The Pfaffian Schur Process

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Abstract

This thesis consists of an introduction and three independent chapters.

In Chapter 2, we define the shifted Schur process as a measure on sequences of strict partitions. This process is a generalization of the shifted Schur measure introduced by Tracy–Widom and Matsumoto, and is a shifted version of the Schur process introduced by Okounkov–Reshetikhin. We prove that the shifted Schur process defines a Pfaffian point process. Furthermore, we apply this fact to compute the bulk scaling limit of the correlation functions for a measure on strict plane partitions which is an analog of the uniform measure on ordinary plane partitions. This allows us to obtain the limit shape of large strict plane partitions distributed according to this measure. The limit shape is given in terms of the Ronkin function of the polynomial $P(z,w) = -1 + z + w + zw$ and is parameterized on the domain representing half of the amoeba of this polynomial. As a byproduct, we obtain a shifted analog of famous MacMahon's formula.

In Chapter 3, we generalize the generating formula for plane partitions known as MacMahon's formula, as well as its analog for strict plane partitions. We give a 2–parameter generalization of these formulas related to Macdonald's symmetric functions. Our formula is especially simple in the Hall–Littlewood case. We also give a bijective proof of the analog of MacMahon's formula for strict plane partitions.

In Chapter 4, generating functions of plane overpartitions are obtained using various methods: nonintersecting paths, RSK type algorithms and symmetric functions. We give t-generating formulas for cylindric partitions. We also show that overpartitions correspond to domino tilings and give some basic properties of this correspondence. This is a joint work with Sylvie Corteel and Cyrille Savelief.

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Chapter 1

Introduction

1.1 Plane partitions and symmetric functions

The main object of study of this thesis is a plane partition, which is a Young diagram filled with positive integers that form nonincreasing rows and columns. A plane partition can be seen as a 3–dimensional object obtained from the graph of the height function where the heights are given by the filling numbers. The sum of all entries of a plane partition Π is denoted with $|\Pi|$; it is also the volume of Π when Π is seen as a 3-dimensional object. Diagonals of a plane partition are ordinary partitions and so each plane partition can be seen as a two-sided sequence of ordinary partitions.

5	3	2	1	1
4	3	2	1	
3	3	2		
2	2	1		

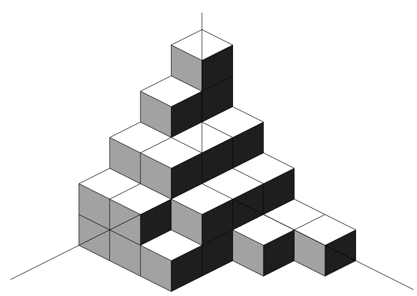

Figure 1.1: A plane partition

A (diagonally) strict plane partition is a plane partition whose diagonal partitions are strict partitions, i.e., filling numbers along diagonals are strictly decreasing. The figure above is an example of a strict plane partition. The number of border strips (connected boxes filled with the same number) of a strict plane partition Π is denoted with $k(\Pi)$.

A plane overpartition is a plane partition where in each row the last occurrence of an integer can be overlined or not and in each column the first occurrence of an integer can be overlined or not and all others are overlined. By deleting overlines one obtains a strict plane partition. There are $2^{k(\Pi)}$ different plane overpartitions corresponding to the same strict plane partition Π.

One can introduce various weights on the set of all plane partitions and study measures that they induce. The uniform measure on plane partitions is the one where the weight of a plane partition Π is proportional to $q^{|\Pi|}$. It is called uniform because all plane partitions of the same volume have the same probability. In this thesis we study a measure on the set of all strict plane partitions where the weights are proportional to $2^{k(\Pi)}q^{|\Pi|}$. This measure can also be seen as a uniform measure on plane overpartitions or, as we show, a measure on domino tilings. We study this measure using the shifted Schur process which will be introduced in Section 1.3. In our study of the measure on strict plane partitions we are particulary interested in limiting properties of this measure.

Introduction of different q weights on plane partitions automatically imposes a question of generating functions (they are inverses of the normalization constants for these measures). For example, the normalization constant of the uniform measure on plane partitions is given by the famous MacMahon generating formula. We give an analog of this formula for strict plane partitions. We further generalize this in several different directions. We describe this in more detail in Section 1.4.

In our study of measures on plane partitions we often use symmetric functions. In particular, we use Schur functions which are import for the representation theory of symmetric groups. Their analogs for projective representations of symmetric groups are called Schur P and Q functions. The Schur process and shifted Schur process, that will be introduced later, are defined using Schur, respectively Schur P and Q functions. Both standard Schur and Schur P and Q functions are special cases of a 1-parameter family of functions called Hall-Littlewood symmetric functions which are further a special case of a 2-parameter family called Macdonald functions. We utilize these functions via various specializations which are algebra homomorphisms between algebras of symmetric functions and complex numbers. The book [Mac95] is a good reference on symmetric functions.

1.2 Determinantal and Pfaffian point processes

Let X be $\mathbb{Z}^d$ or $\mathbb{R}^d$. A configuration ξ is a finite or countable subset of X such that its intersection with any bounded set is finite. Let *Conf* be the set of all configurations.

Let $B \subset X$ be a bounded Borel set and let $n \geq 0$. We define $\mathcal{B}$ to be a σ-algebra on *Conf* generated by all sets $\{\xi \in Conf : |\xi \cap B| = n\}$.

A random point process or a random point field is a probability measure on $(Conf, \mathcal{B})$. Sometimes the most convenient way to define the probability measure on $(Conf, \mathcal{B})$ is via the point correlation functions. The k-th point correlation function ρ_k is a map from X^k to $\mathbb{R}_0^+$. For $X = \mathbb{Z}^d$ the k-th point correlation function $\rho_k(x_1, x_2, \ldots, x_k)$ is a probability that a configuration ξ contains all x_is. For $X = \mathbb{R}^d$ with the underlying Lebesgue measure, the k-th point correlation function $\rho_k(x_1, x_2, \ldots, x_k)$ is a probability that a configuration ξ intersects infinitesimal volumes dx_i around x_i, for every i, divided by $dx_1 dx_2 \cdots dx_k$.

A random point process is determinantal if there exists a function K on $X \times X$ such that

$$\rho_k(x_1, x_2, \ldots, x_k) = \det(K(x_i, x_j))_{1 \leq i,j \leq k}.$$

Function K is called the correlation kernel of the determinantal random point process.

Examples of determinantal processes are many: fermion gas, Coulomb gas, Gaussian unitary ensemble and many other random matrix models, non-intersecting paths of a Markov process, Plancherel measure on partitions. The Schur process is an example of a determinantal point process.

A random point process is called Pfaffian if there exists a 2×2 skew-symmetric matrix valued function on $X \times X$

$$K(x, y) = \begin{pmatrix} K_{11}(x, y) & K_{12}(x, y) \\ K_{21}(x, y) & K_{22}(x, y) \end{pmatrix}$$

such that

$$\rho_k(x_1, x_2, \ldots, x_k) = \mathrm{Pf}(K(x_i, x_j))_{1 \leq i,j \leq k}.$$

Function K is called the correlation kernel of the Pfaffian random point process.

Examples of a Pfaffian point process are $\beta = 1$ and $\beta = 4$ polynomial ensembles of random matrices and some dimer models. We show that the shifted Schur process is a

Pfaffian point process and thus also the measure on strict plane partitions introduced in Section 1.1.

A good reference on random point processes is a book of Daley and Vere-Jones [DVJ03] and on determinantal and Pfaffian point processes articles [Sos00] and [BR05].

1.3 Schur process

The Schur process is a general recipe for assigning weights to sequences of partitions. It was introduced and studied by Okounkov and Reshetikhin in 2003, see [OR03]. Parameters of the Schur process are sequences of specializations and one obtains various measures for different choices of these specializations. The Schur process has been used for analyzing uniform measures on plane partitions, harmonic analysis of the infinite symmetric group, Szegö–type formulas for Toeplitz determinants, relative Gromov–Witten theory of $\mathbb{C}^*$, random domino tilings of the Aztec diamond, polynuclear growth processes.

Okounkov and Reshetikhin showed that the Schur process is a determinantal process. They used a Fock space formalism to give a formula for the correlation functions. They further used the formula to study asymptotics of the uniform measure on plane partitions. In particular, they obtained the limit shape of large uniformly distributed plane partitions, previously derived by Cerf and Kenyon in a different way.

By analogy with the Schur process, we introduce and study the shifted Schur process that is a measure on sequences of strict partitions. This is also a generalization of the shifted Schur measure introduced by Tracy and Widom. This process is defined using skew Schur P and Q symmetric functions. Precisely, the shifted Schur process is a measure that depends on a sequence of specializations $\rho = (\rho_0^+, \rho_1^-, \rho_1^+, \dots, \rho_T^-)$ and that to a sequence of strict plane partitions $\lambda = (\lambda^1, \lambda^2, \dots, \lambda^T)$ assigns

$$\operatorname{Prob}(\lambda) = \frac{1}{Z(\rho)} \sum_{\mu} \prod_{n=1}^{T} Q_{\lambda^n/\mu^{n-1}}(\rho_{n-1}^+) P_{\lambda^n/\mu^n}(\rho_n^-),$$

where $Z(\rho)$ is the partition function, $\mu^0 = \mu^T = \emptyset$, and the sum goes over all sequences of strict partitions $\mu = (\mu^1, \mu^2, \dots, \mu^{T-1})$. For clarification see Chapter 2.

We show that this process is a Pfaffian point process using a Fock space formalism associated with strict partitions due to Jimbo, Kashiwara and Miwa. The Pfaffian formula comes from a Wick–type lemma.

Theorem 1.3.1. *Let $X \subset \mathbb{N} \times [1, 2, \dots T]$ with $|X| = n$. The correlation function has the form*

$$\rho(X) = \mathrm{Pf}(M_X)$$

where M_X is a skew-symmetric $2n \times 2n$ matrix

$$M_X(i,j) = \begin{cases} K_{x_i,x_j}(t_i,t_j) & 1 \le i < j \le n, \\ (-1)^{x_{j'}} K_{x_i,-x'_j}(t_i,t_{j'}) & 1 \le i \le n < j \le 2n, \\ (-1)^{x_{i'}+x_{j'}} K_{-x_{i'},-x_{j'}}(t_{i'},t_{j'}) & n < i < j \le 2n, \end{cases}$$

where $i' = 2n - i + 1$ and $K_{x,y}(t_i,t_j)$ is the coefficient of $z^x w^y$ in the formal power series expansion of

$$\frac{z-w}{2(z+w)} J(z,t_i) J(w,t_j)$$

in the region $|z| > |w|$ if $t_i \ge t_j$ and $|z| < |w|$ if $t_i < t_j$.

Here $J(z,t)$ is given with

$$J(t,z) = \prod_{t \le m} F(\rho_m^-; z) \prod_{m \le t-1} F(\rho_m^+; z^{-1}),$$

where $F(x;z) = \prod_i (1 + x_i z)/(1 - x_i z)$.

A special case of the shifted Schur process is a measure on (diagonally) strict plane partitions weighted by $2^{k(\Pi)} q^{|\Pi|}$ (also a uniform measure on overpartitions). We use saddle point analysis to study the bulk scaling limit of the correlation kernel. In the limit, the Pfaffian reduces to a determinant for points away from the boundary and remains a Pfaffian on the boundary. Away from the boundary, this limiting correlation kernel is the same as the kernel for the α–β paths defined by Borodin and Shlosman, which is a more general kernel than the incomplete beta kernel defined by Okounkov and Reshetikhin. On the boundary the kernel is similar, but lacks the translation invariant property possessed by the kernel for the α–β paths. The result is stated precisely below.

A strict plane partition Π is represented by a point configuration in

$$\mathfrak{X} = \{(t,x) \in \mathbb{Z} \times \mathbb{Z} \mid x > 0\}.$$

The configuration is the set of all $(j-i,\Pi(i,j))$, where $\Pi(i,j)$ is the (i,j)th entry of Π.

Let $\gamma^+_{R,\theta}$, respectively $\gamma^-_{R,\theta}$, be the counterclockwise, respectively clockwise, arc of $|z|=R$ from $Re^{-i\theta}$ to $Re^{i\theta}$.

Theorem 1.3.2. *Let $X=\{(t_i,x_i):i=1,\dots,n\}\subset\mathfrak{X}$ be such that*

$$rt_i\to\tau,\quad rx_i\to\chi\quad \text{as } r\to+0,$$

$$t_i-t_j=\Delta t_{ij}=const,\quad x_i-x_j=\Delta x_{ij}=const.$$

a) If $\chi>0$ then

$$\lim_{r\to+0}\rho(X)=\det[K(i,j)]_{i,j=1}^n,$$

where

$$K(i,j)=\frac{1}{2\pi i}\int_{\gamma^{\pm}_{R,\theta}}\left(\frac{1-z}{1+z}\right)^{\Delta t_{ij}}\frac{1}{z^{\Delta x_{ij}+1}}dz,$$

where we choose $\gamma^+_{R,\theta}$ if $\Delta t_{ij}\geq 0$ and $\gamma^-_{R,\theta}$ otherwise, where $R=e^{-|\tau|/2}$ and

$$\theta=\begin{cases}\arccos\dfrac{(e^{|\tau|}+1)(e^{\chi}-1)}{2e^{|\tau|/2}(e^{\chi}+1)}, & \dfrac{(e^{|\tau|}+1)(e^{\chi}-1)}{2e^{|\tau|/2}(e^{\chi}+1)}\leq 1,\\ 0, & otherwise,\end{cases}$$

b) If $\chi=0$ and in addition to the above conditions we assume

$$x_i=const$$

then

$$\lim_{r\to+0}\rho(X)=\mathrm{Pf}[M(i,j)]_{i,j=1}^{2n},$$

where M is a skew symmetric matrix given by

$$M(i,j)=\begin{cases}\dfrac{(-1)^{x_j}}{2\pi i}\displaystyle\int_{\gamma^{\pm}_{R,\theta}}\left(\frac{1-z}{1+z}\right)^{\Delta t_{ij}}\frac{dz}{z^{x_i+x_j+1}} & 1\leq i<j\leq n,\\ \dfrac{1}{2\pi i}\displaystyle\int_{\gamma^{\pm}_{R,\theta}}\left(\frac{1-z}{1+z}\right)^{\Delta t_{ij'}}\frac{dz}{z^{x_i-x_{j'}+1}} & 1\leq i\leq n<j\leq 2n,\\ \dfrac{(-1)^{x_{i'}}}{2\pi i}\displaystyle\int_{\gamma^{\pm}_{R,\theta}}\left(\frac{1-z}{1+z}\right)^{\Delta t_{i'j'}}\frac{dz}{z^{-(x_{i'}+x_{j'})+1}} & n<i<j\leq 2n,\end{cases}$$

where $i' = 2n - i + 1$ *and we choose* $\gamma^{+}_{R,\theta}$ *if* $\Delta t_{ij} \geq 0$ *and* $\gamma^{-}_{R,\theta}$ *otherwise, where* $R = e^{-|\tau|/2}$ *and* $\theta = \pi/2$.

Using the 1–point correlation function, we compute the limit shape of large random strict plane partitions distributed according to this measure. The limit shape is given in terms of the Ronkin function of the polynomial $P(z, w) = -1 + z + w + zw$ and is parameterized on the domain representing half of the amoeba of this polynomial. The limit shape and the corresponding amoeba are shown below.

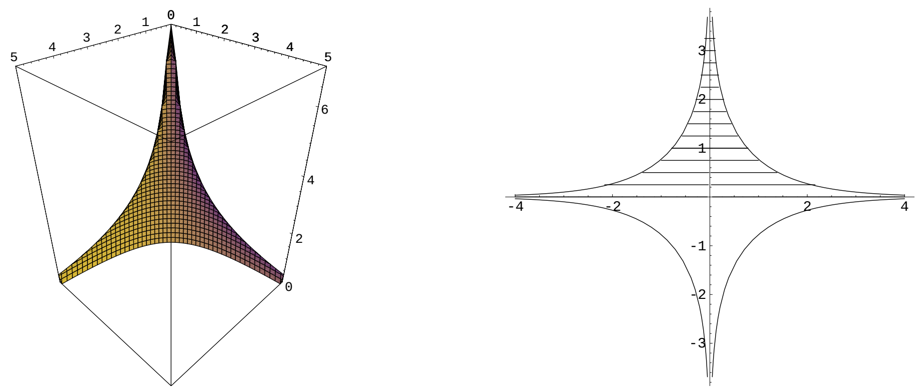

Figure 1.2: The limit shape

We show using nonintersecting paths that the measure on strict plane partitions is related to a measure on domino tilings. This relationship was expected because of similarities between correlation kernels, limit shapes as well as some other features, but the bijection is not as straightforward as in the case of plane partitions and lozenge tilings. In this case one just needs to look at the 3–dimensional diagram of a plane partition and see it as a two dimensional figure to see the lozenge tiling.

1.4 MacMahon's generating formula

MacMahon's formula from the beginning of last century is a well known generating formula for plane partitions:

$$\sum_{\substack{\Pi \text{ is a} \\ \text{plane partition}}} q^{|\Pi|} = \prod_{n=1}^{\infty} \left(\frac{1}{1-q^n} \right)^n .$$

We give its analog for plane overpartitions:

Theorem 1.4.1. *The generating function of plane overpartitions is*

$$\sum_{\substack{\Pi \text{ is a} \\ \text{plane overpartition}}} q^{|\Pi|} = \prod_{n=1}^{\infty} \left(\frac{1+q^n}{1-q^n} \right)^n .$$

We give several different proofs of this formula. This theorem was first proved in [FW07, Vul07] using Schur P and Q symmetric functions and a suitable Fock space. We later gave a bijective proof of this formula. We refer to it as the shifted MacMahon formula.

We further generalize both of these formulas to obtain

Theorem 1.4.2.

$$\sum_{\substack{\Pi \text{ is a} \\ \text{plane partition}}} A_\Pi(t) q^{|\Pi|} = \prod_{n=1}^{\infty} \left(\frac{1-tq^n}{1-q^n} \right)^n ,$$

where the weight $A_\Pi(t)$ is a polynomial in t that we describe below.

Given a plane partition Π, we decompose each connected component (a connected set of boxes filled with a same number) of its diagram into border components (i.e., rim hooks) and assign to each border component a level which is its diagonal distance to the end of the component. We associate to each border component of level i, the weight $(1-t^i)$. The weight $A_\Pi(t)$ is the product of the weights of its border components. The figure below corresponds to $A_\pi(t) = (1-t)^{10}(1-t^2)^3(1-t^3)^2$.

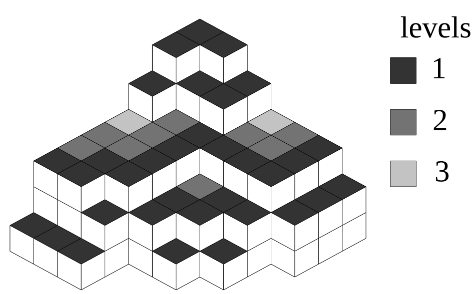

Figure 1.3: Levels

The proof of Theorem 1.4.2 is related to Hall-Littlewood symmetric functions. It can be further generalized to a 2-parameter formula related to Macdonald functions.

This formula can be naturally extended to other objects similar to plane partitions: reverse plane partitions and cylindric partitions. In short, reverse plane partitions are plane partitions where nonincreasing property is substituted with nondecreasing property and cylindric partitions are skew plane partitions on a cylinder. These extensions are given in Chapter 4.

We also generalize the shifted MacMahon formula in another direction using results of Gessel-Viennot and Stembridge on nonintersecting paths. The main results are a hook-content formula for q–enumeration of plane overpartitions of a given shape and hook formula for reverse plane overpartitions. The results are stated below.

Let λ be a partition. Let $\mathcal{S}(\lambda)$, respectively $\mathcal{S}^R(\lambda)$ be all plane overpartitions, respectively reverse plane overpartitions, of shape λ (i.e., they are supported on the Young diagram of λ). Stanley's hook-length formula says that the generating function of column strict plane partitions of shape λ is

$$q^{\sum_i i\lambda_i} \prod_{(i,j)\in\lambda} \frac{1-q^{n+c_{i,j}}}{1-q^{h_{i,j}}}, \tag{1.4.1}$$

where $h_{i,j} = \lambda_i - i + 1 + \lambda'_j - j$ is the hook length of the cell (i,j) and $c_{i,j} = j - i$ is the content of the cell (i,j).

We prove the following hook-length formula for plane overpartitions:

Theorem 1.4.3. *The generating function of plane overpartitions of shape λ is*

$$\sum_{\Pi\in\mathcal{S}(\lambda)} q^{|\Pi|} = q^{\sum_i i\lambda_i} \prod_{(i,j)\in\lambda} \frac{1+q^{c_{i,j}}}{1-q^{h_{i,j}}}. \tag{1.4.2}$$

For reverse plane partitions, it was proved by Gansner [Gan81] that the generating function of reverse plane partitions of a given shape λ is

$$\prod_{(i,j)\in\lambda} \frac{1}{1-q^{h_{i,j}}}. \tag{1.4.3}$$

We prove the following hook formula for reverse plane overpartitions:

Theorem 1.4.4. *The generating function of reverse plane overpartitions of shape λ is*

$$\sum_{\Pi\in\mathcal{S}^R(\lambda)} q^{|\Pi|} = \prod_{(i,j)\in\lambda} \frac{1+q^{h_{i,j}}}{1-q^{h_{i,j}}}.$$

1.5 Organization of thesis

This thesis consists of three separate papers. Chapter 2, which appeared in [Vul07], studies the shifted Schur process. We give proofs of Theorems 1.3.1 and 1.3.2 and find the limit shape of the measure on strict plane partitions there. We also give a symmetric function proof of Theorem 1.4.1. In Chapter 3, which appeared in [Vul09], we prove a generalized MacMahon's formula for the Macdonald and Hall-Littlewood case, namely Theorem 1.4.2. We also give a bijective proof of Theorem 1.4.1. Finally, Chapter 4 is a joint work with Sylvie Corteel and Cyrille Savelief. Parts of it are also included in Savelief's master thesis [Sav07]. In Chapter 4 we prove Theorems 1.4.3 and 1.4.4, as well as generalizations of Theorem 1.4.2 for reverse plane partitions and cylindric partitions.

Each chapter has its own introduction motivating the work and discussing relevant background.

Chapter 2

The shifted Schur process and asymptotics of large random strict plane partitions

2.1 Introduction

The basic object of this paper is the *shifted Schur process* that we introduce below.

Consider the following two figures:

5	3	2	1	1
4	3	2	1	
3	3	2		
2	2	1		

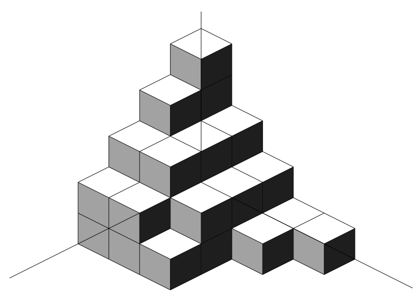

Figure 2.1: A plane partition

Both these figures represent a *plane partition*– an infinite matrix with nonnegative integer entries that form nonincreasing rows and columns with only finitely many nonzero entries. The second figure shows the plane partition as a 3-dimensional object where the height of the block positioned at (i, j) is given with the $(i, j)^{th}$ entry of our matrix.

Each diagonal: $((1, k), (2, k+1), (3, k+2), \dots)$ or $((k, 1), (k+1, 2), (k+2, 3), \dots)$ of a plane partition is an (ordinary) partition– a nonincreasing sequence of nonnegative integers with only finitely many nonzero elements. A *strict* partition is an ordinary partition with distinct positive elements. The example given above has all diagonals strict and so it is a *strict plane partition*– a plane partition whose diagonals are strict partitions.

For a plane partition π one defines the volume $|\pi|$ to be the sum of all entries of the corresponding matrix, and $A(\pi)$ to be the number of "white islands"– connected components of white rhombi of the 3-dimensional representation of π. [1] Then $2^{A(\pi)}$ is equal to the number of ways to color these islands with two colors.

For a real number q, $0 < q < 1$, we define a probability measure $\mathfrak{M}_q$ on strict plane partitions as follows. If π is a strict plane partition we set $\mathrm{Prob}(\pi)$ to be proportional to $2^{A(\pi)}q^{|\pi|}$. The normalization constant is the inverse of the partition function of these weights, which is explicitly given by

Proposition. (Shifted MacMahon's formula)

$$\sum_{\substack{\pi \text{ is a strict} \\ \text{plane partition}}} 2^{A(\pi)} q^{|\pi|} = \prod_{n=1}^{\infty} \left(\frac{1+q^n}{1-q^n} \right)^n .$$

This formula has appeared very recently in [FW07] and we were not able to locate earlier references. This is a shifted version of the famous MacMahon's formula:

$$\sum_{\substack{\pi \text{ is a plane} \\ \text{partition}}} q^{|\pi|} = \prod_{n=1}^{\infty} \left(\frac{1}{1-q^n} \right)^n .$$

A purely combinatorial proof of the shifted MacMahon's formula will appear in a subsequent publication.

The measure described above is a special case of the *shifted Schur process*. This is a measure on sequences of strict (ordinary) partitions. The idea came from an analogy with the *Schur process* introduced in [OR03] that is a measure on sequences of (ordinary) partitions. The Schur process is a generalization of the *Schur measure* on partitions introduced earlier in [Oko01]. The shifted Schur process we define is the generalization of the *shifted Schur measure* that was introduced and studied in [TW04] and [Mat05].

The Schur process and the Schur measure have been extensively studied in recent years and they have various applications, see e.g., [Bor07], [BO00a], [BO00b], [BR05], [IS07], [Joh03], [Joh05], [OR07], [PS02].

In this chapter we define the shifted Schur process and we derive the formulas for its

[1] For the given example $|\pi| = 35$ and $A(\pi) = 7$.

correlation functions in terms of Pfaffians of a correlation kernel. We show that $\mathfrak{M}_q$ described above can be seen as a special shifted Schur process. This allows us to compute the correlation functions for this special case and further to obtain their bulk scaling limit when $q \to 1$ (partitions become large).

The shifted Schur process is defined using *skew Schur P* and *Q functions*. These are symmetric functions that appear in the theory of projective representations of the symmetric groups.

As all other symmetric functions, Schur $P_\lambda(x_1, x_2, \dots)$ function, where λ is a strict partition, is defined by a sequence of polynomials $P_\lambda(x_1, x_2, \dots x_n)$, $n \in \mathbb{N}$, with a property that $P_\lambda(x_1, x_2, \dots x_n, 0) = P_\lambda(x_1, x_2, \dots x_n)$.

For a partition λ one defines the length $l(\lambda)$ to be the number of nonzero elements. Let $\lambda = (\lambda_1, \lambda_2, \dots)$ be a strict partition and $n \geq l$, where l is the length of λ. Then

$$P_\lambda(x_1, \dots, x_n) = \sum_{w \in S_n / S_{n-l}} w\Big(x_1^{\lambda_1} \cdots x_l^{\lambda_l} \prod_{i=1}^{l} \prod_{j>i} \frac{x_i + x_j}{x_i - x_j}\Big),$$

where S_{n-l} acts on $x_{l+1}, \dots, x_n$. Schur Q_λ function is defined as $2^l P_\lambda$.

Both $\{P_\lambda : \lambda \text{ strict}\}$ and $\{Q_\lambda : \lambda \text{ strict}\}$ are bases for $\mathbb{Q}[p_1, p_3, p_5, \dots]$, where $p_r = \sum x_i^r$ is the r^{th} power sum. A scalar product in $\mathbb{Q}[p_1, p_3, p_5, \dots]$ is given with $\langle P_\lambda, Q_\mu \rangle = \delta_{\lambda,\mu}$.

For strict partitions $\lambda = (\lambda_1, \lambda_2, \dots)$ and $\mu = (\mu_1, \mu_2, \dots)$, skew Schur functions are defined by

$$P_{\lambda/\mu} = \begin{cases} \sum_\nu \langle P_\lambda, Q_\lambda Q_\nu \rangle P_\nu & \lambda \supset \mu, \\ 0 & \text{otherwise}, \end{cases} \qquad \text{and} \qquad Q_{\lambda/\mu} = 2^{l(\lambda) - l(\mu)} P_{\lambda/\mu},$$

where $\lambda \supset \mu$ if $\lambda_i \geq \mu_i$ for every i. Note that $P_{\lambda/\emptyset} = P_\lambda$ and $Q_{\lambda/\emptyset} = Q_\lambda$.

This is just one of many ways to define skew Schur P and Q functions (see Chapter 3 of [Mac95]). We give another definition in the paper that is more convenient for us.

We use Λ to denote the algebra of symmetric functions. A *specialization* of the algebra of symmetric functions is an algebra homomorphism $\Lambda \to \mathbb{C}$. If ρ is a specialization and $f \in \Lambda$ then we use $f(\rho)$ to denote the image of f under ρ.

Let $\rho = (\rho_0^+, \rho_1^-, \rho_1^+, \dots, \rho_T^-)$ be a finite sequence of specializations. For two sequences of strict partitions $\lambda = (\lambda^1, \lambda^2, \dots, \lambda^T)$ and $\mu = (\mu^1, \mu^2, \dots, \mu^{T-1})$ we define

$$W(\lambda, \mu) = Q_{\lambda^1}(\rho_0^+) P_{\lambda^1/\mu^1}(\rho_1^-) Q_{\lambda^2/\mu^1}(\rho_1^+) \dots Q_{\lambda^T/\mu^{T-1}}(\rho_{T-1}^+) P_{\lambda^T}(\rho_T^-).$$

Then $W(\lambda, \mu) = 0$ unless

$$\emptyset \subset \lambda^1 \supset \mu^1 \subset \lambda^2 \supset \mu^2 \subset \dots \supset \mu^{T-1} \subset \lambda^T \supset \emptyset.$$

The shifted Schur process is a measure that to a sequence of strict plane partitions $\lambda = (\lambda^1, \lambda^2, \dots, \lambda^T)$ assigns

$$\mathrm{Prob}(\lambda) = \frac{1}{Z(\rho)} \sum_{\mu} W(\lambda, \mu),$$

where $Z(\rho)$ is the partition function, and the sum goes over all sequences of strict partitions $\mu = (\mu^1, \mu^2, \dots, \mu^{T-1})$.

Let $X = \{(x_i, t_i) : i = 1, \dots, n\} \subset \mathbb{N} \times [1, 2, \dots, T]$ and let $\lambda = (\lambda^1, \lambda^2, \dots, \lambda^T)$ be a sequence of strict partitions. We say that $X \subset \lambda$ if x_i is a part (nonzero element) of the partition λ^{t_i} for every $i = 1, \dots, n$. Define the *correlation function* of the shifted Schur process by

$$\rho(X) = \mathrm{Prob}(X \subset \lambda).$$

The first main result of this chapter is that the shifted Schur process is a Pfaffian process, i.e., its correlation functions can be expressed as Pfaffians of a certain kernel.

Theorem A. Let $X \subset \mathbb{N} \times [1, 2, \dots T]$ with $|X| = n$. The correlation function has the form

$$\rho(X) = \mathrm{Pf}(M_X)$$

where M_X is a skew-symmetric $2n \times 2n$ matrix

$$M_X(i,j) = \begin{cases} K_{x_i, x_j}(t_i, t_j) & 1 \le i < j \le n, \\ (-1)^{x_{j'}} K_{x_i, -x'_j}(t_i, t_{j'}) & 1 \le i \le n < j \le 2n, \\ (-1)^{x_{i'} + x_{j'}} K_{-x_{i'}, -x_{j'}}(t_{i'}, t_{j'}) & n < i < j \le 2n, \end{cases}$$

where $i' = 2n - i + 1$ and $K_{x,y}(t_i, t_j)$ is the coefficient of $z^x w^y$ in the formal power series expansion of

$$\frac{z-w}{2(z+w)} J(z, t_i) J(w, t_j)$$

in the region $|z| > |w|$ if $t_i \geq t_j$ and $|z| < |w|$ if $t_i < t_j$.

Here $J(z, t)$ is given with

$$J(t, z) = \prod_{t \leq m} F(\rho_m^-; z) \prod_{m \leq t-1} F(\rho_m^+; z^{-1}),$$

where $F(x; z) = \prod_i (1 + x_i z)/(1 - x_i z)$.

Our approach is similar to that of [OR03]. It relies on two tools. One is the Fock space associated to strict plane partitions and the other one is a Wick type formula that yields a Pfaffian.

Theorem A can be used to obtain the correlation functions for $\mathfrak{M}_q$ that we introduced above. In that case

$$J_q(t, z) = \begin{cases} \dfrac{(q^{1/2} z^{-1}; q)_\infty (-q^{t+1/2} z; q)_\infty}{(-q^{1/2} z^{-1}; q)_\infty (q^{t+1/2} z; q)_\infty} & t \geq 0, \\ \\ \dfrac{(-q^{1/2} z; q)_\infty (q^{-t+1/2} z^{-1}; q)_\infty}{(q^{1/2} z; q)_\infty (-q^{-t+1/2} z^{-1}; q)_\infty} & t < 0, \end{cases}$$

where

$$(z; q)_\infty = \prod_{n=0}^{\infty} (1 - q^n z)$$

is the quantum dilogarithm function. We use the Pfaffian formula to study the bulk scaling limit of the correlation functions when $q \to 1$. We scale the coordinates of strict plane partitions by $r = \log q$. This scaling assures that the scaled volume of strict plane partitions tends to a constant.[2]

Before giving the statement, we need to say that a strict plane partition is uniquely represented by a point configuration (subset) in

$$\mathfrak{X} = \{(t, x) \in \mathbb{Z} \times \mathbb{Z} \mid x > 0\}$$

[2]The constant is equal to $7\zeta(3)/2$.

as follows: $(i,j,\pi(i,j)) \to (j-i,\pi(i,j))$, where $\pi(i,j)$ is the $(i,j)^{th}$ entry of π.

Let $\gamma^+_{R,\theta}$, respectively $\gamma^-_{R,\theta}$, be the counterclockwise, respectively clockwise, arc of $|z|=R$ from $Re^{-i\theta}$ to $Re^{i\theta}$.

Theorem B. Let $X=\{(t_i,x_i): i=1,\dots,n\}\subset\mathfrak{X}$ be such that

$$rt_i\to\tau,\quad rx_i\to\chi\quad\text{as } r\to+0,$$

$$t_i-t_j=\Delta t_{ij}=\text{const},\quad x_i-x_j=\Delta x_{ij}=\text{const}.$$

a) If $\chi>0$ then

$$\lim_{r\to+0}\rho(X)=\det[K(i,j)]_{i,j=1}^n,$$

where

$$K(i,j)=\frac{1}{2\pi i}\int_{\gamma^{\pm}_{R,\theta}}\left(\frac{1-z}{1+z}\right)^{\Delta t_{ij}}\frac{1}{z^{\Delta x_{ij}+1}}dz,$$

where we choose $\gamma^+_{R,\theta}$ if $\Delta t_{ij}\ge 0$ and $\gamma^-_{R,\theta}$ otherwise, where $R=e^{-|\tau|/2}$ and

$$\theta=\begin{cases}\arccos\dfrac{(e^{|\tau|}+1)(e^{\chi}-1)}{2e^{|\tau|/2}(e^{\chi}+1)}, & \dfrac{(e^{|\tau|}+1)(e^{\chi}-1)}{2e^{|\tau|/2}(e^{\chi}+1)}\le 1,\\ 0, & \text{otherwise},\end{cases}$$

b) If $\chi=0$ and in addition to the above conditions we assume

$$x_i=\text{const}$$

then

$$\lim_{r\to+0}\rho(X)=\text{Pf}[M(i,j)]_{i,j=1}^{2n},$$

where M is a skew symmetric matrix given by

$$M(i,j)=\begin{cases}\dfrac{(-1)^{x_j}}{2\pi i}\displaystyle\int_{\gamma^{\pm}_{R,\theta}}\left(\frac{1-z}{1+z}\right)^{\Delta t_{ij}}\frac{dz}{z^{x_i+x_j+1}} & 1\le i<j\le n,\\ \dfrac{1}{2\pi i}\displaystyle\int_{\gamma^{\pm}_{R,\theta}}\left(\frac{1-z}{1+z}\right)^{\Delta t_{ij'}}\frac{dz}{z^{x_i-x_{j'}+1}} & 1\le i\le n<j\le 2n,\\ \dfrac{(-1)^{x_{i'}}}{2\pi i}\displaystyle\int_{\gamma^{\pm}_{R,\theta}}\left(\frac{1-z}{1+z}\right)^{\Delta t_{i'j'}}\frac{dz}{z^{-(x_{i'}+x_{j'})+1}} & n<i<j\le 2n,\end{cases}$$

where $i' = 2n - i + 1$ and we choose $\gamma^+_{R,\theta}$ if $\Delta t_{ij} \geq 0$ and $\gamma^-_{R,\theta}$ otherwise, where $R = e^{-|\tau|/2}$ and $\theta = \pi/2$.

For the equal time configuration (points on the same vertical line) we get the discrete sine kernel and thus, the kernel of the theorem is an extension of the sine kernel. This extension has appeared in [Bor07], but there it did not come from a "physical" problem.

The theorem above allows us to obtain the limit shape of large strict plane partitions distributed according to $\mathfrak{M}_q$, but we do not prove its existence. The limit shape is parameterized on the domain representing a half of the amoeba[3] of the polynomial $P(z,w) = -1+z+w+zw$ (see Section 4 for details).

In the proof of Theorem B we use the saddle point analysis. We deform contours of integral that defines elements of the correlation kernel in a such a way that it splits into an integral that vanishes when $q \to 1$ and another nonvanishing integral that comes as a residue.

The paper is organized as follows. In Section 2 we introduce the shifted Schur process and compute its correlation function (Theorem A). In Section 3 we introduce the strict plane partitions and the measure $\mathfrak{M}_q$. We prove the shifted MacMahon's formula and use Theorem A to obtain the correlation functions for this measure. In Section 4 we compute the scaling limit (Theorem B). Section 5 is an appendix where we give a summary of definitions we use and explain the Fock space formalism associated to strict plane partitions.

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2.2 The shifted Schur process

2.2.1 The measure

Recall that a nonincreasing sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of nonnegative integers with a finite number of parts (nonzero elements) is called a partition. A partition is called strict if all

[3] The amoeba of a polynomial $P(z,w)$ is

$$\{(\xi,\omega) = (\log|z|, \log|w|) \in \mathbb{R}^2 \mid (z,w) \in (\mathbb{C}\backslash\{0\})^2,\ P(z,w) = 0\}.$$

parts are distinct. More information on strict partitions can be found in [Mac95], [Mat05]. Some results from these references that we are going to use are summarized in Appendix to this chapter.

The shifted Schur process is a measure on a space of sequences of strict partitions. This measure depends on a finite sequence $\rho = (\rho_0^+, \rho_1^-, \rho_1^+, \dots, \rho_T^-)$ of specializations of the algebra Λ of symmetric functions[4].

Let $\lambda = (\lambda^1, \lambda^2, \dots, \lambda^T)$ and $\mu = (\mu^1, \mu^2, \dots, \mu^{T-1})$ be two sequences of strict partitions. Set

$$W(\lambda, \mu) = Q_{\lambda^1}(\rho_0^+) P_{\lambda^1/\mu^1}(\rho_1^-) Q_{\lambda^2/\mu^1}(\rho_1^+) \dots Q_{\lambda^T/\mu^{T-1}}(\rho_{T-1}^+) P_{\lambda^T}(\rho_T^-).$$

Here $P_{\lambda/\mu}(\rho)$'s and $Q_{\lambda/\mu}(\rho)$'s denote the skew Schur P and Q-functions, see Appendix.

Note that $W(\lambda, \mu) = 0$ unless

$$\emptyset \subset \lambda^1 \supset \mu^1 \subset \lambda^2 \supset \mu^2 \subset \ldots \supset \mu^{T-1} \subset \lambda^T \supset \emptyset.$$

Proposition 2.2.1. *The sum of the weights $W(\lambda, \mu)$ over all sequences of strict partitions $\lambda = (\lambda^1, \lambda^2, \dots, \lambda^T)$ and $\mu = (\mu^1, \mu^2, \dots, \mu^{T-1})$ is equal to*

$$Z(\rho) = \prod_{0 \le i < j \le T} H(\rho_i^+, \rho_j^-), \tag{2.2.1}$$

where

$$H(\rho_i^+, \rho_j^-) = \sum_{\lambda \text{ strict}} Q_\lambda(\rho_i^+) P_\lambda(\rho_j^-).$$

We give two proofs of this statement.

Proof. **1.** From (2.5.16) and (2.5.17) it follows that

$$Z(\rho) = \sum_{\lambda, \mu} W(\lambda, \mu) = \langle \Gamma_+(\rho_T^-) \Gamma_-(\rho_{T-1}^+) \cdots \Gamma_-(\rho_1^+) \Gamma_+(\rho_1^-) \Gamma_-(\rho_0^+) v_\emptyset, v_\emptyset \rangle.$$

We can move all Γ_+'s to the right and all Γ_-'s to the left using (2.5.14). We obtain

$$Z(\rho) = \prod_{0 \le i < j \le T} H(\rho_i^+, \rho_j^-) \langle \Gamma_-(\rho_{T-1}^+) \cdots \Gamma_-(\rho_0^+) \Gamma_+(\rho_T^-) \cdots \Gamma_+(\rho_1^-) v_\emptyset, v_\emptyset \rangle.$$

---

[4] A specialization of Λ is an algebra homomorphism $\Lambda \to \mathbb{C}$.

Then (2.5.13) implies (2.2.1). □

Proof. **2.** We use Proposition 2.5.1. The idea is the same as in Proposition 2.1 of [BR05]. The proof goes by induction on T.

Using the formula from Proposition 2.5.1 we substitute sums over λ^i's with sums over τ^{i-1}'s. This gives

$$\prod_{i=0}^{T-1} H(\rho_i^+, \rho_{i+1}^-) \sum_{\mu,\tau} Q_{\mu^1}(\rho_0^+) P_{\mu^1/\tau^1}(\rho_2^-) Q_{\mu^2/\tau^1}(\rho_1^+) \dots P_{\mu^{T-1}}(\rho_T^-).$$

This is the sum of $W(\mu,\tau)$ with $\mu = (\mu^1, \dots, \mu^{T-1})$ and $\tau = (\tau^1, \dots, \tau^{T-2})$. Inductively, we obtain (2.2.1). □

Definition. The shifted Schur process is a measure on the space of finite sequences that to $\lambda = (\lambda^1, \lambda^2, \dots, \lambda^T)$ assigns

$$\text{Prob}(\lambda) = \frac{1}{Z(\rho)} \sum_{\mu} W(\lambda, \mu),$$

where the sum goes over all $\mu = (\mu^1, \mu^2, \dots, \mu^{T-1})$.

2.2.2 Correlation functions

Our aim is to compute the correlation functions for the shifted Schur process.

Let $X = \{(x_i, t_i) : i = 1, \dots, n\} \subset \mathbb{N} \times [1, 2, \dots, T]$ and let $\lambda = (\lambda^1, \lambda^2, \dots, \lambda^T)$ be a sequence of strict partitions. We will say that $X \subset \lambda$ if x_i is a part of the partition λ^{t_i} for every $i = 1, \dots, n$.

Definition. Let $X \subset \mathbb{N} \times [1, 2, \dots T]$. Then the correlation function of the shifted Schur process corresponding to X is

$$\rho(X) = \text{Prob}(X \subset \lambda).$$

We are going to show that the shifted Schur process is a Pfaffian process in the sense of [BR05], that is, its correlation functions are Pfaffians of submatrices of a fixed matrix called the correlation kernel. This is stated in the following theorem.

Theorem 2.2.2. *Let $X \subset \mathbb{N} \times [1, 2, \dots T]$ with $|X| = n$. The correlation function is given with*

$$\rho(X) = \mathrm{Pf}(M_X)$$

where M_X is a skew-symmetric $2n \times 2n$ matrix

$$M_X(i,j) = \begin{cases} K_{x_i,x_j}(t_i,t_j) & 1 \le i < j \le n, \\ (-1)^{x_{j'}} K_{x_i,-x'_j}(t_i,t_{j'}) & 1 \le i \le n < j \le 2n, \\ (-1)^{x_{i'}+x_{j'}} K_{-x_{i'},-x_{j'}}(t_{i'},t_{j'}) & n < i < j \le 2n, \end{cases}$$

where $i' = 2n - i + 1$, $j' = 2n - j + 1$ and $K_{x,y}(t_i, t_j)$ is the coefficient of $z^x w^y$ in the formal power series expansion of

$$K((z,t_i),(w,t_j)) := \sum_{x,y \in \mathbb{Z}} K_{x,y}(t_i,t_j) z^x w^y = \frac{z-w}{2(z+w)} J(z,t_i) J(w,t_j)$$

in the region $|z| > |w|$ if $t_i \ge t_j$ and $|z| < |w|$ if $t_i < t_j$.

Here $J(z,t)$ is given with

$$J(t,z) = \prod_{t \le m} F(\rho_m^-; z) \prod_{m \le t-1} F(\rho_m^+; z^{-1}), \tag{2.2.2}$$

where F is defined with (2.5.3).

Proof. The proof consists of two parts. In the first part we express the correlation function via the operators Γ_+ and Γ_- (see Appendix). In the second part we use a Wick type formula to obtain the Pfaffian.

Let $\rho_T^+ = \rho_0^- = 0$ and let $t_0 = 0$.

First, we assume $1 \le t_1 \le t_2 \le \dots \le t_n \le T$. Using formulas (2.5.16), (2.5.17) and (2.5.11) we get that the correlation function is

$$\frac{1}{Z(\rho)} \left\langle \prod_{m=t_n}^{T} \Gamma_-(\rho_m^+)\Gamma_+(\rho_m^-) \prod_{i=1}^{n} \left(2\psi_{x_i}\psi_{x_i}^* \prod_{m=t_{i-1}}^{t_i-1} \Gamma_-(\rho_m^+)\Gamma_+(\rho_m^-) \right) v_\emptyset, v_\emptyset \right\rangle,$$

where the products of the operators should be read from right to left in an increasing time order.

Thus, the correlation function is equal to $\prod_{i=1}^{n}(-1)^{x_i}$ times the coefficient of $\prod_{i=1}^{n} u_i^{x_i} v_i{}^{-x_i}$ in the formal power series

$$\frac{1}{Z(\rho)}\Big\langle \prod_{m=t_n}^{T} \Gamma_-(\rho_m^+)\Gamma_+(\rho_m^-) \prod_{i=1}^{n} \Big(2\psi(u_i)\psi(v_i) \prod_{m=t_{i-1}}^{t_i-1} \Gamma_-(\rho_m^+)\Gamma_+(\rho_m^-)\Big) v_\emptyset, v_\emptyset \Big\rangle,$$

where ψ is given by (2.5.12).

We use formulas (2.5.14) and (2.5.15) to put all Γ_-'s on the left and Γ_+'s on the right side of $\prod_{i=1}^{n} 2\psi(u_i)\psi(v_i)$. Since $\Gamma_+ v_\emptyset = v_\emptyset$, see (2.5.13), we obtain

$$\prod_{i=1}^{n}(J(t_i,u_i)J(t_i,v_i))\Big\langle \prod_{i=1}^{n} 2\psi(u_i)\psi(v_i)v_\emptyset, v_\emptyset\Big\rangle. \tag{2.2.3}$$

Thus, if $1 \le t_1 \le t_2 \le \cdots \le t_n \le T$ then the correlation function $\rho(X)$ is equal to $\prod_{i=1}^{n}(-1)^{x_i}$ times the coefficient of $\prod_{i=1}^{n} u_i^{x_i} v_i{}^{-x_i}$ in the formal power series (2.2.3).

Now, more generally, let $\pi \in S_n$ such that $1 \le t_{\pi(1)} \le t_{\pi(2)} \le \cdots \le t_{\pi(n)} \le T$, then the correlation function $\rho(X)$ is equal to $\prod_{i=1}^{n}(-1)^{x_i}$ times the coefficient of $\prod_{i=1}^{n} u_i^{x_i} v_i{}^{-x_i}$ in the formal power series

$$\prod_{i=1}^{n}(J(t_i,u_i)J(t_i,v_i))\Big\langle \prod_{i=1}^{n} 2\psi(u_{\pi(i)})\psi(v_{\pi(i)})v_\emptyset, v_\emptyset\Big\rangle. \tag{2.2.4}$$

The above inner product is computed in Lemma 2.2.4 and Lemma 2.2.5, that will be proved below. Then by Lemma 2.2.5 we obtain that in the region $|u_{\pi(n)}| > |v_{\pi(n)}| > \cdots > |u_{\pi(1)}| > |v_{\pi(1)}|$ expression (2.2.4) is equal to $\mathrm{Pf}(A)$ with

$$A(i,j) = \begin{cases} K((u_i,t_i),(u_j,t_j)) & 1 \le i < j \le n, \\ K((u_i,t_i),(v_{j}',t_{j}')) & 1 \le i \le n < j \le 2n, \\ K((v_{i}',t_{i}'),(v_{j}',t_{j}')) & n < i < j \le 2n, \end{cases}$$

where $i' = 2n - i + 1$, $j' = 2n - j + 1$ and K is as above.

Let $(y_1,\ldots,y_{2n}) = (x_1,\ldots,x_n,-x_n,\ldots,-x_1)$ and $(z_1,\ldots,z_{2n}) = (u_1,\ldots,u_n,v_n,\ldots,v_1)$. By definition of the Pfaffian,

$$\mathrm{Pf}(M_X) = \prod_{i=1}^{n}(-1)^{x_i} \sum_{\alpha} \mathrm{sgn}(\alpha) \prod_{s} K_{y_{i_s},y_{j_s}},$$

where the sum is taken over all permutations $\alpha = \begin{pmatrix} 1 & 2 & \cdots & 2n-1 & 2n \\ i_1 & j_1 & \cdots & i_n & j_n \end{pmatrix}$, such that $i_1 < \cdots < i_n$ and $i_s < j_s$ for every s.

Also,

$$\begin{aligned} \mathrm{Pf}(A) &= \sum_{\alpha} \mathrm{sgn}(\alpha) \prod_{s} K(z_{i_s}, z_{j_s}) \\ &= \sum_{y_1,\ldots,y_{2n}\in\mathbb{Z}} \sum_{\alpha} \mathrm{sgn}(\alpha) \prod_{s} K_{y_{i_s},y_{j_s}} \prod_{i=1}^{2n} z_i^{y_i}. \end{aligned}$$

Finally, since

$$\rho(X) = \prod_{i=1}^{n} (-1)^{x_i} [\prod_{i=1}^{n} u_i^{x_i} v_i{}^{-x_i} : \mathrm{Pf}(A)]$$

we get

$$\rho(X) = \mathrm{Pf}(M_X).$$

□

Lemma 2.2.3.

$$\langle \psi_{k_{2n}} \cdots \psi_{k_1} v_\emptyset, v_\emptyset \rangle = \sum_{i=2}^{2n} (-1)^i \langle \psi_{k_{2n}} \cdots \hat{\psi}_{k_i} \cdots \psi_{k_2} v_\emptyset, v_\emptyset \rangle \langle \psi_{k_i} \psi_{k_1} v_\emptyset, v_\emptyset \rangle \tag{2.2.5}$$

Equivalently,

$$\langle \psi_{k_{2n}} \cdots \psi_{k_1} v_\emptyset, v_\emptyset \rangle = \mathrm{Pf}[A_{ij}]_{i,j=1}^{2n},$$

where A is a skew symmetric matrix with $A_{ij} = \langle \psi_{k_j} \psi_{k_i} v_\emptyset, v_\emptyset \rangle$ for $i < j$.

Proof. Throughout the proof we use $\langle \cdots v_\emptyset, v_\emptyset \rangle = \langle \cdots \rangle$ to shorten the notation.

We can simplify the proof if we use the fact that the operators ψ_k and ψ_k^* add, respectively remove e_k, but the proof we are going to give will apply to a more general case, namely, we consider any ψ_k's that satisfy

$$\psi_k v_\emptyset = 0, \quad k < 0, \tag{2.2.6}$$

$$\psi_0 v_\emptyset = a_0 v_\emptyset, \tag{2.2.7}$$

$$\psi_k^* = b_k \psi_{-k}, \tag{2.2.8}$$

$$\psi_k \psi_l + \psi_l \psi_k = c_k \delta_{k,-l}, \tag{2.2.9}$$

for some constants a_0, b_k and c_k. These properties are satisfied for ψ_k we are using (see Appendix).

The properties (2.2.6)-(2.2.9) immediately imply that

$$\psi_k \psi_k = 0, \quad k \neq 0 \tag{2.2.10}$$

and

$$\langle \psi_k \psi_l \rangle = 0, \quad k \neq -l \tag{2.2.11}$$

because for $k \neq -l$

$$\langle \psi_k \psi_l \rangle = \begin{cases} \langle \psi_k \psi_l v_\emptyset, v_\emptyset \rangle & l < 0, \\ -b_l \langle \psi_k v_\emptyset, \psi_{-l} v_\emptyset \rangle & l > 0, \\ a_0 \langle \psi_k v_\emptyset, v_\emptyset \rangle & l = 0. \end{cases}$$

Observe that if $k_1, \ldots, k_{2n-1}$ are all different from 0 then

$$\langle \psi_{k_{2n-1}} \cdots \psi_{k_1} \rangle = 0. \tag{2.2.12}$$

This is true because there is $k > 0$ such that $(k_1, \ldots, k_{2n-1})$ has odd number of elements whose absolute value is k. Let those be $k_{j_1}, \ldots, k_{j_{2s-1}}$. If there exist k_{j_i} and $k_{j_{i+1}}$ both equal either to k or $-k$ then $\psi_{k_{2n-1}} \cdots \psi_{k_1} = 0$ by (2.2.9) and (2.2.10). Otherwise, $(k_{j_1}, \ldots, k_{j_{2s-1}})$ is either $(k, -k, \ldots, -k, k)$ or $(-k, k, \ldots, k, -k)$. In the first case we move ψ_k to the left and use (2.2.6) and (2.2.8) and in the second we move ψ_{-k} to the right and use (2.2.6).

Also, observe that if $(k_n, \ldots, k_1)$ contains t zeros and if those appear on places $j_t, \ldots, j_1$ then

$$\langle \psi_{k_n} \cdots \psi_{k_1} \rangle = a_0^t \langle \psi_{k_n} \cdots \hat{\psi}_{k_{j_t}} \cdots \hat{\psi}_{k_{j_1}} \cdots \psi_{k_1} \rangle (-1)^{\sum_{m=1}^t j_m - m}. \tag{2.2.13}$$

This follows from (2.2.7) and (2.2.9) by moving $\psi_{k_{j_m}}$ to the m^{th} place from the right.

Now, we proceed to the proof of the lemma. We use induction on n. We need to show that $L(k_{2n}, \ldots, k_1) = R(k_{2n}, \ldots, k_1)$ where L and R are the left, respectively right hand side of (2.2.5). Obviously it is true for $n = 1$. We show it is true for n.

If $k_1 < 0$ then $L = R = 0$ by (2.2.6).

If $k_1 = 0$ then let t be the number of zeros in $k_{2n}, \ldots, k_1$ and let those appear on places

$j_t, \ldots, j_1 = 1$. Then

$$L = a_0^t \langle \psi_{k_{2n}} \cdots \hat{\psi}_{k_{j_t}} \cdots \hat{\psi}_{k_{j_1}} \cdots \psi_{k_1} \rangle \prod_{m=1}^{t} (-1)^{j_m - m},$$

while

$$\begin{aligned} R &= \sum_{i=2}^{t} (-1)^{j_i} \langle \psi_{k_{2n}} \cdots \hat{\psi}_{k_{j_i}} \cdots \psi_{k_2} \rangle \langle \psi_{k_{j_i}} \psi_{k_1} \rangle \\ &= \sum_{i=2}^{t} (-1)^{j_i} a_0^t \langle \psi_{k_{2n}} \cdots \hat{\psi}_{k_{j_t}} \cdots \hat{\psi}_{k_{j_1}} \cdots \psi_{k_1} \rangle \prod_{\substack{m=2 \\ m \neq i}}^{t} (-1)^{j_m - m} \\ &= a_0^t \langle \psi_{k_{2n}} \cdots \hat{\psi}_{k_{j_t}} \cdots \hat{\psi}_{k_{j_1}} \cdots \psi_{k_1} \rangle \prod_{m=1}^{t} (-1)^{j_m - m} \sum_{i=2}^{t} (-1)^i. \end{aligned}$$

If t is even then $L = R$. If t is odd then $2n - t$ is odd then $\langle \psi_{k_{2n}} \cdots \hat{\psi}_{k_{j_t}} \cdots \hat{\psi}_{k_{j_1}} \cdots \psi_{k_1} \rangle = 0$ by (2.2.12) and thus $L = R = 0$.

Finally, we can assume $k_1 > 0$. If $k_i \neq -k_1$ for every $i \in \{2, \ldots, n\}$ then

$$L(k_{2n}, \ldots, k_1) = R(k_{2n}, \ldots, k_1) = 0.$$

This is true because $R = 0$ by (2.2.11) and $L = 0$ since it is possible to move ψ_{k_1} to the left and then use (2.2.6) and (2.2.8).

So, we need to show that $L = R$ if $k_1 > 0$ and there exists $i \in \{2, \ldots, n\}$ such that $k_i = -k_1$.

First, we assume $k_2 = -k_1$. Then

$$L = \langle \psi_{k_{2n}} \cdots \psi_{k_3} \rangle \langle \psi_{k_2} \psi_{k_1} \rangle,$$

because $\psi_{-k} \psi_k v_\emptyset = \langle \psi_{-k} \psi_k \rangle v_\emptyset$. On the other hand

$$R = \langle \psi_{k_{2n}} \cdots \psi_{k_3} \rangle \langle \psi_{k_2} \psi_{k_1} \rangle,$$

because for every $i > 2$ we have that $\langle \psi_{k_{2n}} \cdots \hat{\psi}_{k_i} \cdots \psi_{k_2} \rangle = 0$ by (2.2.6). Thus, $L = R$.

To conclude the proof we show that

$$\begin{array}{rcl} L(k_{2n},\dots,k_s,k_{s-1},\dots,k_1) &=& R(k_{2n},\dots,k_s,k_{s-1},\dots,k_1) \\ && \Downarrow \\ L(k_{2n},\dots,k_{s-1},k_s,\dots,k_1) &=& R(k_{2n},\dots,k_{s-1},k_s,\dots,k_1). \end{array} \tag{2.2.14}$$

Then we use this together with the fact that there is i such that $k_i = -k_1$ and that $L = R$ if $i = 2$ to prove the induction step.

$$\begin{aligned} &L(k_{2n},\dots,k_{s-1},k_s,\dots,k_1) = \langle \psi_{k_{2n}} \cdots \psi_{k_{s-1}} \psi_{k_s} \cdots \psi_{k_1} \rangle = \\ &= -\langle \psi_{k_{2n}} \cdots \psi_{k_s} \psi_{k_{s-1}} \cdots \psi_{k_1} \rangle + c_{k_s} \delta_{k_s,-k_{s-1}} \langle \psi_{k_{2n}} \cdots \hat{\psi}_{k_s} \hat{\psi}_{k_{s-1}} \cdots \psi_{k_1} \rangle \\ &= -L(k_{2n},\dots,k_s,k_{s-1},\dots,k_1) + c_{k_s} \delta_{k_s,-k_{s-1}} \langle \psi_{k_{2n}} \cdots \hat{\psi}_{k_s} \hat{\psi}_{k_{s-1}} \cdots \psi_{k_1} \rangle \end{aligned}$$

$$\begin{aligned} & R(k_{2n},\dots,k_{s-1},k_s,\dots,k_1) = \\ = & \sum_{\substack{i=2 \\ i \neq s,s-1}}^{2n} (-1)^i \left[-\langle \psi_{k_{2n}} ... \hat{\psi}_{k_i} ... \psi_{k_1} \rangle + c_{k_s} \delta_{k_s,-k_{s-1}} \langle \psi_{k_{2n}} ... \hat{\psi}_{k_i} ... \hat{\psi}_{k_s} \hat{\psi}_{k_{s-1}} ... \psi_{k_1} \rangle \right] \langle \psi_{k_i} \psi_{k_1} \rangle \\ & + (-1)^{s-1} \langle \psi_{k_{2n}} ... \hat{\psi}_{k_s} ... \psi_{k_1} \rangle \langle \psi_{k_s} \psi_{k_1} \rangle + (-1)^s \langle \psi_{k_{2n}} ... \hat{\psi}_{k_{s-1}} ... \psi_{k_1} \rangle \langle \psi_{k_{s-1}} \psi_{k_1} \rangle \\ = & -R(k_{2n},\dots,k_s,k_{s-1},\dots,k_1) \\ & + c_{k_s} \delta_{k_s,-k_{s-1}} \sum_{\substack{i=2 \\ i \neq s,s-1}}^{2n} (-1)^i \langle \psi_{k_{2n}} ... \hat{\psi}_{k_i} ... \hat{\psi}_{k_s} \hat{\psi}_{k_{s-1}} ... \psi_{k_1} \rangle \langle \psi_{k_i} \psi_{k_1} \rangle. \end{aligned}$$

Since, by the inductive hypothesis

$$\langle \psi_{k_{2n}} ... \hat{\psi}_{k_s} \hat{\psi}_{k_{s-1}} ... \psi_{k_1} \rangle = \sum_{\substack{i=2 \\ i \neq s,s-1}}^{2n} (-1)^i \langle \psi_{k_{2n}} ... \hat{\psi}_{k_i} ... \hat{\psi}_{k_s} \hat{\psi}_{k_{s-1}} ... \psi_{k_1} \rangle \langle \psi_{k_i} \psi_{k_1} \rangle$$

we conclude that (2.2.14) holds.

□

Lemma 2.2.4. *Let* $\pi \in S_n$.

$$\left\langle \prod_{i=1}^{n} \psi(u_{\pi(i)}) \psi(v_{\pi(i)}) v_\emptyset, v_\emptyset \right\rangle = \mathrm{Pf}(\Psi_\pi),$$

where Ψ_π is a skew-symmetric $2n$-matrix given with

$$\Psi_\pi(i,j) = \begin{cases} \psi_\pi((u_i,i),(u_j,j)) & 1 \le i < j \le n, \\ \psi_\pi((u_i,i),(v'_j,j')) & 1 \le i \le n < j \le 2n, \\ \psi_\pi((v'_i,i'),(v'_j,j')) & n < i < j \le 2n, \end{cases}$$

where $i' = 2n - i + 1$, $j' = 2n - j + 1$ and

$$\psi_\pi((z,i),(w,j)) = \begin{cases} \langle \psi(z)\psi(w)v_\emptyset, v_\emptyset\rangle & \pi^{-1}(i) \ge \pi^{-1}(j), \\ -\langle \psi(w)\psi(z)v_\emptyset, v_\emptyset\rangle & \pi^{-1}(i) < \pi^{-1}(j). \end{cases}$$

Proof. First we show that the statement is true for $\pi = id$. From Lemma 2.2.3 we have

$$\begin{aligned} &\langle \psi(u_n)\psi(v_n)\cdots\psi(u_1)\psi(v_1)v_\emptyset, v_\emptyset\rangle = \langle \psi_{k_{2n}}\cdots\psi_{k_1}v_\emptyset, v_\emptyset\rangle \prod_{i=1}^{n} u_i v_i \\ &= \sum_{i=2}^{2n} (-1)^i \langle \psi_{k_{2n}}\cdots\hat{\psi}_{k_i}\cdots\psi_{k_1}v_\emptyset, v_\emptyset\rangle\langle \psi_{k_i}\psi_{k_1}v_\emptyset, v_\emptyset\rangle \prod_{i=1}^{n} u_i v_i \\ &= \sum_{i=1}^{n} \langle \psi(u_n)\psi(v_n)\cdots\widehat{\psi(u_i)}\cdots\psi(u_1)v_\emptyset, v_\emptyset\rangle\langle\psi(u_i)\psi(v_1)v_\emptyset, v_\emptyset\rangle \\ &- \sum_{i=2}^{n} \langle \psi(u_n)\psi(v_n)\cdots\widehat{\psi(v_i)}\cdots\psi(u_1)v_\emptyset, v_\emptyset\rangle\langle\psi(v_i)\psi(v_1)v_\emptyset, v_\emptyset\rangle. \end{aligned}$$

Then by the expansion formula for the Pfaffian we get that

$$\langle \psi(u_n)\psi(v_n)\cdots\psi(u_1)\psi(v_1)v_\emptyset, v_\emptyset\rangle = \mathrm{Pf}(A(u_n, v_n, \ldots, u_1, v_1)),$$

where $A(u_n, v_n, \ldots, u_1, v_1)$ is a skew symmetric $2n \times 2n$ matrix

$$\begin{bmatrix} 0 & a(u_n,v_n) & \ldots & a(u_n,u_1) & a(u_n,v_1) \\ -a(u_n,v_n) & 0 & \ldots & a(v_n,u_1) & a(v_n,v_1) \\ \vdots & & \ddots & & \ldots \\ -a(u_n,u_1) & -a(v_n,u_1) & \ldots & 0 & a(u_1,v_1) \\ -a(u_n,v_1) & -a(v_n,v_1) & \ldots & -a(u_1,v_1) & 0 \end{bmatrix},$$

with

$$a(z,w) = \langle \psi(z)\psi(w)v_\emptyset, v_\emptyset\rangle.$$

Thus, the columns and rows of A appear "in order" $u_n, v_n, \ldots, u_1, v_1$. Rearrange these columns and columns "in order" $u_1, u_2, \ldots, u_n, v_n, \ldots, v_2, v_1$. Let B be the new matrix. Since the number of switches we have to make to do this rearrangement is equal to $n(n-1)$ and that is even, we have that $\mathrm{Pf}(A) = \mathrm{Pf}(B)$ and

$$B = \begin{bmatrix} 0 & -a(u_2,u_1) & \ldots & -a(v_2,u_1) & a(u_1,v_1) \\ a(u_2,u_1) & 0 & \ldots & a(u_2,v_2) & a(u_2,v_1) \\ \vdots & & \ddots & & \ldots \\ a(v_2,u_1) & -a(u_2,v_2) & \ldots & 0 & a(v_2,v_1) \\ -a(u_1,v_1) & -a(u_2,v_1) & \ldots & -a(v_2,v_1) & 0 \end{bmatrix},$$

This shows that $B = \Psi_{id}$.

Now, we want to show that the statement holds for any $\pi \in S_n$. We have just shown that

$$\langle \psi(u_n)\psi(v_n)\cdots\psi(u_1)\psi(v_1)v_\emptyset, v_\emptyset\rangle = \mathrm{Pf}(B(u_n, v_n, \ldots, u_1, v_1)),$$

where B is given above, i.e.

$$B(i,j) = \begin{cases} B((u_i,i),(u_j,j)) & 1 \le i < j \le n, \\ B((u_i,i),(v'_j,j')) & 1 \le i \le n < j \le 2n, \\ B((v'_i,i'),(v'_j,j')) & n < i < j \le 2n, \end{cases}$$

where $i' = 2n - i + 1, j' = 2n - j + 1$ and

$$B((z,i),(w,j)) = \begin{cases} \langle \psi(z)\psi(w)v_\emptyset, v_\emptyset\rangle & i \ge j, \\ -\langle \psi(w)\psi(z)v_\emptyset, v_\emptyset\rangle & i < j. \end{cases}$$

Then

$$\langle \psi(u_{\pi(n)})\psi(v_{\pi(n)})\cdots\psi(u_{\pi(1)})\psi(v_{\pi(1)})v_\emptyset, v_\emptyset\rangle = \mathrm{Pf}(B(u_{\pi(n)}, v_{\pi(n)}, \ldots, u_{\pi(1)}, v_{\pi(1)})).$$

Change the order of rows and columns in $B(u_{\pi(n)}, v_{\pi(n)}, \ldots, u_{\pi(1)}, v_{\pi(1)})$ in such a way that the rows and columns of the new matrix appear "in order" $u_1, u_2, \ldots, u_n, v_n, \ldots, v_2, v_1$. Let C be that new matrix. The number of switches we are making is even because we can first change the order to $u_n, v_n, \ldots, u_1, v_1$ by permuting pairs $(u_j v_j)$, and the number of switches from this order to $u_1, u_2, \ldots, u_n, v_n, \ldots, v_2, v_1$ is $n(n-1)$ as noted above. Thus $\mathrm{Pf}(C) = \mathrm{Pf}(B)$. Then $C = \Psi_\pi$ because

$$C(i,j) = \begin{cases} B((u_i, \pi^{-1}(i)), (u_j, \pi^{-1}(j))) & 1 \le i < j \le n, \\ B((u_i, \pi^{-1}(i)), (v'_j, \pi^{-1}(j'))) & 1 \le i \le n < j \le 2n, \\ B((v'_i, \pi^{-1}(i')), (v'_j, \pi^{-1}(j'))) & n < i < j \le 2n. \end{cases}$$

□

Lemma 2.2.5. *Let $\pi \in S_n$. In the domain $|u_{\pi(n)}| > |v_{\pi(n)}| > \cdots > |u_{\pi(1)}| > |v_{\pi(1)}|$*

$$\langle \prod_{i=1}^{n} \psi(u_{\pi(i)}) \psi(v_{\pi(j)}) v_\emptyset, v_\emptyset \rangle = \mathrm{Pf}(\Psi),$$

where Ψ is a skew-symmetric $2n$-matrix given with

$$\Psi(i,j) = \begin{cases} \psi(u_i, u_j) & 1 \le i < j \le n, \\ \psi(u_i, v'_j) & 1 \le i \le n < j \le 2n, \\ \psi(v'_i, v'_j) & n < i < j \le 2n, \end{cases}$$

where $i' = 2n - i + 1, j' = 2n - j + 1$ and

$$\psi(z,w) = \frac{z-w}{4(z+w)}.$$

Proof. This is a direct corollary of Lemma 2.2.4 and a formula given in Appendix:

$$\langle \psi(z)\psi(w) v_\emptyset, v_\emptyset \rangle = \frac{z-w}{4(z+w)} \quad \text{for } |z| > |w|.$$

It is enough to check for $1 \le i < j \le n$, because other cases are similar. If $\pi^{-1}(i) >$

$\pi^{-1}(j)$, respectively $\pi^{-1}(i) < \pi^{-1}(j)$ then

$$\psi_\pi((u_i, i), (u_j, j)) = \psi(u_i, u_j). \tag{2.2.15}$$

in the region $|u_i| > |u_j|$, respectively $|u_i| < |u_j|$. This means that (2.2.15) holds for the given region $|u_{\pi(n)}| > |v_{\pi(n)}| > \cdots > |u_{\pi(1)}| > |v_{\pi(1)}|$. □

Remark. Equation (2.2.5) appears in [Mat05]. Lemmas 2.2.4 and 2.2.5 are not of interest in [Mat05] for $\pi \neq id$ because in the case of the shifted Schur measure one does not need to consider the time order.

2.3 Measure on strict plane partitions

In this section we introduce strict plane partitions and a measure on them. This measure can be obtained as a special case of the shifted Schur process by a suitable choice of specializations of the algebra of symmetric functions. Then the correlation functions for this measure can be obtained as a corollary of Theorem 2.2.2. Using this result we compute the (bulk) asymptotics of the correlation functions as the partitions become large. One of the results we get along the way is an analog of MacMahon's formula for the strict plane partitions.

2.3.1 Strict plane partitions

A plane partition π can be viewed in different ways. One way is to fix a Young diagram, the support of the plane partition, and then to associate a positive integer to each box in the diagram such that integers form nonincreasing rows and columns. Thus, a plane partition is a diagram with row and column nonincreasing integers associated to its boxes. It can also be viewed as a finite two-sided sequence of ordinary partitions, since each diagonal in the support diagram represents a partition. We write $\pi = (\lambda^{-T_L}, \ldots, \lambda^{-1}, \lambda^0, \lambda^1, \ldots, \lambda^{T_R})$, where the partition λ^0 corresponds to the main diagonal and λ^k corresponds to the diagonal that is shifted by $\pm k$, see Figure 2.2. Every such two-sided sequence of partitions represents

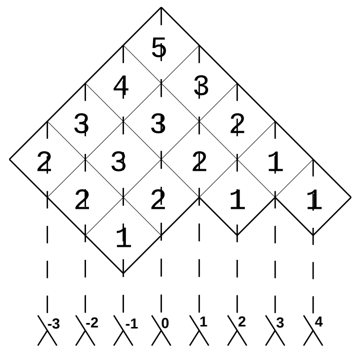


Figure 2.2: A strict plane partition $\pi = ((2), (3, 2), (4, 3, 1), (5, 3, 2), (3, 2), (2, 1), (1), (1))$

a plane partition if and only if

$$\begin{gathered} \lambda^{-T_L} \subset \cdots \subset \lambda^{-1} \subset \lambda^0 \supset \lambda^1 \supset \cdots \supset \lambda^{T_R}, \\ \text{skew diagrams } \lambda^{k+1} - \lambda^k \ (\lambda^k - \lambda^{k+1}) \text{ are horizontal strips.} \end{gathered} \tag{2.3.1}$$

A strict plane partition is a finite two-sided sequence of strict partitions with property (2.3.1). In other words, it is a plane partition whose diagonals are strict partitions. As an example, a strict plane partition corresponding to

$$\pi = ((2), (3, 2), (4, 3, 1), (5, 3, 2), (3, 2), (2, 1), (1), (1))$$

is given in Figure 2.2.

We give one more representation of a plane partition as a 3-dimensional diagram that is defined as a collection of $1 \times 1 \times 1$ boxes packed in a 3-dimensional corner in such a way that the column heights are given by the filling numbers of the plane partition. A 3-dimensional diagram corresponding to the example above is shown in Figure 2.3.

The number $|\pi|$ is the norm of π and is equal to the sum of the filling numbers. If π is seen as a 3-dimensional diagram then $|\pi|$ is its volume.

We introduce a number $A(\pi)$ that we call the alternation of π as

$$A(\pi) = \sum_{i=1}^{T_R+1} a(\lambda^{i-1} - \lambda^i) + \sum_{i=0}^{-T_L} a(\lambda^i - \lambda^{i-1}) - l(\lambda^0), \tag{2.3.2}$$

where $\lambda^{T_R+1} = \lambda^{-T_L-1} = \emptyset$, and l and a are as defined in Appendix, namely $l(\lambda)$ is the

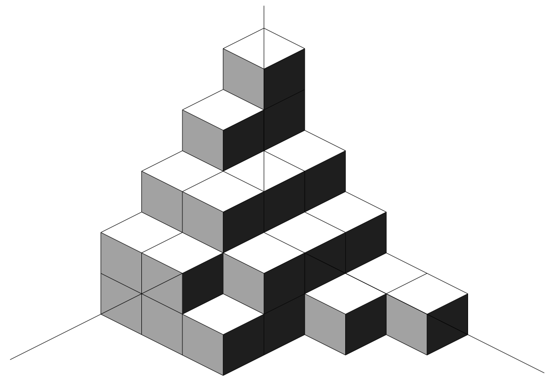


Figure 2.3: The 3-dimensional diagram of $\pi = ((2), (3,2), (4,3,1), (5,3,2), (3,2), (2,1), (1), (1))$

number of (nonzero) parts of λ and $a(\lambda - \mu)$ is the number of connected components of the shifted skew diagram $\lambda - \mu$.

This number is equal to the number of white islands (formed by white rhombi) of the 3-dimensional diagram of the strict plane partition. For the given example $A(\pi)$ is 7 (see Figure 2.4). In other words, this number is equal to the number of connected components of a plane partition, where a connected component consists of boxes associated to a same integer that are connected by sides of the boxes (i.e., it is a border strip filled with a same integer). For the given example there are two connected components associated to the number 2 (see Figure 2.4).

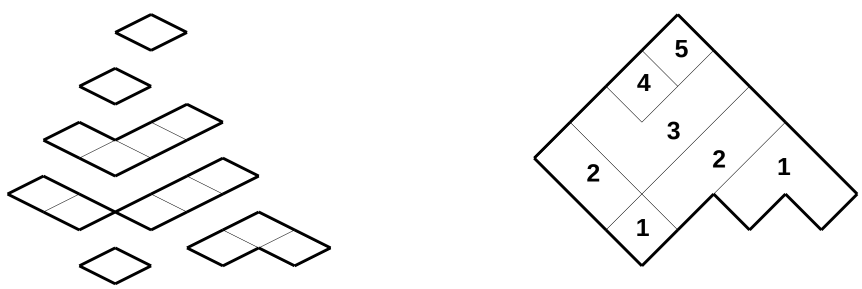


Figure 2.4: Connected components of $\pi = ((2), (3,2), (4,3,1), (5,3,2), (3,2), (2,1), (1), (1))$

It is not obvious that $A(\pi)$ defined by (2.3.2) is equal to the number of the connected components. We state this fact as a proposition and prove it.

Proposition 2.3.1. *Let π be a strict plane partition. Then the alternate $A(\pi)$ defined with (2.3.2) is equal to the number of connected components of π.*

Proof. We show this inductively. Denote the last nonzero part in the last row of the support of π by x. Denote a new plane partition obtained by removing the box containing x with π'.

We want to show that $A(\pi)$ and $A(\pi')$ satisfy the relation that does not depend whether we choose (2.3.2) or the number of connected components for the definition of the alternate.

We divide the problem in four cases I, II, III and IV shown in Figure 2.5 and we further divide these cases in several new ones. The cases depend on the position and the value of x_L and x_R.

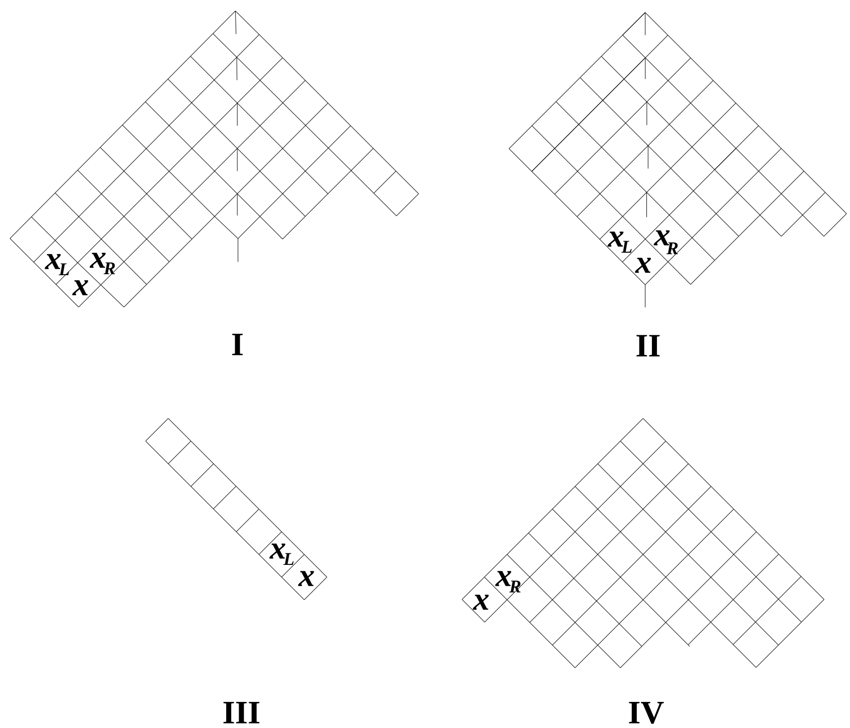


Figure 2.5: Cases I, II, III and IV

Then using (2.3.2) we get

$$A(\pi') = A(\pi) + \text{contribution from } x_L + \text{contribution from } x_R + \text{change of } l(\lambda^0).$$

Let us explain this formula for case I when $x_L = x_R = x$ in more detail.

Let λ_{x_L}, λ_x and λ_{x_R} be the diagonal partitions of π containing x_L, x and x_R, respectively.

Let λ'_x be a partition obtained from λ_x by removing x. Then

$$\begin{aligned}\text{contribution from } x_L &= a(\lambda'_x - \lambda_{x_L}) - a(\lambda_x - \lambda_{x_L}) = 0,\\ \text{contribution from } x_R &= a(\lambda_{x_R} - \lambda'_x) - a(\lambda_{x_R} - \lambda_x) = 1,\\ \text{change of } l(\lambda^0) &= l(\lambda^0(\pi)) - l(\lambda^0(\pi')) = 0.\end{aligned}$$

For all the cases the numbers are

$A(\pi') = A(\pi) +$

I	II	III	IV	
0+1+0	0+0+1	0+∅+0	∅+0+0	$x_L = x, x_R = x$
-1+0+0	-1-1+1	1+∅+0	∅-1+0	$x_L > x, x_R > x$
0+0+0	0-1+1			$x_L = x, x_R > x$
-1+1+0	-1+0+1			$x_L > x, x_R = x$

It is easy to verify that we get the same value for $A(\pi')$ in terms of $A(\pi)$ using the connected component definition. Then inductively this gives a proof that the two definitions are the same. □

We can give a generalization of Lemma 2.3.1. Instead of starting from a diagram of a partition we can start from a type of a diagram shown in Figure 2.6 (connected skew Young diagram) and fill the boxes of this diagram with row and column nonincreasing integers such that integers on the same diagonal are distinct. Call this object a skew strict plane partition

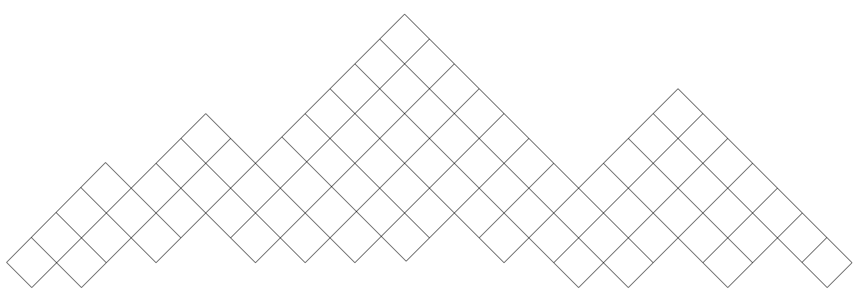


Figure 2.6: SSPP

(SSPP). We define connected components of SSPP in the same way as for the strict plane partitions. Then we can give an analog of Lemma 2.3.1 for SSPP. We will not use this result further in the chapter.

Lemma 2.3.2. *Let π be a SSPP*

$$\pi = (\cdots \subset \lambda^{t_{-2}} \supset \cdots \supset \lambda^{t_{-1}} \subset \cdots \subset \lambda^{t_0} \supset \cdots \supset \lambda^{t_1} \subset \cdots \subset \lambda^{t_2} \supset \cdots).$$

Then the alternate $A(\pi)$ defined by

$$A(\pi) = \sum_{i=-\infty}^{\infty} a(\lambda^i, \lambda^{i-1}) - \sum_{i=-\infty}^{\infty} l(\lambda^{t_{2i}}) + \sum_{i=-\infty}^{\infty} l(\lambda^{t_{2i+1}}),$$

with

$$a(\lambda,\mu) = \begin{cases} a(\lambda-\mu) & \lambda \supset \mu \\ a(\mu-\lambda) & \mu \supset \lambda \end{cases}$$

is equal to the number of connected components of π.

Proof. The proof is similar to that of Lemma 2.3.1. □

2.3.2 The measure

For q such that $0 < q < 1$ we define a measure on strict plane partitions by

$$P(\pi) = \frac{2^{A(\pi)} q^{|\pi|}}{Z}, \tag{2.3.3}$$

where Z is the partition function, i.e $Z = \sum_\pi 2^{A(\pi)} q^{|\pi|}$. This measure has a natural interpretation since $2^{A(\pi)}$ is equal to the number of colorings of the connected components of π.

This measure can be obtained as a special shifted Schur process for an appropriate choice of the specializations ρ. Then Proposition 2.2.1 can be used to obtain the partition function:

Proposition 2.3.3.

$$\sum_{\substack{\pi \text{ is a strict} \\ \text{plane partition}}} 2^{A(\pi)} q^{|\pi|} = \prod_{n=1}^{\infty} \left(\frac{1+q^n}{1-q^n} \right)^n.$$

This is the generating formula for the strict plane partitions and we call it the shifted MacMahon's formula since it can be viewed as an analog of MacMahon's generating formula for the plane partitions that says

$$\sum_{\substack{\pi \text{ is a plane} \\ \text{partition}}} q^{|\pi|} = \prod_{n=1}^{\infty} \left(\frac{1}{1-q^n} \right)^n .$$

Before we show that this measure is a special shifted Schur process let us recall some facts that can be found in Chapter 3 of [Mac95]. We need the values of skew Schur P and Q functions for some specializations of the algebra of symmetric functions. They can also be computed directly using (2.5.1).

If ρ is a specialization of Λ where $x_1 = s$, $x_2 = x_3 = \ldots = 0$ then

$$Q_{\lambda/\mu}(\rho) = \begin{cases} 2^{a(\lambda-\mu)} s^{|\lambda|-|\mu|} & \lambda \supset \mu,\ \lambda - \mu \text{ is a horizontal strip,} \\ 0 & \text{otherwise,} \end{cases}$$
$$P_{\lambda/\mu}(\rho) = \begin{cases} 2^{a(\lambda-\mu)-l(\lambda)+l(\mu)} s^{|\lambda|-|\mu|} & \lambda \supset \mu,\ \lambda - \mu \text{ is a horizontal strip,} \\ 0 & \text{otherwise,} \end{cases}$$

where, as before, $a(\lambda-\mu)$ is the number of connected components of the shifted skew diagram $\lambda - \mu$. In particular, if ρ is a specialization given with $x_1 = x_2 = \ldots = 0$ then

$$Q_{\lambda/\mu}(\rho) = \begin{cases} 1 & \lambda = \mu, \\ 0 & \text{otherwise,} \end{cases}$$
$$P_{\lambda/\mu}(\rho) = \begin{cases} 1 & \lambda = \mu, \\ 0 & \text{otherwise.} \end{cases}$$

In order to obtain the measure (2.3.3) as a special shifted Schur process we set

$$\begin{aligned} &\rho_n^+ : x_1 = q^{-(2n+1)/2},\ x_2 = x_3 = \ldots = 0 && n \le -1, \\ &\rho_n^- : x_1 = x_2 = \ldots = 0 && n \le -1, \\ &\rho_n^- : x_1 = q^{(2n+1)/2},\ x_2 = x_3 = \ldots = 0 && n \ge 0, \\ &\rho_n^+ : x_1 = x_2 = \ldots = 0 && n \ge 0. \end{aligned} \tag{2.3.4}$$

This measure is supported on strict plane partitions viewed as a two-sided sequence $(\ldots, \lambda^{-n}, \ldots, \lambda^0, \ldots, \lambda^n, \ldots)$. Indeed, for any two sequences $\lambda = (\ldots, \lambda^{-n}, \ldots, \lambda^0, \ldots, \lambda^n, \ldots)$

and $\mu = (\dots, \mu^{-n}, \dots, \mu^0, \dots, \mu^n, \dots)$ we have that the weight is given with

$$W(\lambda, \mu) \quad = \quad \prod_{n=-\infty}^{\infty} Q_{\lambda^n/\mu^{n-1}}(\rho_{n-1}^+) P_{\lambda^n/\mu^n}(\rho_n^-),$$

where only finitely many terms contribute. Then $W(\lambda, \mu) = 0$ unless

$$\mu^n = \begin{cases} \lambda^n & n < 0, \\ \lambda^{n+1} & n \geq 0, \end{cases}$$

$$\cdots \subset \lambda^{-n} \subset \cdots \subset \lambda^0 \supset \cdots \supset \lambda^n \supset \cdots ,$$

skew diagrams $\lambda^{n+1} - \lambda^n$ for $n < 0$ and

$\lambda^n - \lambda^{n+1}$ for $n \geq 0$ are horizontal strips,

and in that case

$$\begin{aligned} W(\lambda, \mu) \quad &= \quad \prod_{n=-\infty}^{0} 2^{a(\lambda^n - \lambda^{n-1})} q^{(-2n+1)(|\lambda^n| - |\lambda^{n-1}|)/2} \\ &\quad \cdot \prod_{n=1}^{\infty} 2^{a(\lambda^{n-1} - \lambda^n) - l(\lambda^{n-1}) + l(\lambda^n)} q^{(2n-1)(|\lambda^{n-1}| - |\lambda^n|)/2} \\ &= \quad 2^{A(\lambda)} q^{|\lambda|}. \end{aligned}$$

Thus, the given choice of ρ's defines a shifted Schur process (or a limit of shifted Schur processes as we explain in a remark below) that is indeed equal to the measure on the strict plane partitions given with (2.3.3).

Proposition 2.2.1 allows us to obtain the shifted MacMahon's formula. If ρ^+ is $x_1 = s$, $x_2 = x_3 = \ldots = 0$ and ρ^- is $x_1 = t$, $x_2 = x_3 = \ldots = 0$ then

$$H(\rho^+, \rho^-) = \prod_{i,j} \frac{1 + x_i y_j}{1 - x_i y_j}\bigg|_{x=\rho^+,\, y=\rho^-} = \frac{1 + st}{1 - st}.$$

Thus, for the given specializations of ρ_i^+'s and ρ_i^-'s we have

$$\begin{aligned} Z(\rho) = \prod_{i<j} H(\rho_i^+, \rho_j^-) &= \frac{1+q}{1-q} \cdot \frac{1+q^2}{1-q^2} \cdot \frac{1+q^3}{1-q^3} \cdots \\ & \frac{1+q^2}{1-q^2} \cdot \frac{1+q^3}{1-q^3} \cdots \\ & \frac{1+q^3}{1-q^3} \cdot \frac{1+q^4}{1-q^4} \cdots \\ &= \prod_{n=1}^{\infty} \left(\frac{1+q^n}{1-q^n} \right)^n . \end{aligned}$$

Remark. A shifted Schur process depends on finitely many specializations. For that reason, measure (2.3.3) is a limit of measures defined as shifted Schur processes rather than a shifted Schur process itself. For every T let specializations $\rho_n^{\pm}$ be as in (2.3.4) if $|n| \leq T$ and zero otherwise. They define a shifted Schur process whose support is S_T that is the set of strict plane partitions with $\lambda^n = \emptyset$ for every $|n| > T$. The partition function for this measure is $\sum_{\pi \in S_T} 2^{A(\pi)} q^{|\pi|}$ and is bounded by $\prod(\frac{1+q^n}{1-q^n})^n$. Let S be the set of all strict plane partitions. Then the T^{th} partial sum (sum of all terms that involve q^m for $m \leq T$) of $\sum_{\pi \in S} 2^{A(\pi)} q^{|\pi|}$, which is equal to the T^{th} partial sum of $\sum_{\pi \in S_T} 2^{A(\pi)} q^{|\pi|}$, is bounded by $\prod(\frac{1+q^n}{1-q^n})^n$. Hence, $\sum_{\pi \in S} 2^{A(\pi)} q^{|\pi|}$ converges. Therefore, $\sum_{\pi \in S_T} 2^{A(\pi)} q^{|\pi|} \to \sum_{\pi \in S} 2^{A(\pi)} q^{|\pi|}$ as $T \to \infty$. Thus, the correlation function of the measure (2.3.3) is the limit of the correlation functions of the approximating shifted Schur processes as $T \to \infty$.

Our next goal is to find the correlation function for the measure (2.3.3). For that we need to restate Theorem 2.2.2 for the given specializations. In particular, we need to determine $J(t, z)$. When ρ is such that $x_1 = s, x_2 = x_3 = \cdots = 0$ then (see (2.5.3))

$$F(\rho; z) = \frac{1+sz}{1-sz}.$$

Thus, for our specializations (see (2.2.2))

$$J(t,z)=\begin{cases}\dfrac{\prod\limits_{m\geq t}\dfrac{1+q^{m+1/2}z}{1-q^{m+1/2}z}}{\prod\limits_{m\geq 0}\dfrac{1+q^{m+1/2}z^{-1}}{1-q^{m+1/2}z^{-1}}}=\dfrac{(q^{1/2}z^{-1};q)_\infty(-q^{t+1/2}z;q)_\infty}{(-q^{1/2}z^{-1};q)_\infty(q^{t+1/2}z;q)_\infty} & t\geq 0\\[4ex] \dfrac{\prod\limits_{m\geq 0}\dfrac{1+q^{m+1/2}z}{1-q^{m+1/2}z}}{\prod\limits_{m\geq -t}\dfrac{1+q^{m+1/2}z^{-1}}{1-q^{m+1/2}z^{-1}}}=\dfrac{(-q^{1/2}z;q)_\infty(q^{-t+1/2}z^{-1};q)_\infty}{(q^{1/2}z;q)_\infty(-q^{-t+1/2}z^{-1};q)_\infty} & t<0,\end{cases}\tag{2.3.5}$$

where

$$(z;q)_\infty=\prod_{n=0}^{\infty}(1-q^n z)$$

is the quantum dilogarithm function.

It is convenient to represent a strict plane partition $\pi=(\ldots,\lambda^{-1},\lambda^0,\lambda^1,\ldots)$ as a subset of

$$\mathfrak{X}=\{(t,x)\in\mathbb{Z}\times\mathbb{Z}\mid x>0\}\,,$$

where (t,x) belongs to this subset if and only if x is a part of λ^t. We call this subset the plane diagram of the strict plane partition π.

Corollary 2.3.4. *For a set $X=\{(t_i,x_i):i=1,\ldots,n\}\subset\mathfrak{X}$ representing a plane diagram, the correlation function is given with*

$$\rho(X)=\mathrm{Pf}(M_X^{3D})\tag{2.3.6}$$

where M_X^{3D} is a skew-symmetric $2n\times 2n$ matrix and $M_X^{3D}(i,j)$ is given with

$$\begin{cases}K_{x_i,x_j}(t_i,t_j) & 1\leq i<j\leq n,\\ (-1)^{x_{j'}}K_{x_i,-x_{j'}}(t_i,t_{j'}) & 1\leq i\leq n<j\leq 2n,\\ (-1)^{x_{i'}+x_{j'}}K_{-x_{i'},-x_{j'}}(t_{i'},t_{j'}) & n<i<j\leq 2n,\end{cases}\tag{2.3.7}$$

where $i'=2n-i+1$ and $K((t_i,x),(t_j,y))$ is the coefficient of $z^x w^y$ in the formal power *series expansion of*

$$K((t_i,z),(t_j,w)) = \frac{z-w}{2(z+w)} J(t_i,z)J(t_j,w)$$

in the region $|z|>|w|$ *if* $t_i \geq t_j$ *and* $|z|<|w|$ *if* $t_i < t_j$. *Here* $J(t,z)$ *is given with (2.3.5).*

2.4 Asymptotics of large random strict plane partitions

In this section we compute the bulk limit for shifted plane diagrams with a distribution proportional to $2^{A(\pi)}q^{|\pi|}$, where π is the corresponding strict plane partition. In order to determine the correct scaling we first consider the asymptotic behavior of the volume $|\pi|$ of our random strict plane partitions.

2.4.1 Asymptotics of the volume

The scaling we are going to choose when computing the limit of the correlation functions will be $r=-\log q$ for all directions. One reason for that lies in the fact that $r^3|\pi|$ converges in probability to a constant. Thus, the scaling assures that the volume tends to a constant. Our argument is similar to that of Lemma 2 of [OR03].

Proposition 2.4.1. *If the probability of* π *is given with (2.3.3) then*

$$r^3|\pi| \to \frac{7}{2}\zeta(3), \quad r \to +0.$$

The convergence is in probability.

Proof. We recall that if $E(X_n)\to c$ and $Var(X_n)\to 0$ then $X_n \to c$ in probability. Thus, it is enough to show that

$$E(r^3|\pi|) \to \frac{7}{2}\zeta(3), \quad r \to +0$$

and

$$Var(r^3|\pi|) \to 0, \quad r \to +0.$$

First, we observe that

$$E(|\pi|) = \frac{\sum_\pi 2^{A(\pi)}q^{|\pi|}|\pi|}{Z} = \frac{q\frac{d}{dq}Z}{Z}$$

and

$$Var(|\pi|) = E(|\pi|^2) - E(|\pi|)^2 = \frac{\sum_\pi 2^{A(\pi)} q^{|\pi|} |\pi|^2}{Z} - \frac{q^2(\frac{d}{dq}Z)^2}{Z^2} = q\frac{d}{dq}E(|\pi|).$$

Since $Z = \prod_{n\geq 1}(\frac{1+q^n}{1-q^n})^n$, we have

$$\frac{d}{dq}Z = \prod_{n\geq 1}\left(\frac{1+q^n}{1-q^n}\right)^n \sum_{m\geq 1} \frac{m\left(\frac{1+q^m}{1-q^m}\right)^{m-1}\frac{2mq^{m-1}}{(1-q^m)^2}}{\left(\frac{1+q^m}{1-q^m}\right)^m} = Z\sum_{m\geq 1}\frac{2m^2q^{m-1}}{1-q^{2m}}.$$

Then

$$E(|\pi|) = \frac{q\frac{d}{dq}Z}{Z} = \sum_{m\geq 1}\frac{2m^2q^m}{1-q^{2m}} = \sum_{m\geq 1, k\geq 0} 2m^2 q^{m(2k+1)} = 2\sum_{k\geq 0}\frac{q^{2k+1}(1+q^{2k+1})}{(1-q^{2k+1})^3},$$

because

$$\sum_{m\geq 1} m^2 q^m = \frac{q(1+q)}{(1-q)^3}, \quad |q|<1.$$

Now,

$$r^3\frac{q^{2k+1}(1+q^{2k+1})}{(1-q^{2k+1})^3} = \frac{e^{-r(2k+1)}(1+e^{-r(2k+1)})}{\left(\frac{r(2k+1)+o(r(2k+1))}{r}\right)^3} \nearrow \frac{2}{(2k+1)^3} \quad r\to +0.$$

Then by the uniform convergence

$$\lim_{r\to +0} r^3 E(|\pi|) = 2\sum_{k\geq 0}\lim_{r\to +0} r^3\frac{q^{2k+1}(1+q^{2k+1})}{(1-q^{2k+1})^3} = 4\sum_{k\geq 0}\frac{1}{(2k+1)^3}.$$

Finally, since

$$\sum_{k\geq 0}\frac{1}{(2k+1)^3} = (1-\frac{1}{2^3})\zeta(3)$$

it follows that

$$E(r^3|\pi|) \to \frac{7}{2}\zeta(3), \quad r\to +0.$$

For the variance we have

$$Var(|\pi|) = q\frac{d}{dq}E(|\pi|) = -\frac{d}{dr}E(|\pi|) \sim \frac{21\zeta(3)}{2r^4},$$

since by L'Hôpital's rule

$$\frac{7}{2}\zeta(3) = \lim_{r\to+0} \frac{E(|\pi|)}{\frac{1}{r^3}} = \lim_{r\to+0} \frac{\frac{d}{dr}E(|\pi|)}{-3\frac{1}{r^4}}.$$

Thus,

$$Var(r^3|\pi|) \to 0, \quad r \to +0.$$

□

2.4.2 Bulk scaling limit of the correlation functions

We compute the limit of the correlation function for a set of points that when scaled by $r = -\log q$ tend to a fixed point in the plane as $r \to +0$. Namely, we compute the limit of (2.3.7) for

$$rt_i \to \tau, \quad rx_i \to \chi \quad \text{as } r \to +0,$$

$$t_i - t_j = \Delta t_{ij} = \text{const}, \quad x_i - x_j = \Delta x_{ij} = \text{const},$$

where $\chi \geq 0$.

In Theorem 2.4.2 we show that in the limit the Pfaffian (2.3.6) turns into a determinant whenever $\chi > 0$ and it remains a Pfaffian on the boundary $\chi = 0$.

Throughout this chapter $\gamma^+_{R,\theta}$ ($\gamma^-_{R,\theta}$) stands for the counterclockwise (clockwise) oriented arc on $|z| = R$ from $Re^{-i\theta}$ to $Re^{i\theta}$.

More generally, if γ is a curve parameterized by $R(\phi)e^{i\phi}$ for $\phi \in [-\pi, \pi]$ then γ^+_θ (γ^-_θ) stands for the counterclockwise (clockwise) oriented arc on γ from $R(\theta)e^{-i\theta}$ to $R(\theta)e^{i\theta}$.

Recall that our phase space is $\mathfrak{X} = \{(t,x) \in \mathbb{Z} \times \mathbb{Z} \,|\, x > 0\}$.

Theorem 2.4.2. *Let $X = \{(t_i, x_i) : i = 1, \ldots, n\} \subset \mathfrak{X}$ be such that*

$$rt_i \to \tau, \quad rx_i \to \chi \quad \textit{as } r \to +0,$$

$$t_i - t_j = \Delta t_{ij} = \textit{const}, \quad x_i - x_j = \Delta x_{ij} = \textit{const}.$$

a) If $\chi > 0$ then

$$\lim_{r\to+0} \rho(X) = \det[K(i,j)]^n_{i,j=1},$$

where

$$K(i,j) = \frac{1}{2\pi i}\int_{\gamma^{\pm}_{R,\theta}} \left(\frac{1-z}{1+z}\right)^{\Delta t_{ij}} \frac{1}{z^{\Delta x_{ij}+1}} dz,$$

where we choose $\gamma^{+}_{R,\theta}$ *if* $\Delta t_{ij} \geq 0$ *and* $\gamma^{-}_{R,\theta}$ *otherwise, where* $R = e^{-|\tau|/2}$ *and*

$$\theta = \begin{cases} \arccos \frac{(e^{|\tau|}+1)(e^{\chi}-1)}{2e^{|\tau|/2}(e^{\chi}+1)}, & \frac{(e^{|\tau|}+1)(e^{\chi}-1)}{2e^{|\tau|/2}(e^{\chi}+1)} \leq 1, \\ 0, & \textit{otherwise}, \end{cases} \tag{2.4.1}$$

b) If $\chi = 0$ *and in addition to the above conditions we assume*

$$x_i = const$$

then

$$\lim_{r\to+0} \rho(X) = \mathrm{Pf}[M(i,j)]_{i,j=1}^{2n},$$

where M *is a skew symmetric matrix given by*

$$M(i,j) = \begin{cases} \frac{(-1)^{x_j}}{2\pi i}\int_{\gamma^{\pm}_{R,\theta}} \left(\frac{1-z}{1+z}\right)^{\Delta t_{ij}} \frac{dz}{z^{x_i+x_j+1}} & 1 \leq i < j \leq n, \\ \frac{1}{2\pi i}\int_{\gamma^{\pm}_{R,\theta}} \left(\frac{1-z}{1+z}\right)^{\Delta t_{ij'}} \frac{dz}{z^{x_i-x_{j'}+1}} & 1 \leq i \leq n < j \leq 2n, \\ \frac{(-1)^{x_{i'}}}{2\pi i}\int_{\gamma^{\pm}_{R,\theta}} \left(\frac{1-z}{1+z}\right)^{\Delta t_{i'j'}} \frac{dz}{z^{-(x_{i'}+x_{j'})+1}} & n < i < j \leq 2n, \end{cases}$$

where $i' = 2n - i + 1$ *and we choose* $\gamma^{+}_{R,\theta}$ *if* $\Delta t_{ij} \geq 0$ *and* $\gamma^{-}_{R,\theta}$ *otherwise, where* $R = e^{-|\tau|/2}$ *and* $\theta = \pi/2$.

This implies that for an equal time configuration (points on the same vertical line) we get

Corollary 2.4.3. *For* $X = \{(t, x_i), i = 1, \ldots, n\} \subset \mathfrak{X}$ *such that*

$$rt \to \tau, \quad rx_i \to \chi \quad \textit{as } r \to +0,$$

$$x_i - x_j = \Delta x_{ij} = const,$$

where $\chi \geq 0$,

$$\lim_{r\to+0} \rho(X) = \det\left[\frac{\sin(\theta\Delta x_{ij})}{\pi\Delta x_{ij}}\right],$$

where θ *is given with (2.4.1).*

Remark. The kernel of Theorem 2.4.2 can be viewed as an extension of the discrete sine kernel of Corollary 2.4.3. This is one of the extensions constructed in Section 4 of [Bor07], but it is the first time that this extension appears in a "physical" problem.

The limit of the 1-point correlation function gives a density for the points of the plane diagram of strict plane partitions. We state this as a corollary.

Corollary 2.4.4. *The limiting density of the point* (τ,χ) *of the plane diagram of strict plane partitions is given with*

$$\rho(\tau,\chi) = \lim_{\substack{rt\to\tau, rx\to\chi,\\ r\to+0}} K_{3D}((t,x),(t,x)) = \frac{\theta}{\pi},$$

where θ *is given with (2.4.1).*

Remark. Using Corollary 2.4.4 we can determine the hypothetical limit shape of a typical 3-dimensional diagram. This comes from the observation that for a strict plane partition

$$\begin{aligned} x(\tau,\chi) &= \int_{\chi}^{\infty} \rho(\tau,s)ds, \\ y(\tau,\chi) &= \int_{\chi}^{\infty} \rho(\tau,s)ds + \tau, \\ z(\tau,\chi) &= \chi, \end{aligned}$$

if $\tau \geq 0$ and

$$\begin{aligned} x(\tau,\chi) &= \int_{\chi}^{\infty} \rho(\tau,s)ds - \tau, \\ y(\tau,\chi) &= \int_{\chi}^{\infty} \rho(\tau,s)ds, \\ z(\tau,\chi) &= \chi \end{aligned}$$

if $\tau < 0$. Indeed, for a point (x,y,z) of a 3-dimensional diagram, where $z = z(x,y)$ is the height of the column based at (x,y), the corresponding point in the plane diagram is $(\tau,\chi) = (y-x,z)$. Also, for the right (respectively left) part of a 3-dimensional diagram the coordinate x (respectively y) is given by the number of points of the plane diagram above (τ, χ) (i.e. all (τ, s) with $s \geq \chi$) and that is in the limit equal to $\int_{\chi}^{\infty} \rho(\tau, s) ds$.

The hypothetical limit shape is shown in Figure 2.4.2.

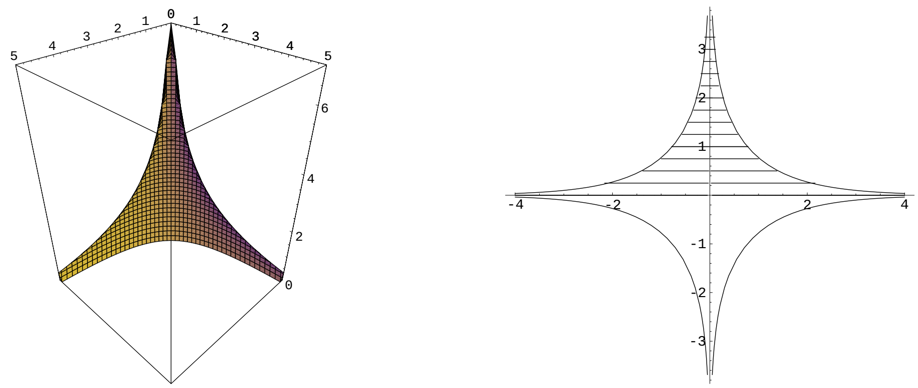


Figure 2.7: Left: The limit shape; Right: The amoeba of $-1 + z + w + zw$

Recall that the amoeba of a polynomial $P(z, w)$ where $(z, w) \in \mathbb{C}^2$ is

$$\{(\xi, \omega) = (\log |z|, \log |w|) \in \mathbb{R}^2 \mid (z, w) \in (\mathbb{C}\backslash\{0\})^2,\ P(z, w) = 0\}.$$

The limit shape of the shifted Schur process can be parameterized with $(\xi, \omega) = (\tau/2, \chi/2)$ for $(\xi, \omega) \in \mathfrak{D}$ where $\mathfrak{D}$ is the shaded region given in Figure 2.4.2. The boundaries of this region are $\omega = 0$, $\omega = \log[(e^{\xi} + 1)/(e^{\xi} - 1)]$ for $\xi > 0$ and $\omega = \log[(e^{-\xi} + 1)/(e^{-\xi} - 1)]$ for $\xi < 0$. This region is the half of the amoeba of $-1 + z + w + zw$ for $\omega = \log |w| \geq 0$.

Proof. (Theorem 2.4.2) Because of the symmetry it is enough to consider only $\tau \geq 0$.

In order to compute the limit of (2.3.7) we need to consider three cases

$$\begin{aligned} 1)\ K^{1,2}_{++} &= K_{x_1,x_2}(t_1, t_2) \\ 2)\ K^{1,2}_{+-} &= (-1)^{x_2} K_{x_1,-x_2}(t_1, t_2) \\ 3)\ K^{1,2}_{--} &= (-1)^{x_1+x_2} K_{-x_1,-x_2}(t_1, t_2), \end{aligned}$$

when $rx_i \to \chi$, $rt_i \to \tau$, $r \to +0$, $q = e^{-r}$ and $\Delta t_{1,2}$ and $\Delta x_{1,2}$ are fixed.

Since we are interested in the limits of 1), 2) and 3) we can assume that $rt_i \geq e^{\tau/2}$, for every i.

We start with 2). By the definition,

$$K^{1,2}_{+-} = \frac{(-1)^{x_2}}{(2\pi i)^2} \iint\limits_{\substack{|z|=(1\pm\epsilon)e^{\tau/2}\\|w|=(1\mp\epsilon)e^{\tau/2}}} \frac{z-w}{2(z+w)} J(t_1,z)J(t_2,w)\frac{1}{z^{x_1+1}w^{-x_2+1}}dzdw,$$

where we take the upper signs if $t_1 \geq t_2$ and the lower signs otherwise. Here, we choose $\epsilon \in (0, 1-q^{1/2})$ since $J(t,z)$ is equal to its power series expansion in the region $q^{1/2} < |z| < q^{-t-1/2}$. With the change of variables $w \mapsto -w$ we get

$$K^{1,2}_{+-} = e^{-\tau(x_1-x_2)/2} \cdot I_{+-},$$

where

$$I_{+-} = \frac{1}{(2\pi i)^2} \iint\limits_{\substack{|z|=(1\pm\epsilon)e^{\tau/2}\\|w|=(1\mp\epsilon)e^{\tau/2}}} \frac{z+w}{2zw(z-w)} \frac{J(t_1,z)}{J(t_2,w)} \left(\frac{z}{e^{\tau/2}}\right)^{-x_1} \left(\frac{w}{e^{\tau/2}}\right)^{x_2} dzdw.$$

We consider cases 1) and 3) together because terms in the Pfaffian contain K_{++} and K_{--} in pairs.

Using the definition and a simple change of coordinates $w \mapsto -w$ we have that

$$K^{1,2}_{++} \cdot K^{3,4}_{--} = e^{-\tau(x_1+x_2-x_3-x_4)/2}(-1)^{x_2+x_3} \cdot I_{++} \cdot I_{--} \tag{2.4.2}$$

where

$$I_{++} = \frac{1}{(2\pi i)^2} \iint\limits_{\substack{|z|=(1\pm\epsilon)e^{\tau/2}\\|w|=(1\mp\epsilon)e^{\tau/2}}} \frac{z+w}{2zw(z-w)} \frac{J(t_1,z)}{J(t_2,w)} \left(\frac{z}{e^{\tau/2}}\right)^{-x_1} \left(\frac{w}{e^{\tau/2}}\right)^{-x_2} dzdw$$

and

$$I_{--} = \frac{1}{(2\pi i)^2} \iint\limits_{\substack{|z|=(1\pm\epsilon)e^{\tau/2}\\|w|=(1\mp\epsilon)e^{\tau/2}}} \frac{z+w}{2zw(z-w)} \frac{J(t_3,z)}{J(t_4,w)} \left(\frac{z}{e^{\tau/2}}\right)^{x_3} \left(\frac{w}{e^{\tau/2}}\right)^{x_4} dzdw,$$

where the choice of signs is as before: in the first integral we choose the upper signs if $t_1 \geq t_2$ and the lower signs otherwise; similarly in the second integral with t_3 and t_4.

In the first step we will prove the following claims.

Claim 1.

$$\lim_{r\to+0} I_{+-} = \frac{1}{2\pi i}\int_{\gamma^{\pm}_{e^{\tau/2},\theta}} \frac{1}{z}\left(\frac{1-e^{-\tau}z}{1+e^{-\tau}z}\right)^{\Delta t_{1,2}} \left(\frac{z}{e^{\tau/2}}\right)^{-\Delta x_{1,2}} dz,$$

where we choose the plus sign if $t_1 \geq t_2$ and minus otherwise and θ is given with (2.4.1).

Claim 2.

$$\lim_{r\to+0} I_{++} = \lim_{r\to+0}\frac{1}{2\pi i}(-1)^{x_1+x_2+1}\int_{\gamma^{\mp}_{e^{\tau/2},\theta}} \frac{1}{z}\frac{J(t_2,z)}{J(t_1,z)}\left(\frac{z}{e^{\tau/2}}\right)^{-(x_1+x_2)} dz,$$

where we choose the minus sign if $t_1 \geq t_2$ and plus otherwise, where θ is given with (2.4.1). (We later show that the limit in the right hand side always exists.)

Claim 3.

$$\lim_{r\to+0} I_{--} = \lim_{r\to+0}\frac{1}{2\pi i}\int_{\gamma^{\pm}_{e^{\tau/2},\theta}} \frac{1}{z}\frac{J(t_3,z)}{J(t_4,z)}\left(\frac{z}{e^{\tau/2}}\right)^{x_3+x_4} dz,$$

where we choose the plus sign if $t_3 \geq t_4$ and minus otherwise, where θ is given with (2.4.1). (We later show that the limit in the right hand side always exists.)

In the next step we will show that Claims 2 and 3 imply that the limits of I_{++} and I_{--} vanish when $r \to +0$ unless $\chi = 0$. We state this in two claims.

Claim 4. If $\chi > 0$ then $\lim_{r\to+0} I_{++} = 0$ and if $\chi = 0$ then

$$\lim_{r\to+0} I_{++} = \frac{1}{2\pi i}\int_{\gamma^{\pm}_{e^{\tau/2},\pi/2}} \frac{1}{z}\left(\frac{1-e^{-\tau}z}{1+e^{-\tau}z}\right)^{\Delta t_{1,2}} \left(\frac{z}{e^{\tau/2}}\right)^{-(x_1+x_2)} dz,$$

where we pick $\gamma^{+}_{e^{\tau/2},\pi/2}$ if $\Delta t_{1,2} \geq 0$ and $\gamma^{-}_{e^{\tau/2},\pi/2}$ otherwise.

Claim 5. If $\chi > 0$ then $\lim_{r\to+0} I_{--} = 0$ and if $\chi = 0$ then

$$\lim_{r\to+0} I_{--} = \frac{1}{2\pi i}\int_{\gamma^{\pm}_{e^{\tau/2},\pi/2}} \frac{1}{z}\left(\frac{1-e^{-\tau}z}{1+e^{-\tau}z}\right)^{\Delta t_{3,4}} \left(\frac{z}{e^{\tau/2}}\right)^{(x_3+x_4)} dz,$$

where we pick $\gamma^{+}_{e^{\tau/2},\pi/2}$ if $\Delta t_{3,4} \geq 0$ and $\gamma^{-}_{e^{\tau/2},\pi/2}$ otherwise.

We postpone the proof of these claims and proceed with the proof of Theorem 2.4.2.

a) If $\chi > 0$ then the part of the Pfaffian coming from $++$ and $--$ blocks (two $n \times n$ blocks on the main diagonal) will not contribute to the limit. This is because every term in the Pfaffian contains equally many K_{++} and K_{--} factors. Now, by (2.4.2) we have that

$K_{++}^{1,2} \cdot K_{--}^{3,4} = \text{const}(-1)^{x_2+x_3} \cdot I_{++}I_{--}$ and then by Claim 4 and Claim 5 we have that $K_{++}^{1,2} \cdot K_{--}^{3,4} \to 0$ since $I_{++} \to 0$ and $I_{--} \to 0$.

This means that the Pfaffian reduces to the determinant of $+-$ block, because

$$\text{Pf}\begin{bmatrix} 0 & A \\ -A & 0 \end{bmatrix} = \det A.$$

By Claim 1 we have that

$$K_{+-}^{1,2} = e^{-\tau(x_1-x_2)}\frac{1}{2\pi i}\int_{\gamma^{\pm}_{e^{-\tau/2},\theta}} \left(\frac{1-z}{1+z}\right)^{\Delta t_{1,2}} \frac{1}{z^{\Delta x_{1,2}+1}}dz,$$

and it is easily verified that $e^{\tau(\dots)}$ prefactors cancel out in the determinant.

Thus, if $\chi > 0$ then for

$$rt_i \to \tau, \quad rx_i \to \chi \quad \text{as } r \to +0,$$

$$t_i - t_j = \Delta t_{ij} = \text{const}, \quad x_i - x_j = \Delta x_{ij} = \text{const},$$

we have that

$$\rho(X) \to \det[K(i,j)]_{i,j=1}^n,$$

where $K(i,j)$ is given in the statement of the theorem.

b) Now, if $\chi = 0$ and x_i does not depend on r, then by Claims 1, 4 and 5 we have that

$$\begin{aligned} K_{++}^{1,2} &= (-1)^{x_2}e^{-\tau(x_1+x_2)/2}\frac{1}{2\pi i}\int_{\gamma^{\pm}_{e^{\tau/2},\theta}} \frac{1}{z}\left(\frac{1-e^{-\tau}z}{1+e^{-\tau}z}\right)^{\Delta t_{1,2}}\left(\frac{z}{e^{\tau/2}}\right)^{-(x_1+x_2)}dz \\ &= (-1)^{x_2}e^{-\tau(x_1+x_2)}\frac{1}{2\pi i}\int_{\gamma^{\pm}_{e^{-\tau/2},\theta}} \left(\frac{1-z}{1+z}\right)^{\Delta t_{1,2}} \frac{1}{z^{x_1+x_2+1}}dz, \end{aligned}$$

$$\begin{aligned} K_{--}^{3,4} &= (-1)^{x_3}e^{\tau(x_3+x_4)/2}\frac{1}{2\pi i}\int_{\gamma^{\pm}_{e^{\tau/2},\theta}} \frac{1}{z}\left(\frac{1-e^{-\tau}z}{1+e^{-\tau}z}\right)^{\Delta t_{3,4}}\left(\frac{z}{e^{\tau/2}}\right)^{x_3+x_4}dz \\ &= (-1)^{x_3}e^{\tau(x_3+x_4)}\frac{1}{2\pi i}\int_{\gamma^{\pm}_{e^{-\tau/2},\theta}} \left(\frac{1-z}{1+z}\right)^{\Delta t_{3,4}} \frac{1}{z^{-(x_3+x_4)+1}}dz \end{aligned}$$

and

$$K^{1,2}_{+-} = e^{-\tau(x_1-x_2)}\frac{1}{2\pi i}\int_{\gamma^{\pm}_{e^{-\tau/2},\theta}}\left(\frac{1-z}{1+z}\right)^{\Delta t_{1,2}}\frac{1}{z^{(x_1-x_2)+1}}dz.$$

As before, it is easily verified that $e^{\tau(\dots)}$ prefactors cancel out in the Pfaffian.

Hence, if $\chi = 0$ then for

$$rt_i \to \tau, \quad rx_i \to \chi \quad \text{as } r \to +0,$$

$$t_i - t_j = \Delta t_{ij} = \text{const}, \quad x_i - x_j = \Delta x_{ij} = \text{const}, \quad x_i = \text{const}$$

we have that

$$\rho(X) \to \text{Pf}[M(i,j)]^{2n}_{i,j=1},$$

where $M(i,j)$ is given in the statement of the theorem.

It remains to prove Claims 1 through 5.

Proof. (Claim 1) In order to compute the limit of I_{+-} we focus on its exponentially large term. Since $(z+w)/2zw(z-w)$ remains bounded away from $z = w$, $z = 0$ and $w = 0$, the exponentially large term comes from

$$\frac{J(t_1,z)}{J(t_2,w)}\left(\frac{z}{e^{\tau/2}}\right)^{-x_1}\left(\frac{w}{e^{\tau/2}}\right)^{x_2} = \frac{\exp\left[\log J(t_1,z) - x_1(\log z - \tau/2)\right]}{\exp\left[\log J(t_2,w) - x_2(\log w - \tau/2)\right]}. \tag{2.4.3}$$

To determine a behavior of this term we need to know the asymptotics of $\log(z;q)_\infty$ when $r \to +0$. Recall that the dilogarithm function is defined by

$$\text{dilog}(z) = \sum_{n=1}^{\infty}\frac{(1-z)^n}{n^2}, \quad |1-z| \le 1,$$

with the analytic continuation given by

$$\text{dilog}(1-z) = \int_1^z \frac{\log(t)}{1-t}dt,$$

with the negative real axis as a branch cut. Then

$$\log(z;q)_\infty = -\frac{1}{r}\text{dilog}(1-z) + O(\text{dist}(1, rz|0 \le r \le 1)^{-1}), \quad r \to +0.$$

(see e.g., [Bor07]). Hence, (2.4.3) when $r \to +0$ behaves like

$$\exp \frac{1}{r} \left[S(z,\tau,\chi) - S(w,\tau,\chi) \right] + O(1),$$

where

$$\begin{aligned} S(z,\tau,\chi) &= -\text{dilog}(1+e^{-\tau}z) - \text{dilog}(1-\frac{1}{z}) + \text{dilog}(1-e^{-\tau}z) \\ &\quad +\text{dilog}(1+\frac{1}{z}) - \chi(\log z - \tau/2). \end{aligned}$$

The real part of $S(z,\tau,\chi)$ vanishes for $|z| = e^{\tau/2}$. We want to find the way it changes when we move from the circle $|z| = e^{\tau/2}$. For that we need to find

$$\frac{d}{dR}\text{Re}S(z,\tau,\chi).$$

On the circle, $\text{Re}S(z,\tau,\chi) = 0$ implies that the derivative in the tangent direction vanishes, i.e.,

$$y\frac{d\text{Re}S}{dx} - x\frac{d\text{Re}S}{dy} = 0.$$

The Cauchy-Riemann equations on $|z| = e^{\tau/2}$ then yield

$$R\frac{d}{dR}\text{Re}S(z,\tau,\chi) = x\frac{d\text{Re}S}{dx} + y\frac{d\text{Re}S}{dy} = z\frac{d}{dz}S(z,\tau,\chi).$$

Simple calculation gives

$$z\frac{d}{dz}S(z,\tau,\chi) = -\chi + \log\frac{(1+e^{-\tau}z)(z+1)}{(1-e^{-\tau}z)(z-1)}$$

which implies that

$$z\frac{d}{dz}S(z,\tau,\chi) = -\chi + \log\left|\frac{z+1}{z-1}\right|^2, \quad \text{for } |z| = e^{\tau/2}.$$

Then

$$z\frac{d}{dz}S(z,\tau,\chi) > 0 \quad \text{iff} \quad e^{\chi/2} < \left|\frac{z+1}{z-1}\right|, \quad \text{for } |z| = e^{\tau/2}. \tag{2.4.4}$$

One easily computes that $e^{\chi/2} = \left|\frac{z+1}{z-1}\right|$ is a circle with center $\frac{R^2+1}{R^2-1}$ and radius $\frac{2R}{R^2-1}$ where

$R = e^{\chi/2}$. If

$$\frac{(e^{\tau}+1)(e^{\chi}-1)}{2e^{\tau/2}(e^{\chi}+1)} > 1 \tag{2.4.5}$$

then this circle does not intersect $|z| = e^{\tau/2}$ and $\frac{d}{dR}\mathrm{Re}S(z,\tau,\chi)$ is negative for every point z such that $|z| = e^{\tau/2}$. Otherwise, this circle intersects $|z| = e^{\tau/2}$ at $z = e^{\tau/2}e^{\pm i\theta}$ where θ is given with (2.4.1). Thus, the sign of $\frac{d}{dR}\mathrm{Re}S(z,\tau,\chi)$ changes at these points.

Let γ_z and γ_w be as in Figure 2.8. We pick them in such a way that the real parts of $S(z,\tau,\chi)$ and $S(w,\tau,\chi)$ are negative everywhere on these contours except at the two points with the argument equal to $\pm\theta$ (if (2.4.5) is satisfied they are negative at these points too). This is possible by (2.4.4).

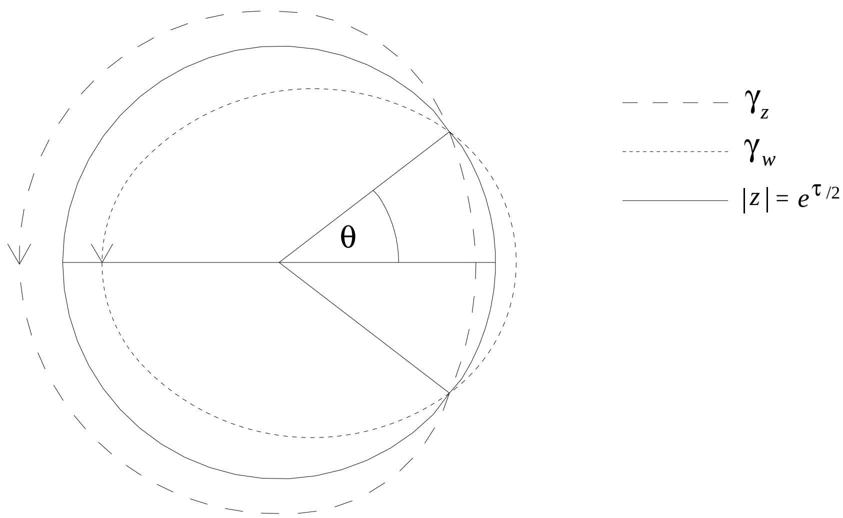


Figure 2.8: Contours γ_z and γ_w

The reason we are introducing γ_z and γ_w is that we want to deform the contours in I_{+-} to these two and then use that

$$\int_{\gamma} e^{\frac{S(z)}{r}} dz \to 0, \quad r \to +0.$$

whenever S has a negative real part for all but finitely many points on γ . When deforming contours we will need to include the contribution coming from the poles and that way the integral I_{+-} will be a sum of integrals where the first one vanishes as $r \to +0$ and the other nonvanishing one comes as a residue.

Let $f(z,w)$ be the integrand in I_{+-}. We consider two cases a) $t_1 \geq t_2$ and b) $t_1 < t_2$ separately.

Case a) $t_1 \geq t_2$. We omit the integrand $f(z,w)dzdw$ in the formulas below.

$$\begin{aligned}
I_{+-} &= \frac{1}{(2\pi i)^2}\int\limits_{|z|=(1+\epsilon)e^{\tau/2}}\ \int\limits_{|w|=(1-\epsilon)e^{\tau/2}} = \frac{1}{(2\pi i)^2}\int\limits_{\gamma_z}\ \int\limits_{|w|=(1-\epsilon)e^{\tau/2}} \\
&= \frac{1}{(2\pi i)^2}\left[-\int_{\gamma^-_{z,\theta}}\int_{|w|=(1-\epsilon)e^{\tau/2}} + \int_{\gamma^+_{z,\theta}}\int_{|w|=(1-\epsilon)e^{\tau/2}}\right] \\
&= \frac{1}{(2\pi i)^2}\left[-\int_{\gamma^-_{z,\theta}}\int_{\gamma_w} + \int_{\gamma^+_{z,\theta}}\int_{\gamma_w} - 2\pi i\int_{\gamma^+_{z,\theta}} \mathrm{Res}_{w=z} f(z,w)dz\right] \\
&= \frac{1}{(2\pi i)^2}\left[\int_{\gamma_z}\int_{\gamma_w} - 2\pi i\int_{\gamma^+_{z,\theta}} \mathrm{Res}_{w=z} f(z,w)dz\right] \\
&= I_1 - \frac{1}{2\pi i}\int_{\gamma^+_{e^{\tau/2},\theta}} \frac{2z}{-2z^2}\frac{J(t_1,z)}{J(t_2,z)}\cdot\left(\frac{z}{e^{\tau/2}}\right)^{-x_1}\cdot\left(\frac{z}{e^{\tau/2}}\right)^{x_2} dz \\
&= I_1 + \frac{1}{2\pi i}\int_{\gamma^+_{e^{\tau/2},\theta}} \frac{1}{z}\frac{J(t_1,z)}{J(t_2,z)}\cdot\left(\frac{z}{e^{\tau/2}}\right)^{-\Delta x_{1,2}} dz.
\end{aligned}$$

The first integral denoted with I_1 vanishes as $r \to +0$, while the other one goes to the integral from our claim since

$$\lim_{r\to+0}\frac{J(t_1,z)}{J(t_2,z)} = \left(\frac{1-e^{-\tau}z}{1+e^{-\tau}z}\right)^{\Delta t_{1,2}}.$$

Case b) $t_1 < t_2$ is handled similarly

$$\begin{aligned}
I_{+-} &= \frac{1}{(2\pi i)^2}\int\limits_{|z|=(1-\epsilon)e^{\tau/2}}\ \int\limits_{|w|=(1+\epsilon)e^{\tau/2}} = \frac{1}{(2\pi i)^2}\int\limits_{\gamma_z}\ \int\limits_{|w|=(1+\epsilon)e^{\tau/2}} \\
&= \frac{1}{(2\pi i)^2}\left[\int_{\gamma_z}\int_{\gamma_w} - 2\pi i\int_{\gamma^-_{e^{\tau/2},\theta}} \mathrm{Res}_{w=z} f(z,w)dz\right] \\
&= I_2 - \frac{1}{2\pi i}\int_{\gamma^-_{e^{\tau/2},\theta}} \frac{2z}{-2z^2}\frac{J(t_1,z)}{J(t_2,z)}\cdot\left(\frac{z}{e^{\tau/2}}\right)^{-x_1}\cdot\left(\frac{z}{e^{\tau/2}}\right)^{x_2} dz \\
&= I_2 + \frac{1}{2\pi i}\int_{\gamma^-_{e^{\tau/2},\theta}} \frac{1}{z}\frac{J(t_1,z)}{J(t_2,z)}\cdot\left(\frac{z}{e^{\tau/2}}\right)^{-\Delta x_{1,2}} dz.
\end{aligned}$$

□

Proof. (Claim 2) The exponentially large term of I_{++} comes from

$$\frac{J(t_1,z)}{J(t_2,w)}\left(\frac{z}{e^{\tau/2}}\right)^{-x_1}\left(\frac{w}{e^{\tau/2}}\right)^{-x_2}=\frac{\exp\left[\log J(t_1,z)-x_1(\log z-\tau/2)\right]}{\exp\left[\log J(t_2,w)+x_2(\log w-\tau/2)\right]}$$

that when $r\to +0$ behaves like

$$\exp\frac{1}{r}\left[S(z,\tau,\chi)+T(w,\tau,\chi)\right]+O(1),$$

where

$$\begin{aligned} S(z,\tau,\chi) \;&=\; -\mathrm{dilog}(1+e^{-\tau}z)-\mathrm{dilog}(1-\frac{1}{z})+\mathrm{dilog}(1-e^{-\tau}z)\\ &\quad +\mathrm{dilog}(1+\frac{1}{z})-\chi(\log z-\tau/2)\end{aligned}$$

and

$$\begin{aligned} T(w,\tau,\chi) \;&=\; \mathrm{dilog}(1+e^{-\tau}w)+\mathrm{dilog}(1-\frac{1}{w})-\mathrm{dilog}(1-e^{-\tau}w)\\ &\quad -\mathrm{dilog}(1+\frac{1}{w})-\chi(\log w-\tau/2).\end{aligned}$$

Real parts of $S(z,\tau,\chi)$ and $T(w,\tau,\chi)$ vanish for $|z|=e^{\tau/2}$ and $|w|=e^{\tau/2}$, respectively. To see the way they change when we move from these circles we need $\frac{d}{dR}\mathrm{Re}S(z,\tau,\chi)$ and $\frac{d}{dR}\mathrm{Re}T(w,\tau,\chi)$. For $|z|=e^{\tau/2}$ and $|w|=e^{\tau/2}$ they are equal to $\frac{z}{R}\frac{d}{dz}S(z,\tau,\chi)$ and $\frac{w}{R}\frac{d}{dz}T(w,\tau,\chi)$, respectively.

The needed estimate for S is (2.4.4) above.

Similarly,

$$w\frac{d}{dw}T(w,\tau,\chi)>0 \quad \text{iff} \quad e^{\chi/2}<\left|\frac{w-1}{w+1}\right|, \quad \text{for } |w|=e^{\tau/2}.$$

Hence, if (2.4.5) holds then both $\frac{d}{dR}\mathrm{Re}S(z,\tau,\chi)$ and $\frac{d}{dR}\mathrm{Re}T(z,\tau,\chi)$ are negative for every point z such that $|z|=e^{\tau/2}$. Otherwise, the sign of $\frac{d}{dR}\mathrm{Re}S(z,\tau,\chi)$ changes at $z=e^{\tau/2}e^{\pm i\theta}$, while the sign of $\frac{d}{dR}\mathrm{Re}T(w,\tau,\chi)$ changes at $w=e^{\tau/2}e^{\pm i(\pi-\theta)}$.

We deform the contours in I_{++} to γ_z and γ_w shown in Figure 2.9 because the real parts of S and T are negative on these contours (except for finitely many points).

As before, we distinguish two cases a) $t_1\geq t_2$ and b) $t_1<t_2$.

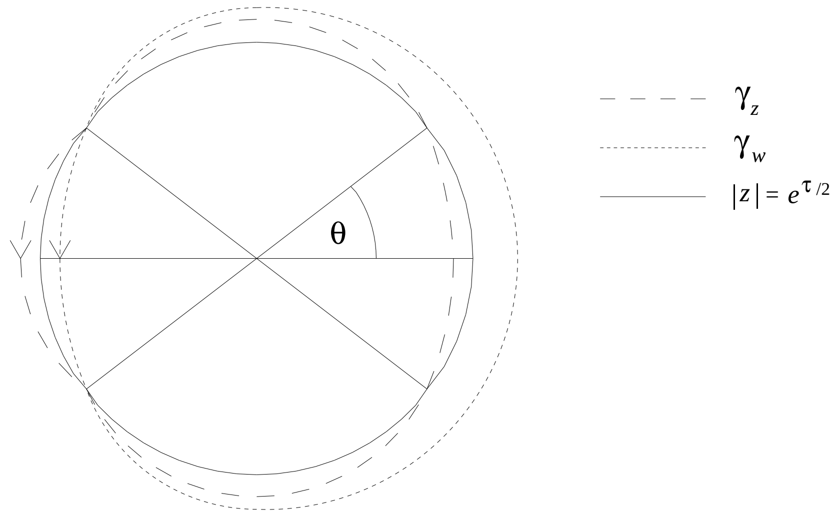


Figure 2.9: Contours γ_z and γ_w

For a) $t_1 \geq t_2$ (where we again omit $f(z,w)dzdw$)

$$
\begin{aligned}
I_{++} &= \frac{1}{(2\pi i)^2} \int_{|z|=(1+\epsilon)e^{\tau/2}} \int_{|w|=(1-\epsilon)e^{\tau/2}} = \frac{1}{(2\pi i)^2} \int_{\gamma_z} \int_{|w|=(1-\epsilon)e^{\tau/2}} \\
&= \frac{1}{(2\pi i)^2} \left[-\int_{\gamma^-_{z,\pi-\theta}} \int_{|w|=(1-\epsilon)e^{\tau/2}} + \int_{\gamma^+_{z,\pi-\theta}} \int_{|w|=(1-\epsilon)e^{\tau/2}} \right] \\
&= \frac{1}{(2\pi i)^2} \left[-\int_{\gamma^-_{z,\pi-\theta}} \int_{\gamma_w} + \int_{\gamma^+_{z,\pi-\theta}} \int_{\gamma_w} - 2\pi i \int_{\gamma^+_{z,\pi-\theta}} \mathrm{Res}_{w=z} f(z,w) dz \right] \\
&= \frac{1}{(2\pi i)^2} \left[\int_{\gamma_z} \int_{\gamma_w} - 2\pi i \int_{\gamma^+_{e^{\tau/2},\pi-\theta}} \mathrm{Res}_{w=z} f(z,w) dz \right] \\
&= I_1 - \frac{1}{2\pi i} \int_{\gamma^+_{e^{\tau/2},\pi-\theta}} \frac{2z}{-2z^2} \frac{J(t_1,z)}{J(t_2,z)} \cdot \left(\frac{z}{e^{\tau/2}} \right)^{-x_1} \cdot \left(\frac{z}{e^{\tau/2}} \right)^{-x_2} dz.
\end{aligned}
$$

For the same reason as in the proof of Claim 1 we have that $I_1 \to 0$ when $r \to +0$. Thus,

$\lim_{r\to+0} I_{++} = \lim_{r\to+0} I_2$, where

$$\begin{aligned} I_2 &= \frac{1}{2\pi i}\int_{\gamma^+_{e^{\tau/2},\pi-\theta}} \frac{1}{z}\frac{J(t_1,z)}{J(t_2,z)}\cdot\left(\frac{z}{e^{\tau/2}}\right)^{-x_1}\cdot\left(\frac{z}{e^{\tau/2}}\right)^{-x_2} dz \\ &= \frac{1}{2\pi i}\int_{\gamma^-_{e^{\tau/2},\theta}} \frac{1}{-z}\frac{J(t_1,-z)}{J(t_2,-z)}\cdot\left(\frac{-z}{e^{\tau/2}}\right)^{-x_1}\cdot\left(\frac{-z}{e^{\tau/2}}\right)^{-x_2} dz \\ &= (-1)^{x_1+x_2+1}\cdot\frac{1}{2\pi i}\int_{\gamma^-_{e^{\tau/2},\theta}} \frac{1}{z}\frac{J(t_2,z)}{J(t_1,z)}\left(\frac{z}{e^{\tau/2}}\right)^{-(x_1+x_2)} dz. \end{aligned}$$

For b) $t_1 < t_2$

$$\begin{aligned} I_{++} &= \frac{1}{(2\pi i)^2}\int_{|z|=(1-\epsilon)e^{\tau/2}}\int_{|w|=(1+\epsilon)e^{\tau/2}} = \frac{1}{(2\pi i)^2}\int_{\gamma_z}\int_{|w|=(1+\epsilon)e^{\tau/2}} \\ &= \frac{1}{(2\pi i)^2}\left[\int_{\gamma_z}\int_{\gamma_w} -2\pi i\int_{\gamma^-_{e^{\tau/2},\pi-\theta}} \mathrm{Res}_{w=z} f(z,w)dz\right] \\ &= I_3 - \frac{1}{2\pi i}\int_{\gamma^-_{e^{\tau/2},\pi-\theta}} \frac{2z}{-2z^2}\frac{J(t_1,z)}{J(t_2,z)}\cdot\left(\frac{z}{e^{\tau/2}}\right)^{-x_1}\cdot\left(\frac{z}{e^{\tau/2}}\right)^{-x_2} dz. \end{aligned}$$

Thus, $\lim_{r\to+0} I_{++} = \lim_{r\to+0} I_4$, where

$$\begin{aligned} I_4 &= \frac{1}{2\pi i}\int_{\gamma^-_{e^{\tau/2},\pi-\theta}} \frac{1}{z}\frac{J(t_1,z)}{J(t_2,z)}\cdot\left(\frac{z}{e^{\tau/2}}\right)^{-x_1}\cdot\left(\frac{z}{e^{\tau/2}}\right)^{-x_2} dz \\ &= \frac{1}{2\pi i}\int_{\gamma^+_{e^{\tau/2},\theta}} \frac{1}{-z}\frac{J(t_1,-z)}{J(t_2,-z)}\cdot\left(\frac{-z}{e^{\tau/2}}\right)^{-x_1}\cdot\left(\frac{-z}{e^{\tau/2}}\right)^{-x_2} dz \\ &= (-1)^{x_1+x_2+1}\cdot\frac{1}{2\pi i}\int_{\gamma^+_{e^{\tau/2},\theta}} \frac{1}{z}\frac{J(t_2,z)}{J(t_1,z)}\left(\frac{z}{e^{\tau/2}}\right)^{-(x_1+x_2)} dz. \end{aligned}$$

Therefore, we conclude that

$$\lim_{r\to+0} I_{++} = \lim_{r\to+0} I,$$

with

$$I = (-1)^{x_1+x_2+1}\cdot\frac{1}{2\pi i}\int_{\gamma^{\mp}_{e^{\tau/2},\theta}} \frac{1}{z}\frac{J(t_2,z)}{J(t_1,z)}\left(\frac{z}{e^{\tau/2}}\right)^{-(x_1+x_2)} dz,$$

where we choose $\gamma^-_{e^{\tau/2},\theta}$ if $t_1 \geq t_2$ and $\gamma^+_{e^{\tau/2},\theta}$ otherwise. $\square$

Proof. (Claim 3) The exponentially large term of I_{--} comes from

$$\frac{J(t_3,z)}{J(t_4,w)}\left(\frac{z}{e^{\tau/2}}\right)^{x_3}\left(\frac{w}{e^{\tau/2}}\right)^{x_4}=\frac{\exp\left[\log J(t_3,z)+x_3(\log z-\tau/2)\right]}{\exp\left[\log J(t_4,w)-x_4(\log w-\tau/2)\right]}$$

that when $r\to+0$ behaves like

$$\exp\frac{1}{r}\left[S(z,\tau,\chi)+T(w,\tau,\chi)\right]+O(1),$$

where

$$\begin{aligned}S(z,\tau,\chi) &= -\mathrm{dilog}(1+e^{-\tau}z)-\mathrm{dilog}(1-\frac{1}{z})+\mathrm{dilog}(1-e^{-\tau}z)\\ &\quad+\mathrm{dilog}(1+\frac{1}{z})+\chi(\log z-\tau/2)\end{aligned}$$

and

$$\begin{aligned}T(w,\tau,\chi) &= \mathrm{dilog}(1+e^{-\tau}w)+\mathrm{dilog}(1-\frac{1}{w})-\mathrm{dilog}(1-e^{-\tau}w)\\ &\quad-\mathrm{dilog}(1+\frac{1}{w})+\chi(\log w-\tau/2).\end{aligned}$$

As before, real parts of $S(z,\tau,\chi)$ and $T(w,\tau,\chi)$ vanish for $|z|=e^{\tau/2}$ and $|w|=e^{\tau/2}$, respectively.

Since

$$z\frac{d}{dz}S(z,\tau,\chi)>0 \quad \text{iff} \quad e^{\chi/2}>\left|\frac{z-1}{z+1}\right| \quad \text{for } |z|=e^{\tau/2}$$

and

$$w\frac{d}{dw}T(w,\tau,\chi)>0 \quad \text{iff} \quad e^{\chi/2}>\left|\frac{w+1}{w-1}\right| \quad \text{for } |w|=e^{\tau/2},$$

we deform contours in I_{--} to γ_z and γ_w shown in Figure 2.10.

Using the same reasoning as in the proof of Claims 1 and 2 we get that

$$\lim_{r\to+0}I_{--}=\lim_{r\to+0}I,$$

with

$$I=\frac{1}{2\pi i}\int_{\gamma^{\pm}_{e^{\tau/2},\theta}}\frac{1}{z}\frac{J(t_3,z)}{J(t_4,z)}\left(\frac{z}{e^{\tau/2}}\right)^{x_3+x_4}dz,$$

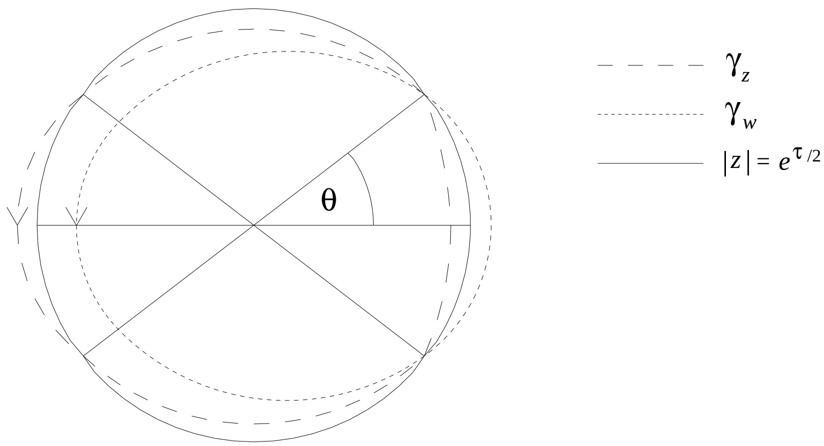


Figure 2.10: Contours γ_z and γ_w

where we choose $\gamma^+_{e^{\tau/2},\theta}$ if $t_3 \geq t_4$ and $\gamma^-_{e^{\tau/2},\theta}$ otherwise. □

Proof. (Claim 4) We start from the integral on the right hand side in Claim 2.

Because

$$\lim_{r \to +0} \frac{J(z,t_1)}{J(z,t_2)} = \left(\frac{1 - e^{-\tau} z}{1 + e^{-\tau} z} \right)^{\Delta t_{1,2}}, \tag{2.4.6}$$

we focus on

$$\lim_{r \to +0} \left(\frac{z}{e^{\tau/2}} \right)^{-(x_1+x_2)} = \exp\left[\frac{S(z,\tau,\chi)}{r} \right],$$

where

$$S(z,\tau,\chi) = -2\chi \log z + \chi\tau.$$

Assume $\chi > 0$. Then $\mathrm{Re} S(z,\tau,\chi) > 0$ if and only if $|z| < e^{\tau/2}$.

For $\Delta t_{1,2} \geq 0$ we deform the contour $\gamma^-_{e^{\tau/2},\theta}$ to γ_z (see Figure 2.11). Using the same argument as in the proof of Claim 1 we get that $\lim_{r \to +0} I_{++} = 0$ for $\Delta t_{1,2} \geq 0$. A similar argument can be given for $\Delta t_{1,2} < 0$.

For $\chi = 0$ the claim follows directly from Claim 2 changing $z \mapsto -z$ and using (2.4.6). □

Proof. (Claim 5) The proof is similar to the proof of Claim 4. □

The proof of Theorem 2.4.2 is now complete. □

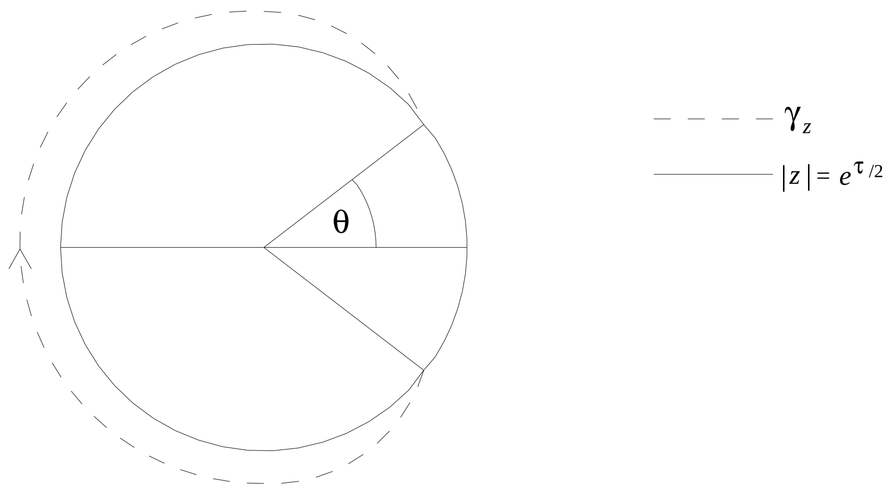


Figure 2.11: Contour γ_z

2.5 Appendix

2.5.1 Strict partitions

A strict partition is a sequence of strictly decreasing integers such that only finitely many of them are nonzero. The nonzero integers are called parts. Let throughout this section $\lambda = (\lambda_1, \lambda_2, \dots)$ and $\mu = (\mu_1, \mu_2, \dots)$ be strict partitions.

The empty partition is $\emptyset = (0, 0, \dots)$.

The weight of λ is $|\lambda| = \sum \lambda_i$.

The length of λ is $l(\lambda) = \#$ of parts of λ.

The diagram of λ is the set of points $(i, j) \in \mathbb{Z}^2$ such that $1 \leq j \leq \lambda_i$. The points of the diagram are represented by 1×1 boxes. The shifted diagram of λ is the set of points $(i, j) \in \mathbb{Z}^2$ such that $i \leq j \leq \lambda_i + i - 1$.

The diagram and the shifted diagram of $\lambda = (5, 3, 2)$ are shown in Figure 2.12.

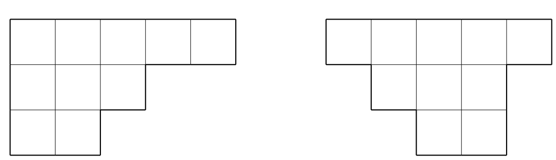


Figure 2.12: Diagram and shifted diagram of $(5, 3, 2)$

A partition μ is a subset of λ if $\mu_i \leq \lambda_i$ for every i. We write $\mu \subset \lambda$ or $\lambda \supset \mu$ in that case. This means that the diagram of λ contains the diagram of μ.

If $\lambda \supset \mu$, the (shifted) skew diagram $\lambda - \mu$ is the set theoretic difference of the (shifted) diagrams of λ and μ.

If $\lambda = (5, 3, 2)$ and $\mu = (4, 1)$ then $\lambda \supset \mu$ with the skew and the shifted skew diagram $\lambda - \mu$ as in Figure 2.13.

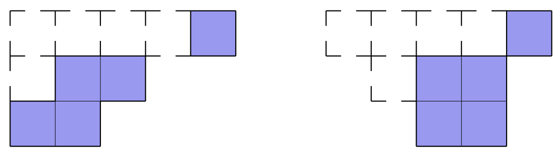


Figure 2.13: Skew diagram and shifted skew diagram of $(5, 3, 2) - (4, 1)$

A skew diagram is a horizontal strip if it does not contain more than one box in each column. The given example is not a horizontal strip.

A connected part of a (shifted) skew diagram that contains no 2×2 block of boxes is called a border strip. The skew diagram of the example above has two border strips.

If $\lambda \supset \mu$ and $\lambda - \mu$ is a horizontal strip then we define $a(\lambda - \mu)$ as the number of integers $i \geq 1$ such that the skew diagram $\lambda - \mu$ has a box in the i^{th} column but not in the $(i+1)$st column or, equivalently, as the number of mutually disconnected border strips of the shifted skew diagram $\lambda - \mu$.

Let P be a totally ordered set

$$P = \{1 < 1' < 2 < 2' < \cdots\}.$$

We distinguish elements in P as unmarked and marked, the latter being one with a prime. We use $|p|$ for the unmarked number corresponding to $p \in P$.

A marked shifted (skew) tableau is a shifted (skew) diagram filled with row and column nonincreasing elements from P such that any given unmarked element occurs at most once in each column whereas any marked element occurs at most once in each row. Examples of a marked shifted tableau and a marked shifted skew tableau are

$$\begin{array}{ccccc} 5 & 3' & 2' & 1' & 1 \\ & 3 & 2' & 1' & \\ & & 1 & 1 & \end{array} \qquad\qquad \begin{array}{ccc} & 1' & 1 \\ 2' & 1' & \\ 1 & 1 & \end{array}$$

For $x = (x_1, x_2, \ldots, x_n)$ and a marked shifted (skew) tableau T we use x^T to denote

$x_1^{a_1}x_2^{a_2}\dots x_n^{a_n}$, where a_i is equal to the number of elements p in T such that $|p| = i$.

The skew Schur function $Q_{\lambda/\mu}$ is a symmetric function defined as

$$Q_{\lambda/\mu}(x_1, x_2, \dots, x_n) = \begin{cases} \sum_T x^{|T|}, & \lambda \supset \mu, \\ 0, & \text{otherwise}, \end{cases} \tag{2.5.1}$$

where the sum is taken over all marked skew shifted tableaux of shape $\lambda - \mu$ filled with elements $p \in P$ such that $|p| \le n$. Skew Schur $P_{\lambda/\mu}$ function is defined as $P_{\lambda/\mu} = 2^{l(\mu)-l(\lambda)}Q_{\lambda/\mu}$. Also, one denotes $P_\lambda = P_{\lambda/\emptyset}$ and $Q_\lambda = Q_{\lambda/\emptyset}$.

This is just one of many ways of defining Schur P and Q functions (see Chapter 3 of [Mac95]).

We set

$$H(x, y) = \prod_{i,j} \frac{1 + x_i y_j}{1 - x_i y_j} \tag{2.5.2}$$

and

$$F(x; z) = \prod_{i=1}^{\infty} \frac{1 + x_i z}{1 - x_i z}. \tag{2.5.3}$$

$F(x; z)$ is denoted with $Q_x(z)$ in [Mac95].

Let Λ be the algebra of symmetric functions. A specialization of Λ is an algebra homomorphism $\Lambda \to \mathbb{C}$. If ρ is a specialization of Λ then we write $P_{\lambda/\mu}(\rho)$ and $Q_{\lambda/\mu}(\rho)$ for the images of $P_{\lambda/\mu}$ and $Q_{\lambda/\mu}$, respectively. Every map $\rho_x : (x_1, x_2, \dots) \to (a_1, a_2, \dots)$ where $a_i \in \mathbb{C}$ and only finitely many a_i's are nonzero defines a specialization. For that case the definition (2.5.1) is convenient for determining $Q_{\lambda/\mu}(\rho_x)$. We use $H(\rho_x, \rho_y)$ and $F(\rho_x; z)$ for the images of (2.5.2) and (2.5.3) under $\rho_x \otimes \rho_y$ and ρ_x, respectively.

We recall some facts that can be found in Chapter 3 of [Mac95]:

$$H(x, y) = \sum_{\lambda \text{ strict}} Q_\lambda(x) P_\lambda(y) = \prod_{i,j} \frac{1 + x_i y_j}{1 - x_i y_j}, \tag{2.5.4}$$

$$Q_\lambda(x, z) = \sum_{\mu \text{ strict}} Q_{\lambda/\mu}(x) Q_\mu(z), \tag{2.5.5}$$

$$P_\lambda(x, z) = \sum_{\mu \text{ strict}} P_{\lambda/\mu}(x) P_\mu(z), \tag{2.5.6}$$

$$Q_{\lambda/\mu}(x) = \sum_{\nu \text{ strict}} f^{\lambda}_{\mu\nu} Q_\nu(x), \tag{2.5.7}$$

$$P_\mu(x)P_\nu(x) = \sum_{\lambda \text{ strict}} f^{\lambda}_{\nu\mu} P_\lambda(x), \tag{2.5.8}$$

for $f^{\lambda}_{\mu\nu} \in \mathbb{Z}$.

Proposition 2.5.1.

$$\sum_{\lambda \text{ strict}} Q_{\lambda/\mu}(x)P_{\lambda/\nu}(y) = H(x,y) \sum_{\tau \text{ strict}} Q_{\nu/\tau}(x)P_{\mu/\tau}(y)$$

Proof. Let $H = H(x,y)H(x,u)H(z,y)H(z,u)$. Then by (2.5.4) we have

$$H = \sum_{\lambda} Q_\lambda(x,z)P_\lambda(y,u).$$

We can compute H in two different ways using (2.5.5), (2.5.6), (2.5.7) and (2.5.8). One way we get

$$\begin{aligned} H &= \sum_{\lambda} Q_\lambda(x,z)P_\lambda(y,u) = \sum_{\lambda,\mu,\nu} Q_{\lambda/\mu}(x)Q_\mu(z)P_{\lambda/\nu}(y)P_\nu(u) \\ &= \sum_{\mu,\nu}\left[\sum_{\lambda} Q_{\lambda/\mu}(x)P_{\lambda/\nu}(y)\right] Q_\mu(z)P_\nu(u). \end{aligned} \tag{2.5.9}$$

The other way we get

$$\begin{aligned} H &= H(x,y)\sum_{\sigma,\rho,\tau} Q_\sigma(x)P_\sigma(u)Q_\rho(z)P_\rho(y)Q_\tau(z)P_\tau(u) \\ &= H(x,y)\sum_{\sigma,\rho,\tau} Q_\sigma(x)P_\rho(y)(Q_\rho(z)Q_\tau(z))(P_\sigma(u)P_\tau(u)) \\ &= H(x,y)\sum_{\sigma,\rho,\tau} Q_\sigma(x)P_\rho(y)\sum_{\mu} 2^{-l(\mu)}2^{l(\rho)}2^{l(\tau)} f^{\mu}_{\rho\tau} Q_\mu(z)\sum_{\nu} f^{\nu}_{\sigma\tau}P_\nu(u) \\ &= H(x,y)\sum_{\tau,\mu,\nu} Q_\mu(z)P_\nu(u)(\sum_{\rho} 2^{l(\tau)}2^{l(\rho)}2^{-l(\mu)} f^{\mu}_{\rho\tau}P_\rho(y))(\sum_{\sigma} f^{\nu}_{\sigma\tau}Q_\sigma(x)) \\ &= H(x,y)\sum_{\tau,\mu,\nu} Q_\mu(z)P_\nu(u)P_{\mu/\tau}(y)Q_{\nu/\tau}(x) \\ &= \sum_{\mu,\nu}\left[H(x,y)\sum_{\tau} P_{\mu/\tau}(y)Q_{\nu/\tau}(x)\right] Q_\mu(z)P_\nu(u). \end{aligned} \tag{2.5.10}$$

Now, the proposition follows from (2.5.9) and (2.5.10). □

We give another proof of this proposition at the end of Section 2.5.2.

2.5.2 Fock space associated with strict partitions

We introduce a Fock space associated with strict partitions. We follow [Mat05]. See also [DJKM82].

Let V be a vector space generated by vectors

$$v_\lambda = e_{\lambda_1} \wedge e_{\lambda_2} \wedge \cdots \wedge e_{\lambda_l},$$

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ is a strict partition. In particular, $v_\emptyset = 1$ and is called the vacuum.

For every $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ we define two (creation and annihilation) operators ψ_k and ψ_k^*. These two operators are adjoint to each other for the inner product defined by

$$\langle v_\lambda, v_\mu \rangle = 2^{-l(\lambda)} \delta_{\lambda,\mu}.$$

For $k = 0$, let

$$\psi_0 v_\lambda = \psi_0^* v_\lambda = \frac{(-1)^{l(\lambda)}}{2} v_\lambda.$$

For $k \geq 1$, the operator ψ_k adds e_k, on the left while ψ_k^* removes e_k, on the left and divides by 2. More precisely,

$$\begin{aligned} \psi_k v_\lambda &= e_k \wedge v_\lambda, \\ \psi_k^* v_\lambda &= \sum_{i=1}^{l(\lambda)} \frac{(-1)^{i-1}}{2} \delta_{k,\lambda_i} e_{\lambda_1} \wedge \cdots \wedge \widehat{e}_{\lambda_i} \wedge \cdots e_{\lambda_l}. \end{aligned}$$

These operators satisfy the following anti-commutation relations

$$\begin{aligned} \psi_i \psi_j^* + \psi_j^* \psi_i &= \frac{1}{2}\delta_{i,j}, & (i,j) &\in \mathbb{N}_0^2, \\ \psi_i \psi_j + \psi_j \psi_i &= 0, & (i,j) &\in \mathbb{N}_0^2 \backslash \{(0,0)\}, \\ \psi_i^* \psi_j^* + \psi_j^* \psi_i^* &= 0, & (i,j) &\in \mathbb{N}_0^2 \backslash \{(0,0)\}. \end{aligned}$$

Observe that for any $i \in \mathbb{N}$

$$\psi_i \psi_i^* v_\lambda = \begin{cases} v_\lambda/2 & \text{if } i \in \lambda, \\ 0 & \text{otherwise.} \end{cases} \tag{2.5.11}$$

Let $\psi(z)$ be a generating function for ψ_k and ψ_k^*:

$$\psi(z) = \sum_{k \in \mathbb{Z}} \widetilde{\psi}_k z^k, \tag{2.5.12}$$

where

$$\widetilde{\psi}_k = \begin{cases} \psi_k & k \geq 0, \\ (-1)^k \psi_{-k}^* & k \leq -1. \end{cases}$$

Then the following is true

$$\langle \widetilde{\psi}_k \widetilde{\psi}_l v_\emptyset, v_\emptyset \rangle = \begin{cases} (-1)^k/2 & k = -l \geq 1, \\ 1/4 & k = l = 0, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\begin{aligned} \langle \psi(z)\psi(w) v_\emptyset, v_\emptyset \rangle &= \sum_{k=0}^{\infty} \langle \psi_k^* \psi_k v_\emptyset, v_\emptyset \rangle \left(-\frac{w}{z}\right)^k \\ &= \frac{z-w}{4(z+w)}, \qquad \text{for } |z| > |w|. \end{aligned}$$

For every odd positive integer n we introduce an operator α_n and its adjoint $\alpha_{-n} = \alpha_n^*$:

$$\alpha_n = \sum_{k \in \mathbb{Z}} (-1)^k \widetilde{\psi}_{k-n} \widetilde{\psi}_{-k},$$

$$\alpha_{-n} = \sum_{k \in \mathbb{Z}} (-1)^k \widetilde{\psi}_{n-k} \widetilde{\psi}_k.$$

Then the following commutation relations are true

$$[\alpha_n, \alpha_m] = \frac{n}{2} \delta_{n,-m},$$

$$[\alpha_n, \psi(z)] = z^n \psi(z),$$

for all odd integers n and m, where $[\alpha, \beta] = \alpha\beta - \beta\alpha$.

We introduce two operators Γ_+ and its adjoint Γ_-. They are defined by

$$\Gamma_\pm(x) = \exp\left[\sum_{n=1,3,5,\ldots} \frac{2p_n(x)}{n}\alpha_{\pm n}\right].$$

Here $p_n \in \Lambda$ are the power sums. Then one has:

$$\Gamma_+(x)v_\emptyset = v_\emptyset, \tag{2.5.13}$$

$$\Gamma_+(y)\Gamma_-(x) = H(x,y)\Gamma_-(x)\Gamma_+(y), \tag{2.5.14}$$

$$\Gamma_\pm(x)\psi(z) = F(x; z^{\pm 1})\psi(z)\Gamma_\pm(x), \tag{2.5.15}$$

where $H(x,y)$ and $F(x;z)$ are given with (2.5.2) and (2.5.3), respectively.

The connection between the described Fock space and skew Schur P and Q functions comes from the following:

$$\Gamma_-(x)v_\mu = \sum_{\lambda \text{ strict}} Q_{\lambda/\mu}(x)v_\lambda, \tag{2.5.16}$$

$$\Gamma_+(x)v_\lambda = \sum_{\mu \text{ strict}} P_{\lambda/\mu}(x)v_\mu. \tag{2.5.17}$$

We conclude this appendix with another proof of Proposition 2.5.1.

Proof. (Proposition 2.5.1)

$$\begin{aligned}\langle \Gamma_-(x)v_\mu, \Gamma_-(y)v_\lambda\rangle &= \left\langle \sum_{\lambda \text{ strict}} Q_{\lambda/\mu}(x)v_\lambda, \sum_{\lambda \text{ strict}} Q_{\lambda/\nu}(y)v_\lambda \right\rangle \\ &= 2^{-l(\nu)} \sum_{\lambda \text{ strict}} Q_{\lambda/\mu}(x)P_{\lambda/\nu}(y)\end{aligned}$$

$$\begin{aligned}\langle \Gamma_+(y)v_\mu, \Gamma_+(x)v_\nu\rangle &= \left\langle \sum_{\tau \text{ strict}} P_{\mu/\tau}(y)v_\tau, \sum_{\tau \text{ strict}} P_{\nu/\tau}(x)v_\tau \right\rangle \\ &= 2^{-l(\nu)} \sum_{\tau \text{ strict}} P_{\mu/\tau}(y)Q_{\nu/\tau}(x)\end{aligned}$$

By (2.5.14) we have

$$\langle \Gamma_+(y)\Gamma_-(x)v_\mu, v_\nu\rangle = H(x,y)\langle \Gamma_-(x)\Gamma_+(y)v_\mu v_\nu\rangle$$

which implies

$$\sum_{\lambda \text{ strict}} Q_{\lambda/\mu}(x)P_{\lambda/\nu}(y) = H(x,y) \sum_{\tau \text{ strict}} Q_{\nu/\tau}(x)P_{\mu/\tau}(y)$$

□

Chapter 3

A generalization of MacMahon's formula

3.1 Introduction

A plane partition is a Young diagram filled with positive integers that form nonincreasing rows and columns. Each plane partition can be represented as a finite two sided sequence of ordinary partitions $(\dots, \lambda^{-1}, \lambda^0, \lambda^1, \dots)$, where λ^0 corresponds to the ordinary partition on the main diagonal and λ^k corresponds to the diagonal shifted by k. A plane partition whose all diagonal partitions are strict ordinary partitions (i.e., partitions with all distinct parts) is called a *strict* plane partition. Figure 3.1 shows two standard ways of representing a plane partition. Diagonal partitions are marked on the figure on the left.

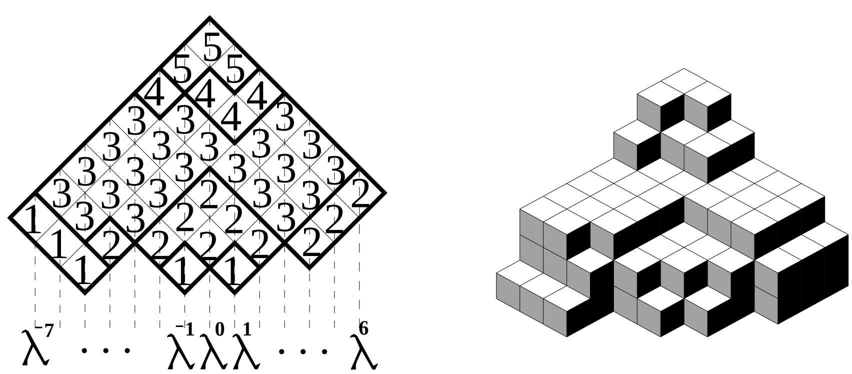


Figure 3.1: A plane partition

For a plane partition π one defines the weight $|\pi|$ to be the sum of all entries. A *connected component* of a plane partition is the set of all rookwise connected boxes of its

Young diagram that are filled with the same number. We denote the number of connected components of π with $k(\pi)$. For the example from Figure 3.1 we have $k(\pi) = 10$ and its connected components are shown in Figure 3.1 (left- bold lines represent boundaries of these components, right- white terraces are connected components).

Denote the set of all plane partitions with $\mathcal{P}$ and with $\mathcal{P}(r,c)$ we denote those that have zero $(i,j)^{th}$ entry for $i > r$ and $j > c$. Denote the set of all strict plane partitions with $\mathcal{SP}$.

A generating function for plane partitions is given by the famous MacMahon's formula (see e.g., 7.20.3 of [Sta99]):

$$\sum_{\pi \in \mathcal{P}} s^{|\pi|} = \prod_{n=1}^{\infty} \left(\frac{1}{1-s^n} \right)^n . \tag{3.1.1}$$

Recently, a generating formula for the set of strict plane partitions was found in [FW07] and [Vul07]:

$$\sum_{\pi \in \mathcal{SP}} 2^{k(\pi)} s^{|\pi|} = \prod_{n=1}^{\infty} \left(\frac{1+s^n}{1-s^n} \right)^n . \tag{3.1.2}$$

We refer to it as the shifted MacMahon's formula.

In this chapter, we generalize both formulas (3.1.1) and (3.1.2). Namely, we define a polynomial $A_\pi(t)$ that gives a generating formula for plane partitions of the form

$$\sum_{\pi \in \mathcal{P}(r,c)} A_\pi(t) s^{|\pi|} = \prod_{i=1}^{r} \prod_{j=1}^{c} \frac{1 - ts^{i+j-1}}{1 - s^{i+j-1}}$$

with the property that $A_\pi(0) = 1$ and

$$A_\pi(-1) = \begin{cases} 2^{k(\pi)}, & \pi \text{ is a strict plane partition,} \\ 0, & \text{otherwise.} \end{cases}$$

We further generalize this and find a rational function $F_\pi(q,t)$ that satisfies

$$\sum_{\pi \in \mathcal{P}(r,c)} F_\pi(q,t) s^{|\pi|} = \prod_{i=1}^{r} \prod_{j=1}^{c} \frac{(ts^{i+j-1};q)_\infty}{(s^{i+j-1};q)_\infty},$$

where

$$(s;q)_\infty = \prod_{n=0}^{\infty} (1 - sq^n)$$

and $F_\pi(0,t) = A_\pi(t)$. We describe $A_\pi(t)$ and $F_\pi(q,t)$ below.

In order to describe $A_\pi(t)$ we need more notation. If a box (i,j) belongs to a connected component C then we define its *level* $h(i,j)$ as the smallest positive integer such that $(i+h, j+h)$ does not belong to C. In other words, levels represent distance from the "rim", distance being measured diagonally. A *border component* is a rookwise connected subset of a connected component where all boxes have the same level. We also say that this border component is of this level. For the example above, border components and their levels are shown in Figure 3.2.

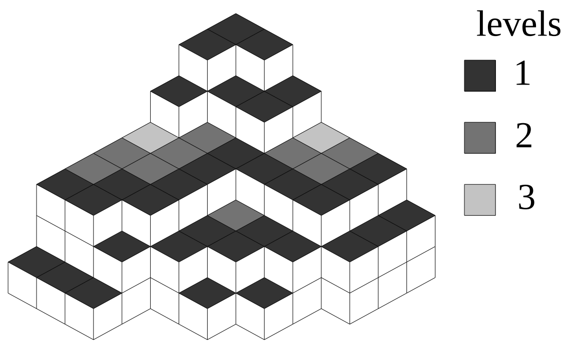


Figure 3.2: Border components

For each connected component C we define a sequence $(n_1, n_2, \dots)$ where n_i is the number of i-level border components of C. We set

$$P_C(t) = \prod_{i \geq 1} (1-t^i)^{n_i}.$$

Let $C_1, C_2, \dots, C_{k(\pi)}$ be connected components of π. We define

$$A_\pi(t) = \prod_{i=1}^{k(\pi)} P_{C_i}(t).$$

For the example above $A_\pi(t) = (1-t)^{10}(1-t^2)^3(1-t^3)^2$.

$F_\pi(q,t)$ is defined as follows. For nonnegative integers n and m let

$$f(n,m) = \begin{cases} \prod_{i=0}^{n-1} \frac{1-q^i t^{m+1}}{1-q^{i+1}t^m}, & n \geq 1, \\ 1, & n = 0. \end{cases}$$

Here q and t are parameters.

Let $\pi \in \mathcal{P}$ and let (i,j) be a box in its support (where the entries are nonzero). Let λ, μ and ν be ordinary partitions defined by

$$\begin{aligned} \lambda &= (\pi(i,j), \pi(i+1,j+1), \dots), \\ \mu &= (\pi(i+1,j), \pi(i+2,j+1), \dots), \\ \nu &= (\pi(i,j+1), \pi(i+1,j+2), \dots). \end{aligned} \tag{3.1.3}$$

To the box (i,j) of π we associate

$$F_\pi(i,j)(q,t) = \prod_{m=0}^{\infty} \frac{f(\lambda_1 - \mu_{m+1}, m) f(\lambda_1 - \nu_{m+1}, m)}{f(\lambda_1 - \lambda_{m+1}, m) f(\lambda_1 - \lambda_{m+2}, m)}.$$

Only finitely many terms in this product are different from 1.

To a plane partition π we associate a function $F_\pi(q,t)$ defined by

$$F_\pi(q,t) = \prod_{(i,j)\in\pi} F_\pi(i,j)(q,t). \tag{3.1.4}$$

For the example above

$$F_\pi(0,0)(q,t) = \frac{1-q}{1-t} \cdot \frac{1-q^3t^2}{1-q^2t^3} \cdot \frac{1-q^5t^4}{1-q^4t^5} \cdot \frac{1-q^3t^5}{1-q^4t^4}.$$

Two main results of this chapter are

Theorem A. (Generalized MacMahon's formula; Macdonald's case)

$$\sum_{\pi\in\mathcal{P}(r,c)} F_\pi(q,t) s^{|\pi|} = \prod_{i=1}^{r} \prod_{j=1}^{c} \frac{(ts^{i+j-1};q)_\infty}{(s^{i+j-1};q)_\infty},$$

In particular,

$$\sum_{\pi\in\mathcal{P}} F_\pi(q,t)s^{|\pi|}=\prod_{n=1}^{\infty}\left[\frac{(ts^n;q)_\infty}{(s^n;q)_\infty}\right]^n.$$

Theorem B. (Generalized MacMahon's formula; Hall-Littlewood's case)

$$\sum_{\pi\in\mathcal{P}(r,c)} A_\pi(t)s^{|\pi|}=\prod_{i=1}^{r}\prod_{j=1}^{c}\frac{1-ts^{i+j-1}}{1-s^{i+j-1}}.$$

In particular,

$$\sum_{\pi\in\mathcal{P}} A_\pi(t)s^{|\pi|}=\prod_{n=1}^{\infty}\left(\frac{1-ts^n}{1-s^n}\right)^n.$$

Clearly, the second formulas (with summation over $\mathcal{P}$) are limiting cases of the first ones as $r,c\to\infty$.

The proof of Theorem A was inspired by [OR03] and [Vul07]. It uses a special class of symmetric functions called skew Macdonald functions. For each $\pi\in\mathcal{P}$ we introduce a weight function depending on several specializations of the algebra of symmetric functions. For a suitable choice of these specializations the weight functions become $F_\pi(q,t)$.

We first prove Theorem A, and then Theorem B is obtained as a corollary of Theorem A after we show that $F_\pi(0,t)=A_\pi(t)$.

Proofs of formula (3.1.2) appeared in [FW07] and [Vul07]. Both these proofs rely on skew Schur functions and a Fock space corresponding to strict plane partitions. In this paper we also give a bijective proof of (3.1.2) that does not involve symmetric functions.

We were informed by Sylvie Corteel that a generalization of (3.1.2) was found in her joint work with Cyrille Savelief. This generalization gives a generating formula for strict plane partitions with bounded entries. It will appear in the master thesis of Cyrille Savelief, Combinatoire des overpartitions planes, Universite Paris 7, 2007.

There are different generalizations of MacMahon's formula in literature. Note that the paper [Ciu05] with a similar title to ours gives a different generalization.

The paper is organized as follows. Section 2 consists of two subsections. In Subsection 3.2.1 we prove Theorem A. In Subsection 3.2.2 we prove Theorem B by showing that $F_\pi(0,t)=A_\pi(t)$. In Section 3.3 we give a bijective proof of (3.1.2).

Acknowledgement. This work is a part of my doctoral dissertation at California Institute of Technology and I thank my advisor Alexei Borodin for all his help. Also, I thank Sylvie

Corteel for informing me about the generalization of the shifted MacMahon's formula.

3.2 Generalized MacMahon's formula

3.2.1 Macdonald's case

We recall a definition of a plane partition. For basics, such as ordinary partitions and Young diagrams see Chapter I of [Mac95].

A plane partition π can be viewed in different ways. One way is to fix a Young diagram, the support of the plane partition, and then to associate a positive integer to each box in the diagram such that integers form nonincreasing rows and columns. Thus, a plane partition is a diagram with row-wise and column-wise nonincreasing integers. It can also be viewed as a finite two-sided sequence of ordinary partitions, since each diagonal in the support diagram represents a partition. We write $\pi = (\dots, \lambda^{-1}, \lambda^0, \lambda^1, \dots)$, where the partition λ^0 corresponds to the main diagonal and λ^k corresponds to the diagonal that is shifted by k, see Figure 3.1. Every such two-sided sequence of partitions represents a plane partition if and only if

$$\begin{gathered} \cdots \subset \lambda^{-1} \subset \lambda^0 \supset \lambda^1 \supset \cdots \text{ and} \\ [\lambda^{n-1}/\lambda^n] \text{ is a horizontal strip (a skew diagram with} \\ \text{at most one square in each column) for every } n, \end{gathered} \tag{3.2.1}$$

where

$$[\lambda/\mu] = \begin{cases} \lambda/\mu & \text{if } \lambda \supset \mu, \\ \mu/\lambda & \text{if } \mu \supset \lambda \end{cases}.$$

The weight of π, denoted with $|\pi|$, is the sum of all entries of π.

We denote the set of all plane partitions with $\mathcal{P}$ and its subset containing all plane partitions with at most r nonzero rows and c nonzero columns with $\mathcal{P}(r, c)$. Similarly, we denote the set of all ordinary partitions (Young diagrams) with $\mathcal{Y}$ and those with at most r parts with $\mathcal{Y}(r) = \mathcal{P}(r, 1)$.

We use the definitions of $f(n, m)$ and $F_\pi(q, t)$ from the Introduction. To a plane partition π we associate a rational function $F_\pi(q, t)$ that is related to Macdonald symmetric functions (for reference see Chapter VI of [Mac95]).

In this section we prove Theorem A. The proof consists of a few steps. We first define weight functions on sequences of ordinary partitions (Section 3.2.1.1). These weight functions are defined using Macdonald symmetric functions. Second, for suitably chosen specializations of these symmetric functions we obtain that the weight functions vanish for every sequence of partitions except if the sequence corresponds to a plane partition (Section 3.2.1.2). Finally, we show that for $\pi \in \mathcal{P}$ the weight function of π is equal to $F_\pi(q,t)$(Section 3.2.1.3).

Before showing these steps we first comment on a corollary of Theorem A.

Fix $c = 1$. Then, Theorem A gives a generating function formula for ordinary partitions since $\mathcal{P}(r,1) = \mathcal{Y}(r)$. For $\lambda = (\lambda_1, \lambda_2, \dots) \in \mathcal{Y}(r)$ we define $d_i = \lambda_i - \lambda_{i+1}$, $i = 1, \dots, r$. Then

$$F_\lambda(q,t) = \prod_{i=1}^{r} f(d_i, 0) = \prod_{i=1}^{r} \prod_{j=1}^{d_i} \frac{1 - tq^{j-1}}{1 - q^j}.$$

Note that $F_\lambda(q,t)$ depends only on the set of distinct parts of λ.

Corollary 3.2.1.

$$\sum_{\lambda \in \mathcal{Y}(r)} F_\lambda(q,t) s^{|\lambda|} = \prod_{i=1}^{r} \frac{(ts^i;q)_\infty}{(s^i;q)_\infty}.$$

In particular,

$$\sum_{\lambda \in \mathcal{Y}} F_\lambda(q,t) s^{|\lambda|} = \prod_{i=1}^{\infty} \frac{(ts^i;q)_\infty}{(s^i;q)_\infty}.$$

This corollary is easy to show directly.

Proof. First, we expand $(ts;q)_\infty/(s;q)_\infty$ into the power series in s. Let $a_d(q,t)$ be the coefficient of s^d. Observe that

$$\frac{(ts;q)_\infty}{(s;q)_\infty} := \sum_{d=0}^{\infty} a_d(q,t) s^d = \frac{1-ts}{1-s} \sum_{d=0}^{\infty} a_d(q,t) s^d q^d.$$

By identifying coefficients of s^d, and doing induction on d, this implies that

$$a_d(q,t) = f(d,0).$$

Every $\lambda = (\lambda_1, \dots, \lambda_r) \in \mathcal{Y}(r)$ is uniquely determined by $d_i \in \mathbb{N} \cup \{0\}$, $i = 1, \dots, r$, where $d_i = \lambda_i - \lambda_{i+1}$. Then $\lambda_i = \sum_{j\geq 0} d_{i+j}$ and $|\lambda| = \sum_{i=1}^{r} id_i$. Therefore,

$$\begin{aligned}\prod_{i=1}^{r} \frac{(ts^i;q)_\infty}{(s^i;q)_\infty} &= \prod_{i=1}^{r}\sum_{d_i=0}^{\infty} a_{d_i}(q,t)s^{id_i} \\ &= \sum_{d_1,\dots,d_r}\left[\prod_{i=1}^{r} a_{d_i}(q,t)\right]\cdot\left[s^{\sum_{i=1}^{r} id_i}\right] = \sum_{\lambda\in\mathcal{Y}(r)} F_\lambda(q,t)s^{|\lambda|}.\end{aligned}$$

□

3.2.1.1 The weight functions

The weight function is defined as a product of Macdonald symmetric functions P and Q. We assume familiarity with Chapter VI of [Mac95] and we follow the notation used there.

Recall that Macdonald symmetric functions $P_{\lambda/\mu}(x;q,t)$ and $Q_{\lambda/\mu}(x;q,t)$ depend on two parameters q and t and are indexed by pairs of ordinary partitions λ and μ. In the case when $q = t = 0$ they are equal to ordinary Schur functions, in the case when $q = 0$ and $t = -1$ to Schur P and Q functions and for $q = 0$ to Hall-Littlewood symmetric functions.

For an ordinary partition λ and a box $s = (i,j)$ we write $s \in \lambda$ if s is a box in the Young diagram of λ. Let

$$b_\lambda(s) = b_\lambda(s;q,t) = \begin{cases} \dfrac{1-q^{\lambda_i-j}t^{\lambda'_j-i+1}}{1-q^{\lambda_i-j+1}t^{\lambda'_j-i}}, & s\in\lambda, \\ 1 & \text{otherwise,}\end{cases}$$

where λ' is the conjugate partition of λ and

$$b_\lambda(q,t) = \prod_{s\in\lambda} b_\lambda(s). \tag{3.2.2}$$

The relationship between P and Q functions is given with (see (7.8) of [Mac95, Chapter VI])

$$Q_{\lambda/\mu} = \frac{b_\lambda}{b_\mu}P_{\lambda/\mu}. \tag{3.2.3}$$

Recall that (by (7.7(i)) of [Mac95, Chapter VI] and (3.2.3))

$$P_{\lambda/\mu} = Q_{\lambda/\mu} = 0 \quad \text{unless } \lambda \supset \mu. \tag{3.2.4}$$

We set $P_\lambda = P_{\lambda/\emptyset}$ and $Q_\lambda = Q_{\lambda/\emptyset}$. Recall that ((2.5) and (4.13) of [Mac95, Chapter VI])

$$\Pi(x,y;q,t) = \sum_{\lambda\in\mathcal{Y}} Q_\lambda(x;q,t)P_\lambda(y;q,t) = \prod_{i,j} \frac{(tx_iy_j;q)_\infty}{(x_iy_j;q)_\infty}.$$

Let Λ be the algebra of symmetric functions. A specialization of Λ is an algebra homomorphism $\Lambda \to \mathbb{C}$. If ρ and σ are specializations of Λ then we write $P_{\lambda/\mu}(\rho;q,t)$, $Q_{\lambda/\mu}(\rho;q,t)$ and $\Pi(\rho,\sigma;q,t)$ for the images of $P_{\lambda/\mu}(x;q,t)$, $Q_{\lambda/\mu}(x;q,t)$ and $\Pi(x,y;q,t)$ under ρ, respectively $\rho\otimes\sigma$. Every map $\rho : (x_1,x_2,\dots) \to (a_1,a_2,\dots)$ where $a_i \in \mathbb{C}$ and only finitely many a_i's are nonzero defines a specialization.

Let

$$\varphi_{\lambda/\mu}(q,t) = \prod_{s\in C_{\lambda/\mu}} \frac{b_\lambda(s)}{b_\mu(s)}, \tag{3.2.5}$$

where $C_{\lambda/\mu}$ is the union of all columns that intersect λ/μ. If ρ is a specialization of Λ where $x_1 = a$, $x_2 = x_3 = \ldots = 0$ then by (7.14) of [Mac95, Chapter VI]

$$Q_{\lambda/\mu}(\rho;q,t) = \begin{cases} \varphi_{\lambda/\mu}(q,t)a^{|\lambda|-|\mu|} & \lambda\supset\mu,\ \lambda/\mu \text{ is a horizontal strip,} \\ 0 & \text{otherwise.} \end{cases} \tag{3.2.6}$$

Let $\rho = (\rho_0^+, \rho_1^-, \rho_1^+, \dots, \rho_T^-)$ be a finite sequence of specializations. For two sequences of partitions $\lambda = (\lambda^1,\lambda^2,\dots,\lambda^T)$ and $\mu = (\mu^1,\mu^2,\dots,\mu^{T-1})$ we set the weight function $W(\lambda,\mu;q,t)$ to be

$$W(\lambda,\mu;q,t) = \prod_{n=1}^{T} Q_{\lambda^n/\mu^{n-1}}(\rho_{n-1}^+;q,t)P_{\lambda^n/\mu^n}(\rho_n^-;q,t),$$

where $\mu^0 = \mu^T = \emptyset$. Note that by (3.2.4) it follows that $W(\lambda,\mu;q,t) = 0$ unless

$$\emptyset \subset \lambda^1 \supset \mu^1 \subset \lambda^2 \supset \mu^2 \subset \ldots \supset \mu^{T-1} \subset \lambda^T \supset \emptyset.$$

Proposition 3.2.2. *The sum of the weights $W(\lambda,\mu;q,t)$ over all sequences of partitions $\lambda = (\lambda^1,\lambda^2,\dots,\lambda^T)$ and $\mu = (\mu^1,\mu^2,\dots,\mu^{T-1})$ is equal to*

$$Z(\rho;q,t) = \prod_{0\le i<j\le T} \Pi(\rho_i^+,\rho_j^-;q,t). \tag{3.2.7}$$

Proof. We use

$$\sum_{\lambda\in\mathcal{Y}} Q_{\lambda/\mu}(x)P_{\lambda/\nu}(y) = \Pi(x,y)\sum_{\tau\in\mathcal{Y}} Q_{\nu/\tau}(x)P_{\mu/\tau}(y).$$

The proof of this is analogous to the proof of Proposition 5.1 that appeared in our earlier paper [Vul07]. Also, see Example 26 of Chapter I, Section 5 of [Mac95].

We prove (3.2.7) by induction on T. Using the formula above we substitute sums over λ^i's with sums over τ^{i-1}'s as in the proof of Proposition 2.1 of [BR05]. This gives

$$\prod_{i=0}^{T-1}\Pi(\rho_i^+,\rho_{i+1}^-)\sum_{\mu,\tau} Q_{\mu^1}(\rho_0^+)P_{\mu^1/\tau^1}(\rho_2^-)Q_{\mu^2/\tau^1}(\rho_1^+)\dots P_{\mu^{T-1}}(\rho_T^-).$$

This is the sum of $W(\mu,\tau)$ with $\mu=(\mu^1,\dots,\mu^{T-1})$ and $\tau=(\tau^1,\dots,\tau^{T-2})$. Inductively, we obtain (3.2.7). □

3.2.1.2 Specializations

For $\pi=(\dots,\lambda^{-1},\lambda^0,\lambda^1,\dots)\in\mathcal{P}$ we define a function $\Phi_\pi(q,t)$ by

$$\Phi_\pi(q,t)=\frac{1}{b_{\lambda^0}(q,t)}\prod_{n=-\infty}^{\infty}\varphi_{[\lambda^{n-1}/\lambda^n]}(q,t), \tag{3.2.8}$$

where b and φ are given with (3.2.2) and (3.2.5). Only finitely many terms in the product are different than 1 because only finitely many λ^n are nonempty partitions.

We show that for a suitably chosen specializations the weight function vanishes for every sequence of ordinary partitions unless this sequence represents a plane partition, in which case it becomes (3.2.8). This, together with Proposition 3.2.2, will imply the following

Proposition 3.2.3.

$$\sum_{\pi\in\mathcal{P}(r,c)}\Phi_\pi(q,t)s^{|\pi|}=\prod_{i=1}^{r}\prod_{j=1}^{c}\frac{(ts^{i+j-1};q)_\infty}{(s^{i+j-1};q)_\infty}.$$

Proof. Let

$$\begin{aligned}
&\rho_n^+ : x_1 = s^{-n-1/2},\ x_2 = x_3 = \ldots = 0 && -r \le n \le -1,\\
&\rho_n^- : x_1 = x_2 = \ldots = 0 && -r+1 \le n \le -1,\\
&\rho_n^- : x_1 = s^{n+1/2},\ x_2 = x_3 = \ldots = 0 && 0 \le n \le c-1,\\
&\rho_n^+ : x_1 = x_2 = \ldots = 0 && 0 \le n \le c-2.
\end{aligned}$$

Then for any two sequences

$$\begin{aligned}
\lambda &= (\lambda^{-r+1} \ldots, \lambda^{-1}, \lambda^0, \lambda^1, \ldots \lambda^{c-1})\\
\mu &= (\mu^{-r+1} \ldots, \mu^{-1}, \mu^0, \mu^1, \ldots \mu^{c-2})
\end{aligned}$$

the weight function is given with

$$W(\lambda,\mu) = \prod_{n=-r+1}^{c-1} Q_{\lambda^n/\mu^{n-1}}(\rho_{n-1}^+) P_{\lambda^n/\mu^n}(\rho_n^-),$$

where $\mu^{-r} = \mu^{c-1} = \emptyset$. By (3.2.3), (3.2.4) and (3.2.6) we have that $W(\lambda,\mu) = 0$ unless

$$\mu^n = \begin{cases} \lambda^n & n < 0,\\ \lambda^{n+1} & n \ge 0, \end{cases}$$

$$\cdots \lambda^{-1} \subset \lambda^0 \supset \lambda^1 \supset \cdots,$$

$$[\lambda^{n-1}/\lambda^n] \text{ is a horizontal strip for every } n,$$

i.e., $\lambda \in \mathcal{P}$ and in that case

$$\begin{aligned}
W(\lambda,\mu) &= \prod_{n=-r+1}^{0} \varphi_{\lambda^n/\lambda^{n-1}}(q,t) s^{(-2n+1)(|\lambda^n|-|\lambda^{n-1}|)/2}\\
&\quad \cdot \prod_{n=1}^{c} \frac{b_{\lambda^n}(q,t)}{b_{\lambda^{n-1}}(q,t)} \varphi_{\lambda^{n-1}/\lambda^n}(q,t) s^{(2n-1)(|\lambda^{n-1}|-|\lambda^n|)/2}\\
&= \frac{1}{b_{\lambda^0}(q,t)} \prod_{n=-r+1}^{c} \varphi_{[\lambda^{n-1}/\lambda^n]}(q,t) s^{|\lambda|} = \Phi_\lambda(q,t) s^{|\lambda|}.
\end{aligned}$$

If ρ^+ is $x_1 = s$, $x_2 = x_3 = \ldots = 0$ and ρ^- is $x_1 = r$, $x_2 = x_3 = \ldots = 0$ then

$$\Pi(\rho^+, \rho^-) = \prod_{i,j} \frac{(tx_iy_j; q)_\infty}{(x_iy_j; q)_\infty}\Bigg|_{x=\rho^+,\, y=\rho^-} = \frac{(tsr; q)_\infty}{(sr; q)_\infty}.$$

Then, by Proposition 3.2.2, for the given specializations of ρ_i^+'s and ρ_j^-'s we have

$$Z = \prod_{i=-1}^{-r} \prod_{j=0}^{c-1} \Pi(\rho_i^+, \rho_j^-) = \prod_{i=1}^{r} \prod_{j=1}^{c} \frac{(ts^{i+j-1}; q)_\infty}{(s^{i+j-1}; q)_\infty}.$$

□

3.2.1.3 Final step

We show that $F_\pi(q,t) = \Phi_\pi(q,t)$. Then Proposition 3.2.3 implies Theorem A.

Proposition 3.2.4. *Let* $\pi \in \mathcal{P}$. *Then*

$$F_\pi(q,t) = \Phi_\pi(q,t).$$

Proof. We show this by induction on the number of boxes in the support of π. Denote the last nonzero part in the last row of the support of π by x. Let λ be a diagonal partition containing it and let x be its k^{th} part. Because of the symmetry with respect to the transposition we can assume that λ is one of diagonal partitions on the left.

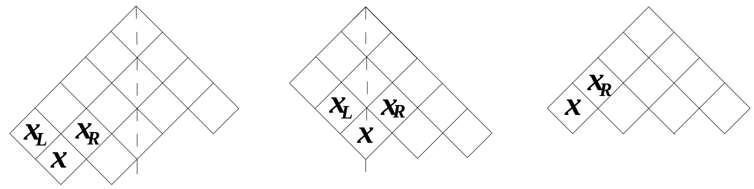
Let π' be the plane partition obtained from π by removing x. We want to show that F_π and $F_{\pi'}$ satisfy the same recurrence relation as Φ_π and $\Phi_{\pi'}$. The verification uses the explicit formulas for b_λ and $\varphi_{\lambda/\mu}$ given by (3.2.2) and (3.2.5).

We divide the problem into several cases depending on the position of the box containing x. Let I, II and III be the cases shown in Figure 3.3.

Let (i,j) be the box containing x. We denote the filling number of $(i, j-1)$, respectively $(i-1, j)$, with x_L, respectively x_R. Let λ^L and λ^R be the diagonal partitions of π containing x_L and x_R, respectively. Let λ' be the partition obtained from λ by removing x.

If III then $k = 1$ and one checks easily that

$$\frac{F_{\pi'}}{F_\pi} = \frac{\Phi_{\pi'}}{\Phi_\pi} = \frac{f(\lambda_1^R, 0)}{f(\lambda_1^R - \lambda_1, 0) f(\lambda_1, 0)}.$$



I II III

Figure 3.3: Cases I, II and III

Assume I or II. Then

$$\Phi_{\pi'} = \Phi_\pi \cdot \frac{\varphi_{[\lambda'/\lambda^L]}}{\varphi_{[\lambda/\lambda^L]}} \cdot \frac{\varphi_{[\lambda'/\lambda^R]}}{\varphi_{[\lambda/\lambda^R]}} \cdot \frac{b_{\lambda^0(\pi)}}{b_{\lambda^0(\pi')}} = \Phi_\pi \cdot \Phi_L \cdot \Phi_R \cdot \Phi_0.$$

Thus, we need to show that

$$\Phi_L \cdot \Phi_R \cdot \Phi_0 = F := \frac{F_{\pi'}}{F_\pi}. \tag{3.2.9}$$

If I then $\lambda^L_{k-1} = x_L$ and $\lambda^R_k = x_R$. From the definition of φ we have that

$$\Phi_L = \prod_{i=0}^{k-1} \frac{f(\lambda_{k-i} - \lambda_k, i)}{f(\lambda_{k-i}, i)} \cdot \prod_{i=0}^{k-2} \frac{f(\lambda^L_{k-1-i}, i)}{f(\lambda^L_{k-1-i} - \lambda_k, i)}. \tag{3.2.10}$$

Similarly,

$$\Phi_R = \prod_{i=0}^{k-2} \frac{f(\lambda_{k-1-i} - \lambda_k, i)}{f(\lambda_{k-1-i}, i)} \cdot \prod_{i=0}^{k-1} \frac{f(\lambda^R_{k-i}, i)}{f(\lambda^R_{k-i} - \lambda_k, i)}.$$

If II then $\lambda^L_{k-1} = x_L$ and $\lambda^R_{k-1} = x_R$ and both Φ_L and Φ_R are given with (3.2.10), substituting L with R for Φ_R, while

$$\Phi_0 = \prod_{i=0}^{k-1} \frac{f(\lambda_{k-i}, i)}{f(\lambda_{k-i} - \lambda_k, i)} \cdot \prod_{i=0}^{k-2} \frac{f(\lambda_{k-1-i} - \lambda_k, i)}{f(\lambda_{k-1-i}, i)}.$$

From the definition of F one can verify that (3.2.9) holds. □

3.2.2 Hall-Littlewood's case

We analyze the generalized MacMahon's formula in Hall-Littlewood's case, i.e., when $q = 0$, in more detail. Namely, we describe $F_\pi(0, t)$.

We use the definition of $A_\pi(t)$ from the Introduction. In Proposition 3.2.6 we show that

$F_\pi(0,t) = A_\pi(t)$. This, together with Theorem A, implies Theorem B.

Note that the result implies the following simple identities. If $\lambda \in \mathcal{Y} = \bigcup_{r\geq 1} \mathcal{P}(r,1)$ then $k(\lambda)$ becomes the number of distinct parts of λ.

Corollary 3.2.5.

$$\sum_{\lambda\in\mathcal{Y}(r)} (1-t)^{k(\lambda)} s^{|\lambda|} = \prod_{i=1}^{r} \frac{1-ts^i}{1-s^i}.$$

In particular,

$$\sum_{\lambda\in\mathcal{Y}} (1-t)^{k(\lambda)} s^{|\lambda|} = \prod_{i=1}^{\infty} \frac{1-ts^i}{1-s^i}.$$

These formulas are easily proved by the argument used in the proof of Corollary 3.2.1.

We now prove

Proposition 3.2.6. *Let $\pi \in \mathcal{P}$. Then*

$$F_\pi(0,t) = A_\pi(t).$$

Proof. Let B be a h-level border component of π. Let $F(i,j) = F_\pi(i,j)(0,t)$. It is enough to show that

$$\prod_{(i,j)\in B} F(i,j) = 1 - t^h. \tag{3.2.11}$$

Let

$$c(i,j) = \chi_B(i+1,j) + \chi_B(i,j+1),$$

where χ_B is the characteristic function of B taking value 1 on the set B and 0 elsewhere. If there are n boxes in B then

$$\sum_{(i,j)\in B} c(i,j) = n - 1. \tag{3.2.12}$$

Let $(i,j) \in B$. We claim that

$$F(i,j) = (1-t^h)^{1-c(i,j)}. \tag{3.2.13}$$

Then (3.2.12) and (3.2.13) imply (3.2.11).

To show (3.2.13) we observe that

$$f(l,m)(0,t) = \begin{cases} 1 & l = 0 \\ 1 - t^{m+1} & l \geq 1. \end{cases}$$

With the same notation as in (3.1.3) we have that μ_m, ν_m, λ_m, λ_{m+1} are all equal to λ_1 for every $m < h$, while for every $m > h$ they are all different from λ_1. Then

$$\begin{aligned} F(i,j) &= \prod_{m=0}^{\infty} \frac{f(\lambda_1 - \mu_{m+1}, m)(0,t) f(\lambda_1 - \nu_{m+1}, m)(0,t)}{f(\lambda_1 - \lambda_{m+1}, m)(0,t) f(\lambda_1 - \lambda_{m+2}, m)(0,t)} \\ &= \frac{f(\lambda_1 - \mu_h, h-1)(0,t) f(\lambda_1 - \nu_h, h-1)(0,t)}{f(\lambda_1 - \lambda_h, h-1)(0,t) f(\lambda_1 - \lambda_{h+1}, h-1)(0,t)} \\ &= \frac{(1-t^h)^{1-\chi_B(i+1,j)}(1-t^h)^{1-\chi_B(i,j+1)}}{1 \cdot (1-t^h)} = (1-t^h)^{1-c(i,j)}. \end{aligned}$$

□

3.3 A bijective proof of the shifted MacMahon's formula

In this section we are going to give another proof of the shifted MacMahon's formula (3.1.2). More generally, we prove

Theorem 3.3.1.

$$\sum_{\pi \in \mathcal{SP}(r,c)} 2^{k(\pi)} x^{\mathrm{tr}(\pi)} s^{|\pi|} = \prod_{i=1}^{r} \prod_{j=1}^{c} \frac{1 + xs^{i+j-1}}{1 - xs^{i+j-1}}.$$

Here $\mathcal{SP}(r,c)$ is the set of strict plane partitions with at most r rows and c columns. The trace of π, denoted with $\mathrm{tr}(\pi)$, is the sum of diagonal entries of π.

The x-version of the shifted MacMahon's formula given in Theorem 3.3.1 is, as far as we know, a new result. Its proof is mostly independent of the rest of the paper. It is similar in spirit to the proof of MacMahon's formula given in Section 7.20 of [Sta99]. It uses two bijections. One correspondence is between strict plane partitions and pairs of shifted tableaux. The other one is between pairs of marked shifted tableaux and marked matrices and it is obtained by the shifted Knuth's algorithm.

We recall the definitions of a marked tableau and a marked shifted tableau (see e.g. Chapter XIII of [HH92]).

Let P be a totally ordered set

$$P = \{1 < 1' < 2 < 2' < \cdots\}.$$

We distinguish elements in P as marked and unmarked, the former being the one with a prime. We use $|p|$ for the unmarked number corresponding to $p \in P$.

A marked (shifted) tableau is a (shifted) Young diagram filled with row and column nonincreasing elements from P such that any given unmarked element occurs at most once in each column whereas any marked element occurs at most once in each row. Examples of a marked tableau and a marked shifted tableau are given in Figure 3.4.

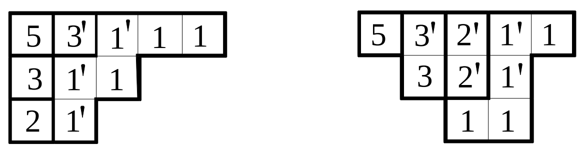


Figure 3.4: A marked tableau and a marked shifted tableau

An unmarked (shifted) tableau is a tableau obtained by deleting primes from a marked (shifted) tableau. We can also define it as a (shifted) diagram filled with row and column nonincreasing positive integers such that no 2×2 square is filled with the same number. Unmarked tableaux are strict plane partitions.

We define connected components of a marked or unmarked (shifted) tableau in a similar way as for plane partitions. Namely, a connected component is a set of rookwise connected boxes filled with p or p'. By the definition of a tableau all connected components are border strips (a set of rookwise connected boxes containing no 2×2 block of boxes). Connected components for the examples above are shown in Figure 3.4 (bold lines represent boundaries of these components).

We use $k(S)$ to denote the number of components of a marked or unmarked (shifted) tableau S. For every marked (shifted) tableau there is a corresponding unmarked (shifted) tableau obtained by deleting all the primes. The number of marked (shifted) tableaux corresponding to the same unmarked (shifted) tableau S is equal to $2^{k(S)}$ because there are exactly two possible ways to mark each border strip.

For a tableau S, we use $\mathrm{sh}(S)$ to denote the shape of S that is an ordinary partition with parts equal to the lengths of rows of S. We define $\ell(S) = \ell(\mathrm{sh}(S))$ and $\max(S) = |p_{\max}|$, where $p_{\max}$ is the maximal element in S. For both examples $\mathrm{sh}(S) = (5,3,2)$, $\ell(S) = 3$ and $\max(S) = 5$.

A marked matrix is a matrix with entries from $P \cup \{0\}$. Let $\mathcal{M}(r,c)$ be the set of $r \times c$ marked matrices.

Let $\mathcal{ST}^M(r,c)$, respectively $\mathcal{ST}^U(r,c)$, be the set of ordered pairs (S,T) of marked, respectively unmarked, shifted tableaux of the same shape where $\max(S) = c$, $\max(T) = r$ and T has no marked letters on its main diagonal. There is a natural mapping $i : \mathcal{ST}^M(r,c) \to \mathcal{ST}^U(r,c)$ obtained by deleting primes and for $(S,T) \in \mathcal{ST}^U(r,c)$

$$|i^{-1}[(S,T)]| = 2^{k(S)+k(T)-l(S)}. \tag{3.3.1}$$

The shifted Knuth's algorithm (see Chapter XIII of [HH92]) establishes the following correspondence.

Theorem 3.3.2. *There is a bijective correspondence between matrices $A = [a_{ij}]$ over $P \cup \{0\}$ and ordered pairs (S,T) of marked shifted tableaux of the same shape such that T has no marked elements on its main diagonal. The correspondence has the property that $\sum_i a_{ij}$ is the number of entries s of S for which $|s| = j$ and $\sum_j a_{ij}$ is the number of entries t of T for which $|t| = i$.*
In particular, this correspondence maps $\mathcal{M}(r,c)$ onto $\mathcal{ST}^M(r,c)$ and

$$|\mathrm{sh}(S)| = \sum_{i,j} |a_{ij}|, \qquad |S| = \sum_{i,j} j|a_{ij}|, \qquad |T| = \sum_{i,j} i|a_{ij}|.$$

Remark. The shifted Knuth's algorithm described in Chapter XIII of [HH92] establishes a correspondence between marked matrices and pairs of marked shifted tableaux with row and column *nondecreasing* elements. This algorithm can be adjusted to work for marked shifted tableaux with row and column *nonincreasing* elements. Namely, one needs to change the encoding of a matrix over $P \cup \{0\}$ and two algorithms BUMP and EQBUMP, while INSERT, UNMARK, CELL and *unmix* remain unchanged.

One encodes a matrix $A \in \mathcal{P}(r,c)$ into a two-line notation E with pairs $\begin{matrix} i \\ j \end{matrix}$ repeated $|a_{ij}|$ times, where i is going from r to 1 and j from c to 1. If a_{ij} was marked, then we mark the leftmost j in the pairs $\begin{matrix} i \\ j \end{matrix}$. The example from p. 246 of [HH92]:

$$A = \begin{pmatrix} 1' & 0 & 2 \\ 2 & 1 & 2' \\ 1' & 1' & 0 \end{pmatrix}$$

would be encoded as

$$E = \begin{matrix} 3 & 3 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 \\ 2' & 1' & 3' & 3 & 2 & 1 & 1 & 3 & 3 & 1' \end{matrix}.$$

Algorithms BUMP and EQBUMP insert $x \in P \cup \{0\}$ into a vector v over $P \cup \{0\}$. By BUMP (resp. EQBUMP) one inserts x into v by removing (bumping) the leftmost entry of V that is less (resp. less or equal) than x and replacing it by x or if there is no such entry then x is placed at the end of v.

For the example from above this adjusted shifted Knuth's algorithm would give

$$S = \begin{matrix} 3' & 3 & 3 & 3 & 1' \\ & 2' & 2 & 1 & 1 \\ & & 1' & & \end{matrix} \quad \text{and} \quad T = \begin{matrix} 3 & 3 & 2' & 2 & 2 \\ & 2 & 2 & 1 & 1 \\ & & 1 & & \end{matrix}.$$

The other correspondence between pairs of shifted tableaux of the same shape and strict plane partitions is described in the following theorem. It is parallel to the correspondence from Section 7.20 of [Sta99].

Theorem 3.3.3. *There is a bijective correspondence* Π *between strict plane partitions* π *and ordered pairs* (S,T) *of shifted tableaux of the same shape. This correspondence maps* $\mathcal{SP}(r,c)$ *onto* $\mathcal{ST}^U(r,c)$ *and if* $(S,T) = \Pi(\pi)$ *then*

$$|\pi| = |S| + |T| - |\operatorname{sh}(S)|,$$

$$\operatorname{tr}(\pi) = |\operatorname{sh}(S)| = |\operatorname{sh}(T)|,$$

$$k(\pi) = k(S) + k(T) - l(S).$$

Proof. Every $\lambda \in \mathcal{Y}$ is uniquely represented by Frobenius coordinates $(p_1, \ldots, p_d \,|\, q_1, \ldots, q_d)$ where d is the number of diagonal boxes in the Young diagram of λ and p's and q's correspond to the arm length and the leg length, i.e. $p_i = \lambda_i - i + 1$ and $q_i = \lambda'_i - i + 1$, where $\lambda' \in \mathcal{Y}$ is the transpose of λ.

Let $\pi \in \mathcal{SP}$. Let $(\mu_1, \mu_2, \ldots)$ be a sequence of ordinary partitions whose diagrams are obtained by horizontal slicing of the 3-dimensional diagram of π (see Figure 3.5). The Young diagram of μ_1 corresponds to the first slice and is the same as the support of π, μ_2 corresponds to the second slice etc. More precisely, the Young diagram of μ_i consists of all boxes of the support of π filled with numbers greater or equal to i. For example, if

$$\pi = \begin{array}{ccccc} 5 & 3 & 2 & 1 & 1 \\ 4 & 3 & 2 & 1 & \\ 3 & 3 & 2 & & \\ 2 & 2 & 1 & & \end{array},$$

then $(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5)$ are

$$\mu_1 = \begin{array}{ccccc} * & * & * & * & * \\ * & * & * & * & \\ * & * & * & & \\ * & * & * & & \end{array}, \quad \mu_2 = \begin{array}{ccc} * & * & * \\ * & * & * \\ * & * & * \\ * & * & \end{array}, \quad \mu_3 = \begin{array}{cc} * & * \\ * & * \\ * & * \end{array}, \quad \mu_4 = \begin{array}{c} * \\ * \end{array}, \quad \mu_5 = * \, .$$

Let S, respectively T, be an unmarked shifted tableau whose i^{th} diagonal is equal to p, respectively q, Frobenius coordinate of μ_i. For the example above

$$S = \begin{array}{ccccc} 5 & 3 & 2 & 1 & 1 \\ & 3 & 2 & 1 & \\ & & 1 & 1 & \end{array} \quad \text{and} \quad T = \begin{array}{ccccc} 4 & 4 & 3 & 2 & 1 \\ & 3 & 3 & 2 & \\ & & 2 & 1 & \end{array} .$$

It is not hard to check that Π is a bijection between pairs of unmarked shifted tableaux of the same shape and strict plane partitions.

We only verify that

$$k(\pi) = k(S) + k(T) - l(S). \tag{3.3.2}$$

Other properties are straightforward implications of the definition of Π.

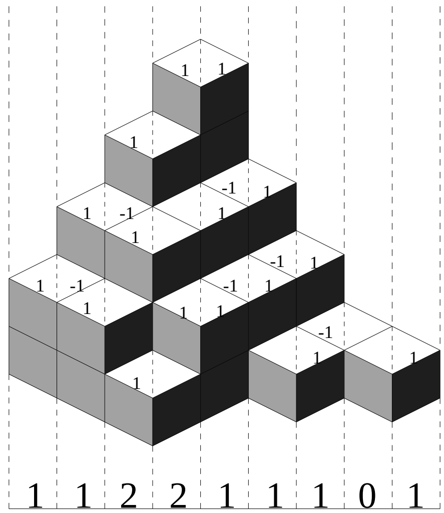


Figure 3.5: 3-dimensional diagram of a plane partition

Consider the 3-dimensional diagram of π (see Figure 3.5) and fix one of its vertical columns on the right (with respect to the main diagonal). A rhombus component consists of all black rhombi that are either directly connected or that have one white space between them. For the columns on the left we use gray rhombi instead of black ones. The number at the bottom of each column in Figure 3.5 is the number of rhombus components for that column. Let b, respectively g, be the number of rhombus components for all right, respectively left, columns. For the given example $b = 4$ and $g = 6$.

One can obtain b by a different counting. Consider edges on the right side. Mark all the edges with 0 except the following ones. Mark a common edge for a white rhombus and a black rhombus where the black rhombus is below the white rhombus with 1. Mark a common edge for two white rhombi that is perpendicular to the plane of black rhombi with -1. See Figure 3.5. One obtains b by summing these numbers over all edges on the right side of the 3-dimensional diagram. One recovers c in a similar way by marking edges on the left.

Now, we restrict to a connected component (one of the white terraces, see Figure 3.5) and sum all the number associated to its edges. If a connected component does not intersect the main diagonal then the sum is equal to 1. Otherwise this sum is equal to 2. This implies that

$$k(\pi) = b + g - l(\lambda^0).$$

Since $l(S) = l(\lambda^0)$ it is enough to show that $k(S) = b$ and $k(T) = g$ and (3.3.2) follows.

Each black rhombus in the right i^{th} column of the 3-dimensional diagram corresponds to an element of a border strip of S filled with i and each rhombus component corresponds to a border strip component. If two adjacent boxes from the same border strip are in the same row then the corresponding rhombi from the 3-dimensional diagram are directly connected and if they are in the same column then there is exactly one white space between them. This implies $k(S) = b$. Similarly, we get $k(T) = g$. □

Now, using the described correspondences sending $\mathcal{SP}(r,c)$ to $\mathcal{ST}^U(r,c)$ and $\mathcal{ST}^M(r,c)$ to $\mathcal{M}(r,c)$ we can prove Theorem 3.3.1.

Proof.

$$\begin{aligned}
\sum_{\pi \in \mathcal{SP}(r,c)} 2^{k(\pi)} x^{\mathrm{tr}(\pi)} s^{|\pi|} &\overset{\text{Thm 3.3.3}}{=} \sum_{(S,T) \in \mathcal{ST}^U(r,c)} 2^{k(S)+k(T)-l(S)} x^{|\mathrm{sh} S|} s^{|S|+|T|-|\mathrm{sh} S|} \\
&\overset{\text{by (3.3.1)}}{=} \sum_{(S,T) \in \mathcal{ST}^M(r,c)} x^{|\mathrm{sh} S|} s^{|S|+|T|-|\mathrm{sh} S|} \\
&\overset{\text{Thm 3.3.2}}{=} \sum_{A \in \mathcal{M}(r,c)} x^{\sum_{i,j} |a_{ij}|} s^{\sum_{i,j} (i+j-1)|a_{ij}|} \\
&= \prod_{i=1}^{r} \prod_{j=1}^{c} \sum_{a_{ij} \in P \cup 0} x^{|a_{ij}|} s^{(i+j-1)|a_{ij}|} \\
&= \prod_{i=1}^{r} \prod_{j=1}^{c} \frac{1 + x s^{i+j-1}}{1 - x s^{i+j-1}}.
\end{aligned}$$

□

Letting $r \to \infty$ and $c \to \infty$ we get

Corollary 3.3.4.

$$\sum_{\pi \in \mathcal{SP}} 2^{k(\pi)} x^{tr(\pi)} s^{|\pi|} = \prod_{n=1}^{\infty} \left(\frac{1 + x s^n}{1 - x s^n} \right)^n.$$

At $x = 1$ we recover the shifted MacMahon's formula.

Chapter 4

Plane overpartitions and cylindric partitions

4.1 Introduction

The goal of the first part of this chapter is to introduce a new object: the plane overpartitions and to give several enumeration formulas for these plane overpartitions. A *plane overpartition* is a plane partition where in each row the last occurrence of an integer can be overlined or not and in each column the first occurrence of an integer can be overlined or not and all the others are overlined. An example of a plane overpartition is

$$\begin{array}{cccc} 4 & 4 & \bar{4} & \bar{3} \\ \bar{4} & 3 & 3 & \bar{3} \\ \bar{4} & \bar{3} & & \\ 3 & & & \end{array}$$

This is an overpartition of the shape $(4, 4, 2, 1)$, with the weight equal to 38 and 6 overlined parts.

This chapter takes its place in the series of papers on overpartitions started by Corteel and Lovejoy [CL04]. The motivation is to show that the generating function of plane overpartitions is:

$$\prod_{n \geq 1} \left(\frac{1+q^n}{1-q^n} \right)^n .$$

In this chapter, we give several proofs of this result and several refinements and generalizations. Namely, we give

Theorem 1. *the hook–length formula for the generating function of plane overpartitions of a given shape, see Theorem 4.1.3;*

Theorem 2. *the hook formula for the generating function for reverse plane overpartitions, see Theorem 4.1.5;*

Theorem 3. *the generating formula for plane overpartitions with bounded parts, see Theorem 4.1.6.*

The goal of the second part of this chapter is to extend the generating formula for cylindric partitions due to Borodin and the 1-parameter generalized MacMahon's formula due to Vuletić:

$$\sum_{\substack{\Pi \text{ is a} \\ \text{plane partition}}} A_\Pi(t) q^{|\Pi|} = \prod_{n=1}^{\infty} \left(\frac{1-tq^n}{1-q^n} \right)^n, \tag{4.1.1}$$

where the weight $A_\Pi(t)$ is a polynomial in t that we describe below.

Given a plane partition Π, we decompose each connected component (a connected set of boxes filled with a same number) of its diagram into border components (i.e., rim hooks) and assign to each border component a level which is its diagonal distance to the end of the component. We associate to each border component of level i, the weight $(1-t^i)$. The weight of the plane partition Π is $A_\Pi(t)$ the product of the weights of its border components. See [Vul09] for further details and Figure 4.1 for an example of a plane partition of weight $(1-t)^{10}(1-t^2)^2(1-t^3)^2$.

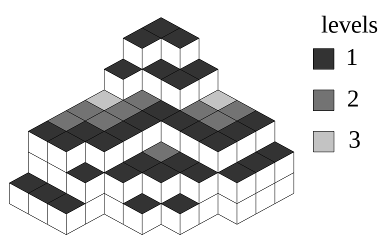


Figure 4.1: Weight of a plane partition

We give a new proof of the 1-parameter generalized MacMahon's formula. We also extend this formula to two more general objects: skew plane partitions and cylindric partitions. We give

Theorem 4. *1-parameter generalized formula for skew plane partitions, see Theorem 4.1.7;*

Theorem 5. *1-parameter generalized formula for cylindric partitions, see Theorem 4.1.8.*

In the rest of this section we give definitions and explain our results in more detail.

A partition λ is a nonincreasing sequence of positive integers $(\lambda_1, \dots, \lambda_k)$. The λ_is are called parts of the partition and the number of parts is denoted by $\ell(\lambda)$. The weight $|\lambda|$ of λ is the sum of its parts. A partition λ can be graphically represented by the Ferrers diagram that is a diagram formed of $\ell(\lambda)$ left justified rows, where the i^{th} row consists of λ_i cells (or boxes). The conjugate of a partition λ, denoted with λ', is a partition that has the Ferrers diagram equal to the transpose of the Ferrer diagram of λ. For a cell (i,j) of the Ferrers diagram of λ the hook length of this cell is $h_{i,j} = \lambda_i + \lambda'_j - i - j + 1$ and the content $c_{i,j} = j - i$. It is well known that the generating function of partitions that have at most n parts is $1/(q)_n$, where $(a)_n = (a;q)_n = \prod_{i=0}^{n-1}(1 - aq^i)$. More definitions on partitions can be found, for example, in [And76] or [Mac95].

An *overpartition* is a partition where the last occurrence of an integer can be overlined [CL04]. Last occurrences in an overpartition are in one–to–one correspondence with corners of the Ferrers diagram and overlined parts can be represented by marking the corresponding corners. The generating function of overpartitions that have at most n parts is $(-q)_n/(q)_n$.

Let λ be a partition. A *plane partition* of shape λ is a filling of cells of the Ferrers diagram of λ with positive integers that form a nonincreasing sequence along each row and each column. We denote the shape of a plane partition Π with $\mathrm{sh}(\Pi)$ and the sum of all entries with $|\Pi|$, called the weight of Π. It is well known, under the name of MacMahon's formula, that the generating function of plane partitions is

$$\sum_{\substack{\Pi \text{ is a} \\ \text{plane partition}}} q^{|\Pi|} = \prod_{n=1}^{\infty} \left(\frac{1}{1-q^n} \right)^n .$$

One way to prove this is to construct a bijection between plane partitions and pairs of semi–standard reverse Young tableaux of a same shape and to use the inverse of the RSK algorithm, that gives a bijection between those pairs of those tableaux [BK72].

Recall that a plane overpartition is a plane partition where in each row the last occurrence of an integer can be overlined or not and in each column the first occurrence of an integer can be overlined or not and all the others are overlined. This definition implies that the entries strictly decrease along diagonals. Therefore a plane overpartition is a plane partition made of rim hooks (connected skew shapes containing no 2×2 block of squares) where some entries are overlined. More precisely, it is easy to check that inside a rim hook only one entry can be chosen to be overlined or not and this entry is the upper right entry.

Plane overpartitions are therefore in bijection with rim hook plane partitions where each rim hook can be overlined or not (or weighted by 2). Recently, those weighted rim hook plane partitions were studied in [FW07, FW09, Vul07, Vul09]. The first result obtained was the shifted MacMahon's formula that says that the generating function of plane overpartitions is

$$\sum_{\substack{\Pi \text{ is a} \\ \text{plane overpartition}}} q^{|\Pi|} = \prod_{n=1}^{\infty} \left(\frac{1+q^n}{1-q^n}\right)^n.$$

This was obtained as a limiting case of the generating formula for plane overpartitions which fit into a $r \times c$ box, i.e., whose shapes are contained in the rectangular shape with r rows and c columns.

Theorem 4.1.1. *[FW07, Vul07] The generating function of plane overpartitions which fit in a $r \times c$ box is*

$$\prod_{i=1}^{r}\prod_{j=1}^{c} \frac{1+q^{i+j-1}}{1-q^{i+j-1}}.$$

This theorem was proved in [FW07, Vul07] using Schur P and Q symmetric functions and a suitable Fock space. In [Vul09] the theorem was proved in a bijective way where a RSK–type algorithm (due to Sagan [Sag87], see also Chapter XIII of [HH92]) was used to construct a bijection between plane overpartitions and matrices of nonnegative integers where positive entries can be overlined.

In Section 2, we give a mostly combinatorial proof of the generalized MacMahon formula [Vul09]. Namely, we prove

Theorem 4.1.2. *[Vul09]*

$$\sum_{\Pi \in \mathcal{P}(r,c)} A_{\Pi}(t) q^{|\Pi|} = \prod_{i=1}^{r}\prod_{j=1}^{c} \frac{1-tq^{i+j-1}}{1-q^{i+j-1}}. \tag{4.1.2}$$

where $\mathcal{P}(r,c)$ is the set of plane partitions with at most r rows and c columns. When we set $t = -1$, only the border components of level 1 have a non-zero weight and we get back Theorem 4.1.1.

The main result of Section 3 is a hook-content formula for the generating function of plane overpartitions of a given shape. More generally, we give a weighted generating function where overlined parts are weighted by some parameter a. Throughout Sections 3, 4 and 5 we assume these weights, unless otherwise stated.

Let $\mathcal{S}(\lambda)$ be the set of all plane overpartitions of shape λ. The number of overlined parts of an overpartition Π is denoted with $o(\Pi)$.

Theorem 4.1.3. *Let λ be a partition. The weighted generating function of plane overpartitions of shape λ is*

$$\sum_{\Pi \in \mathcal{S}(\lambda)} a^{o(\Pi)} q^{|\Pi|} = q^{\sum_i i\lambda_i} \prod_{(i,j)\in\lambda} \frac{1 + aq^{c_{i,j}}}{1 - q^{h_{i,j}}}. \tag{4.1.3}$$

We prove this Theorem using non intersecting paths. Another way to prove this result is to show that plane overpartitions of shape λ are in bijection with super semi-standard tableaux of shape λ. This is done in the thesis of the second author [Sav07] using a modification of the jeu de taquin. The super semi-standard tableaux of given shape were enumerated by Krattenthaler in [Kra96] using the Hillman-Grassl algorithm and a modified version of the jeu de taquin.

We also give the weighted generating formula for plane overpartitions "bounded" by λ, where by that we mean plane overpartitions such that the i^{th} row of the plane overpartition is an overpartition that has at most λ_i parts and at least λ_{i+1} parts. Let $\mathcal{B}(\lambda)$ be the set of all such plane overpartitions.

Theorem 4.1.4. *Let λ be a partition. The weighted generating function of plane overpartitions such that the i^{th} row of the plane overpartition is an overpartition that has at most λ_i parts and at least λ_{i+1} parts is*

$$\sum_{\Pi \in \mathcal{B}(\lambda)} a^{o(\Pi)} q^{|\Pi|} = q^{\sum_i (i-1)\lambda_i} \prod_{(i,j)\in\lambda} \frac{1 + aq^{c_{i,j}+1}}{1 - q^{h_{i,j}}}. \tag{4.1.4}$$

where $h_{i,j} = \lambda_i - i + 1 + \lambda'_j - j$ is the hook length of the cell (i,j) and $c_{i,j} = j - i$ is the content of the cell (i,j).

Note that it is enough to assign weights to overlined (or nonoverlined) parts only because generating functions where overlined and nonoverlined parts are weighted by a and b, respectively, follow trivially from the above formulas.

We prove these two theorems using a correspondence between plane overpartitions and sets of nonintersecting paths that use three kinds of steps. The work of Brenti used similar paths to compute super Schur functions [Bre93]. Using Gessel–Viennot results [GV85] we obtain determinantal formulas for the generating functions above. Then we evaluate the determinants to obtain the hook–content formulas for these determinants. We use a simple involution to show that the Stanley hook–content formula (Theorem 7.21.2 of [Sta99]) follows from our formula.

We connect these formulas to some special values of symmetric functions that are not obtained by evaluations.

The end of Section 3 is devoted to *reverse* plane overpartitions. A reverse plane (over) partition is defined like a plane (over)partition with nonincreasing property substituted with nondecreasing property and allowing 0's as entries. Precisely, a reverse plane partition of shape λ is a filling of cells of the Ferrers diagram of λ with nonnegative integers that form a nondecreasing sequence along each row and each column. A reverse plane overpartition is a reverse plane partition where only positive entries can be overlined and in each row the last occurrence of an integer can be overlined or not and in each column the first occurrence of a positive integer can be overlined or not and all others (if positive) are overlined. An example of a reverse plane overpartition is

$$\begin{array}{cccccc} 0 & 0 & 3 & 4 & 4 & \bar{4} \\ 0 & 0 & 4 & \bar{4} & & \\ 1 & \bar{3} & & & & \\ 3 & \bar{3} & & & & \end{array}$$

It was proved by Gansner [Gan81] that the generating function of reverse plane partitions of a given shape λ is

$$\prod_{(i,j)\in\lambda} \frac{1}{1-q^{h_{i,j}}}. \tag{4.1.5}$$

Let $\mathcal{S}^R(\lambda)$ be the set of all reverse plane overpartitions of shape λ. The generating function of reverse plane overpartitions is given by the following hook–formula.

Theorem 4.1.5. *Let λ be a partition. The generating function of reverse plane overpartitions of shape λ is*

$$\sum_{\Pi \in \mathcal{S}^R(\lambda)} q^{|\Pi|} = \prod_{(i,j)\in\lambda} \frac{1+q^{h_{i,j}}}{1-q^{h_{i,j}}}.$$

We construct a bijection between reverse plane overpartitions of a given shape and sets of nonintersecting paths whose endpoints are not fixed. Using results of [Ste89] we obtain a Pfaffian formula for the generating function of reverse plane partitions of a given shape. We further evaluate the Pfaffian and obtain a bijective proof of the hook formula. When λ is the partition with r parts equal to c, this result is the generating formula for plane overpartitions fitting in a box $r \times c$, namely Theorem 4.1.1.

In Section 4 we make a connection between plane overpartitions and domino tilings. We give some basic properties of this correspondence, like how removing a box or changing a mark to unmark changes the corresponding tiling. This, correspondence connects a measure on strict plane partitions studied in [Vul09] to a measure on domino tilings. This connection was expected by similarities in correlation kernels, limit shapes and some other features of these measures, but the connection was not established before.

In Section 5 we propose a bijection between matrices and pairs of plane overpartitions based on Berele and Remmel [BR85] which gives a bijection between matrices and pairs of (k,l)–semistandard tableaux. This bijection is based on the jeu de taquin. We give another stronger version of the shifted MacMahon's formula, as we give a weighted generating function of plane overpartitons with bounded entries. Let $\mathcal{L}(n)$ be the set of all plane overpartitions with the largest entry at most n.

Theorem 4.1.6. *The weighted generating functions of plane overpartitions where the largest entry is at most n is*

$$\sum_{\Pi \in \mathcal{L}(n)} a^{o(\Pi)} q^{|\Pi|} = \frac{\prod_{i,j=1}^{n}(1+aq^{i+j})}{\prod_{i=1}^{n}\prod_{j=0}^{i-1}(1-q^{i+j})(1-aq^{i+j})}.$$

In Section 6 we study interlacing sequences and cylindric partitions. We say that a sequence of partitions $\Lambda = (\lambda^1, \ldots, \lambda^T)$ is *interlacing* if λ^i/λ^{i+1} or λ^{i+1}/λ^i is a horizontal strip, i.e. a skew shape having at most one cell in each column. Let $A = (A_1, \ldots, A_{T-1})$ be a sequence of 0's and 1's. We say that an interlacing sequence $\Lambda = (\lambda^1, \ldots, \lambda^T)$ has *profile*

A if when $A_i = 1$, respectively $A_i = 0$ then λ^i/λ^{i+1}, respectively λ^{i+1}/λ^i is a horizontal strip. The *diagram* of an interlacing sequence is the set of boxes filled with parts of λ^i's

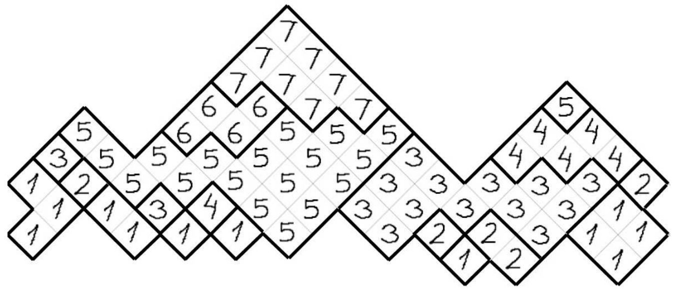


Figure 4.2: The diagram of an interlacing sequence

as shown in Figure 4.2. Namely, the i^{th} diagonal represents λ^i and if the first part of λ^i is placed at (i, j) then the first part of λ^{i+1} is placed at $(i+1, j-1)$ if $A_i = 1$ or $(i+1, j+1)$ otherwise. Observe that the profile A defines the upper border of the diagram (part of the border with a positive slope corresponds to 0's and with negative to 1's).

A (skew) plane partition and cylindric partition are examples of interlacing sequences. A plane partition can be written as $\Lambda = (\emptyset, \lambda^1, \dots, \lambda^T, \emptyset)$ with profile $A = (0, 0, \dots, 0, 1, \dots, 1, 1)$ and λ^is are diagonals of the plane partition. A skew plane partition is an interlacing sequence $\Lambda = (\emptyset, \lambda^1, \dots, \lambda^T, \emptyset)$ with a profile $A = (A_0, A_1, \dots, A_{T-1}, A_T)$, where $A_0 = 0$ and $A_T = 1$. A cylindric partition is an interlacing sequence $\Lambda = (\lambda^0, \lambda^1, \dots, \lambda^T)$ where $\lambda^0 = \lambda^T$, and T is called the period of Λ. A cylindric partitions can be represented by the *cylindric diagram* that is obtained from the ordinary diagram by identification of the first and last diagonal.

A *connected component* of an interlacing sequence Λ is the set of all rookwise connected boxes of its diagram that are filled with a same number. We denote the number of connected components of Λ with $k(\Lambda)$. For the example from Figure 4.2 we have $k(\Lambda) = 19$ and its connected components are shown in Figure 4.2 (bold lines represent boundaries of these components).

If a box (i, j) belongs to a connected component C then we define its *level* $\ell(i, j)$ as the smallest positive integer such that $(i + \ell, j + \ell)$ does not belong to C. In other words, a level represents the distance from the "rim", distance being measured diagonally. A *border component* is a rookwise connected subset of a connected component where all boxes have the same level. We also say that this border component is of this level. All border components are rim hooks. For the example above, border components and their levels are shown in

Figure 4.3.

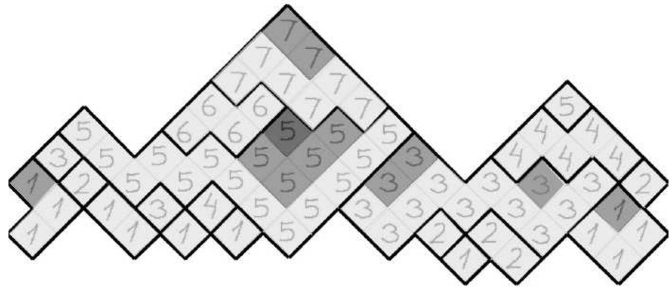


Figure 4.3: Border components with levels

For a cylindric partition we define *cylindric connected components* and *cylindric border components* in the same way but connectedness is understood on the cylinder, i.e., boxes are connected if they are rookwise connected in the cylindric diagram.

Let $(n_1, n_2, \dots)$ be a sequence of nonnegative integers where n_i is the number of i–level border components of Λ. We set

$$A_\Lambda(t) = \prod_{i\geq 1}(1-t^i)^{n_i}. \tag{4.1.6}$$

For the example above $A_\Lambda(t) = (1-t)^{19}(1-t^2)^6(1-t^3)$.

Similarly, for a cylindric partition Π we define

$$A_\Pi^{\mathrm{cyl}}(t) = \prod_{i\geq 1}(1-t^i)^{n_i^{\mathrm{cyl}}},$$

where n_i^{cyl} is the number of cylindric border components of level i.

In Section 6 we give a generating formula for skew plane partitions. Let $\mathrm{Skew}(T,A)$ be the set of all skew plane partitions $\Lambda = (\emptyset, \lambda^1, \dots, \lambda^T, \emptyset)$ with profile $A = (A_0, A_1, \dots, A_{T-1}, A_T)$, where $A_0 = 0$ and $A_T = 1$.

Theorem 4.1.7. *(Generalized MacMahon's formula for skew plane partitions; Hall-Littlewood's case)*

$$\sum_{\Pi\in Skew(T,A)} A_\Pi(t) q^{|\Pi|} = \prod_{\substack{0\leq i<j\leq T\\ A_i=0,\, A_j=1}} \frac{1-tq^{j-i}}{1-q^{j-i}}.$$

Note that as profiles are words in $\{0,1\}$, a profile $A = (A_0, \dots, A_T)$ encodes the border of a Ferrers diagram λ. Skew plane partitions of profile A are in one–to–one correspondence with reverse plane partitions of shape λ. Moreover, one can check that

$$\prod_{\substack{0\le i<j\le T\\ A_i=0,\,A_j=1}} \frac{1-tq^{j-i}}{1-q^{j-i}} = \prod_{(i,j)\in\lambda} \frac{1-tq^{h_{i,j}}}{1-q^{h_{i,j}}}.$$

Therefore the Theorem of Gansner (equation (4.1.5)) is Theorem 4.1.7 with $t=0$ and our Theorem 4.1.5 on reverse plane overpartitions is Theorem 4.1.7 with $t=-1$.

This theorem is also a generalization of results of Vuletić [Vul09]. In [Vul09] a 2–parameter generalization of MacMahon's formula related to Macdonald symmetric functions was given and the formula is especially simple in the Hall-Littlewood case. In the Hall-Littlewood case, this is a generating formula for plane partitions weighted by $A_\Pi(t)$. Theorem 4.1.7 can be naturally generalized to Macdonald case, but we do not pursue this here.

Let $\mathrm{Cyl}(T,A)$ be the set of all cylindric partitions with period T and profile A. The main result of Section 6 is:

Theorem 4.1.8. *(Generalized MacMahon's formula for cylindric partitions; Hall-Littlewood's case)*

$$\sum_{\Pi\in Cyl(T,A)} A_\Pi^{cyl}(t)q^{|\Pi|} = \prod_{n=1}^{\infty} \frac{1}{1-q^{nT}} \prod_{\substack{1\le i,j\le T\\ A_i=0,\,A_j=1}} \frac{1-tq^{(j-i)_{(T)}+(n-1)T}}{1-q^{(j-i)_{(T)}+(n-1)T}},$$

where $i_{(T)}$ is the smallest positive integer such that $i\equiv i_{(T)}$ mod T.

The case $t=0$ is due to Borodin and represents a generating formula for cylindric partitions. Cylindric partitions were introduced and enumerated by Gessel and Krattenthaler [GK97]. The result of Borodin could be also proven using Theorem 5 of [GK97] and the $\mathrm{SU}(r)$-extension of Bailey's ${}_6\psi_6$ summation due to Gustafson (equation (7.9) in [GK97]) [Kra08]. Again Theorem 4.1.8 can be naturally generalized to Macdonald case. The trace generating function of those cylindric partitions could also be easily derived from our proof, as done by Okada [Oka09] for the reverse plane partitions case.

The chapter is organized as follows. In Section 2 we give a mostly combinatorial proof of the generalized MacMahon formula. In Section 3 we use nonintersecting paths and Gessel–Viennot result's to obtain the hook–length formulas for plane overpartitions for reverse plane partitions of a given shape. In Section 4 we make the connection between tilings and plane overpartitions. In Section 5 we construct a bijection between matrices and pairs of plane overpartitions and obtain a generating formula for plane overpartitions with bounded part size. In Section 6 we give the hook formula for reverse plane partitions contained in a given shape and the 1–parameter generalization of the generating formula for cylindric partitions.

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4.2 Plane partitions and Hall–Littlewood functions

In this Section, we give an alternative proof of the generalization of MacMahon formula due to the third author [Vul09]. Our proof is mostly combinatorial as it uses a bijection between plane partitions and pairs of strict plane partitions of the same shape and the combinatorial description of the Hall–Littlewood functions [Mac95].

Let $\mathcal{P}(r,c)$ be the set of plane partitions with at most r rows and c columns. Given a plane partition Π, let $A_\Pi(t)$ be the polynomial defined in (4.1.6), as $A_\Pi(t) = \prod_{r \text{ border component}}(1-t^{\text{level}(r)})$.

Recall that Theorem 4.1.2 states that

$$\sum_{\Pi\in\mathcal{P}(r,c)} A_\Pi(t) q^{|\Pi|} = \prod_{i=1}^{r}\prod_{j=1}^{c} \frac{1-tq^{i+j-1}}{1-q^{i+j-1}}.$$

Any plane partition Π is in bijection with a sequence of partitions $(\pi^{(1)},\pi^{(2)},\ldots)$. This sequence is such that $\pi^{(i)}$ is the shape of the entries greater than or equal to i in Π. For example if

$$\Pi = \begin{matrix} 4433 \\ 3332 \\ 1 \end{matrix}\ ,$$

the corresponding sequence is $((4,4,1),(4,4),(4,3),(2))$.

Note that the plane partition Λ is column strict if and only if $\lambda^{(i)}/\lambda^{(i+1)}$ is a horizontal strip.

We use a bijection between pairs of column strict plane partitions (Σ, Λ) and plane partitions Π due to Bender and Knuth [BK72]. We suppose that (Σ, Λ) are of the same shape λ and that the corresponding sequences are $(\sigma^{(1)}, \sigma^{(2)}, \ldots)$ and $(\lambda^{(1)}, \lambda^{(2)}, \ldots)$.

Given a plane partition $\Pi = (\Pi_{i,j})$, we define the entries of diagonal x to be the partition $(\Pi_{i,j})$ with $i, j \geq 1$ and $j - i = x$. The bijection is such that the entries of diagonal x of Π are $\sigma^{(x+1)}$ if $x \geq 0$ and $\lambda^{(-x-1)}$ otherwise. Note that as Λ and Σ have the same shape, the entries of the main diagonal $(x = 0)$ are $\sigma^{(1)} = \lambda^{(1)}$.

For example, start with

$$\Sigma = \begin{array}{l} 4444 \\ 2221 \\ 111 \end{array} \text{ and } \Lambda = \begin{array}{l} 4433 \\ 3322 \\ 111 \end{array},$$

whose sequences are $((4,4,3),(4,3),(4),(4))$ and $((4,4,3),(4,4),(4,2),(2))$, respectively and get

$$\Pi = \begin{array}{l} 4444 \\ 443 \\ 443 \\ 22 \end{array}.$$

This construction implies that:

$$\begin{aligned} |\Pi| &= |\Sigma| + |\Lambda| - |\lambda|; \\ A_\Pi(t) &= \frac{\varphi_\Sigma(t)\varphi_\Lambda(t)}{b_\lambda(t)}; \end{aligned}$$

with

$$b_\lambda(t) = \prod_{i \geq 1} \phi_{m_i(\lambda)}(t); \quad \phi_r(t) = \prod_{j=1}^{r} (1 - t^j);$$

and

$$\phi_\Lambda(t) = \prod_{i \geq 1} \phi_{\lambda^{(i)}/\lambda^{(i+1)}}(t);$$

given a horizontal strip $\theta = \lambda/\mu$,

$$\phi_\theta(t) = \prod_{i \in I} (1 - t^{m_i(\lambda)}),$$

where $m_i(\lambda)$ is the multiplicity of i in λ and I is the set of integers such that $\theta'_i = 1$ and $\theta'_{i+1} = 0$. See [Mac95], Chapter III Sections 2 and 5.

Indeed

- Each factor $(1-t^i)$ in $b_\lambda(t)$ is in one-to-one correspondence with a border component of level i that goes through the main diagonal of Π.
- Each factor $(1-t^i)$ in $\phi_\Sigma(t)$ is in one-to-one correspondence with a border component of level i that ends in a non-negative diagonal.
- Each factor $(1-t^i)$ in $\phi_\Lambda(t)$ is in one-to-one correspondence with a border component of level i that starts in a non-positive diagonal.

Continuing with our example, we have

$$\phi_\Sigma(t) = (1-t)^2(1-t^2); \quad \phi_\Lambda(t) = (1-t)^3(1-t^2); \quad b_\lambda(t) = (1-t)^2(1-t^2);$$

and

$$A_\Pi(t) = (1-t)^3(1-t^2) = \frac{(1-t)^2(1-t^2)(1-t)^3(1-t^2)}{(1-t)^2(1-t^2)}.$$

We recall the combinatorial definition of the Hall–Littlewood functions as done in the book of Macdonald [Mac95]. The Hall-Littlewood function $Q_\lambda(x;t)$ can be defined as

$$Q_\lambda(x;t) = \sum_{\substack{\Lambda \\ \mathrm{sh}(\Lambda)=\lambda}} \phi_\Lambda(t) x^\Lambda;$$

where $x^\Lambda = x_1^{\alpha_1} x_2^{\alpha_2} \dots$ and α_i is the number of entries equal to i in Λ. See [Mac95] Chapter III, equation (5.11).

A direct consequence of the preceding bijection is that, the entries of Σ are less or equal to r and the entries of Λ are less or equal to c if and only if Π is in $\mathcal{P}(r,c)$. Therefore:

$$\sum_{\Pi \in \mathcal{P}(r,c)} A_\Pi(t) q^{|\Pi|} = \sum_\lambda \frac{Q_\lambda(q,\dots,q^r,0,\dots;t) Q_\lambda(q^0,\dots,q^{c-1},0,\dots;t)}{b_\lambda(t)}.$$

Finally, we need equation (4.4) in Chapter III of [Mac95].

$$\sum_{\lambda} \frac{Q_\lambda(x;t)Q_\lambda(y;t)}{b_\lambda(t)} = \prod_{i,j} \frac{1-tx_iy_j}{1-x_iy_j}; \tag{4.2.1}$$

With the substitutions $x_i = q^i$ for $1 \le i \le r$ and 0 otherwise and $y_j = q^{j-1}$ for $1 \le j \le c$ and 0 otherwise, we get the result.

4.3 Nonintersecting paths

4.3.1 Plane overpartitions of a given shape

In this section we represent plane overpartitions as nonintersecting paths and using results from [GV85] we prove Theorems 4.1.3 and 4.1.4. A similar approach was used for example in [Bre93] to compute Super–Schur functions.

We construct a bijection between the set of paths from $(0,0)$ to (x,k) and the set of overpartitions with at most k parts less or equal to x. Given an overpartition the corresponding path consists of North and East edges that form the border of the Ferrer diagram of the overpartition except for corners containing an overlined entry where we substitute a pair of North and East edge with an North–East edge. For example, the path corresponding to the overpartition $(5, \bar{5}, 3, 3, \bar{2})$ is shown on Figure 4.4.

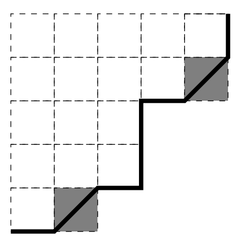

Figure 4.4: Paths and overpartitions

To each overpartition λ we associate a weight equal to $a^{o(\lambda)}q^{|\lambda|}$, where $o(\lambda)$ is the number of overlined parts. To have the same weight on the corresponding path we introduce the following weights on edges. We assign weight 1 to East edges, q^i to North edges on level i and weight aq^{i+1} to North-East edges joining levels i and $i+1$. For convenience we use these paths and weights to construct a bijection between plane overpartitions and sets of nonintersecting paths.

Lemma 4.3.1. *[CL04] The generating functions of overpartitions with at most k parts is given by*

$$\sum_{l(\lambda)\le k} a^{o(\lambda)}q^{|\lambda|} = \frac{(-aq)_k}{(q)_k} \tag{4.3.1}$$

and exactly k parts is given by

$$\sum_{l(\lambda)=k} a^{o(\lambda)}q^{|\lambda|} = q^k\frac{(-a)_k}{(q)_k}. \tag{4.3.2}$$

Proof. Let λ be an overpartition seen as the corresponding path. Let $r_i(\lambda)$ be the number of North–East edges joining levels i and $i+1$. Obviously, it takes values 0 or 1. Let $s_i(\lambda)$ be the number of East edges on level i. Then

$$\sum_{l(\lambda)\le k} a^{o(\lambda)}q^{|\lambda|} = \prod_{i=1}^{k}\sum_{r_i=0,1}(aq^i)^{r_i}\prod_{i=1}^{k}\sum_{s_i=0}^{\infty}\left(q^i\right)^{s_i} = \frac{(-aq)_k}{(q)_k}.$$

Now, if λ is an overpartition with exactly k parts then $r_k(\lambda)$ or $s_k(\lambda)$ must be nonzero. Then using (4.3.1) we obtain

$$\sum_{l(\lambda)=k} a^{o(\lambda)}q^{|\lambda|} = q^k\left[\frac{(-aq)_k}{(q)_k} + a\frac{(-aq)_{k-1}}{(q)_{k-1}}\right] = q^k\frac{(-a)_k}{(q)_k}.$$

□

For a plane overpartition Π of shape λ we construct a set of nonintersecting paths using paths from row overpartitions where the starting point of the path corresponding to the i^{th} row is shifted upwards by $\lambda_1-\lambda_i+i-1$ so that the starting point is $(0,\lambda_1-\lambda_i+i-1)$. That way we obtain a bijection between the set of nonintersecting paths from $(0,\lambda_1-\lambda_i+i-1)$ to (x,λ_1+i-1), where i goes from 1 to $\ell(\lambda)$ and the set of plane overpartitions whose i^{th} row overpartition has at most λ_i and at least λ_{i-1} parts with x greater or equal to the largest part. The weights on the edges correspond to the weight $a^{o(\Pi)}q^{|\Pi|}$. Figure 4.5 (see also Figure 4.8) shows the corresponding set of nonintersecting paths for $x=8$ and the plane overpartition

$$\begin{array}{ccccc} 7 & 4 & \bar{3} & 2 & \bar{2} \\ 3 & 3 & \bar{3} & \bar{2} & \\ \bar{3} & 2 & \bar{2} & & \\ 2 & & & & \end{array}.$$

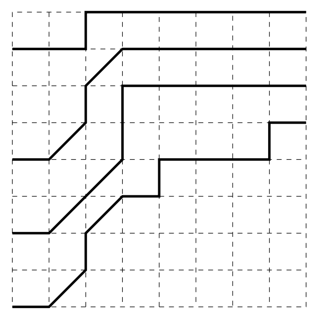

Figure 4.5: Nonintersecting paths

Definition. For a partition λ we define $M_\lambda(a;q)$ to be a $\ell(\lambda) \times \ell(\lambda)$ matrix whose $(i,j)^{th}$ entry is given by

$$\frac{(-a)_{\lambda_j+i-j}}{(q)_{\lambda_j+i-j}}.$$

For a partition λ let $\mathcal{B}(\lambda)$ and $\mathcal{S}(\lambda)$ be as in the introduction, i.e., the sets of all plane overpartitions bounded by shape λ and of the shape λ, respectively.

Proposition 4.3.2. *Let λ be a partition. The weighted generating function of plane overpartitions such that the i^{th} row of the plane overpartition is an overpartition that has at most λ_i parts and at least λ_{i+1} parts is given by*

$$\sum_{\Pi \in \mathcal{B}(\lambda)} a^{o(\Pi)} q^{|\Pi|} = \det M_\lambda(aq;q).$$

Proof. From (4.3.1) we have that the limit when x goes to infinity of the number of paths from $(0,0)$ to (x,k) is $(-aq)_k/(q)_k$. Using Corollary 2 of [GV85] we have that $\det M_\lambda(aq;q)$ is the limit when x goes to infinity of the generating function of $\ell(\lambda)$ nonintersecting paths going from $(0, \lambda_1 + i - 1 - \lambda_i)$ to $(x, \lambda_1 + i - 1)$. Thanks to the bijection between paths and overpartitions this is also the generating function for overpartitions whose i^{th} row overpartition has at most λ_i and at least λ_{i+1} parts. □

Proposition 4.3.3. *Let λ be a partition. The weighted generating function of plane overpartitions of shape λ is given by*

$$\sum_{\Pi \in \mathcal{S}(\lambda)} a^{o(\Pi)} q^{|\Pi|} = q^{|\lambda|} \det M_\lambda(a;q).$$

Proof. The proof is the same as the proof of Proposition 4.3.2. Here we just use (4.3.2) instead of (4.3.1). □

We now prove a lemma that will allow us to give a product formula for these generating functions.

Lemma 4.3.4.

$$\det M_\lambda(a;q) = q^{\sum_i (i-1)\lambda_i} \prod_{(i,j)\in\lambda} \frac{1+aq^{c_{i,j}}}{1-q^{h_{i,j}}}. \tag{4.3.3}$$

Proof. We prove the lemma by induction on the number of columns of λ. For the empty partition the lemma is true. So we assume that λ is nonempty and that for all μ having less columns then λ the lemma holds, i.e.,

$$\det M_\mu(a;q) = q^{\sum_i (i-1)\mu_i} \prod_{(i,j)\in\mu} \frac{1+aq^{c_{i,j}}}{1-q^{h_{i,j}}}. \tag{4.3.4}$$

Suppose that $\ell(\lambda) = k$. Let $M = M_\lambda(a;q)$ and $Q = M_{\overline{\lambda}}(aq;q)$, where $\overline{\lambda} = (\lambda_1 - 1, \ldots, \lambda_k - 1)$. Therefore,

$$Q(i,j) = \frac{(-aq)_{\lambda_j+i-j-1}}{(q)_{\lambda_j+i-j-1}}.$$

Our goal is to show that

$$\det M = q^{\binom{k}{2}} \prod_{i=1}^{k} \frac{1+aq^{c_{i,1}}}{1-q^{h_{i,1}}} \det Q, \tag{4.3.5}$$

because then (4.3.3) follows by induction.

We introduce two $k \times k$ matrices N and P with the following properties:

$$d_{M/N} := \frac{\det M}{\det N} = \prod_{i=1}^{k} \frac{(-a)_{h_{i,1}}}{(q)_{h_{i,1}}}, \tag{4.3.6}$$

$$d_{N/P} := \frac{\det N}{\det P} = \frac{\prod_{i=0}^{k-1}(a+q^i)}{(1+a)^k} = q^{\binom{k}{2}} \frac{\prod_{i=1}^{k}(1+aq^{c_{i,1}})}{(1+a)^k}, \tag{4.3.7}$$

$$d_{P/Q} := \frac{\det P}{\det Q} = \prod_{i=1}^{k} \frac{(q)_{h_{i,1}-1}}{(-aq)_{h_{i,1}-1}}. \tag{4.3.8}$$

These properties imply (4.3.5).

Matrices N and P are given by

$$N(i,j) = M(i,j)\frac{(q)_{h_{j,1}}}{(-a)_{h_{j,1}}} = \frac{(-a)_{\lambda_j+i-j}(q)_{h_{j,1}}}{(q)_{\lambda_j+i-j}(-a)_{h_{j,1}}}$$

and

$$P(i,j) = Q(i,j)\frac{(q)_{h_{j,1}-1}}{(-aq)_{h_{j,1}-1}} = \frac{(-aq)_{\lambda_j+i-j-1}(q)_{h_{j,1}-1}}{(q)_{\lambda_j+i-j-1}(-aq)_{h_{j,1}-1}}.$$

Properties (4.3.6) and (4.3.8) follow immediately. We will prove that

$$P(i,j) = \frac{(1+a)N(i,j) - (1-q^{k-i})P(i+1,j)}{a+q^{k-i}}, \qquad i < k, \tag{4.3.9}$$

which will imply (4.3.7).

Let

$$S = (1+a)N(i,j) - (1-q^{k-i})P(i+1,j).$$

Note that $h_{j,1} = \lambda_j - j + k$. Then

$$\begin{aligned} S &= \frac{(1+a)(-a)_{\lambda_j+i-j}(q)_{h_{j,1}}}{(q)_{\lambda_j+i-j}(-a)_{h_{j,1}}} - \frac{(1-q^{k-i})(-aq)_{\lambda_j+i-j}(q)_{h_{j,1}-1}}{(q)_{\lambda_j+i-j}(-aq)_{h_{j,1}-1}} \\ &= \left[\frac{(1+a)(1-q^{h_{j,1}})}{1-q^{\lambda_j+i-j}} - \frac{(1-q^{k-i})(1+aq^{\lambda_j+i-j})}{1-q^{\lambda_j+i-j}}\right] P(i,j) \\ &= (a+q^{k-i})P(i,j) \end{aligned}$$

and thus (4.3.9) holds.

□

Then using this lemma we obtain the product formulas for the generating functions from

Propositions 4.3.2 and 4.3.3. Those are the hook–length formulas given in Theorems 4.1.3 and 4.1.4.

Stanley's hook-content formula states the generating function of column strict plane partitions of shape λ where the entries are less or equal to n is

$$\prod_{(i,j)\in\lambda} \frac{1+q^{n+c_{i,j}}}{1-q^{h_{i,j}}}.$$

See Theorem 7.21.2 of [Sta99]. It can be seen as a special case of the hook–content formula from Theorem 4.1.3 for $a=-q^n$. Indeed there is an easy sign reversing involution on plane overpartitions where the overlined entries are larger than n and where the sign of each entry is $-$ if the entry is overlined (and greater than n) and $+$ otherwise and the sign of the overpartition is the product of the signs of the entries. We define now this involution. If the largest entry of the plane overpartition is greater than n then if it appears overlined in the first row then take off the overline and otherwise overline the rightmost occurrence of the largest entry in the first row. Note that this changes the sign of the plane overpartition. If the largest entry is at most n then do nothing.

Now, we want to obtain a formula for plane overpartitions with at most r rows and c columns

Proposition 4.3.5. *The weighted generating function of plane overpartitions with at most r rows and c columns*

$$\sum_{c\geq\lambda_1\geq\ldots\geq\lambda_{(r-1)/2}\geq 0} \det M_{(c,\lambda_1,\lambda_1,\ldots,\lambda_{(r-1)/2},\lambda_{(r-1)/2})}(a;q)$$

if r is odd and

$$\sum_{n\geq\lambda_1\geq\ldots\geq\lambda_{r/2}\geq 0} \det M_{(\lambda_1,\lambda_1,\ldots,\lambda_{r/2},\lambda_{r/2})}(a;q)$$

otherwise.

In particular, the weighted generating function of all overpartitions is:

$$\sum_{\lambda_1\geq\ldots\geq\lambda_k} \det M_{(\lambda_1,\lambda_1,\ldots,\lambda_k,\lambda_k)}(a;q)$$

Proof. This is a direct consequence of Proposition 4.3.2. □

We will use this result to get another "symmetric function" proof of the shifted MacMahon formula.

4.3.2 The shifted MacMahon formula

The shifted MacMahon formula obtained in [FW07, Vul07, Vul09]:

$$\sum_{\substack{\Pi \text{ is a plane} \\ \text{overpartition}}} q^{|\Pi|} = \prod_{n=1}^{\infty} \left(\frac{1+q^n}{1-q^n}\right)^n$$

can be obtained from symmetric functions using the results of the preceeding Subsection.

We showed how Stanley's hook-content formula follows from Theorem 4.1.3. Alternatively, we could use Stanley's hook–content formula and symmetric functions to prove Lemma 4.3.4 that further implies Theorems 4.1.3 and 4.1.4.

A specialization is an algebra homomorphism between the algebra of symmetric functions and $\mathbb{C}$. We use three standard bases for the algebra of symmetric functions: Schur functions s, complete symmetric functions h and power sums p (see [Mac95]). If ρ is a specialization we denote their images with $s|_\rho$, $h|_\rho$ and $p|_\rho$.

Let $\rho(a) : \Lambda \to \mathbb{C}[a]$ be an algebra homomorphism given with

$$h_n|_{\rho(a)} = \frac{(-a)_n}{(q)_n}.$$

Since, for $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$

$$s_\lambda = \det(h_{\lambda_i - i + j})$$

we have from Definition 4.3.1 that

$$s_\lambda|_{\rho(a)} = M_\lambda(a; q). \tag{4.3.10}$$

Now, we show that

$$p_n|_{\rho(a)} = \frac{1-(-a)^n}{1-q^n}.$$

This follows from (2.2.1) of [And76]:

$$1+\sum_{n=1}^{\infty}\frac{(-a)_n}{(q)_n}t^n=\prod_{n=0}^{\infty}\frac{1+atq^n}{1-tq^n}$$

and

$$P(t)=\frac{H'(t)}{H(t)},$$

where H and P are generating functions for h_n and p_n.

Recall the well known formula (see for example Theorem 7.21.2 of [Sta99]):

$$s_\lambda(1,q,\dots,q^n,0,\dots)=q^{b(\lambda)}\prod_{x\in\lambda}\frac{1-q^{n+c(x)+1}}{1-q^{h(x)}}. \tag{4.3.11}$$

Observe that $\rho(q^n)$ is a specialization given with the evaluation $x_i=q^i$ for $i=1,\dots,n$ and 0 otherwise. Then (4.3.11) says

$$s_\lambda|_{\rho(q^n)}=q^{b(\lambda)}\prod_{x\in\lambda}\frac{1-q^nq^{c(x)+1}}{1-q^{h(x)}},\qquad \text{for every } n.$$

This implies that

$$s_\lambda|_{\rho(a)}=q^{b(\lambda)}\prod_{x\in\lambda}\frac{1+aq^{c(x)+1}}{1-q^{h(x)}},$$

because we have polynomials in a on both sides and equality is satisfied for infinitely many values ($a=q^n$, for every n). So, this gives an alternate proof of Lemma 4.3.4.

The shifted MacMahon formula can be obtained from (4.3.10) and Proposition 4.3.5. We have that

$$\sum_{\substack{\Pi \text{ is a plane}\\ \text{overpartition}}}q^{|\Pi|}=\sum_{\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_k}s_{(\lambda_1,\lambda_1,\lambda_2,\lambda_2,\dots,\lambda_k,\lambda_k)}|_{\rho_1}=\sum_{\lambda' \text{ even}}s_\lambda|_{\rho_1},$$

where λ' is the transpose of λ and a partition is even if it has even parts and $\rho_1=\rho(1)$.

By Ex. 5 b) on p. 77 of [Mac95] we have that

$$\sum_{\lambda' \text{ even}}s_\lambda=\prod_{i<j}\frac{1}{x_i-x_j}=\exp\left[\sum_{n=1}^{\infty}\frac{1}{2n}(p_n^2-p_{2n})\right].$$

So,

$$\begin{aligned}\sum_{\lambda' \text{ even}} s_\lambda|_{\rho_1} &= \exp\left[\sum_{n=1,3,5,\ldots} \frac{1}{2n}\left(\left(\frac{1+q^n}{1-q^n}\right)^2 - 1\right)\right] \\ &= \exp\left[\sum_{n=1,3,5,\ldots} \frac{2q^n}{n(1-q^n)^2}\right] \\ &= \exp\left[\sum_{n=1,3,5,\ldots} \frac{2}{n} p_n^2|_{\rho_2}\right],\end{aligned}$$

where the specialization ρ_2 is given with

$$p_n|_{\rho_2} = \frac{q^{n/2}}{1-q^n}.$$

This means that the specialization ρ_2 is actually an evaluation given with $x_i = q^{(2i+1)/2}$. Since

$$\exp\left[\sum_{n=1,3,5,\ldots} \frac{2}{n} p_n^2\right] = \prod_{i,j=1}^{\infty} \frac{1+x_i x_j}{1-x_i x_j}$$

we obtain

$$\exp\left[\sum_{n=1,3,5,\ldots} \frac{2}{n} p_n^2|_{\rho_2}\right] = \prod_{n=1}^{\infty} \left(\frac{1+q^n}{1-q^n}\right)^n$$

which proves the shifted MacMahon formula.

4.3.3 Reverse plane overpartitons

In this Section we construct a bijection between the set of all reverse plane overpartitions and sets of nonintersecting paths whose endpoints are not fixed. We use this bijection and Stembridge's results [Ste89] to obtain a Pfaffian formula for the generating function of reverse plane overpartitions of a given shape. Evaluating the Pfaffian we obtain the hook formula for reverse plane overpartitions due to Okada. Namely, if $\mathcal{S}^R(\lambda)$ is the set of all reverse plane partitions of shape λ then

Theorem 4.3.6. *The generating function of reverse plane overpartitions of shape λ is*

$$\sum_{\Pi \in \mathcal{S}^R(\lambda)} q^{|\Pi|} = \prod_{(i,j)\in\lambda} \frac{1+q^{h_{i,j}}}{1-q^{h_{i,j}}}.$$

We construct a weight preserving bijection between reverse plane overpartitions and sets of nonintersecting paths on a triangular lattice in a similar fashion as in Section 2. The lattice consists of East, North and North–East edges. East edges have weight 1, North edges on level i have weight q^i and North–East edges joining levels i and $i+1$ have weight q^i. The weight of a set of nonintersecting paths p is the product of the weights of their edges and is denoted with $w(p)$. Let Π be a reverse plane overpartition whose positive entries form a skew shape λ/μ and let $\ell = \ell(\lambda)$. Then Π can be represented by a set of n nonintersecting lattice paths such that

- the departure points are $(0, \mu_i + \ell - i)$ and
- the arrivals points are $(x, \lambda_i + \ell - i)$,

for a large enough x and $i = 1, \ldots, \ell$. For example let $x = 8$, $\lambda = (5,4,2,2)$ and $\mu = (2,1)$. Figure 4.6 shows the corresponding set of nonintersecting paths for the reverse plane overpartition of shape $\lambda\backslash\mu$

$$\begin{array}{ccccc} & & 3 & 4 & 4 \\ & 3 & 4 & \bar{4} & \\ 1 & \bar{3} & & & \\ 3 & \bar{3} & & & \end{array} .$$

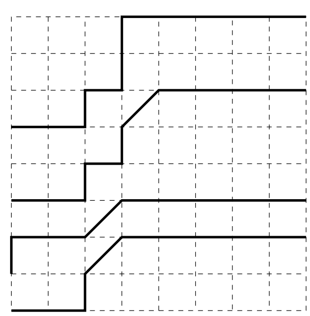

Figure 4.6: Nonintersecting paths and reverse plane overpartitions

This implies that all reverse plane overpartitions of shape λ can be represented by non-intersecting lattice paths such that

- the departure points are a l–element subset of $\{(0,i) \mid i \geq 0\}$ and
- the arrivals points are $(x, \lambda_i + l - i)$,

with $x \to \infty$.

Now, for $r_1 > r_2 > \cdots > r_n \geq 0$ we define

$$W(r_1, r_2, \dots, r_n) = \lim_{x\to\infty} \sum_{p \in P(x; r_1, \dots, r_n)} w(p),$$

where $P(x; r_1, r_2, \dots, r_n)$ is the set of all nonintersecting paths joining a n-element subset of $\{(0,i) | i \geq 0\}$ with $\{(x, r_1), \dots, (x, r_n)\}$. Note that for $r_1 > r_2 > \cdots > r_n > 0$

$$W(r_1, \dots, r_n, 0) = W(r_1 - 1, \dots, r_n - 1). \tag{4.3.12}$$

By Stembridge's Pfaffian formula for the sum of the weights of nonintersecting paths where departure points are not fixed [Ste89] we obtain

$$W(r_1, r_2, \dots, r_n) = \mathrm{Pf}(D),$$

where if n is even D is $n \times n$ skew–symmetric matrix defined by $D_{i,j} = W(r_i, r_j)$ for $1 \leq i < j \leq n$ and if n is odd D is $(n+1) \times (n+1)$ skew–symmetric matrix defined by $D_{i,j} = W(r_i, r_j)$ and $D_{i,n+1} = W(r_i)$ for $1 \leq i < j < n+1$.

Lemma 4.3.7. *Let $r > s \geq 0$. Then*

$$W(s) = \frac{(-q)_s}{(q)_s}, \tag{4.3.13}$$

$$W(r, s) = \frac{(-q)_r}{(q)_r} \cdot \frac{(-q)_s}{(q)_s} \cdot \frac{1 - q^{r-s}}{1 + q^{r-s}}. \tag{4.3.14}$$

Proof. From Lemma 4.3.1 we have that the generating function of overpartitions with at most n parts is $M(n) = (-q)_n/(q)_n$ and with exactly n parts is $P(n) = q^n(-1)_n/(q)_n$. We set $M(n) = P(n) = 0$, for negative n.

From this lemma it follows:

$$\sum_{i=0}^{n} P(i) = M(n). \tag{4.3.15}$$

Thus,

$$W(s) = M(s) = \frac{(-q)_s}{(q)_s} \quad \text{and} \quad W(r, 0) = M(r-1) = \frac{(-q)_{r-1}}{(q)_{r-1}}.$$

We prove (4.3.14) by induction on s. The formula for the base case $s = 0$ holds by above. So, we assume $s \geq 1$.

By Gessel-Viennot's determinantal formula [GV85] we have that

$$W(r,s) = \sum_{i=0}^{s} \sum_{j=i+1}^{r} P(r-j)P(s-i) - P(r-i)P(s-j).$$

Summing over j and using (4.3.15) we obtain

$$W(r,s) = \sum_{i=0}^{s} P(s-i)M(r-i-1) - P(r-i)M(s-1-i).$$

Then

$$W(r,s) = W(r-1,s-1) + P(s)M(r-1) - P(r)M(s-1).$$

It is enough to prove that

$$\frac{P(s)M(r-1) - P(r)M(s-1)}{W(r-1,s-1)} = \frac{2(q^r+q^s)}{(1-q^r)(1-q^s)} \tag{4.3.16}$$

and (4.3.14) follows by induction. Now,

$$\begin{aligned} &P(s)M(r-1) - P(r)M(s-1) = \\ &= q^s \frac{(-1)_s}{(q)_s} \cdot \frac{(-q)_{r-1}}{(q)_{r-1}} - q^r \frac{(-1)_r}{(q)_r} \cdot \frac{(-q)_{s-1}}{(q)_{s-1}} \\ &= \frac{(-q)_{r-1}}{(q)_{r-1}} \cdot \frac{(-q)_{s-1}}{(q)_{s-1}} \cdot \frac{1+q^{r-s}}{1-q^{r-s}} \cdot \frac{2(q^r+q^s)}{(1-q^r)(1-q^s)}. \end{aligned}$$

Using inductive hypothesis for $W(r-1,s-1)$ we obtain (4.3.16).

□

Let F_λ be the generating function of reverse plane overpartitions of shape λ. Then using the bijection we have constructed we obtain that

$$F_\lambda = W(\lambda_1 + l - 1, \lambda_2 + l - 2, \ldots, \lambda_l)$$

which, after applying Stembridge's result, gives us the Pfaffian formula. This Pfaffian formula can be expressed as a product after the following observations.

Let M be $2k \times 2k$ skew–symmetric matrix. One of definitions of the Pfaffian is the following:

$$\mathrm{Pf}(M) = \sum_{\Pi=(i_1,j_1)\dots(i_n,j_n)} \mathrm{sgn}(\Pi) M_{i_1,j_1} M_{i_2,j_2} \cdots M_{i_n,j_n},$$

where the sum is over all of perfect matchings (or fixed point free involutions) of $[2n]$. Also, for $r > s > 0$

$$\begin{aligned} W(s) &= \frac{1+q^s}{1-q^s} W(s-1) \\ W(r,s) &= \frac{1+q^r}{1-q^r} \cdot \frac{1+q^s}{1-q^s} W(r-1,s-1). \end{aligned}$$

Then

$$F_\lambda = \prod_{j=1}^{l} \frac{1+q^{h_{j,1}}}{1-q^{h_{j,1}}} \cdot W(\lambda_1+l-2, \lambda_2+l-3, \dots, \lambda_l-1) = F_{\bar\lambda},$$

where $\bar\lambda = (\lambda_1-1,\dots,\lambda_l-1)$ if $\lambda_l > 1$ and $\bar\lambda = (\lambda_1-1,\dots,\lambda_{l-1}-1)$ if $\lambda_l = 1$ (see (4.3.12) in this case). Inductively we obtain Theorem 4.3.6.

4.4 Domino tilings

In [Vul07] a measure on strict plane partitions was studied. Strict plane partitions are plane partitions were all diagonals are strict partitions, i.e., strictly decreasing sequences. They can also be seen as plane overpartitions where all overlines are deleted. There are $2^{k(\Pi)}$ different plane overpartitions corresponding to a same strict plane partition Π.

Alternatively, a strict plane partition can be seen as a subset of $\mathbb{N}\times\mathbb{Z}$ consisting of points (t,x) where x is a part of the diagonal partition indexed by t, see Figure 4.7. We call this set the 2-dimensional diagram of that strict plane partition. The connected components are connected sets (no holes) on the same horizontal line. The 2-dimensional diagram of a plane overpartition is the 2-dimensional diagram of its corresponding strict plane partition.

The measure studied in [Vul07] assigns to each strict plane partition a weight equal to $2^{k(\Pi)} q^{|\Pi|}$. The limit shape of this measure is given in terms of the Ronkin function of the polynomial $P(z,w) = -1 + z + w + zw$ and it is parameterized on the domain representing half of the amoeba of this polynomial. This polynomial is also related to plane tilings with dominoes. This, as well as some other features like similarities in correlation kernels, see [Joh05, Vul07], suggested that the connection between this measure and domino tilings is likely to exist.

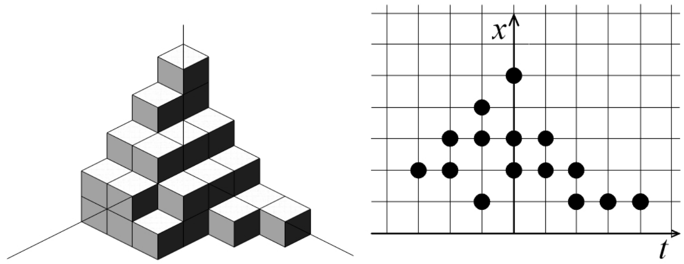

Figure 4.7: A 2-dimensional diagram

Alternatively, one can see this measure as a uniform measure on plane overpartitions, i.e., each plane overpartition Π has a probability proportional to $q^{|\Pi|}$. In Section 4.3 we have constructed a bijection between overpartitions and nonintersecting paths passing through the elements of the corresponding 2-dimensional diagrams, see Figure 4.8. The paths consist of edges of three different kinds: horizontal (joining (t, x) and $(t+1, x)$), vertical (joining $(t, x+1)$ and (t, x)) and diagonal (joining $(t, x+1)$ and $(t+1, x)$). There is a standard way to construct a tiling with dominoes using these paths, see for example [Joh05]. We explain the process below.

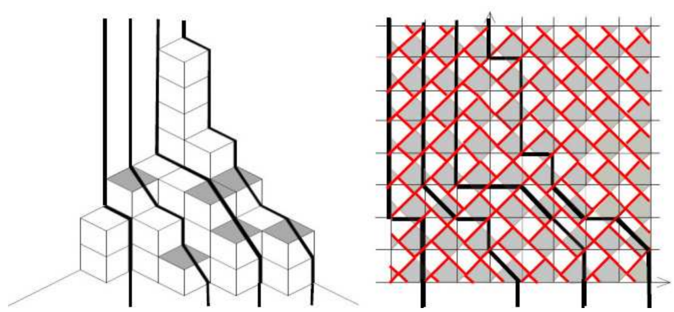

Figure 4.8: A strict plane partition and the domino tiling

We start from $\mathbb{R}^2$ and color it in a chessboard fashion such that $(1/4, 1/4)$, $(-1/4, 3/4)$, $(1/4, 5/4)$ and $(3/4, 3/4)$ are vertices of a white square. So, the axes of this infinite chessboard form angles of 45 and 135 degrees with axes of $\mathbb{R}^2$. A domino placed on this infinite chessboard can be one of the four types: $(1, 1)$, $(-1, -1)$, $(-1, 1)$ or $(1, -1)$, where we say that a domino is of type (x, y) if (x, y) is a vector parallel to the vector whose starting, respectively end point is the middle of the white, respectively black square of that domino.

Now, take a plane overpartition and represent it by a set of nonintersecting paths passing through the elements of the 2-dimensional diagram of this plane overpartition. We cover each edge by a domino that satisfies that the endpoints of that edge are middle points of sides of the black and white square of that domino. That way we obtain a tiling of a part of the plane with dominoes of three types: $(1,1)$, $(-1,-1)$ and $(1,-1)$. More precisely, horizontal edges correspond to $(1,1)$ dominoes, vertical to $(-1,-1)$ and diagonal to $(1,-1)$. To tile the whole plane we fill the rest of it by dominoes of the forth, $(-1,1)$ type. See Figure 4.8 for an illustration.

This way we have established a correspondence between plane overpartitions and plane tilings with dominoes. We now give some of the properties of this correspondence. First, we describe how a tiling changes in the case when we add or remove an overline or we add or remove a box from a plane overpartition. We require that when we add/remove an overline or a box we obtain a plane overpartition again. It is enough to consider only one operation adding or removing since they are inverse of each other. In terms of the 2-dimensional diagram of a plane overpartition, adding an overline can occur at all places where (t,x) is in the diagram and $(t+1,x)$ is not. Removing a box can occur at all places where (t,x) is the diagram and $(t,x-1)$ is not.

Adding an overline means that a pair of horizontal and vertical edges is replaced by one diagonal edge. This means that the new tiling differs from the old one by replacing a pair of $(1,1)$ and $(-1,-1)$ dominoes by a pair of $(1,-1)$ and $(-1,1)$ dominoes, see Figure 4.9.

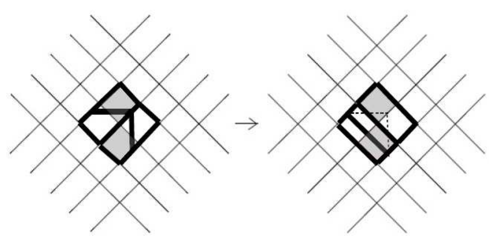

Figure 4.9: Adding an overline

Removing a box from a plane overpartition corresponds to a similar thing. Observe that if a box can be removed then the corresponding part is overlined or it can be overlined to obtain a plane overpartition again. So, it is enough to consider how the tiling changes when we remove a box from an overlined part since we have already considered the case of adding or removing an overline. If we remove a box from an overlined part we change a diagonal edge by a pair of vertical and horizontal edges. This means that the new tiling differs from the old one by replacing a pair of $(1,-1)$ and $(-1,1)$ dominoes by a pair of

$(1,1)$ and $(-1,-1)$ dominoes, see Figure 4.10. The difference between removing an overline and removing a box is just in the position where the flipping of dominoes occurs.

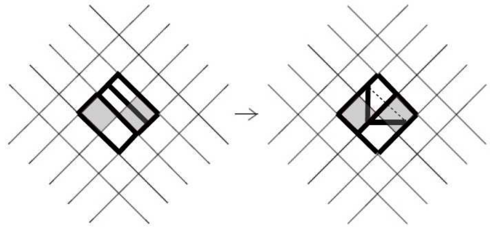

Figure 4.10: Removing a box

We conclude this section by an observation that plane overpartitions of a given shape λ and whose parts are bounded by n are in bijection with domino tilings of a rectangle $[-\ell(\lambda)+1, \lambda_1] \times [0, n]$ with certain boundary conditions. These conditions are imposed by the fact that outside of this rectangle nonintersecting paths are just straight lines. We describe the boundary conditions precisely in the proposition below.

Proposition 4.4.1. *The set $\mathcal{S}(\lambda) \cup \mathcal{L}(n)$, of all plane overpartitions of shape λ and whose largest part is at most n, is in a bijection with plane tilings with dominoes where a point $(t, x) \in \mathbb{Z} \times \mathbb{R}$ is covered by a domino of type $(-1,-1)$ if*

- $t \leq -\ell(\lambda)$,
- $-\ell(\lambda) < t \leq 0$ *and* $x \geq n$,
- $t = \lambda_i - i + 1$ *for some* i *and* $x \leq 0$,

and a domino of type $(-1, 1)$ if

- $t > \lambda_1$,
- $0 < t \leq \lambda_1$ *and* $x \geq n + 1/2$,
- $t \neq \lambda_i - i + 1$ *for all* i *and* $x \leq -1/2$.

For the example from Figure 4.8 the boundary conditions are shown in Figure 4.11.

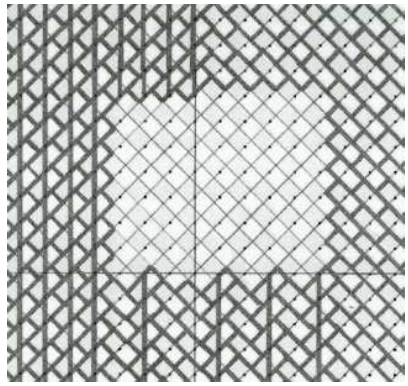

Figure 4.11: Boundary conditions

4.5 Robinson-Schensted-Knuth (RSK) type algorithm for plane overpartitions

In this Section we are going to give a bijection between matrices and pairs of plane overpartitions of the same shape. This bijection is inspired by the algorithm RS2 of Berele and Remmel [BR85] which gives a bijection between matrices and pairs of (k,ℓ)-semistandard tableaux. It is shown in [Sav07] that there is a bijection between (n,n)-semistandard tableaux and plane overpartitions whose largest entry is at most n. This bijection is based on the jeu de taquin.

We then apply properties of this algorithm to enumerate plane overpartitions, as done by Bender and Knuth [BK72] for plane partitions.

4.5.1 The RSK algorithm

We start by describing a bijection between matrices and pairs of numbers. Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be a $2n \times 2n$ matrix, made of four $n \times n$ blocks A, B, C, and D. The blocks A and D are non-negative integer matrices, and B and C are $\{0,1\}$ matrices. The encoding of M into pairs is made using the following rules:

- for each non-zero entry $a_{i,j}$ of A, we create $a_{i,j}$ pairs $\begin{pmatrix} i \\ j \end{pmatrix}$,
- for each non-zero entry $b_{i,j}$ of B, we create one pair $\begin{pmatrix} i \\ \bar{j} \end{pmatrix}$,

- for each non-zero entry $c_{i,j}$ of C, we create one pair $\begin{pmatrix} \bar{i} \\ j \end{pmatrix}$,
- for each non-zero entry $d_{i,j}$ of D, we create $d_{i,j}$ pairs $\begin{pmatrix} \bar{i} \\ \bar{j} \end{pmatrix}$.

It is clear that this encoding defines a one-to-one correspondence between matrices and pairs of numbers.

Example 1. *Let* $M = \begin{pmatrix} 0 & 2 & 1 & 0 \\ 2 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$*. After encoding M, we obtain*

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ \bar{1} \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ \bar{1} \end{pmatrix}, \begin{pmatrix} \bar{1} \\ 1 \end{pmatrix}, \begin{pmatrix} \bar{1} \\ 2 \end{pmatrix}, \begin{pmatrix} \bar{1} \\ \bar{2} \end{pmatrix}, \begin{pmatrix} \bar{2} \\ \bar{1} \end{pmatrix}, \begin{pmatrix} \bar{2} \\ \bar{2} \end{pmatrix}.$$

From now, we fix the order $\bar{1} < 1 < \bar{2} < 2 < \bar{3} < 3 < \dots$, and we sort the pairs to create a two–line array L such that:

- the first line is a non-increasing sequence
- if two entries of the first line are equal and overlined (resp. non-overlined) then the corresponding entries in the second line are in weakly increasing (resp. decreasing) order.

Example 2 (example 1 continued)**.** *After sorting, we obtain the two-line array L*

$$\begin{pmatrix} 2, 2, 2, \bar{2}, \bar{2}, 1, 1, 1, \bar{1}, \bar{1}, \bar{1} \\ 1, 1, \bar{1}, \bar{1}, \bar{2}, 2, 2, 2, \bar{1}, 1, \bar{2}, 2 \end{pmatrix}.$$

We now describe the part of the bijection which is the *insertion algorithm.* It is based on an algorithm proposed by Knuth in [Knu70] and quite similar to the algorithm RS2 of [BR85].

We first define how to insert the entry j into a plane overpartition P.

(a) $i \leftarrow 1$.

(b) $x_i \leftarrow j$.

(c) If x_i is smaller than all entries of the i^{th} row of P, then insert x_i at the end of this row.

(d) Otherwise let j_i the index such that $P_{i,j_1-1} \geq x_i > P_{i,j_i}$.

  (a) If $P_{i,j_1-1} = x_i$ and P_{i,i_j-1} is overlined, then $x_{i+1} \leftarrow x_i$.

  (b) Otherwise

    i. $x_{i+1} \leftarrow P_{i,j_i}$

    ii. $P_{i,j_i} \leftarrow x_i$.

(e) $i \leftarrow i+1$. Go to step 3.

Now we define how to insert a pair $\begin{pmatrix} i \\ j \end{pmatrix}$ into a pair of plane overpartitions of the same shape (P, Q).

(a) Insert j in P.

(b) If the insertion ends in column c and row r of P, then insert i in column c and row r in Q.

Now to go from the two–line array L to pairs of plane overpartitions of the same shape goes as follows: start with two empty plane overpartitions and insert each pair of L going from left to right. This is identitical to the classical RSK algorithm [Knu70].

Continuing the previous example, we get

$$P = \begin{matrix} 2 & 2 & 2 \\ \bar{2} & 1 & 1 \\ \bar{2} & \bar{1} & \\ 1 & & \\ \bar{1} & & \\ \bar{1} & & \end{matrix} \quad ; \quad Q = \begin{matrix} 2 & 2 & 2 \\ \bar{2} & 1 & 1 \\ \bar{2} & \bar{1} & \\ 1 & & \\ \bar{1} & & \\ \bar{1} & & \end{matrix} \quad .$$

Theorem 4.5.1. *There is a one-to-one correspondance between matrices M and pairs of plane overpartitions of the same shape (P, Q). In this correspondance:*

- *k appears in P exactly $\sum_i a_{ik} + c_{ik}$*
- *$\bar{k}$ appears in P exactly $\sum_i b_{ik} + d_{ik}$*
- *k appears in Q exactly $\sum_i a_{ik} + b_{ik}$*
- *$\bar{k}$ appears in Q exactly $\sum_i c_{ik} + d_{ik}$.*

Proof. The proof is really identical the proof of the RSK algorithm [Knu70] or the RS2 algorithm [BR85]. Details are given in [Sav07]. □

As in [Knu70], we can show that:

Theorem 4.5.2. *If* insertion algorithm *produces (P, Q) with input matrix M, then* insertion algorithm *produces (Q, P) with input matrix M^t.*

Proof. The proof is again analogous to [Knu70]. Given a two-line array $\begin{pmatrix} u_1, \ldots, u_N \\ v_1, \ldots, v_N \end{pmatrix}$, we partition the pairs (u_ℓ, v_ℓ) such that (u_k, v_k) and (u_m, v_m) with are in the same class if and only if:

- $u_k \leq u_m$ and if $u_k = u_m$, then u_k is overlined AND
- $v_k \geq v_m$ and if $v_k = v_m$, then v_k is overlined.

Then one can sort each class so that the first entry of each pair appear in non-increasing order and then sort the classes so that the first entries of the first pair of each class are in non-increasing order. For example if the two-line array is

$$\begin{pmatrix} 2, 2, 2, \bar{2}, \bar{2}, 1, 1, 1, \bar{1}, \bar{1}, \bar{1} \\ 1, 1, \bar{1}, \bar{1}, \bar{2}, 2, 2, \bar{1}, 1, \bar{2}, 2 \end{pmatrix},$$

we get the classes

$$C_1 = \{(\bar{2}, \bar{1}), (\bar{2}, 1), (1, \bar{2}), (\bar{1}, \bar{2})\}, \quad C_2 = \{(1, 1)\}, \quad C_3 = \{(1, \bar{1}), (\bar{1}, \bar{1}), (\bar{1}, 1)\}.$$

If the classes are $C_1, \ldots, C_d$ with

$$C_i = \{(u_{i1}, v_{i1}), \ldots, (u_{in_i}, v_{in_i})\}$$

then the first row of P is

$$v_{1n_1}, \ldots, v_{dn_d}$$

and the first row of Q is

$$u_{11}, \ldots, u_{dd}.$$

Moreover one constructs the rest of P and Q using the pairs:

$$\cup_{i=1}^{d} \cup_{j=1}^{n_i-1} \begin{pmatrix} u_{i,j+1} \\ v_{ij} \end{pmatrix}.$$

See Lemma 1 of [Knu70] and [Sav07] for a complete proof. As the two-line array corresponding to M^T is obtained by interchanging the two lines of the array and rearranging the columns, the Theorem follows. □

This implies that $M = M^T$ if and only if $P = Q$. Therefore:

Theorem 4.5.3. *There is a one-to-one correspondance between symmetric matrices M and plane overpartitions P. In this correspondance:*

- *k appears in P exactly $\sum_i a_{ik} + c_{ik}$*
- *$\bar{k}$ appears in P exactly $\sum_i b_{ik} + d_{ik}$.*

4.5.2 Enumeration of plane overpartitions

From Theorem 4.5.3, we can get directly the generating function for plane overpartitions whose largest entry is at most n. These objects are in bijection with symmetric matrix M of size $2n \times 2n$ with blocks A, B, C and D, each of size $n \times n$. The weight of the plane overpartition is $\sum_{i,j} i(a_{i,j} + b_{i,j} + c_{i,j} + d_{i,j})$. As M is symmetric, the weight of the plane overpartition can also be written as $\sum_{i,j}(i+j)b_{i,j} + \sum_i i(a_{i,i} + d_{i,i}) + \sum_{j<i}(i+j)(a_{i,j} + d_{i,j})$.

Let $O_n(m,k)$ be the number of plane overpartitions of m with k overlined parts and entries at most n and $\mathcal{O}_n(q,a) = \sum_{m,k} O_n(m,k) q^m a^k$. Then

Theorem 4.5.4. *The generating function $\mathcal{O}_n(q,a)$ for plane overpartitions whose largest entry is at most n is*

$$\mathcal{O}_n(q,a) = \frac{\prod_{i,j=1}^{n}(1 + aq^{i+j})}{\prod_{i=1}^{n}\prod_{j=0}^{i-1}(1 - q^{i+j})(1 - aq^{i+j})}.$$

Let $\mathcal{O}(q,a)$ be the limit of $\mathcal{O}_n(q,a)$ when n goes to infinity. A direct consequence of the Theorem is the following.

Corollary 4.5.5. *The generating function for plane overpartitions $\mathcal{O}(q,a)$ is*

$$\prod_{i=1}^{\infty} \frac{(1+aq^i)^{i-1}}{(1-q^i)^{\lceil i/2\rceil}(1-aq^i)^{\lceil i/2\rceil}}.$$

We can also get some more general result, as in [BK72]:

Theorem 4.5.6. *The generating function of plane overpartitions whose parts lie in a set S of positive integers is:*

$$\prod_{i\in S}\left(\frac{\prod_{j\in S}(1+aq^{i+j})}{(1-q^i)(1-aq^i)}\prod_{\substack{j\in S\\ j<i}}\frac{1}{(1-q^{i+j})(1-aq^{i+j})}\right). \tag{4.5.1}$$

For example

Corollary 4.5.7. *The generating function for plane overpartitions into odd parts is*

$$\prod_{i=1}^{\infty}\frac{(1+aq^{2i})^{i-1}}{(1-q^{2i-1})(1-aq^{2i-1})(1-q^{2i})^{\lfloor i/2\rfloor}(1-aq^{2i})^{\lfloor i/2\rfloor}}.$$

4.6 Interlacing sequences and cylindric partitions

We want to combine results of [Bor07] and [Vul09] to obtain a 1–parameter generalization of the formula for the generating function of cylindric partitions related to the Hall-Littlewood symmetric functions.

We use definitions of interlacing sequences, profiles, cylindric partitions, polynomials $A_\Pi(t)$ and $A_\Pi^{\mathrm{cyl}}(t)$ given in Introduction.

For an ordinary partition λ we define a polynomial

$$b_\lambda(t)=\prod_{i\geq 1}\varphi_{m_i(\lambda)}(t), \tag{4.6.1}$$

where $m_i(\lambda)$ denotes the number of times i occurs as a part of λ and $\varphi_r(t)=(1-t)(1-t^2)\cdots(1-t^r)$.

For a horizontal strip $\theta = \lambda/\mu$ we define

$$I_{\lambda/\mu} = \left\{i \geq 1 |\, \theta'_i = 1 \text{ and } \theta'_{i+1} = 0\right\}$$
$$J_{\lambda/\mu} = \left\{j \geq 1 |\, \theta'_j = 0 \text{ and } \theta'_{j+1} = 1\right\}.$$

Let

$$\varphi_{\lambda/\mu}(t) = \prod_{i \in I_{\lambda/\mu}} (1 - t^{m_i(\lambda)}) \text{ and } \psi_{\lambda/\mu}(t) = \prod_{j \in J_{\lambda/\mu}} (1 - t^{m_j(\mu)}). \tag{4.6.2}$$

Then

$$\varphi_{\lambda/\mu}/\psi_{\lambda/\mu} = b_\lambda(t)/b_\mu(t).$$

For an interlacing sequence $\Lambda = (\lambda^1, \dots, \lambda^T)$ with profile $A = (A_1, \dots, A_{T-1})$ we define $\Phi_\Lambda(t)$:

$$\Phi_\Lambda(t) = \prod_{i=1}^{T-1} \phi_i(t), \tag{4.6.3}$$

where

$$\phi_i(t) = \begin{cases} \varphi_{\lambda^{i+1}/\lambda^i}(t), & A_i = 0, \\ \psi_{\lambda^i/\lambda^{i+1}}(t), & A_i = 1. \end{cases}$$

For $\Lambda = (\lambda^1, \dots, \lambda^T)$ and $M = (\mu^1, \dots, \mu^S)$ such that $\lambda^T = \mu^1$ we define

$$\Lambda \cdot M = (\lambda^1, \dots, \lambda^T, \mu^2, \dots, \mu^S)$$

and

$$\Lambda \cap M = \lambda^T.$$

Then

$$A_{\Lambda \cdot M} = \frac{A_\Lambda A_M}{b_{\Lambda \cap M}} \quad \text{and} \quad \Phi_{\Lambda \cdot M} = \Phi_\Lambda \Phi_M. \tag{4.6.4}$$

For an interlacing sequence $\Lambda = (\lambda^1, \dots, \lambda^T)$ with profile $A = (A_1, \dots, A_{T-1})$ we define the reverse $\overline{\Lambda} = (\lambda^T, \dots, \lambda^1)$ whose profile is $\overline{A} = (1 - A_{T-1}, \dots, 1 - A_1)$. Then

$$A_{\overline{\Lambda}} = A_\Lambda \quad \text{and} \quad \Phi_{\overline{\Lambda}} = \frac{b_{\lambda^1} \Phi_\Lambda}{b_{\lambda^T}}. \tag{4.6.5}$$

For an ordinary partition λ we construct an interlacing sequence $\langle \lambda \rangle = (\emptyset, \lambda^1, \dots, \lambda^L)$ of length $L + 1 = l(\lambda) + 1$, where λ^i is obtained from λ by truncating the last $L - i$ parts.

Then

$$A_{\langle\lambda\rangle} = \Phi_{\langle\lambda\rangle} = b_\lambda. \tag{4.6.6}$$

In our earlier paper [Vul09] (Propositions 2.4 and 2.6) we have shown that for a plane partition Π

$$\Phi_\Pi = A_\Pi. \tag{4.6.7}$$

Now, more generally

Proposition 4.6.1. *If $\Lambda = (\lambda^1, \dots, \lambda^T)$ is an interlacing sequence then*

$$\Phi_\Lambda = \frac{A_\Lambda}{b_{\lambda^1}}.$$

Proof. If we show that the statement is true for sequences with constant profiles then inductively using (4.6.4) we can show it is true for sequences with arbitrary profile. It is enough to show that the statement is true for sequences with $(0, \dots, 0)$ profile because of (4.6.5). So, let $\Lambda = (\lambda^1, \dots, \lambda^T)$ be a sequence with $(0, \dots, 0)$ profile. Then $\Pi = \langle \lambda^1 \rangle \cdot \Lambda \cdot \overline{\langle \lambda^T \rangle}$ is a plane partition and from (4.6.4), (4.6.5), (4.6.6) and (4.6.7) we obtain that $A_\Pi = A_\Lambda$ and $\Phi_\Pi = b_{\lambda^1} \Phi_\Lambda$. □

For skew plane partitions and cylindric partitions we obtain the following two corollaries.

Corollary 4.6.2. *For a skew plane partition Π we have that $\Phi_\Pi = A_\Pi$.*

Corollary 4.6.3. *If Π is a cylindric partition given with $\Lambda = (\lambda^0, \dots, \lambda^T)$ then $\Phi_\Lambda = A_\Pi^{cyl}$.*

The last corollary comes from the observation that if a cylindric partition Π is given by a sequence $\Lambda = (\lambda^0, \dots, \lambda^T)$ then

$$A_\Pi^{\mathrm{cyl}}(t) = A_\Lambda(t)/b_{\lambda^0}(t). \tag{4.6.8}$$

In the rest of this Section we prove generalized MacMahon's formulas for skew plane partitions and cylindric partitions that are stated in Theorems 4.1.7 and 4.1.8. The proofs of these theorems were inspired by [Bor07], [OR03] and [Vul09]. We use a special class of symmetric functions called Hall-Littlewood functions.

4.6.1 The weight functions

In this Subsection we introduce weights on sequences of ordinary partitions. For that we use Hall-Littlewood symmetric functions P and Q. We recall some of the facts about these functions, but for more details see Chapters III and VI of [Mac95]. We follow the notation used there.

Recall that Hall-Littlewood symmetric functions $P_{\lambda/\mu}(x;t)$ and $Q_{\lambda/\mu}(x;t)$ depend on a parameter t and are indexed by pairs of ordinary partitions λ and μ. In the case when $t=0$ they are equal to ordinary Schur functions and in the case when $t=-1$ to Schur P and Q functions.

The relationship between P and Q functions is given by (see (5.4) of [Mac95, Chapter III])

$$Q_{\lambda/\mu} = \frac{b_\lambda}{b_\mu} P_{\lambda/\mu}, \tag{4.6.9}$$

where b is given with (4.6.1). Recall that (by (5.3) of [Mac95, Chapter III] and (4.6.9))

$$P_{\lambda/\mu} = Q_{\lambda/\mu} = 0 \quad \text{unless } \lambda \supset \mu. \tag{4.6.10}$$

We set $P_\lambda = P_{\lambda/\emptyset}$ and $Q_\lambda = Q_{\lambda/\emptyset}$. Recall that ((4.4) of [Mac95, Chapter III])

$$H(x,y;t) := \sum_\lambda Q_\lambda(x;t) P_\lambda(y;t) = \prod_{i,j} \frac{1-tx_iy_j}{1-x_iy_j}.$$

A specialization of an algebra $\mathcal{A}$ is an algebra homomorphism $\rho : \mathcal{A} \to \mathbb{C}$. If ρ and σ are specializations of the algebra of symmetric functions then we write $P_{\lambda/\mu}(\rho;t)$, $Q_{\lambda/\mu}(\rho;t)$ and $H(\rho,\sigma;t)$ for the images of $P_{\lambda/\mu}(x;t)$, $Q_{\lambda/\mu}(x;t)$ and $H(x,y;t)$ under ρ, respectively $\rho \otimes \sigma$. Every map $\rho : (x_1, x_2, \dots) \to (a_1, a_2, \dots)$ where $a_i \in \mathbb{C}$ and only finitely many a_i's are nonzero defines a specialization. These specializations are called evaluations. A multiplication of a specialization ρ by a scalar $a \in \mathbb{C}$ is defined by its images on power sums:

$$p_n(a \cdot \rho) = a^n p_n(\rho).$$

If ρ is a specialization of Λ where $x_1 = a$, $x_2 = x_3 = \ldots = 0$ then by (5.14) and (5.14') of [Mac95, Chapter VI]

$$Q_{\lambda/\mu}(\rho;t) = \begin{cases} \varphi_{\lambda/\mu}(t)a^{|\lambda|-|\mu|} & \lambda \supset \mu,\ \lambda/\mu \text{ is a horizontal strip,} \\ 0 & \text{otherwise.} \end{cases} \tag{4.6.11}$$

Similarly,

$$P_{\lambda/\mu}(\rho;t) = \begin{cases} \psi_{\lambda/\mu}(t)a^{|\lambda|-|\mu|} & \lambda \supset \mu,\ \lambda/\mu \text{ is a horizontal strip,} \\ 0 & \text{otherwise.} \end{cases} \tag{4.6.12}$$

Let $T \geq 2$ be an integer and $\rho^{\pm} = (\rho_1^{\pm}, \ldots, \rho_{T-1}^{\pm})$ be finite sequences of specializations. For a sequence of partitions $\Lambda = (\lambda^1, \ldots, \lambda^T)$ we set the weight function $W(\Lambda; t)$ to be

$$W(\Lambda) \quad = \quad q^{T|\lambda^T|} \sum_{M} \prod_{n=1}^{T-1} P_{\lambda^n/\mu^n}(\rho_n^-; t) Q_{\lambda^{n+1}/\mu^n}(\rho_n^+; t),$$

where q and t are parameters and the sum ranges over all sequences of partitions $M = (\mu^1, \ldots, \mu^{T-1})$.

We can also define the weights using another set of specializations $R^{\pm} = (R_1^{\pm}, \ldots, R_{T-1}^{\pm})$ were $R_i^{\pm} = q^{\pm i}\rho_i^{\pm}$. Then

$$W(\Lambda) = \sum_{M} W(\Lambda, M, R^-, R^+),$$

where

$$W(\Lambda, M, R^-, R^+) = q^{|\Lambda|} \prod_{n=1}^{T-1} P_{\lambda^n/\mu^n}(R_n^-; t) Q_{\lambda^{n+1}/\mu^n}({R^+}_n; t).$$

We will focus on two special sums

$$Z_{\text{skew}} = \sum_{\Lambda=(\emptyset,\lambda^1,\ldots,\lambda^T,\emptyset)} W(\Lambda)$$

and

$$Z_{\text{cyl}} = \sum_{\substack{\Lambda=(\lambda^0,\lambda^1,\ldots,\lambda^T) \\ \lambda^0=\lambda^T}} W(\Lambda).$$

4.6.2 Specializations

We show that for a suitably chosen specialization the weight function vanishes for every sequence of ordinary partitions unless this sequence represents an interlacing sequence, in which case it becomes (4.6.3).

Let $A^- = (A_1^-, \ldots, A_{T-1}^-)$ and $A^+ = (A_1^+, \ldots, A_{T-1}^+)$ be sequences of 0's and 1's such that $A_k^- + A_k^+ = 1$.

If specializations $R^\pm$ are evaluations given by

$$R_k^\pm : x_1 = A_k^\pm, x_2 = x_3 = \ldots = 0 \tag{4.6.13}$$

then $W(\Lambda)$ vanishes unless Λ is an interlacing sequence of profile A^- and in that case

$$W(\Lambda; t) = \Phi_\Lambda(t) q^{|\Lambda|}.$$

Then from Corollaries 4.6.2 and 4.6.3 we have that

$$Z_{\text{skew}} = \sum_{\Pi \in \text{Skew}(T,A)} A_\Pi(t) q^{|\Pi|}$$

and

$$Z_{\text{cyl}} = \sum_{\Pi \in \text{Cyl}(T,A)} A_\Pi^{\text{cyl}}(t) q^{|\Pi|}$$

where A^- in both formulas is given by the fixed profile A of skew plane partitions, respectively cylindric partitions.

4.6.3 Partition functions

If ρ^+ is $x_1 = s$, $x_2 = x_3 = \ldots = 0$ and ρ^- is $x_1 = r$, $x_2 = x_3 = \ldots = 0$ then

$$H(\rho^+, \rho^-) = \frac{1 + tsr}{1 - sr}.$$

Thus, for specializations given by (4.6.13)

$$H(\rho_i^+, \rho_j^-) = \frac{1 + tq^{j-i} A_i^+ A_j^-}{1 - q^{j-i} A_i^+ A_j^-}. \tag{4.6.14}$$

We have shown in our earlier paper (Proposition 2.2 of [Vul09]) that

Proposition 4.6.4.

$$Z_{skew}(\rho^-,\rho^+)=\prod_{0\le i<j\le T} H(\rho_i^+,\rho_j^-).$$

Then this proposition together with (4.6.14) implies Theorem 4.1.7. The generating formula for skew plane partitions can also be seen as the generating formula for reverse plane partitions as explained in Introduction.

Each skew plane partition can be represented as an infinite sequence of ordinary partitions by adding infinitely many empty partitions to the left and right side. That way the profiles become infinite sequences of 0's and 1's. Theorem 4.1.7 also gives the generating formula for skew plane partitions of infinite profiles $A=(\dots,A_{-1},A_0,A_1,\dots)$:

$$\sum_{\Pi\in\mathrm{Skew}(A)} A_\Pi(t)q^{|\Pi|}=\prod_{\substack{i<j\\ A_i=0,\,A_j=1}}\frac{1-tq^{j-i}}{1-q^{j-i}}. \tag{4.6.15}$$

Similarly for cylindric partitions, using (4.6.14) together with the following proposition we obtain Theorem 4.1.8.

Proposition 4.6.5.

$$Z_{cyl}(R^-,R^+)=\prod_{n=1}^{\infty}\frac{1}{1-q^{nT}}\prod_{k,\,l=1}^{T} H(q^{(k-l)_{(T)}+(n-1)T}R_k^-,R_l^+), \tag{4.6.16}$$

where $i_{(T)}$ is the smallest positive integer such that $i\equiv i_{(T)}$ mod T.

Proof. We use

$$\sum_\lambda Q_{\lambda/\mu}(x)P_{\lambda/\nu}(y)=H(x,y)\sum_\tau Q_{\nu/\tau}(x)P_{\mu/\tau}(y). \tag{4.6.17}$$

The proof of this is analogous to the proof of Proposition 5.1 that appeared in our earlier paper [Vul07]. Also, see Example 26 of Chapter I, Section 5 of [Mac95].

The proof of (4.6.16) uses the same idea as in the proof of Proposition 1.1 of [Bor07].

We start with the definition of $Z_{\text{cyl}}(R^-,R^+)$. We omit index cyl.

$$\begin{aligned} Z(R^-,R^+) &= \sum_{\Lambda,M} W(\Lambda,M,R^-,R^+) = \sum_{\Lambda,M} q^{|\Lambda|}\prod_{n=1}^{T} P_{\lambda^{n-1}/\mu^n}(R_n^-)Q_{\lambda^n/\mu^n}(R_n^+) \\ &= \sum_{\Lambda,M} q^{|\mu|}\prod_{n=1}^{T} P_{\lambda^{n-1}/\mu^n}(qR_n^-)Q_{\lambda^n/\mu^n}(R_n^+). \end{aligned}$$

If $x=(x_1,x_2,\dots,x_T)$ is a vector we define the shift as $\operatorname{sh}(x)=(x_2,\dots,x_T,x_1)$. We set $R_0^\pm=R_T^\pm$, $\mu_0=\mu_T$ and $\tau_0=\tau_T$. If using formula (4.6.17) we substitute sums over λ^i's with sums over τ^{i-1}'s we obtain

$$\begin{aligned} Z(R^-,R^+) &= H(q\operatorname{sh}R^-;R^+)\sum_{\mu,\tau} q^{|\mu|}Q_{\mu^1/\tau^0}(R_T^+),P_{\mu^0/\tau^0}(qR_1^-)\cdot \\ &\quad \cdot Q_{\mu^2/\tau^1}(R_1^+)P_{\mu^1/\tau^1}(qR_2^-)\cdots Q_{\mu^0/\tau^{T-1}}(qR_{T-1}^+)P_{\mu^{T-1}/\tau^{T-1}}(R_T^-) \\ &= H(q\operatorname{sh}R^-;R^+)\sum_{\mu,\tau} W(\operatorname{sh}\mu,\tau,q\operatorname{sh}R^-,R^+) \\ &= H(q\operatorname{sh}R^-;R^+)Z(q\operatorname{sh}R^-,R^+). \end{aligned}$$

Since $\operatorname{sh}^T=\operatorname{id}$ if we apply the same trick T times we obtain

$$\begin{aligned} Z(R^-,R^+) &= \prod_{i=1}^{T} H(q^i\operatorname{sh}^i R^-;R^+)\cdot Z(q^TR^-,R^+) \\ &= \prod_{i=1}^{T} H(q^i\operatorname{sh}^i R^-;R^+)\cdot Z(sR^-,R^+). \end{aligned}$$

where $s=q^T$. Thus,

$$Z=\prod_{n=1}^{\infty}\prod_{i=1}^{T} H(q^{i+(n-1)T}\operatorname{sh}^i R^-;R^+)\lim_{n\to\infty} Z(s^nR^-,R^+).$$

Because

$$\lim_{n\to\infty} Z(s^nR^-,R^+)=\lim_{n\to\infty} Z(\text{trivial},R^+)=\prod_{n=1}^{\infty}\frac{1}{1-s^n}$$

and

$$\prod_{i=1}^{T} H(q^i \,\mathrm{sh}^i\, R^-, R^+) = \prod_{l=1}^{T} \left[\prod_{k=l+1}^{T} H(q^{k-l} R_k^-, R_l^+) \prod_{k=1}^{l} H(q^{T+k-l} R_k^-, R_l^+) \right]$$
$$= \prod_{k,\,l=1}^{T} H(q^{(k-l)_{(T)}} R_k^-, R_l^+),$$

we conclude that (4.6.16) holds. □

Observe that if in Theorem 4.1.8 we let $T \to \infty$, i.e., the circumference of cylinder goes to infinity then we reconstruct (4.6.15).

4.7 Concluding remarks

In this chapter, we determine the generating functions of plane overpartitions with several types of constraints. In particular, we can compute the generating function of plane overpartitions with at most r rows and c columns and the generating function of plane overpartitions with entries at most n. The natural question is therefore to put those constraints together and to compute the generating function of plane overpartitions with at most r rows, c columns and entries at most n. Unfortunately this generating function is not a product.

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