THE REARRANGEMENT OF FUNCTIONS AND MAXIMIZATION OF A CONVOLUTION INTEGRAL

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ABSTRACT

For the additive group of the real numbers, the value, at the origin, of the convolution of three non-negative integrable functions is less than or equal to the corresponding value obtained from the symmetric rearrangements of those functions. That is

$$f * g * h(0) \leq f^* * g^* * h^*(0) .$$

This problem is investigated on general groups and it is shown that under a wide interpretation of the notion of symmetric rearrangement the validity of the inequality implies serious restrictions on the group.

Both new and known results in E^n are developed using methods which in modified form yield the answer to the similar problem on the sphere.

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<u>INTRODUCTION.</u> Let G be a locally compact topological group whose unit element is denoted by 'e'. It is known that such a group has a left invariant measure which is unique up to a multiplicative constant.[1] Left invariance means that $\mu(A) = \mu(sA)$ for every element s of G. The uniqueness of this measure implies the existence of a function $\Delta(s)$ such that $\mu(As) = \Delta(s)\mu(A)$. If this function is identically equal to one, the group G is called unimodular.

The convolution, $f * g$, of two integrable functions f, g is defined by

$$f * g(t) = \int f(x)\, g(x^{-1} t)\, dx = \int f(tx)\, g(x^{-1})\, dx .$$

If one function is bounded the convolution function is defined every where and is continuous.[2]

We shall be concerned with bounded, integrable, non-negative functions. For the additive group of the real numbers, it is known that for any three such functions

$$f * g * h(e) \leq f^* * g^* * h^*(e)$$

where f^* is the symmetric rearrangement of f, and similarly for g and h.[3]

The main objective of this study is to investigate this problem on general groups. It is shown that under a wide interpretation of

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1) Ref. 1.

2) Ref. 2, p. 50.

3) cf. Ref. 3.

'The symmetric rearrangement of a function' the validity of the above inequality implies serious restrictions on the group G. These restrictions are discussed in the first section and a result for the group of rotations in two dimensions is obtained. The second section deals with the group of translations in two dimensions. Although the main result was known, the methods used are more constructive and more direct. Furthermore, properly modified, they are applied in section 3 to a study of functions defined on a sphere. The Appendix contains the proofs of some measure theoretic results which are needed in the main part of the text.

<u>First Section</u>. The family of non-negative integrable functions is designated L^{1+}.

<u>Def</u>. Two functions f, g will be said to be a rearrangement of one another if for every α, $\mu\{x|f(x) > \alpha\} = \mu\{x|g(x) > \alpha\}$.

This notion is obviously an equivalence relation and will be denoted by $f \sim g$.

On the additive group of real numbers, in each equivalence class of functions there exists one symmetrically decreasing function. This member of a class is called the symmetric rearrangement of any function in the class. On a general group it is not evident whether or not a natural notion of symmetric rearrangement exists. To investigate this question the following two hypotheses are introduced.

Hypothesis A. There exists three mappings from L^{1+} into itself taking f into f^i, $i = 1,2,3$, such that $f \sim f^i$, $f \sim g$ implies $f^i = g^i$ and for all $f, g, h \in L^{1+}$

$$f * g * h(e) \leq f^1 * g^2 * h^3(e) .$$[4]

Hypothesis B. Hypothesis A is satisfied by three mappings which are identical.

The additive group of real numbers satisfies the stronger hypothesis B.[3] The additive group of integers is an example where A but not B is satisfied provided the domain of the third mapping is restricted to those functions in L^{1+} which assume their maximum an odd number of times and other values an even number of times.[5]

If a function is in the range of the i-th mapping it will be called symmetric of type i, and if the function is also the characteristic function of a set the corresponding set will also be called symmetric of type i.

THEOREM 1. If G satisfies Hypothesis A then G is unimodular.

Proof: We first prove that if Φ_A, Φ_B, Φ_C are the characteristic functions of sets of positive measure A, B, C and G is not unimodular then

$$\max_{A' \sim A, B' \sim B, C' \sim C} \Phi_{A'} * \Phi_{B'} * \Phi_{C'}(e) = \mu(A)\mu(B) .$$

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4) For f, g, h non-negative and measurable, the convolution is defined everywhere provided $+\infty$ is allowed as a value. Thus $f * g * h(e)$ exists.

5) Cf. Ref. 4 and Ref. 5.

Secondly we prove that this can not always be achieved by symmetric sets.

<u>First Part</u>: Since $\Phi_{C'}(x) \leq 1$ for all x, $\Phi_{A'} * \Phi_{B'} * \Phi_{C'}(e) \leq \mu(A)\,\mu(B)$.

On the other hand the upper bound is obtained as follows. By the corollary to Theorem B (Appendix) there exist sets A_1, B_1 with compact closure such that $\mu(A_1) = \mu(A)$, $\mu(B_1) = \mu(B)$. Define

$$D = \left\{ z \,|\, \mu(A_1 \cap z^{-1} B_1^{-1}) > 0 \right\}.$$

Then $D \subseteq B_1^{-1} A_1^{-1}$ and the compactness of the closure of this last set implies $\mu(D) < \infty$. By considering $\Delta(z^n) = [\Delta(z)]^n$ where n is any positive or negative integer, it is clear that t can be chosen so that $\Delta(t)$ is arbitrarily close to zero. In particular there exists t such that $\mu(Dt) = \Delta(t)\,\mu(D) < \mu(C)$ (since $\mu(C) > 0$). Again by Theorem B there exists C' such that $Dt \subseteq C'$ and $\mu(C') = \mu(C)$. Now consider $\Phi_{t^{-1}A_1}$, Φ_{B_1}, $\Phi_{C'}$. We have

$$\begin{aligned} \Phi_{t^{-1}A_1} * \Phi_{B_1} * \Phi_{C'}(e) &\geq \Phi_{t^{-1}A_1} * \Phi_{B_1} * \Phi_{Dt}(e) \\ &= \Phi_{A_1} * \Phi_{B_1} * \Phi_{D}(e) . \end{aligned}$$

Using the definition of D this becomes

$$\begin{aligned} \iint \Phi_{A_1}(x)\,\Phi_{B_1}(x^{-1}y)\,\Phi_D(y^{-1})\,dx\,dy &= \iint \Phi_{A_1}(x)\,\Phi_{B_1}(x^{-1}y)\,dx\,dy \\ &= \mu(A)\,\mu(B) \end{aligned}$$

proving the first part of the proof.

Second Part: Let A, B, C be symmetric of type 1, 2, 3 respectively and of positive measure. Hypothesis A implies

$$\mu(A)\mu(B) = \Phi_A * \Phi_B * \Phi_C(e) = \int_{C^{-1}} \mu(A \cap zB^{-1})\, dz$$
$$= \int_C \mu(A \cap z^{-1}B^{-1})\, \Delta(z)\, dz \, .$$

On the other hand if $\mu(C) < \mu\{z | \mu(A \cap z^{-1}B^{-1}) > 0\}$ then the above equality is obviously impossible. Furthermore to be non-unimodular the group can not be discrete[6] so that C can be chosen (and thus a symmetric C) so as to have a positive measure which satisfies the above inequality. $(\mu\{z|\mu(A \cap z^{-1}B^{-1}) > 0\} > 0$ by Theorem B.) Q.E.D.

For the additive group of the real numbers, the symmetric sets are $S_\alpha = \{x|\ |x| \leq \alpha/2\}$ and the symmetric rearrangement of a positive integrable function f is defined as follows.[7] Define $\alpha(t) = \mu\{x|f(x) > t\}$ and $f^*(x) = \sup\{t|x \in S_{\alpha(t)}\}$. That f^* is a rearrangement of f can be seen as follows. $x \in S_{\alpha(t_1)}$ and $t_1 > t$ implies $f^*(x) \geq t_1 > t$. Thus $S_{\alpha(t_1)} \subseteq \{x|f^*(x) > t\}$. On the other hand if $f^*(x) > t$ then there exists t_1 such that $t_1 > t$ and $x \in S_{\alpha(t_1)}$. By the monotonicity of the family of sets S_α and the monotonicity of the function $\alpha(t)$, this implies $x \in S_{\alpha(t)}$. Thus $S_{\alpha(t_1)} \subseteq \{x|f^*(x) > t\} \subseteq S_{\alpha(t)}$ and $\alpha(t_1) \leq \mu\{x|f^*(x) > t\} \leq \alpha(t)$.

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6) Haar measure on a discrete group assigns unit mass to each point and thus is right invariant also.

7) Cf. Ref. 5, p. 276.

Since α is monotone decreasing and continuous on the right, $\alpha(t) = \mu\left\{x \mid f^*(x) > t\right\}$ which proves that f^* is a rearrangement of f. That Hypothesis B is satisfied for this symmetrization is a simple consequence of the validity of the inequality for characteristic functions of sets.[8]

The above discussion need not be restricted to the real line. If there are three monotone classes of sets such that Hypothesis A is valid when L^{1+} is replaced by the class of characteristic functions of sets then it can be extended to L^{1+} by the above construction. Furthermore it is a consequence of the corollary to Theorem C (Appendix) that the symmetric functions given by Hypothesis A differ on sets of measure zero from those constructed from symmetric sets. The next theorem shows that if Hypothesis A is valid on a group the three monotone classes exist.

THEOREM 2. If Hypothesis A holds the symmetric sets of type i form a monotone family of open sets to within measure zero. (That is, there exists a family $\left\{O^i_\alpha \mid \alpha \in \text{range of } \mu\right\}$ such that $\alpha < \beta$ implies $O^i_\alpha \subset O^i_\beta$, $\mu(O^i_\alpha) = \alpha$ and $\mu[S^i_\alpha \oplus O^i_\alpha] = 0$.

<u>Proof:</u> The unimodularity implies that cyclic permutations of the three functions in the double convolution leave the value at the origin invariant so that any cyclic permutation of the three symmetrizations also satisfies Hypothesis A and it will suffice to prove that the first symmetrization has the stated property.

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8) Cf. ibid., p. 279-280.

First suppose G discrete, then we can normalize the measure so that each point has one unit weight. Let C be symmetric of type 3 and measure one and let B be symmetric of type 2 and measure n. Then

$$\Phi_A * \Phi_B * \Phi_C(e) = \int_{C^{-1}} \mu[A \cap yB^{-1}]\, dy = \mu[A \cap c^{-1}B^{-1}] .$$

It is obvious that if A is symmetric of type 1 and $\mu(A) = n$ then $A = c^{-1}B^{-1}$ in order for the maximum to be achieved. Likewise if $\mu(A) < n$ then $A \subset c^{-1}B^{-1}$ and if $\mu(A) > n$ then $A \supset c^{-1}B^{-1}$ and the monotonicity is established.

Notice that the proof in the discrete case made use of sets of the form $\{ x | \Phi_B * \Phi_C(x) \geq t \}$ to separate A's with different measure.[9] This same method will be used in the non-discrete case. Let A_1^1, A_2^1 be symmetric of type 1 with $\mu(A_1^1) < \mu(A_2^1)$. Since G is not discrete there exists A_3^1 such that $\mu(A_1^1) < \mu(A_3^1) < \mu(A_2^1)$.[10] Furthermore by Theorem B (Appendix) there exists A_2, A_3 such that $A_1^1 = A_1 \subset A_3 \subset A_2, \mu(A_2) = \mu(A_2^1)$ and $\mu(A_3) = \mu(A_3^1)$. Now let g be an approximate identity for $\Phi_{A_3^{-1}}$, that is, $g \geq 0$, $\int g = 1$ and

$$||g * \Phi_{A_3^{-1}} - \Phi_{A_3^{-1}}||_1 < \varepsilon.$$

Finally let $F(x) = g^2 * \Phi^3_{A_3^{-1}}(x)$ and t_1, t_2, t_3 be the numbers whose existence is asserted by Theorem C (Appendix). Then, to within measure zero $A_1^1 \supset F_{t_1}$, $A_2^1 \supset F_{t_2}$, $A_3^1 \supset F_{t_3}$, and $t_1 \geq t_3 \geq t_2$.[11]

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9) To separate in the sense that if $A \leq B < C$ the B separates A and C.

10) Non-discreteness implies there are sets of arbitrarily small positive measure (see Ref. 6, pp. 123-124) and the assertion then follows from Theorem B.

11) See Theorem C for definition of F_t.

Also by Theorem C if $t_2 < t_1$ then $F_{t_2} \supseteq A_1^1$ to within measure zero and $A_2^1 \supseteq A_1^1$ which proves monotonicity to within measure zero.

That ε can be chosen so as to make $t_2 < t_1$ can be seen as follows. By unimodularity,

$$|\Phi_{A_i} * g * \Phi_{A_3^{-1}}(e) - \Phi_{A_i} * \Phi_{A_3^{-1}}(e)| = |(g * \Phi_{A_3^{-1}} - \Phi_{A_3^{-1}}) * \Phi_{A_i}(e)| .$$

Since $(g * \Phi_{A_3^{-1}} - \Phi_{A_3^{-1}}) * \Phi_{A_i}(x)$ is continuous [see footnote 2] supremum of absolute value and $|| \quad ||_\infty$ are the same so that

$$(1) \quad |\Phi_{A_i} * g * \Phi_{A_3^{-1}}(e) - \Phi_{A_i} * \Phi_{A_3^{-1}}(e)| \leq ||(g * \Phi_{A_3^{-1}} - \Phi_{A_3^{-1}}) * \quad A_i||_\infty$$

$$\leq \varepsilon \, ||\Phi_{A_i}||_\infty = \varepsilon .$$

The last inequality is the theorem in Ref. 6, page 121. Now $\Phi_{A_i} * \Phi_{A_3^{-1}}(e) = \mu[A_i \cap A_3] = \min[\mu(A_i), \mu(A_3)]$ so that (1) implies

$$\min[\mu(A_i), \mu(A_3)] - \varepsilon \leq \Phi_{A_i} * g * \Phi_{A_3^{-1}}(e).$$

Symmetrization does not decrease the double convolution so that

$$(2) \quad \min[\mu(A_i), \mu(A_3)] - \varepsilon \leq \Phi_{A_i^{-1}} * g^2 * \Phi^3_{A_3^{-1}}(e) \leq \min[\mu(A_i), \mu(A_3)].$$

The last inequality is obtained from the facts that $\int g^2 = 1$, and $\Phi_{A_i}, \Phi_{A_3^{-1}} \leq 1$. Now by Theorem C (Appendix),

$$(3)\quad \Phi_{A_i^1} * g^2 * \Phi^3_{A_3^{-1}}(e) = \int_{A_i^1} F(x)\,dx = \int_{F_{t_i}} F(x)\,dx + [\mu(A_1) - \mu(F_{t_i})]t_i .$$

If $t_1 = t_2 = t_3$ then by (2) and (3) applied to $i = 1$ and 3,

$$[\mu(A_3) - \mu(A_1)]t_1 = \Phi_{A_3^1} * g^2 * \Phi^3_{A_3^{-1}}(e) - \Phi_{A_1^1} * g^2 * \Phi_{A_3^{-1}}$$

$$> \mu(A_3) - \mu(A_1) - \varepsilon$$

while (2) and (3) applied to $i = 2$ and 3 yields

$$[\mu(A_2) - \mu(A_3)]t_1 = \Phi_{A_2^1} * g^2 * \Phi_{A_3^{-1}}(e) - \Phi_{A_3^1} * g^2 * \Phi_{A_3^{-1}}(e) < \varepsilon .$$

If $\varepsilon = (1/2)\min[\mu(A_3) - \mu(A_1), \mu(A_2) - \mu(A_3)]$ then the inequalities can not be simultaneously satisfied, so that it must be that $t_2 < t_1$.

$F(x)$ is continuous [see footnote 2] so that F_{t_2} is open. Thus we have shown that if $\mu(A_1^1) < \mu(A^1)$ then there exists an open set, O_i which separates A_1^1 and A^1 to within measure zero. Choosing a sequence A_i^1 such that $\mu(A_i^1) \uparrow \mu(A^1)$ shows that $\bigcup_i O_i \subset A^1$ to within measure zero and $\mu(\bigcup_i O_i) \geq \mu(O_n) \uparrow \mu(A^1)$. Thus $\bigcup_i O_i$ is an open set which differs from A^1 by a set of measure zero.

Q.E.D.

Theorem 2 says that any three mappings of L^{1+} into itself for which Hypothesis A is valid is not essentially different from symmetrizations based on three monotone classes of open sets. The proof can be made to yield more in the discrete case, namely, Hypothesis A implies Hypothesis B. For the proof shows that $A_\alpha = c^{-1}B_\alpha^{-1}$ where

where $\{c\} = C_1$. Applying this result to the cyclic permutations of the three classes gives $B_\alpha = a^{-1}C_\alpha^{-1}$, $C_\alpha = b^{-1}A_\alpha^{-1}$ from which it can be shown that $A_\alpha a^{-1} = aB_\alpha c = c^{-1}C_\alpha$. Furthermore unimodularity gives

$$\Phi_{Aa^{-1}} * \Phi_{aBc} * \Phi_{c^{-1}C}(e) = \Phi_A * \Phi_B * \Phi_C(e).$$

So that the symmetrization based on the class $A_\alpha a^{-1}$ shows Hypothesis B to be satisfied. It is conjectured that in general Hypothesis A is valid if and only if Hypothesis B is valid.

A further restriction on any G for which Hypothesis A is valid is the following.

THEOREM 3. If there exists M and N, subgroups of positive measure, such that $\mu(M) < \mu(N)$ and $\mu(N)/\mu(M)$ is not an integer then Hypothesis A is not valid.

<u>Proof</u>: We assume Hypothesis A and begin by showing that M^1 and N^1 are "almost" subgroups.

Now $\Phi_M * \Phi_M * \Phi_M(e) = \int_M \int_{M^{-1}} \Phi_M(xy^{-1})\, dx\, dy = [\mu(M)]^2$ and,

since this is also an upper bound over all rearrangements,

$$\int_{M^1} \mu[M^2 \cap x(M^3)^{-1}]\, dx = \Phi_{M^1} * \Phi_{M^2} * \Phi_{M^3}(e) = [\mu(M)]^2.$$

Since $\mu[M^2 \cap x(M^3)^{-1}] \leq \mu(M)$, then $M^1 \subseteq \bar{M} = \{x \mid \mu[M^2 \cap x(M^3)^{-1}] = \mu(M)\}$ except for a set of measure zero. Otherwise the upper bound could not be attained. On the other hand

$$\mu(M)\mu(\bar{M}) = \int_{\bar{M}} \mu[M^2 \cap x(M^3)^{-1}]\, dx \leq \int \mu\, [M^2 \cap x(M^3)^{-1}]\, dx = [\mu(M)]^2$$

so that $\mu(\bar{M}) = \mu(M)$ and $M^1 = \bar{M}$ except for a set of measure zero. Next it is claimed that if $m \varepsilon \bar{M}$ then $\bar{M}m^{-1}$ is a group, which will be shown as follows. Now the measure of the symmetric difference, $M^2 \oplus x(M^3)^{-1}$, is $\mu\left\{M^2 \oplus x(M^3)^{-1}\right\} = \mu\left\{(M^3)^{-1}\right\} + \mu\left\{(M^2)\right\} - 2\mu\left\{M^2 \cap x(M^3)^{-1}\right\}$. But unimodularity implies $\mu\left\{(M^3)^{-1}\right\} = \mu\left\{(M^3)\right\} = \mu\left\{(M^2)\right\} = \mu\left\{(M)\right\}$. Thus $x \varepsilon \bar{M}$ if and only if $\mu\left\{M^2 \oplus x(M^3)^{-1}\right\} = 0$. It follows from the triangular inequality for the measure of the symmetric difference of sets that

$$x \varepsilon \bar{M}m^{-1} \qquad \text{if and only if } \mu \left\{M^2 \oplus xM^2\right\} = 0.$$

Finally, if $x, y \varepsilon \bar{M}m^{-1}$ then

$$\mu(M^2 \oplus xy^{-1}M^2) \leq \mu(M^2 \oplus xM^2) + \mu(xM^2 \oplus xy^{-1}M^2)$$

$$= \mu(M^2 \oplus xM^2) + \mu(yM^2 \oplus M^2) = 0,$$

so that $xy^{-1} \varepsilon \bar{M}m^{-1}$. Which proves that $\bar{M}m^{-1}$ is a group. Similar arguments hold for $\bar{N}$. Since $M^1 \subseteq N^1$ and $\bar{M}, \bar{N}$ differ from M^1, N^1 by sets of measure zero, we have $\mu(\bar{M} \cap \bar{N}) = \mu(M) > 0$ and so $\bar{M} \cap \bar{N}$ is non-empty. Let $m \varepsilon \bar{M}$ $\bar{N}$, then $\bar{M}m^{-1}$ and $\bar{N}m^{-1}$ are subgroups of G as also is $(\bar{M} \cap \bar{N})\, m^{-1}$. Further, $\mu\left\{(\bar{M} \cap \bar{N})\, m^{-1}\right\} = \mu(M)$. Now $\bar{N}m^{-1}$ consists of a union of cosets of $(\bar{M} \cap \bar{N})\, m^{-1}$ so that $\mu(N) = \mu(\bar{N}m^{-1})$ $= \mu(M)$ times the number of cosets of $(\bar{M} \cap \bar{N})\, m^{-1}$ in $\bar{N}m^{-1}$. But this contradicts the condition of the theorem that $\mu(N)/\mu(M)$ be finite but not an integer.

Q.E.D.

Theorem 3 eliminates from consideration any finite groups other than those whose order is a power of a prime. The latter case is still open.

Examples. The group of translations in Euclidean n-space is an example of a group in which Hypothesis B holds. The case of the real line has already been cited. The higher dimenionsal cases can be obtained from an extension of the work of Gross[12] (or from later parts of this paper). Gross constructed a suitable Steiner sequence[13] in 3-space such that beginning with a given integrable set the successive sets converge to the interior of a sphere. By convergence we mean $\mu(A^i \oplus S) \to 0$ as $i \to \infty$, where $\oplus$ indicates the symmetric difference of two sets. This together with Theorem 5 shows that Hypothesis B holds where symmetric sets are the interiors of concentric spheres.

It is conjectured that Hypothesis B holds on the circle. However the following theorem is the best result available.

THEOREM 4. Let f, g, h be integrable functions on the group of plane rotations (represented as $\left\{\phi \mid -\pi \leq \phi < \pi\right\}$). Furthermore let $0_\alpha = \left\{\phi \mid |\phi| < \alpha/2\right\}$ for $0 \leq \alpha \leq 2\pi$ and f^1 be the symmetrization of f based on the class $\left\{0_\alpha\right\}$ and similarly for g^1, h^1.[14] Then

$$f * g * h^1(0) \leq f^1 * g^1 * h^1(0)$$

12) Ref. 7.

13) See Definition preceding Theorem 5.

14) F^1 is also called the symmetrically decreasing rearrangement in analogy with the corresponding rearrangement on the real line.

Proof: By the remarks preceding Theorem 2, the general result follows from the validity of the inequality in the case where f, g, h are the characteristic functions of sets. Let A, B, C be integrable sets, with measure $2a, 2b, 2c$, respectively (for convenience $\mu(G) = 2\pi$) and define

$$F(\phi) = \Phi_B * \Phi_C(-\phi)$$

Then $\Phi_A * \Phi_B * \Phi_C(0) = \int_A F(\phi)\, d\phi$.

Theorem C (Appendix) asserts the existence of t such that

$$\mu(F_t) \leq 2a \leq \mu\left\{\phi \mid F(\phi) \geq t\right\}$$

where

$$F_t = \left\{\phi \mid F(\phi) > t\right\}.$$

For each a designate by $t(a)$ the infimum of all such t. Notice that $F^1(\phi) = t(|\phi|)$ and

$$\int_A F(\phi)\, d\phi \leq \int_{F_{t(a)}} F(\phi)\, d\phi + t(a)[2a - \mu(F_{t(a)})] = \int_{-a}^{a} F^1(\phi)\, d\phi.$$

The proof of the theorem reduces to showing

$$\int_{-a}^{a} F^1(\phi)\, d\phi \leq \int_{-a}^{a} \Phi_B^1 * \Phi_C^1(-\phi)\, d\phi$$

at this point we introduce Lemma 2 which will be proved later.

LEMMA 2. $|t(a_1) - t(a_2)| \leq |a_1 - a_2|$

Let us now examine

$$F^*(\phi) \equiv_{\text{DEF.}} \Phi^1_B * \Phi^1_C(-\phi)\, d\phi$$

$$(1) \qquad F^*(\phi) = \begin{cases} \min[2b,2c] & \text{for} \quad |\phi| \leq |b-c| \\ \max[0,\ 2b+2c-2\pi] & \text{for} \quad |\phi| \geq \min[b+c,\ 2\pi-b-c] \\ \text{linear and continuous for} & \text{all} \quad \phi \end{cases}$$

In order for F^* to be linear and continuous the derivative must be $-\text{sgn}\ \phi$ for $|b-c| < |\phi| < \min[b+c,\ 2\pi-b-c]$. With the use of the definition of F and F^1 it can be shown that

$$(2) \qquad \min[2b,2c] \geq F^1(\phi) \geq \max[0,\ 2b+2c-2\pi]$$

and

$$(3) \qquad \int_{-\pi}^{\pi} F^1(\phi)\, d\phi = \int_{-\pi}^{\pi} F^*(\phi)\, d\phi = 4bc.$$

If $a < |b-c|$ or $a > \min[b+c,\ 2\pi-b-c]$ then the inequality

$$\int_{-a}^{a} F^1(\phi)\, d\phi \leq \int_{-a}^{a} F^*(\phi)\, d\phi$$

readily follows from (1), (2) and (3). Now suppose

$$F^1(a) \leq F^*(a)$$

and

$$|b-c| \leq a \leq \min[b+c,\ 2\pi-b-c] .$$

Then by Lemma 2

$$F^1(|x|) - F^1(a) \leq |a - |x||$$

so that for $|b-c| \leq |x| \leq a$

$$F^1(|x|) = F^1(a) + F^1(|x|) - F^1(a) \leq F^*(a) + a - |x| = F^*(x) .$$

Also

$$F^1(x) \leq \min[2b,2c] = F^*(x) \qquad \text{for} \qquad |x| \leq |b-c|$$

so that

$$\int_{-a}^{a} [F^*(x) - F^1(x)]\, dx \geq 0 .$$

On the other hand suppose

$$F^1(a) \geq F^*(a) .$$

Then again using Lemma 2 for $a \leq |x| \leq \min[b+c,\ 2\pi-b-c]$ gives

$$F^1(x) = F^1(a) + F^1(x) - F^1(a) \geq F^*(a) - (|x| - a) = F^*(x) .$$

Also

$$F^1(x) \geq \max[0,\ 2b+2c-2\pi] = F^*(x) \quad \text{for} \quad \min[b+c,\ 2\pi-b-c] \leq |x|$$

so that

$$\int_{-a}^{a} [F^*(x) - F^1(x)]\, dx = -\int_{|x|>a} [F^*(x) - F^1(x)]\, dx \geq 0$$

Q.E.D.

In what follows Φ_B, Φ_C, F, F^1 will be assumed to be periodic functions on the real line with period 2π. To prove Lemma 2 we shall need

LEMMA 1. $|F(x_1) - F(x_2)| \leq |x_1 - x_2|$.

Proof: We can obviously assume $x_1 \geq x_2$. Then

$$|F(x_1) - F(x_2)| = \left| \int_{-\pi}^{\pi} \Phi_B(y)\, \Phi_C^1(-x_1 - y)\, dy - \int_{-\pi}^{\pi} \Phi_B(y)\, \Phi_C^1(-x_2 - y)\, dy \right|$$

$$= \left| \int_{-x_1-c}^{-x_1+c} \Phi_B(y)\, dy - \int_{-x_2-c}^{-x_2+c} \Phi_B(y)\, dy \right|$$

$$= \left| \int_{-x_1}^{-x_2} [\Phi_B(y - c) - \Phi_B(y + c)]\, dy \right|$$

$$\leq \int_{-x_1}^{-x_2} |\Phi_B(y - c) - \Phi_B(y + c)|\, dy$$

$$\leq x_1 - x_2 .$$

LEMMA 2. $|t(a_1) - t(a_2)| \leq |a_1 - a_2|$.

Proof: We may suppose $t(a_1) < t(a_2)$ and thus $a_1 > a_2$. The proof consists of constructing two intervals in $F_{t(a_1)} \cap \bar{F}_{t(a_2)}$, where $\bar{A}$ denotes the complement of A, and using Lemma 1 to estimate their length. Suppose $0 < a_2$ and $a_1 < \pi$. Then by Theorem C (Appendix)

$$\mu\left\{x | F(x) \geq t(a_1)\right\} \geq 2a_i > 0 \qquad i = 1,2$$

and

$$\mu\left\{x | F(x) \leq t(a_i)\right\} = 2\pi - \mu(F_{t(a_i)}) \geq 2\pi - 2a_i > 0.$$

Since these sets have positive measure they are non-empty. Furthermore Lemma 1 shows $F(x)$ to be continuous. Thus there exists x_i, $i = 1,2$ such that

$$F(x_i) = t(a_i) .$$

The cases $a_2 = 0$ or $a_1 = \pi$ can be handled by taking limits and pointing out that the circle is compact. Since translations of F do not affect the statement of the lemma we can assume $x_1 = -\pi$ and $-\pi < x_2 < \pi$.

Let

$$\phi_1 = \sup \left\{ x \middle| -\pi \leq x < x_2 \quad \text{and} \quad F(x) = t(a_1) \right\}$$
$$\phi_2 = \inf \left\{ x \middle| x_2 < x \leq +\pi \quad \text{and} \quad F(x) = t(a_1) \right\}.$$

Both of the sets exhibited are non-empty since $F(-\pi) = F(\pi) = t(a_1)$ so that ϕ_1, ϕ_2 are finite and $F(\phi_1) = F(\phi_2) = t(a_1)$ by continuity of F. Next let

$$\phi_1' = \inf \left\{ x \middle| \phi_1 < x \leq x_2 \quad \text{and} \quad F(x) = t(a_2) \right\}$$
$$\phi_2' = \sup \left\{ x \middle| x_2 \leq x < \phi_2 \quad \text{and} \quad F(x) = t(a_2) \right\}.$$

A similar argument shows that ϕ_1', ϕ_2' are finite and $F(\phi_1') = F(\phi_2') = t(a_2)$.

Now $x \in (\phi_1, \phi_1') \cup (\phi_2', \phi_2)$ implies $t(a_1) < F(x) < t(a_2)$ which tells us that $\left\{ x \middle| F(x) \geq t(a_2) \right\}$, (ϕ_1, ϕ_1'), (ϕ'_2, ϕ_2) are disjoint and their union is contained in $F_{t(a_1)}$. Then it follows from the definition of $t(a_2)$ and $t(a_1)$ that

$$(1) \qquad 2a_2 + \phi_1' - \phi_1 + \phi_2 - \phi_2' \leq \mu(F_{t(a_1)}) \leq 2a_1 .$$

By Lemma 1

$$t(a_1) - t(a_2) \leq \phi_1' - \phi_1, \phi_2 - \phi_2'$$

so that (1) becomes

$$2[t(a_1) - t(a_2)] \leq 2(a_1 - a_2) .$$

Q.E.D.

Second Section. Steiner Symmetrization.

By the direct product of two groups, $G = G_1 \times G_2$, we mean the set of all ordered pairs $\langle g_1, g_2\rangle$ with $g_i \in G_i$ and multiplication defined by $\langle g_1, g_2\rangle \langle h_1, h_2\rangle = \langle g_1h_1, g_2h_2\rangle$. The groups G_1, G_2 are isomorphic to subgroups of G, namely $G_1 \sim G_1 \times e$, $G_2 \sim e \times G_2$. Hereafter G_1, G_2 will be identified with these subgroups and two decompositions will be considered distinct if at least one pair of corresponding subgroups are. If G_1, G_2 have left invariant measures, μ_1 and μ_2, then it is obvious that the product measure is a left invariant measure on G.[15]

Now let $G = H \times R$ where R is a subgroup identified with the real numbers, and let μ_1, μ_2 be the left invaraint measure on H and Lebesque measure respectively. μ_2^* denotes Lebesque outer measure.

DEF. If A is a measurable subset of G then generalized Steiner symmetrization of A with respect to the subgroup H is

$$A_H = \left\{\langle h,r\rangle \middle| h \in H,\ r \in R \quad \text{and} \quad |r| \leq \tfrac{1}{2}\mu_2^*\{r|\langle h,r\rangle \in A\}\right\}$$

If $\mu_2^*\{r|\langle h,r\rangle \in A\} = 0$ then the condition

$$|r| \leq \tfrac{1}{2}\mu_2\{r|\langle h,r\rangle \in A\}$$

is replaced by the condition r is an element of the empty set or $r = 0$ accordingly as $\{r|\langle h,r\rangle \in A\}$ is empty or non-empty.

In more descriptive terms, generalized Steiner symmetrization replaces the intersection of A with the line $h \times R$ by an interval with the same linear measure and centered at $\langle h,0\rangle$. By Fubini's

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15) For definition of product measure see Ref. 6, pp. 44-45.

theorem $\mu(A_H) = \mu(A)$ so that A_H is a rearrangement of A.

The group of translations in Euclidean n-space is the n-dimensional vector space over the real numbers and admits of infinitely many representations of the form $H_i \times R_i$ where R_i is a linear 1-dimensional vector space (and thus isomorphic to the real numbers) and H_i is an n-1-dimensional subgroup. If we introduce a scalar product then we have available the concept of perpendicularity so that the following definition makes sense.

<u>DEF</u>. If V_n is the group of translations in Euclidean n-space, H is an (n-1)-dimensional subgroup and R is a 1-dimensional subgroup perpendicular to H, then the generalized Steiner symmetrization is called the Steiner symmetrization.[16]

Since generalized Steiner symmetrization takes monotone classes of sets into monotone classes of sets, the Steiner symmetrization of non-negative functions can be defined by means of the construction of symmetric functions in terms of symmetric sets which follows Theorem 1.

<u>DEF</u>. If $\{H_i\}$ be a sequence of (n-1)-dimensional subgroups and A an integrable set in V_n and if $A^o = A$, $A^i = (A^{i-1})_{H_i}$, then $\{A^i\}$ is a Steiner sequence.

THEOREM 5. Generalized Steiner symmetrization never increases

$$f * g * h(e)$$

provided f, g, h are non-negative integrable functions.

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16) Cf. Ref. 8, pp. 75-91.

<u>Proof</u>: Let $G = G_1 \times R$. s and t will be elements of G_1 and x and y will be real numbers.

$$f * g * h(e) = \int_{G_1}\int_{G_1}\int_R\int_R f(s,x)\, g(s^{-1}t, y-x)\, h(t^{-1}, -y)\, dx\, dy\, ds\, dt\,.$$

The order of integration has been rearranged using Fubini's theorem. Now for almost all s and t, $f(s, \ )$, $g(s^{-1}t, \ )$ and $h(t^{-1}, \ )$ are measurable functions on R so that the validity of Hypothesis B on the additive group of real numbers implies for almost all s and t

$$\int_R\int_R f(s,x)\, g(s^{-1}t, y-x)\, h(t, -y)\, dx\, dy \leq$$
$$\leq \int_R\int_R f^1(s,x)\, g^1(s^{-1}t, y-x)\, h^1(t^{-1}, -y)\, dx\, dy.$$

$f^1(s, \ )$ is the symmetrically decreasing rearrangement of $f(s, \ )$ and similarly for $g(s^{-1}t, \ )$ and $h(t^{-1}, \ )$. Integrating both sides of the above inequality over $G_1 \times G_1$ and observing that $f^1(s,x)$ is in fact the generalized Steiner symmetrization with respect to G_1 gives the theorem. Q.E.D.

Steiner symmetrization was originally defined for convex sets in the plane in connection with the isoperimetric problem. It was shown in the reference in footnote 16 that Steiner symmetrization does not increase the perimeter of convex polygons. More generally it can be shown for any polygonal Jordan curve, Γ, with a finite number of sides that the symmetrization of the curve and its interior results in a point set whose boundary is a rectifiable curve whose length is not greater than the length of Γ. (It will not necessarily be a Jordan curve.)

By suitable polygonal approximations the results extend to more general rectifiable closed curves. However, the following theorem has a direct proof.

THEOREM 6. Let Γ be a rectifiable closed curve in the plane such that the winding number of every point not on the curve is 0 or 1.[17] Let A be the set of all points which lie on Γ or whose winding number is 1. If A_L is the Steiner symmetrization with respect to the line L, then the boundary of A_L is contained in a curve whose length is less than or equal to the length of Γ and which has the same winding number property as Γ.

The proof is very similar to the proof of Theorem 11 and so will be omitted.

The rest of this section will be restricted to Steiner symmetrization in the plane although some extension to higher dimensions is possible.

Let $\{L_i\}$ be an arbitrary sequence of one dimensional subgroups of V_2 and $\{R_i\}$ the perpendicular subgroups. Further let A be a set with a rectifiable, connected curve for its boundary and $\{A^i\}$ be the Steiner sequence generated by A and $\{L_i\}$. If we rotate the sets A^i such that their axes of symmetry are all L_1, then each A^i can be represented in the form

$$A^i = \{<x,y> | <x,0> \varepsilon L_1 \ <0,y> \varepsilon R_1 \ \text{ and } \ |y| \leq F_i(x)\} .$$

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17) The winding number is also called the topological index or merely index. For the definition see Ref. 9, p. 149.

For the perimeter to be finite and connected, A must be bounded, i.e., there exists M such that $\langle x,y \rangle \varepsilon A$ implies $x^2 + y^2 \leq M^2$. Therefore $F_i(\pm M) = 0$. The perimeter of A^i is a bound for the variation of F_i and this is bounded by the perimeter of A by theorem 6. Thus the functions F_i have a uniform bound on their variation and $F_i(\pm M) = 0$. By Helly's selection principle there exists a subsequence which converges almost everywhere to a function F and the corresponding subsequence of sets converges in measure to the corresponding set. It is not difficult to show that even without rotating the A^i to a common line of symmetry there exists a subsequence which converges in measure. The following theorem shows that the restriction that A have a finite perimeter can be weakened.

THEOREM 7. Let $L_1, R_1, L_2, R_2, F_1, F_2$ be as before and also $L_1 \neq L_2$. If A is a bounded integrable set bounded by M then F_2 is of bounded variation and the perimeter of A^2 is finite.

<u>Proof:</u> We choose two coordinate systems and an orientation in the plane so that

$$L_1 = \left\{(x',y') | y' = 0\right\}, \qquad L_2 = \left\{(x,y) | y = 0\right\}$$

and the angle, α, from L_2 to L_1 is positive but less than or equal to $\pi/2$.

First suppose $\alpha = \pi/2$, then the second symmetrization is seen to be the same as replacing A^1 by the area bounded by the symmetric decreasing rearrangement of $F_1(x')$. The symmetric decreasing rearrangement is monotone increasing in the second quadrant, monotone decreasing in the first quadrant, $F^*(-M) = F^*(M) = 0$ and $F^*(O) \leq M$. Thus var $F^* \leq 2M$. We can define the perimeter as

$$2 \sup \sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + (F^*(x_i) - F^*(x_{i-1}))^2} \; .$$

The supremum is taken over all partitions of the form $-M = x_o < x_1 \cdots < x_n = M$. A bound for the perimeter is $2 \text{ var } F^* + 4M \leq 8M$.

Next suppose $0 < \alpha < \pi/2$. Let μ_1 be Lebesque measure on the line and define

$$F_{21}(x) = \mu_1 \left\{ y \,|\, (x,y) \in A_1 \quad \text{and} \quad y > x \tan \alpha \right\}$$
$$F_{22}(x) = \mu_1 \left\{ y \,|\, (x,y) \in A_1 \quad \text{and} \quad y \leq x \tan \alpha \right\} .$$

The above exhibited sets can be shown measureable for all x and it is obvious that

$$F_2(x) = F_{21}(x) + F_{22}(x) \; .$$

The function $F_{21}(x) + x/\sin \alpha \cos \alpha$ will now be shown to be monotone increasing. Suppose $x_1 > x_2$, then

$$F_{21}(x_1) - F_{21}(x_2) + \frac{x_1 - x_2}{\sin \alpha \cos \alpha}$$

$$= \int_{x_1 \tan \alpha}^{\infty} \Phi_{A^1}(x_1,y)\, dy - \int_{x_2 \tan \alpha}^{\infty} \Phi_{A^1}(x_2,y)\, dy + \frac{x_1 - x_2}{\sin \alpha \cos \alpha}$$

$$= \int_{x_1 \tan \alpha}^{\infty} [\Phi_{A^1}(x_1,y) - \Phi_{A^1}(x_2, y + \frac{x_1 - x_2}{\tan \alpha})]\, dy$$
$$- \int_{x_2 \tan \alpha}^{x_2 \tan \alpha + (x_1 - x_2/\sin \alpha \cos \alpha)} \Phi_{A^1}(x_2,y)\, dy + \frac{x_1 - x_2}{\sin \alpha \cos \alpha}$$

(1)

$$\geq \int_{x_1 \tan \alpha}^{\infty} [\Phi_{A^1}(x_1,y) - \Phi_{A^1}(x_2,y + \frac{x_1 - x_2}{\tan \alpha})]\, dy \; ,$$

Now if $\Phi_{A^1}(x_2, y + \frac{x_1-x_2}{\tan\alpha}) = 1$ then the point $P(x_2, y + \frac{x_1-x_2}{\tan\alpha})$ is in A^1 and its primed coordinates must satisfy

(2) $$|y'| \le F_1(x')$$

where

$$x' = x_2 \cos\alpha + \sin\alpha(y + \frac{x_1-x_2}{\tan\alpha}) = x_1 \cos\alpha + y \sin\alpha$$

$$y' = -x_2 \sin\alpha + \cos\alpha(y + \frac{x_1-x_2}{\tan\alpha}) = -x_1 \sin\alpha + y \cos\alpha + \frac{x_1-x_2}{\sin\alpha} .$$

Expressing (2) in the unprimed system gives

(3) $$-x_1 \sin\alpha + y \cos\alpha + \frac{x_1-x_2}{\sin\alpha} \le F_1(x_1 \cos\alpha + y \sin\alpha) .$$

The last term on the left hand side of the inequality is positive and, in the range of integration of (1), $-x_1 \sin\alpha + y \cos\alpha$ is non-negative so that in the range of integration (3) implies

$$|-x_1 \sin\alpha + y \cos\alpha| \le F_1(x_1 \cos\alpha + y \sin\alpha) .$$

But this says that $\Phi_{A^1}(x_1,y) = 1$. We have just shown that in the range of integration $\Phi_{A^1}(x_2, y + x_1-x_2/\tan\alpha) = 1$ implies $\Phi_{A^1}(x_1,y) = 1$. Therefore the integral in (1) is non-negative and the monotonicity of $F_{21}(x) + x/\sin\alpha\cos\alpha$ is established. Furthermore $F_{21}(-M) = F_{21}(M) = 0$ so that $F_{21}(x) + x/\sin\alpha\cos\alpha$ is of bounded variation in $[-M,M]$. A similar argument is valid for F_{22} so that $F_2 = F_{21} + F_{22}$ is of bounded variation. Defining the perimeter as in the case $\alpha = \pi/2$ completes the theorem.

The existence of subsequences which converge naturally raises the question, "Under what conditions does a Steiner sequence converge?" That not every Steiner sequence of a bounded integrable set converges can be seen by the following example. Let the angle α_n, between L_n and L_{n+1} be $1/n$ and let A be a circle whose center is on L_1 but not <0,0> . Then if r_o is the distance of the center of A from <0,0> then A^n is a circle whose center is at a distance $r_n = r_o \prod_{i=1}^{n-1} \cos \alpha_r$ from <0,0> . The infinite product converges absolutely to a non-zero limit and the lines L_n intersect the circle $\sqrt{x^2+y^2} = r_o \prod_{i=1}^{\infty} \cos \alpha_i$ in an everywhere dense set. Therefore the sequence of centers have all points of the circle as limit points and A^i does not converge. It is conjectured that if $\sum_{n=1}^{\infty} |\alpha_n| < \infty$ or $\sum_{n=1}^{\infty} \alpha_n^2 = \infty$ and if $\bar{A}$ is bounded then the Steiner sequence converges in measure. The following theorem will enable us to show convergence of Steiner sequences provided the sequence $\{L_i\}$ is periodic.

THEOREM 8. Let A be a bounded integrable set in V_2, $\{L_i\}$ a sequence of one dimensional subspaces and R_i the perpendicular subspaces. Then the Steiner sequence obtained from A and $\{L_i\}$ has the property that

$$\sum_{n=1}^{\infty} \mu(A^n \oplus A^{n+1})^3 < \infty .$$

The proof will not be presented here since it is analogous to and somewhat simpler than the proof to Theorem 12.

COROLLARY. $\mu(A^n \oplus A^{n+N}) \to 0$ as $n \to \infty$.

The proof is obvious.

THEOREM 9. If the sequence $\{L_n\}$ is periodic and A is bounded the Steiner sequence converges in measure to a set having all the L_n as lines of Steiner symmetry.

Proof: Let $\{L_i\}$ have period p. According to Theorem 7 and the discussion preceding it there exists a subsequence A^{n_k} converging to a set A^*. At least one of the subgroups $L_1, \dots, L_p$, say L_s, must occur infinitely often among $\{L_{n_k}\}$ and it can be seen that A^* must be Steiner symmetric with respect to L_s to within measure zero. Now consider the sequences L_{n_k+N} for $N = 1, \dots, p-1$. The line L_{s+N} must occur infinitely often in this sequence. But since $\mu(A^n \oplus A^{n+N}) \to 0$ as $n \to \infty$, this sequence also converges to A^*. Thus A^* is Steiner symmetric to within measure zero with respect to all the lines $\{L_i\}$. It can be shown that

$$\mu(A \oplus B) \geq \mu(A_L \oplus B_L),$$

that is, Steiner symmetrization does not increase symmetric differences. This fact coupled with the fact that A^* is invariant under symmetrization with respect to all $\{L_i\}$ proves that the sequence A^i converges.

Q.E.D.

Pólya and Szegö point out in Ref. 11 that a Steiner sequence in 3-space based upon two planes whose included angle is an irrational multiple of π must converge to a solid of revolution about the line of intersection of the two planes <u>if</u> the sequence converges at all.[18]

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18) They also point out that the "if" has been removed in the case of convex solids. Cf. Ref. 10, pp. 86-90.

By observing what happens in each plane perpendicular to that line of intersection and using the preceding results, we can remove the "if" for all bounded integrable sets in 3-space.

Third Section. Steiner Symmetrization on a Sphere.

Although the sphere is not a group manifold there is a symmetrization analogous to Steiner symmetrization in the plane and several of the theorems carry over to the sphere. In particular, the isoperimetric property (Theorem 6), the convergence to zero of symmetric differences in a Steiner sequence (Theorem 8) and the convergence of a Steiner sequence based on a periodic sequence of coordinate systems (Theorem 9) all hold on the sphere. On the other hand, Theorem 5 (the decrease in the value of the convolution of three positive functions) must be modified before it is valid on the sphere.

Let C be a spherical coordinate system. That is, C consists of a mapping $P(\theta,\phi)$, which maps $[-\frac{\pi}{2}, \frac{\pi}{2}] \times [-\pi, \pi)$ onto the unit sphere such that the element of area, dP, is $\cos\theta \, d\theta \, d\phi$. ($\theta$ is the complement of the polar angle and ϕ is the azimuthal angle.) We will designate Lebesgue measure on $[-\pi, \pi)$ by μ_1 and Lebesgue outer measure by μ_1^*.

DEF. Let A be an integrable set on the sphere. Define

$$A_{C,\theta} = \left\{\phi \mid |\phi| \le \tfrac{1}{2}\mu_1^* \left\{\phi \mid P(\theta,\phi) \in A\right\}\right\}$$

if $\mu \left\{\phi \mid P(\theta,\phi) \in A\right\} \neq 0$. Otherwise define $A_{C,\theta}$ to be empty or $\{0\}$ accordingly as $\left\{\phi \mid P(\theta,\phi) \in A\right\}$ is empty or not.

<u>DEF.</u> $A_C = \{P(\theta,\phi) | \phi \in A_{C,\theta}\}$.

In attempting to construct a theorem analogous to Theorem 5, two difficulties occur. The sphere is not a group manifold and the necessary property of the real line is replaced by Theorem 4 which applies to the circle. Theorem 4 is adequate for the following theorem and the first difficulty is avoided.

THEOREM 10. Let $\rho(P,Q)$ be the spherical distance from P to Q, and let $H(\rho)$ be a monotone decreasing function of ρ. Further let F, G be non-negative integrable functions on the sphere. Then the Steiner symmetrization of F, G does not decrease

$$I = \iint F(P)\, G(Q)\, H(\rho(P,Q))\, dP\, dQ.$$

<u>Proof:</u> As before the general case follows from the particular case where F and G are the characteristic functions of measurable sets on the sphere, and H is the characteristic function of an interval $0 \leq \rho \leq h$. Writing the integral in the coordinate system C and interchanging the order of integration

$$I = \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} \left[\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} F(\theta,\phi)\, G(\theta',\phi')\, H(\rho(\theta,\phi,\theta',\phi'))\, d\phi\, d\phi' \right] \cdot \cos\theta \cos\theta'\, d\theta\, d\theta'.$$

Consider $H(\rho(\theta,\phi,\theta',\phi'))$. For fixed θ, ϕ this is the characteristic function of the spherical disc consisting of all points (θ',ϕ') whose distance from (θ,ϕ) is less than or equal to h.

The circle θ' = constant either does not intersect this region or intersects in a connected arc of the form $[\phi - a, \phi + a]$ where a is a function of θ, θ'. Define $H_{\theta,\theta'}(\phi)$ to be the characteristic function of the set, $\left\{ \phi \mid |\phi| \leq a \right\}$. Then

$$I = \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} \left[\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} F(\theta,\phi)\, G(\theta',\phi')\, H_{\theta,\theta'}(\phi-\phi')\, d\phi\, d\phi' \right]$$

$$\cdot \cos\theta \cos\theta'\, d\theta\, d\theta'.$$

Since $H_{\theta,\theta'}(x) = H_{\theta,\theta'}(-x)$ and $G(\theta', -\phi')$ is a rearrangement of $G(\theta',\phi')$, Theorem 4 applies to the bracketed expression and yields

$$I \leq \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} F^1(\theta,\phi)\, G^1(\theta',\phi')\, H(\rho\,(P,Q))$$

$$\cdot \cos\theta \cos\theta'\, d\phi\, d\phi'\, d\theta\, d\theta',$$

where

$$F^1(\theta,\phi) = \begin{cases} 1 & \text{if } |\phi| \leq \frac{1}{2}\mu_1\left\{\phi \mid F(\theta,\phi) = 1\right\} \\ 0 & \text{otherwise} \end{cases}$$

and a similar expression for G^1. Obviously F^1 is the Steiner symmetrization of F with respect to C to within measure zero, and the theorem is proved.

The statement of the analog to Theorem 6 is somewhat complicated by the properties of the sphere. Let the sphere be represented in Euclidean 3-space by $z = \sin\theta$, $z = \cos\theta \cos\phi$, $y = \cos\theta \sin\phi$ and consider the stereographic projection of the sphere from $(-1,0\ 0)$ onto the yz plane. The winding number of a point of the sphere relative to a curve bounded away from $(-1,0,0)$ will be defined as the winding number of the stereographic image relative to the image of the curve.

THEOREM 11. Let C be as before and let Γ be a rectifiable curve, bounded away from the point whose coordinates are $(0,-\pi)$, such that the winding number of every point not in Γ is either 0 or 1. Define A to be the union of Γ with the set of all points whose winding numbers are 1. Then the boundary of A_C is contained in a curve Γ' whose length is not greater than the length of Γ and which has the same winding number property.

Proof: Γ is a closed set and the limit points of a set of points with winding number 1 either have winding number 1 or lie on Γ.[19] Thus A is closed and $\{\phi | P(\theta,\phi) \in A\}$ is a closed set on the unit circle and so is measureable. Thus $F(\theta) \overset{DEF}{=} \frac{1}{2}\mu_1 \{\phi | P(\theta,\phi) \in A\}$ exists for all θ.

Define $\theta_1 = \inf \{\theta | \exists\, \phi$ such that $P(\theta,\phi) \in \Gamma\}$

$\theta_2 =$ sup of the same set.

Now if the cap, $\{P(\theta,\phi) | \theta < \theta_1\}$, possessed both points in A and not in A, there would be points of Γ with θ-coordinates less than θ_1. Thus the caps, $\{P(\theta,\phi) | \theta < \theta_1\}$ and $\{P(\theta,\phi) | \theta > \theta_2\}$ consist entirely of interior points or exterior points of A. By the definition of Steiner symmetrization on the sphere this implies each cap is either interior or exterior to A_C so that A_C has no boundary points with $\theta < \theta_1$ or $\theta > \theta_2$.

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19) Cf. Ref. 9.

A simple argument shows that the boundary points of A_C are contained in B where

$$B = \left\{ P(\theta,\phi) \mid \theta_1 \leq \theta \leq \theta_2,\ \min[F(\theta), \underline{\lim}_{x\to\theta} F(x)] \leq |\phi| \leq \max[F(\theta), \overline{\lim}_{x\to\theta} F(x)] \right\}.$$

In fact the only additional points this set has are of the form (θ,π) and are thrown in to make the boundary connected.

Define
$$L(\theta) = \overline{\lim_{|\sigma|\to 0}} \sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + \cos^2 \xi_i [F(x_i) - F(x_{i-1})]^2}$$

where $\overline{\lim}$ is taken over all partitions, $\sigma = \{x_o < \xi_1 < x_1 = \theta_1 < < \xi_2 < x_2 \cdots < x_n = \theta$ and $|\sigma| = \max_i (x_i - x_{i-1})$. In the event $\theta_1 = -\pi/2$ the first term is omitted. It is obvious that we are trying to "measure" the perimeter and the first term nicely takes care of boundary points of the cap, $\theta < \theta_1$ whether it consists of interior points or exterior points. The cap $\theta > \theta_2$ is taken care of by using $L(\theta_2+) = \lim_{x\to\theta+} L(x)$. The $+$ sign is omitted if $\theta_2 = \pi/2$. First we show that $2L(\theta_2+)$ is less than or equal to the length of Γ. Lastly we will exhibit the curve Γ' which has length $2L(\theta_2+)$, maps onto B, and has the right winding number property.

Proof that $2L(\theta_2+)$ is less than or equal to the length of Γ.

The following lemma is used and its proof will be found in the Appendix.

LEMMA TO THEOREM 11. Let Γ, $F(\theta)$, σ be as before and let Γ have length S and arc length s as parameter. Let

$$I_i(\sigma) = \left\{ s \mid x_{i-1} < \theta(s) < x_i \quad \text{or} \quad \theta(s) = x_{i-1} \quad \text{and} \quad \frac{d\phi}{ds} > 0 \right.$$
$$\left. \quad \text{or} \quad \theta(s) = x_i \quad \text{and} \quad \frac{d\phi}{ds} < 0 \right\}$$

Then $\theta(s)$, $\phi(s)$ are absolutely continuous functions of s and

$$2|F(x_i) - F(x_{i-1})| \leq \int_{I_i(\sigma)} \left|\frac{d\phi}{ds}\right| ds.$$

The lemma is used as follows

$$(1) \quad 4 \cos^2 \xi_i [F(x_i) - F(x_{i-1})]^2 \leq \cos^2 \xi_i \left[\int_{I_i(\sigma)} \left|\frac{d\phi}{ds}\right| ds \right]^2$$
$$\leq \left[\int_{I_i(\sigma)} \cos \theta(s) \left|\frac{d\phi}{ds}\right| ds \right]^2$$
$$+ 2(x_i - x_{i-1}) \left[\int_{I_i(\sigma)} \cos \theta(s) \left|\frac{d\phi}{ds}\right| ds \right]$$
$$\left[\int_{I_i(\sigma)} \left|\frac{d\phi}{ds}\right| ds \right].$$

From (1) it follows that

$$(2) \quad 2\sqrt{[x_i - x_{i-1}]^2 + [F(x_i) - F(x_{i-1})]^2 \cos^2 \xi} \leq$$

$$\leq \sqrt{4[x_i - x_{i-1}]^2 + \left[\int_{I_i(\sigma)} \cos \theta(s) \left|\frac{d\phi}{ds}\right| ds \right]^2} +$$
$$+ \frac{(x_i - x_{i-1}) \left[\int_{I_i(\sigma)} \cos \theta(s) \left|\frac{d\phi}{ds}\right| ds \right] \left[\int_{I_i(\sigma)} \left|\frac{d\phi}{ds}\right| ds \right]}{\sqrt{4[x_i - x_{i-1}]^2 + \left[\int_{I_i(\sigma)} \cos \theta(s) \left|\frac{d\phi}{ds}\right| ds \right]^2}}.$$

The last term is less than $|\sigma| \int_{I_i(\sigma)} |\frac{d\phi}{ds}| ds$. If $a > 0$ and we restrict the partitions to these for which $-\pi/2 + a < x_o$ and $x_n < \pi/2 - a$, then summing (2) over $1 \leq i \leq n$ gives

$$(3) \quad \sum_{i=1}^{n} 2\sqrt{[x_i - x_{i-1}]^2 + [F(x_i) - F(x_{i-1}]^2}$$

$$\leq \sum_{i=1}^{n} \sqrt{4[x_i - x_{i-1}]^2 + [\int_{I_i(\sigma)} \cos \theta(s) |\frac{d\phi}{ds}| ds]^2} + \frac{|\sigma|}{\cos a} S.$$

If $\theta_1 > -\pi/2$ and $\theta < \pi/2$ then (3) implies

$$(4) \quad 2L(\theta) \leq \overline{\lim_{|\sigma| \to 0}} \sum_{i=1}^{n} \sqrt{4[x_i - x_{i-1}]^2 + [\int_{I_i(\sigma)} \cos \theta(s) |\frac{d\phi}{ds}| ds]^2} .$$

The cases $\theta_1 = -\pi/2$ or $\theta = \pi/2$ can be handled by taking limits.

Now if $\theta_1 \leq x_{i-1} < x_i \leq \theta_2$ then the fact that Γ is a simple connected closed curve and the definition of θ_1 and θ_2 imply that there exists two disjoint subsets of $[0,S)$, say $I_i^1(\sigma)$, $I_i^2(\sigma)$, such that $x_{i-1} < x < x_i$ implies $x = \theta(s)$ has a solution in each set. Since $\theta(s)$ is of bounded variation and uniformly continuous, $|x_i - x_{i-1}| \leq \underset{I_i^1(\sigma)}{\text{Var}} \theta(s) = \int_{I_i^1(\sigma)} |\frac{d\theta}{ds}| ds$ with a similar expression for $I_i^2(\sigma)$. Also $I_i^1(\sigma)$ and $I_i^2(\sigma)$ are contained in $I_i(\sigma)$ so that

$$2|x_i - x_{i-1}| \leq \int_{I_i(\sigma)} |\frac{d\theta}{ds}| ds ,$$

Finally (4) becomes

(5) $2L(\theta_2 + \delta)$

$$\leq 2\delta + \overline{\lim_{|\sigma| \to 0}} \sum_{i=1}^{n} \sqrt{\left[\int_{I_i(\sigma)} \left|\frac{d\theta}{ds}\right| ds\right]^2 + \left[\int_{I_i(\sigma)} \cos\theta(s) \left|\frac{d\phi}{ds}\right| ds\right]^2}\ .$$

The 2δ is the contribution to $|x_i - x_{i-1}|$ from that portion of the partition between θ_2 and $\theta_2 + \delta$. By Minkowski's inequality[20]

(6) $$2L(\theta_2 + \delta) \leq 2\delta + \overline{\lim_{|\sigma| \to 0}} \sum_{i=1}^{n} \int_{I_i(\sigma)} \sqrt{\left|\frac{d\theta}{ds}\right|^2 + \cos^2\theta \left|\frac{d\phi}{ds}\right|^2}\, ds$$

Since the $I_i(\sigma)$ are disjoint

$$2L(\theta_2 + \delta) \leq 2\delta + S.$$

Taking the limit as $\delta \to 0+$ completes the first part of the proof in the event $\theta_2 < \pi/2$. If $\theta_2 = \pi/2$ then it can be shown that $2L(\theta_2) = 2 \lim_{x \to \theta_2 -} L(x)$ so that a slight modification of (5) and (6) corresponding to $\delta < 0$ gives the result.

Lastly, B may be represented by the points of a curve of length $2L(\theta_2 +)$ as follows. $L(\theta)$ is a strictly monotone increasing function of θ so that for $0 \leq s \leq L(\theta_2 +)$ there exists a unique θ such that $L(\theta -) \leq s \leq L(\theta +)$.

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20 Ref. 5, formula 6.13.2.

Define $\theta'(s)$ to be that θ.

If $s \leq L(\theta)$ where $\theta = \theta'(s)$ define

$$\phi'(s) = -\left[F(\theta-) + \frac{F(\theta) - F(\theta-)}{L(\theta) - L(\theta-)} (s - L(\theta-)) \right]$$

and if $s > L(\theta)$ where $\theta = \theta'(s)$ define

$$\phi'(s) = -\left[F(\theta) + \frac{F(\theta+) - F(\theta)}{L(\theta+) - L(\theta)} (s - L(\theta)) \right].$$

This represents a curve Γ' in the negative hemisphere $(\phi \leq 0)$. Replacing $\theta'(s)$ by $\theta'(2L(\theta_2+) - s)$ and $\phi'(s)$ by $-\phi'(2L(\theta_2+) - s)$ completes the curve, Γ', in the positive hemisphere $(\phi \geq 0)$. Notice that $\theta(L(\theta_2+)) = \theta(2L(\theta_2+) - L(\theta_2+)) = \theta_2$ and $\phi(L(\theta_2+)) = F(\theta_2+)$ while $-\phi(2L(\theta_2+) - L(\theta_2+)) = -F(\theta_2+)$. Now the cap $\theta > \theta_2$ consists either entirely of interior points of A_C or of exterior points so that $F(\theta_2+) = 0$ or π. In either event $[\theta_2, F(\theta_2+)]$ and $(\theta_2, -F(\theta_2+))$ represent the same point. Thus the two pieces of Γ are connected so that Γ' is a curve. A similar argument at θ_1 proves that Γ' is a closed curve.

We must show that Γ' has the winding number property. That is exterior points of A_C have winding number 0 and interior points have winding number 1. If both caps, $(\theta < \theta_1)$ and $(\theta > \theta_2)$ consist only of exterior points then the mapping $\Gamma' \rightarrow \Gamma'_\alpha$ gives by $\theta'(s) \rightarrow \theta'(s)$ and $\phi(s) \rightarrow (1-\alpha)\,\phi(s)$ is an admissible homotopy relative to exterior points.[21] For $\alpha = 1$, Γ'_α lies entirely on the arc, $\phi = 0$ and the winding number of any point not on Γ'_1 is 0. If a cap is interior then construction of the homotpy must be preceeded by a homotopy

21) See ref. 9 for definition of admissible homotopy.

which frees the cap from the curve.

For interior points the argument is as follows. (θ_o, ϕ_o) is interior to A_C if and only if $|\phi_o| < \min[\lim_{x \to \theta} F(x), F(\theta_o)]$. But if (θ_o, ϕ_o) has this property then so do all points of the arc $|\phi| \leq |\phi_o|$, $\theta = \theta_o$. Since winding number is continuous[15] this implies that $(\theta_o, 0)$ has the same winding number as (θ_o, ϕ_o). Now examine the stereographic image of Γ' in the yz plane. The curve goes from the image of $(\theta_1, F(\theta_1-))$ to $(\theta_2, F(\theta_2+))$ in the negative half plane $(z \leq 0)$ and back again in the positive half plane. It is easily seen that the net change of $\arg[y + iz - y_o]$ for y_o between these two points and not on the image of Γ' must be 2π so that the winding number is +1.[19]

The analyogy between Steiner symmetrizations on the plane and sphere breaks down at Theorem 7 where a finite perimeter was obtained. The proof of Theorem 7 depended upon the fact that lines perpendicular to L_1 meet the lines perpendicular to L_2 in a positive angle. However if C, C' are two coordinate systems on a sphere with common origin $(P(0,0) = P'(0,0))$ then the lines of constant θ are tangent to the lines of constant θ' if they meet along the great circle $\phi = \pm \pi/2$. It seems evident that a set could be constructed which was Steiner symmetric in C but which had infinite perimeter in every neighborhood of the great circle $\phi = \pm \pi/2$ and such that A_C also had infinite perimeter. This is apparently the only point where the analogy breaks down since we have:

THEOREM 12. Let $\{C_i\}$ be a sequence of coordinate systems on the sphere with common origin $(P_i(0,0) \equiv P_1(0,0))$ and let A be a measureable set and $\{A^i\}$ the Steiner sequence generated by A and $\{C_i\}$. Then

$$\sum_{i=1}^{\infty} \mu(A^i \oplus A^{i+1})^3 < \infty .$$

<u>Proof</u>: We shall estimate the terms $\mu(A^i \oplus A^{i+1})^3$ by terms of a convergent series.

Define $D^r = \{ P_i(\theta,\phi) | \cos\theta \cos\phi > r \}$ for $-1 \leq r \leq 1$.

Using the representation of the sphere in 3-space we can say that D^r consists of all points of the sphere whose x-coordinate is greater than r. Since all of the coordinate systems C_i give the same x-axis, D^r is independent of coordinate system. Now

$$\mu_1 \{\phi | P_{i+1}(\theta,\phi) \in A^i \cap D^r\} \leq \min[\mu_1\{\phi | P_{i+1}(\theta,\phi) \in D^r\}, \mu_1(A_{C^{i+1},\theta})].$$

By the definition of A^{i+1} the right hand side of this inequality equals

$$\mu_1 \{\phi | P_{i+1}(\theta,\phi) \in A^{i+1} \cap D^r\} .$$

Multiplying each of these functions by $\cos\theta\, d\theta$ and integrating gives

$$\mu(A^i \cap D^r) \leq \mu(A^{i+1} \cap D^r) \leq \mu(D^r) = (1 - r)\, 2\pi ,$$

Defining

$$G_i = \int_{-1}^{1} \mu(A^i \cap D^r)\, dr \leq \int_{-1}^{1} (1 - r)\, 2\pi\, dr = 4\pi ,$$

we see G_i as a monotone increasing sequence bounded from above so that

$$\sum_{i=1}^{\infty} G_{i+1} - G_i \leq 4\pi - G_1 < \infty .$$

We shall bound $\mu(A^i \oplus A^{i+1})^3$ by a fixed multiple of $G_{i+1} - G_i$. For convenience in the following we use $i = 0$. Define

$$D_\theta^r = \{\phi | P_1(\theta,\phi) \varepsilon D^r\}$$

$$A_\theta = \{\phi | P_1(\theta,\phi) \varepsilon A\} .$$

The corresponding function for $A' = A_C$ is $A_{C,\theta}$ which has been previously defined.

Then

$$G_1 - G_o = \int_{-1}^{1} \int_{-\pi/2}^{\pi/2} \mu_1(A_{C,\theta} \cap D_\theta^r) - \mu_1(A_\theta \cap D_\theta^r) \cos\theta \, d\theta \, dr.$$

Now if $r < -\cos\theta$, then $D_\theta^r = [-\pi,\pi)$ so that $\mu_1(A_{C,\theta} \cap D_\theta^r) = \mu_1(A_{C,\theta}) = \mu_1(A_\theta) = \mu_1(A_\theta \cap D_\theta^r)$. Also if $r > \cos\theta$ then D_θ^r is empty. Thus

$$(1) \quad G_1 - G_o = \int_{-\pi/2}^{\pi/2} \int_{-\cos\theta}^{\cos\theta} [\mu(A_{C,\theta} \cap D_\theta^r) - \mu_1(A_\theta \cap D_\theta^r)] \, dr \cos\theta \, d\theta .$$

Since A_C is Steiner symmetric with respect to the coordinate system C,

$$\mu_1(A_{C,\theta} \cap D_\theta^r) = \min[\mu_1(A_{C,\theta}), \mu_1(D_\theta^r)] = \min[\mu_1(A_\theta), 2 \arccos \frac{r}{\cos\theta}] .$$

Also

$$\mu_1(A_\theta \quad D_\theta^r) \leq \min[\mu_1(A_\theta), \mu_1(D_\theta^r)]$$

so that the integrand is always non-negative and decreasing the range of integration to $-\pi/2 \leq \theta \leq \pi/2$ and $-\cos\theta \leq r \leq \cos\theta \cos(\mu_1(A_\theta)/2)$ decreases the right hand side of (1). In this restricted range

$\mu_1(D_\theta^r) \geq \mu_1(A_\theta)$. With these considerations (1) implies

$$G_1 - G_o \geq \int_{-\pi/2}^{\pi/2} \int_{-\cos\theta}^{\cos\theta\cos\frac{\mu_1(A_\theta)}{2}} [\mu_1(A_\theta) - \mu_1(A_\theta \cap D_\theta^r)] \cos\theta \, dr \, d\theta$$

(2)
$$= \int_{-\pi/2}^{\pi/2} \int_{-\cos\theta}^{\cos\theta\cos\frac{\mu_1(A_\theta)}{2}} \int_{\phi \notin D_\theta^r} \Phi_A(\theta,\phi) \, d\phi \, dr \cos\theta \, d\theta,$$

where Φ_A is the characteristic function of A. Interchanging the order of integration

$$(3) \quad G_1 - G_o \geq \int_{-\pi/2}^{\pi/2} \int_{|\phi| \geq \frac{\mu_1(A_\theta)}{2}} \int_{\cos\theta\cos\phi}^{\cos\theta\cos\frac{\mu_1(A_\theta)}{2}} \Phi_A(\theta,\phi) \, dr \, d\phi \cos\theta \, d\theta.$$

The integration in the variable r can now be performed to give

$$(4) \quad G_1 - G_o \geq \int_{-\pi/2}^{\pi/2} \int_{|\phi| \geq \frac{\mu_1(A_\theta)}{2}} \Phi_A(\theta,\phi) \cos^2\theta [\cos\frac{\mu(A_\theta)}{2} - \cos\phi] \, d\phi \, d\theta,$$

The non-trivial range of integration of ϕ is that portion of A_θ which is disjoint from $A_{C,\theta}$ and the measure of this range is $\mu_1(A_\theta \cap \bar{A}_{C,\theta})$ where $\bar{A}_{C\theta}$ is the complement of $A_{C\theta}$. An immediate consequence of Theorem C (Appendix) is that

$$(5) \quad G_1 - G_o \geq \int_{-\pi/2}^{\pi/2} \int_{\frac{\mu_1(A_\theta)}{2} \leq |\phi| \leq \frac{\mu_1(A_\theta \cap \bar{A}_{C\theta})}{2} + \frac{\mu_1(A_\theta)}{2}} \cos^2\theta [\cos(\mu(A_\theta)/2) - \cos\phi] \, d\phi \, d\theta$$

Performing the integration in the variable $\emptyset$ and using a trigonometric identity for the difference of sines gives

$$(6) \quad G_1 - G_o \geq \int_{-\pi/2}^{\pi/2} \cos^2\theta \left[\frac{\mu_1(A_\theta \cap \bar{A}_{C\theta})}{2} \cos \frac{\mu(A_\theta)}{2} - 2 \sin \frac{\mu_1(A_\theta \cap \bar{A}_{C\theta})}{4} \cos\left(\frac{\mu A_\theta}{2} + \frac{\mu(A_\theta \cap A_{0\theta})}{4} \right) \right] d\theta .$$

Using the fact that for $x > 0$, $-x \leq -\sin x$ and a trigonometric identity for the difference of cosines we obtain from (6)

$$(7) \quad G_1 - G_o \geq \int_{-\pi/2}^{\pi/2} \mu_1(A_\theta \cap \bar{A}_{C\theta}) \sin \frac{\mu_1(A_\theta \cap \bar{A}_{C\theta})}{4} \sin\left[\frac{\mu_1(A_\theta)}{2} + \frac{\mu(A_\theta \cap \bar{A}_{C\theta})}{8} \right] \cos^2\theta \, d\theta .$$

Since

$$\frac{\mu_1(A_\theta \cap \bar{A}_{C\theta})}{4} \leq \frac{\mu_1(A_\theta)}{4} \leq \frac{\pi}{2} \quad \text{and} \quad \frac{\mu_1(A_{C\theta})}{2} + \frac{\mu(A_\theta \cap \bar{A}_{C\theta})}{2} \leq \pi,$$

The trigonometric functions may be estimated from below by linear functions to yield

$$G_1 - G_o \geq \int_{-\pi/2}^{\pi/2} \frac{\mu_1(A_\theta \cap \bar{A}_\theta')^3}{64} \cos^2\theta \, d\theta .$$

Multiplying the integrand by $\cos\theta$ and applying Holder's inequality to the functions $f(\theta) = \mu_1(A_\theta \cap \bar{A}_{C\theta})$, $g(\theta) = \frac{1}{\pi}$ gives

$$G_1 - G_o \geq \frac{\pi^2}{64} [\mu(A \cap \bar{A}_C)]^3 .$$

Since $\mu(A) = \mu(A')$, we have $\mu(\bar{A} \cap A_C) = \mu(A \cap \bar{A}_C) = \frac{1}{2}\mu(A \oplus A_C)$

Thus

$$G_1 - G_0 \geq \frac{\pi^2}{512} [\mu(A \oplus A_C)]^3 .$$

Q.E.D.

Our next theorem is the analogue to Theorem 9.

THEOREM 13. Let A be a measurable set on the sphere and $\{C_i\}$ be a periodic sequence of coordinate systems with a common origin. Then the Steiner sequence converges in measure to a set which is Steiner symmetric in all the coordinate systems.

Proof: If A has a finite perimeter the argument is identical with the proof of Theorem 9. The second observation is that the class $C = \{A \mid A^i \text{ converges in measure}\}$ is closed under complementation. For if $\bar{A}$ is the complement of A then $\bar{A}^i$ is obtained from $\overline{A^i}$ by reflection through the yz plane. And if A^i converges then so does $\overline{A^i}$. The third observation is that C is closed under monotone limits. For let B_n be a monotone sequence of sets of C. Then the B_n converge to a set we shall denote by B_∞. Denote by B_n^∞ the set to which the Steiner sequence B_n^i converges (n fixed). Since Steiner symmetrization takes a monotone family of sets into a family monotone to within measure zero, the sequence B_n^∞ must be monotone up to measure zero and thus converges in measure to a set we shall denote by B_∞^∞. Then

$$(1) \quad \mu(B_\infty^i \oplus B_\infty^\infty) \leq \mu(B_n^i \oplus B_\infty^i) + \mu(B_n^\infty \oplus B_n^i) + \mu(B_n^\infty \oplus B_\infty^\infty) .$$

Since Steiner symmetrization takes monotone sets into monotone sets, it can be shown that $\mu(B_n^i \oplus B_\infty^i) \leq \mu(B_n \oplus B_\infty)$ for all i. By choosing n such that $\mu(B_n \oplus B_\infty) < \frac{1}{4}\varepsilon$ and $\mu(B_n^\infty \oplus B_\infty^\infty) < \frac{1}{4}\varepsilon$ (both sequences converge) (1) becomes

$$\mu(B_\infty^i \oplus B_\infty^\infty) \leq \frac{1}{2}\varepsilon + \mu(B_n^\infty \oplus B_n^i) .$$

But $B_n^i \to B_n^\infty$ as $i \to \infty$, so that for all i sufficiently large

$$\mu(B_\infty^i \oplus B_\infty^\infty) < \varepsilon.$$

Thus B_∞ is in C. If we could show that C was a ring we would be done. Unfortunately there is no direct proof of this and we must proceed differently. It is well known that the sphere is separable under the ordinary topology. And in fact circular discs with rational coordinates for their centers and rational radii form a basis.[16] Since finite unions of circular discs have finite perimeter and every open set is the monotone increasing limit of such sets, every open set is in C. Next every G_δ is in C.[22] Lastly, since convergence is only to within measure zero, any set which differs from a G_δ by a set of measure zero is in C. But this includes all measurable sets.

Q.E.D.

The next theorem is the analogue of Theorem 4 on the sphere.

THEOREM 14. Let F, G be integrable functions on the sphere and H be monotone decreasing on $[0,\pi]$. Let F^*, G^* be the rearrangements

22) See ref. 12, p. 3 for definitions.

of F, G based on the sets $D^r = \{ P(\theta,\phi) | \cos\theta \cos\phi > r \}$. Then

$$\iint F(P)\, G(Q)\, H(\rho(P,Q))\, dP\, dQ \leq \iint F^*(P)\, G^*(Q)\, H(\rho(P,Q))\, dP\, dQ .$$

Proof: As on several preceding occasions the general case follows from the case when F, G are the characteristic functions of measurable sets, A and B. Let C_1, C_2 be two coordinate systems such that the angle between the arcs $\{ P_1(\theta,\phi) | \phi = 0 \}$ and $\{ P_2(\theta,\phi) | \phi = 0 \}$ at their point of intersection, $P_1(0,0) = P_2(0,0)$, is an irrational multiple of π. Now $\text{Lim}\, A^i$, $\text{Lim}\, B^i$ are Steiner symmetric in both coordinate systems and thus must be circular discs (the D^r's).[23] An application of Theorem 10 completes the proof.

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23) Ref. 8, p. 90.

APPENDIX

Theorems A, B, C are simple measure theoretic results which are used in the main part of the text.

THEOREM A. If A, B are integrable sets on G then

$$\mu\{z \mid \mu(A \cap zB) > 0\} = 0 \quad \text{if and only if}$$

$$\mu(A) = 0 \quad \text{or} \quad \mu(B) = 0 .$$

Proof:
$$\mu(A \cap zB) = \int \Phi_A(x)\, \Phi_B(z^{-1}x)\, dx$$

and

$$\int \mu(A \cap zB)\, dz = \iint \Phi_A(x)\, \Phi_B(z^{-1}x)\, dx\, dz$$

$$= \int \Phi_A(x) \int \Phi_B((x^{-1}z)^{-1})\, dz\, dx$$

$$= \int \Phi_A(x)\, \mu(B^{-1})\, dx = \mu(A)\, \mu(B^{-1}) .$$

Since $\mu(A \cap zB)$ is non-negative, $\mu(A)\mu(B^{-1})$ is zero if and only if

$$\mu\{z \mid \mu(A \cap zB) > 0\} = 0.$$

Lastly, $\mu(B^{-1}) = \int \Phi_B(z)\, \Delta(z^{-1})\, dz$. Since $\Phi_B(z)$ is non-negative and $\Delta(z^{-1})$ is positive everywhere, $\mu(B^{-1})$ is zero if and only if $\mu(B) = 0$. The theorem then follows.

THEOREM B. Let A, B, C be integrable sets on G with $\mu(A) \leq \mu(B) \leq \mu(C)$. Then there exists A^1, C^1 such that $\mu(A^1) = \mu(A)$, $\mu(C^1) = \mu(C)$ and $A^1 \subseteq B \subseteq C^1$.

Proof: We will construct A^1. The modification for C^1 is obvious.

If A or B have zero measure we are through. Otherwise consider $\mu\{A \cap z^{-1}B\}$. There exists z_1 such that

$$\mu(A \cap z_1^{-1}B) \geq \frac{1}{2} \sup_{z\varepsilon G} \mu(A \cap z^{-1}B) .$$

Define $A_1 = (z_1 A) \cap B$, $B_1^* = B - A_1$, $A_1^* = A - z_1^{-1}A_1$. Then $A_1 \subseteq B$, and $\mu(A_1^*) + \mu(A_1) = \mu(A)$.

Inductively we choose z_{i+1} such that

$$\mu(A_i^* \cap z_{i+1}^{-1} B_i^*) \geq \frac{1}{2} \sup_{z\varepsilon G} \mu(A_i^* \cap z^{-1}B_i^*)$$

and define

$$A_{i+1} = A_i \cup [z_{i+1} A_i^* \cap B_i^*]$$

$$B_{i+1}^* = B - A_{i+1}$$

$$A_{i+1}^* = A_i^* - A_i^* \cap z_{i+1}^{-1} B_i^* .$$

It is easily shown by induction that

$$A_i \subseteq B$$

(1) $$\mu(A_{i+1}) = \mu(A_i) + \mu[A_i^* \cap z_{i+1}^{-1} B_i^*],$$

(2) $$\mu(A_i^*) + \mu(A_i) = \mu(A),$$

and $\{A_i\}$ is a monotone increasing sequence and $\{B_i^*\}$, $\{A_i^*\}$ are monotone decreasing. Thus

$$A' = \lim_{i\to\infty} A_i, \quad B_\infty^* = \lim_{i\to\infty} B_i^*, \quad \text{and} \quad A_\infty^* = \lim_{i\to\infty} A_i^* \quad \text{exist.}$$

Also $\mu(A_i^* \cap z^{-1}B_i^*)$ decreases monotonely to $\mu(A_\infty^* \cap z^{-1}B_\infty^*)$. Furthermore, since $\{A_i\}$ is monotone and the $\mu(A_i) \leq \mu(B) < \infty$, we have

$$\mu(A_i^* \cap z_{i+1}^{-1}B_i^*) = \mu(A_{i+1}) - \mu(A_i) = \mu(A_{i+1} - A_i) \to 0 .$$

Thus $\sup_{z\in G} \mu(A_\infty^* \cap z^{-1}B_\infty^*) \leq \sup_{z\in G} \mu(A_i^* \cap z^{-1}B_\infty^*) \leq 2\mu(A_i^* \cap z_{i+1}^{-1}B_i^*) \to 0$ so that $\mu(A_\infty^* \cap z^{-1}B_\infty^*) \equiv 0$. By Theorem A either $\mu(A_\infty^*)$ or $\mu(B_\infty^*)$ equals zero. Now $B_\infty^* = B - A^1$ and

(3) $$\mu(B_\infty^*) = \mu(B) - \mu(A^1).$$

Taking the limit of (2) as $i \to \infty$ gives

(4) $$\mu(A_\infty^*) + \mu(A^1) = \mu(A) .$$

That A_∞^* is the one with zero measure follows from (3) and (4). Then from (4) it follows that $\mu(A^1) = \mu(A)$. Since $A_i \subseteq B$ for all i, $A^1 \subseteq B$.

Q.E.D.

COROLLARY. If A is integrable then there exists A^1 such that $\mu(A) = \mu(A^1)$ and the closure of A^1 is compact.

Proof: If G is compact then any closed subsets are compact and the corollary is proved. If G is not compact, since it is locally compact, there exist compact sets of arbitrarily large measure and thus one whose measure is greater than $\mu(A)$. The corollary then follows from the theorem.

THEOREM C. Let $F(x)$ be a positive integrable function on G and let A be an integrable set. Then there exists a real number t such that

$$\mu\{x|F(x) \geq t\} \geq \mu(A) \geq \mu\{x|F(x) > t\} \overset{DEF}{=} F_t$$

and

$$\int_A F(x)\,dx \leq \int_{F_t} F(x)\,dx + t\,[\mu(A) - \mu(F_t)]\,.$$

Furthermore if equality holds then

$$F_t \subseteq A \subseteq \{x|F(x) \geq t\} \qquad \text{except for a set of measure zero.}$$

Proof: Let $t_o = \inf\{t|\mu(F_t) < \mu(A)\}$

The sets F_t are monotone increasing as t decreases and $F_t = \lim\limits_{\to 0+} F_{t+\delta}$. Thus $\mu(F_t)$ is continuous on the right and

$$\mu(F_{t_o}) \leq \mu(A).$$

Also $\{x|F(x) \geq t_o\} = \bigcap\limits_{n=1}^{\infty} F_{t_o - 1/n}$ so that

$$\mu\{x|F(x) \geq t_o\} = \lim_{n\to\infty} \mu(F_{t_o - \frac{1}{2}}) \geq \mu(A).$$

Secondly, let $A_1 = A \cap F_{t_o}$, $A_2 = A \cap [\{x|F(x) \geq t_o\} - A_1]$, and $(A - A_1) - A_2 = A_3$. Then

$$\int_A f(x)\,dx = \int_{A_1} F(x)\,dx + \int_{A_2} F(x)\,dx + \int_{A_3} F(x)\,dx\,.$$

By Theorem B there exists a subset, C, of $A_2 \cup A_3$ whose measure is

$\mu(F_{t_o}) - \mu(A_1)$. Since $F(x) > t_o$ on F_t and $F(x) \leq t_o$ on C we have

$$\int_A F(x)\,dx \leq \int_{F_{t_o}} F(x)\,dx + \int_{A_2 - A_2 \cap C} F(x)\,dx + \int_{A_3 - A_3 \cap C} F(x)\,dx$$

with strict inequality holding unless $\mu(C) = 0$, that is, unless $F_t \subseteq A$ to within measure zero. A similar construction replaces $A_3 - A_3 \cap C$ by a subset of $\{x \mid F(x) = t_o\} - (A_2 - A_2 \cap C)$ and a corresponding strict inequality results unless $(A_3 - A_3 \cap C) = 0$. Combining the two steps yields

$$\int_A F(x)\,dx \leq \int_{F_{t_o}} F(x)\,dx + t_o[\mu(A) - \mu(F_{t_o})]$$

with strict inequality holding unless

$$F_{t_o} \subseteq A \subseteq \{x \mid F(x) \geq t_o\}$$

except for sets of measure zero.

LEMMA TO THEOREM 11.

Let Γ be a rectifiable closed curve on the sphere given by $\{(\theta(s), \phi(s)) \mid 0 \leq s \leq S\}$ where s is the arc length from the point $(\theta(0), \phi(0))$ along the positive orientation of the curve. For $-\pi/2 < a < b < \pi/2$, define

$$I = \left\{ s \mid a < \theta(s) < b \quad \text{or} \quad \theta(s) = a \text{ and } \frac{d\phi}{ds} > 0 \quad \text{or} \quad \theta(s) = b \text{ and } \frac{d\phi}{ds} < 0 \right\}.$$

Statement of Lemma to Theorem 11. If Γ is bounded away from $P(0,-\pi)$ and the winding number of every point not on Γ is either zero or one and

$$F(\theta) = \frac{1}{2}\mu_1 \{\phi | P(\theta,\phi) \text{ is on } \Gamma \text{ or has winding number } 1\}$$

then $\phi(s)$ and $\theta(s)$ are absolutely continuous functions of s on I and

$$2|F(b) - F(a)| \leq \int_I |\frac{d\phi}{ds}| \, ds \, .$$

Proof: We fist show absolute continuity. Choose $\delta > 0$ such that $-\pi/2 + \delta < a$ and $b < \pi/2 - \delta$. Thus

$$I \subseteq \{ s | -\frac{\pi}{2} + \delta < \theta(s) < \frac{\pi}{2} + \delta \}.$$

This last set is open and thus consists of a countable union of disjoint open intervals. On each of these intervals we have if $s_1 > s_o$ and each are in the interval

$$s_1 - s_o = \overline{\lim_{|\sigma|\to 0}} \sum_{i=1}^{n(\sigma)} \sqrt{(\Delta\theta_i)^2 + \cos^2[\theta(\varepsilon_i)] \, (\Delta\phi_i)^2}$$

where σ is a partition of $[s_o, s_1]$, $s_o = x_1 < \varepsilon_1 < x_2 \cdots < x_{n(\sigma)} = s_1$ and $\Delta\theta_i$, $\Delta\phi_i$ are the differences of consecutive values of the corresponding functions, $|\sigma| = \max_i x_i - x_{i-1}$. It follows that

$$s_1 - s_o \geq \overline{\lim_{|\sigma|\to 0}} \sum |\Delta\theta_i| \geq |\theta(s_1) - \theta(s_2)|$$

and

$$s_1 - s_o \geq \overline{\lim_{|\sigma|\to 0}} \sum \cos[\theta(\varepsilon_i)] \, |\Delta\phi_i| \geq \cos(\frac{\pi}{2} - \delta) | \, \phi(s_1) - \phi(s_2)| \, .$$

Now let I_1, I_2 be a sequence of intervals $[a_i, b_i]$ contained in $\{s \mid -\pi/2 + \delta < \theta(s) < \pi/2 - \delta\}$ such that $\sum b_i - a_i < \varepsilon \sin \delta$. Then

$$\sum_{i=1}^{\infty} |\phi(b_i) - \phi(a_i)| \le \frac{\sum_{i=1}^{\infty} (b_i - a_i)}{\cos(\frac{\pi}{2} - \delta)} < \frac{\varepsilon \sin \delta}{\cos(\frac{\pi}{2} - \delta)} = \varepsilon.$$

Therefore $\phi(s)$ is absolutely continuous. A similar computation shows $\theta(s)$ to be absolutely continuous and in fact by letting $\delta \to 0$ the absolute continuity of $\theta(s)$ on the whole interval $[0,S]$ can be shown

Next we will prove $2|F(b) - F(a)| \le \int_I |\frac{d\phi}{ds}| \, ds$. Let $\Phi_a(\phi) = 1$ if $P(a,\phi)$ is on Γ or has winding number 1, and $\Phi_a(\phi) = 0$ otherwise. Similarly for $\Phi_b(\phi)$. Then

$$2|F(b) - F(a)| = \left|\int_{-\pi}^{\pi} [\Phi_b(\phi) - \Phi_a(\phi)] \, d\phi\right|$$

$$\le \int_{-\pi}^{\pi} |\Phi_b(y) - \Phi_a(y)| \, dy \, .$$

If we can show the last integral to be less than or equal to $\int_I |\frac{d\phi}{ds}| ds$ the proof will be complete.

To do this we need two lemmas.

LEMMA 1. Let $N_I(y)$ be the number of solutions to the equation $\phi(s) = y$, $s \in I$. Then

$$\int_{-\pi}^{\pi} N_I(y) \, dy = \int_I |\frac{d\phi}{ds}| \, ds \, .$$

LEMMA 2. $|\Phi_b(y) - \Phi_a(y)| \le N_I(y)$ except for a set of y of measure zero.

That $\int_{-\pi}^{\pi} |\Phi_b(y) - \Phi_a(y)|\, dy \leq \int_I |\frac{d\phi}{ds}|\, ds$ is an immediate consequence of these two lemmas.

Proof of Lemma 1. Consider the class, C, of all sets A for which

$$\int_{-\pi}^{\pi} N_A(y)\, dy = \int_A |\frac{d\phi}{ds}|\, ds.$$

If A is an interval then $\int_A |\frac{d\phi}{ds}|\, ds = \underset{A}{\text{Var}}\, \phi$.[24] But a result of S. Banach shows that intervals are in C.[25] Next, if $A_1 \cup A_2 = A_3$ and $A_1 \cap A_2$ is empty then $N_{A_1}(y) + N_{A_2}(y) = N_{A_3}(y)$, so that

$$\int N_{A_1}(y)\, dy + \int N_{A_2}(y)\, dy = \int N_{A_3}(y)\, dy .$$

Also

$$\int_{A_1} |\frac{d\phi}{ds}|\, ds + \int_{A_2} |\frac{d\phi}{ds}|\, ds = \int_{A_3} |\frac{d\phi}{ds}|\, ds .$$

The integral equalities are to be interpreted as asserting also that if any two of the integrals exist then the third does. From this it follows that C is closed under relative complements and finite unions of disjoint members of C. If $\{A_i\}$ is a monotone sequence of members of C converging to A, then

$$N_{A_i}(y) \text{ converges monotonely to } N_A(y),$$

$$\int_{-\pi}^{\pi} N_{A_i}(y)\, dy \text{ converges monotonely to } \int_{-\pi}^{\pi} N_A(y)\, dy$$

and

$$\int_{A_i} |\frac{d\phi}{ds}|\, ds \text{ converges monotonely to } \int_A |\frac{d\phi}{ds}|\, ds .$$

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24) Ref. 13, Theorem on page 48.

25) Ref. 14, Theorem 2 page, 228.

Thus C is closed under monotone limits. In particular since every open set is the countable union of disjoint opern intervals, every open subset and every closed subset of $\{s|-\pi/2 + \delta < \theta(s) < \pi/2 - \delta\}$ are in C. Finally the countable intersections of open sets (G_δ's) and countable unions of closed set (F_σ's) belong to C. Now if A is measureable there exists F_σ, G_δ such that $F_\sigma \subseteq A \subseteq G_\delta$ and $\mu(G_\delta - F_\sigma) = 0$.[26] We have

$$N_{F_\sigma}(y) \leq N_A(y) \leq N_{G_\delta}$$

and

$$\int_{F_\sigma} \left|\frac{d\phi}{ds}\right| ds = \int_{G_\delta} \left|\frac{d\phi}{ds}\right| ds \quad \text{provided} \quad G_\delta \subset \{s|-\pi/2 + \delta < \theta(s) < \pi/2 - \delta\}.$$

The last statement implies $\int N_{F_\sigma}(y)\,dy = \int N_{G_\delta}(y)\,dy$. Since $N_{F_\sigma} \leq N_{G_\delta}$, this implies N_{F_σ} and N_{G_δ} and thus also N_A must differ on a set of measure zero. Thus $\int N_A(y)\,dy$ exists $\int_F \left|\frac{d\phi}{ds}\right| ds = \int_A \left|\frac{d\phi}{ds}\right| ds$.

Thereforeevery measureable set is in C and in particular I is in C.

Q.E.D.

Proof of Lemma 2. If $\Phi_b(y)$ and $\Phi_a(y)$ are both 1 or both 0, it is trivial that

$$|\Phi_b(y) - \Phi_a(y)| \leq N_I(y).$$

Also if there exists s such that $a < \theta(s) < b$ and $\phi(s) = y$ then

$$|\Phi_b(y) - \Phi_a(y)| \leq 1 \leq N_I(y).$$

That leaves the case where exactly one of the points $P(b,y)$ and $P(a,y)$ has winding number 0 and no point of $\{P(\theta,y)|a < \theta < b\}$ are on the

26) Ref. 12, p. 66, ex. 2.

curve Γ. Suppose (b,y) is the point with winding number 0. Then, by continuity, all of the points $\{P(\theta,y) | a < \theta < b\}$ must have winding number 0 and (a,y) must be a point on Γ. We shall now have to exclude sets of measure zero.

First let $\delta > 0$, $-\pi/2 + \delta < a < b < \pi/2 + \delta$ and

$$A = \left\{ s \middle| -\frac{\pi}{2} + \delta < \theta(s) < \frac{\pi}{2} - \delta \right\}.$$

Then $N_A(y)$ is integrable by the preceding lemma and therefore finite almost everywhere. We exclude those y for which $N_A(y)$ is infinite. Secondly let $s_1, s_2, \ldots, s_n$ be the solutions to $\phi(s) = y$, $s \in A$. Since $\phi(s)$ and $\theta(s)$ are absolutely continuous on A, $d\phi/ds$ and $d\theta/ds$ exist and are finite almost everywhere. We exclude that set of y for which there exists s such that $\phi(s) = y$, $s \in A$ and either $d\phi/ds$ or $d\theta/ds$ do not exist or are not finite. Finally we look at those s_i for which $\theta(s_i) = a$. It is claimed that

$$\mu_1 \left\{ s \middle| \theta(s) = a \text{ and } \frac{d\theta}{ds} \neq 0 \right\} = 0.$$

The reason is as follows

$$\frac{d\theta}{ds} - \varepsilon < \frac{\theta(s + \Delta) - \theta(s)}{\Delta} < \frac{d\theta}{ds} + \varepsilon$$

for all $|\Delta|$ sufficiently small. Thus if $d\theta/ds \neq 0$ and $\theta(s) = a$ then $\theta(s') = a$ can have no other solution in some neighborhood of s. Thus no point of $\{s | \theta(s) = a \text{ and } d\theta/ds \neq 0\}$ can be a density point of this set. It follows that this set is of measure zero.[27] The corresponding set of y must also be of measure zero and will be excluded.

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27) Ref. 15, pp. 123-124, 1st edition; pp. 185-187, 2nd edition.

Now let us examine the set of y remaining. For each y in this set we have: There exists s with $\phi(s) = y$, $\theta(s) = a$. There are only a finite number of solutions to $\phi(s) = y$, $-\pi/2 + \delta < \theta(s) < \pi/2 - \delta$ For each solution to $\phi(s) = y$, $\theta(s) = a$ we have $d\phi/ds$ exists and $d\theta/ds = 0$.

Since $1 = \sqrt{(\frac{d\phi}{ds})^2 + \cos^2\theta\,(\frac{d\phi}{ds})^2}$, $\frac{d\phi}{ds}$ cannot be zero and in fact

$$\frac{d\phi}{ds} = \pm\frac{1}{\cos\theta}.$$

Let $s_1, s_2, \ldots, s_n$ be the solutions to $\phi(s) = y$, $\theta(s) = a$. Since there are only a finite number of solutions to $\phi(s) = y$, $-\pi/2 + \delta < \theta(s) < \pi/2 - \delta$, there is an arc $\{P(\theta,y) \mid a - \varepsilon < \theta < a + \varepsilon\}$ such that the only point in common with Γ is (a,y). For each s_i, $i = 1, \ldots, n$ such that $d\phi/ds > 0$ there exists an ε_i such that $\phi(s) < y$ for $s \in (s_i - \varepsilon_i, s_i)$ and $\phi(s) > y$ for $s \in (s_i, s_i + \varepsilon_i)$. Similarly for $d\phi/ds < 0$. Thus there is a crossing of the arc of Γ. If we look at the stereographic image of the arc and of Γ on the yz plane using the point $P(0,-\pi)$ to project from, we see that if $d\phi/ds > 0$ the contribution to the winding number of $(a - \varepsilon, y)$ minus the winding number of $(a + \varepsilon, y)$ is $+1$, while if $d\phi/ds < 0$ the contribution is -1.

Letting $W(\theta,\phi)$ be the winding number of the point $P(\theta,\phi)$

$$W(a-\delta,y) - Q(a+\delta,y) = \sum_i \operatorname{sgn} \frac{d\phi}{ds}\Big|_{s_i}.$$

But $W(a+\delta,y) = 0$ and $W(a-\delta,y)$ is non-negative so that

$$\sum_i \operatorname{sgn} \frac{d\phi}{ds}\Big|_{s_i} \geq 0 .$$

Since there is at least one term in the sum, there must exist one positive term so that $N_I(y) \geq 1 \geq |\phi_b(y) - \phi_a(y)|$. A similar argument is valid if $\phi_b(y) = 1$ and $\phi_a(y) = 0$ and completes the proof.

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