Localized Non-blowup Conditions for 3D Incompressible Euler Flows and Related Equations

Thesis by

Xinwei Yu

In Partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California

2005

(Submitted 29th May 2005)

Acknowledgements

In retrospect, thinking of the five years I spent at Caltech, I feel deep gratitude to my advisor Prof. Thomas Y. Hou. As an advisor, his knowledge and insights kept guiding me through the uncharted field of research; as a collaborator, his perseverance and optimism never failed to pull me out of despondency; as a top mathematician, his style and devotion set a barely achievable goal for my future academic career. Under his influence, I have changed in ways unimaginable to myself five years ago, and I appreciate these changes. So, first and foremost, my deepest gratitude goes to my advisor, Prof. Thomas Y. Hou.

I would also like to heartily thank Dr. Jian Deng, now at Fudan University, who helped me a lot during our enjoyable collaboration on the problems studied in this thesis.

I would like to thank Prof. Jerrold Marsden for providing me much of the historical background of the Euler singularity research.

I would like to thank Prof. Jerrold Marsden, Prof. Oscar Bruno, and Prof. Dan Meiron for kindly serving on my defense committee.

Before my thanks extend to the non-academic level, I would like to thank Theofilos Strinopoulos, Gaby Stredie, and Lei Zhang, who as my officemates make Firestone 214 an enjoyable place to work and chat. I would also like to thank Prof. Niles Pierce, Prof. Emmanuel Candes, Prof. Danping Yang, Dr. Patrick Guidotti, Dr. Haomin Zhou, Wuan Luo, Ke Wang, and all other ACM people for many helpful discussions/chats. My special thanks go to Chad Schmutze and Sheila Shull, for taking such good care of so many things of whose existence I am almost unaware.

On the non-academic side, my first and foremost thanks go to my parents, Prof.

Dehao Yu and Xiaoci Qin, who taught me so many things, helped me whenever I was in trouble, and whose unwavering love was, is, and will be my source of courage and power. My deep thanks also go to my smart, knowledgeable, beautiful, and lovely wife, Dr. Yindi Jing, who helped me a lot in my life and work. Because of her my life has become so colorful and enjoyable.

I am lucky to have made so many good friends at Caltech: those who live in Braun house from 00 - 02, those who go to Shau May at 6pm sharp, those who play badminton every week, and others too many to name or categorize here. Thank you so much for making my stay at Caltech a happy experience!

Finally, I would like to thank the developers of LyX and the author of its Caltech thesis layout, Ling Li, for saving much of my time by taking care of all the boring details of LaTeX.

Abstract

In this thesis, new results excluding finite time singularities with localized assumptions/conditions are obtained for the 3D incompressible Euler equations.

The 3D incompressible Euler equations are some of the most important nonlinear equations in mathematics. They govern the motion of ideal fluids. After hundreds of years of study, they are still far from being well-understood. In particular, a long-outstanding open problem asks whether finite time singularities would develop for smooth initial values. Much theoretical and numerical study on this problem has been carried out, but no conclusion can be drawn so far.

In recent years, several numerical experiments have been carried out by various authors, with results indicating possible breakdowns of smooth solutions in finite time. In these numerical experiments, certain properties of the velocity and vorticity field are observed in near-singular flows. These properties violate the assumptions of existing theoretical theorems which exclude finite time singularities. Thus there is a gap between current theoretical and numerical results. To narrow this gap is the main purpose of the work presented in this thesis.

In this thesis, a new framework of investigating flows carried by divergence-free velocity fields is developed. Using this new framework, new, localized sufficient conditions for the flow to remain smooth are obtained rigorously. These new results can deal with fast shrinking large vorticity regions and are applicable to recent numerical experiments. The application of the theorems in this thesis reveals new subtleties, and yields new understandings of the 3D incompressible Euler flow.

This new framework is then further applied to a two-dimensional model equation, the 2D quasi-geostrophic equation, for which global existence is still unproved. Under certain assumptions, we obtain new non-blowup results for the 2D quasi-geostrophic equation.

Finally, future plans of applying this new framework to some other PDEs as well as other possibilities of attacking the 3D Euler and 2D quasi-geostrophic singularity problems are discussed.

Contents

List of Figures

List of Tables

Chapter 1

Introduction

1.1 The 3D Incompressible Euler Equations

1.1.1 Derivation

The equations that govern the evolution of ideal fluids are the Euler equations. In the following, we will give a brief derivation. More detailed ones can be found in textbooks such as Chorin-Marsden [CM93] or Majda-Bertozzi [MB02].

Consider a domain (bounded or boundless) Ω that is filled with some fluid such as water. In classical continuum mechanics, the fluid can be seen as consisting of a collection of infinitesimal particles. At each time t, each particle has a one-to-one correspondence to the space positions $x = (x_1, x_2, x_3) \in \Omega$. The fluid can be totally described by the following quantities at each position: the density ρ, the velocity $u = (u_1, u_2, u_3)$, and the pressure p. We denote the position of any particle at time t by $X(\alpha, t)$, where $\alpha \in \Omega$ is the position of this particle at time $t = 0$. The evolution of these particles is governed by the following differential equation:

$$\begin{aligned} \frac{dX(\alpha,t)}{dt} &= u(X(\alpha,t),t), \\ X(\alpha,0) &= \alpha. \end{aligned} \tag{1.1.1}$$

To study the dynamics of the fluid, we must establish relations between u and p. This is achieved by considering two basic mechanical rules: the conservation of mass, and the conservation of momentum.

The conservation of mass gives

$$\frac{d}{dt}\int_W \rho \, dx = -\int_{\partial W} \rho u \cdot \mathbf{n} \, ds,$$

where W is an arbitrary subset of Ω, and $\mathbf{n}$ is the unit outer normal vector. It can be shown that, when the density ρ at the initial time $t = 0$ is constant everywhere, it will remain so for all times $t > 0$, therefore, without loss of generality, ρ can be taken to be 1 everywhere, for all times. In this case, conservation of mass becomes simply

$$0 = \frac{d}{dt}\int_W \, dx = -\int_{\partial W} u \cdot \mathbf{n} \, d\sigma, \tag{1.1.2}$$

where the first equality is because W is fixed. By Gauss's theorem we easily obtain the following incompressibility condition:

$$\nabla \cdot u = 0. \tag{1.1.3}$$

The conservation of momentum implies

$$\frac{d}{dt}\int_{\Omega_t} u \, dx = F\left(\Omega_t\right), \tag{1.1.4}$$

where $\Omega_t = X\left(\Omega_0, t\right)$ is the flow image of some arbitrary $\Omega_0 \subset \Omega$, i.e., Ω_t consists of an arbitrary collection of particles that is carried by the flow. If we restrict ourselves to the case that the fluid is ideal, we have

$$F\left(\Omega_t\right) = \int_{\partial \Omega_t} -p\mathbf{n} \, d\sigma,$$

where p is the pressure and $\mathbf{n}$ the unit outer normal vector. One can show that (1.1.4) is equivalent to

$$\int_{\Omega_t} u_t + u \cdot \nabla u \, dx = -\int_{\Omega_t} \nabla p \, dx,$$

which gives

$$u_t + u \cdot \nabla u = -\nabla p \tag{1.1.5}$$

due to the arbitrariness of Ω_t.

Combining (1.1.3) and (1.1.5), we obtain the Euler equations

$$\begin{aligned} u_t + u \cdot \nabla u &= -\nabla p, \\ \nabla \cdot u &= 0. \end{aligned} \tag{1.1.6}$$

In the following, we will focus on (1.1.6) in the whole space, i.e., $\Omega = \mathbb{R}^3$, with fast decay boundary conditions.

Remark 1.1.1. It is also possible to derive the Euler equations in a variational setting. See, e.g., Marchioro-Pulvirenti [MP94] or Marsden-Ratiu [MR94].

1.1.2 The vorticity formulation

An important equivalent formulation of the 3D Euler equations (1.1.6) is the following vorticity formulation:

By taking $\nabla\times$ on both sides of the momentum equation (1.1.5) and defining the vorticity $\omega \equiv \nabla \times u$, we obtain the following vorticity equation:

$$\omega_t + u \cdot \nabla \omega = (\nabla u) \cdot \omega, \tag{1.1.7}$$

where the vorticity ω and the velocity u are related by the so-called Biot-Savart law

$$u(x,t) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3} \times \omega(y,t)\ dy \tag{1.1.8}$$

when the quantities under consideration have sufficient decay at infinity.

Combining (1.1.7), (1.1.8), and the fact that $\nabla \cdot \omega = 0$, which follows from the definition of vorticity, we obtain the vorticity formulation of the 3D Euler equations in $\mathbb{R}^3$:

$$\begin{aligned} \omega_t + u \cdot \nabla \omega &= (\nabla u) \cdot \omega \\ u(x,t) &= \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3} \times \omega(y,t)\ dy \\ \nabla \cdot \omega &= 0. \end{aligned} \tag{1.1.9}$$

In the following, we will use the notation $\frac{D}{Dt}$ to denote differentiation in time along the Lagrangian trajectory (also called the "material derivative"), i.e.,

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u \cdot \nabla. \tag{1.1.10}$$

Remark 1.1.2. The physical meaning of ω can be seen from the following argument. By considering the Taylor expansion of the velocity at some point x, we have

$$u(x+h,t) = u(x,t) + S(x,t)\,h + \Omega(x,t)\,h + O\left(h^2\right),$$

where $S(x,t) = \frac{1}{2}\left(\nabla u + \nabla u^T\right)$ and $\Omega(x,t) = \frac{1}{2}\left(\nabla u - \nabla u^T\right)$. It is easy to check that

$$\Omega(x,t)\,h = \frac{1}{2}\omega(x,t) \times h.$$

Therefore $\omega(x,t)$ indicates "local rotation." Also note that the first term $u(x,t)$ represents translation, and the second represents deformation (see Majda-Bertozzi [MB02] for details).

1.2 The Euler Singularity Problem

1.2.1 Statement of the problem

One of the most natural and most fundamental questions to ask about any PDE is whether it is "well-posed," that is, do we have existence, uniqueness, and furthermore, continuous dependence on initial and boundary values of the solution. None of these questions has been satisfactorily answered for the Cauchy problem of the 3D Euler equations in $\mathbb{R}^3$. In particular, the following long-standing open problem is of much interest:

The Euler Singularity Problem. Given a smooth enough initial value u_0 (e.g., $u_0 \in H^m(\mathbb{R}^3)$ for $m > 5/2$), will there be a finite time T^* such that the solution $u(x, T^*)$ will cease to be H^m?

Remark 1.2.1. Unlike the Navier-Stokes equations, there is currently no "natural" ways to define weak solutions for the Euler equations. Therefore the standard approach, that is one first obtains weak solutions in certain "natural" function spaces, then bootstraps to get more regularity, does not work well for the Euler equations.[1] In particular, it has been shown that the naïve definition

$$u \in L^2\left(\mathbb{R}^d \times \mathbb{R}^+\right), \ u \text{ satisfies (1.1.6) as distributions,}$$

leads to non-uniqueness even for $d = 2$ (Scheffer [Sch93], Shnirelman [Shn97]), in which case classical solutions are known to exist and be unique. Therefore people have been trying to obtain a reasonable definition by binding the above "naïve" definition with some energy decreasing condition, see, e.g., Shnirelman [Shn98, Shn00]. The physical motivation behind this is that such weak solutions may be the correct ones to describe turbulent flows, see, e.g., Robert [Rob03].

1.2.2 Relation to the onset of turbulence

Among the most important problems in fluid mechanics is the onset of turbulence. There is some hint that the above Euler singularity problem may be related to this phenomenon, although no rigorous relations have been established so far.[2] The argu-

[1] On the other hand, this approach had some success in dealing with the Navier-Stokes equations, see, e.g., Sohr [Soh01] or Ladyzhenskaya [Lad03].

[2] Compare with shock theory for compressible flows: finite time singularities (shocks) indicate the onset of "new types of solutions" that are correctly described by weak solutions with certain entropy conditions that guarantee energy dissipation.

ment is intuitive and based on the following two (seemingly contradictory) observations.

First, from various experiments, the following empirical law of finite energy dissipation is obtained for the Navier-Stokes equations after the flow turned turbulent. We recall that the Navier-Stokes equations read

$$\begin{aligned} u_t + u \cdot \nabla u &= -\nabla p + \nu \triangle u, \\ \nabla \cdot u &= 0, \end{aligned}$$

which are the Euler equations with an extra dissipation term $\nu \triangle u$. In well-developed turbulent flows, the energy dissipation rate $\nu \left\langle |\nabla u|^2 \right\rangle$ is measured for various viscosities ν, and the results are consistent with the following "law of finite energy dissipation" (Frisch [Fri95]):

$$\liminf_{\nu \to 0} \nu \left\langle |\nabla u|^2 \right\rangle = \varepsilon > 0.$$

Here the bracket $\langle \cdot \rangle$ stands for ensemble average, that is, the average of many experiments with identical initial/boundary values. Intuitively, when $\nu \to 0$, the solution u of the Navier-Stokes equations would approach some solution of the Euler equations in some way, while at the same time ∇u should tend to infinity. This implies finite time singularities for the solution of the Euler equations.

On the other hand, one can easily check that the kinetic energy

$$\int_{\mathbb{R}^3} |u|^2 \ dx$$

is conserved for the Euler equations when u is smooth enough.[3]

Therefore, if the Euler equations are able to describe the onset of turbulence at all, intuitively, for some generic initial velocity fields, the corresponding classical solutions to the Euler equations would develop finite time singularities. However, so

[3] The so-called Onsager's conjecture (Onsager [Ons49]) claims that energy will be conserved if and only if u is Hölder continuous with exponent greater than $1/3$. See Eyink [Eyi94] or Constantin-E-Titi [CET94] for proof of the "if" part. For the "only if" part, there are relevant efforts of defining suitable weak solutions, see Remark 1.2.1.

far no rigorous relation between Euler singularities and the onset of turbulence has been found.

1.2.3 Relation to the Navier-Stokes singularity problem

The Navier-Stokes singularity problem, namely, whether the solutions for the Navier-Stokes equations with smooth initial values would develop finite time singularities or not, is also a long-outstanding open problem in applied mathematics: see Fefferman [Fef00] or Ladyzhanskaya [Lad03] for descriptions of this problem.

It is revealed in Constantin [Con86] that the Euler singularity problem is also related to the Navier-Stokes singularity problem. This relation is explored by the following theorem, which claims that, when the solution to the Euler equations stays smooth, so does the solution to the Navier-Stokes equations.

Theorem 1.2.2. *(Constantin 1986). Let $v(x,t)$ be a solution to the 3D Euler equations which is smooth in $[0,T]$, then there exists $\nu_0 = \nu_0(T;v)$ such that when $0 < \nu \le \nu_0$, the solution to the Navier-Stokes equations with the same initial value exists and is smooth in $[0,T]$.*

1.3 Review of Euler Singularity Research

1.3.1 A brief history of theoretical research

The Euler equations were proposed as a mathematical model of fluid motion by Euler in 1755 ([Eul55]). Later it was found that due to its neglect of the viscosity, the system leads to unphysical solutions (e.g., the D'Alembert's paradox). This phenomenon inspired the modification known as the Navier-Stokes equation, which has served predicting fluid motions ever since (see, e.g., Cannone-Friedlander [CF03] for this part of history). However, the 3D Euler equations remain fascinating to mathematicians due to its mathematical sophisticacy.

In the 1920s, local ("local" with respect to time) existence and uniqueness was obtained by Lichtenstein and Gunther ([Licht25], [Gun26, Gun27, Gun28]) for classical solutions defined as $u(x,t) \in C^\lambda$ in space for $\lambda \in (0,1)$ and C^1 in time. In the 1970s, local existence for 3D Euler solutions in Sobolev spaces H^m was obtained by various authors (Ebin-Fischer-Marsden [EFM70], Swann [Swa71], Kato [Kat72]). The main result for the whole space case is the following:

$$u_0 \in H^m\left(\mathbb{R}^3\right), \; m > 5/2 \Rightarrow u \in H^m \text{ at least up to } T_0 = T_0\left(\|u_0\|_{H^m}\right).$$

Unlike the 2D case, the global well-posedness for the 3D Euler equations still remains open.[4] Nonetheless, quite a few interesting results have been obtained, for example Beale-Kato-Majda [BKM84] (which we will discuss in more detail in the next subsection), Caflisch [Caf93], Babin-Mahalov-Nicolaenko [BMN01], and Tadmor [Tad01].

1.3.2 The Beale-Kato-Majda criterion

In 1984, Beale, Kato, and Majda ([BKM84]) obtained the following necessary and sufficient condition for the existence of H^m solutions for $m > 5/2$:

$$u \text{ cease to be in } H^m \text{ at } T^* \Leftrightarrow \int_0^{T^*} \|\omega\|_{L^\infty}(t)\; dt = \infty, \tag{1.3.1}$$

where $\omega = \nabla \times u$ is the vorticity as defined in Subsection 1.1.2. This criterion improves previous ones by Ebin-Fischer-Marsden [EFM70] and Bardos-Frisch [BF76].

Later the BKM criterion (1.3.1) was extended and improved by others. Among them, Ponce ([Pon85]) proved in 1985 that ω can be replaced by the deformation tensor $S = \left(\nabla u + \nabla u^T\right)/2$; Konzono and Taniuchi ([KT00]) in 2000 proved that $\|\omega\|_{L^\infty}$ can be replaced by $\|\omega\|_{BMO}$, where BMO is the bounded mean oscillation space; The BMO norm is further weakened to Besov norm $\|\omega\|_{\dot{B}^0_{\infty,\infty}}$ and Triebel-Lizorkin norm $\|\omega\|_{\dot{F}^0_{\infty,\infty}}$ by Chae ([Cha02, Cha04a]). Another interesting result by

[4] Global well-posedness for classical solutions of the 2D Euler equations was obtained by Wolibner ([Wol33]) in the 1930s, whose result was later modernized and extended by Kato ([Kat67]). Yudovich [Yud63, Yud95] obtained global well-posedness when the L^p-norm of the vorticity $\omega \equiv \nabla \times u$ is carefully controlled for all $p \in (1,\infty)$.

Chae ([Cha04b]) claims that as long as the quantity $\int_0^{T^*} \|\tilde{\omega}\|_{\dot{B}^0_{\infty,1}}$ remains bounded, where $\tilde{\omega} \equiv \omega^1 e_1 + \omega^2 e_2$ are the first two components of ω, the solution can be extended beyond T^*.

Remark 1.3.1. It is interesting to put the following result due to Lions and DiPerna (Lions [Lio96] pp. 150–153, the wording here follows Bardos [Bar01], page 8), proved via construction, in juxtaposition with the BKM criterion.

Theorem 1.3.2. *(Lions-DiPerna) In space dimension* 3*, and for any* $1 < p < \infty$*, there exists no function*

$$\phi(Z,t), \quad \lim_{Z \to 0} \phi(Z,t) = 0$$

such that the vorticity of the incompressible Euler equations satisfies the estimate:

$$\|\omega(t)\|_{L^p} \leq \phi\left(\|\omega(0)\|_{L^p}, t\right).$$

The above theorem greatly reduces the hope of proving global existence of the Euler equations by a priori estimates in functional spaces alone. Furthermore, it seems to some extent complementary to the BKM criterion in the sense that this theorem does not include the case $p = \infty$. In fact, the construction by Lions and DiPerna only yields $O(t)$ growth in $\|\omega\|_{L^\infty}$. Finally, how the combination of both results would help in searching for/excluding finite time Euler singularities remains to be revealed.

1.3.3 Nonlinearity: strong or weak?

The BKM criterion (1.3.1) relates the Euler singularity problem to the evolution of vorticity, which is governed by the vorticity equation (1.1.7):

$$\frac{D\omega}{Dt} \equiv \omega_t + u \cdot \nabla\omega = \nabla u \cdot \omega.$$

Naïvely, since $\nabla u = \nabla\left(\nabla \times (-\triangle)^{-1}\right)\omega$ is related to the vorticity by a singular integral operator (see, e.g., Majda-Bertozzi [MB02]), ∇u and ω are of the same order.

This suggests that (1.1.7) can be modelled by the following ODE:

$$\frac{ds}{dt} = s^2,$$

which implies a finite time singularity.[5]

However, it is revealed in Constantin [Con94] that subtle cancellation may exist. There the equation governing the evolution of the vorticity magnitude $|\omega|$ is found to be

$$\frac{D|\omega|}{Dt} = \alpha |\omega|$$

with the stretching factor (here we make the dependence on time implicit)

$$\begin{aligned} \alpha(x) &\equiv \xi(x) \cdot \nabla u(x) \cdot \xi(x) \\ &= \frac{3}{4\pi} p.v. \int_{\mathbb{R}^3} D(\hat{y}, \xi(x+y), \xi(x)) |\omega(x+y)| \frac{dy}{|y|^3}, \end{aligned}$$

where $\xi \equiv \omega / |\omega|$, $\hat{y} \equiv y / |y|$, and

$$D(a, b, c) \equiv (a \cdot c) \det(a, b, c)$$

for $a, b, c \in \mathbb{R}^3$.

Since a determinant is involved, when $\xi(x+y)$'s are aligned with $\xi(x)$, it is possible that $\alpha(x, t)$ be much smaller than $|\omega|$. Therefore, depletion of nonlinearity may occur, and the formal argument that claims that α and ω are of the same order may not be telling the truth.

1.3.4 Computations guided by the BKM criterion

In the above subsection, two observations with opposite implications are made. What makes things more complicated is that they both seem to be supported by recent

[5]This is the simplest model equation for the 3D Euler equations, which brutally violates important properties such as the global interaction between u and ω. In Subsection 1.3.6 other more sophisticated models will be discussed. There it can be seen that energy conservation, incompressibility, nonlocal interaction in the nonlinear term, and the fact that particles are carried by the flow are really important.

numerical observations. In the following, we will give a brief review of numerical efforts searching for finite time singularities, highlighting those related to the two observations in the previous subsection.

Study of possible Euler singularity by large scale numerical computation started roughly in the early 1980s (Brachet-Meiron-Orszag-Nickel-Morf-Frisch [BMONMF83]), and is followed by computations using various numerical techniques (e.g., Pumir-Siggia [PS90], Bell-Marcus [BM92], Brachet et al [BMVPS92], Kerr [Ker93], etc.). Evidences favoring both existence and singularity were obtained. See Grauer-Sideris [GS95] or Frisch [Fri95] (Section 7.8) for brief reviews of these computations.

More recently, several numerical computations (e.g., Kerr [Ker93, Ker95, Ker96, Ker97, Ker98, Ker04], Pelz [Pel97, Pel01], Grauer-Marliani-Germaschewski [GMG98]) have been performed searching for possible candidates for a finite time blowup, monitoring the growth of $\|\omega\|_{L^\infty}$ under the guidance of the BKM criterion (1.3.1). Most of them suggest a growth rate of $(T^* - t)^{-1}$ for the maximum vorticity, which seems to support the modelling of 3D Euler by $ds/dt = s^2$. However, at the same time, it is also observed that large vorticity resides in small, fast-shrinking regions where certain alignment of vorticity directions occurs.

1.3.5 Non-blowup theorem by Constantin-Fefferman-Majda

In 1996, Constantin-Fefferman-Majda [CFM96] explored rigorously the possibility that alignment of unit vorticity vectors ξ would cause depletion of nonlinearity. Their main result is the following:

Definition 1.3.3. (Smoothly directed). A set W_0 is said to be smoothly directed if there exists $\rho > 0$ and r, $0 < r \leq \rho/2$ such that the following three conditions are satisfied.

1. For every $q \in W_0^* \equiv \{q \in W_0;\ |\omega_0(q)| \neq 0\}$ and all time $t \in [0, T)$, the function $\xi(\cdot, t)$ has a Lipschitz extension (denoted by the same letter) to the Euclidean

ball of radius 4ρ centered at $X(q,t)$, denoted as $B_{4\rho}(X(q,t))$, and

$$M = \sup_{q \in W_0^*} \int_0^{T^*} \|\nabla \xi(\cdot,t)\|^2_{L^\infty(B_{4\rho}(X(q,t)))} \, dt < \infty.$$

2. The maximum vorticity in $B_r(W_t)$ is always comparable of the maximum vorticity in a larger neighborhood $B_{3r}(W_t)$.

$$\sup_{B_{3r}(W_t)} |\omega(x,t)| \le m \sup_{B_r(W_t)} |\omega(x,t)|$$

holds for all $t \in [0,T^*)$ with $m \ge 0$ constant. Here $W_t \equiv X(W_0,t)$.

3. For all $t \in [0,T^*)$,

$$\sup_{B_{4\rho}(W_t)} |u(x,t)| \le U$$

for some constant U.

Theorem 1.3.4. *(Constantin-Fefferman-Majda 1996). Assume W_0 is smoothly directed. Then there exists $\tau > 0$ and Γ such that*

$$\sup_{B_r(W_t)} |\omega(x,t)| \le \Gamma \sup_{B_\rho(W_{t_0})} |\omega(x,t_0)|$$

holds for any $0 \le t_0 < T^*$ and $0 \le t - t_0 \le \tau$.

Remark 1.3.5. In other words, the above result claims that no finite time singularity would exist if in an $O(1)$ neighborhood of some region carried by the flow,

1. $\|\nabla \xi\|_{L^\infty} \in L^2$ in time,

2. the maximum vorticity inside this neighborhood is always comparable to the global maximum,

3. the velocity is uniformly bounded.

Theorem 1.3.4 reveals how alignment of vorticity directions may help deplete nonlinearity. However, since this $O(1)$ neighborhood cannot shrink to a single point, Theorem 1.3.4 does not apply to recent numerical observations claiming finite time singularities.

1.3.6 Model equations

1.3.6.1 Overview

Since it is hard to treat the nonlinearity in the 3D Euler equations, it is natural to consider models that

1. have simpler dynamics, are easier to study, and at the same time
2. keep as many as possible the major characteristics of the Euler dynamics, such that studying the model equations would enhance our understanding of the Euler equations.

The ODE, $ds/dt = s^2$ in Subsection 1.3.3, succeeds in 1 while it fails in 2 due to over-emphasis on 1. There are many model equations in the literature that are far more sophisticated, see, e.g., Constantin-Lax-Majda [CLM85], Constantin [Con86], Liu-Tadmor [LT02], Friedlander-Pavlović [FP04]. They can be categorized into the following three classes:

1. Modelling evolution of ∇u (or its eigenvalues): Constantin [Con86], Liu-Tadmor [LT02], etc. Both models produce finite time singularities. However, the former disregards the fact that quantities are convected by the flow, while in the latter, the nonlocal operator producing the pressure hessian P is replaced by a local one. As a result, these blowups bear little implication for the Euler system.
2. Modelling vorticity evolution: Constantin-Lax-Majda [CLM85], Constantin-Majda-Tabak [CMT94], etc. Finite time singularities have been shown to exist for the former model, which localizes interactions. The latter model is known as the 2D quasi-geostrophic (2D QG) equation and is the most sophisticated model so far. Its singularity problem is still open, and will be discussed in more detail in the remainder of this subsection.

3. Modelling energy transfer in the Fourier space: Dinaburg-Sinai [DS04], Friedlander-Pavlović [FP04], Katz-Pavlović [KP04], etc.[6] In these models, nonlocal interactions of different Fourier modes are replaced by interactions between adjacent modes (neighboring "shells"). As a result, possible depletion of nonlinearity may be lost. For example, in Friedlander-Pavlović [FP04], blowup is obtained with an artificial local nonlinear term that reaches the critical scaling of the Sobolev embedding, which is the strongest nonlinearity possible. Also, energy conservation is broken in some of such models (e.g., Dinaburg-Sinai [DS04]).[7] A nice overview and discussion of the above models can be found in Waleffe [Wal04].

Among them, the most sophisticated and revealing one is the 2D quasi-geostrophic equation(henceforth referred to as the 2D QG equation) proposed as a model for the Euler equations in Constantin-Majda-Tabak [CMT94]. The Cauchy problem of the 2D QG equation is still open today. We will give a brief discussion in the remainder of this subsection, and present our results for the QG equation in 1.4.2.

1.3.6.2 The 2D QG equation

The 2D QG equation (aka surface-quasi-geostrophic, SQG) describes the variation of the density variation θ at the surface of the earth. The symbol θ, which usually represents temperature, is chosen because in the case of the ideal gas, the density variation is proportional to the temperature.

The 2D QG equation is given by

$$\frac{D\theta}{Dt} \equiv \theta_t + u \cdot \nabla\theta = 0,$$

[6] These energy transfer models are inspired by the so-called shell models in turbulence theory, which have been extensively studied by many authors. A good introduction to these models can be found in Bohr et al [BJPV98], Chapter 3, or Biferale [Bif03].

[7] Blowup has been proved when infinite energy is allowed. See e.g. Stuart [Stu87], Ohkitani-Gibbon [OG00], Constantin [Con00, Con03], and Childress et al. [CISY89]. Note that in the last one, singular solutions for 2D Euler flows have been constructed.

where $\theta(x,t)$ is a scalar, and the divergence-free velocity field $u(x,t)$ is defined by

$$\begin{aligned} (-\triangle)^{1/2}\psi &= -\theta \\ u &= \nabla^{\perp}\psi \equiv \left(-\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1}\right)^T \psi \end{aligned}$$

with

$$(-\triangle)^{1/2}\psi = \int e^{2\pi i x\cdot\xi} 2\pi |\xi| \hat{\psi}(\xi)\ d\xi$$

for

$$\psi = \int e^{2\pi i x\cdot\xi} \hat{\psi}(\xi)\ d\xi.$$

A brief derivation of the above equations can be found in Majda-Tabak [MT96].

It is found in Constantin-Majda-Tabak [CMT94] that, the evolution of the tangent vector to the level sets $\nabla^{\perp}\theta$ bears much resemblance to the vorticity evolution for the 3D Euler equations. In particular, we have the following three major analogies:

- The vector $\nabla^{\perp}\theta$ is governed by the following equation

$$\left(\nabla^{\perp}\theta\right)_t + u\cdot\nabla\left(\nabla^{\perp}\theta\right) = (\nabla u)\cdot\left(\nabla^{\perp}\theta\right), \tag{1.3.2}$$

  where ∇u is of the same order as $\nabla^{\perp}\theta$. (1.3.2) is similar to the vorticity evolution equation (1.1.7). Furthermore, the evolution equation for the magnitude of $\left|\nabla^{\perp}\theta\right|$ is

$$\left|\nabla^{\perp}\theta\right|_t + u\cdot\nabla\left|\nabla^{\perp}\theta\right| = S(x,t)\left|\nabla^{\perp}\theta\right|,$$

  where the stretching factor

$$S(x,t) = p.v.\int_{\mathbb{R}^2} \frac{(\hat{y}\cdot\xi(x))\det(\xi(x+y),\xi(x))}{|y|^2}\left|\nabla^{\perp}\theta(x+y)\right|\ dy$$

  with $\hat{y} = y/|y|$ and $\xi(x,t) = \nabla^{\perp}\theta/\left|\nabla^{\perp}\theta\right|$ has striking resemblance to the stretching factor

$$\alpha(x,t) = \frac{3}{4\pi}p.v.\int_{\mathbb{R}^3} \frac{(\hat{y}\cdot\xi(x))\det(\hat{y},\xi(x+y),\xi(x))}{|y|^3}|\omega(x+y)|\ dy$$

in global behaviors and subtle cancellation properties.

- Similar to vortex lines in 3D Euler dynamics, the level sets of θ, which are tangent to $\nabla^{\perp}\theta$ by definition, are also carried by an incompressible flow,

- The kinetic energy $\|u\|_{L^2}^2$ is conserved by the flow. Recall that this conservation also holds for the 3D Euler equations.

Based on the above and other observations, Constantin-Majda-Tabak [CMT94] conclude that the 2D QG equation would serve as a good model of the 3D Euler equations. In particular, the evolution of $\nabla^{\perp}\theta$ in the 2D QG flow resembles that of ω in the 3D Euler flow.

1.3.6.3 Is there a finite time QG singularity?

Since Constantin, Majda, and Tabak's pioneering work [CMT94], many theoretical and numerical results have been obtained for the QG Cauchy problem, see, e.g., Chae [Cha03b], Constantin-Nie-Schörghofer [CNS98, CNS99], Constantin [Con98], Cordoba-Fefferman [CF02a], Cordoba [Cor97, Cor98], Ohkitani-Yamada [OY97], etc. In particular, the QG blowup is also controlled by the following BKM-type criterion (Constantin-Majda-Tabak [CMT94]):

$$\text{Blow-up at time } T^* \Leftrightarrow \int_0^{T^*} \left\|\nabla^{\perp}\theta\right\|_{L^\infty}(t)\ dt = \infty. \tag{1.3.3}$$

It is further proved that when both $\int_0^{T^*} \|\nabla\xi\|_{L^\infty}\ dt$ and $\int_0^{T^*} \|u\|_{L^\infty}^2\ dt$ are finite, no blowup can occur (Constantin-Majda-Tabak [CMT94]).

There are also many computations guided by (1.3.3). In Constantin-Majda-Tabak [CMT94], it is found that when θ admits a saddle point, $|\nabla\xi|$ may be large around this point. Therefore the above result may not apply. In this case it is observed that $\|\nabla\theta\|_{L^\infty}$ grows at the rate $\sim (T^*-t)^{-1.7}$, which indicates finite time blowup. However, later computations (Ohkitani-Yamada [OY97], Constantin-Nie-Schörghofer [CNS98, CNS99]) found that the growth of $\|\nabla\theta\|_{L^\infty}$ can be better fitted by double exponential growth.

In 1998, D. Cordoba [Cor97, Cor98] proved that, even when there is a saddle point, the growth of $\nabla^{\perp}\theta$ would be bounded by a quadruple exponential under certain mild conditions on the flow, and hence no blowup would occur. Later, in Cordoba-Fefferman [CF02a], a double exponential rate was obtained for two level sets approaching each other "semi-uniformly."

Unlike the 3D Euler equations, currently no strong numerical results indicating blowup exists for the 2D QG equation. However, its singularity problem still remains open today, as we have mentioned before. There are even 1D model equations for the 2D QG equation being studied (Chae-Cordoba-Cordoba-Fontelos [CCCF04]). These 1D models develops singularities in finite time. However so far these 1D singularities shed little light on constructing/excluding 2D QG singularities.

1.4 Summary of Main Results

In this section we summarize our main results on the non-blowup of the 3D Euler equations and the 2D QG equation. They will be proved in chapters 2 and 3.

1.4.1 Non-blowup conditions for the 3D Euler equations

As in Deng-Hou-Yu [DHY04, DHY05], we assume that the initial velocity field u_0 is smooth enough and vanishes rapidly at infinity, e.g., $u_0 \in H^3(\mathbb{R}^3)$.

First we recall the definition of vortex lines (Chorin-Marsden [CM93]):

Definition 1.4.1. A vortex line is a curve L that is tangent to the vorticity vector ω at each of its points.

One important property of any vortex line is that it is carried by the flow, as shown by the following theorem. Its proof can be found in, e.g., Chorin-Marsden [CM93].

Theorem 1.4.2. *If a curve moves with the flow and is a vortex line at time $t = 0$, then it remains so for all times.*

Now using $\Omega(t)$ to denote the global maximum vorticity, we consider, at time t, a single vortex line segment L_t along which the maximum vorticity is comparable to $\Omega(t)$. Denote by $L(t)$ the arc length of L_t, ξ the tangential, and $\mathbf{n}$ the normal unit vector of L_t. Note that by the definition of vortex lines, $\xi = \omega/|\omega|$. $U_\xi(t)$ is defined as the maximum of $|(u \cdot \xi)(x,t) - (u \cdot \xi)(y,t)|$ for any two points $x, y \in L_t$, $U_n(t)$ as the maximum normal velocity, $M(t)$ as the maximum of $|\nabla \cdot \xi|$, and $K(t)$ as the maximum of κ the curvature, both along L_t. We should point out that in general L_t is just a subset of $X(L_{t'}, t', t)$, that is $X(L_{t'}, t', t) \supseteq L_t$, for any $t' < t$, where $X(A, s, t)$ denotes the flow image at time $t > s$ of a set of particles that are at position A at time s.

With these notations, we present our first main theorem.

Theorem 1.4.3. *(Deng-Hou-Yu [DHY05]). Assume that there is a family of vortex line segments L_t and $T_0 \in [0, T^*)$, such that $X(L_{t_1}, t_1, t_2) \supseteq L_{t_2}$ for all $T_0 < t_1 < t_2 < T^*$. Also assume that $\Omega(t)$ is monotonically increasing and $\max_{x \in L_t} |\omega(x,t)| \geq c_0 \Omega(t)$ for some $c_0 > 0$ when t is sufficiently close to T^*. Furthermore, we assume there are constants C_U, C_0, c_L such that*

1. *$[U_\xi(t) + U_n(t) K(t) L(t)] \leq C_U (T^* - t)^{-A}$ for some constant $A \in (0,1)$,*
2. *$M(t) L(t),\ K(t) L(t) \leq C_0$,*
3. *$L(t) \geq c_L (T^* - t)^B$ for some constant $B \in (0,1)$.*

Then there will be no blowup in the 3D incompressible Euler flow up to time T^*, as long as $B < 1 - A$.

Note that the same result holds if we replace the first assumption in the above theorem by $U_\xi(t) + U_n(t) \leq C_U (T^* - t)^{-A}$, since this assumption combined with the second assumption 2 will give us the first assumption 1 in the theorem. In most numerical observations, $\Omega(t) \sim (T^* - t)^{-1}$, which bounds the maximum velocity by $(T^* - t)^{-3/5}$ according to Lemma A.1.1 in Appendix A.1. Therefore in these cases, A would be no more than $3/5$.

The intuition of Theorem 1.4.3 is the following. Suppose the region $D(t)$ where vorticity is large (e.g., comparable to $\Omega(t)$) shrinks at the rate $(T^* - t)^B$for some $B \in (0,1)$. Then we take L_{t_0} to be a vortex line segment in $D(t)$, and let $L_t = X(L_{t_0}, t_0, t) \cap D(t)$ to be the intersection of the flow image of L_{t_0} with $D(t)$. Since the vorticity magnitude on L_t is growing fast, intuitively L_t should undergo strong stretching at each point. Therefore the ends of L_t are always carried out of $D(t)$ by the flow, which implies that the length of L_t, namely $L(t)$, would be comparable with the diameter of $D(t)$, which is $(T^* - t)^B$. Now if the velocity and vorticity vector fields satisfy assumptions 1 and 2, no blowup would occur. In particular, one can show that if $\nabla\xi$ and $U(t) \equiv \max_{x\in D(t)} |u(x,t)|$ are bounded by $(T^* - t)^{-B}$ and $(T^* - t)^{-A}$, respectively, then no blowup would occur.

Unlike previous theorems, Theorem 1.4.3 allows the regions where vorticity concentrates to shrink and the maximum velocity to blowup. However, in some numerical computations, the scaling $B = 1 - A = 1/2$ is observed, which lies just beyond Theorem 1.4.3 (e.g., Kerr [Ker93, Ker95, Ker96, Ker97, Ker98, Ker04]). In Deng-Hou-Yu [DHY04], we improved Theorem 1.4.3 to cover this critical case, and revealed new subtleties in the 3D incompressible Euler flow. Namely, no blowup wold occur if the various constants in (1)–(3) satisfy certain conditions.

Theorem 1.4.4. *(Deng-Hou-Yu [DHY04]). Under the same assumptions as in Theorem 1.4.3, there will be no blowup in the 3D incompressible Euler flow up to time T^* in the case $B = 1 - A$, as long as the following condition is satisfied:*

$$R^3 K < y_1\left(R^{A-1}(1-A)^{1-A}/(2-A)^{2-A}\right), \tag{1.4.1}$$

where $R = e^{C_0}/c_0$, $K \equiv \frac{C_U c_0}{c_L(1-A)}$, and $y_1(m)$ denotes the smallest positive y such that

$$\frac{y}{(y+1)^{2-A}} = m.$$

In the following, we briefly discuss how Theorem 1.4.4 may be applied to the recent numerical observations by Kerr ([Ker93, Ker95, Ker96, Ker97, Ker98, Ker04]).

In these computations, one may take $A = B = 1/2$ and C_0 some small constant (see 2.6 for detailed discussion). If we can take L_t such that it goes through the peak vorticity point, i.e., $c_0 = 1$, condition (1.4.1) is equivalent to

$$\frac{2C_U}{c_L} < K_{max}\left(C_0\right)$$

for some function K_{max}. The value of K_{max} for small constant C_0's are of order 1, for example, when $C_0 = 0.1$, $K_{max} > 0.86$. That is, no blowup would occur if

$$\frac{C_U}{c_L} \leq 0.43.$$

In numerical computations (e.g., Kerr [Ker04]), maximum velocity is measured to behave like $\left(T^* - t\right)^{-1/2}$ with $O\left(1\right)$ constant coefficient. Therefore it would be interesting to perform careful numerical experiments to check if this condition is satisfied in near-singular flows.

1.4.2 Non-blowup results for the 2D QG equation

Due to the resemblance between the 3D Euler equations and the 2D QG equation, we can naturally apply the methods developed in Chapter 2 to the 2D QG equation. The results are the following.

We denote by $\Omega\left(t\right)$ the global maximum of the quantity $\left|\nabla^{\perp}\theta\right|$. L_t, $L\left(t\right)$, $M\left(t\right)$, and $K\left(t\right)$ are defined in a similar way as in 1.4.1, with ω replaced by $\nabla^{\perp}\theta$ and "vortex line" replaced by level sets. With these notations, we have the following theorems.

First, a direct adaptation of the proof of Theorem 1.4.3 gives us

Theorem 1.4.5. *Under the same assumptions as in Theorem 1.4.3, there will be no blowup if we replace the first assumption by* $\Omega\left(t\right) \leq \left(T^* - t\right)^{-B}$ *for some* $B \in (0, \infty)$*, and the third assumption by*

$$L\left(t\right) \geq c_L \left(T^* - t\right)^A$$

for some constant $A < 1$.

Remark 1.4.6. Note that in Theorem 1.4.5 we replace the assumption on the growth rate of the maximum velocity by a rather weak assumption on $\Omega(t)$. This is possible due to the estimate $\|u\|_{L^\infty} \lesssim \log \Omega$ when $\Omega \geq e$ obtained by Cordoba ([Cor98], the proof can be found in A.3.). With the help of this estimate, we further obtain the following theorem which yields a triple exponential growth rate estimate on $\Omega(t)$ without any assumption on its growth rate.

Theorem 1.4.7. *Assume that there is a family of level set segments* L_t *and* $T_0 \in [0, T^*)$ *such that* $X(L_{t_0}, t_0, t) \supseteq L_t$ *for all* $T_0 \leq t_0 < t < T^*$. *Also assume that* $\Omega(t)$ *is monotonically increasing and* $\Omega_L(t) \geq c_0 \Omega(t)$ *for some* $0 < c_0 \leq 1$ *for all* $t \in [T_0, T^*)$. *Furthermore, assume that there exist constants* $c_L, C_0 > 0$, *such that*

1. *(H1)* $L(t) \geq \frac{c_L}{\log \log \Omega(t)}$,

2. *(H2)* $M(t) L(t), K(t) L(t) \leq C_0$.

Then there will be no blowup in the 2D QG equation up to $T^* < \infty$. *Furthermore, for* $t \in [0, T^*)$, *we have the following triple exponential estimate.*

$$\log \log \log \Omega(t) \leq C_1 t + C_2 \tag{1.4.2}$$

for some constants $C_1, C_2 > 0$ *independent of* t.

Remark 1.4.8. It is worth mentioning that although assumption (H1) looks quite restrictive if $(T^* - t)^{-1}$ is considered to be generic for possible blowups in $\Omega(t)$, theoretically it is really a weaker assumption since this lower bound for $L(t)$ involves $\Omega(t)$ and is therefore nonlinear. This in fact makes the proof of Theorem 1.4.7 harder than that of Theorem 1.4.5.

So far, in numerical computations, the growth rate of $\Omega(t) = \|\nabla \theta\|_{L^\infty}$ never exceeds a double exponential. We can obtain a double exponential upper bound if the regularity of ξ is better. More precisely, we have the following corollary:

Corollary 1.4.9. *Assume that all the conditions in Theorem 1.4.7 are satisfied, except that (H1) is replaced by*

$$\text{(H1'):}\ L(t) \geq c_L.$$

Then the estimate (1.4.2) can be improved to

$$\log \log \Omega(t) \leq C_1' t + C_2' \tag{1.4.3}$$

for some constants C_1', C_2' independent of time.

Remark 1.4.10. In Section 3.3, we show that for a simple hyperbolic saddle scenario that is similar to the one in Cordoba [Cor97, Cor98], conditions (H1') and (H2) are in fact satisfied.

Chapter 2

Non-blowup of the 3D Euler Equations

In this chapter, we will discuss in detail the relation between local properties of the vorticity/velocity fields and the growth of maximum vorticity, and prove theorems 1.4.3 and 1.4.4, which exclude finite time singularity for the 3D Euler equations by localized non-blowup conditions.

The work presented in this chapter consists of materials from the following two papers:

1. [DHY05]: J. Deng, T. Y. Hou, X. Yu. Geometric properties and non-blowup of 3D incompressible Euler flow. *Comm. PDE*, 30: 225-243, 2005.

2. [DHY04]: J. Deng, T. Y. Hou, X. Yu. Improved geometric conditions for non-blowup of the 3D incompressible Euler equation. Submitted to *Comm. PDE*, 2004.

Before presenting the results, we first fix some notation conventions.

- C or c: generic constants, whose value may change from line to line.

- ξ: the unit vorticity direction, i.e., $\xi(x,t) \equiv \omega(x,t) / |\omega(x,t)|$ whenever the right-hand side ratio is well-defined.

- T^*: the alleged time when the first finite time singularity occurs.

- x, α: Cartesian coordinate variables. Thus x and α are both vectors in $\mathbb{R}^3$.

- s, β: arc length variables along one vortex line.

- $X(\alpha, \tau, t)$: the particle trajectory passing α at time τ. That is, $X(\alpha, \tau, t)$ solves

$$\begin{aligned} \frac{\partial X(\alpha, \tau, t)}{\partial t} &= u(X(\alpha, \tau, t)) \\ X(\alpha, \tau, \tau) &= \alpha. \end{aligned}$$

  For any set $A \subseteq \mathbb{R}^3$, we denote

$$X(A, \tau, t) \equiv \cup_{\alpha \in A} X(\alpha, \tau, t).$$

  When $\tau = 0$, $X(\alpha, 0, t)$ reduces to the conventional Lagrangian representation of the flow (Chorin and Marsden [CM93]), and we will use the conventional notation $X(\alpha, t) \equiv X(\alpha, 0, t)$.

- $\sim$: We write $a(t) \sim b(t)$ if there are absolute constants $c, C > 0$ such that

$$c|a(t)| \leq |b(t)| \leq C|a(t)|.$$

- $\gtrsim (\lesssim)$: We write $a(t) \gtrsim b(t)$ if there is an absolute constant $c > 0$ such that

$$|a(t)| \geq c|b(t)|.$$

  $a(t) \lesssim b(t)$ is defined similarly.

2.1 Two Observations

It has long been observed that at later times, large vorticity regions in incompressible flows of high Reynolds numbers do not have full dimensions, that is, they may be very thin in one or two directions. In particular, many of these regions have the shapes of "one-dimensional" vortex tubes that are long and thin.[1] This phenomenon has also

[1] Whether these vortex tubes determine the behavior of the flow is subject to different opinions among researchers. A discussion of this matter can be found in Frisch [Fri95].

been observed in recent numerical computations (e.g., Kerr [Ker93, Ker95, Ker96, Ker97, Ker98, Ker04], Pelz [Pel97, Pel01]). Recall that a vortex tube is a collection of vortex lines. Since these tubes are very small, it is natural to study the static and dynamic properties of vorticity along one vortex line, which would be "representative" in some sense (This is the philosophy behind the "vortex filament" theory, see, e.g., Chorin [Cho94] Chapter 7, Majda-Bertozzi [MB02] Chapter 7).

In this section, we study both the static and dynamic properties of vorticity along one vortex line. We start with two observations, which relate static and dynamic change of vorticity magnitude with geometric properties of the vorticity direction field.

2.1.1 Direction and magnitude of vorticity

First, we have the following lemma, which relates, through the incompressibility condition, the vortex line geometry to the magnitude of vorticity.

Lemma 2.1.1. *Let* $\xi(x,t) = \frac{\omega(x,t)}{|\omega(x,t)|}$ *be the direction of the vorticity vector. Assume that at a fixed time* $t > 0$ *the vorticity* $\omega(x,t)$ *is* C^1 *in* x. *Then at this time* t*, for any* x *such that* $\omega(x,t) \neq 0$*, there holds*

$$\frac{\partial |\omega|}{\partial s}(x,t) = -\left((\nabla \cdot \xi)\,|\omega|\right)(x,t)\,, \tag{2.1.1}$$

where s *is the arc length variable along the vortex line passing* x. *We denote this vortex line by* l.

Furthermore, for any $y \in l$ *such that* ω *does not vanish at any point in the vortex line segment between* x *and* y*, (2.1.1) then gives*

$$|\omega(y,t)| = |\omega(x,t)|\, e^{\int_x^y (-\nabla\cdot\xi)\; ds}, \tag{2.1.2}$$

where the integration is along the vortex line.

Proof. Notice that $\omega(x,t) = |\omega(x,t)|\,\xi(x,t)$. Since $\omega(x,t) \neq 0$, $\xi(x,t) = \frac{\omega(x,t)}{|\omega(x,t)|}$ is well-defined in a neighborhood of x. The incompressibility condition $\nabla \cdot \omega = 0$ then gives

$$\begin{aligned} 0 = \nabla \cdot \omega &= \nabla \cdot (|\omega| \, \xi) \\ &= (\nabla |\omega|) \cdot \xi + (\nabla \cdot \xi) |\omega| \\ &= (\xi \cdot \nabla) |\omega| + (\nabla \cdot \xi) |\omega| \, . \end{aligned}$$

Further note that the directional derivative $\xi \cdot \nabla$ is actually the arc length derivative along the vortex line, i.e., $\xi \cdot \nabla = \partial/\partial s$, as can be seen from the following argument. We have

$$\xi \cdot \nabla = \frac{dx\,(s)}{ds} \cdot \nabla$$

by the the fact that the unit tangential direction vector ξ can be obtained by differentiating the parametrized formulation $x = x\,(s)$ of the vortex line. Now by the chain rule,

$$\frac{dx}{ds} \cdot \nabla = \frac{\partial}{\partial s}.$$

Therefore we obtain

$$\frac{\partial |\omega|}{\partial s} = -\,(\nabla \cdot \xi) \, |\omega| \, ,$$

which is just (2.1.1).

Now solving the ODE (2.1.1) along l by multiplying $e^{\int (\nabla \cdot \xi)}$ on both sides, and then integrating from x to y, we easily obtain (2.1.2). □

Using Lemma 2.1.1 we can obtain the following theorem.

Theorem 2.1.2. *Consider any 3D incompressible flow (Euler or Navier-Stokes). Let $x(t)$ be a family of points such that $|\omega\,(x(t), t)| \gtrsim \Omega(t)$ (recall that $\Omega(t)$ is the maximum vorticity magnitude $\|\omega\,(\cdot, t)\|_{L^\infty(\mathbb{R}^3)}$ at time t). Assume that for all $t \in [0, T)$ there is another point $y\,(t)$ on the same vortex line as $x\,(t)$ such that the direction of vorticity ξ along this vortex line between $x\,(t)$ and $y\,(t)$ is well-defined. If we further assume that*

$$\left| \int_{x(t)}^{y(t)} (\nabla \cdot \xi)\,(s, t) \; ds \right| \leq C \tag{2.1.3}$$

for some absolute constant C *and*

$$\int_0^T |\omega(y(t), t)| \ dt < \infty,$$

then there will be no blowup up to time T*. Moreover, we have*

$$e^{-C} \leq \frac{|\omega(x(t), t)|}{|\omega(y(t), t)|} \leq e^C.$$

Proof. Using (2.1.2) and (2.1.3) we easily obtain

$$\int_0^T |\omega(x(t), t)| \ dt \leq e^C \int_0^T |\omega(y(t), t)| \ dt < \infty.$$

Then by our assumption on $\omega(x(t), t)$, we have

$$\int_0^T \Omega(t) \ dt \lesssim \int_0^T |\omega(x(t), t)| \ dt < \infty,$$

and our theorem follows directly from the BKM criterion. □

Theorem 2.1.2 gives a practical criterion for judging possible blowups in numerical computations. It also suggests that when searching for a finite time blowup numerically, one has to pay attention to the geometric property of vortex filaments. It is not enough to just track the maximum vorticity magnitude and the point at which this maximum is attained. The vorticity magnitudes at other points are also crucial. In particular, the above theorem implies that if there is a non-vanishing vortex line segment containing the maximum vorticity up to time T^* such that the "weakly regularly orientedness" condition (2.1.3) is satisfied, then no point singularity is possible up to this time T^*. To illustrate, we discuss the relation between Theorem 2.1.2 and the numerical observations by Pelz [Pel97, Pel01].

Example 2.1.3. In [Pel97, Pel01], Pelz studied a class of incompressible flows with strong symmetry and conjectured that such flows can lead to a finite time blowup. In these computations, vorticity is concentrated in small vortex tubes of length scale $(T^* - t)^{1/2}$. After a re-scaling $x \mapsto (T^* - t)^{-1/2} x$, these tubes seem to have a regular shape. This suggests that the length of this inner region scale like $(T^* - t)^{1/2}$ and the scaling of $\nabla \cdot \xi$ within this inner region is of the order $(T^* - t)^{-1/2}$. Let us take the point $x(t)$ to be the point inside one tube where the maximum vorticity is attained, and $y(t)$ to be a point on the same vortex line, but outside the tube. Within this inner region, condition (2.1.3) is likely to be satisfied. Thus by Theorem 2.1.2 we see that if the maximum vorticity *outside* these small tubes is integrable in time, then there is no blowup inside the tubes. It is likely that the maximum vorticity outside these small tubes has a growth rate smaller than that inside these small regions. This casts doubt on the validity of Pelz's claim on the finite time formation of a point singularity. To validate Pelz's claim, one needs to perform more careful numerical study to check whether there exists a non-vanishing vortex line segment within which condition (2.1.3) is satisfied or whether the vorticity within the inner tube-shaped region blows up at the same rate.

Before ending this subsection, we present an interesting application of the philosophy of writing $\nabla \cdot \omega = \xi \cdot \nabla |\omega| + (\nabla \cdot \xi) |\omega|$.

First we note that for any vector field $\tilde{\omega}(x, t)$, which is in the same direction of $\omega(x, t)$, we can write

$$\omega(x, t) = \tilde{\omega}(x, t) \tilde{\Omega}(x, t)$$

for some scalar function $\tilde{\Omega}(x, t) > 0$. Applying the same argument as in the proof of Lemma 2.1.1, we obtain an equivalent form of $\nabla \cdot \omega$:

$$\nabla \cdot \omega = (\tilde{\omega} \cdot \nabla) \tilde{\Omega} + (\nabla \cdot \tilde{\omega}) \tilde{\Omega}. \tag{2.1.4}$$

(2.1.4) becomes handy when checking the divergence-free property in certain cases that arise in preparation of initial values for numerical computations.

In numerical computations of incompressible fluids, people often prefer to construct compactly supported initial vorticity by first constructing a family of vortex lines, and then assigning vorticity magnitude to them (see, e.g., Kerr [Ker93], Hussein-Melander [HM92]). The reason is the following. It is physically highly meaningful to study those initial flows that the initial vorticity concentrates in vortex tubes with certain prescribed geometry, e.g., two anti-parallel tubes. And the aforementioned way of constructing the inital vorticity field just fits this requirement by first fixing the geometry of the vortex tubes with prescribed vortex lines. These vortex lines are usually given in parametrized form $(x(\tau), y(\tau), z(\tau))$, which would make it inconvenient to first compute the vorticity as a function of (x, y, z), and then check $\nabla \cdot \omega = 0$. However, noticing that during the construction of the initial vorticity, one naturally obtains $\tilde{\omega}$ (may not be of unit length) by differentiating $(x(\tau), y(\tau), z(\tau))$ and $\tilde{\Omega}$ by prescription, we readily see that (2.1.4) yields a simple way to check whether the initial value is divergence free. To illustrate this point, we study the following claim:

Claim 2.1.4. The vorticity field $\omega(x, y, z)$ constructed in the following way is always divergence free.

1. Take a "center curve" $(x_0 + x(y), y, z_0 + z(y))$.

2. Take a "bump function" $f(r)$, e.g., $f(r) = e^{-r^2/2}$.

3. Construct initial vorticity along each vortex line $(\tilde{x}_0 + x(y), y, \tilde{z}_0 + z(y))$ by assigning the following (relative) vorticity strength:

$$f\left(\left((x_0 - \tilde{x}_0)^2 + (z_0 - \tilde{z}_0)^2\right)^{1/2}\right)(x'(y), 1, z'(y)).$$

Proof. Take $\tilde{\omega} = (x'(y), 1, z'(y))$ and $\tilde{\Omega} = f\left(\left((x_0 - \tilde{x}_0)^2 + (z_0 - \tilde{z}_0)^2\right)^{1/2}\right)$. Since $\tilde{\Omega}$ is constant along each vortex line, $\tilde{\omega} \cdot \nabla\tilde{\Omega} = 0$ automatically. Further, since $\tilde{\omega}$ only depends on y, $\nabla \cdot \tilde{\omega} = \frac{\partial}{\partial y}1 = 0$. Now applying (2.1.4), we obtain $\nabla \cdot \omega = 0$. □

2.1.2 An alternative formulation of vorticity growth

It is well-known that the evolution of the magnitude of vorticity along any particle path is governed by the following equation (Constantin [Con94]):

$$\begin{aligned}\frac{D\left|\omega\left(x,t\right)\right|}{Dt} &= \xi\left(x,t\right)\cdot\left(\nabla u\left(x,t\right)\cdot\xi\left(x,t\right)\right)\left|\omega\left(x,t\right)\right| \\ &\equiv \alpha\left(x,t\right)\left|\omega\left(x,t\right)\right|,\end{aligned}$$

where $D/Dt \equiv \partial_t + u\cdot\nabla$ is the material derivative. After some simple calculation we have another form of the stretching factor α:

$$\begin{aligned}\alpha &= \xi\cdot\nabla u\cdot\xi \\ &= \left(\xi\cdot\nabla\right)\left(u\cdot\xi\right) - u\cdot\left(\xi\cdot\nabla\right)\xi \\ &= \left(u\cdot\xi\right)_s - \kappa\left(u\cdot\mathbf{n}\right),\end{aligned}$$

where we have used the fact that $\xi\cdot\nabla = \partial/\partial s$ and the well-known basic relation in differential geometry

$$\frac{\partial\xi}{\partial s} = \kappa\mathbf{n}$$

with $\kappa = \left|\xi\cdot\nabla\xi\right|$ the curvature and $\mathbf{n}$ the unit normal vector of the vortex line.

Remark 2.1.5. This formulation of the stretching factor $\alpha = \left(u\cdot\xi\right)_s - \kappa\left(u\cdot\mathbf{n}\right)$ reveals that, instead of the whole gradient of all three components of the velocity vector ∇u, only a partial derivative $\partial/\partial s$ of one particular component $u\cdot\xi$ and the normal velocity $u\cdot\mathbf{n}$ itself (not any of its derivatives!) are involved. This subtle difference with the form $\xi\cdot\nabla u\cdot\xi$ will prove to be vital to our analysis, as will become clear in sections 2.3, 2.4, and 2.5.

2.2 Geometric Description of Vorticity Stretching

Having written the stretching factor into a form that involves the geometric properties of the vorticity and velocity fields for any fixed time, we need to find a way to describe the dynamic growth of the vorticity by quantities "more geometric." It turns out that we can achieve this by studying the stretching of one vortex line segment.

In the following, we will study how the relative rate of arc length stretching along a vortex filament is related to the relative rate of the maximum vorticity growth in time.

2.2.1 Estimation at one point

For any starting time t_0, consider the evolution of a vortex line for $t > t_0$. Let s and β be the arc length parameters of this vortex line at time t and t_0, respectively. We can write, for this very vortex line, $s = s(\beta, t)$. Note that $s(\beta, t_0) = \beta$. Then we have the following lemma.

Lemma 2.2.1. *For any point α at time t_0 such that $\omega(\alpha, t_0) \neq 0$, let $X(\alpha, t_0, t)$ be the position of the same particle at time $t > t_0$. Then we have*

$$\frac{\partial s}{\partial \beta}\left(X(\alpha, t_0, t), t\right) = \frac{\left|\omega\left(X(\alpha, t_0, t), t\right)\right|}{\left|\omega(\alpha, t_0)\right|}. \tag{2.2.1}$$

Proof. Without loss of generality, we prove the lemma for $t_0 = 0$. Denote $\omega_0(\alpha) \equiv \omega(\alpha, 0)$. Now according to our notation convention, we will simply use $X(\alpha, t)$ for $X(\alpha, 0, t)$.

It is well-known that for 3D Euler flows we have (Chorin-Marsden [CM93])

$$\omega\left(X(\alpha, t), t\right) = \nabla_\alpha X(\alpha, t) \cdot \omega_0(\alpha).$$

Then

$$\begin{aligned}\left|\omega\left(X(\alpha, t), t\right)\right| &= \xi\left(X(\alpha, t), t\right) \cdot \omega\left(X(\alpha, t), t\right) \\ &= \xi\left(X(\alpha, t), t\right) \cdot \nabla_\alpha X(\alpha, t) \cdot \xi(\alpha, t_0)\left|\omega_0(\alpha)\right|.\end{aligned}$$

Note that $\xi\left(X(\alpha, t), t\right) = \partial X(\alpha, t) / \partial s$ for any t, where s is the arc length variable of the vortex line that passes $X(\alpha, t)$ at time t. In particular, we have $\xi(\alpha, t_0) = \frac{\partial \alpha}{\partial \beta}$.

Now we can further simplify the above equation as

$$\begin{aligned}
|\omega(X(\alpha,t),t)| &= \frac{\partial X(\alpha,t)}{\partial s}\cdot\nabla_\alpha X(\alpha,t)\cdot\frac{\partial\alpha}{\partial\beta}|\omega_0(\alpha)| \\
&= \frac{\partial X(\alpha,t)}{\partial s}\cdot\frac{\partial X(\alpha,t)}{\partial\beta}|\omega_0(\alpha)| \\
&= \left(\frac{\partial X(\alpha,t)}{\partial s}\cdot\frac{\partial X(\alpha,t)}{\partial s}\right)\frac{\partial s}{\partial\beta}|\omega_0(\alpha)| \\
&= |\xi(X(\alpha,t),t)|^2\frac{\partial s}{\partial\beta}|\omega_0(\alpha)| \\
&= \frac{\partial s}{\partial\beta}|\omega_0(\alpha)|.
\end{aligned}$$

Thus ends the proof. □

2.2.2 Estimation by a vortex line segment

In Subsection 2.2.1, we have related the vorticity growth at one Lagrangian point with the relative stretching of the vortex line at that point. To take advantage of the two observations in 2.1, we need to further relate the growth of vorticity magnitude with the length change of one vortex line segment, instead of the local stretching rate as in (2.2.1). This is done by the following lemma.

Lemma 2.2.2. *For any t_0, let l_t be a vortex line segment that is carried by the flow, i.e., $l_t = X(l_{t_0}, t_0, t)$ for $t \geq t_0$. Denote its length by $l(t)$. Define*

$$m(t) \equiv \max_{x\in l_t} |\nabla\cdot\xi(x,t)|,$$

recalling that $\xi = \omega/|\omega|$ is the unit vorticity direction. If we further denote $\Omega_l(t) \equiv$ $\max_{x\in l_t}|\omega(x,t)|$*, then the following inequality holds.*

$$e^{-m(t)l(t)}\frac{\Omega_l(t)}{\Omega_l(t_0)} \leq \frac{l(t)}{l(t_0)} \leq e^{m(t_0)l(t_0)}\frac{\Omega_l(t)}{\Omega_l(t_0)}. \tag{2.2.2}$$

Proof. Let β denote the arc length parameter at time t_0, and s denote the arc length parameter at time t. Denote the two end points of l_{t_0} by β_1 and β_2. Then we have

$$\begin{aligned} l(t) &= \int_{\beta_1}^{\beta_2} s_\beta \, d\beta \\ &= \int_{\beta_1}^{\beta_2} \frac{|\omega(X(\alpha, t_0, t), t)|}{|\omega(\alpha, t_0)|} \, d\beta \\ &\leq \int_{\beta_1}^{\beta_2} \frac{\Omega_l(t)}{e^{-m(t_0)l(t_0)}\Omega_l(t_0)} \, d\beta \\ &= e^{m(t_0)l(t_0)} \frac{\Omega_l(t)}{\Omega_l(t_0)} l(t_0), \end{aligned}$$

where the second equality is due to (2.2.1), the inequality comes from

$$|\omega(X(\alpha, t_0, t), t)| \leq \Omega_l(t)$$

due to the definition of $\Omega_l(t)$, and

$$|\omega(\alpha, t_0)| \leq e^{-m(t_0)l(t_0)} \Omega_l(t_0)$$

due to (2.1.2).

Similarly, we have

$$\begin{aligned} l(t) &= \int_{\beta_1}^{\beta_2} s_\beta \, d\beta \\ &= \int_{\beta_1}^{\beta_2} \frac{|\omega(X(\alpha, t_0, t), t)|}{|\omega(\alpha, t_0)|} \, d\beta \\ &\geq \int_{\beta_1}^{\beta_2} \frac{e^{-m(t)l(t)}\Omega_l(t)}{\Omega_l(t_0)} \, d\beta \\ &= e^{-m(t)l(t)} \frac{\Omega_l(t)}{\Omega_l(t_0)} l(t_0). \end{aligned}$$

Thus ends the proof. $\square$

2.3 Key Estimate for Vorticity Growth

Now we can combine the understandings developed in subsections 2.2.1 and 2.2.2 and obtain our key estimate of vorticity growth.

By (2.2.1) and recalling that

$$\frac{D|\omega|}{Dt} = \alpha|\omega| = [(u\cdot\xi)_s - \kappa(u\cdot\mathbf{n})]|\omega|,$$

we see that the same equation holds for the growth of s_β:

$$\begin{aligned}\frac{Ds_\beta}{Dt} &= [(u\cdot\xi)_s - \kappa(u\cdot\mathbf{n})]s_\beta \\ &= (u\cdot\xi)_\beta - \kappa(u\cdot\mathbf{n})s_\beta,\end{aligned} \tag{2.3.1}$$

where the last equality is by the chain rule.

Now integrating equation (2.3.1) along one Lagrangian vortex line l_t (by Lagrangian we mean $l_t = X(l_s, s, t)$ for any $t_0 < s < t$ at some fixed time t_0), whose ends are denoted by β_1 and β_2 at time t_0, we have

$$\begin{aligned}\frac{D[s(\beta_2,t) - s(\beta_1,t)]}{Dt} &= (u\cdot\xi)(X(\beta_2,t_0,t),t) - (u\cdot\xi)(X(\beta_1,t_0,t),t) \\ &\quad - \int_{\beta_1}^{\beta_2}\kappa(u\cdot\mathbf{n})s_\beta\, d\beta.\end{aligned} \tag{2.3.2}$$

Further, we integrate (2.3.2) from t_0 to some later time t. We get

$$\begin{aligned}s(\beta_2,t) - s(\beta_1,t) &= s(\beta_2,t_0) - s(\beta_1,t_0) \\ &\quad + \int_{t_0}^{t}[(u\cdot\xi)(X(\beta_2,t_0,\tau),\tau) - (u\cdot\xi)(X(\beta_1,t_0,\tau),\tau)]\, d\tau \\ &\quad - \int_{t_0}^{t}\int_{\beta_1}^{\beta_2}\kappa(u\cdot\mathbf{n})s_\beta\, d\beta d\tau.\end{aligned}$$

Noticing that $s(\beta_2, t) - s(\beta_1, t)$ is just $l(t)$, the length of l_t, we have

$$\begin{aligned} l(t) \quad &\leq \quad l(t_0) + \int_{t_0}^{t} \max_{x,y \in l_\tau} \left| (u \cdot \xi)(x, \tau) - (u \cdot \xi)(y, \tau) \right| \, d\tau \\ &\quad + \int_{t_0}^{t} k(\tau) \max_{l_\tau} |u \cdot \mathbf{n}| \, l(\tau) \, d\tau, \end{aligned} \tag{2.3.3}$$

where $k(\tau) \equiv \max_{x \in l_t} \kappa(x, \tau)$ is the maximum curvature along l_τ at time τ.

Now recall that in Lemma 2.2.2 we have proved that

$$e^{-m(t)l(t)} \frac{\Omega_l(t)}{\Omega_l(t_0)} \leq \frac{l(t)}{l(t_0)} \leq e^{m(t_0)l(t_0)} \frac{\Omega_l(t)}{\Omega_l(t_0)}$$

for a Lagrangian vortex line segment l_t at times t and t_0. Combining this estimate and (2.3.3), we obtain

$$\Omega_l(t) \leq e^{m(t)l(t)} \Omega_l(t_0) \left[1 + \frac{1}{l(t_0)} \int_{t_0}^{t} \left[U_\xi(\tau) + k(\tau) U_n(\tau) l(\tau) \right] \, d\tau \right], \tag{2.3.4}$$

where we have defined $U_\xi(\tau) \equiv \max_{x,y \in l_\tau} |(u \cdot \xi)(x, \tau) - (u \cdot \xi)(y, \tau)|$, $U_n(\tau) \equiv \max_{l_\tau} |u \cdot \mathbf{n}|$, and used the fact that $m(t) = \max_{x \in l_t} |\nabla \cdot \xi|$. We call (2.3.4) our *key estimate.*

2.4 Proof of Theorem 1.4.3

2.4.1 Main idea

The proof relies heavily on our key estimate (2.3.4), which bounds the ratio of the maximum vorticity at two different times by local properties of the vorticity and velocity fields. The main idea is the following.

If the maximum vorticity blew up at a finite time T^*, then for any constant $r > 1$, we could divide the time interval $[0, T^*)$ into an infinite number of subintervals, $[t_k, t_{k+1})$, in which the maximum vorticity increases geometrically, i.e., $\Omega(t_{k+1}) =$

$r\Omega(t_k)$. By using our assumptions, we have

$$\int_{t_0}^{T} \Omega(t)\ dt \leq \sum_{k=0}^{\infty} \Omega(t_{k+1})(t_{k+1} - t_k) \leq \Omega(t_0) \sum_{k=0}^{\infty} r^k (T^* - t_k). \tag{2.4.1}$$

Now the key of our proof is to show the existence of one particular $r > 1$ such that the corresponding t_k converges to T^* so fast that $\limsup_{k\to\infty} \frac{r^{k+1}(T^*-t_{k+1})}{r^k(T^*-t_k)} = \limsup_{k\to\infty} \frac{r(T^*-t_{k+1})}{(T^*-t_k)} < 1$. This makes the summation finite, and thus we get a contradiction.

This main idea lies behind the proof of both of our main theorems, as will be presented in more detail in the following.

2.4.2 The proof

We prove Theorem 1.4.3 by contradiction. First, by translating the initial time we can assume that the assumptions in Theorem 2 hold in $[0, T^*)$. Define

$$r \equiv \frac{R}{c_0} + 1, \tag{2.4.2}$$

where $R \equiv e^{C_0}$with C_0 being the constant in the conditions of Theorem 1.4.3 such that $M(t)L(t) \leq C_0$ for all $t \in [0, T)$, and c_0 is the constant such that $\Omega_L(t) \geq c_0\Omega(t)$. Throughout the proof we denote $\Omega_L(t) \equiv \max_{x\in L_t} |\omega(x,t)|$. The reason for choosing the parameter r this way will become clear later in the proof. If there were a finite time blowup at time T, we would have

$$\int_0^T \Omega(t)\ dt = \infty,$$

or equivalently, for any $t_0 \in [0, T)$,

$$\int_{t_0}^{T^*} \Omega(t)\ dt = \infty.$$

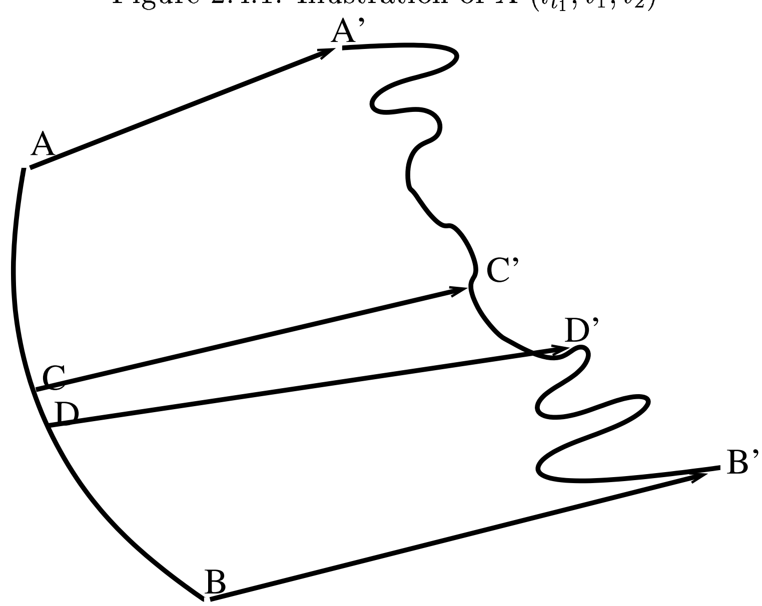
Then necessarily we have $\Omega(t) \nearrow \infty$ as $t \nearrow T^*$. Now we can take a time sequence $t_1, t_2, \ldots, t_n, \ldots$ such that

$$\Omega(t_{k+1}) = r\Omega(t_k),$$

where r is defined in (2.4.2). Since $\Omega(t)$ is monotone and T^* is the smallest time such that $\int_0^{T^*} \Omega(t)\ dt = \infty$, it is obvious that $t_n \nearrow T^*$as $n \to \infty$.

Now we choose $l_{t_2} = L_{t_2}$. By our assumptions on L_t, there exists $l_{t_1} \subset L_{t_1}$ such that $X(l_{t_1}, t_1, t_2) = l_{t_2}$. This is a crucial step to our theorem and we illustrate it in the following graphic:

Figure 2.4.1: Illustration of $X(l_{t_1}, t_1, t_2)$



In the above graphic, segment AB and $C'D'$ are L_{t_1} and L_{t_2}. By our assumptions, L_t may shrink with time. If this is the case, the flow image of AB, denoted by $A'B'$ would be much longer than $C'D'$. Note that the segments $A'C'$ and $D'B'$ do not have to have a good bound for $\nabla \cdot \xi$ or κ. Now our choice of l_{t_1} and l_{t_2} is the following. We take l_{t_2} to be $C'D'$. Then l_{t_1} has to be the pre-image of l_{t_2} and is denoted by CD. It is crucial to notice that the length of l_{t_1}, i.e., $l(t_1)$, is bounded from above by $L(t_2)$ instead of $L(t_1)$. The important task now is to obtain an estimate on the lower bound of $l(t_1)$, which turns out to be of the same order as $L(t_2)$. If we further denote

$$\Omega_l(t_i) \equiv \max_{x \in l_{t_i}} |\omega(x,t)| \quad i = 1, 2,$$

then by taking $t = t_2$ in (2.2.2) we would have

$$l(t_1) \geq l(t_2) \frac{1}{R} \frac{\Omega_L(t_1)}{\Omega_L(t_2)},$$

where the last inequality is due to the assumption $M(t)L(t) \leq C_0$ and $R = e^{C_0}$. Note that by assumption we have $\Omega_L(t) \geq c_0 \Omega(t)$. Thus $l(t_1)$ can be further bounded from below by

$$\begin{aligned} l(t_1) &\geq l(t_2) \frac{c_0}{R} \frac{\Omega(t_1)}{\Omega(t_2)} \\ &= \frac{c_0}{Rr} l(t_2) = \frac{c_0}{Rr} L(t_2) \gtrsim (T - t_2)^B. \end{aligned}$$

On the other hand, we have from (2.3.4)

$$\begin{aligned} \Omega_l(t_2) &\leq e^{M(t_2)l(t_2)} \Omega_l(t_1) \\ &\quad \cdot \left[1 + \frac{C}{l(t_1)} \int_{t_1}^{t_2} \left(U_\xi(\tau) + M(\tau) U_n(\tau) l(\tau) \right) d\tau \right]. \end{aligned}$$

By the assumptions of Theorem 1.4.3, we have

$$M(t_2) l(t_2) \leq M(t_2) L(t_2) \leq C_0$$

and

$$U_\xi(\tau) + U_n(\tau) M(\tau) l(\tau) \leq U_\xi(\tau) + U_n(\tau) M(\tau) L(\tau) \lesssim (T^* - \tau)^{-A}.$$

Then it follows that

$$\Omega_l(t_2) \leq R\Omega_l(t_1) + CR \frac{\Omega_l(t_1)}{(T^* - t_2)^B} \int_{t_1}^{t_2} (T^* - \tau)^{-A} \, d\tau.$$

Note that the constant C here depends on R, r, and c_0.

Since $\Omega_L(t) \geq c_0\Omega(t)$ by assumption, we have

$$\begin{aligned}
\Omega(t_{k+1}) &\leq \frac{1}{c_0}\Omega_L(t_{k+1}) = \frac{1}{c_0}\Omega_l(t_{k+1}) \\
&\leq \frac{R}{c_0}\Omega_l(t_k) + \frac{CR}{c_0}\frac{\Omega_l(t_k)}{(T^*-t_{k+1})^B}\int_{t_k}^{t_{k+1}}(T^*-\tau)^{-A}\ d\tau \\
&\leq \frac{R}{c_0}\Omega(t_k) + \frac{CR}{c_0}\frac{\Omega(t_k)}{(T^*-t_{k+1})^B}\int_{t_k}^{t_{k+1}}(T^*-\tau)^{-A}\ d\tau \\
&\leq (r-1)\Omega(t_k) + \frac{CR}{(1-A)c_0}\frac{\Omega(t_k)}{(T^*-t_{k+1})^B}\left[(T^*-t_k)^{1-A} - (T^*-t_{k+1})^{1-A}\right],
\end{aligned}$$

where $r = (R/c_0)+1$ is defined as in (2.4.2). We still denote $CR/(c_0(1-A))$ by C, note that the generic constant C now depends on R, r, c_0 and is proportional to $(1-A)^{-1}$. Since $(T^*-t_{k+1})^{1-A} > 0$, we can discard it and obtain

$$\Omega(t_{k+1}) \leq (r-1)\Omega(t_k) + C\Omega(t_k)\frac{(T^*-t_k)^{1-A}}{(T^*-t_{k+1})^B}. \tag{2.4.3}$$

Recall that $\Omega(t_{k+1}) = r\Omega(t_k)$ by our construction, so we can cancel $\Omega(t_k)$ from both sides of (2.4.3) and obtain

$$r \leq (r-1) + C\frac{(T^*-t_k)^{1-A}}{(T^*-t_{k+1})^B}, \tag{2.4.4}$$

which gives

$$(T^*-t_{k+1})^B \leq C(T^*-t_k)^{1-A}, \tag{2.4.5}$$

or equivalently

$$(T^*-t_{k+1}) \leq C(T^*-t_k)^{1+2\delta},$$

where

$$\delta \equiv \frac{1}{2}\left(\frac{1-A}{B}-1\right) > 0.$$

Now it is quite clear why we take $\Omega(t_{k+1})/\Omega(t_K) = r > R/c_0$ and choose $r = R/c_0+1$, since otherwise we would not be able to obtain (2.4.3), and consequently we would not be able to get crucial estimates (2.4.4) and (2.4.5).

Now note that C and δ are independent of k. Therefore when k is large enough, $(T^* - t_k)$ becomes so small that $(T^* - t_k)^\delta C < 1$. Thus we have

$$(T^* - t_{k+1}) \le (T^* - t_k)^\delta (T^* - t_k),$$

or equivalently,

$$\frac{r(T^* - t_{k+1})}{(T^* - t_k)} \le r(T^* - t_k)^\delta$$

for k large enough.

Finally, since r is also independent of k, when k is large enough we would have

$$r(T^* - t_k)^\delta < a < 1$$

for some a uniformly in large k. This implies $r^k (T^* - t_k) \le a^k (T - t_0)$ and consequently the convergence of the right-hand side of (2.4.1), which in turn gives

$$\int_{t_0}^{T^*} \Omega(t)\ dt < \infty.$$

The proof ends after invoking the BKM criterion (1.3.1).

2.5 Proof of Theorem 1.4.4

The first half of the proof of Theorem 1.4.4 follows the same line as the proof of Theorem 1.4.3. However, it keeps track of various constants more carefully. Recall that in the proof of Theorem 1.4.3, we divide the time interval $[T_0, T^*)$ into subintervals $[t_k, t_{k+1})$ such that

$$\Omega(t_{k+1}) = r\Omega(t_k)$$

for some constant r to be fixed. Then since

$$\int_{t_0}^{T^*} \Omega(t)\ dt \le \Omega(t_0) \sum_{k=0}^{\infty} r^{k+1} (T - t_k),$$

it is enough to show that

$$\limsup_{k\to\infty} \frac{r\left(T^* - t_{k+1}\right)}{\left(T^* - t_k\right)} < 1$$

to get a contradiction. In the proof of Theorem 1.4.3, we show that $T^* - t_k$ is decreasing faster than geometrically and therefore the upper limit is less than 1. Here we will see that $T^* - t_k$ decreases just geometrically with the ratio $\left(T^* - t_{k+1}\right) / \left(T^* - t_k\right)$ bounded by some certain constant depending on the various constants in the growth rate assumptions of the theorem. Therefore our task is to find conditions for the various constants C_U, c_L, etc., such that there exists $r > e^{C_0}/c_0$ that would guarantee the upper limit to be strictly less than 1. We carry out our investigation now.

By our key estimate, we have

$$\begin{aligned}
\frac{\Omega_L\left(t_{k+1}\right)}{\Omega_L\left(t_k\right)} &\leq e^{C_0}\left[1 + e^{C_0}\frac{\Omega_l\left(t_{k+1}\right)}{\Omega_l\left(t_k\right)}\frac{1}{l\left(t_{k+1}\right)}\int_{t_k}^{t_{k+1}}\left(U_\xi\left(\tau\right) + K\left(\tau\right)U_n\left(\tau\right)L\left(\tau\right)\right)\,d\tau\right] \\
&\leq e^{C_0}\left[1 + e^{2C_0}\frac{\Omega_L\left(t_{k+1}\right)}{\Omega\left(t_k\right)}\frac{1}{L\left(t_{k+1}\right)}\int_{t_k}^{t_{k+1}}\left(U_\xi\left(\tau\right) + K\left(\tau\right)U_n\left(\tau\right)L\left(\tau\right)\right)\,d\tau\right],
\end{aligned}$$

where we have used $\Omega_l\left(t_k\right) \geq e^{-C_0}\Omega_L\left(t_k\right)$, a consequence of Lemma 2.1.1. Now since $\Omega_L\left(t\right) \geq c_0\Omega\left(t\right)$ by assumption, we have

$$\begin{aligned}
r &= \frac{\Omega\left(t_{k+1}\right)}{\Omega\left(t_k\right)} \leq \frac{1}{c_0}\frac{\Omega_L\left(t_{k+1}\right)}{\Omega_L\left(t_k\right)} \\
&\leq \frac{e^{C_0}}{c_0}\left[1 + e^{2C_0}\frac{\Omega\left(t_{k+1}\right)}{\Omega\left(t_k\right)}\frac{1}{L\left(t_{k+1}\right)c_0}\int_{t_k}^{t_{k+1}}\left(U_\xi\left(\tau\right) + K\left(\tau\right)U_n\left(\tau\right)L\left(\tau\right)\right)\,d\tau\right] \\
&\leq R\left[1 + R^2 r\frac{c_0}{c_L\left(T^* - t_{k+1}\right)^{1-A}}\int_{t_k}^{t_{k+1}} C_U\left(T^* - \tau\right)^{-A}\,d\tau\right] \\
&= R + R^3 K r\left[\left(\frac{T^* - t_k}{T^* - t_{k+1}}\right)^{1-A} - 1\right],
\end{aligned}$$

where we have used the assumptions on the growth rates and have defined

$$R \equiv \frac{e^{C_0}}{c_0} \text{ and } K \equiv \frac{C_U c_0}{c_L\left(1 - A\right)}.$$

Now the above can be rewritten as

$$\left(\frac{T^* - t_k}{T^* - t_{k+1}}\right)^{1-A} \geq \frac{\left(R^3 K r + 1\right) r - R}{R^3 K r},$$

which gives

$$\frac{T^* - t_{k+1}}{T^* - t_k} \leq \left(\frac{R^3 K r}{\left(R^3 K r + 1\right) r - R}\right)^{1/(1-A)}.$$

Recall that it is enough to find one $r > R \equiv e^{C_0}/c_0$ such that

$$\limsup_{k\to\infty} \frac{r\left(T^* - t_{k+1}\right)}{T^* - t_k} < 1.$$

Therefore it is sufficient to find $r > R$ such that

$$\frac{r^{2-A} R^3 K}{\left(R^3 K + 1\right) r - R} < 1. \tag{2.5.1}$$

In the following we will show that the existence of such r is equivalent to the condition on R and K in the theorem, i.e.,

$$R^3 K < y_1\left(R^{A-1}\left(1 - A\right)^{1-A} / \left(2 - A\right)^{2-A}\right),$$

where $y_1\left(m\right)$ denotes the smallest positive solution of

$$\frac{y}{\left(y + 1\right)^{2-A}} = m.$$

We summarize this as the following lemma, whose proof is postponed to A.2.

Lemma 2.5.1. *There exists $r > R$ such that (2.5.1) is satisfied if and only if*

$$R^3 K < y_1\left(R^{A-1}\left(1 - A\right)^{1-A} / \left(2 - A\right)^{2-A}\right),$$

where $y_1\left(m\right)$is as defined above.

Clearly, the proof of Theorem 1.4.4 also ends with the application of the lemma.

2.6 Applications of Theorems 1.4.3 and 1.4.4

In most numerical studies of 3D Euler singularities, it has been observed that the maximum vorticity, if it blows up at all, grows as $(T^* - t)^{-1}$. On the other hand, Kelvin's circulation theorem suggests that the maximum velocity be bounded from above by the square root of maximum vorticity. Thus $A = 1/2$ is in some sense the worst blowup scenario for the velocity field if we consider the $(T^* - t)^{-1}$ blowup for the vorticity field as generic. If we follow the vortex filament along which the maximum vorticity is attained, then we have $c_0 = 1$. Thus it is of practical interest to study Condition (1.4.1) for the case $A = 1/2$ and $c_0 = 1$ and investigate the parameter range for C_0, C_U, c_L in which no finite time blowup would occur.

In the case $A = 1/2$ and $c_0 = 1$, Theorem 1.4.4 implies that if

$$e^{3C_0} K < y_1 \left(e^{-C_0/2} \frac{2}{3^{3/2}} \right),$$

then there will be no finite time singularity up to time T^*. We can rewrite the condition as

$$K < K_{max}\left(C_0\right) \equiv e^{-3C_0} y_1 \left(2e^{-C_0/2}/3^{3/2}\right),$$

where y_1 is the smallest positive number such that

$$\frac{y}{(y+1)^{3/2}} = \frac{2}{3^{3/2}} e^{-C_0/2}.$$

One can obtain K_{max} easily by solving either numerically or analytically the cubic equation

$$\frac{2}{3^{3/2}} e^{-C_0/2} \left(y+1\right)^3 - y^2 = 0$$

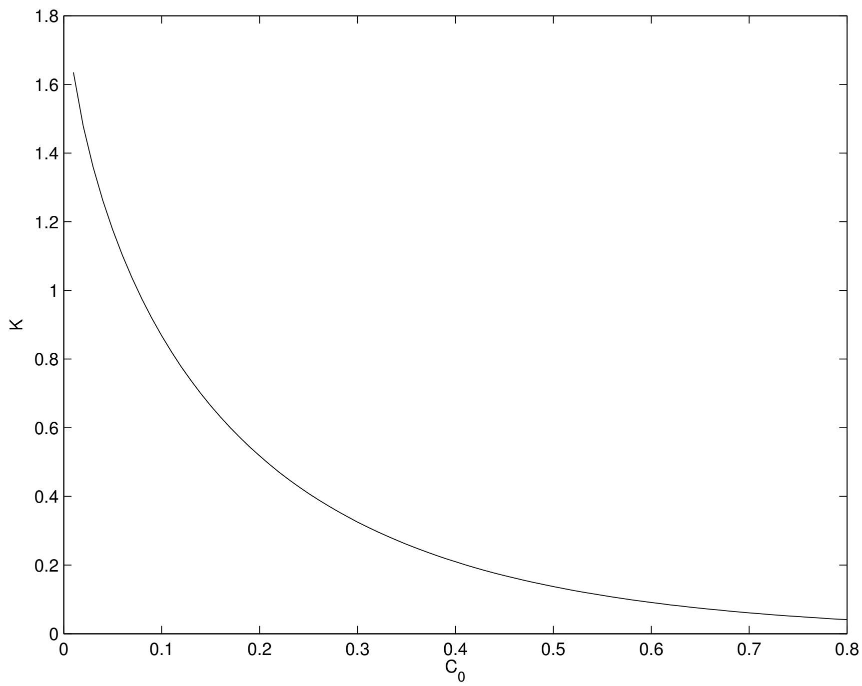
for each $C_0 > 0$. See Figure 2.6.1 and Table 2.1 for a plot of K_{max} as well as a table of K_{max} for some representative values of C_0.

Next we apply Theorem 1.4.4 to Kerr's computations. In a sequence of papers (Kerr [Ker93, Ker95, Ker96, Ker97, Ker98, Ker04]), computations for the perturbed

Table 2.1: $K_{max}(C_0)$ for small C_0's

C_0	0.05	0.1	0.15	0.2	0.25	0.3
$K_{max}(C_0)$	1.1770	0.8682	0.6644	0.5180	0.4088	0.3253

Figure 2.6.1: $K_{max}(C_0)$



anti-parallel vortex tube setting (Figure 2.6.2) are performed.[2] Kerr observed that when t is close enough to the alleged blowup time T^*, the region bounded by the contour of $0.6\,\|\omega\|_{L^\infty}$, aka the active region ([Ker97]), looks like two vortex sheets with thickness $\sim (T^* - t)$ meeting at an angle ([Ker96], see Figure 2.6.3, and Figure 2.6.4 for a diagram). This active region has length scale $(T^* - t)^{1/2}$ in the vorticity direction. The maximum vorticity resides in the small tube-like region with scaling

[2] Using a pair of perturbed anti-parallel vortex tubes as the initial value when searching for possible Euler singularities has been the strategy of many numerical computations. One of the reasons is that this initial setting leads to vortex reconnection, which changes the topology of the vorticity vector field, in Navier-Stokes simulations (e.g., Hussein-Melander [HM92], Shelley-Meiron-Orszag [SMO93]). The idea is to trigger the Crow instability and thus obtain fast growth of vorticity. Both blowup and non-blowup (more specifically, exponential growth of vorticity) behaviors have been observed by different authors, and also are the same authors when using different resolutions. Recently, H. K. Moffatt proposed two orthogonally aligned vortex dipoles as another candidate for point collapsing blowup, see [Mof00].

$(T^* - t)^{1/2} \times (T^* - t) \times (T^* - t)$, which is the intersection of the two sheets. Inside the active region, vortex lines are "relatively straight" ([Ker97]). Thus, assumption 3 in the theorem is verified, and therefore we have $L(t) \geq c_L (T^* - t)^{1/2}$ for some $c_L > 0$. Since this observation is made according to the re-scaled picture of vortex lines, it is likely that both the curvature κ and $\nabla \cdot \xi$ in this region are bounded by the order $(T^* - t)^{-1/2}$. In this case, assumption 2 is verified. It is also observed that the maximum velocity of the flow is located on the boundary of the active region, that is $(T^* - t)^{1/2}$ away from the maximum vorticity, and grows like $(T^* - t)^{-1/2}$ ([Ker04]). If we take the worst scenario that $U_\xi(t)$ also blows up like $(T^* - t)^{-1/2}$, then we will have $A = 1/2$ in our theorem and assumptions 1–3 in the theorem are all verified. Furthermore, since the vortex lines are "relatively straight," we can expect C_0 to be quite small. If we take $C_0 \leq 0.1$ as a reasonable guess, then there will be no finite time singularity if C_U and c_L satisfy the following constraint:

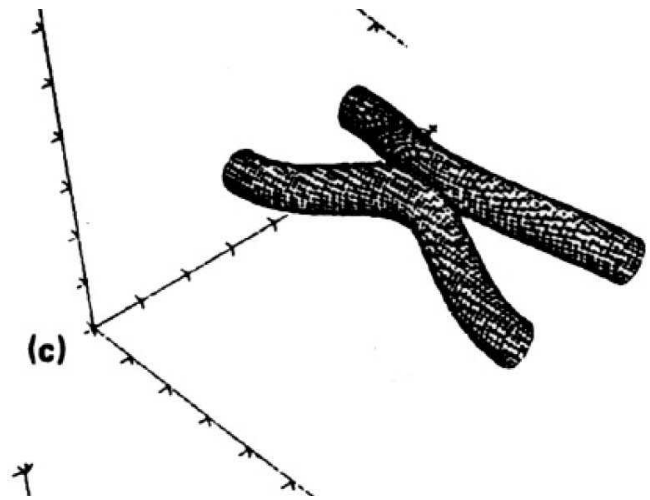
$$\frac{C_U}{c_L} \leq 0.4341.$$

Currently there are no numerical measurements of C_U and c_L available. Whether the scaling constants c_L, C_U, etc., satisfy Condition (1.4.1) in Theorem 1.4.4 is still unknown. We are currently carrying out careful numerical studies to obtain accurate measurements for these scaling constants.

2.7 Conclusion

In this chapter, we derived an estimate for the local vortex line stretching from two simple observations. With this estimate, we are able to prove non-blowup under very localized assumptions on the vorticity and velocity fields. Unlike existing theoretical results, our new theorems are applicable to the observations of recent computations searching for finite time singularities. Application of these theorems showed that, although the flows in recent numerical computations show strong indication of finite

Figure 2.6.2: Anti-parallel vortex tubes (Kerr [Ker03])



time singularities, there is still a large possibility that no blowup would occur in them. More specifically, even if the scalings of various quantities are consistent with each other and suggest blowup, they could still be artificial if the constant coefficients of these scalings do not satisfy certain relations. Thus in this sense, new results derived in this chapter revealed new subtleties in the 3D Euler flow.

Figure 2.6.3: Active region (Kerr [Ker96])

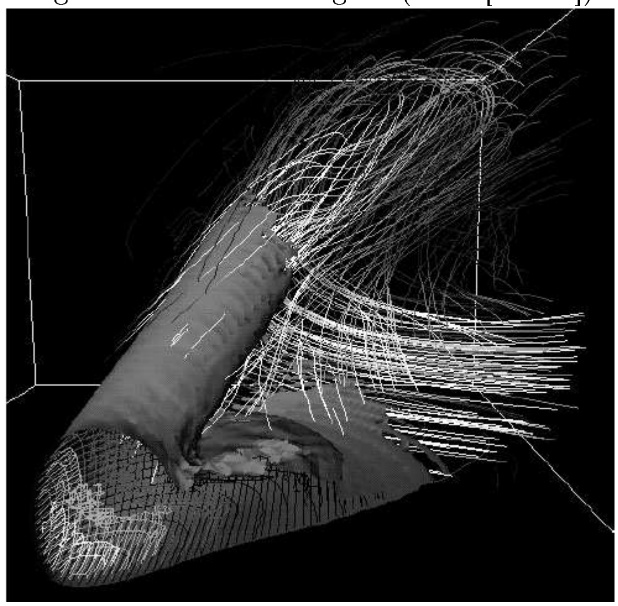
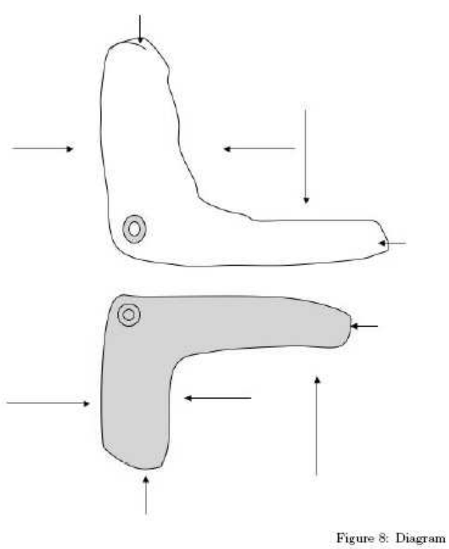


Figure 2.6.4: Diagram of the active region (Kerr [Ker03])



Chapter 3

Non-blowup of the 2D Quasi-geostrophic Equation

3.1 Introduction

The 2D-QG equation has its origin in modelling the transportation of the potential temperature θ by an incompressible flow (Pedlosky [Ped87]) on a 2D surface. The equation reads

$$\frac{D\theta}{Dt} \equiv \frac{\partial \theta}{\partial t} + u \cdot \nabla \theta = 0 \tag{3.1.1}$$

with initial value

$$\theta \mid_{t=0} = \theta_0.$$

The relation between the active scalar $\theta(x,t)$ and the velocity $u(x,t)$ is given by

$$\begin{aligned} u &= \nabla^{\perp} \psi \\ \psi &= (-\triangle)^{-1/2} (-\theta), \end{aligned}$$

where

$$\nabla^{\perp} \psi \equiv \left(-\frac{\partial \psi}{\partial x_2}, \frac{\partial \psi}{\partial x_1} \right).$$

and

$$(-\triangle)^{-1/2} \psi \equiv \int e^{2\pi i x \cdot k} \frac{1}{2\pi |k|} \hat{\psi}(k) \; dk$$

with

$$\psi(x) = \int e^{2\pi i x \cdot k} \hat{\psi}(k)\ dk.$$

A simple derivation of the above system can be found in Majda-Tabak [MT96].

The finite time singularity problem for the 2D QG equation is important both physically and mathematically. The physical importance comes from the fact that the formation of a "sharp front" would imply the "crash" of warm and cold air, which is of interest to meteorologists. The mathematical reason for 2D QG's importance is its close relation to the 3D Euler equations, as we have discussed briefly in Subsection 1.3.6 and will discuss in more detail below.

3.1.1 Analogies between the 2D QG equation and the 3D Euler equations

In their 1994 paper [CMT94], Constantin, Majda, and Tabak pointed out that the 2D QG equation has a striking mathematical and physical analogy to the 3D incompressible Euler equations, and they both exhibit similar geometric/analytic structures. For example, if we take $\nabla^{\perp}$ in both sides of (3.1.1), we obtain

$$\left(\nabla^{\perp}\theta\right)_t + (u \cdot \nabla)\left(\nabla^{\perp}\theta\right) = (\nabla u) \cdot \nabla^{\perp}\theta,$$

where

$$\nabla u = \nabla\nabla^{\perp}(-\triangle)^{-1/2}(-\theta)$$

is of the same order as $\nabla^{\perp}\theta$. Recall that the vorticity equation for the 3D Euler flow reads

$$\omega_t + (u \cdot \nabla)\,\omega = (\nabla u) \cdot \omega,$$

where ∇u is also of the same order as ω. Therefore we may learn more of the evolution of ω by studying the evolution of $\nabla^{\perp}\theta$.

In [CMT94], the analytic resemblance is carried further when the following equation governing the evolution of the magnitude of $\nabla^{\perp}\theta$ is derived:

$$\frac{D\left|\nabla^{\perp}\theta\right|}{Dt}=S\left(x,t\right)\left|\nabla^{\perp}\theta\right|,$$

where the stretching factor

$$S\left(x,t\right)=\int_{\mathbb{R}^2}\frac{\left(\hat{y}\cdot\xi\left(x,t\right)\right)\det\left(\xi\left(x+y,t\right),\xi\left(x,t\right)\right)}{\left|y\right|^2}\left|\nabla^{\perp}\theta\left(x+y,t\right)\right|\,dy$$

with $\xi\left(x,t\right)=\nabla^{\perp}\theta/\left|\nabla^{\perp}\theta\right|$ and $\hat{y}=y/\left|y\right|$. This very much resembles the stretching factor

$$\alpha\left(x,t\right)=\int_{\mathbb{R}^3}\frac{\left(\hat{y}\cdot\xi\left(x\right)\right)\det\left(\hat{y},\xi\left(x+y\right),\xi\left(x\right)\right)}{\left|y\right|^3}\left|\omega\left(x+y\right)\right|\,dy$$

for the 3D Euler equations.

Based on these analytic analogies, Constantin-Majda-Tabak [CMT94] proved the following local existence and BKM criterion for QG. These results were later generalized to Besov/Triebel-Lizorkin spaces (Chae [Cha03b]). A similar result to the one obtained by Constantin, Fefferman, and Majda [CFM96] is also obtained in [CMT94]. We list these results below.

Theorem 3.1.1. *(Constantin-Majda-Tabak [CMT94]). If the initial value $\theta_0\left(x\right)$ belongs to the Sobolev space $H^k\left(\mathbb{R}^2\right)$ for some integer $k\geq 3$, then there is a smooth solution $\theta\left(x,t\right)\in H^k\left(\mathbb{R}^2\right)$ for the 2D QG equation for each time t in a sufficiently small time interval $[0,T^*)$. Furthermore, if T^* the maximal interval of smooth existence is finite, i.e., $T^*<\infty$, then T^* is characterized by*

$$\left\|\theta\left(\cdot,t\right)\right\|_k\nearrow\infty \text{ as } t\nearrow T^*$$

and can be estimated from below by

$$T^*\gtrsim\frac{1}{\left\|\theta_0\right\|_k}.$$

Theorem 3.1.2. *(Constantin-Majda-Tabak [CMT94]). Consider the unique smooth* *solution of the 2D QG active scalar with initial data,* $\theta_0(x) \in H^k(\mathbb{R}^2)$ *with* $k \geq 3$. *The following are equivalent*

1. *The time interval* $[0, T^*)$ *is a maximal interval of* H^k *existence for the 2D QG active scalar.*

2. *The quantity* $\left\|\nabla^{\perp}\theta\right\|_{L^\infty}(t)$ *accumulates so rapidly that*

$$\int_0^{T^*} \left\|\nabla^{\perp}\theta\right\|_{L^\infty}(s)\ ds \to \infty.$$

3. *Let* $S^*(x,t) \equiv \max_{x\in\mathbb{R}^2} S(x,t)$, *then*

$$\int_0^{T^*} S^*(t)\ dt = \infty.$$

Theorem 3.1.3. *(Constantin-Majda-Tabak [CMT94]) A set* Ω_0*is smoothly directed if there exists* $\rho > 0$ *such that*

$$\sup_{q\in\Omega_0}\int_0^T |u(X(q,t),t)|^2\ dt < \infty$$

and

$$\sup_{q\in\Omega_0^*}\int_0^T \|\nabla\xi(\cdot,t)\|_{L^\infty(B_\rho(X(q,t),t))}\ dt < \infty,$$

where $B_\rho(x)$ *is the ball of radius* ρ *centered at* x *and*

$$\Omega_0^* = \{q \in \Omega_0 \mid \omega_0(q) \neq 0\}.$$

If we denote

$$\Omega_t = X(\Omega_0, t)$$

and

$$O_T(\Omega_0) = \{(x,t) \mid x \in \Omega_t,\ 0 \leq t \leq T\},$$

then under the assumption that Ω_0 *is smoothly directed, we have*

$$\sup_{O_T(\Omega_0)} |\nabla\theta\,(x,t)| < \infty,$$

i.e., there can be no blowup in $O_T\,(\Omega_0)$.

Remark 3.1.4. It is worth mentioning that Theorem 3.1.3's assumptions are weaker than the one for the 3D Euler equations in Constantin-Fefferman-Majda [CFM96] (Theorem 1.3.4), since the velocity is no longer required to be uniformly bounded.

Besides these analytic analogies, it is also observed in [CMT94] that geometrically, if we view $\nabla^{\perp}\theta$ as corresponding to ω in the 3D Euler equations, the level sets of θ are a QG analogy to the vortex lines in the 3D Euler equations in the sense that these level sets are tangent to $\nabla^{\perp}\theta$ and are carried by the flow.

3.1.2 The QG singularity problem

Due to the above analogies between the 2D QG equation and 3D Euler equations, the 2D QG singularity problem received much interest in the recent 10 years, with the hope that careful study of this problem would shed some light on the Euler singularity problem.

However, it turned out that the 2D QG singularity problem is also beyond reach of current mathematical techniques. Nonetheless, although unable to thoroughly solve the problem, a symbiosis of theory and numerics proved to be much more successful in attacking this problem than the 3D Euler singularity problem.

By Theorem 3.1.3, as long as the direction field of the level sets remains smooth enough around the maximum stretching point, it is not likely to have any finite time blowup in the 2D QG equation. This claim is supported by the numerical results in the pioneering paper [CMT94] with various initial values. On the other hand, the authors did find a possible candidate for finite time singularities, that is the "hyperbolic saddle" case, where $\theta\,(x,t)$ admits a saddle point, around which the smoothness of the level sets deteriorates rapidly. In this case, the growth of $\max|\nabla\theta|$

is fitted by $1/(8.25-t)^{1.7}$, which implies blowup according to Theorem 3.1.2.

In 1997, Ohkitani and Yamada re-did the simulations and pushed further to higher resolutions (8192×8192). They found that the same result can be fitted equally well by double exponential growth ([OY97]), indicating that no finite time blowup, at least up to the time of their computations, would occur.

Following that, Constantin-Nie-Schörghofer [CNS98, CNS99] found that the double exponential rate is in several aspects a better fit, thus implying that there would be no finite time singularity for the 2D QG equation.

Around the same time, D. Cordoba [Cor97, Cor98] proved that under some mild assumptions, the hyperbolic saddles would not cause a finite time singularity, instead the growth of $\left|\nabla^{\perp}\theta\right|$ is bounded by quadruple exponential. Later in Cordoba-Fefferman [CF02a], the so-called "semi-uniform collapse" scenario, which covers most of the scenarios considered by Constantin-Majda-Tabak [CMT94] and Cordoba [Cor97, Cor98], is considered. Double exponential rate is obtained for the approaching of the two collapsing level sets.

Before presenting our main results and the proofs, we first fix the notations.

- C or c: generic constants, whose value may change from line to line.
- ξ: the direction of the $\nabla^{\perp}\theta \equiv (-\partial_2\theta, \partial_1\theta)$. In other words, ξ is the tangent direction of the level sets.
- T^*: the alleged time when the first finite time singularity occurs.
- x, α: Cartesian coordinate variables. Thus x and α are both vectors in $\mathbb{R}^2$.
- s, β: arc length variables along one level set.
- $X(\alpha, t_1, t_2)$: the particle trajectory passing α at time τ. That is, $X(\alpha, \tau, t)$ solves

$$\begin{aligned}\frac{\partial X(\alpha,\tau,t)}{\partial t} &= u(X(\alpha,\tau,t))\\ X(\alpha,\tau,\tau) &= \alpha.\end{aligned}$$

For any set $A \subseteq \mathbb{R}^2$, we denote

$$X(A,\tau,t) \equiv \cup_{\alpha\in A} X(\alpha,\tau,t).$$

When $\tau = 0$, we use $X(\alpha,t) \equiv X(\alpha,0,t)$.

- We also use "$\sim$", "$\lesssim$", and "$\gtrsim$" as defined in the beginning of Chapter 2.

3.2 Proof of Main Results

For the 2D QG equation, thanks to the better velocity bound in Cordoba [Cor98] (presented in Appendix A.3), we are able to apply the method in the previous chapter and obtain theorems 1.4.7 and 1.4.5.

3.2.1 Preparation of the proof

The preparation follows the same lines as in Chapter 2. First, since $\nabla^\perp\theta$ is divergence free, we have the following similar result for the relation between $|\nabla\theta|$ and the direction field $\xi \equiv \nabla^\perp\theta / \left|\nabla^\perp\theta\right|$.

Lemma 3.2.1. *Let $\xi(x,t) = \frac{\nabla^\perp\theta(x,t)}{\left|\nabla^\perp\theta(x,t)\right|}$ be the direction of the vorticity vector. Assume at a fixed time $t > 0$ the vorticity $\nabla^\perp\theta(x,t)$ is C^1 in x. Then at this time t, for any x such that $\nabla\theta(x,t) \neq 0$, there holds*

$$\frac{\partial|\nabla\theta|}{\partial s}(x,t) = -\left(\left(\nabla\cdot\xi\right)|\nabla\theta|\right)(x,t), \tag{3.2.1}$$

where s is the arc length variable along the vortex line passing x. We denote this vortex line by l.

Furthermore, for any $y \in l$ such that ω does not vanish at any point in the vortex line segment between x and y, (3.2.1) then gives

$$|\nabla\theta(y,t)| = |\nabla\theta(x,t)|\, e^{\int_x^y(-\nabla\cdot\xi)\; ds}, \tag{3.2.2}$$

where the integration is along the vortex line.

Proof. The proof is similar to that of Lemma 2.1.1 and is thus omitted. □

In the QG case, we don't have at hand the formula

$$\omega\left(X\left(\alpha,t\right),t\right)=\nabla_{\alpha}X\left(\alpha,t\right)\cdot\omega_{0}\left(\alpha\right),$$

which gives us

$$\frac{\partial s}{\partial\beta}\left(X\left(\alpha,t_{0},t\right),t\right)=\frac{\left|\omega\left(X\left(\alpha,t_{0},t\right),t\right)\right|}{\left|\omega\left(\alpha,t_{0}\right)\right|}.$$

However, the QG version is easily derived by the following calculation:

$$\begin{aligned}
\nabla^{\perp}\theta\mid_{t=0} &= \nabla_{\alpha}^{\perp}\theta_{0}\left(\alpha\right)\\
&= \nabla_{\alpha}^{\perp}\theta\left(X\left(\alpha,t\right),t\right)\\
&= \left(\nabla_{\alpha}\theta\left(X\left(\alpha,t\right),t\right)\right)^{\perp}\\
&= \left(\nabla_{\alpha}X\cdot\nabla_{x}\theta\right)^{\perp}\\
&= \left(\begin{bmatrix} X_{1,1} & X_{1,2}\\ X_{2,1} & X_{2,2}\end{bmatrix}\begin{bmatrix}\partial_{1}\theta\\ \partial_{2}\theta\end{bmatrix}\right)^{\perp}\\
&= \left(\begin{bmatrix} -X_{2,1} & -X_{2,2}\\ X_{1,1} & X_{1,2}\end{bmatrix}\begin{bmatrix}\partial_{1}\theta\\ \partial_{2}\theta\end{bmatrix}\right)\\
&= \begin{bmatrix} X_{2,2} & -X_{2,1}\\ -X_{1,2} & X_{1,1}\end{bmatrix}\begin{bmatrix}-\partial_{2}\theta\\ \partial_{1}\theta\end{bmatrix}\\
&= \left(\nabla_{\alpha}X\right)^{-1}\nabla^{\perp}\theta\left(X\left(\alpha,t\right),t\right),
\end{aligned}$$

where $X_{i,j}\equiv\frac{\partial X_{i}}{\partial\alpha_{j}}$.

Now by the same argument as in Subsection 2.2.1, we have

$$\frac{\partial s}{\partial\beta}\left(X\left(\alpha,t_{0},t\right),t\right)=\frac{\left|\nabla\theta\left(X\left(\alpha,t_{0},t\right),t\right)\right|}{\left|\nabla\theta\left(\alpha,t_{0}\right)\right|}. \tag{3.2.3}$$

Further, by arguing in a similar way as in 2.2.2, we have the following lemma.

Lemma 3.2.2. *For any* t_0*, let* l_t *be a level set segment that is carried by the flow, i.e.,* $l_t = X(l_{t_0}, t_0, t)$ *for* $t \geq t_0$*. Denote its length by* $l(t)$ *and define*

$$m(t) \equiv \max_{x \in l_t} |\nabla \cdot \xi(x,t)|,$$

where $\xi = \nabla^{\perp}\theta / \left|\nabla^{\perp}\theta\right|$ *is the unit tangent direction. If we further denote* $\Omega_l(t) \equiv \max_{x \in l_t} \left|\nabla^{\perp}\theta(x,t)\right|$*, then the following inequality holds:*

$$e^{-m(t)l(t)} \frac{\Omega_l(t)}{\Omega_l(t_0)} \leq \frac{l(t)}{l(t_0)} \leq e^{m(t_0)l(t_0)} \frac{\Omega_l(t)}{\Omega_l(t_0)}. \tag{3.2.4}$$

In Constantin-Majda-Tabak [CMT94], it is derived that

$$\frac{D|\nabla\theta|}{Dt} = (\xi \cdot \nabla u \cdot \xi)|\nabla\theta|.$$

Now, similar to the argument in Subsection 2.1.2, we can easily obtain the evolution equation of s_β as

$$\frac{ds_\beta}{dt} = (u \cdot \xi)_\beta - \kappa(u \cdot \mathbf{n}) s_\beta.$$

Integrating along l_t and then from t_0 to t, we obtain the following estimate

$$\Omega_l(t) \leq e^{m(t)l(t)} \Omega_l(t_0) \left[1 + \frac{2}{l(t_0)} \int_{t_0}^{t} (1 + K(\tau) l(\tau)) U(\tau) \; d\tau\right], \tag{3.2.5}$$

where $U(\tau) \equiv \|u\|_\infty$. Note that here the key estimate makes things less sharp by replacing both U_ξ and U_n by an overestimate U. The reason is that we have the following estimate obtained by Cordoba ([Cor98]):

Lemma 3.2.3. *There exists a generic constant* $C > 0$ *such that for* $t > 0$*,*

$$\|u\|_\infty \leq C \log \Omega(t) + C. \tag{3.2.6}$$

For completeness, we sketch its proof in Appendix A.3.

Now combining (3.2.5) and (3.2.6), we have the following key estimate

$$\Omega_l(t) \leq e^{m(t)l(t)}\Omega_l(t_0)\left[1+\frac{C}{l(t_0)}\int_{t_0}^{t}(1+K(\tau)\,l(\tau))(\log\Omega(\tau)+1)\;d\tau\right], \quad (3.2.7)$$

where C is an absolute constant independent of any parameters.

3.2.2 Proof of Theorem 1.4.7

The proof relies heavily on (3.2.7), which gives a Grönwall-type estimate for the growth of magnitude of $\nabla\theta$ at two different times.

By the assumptions of Theorem 1.4.7, we have the following estimate for the growth of the maximum $\nabla\theta$:

$$\Omega(t) \leq R(t)\,\Omega(t_0)\left[1+\frac{C}{l(t_0)}\int_{t_0}^{t}(1+K(\tau)\,l(\tau))(\log\Omega(\tau)+1)\;d\tau\right], \quad (3.2.8)$$

where $R(t) \equiv e^{m(t)l(t)}/c_0$. Let $R = e^{C_0}/c_0$ with C_0 and c_0 defined in Theorem 1.4.7. Then we have $R > R(t)$ for all $t \in [0,T)$. Now (3.2.8) gives

$$\Omega(t) \leq R\Omega(t_0)\left[1+\frac{C}{l(t_0)}\int_{t_0}^{t}(1+C_0)(\log\Omega(\tau)+1)\;d\tau\right]. \quad (3.2.9)$$

Heuristically, after taking one derivative in time and setting $t_0 = t$, we would get

$$\Omega'(t) \leq C\Omega(t)\log\log\Omega(t)\log\Omega(t).$$

This would yield a triple exponential upper bound for $\Omega(t)$. However, this procedure is not mathematically correct. An estimate on the lower bound of $l(t_0)$ is needed. In the following, we will obtain this lower bound and establish this triple exponential upper bound rigorously.

3.2.2.1 Outline

We will prove Theorem 1.4.7 in four steps.

1. Divide $[T_0, T)$ into intervals $[t_k, t_{k+1})$ such that

$$\frac{\Omega(t_{k+1})}{\Omega(t_k)} = r \tag{3.2.10}$$

for some constant $r > R$. One of the reasons for this partition is to obtain an sharp lower bound estimate for $l(t_k)$ within each time interval $[t_k, t_{k+1})$ using our relationship between the relative growth of maximum $\nabla\theta$ and the relative growth of arc length stretching between two different times.

2. Use (3.2.4) to estimate a lower bound on $l(t_k)$, which in turn gives an upper bound for $\Omega(t_{k+1})$:

$$\Omega(t_{k+1}) \leq R\Omega(t_k)\left[1 + C\frac{(1+C_0)Rr}{c_L}\log\log\Omega(t_k)\int_{t_k}^{t_{k+1}}(\log\Omega(\tau)+1)\ d\tau\right]. \tag{3.2.11}$$

3. Use (3.2.11) to obtain a local estimate on the triple exponential growth estimate of $\Omega(t_{k+1})$ in $[t_k, t_{k+1})$:

$$\begin{aligned}\log\log\log\Omega(t_{k+1}) \quad &\leq \quad \log\log\log\Omega(t_k) + C\frac{R^2 r(1+C_0)}{c_L}(t_{k+1}-t_k)\\ &+\frac{\log R}{\log\Omega(t_k)\log\log\Omega(t_k)}.\end{aligned} \tag{3.2.12}$$

4. Sum up the estimates for each $[t_k, t_{k+1})$ to obtain

$$\begin{aligned}\log\log\log\Omega(t_n) \quad &\leq \quad \log\log\log\Omega(t_0) + C\frac{R^2 r(1+C_0)}{c_L}(t_n-t_0)\\ &+\sum_{i=0}^{n-1}\frac{\log R}{\log\Omega(t_i)\log\log\Omega(t_i)}.\end{aligned} \tag{3.2.13}$$

   It can be shown that the sum in the RHS of (3.2.13) can be bounded as follows:

$$\sum_{i=0}^{n-1}\frac{\log R}{\log\Omega(t_i)\log\log\Omega(t_i)} \leq \frac{\log R}{\log r}\log\log\log\Omega(t_n) + C \tag{3.2.14}$$

   for some constant C. Now substituting (3.2.14) into (3.2.13) would give the desired triple exponential estimate for $\Omega(t_n)$:

$$\log\log\log\Omega(t_n) \le \frac{\log r}{\log r - \log R}\left[C\frac{R^2 r(1+C_0)}{c_L}(t_n - t_0) + C'\right]. \tag{3.2.15}$$

3.2.2.2 Proof of the theorem

We now carry out in detail the above four steps.

1. **Partition of the time interval.**

   Let r be any constant such that $r > R$ and $t_0 \in [T_0, T)$ close enough to T so that $\Omega(t_0) > 2e$ and $\log\log(r\Omega(t_0)) \le 2\log\log\Omega(t_0)$. Define $t_0 < t_1 < \cdots < t_k < \cdots < T$ by (3.2.10), which is copied here.

   $$\frac{\Omega(t_{k+1})}{\Omega(t_k)} = r. \tag{3.2.16}$$

   If there exists $n \in \mathbb{N}$ such that we cannot find t_{n+1} using (3.2.16), or equivalently, such that for any $t \in (t_n, T^*)$,

   $$\frac{\Omega(t)}{\Omega(t_n)} < r,$$

   then $\Omega(t)$ remains bounded in $[0, T^*]$, and thus no blowup can occur. Now we assume that for all $k \in \mathbb{N}$ we can find t_k iteratively such that (3.2.16) is satisfied. Since $\lim_{k\to\infty}\Omega(t_k) = \infty$ and T^* is the smallest time such that $\int_0^{T^*}\Omega(\tau)\ d\tau = \infty$, we must have $t_k \nearrow T^*$.

2. **Estimate of the lower bound for $l(t_k)$**

   We apply (3.2.9) to the time interval $[t_k, t_{k+1}]$. For $t \in [t_k, t_{k+1}]$, choose $l_{t_{k+1}} \subset L_{t_{k+1}}$ so that $\Omega_l(t_{k+1}) = \Omega_L(t_{k+1})$ and $l(t_{k+1}) = \frac{c_L}{\log\log\Omega(t_{k+1})}$, and let l_t be such that $l_{t_{k+1}} = X(l_t, t, t_{k+1})$, i.e., l_t is the pullback of $l_{t_{k+1}}$ to time $t \in [t_k, t_{k+1}]$. By the assumptions of Theorem 1.4.7 we have $l_t \subset L_t$ for $t \in [t_k, t_{k+1}]$. Therefore,

   $$\Omega(t) \le R\Omega(t_k)\left[1 + C\frac{(1+C_0)}{l(t_k)}\int_{t_k}^{t}(\log\Omega(\tau) + 1)\ d\tau\right].$$

Next we obtain a lower bound for $l(t_k)$. Using (3.2.2), we have

$$\frac{l(t_{k+1})}{l(t_k)} \le R\frac{\Omega(t_{k+1})}{\Omega(t_k)} = Rr,$$

which gives

$$\frac{1}{l(t_k)} \le \frac{Rr}{l(t_{k+1})} = \frac{Rr \log\log\Omega(t_{k+1})}{c_L} \le \frac{2Rr\log\log\Omega(t_k)}{c_L}$$

since $\Omega(t_k) > \Omega(t_0)$ is large enough by our choice of t_0. Thus, we obtain the upper bound (3.2.11) for $\Omega(t)$:

$$\Omega(t) \le R\Omega(t_k)\left[1 + C\frac{(1+C_0)Rr}{c_L}\log\log\Omega(t_k)\int_{t_k}^{t}(\log\Omega(\tau)+1)\ d\tau\right] \tag{3.2.17}$$

for all $t \in [t_1, t_2]$, where C is some absolute constant independent of any parameters.

3. **Local triple exponential growth estimate**

Define $\tilde{\Omega}(t)$, $t \in [t_k, t_{k+1}]$ as

$$\tilde{\Omega}(t) = R\Omega(t_k)\left[1 + C\frac{(1+C_0)Rr}{c_L}\log\log\Omega(t_k)\int_{t_k}^{t}\left(\log\tilde{\Omega}(\tau)+1\right)\ d\tau\right]. \tag{3.2.18}$$

First we prove that $\Omega(t) < \tilde{\Omega}(t)$, for $t \in [t_k, t_{k+1}]$. When $t = t_k$ we have $\tilde{\Omega}(t_k) = R\Omega(t_k) > \Omega(t_k)$. Now suppose there exists $\delta \in (0, t_{k+1} - t_k]$ so that $\tilde{\Omega}(t) > \Omega(t)$ when $t \in [t_k, t_k + \delta)$, and $\tilde{\Omega}(t_k+\delta) = \Omega(t_k+\delta)$. Using (3.2.17) and substituting $\tilde{\Omega}(t_k+\delta) = \Omega(t_k+\delta)$ into (3.2.18), we obtain

$$\int_{t_k}^{t_k+\delta}\log\tilde{\Omega}(\tau)\ d\tau \le \int_{t_k}^{t_k+\delta}\log\Omega(\tau)\ d\tau,$$

which contradicts the assumption that $\tilde{\Omega}(t) > \Omega(t)$ when $t \in [t_k, t_k+\delta)$! Therefore, such δ cannot exist, which is equivalent to $\Omega(t) < \tilde{\Omega}(t)$ for all $t \in [t_k, t_{k+1}]$.

Next we differentiate (3.2.18) with respect to t and get

$$\tilde{\Omega}'(t) = C\frac{R^2 r(1+C_0)}{c_L}\Omega(t_k)\log\log\Omega(t_k)\left(\log\tilde{\Omega}(t)+1\right).$$

By using $\tilde{\Omega}(t) > \Omega(t)$, we easily obtain

$$\begin{aligned}\left(\log\log\log\tilde{\Omega}(t)\right)' &= C\frac{R^2 r(1+C_0)}{c_L}\frac{\Omega(t_k)\log\log\Omega(t_k)\left(\log\tilde{\Omega}(t)+1\right)}{\tilde{\Omega}(t)\log\log\tilde{\Omega}(t)\log\tilde{\Omega}(t)} \\ &\leq C\frac{R^2 r(1+C_0)}{c_L}. \end{aligned} \tag{3.2.19}$$

Now integrating (3.2.19) over t, we obtain the triple exponential growth estimate.

Finally, noticing that $\tilde{\Omega}(t_k) = R\Omega(t_k)$ and $\log\log x$ is a concave function for $x > e^{-1}$, we get

$$\begin{aligned}\log\log\log\tilde{\Omega}(t_k) &= \log\log(\log R + \log\Omega(t_k)) \\ &\leq \log\log\log\Omega(t_k) + (\log\log)'(\log\Omega(t_k))\log R \\ &= \log\log\log\Omega(t_k) + \frac{\log R}{\log\Omega(t_k)\log\log\Omega(t_k)}. \end{aligned} \tag{3.2.20}$$

Combining (3.2.20) with the triple exponential estimate (3.2.19) for $\tilde{\Omega}(t)$ and using $\Omega(t) < \tilde{\Omega}(t)$ for $t \in [t_k, t_{k+1}]$, we obtain (3.2.12) by taking $t = t_{k+1}$.

4. **Global estimate**

In the last step we obtain

$$\begin{aligned}\log\log\log\Omega(t_{k+1}) \leq\; &\log\log\log\Omega(t_k) + C\frac{R^2 r(1+C_0)}{c_L}(t_{k+1}-t_k) \\ &+\frac{\log R}{\log\Omega(t_k)\log\log\Omega(t_k)}.\end{aligned}$$

Summing over $k=0$ to $n-1$, we obtain

$$\begin{aligned}\log\log\log\Omega\left(t_n\right) &\leq \log\log\log\Omega\left(t_0\right)+C\frac{R^2 r\left(1+C_0\right)}{c_L}\left(t_n-t_0\right)\\ &\quad+\sum_{k=0}^{n-1}\frac{\log R}{\log\Omega\left(t_k\right)\log\log\Omega\left(t_k\right)}. \end{aligned}\tag{3.2.21}$$

Next we will estimate the sum on the RHS of (3.2.21) and prove that

$$\sum_{k=0}^{n-1}\frac{\log R}{\log\Omega\left(t_k\right)\log\log\Omega\left(t_k\right)}\leq\frac{\log R}{\log r}\log\log\log\Omega\left(t_n\right)+C \tag{3.2.22}$$

for some constant $C>0$, $n\geq 2$.

Since $\Omega\left(t_n\right)=r^n\Omega\left(t_0\right)$, we have

$$\begin{aligned}\sum_{k=0}^{n-1}\frac{\log R}{\log\Omega\left(t_k\right)\log\log\Omega\left(t_k\right)} &= \sum_{k=0}^{n-1}\frac{\log R}{\log\left(r^k\Omega_0\right)\log\log\left(r^k\Omega_0\right)}\\ &= \sum_{k=0}^{n-1}\frac{\log R}{\left(k\log r+\log\Omega_0\right)\log\left(k\log r+\log\Omega_0\right)}\\ &= \frac{\log R}{\log r}\sum_{k=0}^{n-1}\frac{\log r}{\left(k\log r+\log\Omega_0\right)\log\left(k\log r+\log\Omega_0\right)}.\end{aligned}$$

Note that the sum

$$\sum_{k=0}^{n-1}\frac{\log r}{\left(k\log r+\log\Omega_0\right)\log\left(k\log r+\log\Omega_0\right)}$$

is in the form of a Riemann sum of the function $\left(x\log x\right)^{-1}$. Since $\left(x\log x\right)^{-1}$ is decreasing for $x>e^{-1}$, this sum can be bounded by

$$\begin{aligned}\int_{\log\Omega_0}^{n\log r+\log\Omega_0}\frac{1}{x\log x}\,dx &= \log\log\log\left(r^n\Omega_0\right)-\log\log\log\Omega_0\\ &= \log\log\log\Omega\left(t_n\right)-\log\log\log\Omega_0\end{aligned}$$

since $\Omega\left(t_n\right)=r^n\Omega_0$. This proves (3.2.22).

Now using the fact that $r > R$, we get

$$\log\log\log\Omega\left(t_n\right) \leq \frac{\log r}{\log r - \log R}\left[C\frac{R^2 r\left(1+C_0\right)}{c_L}\left(t_n - t_0\right) + C'\right].$$

Therefore, the growth of $\Omega\left(t\right)$ is bounded by the triple exponential for $t < T$, implying that no blowup can occur at time T. This completes the proof of Theorem 1.4.7.

3.2.3 Proof of Corollary 1.4.9

The proof is similar to that of Theorem 1.4.7. The only difference is that the estimate in Step 2 is replaced by

$$\Omega\left(t_{k+1}\right) \leq R\Omega\left(t_k\right)\left[1 + C\frac{\left(1+C_0\right)Rr}{c_L}\int_{t_k}^{t_{k+1}}\left(\log\Omega\left(\tau\right)+1\right)\,d\tau\right].$$

Thus the $\log\log\log\Omega\left(t\right)$ in Step 3 is replaced by $\log\log\Omega\left(t\right)$. Since for $x > e^{-1}$, $\log x$ is also concave, all the steps can be carried out in the same way. We finally obtain a double exponential growth estimate

$$\log\log\Omega\left(t_n\right) \leq \frac{\log r}{\log r - \log R}\left[C\frac{R^2 r\left(1+C_0\right)}{c_L}\left(t_n - t_0\right) + C'\right].$$

This completes the proof of Corollary 1.4.9.

3.2.4 Proof of Theorem 1.4.5

The proof of Theorem 1.4.5 follows exactly the same line as the proof of Theorem 1.4.3 in Chapter 2. To see this, one only need to notice that by assumption $\Omega\left(t\right) \leq \left(T^* - t\right)^{-B}$, we can easily obtain

$$U\left(t\right) \lesssim \left|\log\left(T^* - t\right)\right| \lesssim \left(T^* - t\right)^{B'}$$

for any $B' > 0$. In particular, we can choose B' such that $B' + A < 1$.

3.3 Application to a Hyperbolic Saddle Scenario

In this section we discuss the regularity assumptions (H1), (H1'), and (H2) in Theorem 1.4.7 and Corollary 1.4.9 in the context of the simple hyperbolic saddle scenario similar to that given in Cordoba [Cor97, Cor98]. By assuming that the level sets of θ take the form of a simple hyperbolic saddle, Cordoba showed in [Cor98] that the angle of the saddle cannot close faster than double exponentially, and then proved that the growth rate of $|\nabla\theta|$ is bounded by a quadruple exponential. Therefore, the possibility of a finite time singularity in the simple hyperbolic saddle case is excluded. In the following, we will study the implications of the simple hyperbolic saddle assumption, and show that in this case, Corollary 1.4.9 applies and can bound the growth of $|\nabla\theta|$ by a double exponential, which is consistent with numerical observations ([OY97], [CNS98, CNS99]).

First we give our definition of a simple hyperbolic saddle, which is slightly different from the one given in Cordoba [Cor97]. The assumptions are based on inspection of the computational result in Constantin-Majda-Tabak [CMT94]. Figure 3.3.1 is the plot of their level sets at various times.

From Figure 3.3.1 one can easily see that a "hyperbolic saddle" seems to be collapsing in the middle of the domain. Similar to Cordoba [Cor97], we make the following definition.

Definition 3.3.1. By a simple hyperbolic saddle ansatz up to time T^*, we mean there exists an $O(1)$ region U of the origin so that

$$\theta(x,t) = \Theta(\rho(x,t),t) \text{ in } U$$

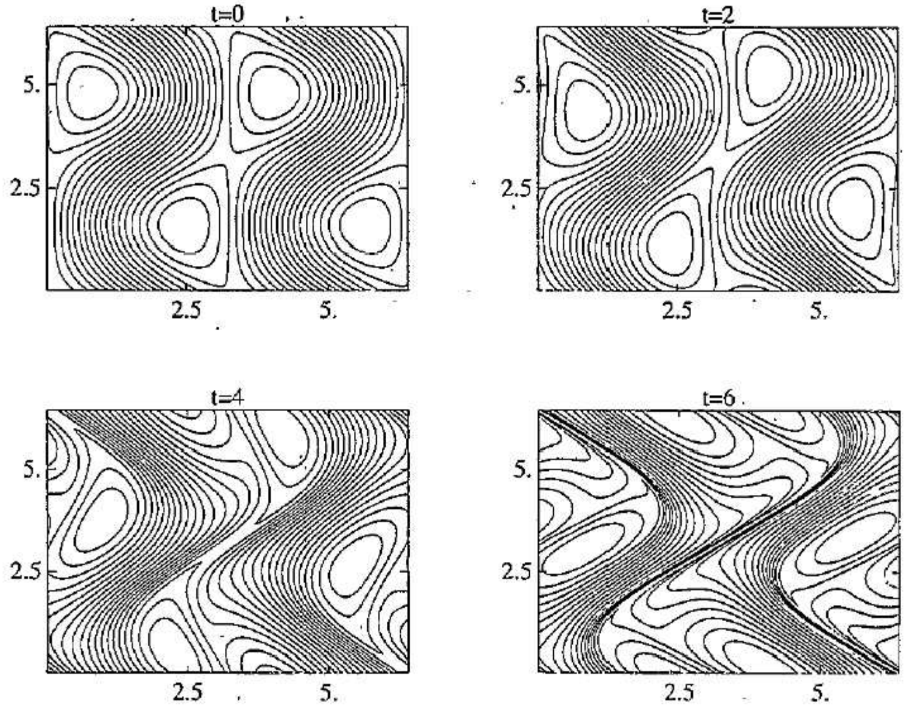
with

$$\rho(x,t) = \delta(t)^2 y_1^2 - y_2^2,$$

where

$$y_i = F_i(x,t), \quad i = 1, 2.$$

Figure 3.3.1: Evolution of θ (Constantin-Majda-Tabak [CMT94])



We further assume that $\delta(t) \in C^1[0,T^*)$, $F_i \in C^2\left(\bar{U} \times [0,T^*]\right)$ for $i = 1, 2$, and

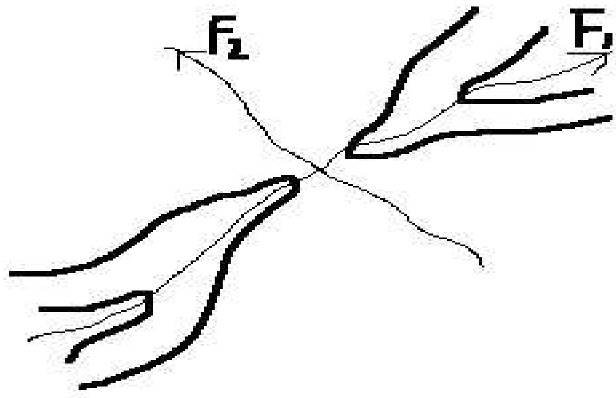
$$\det\left(\frac{\partial F_i}{\partial x_j}\right) \geq c_0 > 0$$

for all $x \in U$ and $t \in [0,T^*]$. This definition is illustrated in Figure 3.3.2.

Remark 3.3.2. The definition here is slightly different from that in Cordoba [Cor97] or [Cor98], where $\rho(x,t) = y_1 y_2 - \cot\alpha(t) \cdot y_2^2$ and $\rho(x,t) = (\alpha(t)\, y_1 - y_2)(\beta(t)\, y_1 + y_2)$ respectively. If we further assume that the small parameters α, β are in $C^2[0,T^*]$, then by adding a rotation to the mapping $(x_1,x_2) \mapsto (y_1,y_2)$, we see that all three definitions are actually equivalent. Since it is not likely that the angle of the hyperbolic saddle would close in a wiggly manner, in the following, we will refer to our hyperbolic saddle ansatz as simply "the hyperbolic saddle case."

We denote the maximum $|\nabla\theta|$ location by $P(t)$ and make one further assumption.

Figure 3.3.2: Simple hyperbolic saddle



- (Ha). There is $d_0 > 0$ such that

$$dist\left(P\left(t\right),\partial U\right)\geq d_0$$

for all $t \in [0, T^*]$.

Remark 3.3.3. In other words, we assume that the location of the maximum $|\nabla\theta|$ is strictly inside U, at least for t sufficiently close to T^*, since we can always redefine the starting time. This is reasonable since otherwise the blowup, if any, would happen at the boundary ∂U and the hyperbolic saddle assumption, which specifies the behavior of the level sets inside U, would be irrelevant.

Finally, we denote by $U^+(t)$ the region between the closing separatrix and $U^-(t)$, the region above and below them. More specifically,

$$\begin{aligned} U^+(t) &\equiv \left\{x \in \bar{U} \mid \rho(x,t) > 0\right\} \text{ and} \\ U^-(t) &\equiv \left\{x \in \bar{U} \mid \rho(x,t) < 0\right\}. \end{aligned}$$

Now we are ready to present our main result of this section.

Proposition 3.3.4. *For any $t \in [0, T^*]$, there is an $O(1)$ segment of the level set passing through $P(t)$ such that $\nabla\xi$ is bounded along this level set segment.*

Remark 3.3.5. Before proving it, we note that when Proposition 3.3.4 holds, Corollary 1.4.9 would give us the desired double exponential growth bound.

Proof. First note that θ and ρ share the same level sets. Thus, for any c, $|\nabla\theta|$ is proportional to $|\nabla\rho|$ along the level set $\rho \equiv c$. Therefore, the point $P(t)$ must also maximize $|\nabla\rho|$ along the level set passing through it.

Our strategy is the following. First we compute $|\nabla\rho|$ explicitly along this level set. Then we show that the maximum of $|\nabla\rho|$ is of the same order as δ by estimating the maximum of an auxiliary function that is smaller than $|\nabla\rho|$ along this level set. It follows from this and the explicit formula of $|\nabla\rho|$ that the maximum must be attained at some point that is $O(1)$ away from the tip of the level set. Finally, by explicitly computing the magnitude of $\nabla\xi$, we show that for points that are $O(1)$ away from the tip of the level set, $|\nabla\xi|$ is bounded, and thus ends the proof.

First we compute $|\nabla\rho|$. We have

$$\begin{aligned} |\nabla\rho| &= 2\left|\delta^2 F_1 \nabla F_1 - F_2 \nabla F_2\right| \\ &= 2\sqrt{\delta^4 L \cdot F_1^2 - 2\delta^2 M \cdot F_1 F_2 + N \cdot F_2^2}, \end{aligned}$$

where

$$L = |\nabla F_1|^2, \quad M = \nabla F_1 \cdot \nabla F_2, \text{ and } N = |\nabla F_2|^2$$

are functions of (x_1, x_2), or equivalently, (F_1, F_2).

Recall that in Definition 3.3.1 we have assumed

$$\left|\det\left(\frac{\partial F_i}{\partial x_j}\right)\right| \geq c_0 > 0,$$

which is equivalent to

$$\left|\nabla^{\perp} F_1 \cdot \nabla F_2\right| \geq c_0 > 0,$$

and in turn yields

$$|\nabla F_1| \cdot |\nabla F_2| \geq c_0 > 0.$$

Since by assumption $F_{1,2} \in C^2\left(\bar{U} \times [0, T^*]\right)$, $|\nabla F_i| \leq C_0 < \infty$ for $i = 1, 2$.

Combining with $|\nabla F_1| \cdot |\nabla F_2| \geq c_0 > 0$, we see that L and N are bounded from below by some positive constant. Therefore, there are constants $b, B > 0$, such that

$$b \leq L \leq B, \quad b \leq N \leq B, \text{ and } -B \leq M \leq B. \tag{3.3.1}$$

Now we have

$$\begin{aligned} \nabla \xi &= \nabla \left(\frac{\nabla^{\perp}\theta}{|\nabla^{\perp}\theta|} \right) = \nabla \left(\frac{\nabla^{\perp}\rho}{|\nabla^{\perp}\rho|} \right) \\ &= \nabla \left(\frac{\delta^2 F_1 \nabla^{\perp} F_1 - F_2 \nabla^{\perp} F_2}{|\delta^2 F_1 \nabla^{\perp} F_1 - F_2 \nabla^{\perp} F_2|} \right). \end{aligned} \tag{3.3.2}$$

By (3.3.1), both $|\nabla^{\perp} F_1|$ and $|\nabla^{\perp} F_2|$ are bounded from above and below. Thus if along the level set passing $P(t)$ of length $O(1)$ we can show that $F_2 \gg \delta^2$ (see Lemma 3.3.6 below), then the δ^2 terms would be negligible as $\delta \to 0$. As a result, we would have

$$\nabla \xi \sim \nabla \left(\frac{F_2 \nabla^{\perp} F_2}{|F_2 \nabla^{\perp} F_2|} \right) = \nabla \left(\frac{\nabla^{\perp} F_2}{|\nabla^{\perp} F_2|} \right),$$

which is bounded since $F_2 \in C^2 \left(\bar{U} \times [0, T^*] \right)$ and $|\nabla^{\perp} F_2|$ is bounded from below away from zero. Furthermore, this bound does not depend on δ. Thus ends the proof. □

It remains to prove the following technical lemma.

Lemma 3.3.6. *There exists in U a level set segment passing $P(t)$ of length $O(1)$, such that $F_2 \gg \delta^2$ along this level set segment as $\delta \to 0$.*

Proof. First we prove the case that $P(t) \in U^+(t)$. We present the proof in several steps. In Step 1 we compute explicitly the magnitude of $\nabla \rho$ along the level set passing $P(t)$. In Step 2 we construct a function G such that $|\nabla \rho| \geq 2\delta\sqrt{G}$, and show that the maximum of G along this level set is bounded away from 0, which implies that the maximum of $|\nabla \rho|$ is of order δ. Finally, in Step 3 we estimate $|\nabla \rho|$ using the explicit formula obtained in Step 1, and show that $|\nabla \rho|$ can be of order δ if and only if it is evaluated at points $O(1)$ away from the tip of the hyperbolic level set. This gives $F_2 \gg \delta^2$ and ends the proof.

- **Step 1.** Evaluation of $|\nabla \rho|$.

  We define $c^u(t)$ to be the upper bound of c such that the level set

$$\delta^2 y_1^2 - y_2^2 = c^2 \delta^2$$

  intersects U. The right-hand side cannot be larger than $c^2\delta^2$ for some constant c because (y_1, y_2) has to stay inside the bounded region U as $\delta \to 0$. In the y-coordinate, the point $(c^u(t), 0)$ lies on ∂U, thus it is easy to see that there is a positive constant C^u such that $c^u(t) \leq C^u$ for all $t \in [0, T^*]$ since the region U is $O(1)$. Also, we choose $c_p(t)$ such that the level set $\delta^2 y_1^2 - y_2^2 = c_p(t)^2 \delta^2$ passes through $P(t)$. By our assumption (Ha) and the regularity of the mapping (F_1, F_2), there exists some positive constant c_b such that $c^u(t) > c_p(t) + c_b$.

  Now we consider the level set $\rho = c_p(t)^2 \delta^2$ on which $P(t)$ is located. For simplicity, we will write c_p instead of $c_p(t)$ in the following argument.

  Substituting $\rho = \delta^2 y_1^2 - y_2^2$ into $\rho = c_p^2 \delta^2$, we have

$$\delta^2 F_1^2 - F_2^2 = c_p^2 \delta^2,$$

  which gives

$$F_2 = \pm\delta\sqrt{F_1^2 - c_p^2}.$$

  Without loss of generality, we just consider the positive branch. Thus, we have

$$|\nabla \rho| = 2\delta\sqrt{\delta^2 L F_1^2 - 2\delta M F_1 \sqrt{F_1^2 - c_p^2} + N\left(F_1^2 - c_p^2\right)}.$$

  This ends Step 1.

- **Step 2.** Estimation of $|\nabla \rho|$ via a lower bound function G.

Let $h \equiv \sqrt{F_1^2 - c_p^2}$, and define $G(h)$ as follows:

$$G(h) \equiv \delta^2 b \left(c_p^2 + h^2\right) - 2\delta B \sqrt{c_p^2 + h^2} \cdot h + bh^2.$$

We have

$$\begin{aligned} |\nabla \rho| &= 2\delta \sqrt{\delta^2 L \left(c_p^2 + h^2\right) - 2\delta h M \sqrt{c_p^2 + h^2} + N h^2} \\ &\geq 2\delta \sqrt{\delta^2 b \left(c_p^2 + h^2\right) - 2\delta B \sqrt{c_p^2 + h^2} \cdot h + bh^2} \\ &= 2\delta \sqrt{G(h)}. \end{aligned}$$

Here $h \equiv \sqrt{F_1^2 - c_p^2}$ is defined in $[0, h^u(t)]$ such that the level set segment between $h = 0$ and $h = h^u(t)$ is inside $U^+(t)$. Note that since $c^u(t) > c_p + c_b$ for some constant c_b, $h^u \sim \sqrt{(c^u)^2 - c_p^2}$ is bounded from below, i.e., there is $h_b > 0$, independent of time, so that $h^u(t) \geq h_b$ for $t \in [0, T^*]$.

To estimate the maximum of $G(h)$, we compute $G'(h)$.

$$\begin{aligned} G'(h) &= 2b\left(\delta^2 + 1\right) h - 2\delta B \left(\sqrt{c_p^2 + h^2} + \frac{h^2}{\sqrt{c_p^2 + h^2}} \right) \\ &\geq 2bh - 2\delta B \left(\sqrt{(C^u)^2 + h^2} + \frac{h^2}{\sqrt{(C^u)^2 + h^2}} \right), \end{aligned} \tag{3.3.3}$$

where C^u is the upper bound of all $c^u(t)$. From (3.3.3) we see that $G'(h) \sim h$ with $\delta = o(1)$ and $h \gg \delta$. Since G is always defined at least in an $O(1)$ interval $[0, h_b]$, we know that when δ is small enough, $G(h)$ increases monotonically when $h \sim O(1)$ and takes its maximum at the boundary. Thus, we get the following lower bound on the maximum of $|\nabla \rho|$:

$$\max |\nabla \rho| \geq 2\delta \sqrt{G(h^u)} \geq c\delta \tag{3.3.4}$$

for some constant c, which only depends on the bounds b, B, and d_0.

- **Step 3.** Final estimate on the lower bound of $|\nabla\rho|$.

On the other hand, for $h \in [0, h^u(t)]$, we have

$$\begin{aligned}|\nabla\rho|(h) &= 2\delta\sqrt{\delta^2 L\left(c_p^2+h^2\right) - 2\delta M h\sqrt{c_p^2+h^2} + Nh^2}\\ &\leq 2\delta\sqrt{B}\sqrt{\left(1+\delta^2\right)h^2 + 2h\sqrt{c_p^2+h^2} + \delta^2 c_p^2}\\ &\leq C\delta^2 + C'\delta^{3/2} + C''\delta\sqrt{1+\delta^2}h, \end{aligned} \tag{3.3.5}$$

where in the last inequality we have used $\sqrt{a^2+b^2+c^2} \leq a+b+c$ for positive numbers a, b, and c.

Let $h_p(t) = \sqrt{F_1^2(P(t)) - c_p^2}$, i.e., the h-value corresponds to $P(t)$. Combining (3.3.4) and (3.3.5), we get

$$\delta\sqrt{1+\delta^2}h_p \geq c\delta - C'\delta^2 - C''\delta^{3/2},$$

which gives $h_p \geq c$ for some constant $c > 0$, provided that δ is small enough. This means that at the point $P(t)$ we have

$$F_2^2 = \delta^2 F_1^2 - \delta^2 c_p^2 = \delta^2 h_p^2 \gtrsim \delta^2,$$

which gives

$$|F_2| \gtrsim \delta.$$

Since h_p is bounded from below away from 0, for $h \in [h_p/2, h_p]$, we have

$$|\nabla\rho|(h) \geq 2\delta\sqrt{G(h)} \geq 2\delta\sqrt{G(h_p/2)} \gtrsim \delta.$$

Applying the estimate above once again for the points corresponding to $h \in [h_p/2, h_p]$, we still have $F_2 \gtrsim \delta$ at those points. Thus we complete the proof for Lemma 3.3.6 when $P(t) \in U^+(t)$.

Next we prove the case when $P(t) \in U^-(t)$. In this case, we are considering

$$\delta^2 F_1^2 - F_2^2 = -\delta^2 c_p^2,$$

which leads to

$$F_2 = \pm\delta\sqrt{F_1^2 + c_p^2}$$

and

$$|\nabla\rho| = 2\delta\sqrt{\delta^2 L F_1^2 - 2\delta M F_1\sqrt{F_1^2 + c_p^2} + N\left(F_1^2 + c_p^2\right)}.$$

Now take $h \equiv F_1$ and

$$G(h) = \delta^2 L h^2 - 2\delta M h\sqrt{h^2 + c_p^2} + N\left(h^2 + c_p^2\right).$$

It is easy to see that $G(h)$ takes its maximum at the boundary ∂U, which corresponds to $h = O(1)$, and leads to $G(h) \gtrsim \delta^2$. As a consequence, we have

$$\max|\nabla\rho| \gtrsim \delta.$$

This implies that

$$h^2 + c_p^2 \gtrsim 1,$$

and therefore

$$|F_2| = \delta\sqrt{h^2 + c_p^2} \gtrsim \delta.$$

Thus ends the proof for Lemma 3.3.6.

□

3.4 A Generalization of Theorem 1.4.7

In Theorem 1.4.7 and Corollary 1.4.9, we assume that the maximum gradient on one level set segment is always comparable to the global maximum of $|\nabla\theta|$, and that the maximum gradient, $\Omega(t)$, is monotonically increasing, at least for later times. In practice, these two conditions may not be satisfied. In this section, we show that these two conditions can be relaxed if the maximum gradient does not change too frequently from one level set to another. More precisely, we have the following theorem.

Theorem 3.4.1. *Assume that there are* $T_0 < T^*$ *and* $t_k, t'_k \nearrow T^*$ *such that* $T_0 < t_1 < t'_1 < \cdots < t_k < t'_k < \cdots < T^*$, $\Omega(t'_k) = \Omega(t_{k+1})$, *and* $\Omega(t)$ *is monotonically increasing in each* $[t_k, t'_k]$. *Further assume that there is a family of level set segments* L_t *such that* $X(L_{t'}, t', t'') \supseteq L_{t''}$ *if* t' *and* t'' *are in the same time interval* $[t_k, t'_k]$ *for some* k. *Also assume that* $\Omega_L(t) > c_0\Omega(t)$ *for some constant* $c_0 > 0$. *If there are constants* $c_L, C_0 > 0$ *such that*

1. $L(t) \geq \frac{c_L}{\log\log\Omega(t)}$,

2. $M(t)L(t), K(t)L(t) \leq C_0$,

3. $\liminf_{k\to\infty} \frac{\Omega(t'_k)}{\Omega(t_k)} > R \equiv \frac{e^{C_0}}{c_0}$,

then there will be no finite time singularity in the 2D QG equation up to time T^*. *Furthermore, we have the triple exponential growth estimate*

$$\log\log\log\Omega(t) \leq Ct + C'$$

for some positive constants C *and* C'*independent of time.*

Before proving Theorem 3.4.1, we give some geometric interpretation of the assumptions in the theorem.

First, instead of making the assumptions in Theorem 1.4.7 for all t after some T_0, we only make these assumptions for each time interval $[t_k, t'_k]$. Intuitively, the t'_ks are "hopping" times. For example, there may be one level set l_t on which the maximum gradient is comparable to $\Omega(t)$ for $t \in [t_1, t'_1]$ but not after t'_1. Thus, at t'_1, the maximum gradient "hops" to another level set l'_t, and stays there till t'_2. If $\Omega(t)$ keeps increasing in $[t_1, t'_2]$, we have $t_1 < t'_1 = t_2 < t'_2$. On the other hand, if $\Omega(t)$ is decreasing in $[t'_1, \tilde{t}]$ for some $\tilde{t} < t'_2$, but increasing after $\tilde{t}$, we can take t_2 to be the last time before t'_2 such that $\Omega(t) = \Omega(t'_1)$. In this case, we have $t_1 < t'_1 < \tilde{t} < t_2 < t'_2$.

Secondly, since the conditions in Theorem 1.4.7 are satisfied in each interval $[t_k, t_k']$, we can apply Theorem 1.4.7 to each interval and obtain a triple exponential growth bound. But to put these bounds together and exclude blowup, we need the technical assumption 3. Basically, Assumption 3 guarantees that the large gradient is not "hopping" too frequently among level sets. In other words, each level set whose maximum gradient is comparable to $\Omega(t)$ will experience enough stretching before the stretching along this level set slows down.

Remark 3.4.2. In practice, we may expect that $L(t) \geq c_L$ holds for those L_t with $c_0 = 1$, as can be seen in Section 3.3. Then, by taking a sub-segment of L_t whose length tends to 0 as $t \nearrow T^*$, we can see that as long as

$$\liminf_{k\to\infty} \frac{\Omega(t_k')}{\Omega(t_k)} > 1,$$

Theorem 3.4.1 applies and no blowup could occur.

Now we present the proof of Theorem 3.4.1.

Proof. Since the $X(L_{t'}, t', t'') \supseteq L_{t''}$ and the monotonicity of $\Omega(t)$ holds in each $[t_k, t_k']$, we can choose $r = r_k > R$ for each time interval and carry out the same steps as in the proof of Theorem 1.4.7.

First, we have, from (H3),

$$\liminf_{k\to\infty} \frac{\Omega(t_k')}{\Omega(t_k)} > R.$$

Thus there exists $R_1 > R$ and $K_0 \in \mathbb{N}$ such that

$$\frac{\Omega(t_k')}{\Omega(t_k)} > R_1$$

for all $k \geq K_0$. Fix this R_1. Since we can always choose t_0 to be this particular t_{K_0}, in the following we will assume that the above holds for all $k \geq 0$.

Now for any fixed k, we choose r_k in each time interval $[t_k, t'_k]$ such that

$$r_k^{m_k} = \frac{\Omega(t'_k)}{\Omega(t_k)},$$

where

$$m_k = \max\left\{ m \in \mathbb{N} \mid m \leq \log_{R_1} \frac{\Omega(t'_k)}{\Omega(t_k)} \right\}.$$

It is easy to see that we have $R_1 \leq r_k < R_1^2$. Denote by Ω_k the common value of $\Omega(t_k)$ and $\Omega\left(t'_{k-1}\right)$. Then, by the same argument as in the proof of Theorem 1.4.7 we have

$$\begin{aligned}
&\log\log\log \Omega_{k+1} - \log\log\log \Omega_k \\
\leq\ & C\frac{R^2 r_k (1 + C_0)}{c_L}(t'_k - t_k) + \sum_{j=0}^{m_k - 1} \frac{\log R}{\log r_k^j \Omega_k \log\log r_k^j \Omega_k} \\
\leq\ & C\frac{R_1^4 (1 + C_0)}{c_L}(t_{k+1} - t_k) + \sum_{j=0}^{m_k - 1} \frac{\log R}{\log R_1^j \Omega_k \log\log R_1^j \Omega_k}.
\end{aligned}$$

Now summing up with respect to k, we get

$$\begin{aligned}
&\log\log\log \Omega_{n+1} - \log\log\log \Omega_1 \\
\leq\ & C\frac{R_1^4 (1 + C_0)}{c_L}(t_{n+1} - t_1) + \sum_{j=0}^{m_1 - 1} \frac{\log R}{\log R_1^j \Omega_1 \log\log \Omega_1} \\
&+ \cdots + \\
&+ \sum_{j=0}^{m_n - 1} \frac{\log R}{\log R_1^j \Omega_n \log\log R_1^j \Omega_n} \\
\leq\ & C\frac{R_1^4 (1 + C_0)}{c_L}(t_{n+1} - t_1) + \sum_{j=0}^{M_n - 1} \frac{\log R}{\log R_1^j \Omega_1 \log\log R_1^j \Omega_1},
\end{aligned}$$

where $M_n = \sum_1^n m_i$ and the last inequality follows from

$$\frac{\Omega_{k+1}}{\Omega_1} = \prod_{j=1}^{k} r_j^{m_j} \geq R_1^{M_k}.$$

Finally, by using the same estimate as the one used in obtaining (3.2.22) in the proof of Theorem 1.4.7, we can bound the sum in the above estimate by

$$\frac{\log R}{\log R_1} \log\log\log R_1^{M_n} \Omega_1 \leq \frac{\log R}{\log R_1} \log\log\log \Omega_{n+1}.$$

Since $R_1 > R$, we obtain a triple exponential upper bound for $\Omega(t_{n+1})$. Thus ends the proof. □

Remark 3.4.3. Similarly to Corollary 1.4.9, when (H1'), (H2), and (H3) are satisfied, we can obtain a double exponential bound for $\Omega(t)$. Thus, Theorem 3.4.1 can also be applied to the hyperbolic saddle scenario.

Remark 3.4.4. The assumption (H3) is necessary to our proof. Without it, the key estimate (3.2.7)

$$\Omega_l(t) \leq e^{m(t)l(t)} \Omega_l(t_0) \left[1 + \frac{C}{l(t_0)} \int_{t_0}^{t} (1 + K(\tau) l(\tau)) (\log \Omega(\tau) + 1) \; d\tau \right] \quad (3.4.1)$$

will be trivially satisfied in the time interval $[t_k, t_k']$, and we will not be able to obtain useful estimates on the lower bound for $l(t_k)$ or the upper bound for $\Omega(t_k')$ from (3.4.1).

3.5 Conclusion

In this chapter, we apply the new framework developed in Chapter 2 to the 2D QG equation, which is the most sophisticated low-dimensional model equation. By taking advantage of the better velocity estimate, we are able to exclude finite time QG singularity with assumptions that are weaker than those in Chapter 2. Furthermore, we show that these assumptions tend to hold in practical cases. And the application of our new results to these cases, which has been proven to remain smooth, offers new understanding of the underlying reason why finite time singularities would not form in these cases.

Chapter 4

Discussion and Future Work

In the final chapter, we first discuss the possibility of improving the results obtained in chapters 2 and 3, as well as the applicability of the new framework to other PDE systems, in Section 4.1. Then, in Section 4.2, we discuss in more detail research plans for attacking the singularity problems for various PDEs related to incompressible fluids.

4.1 Overview

4.1.1 Possible improvements

In Chapter 2 we developed new non-blowup theorems for the 3D Euler equations. In the near future, the following theoretical and numerical work should be carried out:

- Sharper estimates of U_ξ. Recall that in the key estimates (2.3.4) and (3.2.7), we overestimate the difference of u_ξ at the end points of l_t by $U_\xi = \max_{\beta_1, \beta_2 \in l_t} \left| (u \cdot \xi) \,|_{\beta_1}^{\beta_2} \right|$. Furthermore, we have not really obtained an estimate for U_ξ, instead, in applications we just bound it by $2 \left\| u \right\|_{L^\infty}$, which may be a severe overestimate.

- Estimates for $\nabla \cdot \xi$ and κ. In the current results, upper bounds are assumed for geometric quantities $\nabla \cdot \xi$ and κ. For deeper understanding of the 3D Euler and the 2D QG dynamics, it is important to develop dynamical estimates for these quantities.

- Numerical computations. It is important to carry out numerical computations of both the 3D Euler and the 2D QG flows to monitor the quantities appeared in theorems 1.4.3, 1.4.4, and 1.4.7, and consequently gain more understanding of their dynamics.

- Recast the current results in an Eulerian setting. The current results are in a pure Lagrangian setting, in the sense that all the analysis is carried out on one vortex line/levet set transported by the flow. It could be enlightening to try to obtain similar non-blowup results in a pure Eulerian or Eulerian-Lagrangian setting, since such adaption may reveal what has been missing in previous unsuccessful pure Eulerian analysis in dealing with fast-shrinking high-vorticity regions.

4.1.2 Applications to other PDEs

In Chapter 3 we successfully applied the new framework, based on estimation of vortex line stretching, to the 2D QG equation. Naturally, one next step would be trying to apply this new framework to other PDEs. It is easy to realize that one can apply this new framework as long as there is some quantity q that governs the blowup (e.g., the vorticity ω for the 3D Euler equations, $\nabla^{\perp}\theta$ for the 2D QG equation), and which at the same time is convected by a divergence free vector field u:

$$q_t + \mathcal{L}_u q = 0. \tag{4.1.1}$$

Here $\mathcal{L}_u$ is the Lie derivative and q can be a function (as ω in the 2D Euler flow), a one-form (as $\nabla^{\perp}\theta$ in the 2D QG flow), or a two-form (as ω in the 3D Euler flow).

The reason for this applicability is that whenever (4.1.1) holds, the level sets/integral curves of q are carried by the flow. Therefore, we can study its stretching via estimates similar to (2.3.4), and consequently obtain theorems similar to theorems 1.4.3 and 1.4.4.

Such flows include the 2D Boussinesq flow, the 3D Lagrangian averaged Euler flow, and the 3D axisymmetric flow. We will discuss these flows in subsections 4.2.3 and 4.2.4.

4.2 Discussions on Some PDEs

4.2.1 The 3D Euler equations

4.2.1.1 Helicity conservation

The conservation of helicity

$$\mathcal{H} \equiv \int_{\mathbb{R}^3} u \cdot \omega \, dx$$

is one of the most important conservation laws in the 3D Euler flow.[1] Therefore, it is worth studying how this conservation would help us in the Euler singularity problem. So far, few results have been obtained in this direction.[2] One of these results is Chae [Cha03c], which claims that $\mathcal{H}$ is conserved as long as $\omega \in C\left([0,T]; L^{3/2}\right) \cap L^3\left([0,T]; B^{\alpha,\infty}_{9/5}\right)$ for any $\alpha > 1/3$.

4.2.1.2 Spectral dynamics

Recall that the vorticity equation in the 3D Euler flow reads:

$$D_t\omega = \omega_t + u \cdot \nabla\omega = \nabla u \cdot \omega.$$

If we denote by S the deformation tensor $\frac{1}{2}\left(\nabla u + \nabla u^T\right)$, it is obvious that $\nabla u \cdot \omega = S \cdot \omega$, since $\frac{\nabla u - \nabla u^T}{2} \cdot \omega = \frac{1}{2}\omega \times \omega = 0$ by the definition of ω. Therefore, the eigenvalues of S control the growth of ω, which is vital to the development of singularities. Recently, a new identity has been found in Chae [Cha05b], which leads immediately

[1] In fact, it is shown in Serre [Ser84] that there are only three basic conservative quantities involving u and ∇u only: the momentum $\int_{\mathbb{R}^3} u \, dx$, the energy $\int_{\mathbb{R}^3} |u|^2 \, dx$, and the helicity $\int_{\mathbb{R}^3} u \cdot \omega \, dx$.

[2] On the other hand, since helicity conservation is a topological invariant in the sense that no metric is needed, results abound in the abstract setting. See e.g. Arnold-Khesin [AK98], Khesin [Khe05] and references therein. However, it is not clear whether these results may help in the study of the Euler singularity problem or not.

to the following condition for the blowup of the enstrophy:

$$\|\omega\|_{L^2} \nearrow \infty \text{ at } T^* \Rightarrow \int_0^{T^*} \left\|\lambda_2^+\right\| \, dt = \infty,$$

where $\lambda_2^+ \equiv \max\{\lambda_2, 0\}$ and λ_2 is the middle eigenvalue of S in the sense that $\lambda_1 \geq \lambda_2 \geq \lambda_3$. Since $\lambda_1 + \lambda_2 + \lambda_3 = 0$ due to $\nabla \cdot u = 0$, one can easily show that $|\lambda_2| = \min_{i=1,2,3}\{|\lambda_i|\}$. Combining with numerical evidence that ω tends to align with λ_2 instead of λ_1 (e.g., Ashurst-Kerstein-Kerr-Gibson [AKKG87]), we see that it is important to study the evolution of λ_2.

4.2.1.3 Clebsch variables

Recently, Hou and Li ([HL05b]) proved the global existence of a 2D QG analogy of the 3D Lagrangian averaged Euler equations, while the global existence for the latter is still open. The authors achieved this by exploring the fact that θ is convected by the 2D QG equation and therefore enjoys a maximum principle. In comparison, no interesting quantity in general 3D Euler flows has been found to have this property.

Nevertheless, for a wide range of initial values, the vorticity ω can be represented by two convected functions. This representation is discovered by Clebsch ([Cle59]) and thus bears the name "Clebsch variables." Namely, if $\omega = \Omega(\phi, \psi) \nabla\phi \times \nabla\psi$ at $t = 0$, it continues to have this representation as long as ϕ and ψ are convected by the flow. Therefore, the function pair (ϕ, ψ) is a 3D analogy of θ for the 2D QG flow. In the same paper, Hou and Li explored this analogy and obtained the global existence for the 3D Lagrangian averaged Euler flow under the condition that either ϕ or ψ is BV in an arbitrary direction. How the Clebsch representation could help in the 3D Euler singularity problem remains to be seen, and is well-worth studying.

Introductions and discussions of the Clebsch representation can be found in Lamb [Lam32], Marsden-Weinstein [MW83], Graham-Henyey [GH00], Constantin [Con03], etc.

4.2.2 The 2D QG equation

So far, in most of the study on the 2D QG equation, only the analogies between it and the 3D Euler equations are emphasized. However, recent results indicate that those properties that only belong to the 2D QG equation may be crucial to solve the QG singularity problem. For example, the estimate $\|u\|_{L^\infty} \lesssim \log \|\omega\|_{L^\infty}$ proved in Cordoba [Cor98], which is crucial to the results there, is not likely to hold in the 3D Euler flow. One more recent example is Hou-Li [HL05b], see 4.2.1.3.

Therefore, it is important to study those conserved quantities that have not been found of any use so far, for example, $\int_{\mathbb{R}^3} (-\triangle)^{-1/2} \theta \cdot \theta$ and $\int_{c_1 \le \theta \le c_2} \kappa \, |\nabla \theta|$ developed in Constantin [Con94]. It would also be interesting to derive more conservation laws via the Hamiltonian structure of the 2D QG equation (see, e.g., Holm-Zeitlin [HZ98] for the derivation of such structure).

4.2.3 The 3D axisymmetric flow and the 2D Boussinesq equations

4.2.3.1 The 3D axisymmetric flow

When an incompressible inviscid flow is axisymmetric, there is no dependence on θ when writing u and p in cylindrical coordinates (r, θ, x_3). This symmetry would reduce the 3D Euler equations into the following form:

$$\begin{aligned} \frac{\tilde{D}}{Dt}\tilde{u} &= -\tilde{\nabla} p + \frac{\left(ru^\theta\right)^2}{r^4} e_r, \\ \tilde{\nabla} \cdot \tilde{u} &= 0, \\ \frac{\tilde{D}}{Dt}\left(ru^\theta\right) &= 0, \end{aligned} \tag{4.2.1}$$

where $\tilde{\nabla} = \left(\frac{\partial}{\partial r}, \frac{\partial}{\partial x_3}\right) \cdot$, $\tilde{u} = (u^r, u^3)$, and $\frac{\tilde{D}}{Dt} = \frac{\partial}{\partial t} + \tilde{u} \cdot \tilde{\nabla}$. Efforts have been made to obtain singularities away from the axis (i.e., $r > r_0 > 0$) either numerically (e.g., Pumir-Siggia [PS92a, PS92b, PS92c]) or theoretically (Caflisch [Caf93]). Since $r > r_0 > 0$, it is likely that $\nabla \cdot \xi$ and κ may not behave as badly as in the non-symmetrical case. Therefore application of our new framework may yield sharper conditions.

Furthermore, it is shown in Chae-Kim [CK96] and Chae-Imanuvilov [CI99], respectively, that as long as

$$\int_0^{T^*} \|\omega_\theta\| \; dt + \int_0^{T^*} \exp\left[\int_0^t \left\{\|\omega_\theta\| \left(1 + \ln^+ \left(\|\omega_\theta\|_{C^\gamma} \|\omega_\theta\|_{L^p}\right)\right) + \left\|\omega_\theta \ln^- r\right\|\right\} \; ds\right] dt < \infty,$$

where $\omega_\theta = \frac{\partial u^r}{\partial x_3} - \frac{\partial u^3}{\partial x_r}$ and $\|\cdot\|$ denotes $\|\cdot\|_{L^\infty}$ when no subscript is specified, or

$$\int_0^{T^*} \left\|\tilde{\nabla} u^\theta\right\|_{L^\infty} + \left\|\frac{\partial u^\theta}{\partial r}\right\|_{L^\infty} \left\|\frac{1}{r}\frac{\partial u^\theta}{\partial x_3}\right\|_{L^\infty} \; dt < \infty,$$

there will be no singularity in the 3D axisymmetric Euler flow up to time T^*. It is also possible to get improved results in light of these sharper blowup criteria.

4.2.3.2 The 2D Boussinesq equations

When r is away from the axis, the 3D axisymmetric Euler equations are analogous to the following 2D Boussinesq equations:

$$\begin{aligned} u_t + u \cdot \nabla u &= -\nabla p + \begin{pmatrix} 0 \\ \rho \end{pmatrix}, \\ \rho_t + u \cdot \nabla \rho &= 0, \end{aligned}$$

where $u : \mathbb{R}^2 \mapsto \mathbb{R}^2$ and $\rho : \mathbb{R}^2 \mapsto \mathbb{R}$. It turns out that blowup is controlled by $\int_0^{T^*} \int_0^t \|\rho_{x_1}\|_{L^\infty} \; ds \; dt$ (E-Shu [ES94]). Numerical computations (e.g., E-Shu [ES94], Ceniceros-Hou [CH01]) have shown that fast growth of ρ_{x_1} saturates at later times, and therefore no blowup is likely to occur. However, no rigorous global existence theorem has been obtained yet. Since the level sets of ρ are carried by the flow, our framework applies, and may help on the singularity issue.

4.2.4 The Lagrangian averaged Euler equations

The Lagrangian averaged Euler equations have been proposed recently in Holm-Marsden-Ratiu [HMR98a, HMR98b]. They generalize the one-dimensional shallow water theory (Camassa-Holm [CH93]), and have been used as a turbulent closure model (e.g., Chen et al. [CFHOTW99]). They also enjoy the following deep mechanical-geometrical relation with the 3D Euler equations: the Lagrangian averaged Euler equations solve the geodesic on the manifold of area-preserving diffeomorphisms equipped with a weighted H^1 metric, whereas the 3D Euler equations solve the geodesic on the same manifold with a weak L^2 metric. (We refer to Peirce [Pei04] for a detailed introduction to the Lagrangian averaged models and Marsden-Ratiu-Shkoller [MRS00], Gay-Balmaz & Ratiu [GR05] for its geometry.) Therefore, the study of this system may shed light on the 3D Euler dynamics.

The vorticity form of the Lagrangian averaged Euler equation reads

$$q_t + u \cdot \nabla q = \nabla u \cdot q,$$

where $q = (1 - \alpha^2 \triangle)(\nabla \times u)$. Therefore, as discussed in Subsection 4.1.2, our framework should again apply.

Bibliography

[AK98] V. I. Arnold, B. A. Khesin. *Topological methods in hydrodynamics*, Applied Mathematical Sciences, 125, Springer-Verlag New York, 1998.

[AKKG87] Wm. T. Ashurst, A. R. Kerstein, R. M. Kerr, C. H. Gibson. Alignment of vorticity and scalar gradient with strain rate in simulated Navier-Stokes turbulence. *Phys. Fluids*, 30(8): 2343–2353, 1987.

[BMN01] A. Babin, A. Mahalov, B. Nicolaenko. 3D Navier-Stokes and Euler equations with initial data characterized by uniformly large vorticity. *Ind. Univ. Math. Jour.*, 50(1): 1–35, 2001.

[Bar01] C. Bardos. What use for the mathematical theory of the Navier-Stokes equations. *Mathematical fluid mechanics*, 1–25, J. Neustupa, P. Penel eds., Advances in Mathemathcal Fluid Mechanics, Birkhäuser, Basel, 2001.

[BF76] C. Bardos, U. Frisch. Finite-time regularity for bounded and unbounded ideal incompressible fluids using Hölder estimates. In *Turbulence and the Navier-Stokes Equations*, R. Temam ed., Lecture Notes in Math. 565: 1–13, 1976.

[BKM84] J. T. Beale, T. Kato, A. Majda. Remarks on the breakdown of smooth solutions for the 3-D Euler equations. *Comm. Math. Phys.*, 94: 61–66, 1984.

[BM92] J. B. Bell, D. L. Marcus. Vorticity intensification and transition to turbulence in the three-dimensional Euler equations. *Comm. Math. Phys.*, 147: 371–394, 1992.

[Bif03] L. Biferale. Shell models of energy cascade in turbulence. *Annu. Rev. Fluid Mech.*, 35:441–468, 2003.

[BJPV98] T. Bohr, M. Jensen, G. Paladin, A. Vulpiani. *Dynamical systems approach to turbulence.* Cambridge Nonlinear Science Series, 8. Cambridge University Press, Cambridge, 1998.

[BP94] O. N. Boratav, R. B. Pelz. Direct numerical simulation of transition to turbulence from a high-symmetry initial condition. *Phys. Fluids*, 6(8): 2757–2784, 1994.

[BP97] O. N. Boratav, R. B. Pelz. Structures and structure functions in the inertial range of turbulence. *Phys. Fluids*, 9: 1400–1415, 1997.

[BMONMF83] M. E. Brachet, D. Meiron, S. Orszag, B. Nickel, R. Morf, U. Frisch. Small-scale structure of the Taylor-Green vortex. *J. Fluid Mech.*, 130: 411–452, 1983.

[BMVPS92] M. E. Brachet, M. Meneguzzi, A. Vincent, H. Politano, P. -L. Sulem. Numerical evidence of smooth selfsimilar dynamics and possibility of subsequent collapse for three-dimensional ideal flows. *Phys. Fluids A*, 4: 2845–2854, 1992.

[Caf93] R. E. Caflisch. Singularity formation for complex solutions of the 3D incompressible Euler equations. *Physica D*, 67: 1–18, 1993.

[CKS97] R. E. Caflisch, I. Klapper, G. Steele. Remarks on singularities, dimension and energy dissipation for ideal hydrodynamics and MHD. *Comm. Math. Phys.*, 184: 443–455, 1997.

[CH93] R. Camassa, D. D. Holm. An integrable shallow water equation with peaked solitons. *Phys. Rev. Lett.*, 71: 1661–1664, 1993.

[CF03] M. Cannone, S. Friedlander. Navier: blowup and collapse. *Notices of the AMS*, 50(1): 7–13, 2003.

[CH01] H. D. Ceniceros, T. Y. Hou. An efficient dynamically adaptive mesh for potentially singular solutions. *J. Comput. Phys.*, 172: 1–31, 2001.

[Cha02] D. Chae. On the well-posedness of the Euler equations in the Triebel-Lizorkin spaces. *Comm. Pure Appl. Math.*, 40: 654–678, 2002.

[Cha03a] D. Chae. On the Euler equations in the critical Triebel-Lizorkin spaces. *Arch. Rat. Mech. Anal.*, 170: 185–210, 2003.

[Cha03b] D. Chae. The quasi-geostrophic equation in the Triebel-Lizorkin spaces. *Nonlinearity*, 16: 479–495, 2003.

[Cha03c] D. Chae. Remarks on the helicity of the 3-D incompressible Euler equations. *Commun. Math. Phys.* 240: 501–507, 2003.

[Cha04a] D. Chae. Local existence and blowup criterion for the Euler equations in the Besov spaces. *Asymptotic Analysis*, 38(3-4): 339–358, 2004.

[Cha04b] D. Chae. Remarks on the blowup of the Euler equations and the related equations. *Comm. Math. Phys.*, 245: 539–550, 2004.

[Cha04c] D. Chae. Global regularity for the 2D Boussinesq equations with partial viscosity terms. *Preprint* CSCAMM-04-06, University of Maryland, College Park, 2004.

[Cha05a] D. Chae. Remarks on the blowup criterion of the 3D Euler equations. *Nonlinearity,* 18: 1021–1029*,* 2005.

[Cha05b] D. Chae. On the spectral dynamics of the deformation tensor and new a priori estimates for the 3D Euler equations. *Preprint*, 2005.

[CCCF04] D. Chae, A. Cordoba, D. Cordoba, M. A. Fontelos. Finite time singularities in a 1D model of the quasi-geostrophic equation. To appear in *Advances in Mathematics*, Preprint, 2004.

[CK96] D. Chae, N. Kim. On the breakdown of axisymmetric smooth solutions for the 3-D Euler equations. *Comm. Math. Phys.*, 178: 391–398, 1996.

[CI99] D. Chae, O. Y. Imanuvilov. Generic solvability of the axisymmetric 3-D Euler equations and the 2-D Boussinesq equations. *J. Differential Equations*, 156: 1–17, 1999.

[CFHOTW99] S. Y. Chen, C. Foias, D. D. Holm, E. J. Olson, E. S. Titi, S. Wynne. The Camassa-Holm equations as a closure model for turbulent channel and pipe flows. *Phys. Fluids*, 11: 2343–2353, 1999.

[CISY89] S. Childress, G. R. Ierly, E. A. Spiegel, W. R. Young. blowup of unsteady two-dimensional Euler and Navier-Stokes solutions having stagnation point form. *J. Fluid Mech.*, 203: 1, 1989.

[Cho94] A. J. Chorin. *Vorticity and Turbulence*, Applied Mathematical Sciences, 103, Springer-Verlag New York, 1994.

[CM93] A. J. Chorin, J. E. Marsden. *A Mathematical Introduction to Fluid Mechanics.* 3rd Ed., New York: Springer-Verlag, 1993.

[Cle59] A. Clebsch. Über eine allgemeine transformation d. hydrodynamischen gleichungen. *Crelle*, liv., 1857 and lvi., 1859.

[Con86] P. Constantin. Note on loss of regularity for solutions of the 3-D incompressible Euler and related equations. *Comm. Math. Phys.*, 104: 311–326, 1986.

[Con94] P. Constantin. Geometric statistics in turbulence. *SIAM Review*, 36(1): 73–98, 1994.

[Con95] P. Constantin. A few results and open problems regarding incompressible fluids. *Notices of the AMS*, 42(6): 658–663, 1995.

[Con98] P. Constantin. Absence of proper nondegenerate generalized self-similar singularities. *J. Stat. Phys.*, 93(3/4): 777–786, 1998.

[Con00] P. Constantin. The Euler equations and nonlocal conservative Riccati equations. *Intern. Math. Res. Notes*, 9: 455–465, 2000.

[Con03] P. Constantin. *Euler Equations, Navier-Stokes Equations and Turbulence*, C.I.M.E. Lectures, 2003.

[CET94] P. Constantin, W. E, E. S. Titi. Onsager's conjecture on the energy conservation for solutions of Euler's equation. *Comm. Math. Phys.*, 165: 207–209, 1994.

[CFM96] P. Constantin, C. Fefferman, A. Majda. Geometric constraints on potentially singular solutions for the 3-D Euler equation. *Comm. PDE*, 21: 559–571, 1996.

[CLM85] P. Constantin, P. Lax, A. Majda. A simple one-dimensional model for the three dimensional vorticity equation. *Comm. Pure Appl. Math.*, 38: 715–724, 1985.

[CMT94] P. Constantin, A. Majda, E. Tabak. Formation of strong fronts in the 2-D quasi-geostrophic thermal active scalar. *Nonlinearity*, 7: 1495–1533, 1994.

[CNS98] P. Constantin, Q. Nie, N. Schörghofer. Nonsingular surface quasi-geostrophic flow. *Phys. Let. A*, 241: 168–172.

[CNS99] P. Constantin, Q. Nie, N. Schörghofer. Front formation in an active scalar equation. *Phys. Rev. E*, 60(3): 2858–2863, 1999.

[Cor97] D. Cordoba. On the geometry of solutions of the quasi-geostrophic and Euler equations. *Proc. Natl. Acad. Sci.*, 94: 12769–12770, 1997.

[Cor98] D. Cordoba. Nonexistence of simple hyperbolic blowup for the quasi-geostrophic and Euler equations. *Ann. Math.*, 148: 1135–1152, 1998.

[CF01] D. Cordoba, C. Fefferman. On the collapse of tubes carried by 3D incompressible flows. *Comm. Math. Phys.*, 222(2): 293–298, 2001.

[CF02a] D. Cordoba, C. Fefferman. Growth of solutions for QG and 2D Euler equations. *J. AMS*, 15(3): 665–670, 2002.

[CF02b] D. Cordoba, C. Fefferman. Scalars convected by a two-dimensional incompressible flow. *Comm. Pure Appl. Math.*, 55(2): 255–260, 2002.

[CFL04] D. Cordoba, C. Fefferman, R. De la llave. On squirt singularities in hydrodynamics. *SIAM Jour. Math. Anal.*, 36(1): 204–213, 2004.

[CFR04] D. Cordoba, C. Fefferman, J. L. Rodrigo. Almost sharp fronts for the surface quasi-geostrophic equation. *Proc. Natl. Acad. Sci.*, 101(9): 2687–2691, 2004.

[DHY05] J. Deng, T. Y. Hou, X. Yu. Geometric properties and nonblowup of 3D incompressible Euler flow. *Comm. PDE*, 30: 225–243, 2005.

[DHY04] J. Deng, T. Y. Hou, X. Yu. Improved geometric conditions for non-blowup of the 3D incompressible Euler equation. Submitted to *Comm. PDE*, 2004.

[DS04] E. I. Dinaburg, Ya G. Sinai. On some approximation of the 3D Euler system. *Ergod. Th. & Dynam. sys.*, 24: 1443–1450, 2004.

[ES94] W. E, C. Shu. Small-scale structures in Boussinesq convection. *Phys. Fluids*, 6(1): 49–58, 1994.

[EM70] D. G. Ebin, J. E. Marsden. Groups of diffeomorphisms and the motion of an incompressible fluid. *Ann. Math.*, 92: 102–163. 1970.

[EFM70] D. G. Ebin, A. E. Fischer, J. E. Marsden. Diffeomorphism groups, hydrodynamics and relativity. In *Proceedings of the 13th Biennial Seminar of the Canadian Mathematical Congress*, J. Vanstone ed., Montreal. 135–279, 1970.

[Eul55] L. Euler. Principes généraux du mouvement des fluides. *Hist. de L'Acad. de Berlin*, 1755.

[Eyi94] G. Eyink. Energy dissipation without viscosity in ideal hydrodynamics, I. Fourier analysis and local energy transfer. *Physica D*, 78(3-4): 222–240, 1994.

[Fef00] C. Fefferman. Existence & smoothness of the Navier-Stokes equations. *Official millennium problem description.* http://www.claymath.org/millennium/Navier-Stokes_Equations/Official_Problem_Description.pdf

[Fri99] S. Friedlander, Lectures on Stability of an Ideal Fluid. *Hyperbolic Equations and Frequency Interactions*, L. Caffarelli, W. E, eds., IAS/Park City Mathematics Series 5: 229–304. 1999.

[Fri95] U. Frisch. *Turbulence, the legacy of A. N. Kolmogorov*, Cambridge University Press, 1995.

[FP04] S. Friedlander, N. Pavlović. Blowup in a three-dimensional vector model for the Euler equations. *Comm. Pure Appl. Math.*, 42: 705–725, 2004.

[GGH97a] B. Galanti, J. D. Gibbon, M. Heritage. Vorticity alignment results for the three-dimensional Euler and Navier-Stokes equations. *Nonlinearity*, 10: 1675–1694, 1997.

[GR05] F. Gay-Balmaz, T. S. Ratiu. The Lie-Poisson structure of the LAE-α equation. *arXiv:math.DG/0504381* v1, 19 Apr. 2005.

[Gib99] J. D. Gibbon. Vorticity alignment results for the 3D Euler and Navier-Stokes equations. In *Trends in mathematics: fundamental problematic issues in turbulence*, 25-32, A. Gyr, W. Kinzelbach, A. Tsinober eds., Birkhäuser Verlag Basel/Switzerland, 1999.

[Gib02] J. D. Gibbon. A quaternionic structure in the three-dimensional Euler and ideal magneto-hydrodynamics equations. *Physica D*, 166: 17–28, 2002.

[GGK00] J. D. Gibbon, B. Galanti, R. M. Kerr. Stretching and compression of vorticity in the 3D Euler equations. In *Turbulence structure and vortex dynamics*, 23-34, J. C. R. Hunt, J. C. Vassilicos eds., Cambridge University Press, 2000.

[GGH97b] J. D. Gibbon, J. Gibbons, M. Heritage. A logarithmic 3d Euler inequality. *Phys. Fluids*, 9(2): 471–472, 1997.

[GH97] J. D. Gibbon, M. Heritage. Angular dependence and growth of vorticity in the three-dimensional Euler equations. *Phys. Fluids*, 9(4): 901–909, 1997.

[GMS03] J. D. Gibbon, D. R. Moore, J. T. Stuart. Exact, infinite energy, blowup solutions of the three-dimensional Euler equations. *Nonlinearity*, 16: 1823–1831, 2003.

[GH00] C. R. Graham, F. S. Henyey. Clebsch representation near points where the vorticity vanishes. *Phys. Fluids*, 12(4): 744–746, 2000.

[GMG98] R. Grauer, C. Marliani, K. Germaschewski. Adaptive mesh refinement for singular solutions of the incompressible Euler equations. *Phys. Rev. Lett.*, 80(19), 4177–4180, 1998.

[GS95] R. Grauer, T. C. Sideris. Finite time singularities in ideal fluids with swirl. *Physica D*, 88: 116–132, 1995.

[Gun26] N. Gunther. On the motion of fluid in the moving container. *Izvestiya Akademii Nauk SSSR Seriya Fiziko-Matemicheskikh*, 20: 1323–1348, 1503–1532. 1926.

[Gun27] N. Gunther. On the motion of fluid in the moving container. *Izvestiya Akademii Nauk SSSR Seriya Fiziko-Matemicheskikh*, 21: 621–656, 735–756, 1139–1162. 1927.

[Gun28] N. Gunther. On the motion of fluid in the moving container. *Izvestiya Akademii Nauk SSSR Seriya Fiziko-Matemicheskikh*, 22: 9–30. 1928.

[He00] X. He. Self-similar singularities of the 3D Euler equations. *Appl. Math. Lett.*, 13: 41–46, 2000.

[He02] X. He. A sufficient condition for a finite-time L_2 singularity of the 3D Euler equation. *Preprint arXiv:math.AP/0209323 v1*, 2002.

[He04] X. He. Existence of singular self-similar solutions of the three-dimensional Euler equations in a bounded domain. *J. Math. Fluid Mech.*, 6: 389–404, 2004.

[Hel67] H. Helmholtz. On the integrals of the hydrodynamical equations which express vortex-motion. *Philosophical Magazine, Series 4*, 33: 485–512, 1867. Translation.

[HMR98a] D. D. Holm, J. E. Marsden, T. S. Ratiu. Euler-Poincaré models of ideal fluids with nonlinear dispersion. *Phys. Rev. Lett.*, 349: 4173–4177, 1998.

[HMR98b] D. D. Holm, J. E. Marsden, T. S. Ratiu. Euler-Poincaré equations and semidirect products with applications to continuum theories. *Adv. Math.*, 137: 1–81, 1998.

[HZ98] D. D. Holm, V. Zeitlin. Hamilton's principle for quasigeostrophic motion. *Phys. Fluids*, 10(4): 800–806, 1998.

[HL05a] T. Y. Hou, C. Li. Global well-posedness of the viscous Boussinesq equations. *Discrete and Continuous Dynamical Systems*, 12(1): 1–12, 2005.

[HL05b] T. Y. Hou, C. Li. On global well-posedness of the Lagrangian averaged Euler equations. Preprint. 2005.

[HM92] F. Hussain, M. V. Melander. New aspects of vortex dynamics: helical waves, core dynamics, viscous helicity generation, and interaction with turbulence. In *Topological aspects of the dynamics of fluids and plasmas*, H. K. Moffatt et al eds., 293–302, Kluwer Academic Publishers, Netherlands, 1992.

[Kat67] T. Kato. On classical solutions of the two-dimensional non-stationary Euler equations. *Archive for Rational Mechanics & Analysis*, 25: 188–200, 1967.

[Kat72] T. Kato. Nonstationary flows of viscous and ideal fluids in $\mathbb{R}^3$. *J. Funct. Anal.*, 9: 296–305, 1972.

[Kat00] T. Kato. On the smoothness of trajectories in incompressible perfect fluids. *Contemporary Mathematics*, 263: 109–130, 2000.

[KP04] N. Katz, N. Pavlovic. Finite time blowup for a dyadic model of the Euler equations. *AMS Tran.* S 0002-9947(04)03532-9, March 12, 2004.

[Ker93] R. M. Kerr. Evidence for a singularity of the three-dimensional incompressible Euler equations. *Phys. Fluids A*, 5: 1725–1746. 1993.

[Ker95] R. M. Kerr. The role of singularities in Euler. In *Small-scale structure in hydro and magnetohydrodynamic turbulence*, M. Meneguzzi, A. Pouquet, P. L. Sulem eds., Lecture Notes in Physics 462, Springer-Verlag, 1995.

[Ker96] R. M. Kerr. Cover illustration: vortex structure of Euler collapse. *Nonlinearity*, 9: 271–272, 1996.

[Ker97] R. M. Kerr. Euler Singularities and Turbulence. In *19th ICTAM Kyoto'96*, T. Tatsumi, E. Watanabe, T. Kambe eds. Elsevier Science, 1997.

[Ker98] R. M. Kerr. The Outer Regions in Singular Euler. In *Fundamental Problematic Issues in Turbulence*, Tsinobyr and Gyr eds. Birkhäuser, 1998.

[Ker03] R. M. Kerr. http://www.eng.warwick.ac.uk/staff/rmk/rmk_files/ euler_22mar03.pdf, 2003.

[Ker04] R. M. Kerr. Velocity and scaling of collapsing Euler vortices. To appear in *Phys. Fluids*, Preprint, 2004.

[Khe05] B. Khesin. Topological fluid dynamics. *Notices of the AMS*, 52(1): 9–19, 2005.

[KT00] H. Kozono, Y. Taniuchi. Limiting case of the Sobolev inequality in BMO, with application to the Euler equations. *Commun. Math. Phys.*, 214: 191–200, 2000.

[KO00] S. Kouranbaeva, M. Oliver. Global well-posedness for the averaged Euler equations in two dimensions. *Physica D*, 138: 197–209, 2000.

[Lad03] O. A. Ladyzhenskaya. Sixth problem of the millennium: Navier-Stokes equations, existence and smoothness. *Russian Math. Surveys*, 58(2): 251–286 (*Uspekhi Mat. Nauk*, 58(2): 45–78), 2003.

[Lam32] Sir Horace Lamb. *Hydrodynamics*. Dover Publications, New York, 1945.

[Licht25] L. Lichtenstein. Über einige Existenzprobleme der Hydrodynamik homogener, unzusammendrückbarer, reibungsloser Flüssigkeiten und die

Helmholtzschen Wirbelsätze, zweite Abhandlung, vierte Abhandlung. *Math. Z.*, 23: 89–154, 1925; 26: 196–323, 387–415, 725. 1927; 32: 608, 1930.

[Lio96] P. L. Lions. *Mathematical Topics in Fluid Mechanics, Vol. 1, Incompressible Models*, Oxford Lecture Series in Mathematics and its Applications, Oxford, 1996.

[LT02] H. Liu, E. Tadmor. Spectral dynamics of the velocity gradient field in restricted flows. *Comm. Math. Phys.*, 228: 435–466, 2002.

[LNZ99] M. C. Lopes Filho, H. J. Nussenzveig Lopes, Y. Zheng. *Weak solutions for the equations of incompressible and inviscid fluid dynamics.* Text of minicourse for the 22nd Brazilian Colloquium of Mathematics, 1999. Preprint.

[MB02] A. Majda, A. Bertozzi. *Vorticity and Incompressible Flow*, Cambridge University Press, 2002.

[MT96] A. Majda, E. G. Tabak. A two-dimensional model for quasigeostrophic flow: comparison with the two dimensional Euler flow. *Physica D*, 98: 515–522, 1996.

[MP94] C. Marchioro, M. Pulvirenti, *Mathematical Theory of Incompressible Nonviscous Fluids*, Cambridge University Press, 1994.

[MR94] J. E. Marsden, T. S. Ratiu. *Introduction to Mechanics and Symmetry*, TAM 17, Springer-Verlag, 1994.

[MRR00] J. E. Marsden, T. S. Ratiu, G. Raugel. The Euler equations on thin domains. *International Conference on Differential Equations*, Berlin 1999. B. Fiedler, K. Gröger and J. Sprekels eds., World Scientific, 2000, 1198–1203.

[MRS00] J. E. Marsden, T. S. Ratiu, S. Shkoller. The geometry and analysis of the averaged Euler equations and a new diffeomorphism group. *Geom. Funct. Analysis*, 10: 582–599, 2000.

[MS01] J. E. Marsden, S. Shkoller. Global well-posedness for the Lagrangian averaged Navier-Stokes (LANS-α) equations on bounded domains. *Phil. Trans R. Soc Lond. A*, 359: 1449–1468, 2001.

[MW83] J. E. Marsden, A. Weinstein. Coadjoint orbits, vortices, and Clebsch variables for incompressible fluids. *Physica D*, 7(1-3): 305–323, 1983.

[Mof00] H. K. Moffatt. The interaction of skewed vortex pairs: a model for blowup of the Navier-Stokes equations. *J. Fluid Mech.*, 409, 51–68, 2000.

[MOF80] R. H. Morf, S. Orszag, U. Frisch. Spontaneous singularity in three-dimensional inviscid, incompressible flow. *Phys. Rev. Lett.*, 44: 572–575, 1980.

[NB02] C. S. Ng, A. Bhattacharjee. Sufficient condition for finite-time singularity and tendency towards self-similarity in a high-symmetry flow. In *Tubes, sheets and singularities in fluid dynamics*, K. Bajer, H. K. Moffatt eds., 317-328, Kluwer Academic Publishers, Netherlands, 2002.

[Ohk93] K. Ohkitani. Eigenvalue problems in three-dimensional Euler flows. *Phys. Fluids A*, 5(10): 2570–2572, 1993.

[Ohk94] K. Ohkitani. Kinematics of vorticity: vorticity-strain conjugation in incompressible fluid flows. *Phys. Rev. E*, 50(6): 5107–5110, 1994.

[OG00] K. Ohkitani, J. Gibbon. Numerical study of singularity formation in a class of Euler and Navier-Stokes flows. *Phys. Fluids*, 12: 3181–3194, 2000.

[OK95] K. Ohkitani, S. Kishiba. Nonlocal nature of vortex stretching in an inviscid fluid. *Phys. Fluids*, 7(2): 411–421, 1995.

[OY97] K. Ohkitani, M. Yamada, Inviscid and inviscid-limit behavior of a surface quasi-geostrophic flow. *Phys. Fluids*, 9: 876–882, 1997.

[OS01] M. Oliver, S. Shkoller. The vortex blob method as a second-grade non-Newtonian fluid. *Commun. PDEs*, 26: 91–110, 2001.

[Ons49] L. Onsager. Statistical hydrodynamics. *Nuovo Cimento Suppl.* 6: 279–287, 1949.

[Ped87] J. Pedlosky. *Geophysical Fluid Dynamics*, Springer-Verlag, 1987.

[Pei04] J. P. Peirce. Well-posedness of the three-dimensional Lagrangian averaged Navier-Stokes equations. *Ph. D. thesis*, UC Davis, 2004. Available at arXiv:math.AP/0406205 v1, 9, June 2004.

[Pel97] R. B. Pelz. Locally self-similar, finite-time collapse in a high-symmetry vortex filament model. *Phys. Rev. E*, 55(2): 1617–1620, 1997.

[Pel01] R. B. Pelz. Symmetry and the hydrodynamic blowup problem. *J. Fluid Mech.*, 444: 299–320, 2001.

[Pel02] R. B. Pelz. Discrete groups, symmetric flows and hydrodynamic blowup. In *Tubes, sheets and singularities in fluid dynamics*, K. Bajer, H. K. Moffatt eds., 269–283, Kluwer Academic Publishers, Netherlands, 2002.

[PB95] R. B. Pelz, O. N. Boratav. On a possible Euler singularity during transition in a high-symmetry flow. In *Small-scale structures in three-dimensional hydrodynamic and magnetohydrodynamic turbulence, proceedings Nice, France, 1995*, M. Meneguzzi, A. Pouquet, P. L. Sulem eds., 25–32, Springer-Verlage Berlin Heidelberg, 1995.

[PGGB99] R. B. Pelz, Y. Gulak, J. M. Greene, O. N. Boratav. On the finite-time singularity problem in hydrodynamics. In *Trends in mathematics: fundamental problematic issues in turbulence*, A. Gyr, W. Kinzelbach, A. Tsinober eds., 33-40, Birkhäuser Verlag Basel/Switzerland, 1999.

[Pon85] G. Ponce. Remarks on a paper by J. T. Beale, T. Kato, and A. Majda. *Commun. Math. Phys.*, 98: 349–353, 1985.

[PS90] A. Pumir, E. D. Siggia. Collapsing solutions to the 3-D Euler equations. *Phys. Fluids A*, 2(2):220–241, 1990.

[PS92a] A. Pumir, E. D. Siggia. Blow up in axisymmetric Euler flows. In *Topological aspects of the dynamics of fluids and plasmas*, H. K. Moffatt et al. eds., 293–302, Kluwer Academic Publishers, Netherlands, 1992.

[PS92b] A. Pumir, E. D. Siggia. Development of singular solutions to the axisymmetric Euler equations. *Phys. Fluids A*, 4(7): 1472–1491, 1992.

[PS92c] A. Pumir, E. D. Siggia. Finite-time singularities in the axisymmetric three-dimensional Euler equations. *Phys. Rev. Lett.*, 68(10): 1511–1514, 1992.

[Rob03] R. Robert. Statistical hydrodynamics (Onsager revisited). In *Handbook of Mathematical Fluid Dynamics, Vol. II*, S. Friedlander, D. Serre eds., Elsevier Science, 1–54, 2003.

[Rod04] J. L. Rodrigo. The vortex patch problem for the surface quasi-geostrophic equation. *Proc. Nat. Acad. Sci.*, 101(9): 2684–2686.

[Rod05] J. L. Rodrigo. On the evolution of sharp fronts for the quasi-geostrophic equations. *Comm. Pure Appl. Math.* , 43: 821–866, 2005.

[Sch93] V. Scheffer. An inviscid flow with compact support in space-time. *J. Geom. Anal.*, 3(4): 343–401, 1993.

[Ser84] D. Serre. Les invariants du premier ordre de l'equation d'Euler en dimension trois. *Physica D*, 13(1-2): 105–136, 1984.

[SMO93] M. J. Shelley, D. Meiron, S. A. Orszag. Dynamical aspects of vortex reconnection of perturbed anti-parallel vortex tubes. *J. Fluid Mech.*, 246: 613–652, 1993.

[Shn97] A. Shnirelman. On the nonuniqueness of weak solution of the Euler equation. *Comm. Pure Appl. Math.*, Vol. L, 1261–1286, 1997.

[Shn98] A. Shnirelman. Weak solutions of incompressible Euler equations with decreasing energy. *C. R. Acad. Sci. Paris I, Math*, 326: 329–334, 1998.

[Shn00] A. Shnirelman. Weak solutions with decreasing energy of incompressible Euler equations. *Commun. Math. Phys.*, 210: 541–603. 2000.

[Soh01] H. Sohr. *The Navier-Stokes equations : an elementary functional analytic approach*, Birkhäuser Verlag, 2001.

[Stu87] J. T. Stuart. Nonlinear Euler partial differential equations: Singularities in their solutions. In *Proceedings in Honor of C. C. Lin*, D. J. Benney, C. Yuan, F. H. Shu eds., 81-95, World Scientific, Singapore, 1987.

[Swa71] H. Swann. The convergence with vanishing viscosity of nonstationary Navier-Stokes flow to ideal flow in $\mathbb{R}_3$. *Trans. Amer. Math. Soc.*, 157: 373–397, 1971.

[Tab92] M. Tabor. Stretching and alignment in generral flow fields: classical trajectories from Reynolds number zero to infinity. In *Topological aspects of the dynamics of fluids and plasmas*, H. K. Moffatt et al. eds., 83-110, Kluwer Academic Publishers, Netherlands, 1992.

[Tad01] E. Tadmor. On a new scale of regularity spaces with applications to Euler's equations. *Nonlinearity*, 14: 513-532, 2001.

[TSSV95] A. Tsinober, L. Shtilman, A. Sinyavskii, H. Vaisburd. Vortex stretching and enstrophy generation in numerical and laboratory turbulence. In *Small-scale structures in three-dimensional hydrodynamics and magnetohydrodynamic turbulence*, M. Meneguzzi, A. Pouquet, P. L. Sulem eds., 9–16, Springer-Verlag, Berlin-Heidelberg, 1995.

[Vis98] M. Vishik. Hydrodynamics in Besov spaces. *Arch. Rat. Mech. Anal.*, 145: 197–214, 1998.

[Wal04] F. Waleffe. On some dyadic models of the Euler equations. *Preprint.* arXiv:math.AP/0410380, 19 Oct 2004.

[WB92] X. Wang, A. Bhattacharjee. Is there a finite-time singularity in axisymmetric Euler flows? In *Topological aspects of the dynamics of fluids and plasmas*, H. K. Moffatt et al. eds., 293–302, Kluwer Academic Publishers, Netherlands, 1992.

[Wol33] W. Wolibner. Un théorème sur l'existence du mouvement plan d'un fluide parfait, homogéne, incompressible, pendant untemps infiniment long. *Math. Zam.*, 37: 698– 726, 1933.

[Yud63] V. I. Yudovich. Non-stationary flow of an ideal incompressible fluid. *Akademiya Nauk SSSR. Zhurnal Vychislitelnol Matematiki I Matematicheskoi Fiziki*, 3: 1032–1066, 1963. (In Russian.)

[Yud95] V. I. Yudovich. Uniqueness theorem for the basic nonstationary problem in the dynamics of an ideal incompressible fluid. *Math. Res. Letters*, 2: 27–38, 1995.

Appendix A

Proofs of Technical Lemmas

A.1 Lemma 4 in Deng-Hou-Yu [DHY05]

Now we prove Lemma 4 in Deng-Hou-Yu [DHY05], which bounds $\|u\|_\infty$ by $\|\omega\|_{L^\infty}^{3/5}$.

Lemma A.1.1. *Let $u(x,t)$ be the solution to 3D Euler equations, and let $\omega(x,t) \equiv \nabla\times u(x,t)$ be the vorticity. Denote $\Omega(t) \equiv \|\omega(\cdot,t)\|_{L^\infty(\mathbb{R}^3)}$ and $U(t) \equiv \|u(\cdot,t)\|_{L^\infty(\mathbb{R}^3)}$. Then the following inequality holds:*

$$U(t) \lesssim \Omega(t)^{3/5}.$$

Proof. By the Biot-Savart law (1.1.8), we have

$$u(x,t) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{y}{|y|^3} \times \omega(x+y,t)\ dy.$$

Take a smooth cutoff function $\chi : \{0\}\cup\mathbb{R}^+ \mapsto [0,1]$ such that $\chi(r) = 1$ for $r \le 1$ and $\chi(r) = 0$ for $r \ge 2$. Let $\rho > 0$ be a small positive parameter to be determined later. Then we have

$$\begin{aligned} |u(x,t)| &= \left|\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{y}{|y|^3} \times \omega(x+y,t)\ dy\right| \\ &\le \left|\frac{1}{4\pi}\int_{\mathbb{R}^3} \chi\left(\frac{|y|}{\rho}\right) \frac{y}{|y|^3} \times \omega(x+y,t)\ dy\right| \\ &\quad + \left|\frac{1}{4\pi}\int_{\mathbb{R}^3} \left(1-\chi\left(\frac{|y|}{\rho}\right)\right) \frac{y}{|y|^3} \times \omega(x+y,t)\ dy\right|. \end{aligned}$$

Invoking integration by parts in the second integral using $\omega = \nabla \times u$, we have

$$\begin{aligned} |u(x,t)| &\lesssim \Omega(t) \int_{|y|\le 2\rho} \frac{1}{|y|^2}\, dy \\ &+ \int_{|y|\ge\rho} \frac{1}{|y|^3} |u(x+y,t)|\, dy \\ &+ \frac{1}{\rho} \int_{|y|\ge\rho} \frac{1}{|y|^2} |u(x+y,t)|\, dy. \end{aligned}$$

Using polar coordinates in the first integral, and the Schwarz inequality in the other two, we obtain

$$|u(x,t)| \lesssim \Omega(t)\,\rho + \left(\int_{|y|\ge\rho} \frac{1}{|y|^6}\, dy\right)^{1/2} + \frac{1}{\rho}\left(\int_{|y|\ge\rho} \frac{1}{|y|^4}\, dy\right)^{1/2}$$

where we have used the fact that the total energy $\|u\|_{L^2(\mathbb{R}^3)}$ is conserved (Chorin-Marsden [CM93]), i.e., $\|u(\cdot,t)\|_{L^2(\mathbb{R}^3)} = \|u(\cdot,0)\|_{L^2(\mathbb{R}^3)}$.

Finally, we use polar coordinates in the last two integrals, and get

$$\begin{aligned} |u(x,t)| &\lesssim \Omega(t)\,\rho + \left(\int_\rho^\infty \frac{1}{r^4}\, dr\right)^{1/2} + \frac{1}{\rho}\left(\int_\rho^\infty \frac{1}{r^2}\, dr\right)^{1/2} \\ &\lesssim \Omega(t)\,\rho + \rho^{-3/2}. \end{aligned}$$

By taking $\rho = \Omega(t)^{-2/5}$, we obtain

$$|u(x,t)| \lesssim \Omega(t)^{3/5}.$$

The proof ends by noticing the arbitrariness of x. □

A.2 Lemma 2.5.1

Define

$$f(r) \equiv \frac{r^{2-A} R^3 K}{(R^3 K + 1)\, r - R},$$

and

$$r_c \equiv \frac{2-A}{1-A}\frac{R}{R^3K+1}.$$

Since

$$f'(r) = \frac{r^{1-A}R^3K}{\left((R^3K+1)\,r-R\right)^2}\left[(1-A)\left(R^3K+1\right)r-(2-A)\,R\right]$$

is negative for $r < r_c$ and positive for $r > r_c$, we conclude that r_c is the unique minimizer of $f(r)$ in (R,∞). Furthermore, since

$$f(R) = \frac{R^{2-A}R^3K}{(R^3K+1)\,R-R} = R^{1-A} > 1$$

due to $R = e^{C_0}/c_0 > 1$, condition (2.5.1), that is $f(r) < 1$, is equivalent to

$$r_c > R \text{ and } f(r_c) < 1.$$

Next we study the above two conditions. The first condition $r_c > R$ is just

$$\frac{2-A}{1-A}\frac{R}{R^3K+1} > R,$$

which reduces to

$$R^3K < \frac{1}{1-A}. \tag{A.2.1}$$

As to the second condition $f(r_c) < 1$, after some algebra we can rewrite it as

$$\frac{R^3K}{(R^3K+1)^{2-A}} < R^{A-1}\frac{(1-A)^{1-A}}{(2-A)^{2-A}}. \tag{A.2.2}$$

Now let $y = R^3K$ and consider $g(y) \equiv \frac{y}{(y+1)^{2-A}}$. We study its behavior on $\mathbb{R}^+$. It is easy to see that $g(0) = g(+\infty) = 0$. Furthermore, by simple calculations, we have

$$\begin{aligned} g'(y) &= \frac{1}{(y+1)^{4-2A}}\left[(y+1)^{2-A}-(2-A)\,y\,(y+1)^{1-A}\right] \\ &= \frac{(y+1)^{1-A}}{(y+1)^{4-2A}}\left[1-(1-A)\,y\right]. \end{aligned}$$

Thus it is clear that $g(y)$ is increasing in $\left(0, \frac{1}{1-A}\right)$, decreasing in $\left(\frac{1}{1-A}, +\infty\right)$, and reaches its unique maximum at $y = \frac{1}{1-A}$. Since

$$g\left(\frac{1}{1-A}\right) = \frac{(1-A)^{1-A}}{(2-A)^{2-A}}$$

and $R^{1-A} > 1$, there exist exactly two numbers, y_1 and y_2, satisfying $y_2 > \frac{1}{1-A} > y_1 > 0$, such that

$$g(y_1) = g(y_2) = R^{A-1}\frac{(1-A)^{1-A}}{(2-A)^{2-A}}$$

and $g(y) \neq R^{A-1}\frac{(1-A)^{1-A}}{(2-A)^{2-A}}$ for all other $y > 0$.

Now it is easy to see that the two conditions (A.2.1) and (A.2.2) are equivalent to

1. $R^3 K < \frac{1}{1-A}$, and

2. $R^3 K < y_1$ or $R^3 K > y_2$.

Since $y_1 < \frac{1}{1-A} < y_2$, conditions 1 and 2 above are equivalent to the following single condition:

$$R^3 K < y_1,$$

where y_1 is just the smallest $y > 0$ such that $f(y) \equiv \frac{y}{(y+1)^{2-A}} = R^{A-1}\frac{(1-A)^{1-A}}{(2-A)^{2-A}}$. This completes the proof.

A.3 Velocity Bound for the 2D QG Equation

In this section, we briefly go over the velocity bound established in Cordoba [Cor98].

Lemma A.3.1. *Let $\Omega(t) \equiv \left\|\nabla^{\perp}\theta\right\|_{L^\infty(\mathbb{R}^2)}$ denote the maximum gradient of the solution to the 2D QG equation, and let $U(t) \equiv \|u(\cdot, t)\|_{L^\infty}$ denote the maximum velocity. We have the following estimate.*

$$U(t) \lesssim \log\Omega(t) + 1.$$

Proof. For the 2D QG equation, we have

$$u(x,t) = \int_{\mathbb{R}^2} \frac{y^{\perp}\theta(x+y)}{|y|^3}\,dy.$$

Let $\chi(r)$ be a smooth cutoff function such that $\chi(r) = 1$ for $0 \le r \le 1$ and $\chi(r) = 0$ for $r \ge 2$. Let $\rho > 0$ be fixed later. We have

$$\begin{aligned} u(x,t) &= \int \chi\left(\frac{|y|}{\rho}\right) \frac{y^{\perp}}{|y|^3}\theta(x+y)\,dy \\ &\quad + \int \left(1-\chi\left(\frac{|y|}{\rho}\right)\right) \frac{y^{\perp}}{|y|^3}\theta(x+y)\,dy \\ &\equiv I + II. \end{aligned}$$

By integration by parts, we have $|I| \lesssim \rho\Omega(t) + 1$ since $\frac{y^{\perp}}{|y|^3} = \nabla^{\perp}\left(\frac{1}{|y|}\right)$. For II we have

$$\begin{aligned} |II| &\le \int_{|y|\ge\rho} \frac{|\theta(x+y)|}{|y|^2} \\ &= \int_{\rho\le|y|<R} \frac{|\theta(x+y)|}{|y|^2}\,dy + \int_{R\le|y|} \frac{|\theta(x+y)|}{|y|^2}\,dy \\ &\lesssim \|\theta\|_{L^\infty} \log\frac{R}{\rho} + \|\theta\|_{L^2} R^{-1} \\ &\lesssim |\log\rho| + 1. \end{aligned}$$

Where we have used the fact that $\|\theta(\cdot,t)\|_{L^p} = \|\theta(\cdot,0)\|_{L^p}$ for all $p \in [1,\infty]$, and have taken R to be a large fixed constant.

Now we have

$$|u(x,t)| \lesssim \Omega(t)\,\rho + |\log\rho| + 1.$$

Taking $\rho = \Omega(t)^{-1}$, we have

$$U(t) \lesssim \log\Omega(t) + 1$$

due to the arbitrariness of x. □