{A_1}(x) \Phi_{B_1}(x^{-1}y) \Phi_D(y^{-1}) \, dx \, dy = \int \int \Phi_{A_1}(x) \Phi_{B_1}(x^{-1}y) \, dx \, dy \\
= \mu(A) \mu(B)
\]

proving the first part of the proof.
Second Part: Let $A, B, C$ be symmetric of type 1, 2, 3 respectively and of positive measure. Hypothesis A implies

$$\mu(A) \mu(B) = \Phi_A * \Phi_B * \Phi_C(e) = \int_{C^{-1}} \mu(A \cap zB^{-1}) \, dz$$

$$= \int_C \mu(A \cap z^{-1}B^{-1}) \, \Delta(z) \, dz.$$ 

On the other hand if $\mu(C) < \mu\{z | \mu(A \cap z^{-1}B^{-1}) > 0\}$ then the above equality is obviously impossible. Furthermore to be non-unimodular the group can not be discrete so that $C$ can be chosen (and thus a symmetric $C$) so as to have a positive measure which satisfies the above inequality. ($\mu\{z | \mu(A \cap z^{-1}B^{-1}) > 0\} > 0$ by Theorem B.) Q.E.D.

For the additive group of the real numbers, the symmetric sets are $S_a = \{x | |x| \leq a/2\}$ and the symmetric rearrangement of a positive integrable function $f$ is defined as follows. Define

$$a(t) = \mu\{x | f(x) > t\} \text{ and } f^*(x) = \sup \{t | x \in S_a(t)\}.$$ 

That $f^*$ is a rearrangement of $f$ can be seen as follows. $x \in S_a(t_1)$ and $t_1 > t$ implies $f^*(x) \geq t_1 > t$. Thus $S_a(t_1) \subseteq \{x | f^*(x) > t\}$. On the other hand if $f^*(x) > t$ then there exists $t_1$ such that $t_1 > t$ and $x \in S_a(t_1)$. By the monotonicity of the family of sets $S_a$ and the monotonicity of the function $a(t)$, this implies $x \in S_a(t)$. Thus $S_a(t_1) \subseteq \{x | f^*(x) > t\} \subseteq S_a(t)$ and $a(t_1) \leq \mu\{x | f^*(x) > t\} \leq a(t)$.

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6) Haar measure on a discrete group assigns unit mass to each point and thus is right invariant also.

Since $a$ is monotone decreasing and continuous on the right,

$$a(t) = \mu \left\{ x \mid f^*(x) > t \right\}$$

which proves that $f^*$ is a rearrangement of $f$. That Hypothesis B is satisfied for this symmetrization is a simple consequence of the validity of the inequality for characteristic functions of sets.\(^8\)

The above discussion need not be restricted to the real line.

If there are three monotone classes of sets such that Hypothesis A is valid when $L^{1+}$ is replaced by the class of characteristic functions of sets then it can be extended to $L^{1+}$ by the above construction. Furthermore it is a consequence of the corollary to Theorem C (Appendix) that the symmetric functions given by Hypothesis A differ on sets of measure zero from those constructed from symmetric sets. The next theorem shows that if Hypothesis A is valid on a group the three monotone classes exist.

THEOREM 2. If Hypothesis A holds the symmetric sets of type $i$ form a monotone family of open sets to within measure zero. (That is, there exists a family $\left\{ \zeta^i_\alpha \mid \alpha \in \text{range of } \mu \right\}$ such that $\alpha < \beta$ implies $\zeta^i_\alpha \subset \zeta^i_\beta$, $\mu(\zeta^i_\alpha) = \alpha$ and $\mu[\zeta^i_\alpha \oplus \zeta^i_\alpha] = 0$.

Proof: The unimodularity implies that cyclic permutations of the three functions in the double convolution leave the value at the origin invariant so that any cyclic permutation of the three symmetrizations also satisfies Hypothesis A and it will suffice to prove that the first symmetrization has the stated property.

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\(^8\) Cf. ibid., p. 279–280.
First suppose $G$ discrete, then we can normalize the measure so that each point has one unit weight. Let $C$ be symmetric of type 3 and measure one and let $B$ be symmetric of type 2 and measure $n$. Then

$$
\Phi_A * \Phi_B * \Phi_C(e) = \int_{c^{-1}} \mu[A \cap yB^{-1}] \, dy = \mu[A \cap c^{-1}B^{-1}].
$$

It is obvious that if $A$ is symmetric of type 1 and $\mu(A) = n$ then $A = c^{-1}B^{-1}$ in order for the maximum to be achieved. Likewise if $\mu(A) < n$ then $A \subset c^{-1}B^{-1}$ and if $\mu(A) > n$ then $A \supset c^{-1}B^{-1}$ and the monotonicity is established.

Notice that the proof in the discrete case made use of sets of the form $\{ x| \Phi_B * \Phi_C(x) \geq t \}$ to separate $A$'s with different measure. This same method will be used in the non-discrete case. Let $A_1^1, A_2^1$ be symmetric of type 1 with $\mu(A_1^1) < \mu(A_2^1)$. Since $G$ is not discrete there exists $A_3^1$ such that $\mu(A_1^1) < \mu(A_3^1) < \mu(A_2^1)$.

Furthermore by Theorem B (Appendix) there exists $A_2, A_3$ such that $A_1^1 = A_1 \subset A_3 \subset A_2$, $\mu(A_2) = \mu(A_2^1)$ and $\mu(A_3) = \mu(A_2^1)$. Now let $g$ be an approximate identity for $\Phi_{A_3}^{-1}$, that is, $g \geq 0$, $\int g = 1$ and

$$
||g * \Phi_{A_3}^{-1} - \Phi_{A_3}^{-1}||_1 < \epsilon.
$$

Finally let $F(x) = g^2 * \Phi_{A_3}^3(x)$ and $t_1, t_2, t_3$ be the numbers whose existence is asserted by Theorem C (Appendix). Then, to within measure zero $A_1^1 \supset F_{t_1}, A_2^1 \supset F_{t_2}, A_3^1 \supset F_{t_3}$, and $t_1 \geq t_3 \geq t_2$.

9) To separate in the sense that if $A \leq B < C$ the $B$ separates $A$ and $C$.

10) Non-discreteness implies there are sets of arbitrarily small positive measure (see Ref. 6, pp. 123-124) and the assertion then follows from Theorem B.

11) See Theorem C for definition of $F_t$. 

Also by Theorem C if \( t_2 < t_1 \) then \( F_{t_2} \supseteq A_1^1 \) to within measure zero and \( A_2^1 \supseteq A_1^1 \) which proves monotonicity to within measure zero.

That \( \varepsilon \) can be chosen so as to make \( t_2 < t_1 \) can be seen as follows. By unimodularity,

\[
|\Phi_{A_1} \ast g \ast \Phi_{A_3}^{-1}(e) - \Phi_{A_1} \ast \Phi_{A_3}^{-1}(e)| = |(g \ast \Phi_{A_3}^{-1} - \Phi_{A_3}^{-1}) \ast \Phi_{A_1}^{-1}(e)|.
\]

Since \((g \ast \Phi_{A_3}^{-1} - \Phi_{A_3}^{-1}) \ast \Phi_{A_1}^{-1}(x)\) is continuous [see footnote 2] supremum of absolute value and \( || \cdot ||_\infty \) are the same so that

\[
(1) \quad |\Phi_{A_1} \ast g \ast \Phi_{A_3}^{-1}(e) - \Phi_{A_1} \ast \Phi_{A_3}^{-1}(e)| \leq ||(g \ast \Phi_{A_3}^{-1} - \Phi_{A_3}^{-1}) \ast \Phi_{A_1}^{-1}| \leq \varepsilon \quad ||\Phi_{A_1}^{-1}||_\infty = \varepsilon.
\]

The last inequality is the theorem in Ref. 6, page 121. Now

\[
\Phi_{A_1} \ast \Phi_{A_3}^{-1}(e) = \mu[A_1 \cap \overline{A}_3] = \min[\mu(A_1), \mu(A_3)]
\]

so that (1) implies

\[
\min[\mu(A_1), \mu(A_3)] - \varepsilon \leq \Phi_{A_1} \ast g \ast \Phi_{A_3}^{-1}(e).
\]

Symmetrization does not decrease the double convolution so that

\[
(2) \quad \min[\mu(A_1), \mu(A_3)] - \varepsilon \leq \Phi_{A_1}^{-1} \ast g^2 \ast \Phi_{A_3}^{-1}(e) \leq \min[\mu(A_1), \mu(A_3)].
\]

The last inequality is obtained from the facts that \( \int g^2 = 1 \), and \( \Phi_{A_1} \cdot \Phi_{A_3}^{-1} \leq 1 \). Now by Theorem C (Appendix),
\[ (3) \quad \Phi_{A_1}^1 * g^2 * \Phi_{A_3}^3 (e) = \int_{A_1^1} F(x) \, dx = \int_{F_{t_1}} F(x) \, dx + [\mu(A_1) - \mu(F_{t_1})]_{t_1}. \]

If \( t_1 = t_2 = t_3 \) then by (2) and (3) applied to \( i = 1 \) and 3,

\[ [\mu(A_3) - \mu(A_1)]_{t_1} = \Phi_{A_3}^1 * g^2 * \Phi_{A_3}^3 (e) - \Phi_{A_1}^1 * g^2 * \Phi_{A_3}^3 (e) > \mu(A_3) - \mu(A_1) - \epsilon \]

while (2) and (3) applied to \( i = 2 \) and 3 yields

\[ [\mu(A_2) - \mu(A_3)]_{t_1} = \Phi_{A_2}^1 * g^2 * \Phi_{A_3}^3 (e) - \Phi_{A_2}^1 * g^2 * \Phi_{A_3}^3 (e) < \epsilon. \]

If \( \epsilon = (1/2) \min[\mu(A_2) - \mu(A_1), \mu(A_2) - \mu(A_3)] \) then the inequalities can not be simultaneously satisfied, so that it must be that \( t_2 < t_1 \).

\( F(x) \) is continuous [see footnote 2] so that \( F_{t_2} \) is open. Thus we have shown that if \( \mu(A_1) < \mu(A_1) \) then there exists an open set, \( O_i \) which separates \( A_1 \) and \( A_1 \) to within measure zero. Choosing a sequence \( A_1 \) such that \( \mu(A_1) \uparrow \mu(A_1) \) shows that \( \bigcup_{i} O_i \subset A_1 \) to within measure zero and \( \mu(\bigcup_{i} O_i) \geq \mu(0_n) \uparrow \mu(A_1) \). Thus \( \bigcup_{i} O_i \) is an open set which differs from \( A_1 \) by a set of measure zero.

Q.E.D.

Theorem 2 says that any three mappings of \( L^{1+} \) into itself for which Hypothesis A is valid is not essentially different from symmetrizations based on three monotone classes of open sets. The proof can be made to yield more in the discrete case, namely, Hypothesis A implies Hypothesis B. For the proof shows that \( A_a = c^{-1}B_{a}^{-1} \) where
where \( \{ c \} = C_1 \). Applying this result to the cyclic permutations of the three classes gives \( B_a = a^{-1} C_a^{-1} \), \( C_a = b^{-1} A_a^{-1} \) from which it can be shown that \( A_a a^{-1} = a B_a c = c^{-1} C_a \). Furthermore unimodularity gives

\[
\Phi_{A_a^{-1}} * \Phi_{abc} * \Phi_{c^{-1}C}(e) = \Phi_A * \Phi_B * \Phi_C(e).
\]

So that the symmetrization based on the class \( A_a a^{-1} \) shows Hypothesis B to be satisfied. It is conjectured that in general Hypothesis A is valid if and only if Hypothesis B is valid.

A further restriction on any \( G \) for which Hypothesis A is valid is the following.

**THEOREM 3.** If there exists \( M \) and \( N \), subgroups of positive measure, such that \( \mu(M) < \mu(N) \) and \( \mu(N)/\mu(M) \) is not an integer then Hypothesis A is not valid.

**Proof:** We assume Hypothesis A and begin by showing that \( M^1 \) and \( N^1 \) are "almost" subgroups.

Now

\[
\Phi_M * \Phi_M * \Phi_M(e) = \int_M \int_{M^{-1}} \Phi_M(xy^{-1}) \, dx \, dy = [\mu(M)]^2
\]

and, since this is also an upper bound over all rearrangements,

\[
\int_M \mu[M^2 \cap x(M^3)^{-1}] \, dx = \Phi_M * \Phi_M * \Phi_M(e) = [\mu(M)]^2.
\]

Since \( \mu[M^2 \cap x(M^3)^{-1}] \leq \mu(M) \), then \( M^1 \subseteq \overline{M} = \{ x | \mu[M^2 \cap x(M^3)^{-1}] = \mu(M) \} \) except for a set of measure zero. Otherwise the upper bound could not be attained. On the other hand
\[ \mu(M) \mu(\overline{M}) = \int \mu[M^2 \cap x(M^3)^{-1}] \, dx \leq \int \mu [M^2 \cap x(M^3)^{-1}] \, dx = [\mu(M)]^2 \]

so that \( \mu(\overline{M}) = \mu(M) \) and \( M^1 = \overline{M} \) except for a set of measure zero.

Next it is claimed that if \( m \in \overline{M} \) then \( \overline{Mm^{-1}} \) is a group, which will be shown as follows. Now the measure of the symmetric difference,

\[ M^2 \oplus x(M^3)^{-1}, \]

is \( \mu \left\{ M^2 \oplus x(M^3)^{-1} \right\} = \mu \{ (M^3)^{-1} \} + \mu \{ M^2 \} - 2 \mu \{ M^2 \cap x(M^3)^{-1} \} \).

But unimodularity implies \( \mu \{ (M^3)^{-1} \} = \mu \{ (M^3) \} = \mu \{ (M^2) \} = \mu(M) \).

Thus \( x \in \overline{M} \) if and only if \( \mu \{ M^2 \oplus x(M^3)^{-1} \} = 0 \). It follows from the triangular inequality for the measure of the symmetric difference of sets that

\[ x \in \overline{Mm^{-1}} \quad \text{if and only if} \quad \mu \{ M^2 \oplus xM^2 \} = 0. \]

Finally, if \( x, y \in \overline{Mm^{-1}} \) then

\[ \mu (M^2 \oplus xy^{-1} M^2) \leq \mu (M^2 \oplus xM^2) + \mu (xM^2 \oplus xy^{-1} M^2) \]

\[ = \mu (M^2 \oplus xM^2) + \mu (yM^2 \oplus M^2) = 0, \]

so that \( xy^{-1} \in \overline{Mm^{-1}} \). Which proves that \( \overline{Mm^{-1}} \) is a group. Similar arguments hold for \( \overline{N} \). Since \( M^1 \subseteq N^1 \) and \( \overline{M}, \overline{N} \) differ from \( M^1, N^1 \) by sets of measure zero, we have \( \mu(\overline{M} \cap \overline{N}) = \mu(M) > 0 \) and so \( \overline{M} \cap \overline{N} \) is non-empty. Let \( m \in \overline{M} \overline{N} \), then \( \overline{Mm^{-1}} \) and \( \overline{Nm^{-1}} \) are subgroups of \( G \) as also is \( (\overline{M} \cap \overline{N}) m^{-1} \). Further, \( \mu \{ (\overline{M} \cap \overline{N}) m^{-1} \} = \mu(M) \). Now \( \overline{Nm^{-1}} \) consists of a union of cosets of \( (\overline{M} \cap \overline{N}) m^{-1} \) so that \( \mu(N) = \mu(\overline{Nm^{-1}}) = \mu(M) \) times the number of cosets of \( (\overline{M} \cap \overline{N}) m^{-1} \) in \( \overline{Nm^{-1}} \). But this contradicts the condition of the theorem that \( \mu(N)/\mu(M) \) be finite but not an integer.

Q.E.D.
Theorem 3 eliminates from consideration any finite groups other than those whose order is a power of a prime. The latter case is still open.

**Examples.** The group of translations in Euclidean n-space is an example of a group in which Hypothesis B holds. The case of the real line has already been cited. The higher dimensional cases can be obtained from an extension of the work of Gross \(^{12}\) (or from later parts of this paper). Gross constructed a suitable Steiner sequence \(^{13}\) in 3-space such that beginning with a given integrable set the successive sets converge to the interior of a sphere. By convergence we mean \(\mu(A_i \oplus S) \to 0\) as \(i \to \infty\), where \(\oplus\) indicates the symmetric difference of two sets. This together with Theorem 5 shows that Hypothesis B holds where symmetric sets are the interiors of concentric spheres.

It is conjectured that Hypothesis B holds on the circle.

However the following theorem is the best result available.

**THEOREM 4.** Let \(f, g, h\) be integrable functions on the group of plane rotations (represented as \(\{\emptyset | -\pi \leq \emptyset < \pi\}\)). Furthermore let \(O_a = \{\emptyset | |\emptyset| < a/2\}\) for \(0 \leq a \leq 2\pi\) and \(f^1\) be the symmetrization of \(f\) based on the class \(O_a\) and similarly for \(g^1, h^1\). \(^{14}\) Then

\[
f \ast g \ast h^1(0) \leq f^1 \ast g^1 \ast h^1(0)
\]

\(^{12}\) Ref. 7.

\(^{13}\) See Definition preceding Theorem 5.

\(^{14}\) \(f^1\) is also called the symmetrically decreasing rearrangement in analogy with the corresponding rearrangement on the real line.
Proof: By the remarks preceding Theorem 2, the general result follows from the validity of the inequality in the case where \( f, g, h \) are the characteristic functions of sets. Let \( A, B, C \) be integrable sets, with measure \( 2a, 2b, 2c \), respectively (for convenience \( \mu(G) = 2\pi \)) and define

\[
F(\emptyset) = \Phi_B^* \Phi_C^*(-\emptyset)
\]

Then \( \Phi_A^* \Phi_B^* \Phi_C^*(\emptyset) = \int_A F(\emptyset) \, d\emptyset \).

Theorem C (Appendix) asserts the existence of \( t \) such that

\[
\mu(F_t) \leq 2a \leq \mu \left\{ \emptyset | F(\emptyset) \geq t \right\}
\]

where

\[
F_t = \left\{ \emptyset | F(\emptyset) > t \right\}.
\]

For each \( a \) designate by \( t(a) \) the infimum of all such \( t \). Notice that \( F_1(\emptyset) = t(|\emptyset|) \) and

\[
\int_{A} F(\emptyset) \, d\emptyset \leq \int_{F_t(a)} F(\emptyset) \, d\emptyset + t(a)[2a - \mu(F_t(a))] = \int_{-a}^{a} F_1(\emptyset) \, d\emptyset.
\]

The proof of the theorem reduces to showing

\[
\int_{-a}^{a} F_1(\emptyset) \, d\emptyset \leq \int_{-a}^{a} \Phi_B^1 \Phi_C^1(-\emptyset) \, d\emptyset
\]

at this point we introduce Lemma 2 which will be proved later.

**Lemma 2.** \( |t(a_1) - t(a_2)| \leq |a_1 - a_2| \)
Let us now examine

\[ F^* (\phi) \equiv \text{DEF. } \Phi_B^1 \ast \Phi_C^1 (-\phi) \, d\phi \]

(1) \[ F^* (\phi) = \begin{cases} \min[2b,2c] & \text{for } |\phi| \leq |b-c| \\ \max[0, 2b + 2c - 2\pi] & \text{for } |\phi| \geq \min[b+c, 2\pi-b-c] \\ \text{linear and continuous for all } \phi \end{cases} \]

In order for \( F^* \) to be linear and continuous the derivative must be

\(-\text{sgn } \phi \) for \( |b-c| < |\phi| < \min[b+c, 2\pi-b-c] \). With the use of the definition of \( F \) and \( F^1 \) it can be shown that

(2) \[ \min[2b,2c] \geq F^1 (\phi) \geq \max[0, 2b+2c-2\pi] \]

and

(3) \[ \int_{-\pi}^{\pi} F^1 (\phi) \, d\phi = \int_{-\pi}^{\pi} F^* (\phi) \, d\phi = 4bc. \]

If \( a < |b-c| \) or \( a > \min[b+c, 2\pi-b-c] \) then the inequality

\[ \int_{-a}^{a} F^1 (\phi) \, d\phi \leq \int_{-a}^{a} F^* (\phi) \, d\phi \]

readily follows from (1), (2) and (3). Now suppose

\[ F^1 (a) \leq F^* (a) \]

and

\[ |b-c| \leq a \leq \min[b+c, 2\pi-b-c]. \]

Then by Lemma 2

\[ F^1 (|x|) - F^1 (a) \leq |a - |x|| \]

so that for \( |b-c| \leq |x| \leq a \),
\[ F^1(|x|) = F^1(a) + F^1(|x|) - F^1(a) \leq F^*(a) + a - |x| = F^*(x) . \]

Also
\[ F^1(x) \leq \min[2b,2c] = F^*(x) \quad \text{for} \quad |x| \leq |b-c| \]

so that
\[ \int_{-a}^{a} [F^*(x) - F^1(x)] \, dx \geq 0 . \]

On the other hand suppose
\[ F^1(a) \geq F^*(a) . \]

Then again using Lemma 2 for \( a \leq |x| \leq \min[b+c, 2\pi-b-c] \) gives
\[ F^1(x) = F^1(a) + F^1(x) - F^1(a) \geq F^*(a) - (|x| - a) = F^*(x) . \]

Also
\[ F^1(x) \geq \max[0, 2b+2c-2\pi] = F^*(x) \quad \text{for} \quad \min[b+c, 2\pi-b-c] \leq |x| \]

so that
\[ \int_{-a}^{a} [F^*(x) - F^1(x)] \, dx = -\int_{|x|>a} [F^*(x) - F^1(x)] \, dx \geq 0 \]

Q.E.D.

In what follows \( \Phi_B, \Phi_G, F, F^1 \) will be assumed to be periodic functions on the real line with period \( 2\pi \). To prove Lemma 2 we shall need

**Lemma 1.** \[ |F(x_1) - F(x_2)| \leq |x_1 - x_2| . \]
Proof: We can obviously assume \( x_1 \geq x_2 \). Then

\[
|F(x_1) - F(x_2)| = \left| \int_{-\pi}^{\pi} \Phi_B(y) \Phi_C^{1}(\cdot-x_1 - y) \, dy - \int_{-\pi}^{\pi} \Phi_B(y) \Phi_C^{1}(\cdot-x_2 - y) \, dy \right|
\]

\[
= \left| \int_{-x_1 - \epsilon}^{-x_1} \Phi_B(y) \, dy - \int_{-x_2 - \epsilon}^{-x_2} \Phi_B(y) \, dy \right|
\]

\[
= \left| \int_{-x_2}^{-x_1} [\Phi_B(y - \epsilon) - \Phi_B(y + \epsilon)] \, dy \right|
\]

\[
\leq \int_{-x_2}^{-x_1} |\Phi_B(y - \epsilon) - \Phi_B(y + \epsilon)| \, dy
\]

\[
\leq x_1 - x_2.
\]

**Lemma 2.** \( |t(a_1) - t(a_2)| \leq |a_1 - a_2| \).

Proof: We may suppose \( t(a_1) < t(a_2) \) and thus \( a_1 > a_2 \). The proof consists of constructing two intervals in \( F_{t(a_1)} \cap F_{t(a_2)} \), where \( \bar{A} \) denotes the complement of \( A \), and using Lemma 1 to estimate their length. Suppose \( 0 < a_2 \) and \( a_1 < \pi \). Then by Theorem C (Appendix)

\[
\mu\left( x | F(x) \geq t(a_i) \right) \geq 2a_i > 0 \quad i = 1, 2
\]

and

\[
\mu\left( x | F(x) \leq t(a_i) \right) = 2\pi - \mu(F_{t(a_i)}) \geq 2\pi - 2a_i > 0.
\]

Since these sets have positive measure they are non-empty. Furthermore Lemma 1 shows \( F(x) \) to be continuous. Thus there exists \( x_i \), \( i = 1, 2 \) such that

\[
F(x_i) = t(a_i).
\]
The cases $a_2 = 0$ or $a_1 = \pi$ can be handled by taking limits and pointing out that the circle is compact. Since translations of $F$ do not affect the statement of the lemma we can assume $x_1 = -\pi$ and $-\pi < x_2 < \pi$.

Let

$$\varphi_1 = \sup \left\{ x \mid -\pi \leq x < x_2 \quad \text{and} \quad F(x) = t(a_1) \right\}$$

$$\varphi_2 = \inf \left\{ x \mid x_2 < x \leq +\pi \quad \text{and} \quad F(x) = t(a_1) \right\}.$$ 

Both of the sets exhibited are non-empty since $F(-\pi) = F(\pi) = t(a_1)$ so that $\varphi_1$, $\varphi_2$ are finite and $F(\varphi_1) = F(\varphi_2) = t(a_1)$ by continuity of $F$. Next let

$$\varphi'_1 = \inf \left\{ x \mid \varphi_1 < x \leq x_2 \quad \text{and} \quad F(x) = t(a_2) \right\}$$

$$\varphi'_2 = \sup \left\{ x \mid x_2 \leq x < \varphi_2 \quad \text{and} \quad F(x) = t(a_2) \right\}.$$ 

A similar argument shows that $\varphi'_1$, $\varphi'_2$ are finite and $F(\varphi'_1) = F(\varphi'_2) = t(a_2)$.

Now $x \in (\varphi_1, \varphi'_1) \cup (\varphi_2, \varphi'_2)$ implies $t(a_1) < F(x) < t(a_2)$ which tells us that $\left\{ x \mid F(x) \geq t(a_2) \right\}$, $(\varphi_1, \varphi'_1)$, $(\varphi'_2, \varphi_2)$ are disjoint and their union is contained in $F_{t(a_1)}$. Then it follows from the definition of $t(a_2)$ and $t(a_1)$ that

\[
2a_2 + \varphi'_1 - \varphi_1 + \varphi_2 - \varphi'_2 \leq \mu(F_{t(a_1)}) \leq 2a_1.
\]

By Lemma 1

$$t(a_1) - t(a_2) \leq \varphi'_1 - \varphi_1, \varphi_2 - \varphi'_2$$

so that (1) becomes

$$2[t(a_1) - t(a_2)] \leq 2(a_1 - a_2).$$

Q.E.D.
Second Section. Steiner Symmetrization.

By the direct product of two groups, \( G = G_1 \times G_2 \), we mean the set of all ordered pairs \( <g_1, g_2> \) with \( g_1 \in G_1 \) and multiplication defined by \( <g_1, g_2> \cdot <h_1, h_2> = <g_1h_1, g_2h_2> \). The groups \( G_1, G_2 \) are isomorphic to subgroups of \( G \), namely \( G_1 \sim G_1 \times e, G_2 \sim e \times G_2 \). Hereafter \( G_1, G_2 \) will be identified with these subgroups and two decompositions will be considered distinct if at least one pair of corresponding subgroups are. If \( G_1, G_2 \) have left invariant measures, \( \mu_1 \) and \( \mu_2 \), then it is obvious that the product measure is a left invariant measure on \( G \).

Now let \( G = H \times R \) where \( R \) is a subgroup identified with the real numbers, and let \( \mu_1, \mu_2 \) be the left invariant measure on \( H \) and Lebesque measure respectively. \( \mu_2^* \) denotes Lebesque outer measure.

**DEF.** If \( A \) is a measurable subset of \( G \) then generalized Steiner symmetrization of \( A \) with respect to the subgroup \( H \) is

\[
A_H = \left\{<h, r>| h \in H, r \in R \quad \text{and} \quad |r| \leq \frac{1}{2} \mu_2^* \left\{ r|<h, r> \in A \right\} \right\}
\]

If \( \mu_2^* \left\{ r|<h, r> \in A \right\} = 0 \) then the condition

\[
|r| \leq \frac{1}{2} \mu_2 \left\{ r|<h, r> \in A \right\}
\]

is replaced by the condition \( r \) is an element of the empty set or \( r = 0 \) accordingly as \( \left\{ r|<h, r> \in A \right\} \) is empty or non-empty.

In more descriptive terms, generalized Steiner symmetrization replaces the intersection of \( A \) with the line \( h \times R \) by an interval with the same linear measure and centered at \( <h, 0> \). By Fubini's

15) For definition of product measure see Ref. 6, pp. 44-45.
theorem \( \mu(A_H) = \mu(A) \) so that \( A_H \) is a rearrangement of \( A \).

The group of translations in Euclidean \( n \)-space is the \( n \)-dimensional vector space over the real numbers and admits of infinitely many representations of the form \( H_i \times R_i \) where \( R_i \) is a linear 1-dimensional vector space (and thus isomorphic to the real numbers) and \( H_i \) is an \( n-1 \)-dimensional subgroup. If we introduce a scalar product then we have available the concept of perpendiculaity so that the following definition makes sense.

**DEF.** If \( V_n \) is the group of translations in Euclidean \( n \)-space, \( H \) is an \((n-1)\)-dimensional subgroup and \( R \) is a 1-dimensional subgroup perpendicular to \( H \), then the generalized Steiner symmetrization is called the Steiner symmetrization. 16

Since generalized Steiner symmetrization takes monotone classes of sets into monotone classes of sets, the Steiner symmetrization of non-negative functions can be defined by means of the construction of symmetric functions in terms of symmetric sets which follows Theorem 1.

**DEF.** If \( \{H_i\} \) be a sequence of \((n-1)\)-dimensional subgroups and \( A \) an integrable set in \( V_n \) and if \( A^o = A, A^i = (A^{i-1})_{H_i} \), then \( \{A^i\} \) is a Steiner sequence.

**THEOREM 5.** Generalized Steiner symmetrization never increases

\[ f \ast g \ast h(e) \]

provided \( f, g, h \) are non-negative integrable functions.

---

16) Cf. Ref. 8, pp. 75-91.
Proof: Let \( G = G_1 \times \mathbb{R} \). \( s \) and \( t \) will be elements of \( G_1 \) and \( x \) and \( y \) will be real numbers.

\[
 f * g * h(s) = \int_{G_1} \int_{G_1} \int_{\mathbb{R}} \int_{\mathbb{R}} f(s, x) \ g(s^{-1} t, y-x) \ h(t^{-1}, -y) \ dx \ dy \ ds \ dt.
\]

The order of integration has been rearranged using Fubini's theorem. Now for almost all \( s \) and \( t \), \( f(s, ) \), \( g(s^{-1} t, ) \) and \( h(t^{-1}, ) \) are measurable functions on \( \mathbb{R} \) so that the validity of Hypothesis B on the additive group of real numbers implies for almost all \( s \) and \( t \)

\[
 \int_{\mathbb{R}} \int_{\mathbb{R}} f(s, x) \ g(s^{-1} t, y-x) \ h(t, -y) \ dx \ dy \leq \int_{\mathbb{R}} \int_{\mathbb{R}} f^1(s, x) \ g^1(s^{-1} t, y-x) \ h^1(t^{-1}, -y) \ dx \ dy.
\]

\( f^1(s, ) \) is the symmetrically decreasing rearrangement of \( f(s, ) \) and similarly for \( g(s^{-1} t, ) \) and \( h(t^{-1}, ) \). Integrating both sides of the above inequality over \( G_1 \times G_1 \) and observing that \( f^1(s, x) \) is in fact the generalized Steiner symmetrization with respect to \( G_1 \) gives the theorem.

Q.E.D.

Steiner symmetrization was originally defined for convex sets in the plane in connection with the isoperimetric problem. It was shown in the reference in footnote 16 that Steiner symmetrization does not increase the perimeter of convex polygons. More generally it can be shown for any polygonal Jordan curve, \( \Gamma \), with a finite number of sides that the symmetrization of the curve and its interior results in a point set whose boundary is a rectifiable curve whose length is not greater than the length of \( \Gamma \). (It will not necessarily be a Jordan curve.)
By suitable polygonal approximations the results extend to more general rectifiable closed curves. However, the following theorem has a direct proof.

**THEOREM 6.** Let \( \Gamma \) be a rectifiable closed curve in the plane such that the winding number of every point not on the curve is 0 or 1. Let \( A \) be the set of all points which lie on \( \Gamma \) or whose winding number is 1. If \( A_L \) is the Steiner symmetrization with respect to the line \( L \), then the boundary of \( A_L \) is contained in a curve whose length is less than or equal to the length of \( \Gamma \) and which has the same winding number property as \( \Gamma \).

The proof is very similar to the proof of Theorem 11 and so will be omitted.

The rest of this section will be restricted to Steiner symmetrization in the plane although some extension to higher dimensions is possible.

Let \( \{L_1\} \) be an arbitrary sequence of one dimensional subgroups of \( V_2 \) and \( \{R_1\} \) the perpendicular subgroups. Further let \( A \) be a set with a rectifiable, connected curve for its boundary and \( \{A^1\} \) be the Steiner sequence generated by \( A \) and \( \{L_1\} \). If we rotate the sets \( A^i \) such that their axes of symmetry are all \( L_1 \), then each \( A^i \) can be represented in the form

\[
A^i = \{ <x,y> | <x,0> \in L_1, <0,y> \in R_1 \text{ and } |y| \leq F_i(x) \} .
\]

17) The winding number is also called the topological index or merely index. For the definition see Ref. 9, p. 149.
For the perimeter to be finite and connected, \( A \) must be bounded, i.e., there exists \( M \) such that \( \langle x, y \rangle \in A \) implies \( x^2 + y^2 \leq M^2 \). Therefore \( F_1(\pm M) = 0 \). The perimeter of \( A^1 \) is a bound for the variation of \( F_1 \) and this is bounded by the perimeter of \( A \) by theorem 6. Thus the functions \( F_1 \) have a uniform bound on their variation and \( F_1(\pm M) = 0 \).

By Helly's selection principle there exists a subsequence which converges almost everywhere to a function \( F \) and the corresponding subsequence of sets converges in measure to the corresponding set. It is not difficult to show that even without rotating the \( A^1 \) to a common line of symmetry there exists a subsequence which converges in measure. The following theorem shows that the restriction that \( A \) have a finite perimeter can be weakened.

**THEOREM 7.** Let \( L_1, R_1, L_2, R_2, F_1, F_2 \) be as before and also \( L_1 \neq L_2 \).
If \( A \) is a bounded integrable set bounded by \( M \) then \( F_2 \) is of bounded variation and the perimeter of \( A^2 \) is finite.

**Proof:** We choose two coordinate systems and an orientation in the plane so that

\[
L_1 = \{ (x', y') | y' = 0 \}, \quad L_2 = \{ (x, y) | y = 0 \}
\]

and the angle, \( \alpha \), from \( L_2 \) to \( L_1 \) is positive but less than or equal to \( \pi/2 \).

First suppose \( \alpha = \pi/2 \), then the second symmetrization is seen to be the same as replacing \( A^1 \) by the area bounded by the symmetric decreasing rearrangement of \( F_1(x') \). The symmetric decreasing rearrangement is monotone increasing in the second quadrant, monotone decreasing
in the first quadrant, \( F^*(-M) = F^*(M) = 0 \) and \( F^*(O) \leq M \). Thus \( \var F^* \leq 2M \). We can define the perimeter as
\[
2 \sup \sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + (F^*(x_i) - F^*(x_{i-1}))^2}.
\]
The supremum is taken over all partitions of the form \(-M < x_0 < x_1 \cdots < x_n = M\).
A bound for the perimeter is \( 2 \var F^* + 4M \leq 8M \).

Next suppose \( 0 < a < \pi/2 \). Let \( \mu_1 \) be Lebesque measure on the line and define
\[
F_{21}(x) = \mu_1 \left\{ y \mid (x, y) \in A_1 \quad \text{and} \quad y > x \tan a \right\}.
\]
\[
F_{22}(x) = \mu_1 \left\{ y \mid (x, y) \in A_1 \quad \text{and} \quad y \leq x \tan a \right\}.
\]
The above exhibited sets can be shown measureable for all \( x \) and it is obvious that
\[
F_2(x) = F_{21}(x) + F_{22}(x).
\]
The function \( F_{21}(x) + x/\sin a \cos a \) will now be shown to be monotone increasing. Suppose \( x_1 > x_2 \), then
\[
F_{21}(x_1) - F_{21}(x_2) + \frac{x_1 - x_2}{\sin a \cos a}
\]
\[
= \int_{x_1 \tan a}^{\infty} \Phi_{A_1}(x_1, y) \, dy - \int_{x_2 \tan a}^{\infty} \Phi_{A_1}(x_2, y) \, dy + \frac{x_1 - x_2}{\sin a \cos a}
\]
\[
= \int_{x_1 \tan a}^{\infty} \left[ \Phi_{A_1}(x_1, y) - \Phi_{A_1}(x_2, y + \frac{x_1 - x_2}{\tan a}) \right] \, dy
\]
\[
- \int_{x_2 \tan a}^{\infty} \Phi_{A_1}(x_2, y) \, dy + \frac{x_1 - x_2}{\sin a \cos a}
\]
\( \leq \int_{x_1 \tan a}^{\infty} \left[ \Phi_{A_1}(x_1, y) - \Phi_{A_1}(x_2, y + \frac{x_1 - x_2}{\tan a}) \right] \, dy,
\]
(1)
Now if \( \Phi_{A1} (x_2, y + \frac{x_1 - x_2}{\tan a}) = 1 \) then the point \( P(x_2, y + \frac{x_1 - x_2}{\tan a}) \) is in \( A^1 \) and its primed coordinates must satisfy

\[ |y'| \leq F_1 (x') \]

where
\[
x' = x_2 \cos a + \sin a (y + \frac{x_1 - x_2}{\tan a}) = x_1 \cos a + y \sin a
\]
\[
y' = -x_2 \sin a + \cos a (y + \frac{x_1 - x_2}{\tan a}) = -x_1 \sin a + y \cos a + \frac{x_1 - x_2}{\sin a}.
\]

Expressing (2) in the unprimed system gives

\[ -x_1 \sin a + y \cos a + \frac{x_1 - x_2}{\sin a} \leq F_1 (x_1 \cos a + y \sin a). \]

The last term on the left hand side of the inequality is positive and, in the range of integration of (1), \(-x_1 \sin a + y \cos a\) is non-negative so that in the range of integration (3) implies

\[ |-x_1 \sin a + y \cos a| \leq F_1 (x_1 \cos a + y \sin a). \]

But this says that \( \Phi_{A1} (x_1, y) = 1 \). We have just shown that in the range of integration \( \Phi_{A1} (x_2, y + x_1 - x_2/\tan a) = 1 \) implies \( \Phi_{A1} (x_1, y) = 1 \). Therefore the integral in (1) is non-negative and the monotonicity of \( F_{21}(x) + x/\sin a \cos a \) is established. Furthermore \( F_{21}(-M) = F_{21}(M) = 0 \) so that \( F_{21}(x) + x/\sin a \cos a \) is of bounded variation in \([-M, M]\). A similar argument is valid for \( F_{22} \) so that \( F_2 = F_{21} + F_{22} \) is of bounded variation. Defining the perimeter as in the case \( a = \pi/2 \) completes the theorem.
The existence of subsequences which converge naturally raises
the question, "Under what conditions does a Steiner sequence converge?" That not every Steiner sequence of a bounded integrable set converges can be seen by the following example. Let the angle \( a_n \), between \( L_n \) and \( L_{n+1} \) be \( 1/n \) and let \( A \) be a circle whose center is on \( L_1 \) but not
\(<0,0>\). Then if \( r_o \) is the distance of the center of \( A \) from \(<0,0>\) then \( A^n \) is a circle whose center is at a distance \( r_n = r_o \prod_{i=1}^{n-1} \cos a_i \) from \(<0,0>\). The infinite product converges absolutely to a non-zero limit and the lines \( L_n \) intersect the circle \( \sqrt{x^2+y^2} = r_o \prod_{i=1}^{\infty} \cos a_i \) in an everywhere dense set. Therefore the sequence of centers have all points of the circle as limit points and \( A^n \) does not converge.
It is conjectured that if \( \sum_{n=1}^{\infty} |a_n| < \infty \) or \( \sum_{n=1}^{\infty} a_n^2 = \infty \) and if \( \tilde{A} \) is bounded then the Steiner sequence converges in measure. The following theorem will enable us to show convergence of Steiner sequences provided the sequence \( \{L_i\} \) is periodic.

THEOREM 8. Let \( A \) be a bounded integrable set in \( V_2 \), \( \{L_i\} \) a sequence of one dimensional subspaces and \( R_i \) the perpendicular subspaces.
Then the Steiner sequence obtained from \( A \) and \( \{L_i\} \) has the property that
\[
\sum_{n=1}^{\infty} \mu(A^n \oplus A^{n+1})^3 < \infty ,
\]
The proof will not be presented here since it is analogous to and somewhat simpler than the proof to Theorem 12.

COROLLARY. \( \mu(A^n \oplus A^{n+N}) \to 0 \) as \( n \to \infty \).
The proof is obvious.
THEOREM 9. If the sequence \( \{L_n\} \) is periodic and \( A \) is bounded the Steiner sequence converges in measure to a set having all the \( L_n \) as lines of Steiner symmetry.

Proof: Let \( \{L_1\} \) have period \( p \). According to Theorem 7 and the discussion preceding it there exists a subsequence \( A_k^n \) converging to a set \( A^* \). At least one of the subgroups \( L_1, \ldots, L_p \), say \( L_s \), must occur infinitely often among \( \{L_{n_k}\} \) and it can be seen that \( A^* \) must be Steiner symmetric with respect to \( L_s \) to within measure zero. Now consider the sequences \( L_{n_k+N} \) for \( N = 1, \ldots, p-1 \). The line \( L_{s+N} \) must occur infinitely often in this sequence. But since \( \mu(A^n \oplus A^{n+N}) \to 0 \) as \( n \to \infty \), this sequence also converges to \( A^* \). Thus \( A^* \) is Steiner symmetric to within measure zero with respect to all the lines \( \{L_1\} \).

It can be shown that

\[
\mu(A \oplus B) \geq \mu(A_L \oplus B_L),
\]

that is, Steiner symmetrization does not increase symmetric differences.

This fact coupled with the fact that \( A^* \) is invariant under symmetrization with respect to all \( \{L_1\} \) proves that the sequence \( A^i \) converges.

Q.E.D.

Polya and Szegö point out in Ref. 11 that a Steiner sequence in 3-space based upon two planes whose included angle is an irrational multiple of \( \pi \) must converge to a solid of revolution about the line of intersection of the two planes if the sequence converges at all.\(^{18}\)

\(^{18}\) They also point out that the "if" has been removed in the case of convex solids. Cf. Ref. 10, pp. 86-90.
By observing what happens in each plane perpendicular to that line of intersection and using the preceding results, we can remove the "if" for all bounded integrable sets in 3-space.

**Third Section. Steiner Symmetrization on a Sphere.**

Although the sphere is not a group manifold there is a symmetrization analogous to Steiner symmetrization in the plane and several of the theorems carry over to the sphere. In particular, the isoperimetric property (Theorem 6), the convergence to zero of symmetric differences in a Steiner sequence (Theorem 8) and the convergence of a Steiner sequence based on a periodic sequence of coordinate systems (Theorem 9) all hold on the sphere. On the other hand, Theorem 5 (the decrease in the value of the convolution of three positive functions) must be modified before it is valid on the sphere.

Let $C$ be a spherical coordinate system. That is, $C$ consists of a mapping $P(\theta, \phi)$, which maps $[-\frac{\pi}{2}, \frac{\pi}{2}] \times [-\pi, \pi]$ onto the unit sphere such that the element of area, $dP$, is $\cos \theta \, d\theta \, d\phi$. ($\theta$ is the complement of the polar angle and $\phi$ is the azimuthal angle.) We will designate Lebesgue measure on $[-\pi, \pi]$ by $\mu_1$ and Lebesgue outer measure by $\mu_1^*$.

**Def.** Let $A$ be an integrable set on the sphere. Define

$$A_{C, \theta} = \left\{ \phi \mid \phi \leq \frac{1}{2} \mu_1^* \left\{ \phi \mid P(\theta, \phi) \in A \right\} \right\}$$

if $\mu \left\{ \phi \mid P(\theta, \phi) \in A \right\} \neq 0$. Otherwise define $A_{C, \theta}$ to be empty or $\{0\}$ accordingly as $\left\{ \phi \mid P(\theta, \phi) \in A \right\}$ is empty or not.
DEF. \[ A_C = \{ P(\theta, \rho) : \rho \in A_C, \theta \} \].

In attempting to construct a theorem analogous to Theorem 5, two difficulties occur. The sphere is not a group manifold and the necessary property of the real line is replaced by Theorem 4 which applies to the circle. Theorem 4 is adequate for the following theorem and the first difficulty is avoided.

THEOREM 10. Let \( \rho(P, Q) \) be the spherical distance from \( P \) to \( Q \), and let \( H(\rho) \) be a monotone decreasing function of \( \rho \). Further let \( F, G \) be non-negative integrable functions on the sphere. Then the Steiner symmetrization of \( F, G \) does not decrease

\[ I = \iint F(P)G(Q)H(\rho(P,Q)) \, dP \, dQ. \]

**Proof:** As before the general case follows from the particular case where \( F \) and \( G \) are the characteristic functions of measurable sets on the sphere, and \( H \) is the characteristic function of an interval \( 0 \leq \rho \leq h \). Writing the integral in the coordinate system \( CG \) and interchanging the order of integration

\[ I = \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} \left[ \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} F(\theta, \phi)F(\theta', \phi')H(\rho(\phi, \phi', \phi', \phi')) \, d\phi \, d\phi' \right] \sin \theta \cos \theta \, d\theta \, d\theta'. \]

Consider \( H(\rho(\theta, \phi, \theta', \phi')) \). For fixed \( \theta, \phi \) this is the characteristic function of the spherical disc consisting of all points \( (\theta', \phi') \) whose distance from \( (\theta, \phi) \) is less than or equal to \( h \).
The circle \( \Theta' = \text{constant} \) either does not intersect this region or intersects in a connected arc of the form \([\phi - a, \phi + a]\) where \(a\) is a function of \(\Theta, \Theta'\). Define \(H_{\Theta, \Theta'}(\phi)\) to be the characteristic function of the set, \(\{\phi : |\phi| \leq a\}\). Then

\[
I = \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} \left[ \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} F(\Theta, \phi) \ G(\Theta', \phi') \ H_{\Theta, \Theta'}(\phi - \phi') \ d\phi \ d\phi' \right] 
\cdot \cos \Theta \cos \Theta' \ d\Theta \ d\Theta'.
\]

Since \(H_{\Theta, \Theta'}(x) = H_{\Theta, \Theta'}(-x)\) and \(G(\Theta', -\phi')\) is a rearrangement of \(G(\Theta', \phi')\), Theorem 4 applies to the bracketed expression and yields

\[
I \leq \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} F^1(\Theta, \phi) \ G^1(\Theta', \phi') \ H(\rho (P, Q)) 
\cdot \cos \Theta \cos \Theta' \ d\phi \ d\phi' \ d\Theta \ d\Theta',
\]

where

\[
F^1(\Theta, \phi) = \begin{cases} 
1 & \text{if } |\phi| \leq \frac{1}{2} \mu_1 \{\phi | F(\Theta, \phi) = 1\} \\
0 & \text{otherwise}
\end{cases}
\]

and a similar expression for \(G^1\). Obviously \(F^1\) is the Steiner symmetrization of \(F\) with respect to \(C\) to within measure zero, and the theorem is proved.

The statement of the analog to Theorem 6 is somewhat complicated by the properties of the sphere. Let the sphere be represented in Euclidean 3-space by \(z = \sin \Theta, z = \cos \Theta \cos \phi, y = \cos \Theta \sin \phi\) and consider the stereographic projection of the sphere from \((-1, 0, 0)\) onto
the yz plane. The winding number of a point of the sphere relative to a curve bounded away from \((-1,0,0)\) will be defined as the winding number of the stereographic image relative to the image of the curve.

**Theorem 11.** Let \(C\) be as before and let \(\Gamma\) be a rectifiable curve, bounded away from the point whose coordinates are \((0,-\pi)\), such that the winding number of every point not in \(\Gamma\) is either 0 or 1. Define \(A\) to be the union of \(\Gamma\) with the set of all points whose winding numbers are 1. Then the boundary of \(A\) is contained in a curve \(\Gamma'\) whose length is not greater than the length of \(\Gamma\) and which has the same winding number property.

**Proof:** \(\Gamma\) is a closed set and the limit points of a set of points with winding number 1 either have winding number 1 or lie on \(\Gamma\). Thus \(A\) is closed and \(\{\emptyset|P(\theta,\emptyset) \in A\}\) is a closed set on the unit circle and so is measurable. Thus \(P(\theta) \overset{\text{def}}{=} \frac{1}{2} \mu_1 \{\emptyset|P(\theta,\emptyset) \in A\}\) exists for all \(\theta\).

Define \(\Theta_1 = \inf \left\{\theta | \exists \emptyset \text{ such that } P(\theta,\emptyset) \in \Gamma\right\}\)

\(\Theta_2 = \sup\) of the same set.

Now if the cap, \(\{P(\theta,\emptyset)|\theta < \Theta_1\}\), possessed both points in \(A\) and not in \(A\), there would be points of \(\Gamma\) with \(\theta\)-coordinates less than \(\Theta_1\).

Thus the caps, \(\{P(\theta,\emptyset)|\theta < \Theta_1\}\) and \(\{P(\theta,\emptyset)|\theta > \Theta_2\}\) consist entirely of interior points or exterior points of \(A\). By the definition of Steiner symmetrization on the sphere this implies each cap is either interior or exterior to \(A\) so that \(A\) has no boundary points with \(\theta < \Theta_1\) or \(\theta > \Theta_2\).

---

19) Cf. Ref. 9.
A simple argument shows that the boundary points of \( A_0 \) are contained in \( B \) where

\[
B = \left\{ \{\phi, \theta) \mid \phi_1 \leq \theta \leq \phi_2, \ \min[F(\theta), \lim_{x \to \theta} F(x)] \leq |\phi| \leq \max[F(\theta), \lim_{x \to \theta} F(x)] \right\}
\]

In fact the only additional points this set has are of the form \((\theta, u)\) and are thrown in to make the boundary connected.

Define

\[
L(\theta) = \lim_{|\sigma| \to 0} \sum_{i=1}^{n} \sqrt{(\xi_1 - \xi_{i-1})^2 + \cos^2 \theta_i [F(\xi_i) - F(\xi_{i-1})]^2}
\]

where \( \lim \) is taken over all partitions, \( \sigma = \{x_0 < \xi_1 < x_1 = \theta_1 < \xi_2 < x_2 \cdots < x_n = \theta \} \) and \( |\sigma| = \max_i (x_i - x_{i-1}) \). In the event \( \theta_1 = -u/2 \) the first term is omitted. It is obvious that we are trying to "measure" the perimeter and the first term nicely takes care of boundary points of the cap, \( \theta < \theta_1 \) whether it consists of interior points or exterior points. The cap \( \theta > \theta_2 \) is taken care of by using \( L(\theta_2^+) = \lim_{x \to \theta^+} L(x) \). The + sign is omitted if \( \theta_2 = u/2 \). First we show that \( 2L(\theta_2^+) \) is less than or equal to the length of \( \Gamma \). Lastly we will exhibit the curve \( \Gamma' \) which has length \( 2L(\theta_2^+) \), maps onto \( B \), and has the right winding number property.

Proof that \( 2L(\theta_2^+) \) is less than or equal to the length of \( \Gamma \).

The following lemma is used and its proof will be found in the Appendix.
LEMMA TO THEOREM 11. Let $\Gamma$, $F(\theta)$, $\sigma$ be as before and let $\Gamma$ have length $S$ and arc length $s$ as parameter. Let

$$I_1(\sigma) = \left\{ s \mid x_{i-1} < \theta(s) < x_i \quad \text{or} \quad \theta(s) = x_{i-1} \quad \text{and} \quad \frac{d\theta}{ds} > 0 \right\}$$

or $\theta(s) = x_i$ and $\frac{d\theta}{ds} < 0$.

Then $\theta(s)$, $\sigma(s)$ are absolutely continuous functions of $s$ and

$$2|F(x_i) - F(x_{i-1})| \leq \int_{I_1(\sigma)} \left| \frac{d\sigma}{ds} \right| \, ds.$$

The lemma is used as follows

1. $4 \cos^2 \xi \int_1 |F(x_i) - F(x_{i-1})|^2 \leq \cos^2 \xi \left[ \int_{I_1(\sigma)} \left| \frac{d\sigma}{ds} \right| \, ds \right]^2$

   $$\leq \left[ \int_{I_1(\sigma)} \cos \theta(s) \left| \frac{d\theta}{ds} \right| \, ds \right]^2$$

   $$+ 2(x_i - x_{i-1}) \left[ \int_{I_1(\sigma)} \cos \theta(s) \left| \frac{d\theta}{ds} \right| \, ds \right]$$

   $$\left[ \int_{I_1(\sigma)} \left| \frac{d\sigma}{ds} \right| \, ds \right].$$

From (1) it follows that

2. $2\sqrt{x_i - x_{i-1}} |F(x_i) - F(x_{i-1})|^2 \cos^2 \xi \leq$

   $$\leq \sqrt{4[x_i - x_{i-1}]^2 + \left[ \int_{I_1(\sigma)} \cos \theta(s) \left| \frac{d\theta}{ds} \right| \, ds \right]^2}$$

   $$+ \frac{(x_i - x_{i-1}) \left[ \int_{I_1(\sigma)} \cos \theta(s) \left| \frac{d\theta}{ds} \right| \, ds \right] \left[ \int_{I_1(\sigma)} \left| \frac{d\sigma}{ds} \right| \, ds \right]}{\sqrt{4[x_i - x_{i-1}]^2 + \left[ \int_{I_1(\sigma)} \cos \theta(s) \left| \frac{d\theta}{ds} \right| \, ds \right]^2}}.$$
The last term is less than \(|| \int_{I_1(\sigma)} \frac{d\theta}{ds} ds \| \). If \( a > 0 \) and we restrict the partitions to those for which \(-a < x_n < a\), then summing (2) over \( 1 \leq i \leq n \) gives

\[
(3) \quad \sum_{i=1}^{n} \frac{2 \sqrt{[x_i - x_{i-1}]^2 + [F(x_i) - F(x_{i-1})]^2}}{\int_{I_1(\sigma)} \cos \theta(s) \frac{d\theta}{ds} ds} \leq \sum_{i=1}^{n} \sqrt{4[x_i - x_{i-1}]^2 + \int_{I_1(\sigma)} \cos \theta(s) \frac{d\theta}{ds} ds}^2 + \frac{|\sigma|}{\cos a} S.
\]

If \( \theta_1 > -\theta/2 \) and \( \theta < \theta/2 \) then (3) implies

\[
(4) \quad 2L(\theta) \leq \lim_{|\sigma| \to 0} \sum_{i=1}^{n} \sqrt{4[x_i - x_{i-1}]^2 + \int_{I_1(\sigma)} \cos \theta(s) \frac{d\theta}{ds} ds}^2.
\]

The cases \( \theta_1 = -\theta/2 \) or \( \theta = \theta/2 \) can be handled by taking limits.

Now if \( \theta_1 \leq x_{i-1} < x_i \leq \theta_2 \) then the fact that \( \Gamma \) is a simple connected closed curve and the definition of \( \theta_1 \) and \( \theta_2 \) imply that there exists two disjoint subsets of \([0,3]\), say \( I_1(\sigma), I_2(\sigma) \), such that \( x_{i-1} < x < x_i \) implies \( x = \theta(s) \) has a solution in each set. Since \( \theta(s) \) is of bounded variation and uniformly continuous, \( |x_i - x_{i-1}| \leq \text{Var} \theta(s) = \int_{I_1(\sigma)} \frac{|d\theta|}{ds} ds \) with a similar expression for \( I_2(\sigma) \). Also \( I_1(\sigma) \)

and \( I_2(\sigma) \) are contained in \( I_1(\sigma) \) so that

\[
2|x_i - x_{i-1}| \leq \int_{I_1(\sigma)} \frac{|d\theta|}{ds} ds.
\]
Finally (4) becomes

\[(5) \quad 2L(\theta_2 + \delta) \leq 2 \delta + \lim_{|\sigma| \to 0} \sum_{i=1}^{n} \mathcal{V}[\int_{I_1(\sigma)} |\frac{d\theta}{ds}|^2 + \int_{I_1(\sigma)} \cos \theta(s) |\frac{d\theta}{ds}|^2] \quad \cdot \]

The 2 \(\delta\) is the contribution to \(|x_i - x_i'|\) from that portion of the partition between \(\theta_2\) and \(\theta_2 + \delta\). By Minkowski's inequality\(^{20}\)

\[(6) \quad 2L(\theta_2 + \delta) \leq 2 \delta + \lim_{|\sigma| \to 0} \sum_{i=1}^{n} \int_{I_1(\sigma)} \mathcal{V}|\frac{d\theta}{ds}|^2 + \cos^2 \theta |\frac{d\theta}{ds}|^2 \quad ds \quad \cdot \]

Since the \(I_1(\sigma)\) are disjoint

\[2L(\theta_2 + \delta) \leq 2 \delta + S. \]

Taking the limit as \(\delta \to 0^+\) completes the first part of the proof in the event \(\theta_2 < \pi/2\). If \(\theta_2 = \pi/2\) then it can be shown that \(2L(\theta_2) = 2 \lim_{x \to \theta_2^-} L(x)\) so that a slight modification of (5) and (6) corresponding to \(\delta < 0\) gives the result.

Lastly, \(B\) may be represented by the points of a curve of length \(2L(\theta_2+)\) as follows. \(L(\theta)\) is a strictly monotone increasing function of \(\theta\) so that for \(0 \leq s \leq L(\theta_2+)\) there exists a unique \(\theta\) such that \(L(\theta-) \leq s \leq L(\theta+)\).

---

\(^{20}\) Ref. 5, formula 6.13.2.
Define $\theta'(s)$ to be that $\theta$.

If $s \leq L(\theta)$ where $\theta = \theta'(s)$ define

$$\varphi'(s) = -\left[ F(\theta^-) + \frac{F(\theta) - F(\theta^-)}{L(\theta) - L(\theta^-)} (s - L(\theta^-)) \right]$$

and if $s > L(\theta)$ where $\theta = \theta'(s)$ define

$$\varphi'(s) = -\left[ F(\theta) + \frac{F(\theta^+) - F(\theta)}{L(\theta^+) - L(\theta)} (s - L(\theta)) \right].$$

This represents a curve $\Gamma'$ in the negative hemisphere ($\theta \leq 0$). Replacing $\theta'(s)$ by $\theta'(2L(\theta^+)) - s$ and $\varphi'(s)$ by $-\varphi'(2L(\theta^+) - s)$ completes the curve, $\Gamma'$, in the positive hemisphere ($\theta \geq 0$). Notice that $\theta(L(\theta^+)) = \theta(2L(\theta^+)) - L(\theta^+) = \theta_2$ and $\varphi(L(\theta^+)) = F(\theta_2^+)$ while $-\varphi(2L(\theta^+) - L(\theta^+)) = -F(\theta_2^+)$. Now the cap $\theta > \theta_2$ consists either entirely of interior points of $A_C$ or of exterior points so that $F(\theta_2^+) = 0$ or $\pi$. In either event $[\theta_2, F(\theta_2^+)]$ and $(\theta_2, -F(\theta_2^+))$ represent the same point. Thus the two pieces of $\Gamma$ are connected so that $\Gamma'$ is a curve. A similar argument at $\theta_1$ proves that $\Gamma'$ is a closed curve.

We must show that $\Gamma'$ has the winding number property. That is exterior points of $A_C$ have winding number 0 and interior points have winding number 1. If both caps, $(\theta < \theta_1)$ and $(\theta > \theta_2)$ consist only of exterior points then the mapping $\Gamma' \rightarrow \Gamma_0'$ gives by $\theta'(s) \rightarrow \theta'(s)$ and $\varphi(s) \rightarrow (1-a) \varphi(s)$ is an admissible homotopy relative to exterior points.\(^\text{21}\) For $a = 1$, $\Gamma_0'$ lies entirely on the arc, $\varphi = 0$ and the winding number of any point not on $\Gamma_1'$ is 0. If a cap is interior then construction of the homotopy must be preceded by a homotopy

\(^{21}\) See ref. 9 for definition of admissible homotopy.
which frees the cap from the curve.

For interior points the argument is as follows. \((\theta_o, \phi_o)\) is

interior to \(A C\) if and only if \(\left|\phi\right| < \min_{x \in \Theta} [\lim_{x \to \Theta} F(x), F(\theta_o)]\). But if

\((\theta_o, \phi_o)\) has this property then so do all points of the arc \(\left|\phi\right| \leq \left|\phi_o\right|\), \(\theta = \theta_o\). Since winding number is continuous\(^{15}\) this implies that \((\theta_o, 0)\) has the same winding number as \((\theta_o, \phi_o)\). Now examine the stereographic image of \(\Gamma'\) in the \(yz\) plane. The curve goes from the image of

\((\theta_1, F(\theta_1-))\) to \((\theta_2, F(\theta_2+))\) in the negative half plane \((z \leq 0)\) and

back again in the positive half plane. It is easily seen that the

net change of \(\arg[y + iz - y_o]\) for \(y_o\) between these two points and

not on the image of \(\Gamma'\) must be \(2\pi\) so that the winding number is \(+1\).\(^{19}\)

The analogy between Steiner symmetrizations on the plane and

sphere breaks down at Theorem 7 where a finite perimeter was obtained.

The proof of Theorem 7 depended upon the fact that lines perpendicular
to \(L_1\) meet the lines perpendicular to \(L_2\) in a positive angle. However if \(C, C'\) are two coordinate systems on a sphere with common

origin \((P(0,0) = P'(0,0))\) then the lines of constant \(\theta\) are tangent
to the lines of constant \(\theta'\) if they meet along the great circle

\(\phi = \pm \pi/2\). It seems evident that a set could be constructed which was

Steiner symmetric in \(C\) but which had infinite perimeter in every

neighborhood of the great circle \(\phi = \pm \pi/2\) and such that \(A_0\) also had

infinite perimeter. This is apparently the only point where the analogy

breaks down since we have:

**Theorem 12.** Let \[\{C_1\}\] be a sequence of coordinate systems on the

sphere with common origin \((P_1(0,0) = P_1'(0,0))\) and let \(A\) be a measure-
able set and \( \{ A^i \} \) the Steiner sequence generated by \( A \) and \( \{ C_i \} \).

Then

\[
\sum_{i=1}^{\infty} \mu(A^i \Theta A^{i+1})^3 < \infty.
\]

**Proof:** We shall estimate the terms \( \mu(A^i \Theta A^{i+1})^3 \) by terms of a convergent series.

Define \( D^r = \left\{ p_i(\theta, \phi) \mid \cos \theta \cos \phi > r \right\} \) for \( -1 \leq r \leq 1 \).

Using the representation of the sphere in 3-space we can say that \( D^r \) consists of all points of the sphere whose x-coordinate is greater than \( r \).

Since all of the coordinate systems \( C_i \) give the same x-axis, \( D^r \) is independent of coordinate system. Now

\[
\mu_1 \left\{ \phi \mid p_{i+1}(\theta, \phi) \in A^i \cap D^r \right\} \leq \min \left[ \mu_1 \left\{ \phi \mid p_{i+1}(\theta, \phi) \in D^r \right\}, \mu_1(A^{i+1}, \theta) \right].
\]

By the definition of \( A^{i+1} \) the right hand side of this inequality equals

\[
\mu_1 \left\{ \phi \mid p_{i+1}(\theta, \phi) \in A^{i+1} \cap D^r \right\}.
\]

Multiplying each of these functions by \( \cos \theta \, d\theta \) and integrating gives

\[
\mu(A^i \cap D^r) \leq \mu(A^{i+1} \cap D^r) \leq \mu(D^r) = (1 - r) \, 2\pi,
\]

Defining

\[
G_i = \int_{-1}^{1} \mu(A^i \cap D^r) \, dr \leq \int_{-1}^{1} (1 - r) \, 2\pi \, dr = 4\pi,
\]

we see \( G_i \) as a monotone increasing sequence bounded from above so that

\[
\sum_{i=1}^{\infty} G_{i+1} - G_i \leq 4\pi - G_i < \infty.
\]
We shall bound $\mu(A_i^1 \Theta A_{i+1}^1)^3$ by a fixed multiple of $G_{i+1} - G_i$.
For convenience in the following we use $i = 0$. Define

$$D^r_\theta = \{ \emptyset | P_1(\theta, \emptyset) \in D^r \}$$
$$A^r_\theta = \{ \emptyset | P_1(\theta, \emptyset) \in A \} .$$

The corresponding function for $A^i = A_C$ is $A_C, \theta$ which has been previously defined.

Then

$$G_1 - G_0 = \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} \mu_1(A_C, \emptyset \cap D^r_\theta) - \mu_1(A_\theta \cap D^r_\theta) \cos \theta \, d\theta \, dr .$$

Now if $r < -\cos \theta$, then $D^r_\theta = [-\pi, \pi]$ so that $\mu_1(A_C, \emptyset \cap D^r_\theta) = \mu_1(A_C, \emptyset)$

$$= \mu_1(A_\theta) = \mu_1(A_\theta \cap D^r_\theta) .$$

Also if $r > \cos \theta$ then $D^r_\theta$ is empty.

Thus

(1) $$G_1 - G_0 = \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} \cos \theta \, [\mu_1(A_C, \emptyset \cap D^r_\theta) - \mu_1(A_\theta \cap D^r_\theta)] \, dr \, \cos \theta \, d\theta .$$

Since $A_C$ is Steiner symmetric with respect to the coordinate system $C$,

$$\mu_1(A_C, \emptyset \cap D^r_\theta) = \min[\mu_1(A_C, \emptyset), \mu_1(D^r_\theta)] = \min[\mu_1(A_\theta), 2 \arccos \frac{r}{\cos \theta}] .$$

Also

$$\mu_1(A_\theta \cap D^r_\theta) \leq \min[\mu_1(A_\theta), \mu_1(D^r_\theta)]$$

so that the integrand is always non-negative and decreasing the range of integration to $-\pi/2 \leq \theta \leq \pi/2$ and $-\cos \theta \leq r \leq \cos \theta \cos (\mu_1(A_\theta)/2)$ decreases the right hand side of (1). In this restricted range
\[ \mathcal{M}_1(D_\Theta^R) \geq \mathcal{M}_1(A_\Theta). \] With these considerations (1) implies

\[ G_1 - G_0 \geq \int_{-\pi/2}^{\pi/2} \int_{-\cos \Theta}^{\cos \Theta} \frac{\mathcal{M}_1(A_\Theta)}{2} \cos \Theta \cos \frac{\mathcal{M}_1(A_\Theta)}{2} \left[ \mathcal{M}_1(A_\Theta) - \mathcal{M}_1(A_\Theta \cap D_\Theta^R) \right] \cos \Theta \, dr \, d\Theta. \]

(2)

\[ = \int_{-\pi/2}^{\pi/2} \int_{-\cos \Theta}^{\cos \Theta} \frac{\mathcal{M}_1(A_\Theta)}{2} \cos \Theta \cos \frac{\mathcal{M}_1(A_\Theta)}{2} \int_{\phi \in \Phi_A^r} \Phi_A(\Theta, \phi) \, d\phi \, dr \cos \Theta \, d\Theta, \]

where \( \Phi_A \) is the characteristic function of \( A \). Interchanging the order of integration

(3)

\[ G_1 - G_0 \geq \int_{-\pi/2}^{\pi/2} \int_{-\cos \Theta}^{\cos \Theta} \left[ \mathcal{M}_1(A_\Theta) \cos \Theta \cos \frac{\mathcal{M}_1(A_\Theta)}{2} \right] \frac{\mathcal{M}_1(A_\Theta)}{2} \cos \Theta \cos \Theta \cos \Theta \, r \, d\phi \, d\Theta. \]

The integration in the variable \( r \) can now be performed to give

(4)

\[ G_1 - G_0 \geq \int_{-\pi/2}^{\pi/2} \int_{|\phi| \geq \frac{\mathcal{M}_1(A_\Theta)}{2}} \Phi_A(\Theta, \phi) \cos^2 \Theta \left[ \cos \frac{\mathcal{M}_1(A_\Theta)}{2} - \cos \phi \right] \, d\phi \, d\Theta, \]

The non-trivial range of integration of \( \phi \) is that portion of \( A_\Theta \) which is disjoint from \( A_{\Theta \cap} \Theta \) and the measure of this range is \( \mathcal{M}_1(A_\Theta \cap \overline{A}_{\Theta \cap} \Theta) \), where \( \overline{A}_{\Theta \cap} \Theta \) is the complement of \( A_{\Theta \cap} \Theta \). An immediate consequence of Theorem C (Appendix) is that

(5)

\[ G_1 - G_0 \geq \int_{-\pi/2}^{\pi/2} \int_{|\phi| \leq \frac{\mathcal{M}_1(A_\Theta)}{2}} \cos^2 \Theta \left[ \cos \frac{\mathcal{M}_1(A_\Theta)}{2} - \cos \phi \right] \, d\phi \, d\Theta. \]
Performing the integration in the variable $\theta$ and using a trigonometric identity for the difference of sines gives

$$G_1 - G_0 \geq \int_{-\pi/2}^{\pi/2} \cos^2 \theta \left[ \frac{\mu_1(A_\theta \cap \overline{A}_C \theta)}{2} \cos \frac{\mu(A_\theta)}{2} \right. - \left. 2 \sin \frac{\mu_1(A_\theta \cap \overline{A}_C \theta)}{4} \cos \left( \frac{\mu(A_\theta)}{2} + \frac{\mu(A_\theta \cap \overline{A}_C \theta)}{4} \right) \right] \, d\theta .$$

Using the fact that for $x > 0$, $-x \leq -\sin x$ and a trigonometric identity for the difference of cosines we obtain from (6)

$$G_1 - G_0 \geq \int_{-\pi/2}^{\pi/2} \mu_1(A_\theta \cap \overline{A}_C \theta) \sin \frac{\mu_1(A_\theta \cap \overline{A}_C \theta)}{4}$$

$$\sin \left[ \frac{\mu_1(A_\theta)}{2} + \frac{\mu(A_\theta \cap \overline{A}_C \theta)}{8} \right] \cos^2 \theta \, d\theta .$$

Since

$$\frac{\mu_1(A_\theta \cap \overline{A}_C \theta)}{4} \leq \frac{\mu_1(A_\theta)}{4} \leq \frac{\pi}{2} \quad \text{and} \quad \frac{\mu_1(A_\theta \cap \overline{A}_C \theta)}{2} + \frac{\mu(A_\theta \cap \overline{A}_C \theta)}{2} \leq \pi ,$$

The trigonometric functions may be estimated from below by linear functions to yield

$$G_1 - G_0 \geq \int_{-\pi/2}^{\pi/2} \frac{\mu_1(A_\theta \cap \overline{A}_C \theta)^3}{64} \cos^2 \theta \, d\theta .$$

Multiplying the integrand by $\cos \theta$ and applying Holder's inequality to the functions $f(\theta) = \mu_1(A_\theta \cap \overline{A}_C \theta)$, $g(\theta) = \frac{1}{\pi}$ gives

$$G_1 - G_0 \geq \frac{\pi^2}{64} \left[ \mu(A \cap \overline{A}_C \theta) \right]^3 .$$
Since $\mu(A) = \mu(A')$, we have $\mu(\bar{A} \cap A_C) = \mu(\bar{A} \cap A_C') = \frac{1}{2} \mu(\bar{A} \oplus A_C)$.

Thus

$$G_1 - G_0 \geq \frac{n}{512} \left[ \mu(\bar{A} \oplus A_C) \right]^3.$$  

Q.E.D.

Our next theorem is the analogue to Theorem 9.

**THEOREM 13.** Let $A$ be a measurable set on the sphere and $\{C_1\}$ be a periodic sequence of coordinate systems with a common origin.

Then the Steiner sequence converges in measure to a set which is Steiner symmetric in all the coordinate systems.

**Proof:** If $A$ has a finite perimeter the argument is identical with the proof of Theorem 9. The second observation is that the class $C = \{A|A^i$ converges in measure\} is closed under complementation.

For if $\bar{A}$ is the complement of $A$ then $A^i$ is obtained from $A^i$ by reflection through the yz plane. And if $A^i$ converges then so does $\bar{A}^i$. The third observation is that $C$ is closed under monotone limits.

For let $B_n$ be a monotone sequence of sets of $C$. Then the $B_n$ converge to a set we shall denote by $B_\infty$. Denote by $B_n^\infty$ the set to which the Steiner sequence $B_n^i$ converges ($n$ fixed). Since Steiner symmetrization takes a monotone family of sets into a family monotone to within measure zero, the sequence $B_n^\infty$ must be monotone up to measure zero and thus converges in measure to a set we shall denote by $B_\infty^\infty$. Then

$$(1) \mu(B_n^i \oplus B_\infty) \leq \mu(B_n^i \oplus B_\infty^i) + \mu(B_n^\infty \oplus B_n^i) + \mu(B_n^\infty \oplus B_\infty).$$
Since Steiner symmetrization takes monotone sets into monotone sets, it can be shown that \( \mu(B^i_n \oplus B^i_\infty) \leq \mu(B_n \oplus B_\infty) \) for all \( i \). By choosing \( n \) such that \( \mu(B_n \oplus B_\infty) < \frac{1}{4} \varepsilon \) and \( \mu(B^\infty_n \oplus B^\infty_\infty) < \frac{1}{4} \varepsilon \) (both sequences converge) (1) becomes

\[
\mu(B^i_\infty \oplus B^\infty_\infty) \leq \frac{1}{2} \varepsilon + \mu(B^\infty_n \oplus B^i_n).
\]

But \( B^i_n \to B^\infty_n \) as \( i \to \infty \), so that for all \( i \) sufficiently large

\[
\mu(B^i_\infty \oplus B^\infty_\infty) < \varepsilon.
\]

Thus \( B_\infty \) is in \( C \). If we could show that \( C \) was a ring we would be done. Unfortunately there is no direct proof of this and we must proceed differently. It is well known that the sphere is separable under the ordinary topology. And in fact circular discs with rational coordinates for their centers and rational radii form a basis. Since finite unions of circular discs have finite perimeter and every open set is the monotone increasing limit of such sets, every open set is in \( C \). Next every \( G_\delta \) is in \( C \). Lastly, since convergence is only to within measure zero, any set which differs from a \( G_\delta \) by a set of measure zero is in \( C \). But this includes all measurable sets.

Q.E.D.

The next theorem is the analogue of Theorem 4 on the sphere.

**Theorem 14.** Let \( F, G \) be integrable functions on the sphere and \( H \) be monotone decreasing on \([0, u]\). Let \( F^*, G^* \) be the rearrangements.

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22) See ref. 12, p. 3 for definitions.
of $F, G$ based on the sets $D^r = \{ P(\theta, \phi) | \cos \theta \cos \phi > r \}$. Then

$$\int \int F(P) \, G(Q) \, H(\rho(P, Q)) \, dP \, dQ \leq \int \int F^*(P) \, G^*(Q) \, H(\rho(P, Q)) \, dP \, dQ.$$  

**Proof:** As on several preceding occasions the general case follows from the case when $F, G$ are the characteristic functions of measurable sets, $A$ and $B$. Let $C_1, C_2$ be two coordinate systems such that the angle between the arcs $\{ P_1(\theta, \phi) | \phi = 0 \}$ and $\{ P_2(\theta, \phi) | \phi = 0 \}$ at their point of intersection, $P_1(0, 0) = P_2(0, 0)$, is an irrational multiple of $\pi$. Now $\text{Lim } A^i, \text{Lim } B^i$ are Steiner symmetric in both coordinate systems and thus must be circular discs (the $D^r$s). 23

An application of Theorem 10 completes the proof.

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23) Ref. 8, p. 90.
APPENDIX

Theorems A, B, C are simple measure theoretic results which are used in the main part of the text.

THEOREM A. If $A, B$ are integrable sets on $G$ then

$$\mu \{ z \mid \mu(A \cap zB) > 0 \} = 0 \quad \text{if and only if} \quad \mu(A) = 0 \quad \text{or} \quad \mu(B) = 0.$$ 

Proof:

$$\mu(A \cap zB) = \int \Phi_A(x) \Phi_B(z^{-1}x) \, dx$$

and

$$\int \mu(A \cap zB) \, dz = \iint \Phi_A(x) \Phi_B(z^{-1}x) \, dx \, dz$$

$$= \int \Phi_A(x) \int \Phi_B(x^{-1}z^{-1}) \, dz \, dx$$

$$= \int \Phi_A(x) \mu(B^{-1}) \, dx = \mu(A) \mu(B^{-1}).$$

Since $\mu(A \cap zB)$ is non-negative, $\mu(A) \mu(B^{-1})$ is zero if and only if

$$\mu \{ z \mid \mu(A \cap zB) > 0 \} = 0.$$

Lastly, $\mu(B^{-1}) = \int \Phi_B(z) \Delta(z^{-1}) \, dz$. Since $\Phi_B(z)$ is non-negative and $\Delta(z^{-1})$ is positive everywhere, $\mu(B^{-1})$ is zero if and only if $\mu(B) = 0$. The theorem then follows.

THEOREM B. Let $A, B, C$ be integrable sets on $G$ with $\mu(A) \leq \mu(B) \leq \mu(C)$. Then there exists $A^1, C^1$ such that $\mu(A^1) = \mu(A)$, $\mu(C^1) = \mu(C)$ and $A^1 \subseteq B \subseteq C^1$. 
Proof: We will construct $A^1$. The modification for $C^1$ is obvious.

If $A$ or $B$ have zero measure we are through. Otherwise consider $\mu\{A \cap z^{-1}B\}$. There exists $z_1$ such that

$$\mu(A \cap z^{-1}_1B) \geq \frac{1}{2} \sup_{z \in G} \mu(A \cap z^{-1}B).$$

Define $A_1 = z_1A \cap B$, $B_1^* = B - A_1$, $A_1^* = A - z_1^{-1}A_1$. Then $A_1 \subseteq B$, and $\mu(A_1^*) + \mu(A_1) = \mu(A)$.

Inductively we choose $z_{i+1}$ such that

$$\mu(A_i^* \cap z_{i+1}^{-1}B_1^*) \geq \frac{1}{2} \sup_{z \in G} \mu(A_i^* \cap z^{-1}B_1^*)$$

and define

$$A_{i+1} = A_i \cup [z_{i+1}A_i^* \cap B_1^*]$$

$$B_{i+1}^* = B - A_{i+1}$$

$$A_{i+1}^* = A_i^* - A_i^* \cap z_{i+1}^{-1}B_1^*.$$

It is easily shown by induction that $A_i \subseteq B$

(1) $\mu(A_{i+1}) = \mu(A_i) + \mu[A_i^* \cap z_{i+1}^{-1}B_1^*]$

(2) $\mu(A_i^*) + \mu(A_i) = \mu(A)$,

and $\{A_i\}$ is a monotone increasing sequence and $\{B_1^*\}, \{A_i^*\}$ are monotone decreasing. Thus
\[ A^i = \lim_{i \to \infty} A^i, \quad B^*_\infty = \lim_{i \to \infty} B^*_i, \quad \text{and} \quad A^*_\infty = \lim_{i \to \infty} A^*_1 \text{ exist.} \]

Also \( \mu(A^*_1 \cap z^{-1}B^*_1) \) decreases monotonely to \( \mu(A^*_\infty \cap z^{-1}B^*_\infty) \). Furthermore, since \( \{A_i\} \) is monotone and the \( \mu(A_1) \leq \mu(B) < \infty \), we have

\[
\mu(A^*_1 \cap z^{-1}B^*_1) = \mu(A_{i+1}) - \mu(A_i) = \mu(A_{i+1} - A_i) \to 0.
\]

Thus \( \sup_{z \in G} \mu(A^*_1 \cap z^{-1}B^*_\infty) \leq \mu(A^*_1 \cap z^{-1}B^*_\infty) \leq \mu(A^*_1 \cap z^{-1}B^*_1) \to 0 \) so that

\[
\mu(A^*_\infty \cap z^{-1}B^*_\infty) \equiv 0. \text{ By Theorem A either } \mu(A^*_\infty) \text{ or } \mu(B^*_\infty) \text{ equals zero.}
\]

Now \( B^*_\infty = B - A^1 \) and

\[
(3) \quad \mu(B^*_\infty) = \mu(B) - \mu(A^1).
\]

Taking the limit of (2) as \( i \to \infty \) gives

\[
(4) \quad \mu(A^*_\infty) + \mu(A^1) = \mu(A).
\]

That \( A^*_\infty \) is the one with zero measure follows from (3) and (4). Then from (4) it follows that \( \mu(A^1) = \mu(A) \). Since \( A_i \subseteq B \) for all \( i \), \( A^1 \subseteq B \).

Q.E.D.

COROLLARY. If \( A \) is integrable then there exists \( A^1 \) such that \( \mu(A) = \mu(A^1) \) and the closure of \( A^1 \) is compact.

Proof: If \( G \) is compact then any closed subsets are compact and the corollary is proved. If \( G \) is not compact, since it is locally compact, there exist compact sets of arbitrarily large measure and thus one whose measure is greater than \( \mu(A) \). The corollary then follows from the theorem.
THEOREM C. Let \( F(x) \) be a positive integrable function on \( G \) and let \( A \) be an integrable set. Then there exists a real number \( t \) such that

\[
\mu\{x \mid F(x) \geq t\} \geq \mu(A) \geq \mu\{x \mid F(x) > t\} \overset{\text{DEF}}{=} F_t
\]

and

\[
\int_A F(x) \, dx \leq \int_{F_t} F(x) \, dx + t \left[ \mu(A) - \mu(F_t) \right].
\]

Furthermore if equality holds then

\[
F_t \subseteq A \subseteq \{x \mid F(x) \geq t\}
\]

except for a set of measure zero.

Proof: Let \( t_0 = \inf \{t \mid \mu(F_t) < \mu(A)\} \)

The sets \( F_t \) are monotone increasing as \( t \) decreases and \( F_t = \lim_{t \to t_0} F_t \).

Thus \( \mu(F_t) \) is continuous on the right and

\[
\mu(F_{t_0}) \leq \mu(A).
\]

Also \( \{x \mid F(x) \geq t_0\} = \bigcap_{n=1}^{\infty} F_{t_0 - 1/n} \) so that

\[
\mu\{x \mid F(x) \geq t_0\} = \lim_{n \to \infty} \mu(F_{t_0 - 1/n}) \geq \mu(A).
\]

Secondly, let \( A_1 = A \cap F_{t_0} \), \( A_2 = A \cap \{x \mid F(x) \geq t_0\} - A_1 \), and \( (A - A_1) - A_2 = A_3 \). Then

\[
\int_A f(x) \, dx = \int_{A_1} F(x) \, dx + \int_{A_2} F(x) \, dx + \int_{A_3} F(x) \, dx.
\]

By Theorem B there exists a subset, \( C \), of \( A_2 \cup A_3 \) whose measure is
\[ \mu(F_t) - \mu(A_1). \] Since \( F(x) > t_0 \) on \( F_t \) and \( F(x) \leq t_0 \) on \( C \) we have

\[
\int_A F(x) \, dx \leq \int_{F_{t_0}} F(x) \, dx + \int_{A_2 - A_2 \cap C} F(x) \, dx + \int_{A_3 - A_3 \cap C} F(x) \, dx
\]

with strict inequality holding unless \( \mu(C) = 0 \), that is, unless \( F_t \subseteq A \) to within measure zero. A similar construction replaces \( A_3 - A_3 \cap C \) by a subset of \( \{ x \mid F(x) = t_0 \} \) - \( (A_2 - A_2 \cap C) \) and a corresponding strict inequality results unless \( (A_3 - A_3 \cap C) = 0 \). Combining the two steps yields

\[
\int_A F(x) \, dx \leq \int_{F_{t_0}} F(x) \, dx + t_0 [\mu(A) - \mu(F_{t_0})]
\]

with strict inequality holding unless

\[ F_{t_0} \subseteq A \subseteq \{ x \mid F(x) \geq t_0 \} \]

except for sets of measure zero.

**Lemma to Theorem 11.**

Let \( \Gamma \) be a rectifiable closed curve on the sphere given by

\[ \{ (\theta(s), \varphi(s)) \mid 0 \leq s \leq S \} \]

where \( s \) is the arc length from the point \((\theta(0), \varphi(0))\) along the positive orientation of the curve. For \(-\pi/2 < a < b < \pi/2\), define

\[
I = \left\{ s \mid a < \theta(s) < b \quad \text{or} \quad \theta(s) = a \text{ and } \frac{d\varphi}{ds} > 0 \right. \\
\left. \quad \text{or} \quad \theta(s) = b \text{ and } \frac{d\varphi}{ds} < 0 \right\}
\]
Statement of Lemma to Theorem 11. If $\Gamma$ is bounded away from $P(0,-\pi)$ and the winding number of every point not on $\Gamma$ is either zero or one and

$$F(\theta) = \frac{1}{2} \mu_1 \left\{ \theta | P(\theta,\theta) \text{ is on } \Gamma \text{ or has winding number 1} \right\}$$

then $\theta(s)$ and $\phi(s)$ are absolutely continuous functions of $s$ on $I$ and

$$2|F(b) - F(a)| \leq \int_I \left| \frac{d\theta}{ds} \right| ds.$$ 

Proof: We first show absolute continuity. Choose $\delta > 0$ such that $-\pi/2 + \delta < a$ and $b < \pi/2 - \delta$. Thus

$$I \subset \left\{ s \mid -\frac{\pi}{2} + \delta < \theta(s) < \frac{\pi}{2} + \delta \right\}.$$ 

This last set is open and thus consists of a countable union of disjoint open intervals. On each of these intervals we have if $s_1 > s_0$ and each are in the interval

$$s_1 - s_0 = \lim_{|\sigma| \to 0} \sum_{i=1}^{n(\sigma)} \sqrt{(\Delta \theta_i)^2 + \cos^2[\theta(\epsilon_i)]} (\Delta \phi_i)^2$$

where $\sigma$ is a partition of $[s_0, s_1]$, $s_0 = x_1 < x_2 \cdots < x_n(\sigma) = s_1$ and $\Delta \theta_i, \Delta \phi_i$ are the differences of consecutive values of the corresponding functions, $|\sigma| = \max_1^n x_i - x_{i-1}$. It follows that

$$s_1 - s_0 \geq \lim_{|\sigma| \to 0} \sum |\Delta \theta_i| \geq |\theta(s_1) - \theta(s_2)|$$

and

$$s_1 - s_0 \geq \lim_{|\sigma| \to 0} \sum \cos[\theta(\epsilon_i)] |\Delta \phi_i| \geq \cos\left(\frac{\pi}{2} - \delta\right) |\phi(s_1) - \phi(s_2)|.$$
Now let \( I_1, I_2 \) be a sequence of intervals \([a_i, b_i]\) contained in 
\[ \{ s | -\pi/2 + \delta < \Theta(s) < \pi/2 - \delta \} \] such that \( \sum b_i - a_i < \epsilon \sin \delta \). Then

\[
\sum_{i=1}^{\infty} |\Theta(b_i) - \Theta(a_i)| \leq \sum_{i=1}^{\infty} \frac{(b_i - a_i)}{\cos(\pi/2 - \delta)} < \frac{\epsilon \sin \delta}{\cos(\pi/2 - \delta)} = \epsilon.
\]

Therefore \( \Theta(s) \) is absolutely continuous. A similar computation shows \( \Theta(s) \) to be absolutely continuous and in fact by letting \( \delta \to 0 \) the absolute continuity of \( \Theta(s) \) on the whole interval \([0, S]\) can be shown.

Next we will prove \( 2|F(b) - F(a)| \leq \int_I |d\Theta| ds \). Let

\[ \Phi_a(\Theta) = 1 \] if \( F(a, \Theta) \) is on \( \Gamma \) or has winding number 1, and \( \Phi_a(\Theta) = 0 \) otherwise. Similarly for \( \Phi_b(\Theta) \). Then

\[
2|F(b) - F(a)| = \left| \int_{-\pi}^{\pi} \left[ \Phi_b(\Theta) - \Phi_a(\Theta) \right] d\Theta \right|
\leq \int_{-\pi}^{\pi} |\Phi_b(y) - \Phi_a(y)| dy.
\]

If we can show the last integral to be less than or equal to \( \int_I |d\Theta| ds \) the proof will be complete.

To do this we need two lemmas.

**LEMMA 1.** Let \( N_I(y) \) be the number of solutions to the equation \( \Theta(s) = y \), \( \epsilon \in I \). Then

\[
\int_{-\pi}^{\pi} N_I(y) dy = \int_I |d\Theta| ds.
\]

**LEMMA 2.** \( |\Phi_b(y) - \Phi_a(y)| \leq N_I(y) \) except for a set of \( y \) of measure zero.
That \( \int_{-\pi}^{\pi} |\Phi_b(y) - \Phi_a(y)| \, dy \leq \int_{\mathbb{R}} |\frac{d\theta}{ds}| \, ds \) is an immediate consequence of these two lemmas.

**Proof of Lemma 1.** Consider the class, \( C \), of all sets \( A \) for which

\[
\int_{-\pi}^{\pi} N_A(y) \, dy = \int_{\mathbb{R}} |\frac{d\theta}{ds}| \, ds.
\]

If \( A \) is an interval then \( \int_{\mathbb{R}} |\frac{d\theta}{ds}| \, ds = \text{Var } \theta. \)

But a result of S. Banach shows that intervals are in \( C \). Next, if \( A_1 \cup A_2 = A_3 \) and \( A_1 \cap A_2 \) is empty then \( N_{A_1}(y) + N_{A_2}(y) = N_{A_3}(y) \), so that

\[
\int N_{A_1}(y) \, dy + \int N_{A_2}(y) \, dy = \int N_{A_3}(y) \, dy.
\]

Also

\[
\int_{A_1} |\frac{d\theta}{ds}| \, ds + \int_{A_2} |\frac{d\theta}{ds}| \, ds = \int_{A_3} |\frac{d\theta}{ds}| \, ds.
\]

The integral equalities are to be interpreted as asserting also that if any two of the integrals exist then the third does. From this it follows that \( C \) is closed under relative complements and finite unions of disjoint members of \( C \). If \( \{ A_i \} \) is a monotone sequence of members of \( C \) converging to \( A \), then

\[
N_{A_i}(y) \text{ converges monotonely to } N_A(y),
\]

\[
\int_{-\pi}^{\pi} N_{A_i}(y) \, dy \text{ converges monotonely to } \int_{-\pi}^{\pi} N_A(y) \, dy
\]

and

\[
\int_{A_i} |\frac{d\theta}{ds}| \, ds \text{ converges monotonely to } \int_{A} |\frac{d\theta}{ds}| \, ds.
\]

24) Ref. 13, Theorem on page 48.

25) Ref. 14, Theorem 2 page, 228.
Thus \( C \) is closed under monotone limits. In particular since every open set is the countable union of disjoint open intervals, every open subset and every closed subset of \( \{ s \mid -\pi/2 + \delta < \theta(s) < \pi/2 - \delta \} \) are in \( C \).

Finally the countable intersections of open sets \((G_\delta')s\) and countable unions of closed set \((F_\sigma')s\) belong to \( C \). Now if \( A \) is measurable there exists \( F_\sigma, G_\delta \) such that \( F_\sigma \subseteq A \subseteq G_\delta \) and \( \mu(G_\delta - F_\sigma) = 0 \). We have

\[
N_{F_\sigma}(y) \leq N_A(y) \leq N_{G_\delta}
\]

and

\[
\int_{F_\sigma} \frac{d\theta}{ds} \, ds = \int_{G_\delta} \frac{d\theta}{ds} \, ds \text{ provided } G_\delta \subseteq \{ s \mid -\pi/2 + \delta < \theta(s) < \pi/2 - \delta \}.
\]

The last statement implies \( \int N_{F_\sigma}(y) \, dy = \int N_{G_\delta}(y) \, dy \). Since \( N_{F_\sigma} \leq N_{G_\delta} \), this implies \( N_{F_\sigma} \) and \( N_{G_\delta} \) and thus also \( N_A \) must differ on a set of measure zero. Thus \( \int N_A(y) \, dy \) exists \( \int_F \frac{d\theta}{ds} \, ds = \int_A \frac{d\theta}{ds} \, ds \).

Therefore every measurable set is in \( C \) and in particular \( I \) is in \( C \).

Q.E.D.

**Proof of Lemma 2.** If \( \Phi_b(y) \) and \( \Phi_a(y) \) are both 1 or both 0, it is trivial that

\[
|\Phi_b(y) - \Phi_a(y)| \leq N_I(y).
\]

Also if there exists \( s \) such that \( a < \theta(s) < b \) and \( \theta(s) = y \) then

\[
|\Phi_b(y) - \Phi_a(y)| \leq 1 \leq N_I(y).
\]

That leaves the case where exactly one of the points \( P(b,y) \) and \( P(a,y) \) has winding number 0 and no point of \( \{ P(\theta,y) \mid a < \theta < b \} \) are on the

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26) Ref. 12, p. 66, ex. 2.
curve \( \Gamma \). Suppose \((b,y)\) is the point with winding number 0. Then, by continuity, all of the points \( \{ F(\theta,y) | a < \theta < b \} \) must have winding number 0 and \((a,y)\) must be a point on \( \Gamma \). We shall now have to exclude sets of measure zero.

First let \( \delta > 0, -\pi/2 + \delta < a < b < \pi/2 + \delta \) and 

\[
A = \{ s| -\frac{\pi}{2} + \delta < \theta(s) < \frac{\pi}{2} - \delta \}.
\]

Then \( N_A(y) \) is integrable by the preceding lemma and therefore finite almost everywhere. We exclude those \( y \) for which \( N_A(y) \) is infinite.

Secondly let \( s_1, s_2, \ldots, s_n \) be the solutions to \( \phi(s) = y, s \in A \).

Since \( \phi(s) \) and \( \theta(s) \) are absolutely continuous on \( A \), \( d\phi/ds \) and \( d\theta/ds \) exist and are finite almost everywhere. We exclude that set of \( y \) for which there exists \( s \) such that \( \phi(s) = y \), \( s \in A \) and either \( d\phi/ds \) or \( d\theta/ds \) do not exist or are not finite. Finally we look at those \( s_i \) for which \( \theta(s_i) = a \). It is claimed that 

\[
\mu_1 \{ s| \theta(s) = a \text{ and } \frac{d\theta}{ds} \neq 0 \} = 0.
\]

The reason is as follows

\[
\frac{d\theta}{ds} - \varepsilon < \frac{\theta(s + \Delta) - \theta(s)}{\Delta} < \frac{d\theta}{ds} + \varepsilon
\]

for all \( |\Delta| \) sufficiently small. Thus if \( d\theta/ds \neq 0 \) and \( \theta(s) = a \) then \( \theta(s') = a \) can have no other solution in some neighborhood of \( s \). Thus no point of \( \{ s| \theta(s) = a \text{ and } d\theta/ds \neq 0 \} \) can be a density point of this set. It follows that this set is of measure zero.\(^{27}\) The corresponding set of \( y \) must also be of measure zero and will be excluded.

Now let us examine the set of $y$ remaining. For each $y$ in this set we have: There exists $s$ with $\phi(s) = y$, $\theta(s) = a$. There are only a finite number of solutions to $\phi(s) = y$, $-\pi/2 + \delta < \theta(s) < \pi/2 - \delta$.

For each solution to $\phi(s) = y$, $\theta(s) = a$ we have $d\phi/ds$ exists and $d\theta/ds = 0$.

Since $1 = \sqrt{(d\phi/ds)^2 + \cos^2 \theta (d\theta/ds)^2}$, $d\phi/ds$ cannot be zero and in fact

$$\frac{d\phi}{ds} = \pm \frac{1}{\cos \theta}.$$ 

Let $s_1, s_2, \ldots, s_n$ be the solutions to $\phi(s) = y$, $\theta(s) = a$. Since there are only a finite number of solutions to $\phi(s) = y$, $-\pi/2 + \delta < \theta(s) < \pi/2 - \delta$, there is an arc $\{ P(\theta, y) | a - \epsilon < \theta < a + \epsilon \}$ such that the only point in common with $\Gamma$ is $(a, y)$. For each $s_i$, $i = 1, \ldots, n$ such that $d\phi/ds > 0$ there exists an $\epsilon_1$ such that $\phi(s) < y$ for $s \in (s_i - \epsilon_1, s_i)$ and $\phi(s) > y$ for $s \in (s_i, s_i + \epsilon_1)$. Similarly for $d\phi/ds < 0$. Thus there is a crossing of the arc of $\Gamma$. If we look at the stereographic image of the arc and of $\Gamma$ on the $yz$ plane using the point P(0, $-\pi$) to project from, we see that if $d\phi/ds > 0$ the contribution to the winding number of $(a - \epsilon, y)$ minus the winding number of $(a + \epsilon, y)$ is $+1$, while if $d\phi/ds < 0$ the contribution is $-1$.

Letting $W(\theta, \phi)$ be the winding number of the point $P(\theta, \phi)$

$$W(a - \delta, y) - W(a + \delta, y) = \sum_{i} \text{sgn} \left. \frac{d\phi}{ds} \right|_{s_i}.$$ 

But $W(a + \delta, y) = 0$ and $W(a - \delta, y)$ is non-negative so that

$$\sum_{i} \text{sgn} \left. \frac{d\phi}{ds} \right|_{s_i} \geq 0.$$
Since there is at least one term in the sum, there must exist one positive term so that \( N_1(y) \geq 1 \geq |\varphi_b(y) - \varphi_a(y)| \). A similar argument is valid if \( \varphi_b(y) = 1 \) and \( \varphi_a(y) = 0 \) and completes the proof.
REFERENCES


REFERENCES (cont.)

