STABILITY OF THE COMPRESSIBLE LAMINAR BOUNDARY LAYER

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ABSTRACT

In previous theoretical treatments of the stability of the compressible laminar boundary layer the effect of the temperature fluctuations on the "viscous" (rapidly-varying) disturbances is either ignored (Lees-Lin), or is accounted for incompletely (Dunn-Lin). A thorough reexamination of this problem shows that temperature fluctuations have a profound influence on both the "inviscid" (slowly-varying) and viscous disturbances above a Mach number of about 2.0. The present analysis includes the effect of temperature fluctuations on the viscosity and thermal conductivity, and also introduces the viscous dissipation term that was dropped in the earlier theoretical treatments.

Some important results of the present study are: (1), the rate of conversion of energy from the mean flow to the disturbance flow through the action of viscosity in the vicinity of the wall increases with Mach number; (2), instead of being nearly constant across the boundary layer, the amplitude of inviscid pressure fluctuations for Mach numbers greater than 3 decreases markedly with distance outward from the plate surface. This behavior means that the jump in magnitude of the Reynolds stress in the neighborhood of the critical layer is greatly reduced; (3), at Mach numbers less than about 2 dissipation effects are minor, but they become extremely important at higher Mach numbers since for neutral disturbances they must compensate for the generally destabilizing effects of items (1) and (2).

Numerical examples illustrating the effects of compressibility (including neutral stability characteristics) are obtained and are compared with the experimental results of Laufer and Vrebalovich at M = 2.2, and of Demetriades at M = 5.8.

TABLE OF CONTENTS

LIST OF FIGURES

SYMBOLS

The symbols used in the present report are in general those commonly used in the literature on boundary layer stability. In some regrettable instances a symbol will represent more than one item. To minimize the confusion, the different definitions of the same symbol will have listed the section of the report in which they appear.

	Dimensional Quantities	Dimensionless Quantities	Reference Quantities
Positional Coordinates:			
longitudinal	x^*	x	$\sqrt{\dfrac{\nu_e^* x^*}{u_e^*}}$
normal	y^*	y	$\sqrt{\dfrac{\nu_e^* x^*}{u_e^*}}$
Time	t^*	t	$\frac{1}{u_e^*}\sqrt{\nu_e^* x^*/u_e^*}$
Velocity Components:			
longitudinal	$u^* = \bar{u}^* + u^{*\prime}$	$w(y) + f(y)e^{i\alpha(x-ct)}$	u_e^*
normal	$v^* = \bar{v}^* + v^{*\prime}$	$v(y) + \alpha\phi(y)e^{i\alpha(x-ct)}$	u_e^*
Density	$\rho^* = \bar{\rho}^* + \rho^{*\prime}$	$\rho(y) + r(y)e^{i\alpha(x-ct)}$	ρ_e^*
Pressure	$p^* = \bar{p}^* + p^{*\prime}$	$p(y) + \pi(y)e^{i\alpha(x-ct)}$	p_e^*
Temperature	$T^* = \bar{T}^* + T^{*\prime}$	$T(y) + \theta(y)e^{i\alpha(x-ct)}$	T_e^*
Viscosity Coefficients	$\mu^* = \bar{\mu}^* + \mu^{*\prime}$	$\mu(y) + m(y)e^{i\alpha(x-ct)}$	μ_e^*
	$\mu_2^* = \bar{\mu}_2^* + \mu_2^{*\prime}$	$\mu_2(y) + m_2(y)e^{i\alpha(x-ct)}$	μ_e^*
Thermal Conductivity	$k^* = \bar{k}^* + k^{*\prime}$	$\dfrac{\mu(y)}{\sigma} + \dfrac{m(y)}{\sigma}e^{i\alpha(x-ct)}$	$c_p\,\mu_e^*$
Wave Length	λ^*	λ	$\sqrt{\nu_e^* x^*/u_e^*}$
Wave Number	$\alpha^* = \dfrac{2\pi}{\lambda^*}$	$\alpha = \dfrac{2\pi}{\lambda}$	$\sqrt{u_e^*/\nu_e^* x^*}$
Disturbance Propagation Velocity	c^*	c	u_e^*

A	$\left(\frac{w_c''}{w_c'} - \frac{T_c'}{T_c}\right)$
$\tilde{A}$	$\alpha^2 + \alpha Re\, c_i$ (Appendix H)
a, b	coefficients of general temperature fluctuation boundary condition (Section III. 3, Appendix F)
B	$\left(\frac{w_c''^2}{4w_c'^2} + \frac{w_c'''}{3w_c'} - \frac{T_c''}{2T_c}\right) - A\frac{T_c'}{T_c}$
$\tilde{B}$	$\alpha Re\,(1 - c_r)$ (Appendix H)
F	inviscid f function (Sections III. 3. and III. 4., Appendix F)
F(z)	Tietjens function (Appendix D, Table II)
$\mathcal{F}$	f' (Appendices D and G)
G	$\pi'/\alpha^2\pi$ (Section III, Appendices A and B)
$\tilde{G}(z)$	auxiliary function (Appendix D, Table II)
$\hat{G}'(\zeta)$, $\hat{G}''(\zeta)$, $\hat{H}'(\zeta)$	Schlichting's functions (Section III. 4., Appendix G, Table I)
H	θ'/θ (Section III, Appendices B and E)
J	ϕ/θ (Section III, Appendices B and E)
K	f/θ (Section III, Appendices B and E)
K	$\frac{(\gamma-1)M_e^2c^2}{T_w}$ (Section V, Appendix D)
L	ϕ/f (Section III, Appendices B and E)
ℓ	length of plate (characteristic longitudinal length for mean flow)
M	f'/f (Section III, Appendices B and E)
M_e	local Mach number outside the mean boundary layer
M_{ref}	$M_e\sqrt{\frac{1}{T_{ref}}}$, Mach number evaluated using reference fluid properties

N	θ/f	(Section III, Appendices B and E)
P	ϕ'/ϕ	(Appendix H)
p	$\frac{z^{-3/2}}{\sqrt{2}}$	(Appendix D)
$Q(x, y, t)$	quantity of the total flow	
$\overline{Q}(x, y)$	mean or steady component of flow quantity	
$Q'(x, y, t)$	fluctuating component of flow quantity	
$q(y)$	fluctuation amplitude for nearly parallel flow	
$\mathcal{R}$	right hand side of secular equation (Section III. 3, Appendices D and F)	
R^*	gas constant	
Re	Reynolds number based on reference length	
Re_x^*	length Reynolds number	
Re_θ	Reynolds number based on momentum thickness	
Re_{ref}	Reynolds number based on reference length but with fluid properties evaluated at T_{ref}	
S_1	$\frac{1}{\gamma M_e^2}\frac{\sigma T_c T_c'}{N_c' \nu_c}(\alpha Re)^{1/3}\left(\frac{dY_o}{dy}\right)_c^{-3/2}$	(Appendix G)
S_2	$\frac{1}{\gamma M_e^2}\left(\frac{w_c''}{w_c'}\right)\frac{T_c}{\nu_c}(\alpha Re)^{1/3}\left(\frac{dY}{dy}\right)_c^{-3/2}$	(Appendix G)
T_o	stagnation temperature of external stream	(Section IV)
T_{ref}	reference temperature of mean boundary layer	
Y	Tollmien variable for momentum equation (Section III. 4., Appendices D and G)	
Y_o	Tollmien variable for energy equation (Section III. 4, Appendices D and G)	
$\tilde{y}$	complex normal coordinate	(Appendix B)
z	$-\zeta_w$	(Appendix D)
z_o	$-\zeta_{ow}$	(Appendix D)

α_θ	momentum thickness wave number $2\pi\theta/\lambda$
γ	ratio of specific heats
δ	boundary layer thickness
δ_w	thickness of viscous layer near wall,
δ_c	thickness of viscous layer about critical point, of order $(\alpha Re)^{-1/3}$
ϵ	small parameter $(\alpha Re)^{-1/3}$ for Lees-Lin ordering $(\alpha Re)^{-1/2}$ for Dunn-Lin ordering
$\bar{\epsilon}$	small parameter $(\alpha Re_{ref})^{-\frac{1}{2}}$ for present ordering
ζ	normal variable for Lees-Lin expansion about critical point (Section III. 2)
ζ	$(\alpha Re)^{1/3}$ Y (Section III. 4., Appendices D and G)
ζ_o	$(\alpha Re)^{1/3}$ Y (Section III. 4., Appendices D and G)
η	dimensionless normal distance in physical plane (Section IV)
η	$(y - y_c)$ (Appendices A and B)
$\tilde{\eta}$	$(y - \tilde{y}_c)$ (Appendix B)
$\tilde{\tilde{\eta}}$	Dorodnitsyn-Howarth variable (Section III. 1.)
Θ	inviscid θ function (Sections III. 3. and III. 4., Appendix F)
Θ	$\theta \sqrt{dY_o/dy}$ (Appendices D and G)
θ	boundary layer momentum thickness
λ	wave length of disturbance
λ	In Appendix D defined by Eq. (D-31)
ν	kinematic viscosity
σ	Prandtl number
τ	Reynolds stress (Section V)
$\Phi(y)$	inviscid ϕ function (Sections III. 3., III. 4., Appendix F)
$\Phi(z)$	modified Tietjen's function (Appendix D, Table II)
ψ	$1/(1+\frac{i\omega_w'}{c}R)$

Subscripts

c	quantity evaluated at critical point
corr	inviscid function corrected for viscous effects about critical layer (Section III. 4.)
e	local condition outside mean boundary layer (external)
i	imaginary part of quantity
inv	inviscid
o	outer condition
r	real part of quantity
s	quantity for neutral inviscid disturbance (Sections III. 1. and IV)
v	viscous
w	quantity evaluated at the wall
(0), (1), (2), ...	zero, first, second order quantities, etc. (Appendix G)

A bar over a quantity indicates mean value.

Primes generally denote differentiation with respect to y. The few instances where primes denote a fluctuating quantity should not cause any confusion.

I. INTRODUCTION

The stability of a compressible laminar boundary layer to infinitesimal disturbances was first analyzed by Lees and Lin[1,2*]. Their study was in the form of an extension to compressible flow of the principles and techniques already formulated for the study of the stability of incompressible laminar boundary layers. Lees and Lin uncovered some of the important changes both physical and mathematical which are incurred by considering compressibility in the stability analysis. More recently there have been additional analyses[3-5] and also two experiments[6,7] which have tried to clarify the nature of compressible boundary layer stability. These investigations have clarified the stability picture for subsonic and slightly supersonic boundary layers but have hardly been successful in the case of supersonic and hypersonic laminar boundary layers. The purpose of the present study is to probe into the effects of compressibility on the various physical processes associated with the stability phenomenon.

Two basic problems arise in formulating the stability analysis for supersonic and hypersonic boundary layers. These are: (1) the identification of the important physical phenomena of compressible flow stability (or instability); and (2) obtaining equations and solutions to these equations which can adequately describe these phenomena.

The present study considers only "subsonic" disturbances, that is, disturbances whose propagation velocity is subsonic with respect to the free stream velocity $\left[\left(1 - \frac{1}{M_e} \right) < c < 1 \right]$. Such

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* Superscripts denote references at the end of the text.

disturbances have amplitudes which decay exponentially in the free stream. A disturbance which propagates supersonically with respect to the free stream would be expected to have a non-vanishing amplitude far from the wall. It may be noted that in the recent experiment at Mach number 2.2 by Laufer and Vrebalovich[6], supersonic disturbances were not detected and reasonable agreement was obtained with the theory of infinitesimal subsonic disturbances.

For subsonic and slightly supersonic flows, Lees and Lin[1] concluded that the stability characteristics of a given boundary layer profile are unaffected by the boundary conditions on temperature fluctuations. More specifically, the characteristics are determined by satisfying only the velocity fluctuation boundary conditions. Dunn and Lin[5] found that this conclusion is not valid for moderately high supersonic Mach numbers and they gave some discussion of the thermal boundary condition. However, they did not present any calculations which include consideration of the energy equation and thermal boundary conditions.

The analyses of Lees and Lin and Dunn and Lin are first order asymptotic approximations valid when a parameter (α Re), the product of wave number and Reynolds number, is very large. Among the terms which do not enter into this first asymptotic approximation are terms involving dissipation and terms involving fluctuating viscosity and fluctuating thermal conductivity. Cheng[3] points out that terms involving vertical velocity enter into the second approximation. It will be pointed out later that a dissipation term and some terms involving fluctuating transport properties also enter in the second approximation. Since some of these terms increase in magnitude with increase in Mach number, it is necessary to include them at high Mach number.

In the past, approximate methods have been used to solve the asymptotic equations. These methods were valid only for small values of the wave number and for propagation velocities which are not very close to the free stream velocity. For supersonic and hypersonic Mach numbers, larger values of wave number and propagation velocities approaching free stream velocity are encountered and more exact numerical methods will be considered.

The present study then considers the stability of two-dimensional compressible laminar boundary layers to two-dimensional subsonic disturbances. Because the present study is of a probing nature, only the simplest model of a compressible gas is considered -- namely, one with constant specific heats, constant Prandtl number and viscosity a function of temperature alone. The analysis considers both insulated and non-insulated surfaces; however, the numerical examples will be for insulated surfaces only and comparison will be made with the experimental findings of Laufer and Vrebalovich[6] and of Demetriades[7].

Lastly, in the section entitled "Qualitative Description of Compressible Boundary Layer Stability" the effects of compressibility on the physical concepts of the boundary layer stability phenomenon are discussed with the aid of some approximate calculations. Those readers having some familiarity with the stability problem at low speeds may want to read this section first.

II. FORMULATION OF PROBLEM

II. 1. Differential Equations for Infinitesimal Disturbances

Quantities of the total flow such as velocity and temperature are considered to be composed of a mean or steady component which depends only on the space coordinates, and a space-and-time-dependent fluctuating component of infinitesimal magnitude (Eq. 1). The total flow

$$Q(x, y, t) = \bar{Q}(x, y) + Q'(x, y, t) \tag{1}$$

satisfies the time-dependent conservation laws of mass, momentum and energy, while the mean flow satisfies some set of steady flow equations, for example, the boundary layer equations. Subtraction of the mean flow equations from the total flow equations yields the set of conservation law equations satisfied by the disturbances.

Since it is stipulated that the fluctuation amplitudes are very small compared to the mean flow quantities, products and squares of fluctuation quantities are neglected. The resulting equations are then linear partial differential equations in the variables (x, y, t).

So long as $\bar{Q}$ is independent of time, the coefficients of these linear partial differential equations are also independent of time; in fact, time appears only as the linear operator $\partial/\partial t$.* The assumption of disturbances of the form

$$Q' = \tilde{\tilde{q}}(x, y)\, e^{i\alpha c t} \tag{2a}$$

will then eliminate the time variable from the equations and reduce the

* This remark applies also for unsteady mean flows where the percentage changes of the mean profiles in one oscillation period are very small.

independent variables to just the space variables. The additional assumption that the longitudinal (x) variation of q is composed of a rapidly varying part depending on the frequency of the disturbance, and a slowly varying part depending on the decay or growth of the oscillation amplitude, yields the traveling plane wave disturbance form

$$Q' = \tilde{q}(x, y)\, e^{i\alpha(x-ct)} \tag{2b}$$

Substitution of (2b) into the above-mentioned disturbance equations leads to the equations presented by Cheng[3]. Cheng's equations, however, comprise a more general set than is usually considered.

If the mean flow $\overline{Q}$ is a time-independent parallel flow (no mean normal velocities) and remains unchanged to infinity in both directions (such as for a fully developed flow) then the coefficients of the linear partial differential disturbance equations are independent of both x and t so that a disturbance of the form

$$Q' = q(y)\, e^{i\alpha(x-ct)} \tag{3}$$

will reduce the disturbance equations to ordinary differential equations in y alone. The disturbance amplitude q(y) and the propagation velocity in Eq. (3) are taken to be complex. Disturbances are amplified, neutral or damped according to whether $c_i > 0$, $c_i = 0$, or $c_i < 0$, respectively. The real part c_r is the dimensionless velocity of wave propagation.*

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* Eq. (3) is not the only disturbance form possible for parallel flows. It is of some interest in fact to investigate amplified and damped disturbances of the form

$$Q' = q(y)\, e^{\beta x}\, e^{i\alpha(x-ct)}$$

where α, β, and c are all real. Here $\beta = \frac{1}{\sqrt{Q'^2}} \frac{\partial \sqrt{Q'^2}}{\partial x}$ is the amplification coefficient. For neutral disturbances $\beta = 0$ and the analysis is the same as that for $c_i = 0$ in Eq. (3).

Lees and Lin[1] propose the use of the parallel flow disturbance equations for "nearly-parallel" boundary layer flows. Therefore, they omit terms of the following types from the complete disturbance equations:

(1) terms involving mean normal velocity $\bar{v}^*$. The ratio $\bar{v}^*/\bar{u}^*$ is of order

$$\frac{\bar{v}^*}{\bar{u}^*} \sim \frac{M_e^2}{\sqrt{Re_{x^*}}} \sim \frac{M_e^2}{Re_\theta} \sim \frac{M_e^4}{Re_\delta} ,$$

and is assumed to be small for high Reynolds numbers;

(2) longitudinal derivatives of mean flow quantities as compared to their normal derivatives, that is

$$\frac{\partial \bar{Q}}{\partial x} \Bigg/ \frac{\partial \bar{Q}}{\partial y} \sim \delta/\ell \quad << 1 \quad .$$

This ordering is in accordance with the mean flow boundary layer assumptions.

(3) longitudinal derivatives of disturbance amplitudes, so that $\tilde{q}(x, y) \longrightarrow q(y)$. If the longitudinal scale of the disturbance flow is taken to be the wave length λ , then this deletion corresponds to saying that

$$\frac{\partial \tilde{q}(x, y)}{\partial x} \Bigg/ \frac{\partial \tilde{q}(x, y)}{\partial y} \sim \delta/\lambda \quad << 1 \quad .$$

A remark concerning the relation between the "nearly parallel" flow assumptions and the actual physical situation in flow over a cylinder of arbitrary cross-section is in order. By means of the nearly parallel flow assumptions one calculates the stability of a local mean boundary layer profile as if only that profile existed from $-\infty$ to $+\infty$. Thus the history of any small disturbance moving downstream is found by piecewise integration of local amplification or decay rates. This procedure is directly analogous to the calculation of mean boundary layer flows by the piecewise integration of locally similar solutions.

By means of an order of magnitude analysis of the type used to obtain the steady boundary layer equations Dunn and Lin[5] reduce the disturbance equations still further by deleting all terms of order $1/(Re_x)^{\frac{1}{2}}$ or smaller, compared to those retained. Among the additional terms deleted are the following:

(4) dissipation terms in the disturbance energy equation.*

(5) terms involving fluctuations of viscosity and thermal conductivity; in other words, only the leading fluctuating shear stress gradient and heat flux gradient terms are retained.

The equations considered by Dunn and Lin are valid for moderate supersonic Mach numbers. Later (Section III. 3.) it will be shown that for high supersonic Mach numbers, the dissipation terms and terms involving fluctuating viscosity and thermal conductivity must be reconsidered, since they are comparable to the terms previously retained. At this point therefore, we will consider the disturbance equations to be those of Lees and Lin. For a disturbance of the form (3) the dimensionless equations for infinitesimal disturbances in a nearly parallel mean flow are as follows[1]:

<u>Continuity</u>

$$\phi' + if - \frac{T'}{T}\,\phi + i\,(w - c)\,(\pi - \frac{\theta}{T}) = 0 \qquad (4)$$

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* The absence of dissipation terms makes possible a mathematical transformation of the equations for three dimensional disturbances to those for two dimensional disturbances[5].

Momentum

$$\alpha\rho\left\{i(w-c)f+w'\phi\right\} = \frac{i\alpha\pi}{\gamma M_e^2}+\frac{\mu}{Re}\left\{f''+\alpha^2(i\phi'-2f)\right\}$$
$$+\frac{2}{3}\frac{(\mu_2-\mu)}{Re}\alpha^2\left\{i\phi'-f\right\}$$
$$+\frac{1}{Re}\left\{mw''+m'w'+\mu'(f'+i\alpha^2\phi)\right\} \qquad (5)$$

$$\alpha^2\rho\left\{i(w-c)\phi\right\} = -\frac{\pi'}{\gamma M_e^2}+\frac{\mu\alpha}{Re}\left\{2\phi''+if'-\alpha^2\phi\right\}+\frac{2\alpha}{3}\frac{(\mu_2-\mu)}{Re}\left\{\phi''+if'\right\}$$
$$+\frac{\alpha}{Re}\left\{imw'+2\mu'\phi'+\frac{2}{3}(\mu_2'-\mu')(\phi'+if)\right\} \qquad (6)$$

Energy

$$\alpha\rho\left\{i(w-c)\theta+T'\phi\right\} = i\alpha(w-c)\left(\frac{\gamma-1}{\gamma}\right)\pi+\frac{(\gamma-1)M_e^2}{Re}\left\{mw'^2+2\mu w'(f'+i\alpha^2\phi)\right\}$$
$$+\frac{1}{\sigma Re}\left\{\mu(\theta''-\alpha^2\theta)+(mT')'+\mu'\theta'\right\} \qquad (7)$$

The density fluctuations have been eliminated in the above equations through the equation of state:

$$r/\rho = \pi - (\theta/T) \qquad (8)$$

The energy equation, Eq. (7), differs slightly from that of Lees and Lin in that it is derived from the enthalpy equation rather than the internal energy equation. The fluctuating viscosity can be related to the temperature fluctuation through

$$m = \theta\left[(d\mu)/(dT)\right] \qquad (9)$$

while the normal gradient of mean viscosity can be expressed:

$$\mu' = T' \left[(d\mu) / (dT)\right] \tag{10}$$

II. 2. Boundary Conditions

For flows over non-porous surfaces, the longitudinal and normal velocity components of the total non-steady flow must vanish at the surface. Since the mean flow already satisfies these conditions, the disturbance velocity amplitudes must also vanish at the surface. Thus

$$f_w = 0 \tag{11}$$

$$\phi_w = 0 \tag{12}$$

For compressible flows, the surface boundary conditions on temperature fluctuations must also be specified. For the total non-steady flow, the surface boundary condition is that the instantaneous temperature and heat transfer must be continuous across the solid-gas interface. However, most surface materials are highly conductive compared to gases and so would immediately damp any temperature fluctuations. A reasonable, almost universal boundary condition is that the temperature fluctuation amplitude vanish at the surface; that is

$$\theta_w = 0 \tag{13}$$

A somewhat more general boundary condition is that some linear combination of temperature fluctuation amplitude and its normal derivative must vanish at the surface (Appendix A of Reference 5). The specific combination is a function of the surface material, its thickness, the method of cooling, and the disturbance frequency under consideration. Boundary condition (13) is the one primarily considered herein although there will be some treatment of the general boundary condition (Appendix

F).

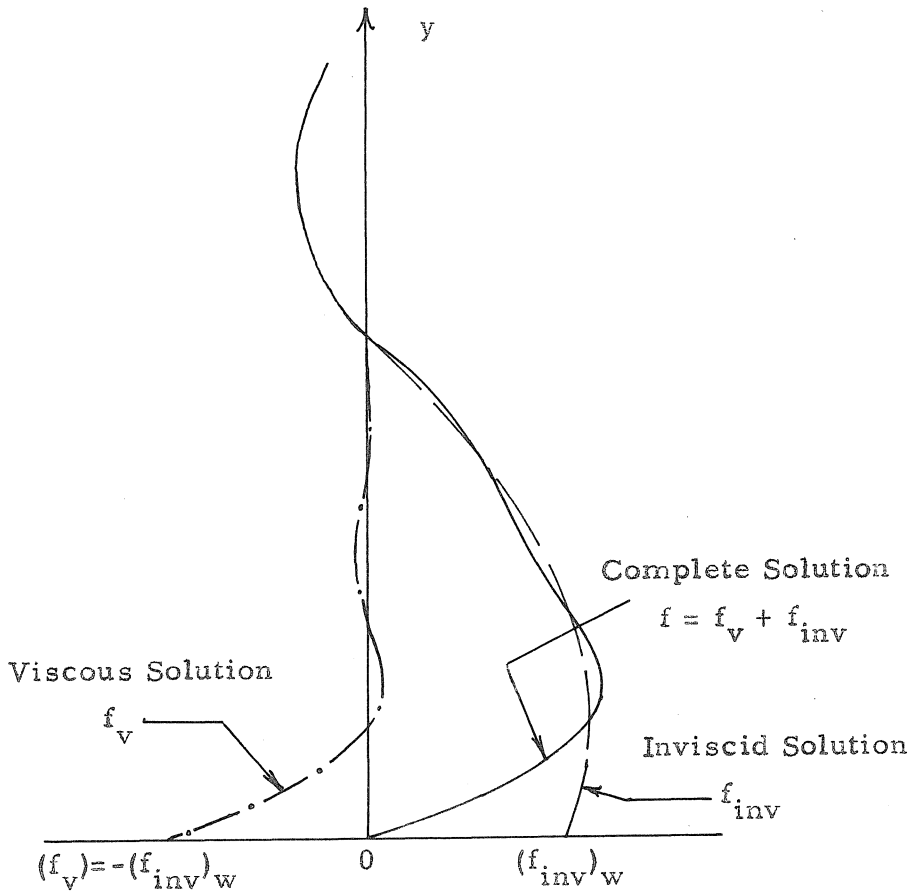
Since only subsonic disturbances are here considered (see Introduction) all disturbance amplitudes vanish far from the wall, i.e.,

$$q(y) \longrightarrow 0 \quad \text{as} \quad y \longrightarrow \infty \tag{14}$$

III. SOLUTION OF DIFFERENTIAL EQUATIONS

As discussed in Reference 1, the disturbance equations [Eqs. (4) - (8)] are regular everywhere except in the limit $y \longrightarrow \infty$; and the solutions of these equations are analytic functions of α, c and Re for all finite values of these parameters. In principle, solutions could be constructed as convergent series expansions around one or more regular points, and these series could be properly joined to exponentially decaying functions as $y \longrightarrow \infty$. However, the quantity $(\alpha Re)^{-1}$ appears in the disturbance equations as a parameter multiplying the highest order derivatives, and it is attractive to consider asymptotic expansions valid for $(\alpha Re) >> 1$.

In particular, we inquire as to the significance of the solutions obtained in the limit of $(\alpha Re) \longrightarrow \infty$. In general, these "inviscid" solutions are certainly incapable of satisfying the boundary conditions $f_w = 0$ and $\theta_w = 0$ at the surface. Thus for any finite value of (αRe) viscous solutions that take on the values $-(f_w)_{inv}$ and $(-\theta_w)_{inv}$ at the surface must always be added (see sketch below) and the situation locally has some of the characteristics of the oscillating plate problem. Since the parameter $(\alpha Re)^{-\frac{1}{2}}$ measures the relative diffusion distance for vorticity during one period (or $(\alpha Re\sigma)^{-\frac{1}{2}}$ measures the corresponding diffusion distance for heat energy), this "inner boundary layer" is thin when $(\alpha Re) >> 1$. Thus we are led to adopt Prandtl's[8] division of the disturbances into slowly-varying solutions that are largely inviscid across the entire flow, and "viscous" rapidly varying functions near the surface. Because of the steep slope of the viscous solutions (see sketch) it is clear that the effect of viscous dissipation at high Mach



SCHEMATIC REPRESENTATION OF
VISCOUS AND INVISCID SOLUTIONS FOR f NEAR SURFACE
(A similar sketch could be drawn for θ.)

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numbers cannot be neglected a priori.

Because the viscous longitudinal velocity fluctuation and density fluctuation amplitudes vary sinusoidally in the x- direction, an incremental normal velocity fluctuation amplitude $\Delta\phi_v$ is induced across the boundary layer, according to the equation of continuity [Eq. (4)]. If we require that all normal velocity fluctuations die out as $y \longrightarrow \infty$ then $\phi_v(0) = -\Delta\phi_v$. The value of ϕ_{inv} at the surface can no longer be zero as it was for $\alpha Re \longrightarrow \infty$. The inviscid solution must adjust slightly giving a value $\phi_{inv}(0)$ such that the wall boundary condition $\phi_v(0) + \phi_{inv}(0) = 0$

is satisfied.* Thus for each value of c the viscous and inviscid solutions and the corresponding values of α and Re are determined uniquely by the conditions $\phi_v(0) = -\phi_{inv}(0)$, $f_v(0) = -f_{inv}(0)$, and $\theta_v(0) = -\theta_{inv}(0)$.

There is another region in which the inviscid solutions cannot be valid in general, and that is the so-called "critical layer" -- the layer of fluid moving at the disturbance propagation velocity. To an observer riding with the disturbance the rates of transport of vorticity and heat in the main flow direction vanish at this layer, but the rates of transport of these quantities in the normal direction do not. Thus viscous diffusion and heat conduction must restore the balance. So long as $[\alpha Re(1-c)] >> 1$ the effects of viscosity and heat conductivity die out rapidly with distance on either side of the critical layer, the amplitude distributions are quickly "smoothed over" there, and the phase shifts in f and θ across this thin layer are given very nearly by the inviscid solutions. On the other hand if (αRe) is of order unity the influence of this inner viscous region spreads inward toward the plate surface, while the viscous layer near the surface spreads out to meet it. In that event the distinction between viscous and inviscid solutions may become meaningless. Difficulties also arise if the quantity $[\alpha Re(1-c)]$ is of order α^2 (or smaller), because the "inviscid" and "viscous" solutions would die out at comparable rates as $y \longrightarrow \infty$,** and the distinction between them would be meaningless in this case also.

Of course the value of αRe is not arbitrary; for a neutral

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* A detailed discussion of this behavior for αRe large but finite is given in Section V.

** This difficulty arises when $c \longrightarrow 1$.

disturbance (for example) it is uniquely determined by the local velocity-temperature profile and the local Mach number. At present there is no way of determining in advance whether the quantity (αRe) is always "sufficiently large" for a neutral disturbance at high Mach numbers. Therefore, we shall proceed provisionally with Prandtl's suggestion, just as previous investigators have done for low speed flow or moderate supersonic Mach numbers. However, this splitting of the solutions must be reexamined a posteriori to determine the conditions under which it is in fact justified; and, conversely, the conditions under which we must return to the complete disturbance equations [Eqs. (4) - (8)].

III. 1. Inviscid Solution

Following Rayleigh, Heisenberg and Lin, a solution is sought to the disturbance equations of the form

$$q(y) = q_o(y) + \left[1 / (\alpha Re)\right] q_1(y) + \ldots \tag{15}$$

The resulting equations for the zeroth approximation (q_o) representing $\lim_{\alpha Re \to \infty} q(y)$ for a fixed y are called the inviscid equations since they are identical with the equations obtained by ignoring viscosity and conductivity altogether. The inviscid equations are:*

Continuity

$$\phi' + if - (T'/T)\,\phi + i(w - c)\left[\pi - (\theta/T)\right] = 0 \tag{16}$$

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* The subscript zero is omitted since the q_o functions are the only ones which will be obtained.

Momentum

$$i \rho (w - c) f + \rho w' \phi = - \frac{i\pi}{\gamma M_e^2} \tag{17}$$

$$i \alpha^2 \rho (w - c) \phi = - \frac{\pi'}{\gamma M_e^2} \tag{18}$$

Energy

$$i \rho (w - c) \theta + \rho T' \phi = i (w - c) \left(\frac{\gamma - 1}{\gamma} \right) \pi \tag{19}$$

The quantities f and θ can be eliminated from Eqs. (16), (17), and (19) so that Eqs. (16) - (19) can be written[1]

$$\left. \begin{aligned} \phi' &= \frac{w'}{w - c} \phi + \frac{i}{\gamma M_e^2} \left[\frac{T - M_e^2 (w - c)^2}{w - c} \right] \pi \\ \pi' &= - i \alpha^2 \gamma M_e^2 \frac{(w - c)}{T} \phi \end{aligned} \right\} \tag{20}$$

Eqs. (20) can be written as a second order linear differential equation in either of the dependent variables ϕ or π. It has been customary in the past to consider the solution of the second order equation in the variable ϕ, which is proportional to the normal velocity fluctuation amplitude. In the present analysis, following a suggestion of Lighthill[9] and some work on panel flutter by Miles[10], the inviscid equation will be written in terms of the pressure fluctuation amplitude π. The boundary condition which the inviscid solution must always satisfy is that the normal velocity fluctuation vanish in the outer inviscid flow. If viscosity and conductivity are ignored ($\alpha Re \longrightarrow \infty$) the inviscid normal velocity fluctuation amplitude must also vanish at the wall; therefore, for $\alpha Re \longrightarrow \infty$, $\pi'(0) = 0$ by Eq. (18). However, for αRe large but finite

$\phi_{inv}(0) = -\phi_v(0) \neq 0$ and thus $\pi'_{inv}(0) \neq 0$.

The inviscid equation and boundary conditions are

$$\pi'' = \left(\frac{2w'}{w-c} - \frac{T'}{T} \right) \pi' + \alpha^2 \left[1 - \frac{M_e^2 (w-c)^2}{T} \right] \pi \tag{21}$$

$$\pi'(0) = 0 \quad (\text{for } \alpha Re \longrightarrow \infty \text{ only}) \tag{22a}$$

$$\pi(\infty) \longrightarrow 0 \quad . \tag{22b}$$

By means of the standard transformation

$$G = \frac{\pi'}{\alpha^2 \pi} \quad , \tag{23}$$

Eq. (21) can be converted into the following first order non-linear equation of the Riccati type

$$G' = \left[1 - \frac{M_e^2 (w-c)^2}{T} \right] + \left(\frac{2w'}{w-c} - \frac{T'}{T} \right) G - \alpha^2 G^2 \quad . \tag{24}$$

The outer boundary condition on G is obtained by considering Eq. (21) for large y:

$$\pi'' - \alpha^2 \left[1 - M_e^2 (1-c)^2 \right] \pi = 0 \quad , \tag{25}$$

whose solutions are

$$\pi \sim e^{\pm \sqrt{1 - M_e^2 (1-c)^2}\, y} \quad . \tag{26}$$

Since $\pi(\infty) \longrightarrow 0$, the negative exponent is chosen in Eq. (26), and the boundary conditions on G are

$$G(0) = 0 \quad (\text{for } \alpha Re \longrightarrow \infty \text{ only}) \tag{27a}$$

$$G(\infty) = - \frac{\sqrt{1 - M_e^2 (1-c)^2}}{\alpha} \quad . \tag{27b}$$

It is instructive at this point to transform the inviscid equation by introducing a Dorodnitsyn-Howarth independent variable. In this form, the boundary layer thickness is normalized.

For $$\tilde{\tilde{\eta}} = \int dy/T \tag{28}$$

and

$$\tilde{\tilde{G}} = \frac{1}{\alpha^2 T^2 \pi} \frac{d\pi}{d\tilde{\tilde{\eta}}} , \tag{29}$$

the inviscid equation Eq. (24) becomes:

$$\frac{d\tilde{\tilde{G}}}{d\tilde{\tilde{\eta}}} = \left[1 - \frac{M_e^2 (w-c)^2}{T}\right] + \left(\frac{2\frac{dw}{d\tilde{\tilde{\eta}}}}{w-c} - \frac{2\frac{dT}{d\tilde{\tilde{\eta}}}}{T}\right)\tilde{\tilde{G}} - (\alpha T_{ref})^2 \left(\frac{T}{T_{ref}}\right)^2 \tilde{\tilde{G}}^2 , \tag{30}$$

where T_{ref} is some representative boundary layer temperature of order 1 for low speed flows but of order M_e^2 for high speed flows.

Following Heisenberg, it has been customary in the past to solve the inviscid equation in the form of a convergent series in powers of α^2. Eq. (30) suggests that the proper expansion parameter for the compressible inviscid solutions is $(\alpha T_{ref})^2$ or $(\alpha M_e^2)^2$ rather than α^2. At high Mach number, even for small α, the quantity (αT_{ref}) may not be small, so that the complete Eq. (30) must be considered.*

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* Of course if the non-linear term in Eq. (24), $\alpha^2 G^2$, is suppressed, we obtain immediately the solution corresponding to the zeroth order inviscid solution of Lees and Lin[1, 2]; namely

$$G_r(y) = \frac{(w-c)^2}{T} \int_{y_c}^{y} \frac{T - M_e^2 (w-c)^2}{(w-c)^2} dy$$

$$G_i(y) = \begin{cases} \frac{A\pi T_c}{w_c'^2} \frac{(w-c)^2}{T} & \text{for } y < y_c \\ 0 & \text{for } y > y_c \end{cases}$$

where

$$A \equiv \frac{w_c''}{w_c'} - \frac{T_c'}{T_c}$$

Eq. (24) is a complex equation. For the purposes of solution it is divided into real and imaginary parts. For neutral disturbances ($c_i = 0$) these equations are:

$$G_r' = \left[1 - \frac{M_e^2(w-c)^2}{T}\right] + \left[\frac{2w'}{w-c} - \frac{T'}{T}\right] G_r - \alpha^2\left(G_r^2 - G_i^2\right) \qquad (31a)$$

$$G_i' = \left[\frac{2w'}{w-c} - \frac{T'}{T}\right] G_i - \alpha^2\left(2G_rG_i\right) \qquad (31b)$$

where

$$G_r = \frac{\pi_r'}{\alpha^2\pi_r} \frac{\left(1 + \frac{\pi_i}{\pi_r}\frac{\pi_i'}{\pi_r'}\right)}{\left(1 + \frac{\pi_i^2}{\pi_r^2}\right)} \qquad (32a)$$

$$G_i = \frac{\pi_i}{\alpha^2\pi_r} \frac{\left(\frac{\pi_i'}{\pi_i} - \frac{\pi_r'}{\pi_r}\right)}{\left(1 + \frac{\pi_i^2}{\pi_r^2}\right)} \qquad (32b)$$

Since in the outer inviscid flow $\pi_i'/\pi_i = \pi_r'/\pi_r$ [from Eqs. (25) and (26)] the outer boundary conditions for Eqs. (31) are

$$\left.\begin{aligned} G_r(\infty) &= -\frac{\sqrt{1 - M_e^2(1-c)^2}}{\alpha} \\ G_i(\infty) &= 0 \end{aligned}\right\} \; . \qquad (33)$$

There is a regular singularity of Eqs. (31) at the point where $w = c$. This point is often called the "critical point". The solution in the neighborhood of this singular point is obtained by series expansion (method of Frobenius), the details of which are given in Appendix A. The resulting behavior of G about the critical point is

$$G_r = -(y-y_c) - A(y-y_c)^2 \ln|y-y_c| + (\text{const})(y-y_c)^2 - A^2(y-y_c)^3 \ln|y-y_c| + \left[A(\text{const}) - \left(2B - 2A^2 + \frac{M_e^2 w_c'^2}{T_c} + \alpha^2\right)\right](y-y_c)^3 + \ldots \tag{34}$$

For $(y - y_c) > 0$

$$G_i = 0 \quad , \tag{35a}$$

For $(y - y_c) < 0$

$$G_i = A\pi (y - y_c)^2 \left[1 + A(y - y_c) + \ldots\right] , \tag{35b}^*$$

where

$$A \equiv \left(\frac{w_c''}{w_c'} - \frac{T_c'}{T_c}\right) = \frac{T_c}{w_c'}\left[\frac{d}{dy}\left(\frac{w'}{T}\right)\right]_c \tag{36}$$

$$B \equiv \left(\frac{w_c''^2}{4w_c'^2} + \frac{w_c'''}{3w_c'} - \frac{T_c''}{2T_c}\right) - A\frac{T_c'}{T_c} \tag{37}$$

It is instructive at this point to examine the nature of the inviscid equation graphically. As an example, the case of the neutral inviscid oscillation ($\alpha Re \longrightarrow \infty$) is chosen. For such an oscillation the boundary condition at the wall is that $G_r(0) = G_i(0) = 0$. From Eq. (31b) it is seen immediately that if G_i is zero at the wall, then G_i must be zero everywhere. Thus from Eq. (35b), a necessary condition for a neutral inviscid oscillation is that

$$A \equiv \left(\frac{w_c''}{w_c'} - \frac{T_c'}{T_c}\right) = 0 \quad .$$

Lees and Lin[1] by a mathematical proof show that this condition is both necessary and sufficient. The value of the propagation velocity for which the condition $A = 0$ is satisfied, is denoted c_s. We are thus concerned

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* The π appearing without subscript in Eq. (35b) is 3.14159... .

with constructing the solution to Eq. (24) for $G = G_r$ with the boundary conditions

$$G(0) = 0$$

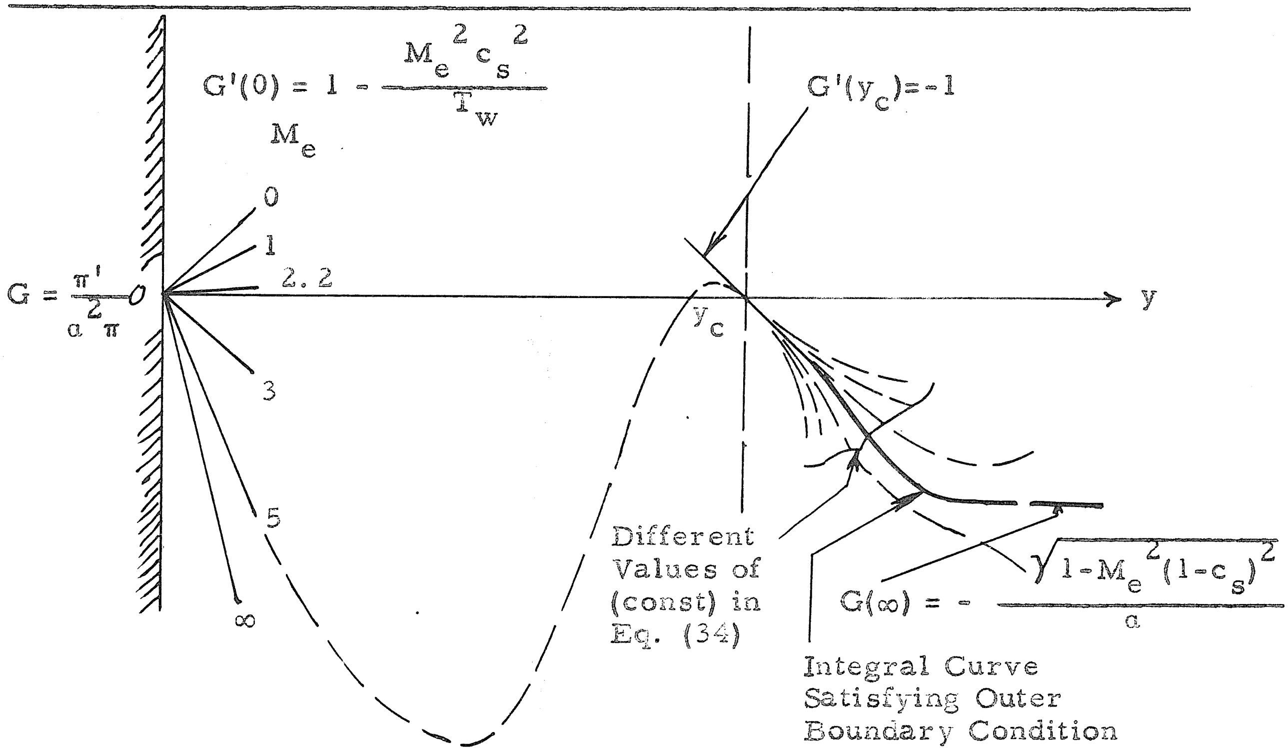
$$G(\infty) = - \frac{\sqrt{1 - M_e^2 (1-c_s)^2}}{\alpha} \quad .$$

The curves in the sketches which follow are drawn for insulated flat plate boundary layers with $\gamma = 1.4$.

Consider first the region about the critical point where the leading terms of the solution [Eq. (34)] are

$$G = -(y - y_c) + (\text{constant})(y - y_c)^2 + \ldots \quad .$$

The slope of G at the critical point is always -1. However, the curvature at this point is an undetermined constant. For a given α only one value of this constant will yield an integral curve that satisfies the outer boundary condition (see following sketch). Once this value of the constant is

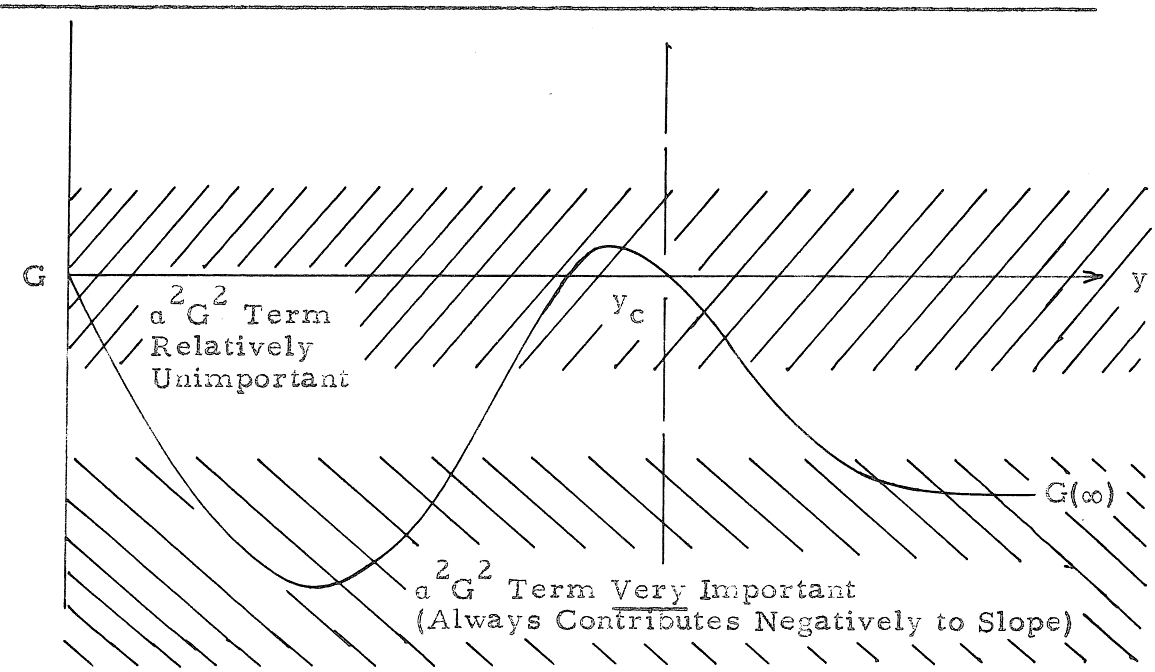


determined for a given α the integration can proceed inward from the critical layer toward the wall. For $c = c_s$, there is only one value of α for which the wall boundary condition $G(0) = 0$ is satisfied. This value will be denoted α_s. The integral curve for α_s approaches the wall boundary condition from above or below according to whether

$$G'(0) = 1 - \frac{M_e^{\,2} c_s}{T_w}$$

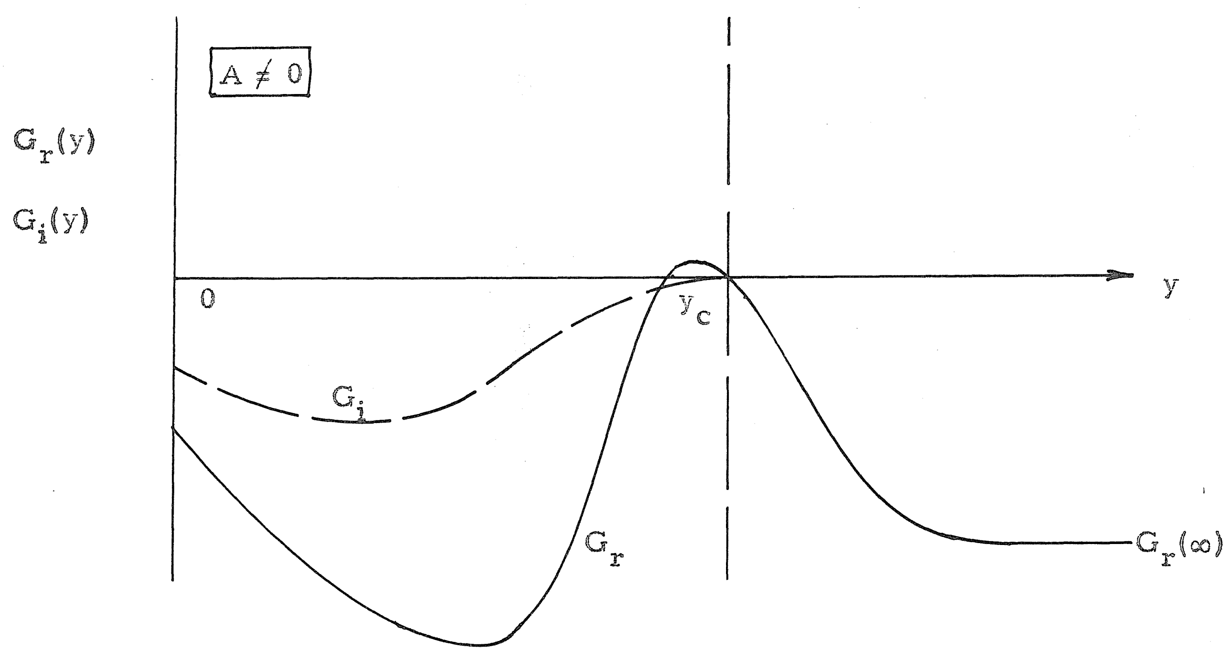
is positive or negative, respectively. Some representative slopes near the wall are shown in the sketch on page 20. For $M_e = 0$, the slope at the wall is + 1; for infinite Mach number, the slope would be -4 for Prandtl number 1. The slope $G'(0)$ is zero for a Mach number slightly above 2.2. Since $\pi/\pi_w = \exp\left[\int_0^y \alpha_s^{\,2}\, G\, dy\right]$, positive areas under the G curve represent increases in pressure fluctuation amplitude relative to the wall value, while negative areas represent decreases.

The non-linear term in Eq. (24) is important only when G is very large (see following sketch). However, it is always important near the



outer boundary, since the outer boundary condition is determined by a balance of the $\left[1 - \frac{M_e^2 (w-c)^2}{T}\right]$ and $\left[-\alpha^2 G^2\right]$ terms of Eq. (24).

For a neutral disturbance with (αRe) finite, the quantity A generally takes on a non-zero value ($A \lesseqgtr 0$ for $c \gtrless c_s$), and G takes on an imaginary part. Also the constant in Eq. (34) is no longer the curvature but only a parameter since the curvature at the critical point is logarithmically singular for $A \neq 0$. The character of the integral curves is shown in the following sketch. The curve for G_r is not very



different from that for $A = 0$, except that $G_r(0) \neq 0$. The imaginary part G_i is always of the same sign as A and also has a non-zero wall value. These non-zero values lead to $(\phi_{inv})_w \neq 0$, $(f_{inv})_w \neq 0$, and $(\theta_{inv})_w \neq 0$, and the action of viscosity is required to cancel these inviscid contributions and satisfy the wall boundary conditions.

The calculation procedure used to obtain the inviscid solution for a given profile $w(y)$, $T(y)$ and value of c is as follows:

Integration from Critical Point to Outer Edge

(1) Evaluate A and B [Eqs. (36) and (37)]. Choose a value of α. Evaluate the outer boundary condition from Eq. (33).

(2) Assume some value for the (const) in Eq. (34). Evaluate G_r for a small positive value of $(y - y_c)$.

(3) Continue calculation of G_r by integration of Eq. (31a) to the outer edge of the mean flow. Compare the result with the value from step (1).

(4) Repeat steps (2) and (3) adjusting the (const) until the outer boundary condition is satisfied.

Integration from Critical Point to Wall

(5) Using the value of (const) from step (4), evaluate G_r and G_i for some small negative value of $(y - y_c)$.

(6) Continue calculation of G_r and G_i by simultaneous integration of Eqs. (31).

The values of G_r and G_i at the wall $(y = 0)$ should be retained since they will be used in satisfying the boundary conditions for (αRe) finite.

The equations presented in this section are for neutral disturbances $(c_i = 0)$. The equations for amplified and damped disturbances $(c_i \neq 0)$ are very similar and are given in Appendix B.

If desired, the distribution of the inviscid disturbance amplitudes can be obtained as follows: Since the function $G(y)$ is defined as $G(y) \equiv \pi'/\alpha^2\pi$, the pressure fluctuation amplitude is obtained by direct integration:

$$\pi = e^{\int \alpha^2 G dy} = e^{\int_{y_c}^{y} \alpha^2 G_r dy} \left[\cos \int_{y_c}^{y} \alpha^2 G_i \, dy + i \sin \int_{y_c}^{y} \alpha^2 G_i \, dy \right]. \tag{38}$$

This integration is carried out starting from the critical layer since the imaginary parts of π and G_i have zero value outward from the critical layer (Appendix A). The real and imaginary parts of π are

$$\pi_r = e^{\int_{y_c}^{y} \alpha^2 G_r dy} \cos \int_{y_c}^{y} \alpha^2 G_i \, dy \tag{39}$$

$$\pi_i = e^{\int_{y_c}^{y} \alpha^2 G_r dy} \sin \int_{y_c}^{y} \alpha^2 G_i \, dy \tag{40}$$

By Eq. (38) the pressure fluctuation amplitudes are referred to their value at the critical layer, that is $\pi_c = \pi_{r_c} = 1$ is the reference pressure fluctuation amplitude.

From the inviscid equations [Eqs. (16) to (19)], the other fluctuation amplitudes can be related to the pressure fluctuation amplitude. Following the notation of Dunn and Lin[5], the inviscid functions are denoted by the capital letters Φ, F, and Ⓗ.

$$\Phi = i \frac{T\pi'}{\gamma M_e^2 \alpha^2 (w-c)} = i \frac{TG\pi}{\gamma M_e^2 (w-c)} \tag{41}$$

$$F = -\frac{T}{\gamma M_e^2 (w-c)} \left[\pi + \frac{w'\pi'}{\alpha^2 (w-c)} \right] = -\frac{T\pi}{\gamma M_e^2 (w-c)} \left[1 + \frac{w'G}{(w-c)} \right] \tag{42}$$

$$\Theta = T\left[\frac{\gamma-1}{\gamma}\pi - \frac{T'\pi'}{\gamma M_e^2\alpha^2(w-c)^2}\right] = T\pi\left[\frac{\gamma-1}{\gamma} - \frac{T'G}{\gamma M_e^2(w-c)^2}\right] \quad (43)$$

For further discussion of amplitude distributions, the reader is referred to Section III. 4.

III. 2. Viscous Solutions

No previous investigator has attempted to obtain analytical solutions of the full viscous equations, even at low Mach numbers. The present analysis is no exception. In order to better understand the present analysis, the assumptions and consequences of the Lees-Lin[1] and Dunn-Lin[4, 5] analyses will be reviewed.

The usual procedure in obtaining viscous solutions has been to solve a set of reduced equations that retain terms up to a certain order, either near the critical point or else near the surface. These sets of reduced equations are the same only if the surface is very close to the critical point. In fact this limitation applies to the Lees-Lin[1] theory in which a solution is obtained by convergent series expansion about the critical point, and is then utilized to satisfy the surface boundary conditions. On the other hand the Dunn-Lin analysis assumes a priori that the wall is far from the critical layer and obtains a set of reduced equations valid near the wall but not at the critical layer.

The reduced equations of Lees and Lin[1] and of Dunn and Lin[4, 5] are obtained by order of magnitude analyses. For the Lees-Lin case the ordering of terms is as follows:

$$\bar{Q},\ \bar{Q}' \sim 1,\quad d/dy \sim 1/\epsilon,\quad (w-c) \sim \epsilon,\quad f \sim 1,\quad \phi \sim \epsilon f,\quad \theta \sim f,\quad \pi \sim \epsilon^3 f \quad (44)$$

This order of magnitude analysis is carried out in Appendix C. It must be remembered that the ordering relation $d/dy \sim 1/\epsilon$ is valid only in a restricted region.

The leading terms of the disturbance equations under assumptions [Eq. (44)] form the following differential equations. These equations will be referred to as the Lees-Lin equations.

$$f''' - \frac{i\,\alpha Re\,(w-c)}{\nu} f' = 0 \tag{45}$$

$$\phi' + if = 0 \tag{46}$$

$$\theta'' - \frac{i\,\alpha Re\,\sigma\,(w-c)}{\nu}\,\theta = \frac{\alpha Re\,\sigma}{\nu} T'\,\phi \tag{47}$$

and

$$\epsilon \sim \frac{1}{(\alpha Re)^{1/3}} \tag{48}$$

Eqs. (45) to (47) are a sixth-order system of ordinary differential equations dependent only on the parameter (αRe). It is to be noted however that the continuity and momentum equations [Eqs. (46) and (45)] are a closed fourth order set independent of the energy equation; in fact it is the same set that is obtained for incompressible flow. As will be shown in the following section on "Eigenvalue Problem", this independence makes possible the determination of the stability characteristics from the velocity fluctuation boundary conditions alone; the temperature fluctuations are irrelevant in this case.

By the convergent series method of Lees and Lin, Eq. (45) is reduced to the form

$$\frac{d^3 f}{d\zeta^3} - i\zeta\,\frac{df}{d\zeta} = 0 \tag{49}$$

where

$$\zeta = \left(\frac{\alpha Re\, w_c'}{\nu_c}\right)^{1/3} (y - y_c) \tag{50}$$

The three solutions are

$$\left.\begin{aligned} f_1 &= \int \zeta^{\frac{1}{2}} H^{(1)}_{\frac{1}{3}}\left[\tfrac{2}{3}(i\zeta)^{\frac{3}{2}}\right] d\zeta \\ f_2 &= \int \zeta^{\frac{1}{2}} H^{(2)}_{\frac{1}{3}}\left[\tfrac{2}{3}(i\zeta)^{\frac{3}{2}}\right] d\zeta \\ f_3 &= 1 \end{aligned}\right\} \tag{51}$$

The four solutions of Eq. (46) are

$$\left.\begin{aligned} \phi_1 &= \left(\frac{\nu_c}{\alpha Re\, w_c'}\right)^{\frac{1}{3}} \int d\zeta \int \zeta^{\frac{1}{2}} H^{(1)}_{\frac{1}{3}}\left[\tfrac{2}{3}(i\zeta)^{\frac{3}{2}}\right] d\zeta \\ \phi_2 &= \left(\frac{\nu_c}{\alpha Re\, w_c'}\right)^{\frac{1}{3}} \int d\zeta \int \zeta^{\frac{1}{2}} H^{(2)}_{\frac{1}{3}}\left[\tfrac{2}{3}(i\zeta)^{\frac{3}{2}}\right] d\zeta \\ \phi_3 &= y - y_c \\ \phi_4 &= 1 \end{aligned}\right\} \tag{52}$$

Of the above solutions only ϕ_1 and f_1 are used. The solutions ϕ_2 and f_2 are rejected since they grow exponentially for large arguments and could not satisfy the outer boundary conditions. The solutions ϕ_3, ϕ_4, and f_3 form the rudiments of the inviscid solution (see Tollmien[12]), and are replaced by the inviscid solution already obtained. Solutions to the energy equation are not pursued in detail by Lees and Lin since they are not relevant to their eigenvalue problem.

Dunn and Lin[4,5] recognized that, for supersonic flows, the propagation velocity c may be some substantial portion of the free stream velocity and the critical point may be relatively far from the wall. Accordingly they ordered the various quantities occurring in the stability equations in the following manner:

$$\begin{aligned} &\bar{Q},\ \bar{Q}' \sim 1,\quad d/dy \sim 1/\epsilon,\quad (w-c) \sim 1,\quad f \sim 1,\quad \phi \sim \epsilon f \\ &\theta \sim f,\quad \pi \sim \epsilon^2 f \end{aligned} \tag{53}$$

The details of this order of magnitude analysis are repeated in Appendix C. The leading terms of the disturbance equations under ordering [Eq. (53)] form the Dunn-Lin viscous equations:

$$f''' - \frac{i\,\alpha Re\,(w-c)}{\nu} f' = 0 \tag{54}$$

$$\phi' + i f = \frac{i\,(w-c)}{T}\,\theta \tag{55}$$

$$\theta'' - \frac{i\,\alpha Re\,\sigma\,(w-c)}{\nu}\,\theta = 0 \tag{56}$$

and

$$\epsilon \sim \frac{1}{(\alpha Re)^{1/2}} \tag{57}$$

These equations also depend only on the one parameter (αRe). The momentum and energy equations [Eqs. (54) and (56)] are mutually independent while the normal velocity fluctuations are related to both the longitudinal velocity fluctuations and temperature fluctuations through the continuity equation [Eq. (55)].

The method of solution of Eqs. (54) to (56) is given by Dunn[4]. He transforms Eqs. (54) and (56) by introducing new variables of the form suggested by Tollmien[11], such that the equations reduce to the form [Eq. (49)] solved by Lees and Lin. The details of the transformation and the solutions of Dunn and Lin are reviewed in Appendix D.*

As the Mach number of the flow increases certain terms of the disturbance equations which are Mach number dependent grow larger than indicated by an ordering procedure based solely on free stream Reynolds number. The normal gradient of mean temperature is of order M_e^2 (Appendix C) and the inviscid amplitude relations [Eqs. (42) and (43)] show that the "inviscid" temperature fluctuations normalized with the free stream temperature are also of order M_e^2, compared to the normalized longitudinal velocity fluctuations. The "viscous" temperature fluctuations must therefore also be of order M_e^2. At the same time the whole mean temperature level in the boundary layer grows like M_e^2, and the free stream static temperature level is no longer relevant. Temperature fluctuations and mean quantities that are temperature dependent should be normalized by some representative temperature $T_{ref} \sim M_e^2 T_e$. As shown by Dunn[4], the new normalizations and new definitions of Mach number and Reynolds number are

$$\theta = \frac{\theta^*}{T_{ref}^*} \qquad\qquad T = \frac{T^*}{T_{ref}^*}$$

$$M_{ref} = \frac{M_e}{\sqrt{T_{ref}}} \qquad\qquad Re_{ref} = \frac{Re}{\nu_{ref}} \tag{58}$$

It is shown in Appendix C that even with the new temperature normalization, normal gradients of mean temperature dependent quantities

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* For ($\alpha Re\, c$) >> 1, it is useful to obtain asymptotic solutions to the viscous disturbance equations. These solutions have in fact already been obtained by Dunn[4]. However, in most cases ($\alpha Re\, c$) is not sufficiently large to warrant the use of the asymptotic solutions, so that they are omitted from the present discussion.

are of magnitude M_{ref}^2 times the normal mean velocity gradient. In general $M_{ref} = \mathcal{O}(1)$ so that the retention of M_{ref} in the ordering relations which follow is mainly for the purpose of identifying the terms depending on mean temperature gradient and the dissipation terms. The ordering relations used in the present analysis are

$$\begin{aligned} &\bar{Q},\ \bar{Q}' \sim 1 \quad \underline{\text{except}} \quad T',\ \mu' \sim M_{ref}^2 \\ &d/dy \sim 1/\bar{\epsilon},\quad f \sim 1,\quad \theta \sim f,\quad \phi \sim \bar{\epsilon} f,\quad \pi \sim \bar{\epsilon}^2 f \end{aligned} \tag{59}$$

As shown in Appendix C the terms in the disturbance equations may be grouped as follows:

(αRe_{ref}) dependence only		α and (αRe_{ref}) dependence		
1, $\bar{\epsilon}$		$\bar{\epsilon}^2$,	$\bar{\epsilon}^3$,	$\bar{\epsilon}^4$
$M_{ref}^2\,\bar{\epsilon}$	$M_{ref}^2\,\bar{\epsilon}^2$		$M_{ref}^2\,\bar{\epsilon}^3$,	$M_{ref}^2\,\bar{\epsilon}^4$

where

$$\bar{\epsilon} \sim \frac{1}{(\alpha Re_{ref})^{1/2}} = \frac{(\gamma_{ref})^{1/2}}{(\alpha Re)^{1/2}} \sim (\gamma_{ref})^{1/2}\epsilon$$

Terms in the disturbance equations arising from the fluctuating viscous stresses, heat flux gradients and viscous dissipation that are regarded of order ϵ in the Lees-Lin and Dunn-Lin analyses are actually of order $\bar{\epsilon}$. For a linear viscosity-temperature relation, $\bar{\epsilon} \sim M_e^2 \epsilon$, and at high supersonic and hypersonic Mach numbers the above-mentioned terms are likely to be comparable in magnitude to the terms of unit order.

Referring to the grouping of ordered terms, no major difficulties arise when terms of order 1, $\bar{\epsilon}$, $M_{ref}^2 \bar{\epsilon}$ and $M_{ref}^2 \bar{\epsilon}^2$ are considered, since these depend only on the single parameter (αRe_{ref}). But if terms of order $M_{ref}^2 \bar{\epsilon}^2$ are to be included in the viscous equations then to be consistent one should also include the pressure fluctuation terms (of order $\bar{\epsilon}^2$) in the continuity and energy equations [Eqs. (E-21) and (E-22)], and also the terms of order $\bar{\epsilon}^2$ with coefficient α^2 arising from the streamwise gradients of fluctuating quantities. In the present analysis only terms of order 1, $\bar{\epsilon}$, and $M_{ref}^2 \bar{\epsilon}$ are retained in order to investigate the first-order effects of viscous dissipation and large mean normal temperature gradients, hitherto neglected.

The viscous disturbance equations here considered are as follows (from Appendix C):

Momentum

$$f''' + \frac{2}{\mu}\frac{d\mu}{dT} T'f'' + \frac{1}{\mu}\frac{d\mu}{dT} w'\theta'' + \frac{i\alpha Re(w-c)}{\nu}\left[\frac{T'}{T}f - f' - \frac{w'}{T}\theta\right] = 0 \quad (60)$$

Continuity

$$\phi' + if - \frac{T'}{T}\phi - \frac{i(w-c)}{T}\theta = 0 \quad (61)$$

Energy

$$\theta'' + \frac{2}{\mu}\frac{d\mu}{dT} T'\theta' + 2(\gamma-1)\sigma M_e^2 w'f' - \frac{i\alpha Re\,\sigma(w-c)}{\nu}\theta = \frac{\sigma\alpha Re\,T'}{\nu}\phi \quad (62)$$

By comparing these equations with the Dunn-Lin equations [Eqs. (E-9) to (C-11)], one sees that all terms formerly assigned to orders 1 and ϵ are included in the present equations. In contrast to the

Lees-Lin and Dunn-Lin viscous equations, the current set [Eqs. (60) to (62)] have no independence properties and the three equations must be solved simultaneously. The linearly independent solutions to Eqs. (60) to (62) are distinguished by their behavior in the outer flow where mean flow quantities have reached their external values. In the outer flow, Eqs. (60) to (62) become

$$f''' - i\,\alpha Re\,(1-c)\,f' = 0 \tag{63}$$

$$\phi' + i\,f = i\,(1-c)\,\theta \tag{64}$$

$$\theta'' - i\,\alpha Re\,\sigma(1-c)\,\theta = 0 \tag{65}$$

One set of solutions to Eqs. (63) to (65) is of the form

$$f' \sim e^{\pm\sqrt{i\,\alpha Re(1-c)}\,y}\ , \quad \theta = 0 \tag{66}$$

while another set is of the form

$$\theta \sim e^{\pm\sqrt{i\,\alpha Re\sigma(1-c)}\,y}\ , \quad f = 0 \tag{67}$$

A third set corresponding to $f' = 0$ is replaced by the inviscid solution and is therefore not considered here. In Eqs. (66) and (67) the solutions with the positive exponents are rejected immediately since they grow exponentially and cannot possibly satisfy the outer boundary conditions [Eq. (14)]. The remaining two sets of solutions corresponding to the negative exponents must now be found. Since Eqs. (60) to (62) are rather tightly coupled, it is likely that their simultaneous solution will have to be obtained numerically. The method is somewhat similar to that used to obtain the inviscid solution.

Consider the solution corresponding to the negative exponent in

Eq. (67). Let

$$H \equiv \theta'/\theta, \qquad J \equiv \phi/\theta, \qquad K \equiv f/\theta \; . \tag{68}$$

In terms of H, J, and K, Eqs. (60) to (62) become

$$H' = -\frac{2}{\mu}\frac{d\mu}{dT}T'H - 2(\gamma-1)\sigma M_e^2 w'(K'+HK) + \frac{i\alpha Re\,\sigma(w-c)}{\nu} + \frac{\sigma\alpha Re\,T'}{\nu}J - H^2 \tag{69}$$

$$J' = -iK + \frac{i(w-c)}{T} + \frac{T'}{T}J - HJ \tag{70}$$

$$\begin{aligned} K''' = & -\frac{2}{\mu}\frac{d\mu}{dT}T'\left[K''+H(2K'+HK)+H'K\right] - \frac{1}{\mu}\frac{d\mu}{dT}w'(H'+H^2) \\ & - \frac{i\alpha Re\,(w-c)}{\nu}\left[\frac{T'}{T}K - (K'+HK) - \frac{w'}{T}\right] \\ & - H\left(3K''+3K'H+3H'K+H^2K\right) - 3K'H' - H''K \end{aligned} \tag{71}$$

with "outer" conditions

$$\left.\begin{aligned} H_o &= -\sqrt{i\,\alpha Re\,\sigma(1-c)} \\ J_o &= -\sqrt{\frac{i(1-c)}{\alpha Re\,\sigma}} \\ K_o &= 0 \end{aligned}\right\} \tag{72}$$

The equations for the set corresponding to the negative exponent in Eq. (66) are obtained in a similar manner. With

$$L \equiv \phi/f, \qquad M \equiv f'/f, \qquad N \equiv \theta/f \tag{73}$$

Eqs. (60) to (62) become

$$L' = -i + \frac{i(w-c)}{T}N + \frac{T'}{T}L - LM \tag{74}$$

$$M'' = -\frac{2}{\mu}\frac{d\mu}{dT}T'\left(M'+M^2\right) - \frac{1}{\mu}\frac{d\mu}{dT}w'\left[N'' + (2N'+NM)M + NM'\right]$$
$$- \frac{i\alpha Re\,(w-c)}{\nu}\left[\frac{T'}{T} - M - \frac{w'}{T}N\right] - M\left[3M'+M^2\right] \qquad (75)$$

$$N'' = -\frac{2}{\mu}\frac{d\mu}{dT}T'\left[N'+NM\right] - 2(\gamma-1)\sigma M_e^2 w'M + \frac{i\alpha Re\,\sigma(w-c)}{\nu}N$$
$$+ \frac{\sigma\alpha Re}{\nu}T'L - (2N'+NM)M - NM' \qquad (76)$$

and the outer conditions are

$$\left.\begin{aligned} L_o &= \sqrt{\frac{i}{\alpha Re(1-c)}} \\ M_o &= -\sqrt{i\,\alpha Re(1-c)} \\ N_o &= 0 \end{aligned}\right\} \qquad (77)$$

The HJK and LMN systems of equations are integrated from the outer edge of the boundary layer to the wall for two main reasons: (1) the outer conditions are known; (2) the outer region behaves as a saddle point in the sense that integral curves other than those satisfying the outer condition diverge from the outer condition. This behavior is shown in Appendix E.

For the purposes of numerical integration, the complex HJK and LMN equations are separated into their component real and imaginary equations. For a neutral oscillation ($c_i = 0$), the HJK and LMN systems are:

$$H_r' = -\frac{2}{\mu}\frac{d\mu}{dT}T'H_r - 2(\gamma-1)\sigma M_e^2 w'\left(K_r' + H_rK_r - H_iK_i\right) + \frac{\sigma\alpha Re}{\nu}T'J_r - \left(H_r^2 - H_i^2\right) \qquad (78a)$$

$$H_i' = -\frac{2}{\mu}\frac{d\mu}{dT} T' H_i - 2(\gamma-1)\sigma M_e^2 w'\left(K_i' + H_r K_i + H_i K_r\right) + \frac{\sigma\alpha Re}{\nu}\left[(w-c) + T' J_i\right] - 2H_r H_i \tag{78b}$$

$$J_r' = K_i + \frac{T'}{T} J_r - H_r J_r + H_i J_i \tag{79a}$$

$$J_i' = -K_r + \frac{T'}{T} J_i + \frac{(w-c)}{T} - H_r J_i - H_i J_r \tag{79b}$$

$$\begin{aligned} K_r''' = & -\frac{2}{\mu}\left(\frac{d\mu}{dT}\right) T'\left[K_r'' + H_r\left(2K_r' + H_r K_r - H_i K_i\right) - H_i\left(2K_i' + H_i K_r + H_r K_i\right) + H_r' K_r - H_i' K_i\right] \\ & -\frac{1}{\mu}\left(\frac{d\mu}{dT}\right) w'\left[H_r' + H_r^2 - H_i^2\right] + \frac{\alpha Re\,(w-c)}{\nu}\left[\frac{T'}{T} K_i - \left(K_i' + H_i K_r + H_r K_i\right)\right] \\ & - H_r\left[3K_r'' + 3K_r' H_r - 3K_i' H_i + 3H_r' K_r - 3H_i' K_i + \left(H_r^2 - H_i^2\right) K_r - 2H_r H_i K_i\right] \\ & + H_i\left[3K_i'' + 3K_r' H_i + 3K_i' H_r + 3H_r' K_i + 3H_i' K_r + 2H_r H_i K_r + \left(H_r^2 - H_i^2\right) K_i\right] \\ & - 3K_r' H_r' + 3K_i' H_i' - H_r'' K_r + H_i'' K_i \end{aligned} \tag{80a}$$

$$\begin{aligned} K_i''' = & -\frac{2}{\mu}\left(\frac{d\mu}{dT}\right) T'\left[K_i'' + H_r\left(2K_i' + H_i K_r + H_r K_i\right) + H_i\left(2K_r' + H_r K_r - H_i K_i\right) + H_r' K_i + H_i' K_r\right] \\ & -\frac{1}{\mu}\left(\frac{d\mu}{dT}\right) w'\left[H_i' + 2H_r H_i\right] - \frac{\alpha Re\,(w-c)}{\nu}\left[\frac{T'}{T} K_r - \left(K_r' + H_r K_r - H_i K_i\right) - \frac{w'}{T}\right] \\ & - H_r\left[3K_i'' + 3K_r' H_i + 3K_i' H_r + 3H_r' K_i + 3H_i' K_r + 2H_r H_i K_r + \left(H_r^2 - H_i^2\right) K_i\right] \\ & - H_i\left[3K_r'' + 3K_r' H_r - 3K_i' H_i + 3H_r' K_r - 3H_i' K_i + \left(H_r^2 - H_i^2\right) K_r - 2H_r H_i K_i\right] \\ & - 3K_r' H_i' - 3K_i' H_r' - H_r'' K_i - H_i'' K_r \end{aligned} \tag{80b}$$

with outer conditions

$$\left.\begin{aligned} H_{r_o} &= -\sqrt{\frac{\alpha Re\,\sigma(1-c)}{2}} & J_{r_o} &= -\sqrt{\frac{(1-c)}{2\alpha Re\,\sigma}} & K_{r_o} &= 0 \\ H_{i_o} &= -\sqrt{\frac{\alpha Re\,\sigma(1-c)}{2}} & J_{i_o} &= -\sqrt{\frac{(1-c)}{2\alpha Re\,\sigma}} & K_{i_o} &= 0 \end{aligned}\right\} \quad (81)$$

and

$$L_r' = -\left(\frac{w-c}{T}\right)N_i + \frac{T'}{T}L_r - L_r M_r + L_i M_i \quad (82a)$$

$$L_i' = -1 + \left(\frac{w-c}{T}\right)N_r + \frac{T'}{T}L_i - L_i M_r - L_r M_i \quad (82b)$$

$$\begin{aligned} M_r'' = &-\frac{2}{\mu}\left(\frac{d\mu}{dT}\right)T'\left[M_r' + M_r^2 - M_i^2\right] \\ &-\frac{1}{\mu}\left(\frac{d\mu}{dT}\right)w'\left[N_r'' + M_r(2N_r' + N_r M_r - N_i M_i) - M_i(2N_i' + N_i M_r + N_r M_i) + N_r M_r' - N_i M_i'\right] \\ &-\frac{\alpha Re(w-c)}{\nu}\left[M_i + \frac{w'}{T}N_i\right] - 3M_r' M_r + 3M_i' M_i - M_r^3 + 3M_r M_i^2 \end{aligned} \quad (83a)$$

$$\begin{aligned} M_i'' = &-\frac{2}{\mu}\left(\frac{d\mu}{dT}\right)T'\left[M_i' + 2M_r M_i\right] \\ &-\frac{1}{\mu}\left(\frac{d\mu}{dT}\right)w'\left[N_i'' + M_r(2N_i' + N_i M_r + N_r M_i) + M_i(2N_r' + N_r M_r - N_i M_i) + N_r M_i' + N_i M_r'\right] \\ &-\frac{\alpha Re(w-c)}{\nu}\left[\frac{T'}{T} - M_r - \frac{w'}{T}N_r\right] - 3M_i' M_r - 3M_r' M_i - 3M_r^2 M_i + M_i^3 \end{aligned} \quad (83b)$$

$$\begin{aligned} N_r'' = &-\frac{2}{\mu}\left(\frac{d\mu}{dT}\right)T'\left[N_r' + N_r M_r - N_i M_i\right] - 2(\gamma-1)\sigma M_e^2 w' M_r - \frac{\sigma\alpha Re}{\nu}\left[(w-c)N_i - T' L_r\right] \\ &-2N_r' M_r + 2N_i' M_i - N_r M_r' + N_i M_i' - N_r(M_r^2 - M_i^2) + 2N_i M_r M_i \end{aligned} \quad (84a)$$

$$\begin{aligned} N_i'' = &-\frac{2}{\mu}\left(\frac{d\mu}{dT}\right)T'\left[N_i' + N_r M_i + N_i M_r\right] - 2(\gamma-1)\sigma M_e^2 w' M_i + \frac{\sigma\alpha Re}{\nu}\left[(w-c)N_r + T' L_i\right] \\ &-2N_i' M_r - 2N_r' M_i - N_r M_i' - N_i M_r' - N_i(M_r^2 - M_i^2) - 2N_r M_r M_i \end{aligned} \quad (84b)$$

with outer conditions

$$\left.\begin{array}{lll} L_{r_0} = \sqrt{\dfrac{1}{2\alpha Re\,(1-c)}} & M_{r_0} = -\sqrt{\dfrac{\alpha Re\,(1-c)}{2}} & N_{r_0} = 0 \\ L_{i_0} = \sqrt{\dfrac{1}{2\alpha Re\,(1-c)}} & M_{i_0} = -\sqrt{\dfrac{\alpha Re\,(1-c)}{2}} & N_{i_0} = 0 \end{array}\right\} \quad (85)$$

The viscous equations for amplified and damped oscillations ($c_i \neq 0$) are given in Appendix B.

III. 3. Eigenvalue Problem

Having indicated the methods of obtaining the various solutions of the disturbance equations, the solutions must be combined to satisfy the boundary conditions. Note that the outer boundary conditions for subsonic disturbances [Eq. (14)] are inherently satisfied by choosing the negative exponents in Eqs. (26), (66), and (67).* The three boundary conditions at the wall remain to be satisfied. They are

$$f_w = 0 \quad (11)$$

$$\phi_w = 0 \quad (12)$$

$$\theta_w = 0 \quad (13)$$

The restricted thermal boundary condition [Eq. (13)] is the one primarily considered. The more general boundary condition $a\theta_w' + b\theta_w = 0$ is discussed in Appendix F.

Following the pattern of Dunn and Lin[5], the inviscid functions in this section will be capitalized, while the functions corresponding to the LMN solution will be given the subscript 3 and those corresponding

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* The process of patching two independent inviscid solutions at the outer boundary by the condition $\phi' + \alpha\sqrt{1 - M_e^2(1-c)^2}\,\phi = 0$, which is required in the Lees-Lin and Dunn-Lin procedures, is here unnecessary.

to the HJK system the subscript 5.

The satisfaction of boundary conditions [Eqs. (11) to (13)] leads to the following determinantal relation

$$\begin{vmatrix} \Phi_w & \phi_{3w} & \phi_{5w} \\ F_w & f_{3w} & F_{5w} \\ \Theta_w & \theta_{3w} & \theta_{5w} \end{vmatrix} = 0 \tag{86}$$

which when expanded yields the secular equation:

$$\frac{\Phi_w}{F_w} = \frac{\phi_{3w}}{f_{3w}} + \frac{\Theta_w}{F_w}\left[\frac{\phi_{5w}}{\theta_{5w}} - \frac{\phi_{3w}}{f_{3w}}\frac{f_{5w}}{\theta_{5w}}\right] + \frac{\Phi_w}{F_w}\frac{\theta_{3w}}{f_{3w}}\frac{f_{5w}}{\theta_{5w}} - \frac{\theta_{3w}}{f_{3w}}\frac{\phi_{5w}}{\theta_{5w}} \tag{87}$$

Note that in the Lees-Lin, Dunn-Lin and present formulations the inviscid solutions depend only on the parameter c while the viscous solutions depend only on (αRe). With the aid of the following identity derived from the inviscid equations [Eqs. (17) and (19)]:

$$\frac{\Theta_w}{F_w} = (\gamma-1)M_e^2 c + i(\gamma-1)M_e^2 c\left\{\frac{w'_w}{c} - \frac{T'_w}{(\gamma-1)M_e^2c^2}\right\}\frac{\Phi_w}{F_w} \tag{88}$$

Eq. (87) can be written:

$$\frac{\Phi_w}{F_w} = \frac{\dfrac{\phi_{3w}}{f_{3w}} + \dfrac{\phi_{5w}}{\theta_{5w}}\left[(\gamma-1)M_e^2 c - \dfrac{\theta_{3w}}{f_{3w}}\right] - (\gamma-1)M_e^2 c\,\dfrac{\phi_{3w}}{f_{3w}}\dfrac{f_{5w}}{\theta_{5w}}}{1 - i(\gamma-1)M_e^2 c\,\dfrac{\phi_{5w}}{\theta_{5w}}\left\{\dfrac{w'_w}{c} - \dfrac{T'_w}{(\gamma-1)M_e^2c^2}\right\} + \dfrac{f_{5w}}{\theta_{5w}}\left[i(\gamma-1)M_e^2 c\,\dfrac{\phi_{3w}}{f_{3w}}\left\{\dfrac{w'_w}{c} - \dfrac{T'_w}{(\gamma-1)M_e^2c^2}\right\} - \dfrac{\theta_{3w}}{f_{3w}}\right]} \tag{89}$$

The Lees-Lin and Dunn-Lin secular equations are also quite simply obtained. For the Lees-Lin viscous solutions $\phi_5 = f_5 = 0$. Since

θ_{5_w} is generally not zero, Eq. (89) becomes

$$\Phi_w / F_w = \phi_{3w} / f_{3w} \quad . \tag{90}$$

This result confirms the irrelevance of the thermal boundary condition in the Lees-Lin case.

For the Dunn-Lin solution, $\theta_3 = f_5 = 0$ (see Appendix D) so that the secular equation is

$$\frac{\Phi_w}{F_w} = \frac{\frac{\phi_{3w}}{f_{3w}} + (\gamma-1) M_e^2 c \frac{\phi_{5w}}{\theta_{5w}}}{1 - i(\gamma-1) M_e^2 c \frac{\phi_{5w}}{\theta_{5w}} \left\{ \frac{w_w'}{c} - \frac{T_w'}{(\gamma-1) M_e^2 c^2} \right\}} \tag{91}$$

For some unstated reason, Dunn and Lin[5] apparently omitted the second term on the right hand side of Eq. (88) so that the denominator of Eq. (91) in their formulation is simply unity. This omission is corrected herein (Appendix D).

In the terminology of the present method (Section III. 2) Eq. (83) may be written

$$\Phi_w / F_w = \mathcal{R} \tag{92}$$

where

$$\mathcal{R} = \frac{L_w + J_w \left[(\gamma-1) M_e^2 c - N_w \right] - (\gamma-1) M_e^2 c L_w K_w}{1 - i(\gamma-1) M_e^2 c J_w \left\{ \frac{w_w'}{c} - \frac{T_w'}{(\gamma-1) M_e^2 c^2} \right\} + K_w \left[i(\gamma-1) M_e^2 c L_w \left\{ \frac{w_w'}{c} - \frac{T_w'}{(\gamma-1) M_e^2 c^2} \right\} - N_w \right]} \tag{93}$$

Again from the inviscid equations Eqs. (16) to (19)

$$\Phi_w / F_w = \frac{i}{\frac{w_w'}{c} - \frac{1}{G_w}} \tag{94}$$

so that relation Eq. (89) may finally be written

$$G_w = (c/w_w')(1 - \psi) \tag{95}$$

where

$$\psi \equiv \frac{1}{1 + i\frac{w_w'}{c}R} \quad . \tag{96}$$

In Eq. (95) G_w depends on α alone while the right-hand side depends on (αRe) alone. The values of α and αRe for which Eq. (95) is satisfied are the desired characteristic values. Some detailed examples of this procedure are given in Section IV.

The secular equation for the general temperature fluctuation boundary condition, $a\,\theta_w' + b\theta_w = 0$, is given in Appendix F.

III. 4. Amplitude Distributions

Once the characteristic values of α and Re are determined, the distributions of the amplitudes of the disturbance quantities across the boundary layer are calculated by obtaining the amplitude distributions for the inviscid solutions, the LMN solutions (solutions 3) and the HJK solutions (solutions 5) and combining them in the manner satisfying the boundary conditions. The discussion in this section concerns the functions π, ϕ, f, and θ. The normal velocity fluctuations are given by $\alpha\phi$ while the density fluctuation amplitude can be obtained by using the equation of state [Eq. (8)].

Before proceeding, it must be recalled that in the process of splitting the solutions into inviscid and viscous types a singularity was artificially introduced into the inviscid equations at the critical point by the complete elimination of viscous and heat conduction effects from these equations (Section III. 1). Accordingly, before the inviscid solutions can be used in composing the amplitude distributions, they must be corrected for the effects of viscosity and thermal conductivity in the neighborhood of the critical layer. For incompressible flow, such corrections were first obtained by Tollmien[11] and Schlichting[13].

Here we seek the leading viscous corrections to the inviscid functions in the region about the critical point. To be more specific the corrected function is given by

$$q_{corr} = q_{inv} - \begin{bmatrix} \text{singular} \\ \text{term} \end{bmatrix} + \begin{bmatrix} \text{viscous} \\ \text{replacement} \\ \text{term} \end{bmatrix}$$

where the viscous replacement function is obtained by solving the disturbance equation containing only the leading viscous terms in the neighborhood of the critical point. This replacement function must satisfy the condition that "far" away from the critical layer it approaches asymptotically the singular portion of the original uncorrected inviscid function (Appendix G).

The behavior of the uncorrected inviscid functions in the neighborhood of the critical layer can be ascertained by obtaining a series expansion of the inviscid solution around the critical point. The series expansion for the pressure fluctuation amplitude has already been obtained (Appendix A). The other inviscid amplitudes are related to π and π' through Eqs. (41) to (43) (Section III.). The results for Φ, F, and Ⓗ are respectively as follows:

$\underline{\Phi}$

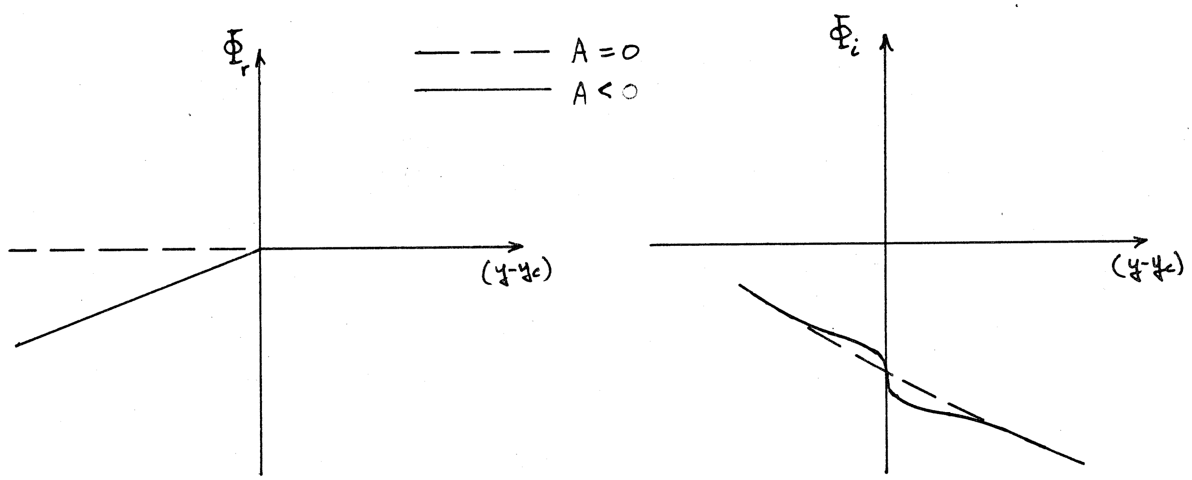
For $(y - y_c) > 0$

$$\Phi = -\frac{i}{\gamma M_e^2} \frac{T_c}{W_c'} \left[1 + A(y-y_c) \ln (y-y_c) + \left\{ \left(\frac{T_c'}{T_c} - \frac{W_c''}{2W_c'} \right) - (\text{const}) \right\} (y-y_c) + \dots \right] \quad (97)$$

For $(y - y_c) < 0$

$$\Phi = -\frac{1}{\gamma M_e^2}\frac{T_c}{w_c'} A\pi (y-y_c) + \ldots - \frac{i}{\gamma M_e^2}\frac{T_c}{w_c'}\left[1 + A(y-y_c)\ln|y-y_c| + \left\{\left(\frac{T_c'}{T_c} - \frac{w_c''}{2w_c'}\right) - (\text{const})\right\}(y-y_c) + \ldots\right] \tag{98}$$

The quantity A is the derivative of the density-vorticity product at the critical point as defined by Eq. (36). The behavior of Φ near the critical point is sketched below



When $A = 0$ there are no discontinuities in Φ. For $A \neq 0$, the values of Φ_r and Φ_i are continuous, but Φ_r is discontinuous in slope at the critical point, while Φ_i has a discontinuity in curvature.

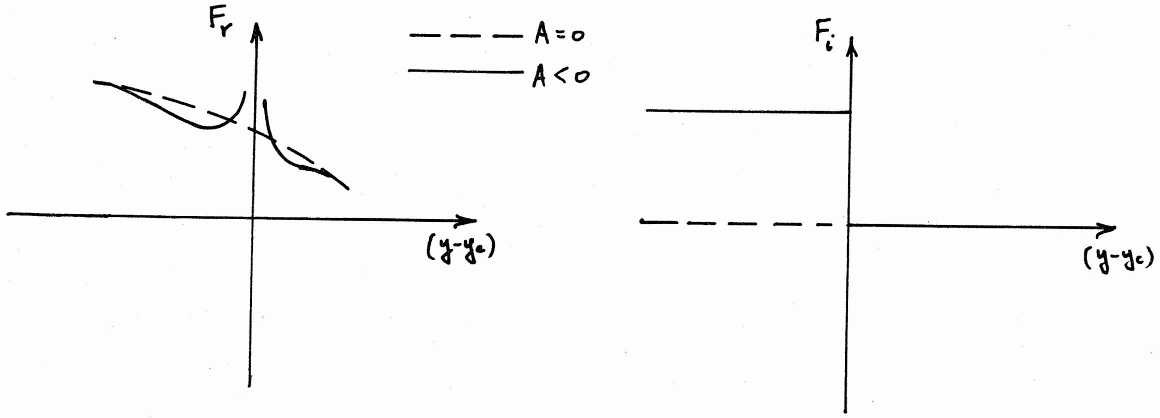
<u>F</u>

For $(y - y_c) > 0$

$$F = \frac{1}{\gamma M_e^2}\frac{T_c}{w_c'}\left[A\ln(y-y_c) + \left\{\frac{w_c''}{2w_c'} - (\text{const})\right\} + \ldots\right] \tag{99}$$

For $(y - y_c) < 0$

$$F = \frac{1}{\gamma M_e^2}\frac{T_c}{w_c'}\left[A \ln|y-y_c| + \left\{\frac{w_c''}{2w_c'} - (\text{const})\right\} + \ldots\right] \qquad (100)$$
$$- \frac{i}{\gamma M_e^2}\frac{T_c}{w_c'} A\pi + \ldots$$



There are no discontinuities in F for A = 0. However, for $A \neq 0$, F_r has a logarithmic infinity while there is a jump discontinuity in F_i.

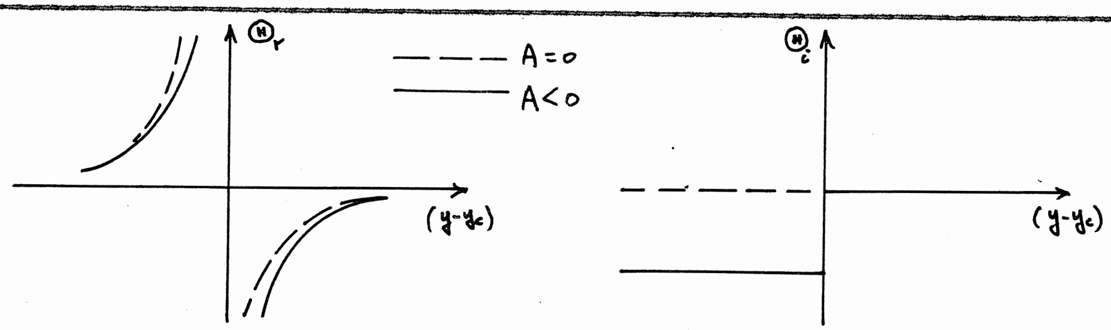
$\textcircled{H}$

For $(y - y_c) > 0$

$$\textcircled{H} = \frac{1}{\gamma M_e^2}\frac{T_c T_c'}{w_c'^2}\left[\frac{1}{(y-y_c)} + A \ln(y-y_c) + \left\{\frac{T_c'}{T_c} + \frac{(\gamma-1)M_e^2 w_c'}{T_c'} - A - (\text{const})\right\} + \ldots\right] \qquad (101)$$

For $(y - y_c) < 0$

$$\textcircled{H} = \frac{1}{\gamma M_e^2}\frac{T_c T_c'}{w_c'^2}\left[\frac{1}{(y-y_c)} + A \ln|y-y_c| + \left\{\frac{T_c'}{T_c} + \frac{(\gamma-1)M_e^2 w_c'}{T_c'} - A - (\text{const})\right\} + \ldots\right]$$
$$- \frac{i}{\gamma M_e^2}\frac{T_c T_c'}{w_c'^2} A\pi + \ldots \qquad (102)$$



To be noted immediately is the discontinuity in Θ_r even for $A = 0$. This irregularity in the distribution of temperature fluctuation amplitude is caused simply by the assumed absence of thermal conduction. For $A \neq 0$, there is an additional logarithmic discontinuity in Θ_r and a jump discontinuity in Θ_i. It may be noted that the inviscid disturbance vorticity given by $(F' + i\alpha^2 \Phi)$ has a discontinuity similar to that of the temperature fluctuations.

The corrections for viscosity and conductivity are introduced logically using the method of convergent series expansion about the critical point as developed by Lees and Lin[1] and Cheng[3]. In the present case (Appendix G) this expansion is carried out in the Tollmien variable. Furthermore the development in Appendix G reduces the correction equations to the forms solved by Schlichting[13], so that the universal functions calculated by Schlichting[13] could be used here. Consider for example the corrections to Θ_r. The function $(\Theta_r)_{corr}$ may be expressed

$$(\Theta_r)_{corr} = \Theta_r - \begin{bmatrix}\text{singular}\\ \text{terms}\end{bmatrix} + \begin{bmatrix}\text{replacement}\\ \text{terms}\end{bmatrix} . \tag{103}$$

The singular terms from Eq. (102) are

$$\frac{1}{\gamma M_e^2}\frac{T_c T_c'}{w_c'^2}\left[\frac{1}{(y-y_c)} + A \ln |y-y_c|\right] \tag{104}$$

The replacement terms from Appendix G are

$$\frac{1}{\gamma M_e^2}\frac{T_c T_c'}{w_c'^2}\left[\frac{1}{(y-y_c)}\zeta_o \hat{G}''(\zeta_o) + A\left\{\hat{G}'(\zeta_o) - \ln\frac{|\zeta_o|}{|y-y_c|}\right\}\right] \tag{105}$$

where

$$\zeta_o = (\alpha Re)^{1/3}\, Y_o \tag{106}$$

The Tollmien variable for the energy equation is defined by

$$Y_0 \equiv \left[\int_{y_c}^{y} \frac{3}{2} \sqrt{\frac{\sigma(w-c)}{\nu}}\, dy \right]^{2/3} \tag{107}$$

and the functions $\hat{G}''(\zeta_0)$ and $\hat{G}'(\zeta_0)$ are the functions calculated and presented by Schlichting[13] and reproduced here (Table I). Combining Eqs. (103), (104), and (105)

$$(\Theta_r)_{corr} = \Theta_r - \frac{1}{\gamma M_e^2} \frac{T_c T_c'}{w_c'^2} \left[\frac{1}{(y-y_c)} \left\{ 1 - \zeta_0 \hat{G}''(\zeta_0) \right\} + A \left\{ \ln|\zeta_0| - \hat{G}'(\zeta_0) \right\} \right] . \tag{108}$$

Schlichting chooses the thickness of the viscous conductive region as $\zeta_0 = \pm 4$. Accordingly the bracketed term in Eq. (108) vanishes for $|\zeta_0| > 4$ so that outside the region $(\Theta_r)_{corr} = \Theta_r$.

Following the same principle, the corrected forms for Θ_i, F_r, and F_i are (replacement terms from Appendix G):

For $(y - y_c) < 0$

$$(\Theta_i)_{corr} = \Theta_i + \frac{A}{\gamma M_e^2} \frac{T_c T_c'}{w_c'^2} \left[\pi + \hat{H}'(\zeta_0) \right] \tag{109a}$$

For $(y - y_c) > 0$

$$(\Theta_i)_{corr} = \frac{A}{\gamma M_e^2} \frac{T_c T_c'}{w_c'^2} \hat{H}'(\zeta_0) \tag{109b}$$

$$(F_r)_{corr} = F_r - \frac{A}{\gamma M_e^2} \frac{T_c}{w_c'} \left\{ \ln|\zeta| - \hat{G}'(\zeta) \right\} \tag{110}$$

For $(y - y_c) < 0$

$$(F_i)_{corr} = F_i + \frac{A}{\gamma M_e^2} \frac{T_c}{w_c'} \left[\pi + \hat{H}'(\zeta) \right] \tag{111a}$$

For $(y - y_c) > 0$

$$(F_i)_{corr} = \frac{A}{\gamma M_e^2} \frac{T_c}{w_c'} \hat{H}'(\zeta) \tag{111b}$$

where

$$\zeta = (\alpha Re)^{1/3} \, Y \tag{112}$$

and

$$Y = \left[\int_{y_c}^{y} \frac{3}{2} \sqrt{\frac{w - c}{\nu}} \, dy \right]^{2/3} \tag{113}$$

The above corrections are the important ones. The largest correction is to the function Θ_r and must be made whether or not A vanishes. All the other corrections are directly proportional to the value of A, which is usually small. Even smaller are the corrections to Φ_r, which is continuous in value but has an infinite slope at the critical point, and the corrections to π_r where the irregularity does not appear until the third derivative. The corrections to Φ and π are not obtained herein. For incompressible flow, temperature fluctuations are irrelevant and all the corrections are likely to be small.

As shown by Eqs. (106) and (112), the thickness of the region about the critical layer where viscosity and conductivity are important is of order $(\alpha Re)^{-1/3}$. For sufficiently large values of (αRe) this thickness is small compared to the distance between the wall and the critical layer. On the other hand, there is some value of (αRe) for which the viscous conductive layer about the critical point extends to the wall, and would in fact "correct" the values of the inviscid functions at the wall. When this occurs, one has reached the point beyond which it is improper in principle to use the wall values of the inviscid solutions in the eigenvalue problem, and the splitting of the solutions into inviscid and viscous types becomes invalid in principle.* The developments of the present section assume that the region about the critical layer where viscosity and conductivity are important does not extend to the wall, so that the eigenvalues α and (αRe) as determined by the procedure of Section III. 3. remain unaffected.

The amplitudes of the viscous solutions are obtained as follows:

$$
\begin{aligned}
f_3 &= e^{\int_0^y M\,dy} \\
&= e^{\int_0^y M_r\,dy}\left[\cos\int_0^y M_i\,dy + i\sin\int_0^y M_i\,dy\right]
\end{aligned}
\tag{114}
$$

or

$$f_{3_r} = e^{\int_0^y M_r\,dy}\cos\int_0^y M_i\,dy \tag{115}$$

$$f_{3_i} = e^{\int_0^y M_r\,dy}\sin\int_0^y M_i\,dy \tag{116}$$

Then

$$\phi_3 = L\,f_3 \tag{117}$$

$$\theta_3 = N\,f_3 \tag{118}$$

Similarly

$$
\begin{aligned}
\theta_5 &= e^{\int_0^y H\,dy} \\
&= e^{\int_0^y H_r\,dy}\left[\cos\int_0^y H_i\,dy + i\sin\int_0^y H_i\,dy\right]
\end{aligned}
\tag{119}
$$

$$\theta_{5_r} = e^{\int_0^y H_r\,dy}\cos\int_0^y H_i\,dy \tag{120}$$

---

* There are no doubt many cases where the viscous corrections to the inviscid functions at the wall would be quite small and hardly affect the eigenvalues. Therefore, the above restrictions are stated "in principle" rather than categorically.

$$\theta_{5_i} = e^{\int_0^y H_r dy} \sin \int_0^y H_i \, dy \tag{121}$$

and

$$\phi_5 = J \, \theta_5 \tag{122}$$

$$f_5 = K \, \theta_5 \tag{123}$$

Since all the solutions satisfy the outer boundary conditions, the proper linear combination is determined by considering the values of the amplitude functions at the wall. These values are

Function	Inviscid	3	5
ϕ	Φ_w	L_w	J_w
f	F_w	1	K_w
θ	Θ_w	N_w	1

If the complex coupling constants for solutions 3 and 5 are respectively b_3 and b_5, then the following relations must be satisfied.

$$\Phi_w + b_3 L_w + b_5 J_w = 0 \tag{124}$$

$$F_w + b_3 + b_5 K_w = 0 \tag{125}$$

$$\Theta_w + b_3 N_w + b_5 = 0 \tag{126}$$

The coupling constants are thus [from Eqs. (125) and (126)]

$$b_3 = \frac{F_w - \Theta_w K_w}{N_w K_w - 1} \tag{127}$$

$$b_5 = \frac{\Theta_w - N_w F_w}{N_w K_w - 1} \quad (128)$$

Relation Eq. (124) is automatically satisfied since the component functions are eigenfunctions. Except for an arbitrary scaling factor the amplitude distributions can now be written:

$$\phi = (\Phi + b_3 \phi_3 + b_5 \phi_5) \quad (129)$$

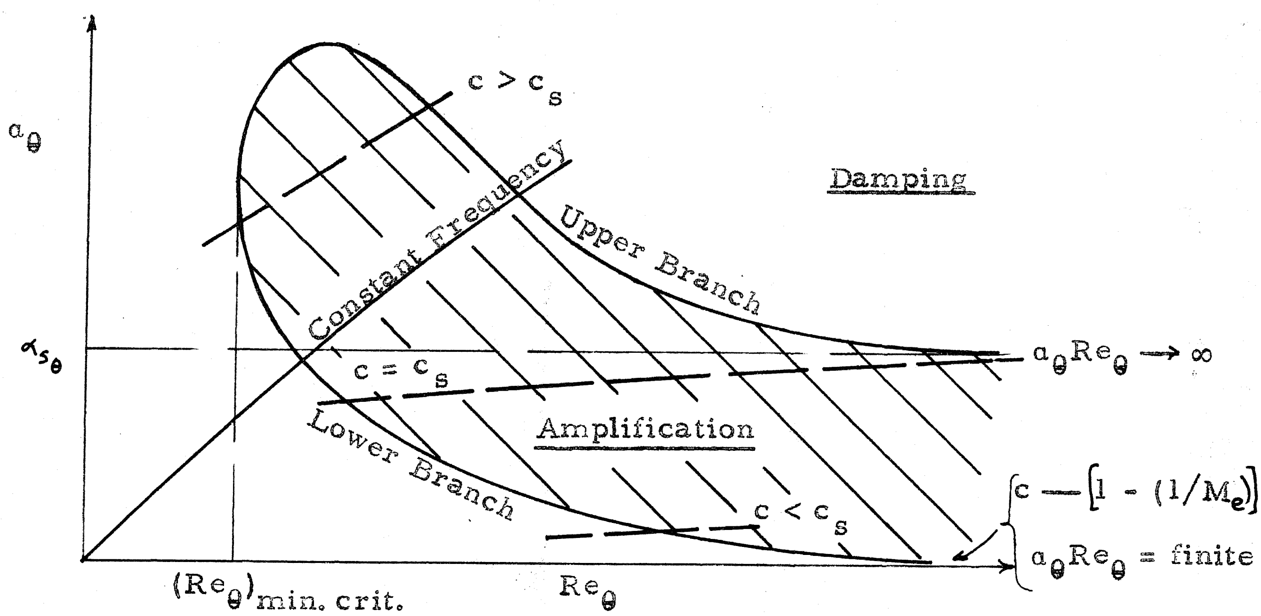
$$f = (F_{corr} + b_3 f_3 + b_5 f_5) \quad (130)$$

$$\theta = (\Theta_{corr} + b_3 \theta_3 + b_5 \theta_5) \quad (131)$$

$$\pi = (\pi_{inv}) \quad (132)$$

IV. EXAMPLES

To illustrate the present methods and to try to obtain an estimate of the validity of these methods and of the reasoning behind them, several numerical examples were obtained. The results for neutral stability are presented in the form of a plot of wave number as a function of Reynolds number, where both quantities are made dimensionless with the boundary layer momentum thickness. The terminology to be used will be described with the aid of the following diagram, which is representative of the stability behavior of subsonic and slightly supersonic insulated boundary layers.



This diagram is often called a stability loop. The upper boundary of the loop is called the "upper branch" and the lower boundary the "lower branch". The locus of constant frequency is a curve through the origin. There is a frequency above which all disturbances are always damped and there is a Reynolds number called the minimum critical Reynolds number below which there is no amplification of infinitesimal disturbances. The limiting wave number of the upper branch α_{s_θ} is that of a neutral inviscid disturbance ($\alpha Re \longrightarrow \infty$) propagating at a velocity c_s. Along the lower branch $\alpha_\theta \longrightarrow 0$ and $c \longrightarrow [1 - (1/Me)]$ as $Re_\theta \longrightarrow \infty$ in such a manner that $(\alpha_\theta \, Re_\theta)$ is finite. For $c = c_s$ there is also a neutral point on the lower branch. Generally for $c > c_s$ there are two neutral points while for $c_s > c > [1 - (1/M_e)]$ there is only a lower branch neutral point.

The calculated values of wave number may be converted to values of dimensionless frequency $(\beta \nu_e / u_e^2)$ (where β is 2π times the frequency) by the relation

$$\frac{\beta \nu_e}{u_e^2} = \frac{\alpha_\theta c}{Re_\theta} \quad .$$

The particular mean flow boundary layer profiles involved in the following examples are those for insulated flat plates recently computed by Mack[14] using real gas fluid properties. Mack's tables are particularly suited for stability calculations in that most of the derivatives of the profile functions required for the stability analysis are presented. The variable η used in this section is that defined by Mack, namely,

$$\eta = y^* \sqrt{\frac{u_e}{\nu_e x^*}} \quad ,$$

and is directly proportional to the physical distance normal to the wall. The values of α and Re quoted in this section are made dimensionless with the height $\eta = 1$. The values of α_θ and Re_θ are made dimensionless with η_θ, where η_θ is a constant whose value varies only slightly with Mach number. For $0 < M_e < 5$, $0.641 < \eta_\theta < 0.664$.[14]

The integrations of the inviscid and viscous equations required for the examples were performed numerically on the Detatron 205 of the Caltech Computing Center using a Runge-Kutta integration method.

IV. 1. Neutral Inviscid Oscillations at $\alpha Re \longrightarrow \infty$

The necessary and sufficient condition for the existence of a neutral purely inviscid oscillation ($\alpha Re \longrightarrow \infty$) is that (Reference 1)

$$A \equiv \left(\frac{w_c''}{w_c'} - \frac{T_c'}{T_c}\right) = 0 \quad \text{for} \quad c > [1 - (1/Me)] \quad . \tag{133}$$

The value of the propagation velocity c for which condition (133) is satisfied is denoted c_s, and this value depends on the particular profile being studied. In addition to the expected sensitivity to Mach number and surface temperature level, it is also quite sensitive to the Prandtl number and viscosity-temperature relationship, as indicated by the calculations of Van Driest[15]. Figure 1 shows the variation of c_s with Mach number at different surface temperature levels. Also shown on Figure 1 is the curve $c = [1 - (1/Me)]$. The disturbance propagation velocity for supersonic and hypersonic boundary layers is a very substantial portion of the free stream velocity and the critical layer (where $w = c$) cannot be thought of as being close to the wall.

For the mean boundary layer profiles of Mack[14], the wave numbers corresponding to $A = 0$, $c = c_s$ were obtained for Mach numbers between 1.3 and 5.6. These values are shown in Figure 2. The value of α_{s_θ} increases above Mach 2, reaches a peak at about Mach number 5, and then decreases with further rise in Mach number. The approximate behavior of α_{s_θ} for very large Mach number was obtained for $\rho\mu$ = constant, Prandtl number 1 by integrating Eq. (30) under the assumption that $M_\infty >> 1$. The result shows that for very large Mach number α_{s_θ} varies as the inverse square of the Mach number -- a trend that seems to be consistent with the calculated points in Figure 2.

The variation of the function $G(y) = \pi'/(\alpha^2\pi)$ is shown in Figure 3. In each case, the largest value of η for which G = 0 is the critical point. These curves have the general behavior described in Section III. 1. The integral under the curves is proportional to the logarithm of the pressure fluctuation amplitude. Thus, if the net area under the G curve is positive, the pressure fluctuation amplitude is higher than the wall value; if the net area is negative, the pressure fluctuation level is below its wall value.

For each of the curves in Figure 3, the pressure fluctuation amplitude at the critical point is calculated by the formula

$$\pi_c/\pi_w = e^{\alpha^2\int_0^{\eta_c} G d\eta} \quad .$$

The results are shown in Figure 4. For Mach numbers up to about 3, the pressure fluctuation level at the critical point is about the same as that at the wall, and is in fact quite constant in the region between the wall and the critical point. Above Mach 3, however, the pressure fluctuation level at the critical point drops quite sharply, and at Mach number 5.6 it is of the order of 6 per cent of the wall pressure fluctuation level. This sharp drop is attributed to the rapid increase with Mach number in the amplitude of normal velocity fluctuations between the wall and the critical point. From normal momentum considerations, this large velocity fluctuation must be counterbalanced by a large gradient in pressure fluctuation amplitude. If the phase of the normal velocity fluctuations leads that of the pressure fluctuations by 90° there is an outward decrease of pressure fluctuation amplitude. If the normal velocity fluctuations lag by 90° there is an outward increase in pressure fluctuation amplitude.

It might be expected that the wave numbers obtained for neutral inviscid disturbances ($\alpha Re \longrightarrow \infty$) at different Mach numbers would be somewhat indicative of the variation of the level of wave number with Mach number for finite Reynolds numbers.

IV. 2. Neutral Stability Characteristics of Insulated Supersonic Boundary Layers

The variation of wave number with Reynolds number for neutral oscillations was calculated at Mach numbers of 2. 2, 3. 2, and 5. 6 for the insulated boundary layer profiles of Mack[14]. The Mach number 2. 2 profile was chosen in order to compare the calculated results with the experimental findings of Laufer and Vrebalovich[6], and the Mach number 5. 6 profile to compare with the Mach number 5. 8 experiment of Demetriades.[7] The calculated results at these two Mach numbers were quite different in character, so that calculations at an intermediate Mach number of 3. 2 were performed in order to clarify the nature of the change between Mach numbers 2. 2 and 5. 6. In this section a comparison will also be made with calculations using the Lees-Lin method and the "corrected" Dunn-Lin method (Appendix D).

IV. 2. 1. Mach Number 2. 2

The neutral stability characteristics at Mach number 2. 2 were calculated by the methods of Section III. Before considering the entire neutral stability diagram, the calculation of the two neutral points at $\eta_c = 3.2$, $c = .61611$ will be examined in some detail.

The inviscid solutions for various α were obtained for $M_\infty = 2.2$, $\eta_c = 3.2$, $c = .61611$ by the method of Section III. 1, and the results are plotted in Figure 5. To be noted is that the imaginary part of $\left(-\frac{w_w'}{c} G_w\right)$ has an almost constant value of 0. 116 independent of α. This value very closely matches the value $v_o(c) = .12639$ suggested for this quantity in the theory of Lees and Lin[1].

The various viscous solutions obtained for $c = .61611$ are also shown in Figure 5. The numbers above the points are the values of αRe for the points. The points for Lees-Lin and Dunn-Lin theories were calculated in two ways: (1) by using the Tietjens and auxiliary functions as described in Appendix D; (2) by making the Lees-Lin and Dunn-Lin approximations in the HJK and LMN systems of Section III. 2. and obtaining the solutions numerically. The results of these calculations differ when αRe becomes small because some terms of order $1/(\alpha Re)$ are omitted in formulating the Tietjens function solutions. For large αRe there is no difference. This agreement at large αRe is a check of the correctness of the present calculation procedure for obtaining the viscous solutions.

The intersections of inviscid and viscous curves of Figure 5 are solutions of the secular equation and give the values of α and αRe for neutral stability. These solutions are listed in the following table:

UPPER BRANCH POINT

Viscous Theory	Symbol Figure 5	α	αRe	α_θ	Re_θ
Lees-Lin	○	.0665	48	.0426	464
Lees-Lin (Tietjens functions)	○'	.0668	48.2	.0429	464
Dunn-Lin	△	.0656	54.5	.0420	534
Dunn-Lin (Tietjens functions)	△'	.0656	54.5	.0420	534
Present	*	.0660	55	.0424	535

LOWER BRANCH POINT

Viscous Theory	Symbol Figure 5	α	αRe	α_θ	Re_θ
Lees-Lin	○	.050	5.7	.032	73
Lees-Lin (Tietjens functions)	○'	.048	6.17	.031	82
Dunn-Lin	△	.047	4.72	.030	65
Dunn-Lin (Tietjens functions)	△'	.040	5.0	.026	80
Present	*	.047	6.5	.030	89

For the point on the upper branch, the two Lees-Lin type calculations agree with each other at a value of αRe of about 48. The two Dunn-Lin calculations are identical to the accuracy of the calculations and give a value of αRe of 55, which is very close to that of the present calculation. The difference between 48 and 55 is the effect of including temperature fluctuations in the determination of the neutral stability characteristics. The close agreement between the Dunn-Lin and present calculations shows that the additional shear, conduction, and dissipation terms included in the present method have a negligible effect on the results at this Mach number and value of αRe. The value of $\bar{\epsilon}$ with $(\alpha Re) = 55$ is 0.187.

For the point on the lower branch the values of (αRe) run between 4.7 and 6.5. Neither the two Lees-Lin solutions nor the two Dunn-Lin solutions agree among themselves. Furthermore, the present calculation with the above-mentioned additional terms gives the highest value of αRe. Without these additional terms, the present method reduces to the Dunn-Lin (Δ) method which happens to give the lowest value of (αRe) in this case. Thus for this particular calculation the inclusion of the $\bar{\epsilon}$ and $M_{ref}^2\ \bar{\epsilon}$ terms, together with the order one terms (Section III. 2.), gives a large percentage increase in αRe. The value of $\bar{\epsilon}$ based on $(\alpha Re) = 6.5$ is 0.543. The fact that the next highest value of αRe for the lower branch points occurs for the Lees-Lin (σ) solution is felt to be fortuitous because this solution omits consideration of the temperature fluctuations altogether.

At Mach number 2.2 the inviscid solutions for different values of c have imaginary parts which are independent of α and whose level increases monotonically with c. The neutral stability diagram is constructed as follows:*

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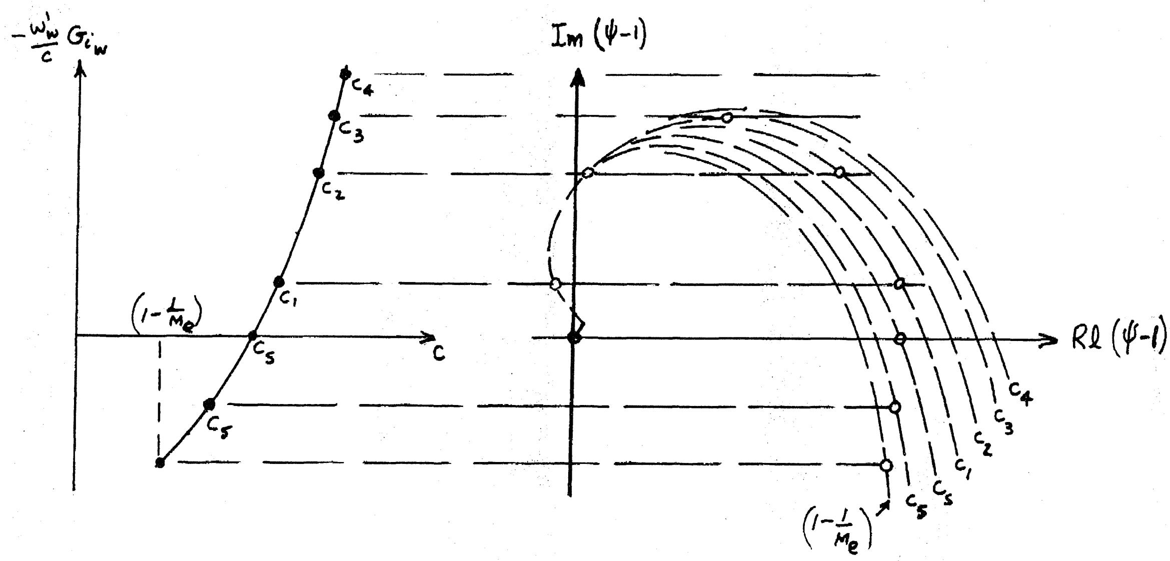
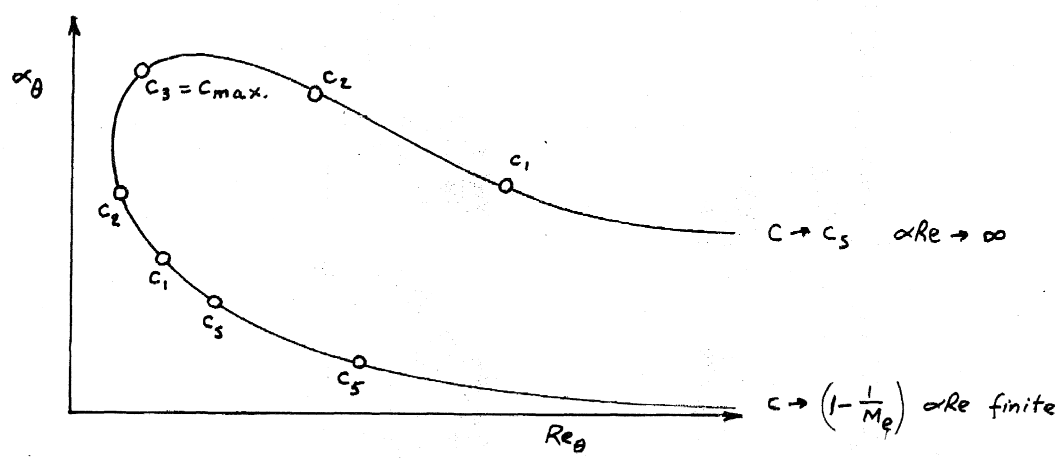
* This construction procedure, which is familiar to those readers who have had occasion to make boundary layer stability calculations, is here presented as background material for describing the related procedures at higher Mach numbers.



Imaginary Part of Inviscid Solutions

Viscous Solutions

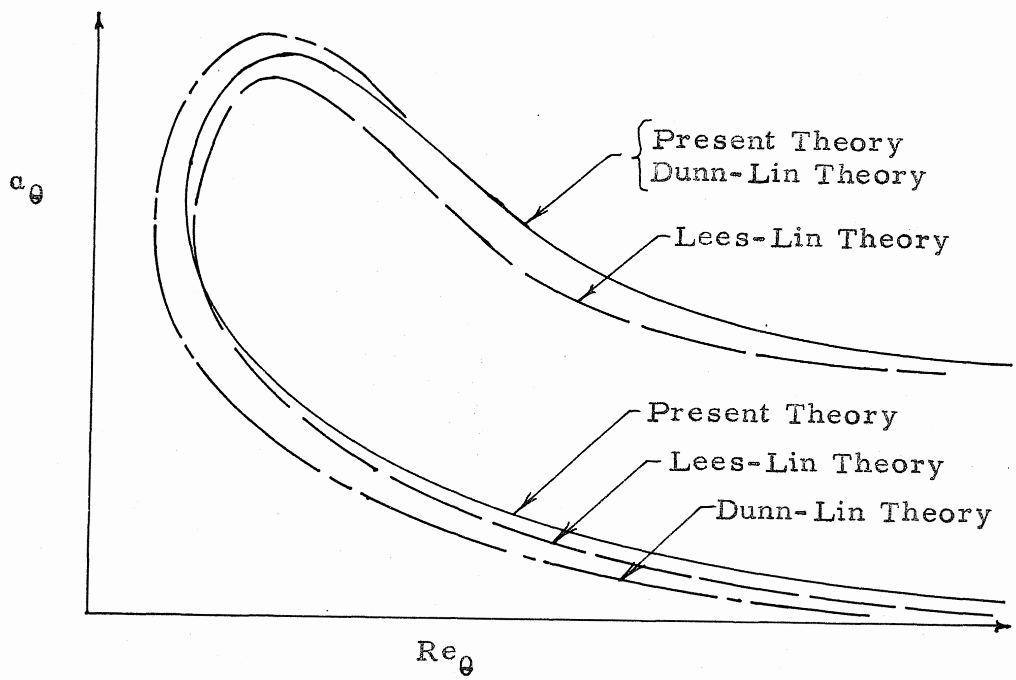


Neutral Stability Diagram

At $c = c_s$, there are two solutions. The solution for which $R\ell(\psi - 1) = Im(\psi - 1) = 0$ represents the neutral inviscid oscillation, while that for $R\ell(\psi - 1) \neq 0$ is a point on the lower branch. For $c > c_s$ (e.g., c_1, c_2, etc.) there are two solutions -- one upper branch and one lower branch -- until for $c = c_3$, the two solutions merge into one. For $c > c_3$ (e.g., c_4) there are no solutions, so that c_3 is the maximum value of c that is obtained. The remainder of the lower branch is composed of the single solutions for $\left[1 - (1/Me)\right] < c < c_s$. The locus of all of these solutions is the familiar neutral stability diagram.

The presently calculated neutral stability diagram for Mach number 2.2 is shown in Figure 6 together with the experimental points of Laufer and Vrebalovich[6]. There is good agreement between theory and experiment on the upper branch. On the lower branch however, the experimental values of $(\alpha_\theta \, Re_\theta)$ are almost twice the theoretical values. The major reason for this difference is probably that $\bar{\epsilon}$ is too large and that the higher order terms omitted in formulating the viscous equations become important. Also shown in Figure 6 as a dotted line is the neutral stability loop calculated using the Lees-Lin viscous solutions. The upper branch of this loop is only slightly below that of the present theory, while the results of the two theories on the lower branch are almost coincident. Although not shown on the figure the upper branch results of the present theory agree with those using the Dunn-Lin viscous solution. On the lower branch, however, the Dunn-Lin results give the lowest values of αRe.

These results relative to the Lees-Lin theory may be explained qualitatively as follows. The improvement introduced in the Dunn-Lin theory is to take proper account of the effect of compressibility on the energy fed into the disturbance flow by the action of viscosity at the wall. As will be shown in the next section the magnitude of this compressibility effect is related to a parameter $(M_e{}^2 c^2)/T_w$. This effect is always destabilizing, so that the Dunn-Lin neutral curve will always



tend to be outside the Lees-Lin loop. In the present theory, however, the effects of including the additional dissipation, shear and conduction terms become noticeable on the lower branch (where αRe is small) and tend to push the lower branch curve back toward the Lees-Lin curve, or even beyond it. At Mach number 2.2 the calculated additional energy production and additional dissipation are probably about equal and opposite, thus explaining the apparent agreement of the present theory with Lees-Lin theory on the lower branch in Figure 6.

A set of disturbance amplitude distributions across the boundary layer was obtained for the upper branch neutral point at $Re_\theta = 535$. These theoretical curves will be compared with a set of experimental distributions obtained by Laufer and Vrebalovich[6] for the upper neutral point at

$Re_\theta = 400$.

The inviscid amplitude distributions calculated by the methods described in Section III. 4. are shown in Figure 7. The boundary layer thickness δ is taken as the height where $w = 0.999$. The critical point for this case is at about $y = 0.4$. The amplitude is chosen so as to match the mass flow fluctuation amplitude at δ with the experimental value of Reference 6. The broken line in each case is the uncorrected inviscid function and is shown only where there is a difference between corrected and uncorrected functions. The region of influence of viscosity about the critical layer is $0.1 < y/\delta < 0.7$. In this case the region does not extend to the wall; therefore, the values of α and αRe obtained using wall values of the uncorrected inviscid functions stand as calculated. The largest correction is to Θ_r. The other corrections depend on the value of $A \equiv \frac{T_c}{w_c'} \left[\frac{d}{dy} \left(\frac{w'}{T} \right) \right]_c$, and are all small since in this case A is small.

The overall amplitude distributions (corrected inviscid plus viscous solutions) are shown in Figure 8 together with the points deduced by Laufer and Vrebalovich from their experimental observations. The agreement outside the critical layer is quite good -- "better than one has a right to expect"* -- while in the neighborhood of the critical layer the agreement is perhaps not quite as good. Laufer[16] indicates that in locally subsonic and transonic flows there is some doubt involved in deducing pressure, velocity, and temperature fluctuation amplitudes

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* Quoting J. Laufer and H. W. Liepmann.

from the mean square hot wire output. At local Mach numbers above 1.2 the calibration of hot-wires is well standarized[17], but such is not the case at transonic speeds. In addition, Laufer and Vrebalovich had to assume the values of pressure fluctuation amplitude in order to deduce velocity and temperature fluctuation amplitudes. It is felt both from some inviscid calculations of Mack (quoted in Reference 6) and from the present results that a value for π of .0007 should have been chosen rather than .0005 for use in reducing the experimental data.

The question of assumed pressure fluctuation level is removed when the mass flow and total temperature fluctuations from theory and experiment are compared. This comparison is shown in Figure 9, and agreement here is quite good.

IV. 2. 2. Mach Number 5.6

The calculation of the neutral stability characteristics at Mach number 2.2 by the present methods gives results which are of the same character as those obtained using the Lees-Lin and Dunn-Lin methods. However, some important differences are noted in the results at Mach number 5.6. In Figure 10 are shown the inviscid and viscous solutions for three values of c. The imaginary parts of the inviscid solutions are very small and certainly very much less than the quantity $v_o(c)$. [Note the difference in ordinate scales between Figures 5 and 10.] As will be shown in Section V., this decrease in $\left(- \frac{w_w'}{c} G_{i_w} \right)$ is related to the decrease in pressure fluctuation amplitude at the critical point relative to that at the wall.

The following special features of Figure 10 are to be noted: (1) the imaginary part of the inviscid solutions is no longer monotonic in

η_c. The level of the quantity $\left(- \frac{w_w'}{c} G_{i_w} \right)$ increases with η_c, reaches a maximum at $\eta_c = 16.5$ and then decreases toward zero as $c \longrightarrow 1$. (2) The values of $\left(- \frac{w_w'}{c} G_{i_w} \right)$ obtained are so small that the viscous solutions to the scale of Figure 10 are just vertical lines. The viscous solutions for large αRe are represented by a line almost coincident with the vertical axis. (3) For each value of $c > c_s$ there are two intersections. The values of αRe for the intersections near the vertical axis are directly dependent on the values of $\left(- \frac{w_w'}{c} G_{i_w} \right)$, and were in fact calculated using the asymptotic relations (D-41) to (D-43) valid for very large αRe. The values of αRe for the intersections to the right of Figure 10 are insensitive to the small value of $\left(- \frac{w_w'}{c} G_{i_w} \right)$ obtained from the inviscid calculation; in fact they are almost identical to the value of αRe that would be obtained for $\left(- \frac{w_w'}{c} G_{i_w} \right) = 0$.

The construction of the neutral stability diagram will now be described. At Mach number 5.6, the quantity $\left(- \frac{w_w'}{c} G_{i_w} \right)$ is slightly dependent on α but in the sketches on the following page, it will be considered as dependent only on c and independent of α. For $c = c_s$ (see sketch on following page), there are two solutions, one of which is that of the neutral inviscid oscillation. As c increases toward unity, two solutions are continually obtained. There is no longer the phenomenon of a maximum value of c above which no neutral oscillations can occur. This behavior occurs because the maximum value of $\left(- \frac{w_w'}{c} G_{i_w} \right)$ from the inviscid solutions (occurring for $c = c_2$ in the above sketches) is much less than the maximum value of $Im(\psi - 1)$ at the pertinent value of c. For $\left[1 - (1/M_e) \right] < c < c_s$ there is only one intersection. The solutions for neutral oscillations form two loops as shown in the sketches.

The two loops obtained at Mach number 5.6 are shown in

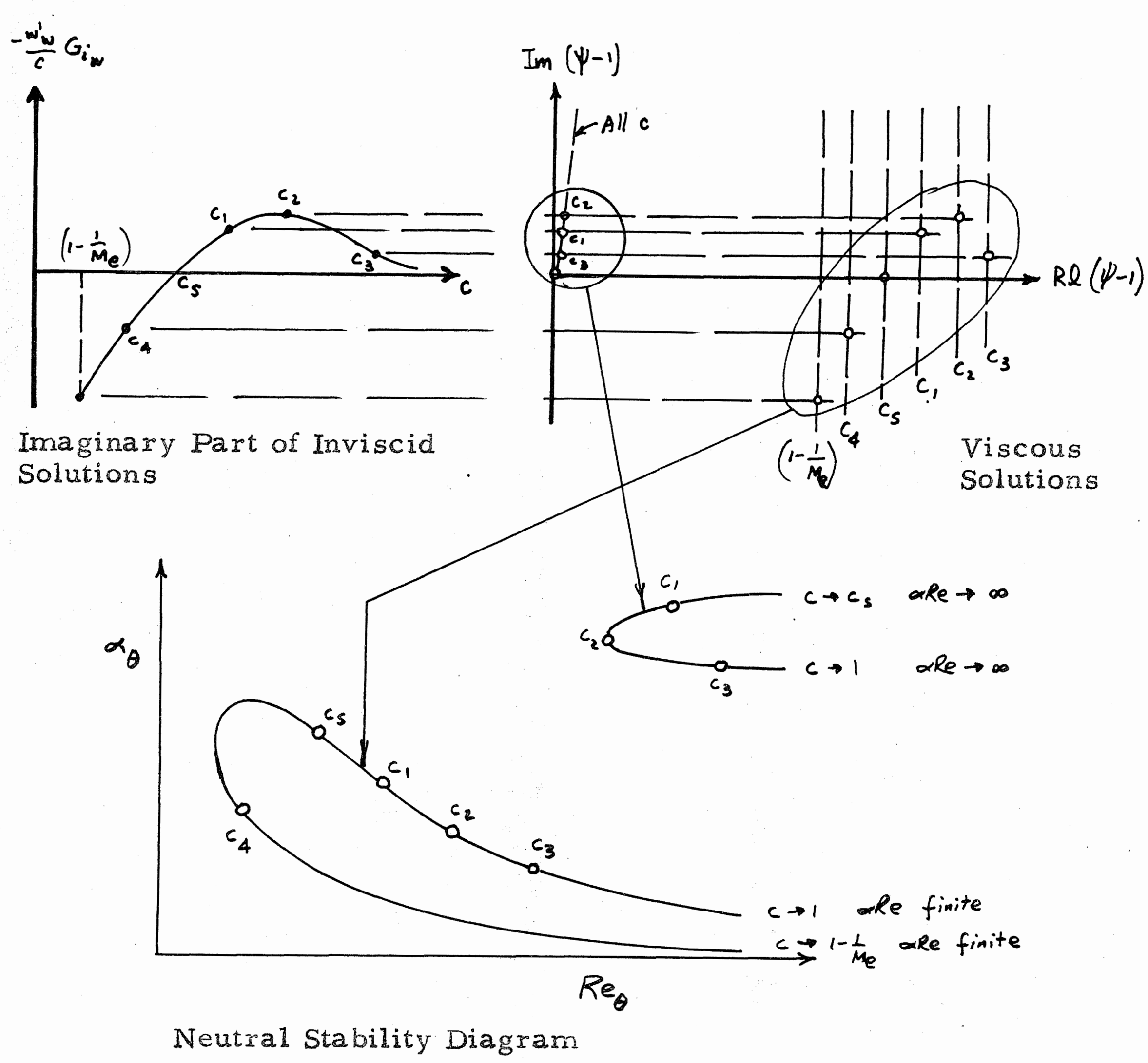


Figure 11. The upper right hand loop has a minimum Reynolds number based on momentum thickness of slightly over 10^5. This value of Re_θ corresponds to a length Reynolds number greater than 10^{10}, and it is not likely that this loop has much practical significance. The data obtained by Demetriades[7] at Mach number 5.8 are identified with the curve in the lower left hand corner of Figure 11. The portions of this

curve drawn in a full unbroken line have about the same shape as Demetriades' data, but are about an order of magnitude lower in Reynolds number than the experimental data. Nevertheless, as shown in Figure 13, a selected point on the curve is about an order of magnitude higher in Reynolds number than obtained using the Lees-Lin and Dunn-Lin theories. This behavior shows the importance of the new dissipation and shear terms presently included. The value of $\bar{\epsilon}$ for this presently calculated test point is 2.0, showing that there is no reason to expect good quantitative agreement between theory and experiment.

The dotted portion of the lower loop is that portion where the propagation velocity c very closely approaches unity. It is suspected that the calculation procedure is inadequate for this portion of the curve, since the splitting of the solutions into inviscid and viscous types is of questionable validity for $c \longrightarrow 1$. It will be shown later in discussing the complete solution to the Orr-Sommerfeld equation (Appendix H) that a necessary condition for splitting the solutions into inviscid and viscous types is that $[\alpha Re\,(1-c)] >> \alpha^2$.

The value of α approaches zero as $Re \longrightarrow \infty$ along the end of the "lower" loop for which $c \longrightarrow [1 - (1/M_e)]$ at a finite value of αRe^1. It is suspected that α also approaches zero as $Re \longrightarrow \infty$ for $c \longrightarrow 1$ but at a different finite value of αRe.

IV. 2. 3. Mach Number 3. 2

Calculations of neutral stability at Mach number 3. 2 were undertaken in order to provide some understanding of the transition from the one loop stability diagram obtained for Mach number 5. 6.

The results of the calculations are however difficult to understand and provide but little insight into the above-mentioned transitional behavior.

Consider the inviscid and viscous solutions for three values of c shown in Figure 12. At the lowest value of c shown ($\eta_c = 6.2$, $c = .79866$) the behavior is much like that at Mach number 2.2; that is, the imaginary part of the inviscid solution at the wall is almost independent of the wave number α. Its level is at about 2/3 $v_o(c)$, where $v_o(c)$ $\left([v_o(c)]_{\eta_c=6.2} = .19170 \right)$ is the value that is used in the Lees-Lin and Dunn-Lin solutions. This decrease in $\left(-\frac{w_w'}{c} G_{i_w} \right)$ is again a manifestation of the decrease in pressure fluctuation amplitude at the critical point relative to that at the wall. (See Section V.) The two intersections of the inviscid and viscous curves for $\eta_c = 6.2$ represent two points of neutral stability.

For $\eta_c = 6.6$, $c = .84087$, the inviscid solutions can no longer be represented as an almost horizontal curve whose imaginary part is independent of α. On the contrary, with increasing α the curve forms a loop, and so three intersections are shown in Figure 12 rather than the two shown for $\eta_c = 6.2$. There is a possible fourth intersection near the origin. Such a fourth intersection was sought, but in performing the numerical calculations large unstable oscillations in the values of G_r and G_i were noted. It is clear from this difficulty that a more careful study should be made of the general integration technique for the inviscid solutions. No statement can be made at this time regarding the existence of such a fourth intersection.

The curve of inviscid solutions for $\eta_c = 6.4$ (not shown) is similar to that for $\eta_c = 6.6$ except that the loop is larger and completely outside that for $\eta_c = 6.6$.

For $\eta_c = 6.8$ (not shown) the loop becomes much smaller and stays completely inside the curve of viscous solutions. An intersection is found at the right of Figure 12, but upon attempting to approach the origin, the inviscid solution again became unstable.

For $\eta_c = 7.0$ and above, the loop seems to have been "pulled tight". Shown for example in Figure 12 is the curve for $\eta_c = 7.2$, $c = .89637$. Here, the inviscid solutions are represented again by an almost horizontal line whose level decreases sharply for high α (near the origin). This behavior is very much like that obtained for Mach number 5.6. There are two intersections for this curve, one at each end, and therefore there are two neutral points. The average level of $\left(-\frac{w_w'}{c} G_{i_w}\right)$, which is about 0.07, is considerably below the value $v_o(c) = 5.63484$, thus indicating the strong effect of the decrease of pressure fluctuation amplitude across the boundary layer.

The neutral stability diagram for Mach number 3.2 is shown in Figure 13. Referring to the small diagram in the upper right hand corner, there are three curves.

The highest curve is made up of the high α intersections near the origin of Figure 12. The solid line represents actual calculated points for $\eta_c = 7.0$, 7.2, and 7.4. For some value of η_c above 7.4 it is expected that the curve would turn back in a manner similar to that obtained for the upper loop at Mach number 5.6. (This expectation is indicated by a broken line in Figure 13.)

The other two curves in the upper right diagram of Figure 13 are drawn to a larger scale in the main body of the figure. The next curve proceeding downward is the remnant of the conventional "upper branch".

It is terminated at the point where the inviscid solutions started "blowing up".

The lowest curve is apparently a narrow closed loop and corresponds to the intersections obtained on the lower right hand side of Figure 12. This loop is very much like the lower loop obtained for Mach number 5.6. The wave number α approaches zero as $Re \longrightarrow \infty$ along the two ends of this loop, but at two different finite values of αRe.

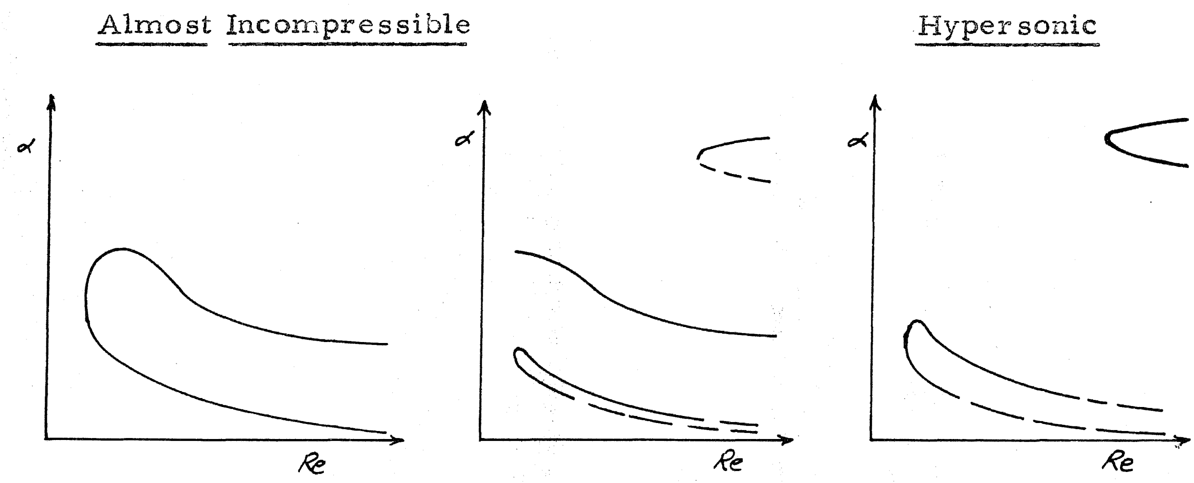
The neutral stability behavior at Mach number 3.2 is evidently not well understood. The rather well defined loop behavior obtained at Mach numbers 2.2 and 5.6 is not as well defined at Mach number 3.2. Numerical difficulties at large α are encountered at Mach number 3.2 but not at the other Mach numbers. Perhaps the only definite statement that can be made is that those portions of Figure 13 shown in solid lines represent calculated neutral oscillations.

IV.2.4. Discussion

From the results obtained so far it is of some interest to consider comparatively the neutral stability behavior of insulated boundary layers over a wide range of Mach numbers. Because these comparisons are based on only a few calculated stability diagrams, some of which are not well understood, parts of the discussion which follows must be considered speculative.

The three types of neutral stability diagrams obtained herein are sketched on the top of the following page.

For Mach numbers of 2.2 and below, only a single stability loop is obtained both theoretically and experimentally[6, 18]. Somewhere



above Mach number 2.2 (represented in the present calculations by Mach number 3.2) two new loops appear while the original one is perhaps starting to disappear. At Mach number 5.6 only the two new loops remain. We may perhaps describe the stability behavior from Mach number zero to about Mach number 2.5 as "almost incompressible"*; that above about Mach number 5 as "hypersonic"; which leaves a very interesting transition region from one type of stability behavior to the other between Mach numbers of about 2.5 and 5.

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* Dr. John Laufer[16] of the Jet Propulsion Laboratory has in fact recently correlated the experimental results of Schubauer and Skramstad[18] for incompressible flow and those of Laufer and Vrebalovich[6] at $M_\infty = 1.6$ and $M_\infty = 2.2$. By plotting $(\beta\nu/u_e^2)$ against Re_δ, where δ is the full boundary layer thickness, he obtains a single diagram for all three Mach numbers. The amplification factors plotted as $(\alpha c_i \delta)/u_e$ vs. Re_δ also correlate for these experiments. These observations support the identification of the neutral stability characteristics at Mach numbers up to about 2.5 as almost incompressible.

The variation of minimum critical Reynolds number with Mach number is also quite interesting. The calculated value of $(Re_\theta)_{min.\ crit.}$ decreases from a value of about 150 at Mach number zero (Reference 2) through a value of about 45 at Mach number 2.2 (Figure 6), to some value below 10 at Mach number 3.2 (Figure 11). All these values are obtained from the conventional "almost incompressible" loop. At Mach number 5.6, the calculated value of $(Re_\theta)_{min.\ crit.}$ is about 45 (Figure 13) and comes from the new lower loop. (It may also be noted that the minimum Reynolds number of the lowest loop at Mach number 3.2 is about 16.) These results indicate that the minimum critical Reynolds number decreases from its Mach number zero value, reaches a minimum somewhere around Mach number 3 and then increases again.

One may speculate about the variation of minimum critical Reynolds number at hypersonic speeds. Note first from Figures 11 and 13 that for the lowest loop, the values of $(\alpha_\theta\ Re_\theta)$ are about the same for both Mach numbers, but the peak value of α_θ is quite a bit lower at Mach number 5.6 than at Mach number 3.2. Following the asymptotic trend of wave number shown in Figure 2, namely that $\alpha_\theta \sim 1/M_e^2$, if $(\alpha_\theta\ Re_\theta)$ remains fairly constant hypersonically then $(Re_\theta)_{min.\ crit.}$ would increase as M_e^2 (and $(Re_x)_{min.\ crit.}$ as M_e^4). We can infer that instability of the laminar boundary layer would move downstream very rapidly with increase in local Mach number. There is some experimental evidence supporting this latter speculation, namely that Bogdonoff[19] reports that he has never observed transition to occur at a local Mach number of 11 even at length Reynolds numbers as high as 10^7.

V. QUALITATIVE DESCRIPTION OF COMPRESSIBLE BOUNDARY LAYER STABILITY

The examples of the previous section (Section IV.) show that the theoretical and experimental neutral stability characteristics and amplitude distributions are in at least qualitative agreement. For a better understanding of the stability phenomenon it is useful to discuss the balance of disturbance energy for a neutral oscillation and the distribution of Reynolds stress through the boundary layer.

The gross character of the stability of a shear flow can be described by the following energy relationship

$$\begin{bmatrix}\text{Net Energy}\\ \text{Change}\\ \text{Per Cycle}\end{bmatrix} = \begin{bmatrix}\text{Net "Production" of}\\ \text{Disturbance Energy}\\ \text{Per Cycle}\end{bmatrix} - \begin{bmatrix}\text{Dissipation}\\ \text{Per Cycle}\\ \end{bmatrix} \qquad (134)$$

For a neutral disturbance the net energy change per cycle is zero. By "production" in Eq. (134) is meant the transferral of energy from the mean flow to the disturbance flow. The net production term is expressed in terms of the Reynolds stress

$$\begin{bmatrix}\text{Net Production of}\\ \text{Disturbance Energy}\\ \text{Per Cycle}\end{bmatrix} = \int_0^{\lambda}\int_0^{\infty} \tau\,(\partial u/dy)\,dy\,dx \qquad (135)$$

where

$$\tau = -\rho\,\overline{u'v'} \qquad (136)^*$$

---

* Of course the "mean flow" is also altered slightly by the action of the Reynolds stress (Stuart[20]), but this effect is of order a^2Re, where a is here the disturbance amplitude, and makes a second order correction to the disturbance flow.

The production and dissipation phenomena both depend on viscosity. For the dissipation this fact is obvious. In the production term it is the action of viscosity which shifts the phase of the disturbance velocities so that the correlation $\overline{u'v'} \neq 0$. It will be shown below that the production term is of order $1/\sqrt{\alpha Re}$ while the dissipation term is approximately linear in the viscosity and is therefore of order $1/(\alpha Re)$. Thus for a neutral oscillation there is some value of (αRe) for which the production and dissipation terms balance. The key quantity in the energy balance is the Reynolds stress. Some knowledge of its behavior will prove desirable in understanding the effects of compressibility.

An expression for Reynolds stress is derived from the expressions for longitudinal and normal velocity fluctuations. All quantities below are dimensionless as described in the list of symbols. The longitudinal and normal velocity fluctuations are

$$\left.\begin{aligned} u' &= R\ell\left[f e^{i\alpha(x-ct)}\right] = \frac{1}{2}\left[f e^{i\alpha(x-ct)} + f^{*} e^{-i\alpha(x-ct)}\right] \\ v' &= \alpha R\ell\left[\phi e^{i\alpha(x-ct)}\right] = \frac{\alpha}{2}\left[\phi e^{i\alpha(x-ct)} + \phi^{*} e^{-i\alpha(x-ct)}\right] \end{aligned}\right\} \quad (137)$$

By utilizing these expressions for u' and v' the Reynolds stress is given by

$$\tau = -\rho\,\overline{u'v'} = -\rho\frac{\alpha}{4}\left[\overline{f\phi e^{2i\alpha(x-ct)}} + \overline{f^{*}\phi^{*} e^{-2i\alpha(x-ct)}} + (f\phi^{*} + f^{*}\phi)\right]$$

But f and ϕ are functions of y only, and over one cycle $\overline{e^{2i\alpha(x-ct)}}$ and $\overline{e^{-2i\alpha(x-ct)}}$ are both zero. Also, $(f\phi^{*} + f^{*}\phi) = 2\,R\ell(f\phi^{*})$ so that

$$\tau = -(\rho/2)\,\alpha\,R\ell\,(f\phi^{*}) \quad (138)$$

In studying the effects of compressibility on Reynolds stress it is useful to consider Prandtl's model of the disturbance flow -- namely --

that the disturbance flow is essentially inviscid (slowly varying) except for two narrow regions where viscosity is important: (1) at the wall where the viscosity causes a shift in phase of the disturbance velocities, and (2) at the critical layer ($w = c$), where there is no longitudinal transport relative to the wave and the vertical transport of vorticity and heat energy can be balanced only by viscous diffusion and heat conduction. Prandtl's model assumes that (αRe) is very large, as is obtained for example along the extremity of the upper branch. In spite of this restriction, Prandtl's model is useful for developing concepts.

Consider first the arguments that were presented together with the sketch on page 12. For a fictitious gas having zero viscosity and zero thermal conductivity ($\alpha Re \longrightarrow \infty$), only the inviscid solutions apply. The normal velocity fluctuation vanishes at the wall and at infinity. The temperature and longitudinal velocity fluctuations also decay to zero far from the wall, but in the absence of viscosity and conductivity, generally take on finite values at the wall.

For a real gas, the longitudinal velocity and temperature fluctuations must vanish at the wall no matter how small the viscosity and thermal conductivity. Thus we must add viscous solutions that take on the wall values $-(f_{inv})_w$ and $-(\theta_{inv})_w$ to the inviscid functions already determined. (See sketch on page 12.) An approximate form of these viscous solutions is obtained by considering the asymptotic form of the viscous equations [Eqs. (54) to (56)]. In a thin layer near the wall ($w \approx 0$), Eqs. (54) to (56) become

$$f_v''' + \frac{i\alpha Rec}{\nu_w} f_v' = 0 \qquad (139)$$

$$\phi_v' = - if_v - \frac{ic}{T_w}\theta_v \tag{140}$$

$$\theta_v'' + \frac{i\alpha Re\sigma c}{\nu_w}\theta_v = 0 \tag{141}$$

The desired solutions to Eqs. (139) and (141) are those which decay to zero far from the wall; they are, respectively,

$$f_v = -(f_{inv})_w \; e^{-(1-i)\, y/\delta_w} \tag{142}$$

$$\theta_v = -(\theta_{inv})_w \; e^{-(1-i)\frac{y\sqrt{\sigma}}{\delta_w}} \tag{143}$$

where $\delta_w = \sqrt{2\nu_w/\alpha Re c}$ is representative of the thickness of the layer near the wall in which the viscosity effects are important.

According to the continuity equation [Eq. (140)], a viscous normal velocity fluctuation is induced by f_v and θ_v. Since $\phi_v(\infty) = 0$, then by utilizing Eqs. (142) and (143) in Eq. (140) one obtains

$$\begin{aligned}\int_\infty^0 \phi_v' \, dy &= (\phi_v)_w - \phi_v(\infty) = (\phi_v)_w \\ &= -\frac{i(1+i)\delta_w}{2}(f_{inv})_w \left(1 + K\sigma^{-\frac{1}{2}}\right)\end{aligned} \tag{144}$$

where

$$K \equiv \frac{(\gamma - 1) M_e^2 c^2}{T_w} \tag{145}$$

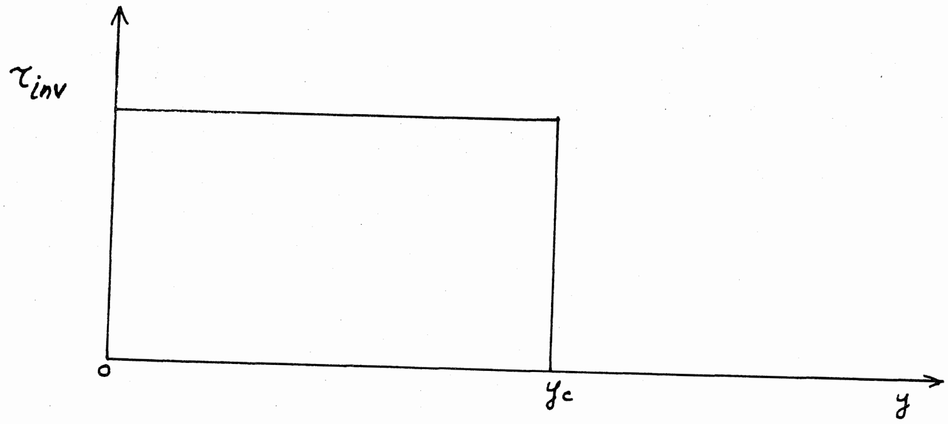
But $\phi_w = (\phi_v)_w + (\phi_{inv})_w = 0$, so that $(\phi_{inv})_w = -(\phi_v)_w$ and the inviscid solutions must now be altered slightly to satisfy the boundary condition at the wall.

In the case of the fictitious inviscid gas the Reynolds stress for a neutral disturbance must vanish everywhere. But for the real gas

$(\phi_{inv})_w \neq 0$ and the Reynolds stress associated with the inviscid solutions is as follows: [Eqs. (138), (144), and (145)] :

$$\tau_{inv}(\delta_w) = \frac{\rho_w \, a \, \delta_w}{4} \left|(f_{inv})_w\right|^2 (1 + K\sigma^{-\frac{1}{2}}) \qquad (146)^*$$

[Note that $\tau_{inv}(\delta_w) > 0$ and is of order $1/\sqrt{Re}$] . Now the Reynolds stress for the inviscid solution is constant in any region where the Wronskian of the two solutions ϖ_r and ϖ_i (or ϕ_r and ϕ_i) is continuous (Reference 1), so this Reynolds stress must be constant in the region between the wall and the critical layer, as depicted in the following sketch. But $\tau \longrightarrow 0$



far from the wall, so the value of τ given by Eq. (146) must be cancelled by an equal and opposite increment in Reynolds stress at the critical layer. This incremental jump in Reynolds stress in the neighborhood of the critical layer is a viscous phenomenon. However, its value can be

* The detailed distribution of the Reynolds stress for the complete solution is discussed on page 78.

calculated from the inviscid solutions, and viscosity has the effect of smoothing the transition from one value to the other. This behavior is analogous to the one dimensional normal shock wave, which is a viscous conductive phenomenon whose gross characteristics can be determined without considering viscosity and conductivity, but whose detailed structure can only be determined through consideration of the effects of viscosity and thermal conductivity.

The magnitude of the jump in Reynolds stress can be obtained directly from the inviscid solutions in the following manner:

$$\begin{aligned}[\tau]_{y_c-0}^{y_c+0} &= -\frac{\rho_c\alpha}{2}\left[Rl\left(f_{inv}\,\phi_{inv}^{*}\right)\right]_{y_c-0}^{y_c+0} \\ &= -\frac{\rho_c\alpha}{2}\left[(f_{inv})_r(\phi_{inv})_r + (f_{inv})_i(\phi_{inv})_i\right]_{y_c-0}^{y_c+0}\end{aligned} \tag{147}$$

Taking the leading terms of f_{inv} and ϕ_{inv} about the critical layer from Eqs. (97) to (100) yields

$$\begin{aligned}[\tau]_{y_c-0}^{y_c+0} &= \frac{\alpha}{2}\,\frac{T_c\,\pi_c^2}{(\gamma M_e^2)^2}\,\frac{A\pi}{w_c'^2} \\ &= -\frac{\alpha}{2}\,\frac{c}{T_w w_w'}\left|(f_{inv})_w\right|^2 v_o(c)\left|\frac{\pi_c}{\pi_w}\right|^2\end{aligned} \tag{148}$$

where

$$v_o(c) \equiv -A\pi\,\frac{cT_c}{w_c'^2}\,\frac{w_w'}{T_w} = -\pi\,\frac{w_w'}{T_w}\,\frac{cT_c^2}{w_c'^3}\left[\frac{d}{dy}\left(\frac{w'}{T}\right)\right]_c \tag{149}$$

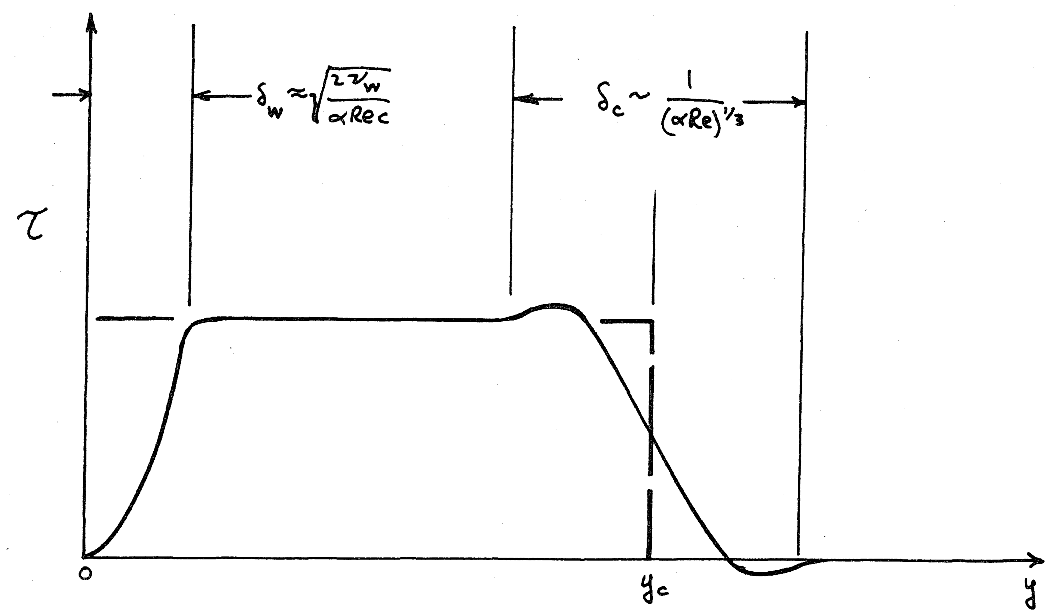
This function arises continually in hydrodynamic stability problems[21] and is in fact included in the compressible boundary layer tabulations of Mack[14]. Along the upper branch $c > c_s$, $A < 0$, $v_o(c) > 0$ so that the jump in Reynolds stress is negative as required. Setting the sum of Eqs. (146)

and (148) equal to zero and remembering that $\delta_w = \sqrt{2\nu_w/\alpha Re\, c}$ yields

$$\sqrt{\alpha Re} = \sqrt{\frac{\nu_w w_w'^2}{2c^3}} \quad \frac{\left(1 + K\sigma^{-\frac{1}{2}}\right)}{V_0(c)\left|\frac{\pi_c}{\pi_w}\right|^2} \quad . \tag{150}$$

The significance of this expression is as follows: As the Mach number increases, K increases [from Eq. (145)] while $|(\pi_c/\pi_w)|^2$ decreases (see Figure 4). Both of these effects tend to extend the upper neutral boundary to larger values of αRe, and are to be interpreted as destabilizing since they enlarge the region of amplified disturbances. It may be noted that expression (150) is exactly the leading term of the secular equation (D-41), which was obtained by formal asymptotic methods.

For neutral disturbances at very large values of αRe, the Reynolds stress has the following distribution through the boundary layer.



A detailed distribution of Reynolds stress for the complete solution near the wall can be obtained using the approximate asymptotic expressions for the disturbance amplitudes given by Eqs. (142) and (143). Thus

$$f \approx f_v + (f_{inv})_w \approx (f_{inv})_w \left[1 - e^{-(1-i)\frac{y}{\delta_w}} \right]$$
$$\approx (f_{inv})_w \left[(1-i)\frac{y}{\delta_w} + i\left(\frac{y}{\delta_w}\right)^2 - \frac{(1+i)}{3}\left(\frac{y}{\delta_w}\right)^3 + \dots \right] \quad (151)$$

$$\theta \approx \theta_v + (\theta_{inv})_w \approx (\theta_{inv})_w \left[1 - e^{-(1-i)\frac{y\sqrt{\sigma}}{\delta_w}} \right]$$
$$\approx (\theta_{inv})_w \left[(1-i)\frac{y\sqrt{\sigma}}{\delta_w} + i\sigma\left(\frac{y}{\delta_w}\right)^2 - \frac{(1+i)\sigma^{3/2}}{3}\left(\frac{y}{\delta_w}\right)^3 + \dots \right] \quad (152)$$

Note in Eq. (88) that for $(\alpha Re) >> 1$, $\phi_{inv}/f_{inv} \sim \mathcal{O}(\delta_w)$, so that

$$(\theta_{inv})_w / (f_{inv})_w = (\gamma - 1) M_e^2 c \left[1 + \mathcal{O}(\delta_w) \right] \quad (153)$$

In the expressions which follow, all terms of order δ_w or smaller will be neglected compared to the unit order terms. Substituting expressions (151) to (153) into the continuity equation [Eq. (4)] yields for ϕ

$$\phi \approx i(f_{inv})_w \delta_w \left[\left(\frac{y}{\delta_w}\right) - \frac{i(1-i)}{2}\left(1 + K\sigma^{+1/2}\right)\left(\frac{y}{\delta_w}\right)^2 - \frac{i}{3}\left(1 + K\sigma\right)\left(\frac{y}{\delta_w}\right)^3 + \dots \right] \quad (154)$$

From Eqs. (138), (151), and (154) the distribution of Reynolds stress near the wall is

$$\tau \approx \rho\frac{\alpha\delta_w}{2} \left| (f_{inv})_w \right|^2 \left\{ \frac{\gamma M_e^2 c^2}{T_w}\left[\left(\frac{y}{\delta_w}\right)^2 - \left(\frac{y}{\delta_w}\right)^3 + \frac{1}{3}\left(\frac{y}{\delta_w}\right)^4 + \dots \right] + \left[\frac{1+K\sigma^{\frac{1}{2}}}{2} - \frac{1+K\sigma}{3} \right]\left(\frac{y}{\delta_w}\right)^4 + \dots \right\} \quad (155)$$

For incompressible flow expression (155) becomes

$$\tau \approx \rho\frac{\alpha\delta_w}{12} \left| (f_{inv})_w \right|^2 \left(\frac{y}{\delta_w}\right)^4 + \dots \quad (156)$$

indicating that the Reynolds stress near the wall is positive and grows as the fourth power of y, as already shown by Lin[21]. For compressible flow, the leading term is a $(y/\delta)^2$ term; furthermore, all coefficients increase with Mach number. Thus for compressible flow, the Reynolds stress grows more rapidly near the wall than for incompressible flow.

The smoothing out about the critical point of the jump discontinutiy in Reynolds stress is evidence of the action of viscosity in a layer about the critical point whose thickness is of order $1/(\alpha Re)^{1/3}$. From Eqs. (97), (98), (110), and (111), this variation for (αRe) large is given approximately by

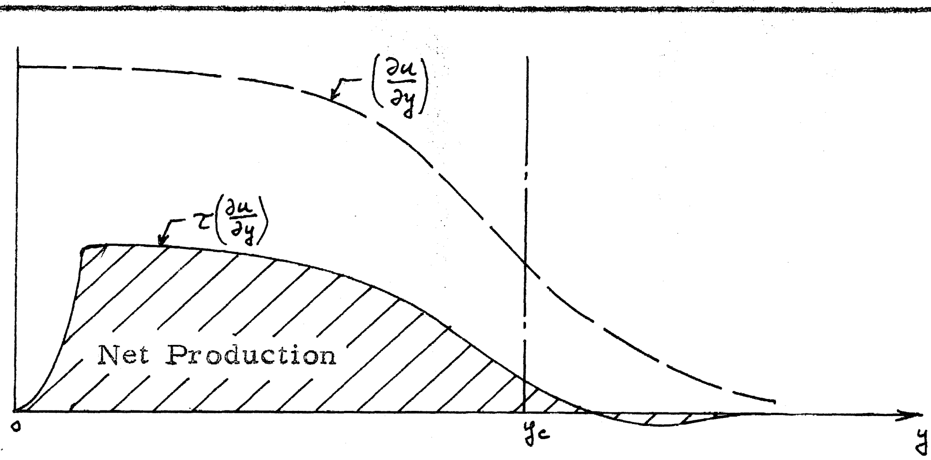
$$\tau(\zeta) - \tau(\zeta = 4) = + \left[\tau\right]_{y_c - 0}^{y_c + 0} \left(\frac{\hat{H}'(\zeta)}{3.14} \right) \tag{157}$$

where

$$\begin{aligned} \zeta &= (\alpha Re)^{1/3}\, Y \\ &= (\alpha Re)^{1/3} \left[\int_{y_c}^{y} 3/2 \sqrt{\frac{w-c}{\nu}}\, dy \right]^{\frac{2}{3}} \end{aligned} \tag{158}$$

and $\hat{H}'(\zeta)$ is the pertinent function of Schlichting[13] (Table I).

The net production of energy may also be sketched.



The shaded area, being the integral under the $\tau(\partial u/\partial y)$ curve, represents the net energy production per cycle. It is this quantity which must be exactly balanced by dissipation for a neutral oscillation.

For (αRe) only moderately large, the two regions in the boundary layer where viscosity is important tend to grow and may even overlap so that a region of constant Reynolds stress may not be observed between the wall and the critical layer. As an example of the Reynolds stress distribution for moderately large (αRe), that calculated from the amplitude distributions of Figures 7 and 8 is shown in Figure 14. The broken line in Figure 14 is the level of the inviscid Reynolds stress between the wall and the critical layer. When $c < c_s$, the overlapping of the two viscous regions must always occur because the jump in Reynolds stress predicted by the inviscid solutions across the critical layer [Eq. (148)] is positive and can never counterbalance the Reynolds stress produced near the wall.

Even when αRe is not large the following qualitative effects remain: The production of disturbance energy at the wall increases with Mach number. The stabilizing or destabilizing effect at the critical layer diminishes in the ratio $|(\pi_c/\pi_w)|^2$, and at high Mach number may be of negligible importance. Both of these tendencies indicate that as the Mach number increases, the net production of disturbance energy also increases so that the dissipation effects must become more important. As indicated by the calculations at Mach number 5.6, this behavior significantly lowers the range of (αRe) for neutral disturbances to a level such that Prandtl's splitting of the disturbance flow into inviscid and viscous parts is no longer appropriate for a proper quantitative estimation of the stability characteristics of a given boundary layer profile.

VI. SUGGESTIONS FOR FURTHER STUDY

The present investigation has probably raised more questions than it has answered, particularly in the sense of pointing out problem areas where it had been thought that no problems existed. These new problems lend themselves both to theoretical and experimental investigation.

VI. 1. Suggested Theoretical Investigations

1. In considering the stability of supersonic and hypersonic boundary layers, asymptotic methods as used in almost every stability investigation through the present do not yield adequate quantitative results. For many of the examples presented, the splitting of solutions into inviscid and viscous types is improper. Also upon a posteriori examination the terms of the disturbance equations omitted on the basis of asymptotic considerations in some cases seem to be as important as those retained.

Accordingly it is suggested that an attempt be made to solve the complete disturbance equations. First, the "complete" equations must be identified. In general they are linear partial differential equations; however, in some cases, for example for fully developed or "parallel" flows, they are ordinary differential equations and are perhaps amenable to solution. The simplest example of a "complete" disturbance equation is the Orr-Sommerfeld equation for incompressible "parallel" flow, and a formulation for its complete solution is given in Appendix H. The method used is similar to that of Section III. 2. for solving the viscous equations.

2. The stability of a boundary layer with respect to three-dimensional disturbances of the oblique plane wave variety was considered by Dunn and Lin[5] within the limitation that dissipation was unimportant. This limitation permitted a mathematical transformation of the equations for three-dimensional disturbances into those for two-dimensional disturbances. Since dissipation is expected to be important at high Mach number it is suggested that the problem be reconsidered at least to the extent considered herein for two-dimensional disturbances. The disturbance equations will not transform to the two-dimensional form but will yet be solvable. This problem is directly analogous to that of steady compressible three-dimensional boundary layers with pressure gradient, where the independence principle does not hold.

3. Because of the numerical difficulties encountered in solving the inviscid equation at Mach number 3.2 for large α, it is suggested that some additional analytical study be made of the inviscid equation. It would be of particular interest to determine if there exists more than one wave number characteristic of a neutral inviscid disturbance. The answer to this question might help clarify the multiloop behavior indicated at Mach number 3.2.

4. The examples considered herein are all for insulated boundary layers. The method presented is applicable as well to non-insulated boundary layers. A reexamination of the effect of surface cooling on stability characteristics in the light of the present theory might be of some interest.

5. Calculations of amplified and damped disturbances using the presented methods would also be of some interest.

VI. 2. Suggested Experimental Investigations

1. The establishment of the multi-loop nature of neutral stability characteristics at a Mach number in the range between 3 and 4. The measurements should include a determination of the disturbance wavelengths so that the propagation velocities of the disturbances can be ascertained.

2. Further measurements of stability characteristics at hypersonic Mach numbers where theoretical results are particularly deficient. Measurements should include neutral stability boundaries, amplification rates, wavelength determinations, and if possible, amplitude distributions.

VII. CONCLUDING REMARKS

As a result of the present study of the stability of the compressible laminar boundary layer, it is concluded that the basic stability mechanisms are the same as for incompressible flow; namely that neutral stability is achieved when the net energy transferred from the mean flow to the disturbance flow by the action of Reynolds stress is exactly dissipated. However, the relative importance of the various mechanisms changes with Mach number, so that the neutral stability diagram for a hypersonic flow might not be recognized in terms of its incompressible counterpart. Among the changes are the following:

1. The rate of conversion of energy from the mean flow to the disturbance flow through the action of the viscosity in the vicinity of the wall increases with Mach number.

2. Instead of being nearly constant across the boundary layer, the amplitude of inviscid pressure fluctuations for Mach numbers greater than 3 decreases markedly with distance outward from the plate surface. This behavior means that the jump in magnitude of the Reynolds stress in the neighborhood of the critical layer is greatly reduced.

3. At Mach numbers less than about 2, dissipation effects are minor, but they become extremely important at higher Mach numbers, since for neutral disturbances, they must compensate for the generally destabilizing effects of items 1 and 2 above.

From the numerical computations some further results are obtained:

4. The wave number of a neutral inviscid disturbance increases with Mach number, reaching a maximum at about Mach number 5. It then decreases and approaches a $1/M_e^2$ variation at very high Mach

numbers.

5. Calculations of neutral stability characteristics and disturbance amplitude distributions at Mach number 2.2 are in substantial agreement with the experimental findings of Laufer and Vrebalovich. The largest differences obtained are on the lower branch, where the asymptotic methods used herein become inadequate.

6. At Mach number 5.6, two loops are obtained, the lower one being identified with the experimental results of Demetriades. Although the present results are an improvement over those obtained using the theories of Lees and Lin and Dunn and Lin, they still underestimate by an order of magnitude the range of Reynolds numbers for which neutral stability is obtained. This deficiency is also attributed to the inadequacy of the asymptotic methods used.

7. The minimum critical Reynolds number for insulated flat plate boundary layers decreases in the range $0 < M < 3$, and then rises very sharply for hypersonic Mach numbers.

Because of the indicated tendency of the asymptotic methods to become inadequate as the Mach number increases (items 5 and 6 above) it is suggested that an attempt be made to solve the complete disturbance equations, perhaps by an extension of the methods used herein.

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APPENDIX A

SERIES SOLUTION OF INVISCID EQUATION ABOUT CRITICAL POINT

In treating the inviscid equation alone, a regular singularity appears at the critical point, (where w = c). The solution of the inviscid equation in the neighborhood of the critical point is obtained by a series expansion that is sometimes called the Method of Frobenius. Although the inviscid equation is treated in the text primarily as a first order non-linear equation, the series expansion will be performed in the manner of Miles[10] on the second-order linear equation which is

$$\pi'' - \frac{T}{(w-c)^2}\left[\frac{(w-c)^2}{T}\right]'\pi' - \alpha^2\left[1 - \frac{M_e^2(w-c)^2}{T}\right]\pi = 0 \qquad (21)$$

Let $\eta = y - y_c$ and assume a solution of the form

$$\pi = \eta^s(a_o + a_1\eta + a_2\eta^2 + \dots) \qquad (A\text{-}1)$$

Also, in the neighborhood of the critical point

$$\frac{T}{(w-c)^2}\left[\frac{(w-c)^2}{T}\right]' = \frac{2}{\eta} + A + (2B - A^2)\eta + \dots \qquad (A\text{-}2)$$

$$1 - \frac{M_e^2(w-c)^2}{T} = 1 - \frac{M_e^2 w_c'^2}{T_c}\eta^2 - \frac{M_e^2 w_c'^2}{T_c}A\eta^3 - \frac{M_e^2 w_c'^2}{T_c}B\eta^4 + \dots \qquad (A\text{-}3)$$

where

$$A = \frac{w_c''}{w_c'} - \frac{T_c'}{T_c} \qquad (A\text{-}4)$$

and

$$B = \frac{w_c''^2}{4w_c'^2} + \frac{w_c'''}{3w_c'} - \frac{T_c''}{2T_c} - A\frac{T_c'}{T_c} \qquad (A\text{-}5)$$

Relations (A-1) to (A-3) are substituted into Eq. (21) and the coefficients of each power of η are made to vanish. The leading power of η is η^{s-2}. Its coefficient vanishes when

$$a_o \left[s(s-1) - 2s \right] = 0$$

Since $a_o \neq 0$, $s = 0, 3$. The coefficient a_o can be set equal to 1 with perfect generality. The values of $a_1, a_2, \ldots$ for the solution corresponding to $s = 3$ (designated π_1) are found from the condition that the coefficients of the η^{s-1}, η^s, ... terms vanish. The resulting solution is

$$\pi_1 = \eta^3 + \frac{3A}{4}\eta^4 + \left(\frac{\alpha^2+6B}{10}\right)\eta^5 + \ldots \tag{A-6}$$

Since the characteristic exponents differ by an integer, the second linearly independent solution of Eq. (21) has the form

$\underline{\eta > 0}$

$$\pi_2 = C\,\pi_1 \ln\eta + 1 + b_1\eta + b_2\eta^2 + b_3\eta^3 + b_4\eta^4 + \ldots$$

$\underline{\eta < 0}$ (A-7)

$$\pi_2 = C\,\pi_1 \,(\ln|\eta| - i\pi) + 1 + b_1\eta + b_2\eta^2 + b_3\eta^3 + b_4\eta^4 + \ldots$$

(The π appearing without subscript in (A-7) is 3.14159.)

Substitution of (A-7) into Eq. (21) yields for π_2:

$$\left.\begin{matrix}\eta > 0 \\ \\ \eta < 0\end{matrix}\right\}: \quad \pi_2 = -\frac{\alpha^2 A}{3}\pi_1 \left\{\begin{matrix}\ln\eta \\ \\ \ln|\eta| - i\pi\end{matrix}\right\} + 1 - \frac{\alpha^2}{2}\eta^2 - \frac{\alpha^2}{4}\left(2B + \frac{M_e^2 W_e'^2}{T_c} + \frac{\alpha^2}{2} - \frac{23}{12}A^2\right)\eta^4 + b_3\pi_1 + \ldots \tag{A-8}$$

The coefficient b_3 is not determined in the procedure and so remains arbitrary. Thus Eq. (A-8) represents the general solution to Eq. (21).

For use in the calculation of the inviscid solutions, Eq. (A-8) is expressed in terms of $G = \pi_2'/\alpha^2\pi_2$.

The real and imaginary parts of G are as follows:

For $\eta > 0$

$$G_r = -\eta - A\eta^2 \ln \eta + \left(\frac{3b_3}{\alpha^2} - \frac{A}{3}\right)\eta^2 - A^2\eta^3 \ln \eta + \left[A\left(\frac{3b_3}{\alpha^2} - \frac{A}{3}\right) - \left(2B - 2A^2 + \frac{M_e^2 w_c'^2}{T_c} + \alpha^2\right)\right]\eta^3 + \ldots \quad \text{(A-9a)}$$

$$G_i = 0 \quad \text{(A-9b)}$$

For $\eta < 0$

$$G_r = -\eta - A\eta^2 \ln|\eta| + \left(\frac{3b_3}{\alpha^2} - \frac{A}{3}\right)\eta^2 - A^2\eta^3 \ln|\eta| + \left[A\left(\frac{3b_3}{\alpha^2} - \frac{A}{3}\right) - \left(2B - 2A^2 + \frac{M_e^2 w_c'^2}{T_c} + \alpha^2\right)\right]\eta^3 + \ldots \quad \text{(A-10a)}$$

$$G_i = A\pi\eta^2(1 + A\eta + \ldots) \quad \text{(A-10b)}$$

APPENDIX B

SOLUTION OF EQUATIONS FOR AMPLIFIED AND DAMPED DISTURBANCES

Just as for neutral disturbances, the inviscid and viscous solutions are obtained separately.

Inviscid Solution

For $c_i \neq 0$, Eq. (24) can be split into the following real and imaginary equations:

$$G_r' = \left[1 - \frac{M_e^2\{(w-c_r)^2 - c_i^2\}}{T}\right] + \left[\frac{2w'(w-c_r)}{(w-c_r)^2+c_i^2} - \frac{T'}{T}\right] G_r - \frac{2w'c_i}{(w-c_r)^2+c_i^2} G_i - \alpha^2(G_r^2 - G_i^2) \tag{B-1a}$$

$$G_i' = \frac{2M_e^2(w-c_r)c_i}{T} + \left[\frac{2w'(w-c_r)}{(w-c_r)^2+c_i^2} - \frac{T'}{T}\right] G_i + \frac{2w'c_i}{(w-c_r)^2+c_i^2} G_r - \alpha^2(2G_rG_i) \tag{B-1b}$$

These equations have a singularity at $w = c_r + i\, c_i$. For amplified disturbances the singularity is in the upper half-plane while for damped disturbances it lies in the lower half plane. It was pointed out by Lin[21] that the proper path of integration is under the singular point. Thus for amplified disturbances it is proper to integrate Eqs. (B-1) from the outer edge to the wall along the real y axis. For damped disturbances the path of integration may proceed along the real y axis except in the neighborhood of the singular point where it must detour under the singular point in such a manner that $-(7\pi/6) < \arg(\tilde{y} - \tilde{y}_c) < -(\pi/6)$, (Reference 1).

Since in all cases $|c_i| << 1$, it is probably not wise to integrate Eqs. (B-1) without a detour even for amplified disturbances, since the coefficients of Eqs. (B-1) will be very large in the neighborhood of $w = c_r$. Therefore the inviscid equations should be integrated in much the same way as done for neutral disturbances. The expansion about the singular point in the complex plane will be carried out to terms linear in c_i.

Let $\tilde{y}$ be the complex independent variable and $\tilde{w}$ the complex mean velocity whose components along the real axis are simply y and w, respectively. The velocity profile is written as an analytic function of $\tilde{y}$ in the following manner

$$\tilde{w} - c = (\tilde{w} - c_r - i\,c_i) \cong w_c{}'(\tilde{y} - \tilde{y}_c) + \ldots \qquad \text{(B-2)}$$

To this approximation all derivatives of the velocity at the critical point are taken as purely real. From Eq. (B-2) it can be shown that the complex coordinate of the critical point, to terms linear in c_i, is

$$\tilde{y}_c = y_c + ic_i/w_c{}' \qquad \text{(B-3)}$$

and that the distance of a point on the real axis from $\tilde{y}_c$ is

$$(y - \tilde{y}_c) = (y - y_c) - ic_i/w_c{}' \qquad \text{(B-4)}$$

Upon defining $\tilde{\eta} \equiv (y - \tilde{y}_c)$ and $\eta \equiv (y - y_c)$, Eq. (B-4) may be written

$$\tilde{\eta} = \eta - ic_i/w_c{}' \qquad \text{(B-5)}$$

Since the developments of Appendix A are not restricted to real variables, the desired result in the present case may be obtained by following exactly the same procedure, but using the present $\tilde{\eta}$ in place of the η of Appendix A. The resulting real and imaginary parts of G for a point on the real y axis are as follows:

For $\eta > 0$

$$G_r = -\eta - A\eta^2 \ln\eta + (\text{const})\eta^2 - A^2\eta^3\ln\eta + \left[A(\text{const}) - \left(2B - 2A^2 + \frac{M_e^2 w_c'^2}{T_c} + \alpha^2\right)\right]\eta^3 + \dots \quad \text{(B-6a)}$$

$$G_i = \frac{c_i}{w_c'}\left[1 - 2A\eta\ln\eta + \{2(\text{const}) + A\}\eta + 3A\eta^2\ln\eta + \left\{3A\left[(\text{const}) + \frac{1}{4}\right] + \frac{31}{4}A^2 - 6B - \frac{3M_e^2 w_c'^2}{T_c}\right\}\eta^2 + \dots\right] \quad \text{(B-6b)}$$

For $\eta < 0$

$$G_r = -\eta - A\eta^2\ln|\eta| + (\text{const})\eta^2 - A^2\eta^3\ln|\eta| + \left[A(\text{const}) - \left(2B - 2A^2 + \frac{M_e^2 w_c'^2}{T_c} + \alpha^2\right)\right]\eta^3 + \dots + \frac{c_i}{w_c'}\left[-2A\eta + 3A\eta^2 + \dots\right] \quad \text{(B-7a)}$$

$$G_i = A\pi\eta^2\left[1 + A\eta + \dots\right] + \frac{c_i}{w_c'}\left[1 - 2A\eta\ln|\eta| + \{2(\text{const}) + A\}\eta + 3A\eta^2\ln|\eta| + \left\{3A\left[(\text{const}) + \frac{1}{4}\right] + \frac{31}{4}A^2 - 6B - \frac{3M_e^2 w_c'^2}{T_c}\right\}\eta^2 + \dots\right] \quad \text{(B-7b)}$$

Retaining only terms linear in c_i, Eqs. (B-1) can be simplified as follows:

$$G_r' = \left[1 - \frac{M_e^2(w - c_r)^2}{T}\right] + \left[\frac{2w'}{w - c_r} - \frac{T'}{T}\right]G_r - \frac{2w'c_i}{(w - c_r)^2}G_i - \alpha^2\left(G_r^2 - G_i^2\right) \quad \text{(B-8a)}$$

$$G_i' = \frac{2M_e^2(w - c_r)c_i}{T} + \left[\frac{2w'}{w - c_r} - \frac{T'}{T}\right]G_i + \frac{2w'c_i}{(w - c_r)^2}G_r - \alpha^2\left(2G_rG_i\right) \quad \text{(B-8b)}$$

The outer boundary conditions for Eqs. (B-8) are

$$G_{r_o} = -\frac{\sqrt{1 - M_e^2(1 - c_r)^2}}{\alpha} \quad \text{(B-9a)}$$

$$G_{i_o} = -\frac{M_e^2(1 - c_r)c_i}{\alpha\sqrt{1 - M_e^2(1 - c_r)^2}} \quad \text{(B-9b)}$$

The method of obtaining the inviscid solutions is the same as described in Section III. 1. The calculation for a given profile and values of c_r and c_i, and an assumed value of α is begun by integrating outward from the critical point using Eqs. (B-6), and continuing with Eq. (B-8) along the real y axis. The(const) in Eqs. (B-6) is adjusted until the outer conditions (B-9) are satisfied for the assumed α. Then the inward integration is performed from the critical point to the wall using Eqs. (B-7) and (B-8).

Viscous Solutions

The equations for HJK (Eqs. (68) to (72)) and LMN (Eqs. (73) to (77)) systems are perfectly proper for amplified and damped disturbances. The equations are completely regular in the finite domain and may be integrated from the outer edge of the boundary layer to the wall along the real y axis. With $c = c_r + i\,c_i$, the real and imaginary equations of the HJK and LMN systems are

HJK:

$$H_r' = \text{RHS}\left[\text{Eq.}(78a)\right]_{c=c_r} + \frac{\alpha Re\,\sigma\,c_i}{\nu} \qquad \text{(B-10a)}$$

$$H_i' = \text{RHS}\left[\text{Eq.}(78b)\right]_{c=c_r} \qquad \text{(B-10b)}$$

$$J_r' = \text{RHS}\left[\text{Eq.}(79a)\right]_{c=c_r} + \frac{c_i}{T} \qquad \text{(B-11a)}$$

$$J_i' = \text{RHS}\left[\text{Eq.}(79b)\right]_{c=c_r} \qquad \text{(B-11b)}$$

$$K_r''' = \text{RHS}\left[\text{Eq.}(80a)\right]_{c=c_r} - \frac{\alpha Re\,c_i}{\nu}\left[\frac{T'}{T}K_r - (K_r' + H_rK_r - H_iK_i) - \frac{w'}{T}\right] \qquad \text{(B-12a)}$$

$$K_i''' = \text{RHS}\left[\text{Eq.}(80b)\right]_{c=c_r} - \frac{\alpha Re\,c_i}{\nu}\left[\frac{T'}{T}K_i - (K_i' + H_iK_r + H_rK_i)\right] \qquad \text{(B-12b)}$$

with outer conditions

$$\left.\begin{aligned} H_{r_o} &= -\sqrt{\frac{\alpha Re\,\sigma}{2}\left[\sqrt{c_i^2+(1-c_r)^2}+c_i\right]} \qquad J_{r_o} = -\sqrt{\frac{\sqrt{c_i^2+(1-c_r)^2}+c_i}{2\alpha Re\,\sigma}} \qquad K_{r_o}=0 \\ H_{i_o} &= -\sqrt{\frac{\alpha Re\,\sigma}{2}\left[\sqrt{c_i^2+(1-c_r)^2}-c_i\right]} \qquad J_{i_o} = -\sqrt{\frac{\sqrt{c_i^2+(1-c_r)^2}-c_i}{2\alpha Re\,\sigma}} \qquad K_{i_o}=0 \end{aligned}\right\} \quad \text{(B-13)}$$

LMN

$$L_r' = \text{RHS}\left[\text{Eq. (82a)}\right]_{c=c_r} + \frac{c_i}{T}N_r \qquad \text{(B-14a)}$$

$$L_i' = \text{RHS}\left[\text{Eq. (82b)}\right]_{c=c_r} + \frac{c_i}{T}N_i \qquad \text{(B-14b)}$$

$$M_r'' = \text{RHS}\left[\text{Eq. (83a)}\right]_{c=c_r} - \frac{\alpha Re\,c_i}{\nu}\left[\frac{T'}{T} - M_r - \frac{w'}{T}N_r\right] \qquad \text{(B-15a)}$$

$$M_i'' = \text{RHS}\left[\text{Eq. (83b)}\right]_{c=c_r} + \frac{\alpha Re\,c_i}{\nu}\left[M_i + \frac{w'}{T}N_i\right] \qquad \text{(B-15b)}$$

$$N_r'' = \text{RHS}\left[\text{Eq. (84a)}\right]_{c=c_r} + \frac{\alpha Re\,\sigma\,c_i}{\nu}N_r \qquad \text{(B-16a)}$$

$$N_i'' = \text{RHS}\left[\text{Eq. (84b)}\right]_{c=c_r} + \frac{\alpha Re\,\sigma\,c_i}{\nu}N_i \qquad \text{(B-16b)}$$

with outer conditions

$$\left.\begin{aligned} L_{r_o} &= \sqrt{\frac{\sqrt{c_i^2+(1-c_r)^2}-c_i}{2\alpha Re\left[(1-c_r)^2+c_i^2\right]}} \qquad M_{r_o} = -\sqrt{\frac{\alpha Re}{2}\left[\sqrt{c_i^2+(1-c_r)^2}+c_i\right]} \qquad N_{r_o}=0 \\ L_{i_o} &= \sqrt{\frac{\sqrt{c_i^2+(1-c_r)^2}+c_i}{2\alpha Re\left[(1-c_r)^2+c_i^2\right]}} \qquad M_{i_o} = -\sqrt{\frac{\alpha Re}{2}\left[\sqrt{c_i^2+(1-c_r)^2}-c_i\right]} \qquad N_{i_o}=0 \end{aligned}\right\} \quad \text{(B-17)}$$

Eigenvalue Problem

The eigenvalue problem is handled exactly as in Section III. 3.

APPENDIX C

APPROXIMATE ORDER OF MAGNITUDE ESTIMATES OF THE TERMS OF THE DISTURBANCE EQUATIONS

In making these order of magnitude estimates, the continuity and energy equations [Eqs. (4) and (7)] are unchanged, but the two momentum equations [Eqs. (5) and (6)] are combined in such a manner that the pressure fluctuation term is eliminated between them. This combined momentum equation is

$$
\begin{aligned}
& i\rho\left[(w-c)(\alpha^2\phi + if') + iw'f + w''\phi + w'\phi'\right] + i\rho'\left[i(w-c)f + w'\phi\right] \\
&= \frac{\mu}{\alpha Re}\left[2\alpha^2\phi'' + i\alpha^2 f' - \alpha^4\phi + if''' - \alpha^2(\phi'' + 2if')\right] + \frac{2i\mu'}{\alpha Re}\left[f'' - \alpha^2 f\right] \\
&+ \frac{i}{\alpha Re}\left[\alpha^2\theta\frac{d\mu}{dT}w' + \theta\frac{d\mu}{dT}w''' + 2\theta'\frac{d\mu}{dT}w'' + 2\theta\left(\frac{d\mu}{dT}\right)'w'' + 2\theta'\left(\frac{d\mu}{dT}\right)'w' + \theta''\frac{d\mu}{dT}w'\right. \\
&\qquad \left. + \theta\left(\frac{d\mu}{dT}\right)''w' + \mu''(f' + i\alpha^2\phi)\right]
\end{aligned}
\qquad \text{(C-1)}
$$

A significant feature of this combined momentum equation is that terms containing the second viscosity coefficient vanish identically.

Lees-Lin Ordering

A set of ordering relations consistent with the convergent series expansion of Lees and Lin[1] are

$$
\begin{aligned}
&\overline{Q},\ \overline{Q}' \sim 1, \quad (w-c) \sim \epsilon, \quad f \sim 1, \quad \phi \sim \epsilon f, \quad \theta \sim f \\
&d/dy \sim 1/\epsilon, \quad \pi \sim \epsilon^3 f
\end{aligned}
\qquad \text{(C-2)}
$$

The leading terms of the left and right sides of Eq. (C-1) under ordering Eq. (C-2) are respectively

$-\rho(w-c)f', \quad (i\mu/\alpha Re) f'''$

For these terms to be of the same order

$$\epsilon \sim 1/(\alpha Re)^{1/3} \tag{C-3}$$

The first and second order terms of Eqs. (C-1), (4), and (7) under ordering (C-2) and (C-3) are

<u>Momentum</u>

$$\overset{1}{-\rho(w-c)f'} + i\rho w'(\overset{1}{\phi'} + \overset{1}{if}) + i\rho'\left[\overset{\epsilon}{i(w-c)f} + \overset{\epsilon}{w'\phi}\right] + \overset{\epsilon}{i\rho w''\phi}$$

$$= \overset{1}{\frac{\mu}{\alpha Re}\left(if'''\right)} + \overset{\epsilon}{\frac{2i\mu'}{\alpha Re} f''} + \overset{\epsilon}{\frac{iw'}{\alpha Re}\left(\frac{d\mu}{dT}\right)\theta''} \tag{C-4}$$

<u>Continuity</u>

$$\overset{1}{\phi'} + \overset{1}{if} = \overset{\epsilon}{\frac{T'}{T}\phi} + \overset{\epsilon}{i(w-c)\frac{\theta}{T}} \tag{C-5}$$

<u>Energy</u>

$$\rho\left[\overset{\epsilon}{i(w-c)\theta} + \overset{\epsilon}{T'\phi}\right] = \frac{1}{\sigma\alpha Re}\left[\overset{\epsilon}{\mu\theta''} + \overset{\epsilon^2}{2\theta' T'\left(\frac{d\mu}{dT}\right)}\right] + \overset{\epsilon^2}{\frac{(\gamma-1)M_e^2}{\alpha Re} 2\mu w' f'} \tag{C-6}$$

The leading terms of Eqs. (C-4) to (C-6) will be called the Lees-Lin equations and can be written

$$f''' - \frac{i\alpha Re(w-c)}{\nu} f' = 0 \tag{45}$$

$$\phi' + if = 0 \tag{46}$$

$$\theta'' - \frac{i\alpha Re\,\sigma(w-c)}{\nu}\theta = \frac{\sigma\alpha Re}{\nu} T'\phi \tag{47}$$

The method of solution of these equations is outlined in the text of this report. Lees and Lin suggest that further terms of the solution be obtained by quadrature if desired. (See Reference 1.)

Dunn-Lin Ordering

Since the critical layer is far from the wall at supersonic speeds, the quantity (w-c) is of unit order near the wall. Dunn and Lin[4, 5] therefore ordered as follows:

$$\bar{Q},\ \bar{Q}' \sim 1, \quad (w-c) \sim 1, \quad d/dy \sim 1/\epsilon, \quad f \sim 1, \quad \phi \sim \epsilon f, \quad \theta \sim f, \quad \pi \sim \epsilon^2 f \qquad \text{(C-7)}$$

The leading terms of the left and right sides of Eq. (C-1) under ordering Eq. (C-7) are again

$$-\rho\,(w-c)\,f', \qquad i\mu/\alpha Re)\,f'''$$

But since $(w-c) \sim 1$,

$$\epsilon \sim 1/(\alpha Re)^{\frac{1}{2}} \qquad \text{(C-8)}$$

The first and second order terms of Eqs. (C-1), (4), and (7) under ordering Eqs. (C-7) and (C-8) are

Momentum

$$\overset{\frac{1}{\epsilon}}{-\rho(w-c)f'} + i\rho w'(\overset{1}{\phi'} + \overset{1}{if}) + \overset{1}{i\rho'\left[i(w-c)f\right]}$$

$$= \overset{\frac{1}{\epsilon}}{\frac{\mu}{\alpha Re}\left(if'''\right)} + \overset{1}{\frac{2i\mu'}{\alpha Re} f''} + \overset{1}{\frac{i\omega'}{\alpha Re}\left(\frac{d\mu}{dT}\right)\theta''} \qquad \text{(C-9)}$$

Continuity

$$\overset{1}{\phi'} + \overset{1}{if} = \overset{\epsilon}{\frac{T'}{T}\phi} + \overset{1}{i(w-c)\frac{\theta}{T}} \qquad \text{(C-10)}$$

Energy

$$\rho\left[\overset{1}{i(w-c)\theta} + \overset{\epsilon}{T'\phi}\right] = \frac{1}{\sigma\alpha Re}\left[\overset{1}{\mu\theta''} + \overset{\epsilon}{2\theta' T'\left(\frac{d\mu}{dT}\right)}\right] + \overset{\epsilon}{\frac{(\gamma-1)M_e^2}{\alpha Re} 2\mu w' f'} \qquad \text{(C-11)}$$

The leading terms of Eqs. (C-9) through (C-11) form the Dunn-Lin viscous equations

$$f''' - \frac{i\alpha Re(\omega - c)}{\nu} f' = 0 \tag{54}$$

$$\phi' + i f = i(\omega - c)\frac{\theta}{T} \tag{55}$$

$$\theta'' - \frac{i\alpha Re\,\sigma(\omega - c)}{\nu}\theta = 0 \tag{56}$$

The major difference between the Lees-Lin and Dunn-Lin systems is in the continuity equation [Eq. (55)]. There is now the possibility of an effect of temperature fluctuations on normal velocity fluctuations, so that the energy equation is quite relevant. A detailed description of the method of solving Eqs. (54) to (56) is given in Appendix D.

Present Method

As the Mach number of the compressible flow increases the free-stream static temperature is no longer indicative of the temperature level in the boundary layer so that the ordering procedure must be revised somewhat as discussed in Section III. 2. of the text. Mean quantities and disturbance quantities which are temperature dependent (temperature, viscosity, conductivity, density) should be referred to their magnitude at some representative reference temperature T_{ref}. As shown by Dunn[4] the effect of this change is to revise the definition of Mach number and Reynolds number in the basic disturbance equations. The new definitions are

$$M_{ref} = \frac{u_e^*}{\sqrt{\gamma R^* T_{ref}^*}} = M_e\sqrt{\frac{1}{T_{ref}}} \tag{C-12}$$

$$Re_{ref} = \frac{u_e^* \ell}{\nu_{ref}^*} = \frac{Re}{\nu_{ref}} \tag{C-13}$$

where M_{ref} is nominally of unit order. This effect has already been pointed out by Dunn[4]. However, the term, M_{ref} also appears in the ordering of terms involving the gradient of mean temperature. For a flat plate with Prandtl number one, the temperature and velocity profiles are related through the Crocco integral

$$\frac{T^*}{T_e^*} = 1 + \frac{\gamma-1}{2} M_e^2 (1-w^2) + \left(\frac{T_w}{T_o} - 1\right)\left(1 + \frac{\gamma-1}{2} M_e^2\right)(1-w) \tag{C-14}$$

or

$$\frac{T^*}{T_{ref}^*} = \frac{1}{T_{ref}} + \frac{\gamma-1}{2} M_{ref}^2 (1-w^2) + \left(\frac{T_w}{T_o} - 1\right)\left(\frac{1}{T_{ref}} + \frac{\gamma-1}{2} M_{ref}^2\right)(1-w) \tag{C-15}$$

Thus

$$\frac{\partial\left(\frac{T^*}{T_{ref}^*}\right)}{\partial y} = -w' \left\{ (\gamma-1) M_{ref}^2 \left[w + \frac{1}{2}\left(\frac{T_w}{T_o} - 1\right)\right] + \frac{1}{T_{ref}}\left(\frac{T_w}{T_o} - 1\right)\right\} \tag{C-16}$$

For ordering purposes

$$T', \mu' \sim M_{ref}^2 w' \tag{C-17}$$

We are now in a position to order the terms of the disturbance differential equations taking into account some of the expected effects of Mach number. The ordering relations are

$$\left.\begin{array}{l} \bar{Q}, \bar{Q}' \sim 1 \quad \underline{\text{except}} \quad \mu', T', \rho' \sim M_{ref}^2 \\ d/dy \sim 1/\bar{\epsilon}, \quad f \sim 1, \quad \theta \sim f, \quad \phi \sim \bar{\epsilon} f, \quad \pi \sim \bar{\epsilon}^2 f \end{array}\right\} \tag{C-18}$$

The leading terms of the momentum equation are the same as for Dunn-Lin ordering so that

$$\bar{\epsilon} \sim 1/(\alpha Re_{ref})^{\frac{1}{3}} \tag{C-19}$$

The terms of the disturbance equations under the considerations (C-18) and (C-19) have the following orders of magnitude:

Momentum

$$\begin{aligned}
&i\rho\left[(w-c)(\overset{\bar{\epsilon}}{\alpha^2\phi} + \overset{\frac{1}{\bar{\epsilon}}}{if'}) + w'(\overset{1}{\phi'} + \overset{1}{if}) + \overset{\bar{\epsilon}}{w''\phi}\right] + i\rho'\left[\overset{M_{ref}^2}{i(w-c)f} + \overset{M_{ref}^2\bar{\epsilon}}{w'\phi}\right] \\
&= \frac{\mu}{\alpha Re_{ref}}\left[\overset{\bar{\epsilon}}{\alpha^2\phi''} - \overset{\bar{\epsilon}}{i\alpha^2 f'} - \overset{\bar{\epsilon}^3}{\alpha^4\phi} + \overset{\frac{1}{\bar{\epsilon}}}{if'''}\right] + \frac{2i\mu'}{\alpha Re_{ref}}\left[\overset{M_{ref}^2}{f''} - \overset{M_{ref}^2\bar{\epsilon}^2}{\alpha^2 f}\right] \\
&+ \frac{i}{\alpha Re_{ref}}\left[\overset{\bar{\epsilon}^2}{\alpha^2\theta\left(\frac{d\mu}{dT}\right)w'} + \overset{\bar{\epsilon}^2}{\theta\left(\frac{d\mu}{dT}\right)w'''} + \overset{\bar{\epsilon}}{2\theta'\frac{d\mu}{dT}w''} + \overset{\bar{\epsilon}^2}{2\theta\left(\frac{d\mu}{dT}\right)'w''} + \overset{\bar{\epsilon}}{2\theta'\left(\frac{d\mu}{dT}\right)'w'}\right. \\
&\left. + \overset{1}{\theta''\left(\frac{d\mu}{dT}\right)w'} + \overset{\bar{\epsilon}^2}{\theta\left(\frac{d\mu}{dT}\right)''w'} + \mu''(\overset{M_{ref}^2\bar{\epsilon}}{f'} + \overset{M_{ref}^2\bar{\epsilon}^3}{i\alpha^2\phi})\right]
\end{aligned} \tag{C-20}$$

Continuity

$$\overset{1}{\phi'} + \overset{1}{if} - \overset{M_{ref}^2\bar{\epsilon}}{\frac{T'}{T}\phi} + i(w-c)\left(\overset{\bar{\epsilon}^2}{\pi} - \overset{1}{\frac{\theta}{T}}\right) = 0 \tag{C-21}$$

Energy

$$\begin{aligned}
&\rho\left[\overset{1}{i(w-c)\theta} + \overset{M_{ref}^2\bar{\epsilon}}{T'\phi}\right] = \overset{\bar{\epsilon}^2}{i(w-c)\left(\frac{\gamma-1}{\gamma}\right)\pi} \\
&+ \frac{1}{\sigma\alpha Re_{ref}}\left[\mu(\overset{1}{\theta''} - \overset{\bar{\epsilon}^2}{\alpha^2\theta}) + \overset{M_{ref}^2\bar{\epsilon}}{2\frac{d\mu}{dT}\theta'T'} + \overset{M_{ref}^2\bar{\epsilon}^2}{\theta\frac{d\mu}{dT}T''}\right] \\
&+ \frac{(\gamma-1)M_{ref}^2}{\alpha Re_{ref}}\left[\overset{M_{ref}^2\bar{\epsilon}^2}{\theta\frac{d\mu}{dT}w'^2} + 2\mu w'(\overset{M_{ref}^2\bar{\epsilon}}{f'} + \overset{M_{ref}^2\bar{\epsilon}^3}{i\alpha^2\phi})\right]
\end{aligned} \tag{C-22}$$

The leading terms of Eqs. (C-20) to (C-22) comprise the Dunn-Lin equations. This conclusion was reached by Dunn[4]. Let us now consider the magnitude of the "higher order" terms.

Relative to the leading terms in each equation, the ordering of terms in Eqs. (C-20) to (C-22) is

$1, \quad \bar{\epsilon}$	$\bar{\epsilon}^2, \bar{\epsilon}^3, \bar{\epsilon}^4$
$M_{ref}^2 \bar{\epsilon}, \quad M_{ref}^2 \bar{\epsilon}^2$	$M_{ref}^2 \bar{\epsilon}^3, \quad M_{ref}^2 \bar{\epsilon}^4$
no α dependence	α and (αRe_{ref}) dependence
(αRe_{ref}) dependence only	

In addition to the conventional ordering of terms: $1, \bar{\epsilon}, \bar{\epsilon}^2, \ldots,$ there are terms of order $M_{ref}^2 \bar{\epsilon}, M_{ref}^2 \bar{\epsilon}^2, \ldots,$ etc. For terms of order, $1, \bar{\epsilon}, M_{ref}^2 \bar{\epsilon}, M_{ref}^2 \bar{\epsilon}^2$, the equations are functions only of one parameter (αRe_{ref}). The next order terms bring in longitudinal derivatives of fluctuating quantities and therefore α dependence as well. It is to be noted that for a linear viscosity-temperature relation

$$M_{ref}^2/(\alpha Re_{ref})^{\frac{1}{2}} = M_e^2/(\alpha Re)^{\frac{1}{2}} \quad .$$

It is therefore quite feasible that for supersonic and hypersonic boundary layer stability analyses, terms of order $\bar{\epsilon}$ and $M_{ref}^2 \bar{\epsilon}$ may be comparable to unit order terms.*

The equations considered in the present analysis consist of those terms in Eqs. (C-20) to (C-22) which are of order $1, \bar{\epsilon}, M_{ref}^2 \bar{\epsilon}$ relative to the leading terms. These equations are

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* It is also important to recognize that the leading terms containing the mean vertical velocity enter to order $M_{ref}^2 \bar{\epsilon}$. These terms are omitted in the present analysis because of the parallel flow assumptions.

Momentum

$$f''' + \frac{2}{\mu}\left(\frac{d\mu}{dT}\right) T'f'' + \frac{1}{\mu}\left(\frac{d\mu}{dT}\right) w'\theta'' + \frac{i\alpha Re_{ref}(w-c)}{\nu}\left[\frac{T'}{T}f - f' - \frac{w'}{T}\theta\right] = 0 \qquad \text{(C-23)}$$

Continuity

$$\phi' + if - \frac{T'}{T}\phi - i(w-c)\frac{\theta}{T} = 0 \qquad \text{(C-24)}$$

Energy

$$\theta'' + \frac{2}{\mu}\left(\frac{d\mu}{dT}\right) T'\theta' + 2(\gamma-1)\sigma M_{ref}^2 w'f' - \frac{i\alpha Re_{ref}\,\sigma(w-c)}{\nu}\theta = \frac{\sigma\alpha Re_{ref}}{\nu} T'\phi \qquad \text{(C-25)}$$

Once the pertinent equations have been chosen, it is allowable to refer temperatures again to the external static temperature. Thus Eqs. (C-23) to (C-25) are quoted in the text with M_{ref} replaced by M_e, and Re_{ref} replaced by Re.

APPENDIX D

THE CORRECTED DUNN-LIN METHOD

In their analysis of boundary layer stability, Dunn and Lin[4,5] recognized that for supersonic flows, the disturbance propagation velocity becomes an appreciable fraction of the free-stream velocity ($c > 1 - 1/M$) and the critical layer where $w = c$ lies outside the linear portion of the mean flow profiles. Accordingly they revised the method of analysis from that of Lees and Lin[1,2] in a number of ways. Of significance is their recognition of the possible importance of the temperature fluctuations in the boundary layer and the wall boundary condition on temperature fluctuations. They also introduced into the solution of the viscous equations a transformation which Tollmien[11] used in the case of incompressible flow, and which eliminates the undesirable approximation of the velocity by a straight line through the critical point.

In formulating the eigenvalue problem, however, Dunn and Lin[5] apparently omitted the second term on the right hand side of Eq. (88). This omission is here corrected. Since the Dunn-Lin method has not yet been presented in detail in the literature, it will be described here in some detail including a corrected formulation of the eigenvalue problem.

The Dunn-Lin viscous equations [Eqs. (54) to (56)] are of sufficient simplicity so that the solutions can be written in terms of universal functions.

The Dunn-Lin viscous equations are (Section III. 2.)

$$f''' - \frac{i\alpha Re\,(w-c)}{\nu} f' = 0 \tag{54}$$

$$\phi' + if = i\,(w-c)\frac{\theta}{T} \tag{55}$$

$$\theta'' - \frac{i\alpha Re\,\sigma(w-c)}{\nu}\theta = 0 \tag{56}$$

where primes denote differentiation with respect to y.

Of the six linearly independent sets of solutions to Eqs. (54) to (56), solutions 1 and 2 are identified with the inviscid solutions, solutions 3 and 4 are those where $f' \neq 0$ but $\theta = 0$ and solutions 5 and 6 are those for which $f = 0$ and $\theta \neq 0$. The aforementioned numbers will appear as subscripts to identify the solutions.

Consider first the momentum equation [Eq. (54)]. Following Tollmien[11] make the transformations

$$Y = \left[\int_{y_c}^{y} \frac{3}{2}\sqrt{\frac{w-c}{\nu}}\, dy\right]^{2/3} \tag{D-1}$$

and

$$\mathcal{F} = f'\sqrt{\frac{dY}{dy}} = f'\left(\frac{w-c}{\nu Y}\right)^{1/4} \tag{D-2}$$

Then Eq. (54) becomes

$$\frac{d^2\mathcal{F}}{dY^2} - \left[i\alpha Re\, Y + \mathcal{O}(1)\right]\mathcal{F} = 0 \tag{D-3}$$

The second term in the bracket of Eq. (D-3) is of order $1/\alpha Re$ compared to the first bracketed term and may be omitted. Thus Eq. (D-3) becomes

$$\frac{d^2\mathcal{F}}{dY^2} - i\alpha Re\, Y\mathcal{F} = 0 \tag{D-4}$$

Now let

$$\zeta = (\alpha Re)^{1/3} Y \tag{D-5}$$

so that Eq. (D-4) becomes

$$\frac{d^2\mathcal{F}}{d\zeta^2} - i\zeta\mathcal{F} = 0 \tag{D-6}$$

which is identical in form with the Lees-Lin[1] formulation. [See Eq. (49).] Eq. (D-6) has two linearly independent solutions in terms of Hankel functions

$$\mathcal{F}_3 = \zeta^{1/2} H^{(1)}_{1/3}\left[\frac{2}{3}(i\zeta)^{3/2}\right] \tag{D-7a}$$

and

$$\mathcal{F}_4 = \zeta^{1/2} H^{(2)}_{1/3}\left[\frac{2}{3}(i\zeta)^{3/2}\right] \tag{D-7b}$$

Solution (D-7b) is rejected immediately since it grows exponentially for large ζ (large y) and cannot possible satisfy the outer boundary conditions on f and ϕ.

The energy equation [Eq. (56)] can be written in a form similar to Eq. (D-3) by the transformations

$$Y_o = \left[\int_{y_c}^{y} \frac{3}{2}\sqrt{\frac{\sigma(\omega-c)}{\nu}}\, dy\right]^{2/3} \tag{D-8}$$

and

$$\Theta = \theta\sqrt{\frac{dY_o}{dy}} = \theta\left[\frac{\sigma(\omega-c)}{\nu Y_o}\right]^{1/4} \tag{D-9}$$

After the term of order unity is dropped compared to the term of order (αRe), one lets

$$\zeta_o = (\alpha Re)^{1/3} Y_o \tag{D-10}$$

Thus the energy equation becomes

$$\frac{d^2 \Theta}{d \zeta_o^2} - i \zeta_o \Theta = 0 \tag{D-11}$$

which has the solutions

$$\Theta_5 = \zeta_o^{1/2} H_{1/3}^{(1)} \left[\frac{2}{3} (i \zeta_o)^{3/2} \right] \tag{D-12a}$$

and

$$\Theta_6 = \zeta_o^{1/2} H_{1/3}^{(2)} \left[\frac{2}{3} (i \zeta_o)^{3/2} \right] \tag{D-12b}$$

The solution Θ_6 grows exponentially for large ζ_o (large y) so that it too is dropped because it cannot satisfy the outer boundary conditions.

For the Dunn and Lin assumptions, the pertinent solutions to the disturbance equations are as follows:

A. Inviscid Solution

Dunn and Lin[4,5] obtained inviscid solutions by the Heisenberg expansion in powers of α^2. However, the inviscid solution in a form automatically satisfying the outer boundary conditions can be obtained by the method presented in the present text in Section III. 1. entitled "Inviscid Solution".

B. Solutions 3

$$f_3 = \left(\frac{dy}{dY}\right)^{3/2} \left(\frac{dY}{d\zeta}\right) \int_{\infty}^{\zeta} \zeta^{1/2} H_{1/3}^{(1)} \left[\frac{2}{3} (i \zeta)^{3/2} \right] d\zeta \tag{D-13}$$

$$\phi_3 = -i\left(\frac{dy}{dY}\right)^{5/2}\left(\frac{dY}{d\zeta}\right)^2 \int_\infty^\zeta \int_\infty^\zeta \zeta^{1/2} H_{1/3}^{(1)}\left[\frac{2}{3}(i\zeta)^{3/2}\right] d\zeta\, d\zeta \tag{D-14}$$

$$\theta_3 = 0 \tag{D-15}$$

C. Solutions 5

$$f_5 = 0 \tag{D-16}$$

$$\phi_5 = \frac{i(\omega-c)}{T}\left(\frac{dy}{dY_o}\right)^{3/2}\left(\frac{dY_o}{d\zeta_o}\right)\int_\infty^{\zeta_o} \zeta_o^{1/2} H_{1/3}^{(1)}\left[\frac{2}{3}(i\zeta_o)^{3/2}\right] d\zeta_o \tag{D-17}$$

$$\theta_5 = \left(\frac{dy}{dY_o}\right)^{1/2} \zeta_o^{1/2} H_{1/3}^{(1)}\left[\frac{2}{3}(i\zeta_o)^{3/2}\right] \tag{D-18}$$

In performing the integrations to obtain f_3, ϕ_3, and ϕ_5, quantities such as (dy/dY), $(dY/d\zeta)$ and mean flow functions were taken outside the integral sign since they are slowly varying compared to the balance of the integrand. Slowly varying functions are those for which $d/dy \sim \mathcal{O}(1)$. This simplification incurs errors no larger than order $(\alpha Re)^{-1/3}$.

The corrected secular equation for the Dunn-Lin assumptions is (Section III. 3.)

$$\frac{\Phi_w}{F_w} = \frac{\dfrac{\phi_{3w}}{f_{3w}} + (\gamma-1)M_e^2 c\,\dfrac{\phi_{5w}}{\theta_{5w}}}{1 - i(\gamma-1)M_e^2 c\,\dfrac{\phi_{5w}}{\theta_{5w}}\left[\dfrac{\omega'_w}{c} - \dfrac{T'_w}{(\gamma-1)M_e^2 c^2}\right]} \tag{D-19}$$

Appearing in (D-19) are the ratios ϕ_{3w}/f_{3w} and ϕ_{5w}/θ_{5w}, which upon evaluation from the viscous solutions are

$$\frac{\phi_{3w}}{f_{3w}} = -i\left(\frac{dy}{dY}\right)_w\left(\frac{dY}{d\zeta}\right)_w \frac{\int_{\infty}^{\zeta_w}\int_{\infty}^{\zeta} \zeta^{1/2} H_{1/3}^{(1)}\left[\frac{2}{3}(i\zeta)^{3/2}\right] d\zeta\, d\zeta}{\int_{\infty}^{\zeta_w} \zeta^{1/2} H_{1/3}^{(1)}\left[\frac{2}{3}(i\zeta)^{3/2}\right] d\zeta} \tag{D-20}$$

and

$$\frac{\phi_{5w}}{\theta_{5w}} = -\frac{ic}{T_w}\left(\frac{dy}{dY_0}\right)_w\left(\frac{dY_0}{d\zeta_0}\right)_w \frac{\int_{\infty}^{\zeta_{0w}} \zeta_0^{1/2} H_{1/3}^{(1)}\left[\frac{2}{3}(i\zeta_0)^{3/2}\right] d\zeta_0}{\zeta_0^{1/2} H_{1/3}^{(1)}\left[\frac{2}{3}(i\zeta_0)^{3/2}\right]} \tag{D-21}$$

Let

$$z = -\zeta_w \tag{D-22}$$

$$z_0 = -\zeta_{0w} \tag{D-23}$$

Also define the Tietjens function

$$F(z) \equiv \frac{-\int_{\infty}^{-z}\int_{\infty}^{\zeta} \zeta^{1/2} H_{1/3}^{(1)}\left[\frac{2}{3}(i\zeta)^{3/2}\right] d\zeta\, d\zeta}{z\int_{\infty}^{-z} \zeta^{1/2} H_{1/3}^{(1)}\left[\frac{2}{3}(i\zeta)^{3/2}\right] d\zeta} \tag{D-24}$$

and an auxiliary function

$$\tilde{G}(z_0) \equiv \frac{-\int_{\infty}^{-z_0} \zeta_0^{1/2} H_{1/3}^{(1)}\left[\frac{2}{3}(i\zeta_0)^{3/2}\right] d\zeta_0}{i\, z_0^{3/2} H_{1/3}^{(1)}\left[\frac{2}{3}(i\zeta_0)^{3/2}\right]} \tag{D-25}$$

Eqs. (D-20) and (D-21) can now be written

$$\frac{\phi_{3w}}{f_{3w}} = i\, z\left(\frac{dy}{dY}\right)_w\left(\frac{dY}{d\zeta}\right)_w F(z) \tag{D-26}$$

and

$$\frac{\phi_{sw}}{\theta_{sw}} = \frac{ic}{T_w} z_o \left(\frac{dy}{dY_o}\right)_w \left(\frac{dY_o}{d\zeta_o}\right)_w \tilde{G}(z_o) \tag{D-27}$$

Note here that $z_o = \sigma^{1/3} z$ and that

$$z\left(\frac{dy}{dY}\right)_w\left(\frac{dY}{d\zeta}\right)_w = z_o\left(\frac{dy}{dY_o}\right)_w\left(\frac{dY_o}{d\zeta_o}\right)_w$$

In terms of $F(z)$ and $\tilde{G}(Z_o)$, Eq. (D-19) becomes

$$\frac{\Phi_w}{F_w} = \frac{iz\left(\frac{dy}{dY}\right)_w\left(\frac{dY}{d\zeta}\right)_w\left[F(z) + \frac{(\gamma-1)M_e^2c^2}{T_w}\tilde{G}(z_o)\right]}{1 + \frac{(\gamma-1)M_e^2c^2}{T_w}\tilde{G}(z_o)\left[\frac{w_w'}{c} - \frac{T_w'}{(\gamma-1)M_e^2c^2}\right]\left[z\left(\frac{dy}{dY}\right)_w\left(\frac{dY}{d\zeta}\right)_w\right]} \tag{D-28}$$

Eq. (D-28) is a proper secular equation. However to avoid some of the difficulties involved in the expressions for $(dy/dY)_w\,(dY/d\zeta)_w$, the equation is further reduced as follows: Let

$$z\left(\frac{dy}{dY}\right)_w\left(\frac{dY}{d\zeta}\right)_w \equiv (1+\lambda)\frac{c}{w_w'} \tag{D-29}$$

where it can be shown that

$$1+\lambda = \frac{w_w'}{c}\sqrt{\frac{\nu_w}{c}}\int_0^{y_c}\frac{3}{2}\sqrt{\frac{c-w}{\nu}}\,dy \tag{D-30}$$

so that

$$\lambda = \frac{w_w'}{c}\sqrt{\frac{\nu_w}{c}}\int_0^{y_c}\frac{3}{2}\sqrt{\frac{c-w}{\nu}}\,dy - 1 \tag{D-31}$$

Further, define

$$K \equiv \frac{(\gamma-1) M_e^2 c^2}{T_w} \tag{D-32}$$

Since from the inviscid solutions

$$\frac{\Phi_w}{F_w} = \frac{i}{\frac{\omega_w'}{c} - \frac{1}{G_w}} \tag{94}$$

Eq. (D-28) can after some manipulation be written

$$G_w = -\frac{c}{w_w'}\left[\frac{\Phi(z)-1}{1-\frac{\lambda}{\lambda+1}\Phi(z)}\right]\frac{\left[1 + K\frac{\tilde{G}(z_0)}{F(z)}\right]}{\left[1 - \frac{K\tilde{G}(z_0)\,\Phi(z)}{1-\frac{\lambda}{\lambda+1}\Phi(z)}\;\frac{T_w'}{(\gamma-1)M_e^2 w_w' c}\right]} \tag{D-33}$$

where

$$\Phi(z) \equiv \frac{1}{1-F(z)} \tag{D-34}$$

is the modified Tietjens function. Also, as can be shown by manipulating with the Tietjens function (D-24) and its derivative with respect to z,

$$\tilde{G}(z) = \frac{F(z)}{1-F(z) - zF'(z)} \tag{D-35}$$

The functions F(z), $\tilde{G}$(z), and Φ(z) have been recently recomputed to z = 8 by Dr. L. Mack of the Jet Propulsion laboratory and are presented here in Table II. For z > 8, an asymptotic form of the secular equation [Eq. (D-33)] can be obtained. In terms of F and $\tilde{G}$ alone, Eq. (D-33) is expressed

$$G_w = -\frac{c}{w'_w}\left[\frac{F + K\tilde{G}}{\frac{1}{1+\lambda} - F - \frac{K\tilde{G}\,T'_w}{(\gamma-1)M_e^2\,w'_w\,c}}\right] \tag{D-36}$$

The asymptotic variation of F(z) from Lin[21] is

$$F(z) \approx \frac{1}{z^{3/2}\,e^{-i\frac{\pi}{4}} - \frac{5}{4}} \tag{D-37}$$

Let

$$p = \frac{z^{-3/2}}{\sqrt{2}} \tag{D-38}$$

so that

$$z^{-3/2}\,e^{i\frac{\pi}{4}} = p + ip$$

The asymptotic forms of F(z) and $\tilde{G}(z_o)$ in terms of p to order p^2 are

$$F(z) \approx p + ip\,(1 + 5/2\;\; p) \tag{D-39}$$

$$\tilde{G}(z_o) = \frac{F(z_o)}{1 - F(z_o) - z_o F'(z_o)} \approx \sigma^{-1/2}\left[p + ip\left(1 + \frac{3}{2}\sigma^{-1/2}p\right)\right] \tag{D-40}$$

Upon substituting Eqs. (D-39) and (D-40) into Eq. (D-36), the secular equation for large z (small p) is written

$$G_W = -\frac{c}{w'_w}\,(1+\lambda)\left(1 + K\sigma^{-1/2}\right)p\left[1 + i\left\{1 + 2(1+\lambda)p\left(1 + \frac{T'_w}{T_w}\frac{c}{w'_w}\sigma^{-1/2}\right) + \left(\frac{5 + 3K\sigma^{-1}}{2 + 2K\sigma^{-1/2}}\right)p\right\}\right] \tag{D-41}$$

Note in Eq. (D-41) that to order p, the real and imaginary parts of the right hand side are equal. The quantity z is related to p by

$$z = \left(\frac{1}{p\sqrt{2}}\right)^{2/3} \tag{D-42}$$

Neutral stability characteristics can now be computed in the following manner for a given boundary layer profile:

For a chosen value of c:

(1) Obtain the inviscid solution and record the values of G_w for various α.

(2) Find the values of z and α for which Eq. (D-33) or Eq. (D-41) is satisfied.

(3) Compute (αRe) from the relation

$$\alpha Re = \frac{z^3}{\left[\frac{3}{2}\int_0^{y_c}\sqrt{\frac{c-w}{\nu}}\,dy\right]^2} \tag{D-43}$$

(4) Knowing α and αRe, the Reynolds number can be computed.

APPENDIX E

OUTER BEHAVIOR OF VISCOUS SOLUTIONS

An examination of the viscous equations [Eqs. (69) to (77)] shows that they are singular far away from the wall outside the mean boundary layer. It is important to determine the nature of this singularity.

Consider first the HJK system [Eqs. (69) to (72)]. Outside the mean boundary layer equations Eqs. (69) to (71) become:

$$H' = i\alpha Re\,\sigma(1-c) - H^2 \tag{E-1}$$

$$J' = -iK + i(1-c) - HJ \tag{E-2}$$

$$K''' = i\alpha Re\,(1-c)\left[K' + HK\right] - H\left[3K'' + 3K'H + 3H'K + H^2K\right] - 3K'H' - H''K \tag{E-3}$$

The outer boundary conditions on H, J, and K are given by

$$\left.\begin{aligned} H_o &= -\sqrt{i\alpha Re\,\sigma(1-c)} \\ J_o &= -\sqrt{\frac{i(1-c)}{\sigma\alpha Re}} \\ K_o &= 0 \end{aligned}\right\} \tag{72}$$

Now let

$$\left.\begin{aligned} H &= H_o + h \\ J &= J_o + j \\ K &= k \end{aligned}\right\} \tag{E-4}$$

where h, j, and k are small quantities such that products of these are negligible. The differential equations for h, j, and k are

$$h' = -2 H_o h \tag{E-5}$$

$$j' = -i k - H_o j - J_o h \tag{E-6}$$

$$k''' + 3 H_o k'' + 2 H_o^2 k' + H_o^2 (1 - \frac{1}{\sigma}) k' + H_o^3 (1 - \frac{1}{\sigma}) k = 0 \tag{E-7}$$

The solution for h is

$$h = c_1 e^{-2H_o y} \tag{E-8}$$

or in terms of real and imaginary parts

$$\begin{aligned} h_r &= c_1 e^{-2H_{o_r} y} \cos(-2H_{o_i}) y \\ h_i &= c_1 e^{-2H_{o_r} y} \sin(-2H_{o_i}) y \end{aligned} \tag{E-9}$$

Since H_{o_r} is negative [Eq. (81)], these are sinusoidal oscillations within an exponentially <u>growing</u> envelope. The only way h can approach zero for $y \longrightarrow \infty$ is for the constant c_1 in Eq. (E-8) to be identically zero, so that h is identically zero.

The solution for k is

$$k = c_2 e^{-H_o y} + c_3 e^{-H_o y \left(1 + \frac{1}{\sqrt{\sigma}}\right)} + c_4 e^{-H_o y \left(1 - \frac{1}{\sqrt{\sigma}}\right)} \tag{E-10}$$

The constants c_2 and c_3 must be zero by the same argument used in showing that $c_1 = 0$. The c_4 term in Eq. (E-10) seems to satisfy the condition that $k \longrightarrow 0$ as $y \longrightarrow \infty$ for $\sigma < 1$, and so cannot be immediately excluded. Now $k = f/\theta$ in the outer flow and

$$\theta = e^{-\sqrt{i \alpha Re\, \sigma (1-c)}\; y} = e^{H_o y},$$

so we recognize that the term containing c_4 in Eq. (E-10) corresponds to a variation of f in the outer flow of the type

$$f \sim e^{-\sqrt{i\,\alpha Re\,(1-c)}\;y} \quad .$$

But this function is exactly the outer variation of f in the LMN solutions [See Eq. (66).] . Since the LMN solutions are being obtained separately, the constant c_4 in Eq. (E-10) can be set equal to zero with perfect generality. This is a manifestation of the linear independence of the HJK and LMN systems.

With h and k identically zero the solution for j is now

$$j = c_5 \; e^{-H_o y} \qquad \text{(E-11)}$$

and c_5 must be zero by the above arguments.

Thus in the outer flow (where mean flow quantities have reached their outer values), H, J, and K must identically take on their outer values [Eq. (72)]. Any departure will diverge from these values and not satisfy the outer boundary conditions.

Consider now the LMN system in the outer flow. Outside the mean boundary layer, Eqs. (74) to (76) are

$$L' = -i + i\,(1-c)\,N - L\,M \qquad \text{(E-12)}$$

$$M'' = i\,\alpha Re\,(1-c)\,M - 3\,M\,M' - M^3 \qquad \text{(E-13)}$$

$$N'' = i\,\alpha Re\,\sigma(1-c)\,N - (2N' + N\,M)\,M - N\,M' \qquad \text{(E-14)}$$

The outer boundary conditions on L, M, and N are

$$\left.\begin{aligned} L_o &= \sqrt{\frac{i}{\alpha Re\,(1-c)}} \\ M_o &= -\sqrt{i\,\alpha Re\,(1-c)} \\ N_o &= 0 \end{aligned}\right\} \qquad (77)$$

Letting

$$\left.\begin{aligned} L &= L_o + \ell \\ M &= M_o + m \\ N &= n \end{aligned}\right\} \qquad \text{(E-15)}$$

where ℓ, m, and n are small quantities, the differential equations for ℓ, m, and n are

$$\ell' = i\,(1-c)\,n - L_o\,m - \ell M_o \qquad \text{(E-16)}$$

$$m'' + 3\,M_o\,m' + 2\,M_o^{\,2}\,m = 0 \qquad \text{(E-17)}$$

$$n'' + 2\,M_o\,n' + M_o^{\,2}\,(1-\sigma)\,n = 0 \qquad \text{(E-18)}$$

The solution for m is

$$m = c_6\, e^{-2M_o y} + c_7\, e^{-M_o y} \qquad \text{(E-19)}$$

Since the real part of M_o is negative, c_6 and c_7 must be zero in order that $m \longrightarrow 0$ for $y \longrightarrow \infty$. Thus m is identically zero. The solution for n is

$$n = c_8\, e^{-M_o(1+\sqrt{\sigma})y} + c_9\, e^{-M_o(1-\sqrt{\sigma})y} \qquad \text{(E-20)}$$

The c_8 term in Eq. (E-20) vanishes by the argument used continually in this Appendix. The c_9 term is identified as representing the outer θ behavior of the HJK system and so may be omitted here with perfect generality.

With m and n now identically zero, the solution for ℓ is

$$\ell = c_{10}\, e^{-M_o y} \qquad \text{(E-21)}$$

and c_{10} must vanish identically.

Thus ℓ, m, and n are identically zero in the outer flow as are h, j, and k. The outer boundary conditions of the JHK and LMN systems are known and must be satisfied where the outer mean flow is uniform to the desired accuracy. It is thus reasonable to integrate the LMN and HJK systems from the outer edge of the boundary layer inward to the wall. Since the disturbance equations [Eqs. (60) to (62)] are regular everywhere else, no difficulty is anticipated in performing these integrations.

APPENDIX F

EIGENVALUE DETERMINATION FOR GENERAL BOUNDARY CONDITION ON TEMPERATURE FLUCTUATIONS

The general thermal boundary condition to be considered is

$$a\theta_w' + b\theta_w = 0 \tag{F-1}$$

where the coefficients a and b are dependent on the surface material, its thickness, the method of cooling and the disturbance frequency under consideration (Appendix A of Reference 5). The boundary conditions on velocity fluctuations are as before; namely, $f_w = \phi_w = 0$.

The determinantal statement of the eigenvalue relation is

$$\begin{vmatrix} \Phi_w & \phi_{3w} & \phi_{5w} \\ F_w & f_{3w} & f_{5w} \\ a\Theta_w' + b\Theta_w & a\theta_{3w}' + b\theta_{3w} & a\theta_{5w}' + b\theta_{5w} \end{vmatrix} = 0 \tag{F-2}$$

which when expanded becomes

$$\frac{\Phi_w}{F_w} = \frac{\phi_{3w}}{f_{3w}} + \left(a\frac{\Theta_w'}{F_w} + b\frac{\Theta_w}{F_w}\right)\frac{\left[\frac{\phi_{5w}}{\theta_{5w}} - \frac{\phi_{3w}}{f_{3w}}\frac{f_{5w}}{\theta_{5w}}\right]}{\left(a\frac{\theta_{5w}'}{\theta_{5w}} + b\right)} + \frac{\Phi_w}{F_w}\frac{f_{5w}}{\theta_{5w}}\frac{\left[a\frac{\theta_{3w}'}{f_{3w}} + b\frac{\theta_{3w}}{f_{3w}}\right]}{\left(a\frac{\theta_{5w}'}{\theta_{5w}} + b\right)} - \frac{\phi_{5w}}{\theta_{5w}}\frac{\left[a\frac{\theta_{3w}'}{f_{3w}} + b\frac{\theta_{3w}}{f_{3w}}\right]}{\left(a\frac{\theta_{5w}'}{\theta_{5w}} + b\right)} \tag{F-3}$$

From the inviscid equations

$$\frac{\Theta_w}{F_w} = (\gamma-1)M_e^2 c + i(\gamma-1)M_e^2 c\left[\frac{w_w'}{c} - \frac{T_w'}{(\gamma-1)M_e^2c^2}\right]\frac{\Phi_w}{F_w} \tag{88}$$

and

$$\frac{\Theta_w'}{F_w} = \frac{T_w'}{T_w}\left[\gamma M_e^2 c - \frac{T_w}{c}\right] + i\,(\gamma-1)M_e^2 c\left[\alpha^2 - \frac{T_w' w_w'}{(\gamma-1)M_e^2c^3} - \frac{T_w''}{(\gamma-1)M_e^2c^2} + \left(\frac{\gamma}{\gamma-1}\right)\frac{T_w'}{T_w}\frac{w_w'}{c}\right]\frac{\Phi_w}{F_w} \qquad \text{(F-4)}$$

so that relation (F-3) becomes:

$$\frac{\Phi_w}{F_w} = \frac{\dfrac{\phi_{3w}}{f_{3w}} + \left\{a\dfrac{T_w'}{T_w}\left(\gamma M_e^2 c - \dfrac{T_w}{c}\right) + b(\gamma-1)M_e^2 c\right\}\dfrac{\left[\dfrac{\phi_{5w}}{\theta_{5w}} - \dfrac{\phi_{3w}}{f_{3w}}\dfrac{f_{5w}}{\theta_{5w}}\right]}{\left(a\dfrac{\theta_{5w}'}{\theta_{5w}} + b\right)} - \dfrac{\phi_{5w}}{\theta_{5w}}\dfrac{\left[a\dfrac{\theta_{3w}'}{f_{3w}} + b\dfrac{\theta_{3w}}{f_{3w}}\right]}{\left(a\dfrac{\theta_{5w}'}{\theta_{5w}} + b\right)}}{1 - i(\gamma-1)M_e^2 c\left[a\left\{\alpha^2 - \dfrac{T_w'w_w'}{(\gamma-1)M_e^2c^3} - \dfrac{T_w''}{(\gamma-1)M_e^2c^2} + \left(\dfrac{\gamma}{\gamma-1}\right)\dfrac{T_w'w_w'}{T_w c}\right\} + b\left\{\dfrac{w_w'}{c} - \dfrac{T_w'}{(\gamma-1)M_e^2c^2}\right\}\right]\dfrac{\left[\dfrac{\phi_{5w}}{\theta_{5w}} - \dfrac{\phi_{3w}}{f_{3w}}\dfrac{f_{5w}}{\theta_{5w}}\right]}{\left(a\dfrac{\theta_{5w}'}{\theta_{5w}} + b\right)} - \dfrac{f_{5w}}{\theta_{5w}}\dfrac{\left[a\dfrac{\theta_{3w}'}{f_{3w}} + b\dfrac{\theta_{3w}}{f_{3w}}\right]}{\left(a\dfrac{\theta_{5w}'}{\theta_{5w}} + b\right)}} \qquad \text{(F-5)}$$

For the Lees-Lin solution where $f_5 = \phi_5 = 0$ everywhere, the thermal boundary condition is still irrelevant and Eq. (F-5) remains

$$\Phi_w/F_w = \phi_{3w}/f_{3w} \qquad \text{(F-6)}$$

For the Dunn-Lin solution where $\theta_3 = f_5 = 0$ everywhere, Eq. (F-5) becomes

$$\frac{\Phi_w}{F_w} = \frac{\dfrac{\phi_{3w}}{f_{3w}} + \left\{a\dfrac{T_w'}{T_w}\left(\gamma M_e^2 c - \dfrac{T_w}{c}\right) + b(\gamma-1)M_e^2 c\right\}\dfrac{\dfrac{\phi_{5w}}{\theta_{5w}}}{\left(a\dfrac{\theta_{5w}'}{\theta_{5w}} + b\right)}}{1 - i(\gamma-1)M_e^2 c\left[a\left\{\alpha^2 - \dfrac{T_w'w_w'}{(\gamma-1)M_e^2c^3} - \dfrac{T_w''}{(\gamma-1)M_e^2c^2} + \left(\dfrac{\gamma}{\gamma-1}\right)\dfrac{T_w w_w'}{T_w c}\right\} + b\left\{\dfrac{w_w'}{c} - \dfrac{T_w'}{(\gamma-1)M_e^2c^2}\right\}\right]\dfrac{\dfrac{\phi_{5w}}{\theta_{5w}}}{\left(a\dfrac{\theta_{5w}'}{\theta_{5w}} + b\right)}} \qquad \text{(F-7)}$$

For the present solutions, Eq. (F-5) cannot be simplified. In the language of the present report

$$\Phi_w / F_w = \mathcal{R} \qquad \text{(F-8)}$$

where

$$\mathcal{R} = \frac{L_w + \left\{ a \frac{T_w'}{T_w}\left(\gamma M_e^2 c - \frac{T_w}{c}\right) + b(\gamma-1)M_e^2 c \right\} \frac{[J_w - L_w K_w]}{[aH_w + b]} - J_w \frac{[a(N_w' + N_w M_w) + bN_w]}{[aH_w + b]}}{1 - i(\gamma-1)M_e^2 c \left[a \left\{ \alpha^2 - \frac{T_w' w_w'}{(\gamma-1)M_e^2 c^3} - \frac{T_w''}{(\gamma-1)M_e^2 c^2} + \left(\frac{\gamma}{\gamma-1}\right)\frac{T_w' w_w'}{T_w c} \right\} + b \left\{ \frac{w_w'}{c} - \frac{T_w'}{(\gamma-1)M_e^2 c^2} \right\} \right] \frac{[J_w - L_w K_w]}{[aH_w + b]} - K_w \frac{[a(N_w' + N_w M_w) + bN_w]}{[aH_w + b]}} \qquad \text{(F-9)}$$

since

$$\frac{\Phi_w}{F_w} = \frac{i}{\frac{w_w'}{c} - \frac{1}{G_w}} \qquad (94)$$

the secular equation is written

$$G_w = \left(c/w_w'\right)(1 - \psi)$$

where

$$\psi = \frac{1}{1 + i \frac{w_w'}{c} \mathcal{R}}$$

and $\mathcal{R}$ is given by relation (F-9). For this more general boundary condition, the function $\mathcal{R}$ depends on both α and (αRe) even though the viscous solutions themselves are dependent on (αRe) alone. This dependence makes the actual numerical work slightly more tedious. Taking α^2 to be zero in $\mathcal{R}$ [Eq. (F-9)] will generally give a good first approximation to the eigenvalues.

The eigenvalue relations for the special thermal boundary condition $\theta_w' = 0$ are obtained by setting $b = 0$ in Eqs. (F-2), (F-3), (F-5), (F-7), and (F-9).

APPENDIX G

VISCOUS AND CONDUCTIVE CORRECTIONS TO INVISCID FUNCTIONS ABOUT THE CRITICAL LAYER

A most desirable method of obtaining the variation of disturbance amplitudes about the critical layer is to work out a number of the leading terms of a convergent series expansion of the complete equations about the critical point ("inner" solutions), compare them with the leading terms of the series solution of the inviscid equation about the critical point ("outer solution"), and then construct a uniformly valid series solution by proper matching of the two series in a region where both series are valid.

The present method is a crude patching approximation to the above; i. e., only the leading viscous terms are obtained and patched to the inviscid solution, without carefully studying the ranges of validity of the two solutions. The result should certainly be qualitatively correct and perhaps may also be sufficient to describe the quantitative variation of disturbance amplitudes about the critical layer.

The convergent solution about the critical point is obtained by the method of Lees and Lin[1], as carried out to higher order terms by Cheng[3]. However, in the present case, this solution is obtained in terms of the Tollmien variable (Appendix D) rather than the physical variable.

An examination of the inviscid solutions about the critical point [Eqs. (97) to (102)] shows that the leading discontinuity is in the temperature fluctuation amplitude, that is, $\Theta \sim 1/(y-y_c)$. This discontinuity is of zero order and is smoothed out by the zero order correction function obtained in this Appendix. For $A \neq 0$, all the amplitude functions (including the temperature fluctuation amplitude) have a logarithmic discontinuity either in value or in a derivative, and the corrections to these discontinuities are of first order or higher.

The correction terms are obtained as follows: Let

$$\left.\begin{aligned} f &= f^{(0)} + \epsilon f^{(1)} + \cdots \\ \phi &= \phi^{(0)} + \epsilon \phi^{(1)} + \cdots \\ \theta &= \theta^{(0)} + \epsilon \theta^{(1)} + \cdots \end{aligned}\right\} \qquad \text{(G-1)}$$

Substituting these relations into Eqs. (C-4) through (C-6) results in the equations for the quantities with superscript zero, namely the Lees-Lin equations.

$$\left.\begin{aligned} & f^{(0)'''} - \frac{i\alpha Re\,(w-c)}{\nu} f^{(0)'} = 0 \\ & \phi^{(0)'} + i f^{(0)} = 0 \\ & \theta^{(0)''} - \frac{i\alpha Re\,\sigma(w-c)}{\nu} \theta^{(0)} = \frac{\sigma\alpha Re\, T'}{\nu} \phi^{(0)} \end{aligned}\right\} \qquad \text{(G-2)}$$

The solution to Eq. (G-2) which corresponds to the leading term of the inviscid solution is

$$f^{(0)} = 0 \qquad \text{(G-3a)}$$

$$\phi^{(0)} = \text{const.} = -\frac{i}{\gamma M_e^2}\frac{T_c}{w_c'} \qquad \text{(G-3b)}$$

The value of the constant in Eq. (G-3b) is the leading term of Eq. (98). To obtain the function $\theta^{(0)}$ we solve the last of Eqs. (G-2) using Eq. (G-3b) for $\phi^{(0)}$, that is

$$\theta^{(0)''} - \frac{i\alpha Re\,\sigma(w-c)}{\nu}\,\theta^{(0)} = -\frac{i}{\gamma M_e^2}\,\frac{\sigma\alpha Re}{\nu_c}\,\frac{T_c T_c'}{w_c'} \tag{G-4}$$

Using the transformations (D-8) and (D-9) and letting

$$S_1 \equiv \frac{1}{\gamma M_e^2}\,\frac{\sigma T_c T_c'}{w_c' \nu_c}\,(\alpha Re)^{1/3}\left(\frac{dY_0}{dy}\right)_c^{-3/2} \tag{G-5}$$

yields in place of Eq. (G-4)

$$\frac{d^2 \Theta^{(0)}}{d\zeta_0^2} - i\zeta_0\,\Theta^{(0)} = -iS_1 \tag{G-6}$$

The solution to Eq. (G-6) obtained by comparison with Eqs. (21) and (24) of Schlichting (TM 1265)[13] is

$$\begin{aligned}\Theta^{(0)} &= S_1\,\hat{G}''(\zeta_0) \\ &= \frac{1}{\gamma M_e^2}\,\frac{T_c T_c'}{w_c'^2}\,\frac{\zeta_0 \hat{G}''(\zeta_0)}{(y-y_c)}\left(\frac{dY_0}{dy}\right)_c^{1/2}\end{aligned} \tag{G-7}$$

or since $\Theta = \theta\sqrt{dY_0/dy}$,

$$\theta^{(0)} = \frac{1}{\gamma M_e^2}\,\frac{T_c T_c'}{w_c'^2}\,\frac{\zeta_0 \hat{G}''(\zeta_0)}{(y-y_c)} \quad , \tag{G-8}$$

where $\hat{G}''(\zeta_0)$ is a function calculated by Schlichting[13] and here reproduced in Table I. Schlichting takes the thickness of the correction layer to be $|\zeta_0| \leq 4$ and in fact forces the correction to be zero at $|\zeta_0| = 4$ rather than let it become exponentially small with increasing ζ_0. Thus for $|\zeta_0| \geq 4$, $\zeta_0 \hat{G}''(\zeta_0) = 1$ and the function Eq. (G-8) exactly matches the leading term of the inviscid solution [Eqs. (101) and (102)].

We now proceed to the next-order terms. The equations governing the next order solutions are from Eqs. (G-1), (G-3), (C-4), (C-5), and (C-6)

$$f^{(1)'''} - \frac{i\alpha Re\,(w-c)}{\nu} f^{(1)'} = \frac{\alpha Re\, w''}{\nu}\phi^{(0)} + \frac{i\alpha Re}{\nu}\frac{w'(w-c)}{T}\theta^{(0)} - \frac{1}{\mu}\left(\frac{d\mu}{dT}\right) w'\theta^{(0)''} \qquad \text{(G-9a)}$$

$$\phi^{(1)'} + i f^{(1)} = \frac{T'}{T}\phi^{(0)} + \frac{i(w-c)}{T}\theta^{(0)} \qquad \text{(G-9b)}$$

$$\theta^{(1)''} - \frac{i\alpha Re\,\sigma(w-c)}{\nu}\theta^{(1)} = \frac{\sigma\alpha Re\, T'}{\nu}\phi^{(1)} - \frac{2}{\mu}\left(\frac{d\mu}{dT}\right) T'\theta^{(0)'} \qquad \text{(G-9c)}$$

Eqs. (G-9) are still a bit unwieldy. The first-order solutions to be obtained will therefore be the particular solution corresponding only to the leading zero-order terms in Eqs. (G-9). From symmetry considerations $\left[\theta^{(0)}\right]''_c = 0$. Since $\theta^{(0)}$ is linear in $(y - y_c)$, the second term on the right-hand side of Eq. (G-9a) varies as $(y - y_c)^2$, while the first term on the right hand side is a constant and therefore the leading term. The reduced form of (G-9a) is

$$f^{(1)'''} - \frac{i\alpha Re\,(w-c)}{\nu} f^{(1)'} = -\frac{i}{\gamma M_e^2}\,\frac{\alpha Re\, T_c}{\nu_c}\left(\frac{w_c''}{w_c'}\right) \qquad \text{(G-10)}$$

Using transformations (D-1) and (D-2) and letting

$$S_2 \equiv \frac{1}{\gamma M_e^2}\left(\frac{w_c''}{w_c'}\right)\frac{T_c}{\nu_c}\left(\alpha Re\right)^{1/3}\left(\frac{dY}{dy}\right)_c^{-3/2} \qquad \text{(G-11)}$$

yields in place of Eq. (G-10)

$$\frac{d^2 \mathcal{F}^{(1)}}{d\zeta^2} - i\zeta\,\mathcal{F}^{(1)} = -i\,S_2 \qquad \text{(G-12)}$$

whose solution by comparison with Eqs. (21) and (24) of Schlichting is

$$\mathcal{F}_r^{(1)} = S_2 \hat{G}''(\zeta)$$

$$= \frac{1}{\gamma M_e^2}\left(\frac{W_c''}{W_c'}\right)\frac{T_c}{W_c'} \quad \frac{\zeta \hat{G}''(\zeta)}{(y-y_c)}\left(\frac{dY}{dy}\right)_c^{1/2} \qquad \text{(G-13a)}$$

$$\mathcal{F}_i^{(1)} = S_2 \hat{H}''(\zeta)$$

$$= \frac{1}{\gamma M_e^2}\left(\frac{W_c''}{W_c'}\right)\frac{T_c}{W_c'} \quad \frac{\zeta \hat{H}''(\zeta)}{(y-y_c)}\left(\frac{dY}{dy}\right)_c^{1/2} \qquad \text{(G-13b)}$$

or

$$f_r^{(1)\prime} = \frac{1}{\gamma M_e^2}\left(\frac{W_c''}{W_c'}\right)\frac{T_c}{W_c'} \quad \frac{\zeta \hat{G}''(\zeta)}{(y-y_c)} \qquad \text{(G-14a)}$$

$$f_i^{(1)\prime} = \frac{1}{\gamma M_e^2}\left(\frac{W_c''}{W_c'}\right)\frac{T_c}{W_c'} \quad \frac{\zeta \hat{H}''(\zeta)}{(y-y_c)} \qquad \text{(G-14b)}$$

According to the approximations of this Appendix, $(d\zeta/dy) \approx (d\zeta/dy)_c$ so that ζ is linear in $(y - y_c)$. With this consideration, Eqs. (G-14) are integrated with respect to y and yield the following expressions for $f_r^{(1)}$ and $f_i^{(1)}$:

$$f_r^{(1)} = \frac{1}{\gamma M_e^2}\left(\frac{W_c''}{W_c'}\right)\frac{T_c}{W_c'}\left[\hat{G}'(\zeta) - \ln\frac{|\zeta|}{|y-y_c|}\right] \qquad \text{(G-15)}$$

for $(y - y_c) < 0$

$$f_i^{(1)} = \frac{1}{\gamma M_e^2}\left(\frac{W_c''}{W_c'}\right)\frac{T_c}{W_c'}\left[\pi + \hat{H}'(\zeta)\right] \qquad \text{(G-16a)}$$

for $(y - y_c) > 0$

$$f_i^{(1)} = \frac{1}{\gamma M_e^2}\left(\frac{W_c''}{W_c'}\right)\frac{T_c}{W_c'}\hat{H}'(\zeta) \qquad \text{(G-16b)}$$

For $|\zeta| \gtrapprox 4$, $\hat{G}'(\zeta) = \ln|\zeta|$ so that (G-15) asymptotically matches the logarithmic behavior of the leading inviscid terms of Eqs. (99) and (100). The coefficients however do not quite match. In place of the coefficient A which appears in Eqs. (99) and (100) only part of A, namely w_c''/w_c', appears in Eqs. (G-15) and (G-16). It is presumed that the contribution from the other part of A, namely $-T_c'/T_c$, would come from $f^{(2)}$ or some higher order term. Accordingly, the correction to the inviscid longitudinal velocity fluctuation will be applied with coefficient A so that the two solutions are patched in magnitude as well as in shape. The adjusted forms of Eqs. (G-15) and (G-16) are

$$f_r^{(1)} = \frac{A}{\gamma M_e^2} \frac{T_c}{w_c'} \left[\hat{G}'(\zeta) - \ln \frac{|\zeta|}{|y-y_c|} \right] \qquad \text{(G-17)}$$

for $(y - y_c) < 0$

$$f_i^{(1)} = \frac{A}{\gamma M_e^2} \frac{T_c}{w_c'} \left[\pi + \hat{H}'(\zeta) \right] \qquad \text{(G-18a)}$$

for $(y - y_c) > 0$

$$f_i^{(1)} = \frac{A}{\gamma M_e^2} \frac{T_c}{w_c'} \hat{H}'(\zeta) \qquad \text{(G-18b)}$$

Turning now to the continuity equation [Eq. (G-9b)] we find that the $f^{(1)}$ and $\phi^{(0)}$ terms are constant in a small neighborhood of the critical layer while $(w - c)\ \theta^{(0)}$ varies initially as $(y - y_c)^2$. An integration of Eq. (G-9b) would show that the first order variation to ϕ would initially be linear in $(y - y_c)$, and asymptotically approach the form $(y - y_c) \ln (y - y_c)$. Since there are no discontinuities in the value of ϕ about the critical layer, the correction to this function will be pursued no further.

From Eqs. (101) and (102) it is seen that the logarithmic singularity in temperature fluctuation amplitude is identical to that considered above for the longitudinal velocity fluctuation. The correction will be found analogously. Consider the leading terms resulting from differentiating Eq. (G-9c) remembering that $\bar{Q}' \sim 1$ and that d/dy of a fluctuating quantity is of order $1/\epsilon$.

$$\theta^{(1)'''} - \frac{i\alpha Re\, \sigma(w-c)}{\nu}\theta^{(1)'} = \frac{\sigma\alpha Re T'}{\nu}\phi^{(1)'} - \frac{2}{\mu}\left(\frac{d\mu}{dT}\right)T'\theta^{(0)''}$$

$$= \frac{\sigma\alpha Re T'}{\nu}\left[\frac{T'}{T}\phi^{(0)} + \frac{i(w-c)}{T}\theta^{(0)} - if^{(1)}\right] - \frac{2}{\mu}\left(\frac{d\mu}{dT}\right)T'\theta^{(0)''} \qquad \text{(G-19)}$$

The leading term on the right hand side of Eq. (G-19) is represented by the $\phi^{(0)}$ term, because $\theta_c^{(0)''} = 0$ and $(w - c)\ \theta^{(0)'} \sim (y - y_c)^2$. The function $f^{(1)}$ is a constant at the critical layer but since the final coefficient will be adjusted anyway, the $f^{(1)}$ term is not considered here. Following the same transformations as used in proceeding from Eqs. (G-4) to (G-6), Eq. (G-19) is written

$$\frac{d^2\Theta^{(1)'}}{d\zeta_0^2} - i\zeta_0\Theta^{(1)'} = -iS_1\frac{T_c'}{T_c} \qquad \text{(G-20)}$$

The solution of (G-20) adjusted to the coefficient A is obtained by steps identical to those used in proceeding from Eqs. (G-12) to (G-18). The adjusted solution is

$$\theta_r^{(1)} = \frac{A}{\gamma M_e^2}\frac{T_c T_c'}{w_c'^2}\left[\hat{G}'(\zeta_0) - \ln\frac{|\zeta_0|}{|y-y_c|}\right] \qquad \text{(G-21)}$$

for $(y - y_c) < 0$

$$\theta_i^{(1)} = \frac{A}{\gamma M_e^2}\frac{T_c T_c'}{w_c'^2}\left[\pi + \hat{H}'(\zeta_0)\right] \qquad \text{(G-22a)}$$

for $(y - y_c) > 0$

$$\theta_i^{(1)} = \frac{A}{\gamma M_e^2}\frac{T_c T_c'}{w_c'^2}\hat{H}'(\zeta_0) \qquad \text{(G-22b)}$$

APPENDIX H

A FORMULATION FOR EXACT NUMERICAL INTEGRATION OF THE ORR-SOMMERFELD EQUATION

The Orr-Sommerfeld equation is the exact differential equation of infinitesimal two-dimensional disturbances for an incompressible flow in which all streamlines are parallel and the mean flow does not change in the longitudinal direction. An example of such a flow is plane Poiseulle flow. The Orr-Sommerfeld equation is often applied to "almost parallel" flows such as boundary layer flows. This application is not exact but is quite acceptable at large Reynolds numbers, more specifically when $(\alpha Re)^{-1/3} << 1$. (The vertical mean velocity introduces terms which may be of order $(\alpha Re)^{-1/3}$ compared to the leading terms of the Orr-Sommerfeld equation).

The equations of infinitesimal disturbances for an incompressible "parallel flow" wherein disturbances of the form $Q'(x, y, t) = q(y)\, e^{i\alpha(x-ct)}$ are considered, are [from Eqs. (4), (5), and (6)]

Continuity

$$\phi' + if = 0 \qquad \text{(H-1)}$$

Longitudinal Momentum

$$i(w-c)\, f + w'\, \phi = -\frac{i\pi}{\gamma M_e^2} + \frac{1}{\alpha Re}\left(f'' - \frac{4}{3}\alpha^2 f + i\alpha^2 \frac{\phi'}{3}\right) \qquad \text{(H-2)}$$

Normal Momentum

$$i\alpha^2 (w-c)\, \phi = -\frac{\pi'}{\gamma M_e^2} + \frac{1}{\alpha Re}\left(i\alpha^2 \frac{f'}{3} - \alpha^4 \phi + \frac{4}{3}\alpha^2 \phi''\right) \qquad \text{(H-3)}$$

The boundary conditions at a solid surface are $\phi = f = 0$. In the outer flow both ϕ and f decay exponentially to zero.

Using Eq. (H-1), f is eliminated from Eqs. (H-2) and (H-3) yielding:

$$-(w-c)\,\phi' + w'\,\phi = -\frac{i\pi}{\gamma M_e^2} + \frac{1}{\alpha Re}\left(i\phi''' - i\alpha^2\phi'\right) \qquad \text{(H-4)}$$

$$i\alpha^2 (w-c)\,\phi = -\frac{\pi'}{\gamma M_e^2} + \frac{\alpha^2}{\alpha Re}\left(\phi'' - \alpha^2\phi\right) \qquad \text{(H-5)}$$

Upon eliminating $(\pi/\gamma M_e^2)$ between Eqs. (H-4) and (H-5), the Orr-Sommerfeld equation is obtained.

$$\phi^{iv} - 2\alpha^2\phi'' + \alpha^4\phi = i\alpha Re\left[(w-c)(\phi'' - \alpha^2\phi) - w''\phi\right] \qquad \text{(H-6)}$$

This equation is everywhere regular except at infinity. The eigenvalues of this equation at $y \longrightarrow \infty$ are

$$\pm\alpha, \quad \pm\sqrt{\alpha^2 + i\alpha Re\,(1-c)} \qquad \text{(H-7)}$$

Of these four, only the two with negative signs are considered since only these represent solutions which satisfy the outer boundary conditions. We therefore seek two linearly independent solutions ϕ_1 and ϕ_2 corresponding to the negative eigenvalues of Eq. (H-7).

The development which follows is written without subscripts but applies <u>separately</u> to solutions 1 and 2. Following the procedures used in the text for both the inviscid and viscous solutions, define a Riccati type variable. Let

$$\phi'/\phi \equiv P \qquad \text{(H-8)}$$

that is *

$$\phi_1'/\phi_1 \equiv P_1 \quad \text{and} \quad \phi_2'/\phi_2 \equiv P_2 \quad .$$

Then

$$\phi''/\phi = P' + P^2 \tag{H-9}$$

$$\phi^{iv}/\phi = P''' + P\left[4P'' + 6\,PP'\right] + 3\,P'^2 + P^4 \tag{H-10}$$

In terms of the new variable P, the Orr-Sommerfeld equation is written

$$\begin{aligned} P''' = (P' + P^2)\left[2\alpha^2 + i\alpha Re\,(w-c)\right] - \left[\alpha^4 + i\alpha Re\,(w-c)\,\alpha^2 + i\alpha Re\,w''\right] \\ - P\left[4P'' + 6\,PP'\right] - 3P'^2 - P^4 \end{aligned} \tag{H-11}$$

In the outer flow P is a constant and from Eq. (H-11) satisfies the following equation

$$\left[P_o^{\,2} - \alpha^2\right]\left[P_o^{\,2} - \left\{\alpha^2 + i\alpha Re\,(1-c)\right\}\right] = 0 \tag{H-12}$$

Upon consideration of the discussion following Eq. (H-7), the pertinent values of P_o for solutions 1 and 2 are respectively

$$(P_1)_o = -\alpha \tag{H-13a}$$

$$(P_2)_o = -\sqrt{\alpha^2 + i\alpha Re\,(1-c)} \tag{H-13b}$$

The detailed formulation follows the procedure employed in obtaining the HJK and LMN systems of Section III. 2. First Eq. (H-11) is split into real and imaginary equations:

---

* Note that for the total solution $(\phi'/\phi)_w \longrightarrow \infty$. The method of this Appendix is based on the premise that the real and imaginary parts of the individual eigenfunctions ϕ_1 and ϕ_2 never vanish simultaneously.

$$P_r''' = \left[P_r' + P_r^2 - P_i^2\right]\left[2\alpha^2 + \alpha Re\, c_i\right] - \left[P_i' + 2P_rP_i\right]\alpha Re\,(1-c_r)$$
$$- \left[\alpha^4 + \alpha^2 \alpha Re\, c_i\right] - P_r\left[4P_r'' + 6P_rP_r' - 6P_iP_i'\right]$$
$$+ P_i\left[4P_i'' + 6P_rP_i' + 6P_iP_r'\right] - 3\left[P_r'^2 - P_i'^2\right] \qquad \text{(H-14a)}$$
$$- \left[P_r^4 - 6P_r^2P_i^2 + P_i^4\right]$$

$$P_i''' = \left[P_r' + P_r^2 - P_i^2\right]\alpha Re\,(1-c_r) + \left[P_i' + 2P_rP_i\right]\left[2\alpha^2 + \alpha Re\, c_i\right]$$
$$- \left[\alpha Re\,(1-c_r) + \alpha Re\, w''\right] - P_i\left[4P_r'' + 6P_rP_r' - 6P_iP_i'\right]$$
$$- P_r\left[4P_i'' + 6P_rP_i' + 6P_iP_r'\right] - 6P_r'P_i' \qquad \text{(H-14b)}$$
$$-4\left[P_r^3P_i - P_rP_i^3\right]$$

The outer conditions to Eqs. (H-14) are

<u>For solution P_1</u>

$$\left.\begin{aligned} (P_{1_r})_o &= -a \\ (P_{1_i})_o &= 0 \end{aligned}\right\} \qquad \text{(H-15)}$$

<u>For solution P_2</u>

$$\left.\begin{aligned} (P_{2_r})_o &= -\left[\frac{\sqrt{\tilde{A}^2 + \tilde{B}^2} + \tilde{A}}{2}\right]^{1/2} \\ (P_{2_i})_o &= -\left[\frac{\sqrt{\tilde{A}^2 + \tilde{B}^2} - \tilde{A}}{2}\right]^{1/2} \end{aligned}\right\} \qquad \text{(H-16)}$$

where

$$\left.\begin{aligned} \tilde{A} &\equiv a^2 + aRe\, c_i \\ \tilde{B} &\equiv aRe\,(1 - c_r) \end{aligned}\right\} \qquad \text{(H-17)}$$

Since the Orr-Sommerfeld equation is regular everywhere except at infinity and the outer conditions are known, Eqs. (H-14) are simultaneously integrated starting from the outer edge and proceeding to the wall.

The boundary conditions at the wall namely $\phi = \phi' = 0$ are satisfied when

$$(P_1)_w = (P_2)_w \tag{H-18}$$

Thus for a given profile and chosen values of c_r and c_i the values of α and αRe are adjusted until condition (H-18) is satisfied.

The method of this appendix is meant to be used in cases where the splitting of solutions into inviscid and viscous types is objectionable. For the situation

$$\left| \frac{\alpha Re\,(1 - c_r)}{\alpha^2 + \alpha Re\, c_i} \right| >> 1 \;,$$

the asymptotic methods are recommended for two reasons: (1) the asymptotic methods are simpler to use; and (2) for (αRe) large, great care must be exercised in performing the numerical integrations, because rapid changes in the behavior of the functions may be expected where the action of viscosity is important.

TABLE I

SCHLICHTING'S CORRECTION FUNCTIONS

(from Reference 13)

ζ	$\hat{G}''(\zeta)$	$\hat{G}'(\zeta)$	$\hat{H}'(\zeta)$
-4.0	-.250	1.386	-3.140
-3.5	-.325	1.243	-3.153
-3.0	-.438	1.056	-3.188
-2.5	-.589	.798	-3.213
-2.0	-.767	.458	-3.175
-1.5	-.839	.051	-3.022
-1.0	-.746	- .354	-2.692
- .5	-.443	- .659	-2.187
0	0	- .774	-1.570
.5	.443	- .659	- .953
1.0	.746	- .354	- .448
1.5	.839	.051	- .118
2.0	.767	.458	.035
2.5	.589	.798	.073
3.0	.438	1.056	.048
3.5	.325	1.243	.013
4.0	.250	1.386	0

TABLE II

TIETJENS AND AUXILIARY FUNCTIONS*

z	F_r	F_i	Φ_r	Φ_i	$\tilde{G}_r$	$\tilde{G}_i$
.04	16.98520	-9.70547	-.28788	-.17479	20.64871	-11.73373
.08	8.58153	-4.85142	-.09358	-.05988	10.48902	-5.86208
.12	5.78146	-3.23248	-.14354	-.09704	7.10585	-3.90332
.16	4.38231	-2.42257	-.19541	-.13996	5.41699	-2.92183
.20	3.54347	-1.93612	-.24893	-.18949	4.40567	-2.33144
.24	2.98484	-1.61148	-.30366	-.24654	3.73341	-1.93677
.28	2.58634	-1.37917	-.35901	-.31213	3.25416	-1.65308
.32	2.28794	-1.20460	-.41415	-.38735	2.89670	-1.43942
.36	2.05623	-1.06849	-.46792	-.47335	2.61980	-1.27192
.40	1.87127	-.95934	-.51878	-.57122	2.39924	-1.13664
.44	1.72027	-.86972	-.56483	-.68203	2.22005	-1.02502
.48	1.59475	-.79477	-.60357	-.80655	2.07170	-.93069
.52	1.48886	-.73108	-.63204	-.94520	1.94710	-.84978
.56	1.39837	-.67621	-.64674	-1.09782	1.84105	-.77927
.60	1.32021	-.62841	-.64372	-1.26330	1.75001	-.71696
.64	1.25208	-.58634	-.61884	-1.43945	1.67093	-.66130
.68	1.19220	-.54895	-.56816	-1.62274	1.60200	-.61097
.72	1.13920	-.51548	-.48827	-1.80810	1.54125	-.56497
.76	1.09200	-.48527	-.37714	-1.98919	1.48743	-.52259
.80	1.04973	-.45784	-.23449	-2.15869	1.43954	-.48317
.84	1.01168	-.43277	-.06233	-2.30900	1.39658	-.44622
.88	.97729	-.40975	+.13486	-2.43307	1.35794	-.41127
.92	.94607	-.38846	.35065	-2.52556	1.32289	-.37798
.96	.91762	-.36871	.57718	-2.58320	1.29104	-.34611
1.00	.89162	-.35029	.80612	-2.60534	1.26190	-.31541
1.04	.86778	-.33303	1.02983	-2.59383	1.23504	-.28558
1.08	.84585	-.31680	1.24187	-2.55230	1.21019	-.25651
1.12	.82565	-.30148	1.43751	-2.48566	1.18696	-.22805
1.16	.80698	-.28695	1.61397	-2.39928	1.16514	-.20007
1.20	.78968	-.27313	1.76988	-2.29843	1.14448	-.17245
1.24	.77363	-.25993	1.90532	-2.18785	1.12471	-.14512
1.28	.75871	-.24730	2.02123	-2.07154	1.10566	-.11802
1.32	.74480	-.23516	2.11911	-1.95271	1.08715	-.09107
1.36	.73182	-.22347	2.20078	-1.83384	1.06892	-.06426
1.40	.71969	-.21216	2.28814	-1.71667	1.05084	-.03759

* These functions as here tabulated were computed by Dr. L. M. Mack of the Jet Propulsion Laboratory, California Institute of Technology. The author is grateful to Dr. Mack for permission to include these tables herein.

z	F_r	F_i	Φ_r	Φ_i	$\tilde{G}_r$	$\tilde{G}_i$
1.44	.70832	-.20120	2.32304	-1.60244	1.03274	-.01103
1.48	.69765	-.19055	2.36720	-1.49191	1.01448	+.01541
1.52	.68763	-.18018	2.40215	-1.38558	.99590	.04163
1.56	.67820	-.17004	2.42925	-1.28360	.97689	.06760
1.60	.66930	-.16011	2.44968	-1.18603	.95733	.09327
1.64	.66090	-.15037	2.46440	-1.09278	.93714	.11853
1.68	.65294	-.14078	2.47425	-1.00366	.91622	.14331
1.72	.64540	-.13133	2.47993	- .91848	.89451	.16749
1.76	.63824	-.12200	2.48199	- .83700	.87196	.19096
1.80	.63142	-.11278	2.48090	- .75897	.84859	.21360
1.84	.62491	-.10360	2.47706	- .68415	.82435	.23527
1.88	.61868	-.09450	2.47074	- .61230	.79931	.25586
1.92	.61271	-.08545	2.46220	- .54322	.77349	.27524
1.96	.60697	-.07642	2.45163	- .47671	.74697	.29330
2.00	.60143	-.06742	2.43917	- .41260	.71984	.30995
2.04	.59607	-.05842	2.42495	- .35074	.69220	.32509
2.08	.59087	-.04942	2.40905	- .29099	.66417	.33865
2.12	.58580	-.04040	2.39154	- .23326	.63588	.35058
2.16	.58085	-.03136	2.37248	- .17747	.60747	.36085
2.20	.57598	-.02227	2.35189	- .12355	.57908	.36945
2.24	.57119	-.01315	2.32982	- .07144	.55086	.37638
2.28	.56644	-.00397	2.30629	- .02112	.52294	.38167
2.32	.56172	+.00527	2.28132	+ .02741	.49545	.38539
2.36	.55701	.01457	2.25495	.07416	.46853	.38757
2.40	.55229	.02394	2.22720	.11910	.44227	.38831
2.44	.54753	.03339	2.19811	.16221	.41678	.38770
2.48	.54271	.04292	2.16771	.20345	.39215	.38582
2.52	.53782	.05253	2.13607	.24279	.36843	.38278
2.56	.53283	.06223	2.10323	.28016	.34570	.37870
2.60	.52772	.07201	2.06927	.31552	.32398	.37367
2.64	.52246	.08189	2.03426	.34882	.30331	.36781
2.68	.51704	.09184	1.99829	.38000	.28371	.36122
2.72	.51142	.10188	1.96146	.40901	.26518	.35401
2.76	.50559	.11200	1.92388	.43582	.24771	.34626
2.80	.49951	.12219	1.88566	.46037	.23130	.33807
2.84	.49317	.13245	1.84692	.48265	.21592	.32953
2.88	.48653	.14276	1.80780	.50263	.20154	.32071
2.92	.47958	.15312	1.76843	.52030	.18814	.31168
2.96	.47227	.16350	1.72894	.53566	.17569	.30252
3.00	.46458	.17390	1.68948	.54873	.16414	.29327
3.04	.45649	.18429	1.65018	.55953	.15346	.28400
3.08	.44797	.19464	1.61119	.56809	.14360	.27474
3.12	.43898	.20493	1.57264	.57446	.13453	.26555
3.16	.42951	.21513	1.53465	.57871	.12620	.25645
3.20	.41952	.22519	1.49736	.58090	.11857	.24747

z	F_r	F_i	Φ_r	Φ_i	$\tilde{G}_r$	$\tilde{G}_i$
3.24	.40900	.23509	1.46089	.58111	.11161	.23866
3.28	.39791	.24476	1.42534	.57943	.10527	.23002
3.32	.38625	.25417	1.39082	.57596	.09951	.22158
3.36	.37400	.26324	1.35741	.57081	.09430	.21336
3.40	.36115	.27193	1.32521	.56408	.08960	.20536
3.44	.34769	.28016	1.29427	.55588	.08538	.19761
3.48	.33364	.28787	1.26467	.54633	.08160	.19011
3.52	.31900	.29497	1.23645	.53556	.07823	.18286
3.56	.30380	.30139	1.20966	.52367	.07524	.17588
3.60	.28807	.30706	1.18432	.51080	.07260	.16916
3.64	.27186	.31188	1.16046	.49705	.07028	.16271
3.68	.25522	.31579	1.13808	.48255	.06825	.15652
3.72	.23824	.31871	1.11719	.46741	.06650	.15060
3.76	.22098	.32057	1.09778	.45173	.06499	.14495
3.80	.20356	.32131	1.07983	.43564	.06370	.13956
3.84	.18607	.32089	1.06333	.41922	.06261	.13442
3.88	.16865	.31928	1.04825	.40258	.06170	.12954
3.92	.15142	.31646	1.03456	.38581	.06096	.12491
3.96	.13450	.31243	1.02221	.36900	.06035	.12052
4.00	.11805	.30721	1.01116	.35222	.05987	.11636
4.04	.10218	.30086	1.00136	.33556	.05951	.11243
4.08	.08703	.29343	.99277	.31908	.05923	.10872
4.12	.07272	.28501	.98534	.30285	.05904	.10522
4.16	.05934	.27569	.97899	.28693	.05892	.10193
4.20	.04699	.26560	.97368	.27136	.05885	.09883
4.24	.03574	.25486	.96935	.25621	.05883	.09593
4.28	.02564	.24361	.96594	.24150	.05885	.09320
4.32	.01671	.23197	.96338	.22728	.05889	.09064
4.36	.00896	.22010	.96161	.21357	.05896	.08824
4.40	.00239	.20813	.96058	.20040	.05903	.08600
4.44	-.00306	.19617	.96023	.18780	.05911	.08391
4.48	-.00740	.18436	.96048	.17577	.05920	.08195
4.52	-.01072	.17279	.96130	.16434	.05927	.08012
4.56	-.01308	.16155	.96261	.15350	.05933	.07841
4.60	-.01455	.15072	.96437	.14327	.05938	.07681
4.64	-.01523	.14037	.96653	.13364	.05942	.07532
4.68	-.01519	.13054	.96902	.12461	.05943	.07393
4.72	-.01452	.12128	.97180	.11617	.05942	.07263
4.76	-.01331	.11259	.97483	.10832	.05939	.07141
4.80	-.01164	.10451	.97806	.10104	.05933	.07027
4.84	-.00959	.09702	.98144	.09432	.05925	.06920
4.88	-.00722	.09013	.98495	.08814	.05913	.06819
4.92	-.00460	.08382	.98853	.08248	.05900	.06725
4.96	-.00180	.07808	.99217	.07733	.05883	.06636

z	F_r	F_i	Φ_r	Φ_i	$\tilde{G}_r$	$\tilde{G}_i$
5.00	+.00113	.07289	.99583	.07267	.05864	.06552
5.04	.00414	.06822	.99947	.06846	.05842	.06472
5.08	.00720	.06404	1.00308	.06470	.05817	.06397
5.12	.01026	.06032	1.00662	.06135	.05790	.06325
5.16	.01328	.05704	1.01009	.05839	.05761	.06256
5.20	.01626	.05416	1.01345	.05580	.05729	.06190
5.24	.01915	.05166	1.01670	.05355	.05695	.06127
5.28	.02194	.04951	1.01982	.05162	.05660	.06066
5.32	.02462	.04766	1.02280	.04998	.05622	.06007
5.36	.02717	.04611	1.02562	.04861	.05582	.05949
5.40	.02958	.04482	1.02829	.04749	.05541	.05894
5.44	.03185	.04376	1.03079	.04659	.05499	.05839
5.48	.03397	.04292	1.03313	.04590	.05455	.05786
5.52	.03594	.04226	1.03529	.04538	.05410	.05734
5.56	.03776	.04177	1.03728	.04503	.05365	.05682
5.60	.03942	.04143	1.03911	.04482	.05318	.05631
5.64	.04093	.04121	1.04076	.04472	.05270	.05581
5.68	.04230	.04111	1.04224	.04474	.05222	.05531
5.72	.04351	.04110	1.04357	.04484	.05174	.05482
5.76	.04459	.04117	1.04473	.04502	.05125	.05433
5.80	.04553	.04130	1.04574	.04525	.05076	.05385
5.84	.04633	.04149	1.04660	.04553	.05026	.05337
5.88	.04701	.04172	1.04733	.04585	.04977	.05289
5.92	.04757	.04198	1.04791	.04618	.04928	.05241
5.96	.04802	.04226	1.04838	.04653	.04879	.05194
6.00	.04836	.04255	1.04872	.04689	.04830	.05147
6.04	.04860	.04285	1.04895	.04724	.04781	.05100
6.08	.04874	.04314	1.04908	.04758	.04733	.05053
6.12	.04880	.04343	1.04912	.04790	.04685	.05006
6.16	.04878	.04371	1.04906	.04820	.04638	.04960
6.20	.04868	.04397	1.04893	.04848	.04591	.04914
6.24	.04851	.04421	1.04872	.04872	.04545	.04868
6.28	.04828	.04442	1.04845	.04893	.04499	.04822
6.32	.04800	.04460	1.04812	.04911	.04454	.04777
6.36	.04767	.04476	1.04774	.04925	.04410	.04732
6.40	.04730	.04489	1.04732	.04935	.04366	.04687
6.44	.04688	.04498	1.04686	.04941	.04323	.04643
6.48	.04643	.04504	1.04636	.04943	.04281	.04599

z	F_r	F_i	Φ_r	Φ_i	$\tilde{G}_r$	$\tilde{G}_i$
6.52	.04596	.04507	1.04584	.04941	.04240	.04556
6.56	.04546	.04507	1.04530	.04936	.04199	.04512
6.60	.04495	.04503	1.04474	.04926	.04159	.04469
6.64	.04442	.04497	1.04417	.04913	.04119	.04427
6.68	.04387	.04487	1.04359	.04897	.04080	.04385
6.72	.04332	.04474	1.04300	.04878	.04042	.04344
6.76	.04277	.04458	1.04242	.04855	.04005	.04303
6.80	.03983	.04278	1.03941	.04631	.03968	.04262
6.84	.03956	.04253	1.03915	.04601	.03932	.04222
6.88	.03929	.04228	1.03888	.04572	.03897	.04183
6.92	.03900	.04202	1.03859	.04541	.03862	.04144
6.96	.03870	.04177	1.03829	.04511	.03828	.04106
7.00	.03839	.04151	1.03799	.04480	.03794	.04068
7.04	.03807	.04124	1.03767	.04449	.03762	.04031
7.08	.03775	.04097	1.03735	.04417	.03729	.03994
7.12	.03742	.04070	1.03702	.04385	.03697	.03957
7.16	.03710	.04043	1.03670	.04352	.03666	.03922
7.20	.03676	.04014	1.03637	.04319	.03635	.03886
7.24	.03643	.03986	1.03604	.04286	.03605	.03852
7.28	.03610	.03957	1.03571	.04252	.03575	.03817
7.32	.03577	.03928	1.03538	.04218	.03546	.03784
7.36	.03545	.03898	1.03506	.04183	.03517	.03751
7.40	.03512	.03868	1.03474	.04148	.03489	.03718
7.44	.03480	.03838	1.03442	.04113	.03461	.03686
7.48	.03448	.03808	1.03411	.04078	.03434	.03654
7.52	.03417	.03777	1.03380	.04043	.03407	.03623
7.56	.03386	.03747	1.03350	.04008	.03380	.03592
7.60	.03356	.03716	1.03320	.03972	.03354	.03562
7.64	.03327	.03685	1.03291	.03937	.03328	.03532
7.68	.03297	.03654	1.03262	.03902	.03302	.03502
7.72	.03269	.03624	1.03234	.03867	.03277	.03473
7.76	.03241	.03593	1.03207	.03833	.03252	.03445
7.80	.03214	.03563	1.03180	.03798	.03227	.03417
7.84	.03187	.03533	1.03154	.03764	.03203	.03389
7.88	.03161	.03503	1.03129	.03730	.03179	.03362
7.92	.03135	.03473	1.03104	.03696	.03155	.03335
7.96	.03110	.03443	1.03080	.03663	.03132	.03308
8.00	.03085	.03414	1.03056	.03631	.03109	.03282

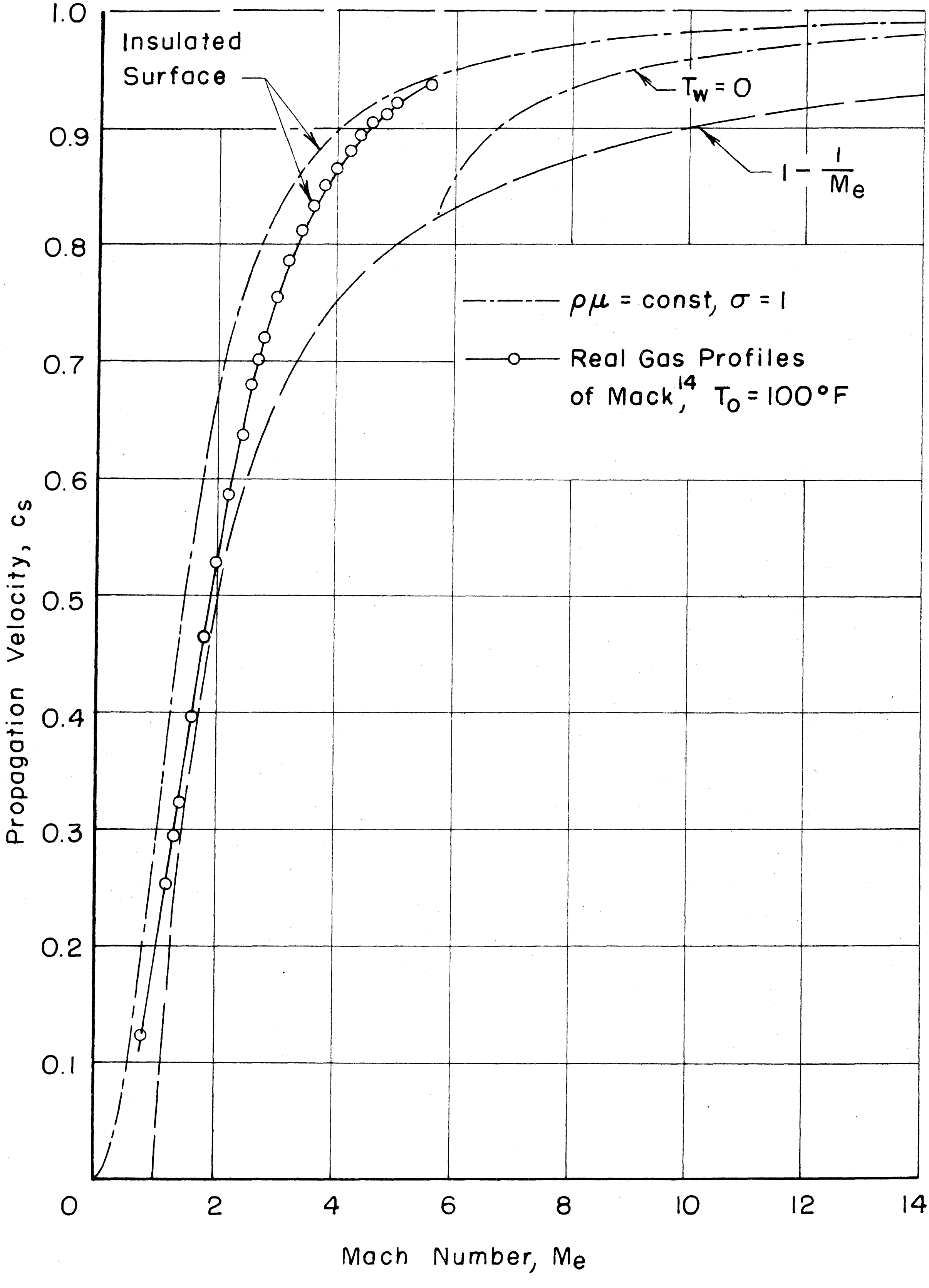

FIG. 1 - PROPAGATION VELOCITY OF NEUTRAL INVISCID DISTURBANCES FOR FLAT PLATE BOUNDARY LAYERS.

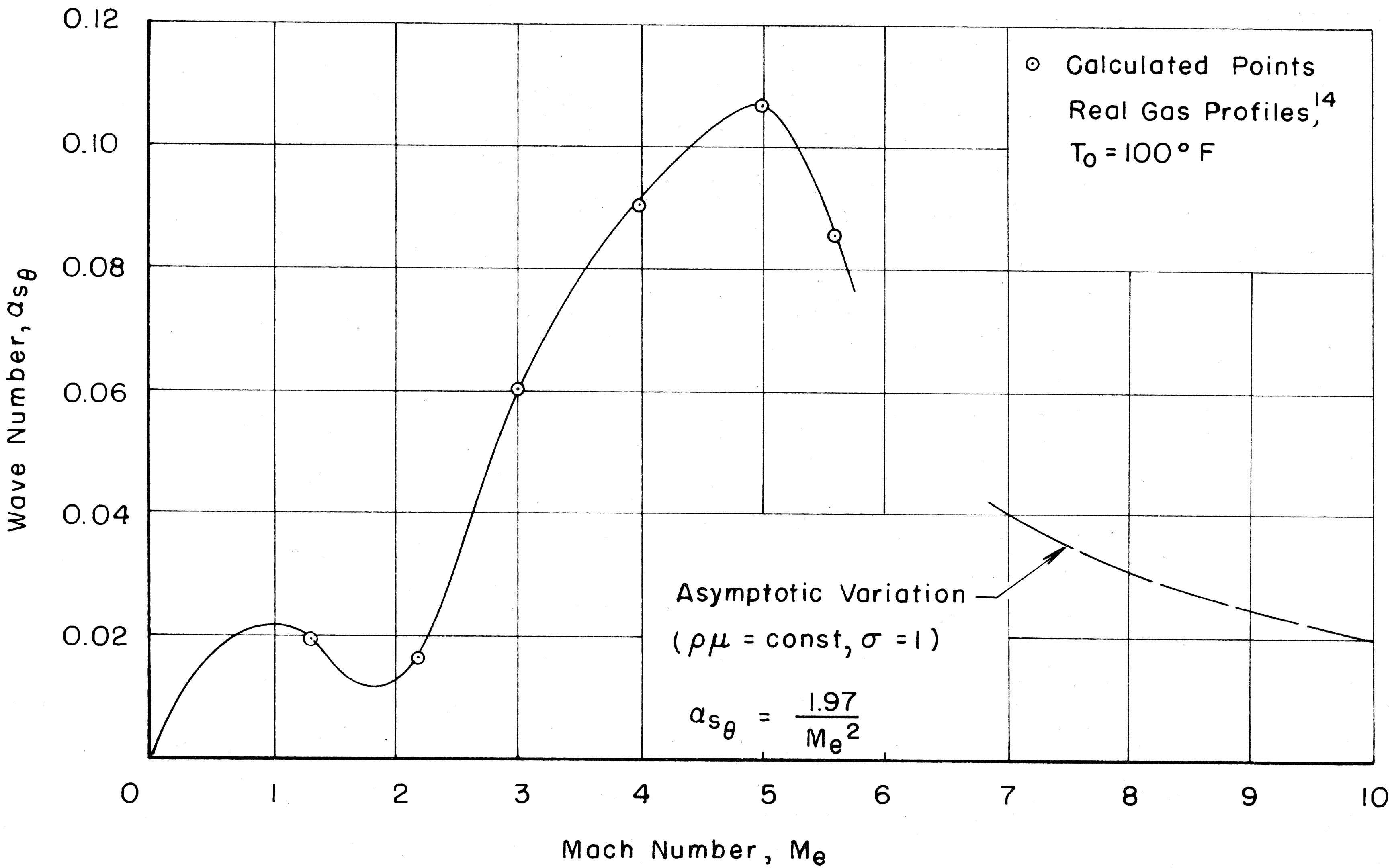

FIG. 2 – WAVE NUMBER OF NEUTRAL INVISCID OSCILLATION.

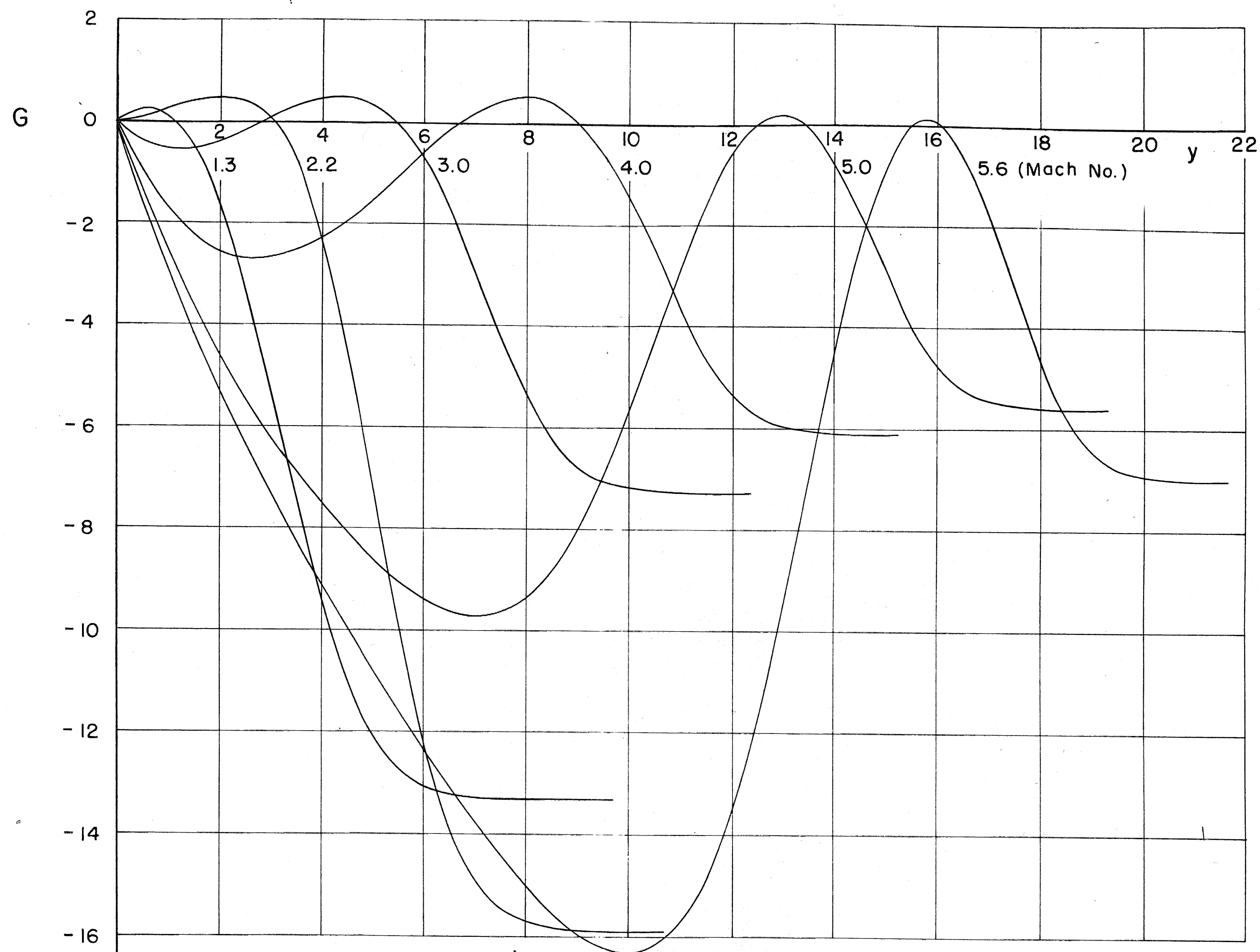

FIG. 3 — DISTRIBUTION OF $G = \frac{\pi'}{\alpha^2 \pi}$ FOR NEUTRAL INVISCID DISTURBANCE

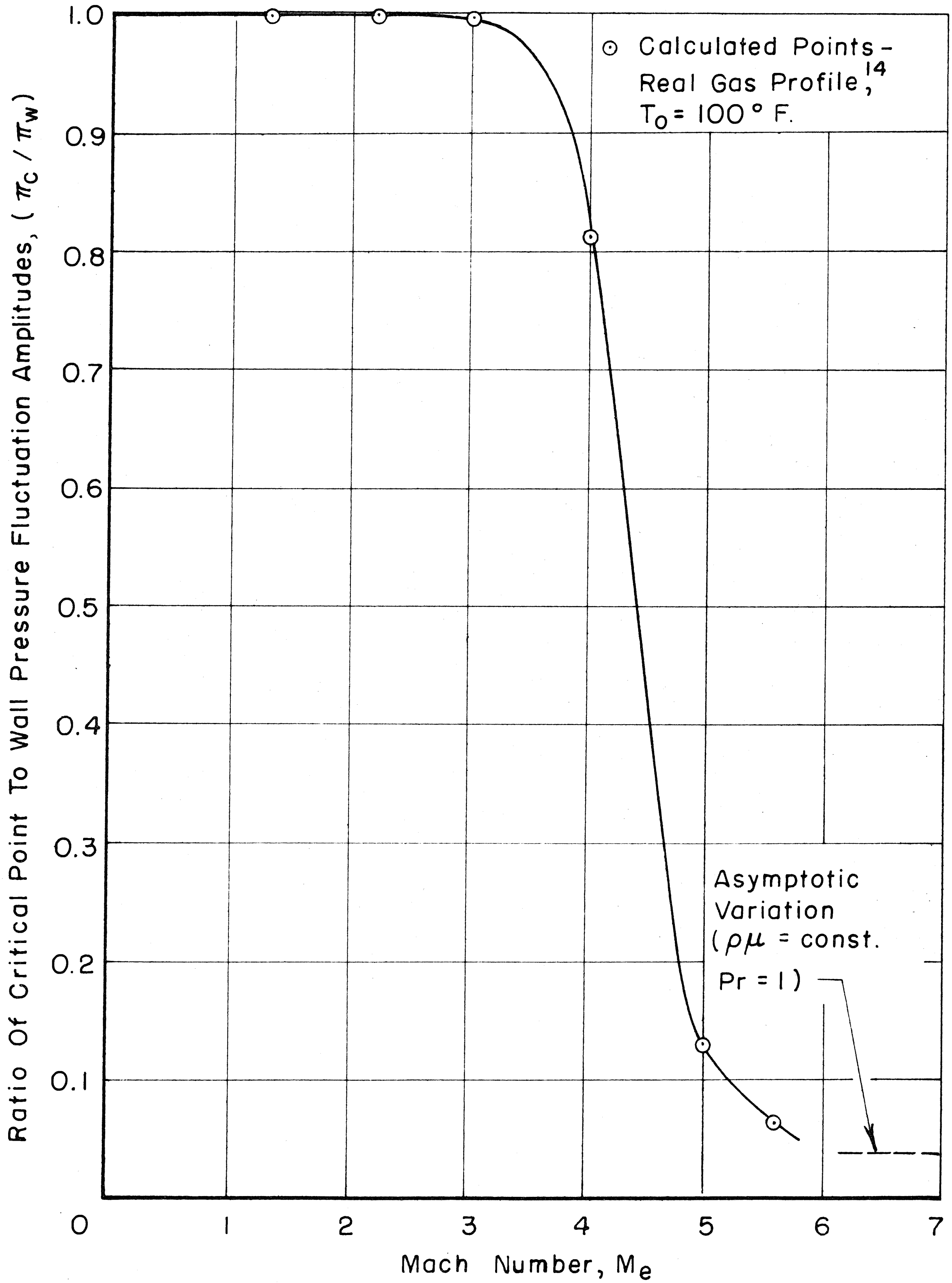

FIG. 4 - PRESSURE FLUCTUATION AMPLITUDE AT CRITICAL POINT, NEUTRAL INVISCID OSCILLATION.

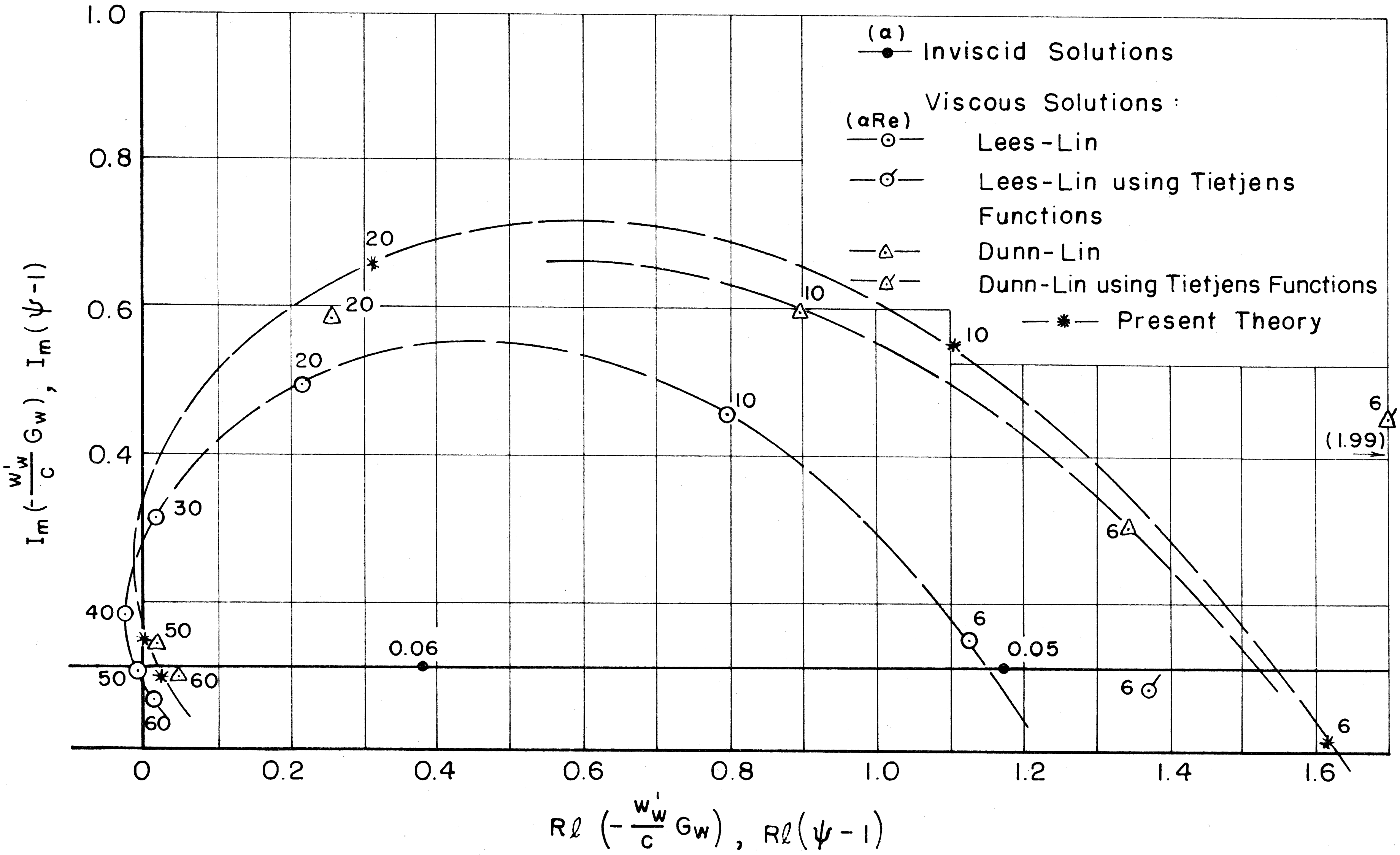

FIG. 5 – DETERMINATION OF α AND αRe FOR NEUTRAL STABILITY, M_e = 2.2, c = 0.61611.

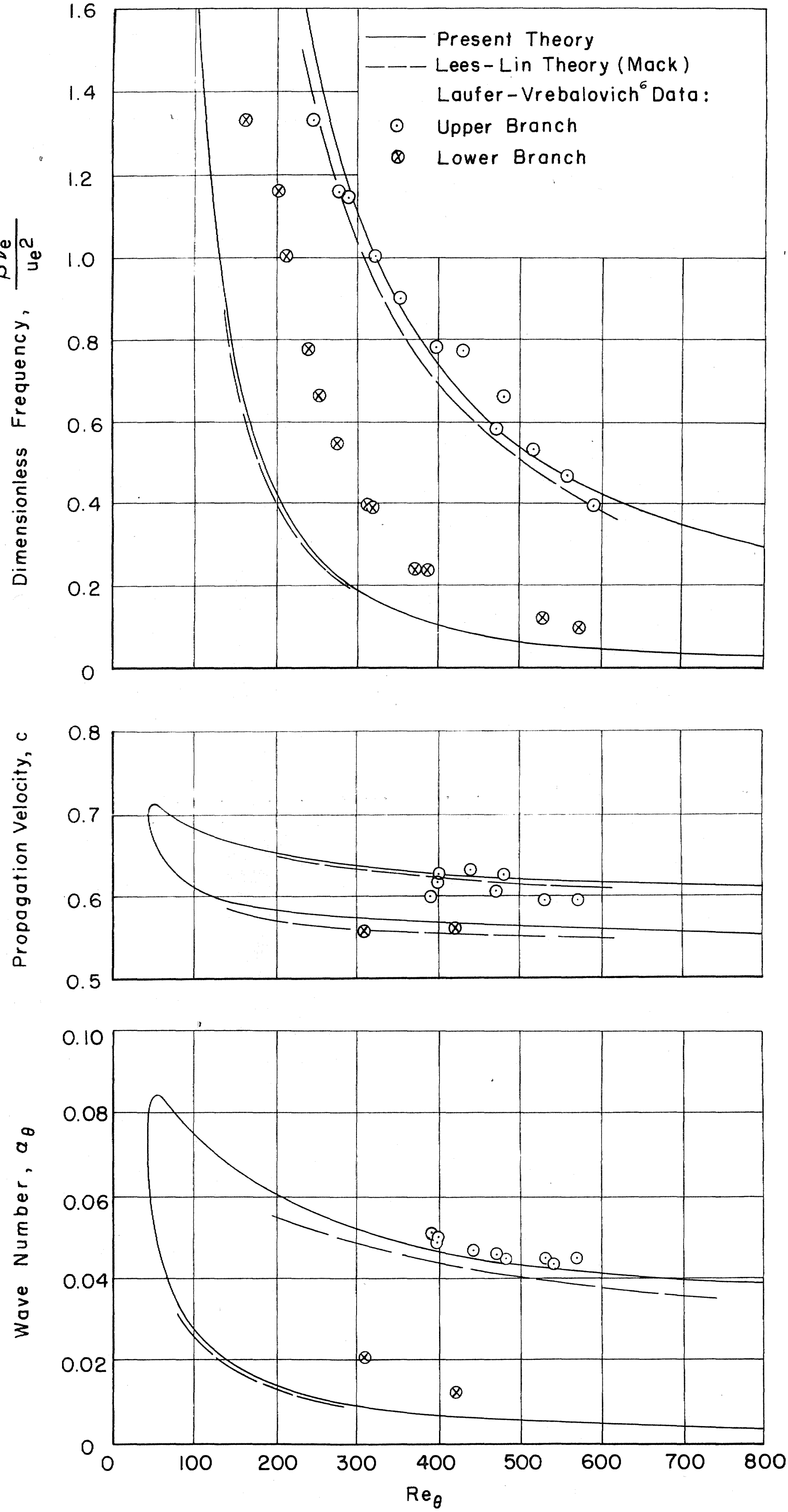

FIG. 6 – NEUTRAL STABILITY CHARACTERISTICS, $M_\infty = 2.2$

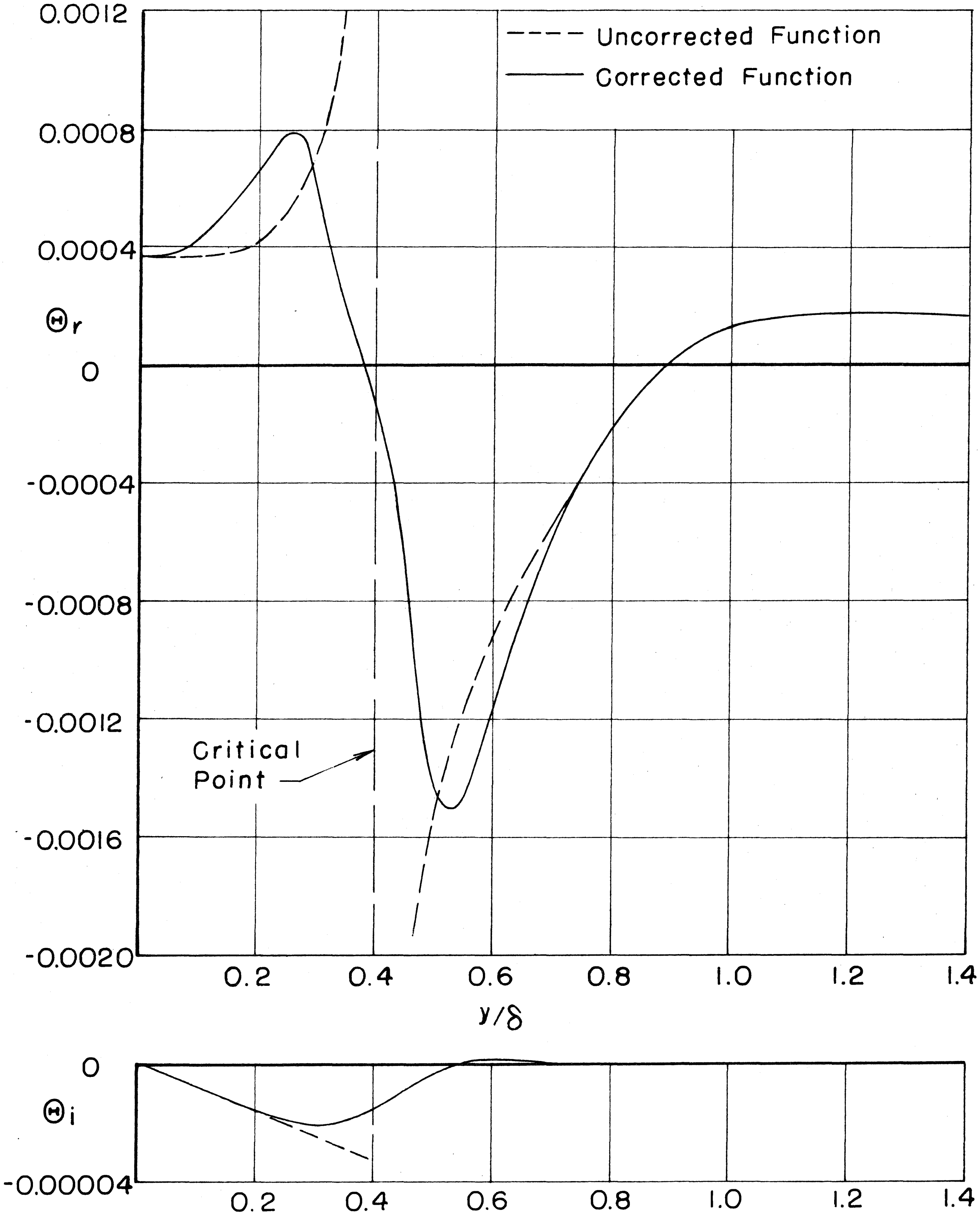

FIG. 7 – INVISCID AMPLITUDE DISTRIBUTIONS FOR NEUTRAL DISTURBANCE, $M_e = 2.2$, $c = 0.616$, $Re_\theta = 535$

(a) Temperature Fluctuations.

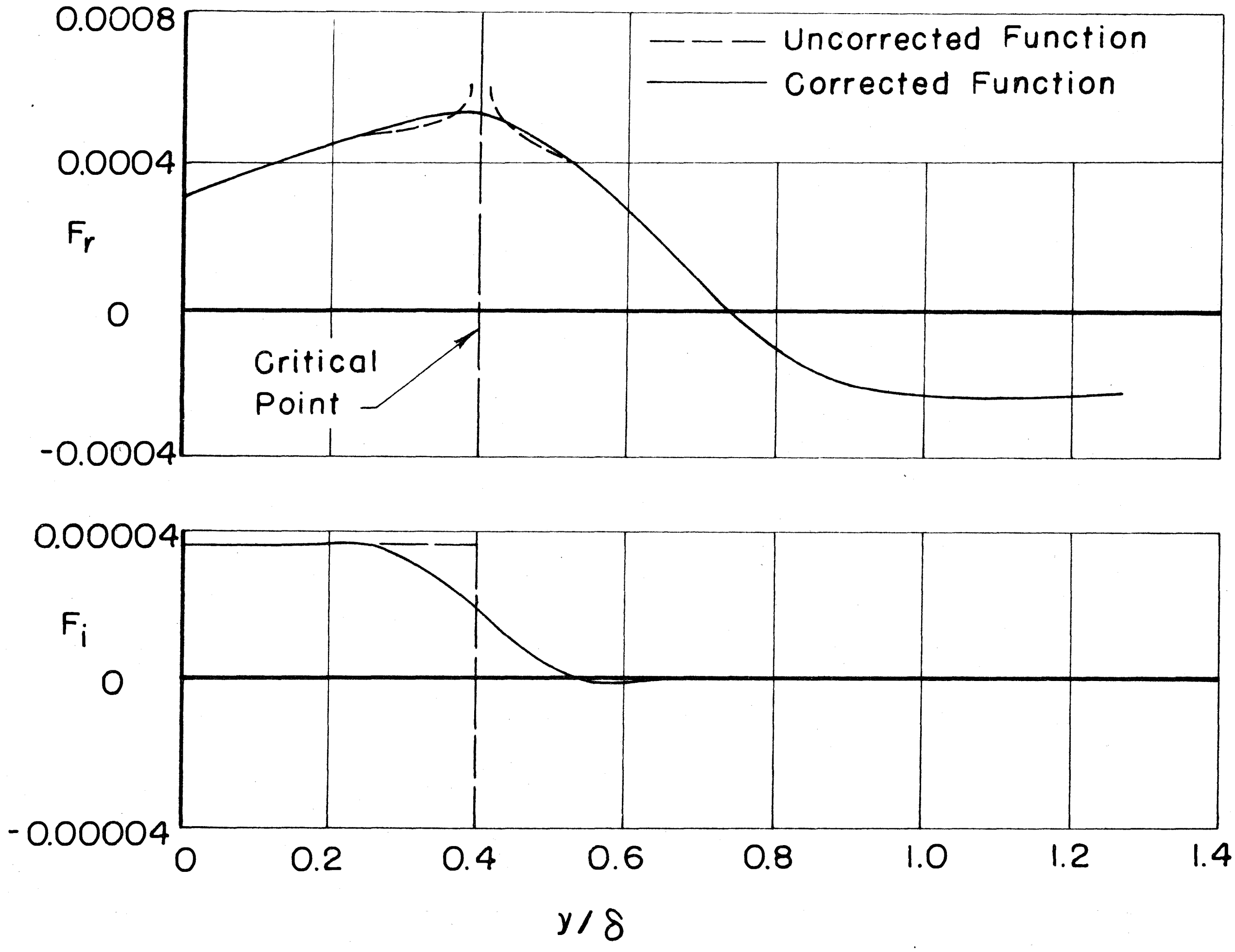

FIG. 7 – INVISCID AMPLITUDE DISTRIBUTIONS FOR NEUTRAL DISTURBANCE, $M_e = 2.2$, $c = 0.616$, $Re_\theta = 535$.

(b) Longitudinal Velocity Fluctuations.

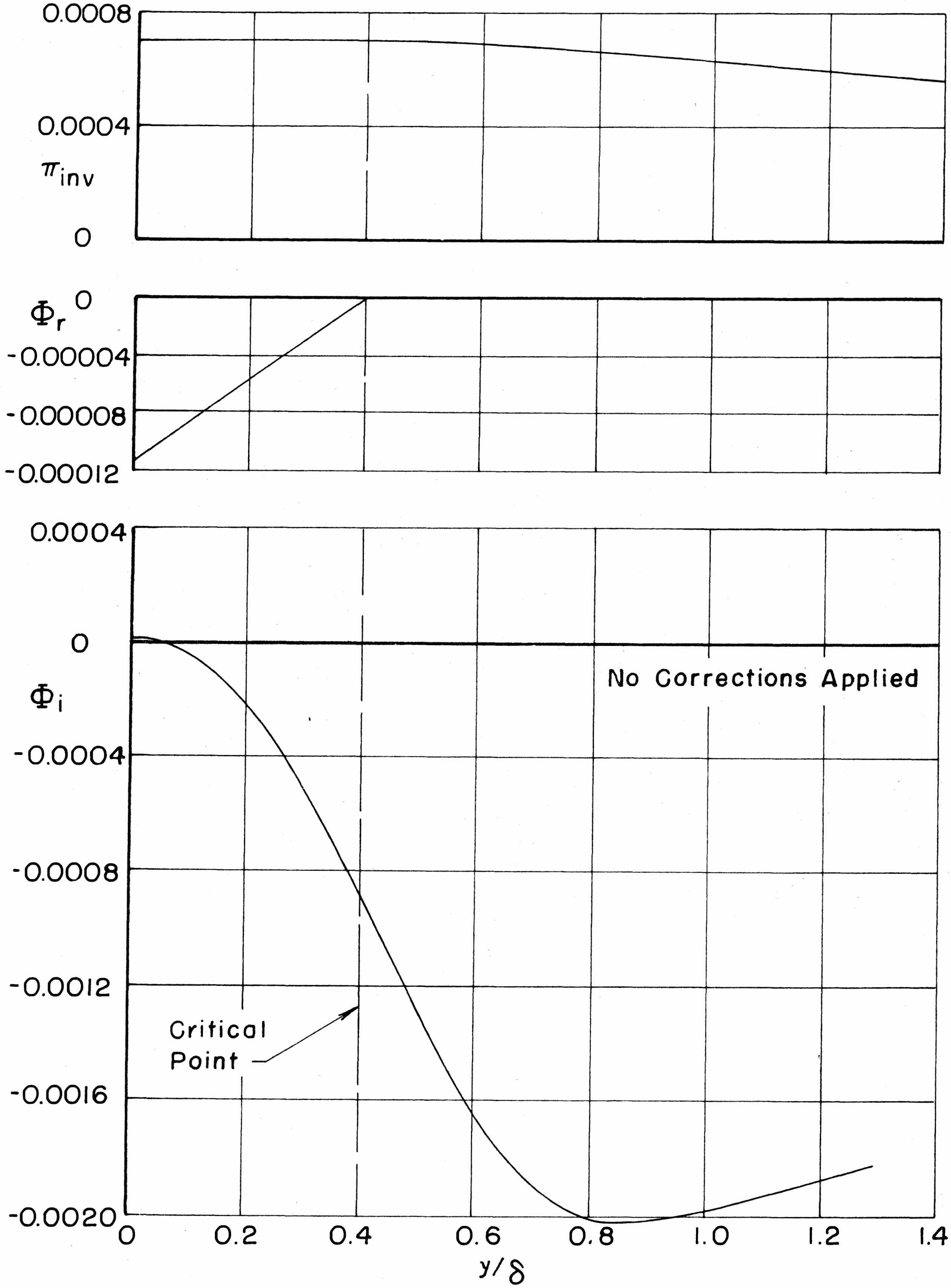

FIG. 7-INVISCID AMPLITUDE DISTRIBUTIONS FOR NEUTRAL DISTURBANCE, $M_e = 2.2$, $c = 0.616$, $Re_\theta = 535$

(c) Pressure And Φ (Normal Velocity) Fluctuations

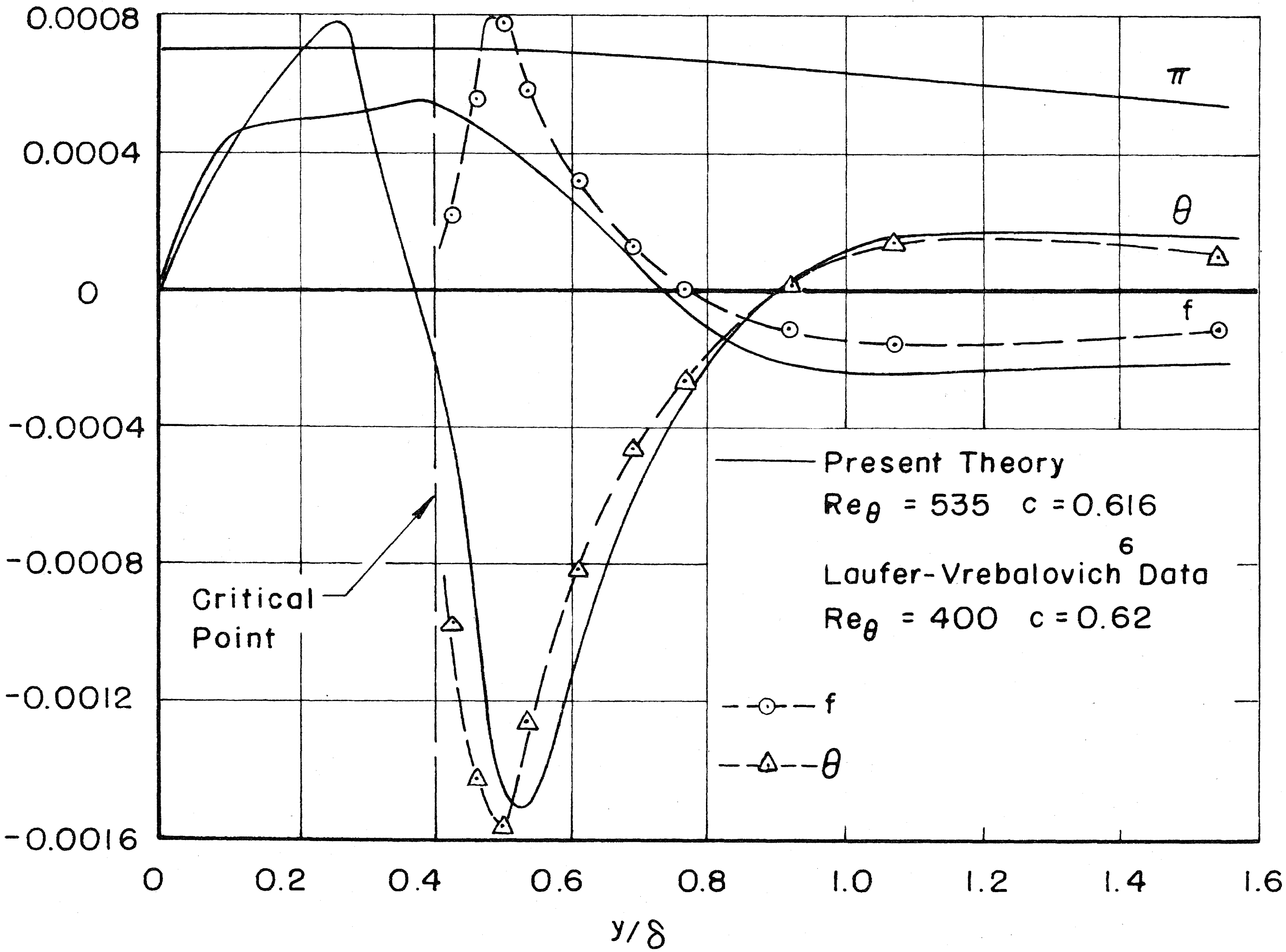

FIG. 8 – AMPLITUDE DISTRIBUTIONS FOR NEUTRAL OSCILLATION, $M_e = 2.2$.

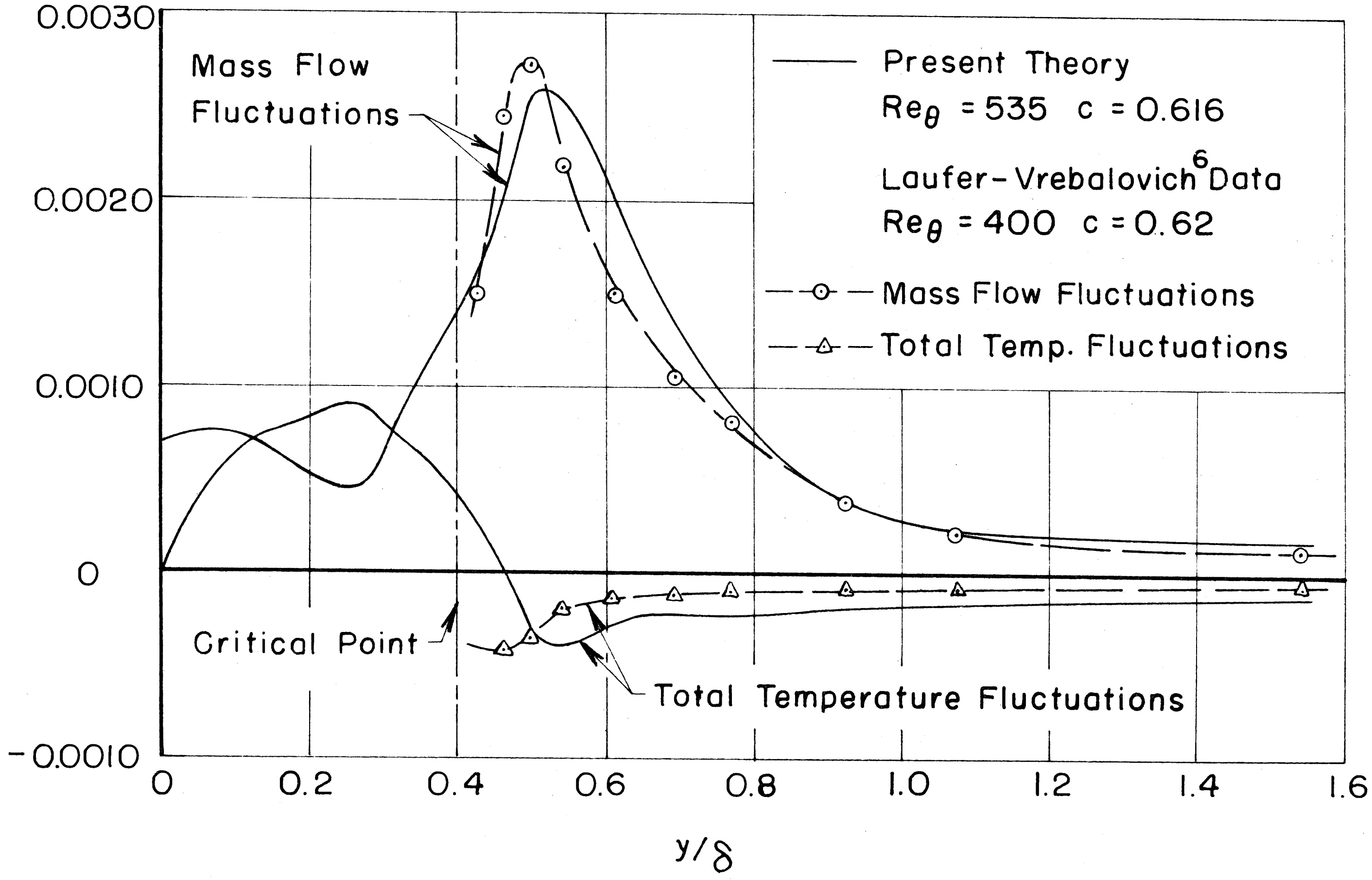

FIG. 9 - MASS FLOW AND TOTAL TEMPERATURE DISTRIBUTIONS FOR NEUTRAL OSCILLATION, $M_e = 2.2$.

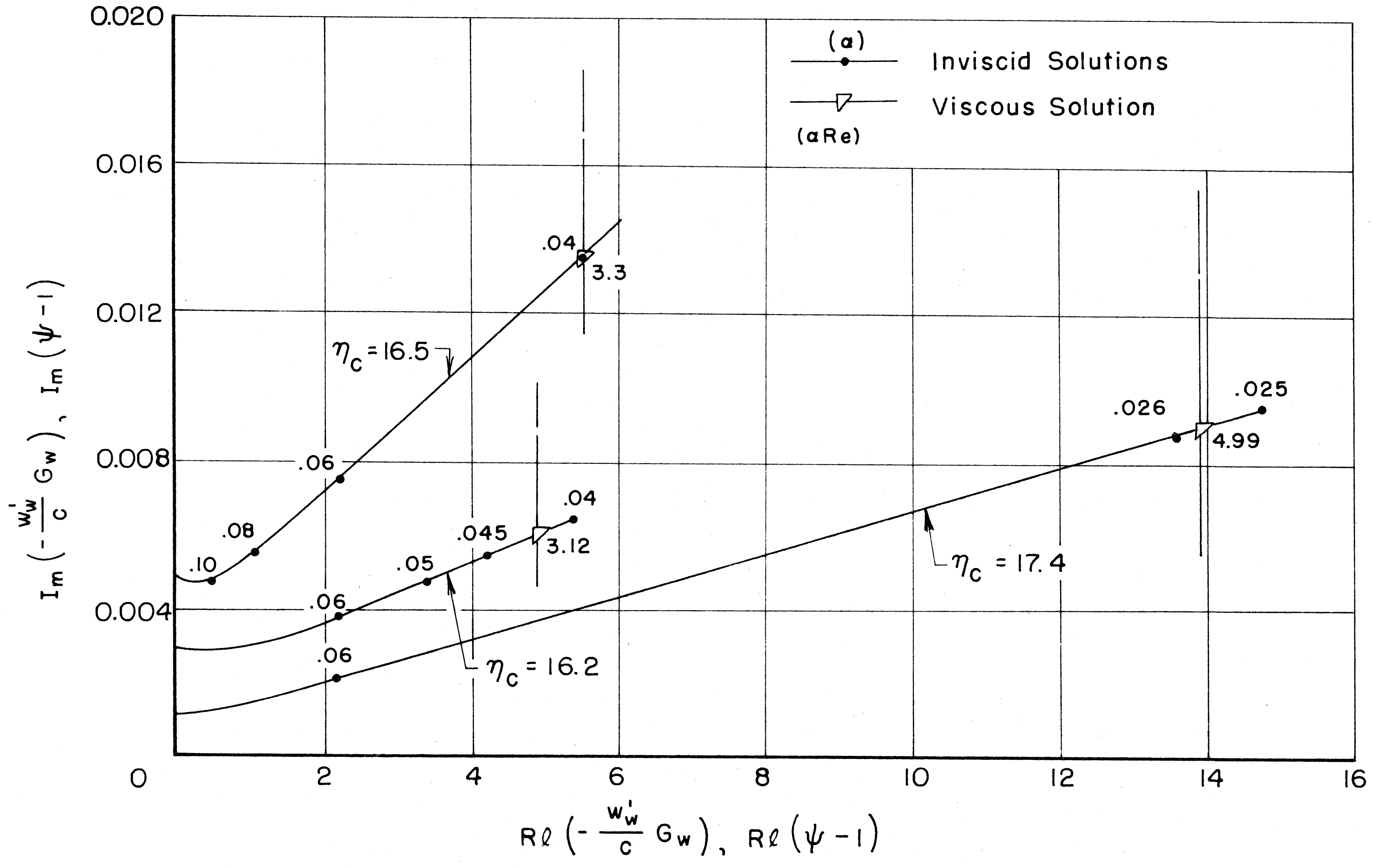

FIG.10-DETERMINATION OF α AND αRe FOR NEUTRAL STABILITY, $M_e = 5.6$

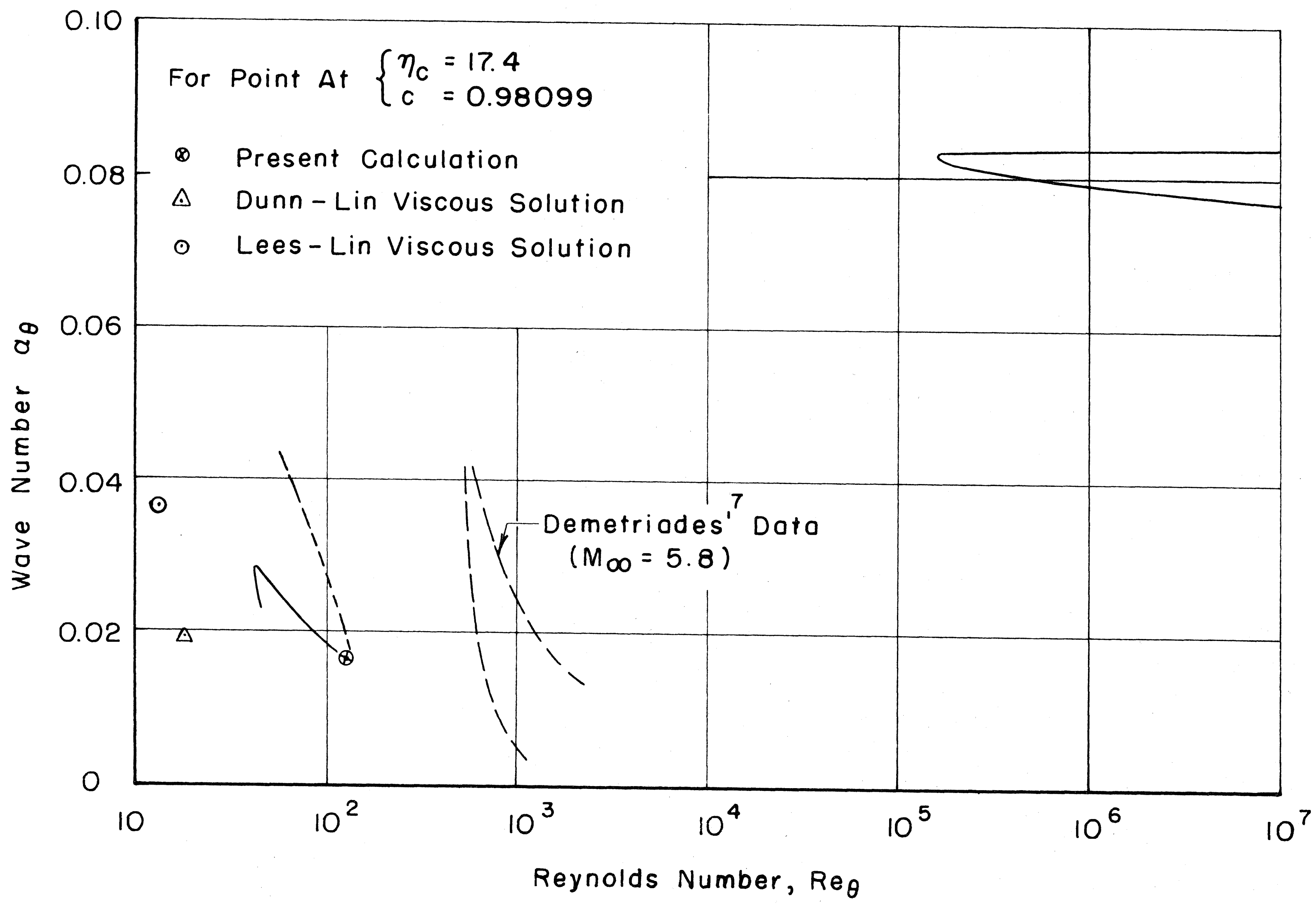

FIG. 11 — NEUTRAL STABILITY CHARACTERISTICS, M_e = 5.6

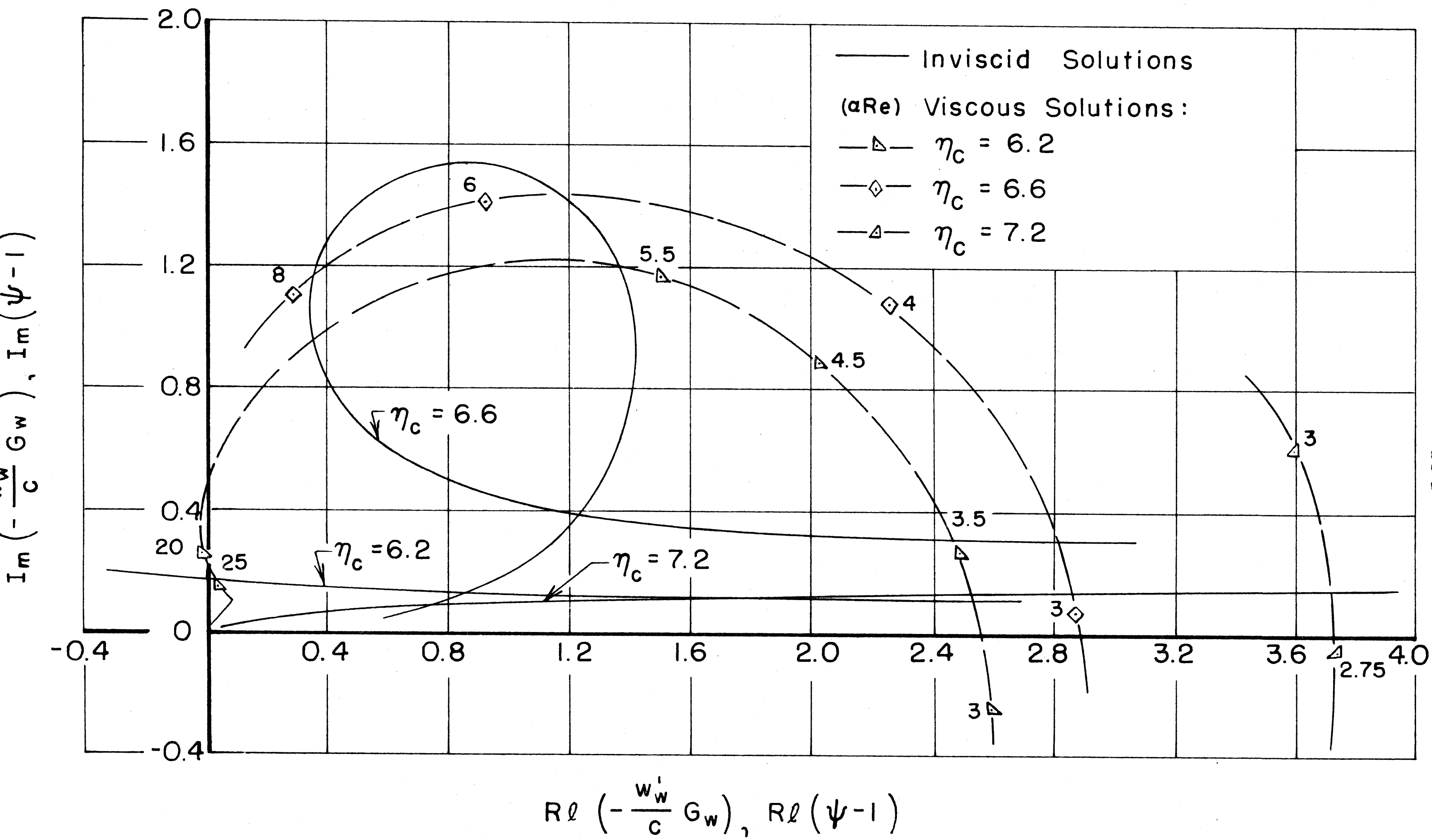

FIG. 12 — DETERMINATION OF α AND αRe FOR NEUTRAL STABILITY, M_e = 3.2

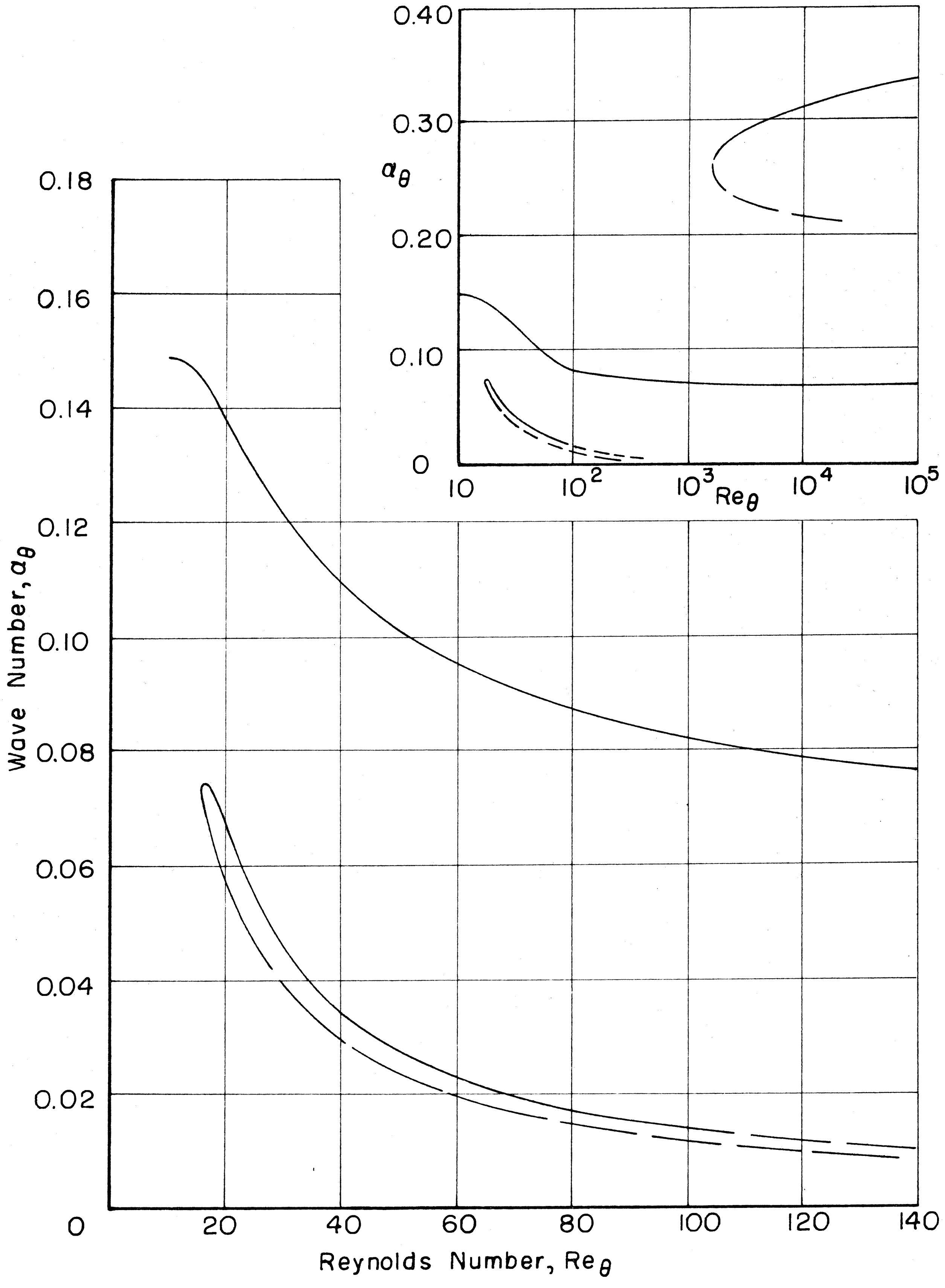

FIG.13- NEUTRAL STABILITY CHARACTERISTICS, $M_e = 3.2$

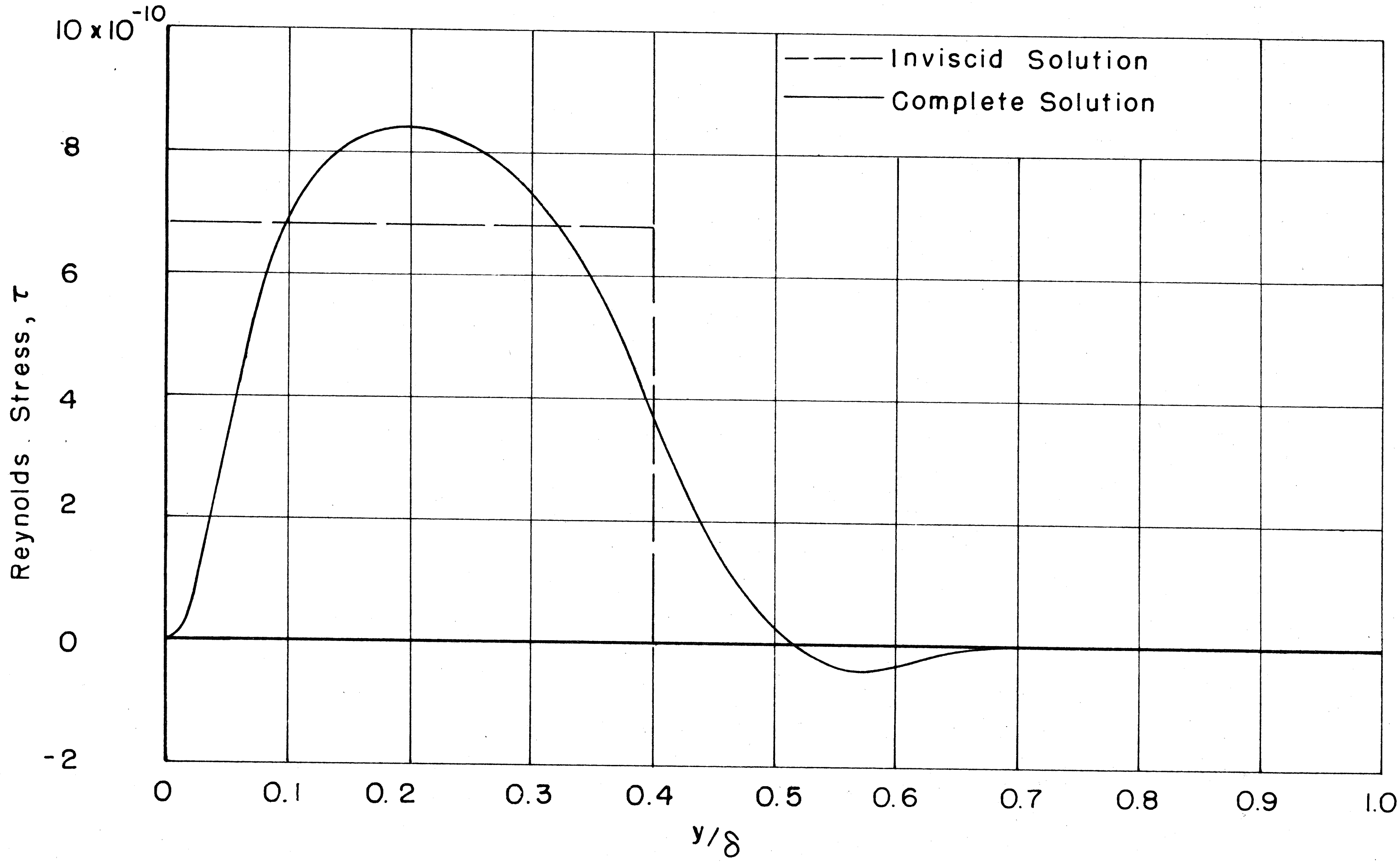

FIG. 14 – REYNOLDS STRESS DISTRIBUTION FOR NEUTRAL OSCILLATION

$M_e = 2.2$, $c = 0.616$, $Re_\theta = 535$