Exact Solutions for Two-dimensional Stokes Flow

Thesis by

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Abstract

This thesis comprises three parts. The principal topic is presented in Part I and concerns the problem of the free-boundary evolution of two dimensional, slow, viscous (Stokes) fluid driven by capillarity. A new theory of exact solutions is presented using a novel global approach involving complex line integrals around the fluid boundaries. It is demonstrated how the consideration of appropriate sets of geometrical line integral quantities leads to a concise theoretical reformulation of the problem. All previously known results for simply-connected regions are retrieved and the analytical form of the exact solutions formally justified. For appropriate initial conditions, an infinite number of *conserved quantities* is identified. An important new general result (herein called the *theorem of invariants*) is also demonstrated.

Further, using the new theoretical reformulation, an extension to the case of *doubly-connected* fluid regions with surface tension is made. A large class of exact solutions for doubly-connected fluid regions is found. The method combines the new theoretical approach with elements of loxodromic function theory. To the best of the author's knowledge, this thesis provides the first known examples of exact solutions for Stokes flow in a doubly-connected topology. The *theorem of invariants* is extended to the doubly-connected case.

Finally analytical arguments are presented to demonstrate the existence, in principle, of a class of exact solutions for geometrically symmetrical *four*-bubble configurations.

In Part II, the *most* general representation for local solutions to the two dimensional elliptic and hyperbolic Liouville equations is formally derived.

In Part III, some analytical observations are presented on solutions to the linearized equations for small disturbances to the axisymmetric Burgers vortex. The relevance to the (as yet unsolved and little studied) problem of the linear stability of Burgers vortex to axially-dependent perturbations is argued and discussed.

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PART I

Chapter 1

Stokes Flow of a Simply-Connected Viscous Blob

1.1 Introduction

This chapter[1] presents a novel formulation of the problem of the slow viscous quasi-steady flow of a two-dimensional simply-connected fluid blob with surface tension on the free boundary. Many exact solutions for special cases of this problem have already appeared in the literature [1]–[11] and rely on a complexification of the problem first exploited by Richardson [5]. The closely related problem of the Stokes flow around a single bubble in a strain field has also recently received much attention [8] [10] [11]. The approach adopted in this paper, while employing the same formulation in terms of complex analytic functions, is essentially different in that the problem for the boundary evolution of the blob is studied by considering a very general set of line integral quantities defined around the boundary of the blob. This approach greatly simplifies much of the unwieldy analysis that has characterized previous treatments, and the interesting mathematical structure underlying the existence of the exact solutions becomes more apparent.

The previous methods, while leading (essentially by inspection) to the exact solutions, have been somewhat *ad hoc*, although Tanveer and Vasconcelos [8] recently presented some mathematical justification for the existence of such solutions. These previous methods essentially rely on hypothesizing a form (or ansatz) for the exact solutions and then demonstrating by direct substitution into the equations that the ansatz is such as to satisfy all the necessary boundary conditions while simultaneously respecting all the necessary analyticity properties of the solution inside the

[1]This chapter is based on material from an article entitled "A Theory of Exact Solutions for Plane Viscous Blobs" by D.G. Crowdy and S. Tanveer to appear in *Journal of Nonlinear Science.* It is reproduced here with the kind permission of Springer Verlag New York Inc., 175 Fifth Avenue, New York, NY 10010.

fluid region. The reader is referred to the papers cited above for more details. The new approach expounded here is not only mathematically more appealing but also reveals important properties of the equations of motion, in particular, the existence of an infinite number of conserved quantities associated with a very general class of exact solutions. The existence of such conserved quantities has not been generally recognized using previous methods. The reformulation also leads to what is referred to herein as a "theorem of invariants" which automatically provides a further finite set of invariants (or 'first integrals') for a subset of the exact solutions. It is also noted that the general theory presented here is readily extended to the case of the Stokes flow around a single bubble with only minor changes in detail. This is also a simply-connected fluid region.

Since many examples of the slow viscous flow of simply-connected fluid blobs have already been explicitly calculated using alternative solution methods, we do not calculate further examples in this chapter. We do, however, give details of a special class of exact solutions with a particularly appealing mathematical structure that comes to light as a result of the reformulation presented here. This example is presented as a case study and represents a generalization of solutions found by Richardson [1].

1.2 Mathematical Formulation

Consider the unsteady evolution of a general simply-connected plane blob of fluid of viscosity μ under the assumptions of no inertial effects, no gravitational effects or effects from other body forces. The equations of motion of the fluid are

$$\mu\nabla^2\mathbf{u} = \nabla p \tag{1.1}$$

$$\nabla.\mathbf{u} = 0 \tag{1.2}$$

where $\mathbf{u}(x, y)$ is the fluid velocity, $p(x, y)$ is the pressure and μ the fluid viscosity. We choose to non-dimensionalize the problem using a as a typical length-scale (e.g. an effective radius where πa^2 is the initial area of the blob). If σ is the surface tension parameter we non-dimensionalize velocities by $\frac{\sigma}{\mu}$, the pressure by $\frac{\sigma}{a}$, length by a and time by $\frac{a\mu}{\sigma}$. Introducing a streamfunction $\psi(x, y)$ such that

$$\mathbf{u} = \nabla^{\perp}\psi \tag{1.3}$$

it is well-known that two-dimensional Stokes flow can be reformulated in terms of this streamfunction which satisfies a **biharmonic equation** in the fluid region i.e.

$$\nabla^4\psi = 0 \tag{1.4}$$

On the blob boundary we must ensure continuity of the shear stress and satisfy the requirement that the jump in the normal stress across the interface equals the product of the surface tension σ and the curvature κ. These two conditions can be written, in non-dimensionalized form, as

$$-pn_j + 2e_{jk}n_k = -\kappa n_j \tag{1.5}$$

where e_{jk} are given by

$$e_{jk} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}\right) \tag{1.6}$$

Additionally, there is a kinematic boundary condition that the normal velocity V_n of a point on the boundary equals the normal fluid velocity at that point, that is,

$$\mathbf{u.n} = V_n \tag{1.7}$$

It is also expected from physical considerations that, in the event that there are flow singularities in the fluid, the strength and position of the singularities should be externally specifiable.

To complexify the problem, all fields are written as functions of $z_1 = x + iy$ and $\bar{z}_1 = x - iy$. According to the well-known Goursat representation for a general biharmonic function we can then write

$$\psi(z_1, \bar{z}_1) = Im[\bar{z}_1 f_1(z_1) + g_1(z_1)] \tag{1.8}$$

where $f_1(z_1)$ and $g_1(z_1)$ are two functions which are analytic in the fluid region. Note that since the blob boundary evolves with time, each of f_1 and g_1 also depend on time t, though this dependence in suppressed in (1.8) for purposes of brevity. All physically relevant quantities can now be written in terms of these two functions $f_1(z_1)$ and $g_1(z_1)$. In particular,

$$\frac{p}{\mu} - i\omega = 4f_1'(z_1) \tag{1.9}$$

$$u_1 + iv_1 = -f_1(z_1) + z_1\bar{f}_1'(\bar{z}_1) + \bar{g}_1'(\bar{z}_1) \tag{1.10}$$

$$e_{11} + ie_{12} = z_1\bar{f}_1''(\bar{z}_1) + \bar{g}_1''(\bar{z}_1) \tag{1.11}$$

where $\bar{f}_1$ denotes the conjugate function: $\bar{f}_1(z_1) = \overline{f_1(\bar{z}_1)}$ and u_1, v_1 represent the components of velocity in the x and y directions respectively. ω is the vorticity of the fluid.

The stress condition must be rewritten in a more convenient form. To do this we define a complex normal as

$$N \equiv n_1 + in_2 = -i(x_s + iy_s) = -iz_{1_s} = -i\exp(i\theta) \tag{1.12}$$

where s is the arclength around the blob traversed in the anticlockwise direction and θ is the angle between the tangent and the real positive axis. The stress condition can then be rewritten in complex form as

$$-pN + 2(e_{11} + ie_{12})\bar{N} = -\kappa N \tag{1.13}$$

Substituting for the various quantities in this equation, and using the fact that $\kappa = \theta_s$, a straightforward calculation reveals that it can be written

$$\frac{\partial S(z_1, \bar{z}_1)}{\partial z_1} z_{1_s} + \frac{\partial S(z_1, \bar{z}_1)}{\partial \bar{z}_1} \bar{z}_{1_s} = -i\frac{z_{1_{ss}}}{2} \tag{1.14}$$

where

$$S(z_1, \bar{z}_1) \equiv f_1(z) + z_1\bar{f}_1'(\bar{z}_1) + \bar{g}_1'(\bar{z}_1) \tag{1.15}$$

Equation (1.14) can be integrated immediately to give

$$f_1(z_1) + z_1\bar{f}_1'(\bar{z}_1) + \bar{g}_1'(\bar{z}_1) = -i\frac{z_{1_s}}{2} + B(t) \tag{1.16}$$

where $B(t)$ is a complex constant of integration.

There is a certain amount of arbitrariness in the functions $f_1(z), g_1'(z)$ which provide a given stress distribution on the blob boundary. In order to derive a simple form of the final evolution equations, it is necessary to exploit this arbitrariness. Physically, the differing choices of $f_1(z)$ and $g_1'(z)$ that leave the stresses invariant give rise to different velocity fields. Thus it is necessary to make specific choices of the available degrees of freedom in the problem to determine a unique velocity field. This is done such as to provide the most convenient form for the evolution equations.

Consider the following (time-dependent) change of origin in physical space and rotation of the physical plane expressed via

$$z_1 = z_0(t) + e^{i\phi(t)}\, z \tag{1.17}$$

where $z_0(t)$ is a complex function of time, and $\phi(t)$ is a real function of time. Given this transformation of z_1, the corresponding transformations of $f_1(z_1), g_1'(z_1)$ that leave the stress distribution invariant can be written

$$f_1(z_1) = e^{i\phi}\left[f(z) + iCz\right] + \gamma(t) \tag{1.18}$$

$$g_1'(z_1) = e^{-i\phi}g'(z) - \bar{z}_0 f_1'(z_1) - \bar{\gamma} + \bar{B} \tag{1.19}$$

where C is a real function of time, while $\gamma(t)$ is a complex function of time. The boundary condition (1.16) then becomes

$$f(z) + z\bar{f}'(\bar{z}) + \bar{g}'(\bar{z}) = -i\frac{z_s}{2} \tag{1.20}$$

Under this same transformation the velocity field becomes

$$\begin{aligned} u_1 + iv_1 &= e^{i\phi}\left[-f(z) + z\bar{f}'(\bar{z}) + \bar{g}'(\bar{z}) - 2iCz\right] - 2\gamma + B \\ &= e^{i\phi}\left[u + iv - 2iCz\right] - 2\gamma + B \end{aligned} \tag{1.21}$$

where $u + iv$ denotes the velocity field in the new variables. It is clear from (1.21) that the arbitrariness expressed by the transformation above corresponds to a velocity field that is determined only up to a rigid body motion i.e. an arbitrary translation and rotation. The suitability of the transformations (1.17)-(1.19) and the choices of the remaining degrees of freedom will become fully clear once the kinematic condition (1.7) is recast in terms of a conformal mapping representation as in the following section.

1.3 Conformal Mapping Representation

Consider the conformal map $z_1(\zeta, t)$ from the interior of the unit circle in the ζ plane into the simply-connected region occupied by the fluid so that $\zeta = 0$ is mapped

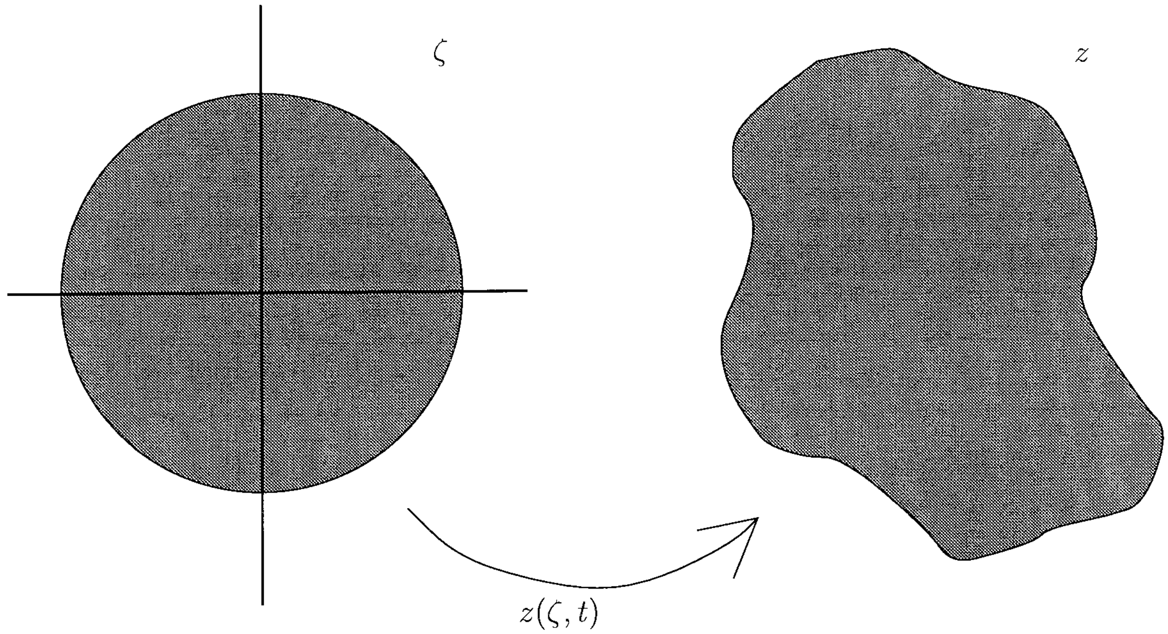

Figure 1.1: Conformal mapping domains

to a point $z_0(t)$ inside the fluid blob. The existence of such a map is guaranteed by Riemann's Theorem. We choose $z_0(0)$ to be any convenient point inside the blob initially. The choice of $\dot{z}_0(t)$ will be made to simplify the problem appropriately as will be seen shortly. It is clear that for sufficiently small time t, $z_0(t)$ will remain inside the blob when $\dot{z}_0(t)$ is finite. *A priori*, that is all that is needed to derive the dynamical equations and the exact solutions – examination of the exact solutions themselves will then determine the time of validity of a particular solution. The remaining rotational degree of freedom of the Riemann mapping theorem will be used later by fixing a rotational freedom in the ζ plane in a convenient way.

The kinematic boundary condition on the bubble can be written as the following boundary condition on the unit circle, $\zeta = e^{i\nu}$:

$$Im\left[\frac{(z_{1_t} - (u_1 + iv_1))}{z_{1_\nu}}\right] = 0 \qquad (1.22)$$

If we now use the substitution (1.17), where z is now viewed as a function of t and ζ

(or ν on the circular boundary), then it is clear that (1.22) is equivalent to

$$Im\left[\frac{(z_t + i(\dot{\phi} + 2C)z - e^{-i\phi}(u + iv + 2\gamma - B) + e^{-i\phi}(\dot{z}_0 + 2\gamma - B))}{z_\nu}\right] = 0 \tag{1.23}$$

We now choose

$$\dot{\phi}(t) = -2C(t) \tag{1.24}$$

$$\dot{z}_0(t) = B(t) - 2\gamma(t) \tag{1.25}$$

so that on using (1.21), equation (1.23) simplifies to

$$Im\left[\frac{z_t - \{-f(z) + z\bar{f}'(\bar{z}) + \bar{g}'(\bar{z})\}}{z_\nu}\right] = 0 \tag{1.26}$$

Note that since the function $z(\zeta, t)$ is simply a translation and rotation of $z_1(\zeta, t)$ then $z(\zeta, t)$ is also a conformal map. Since $\zeta = 0$ corresponds to $z_1 = z_0$, it follows from (1.17) that

$$z(0, t) = 0 \tag{1.27}$$

We further make the arbitrary but convenient specification that

$$\gamma(t) = f_1(z_0(t)) \tag{1.28}$$

then it is clear from (1.18) that

$$f(0) = 0 \tag{1.29}$$

Note also that the specific choice of real function $C(t)$ is unimportant in the simplification to (1.20) and (1.26), with auxiliary conditions (1.27) and (1.29), provided $\phi(t)$

evolves according to $\dot{\phi} = -2C$. It is found that the above conditions are enough to uniquely determine the velocity field, with the evolution equations given by (1.20) and (1.26).

Previous authors ([1] [9]) have suggested various physical arguments that might be used to uniquely specify the velocity field rather than the purely mathematical condition (1.28). One suggestion is the requirement of conservation of global momentum. Although we are considering the zero Reynolds number asymptotic limit of the Navier-Stokes equation where, *locally*, inertial effects (momentum transfer) have been neglected in comparison with the viscous stresses, it is argued [9] that this does not obviate the need to respect *global* conservation of momentum. *Assuming* that global momentum conservation is the appropriate physical principle to invoke, unless the solutions are suitably symmetric, in general the mathematical condition leading to (1.29) above does *not* provide conservation of global momentum. However, in the case when there are no flow singularities in the blob this is of no consequence as there are then no special points in the fluid and an appropriate rigid body motion can be added *a posteriori* to the solution (so that global momentum is conserved) without affecting any other aspect of the flow. Thus, in that case, there is really no need to appeal to any physical principle to uniquely specify the velocity field, and the convenient mathematical condition above serves perfectly well. The case where there does exist a distribution of singularities in the flow is discussed in later sections.

Using (1.20) and the fact that on $|\zeta| = 1$

$$z_s = \frac{i\zeta z_\zeta}{|z_\zeta|} \tag{1.30}$$

the kinematic boundary condition (1.26) becomes the following condition on $|\zeta| = 1$:

$$Re\left[\frac{z_t + 2F(\zeta,t)}{\zeta z_\zeta(\zeta,t)}\right] = \frac{1}{2|z_\zeta|} \tag{1.31}$$

where we define

$$F(\zeta,t) \equiv f(z(\zeta,t),t) \tag{1.32}$$

We also define

$$G(\zeta, t) \equiv g'(z(\zeta, t)) \tag{1.33}$$

Formally, in the following analysis, we assume that $F(\zeta, t)$ is analytic in $|\zeta| \leq 1$ but we allow $G(\zeta, t)$ to possibly have a pole of order r_0 at $\zeta = 0$ and poles of order r_j at $\zeta = \bar{\zeta}_j^{-1}$ inside the unit circle, with

$$0 \leq r_0 \leq M - M_0, \; 0 \leq r_j \leq \gamma_j, \; j = 1..N \tag{1.34}$$

where ζ_j, $j = 1..N$, are the poles of order γ_j of the conformal map $z(\zeta, t)$ outside the unit circle (see (1.39)) and we define

$$M_0 = \sum_{j=1}^{N} \gamma_j \tag{1.35}$$

M is taken to be an arbitrary integer such that $M \geq M_0$. Physically these singularities correspond to general multipoles (e.g. a source/sink, dipole) at z_0 and at z-locations corresponding to $\zeta = \bar{\zeta}_j^{-1}$.

We now convert the boundary condition (1.31) into a differential equation for z valid everywhere in $|\zeta| \leq 1$. Because of the restriction (1.29) (which implies $F(0, t) = 0$) and (1.27), it is easily seen that $\zeta = 0$ is a removable singularity of the expression within the square parentheses on the left hand side of (1.31). The left hand side of (1.31) is clearly the real part of an analytic function in $|\zeta| \leq 1$. Using the Poisson integral formula for $|\zeta| < 1$,

$$z_t + 2F = \zeta I(\zeta, t) z_\zeta \tag{1.36}$$

where

$$I(\zeta, t) = \frac{1}{2\pi i} \oint_{|\zeta'|=1} \frac{d\zeta'}{\zeta'} \left[\frac{\zeta' + \zeta}{\zeta' - \zeta} \right] \frac{1}{2 z_\zeta^{1/2}(\zeta, t) \bar{z}_\zeta^{1/2}(\zeta^{-1}, t)} + iD(t) \tag{1.37}$$

and $D(t)$ is a real function of time. The remaining rotational degree of freedom of the Riemann mapping theorem is used by insisting $D(t) = 0$. That such a freedom exists can be readily observed by replacing ζ by $\zeta e^{i\theta(t)}$ in (1.36), with $\dot{\theta} = D(t)$. Thus, without any loss of generality,

$$I(\zeta, t) = \frac{1}{2\pi i} \oint_{|\zeta'|=1} \frac{d\zeta'}{\zeta'} \left[\frac{\zeta' + \zeta}{\zeta' - \zeta}\right] \frac{1}{2z_\zeta^{1/2}(\zeta, t)\bar{z}_\zeta^{1/2}(\zeta^{-1}, t)} \tag{1.38}$$

Since $z(\zeta, t)$ must be analytic in $|\zeta| \leq 1$ it is possible to express it in the form

$$z(\zeta, t) = \frac{h(\zeta, t)}{\prod_{j=1}^{N}(\zeta - \zeta_j(t))^{\gamma_j}} \tag{1.39}$$

where N, γ_j are arbitrary positive integers and the corresponding poles ζ_j are all outside the unit circle (i.e. $|\zeta_j| > 1$), while $h(\zeta, t)$ is analytic for $|\zeta| \leq 1$.

In the next sections, it will be shown that if $h(\zeta, 0)$ is an arbitrary **polynomial** of sufficiently high order, it remains so later in time, provided the poles $\zeta_j(t)$ evolve in an appropriate manner. It is in this sense that the problem under consideration will be said to have *exact* solutions - i.e. the evolution of the free boundary will have been reduced to the evolution of a *finite* set of parameters. In this way, the nonlinear, nonlocal, free boundary problem (for both simply-connected and, later, for doubly-connected fluid regions) will be reduced to a consistent **finite** nonlinear system.

The success of the method relies on the consideration of classes of *purely geometrical* line integral quantities defined around the unit circle and which involve only the conformal mapping function. It will be shown that, under the dynamics of Stokes flow, these line integral quantities evolve according to a set of evolution equations with a special mathematical structure. From this set of evolution equations many of the intriguing mathematical properties of the flow can be understood.

1.4 Conservation Laws and Exact Solutions

To demonstrate the existence of exact solutions and the conserved quantities associated with them, the problem is now reformulated in terms of a set of general line integral quantities given by

$$J_K(t) = \oint_C K(\zeta,t)\bar{z}(\bar{\zeta},t)z_\zeta(\zeta,t)d\zeta \tag{1.40}$$

where $K(\zeta, t)$ is a general function of ζ and t which will be taken to be analytic on and within the unit circle and C denotes the boundary of the unit circle $|\zeta| = 1$ traversed anticlockwise. Later, special choices of the function $K(\zeta, t)$ will be made in order to establish various results. First we state and prove a theorem about how $J_K(t)$ evolves in time.

Theorem 1.4.1

For $J_K(t)$ defined as in (1.40), where $z(\zeta, t)$ is the conformal mapping function as defined earlier,

$$\dot{J}_K(t) = \oint_C K(\zeta,t)2G(\zeta,t)z_\zeta(\zeta,t)d\zeta + \oint_C \left[K_t(\zeta,t) - \zeta I(\zeta,t)K_\zeta(\zeta,t)\right]\bar{z}(\bar{\zeta},t)z_\zeta(\zeta,t)d\zeta \tag{1.41}$$

Proof: Differentiating $J_K(t)$ with respect to time gives

$$\frac{d}{dt}\oint_C K(\zeta,t)\bar{z}(\bar{\zeta},t)z_\zeta(\zeta,t)\ d\zeta \ = \ \oint_C K(\zeta,t)\left[\bar{z}_t z_\zeta + \bar{z}z_{\zeta t}\right] + K_t(\zeta,t)\bar{z}z_\zeta\ d\zeta \tag{1.42}$$

Using (1.36) (and its complex conjugate) to substitute for $z_t, \bar{z}_t$ gives

$$\begin{aligned}\dot{J}_K(t) = \oint_C K(\zeta,t)\Big[&-2\bar{F}(\bar{\zeta},t)z_\zeta + \frac{1}{\zeta}\bar{I}(\bar{\zeta},t)\bar{z}_\zeta(\bar{\zeta},t)z_\zeta(\zeta,t) \\ &+ \bar{z}[-2F + \zeta I(\zeta,t)z_\zeta]_\zeta\Big] + K_t\bar{z}z_\zeta d\zeta\end{aligned} \tag{1.43}$$

Rearranging terms and integrating one of the terms by parts, this becomes

$$\dot{J}_K(t) = \oint_C K(\zeta,t)\left[-2\bar{F}z_\zeta - 2\bar{z}F_\zeta + \frac{1}{\zeta}[I(\zeta,t)+\bar{I}(\bar{\zeta},t)]z_\zeta\bar{z}_\zeta\right]d\zeta$$
$$+ \oint_C [K_t - \zeta I(\zeta,t)K_\zeta]\,\bar{z}z_\zeta d\zeta \qquad (1.44)$$

Using the stress condition (1.20) which can be written

$$\bar{F}(\bar{\zeta},t)z_\zeta + \bar{z}F_\zeta(\zeta,t) + G(\zeta,t)z_\zeta = \frac{1}{2\zeta}z_\zeta^{1/2}\bar{z}_\zeta^{1/2} \qquad (1.45)$$

and the fact that on C

$$I(\zeta,t) + \bar{I}(\bar{\zeta},t) = \frac{1}{z_\zeta^{1/2}\bar{z}_\zeta^{1/2}} \qquad (1.46)$$

we then obtain the required result. □

In order to demonstrate the existence of exact solutions of the form (1.39), with $h(\zeta,t)$ a polynomial, we will make special choices of the function $K(\zeta,t)$.

Definition: Define a special subclass of the line integrals (1.40) (denoted $J^0_{k_0}(t)$ for each $k_0 = 0,1,2,...$) as

$$J^0_{k_0}(t) = \oint_C K_0(\zeta,t;k_0)\bar{z}(\bar{\zeta},t)z_\zeta(\zeta,t)d\zeta, \qquad (1.47)$$

where

$$K_0(\zeta,t;k_0) = \zeta^{k_0}\prod_{p=1}^{N}(\zeta - \bar{\zeta}_p^{-1})^{\gamma_p} \qquad (1.48)$$

We now state a theorem that connects the properties of the function $h(\zeta,t)$ to the properties of $J^0_{k_0}(t)$.

Theorem 1.4.2

Assume M is an integer such that $M \geq M_0$. Then,

$$J^0_{k_0}(t) = 0 \text{ for all } k_0 \geq M - M_0 \tag{1.49}$$

if and only if $h(\zeta, t)$ is a polynomial of degree at most M.

Proof: The proof of this theorem is given in Appendix A. □

Using Theorem 1.4.1, the following theorem concerning $\dot{J}^0_{k_0}$ is useful:

Theorem 1.4.3

Define $\{d_j | j \geq 0\}$ as the Taylor series coefficients of the following analytic function in $|\zeta| \leq 1$:

$$-k_0 I(\zeta, t) + \sum_{p=1}^{N} \gamma_p \frac{\zeta I(\zeta, t) - \bar{\zeta}_p^{-1} I(\bar{\zeta}_p^{-1}, t)}{\zeta - \bar{\zeta}_p^{-1}} = \sum_{j=0}^{\infty} d_j \zeta^j \tag{1.50}$$

Also, assume that each $\zeta_j(t)$ evolves according to

$$\frac{d}{dt} \zeta_j^{-1} = -\zeta_j^{-1}(t) \; I(\zeta_j^{-1}(t), t) \tag{1.51}$$

Then, for each integer $k_0 \geq 0$,

$$\dot{J}^0_{k_0} = \sum_{j=0}^{\infty} d_j \; J^0_{(k_0+j)} + \oint_{|\zeta|=1} d\zeta \; K_0 \; 2 \; G \; z_\zeta \tag{1.52}$$

Further, if $k_0 \geq r_0$,

$$\dot{J}^0_{k_0} = \sum_{j=0}^{\infty} d_j \; J^0_{(k_0+j)} \tag{1.53}$$

Proof: On substituting $K_0(\zeta, t; k_0)$ for K in Theorem 1.4.1, it follows that

$$\begin{aligned} \dot{J}^0_{k_0} &= \oint_{|\zeta|=1} d\zeta K_0 \bar{z} z_\zeta \sum_{p=1}^{N} \frac{\gamma_p}{(\zeta - \bar{\zeta}_p^{-1})} \left[-\frac{d}{dt}\bar{\zeta}_p^{-1} - \bar{\zeta}_p^{-1} I(\bar{\zeta}_p^{-1}, t) \right] \\ &+ \oint_{|\zeta|=1} d\zeta K_0 \bar{z} z_\zeta \sum_{p=1}^{N} \frac{\gamma_p}{\zeta - \bar{\zeta}_p^{-1}} \left[-\zeta I(\zeta, t) + \bar{\zeta}_p^{-1} I(\bar{\zeta}_p^{-1}, t) \right] \\ &+ \oint_{|\zeta|=1} d\zeta \; K_0 z_\zeta \; (2G - k_0 I \bar{z}) \end{aligned} \tag{1.54}$$

On taking the complex conjugate of (1.51) and using the property that the complex conjugate of $I(\zeta_p^{-1}, t)$ is $I(\bar{\zeta}_p^{-1}, t)$ (which follows from (1.38)), the first integral in (1.54) vanishes. Using the series representation (1.50) (which is uniformly convergent for $|\zeta| \leq 1$) in (1.54) the result (1.52) immediately follows. It is readily seen that if $k_0 \geq r_0$ then $K_0 \; z_\zeta \; G$ must be analytic for $|\zeta| \leq 1$. So, the result (1.53) follows. □

Remark 1.4.1

It is crucial to observe the "upper-triangular" nature of the evolution equations of these line integral quantities – in particular, note that the time derivative of $J^0_p(t)$ for some given integer p depends **only** on the values of $J^0_j(t)$ for $j \geq p$. This is what is meant by the description "upper-triangular". It is this important property of the evolution equations for the $J^0_{k_0}$ that underlies the existence of the exact solutions.

Remark 1.4.2

From the definition of $I(\zeta, t)$ in (1.38), it is clear that on $|\zeta| = 1$, *Re* I is given by the right hand side of (1.31), which is always positive. Since *Re* I is a harmonic function for $|\zeta| \leq 1$, it follows from the maximum principle that *Re* $I(\zeta, t) > 0$ in that domain for as long as the integral (1.38) exists. From (1.51), this immediately implies that *Re* $\left[\dot{\zeta}_j / \zeta_j\right] > 0$ which shows that all pole singularities of the conformal mapping function (1.39) move away from $|\zeta| = 1$. Earlier, Tanveer & Vasconcelos [8] presented a more general argument to show that any initial singularity of $z(\zeta, t)$ in $|\zeta| > 1$ moves outward with time.

Remark 1.4.3

If surface tension effects are **ignored** in the analysis, then it is clear that $I(\zeta, t) \equiv 0$ and, in that case, all the d_k coefficients are zero. Therefore from (1.51)-(1.53), it follows that the singularities $\zeta_j(t)$ and all but a finite number of the line integral quantities are time invariant even when $h(\zeta, t)$ is **not** restricted to a polynomial. Such results for **zero** surface tension when $z(\zeta, t)$ is a general analytic function have been systematically derived by Cummings et al [7] in a manner similar to Theorem 1.4.1, although these results follow directly from earlier work of Tanveer and Vasconcelos [8] (X_k in the notation of section 4 of [8]) who found such invariants in an *ad hoc* manner for the closely related problem of a single bubble in an arbitrary strain field.

Theorem 1.4.4

(Dynamics) If $J^0_{k_0}(0) = 0$ *for* $k_0 \geq M - M_0$, *then* $J^0_{k_0}(t) = 0$ *for* $t > 0$.

Proof: The proof of this important theorem is given in Appendix B. □

Note that Theorem 1.4.4 represents the crucial result of this chapter and contains the quintessential dynamics of the problem. The proof is non-trivial.

Remark 1.4.4

If $J^0_{k_0}(0) = 0$ for $k \geq M - M_0$, as is true when $h(\zeta, t)$ is a polynomial of degree M, then the summation index j in (1.52) ranges only from 0 to $M - M_0 - k_0 - 1$

Theorem 1.4.5

If $h(\zeta, 0)$ *is a polynomial of degree at most* M, *then so is* $h(\zeta, t)$.

Proof: If $h(\zeta, 0)$ is a polynomial of degree at most M, it follows from Theorem 1.4.2 that $J^0_{k_0}(0) = 0$ for $k_0 \geq M - M_0$. From Theorem 1.4.4, it follows that $J^0_{k_0}(t) = 0$ for $t > 0$. The converse of Theorem 1.4.2 then implies $h(\zeta, t)$ is a polynomial of degree at most M. The proof is then complete. □

1.5 Evolution Equations

From this point, we will only be concerned with initial conditions for which $h(\zeta, 0)$ is a polynomial of order M, where $M \geq M_0$. From Theorem 1.4.5 it follows that as long as the solution exists, $h(\zeta, t)$ will remain a polynomial of degree M and this will be assumed henceforth.

It remains to determine the time evolution of the finite set of coefficients of the polynomial $h(\zeta, t)$. To do this, we define a further subclass of line integrals in the following way:

Definition: For each integer j between 1 and N, and integer $k_j = 0, 1, 2, ...$, we define $J^j_{k_j}(t)$ as:

$$J^j_{k_j}(t) = \oint_C K_j(\zeta, t; k_j) \bar{z}(\bar{\zeta}, t) z_\zeta(\zeta, t) d\zeta \tag{1.55}$$

where

$$K_j(\zeta, t; k_j) = \zeta^{M-M_0}(\zeta - \bar{\zeta}_j^{-1})^{k_j} \prod_{\substack{p=1 \\ p \neq j}}^{N} (\zeta - \bar{\zeta}_p^{-1})^{\gamma_p}, \tag{1.56}$$

We now introduce a theorem about the evolution of $J^j_{k_j}(t)$:

Theorem 1.5.1

Assume that $\{\hat{d}^j_n | n \geq 0\}$ are defined as the Taylor series coefficients of the following analytic function around $\zeta = \bar{\zeta}_j^{-1}$

$$-(M - M_0)\ I(\zeta, t) + k_j \left[\frac{-\frac{d}{dt}\bar{\zeta}_j^{-1} - \zeta I(\zeta, t)}{\zeta - \bar{\zeta}_j^{-1}} \right] + \sum_{\substack{p=1 \\ p \neq j}}^{N} \gamma_p \frac{-\zeta I(\zeta, t) + \bar{\zeta}_p^{-1} I(\bar{\zeta}_p^{-1}, t)}{\zeta - \bar{\zeta}_p^{-1}}$$
$$= \sum_{n=0}^{\infty} \hat{d}^j_n\ (\zeta - \bar{\zeta}_j^{-1})^n \tag{1.57}$$

Also, assume that each $\zeta_j(t)$ evolves according to (1.51). Then, for each integer

$k_j \geq 0$,

$$\dot{J}^j_{k_j} = \sum_{n=0}^{\infty} \hat{d}^j_n \; J^j_{(k_j+n)} + \oint_{|\zeta|=1} K_j(\zeta, t; k_j) \; 2 \; G \; z_\zeta \; d\zeta \tag{1.58}$$

Further, if $k_j \geq r_j$,

$$\dot{J}^j_{k_j} = \sum_{n=0}^{\infty} \hat{d}^j_n \; J^j_{(k_j+n)} \tag{1.59}$$

Proof: The proof of this theorem is given in Appendix C. □

We now state a lemma about $J^j_{k_j}(t)$ for $k_j \geq \gamma_j$.

Lemma 1.5.1

$J^j_{k_j}(t) = 0$ *for* $k_j \geq \gamma_j$.

Proof: On substituting (1.39) into (1.55) and using $\bar{\zeta} = 1/\zeta$ on $|\zeta| = 1$, as well as the definition of K_j in (1.56), it is easily observed that the integrand in (1.55) is analytic in $|\zeta| \leq 1$ for $k_j \geq \gamma_j$ and therefore the lemma follows by Cauchy's theorem. □

Remark 1.5.1

Note that the result in the lemma is consistent with (1.59).

Remark 1.5.2

Because of lemma 1.5.1, the summation index n in (1.58) ranges from 0 to $\gamma_j - k_j - 1$.

We now discuss some ramifications of all the theorems above. An immediate observation is that we have identified an infinite set of integral invariants associated with solutions for which $h(\zeta, 0)$ is a polynomial of degree M. Only a *finite* set of integral quantities will be non-zero and time-evolving, namely

$$\{J^j_{k_j} | k_j = 0, 1 ... \gamma_j - 1\}; \; j = 1..N \tag{1.60}$$

$$\{J^0_{k_0} | k_0 = 0, 1...(M - M_0 - 1)\} \tag{1.61}$$

These are determined by solving the differential equations (1.52) and (1.58). (Note simplifications due to Remark 1.4.4 and Remark 1.5.2 above). Thus there are in general $\sum_{p=1}^{N} \gamma_p + M - M_0 = M$ non-zero time-evolving line integral quantities. Writing the polynomial $h(\zeta, t)$ as follows,

$$h(\zeta, t) = \sum_{n=0}^{M} h_n(t)\zeta^n \tag{1.62}$$

condition (1.27) then implies that $h_0(t) \equiv 0$, leaving only M as yet undetermined functions $h_1(t)..h_M(t)$. We now state a conjecture that is so far supported only by numerical evidence:

Conjecture: For given $\zeta_1(t)$, $\zeta_2(t)$, ... $\zeta_N(t)$ outside the unit ζ circle, the evolution of the set of M quantities in (1.60)-(1.61), as defined in (1.47) and (1.55), implicitly determine the evolution of the M quantities $h_1(t)$ through $h_M(t)$.

Remark 1.5.3

Note that the conjecture amounts to no more than an assertion that the "change of coordinates" that we have just employed (by defining the line integrals) can be locally inverted. Consideration of the finite set of line integrals as opposed to the finite set of parameters appearing in the conformal map might be viewed as a "change of coordinates" in which the dynamics of the problem can be seen more clearly.

Remark 1.5.4

It is clear from the definition of $J^0_{k_0}(t)$ and $J^j_{k_j}(t)$ in (1.47), (1.55) and the relations (1.39), (1.62) that these are quadratically dependent on $h_1(t)$ through $h_M(t)$; hence a globally unique relation between the set of J's and h's is unlikely. However, a Newton iterative procedure gives a unique solution locally when subjected to the constraint that $h_j(0)$ are as specified.

1.6 A Theorem of Invariants

For a certain subset of the solutions (1.39), it is possible to deduce immediately a further finite set of invariants which greatly facilitates the calculation of such solutions. We now state and prove a remarkable theorem involving solutions in which the mapping function $z(\zeta, t)$ has *simple* poles outside the unit circle. This theorem turns out to be highly useful in providing immediate "first integrals" of the finite system of first order ordinary differential equations to which the problem has been reduced.

Theorem 1.6.1

(Theorem of Invariants) If the initial conformal map for a viscous blob has the form

$$z(\zeta, 0) = \frac{h(\zeta, 0)}{\prod_{j=1}^{N}(\zeta - \zeta_j(0))^{\gamma_j}} \tag{1.63}$$

where $h(\zeta, 0)$ is a polynomial of degree $M \geq M_0$, then for any j for which $\gamma_j = 1$ and $r_j = 0$ (so that $G(\zeta, t)$ has no singularity at $\zeta = \bar{\zeta}_j^{-1}$) there exists an invariant of the motion given by

$$B_j = \frac{J_0^j(t)\bar{\zeta}_j^{M-M_0}}{\prod_{\substack{p=1 \\ p\neq j}}^{N}(\bar{\zeta}_j^{-1} - \bar{\zeta}_p^{-1})^{\gamma_p}} \tag{1.64}$$

Proof The proof of this theorem is given in Appendix D. □

1.7 Case Study

Since the aim of this chapter is to present a reformulation of the theory of exact solutions for the problem of Stokes flow of a simply-connected viscous blob and since previous studies in the literature have already computed specific examples illustrating the behaviour of viscous fluid blobs, we do not intend to compute further examples here. We do, however, include details of a case study with a particularly appealing mathematical structure that becomes clear as a result of the preceding analysis. We consider the special class of solutions having n *simple* poles (and no other poles) outside the unit circle, i.e. $\{\gamma_j = 1 | j = 1..n\}$ giving

$$z(\zeta, t) = \frac{h(\zeta, t)}{\prod_{j=1}^{n}(\zeta - \zeta_j)} \tag{1.65}$$

where $h(\zeta, 0)$ is taken as a polynomial of degree n. [Note that we could equally well find a solution with $h(\zeta, t)$ any polynomial of degree at least n by taking a suitable initial condition]. We assume the flow is driven purely by surface tension so that there are no flow singularities in the blob and $r_j = 0$ for all $j = 1..n$. The results of this chapter allow the evolution equations for the parameters in this map to be written down in a particularly concise way. From Theorem 1.4.3 we deduce that provided the poles $\{\zeta_j | j = 1..n\}$ evolve according to the equations,

$$\frac{d}{dt}\zeta_j^{-1} = -\zeta_j^{-1} I(\zeta_j^{-1}, t), \; j = 1..n \tag{1.66}$$

then a solution of the form (1.65) can be found. It only remains to determine the n coefficients of $h(\zeta, t)$ i.e. $\{h_k(t) | k = 1..n\}$ (since we know $h_0(t) \equiv 0$). However, Theorem 1.6.1 tells us that there are n invariants (or first integrals) of the motion associated with this solution given by

$$B_j = \frac{J_0^j(t)}{\prod_{\substack{p=1 \\ p \neq j}}^{n}(\bar{\zeta}_j^{-1} - \bar{\zeta}_p^{-1})}, \; j = 1..n \tag{1.67}$$

The invariants $\{B_j | j = 1..n\}$ are determined by initial conditions. These n equations then provide n nonlinear *algebraic* equations for the coefficients $\{h_k(t) | k = 1..n\}$ once the pole positions are known. Thus the $2n$ equations (1.66) and (1.67) provide a complete and concise set of equations for the $2n$ parameters in (1.65).

Finally, we remark that the special case of this example where $n = 2$ includes the problem of the coalescence of 2 viscous cylinders of unequal radius analyzed by Richardson [1] using a direct approach of combining the kinematic boundary condition and the stress condition and adjusting the time evolution of the parameters in the map $z(\zeta, t)$ to give the required analyticity properties of $G(\zeta, t)$ in the unit circle. Such a solution is obtained by making appropriate choices of initial conditions. After extensive algebraic manipulation, Richardson [1] also deduces the existence of 2 invariant quantities which can be shown to be equivalent to (1.67) in the case $n = 2$. He also deduces 2 evolution equations for the poles of the mapping, which can be shown to be equivalent to the more concise equations (1.66). The above case study represents a generalization of these results to general n. After this work was completed, the author became aware that Richardson [6] has also recently identified the generalization of Richardson [1] presented in the case study above by studying partial fraction maps of the form

$$z(\zeta, t) = \sum_{j=1}^{n} \frac{\beta_j \zeta}{1 - \gamma_j \zeta} \tag{1.68}$$

However, the method Richardson used is very different to that presented here and is, in essence, a simplified version of the original method used in Richardson [1]. Richardson [6] goes on to study numerically a class of solutions with initial conditions corresponding to n touching circular cylinders.

1.8 Discussion

With no flow singularities present in the blob, the results of this chapter provide exact solutions, describable in terms of a finite set of parameters, for the physical problem of the time evolution of certain initial boundary shapes driven by surface tension. Mathematically, the analysis also allows for a distribution of multipole singularities to exist within the blob, and again exact solutions for the evolution can be found in this case. It has been found that while it is possible to externally specify the nature and strength of such singularities (i.e. specify the strength of the residue contributions from the last integrals in (1.52) and (1.58)), it is not in general possible to externally specify the singularity positions after the initial time (the singularities necessarily evolve according to (1.51)). Thus, except in very special cases (for example, a single singularity at the origin or at infinity [4] [7] [8] [10] [11]) these mathematical solutions are physically untenable in that they are solutions to a problem where the singularities must move in very special ways determined implicitly by the solution itself. While this is something of a drawback in the use of these solutions to solve particular initial value problems with a given distribution of known singularities at specified points in the flow, such solutions may be instructive *qualitative* models of this physical scenario.

In summary, a novel global approach to the theoretical problem of the slow quasi-steady viscous flow of a two dimensional simply-connected blob of fluid with surface tension has been presented. The new approach renders the mathematical structure underlying the existence of exact solutions more transparent. The approach also simplifies much of the unwieldy algebra which characterizes the actual calculation of solutions using previously-known methods. A central result is that it is possible to find an infinite set of conserved quantities associated with a very general class of initial conformal maps describing the boundary evolution and a finite set of non-trivially evolving line integral quantities which implicitly determine the evolution of such maps.

Chapter 2

Exact Solutions for Annular Viscous Blobs

2.1 Introduction

This chapter[1] presents a theory of exact solutions for the quasi-steady evolution of a plane *annular* viscous blob of fluid driven by surface tension. Although, as described in chapter 1, many exact solutions have been identified for the Stokes flow of a simply-connected fluid region both with and without surface tension (e.g [1]-[10]), we present here the first successful attempt to extend the solution techniques to a **doubly-connected** topology. All of the methods used by other authors up to now rely on conformal mapping techniques and most ([1]-[7], [9]-[10]) consist of conjecturing a form for the conformal map in terms of a finite set of time-evolving parameters and showing that the time evolution of these parameters can be adjusted such that the appropriate analyticity properties of the solution hold inside the fluid region. In chapter 1, an alternative and less cumbersome approach was devised using a reformulation of the problem in terms of a general set of line integrals. This theory, which includes surface tension, was presented in the context of the evolution of a single simply-connected fluid blob, but with minor modifications the same theory can be shown to apply to the case of an infinite expanse of fluid with a single bubble or a semi-infinite expanse of fluid with an infinite free surface. Extension to these cases is routine and will not be expounded here.

In this chapter, the new theoretical approach developed in chapter 1 is extended in a natural way to deal with the problem of an annular blob, which constitutes a doubly-connected fluid region requiring non-trivial adjustments of the solution method.

[1]This chapter is based on material from an article entitled "A Theory of Exact Solutions for Annular Viscous Blobs" by D.G. Crowdy and S. Tanveer to appear in *Journal of Nonlinear Science*. It is reproduced here with the kind permission of Springer Verlag New York Inc., 175 Fifth Avenue, New York, NY 10010.

Although we concentrate here purely on presenting the relevant mathematical theory, the results of this chapter represent a significant step forward and are likely to have great utility in problems of real physical interest. The original motivation for the recent revival of interest in the Stokes flow of a two-dimensional fluid region was *sintering*, a term loosely referring to the consolidation of an assemblage of particles in which surface tension provides the principal mechanism for mass transport. Sintering is a complex topic with a huge literature, and the study of sintering is difficult owing to geometrical complexities. It is therefore natural to isolate parts of the overall problem for individual study, and one of the basic paradigms in the study of sintering is the coalescence of two viscous cylinders (particles). Exact solutions for the coalescence (under Stokes approximation) of two viscous cylinders (particles) driven purely by surface tension have recently been identified by Richardson [1], with further developments to the case of the coalescence of multiple touching cylinders made even more recently [6]. Earlier, Hopper [2] studied the evolution of two touching cylinders of equal size. These studies only cover the case of a simply-connected fluid domain, and Richardson's study [6] of the coalescence of multiple cylinders deals only with an arbitrary number of cylinders in a linear concatenation (so that no cylinder touches more than two other cylinders, the overall fluid region being simply-connected). The relevance of such solutions (even as a model) in a study of the more useful scenario of, say, a general collection (say, a pile) of cylinders/particles, where the fluid region has a connectivity greater than one, is not clear.

This chapter presents a theory of exact solutions for the case of a doubly-connected fluid region for a certain general class of initial conditions. A general class of conformal maps representing initial annular blob configurations is shown, under the evolution equations for Stokes flow, to be such that each member of the class *retains its functional form under evolution.* In this sense, the nonlinear free boundary value problem for this wide class of initial conditions, is reduced (exactly) to the study of a finite first-order system of ordinary differential equations.

Such results then facilitate the (exact) study of the physical scenario where two coalescing cylinders have, say, a small air bubble between them. This would seem to be a natural paradigm for the study of the coalescence of a *general* assemblage of cylinders/particles (rather than a linear concatenation) where, of course, there would inevitably be small air bubbles between the cylinders if they were arbitrarily piled together. As a simple example calculation to verify the validity of the theory presented in this chapter, the simple paradigm of the coalescence of two (unequal) touching cylinders is extended in a natural way to include the case of the evolution of two (unequal) touching cylindrical blobs but now with a small air bubble between them. More involved calculations using the general theory developed here will be presented in future work.

2.2 Mathematical Formulation

Consider the slow viscous flow of an arbitrary annular blob of fluid. The equations of motion in the fluid are given by

$$-\nabla p + \nabla^2 \mathbf{u} = 0 \tag{2.1}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{2.2}$$

Length and time scales have been non-dimensionalized with respect to a and $\frac{a\mu}{\sigma}$ respectively, where a is an effective radius (with πa^2 a measure of the initial area of the blob), σ is the surface tension parameter and μ is the viscosity. Velocities have been rescaled by $\frac{\sigma}{\mu}$ and pressures by $\frac{\sigma}{a}$. The blob now has **two** boundaries, the boundary conditions on each consisting of a stress condition which can be written as

$$-pn_j + 2e_{jk}n_k = -\kappa n_j \tag{2.3}$$

where n_1 and n_2 are the x and y components of the unit normal vector pointing outwards from the bubble boundary and κ is the curvature. e_{jk} are the components of the non-dimensionalized stress tensor given by

$$e_{jk} = \frac{1}{2}\left[\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}\right] \tag{2.4}$$

In addition, there is a kinematic boundary condition that the normal velocity of a point on the blob boundary is the same as the normal component of the fluid velocity at that point.

Again the method proceeds via a Goursat representation of the stream-function. The difficulty now arises with the existence of *two* boundaries on which the stress and kinematic boundary conditions are required to hold. Intuitively, the presence of **two** disjoint free surfaces on which the nonlinear boundary conditions are required to hold seems likely to destroy the mathematical structure that led to the identification of exact solutions in the simply-connected case. In general, this seems to be the case. However, we now show that it is possible to identify a special class of exact solutions for which exactly the same mathematical structure exists.

Consider the two boundary conditions. Firstly, the stress conditions on each boundary can be written in complex form as

$$-pN + 2(e_{11} + ie_{12})\bar{N} = -\kappa N \tag{2.5}$$

In deriving (2.5), the fluid flow on either side of the annular region is neglected – an asymptotically valid assumption when the viscosity ratio between fluids is small. Further, the same constant pressure on either side of the annular blob is assumed. There is some loss of generality in this assumption; for instance, it is not true for a steady annular blob with no motion that lies between two concentric circles. However, there is no additional loss of generality in assuming that the constant pressure on either side is zero, as assumed in deriving (2.5).

Using similar manipulations to those in the previous chapter, the stress conditions on the two boundaries can be integrated with respect to the arc-length parameter s

to yield

$$f(z) + z\bar{f}'(\bar{z}) + \bar{g}'(\bar{z}) = -i\frac{z_s}{2} + A_O(t) \tag{2.6}$$

$$f(z) + z\bar{f}'(\bar{z}) + \bar{g}'(\bar{z}) = -i\frac{z_s}{2} + A_I(t) \tag{2.7}$$

on the outer and inner boundaries of the blob respectively where $A_O(t)$ and $A_I(t)$ are constants of integration which are, in general, functions of time. It follows, using (1.10), that on the outer blob boundary,

$$u + iv = -i\frac{z_s}{2} + A_O(t) - 2f(z) \tag{2.8}$$

while on the inner blob boundary,

$$u + iv = -i\frac{z_s}{2} + A_I(t) - 2f(z) \tag{2.9}$$

Although the main problem of physical interest in this paper is that where the evolution of the annular blob is driven purely by surface tension (corresponding to the function $g'(z)$ having no singularities in the fluid region), *mathematically* the formulation developed in this chapter can be extended to find mathematical solutions corresponding to $g'(z)$ having an arbitrary distribution of poles in the fluid region. This extension was incorporated explicitly in chapter 1, but we do not include details in this chapter – we simply state that the extension is possible here too. Physically, it is well-known that an nth order multipole singularity at a point z_{sing} in the fluid corresponds to $g'(z)$ having an nth-order pole at z_{sing}. As mentioned earlier, it is to be expected, from a physical standpoint, that the strengths and locations of any singularities in the fluid should be externally specifiable, and a corresponding flow field found. Unfortunately, at present we are only able to get a restricted set of (exact) solutions for which it seems to be possible to specify only the *strengths* of the multipole singularities and their *initial* positions. For the exact solutions obtained

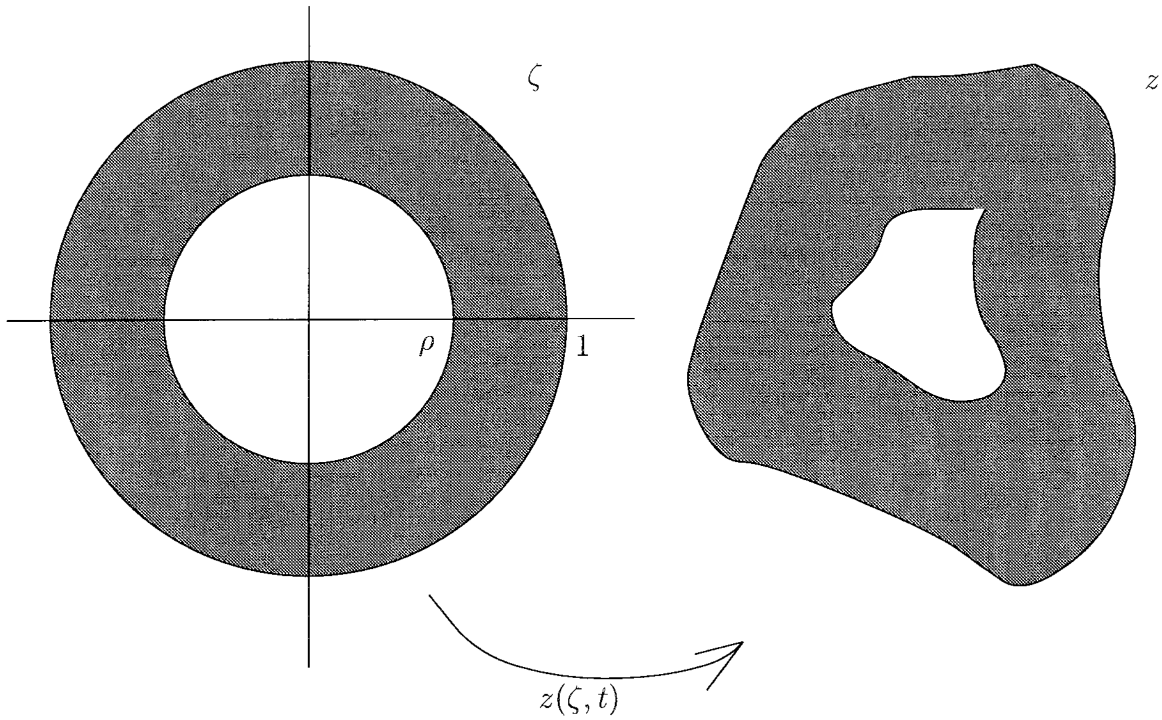

Figure 2.1: Conformal mapping domains

here, the subsequent *positions* of the singularities evolve in a way determined by the solution itself and cannot be externally specified. The physical relevance of such mathematical solutions appears to be limited, except possibly as a convenient model of qualitative phenomena.

2.3 Conformal Mapping

The analysis proceeds by defining a conformal map from an annulus C in a complex ζ-plane where C is $\rho < |\zeta| < 1$ to the fluid region in physical space, the circle $|\zeta| = 1$ mapping to the outer boundary of the fluid annulus, the circle $|\zeta| = \rho$ mapping to the inner boundary. This conformal map will be called $z(\zeta, t)$. It is known, by Riemann's theorem, that any *given* doubly-connected fluid domain can be mapped to such an annulus C for *some* ρ. For a general time-evolving domain in physical space, the *conformal modulus* of the region (see [15] for a definition) must therefore be assumed, *a priori*, to change in time. Thus we suppose that $\rho(t)$ is a function of time to be determined as part of the solution. The remaining degree of freedom of the Riemann Mapping Theorem will be fixed in a convenient way later in the analysis. We will seek solutions for which $z(\zeta, t)$ is analytic in C and, for blobs with smooth boundaries with no corners or cusps, has the property that $|z_\zeta| \neq 0$ everywhere inside C and on the boundary ∂C. Further, in order for the solutions to be physically relevant, $z(\zeta, 0)$ will be restricted to functions that are univalent in C. *A posteriori* examination of the exact solutions obtained clarifies if and when $z(\zeta, t)$ fails to be univalent beyond a certain time. The solutions fail to be physically relevant beyond such a time (if it exists).

The kinematic boundary condition on both blob boundaries can be written

$$Im\left[\frac{\frac{dz}{dt} - (u + iv)}{z_s}\right] = 0 \tag{2.10}$$

Using the facts that

$$z_s = \frac{i\zeta z_\zeta}{|z_\zeta|} \text{ on } |\zeta| = 1 \tag{2.11}$$

$$z_s = -\frac{i\zeta z_\zeta}{\rho|z_\zeta|} \text{ on } |\zeta| = \rho(t) \tag{2.12}$$

then substituting these expressions into the kinematic conditions and using (2.8) and (2.9) yields

$$Re\left[\frac{z_t + 2F(\zeta, t)}{\zeta z_\zeta}\right] = \frac{1}{2|z_\zeta|} + Re\left[\frac{A_O}{\zeta z_\zeta}\right] \tag{2.13}$$

on the outer boundary, where we define

$$F(\zeta, t) \equiv f(z(\zeta, t), t) \tag{2.14}$$

and on the inner boundary,

$$Re\left[\frac{z_t + 2F(\zeta,t)}{\zeta z_\zeta}\right] = -\frac{1}{2\rho|z_\zeta|} - \frac{\dot{\rho}}{\rho} + Re\left[\frac{A_I}{\zeta z_\zeta}\right] \tag{2.15}$$

We also define the function

$$G(\zeta,t) \equiv g'(z(\zeta,t),t) \tag{2.16}$$

Our solution method and results are so far restricted to the case where $A_O = A_I$. It is emphasized that this special choice involves a definite *loss of generality* in the class of solutions being considered. However, it is clear that *without* any further loss of generality, the value of A_O (and hence A_I) can be taken to be zero. This can be seen by redefining $f(z)$ and $g'(z)$ as follows:

$$f(z) \mapsto f(z) + \frac{A_O}{2} \tag{2.17}$$

$$g'(z) \mapsto g'(z) + \frac{\bar{A}_O}{2} \tag{2.18}$$

– a transformation that does not alter the velocity field, but which effectively removes the constants of integration in (2.6) and (2.7) once the choice $A_O = A_I$ has been made. It remains to specify the rotational degree of freedom in the problem but once that is done, the evolution of the annular blob is uniquely determined, as shall be seen later. We remark that since it is only the geometrical evolution of the blob boundaries that is of interest, it is of no importance if the solution is such that the global momentum of the blob is not conserved. Any overall translation or rotation of the blob can be subtracted *a posteriori*, without altering the validity of the solution for the blob shape.

It is immediately clear that the function in square brackets on the left hand sides of (2.13) and (2.15) is an analytic function inside C. Thus, using the Dirichlet formula (or Poisson integral formula) for a harmonic function in terms of the values of its real part on the boundary of C (see Appendix E), we deduce that for ζ within C

$$z_t(\zeta,t) + 2F(\zeta,t) = \zeta I(\zeta,t) z_\zeta(\zeta,t) \tag{2.19}$$

where $I(\zeta,t)$ is given by

$$I(\zeta,t) = I^+(\zeta,t) - I^-(\zeta,t) + C_1(t) + iC_2(t) \tag{2.20}$$

where

$$I^+(\zeta,t) = \frac{1}{4\pi i}\oint_{|\zeta'|=1} \frac{d\zeta'}{\zeta'}\left(1 - 2\frac{\zeta}{\zeta'}\frac{P'(\frac{\zeta}{\zeta'})}{P(\frac{\zeta}{\zeta'})}\right)\left[\frac{1}{z_\zeta^{1/2}(\zeta',t)\bar{z}_\zeta^{1/2}(\frac{1}{\zeta'},t)}\right] \tag{2.21}$$

and

$$I^-(\zeta,t) = \frac{1}{4\pi i}\oint_{|\zeta'|=\rho} \frac{d\zeta'}{\zeta'}\left(1 - 2\frac{\zeta}{\zeta'}\frac{P'(\frac{\zeta}{\zeta'})}{P(\frac{\zeta}{\zeta'})}\right)\left[-\frac{1}{\rho z_\zeta^{1/2}(\zeta',t)\bar{z}_\zeta^{1/2}(\frac{\rho^2}{\zeta'},t)} - \frac{2\dot{\rho}}{\rho}\right] \tag{2.22}$$

and where the function $P(\zeta)$ is defined in Appendix F, $C_1(t)$ is a real function of time given by

$$C_1(t) = -\frac{1}{4\pi i}\oint_{|\zeta'|=\rho} \frac{d\zeta'}{\zeta'}\left[-\frac{1}{\rho z_\zeta^{1/2}(\zeta',t)\bar{z}_\zeta^{1/2}(\frac{\rho^2}{\zeta'})} - \frac{2\dot{\rho}}{\rho}\right] \tag{2.23}$$

and $C_2(t)$ is an arbitrary real function of time. The remaining rotational degree of freedom of the Riemann Mapping Theorem is now used up by choosing $C_2(t) = 0$.

Theorem 2.3.1

With the choice $A_0 = A_I = 0$ in the boundary conditions (2.6) and (2.7), the evolution of the conformal modulus of the corresponding class of solutions is given by the real equation

$$\dot{\rho} = -\frac{\rho}{4\pi i}\left(\oint_{|\zeta|=1} \frac{d\zeta}{\zeta}\frac{1}{z_\zeta^{1/2}\bar{z}_\zeta^{1/2}} + \oint_{|\zeta|=\rho} \frac{d\zeta}{\zeta}\frac{1}{\rho z_\zeta^{1/2}\bar{z}_\zeta^{1/2}}\right) \tag{2.24}$$

Proof: This follows from the averaging condition E.2 in Appendix E applied to the harmonic function $Re\ I$ necessary for $I(\zeta, t)$ to be a *single-valued* analytic function everywhere in C. Note that right hand side of (2.13) and (2.15), with $A_0 = 0 = A_I$, determines $Re\ I$ on the two boundaries. □

Remark 2.3.1

For the class of solutions under consideration, the sign of $\dot{\rho}$ is always negative implying that $\rho(t) \to 0$ as time evolves.

As mentioned earlier, for a flow driven purely by surface tension $g'(z)$ is regular everywhere in the fluid. $f(z)$ is also taken to be analytic everywhere in the fluid. By conformality of $z(\zeta, t)$ in C, this implies that both $G(\zeta, t)$ and $F(\zeta, t)$ are analytic everywhere in C

2.4 Conserved Quantities and Exact Solutions

The analysis will again proceed by considering the time evolution of a number of general (purely geometrical) line integral quantities defined thus

$$J_K(t) = \oint_{\partial C} K(\zeta, t)\bar{z}(\bar{\zeta}, t) z_\zeta(\zeta, t)\ d\zeta \tag{2.25}$$

where ∂C now denotes the boundary of the *annulus* C with $|\zeta| = 1$ traversed anticlockwise and the boundary $|\zeta| = \rho$ traversed clockwise. The function $K(\zeta, t)$ appearing in the integrand is again any function of ζ and t which will be taken to be analytic in the annulus C. Later, special choices will be made for this function in order to establish the required results. We now show that, *under the assumptions made so far*, the equations giving the time evolution of these line integrals look structurally similar to those given for the analogous line integrals defined around the simply-connected region in chapter 1.

Theorem 2.4.1

The time evolution of the integral quantity $J_K(t)$ defined above under the equations of motion for the Stokes flow of an annular blob of fluid is given by

$$\dot{J}_K(t) = \oint_{\partial C} 2K(\zeta,t)G(\zeta,t)z_\zeta(\zeta,t)\ d\zeta + \oint_{\partial C} \left(K_t(\zeta,t) - \zeta I(\zeta,t)K_\zeta\right) \bar{z}(\bar{\zeta},t)z_\zeta(\zeta,t)\ d\zeta \tag{2.26}$$

Proof: The proof of this theorem is given in Appendix G. □

From chapter 1 it is known that in the case of a single *simply-connected* blob of fluid with surface tension, exact solutions in the form of an arbitrary rational function, conformal in the unit circle in the ζ plane, have been identified. By analogy with these solutions, we now seek solutions which are again meromorphic in the ζ plane (except at zero and infinity) but with an **infinite** number of poles outside the annulus C.

We introduce our solution by first defining a set of N parameters $\{\zeta_j | \ j = 1..N\}$, each of which satisfy the condition $1 < |\zeta_j| < \rho^{-1}$, at least initially. $\zeta_j(t)$ will evolve in time according to equations to be determined. The solution will cease to be valid if and when the above condition is violated. It is to be noted that each $\bar{\zeta}_j^{-1}$ is within C. We now define the function $P(\zeta)$, through the infinite product expression

$$P(\zeta) = (1-\zeta)\prod_{m=1}^{\infty}(1-\rho^{2m}\zeta)\prod_{n=1}^{\infty}\left(1-\frac{\rho^{2n}}{\zeta}\right) \tag{2.27}$$

Clearly $P(\zeta)$ has zeros at $\zeta = \rho^{2m}$ for any integer m, positive or negative or zero. Other properties of $P(\zeta)$ are well known and those relevant to this thesis are listed in Appendix F. Note that $P(\zeta)$ has already been used in the Dirichlet formula for the annulus (2.21)-(2.22).

We define the analytic function $h(\zeta,t)$ through the relation

$$z(\zeta,t) = \frac{h(\zeta,t)}{\prod_{j=1}^{N}\left[P(\zeta\zeta_j^{-1})\right]^{\gamma_j}} \tag{2.28}$$

where $\{\gamma_j | j = 1..N\}$ are arbitrary non-negative integers. Since the zeros in the denominator on the right hand side are clearly outside C and $z(\zeta, t)$ is analytic within C, it follows that an analytic h in C implies an analytic z in C and *vice versa.* We also define

$$M_0 = \sum_{j=1}^{N} \gamma_j \tag{2.29}$$

where we assume $M_0 \geq 2$. The reason for this last restriction will become clear later.

It is now appropriate to consider a subclass of the general line integral quantities (2.25) given by

$$J^0_{k_0}(t) = \oint_{\partial C} K_0(\zeta, t; k_0)\bar{z}(\bar{\zeta}, t) z_\zeta(\zeta, t) d\zeta \tag{2.30}$$

where

$$K_0(\zeta, t; k_0) \equiv \zeta^{k_0} \prod_{j=1}^{N} \left[P(\zeta\bar{\zeta}_j)\right]^{\gamma_j}, \; k_0 = ... - 2, -1, 0, 1, 2... \tag{2.31}$$

Note that $K_0(\zeta, t; k_0)$ is analytic in C (indeed it is analytic in the entire ζ-plane except for essential singularities at 0 and ∞ – see Appendix F). We now state an important theorem concerning conserved quantities:

Theorem 2.4.2

(Dynamics) If $J^0_{k_0}(0) = 0$ for all k_0 then

$$J^0_{k_0}(t) \;=\; 0 \text{ for all integers } k_0 \tag{2.32}$$

provided

$$\frac{d}{dt}\bar{\zeta}_j^{-1} = -\bar{\zeta}_j^{-1} I\left(\bar{\zeta}_j^{-1}, t\right) \text{ for all } j = 1..N \tag{2.33}$$

Proof: The proof of this crucial theorem is given in Appendix H. □

We now state some theorems which relate the properties of the function $h(\zeta, t)$ to the properties of the infinite set of quantities $\{J^0_{k_0}(t) \mid k_0 \text{ any integer }\}$.

Theorem 2.4.3

Consider the quantities $J^0_{k_0}(t)$ *defined in (2.30)-(2.31). Then*

$$J^0_{k_0}(t) = 0 \;\forall\; k_0 \tag{2.34}$$

if and only if the function $\bar{h}(\zeta, t)$ *as defined in (2.28) is analytic everywhere in the* ζ *plane except at 0 and* ∞ *and satisfies the functional equation*

$$R(t)\zeta^{-M_0}\bar{h}(\zeta, t) = \bar{h}(\rho^2\zeta, t) \tag{2.35}$$

for all $\zeta \neq 0$ *where* $R(t) = \prod_{j=1}^{N}[-\bar{\zeta}_j(t)]^{\gamma_j}$.

Proof: The proof of this theorem is given in Appendix I. □

Theorem 2.4.4

The function $\bar{h}(\zeta, t)$ *satisfies the functional equation*

$$R(t)\zeta^{-M_0}\bar{h}(\zeta, t) = \bar{h}(\rho^2\zeta, t) \tag{2.36}$$

for all $\zeta \neq 0$ *and is analytic everywhere except possibly at* 0 *and* ∞ *if and only if*

$$\bar{h}(\zeta, t) = \bar{S}(t)\prod_{m=1}^{M_0} P\left(\zeta\bar{\eta}_m^{-1}\right) \tag{2.37}$$

for some $\bar{S}(t)$ *and some* $\{\bar{\eta}_m(t) | m = 1..M_0\}$ *satisfying the condition*

$$\prod_{m=1}^{M_0}[-\bar{\eta}_m(t)] = R(t) \tag{2.38}$$

Proof: The proof of this theorem is given in Appendix J. □

Theorem 2.4.5

If initially $h(\zeta, 0)$ *has the form*

$$\bar{h}(\zeta, 0) = \bar{S}(0) \prod_{m=1}^{M_0} P\left(\zeta \bar{\eta}_m(0)^{-1}\right) \tag{2.39}$$

where $\prod_{m=1}^{M_0}[-\bar{\eta}_m(0)] = \prod_{j=1}^{N}[-\bar{\zeta}_j(0)]^{\gamma_j}$ *and* $\zeta_j(0)$ *(and equivalent points) are the positions of the poles of* $z(\zeta, 0)$ *and provided*

$$\frac{d\bar{\zeta}_j^{-1}}{dt} = -\bar{\zeta}_j^{-1} I\left(\bar{\zeta}_j^{-1}, t\right) \text{ for all } j = 1..N \tag{2.40}$$

then

$$\bar{h}(\zeta, t) = \bar{S}(t) \prod_{m=1}^{M_0} P\left(\zeta \bar{\eta}_m(t)^{-1}\right) \tag{2.41}$$

where

$$\prod_{m=1}^{M_0} [-\bar{\eta}_m(t)] = \prod_{j=1}^{N} [-\bar{\zeta}_j(t)]^{\gamma_j} \tag{2.42}$$

is the unique solution for all future times that the solution exists.

Proof: From the initial form for $h(\zeta, 0)$, it follows (from Theorems 2.4.3 and 2.4.4) that

$$J_{k_0}^0(0) = 0 \ \forall k_0 \tag{2.43}$$

By Theorem 2.4.2 it is known that, provided the poles evolve according to (2.33), $J_{k_0}^0(t) = 0 \ \forall \ k_0$ is the **unique** solution for all time that the solution exists. By Theorem 2.4.3, we then deduce that $\bar{h}(\zeta, t)$ satisfies (2.35) which then implies, by Theorem 2.4.4, that $\bar{h}(\zeta, t)$ has the form (2.37) for all times. Hence Theorem 2.4.5 is proved. □

2.5 Evolution Equations

Theorem 2.4.5 provides the crucial result of this chapter and we will now limit the discussion to initial conditions of the form (2.39), thus implying that $z(\zeta, t)$ has the following form for all time that the solution exists, i.e.

$$z(\zeta, t) = S(t)\frac{\prod_{m=1}^{M_0} P(\zeta\eta_m^{-1})}{\prod_{j=1}^{N}\left[P(\zeta\zeta_j^{-1})\right]^{\gamma_j}} \tag{2.44}$$

where $\{\zeta_j(t)\}, \{\eta_j(t)\}$ satisfy (2.42). Note that we immediately see that such functions are *loxodromic* and satisfy the functional equation $z(\rho^2\zeta, t) = z(\zeta, t)$ for all $\zeta \neq 0$. More information on the theory of loxodromic functions is given in Appendix L.

Remark 2.5.1

Since we have deduced $z(\zeta, t)$ is a loxodromic function, it is known from Remark 2 in Appendix L that such a function is uniquely defined once its poles and zeros in the *fundamental annulus* $\rho^2 < |\zeta| \leq 1$ are known, as well as its value at one other point. Note that the fundamental annulus is *not* the same as C.

Remark 2.5.2

To be physically acceptable, $z(\zeta, t)$ must be univalent in C. It is therefore necessary to pick initial values of $\{\eta_m(0)|m = 1..M_0\}$ and $\{\zeta_j(0)|j = 1..N\}$ such that $z(\zeta, 0)$ is a univalent map for $\rho \leq |\zeta| \leq 1$. If the map subsequently evolves such as to violate this condition then the solution will be deemed invalid thereafter. A necessary though not sufficient condition for this is to ascertain that there are no zeros of the derivative $z_\zeta(\zeta, t)$ in this region. Since $z(\zeta, t)$ is a loxodromic function with the fundamental annulus $\rho^2 < |\zeta| \leq 1$, (i.e. an elliptic function in the variable $\log\ \zeta$), it follows from the well-known theory of elliptic functions of order M_0, that any value of z will be taken M_0 times in this fundamental annulus (or fundamental rectangle if $ln\ \zeta$ is the variable). For univalence initially, it is necessary to pick initial values of the parameters so that z attains no value more than once in the subregion $\rho \leq |\zeta| \leq 1$. That such a choice is possible is far from obvious and is illustrated through a numerical example later in this chapter.

Remark 2.5.3

The reason for the restriction on M_0 mentioned in (2.29) is that a nontrivial loxodromic function (or an elliptic function in the variable log ζ) must be at least of order two. See Appendix L for more information.

We now examine how to derive the evolution equations for the finite set of time-evolving parameters appearing in the solution (2.44). To do this, we consider the line integral quantities defined by

$$J^j_{k_j}(t) = \oint_{\partial C} K_j(\zeta, t; k_j)\bar{z}(\bar{\zeta}, t) z_\zeta(\zeta, t) d\zeta \tag{2.45}$$

where

$$K_j(\zeta, t; k_j) \equiv \left[P(\zeta\bar{\zeta}_j)\right]^{k_j} \prod_{\substack{p=1 \\ p\neq j}}^{N} \left[P(\zeta\bar{\zeta}_p)\right]^{\gamma_p} \tag{2.46}$$

and $k_j = 0, 1, 2....$

Theorem 2.5.1

For the class of initial shapes considered, the following property of the line integral quantities defined in (2.45)-(2.46) holds for all $j = 1..N$:

$$J^j_{k_j}(t) = 0 \text{ for } k_j \geq \gamma_j \tag{2.47}$$

Proof: We use the loxodromic nature of $z(\zeta, t)$ (and hence of $\bar{z}(\zeta, t)$) to reduce $J^j_{k_j}(t)$ to the integral around ∂C of the following function of ζ

$$\left[P(\zeta\bar{\zeta}_j)\right]^{k_j} \left(\prod_{\substack{p=1 \\ p\neq j}}^{N} \left[P(\zeta\bar{\zeta}_p)\right]^{\gamma_p} \right) \bar{z}(\zeta^{-1}, t) z_\zeta(\zeta, t) \tag{2.48}$$

which, by use of the form (2.44) for $z(\zeta, t)$ can be seen to be analytic in C for all $k_j \geq \gamma_j$ and the theorem follows by Cauchy's Theorem. □

Theorem 2.5.2

For the class of initial shapes considered, the $J^j_{k_j}(t)$ satisfies the following equation for $k_j = 0, 1, ..\gamma_j - 1$ and $j = 1..N$:

$$\dot{J}^j_{k_j}(t) = \oint_{\partial C} K_j \left[k_j \zeta \frac{P'(\zeta\bar{\zeta}_j)}{P(\zeta\bar{\zeta}_j)} \left(\frac{d\bar{\zeta}_j}{dt} - \bar{\zeta}_j I \right) + k_j \dot{\rho} \frac{P_\rho(\zeta\bar{\zeta}_j)}{P(\zeta\bar{\zeta}_j)} \right.$$
$$\left. + \sum_{\substack{p=1 \\ p \neq j}}^{N} \gamma_p \zeta \frac{P'(\zeta\bar{\zeta}_j)}{P(\zeta\bar{\zeta}_j)} \left(\frac{d\bar{\zeta}_p}{dt} - \bar{\zeta}_p I(\zeta, t) \right) + \gamma_p \dot{\rho} \frac{P_\rho(\zeta\bar{\zeta}_p)}{P(\zeta\bar{\zeta}_p)} \right] \bar{z} z_\zeta \, d\zeta \tag{2.49}$$

Proof: Applying Theorem 2.4.1, with the substitution $K(\zeta, t) = K_j(\zeta, t; k_j)$, it follows that

$$\dot{J}^j_{k_j}(t) = \oint_{\partial C} K_j 2G(\zeta, t) z_\zeta + K_j \left[k_j \zeta \frac{P'(\zeta\bar{\zeta}_j)}{P(\zeta\bar{\zeta}_j)} \left(\frac{d\bar{\zeta}_j}{dt} - \bar{\zeta}_j I \right) + k_j \dot{\rho} \frac{P_\rho(\zeta\bar{\zeta}_j)}{P(\zeta\bar{\zeta}_j)} \right.$$
$$\left. + \sum_{\substack{p=1 \\ p \neq j}}^{N} \gamma_p \zeta \frac{P'(\zeta\bar{\zeta}_j)}{P(\zeta\bar{\zeta}_j)} \left(\frac{d\bar{\zeta}_p}{dt} - \bar{\zeta}_p I(\zeta, t) \right) + \gamma_p \dot{\rho} \frac{P_\rho(\zeta\bar{\zeta}_p)}{P(\zeta\bar{\zeta}_p)} \right] \bar{z} z_\zeta \, d\zeta \tag{2.50}$$

The analyticity of $K_j G z_\zeta$ in C implies that the first term within the integrand on the right hand side of (2.50) contributes nothing and the result (2.49) follows. □

Remark 2.5.4

Note by inspection that for $k_j \geq \gamma_j$ the pole singularity of $\bar{z}$ at $\zeta = \bar{\zeta}_j^{-1}$ is cancelled out by the zeros of K_j at the same point; further, when ζ_j satisfies (2.33), there is only a *removable* singularity at $\zeta = \bar{\zeta}_j^{-1}$. From this, it is easy to see that the integrand in (2.49) is analytic in such cases and therefore $\dot{J}^j_{k_j} = 0$ for $k_j \geq \gamma_j$, which is consistent with Theorem 2.5.1.

Since the evolution of the N poles $\{\zeta_j | \ j = 1..N\}$ is given by the N equations (2.40), it only remains to deduce the evolution of the M_0+1 time-evolving parameters $S(t), \{\eta_m(t) | m = 1..M_0\}$. Note however that the $\{\eta_m(t)\}$ satisfy the constraint that $\prod_{m=1}^{M_0}[-\eta_m(t)] = \prod_{j=1}^{N}[-\zeta_j(t)]^{\gamma_j}$. Thus there remains precisely M_0 (generally complex) parameters to determine. Note however that there are M_0 non-zero line integral quantities, namely

$$\{J^j_{k_j}(t)|k_j = 0..\gamma_j - 1\},\ j = 1..N \tag{2.51}$$

which can be determined from (2.49). However, this requires us to invoke the following conjecture that we believe to be true, but which is so far supported only by numerical evidence.

Conjecture: For a given set of $\zeta_j(t)$, the M_0 quantities in (2.51) at any time t determine uniquely the $M_0 + 1$ parameters $S(t), \{\eta_m(t)|m = 1..M_0\}$ satisfying the constraint (2.42).

Remark 2.5.5

The conjecture above, if true, implies that for a given set of $\zeta_j(t)$, (2.49) can be viewed as a differential equation to determine $J^j_{k_j}$ since quantities appearing in the integrands, such as z, $\bar{z}$ and I are completely determined by the parameters characterizing z in (2.44), which in turn is known for given $J^j_{k_j}$.

Remark 2.5.6

We are not asserting a *globally* unique relation between the quantities in (2.51) and the parameters appearing in (2.44); only a *locally* unique relation.

It is noted however that the "counting" in the statement of the conjecture is consistent, and the validity of the conjecture is supported by the upcoming example calculation, even though this is but a single special case. First we demonstrate the remarkable result that the *theorem of invariants* described in the simply-connected case in chapter 1 also holds in the doubly-connected scenario.

2.6 A Theorem of Invariants

Theorem 2.6.1

(Theorem of Invariants) Suppose that the conformal map $z(\zeta, t)$ has the form (2.44) and that the evolution of the poles is given by (2.40). For each j for which the corresponding $\gamma_j = 1$, there exists an invariant of the motion given by

$$B_j = \frac{J_0^j(t)}{\prod_{\substack{p=1 \\ p\neq j}}^{N} \left[P(\bar{\zeta}_p \bar{\zeta}_j^{-1})\right]^{\gamma_p}} \tag{2.52}$$

Proof: The proof of this theorem is given in Appendix K. □

Remark 2.6.1

Note that each invariant B_j is determined from initial conditions alone. If $\gamma_j = 1$ for all j, between 1 and N, then there will be N invariants B_1 through B_N

2.7 Case Study

A special class of exact solutions having an appealing mathematical structure was found in chapter 1. We now write down the analogous solutions for an annular blob. Consider the class of exact solutions where $z(\zeta, t)$ has the form

$$z(\zeta, t) = S(t) \frac{\prod_{m=1}^{N} P(\zeta \eta_m^{-1})}{\prod_{j=1}^{N} P(\zeta \zeta_j^{-1})} \tag{2.53}$$

with

$$\prod_{m=1}^{N} \eta_m(t) = \prod_{j=1}^{N} \zeta_j(t) \tag{2.54}$$

This corresponds to the special case of the above solutions with $\{\gamma_j = 1 \mid j = 1..N\}$, where N is an arbitrary positive integer, $N \geq 2$. The evolution equations for maps of this form can be written down as a very concise set. Of course, the evolution of $\rho(t)$ is always given by (2.24) but by the previous theorems, for a solution of the form

(2.53) the poles must evolve according to

$$\frac{d\bar{\zeta}_j^{-1}}{dt} = -\bar{\zeta}_j^{-1} I\left(\bar{\zeta}_j^{-1}, t\right), \; j = 1..N \tag{2.55}$$

which provides the evolution of the poles. In addition, by Theorem 2.6.1, there are N invariants of the motion given by

$$B_j = \frac{J_0^j(t)}{\prod_{\substack{p=1 \\ p\neq j}}^{N} P(\zeta_p \bar{\zeta}_j^{-1})} \; j = 1..N \tag{2.56}$$

where $\{B_j |\; j = 1..N\}$ are complex constants determined from initial conditions. Thus, with the simple poles ζ_j determined from the evolution equations (2.55), the $N+1$ parameters $S(t)$ and $\{\eta_m(t)|\; m = 1..N\}$ satisfying the constraint (2.54) are then determined by inverting the N nonlinear algebraic relations (2.56).

Note finally that in this special case study it is easily seen that the total area of the fluid region is directly proportional to the sum of the N invariants of motion, which means that it is also conserved (as it should be given that there are no sources/sinks in the fluid).

2.8 Example Calculation

Since the purpose of this thesis is to develop the mathematical theory, we reserve a full investigation of the physical phenomena exhibited by the class of solutions found here for a future investigation. However, it is necessary to include here at least one explicit sample calculation for two reasons: first, as evidence for the validity of the aforementioned conjecture. Second, to demonstrate, by explicit construction, the existence of conformal maps that are loxodromic functions and which also satisfy the required conformality and univalency conditions in the annulus C.

As discussed in the introduction, the example calculation that has been chosen constitutes a basic paradigm in the study of sintering and represents a natural generalization of the study of the coalescence of two (unequal) cylinders as carried out recently by Richardson [1] [6]. Here we study the evolution of two (unequal) touching near-cylinders which also happen to have a small air bubble in the region where they touch. In the case of just two cylinders, such a bubble may perhaps have been trapped by some mechanism as the cylinders came into contact. More usefully, this example (and more sophisticated versions of it) is expected to represent a basic paradigm for the evolution of a general assemblage of cylinders/particles where there will inevitably be small air bubbles trapped between the cylinders.

The initial state of the two (almost) cylindrical blobs and an air bubble between them is represented in Figure 2.2 and corresponds to the case $N = 3$ with $\gamma_1 = \gamma_2 = \gamma_3 = 1$ so that

$$z(\zeta, t) = S(t)\frac{P(\frac{\zeta}{\eta_1})P(\frac{\zeta}{\eta_2})P(\frac{\zeta}{\eta_3})}{P(\frac{\zeta}{\zeta_1})P(\frac{\zeta}{\zeta_2})P(\frac{\zeta}{\zeta_3})} \tag{2.57}$$

with initial parameters given by

$$\begin{aligned} &\zeta_1 = 2.2, \ \zeta_2 = -1.25, \ \zeta_3 = 1.29039, \ \eta_1 = 2.08 \\ &\eta_2 = 1.4, \ \eta_3 = -1.21861, \ \rho = 0.13, \ S = 1.0 \end{aligned} \tag{2.58}$$

Such a map represents a univalent map from the annulus C to the region shown in Figure 2.1. Note that there are three simple poles of this mapping which means, by the theorem of invariants, that we can automatically find three invariants of the motion.

Since ρ is initially relatively small, and since it is known that $\rho(t) \to 0$ for the class of solutions found here, certain approximations were made which greatly facilitated the computations to the point where Mathematica could reasonably be used to carry them out. Since no numerical pathologies were expected, an obvious and elementary first method was used in which a simple forward-Euler method was employed to time step the evolution of $\rho(t), \zeta_1(t), \zeta_2(t)$ and $\zeta_3(t)$ (with $h = 0.0002$), while a high-accuracy Newton's method was then used at each time step to invert the three invariants of motion for $\eta_1(t)$, $\eta_2(t)$, $S(t)$ (with $\eta_3 = \frac{\zeta_1\zeta_2\zeta_3}{\eta_1\eta_2}$). Mathematica coped well with all the calculations once both the function $P(\zeta)$ and the kernel function in (2.21) and (2.22) were expanded for small ρ and approximated to within $O(\rho^6)$ (the errors due to this approximation are therefore of order 10^{-5} – i.e. smaller than the global error of the simple time-stepping scheme). Given the existence of points of large curvature in the initial configuration, the blob evolved quickly and the configuration after just 30 time steps is shown relative to the initial configuration in Figure 2.3.

It is clear that the global momentum of the blob is not conserved and there is clearly an overall translation of the blob in the positive x-direction. As mentioned earlier, this is unimportant since it is only the geometrical evolution of the blob boundary that is of interest. (If desired, these physically irrelevant translations can be removed by a straightforward shift of the centres of area to a common point before plotting). Note also that the near-cusps of the initial configuration (observe that the initial enclosed air bubble has three points of very high curvature) became smoother under evolution, and the enclosed bubble grew smaller, as expected. It is clear that the points of high curvature have been smoothed out by the effects of surface tension. Clearly the enclosed bubble is already quite small after only 30 time steps and although no further integration was carried out, it is expected that as more time evolves, the inner bubble will simply continue to get smaller (probably tending to a circular shape) while the outer boundary of the blob will evolve into a circular shape as already observed by Richardson [6]

In any event, the principal purpose of including this simple numerical calculation is to verify that the finite set of nonlinear evolution equations derived in the theory above can indeed be solved (at least locally) to provide the time evolution of the parameters in the exact solution (2.57). In this limited example, the physical behaviour was much as expected. Further studies of the physical properties of the mathematical solutions presented here will be left for future study.

2.9 Final Remarks

One of the principal problems in the study of conformal maps of this form is that of finding initial values of the parameters such that the initial map is a univalent map from C to the fluid domain. This is a highly nontrivial task and the present author knows of no systematic method of constructing such functions – the initial configuration found above for the sample calculation was identified after extensive trial and error in the parameter space of poles and zeros.

An intriguing theoretical question that arises is whether the boundary of *any* doubly-connected fluid domain of given conformal modulus $\rho(t)$ can be approximated as closely as required (in some norm) by a map $z(\zeta, t)$ satisfying the loxodromic property $z(\rho^2\zeta, t) = z(\zeta, t)$ which is a conformal and univalent map from C in the complex ζ-plane. If this is true then a further question that arises is whether the subsequent evolution of the approximating loxodromic function (as given in this chapter) will remain a good approximation to the true evolution of that fluid region under Stokes flow driven purely by surface tension. Such mathematical questions require further investigation.

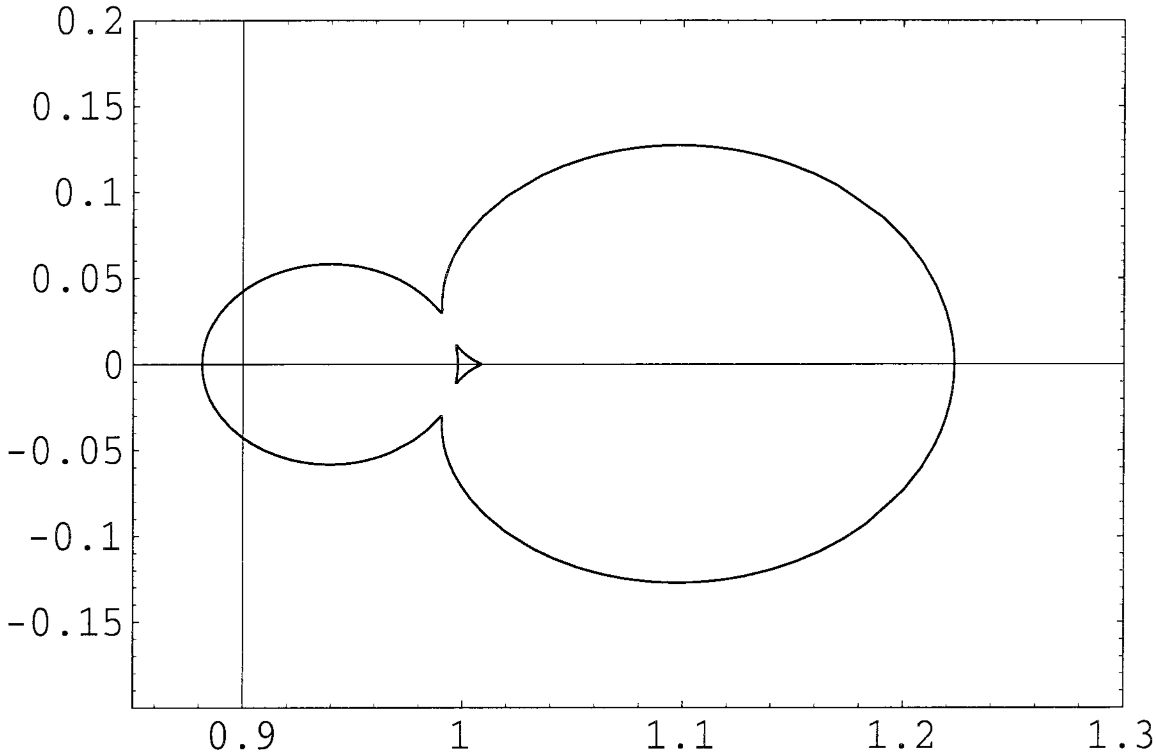

Figure 2.2: Initial configuration

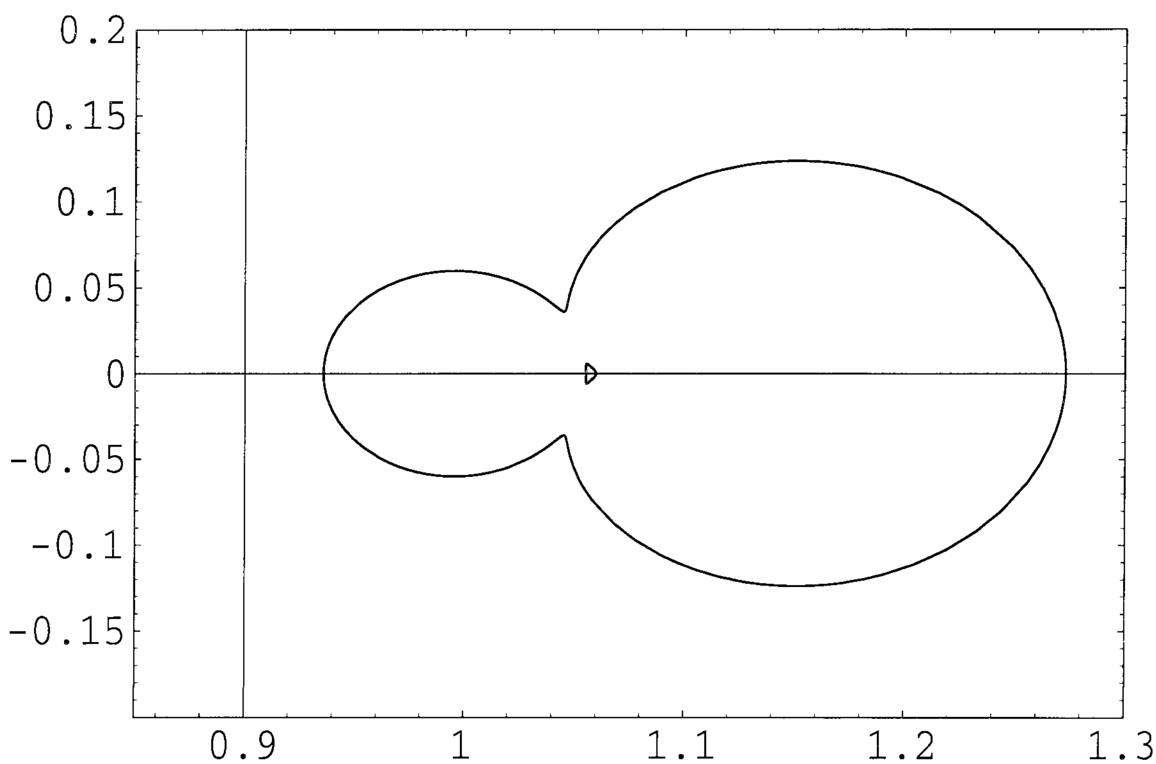

Figure 2.3: Configuration after 30 time steps (h=0.0002)

Chapter 3

Exact Solutions for Stokes Bubbles

3.1 Introduction

In this chapter, the problem of the slow viscous (Stokes) flow of an infinite fluid region containing **two** air bubbles is presented. The methods used are a modification of those presented in the previous chapter. New physical solutions corresponding to the evolution of two bubbles in the presence of a source/sink at infinity are presented. Thus, this chapter extends the analysis of chapter 2 by showing how to incorporate a (simple) pole in the mapping function and how to incorporate a singularity in the flow field.

3.2 Mathematical Formulation

The equations and boundary conditions in this case are almost identical to those of the previous chapter. The differences arise at the point at infinity and also because of the pole in the conformal mapping function. At infinity, we choose to allow for the possibility of a strain field as well as a source/sink of strength $m(t)$. Thus, in general, the flow at infinity will have the following general form

$$\mathbf{u} \sim \Gamma.\mathbf{x} + \mathbf{U}_0(t) + \frac{m(t)}{2\pi}\frac{\hat{\mathbf{x}}}{|\mathbf{x}|} \text{ as } |\mathbf{x}| \to \infty \tag{3.1}$$

where

$$\Gamma = \frac{1}{2}\begin{pmatrix} \delta_1 & \delta_2 - \omega_\infty \\ \delta_2 + \omega_\infty & -\delta_1 \end{pmatrix} \tag{3.2}$$

ω_∞ represents the vorticity at infinity while $\delta_1(t)$ and $\delta_2(t)$ characterize the strain field at infinity. It will turn out that, within the class of solutions presented in this chapter, it is not in general possible to find exact solutions where the strain field, uniform flow field **and** mass flux at infinity are **all** externally specified. This point is discussed in more detail later. However, it **is** possible to find exact solutions for the boundary evolution of two bubbles where there is just a source or sink at infinity (and no strain field), and where the strength of the source/sink at infinity is externally specifiable. In order to incorporate as much generality as possible in the presentation of the theory, the analysis will proceed so as to *include* the possibility of a non-zero straining flow at infinity. In this way it will be explicitly demonstrated precisely why it proves to be impossible (at least using the current methods) to find exact solutions when a strain field and source/sink strength at infinity is specified.

Again we write the general solution of the biharmonic equation as

$$\psi = Im[\bar{z}f(z) + g(z)] \tag{3.3}$$

From (1.10) and the specified conditions at infinity, it follows that

$$f(z) \sim \frac{1}{4}\left[p_\infty(t) - i\omega_\infty\right]z + D(t) + O\left(\frac{1}{z}\right) \text{ as } |z| \to \infty \tag{3.4}$$

where $D(t)$ is generally complex. The real quantity $p_\infty(t)$ represents the pressure at infinity. It can also be shown that

$$g'(z) \sim \frac{1}{2}\left(\delta_1(t) - i\delta_2(t)\right)z + B(t) + \frac{m(t)}{2\pi z} + O\left(\frac{1}{z^2}\right) \text{ as } |z| \to \infty \tag{3.5}$$

where $\delta_1(t), \delta_2(t)$ and $m(t)$ are all real while $B(t)$ is generally complex.

Following the same procedures as in chapter 2, the stress conditions on the boundaries of the two bubbles can be integrated with respect to the arc-length parameter to give

$$f(z) + z\bar{f}'(\bar{z}) + \bar{g}'(\bar{z}) = -i\frac{z_s}{2} + A_1(t) \tag{3.6}$$

$$f(z) + z\bar{f}'(\bar{z}) + \bar{g}'(\bar{z}) = -i\frac{z_s}{2} + A_2(t) \tag{3.7}$$

on the two bubble boundaries where $A_1(t)$ and $A_2(t)$ are constants of integration which are, in general, functions of time. It follows that on the boundary of one of the bubbles (say, bubble 1),

$$u + iv = -i\frac{z_s}{2} + A_1(t) - 2f(z) \tag{3.8}$$

while on the boundary of bubble 2,

$$u + iv = -i\frac{z_s}{2} + A_2(t) - 2f(z) \tag{3.9}$$

It is again assumed that the fluid flow in each of the bubbles is negligible and that the pressure in each bubble is given by the same constant value. There is clearly some loss of generality in this assumption. However, there is no additional loss of generality in assuming that the constant pressure in each bubble is zero.

3.3 Conformal Mapping

We consider a conformal map from the annulus $\rho < |\zeta| < 1$ (denoted C_0) to the **exterior** of the two bubbles. This will require the map to have a simple pole at some point within the annulus (note that the pole **must** be *simple* in order that the map be univalent in the annulus, as required). Let the point in the annulus mapping to physical infinity be denoted by $\zeta_\infty(t)$. By manipulations identical to those in the previous chapter, the kinematic conditions on each boundary can be written

$$Re\left[\frac{z_t + 2F(\zeta,t)}{\zeta z_\zeta}\right] = \frac{1}{2|z_\zeta|} + Re\left[\frac{A_1}{\zeta z_\zeta}\right] \tag{3.10}$$

on the boundary of bubble 1, where we define

$$F(\zeta,t) \equiv f(z(\zeta,t),t) \tag{3.11}$$

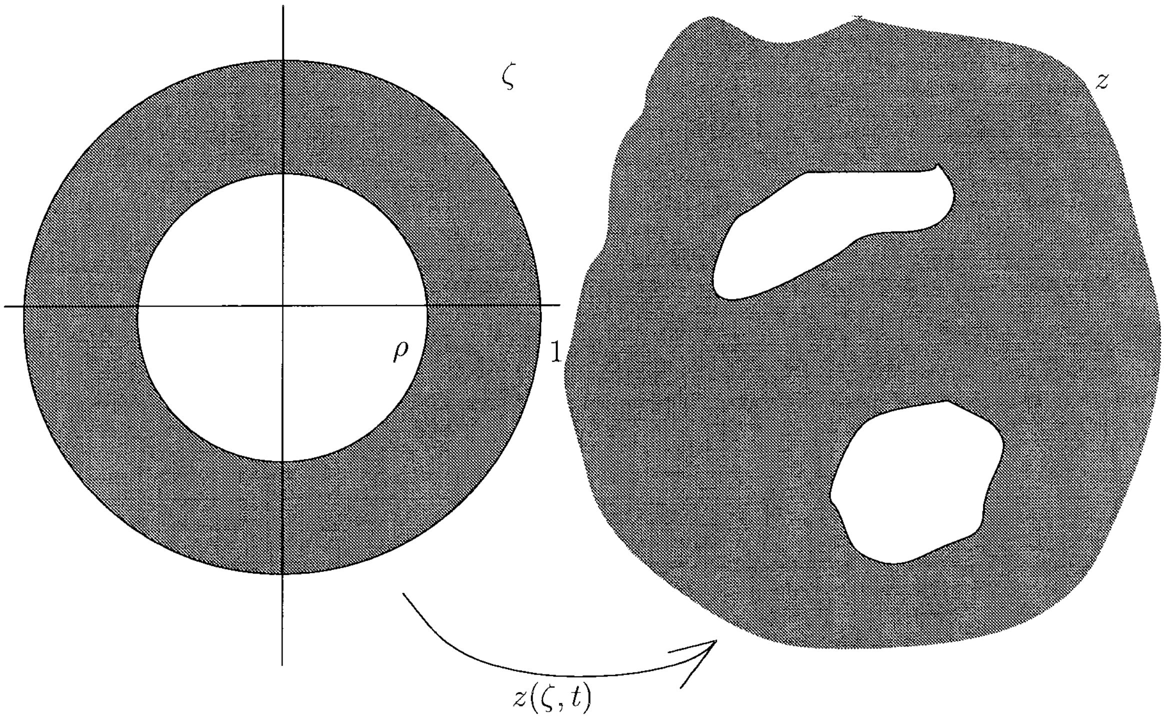

Figure 3.1: Conformal mapping domains

and on the boundary of bubble 2,

$$Re\left[\frac{z_t + 2F(\zeta,t)}{\zeta z_\zeta}\right] = -\frac{1}{2\rho|z_\zeta|} - \frac{\dot{\rho}}{\rho} + Re\left[\frac{A_2}{\zeta z_\zeta}\right] \tag{3.12}$$

The solution method is again restricted to the case where $A_1 = A_2$ – a restriction involving a definite *loss of generality* in the class of solutions being considered. However, without any further loss of generality, we take $A_1 = A_2 = 0$.

It can be seen that the function in square brackets on the left hand sides of (3.10) and (3.12) is an analytic function inside C_0. This is because the singularities in $z(\zeta, t)$ and $F(\zeta, t)$ at ζ_∞ in both the numerator and denominator cancel out leaving a function that is analytic at ζ_∞. Thus, using the integral formulae for a harmonic function in terms of the values of its real part on the boundary of C_0, we deduce that for ζ within C_0

$$z_t(\zeta,t) + 2F(\zeta,t) = \zeta I(\zeta,t) z_\zeta(\zeta,t) \tag{3.13}$$

where $I(\zeta, t)$ is given by the expression defined in the previous chapter. Note that again we make the choice $C_2(t) = 0$. This fixes the remaining rotational degree of freedom.

3.4 Far-field Conditions

Consider the conditions at infinity. Equation (3.13) clearly must hold at the point ζ_∞. Thus, expanding $z(\zeta, t)$ about this point yields

$$z(\zeta, t) = \frac{a_{-1}(t)}{(\zeta - \zeta_\infty)} + a_0(t) + a_1(t)(\zeta - \zeta_\infty) + O(\zeta - \zeta_\infty)^2 \tag{3.14}$$

for some functions $a_{-1}(t), a_0(t)....$ which are all functions of time. Using (3.4) and expanding the function $F(\zeta, t)$ about ζ_∞ in equation (3.13) yields the following two equations (equating the two lowest powers of $(\zeta - \zeta_\infty)$ in equation (3.13))

$$\dot{\zeta}_\infty = -\zeta_\infty I(\zeta_\infty, t) \tag{3.15}$$

$$\dot{a}_{-1}(t) + \frac{a_1(t)}{4}\left(p_\infty(t) - i\omega_\infty\right) = -a_{-1}\zeta_\infty I'(\zeta_\infty) - a_1 I(\zeta_\infty) \tag{3.16}$$

The first of these equations clearly provides an evolution equation for $\zeta_\infty(t)$ while the second equation gives an expression for the pressure and vorticity at infinity in terms of the solution. The pressure at infinity is a quantity that is not externally specifiable but is determined by the solution itself. We also remark that, in the special class of solutions which are symmetrical about the x-axis and for which the vorticity at infinity is initially taken to be zero, then with the choice $C_2 = 0$, the vorticity at infinity remains zero for all time. The example calculation carried out later in this chapter falls within this special class of solutions. The following theorem gives the evolution equation for the conformal modulus $\rho(t)$:

Theorem 3.4.1

With the choice $A_1 = A_2 = 0$ in the boundary conditions (3.6) and (3.7), the evolution of the conformal modulus of the corresponding class of solutions is given by the real equation

$$\dot{\rho} = -\frac{\rho}{4\pi i}\left(\oint_{|\zeta|=1} \frac{d\zeta}{\zeta}\frac{1}{z_\zeta^{1/2}\bar{z}_\zeta^{1/2}} + \oint_{|\zeta|=\rho} \frac{d\zeta}{\zeta}\frac{1}{\rho z_\zeta^{1/2}\bar{z}_\zeta^{1/2}}\right) \tag{3.17}$$

Proof: See corresponding proof in the previous chapter.

3.5 Conservation Laws and Exact Solutions

Again we consider the class of line integrals:

$$J_K(t) = \oint_{\partial C_0} K(\zeta,t)\bar{z}(\bar{\zeta},t)z_\zeta(\zeta,t)\ d\zeta \tag{3.18}$$

where ∂C_0 denotes the boundary of the annulus C_0 with $|\zeta| = 1$ traversed anticlockwise and the boundary $|\zeta| = \rho$ traversed clockwise. $K(\zeta, t)$ is any function of ζ and t taken to be an analytic function of ζ in C_0. The following theorems can be proved in ways analogous to those in the previous chapter, with more or less trivial rearrangements of details. The theorems will therefore be presented without proof.

Theorem 3.5.1

The time evolution of the integral quantity $J_K(t)$ defined above under the equations of motion for the Stokes flow region is given by

$$\dot{J}_K(t) = \oint_{\partial C_0} 2K(\zeta,t)G(\zeta,t)z_\zeta(\zeta,t)\ d\zeta + \oint_{\partial C_0} \left(K_t(\zeta,t) - \zeta I(\zeta,t)K_\zeta\right)\bar{z}(\bar{\zeta},t)z_\zeta(\zeta,t)\ d\zeta \tag{3.19}$$

We define a set of N parameters $\{\zeta_j |\ j = 1..N\}$, each of which satisfy the condition $1 < |\zeta_j| < \rho^{-1}$, at least initially. $\zeta_j(t)$ will evolve in time according to equations to be determined. The solution will cease to be valid if and when the above condition is violated. The function $h(\zeta, t)$ is defined through the relation

$$z(\zeta,t) = \frac{h(\zeta,t)}{P(\zeta\zeta_\infty^{-1})P(\zeta\bar{\zeta}_\infty)\prod_{j=1}^{N}\left[P(\zeta\zeta_j^{-1})\right]^{\gamma_j}} \tag{3.20}$$

where $\{\gamma_j | j = 1..N\}$ are arbitrary non-negative integers. Since ζ_∞ is known to be the only pole of $z(\zeta, t)$ it follows that an analytic h in C_0 implies an analytic z in C_0 and *vice versa*. The parameter $M_0 \geq 0$ is defined

$$M_0 = \sum_{j=1}^{N} \gamma_j \tag{3.21}$$

It is now appropriate to consider integral quantities of the form

$$J_{k_0}^0(t) = \oint_{\partial C_0} K_0(\zeta,t;k_0)\bar{z}(\bar{\zeta},t)z_\zeta(\zeta,t)\ d\zeta \tag{3.22}$$

where

$$K_0(\zeta,t;k_0) \equiv \zeta^{k_0}[P(\zeta{\zeta_\infty}^{-1})]^3 \prod_{j=1}^{N}\left[P(\zeta\bar{\zeta}_j)\right]^{\gamma_j},\ k_0 = ... -2, -1, 0, 1, 2... \tag{3.23}$$

Note that $K_0(\zeta, t; k_0)$ is analytic in C_0. We now state the crucial theorem underlying the existence of exact solutions:

Theorem 3.5.2

(Dynamics) If $J_{k_0}^0(0) = 0$ for all k_0 then

$$J_{k_0}^0(t) = 0 \text{ for all integers } k_0 \tag{3.24}$$

provided

$$\frac{d}{dt}\bar{\zeta}_j^{-1} = -\bar{\zeta}_j^{-1} I\left(\bar{\zeta}_j^{-1}, t\right) \text{ for all } j = 1..N \tag{3.25}$$

and

$$\frac{d\zeta_\infty}{dt} = -\zeta_\infty I(\zeta_\infty, t) \tag{3.26}$$

Note that the evolution equation (3.26) for ζ_∞ is consistent with that already deduced (i.e. (3.15)) from a local analysis.

The following two theorems relate the properties of the function $h(\zeta, t)$ to the properties of the quantities $\{J^0_{k_0}(t) \mid k_0 \text{ any integer }\}$.

Theorem 3.5.3

Consider the quantities $J^0_{k_0}(t)$ defined in (3.22)-(3.23). Then

$$J^0_{k_0}(t) = 0 \;\forall\; k_0 \tag{3.27}$$

if and only if the function $\bar{h}(\zeta, t)$ as defined in (3.20) is analytic everywhere in the ζ plane except at 0 and ∞ and satisfies the functional equation

$$R(t)\zeta^{-M_0-2}\bar{h}(\zeta, t) = \bar{h}(\rho^2\zeta, t) \tag{3.28}$$

for all $\zeta \neq 0$ where $R(t) = \prod_{j=1}^{N}[-\bar{\zeta}_j(t)]^{\gamma_j}\frac{\bar{\zeta}_\infty}{\zeta_\infty}$.

Theorem 3.5.4

The function $\bar{h}(\zeta, t)$ satisfies the functional equation

$$R(t)\zeta^{-M_0-2}\bar{h}(\zeta, t) = \bar{h}(\rho^2\zeta, t) \tag{3.29}$$

for all $\zeta \neq 0$ and is analytic everywhere except possibly at 0 and ∞ if and only if

$$\bar{h}(\zeta, t) = \bar{S}(t)\prod_{m=1}^{M_0+2} P\left(\zeta\bar{\eta}_m^{-1}\right) \tag{3.30}$$

for some $\bar{S}(t)$ *and some* $\{\bar{\eta}_m(t)|m = 1..(M_0+2)\}$ *satisfying the condition*

$$\prod_{m=1}^{M_0+2} [-\bar{\eta}_m(t)] = R(t) \tag{3.31}$$

Theorem 3.5.5

If initially $h(\zeta, 0)$ *has the form*

$$\bar{h}(\zeta, 0) = \bar{S}(0) \prod_{m=1}^{M_0+2} P\left(\zeta \bar{\eta}_m(0)^{-1}\right) \tag{3.32}$$

where $\prod_{m=1}^{M_0}[-\bar{\eta}_m(0)] = \prod_{j=1}^{N}[-\bar{\zeta}_j(0)]^{\gamma_j} \frac{\bar{\zeta}_\infty(0)}{\zeta_\infty(0)}$ *and* $\zeta_j(0)$ *(and equivalent points) are the positions of the poles of* $z(\zeta, 0)$ *and provided*

$$\frac{d\bar{\zeta}_j^{-1}}{dt} = -\bar{\zeta}_j^{-1} I\left(\bar{\zeta}_j^{-1}, t\right) \text{ for all } j = 1..N \tag{3.33}$$

and

$$\frac{d\zeta_\infty}{dt} = -\zeta_\infty I(\zeta_\infty, t) \tag{3.34}$$

then

$$\bar{h}(\zeta, t) = \bar{S}(t) \prod_{m=1}^{M_0+2} P\left(\zeta \bar{\eta}_m(t)^{-1}\right) \tag{3.35}$$

where

$$\prod_{m=1}^{M_0+2} [-\bar{\eta}_m(t)] = \prod_{j=1}^{N}[-\bar{\zeta}_j(t)]^{\gamma_j} \frac{\bar{\zeta}_\infty(t)}{\zeta_\infty(t)} \tag{3.36}$$

for all future times that the solution exists.

3.6 Evolution Equations

Theorem 3.5.5 provides the crucial result and we will now limit the discussion to initial conditions of the form (3.32), thus implying that $z(\zeta, t)$ has the following form for all time that the solution exists, i.e.

$$z(\zeta, t) = S(t) \frac{\prod_{m=1}^{M_0+2} P(\zeta \eta_m^{-1})}{P[\zeta \zeta_\infty^{-1}] P[\zeta \bar{\zeta}_\infty] \prod_{j=1}^{N} \left[P(\zeta \zeta_j^{-1})\right]^{\gamma_j}} \tag{3.37}$$

where $\{\zeta_j(t)\}, \{\eta_j(t)\}$ and $\zeta_\infty(t)$ satisfy (3.36). Note that we immediately see that such functions are *loxodromic* functions.

To derive the evolution equations for the finite set of time-evolving parameters appearing in the solution (3.37), we consider the line integral quantities defined by

$$J_{k_j}^j(t) = \oint_{\partial C_0} K_j(\zeta, t; k_j) \bar{z}(\bar{\zeta}, t) z_\zeta(\zeta, t) d\zeta \tag{3.38}$$

where

$$K_j(\zeta, t; k_j) \equiv \left[P(\zeta \bar{\zeta}_j)\right]^{k_j} P[\zeta \zeta_\infty{}^{-1}]^3 \prod_{\substack{p=1 \\ p \neq j}}^{N} \left[P(\zeta \bar{\zeta}_p)\right]^{\gamma_p} \tag{3.39}$$

and $k_j = 0, 1, 2...$ and those defined by

$$J_{k_\infty}^\infty(t) = \oint_{\partial C_0} K_\infty(\zeta, t; k_\infty) \bar{z}(\bar{\zeta}, t) z_\zeta(\zeta, t) d\zeta \tag{3.40}$$

where

$$K_\infty(\zeta, t; k_\infty) \equiv \left[P(\zeta \zeta_\infty{}^{-1})\right]^{k_\infty} \prod_{p=1}^{N} \left[P(\zeta \bar{\zeta}_p)\right]^{\gamma_p} \tag{3.41}$$

With these definitions, we now identify further sets of conserved quantities as well as a finite set of non-trivially evolving line integral quantities the evolution of which implicitly provides the evolution of the (finite) set of parameters in the conformal mapping function.

Theorem 3.6.1

For the class of initial shapes considered, the following property of the line integral quantities defined in (3.38)-(3.39) holds for all $j = 1..N$:

$$J^j_{k_j}(t) = 0 \text{ for } k_j \geq \gamma_j \tag{3.42}$$

N.B. The above integrals therefore constitute *conserved quantities.*

Theorem 3.6.2

For the class of initial shapes considered, the following property of the line integral quantities defined in (3.40)-(3.41) holds:

$$J^{\infty}_{k_\infty}(t) = 0 \text{ for } k_\infty \geq 3 \tag{3.43}$$

N.B. These integrals constitute further *conserved quantities.*

While we have demonstrated the existence of conserved quantities, the evolution of the finite set of parameters in the mapping function are again given implicitly by the non-trivially evolving line integrals. The only *nontrivial* evolution equations are given in the following two theorems.

Theorem 3.6.3

For the class of initial shapes considered, the $J^{\infty}_{k_\infty}(t)$ satisfies the following nontrivial equation for $k_\infty = 0, 1$ and 2

$$\begin{aligned} \dot{J}^{\infty}_{k_\infty}(t) = \oint_{\partial C_0} K_\infty \Bigg[k_\infty \zeta \frac{P'(\zeta{\zeta_\infty}^{-1})}{P(\zeta{\zeta_\infty}^{-1})} \left(\frac{d{\zeta_\infty}^{-1}}{dt} - {\zeta_\infty}^{-1} I(\zeta, t) \right) + k_\infty \dot{\rho} \frac{P_\rho(\zeta{\zeta_\infty}^{-1})}{P(\zeta{\zeta_\infty}^{-1})} \\ + \sum_{p=1}^{N} \left(\gamma_p \zeta \frac{P'(\zeta\bar{\zeta}_p)}{P(\zeta\bar{\zeta}_p)} \left(\frac{d\bar{\zeta}_p}{dt} - \bar{\zeta}_p I(\zeta, t) \right) + \gamma_p \dot{\rho} \frac{P_\rho(\zeta\bar{\zeta}_p)}{P(\zeta\bar{\zeta}_p)} \right) \Bigg] \bar{z} z_\zeta \, d\zeta \\ + \oint_{\partial C} 2K_\infty G z_\zeta d\zeta \end{aligned} \tag{3.44}$$

Theorem 3.6.4

For the class of initial shapes considered, the $J^j_{k_j}(t)$ satisfies the following nontrivial equation for $k_j = 0, 1, ..\gamma_j - 1$ $(j = 1..N)$:

$$\begin{aligned} \dot{J}^j_{k_j}(t) = \oint_{\partial C_0} K_j \Bigg[& k_j \zeta \frac{P'(\zeta\bar{\zeta}_j)}{P(\zeta\bar{\zeta}_j)} \left(\frac{d\bar{\zeta}_j}{dt} - \bar{\zeta}_j I(\zeta, t) \right) \\ & + k_j \dot{\rho} \frac{P_\rho(\zeta\bar{\zeta}_j)}{P(\zeta\bar{\zeta}_j)} + 3\zeta \frac{P'(\zeta{\zeta_\infty}^{-1})}{P(\zeta{\zeta_\infty}^{-1})} \left[\frac{d}{dt}{\zeta_\infty}^{-1} - {\zeta_\infty}^{-1} I(\zeta, t) \right] + 3\dot{\rho} \frac{P_\rho(\zeta{\zeta_\infty}^{-1})}{P(\zeta{\zeta_\infty}^{-1})} \\ & + \sum_{\substack{p=1 \\ p\neq j}}^{N} \gamma_p \zeta \frac{P'(\zeta\bar{\zeta}_j)}{P(\zeta\bar{\zeta}_j)} \left(\frac{d\bar{\zeta}_p}{dt} - \bar{\zeta}_p I(\zeta, t) \right) + \gamma_p \dot{\rho} \frac{P_\rho(\zeta\bar{\zeta}_p)}{P(\zeta\bar{\zeta}_p)} \Bigg] \bar{z} z_\zeta d\zeta \end{aligned} \tag{3.45}$$

Since the evolution of the N poles $\{\zeta_j | \, j = 1..N\}$ is given by the N equations (3.33) and since the evolution of ζ_∞ is given by (3.34), it only remains to deduce the evolution of the $M_0 + 3$ time-evolving parameters $S(t)$, $\{\eta_m(t) | m = 1..M_0 + 2\}$. Note however that the $\{\eta_m(t)\}$ satisfy the constraint that $\prod_{m=1}^{M_0+2}[-\eta_m(t)] = \frac{\zeta_\infty(t)}{\bar{\zeta}_\infty(t)} \prod_{j=1}^{N}[-\zeta_j(t)]^{\gamma_j}$ Thus there remains precisely $M_0 + 2$ (generally complex) parameters whose time evolution needs to be determined. Note however that there are $M_0 + 3$ non-zero line integral quantities, namely

$$\{J^j_{k_j}(t) | k_j = 0..\gamma_j - 1\}, \; j = 1..N \; ; \; J_0^\infty, \; J_1^\infty, \; J_2^\infty \tag{3.46}$$

the evolution of which are given by (3.44) and (3.45) provided the quantities $\delta_1(t)$, $\delta_2(t)$, $B(t)$ and $m(t)$ are given. This corresponds to specifying the strain field, uniform flow field and source/sink strength at the point at infinity.

The fact that there are only $M_0 + 2$ quantities to be determined from $M_0 + 3$ equations means that, in fact, the problem of finding exact solutions of the form (3.37) while *externally* specifying the strain field, uniform flow and mass flux at infinity cannot in general be achieved. Nevertheless, it **is** possible to find *mathematical* solutions of the form (3.37) where the strain field and uniform flow at infinity are externally specified. In these cases, the mass flux at infinity is governed by the solution itself, and, in this sense the *mathematical* solutions are *physically* somewhat artificial.

Remark 3.6.1

We remark however, that the existence of such mathematical solutions is significant firstly because such solutions are likely to share many of the same behavioural characteristics as the solutions where the mass flux at infinity can be externally specified. The solutions here might therefore be studied to glean information about the general behaviour of two bubbles in an arbitrary strain field. More importantly, recall that it was necessary, in order to find this class of solutions, to impose the condition that $A_1 = A_2$, a specification that involved a loss of generality. It *might* be possible to generalize the mathematical approach presented here to include the case where $A_1 \neq A_2$ and thereby find exact solutions where it is indeed possible to specify not only the strain rate and uniform flow at infinity, but also the mass flux.

In the important case where there is just a source/sink at infinity (and no strain field) it is actually possible to glean some physically realistic exact solutions from the theory presented in this chapter. In this case, there will in general be a uniform flow at infinity and a source/sink of strength $m(t)$. Since it is only the geometrical evolution of the bubble boundaries which is of physical interest, and since the geometrical evolution of the boundaries is completely unaffected by time-dependent uniform translations, it is possible to find exact solutions of the form (3.37) where $m(t)$ can be externally specified. Having specified $m(t)$, the uniform flow at infinity will be determined by the solution itself, however since this will not affect the boundary shapes, this uniform flow at infinity can be subtracted off *a posteriori* (it has no effect on the boundary evolution).

Remark 3.6.2

We also note here that, in the case where there is no strain field present, the solutions need not necessarily have a pole at the point $\bar{\zeta}_\infty^{-1}$ but that solutions can be found of the slightly more general form

$$z(\zeta,t) = S(t)\frac{\prod_{k=1}^{M_0+1} P(\zeta\eta_k^{-1})}{P(\zeta\zeta_\infty^{-1})\prod_{j=1}^{N} P(\zeta\zeta_j^{-1})^{\gamma_j}} \tag{3.47}$$

where

$$\prod_{m=1}^{M_0+1} \eta_m = \zeta_\infty \prod_{j=1}^{N} \zeta_j^{\gamma_j} \tag{3.48}$$

Generalization of the foregoing theory to this alternative class of exact solutions for this case is straightforward.

With regard to the determination of the time-dependent parameters appearing in the conformal map from the time evolving line integrals (3.46), it is necessary to invoke the following conjecture that is believed to be true. The numerical evidence to be presented at the end of this chapter provides some verification of its validity. Since the exact solutions corresponding to two bubbles in a flow with just a source/sink at infinity (and no strain) are the most physically relevant, we therefore state the conjecture in the context of these particular solutions.

Conjecture: For a given set of $\zeta_j(t)$, and given $\zeta_\infty(t)$, then in the case where there is no strain field present and provided the parameter $m(t)$ is externally specified, the $M_0 + 3$ quantities

$$\{J^j_{k_j}(t)|k_j = 0..\gamma_j - 1\},\ j = 1..N\ ;\ J^\infty_0,\ J^\infty_1,\ J^\infty_2 \tag{3.49}$$

at any time t uniquely determine the $M_0 + 3$ parameters $B(t), S(t), \{\eta_m(t)|m = 1..(M_0 + 1)\}$ with η_{M_0+2} being given by the constraint (3.36). (It is emphasized that $B(t)$ only affects the size of an inconsequential uniform flow at infinity and does not affect the evolution of the bubble shapes. Its value is therefore immaterial).

It is noted that the "counting" in the statement of the conjecture is consistent, and the validity of the conjecture is supported by the example calculation carried out later in this chapter, even though this is but a single special case. Before proceeding to an example calculation it is noted that the theorem of invariants also carries over:

3.7 A Theorem of Invariants

Theorem 3.7.1

(Theorem of Invariants) Suppose that the conformal map $z(\zeta, t)$ *has the form (3.37) and that the evolution of the poles is given by (3.33) and (3.34). For each* j *for which the corresponding* $\gamma_j = 1$*, there exists an invariant of the motion given by*

$$B_j = \frac{J_0^j(t)}{P[\bar{\zeta}_j^{-1}\zeta_\infty{}^{-1}]^3 \prod_{\substack{p=1 \\ p\neq j}}^{N} \left[P(\bar{\zeta}_p\bar{\zeta}_j^{-1})\right]^{\gamma_p}} \tag{3.50}$$

3.8 Example Calculation

Reserving a detailed analysis of the behaviour of the new theoretical solutions presented here for a future investigation, it is necessary to include an example to demonstrate the feasibility of the mathematical conjecture just stated. This example also serves as confirmation that loxodromic functions satisfying the necessary conformality and univalency conditions exist.

The initial configuration is given in Figure 3.2. This configuration is given by a mapping of the form

$$z(\zeta, t) = S(t)\frac{P(\zeta\eta_1^{-1})P(\zeta\eta_2^{-1})P(\zeta\eta_3^{-1})}{P(\zeta\zeta_\infty^{-1})P(\zeta\zeta_\infty)P(\zeta\zeta_1^{-1})} \tag{3.51}$$

where all the parameters are real, and have the following initial values:

$$\zeta_\infty = 0.5,\ \zeta_1 = -2.2,\ \eta_1 = -0.85,\ \eta_2 = 2.0,\ \eta_3 = 1.29,\ S = 1.0 \tag{3.52}$$

The evolution of this initial condition in the presence of a general sink at infinity is calculated. It is assumed that there is no strain field at infinity. Since no numerical pathologies were expected and since we do not wish to integrate for particularly long times, an elementary numerical scheme was used. We now briefly describe the method used.

The parameters $\zeta_\infty(t), \zeta_1(t)$ and $\rho(t)$ were time-stepped using an elementary first-order forward Euler, the integrals such as $I(\zeta_\infty, t)$ being calculated using Romberg integration. Similarly, the quantities $J^\infty_{k_\infty},\ k_\infty = 0, 1, 2$ were similarly time-stepped. From the Theorem of Invariants, the presence of the simple pole at ζ_1 implies that there is an invariant of the motion. Once the updated ρ, ζ_∞ and ζ_1 are determined, the four equations given by the invariant of the motion and the updated $J^\infty_{k_\infty},\ k_\infty = 0, 1, 2$ can be used to determine the quantities η_1, η_2, η_3 satisfying (3.36) as well as $S(t)$ and $B(t)$, the value of $m(t)$ having been externally specified. It is noted here that to avoid unnecessary numerical error induced by integrating around C_0, whenever the integrand was a meromorphic function within C_0, the Residue Theorem was employed so that the integral could be computed exactly. It was found that the best way to implement this was to feed a Fortran program with Laurent coefficients imported from Mathematica (which was used to expand the various functions around the appropriate poles). In this way, closed analytical formulae for the integrals were obtained. The configuration after 400 time steps is shown in Figure 3.3. Notice that the presence of a sink at infinity causes the bubbles to move apart, each of them growing bigger, the sharp corners in the initial configuration being caused to smooth out under evolution. This behaviour is much as might intuitively be expected. Note that any overall translation of the entire configuration parallel to the x-axis is immaterial to the evolution of the boundary shapes.

Again the principal difficulty in carrying out a full numerical investigation of the general behaviour of the class of exact solutions found in this chapter is the construction of initial maps satisfying the necessary properties (i.e. loxodromy, conformality, univalence in C_0). Such functions certainly exist, however the present author knows of no systematic method of constructing them.

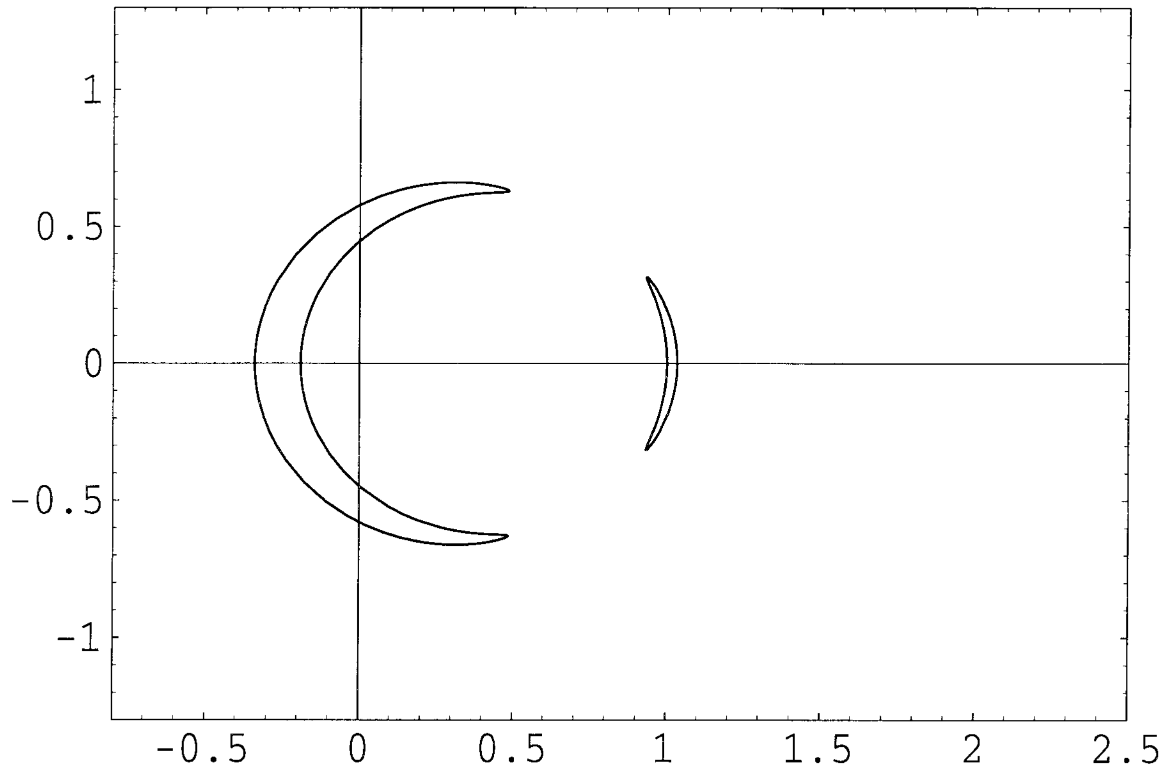

Figure 3.2: Initial two-bubble configuration

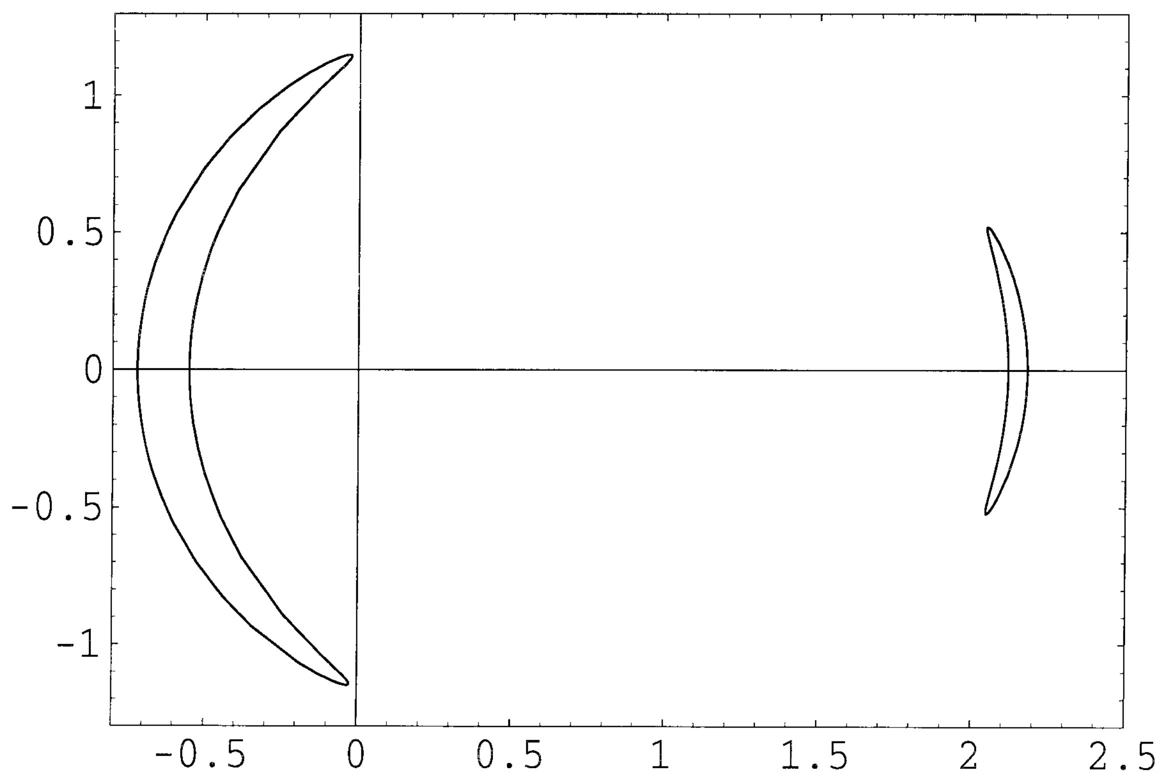

Figure 3.3: Configuration after 400 time steps (h=0.0001)

Chapter 4

Alternative Arguments for the Loxodromic Function Solutions

4.1 Conservation of the Loxodromic Property

It is now demonstrated, by alternative methods to those already employed, that conformal maps that are initially loxodromic functions remain loxodromic functions under Stokes evolution, thereby providing an alternative mathematical justification for the existence of the exact solutions already identified using other methods. In the process, we will demonstrate a more general result concerning *any* conformal map (not necessarily meromorphic) satisfying what shall henceforth be called the *loxodromic property.*

Definition: A function $L(\zeta)$ satisfying the *loxodromic property* is a function which satisfies the functional equation

$$L(\rho^2\zeta) = L(\zeta) \tag{4.1}$$

A *loxodromic function* is understood to be a *meromorphic* function (except at zero and infinity) that also satisfies the *loxodromic property.*

To proceed, first define the following three annular regions:

$C_0 : \{\rho < |\zeta| < 1\}$

$C_1 : \{1 < |\zeta| < \rho^{-1}\}$

$C_2 : \{\rho^2 < |\zeta| < \rho\}$

These regions are illustrated in Figure 4.1. The evolution equation for the conformal

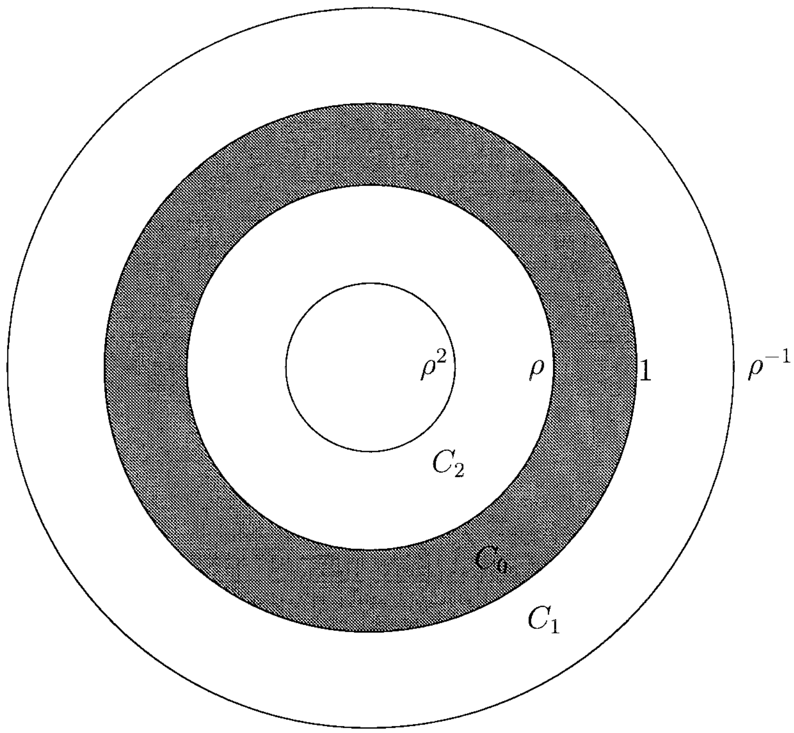

Figure 4.1: Definition of annuli

map $z(\zeta, t)$ in C_0 is known to be given by

$$z_t(\zeta, t) + 2F(\zeta, t) = \zeta I(\zeta, t) z_\zeta(\zeta, t) \tag{4.2}$$

where $I(\zeta, t)$ is defined in chapter 2. By standard methods of contour deformation of the integral function $I(\zeta, t)$ appearing in this equation, the analytic continuation of this equation into C_1 can be deduced to be

$$z_t(\zeta, t) + 2F(\zeta, t) = \zeta I(\zeta, t) z_\zeta(\zeta, t) + \zeta z_\zeta(\zeta, t) \left[\frac{1}{z_\zeta^{1/2}(\zeta, t) \bar{z}_\zeta^{1/2}(\zeta^{-1}, t)} \right] \tag{4.3}$$

Similarly, the analytic continuation of (4.2) into C_2 can be deduced to be

$$z_t(\zeta, t) + 2F(\zeta, t) = \zeta I(\zeta, t) z_\zeta(\zeta, t) + \zeta z_\zeta(\zeta, t) \left[-\frac{1}{\rho z_\zeta^{1/2}(\zeta, t) \bar{z}_\zeta^{1/2}(\rho^2 \zeta^{-1}, t)} - 2\frac{\dot{\rho}}{\rho} \right] \tag{4.4}$$

We first demonstrate the important result that, provided a solution exists, a conformal map that initially satisfies the loxodromic property, continues to satisfy it under evolution. For this purpose it is therefore necessary to *quantify* the loxodromic property of the conformal map in some way. Given the definition of the loxodromic property it is natural to consider the evolution of the following quantity $H(\zeta, t)$ defined by

$$H(\zeta, t) \equiv z(\zeta, t) - z(\rho^2\zeta, t) \tag{4.5}$$

Intuitively, $H(\zeta, t)$ might be considered a measure of the *degree of non-loxodromy* of the conformal map. $H(\zeta, 0) = 0$ for an initial conformal map satisfying the loxodromic property. We shall now proceed to prove the non-trivial result that this function remains zero (under the evolution equations for Stokes flow) if it is zero initially.

Lemma 4.1.1

Define the function $Ker(\zeta, t)$ to be the kernel function of the Dirichlet formula for an annulus (as given in Appendix E) i.e.

$$Ker(\zeta, t) \equiv 1 - 2\zeta\frac{P'(\zeta, t)}{P(\zeta, t)} \tag{4.6}$$

Then the following two identities hold:

$$Ker(\rho^2\zeta, t) = 2 + Ker(\zeta, t) \tag{4.7}$$

and

$$Ker(\zeta^{-1}, t) = -Ker(\zeta, t) \tag{4.8}$$

Proof: First note that

$$\frac{P'(\zeta)}{P(\zeta)} = -\frac{1}{(1-\zeta)} - \sum_{m=1}^{\infty}\frac{\rho^{2m}}{1-\rho^{2m}\zeta} + \sum_{n=1}^{\infty}\frac{\rho^{2n}}{\zeta^2(1-\frac{\rho^{2n}}{\zeta})} \tag{4.9}$$

which means that

$$\rho^2 \frac{P'(\rho^2\zeta)}{P(\rho^2\zeta)} = \frac{1}{\zeta^2(1-\zeta^{-1})} - \sum_{m=1}^{\infty} \frac{\rho^{2m}}{1-\rho^{2m}\zeta} + \sum_{n=1}^{\infty} \frac{\rho^{2n}}{\zeta^2(1-\frac{\rho^{2n}}{\zeta})} \tag{4.10}$$

it is therefore straightforward to show that

$$\rho^2\zeta \frac{P'(\rho^2\zeta)}{P(\rho^2\zeta)} = -1 + \zeta \frac{P'(\zeta)}{P(\zeta)} \tag{4.11}$$

Using this, the first result of the Lemma follows. The second result of the Lemma also follows from similar straightforward manipulations.

Lemma 4.1.2

The function $I(\zeta, t)$, as defined in chapter 2, satisfies the following two equations for ζ strictly inside C_1:

$$I(\zeta^{-1}, t) = -\bar{I}(\zeta, t) \tag{4.12}$$

(note that this equation does **not** *hold either on $|\zeta| = 1$ or $|\zeta| = \rho$)*

$$I(\rho^2\zeta, t) = I(\zeta, t) \tag{4.13}$$

Proof: The proof of the first result follows by simple manipulations of the integral definition of the function $I(\zeta, t)$. In particular, making a change of variable $\eta = \frac{1}{\zeta'}$ in $I^+(\zeta, t)$ and using (4.8) of Lemma 4.1.1 and making a change of variable $\eta = \frac{\rho^2}{\zeta'}$ in $I^-(\zeta, t)$ and using (4.7) of Lemma 4.1.1 will give the required result. The proof of the second result follows from the observation, using (4.7) of Lemma 4.1.1, that

$$\begin{aligned} I(\rho^2\zeta, t) &= \frac{1}{2\pi i} \oint_{|\zeta'|=1} \frac{d\zeta'}{\zeta'} \frac{1}{z_\zeta^{1/2}(\zeta', t)\bar{z}_\zeta^{1/2}(\zeta'^{-1}, t)} \\ &- \frac{1}{2\pi i} \oint_{|\zeta'|=\rho} \frac{d\zeta'}{\zeta'} \left(-\frac{1}{\rho z_\zeta^{1/2}(\zeta', t)\bar{z}_\zeta^{1/2}(\rho^2\zeta'^{-1}, t)} - \frac{2\dot{\rho}}{\rho} \right) + I(\zeta, t) \end{aligned} \tag{4.14}$$

However, by the single-valuedness condition (equivalently, the value of $\dot{\rho}$) the first two line integrals in equation (4.14) both cancel out. This then provides the result (4.13).

Theorem 4.1.1

The function $H(\zeta, t)$ satisfies the following partial differential equation for ζ in the annulus C_1, namely

$$H_t(\zeta,t) - q_1(\zeta,t)H_\zeta(\zeta,t) - q_3(\zeta,t)H(\zeta,t) = 0 \tag{4.15}$$

(provided the two constants of integration arising in the integrated stress conditions are both taken to be zero) where

$$q_1(\zeta,t) = \zeta I(\zeta,t) \tag{4.16}$$

$$q_3(\zeta,t) = 2\frac{\bar{F}_\zeta(\zeta^{-1},t)}{\bar{z}_\zeta(\zeta^{-1},t)} \tag{4.17}$$

Proof: For ζ in C_1 it is known that $z(\zeta, t)$ satisfies

$$z_t(\zeta,t) + 2F(\zeta,t) = \zeta I(\zeta,t)z_\zeta(\zeta,t) + \zeta z_\zeta(\zeta,t)\left[\frac{1}{z_\zeta^{1/2}(\zeta,t)\bar{z}_\zeta^{1/2}(\zeta^{-1},t)}\right] \tag{4.18}$$

If ζ lies in C_1, then $\rho^2\zeta$ lies in C_2 which implies that for ζ in C_1,

$$z_t(\rho^2\zeta,t) + 2F(\rho^2\zeta,t) = \rho^2\zeta I(\rho^2\zeta,t)z_\zeta(\rho^2\zeta,t) - \frac{\rho\zeta z_\zeta^{1/2}(\rho^2\zeta,t)}{\bar{z}_\zeta^{1/2}(\zeta^{-1},t)} - 2\dot{\rho}\rho\zeta z_\zeta(\rho^2\zeta,t) \tag{4.19}$$

This equation is obtained by making the substitution $\zeta \mapsto \rho^2\zeta$ in (4.4). Thus it follows that for ζ in C_1,

$$\begin{aligned}\frac{\partial H(\zeta,t)}{\partial t} &= z_t(\zeta,t) - z_t(\rho^2\zeta,t) - 2\rho\dot{\rho}\zeta z_\zeta(\rho^2\zeta,t)\\ &= -2F(\zeta,t) + \zeta I(\zeta,t)z_\zeta(\zeta,t) + \frac{\zeta z_\zeta^{1/2}(\zeta,t)}{\bar{z}_\zeta^{1/2}(\zeta^{-1},t)}\\ &\quad + 2F(\rho^2\zeta,t) - \rho^2\zeta I(\rho^2\zeta,t)z_\zeta(\rho^2\zeta,t) + \frac{\rho\zeta z_\zeta^{1/2}(\rho^2\zeta,t)}{\bar{z}_\zeta^{1/2}(\zeta^{-1},t)}\end{aligned} \tag{4.20}$$

Now consider the stress conditions on the two boundaries. On $|\zeta| = 1$,

$$-2F(\zeta,t) + \frac{\zeta z_\zeta^{1/2}(\zeta,t)}{\bar{z}_\zeta^{1/2}(\zeta^{-1},t)} = 2\bar{G}(\zeta^{-1},t) + 2\frac{\bar{F}_\zeta(\zeta^{-1},t)}{\bar{z}_\zeta(\zeta^{-1},t)}z(\zeta,t) \tag{4.21}$$

while on $|\zeta| = \rho$

$$\begin{aligned}-2F(\zeta,t) &- \frac{\zeta z_\zeta^{1/2}(\zeta,t)}{\rho\bar{z}_\zeta^{1/2}(\rho^2\zeta^{-1},t)}\\ &= 2\bar{G}(\rho^2\zeta^{-1},t) + 2\frac{\bar{F}_\zeta(\rho^2\zeta^{-1},t)}{\bar{z}_\zeta(\rho^2\zeta^{-1},t)}z(\zeta,t)\end{aligned} \tag{4.22}$$

By the principle of analytic continuation, these two expressions also hold everywhere off the two boundaries of the annulus. In particular, substituting $\zeta \mapsto \rho^2\zeta$ in (4.22) implies

$$\begin{aligned}-2F(\rho^2\zeta,t) &- \frac{\rho\zeta z_\zeta^{1/2}(\rho^2\zeta,t)}{\bar{z}_\zeta^{1/2}(\zeta^{-1},t)}\\ &= 2\bar{G}(\zeta^{-1},t) + \frac{2\bar{F}_\zeta(\zeta^{-1},t)}{\bar{z}_\zeta(\zeta^{-1},t)}z(\rho^2\zeta,t)\end{aligned} \tag{4.23}$$

Subtracting (4.23) from (4.21) yields the expression

$$\begin{aligned}-2F(\zeta,t) + 2F(\rho^2\zeta,t) &+ \frac{\zeta z_\zeta^{1/2}(\zeta,t)}{\bar{z}_\zeta{}^{1/2}(\zeta^{-1},t)} + \frac{\rho\zeta z_\zeta^{1/2}(\rho^2\zeta,t)}{\bar{z}_\zeta^{1/2}(\zeta^{-1},t)}\\ &= 2\frac{\bar{F}_\zeta(\zeta^{-1},t)}{\bar{z}_\zeta(\zeta^{-1},t)}\Big(z(\zeta,t) - z(\rho^2\zeta,t)\Big)\end{aligned} \tag{4.24}$$

Note that the left hand side of (4.24) is exactly the expression appearing in (4.20) so that, making the substitution, (4.20) becomes

$$\frac{\partial H(\zeta,t)}{\partial t} = 2\frac{\bar{F}_\zeta(\zeta^{-1},t)}{\bar{z}_\zeta(\zeta^{-1},t)}\left(z(\zeta,t) - z(\rho^2\zeta,t)\right) + \zeta I(\zeta,t)z_\zeta(\zeta,t) - \rho^2\zeta I(\rho^2\zeta,t)z_\zeta(\rho^2\zeta,t) \tag{4.25}$$

Finally, equation (4.13) of Lemma 4.1.1 then implies that (4.25) becomes

$$\frac{\partial H(\zeta,t)}{\partial t} = 2\frac{\bar{F}_\zeta(\zeta^{-1},t)}{\bar{z}_\zeta(\zeta^{-1},t)}\left(z(\zeta,t) - z(\rho^2\zeta,t)\right) + \zeta I(\zeta,t)\left(z_\zeta(\zeta,t) - \rho^2 z_\zeta(\rho^2\zeta,t)\right) \tag{4.26}$$

equivalently,

$$H_t(\zeta,t) - \zeta I(\zeta,t)H_\zeta(\zeta,t) - 2\frac{\bar{F}_\zeta(\zeta^{-1},t)}{\bar{z}_\zeta(\zeta^{-1},t)}H(\zeta,t) = 0 \tag{4.27}$$

which is the required result.

Corollary 4.1.1

If $H(\zeta,0) = 0$ then $H(\zeta,t) = 0$ for all subsequent times that the solution exists (i.e. the solution retains the loxodromic property under evolution).

Proof: It is important to note that both coefficients $q_1(\zeta,t)$ and $q_3(\zeta,t)$ in (4.15) are analytic functions of ζ in C_1. The result that we are trying to prove, namely that if $H(\zeta,t)$ is initially zero it remains zero under Stokes flow evolution, now follows from the well-known theory of first-order linear partial differential equations with coefficients that are analytic in some domain. Since $H(\zeta,0) = 0$ for ζ in C_1 for an initially loxodromic conformal map, the **unique** solution for $H(\zeta,t)$ as it evolves under Stokes flow is therefore $H(\zeta,t) = 0$. Thus we have proved that a conformal map that initially satisfies the loxodromic property continues to satisfy the loxodromic property under evolution. Note that we have made **no assumption** in the above proofs about whether the mapping function is meromorphic or not.

4.2 Conservation of Singularity Structure

We now demonstrate the separate result that certain initial singularity distributions of the conformal map in C_1 are preserved under evolution. We restrict attention to the class of initial conformal maps having the general form

$$z(\zeta,0) = \sum_{j=1}^{N} \frac{E_j(\zeta,0)}{(\zeta - \zeta_j(0))^{\gamma_j}} + M(\zeta,0) \tag{4.28}$$

where $M(\zeta,0)$ and $E_j(\zeta,0)$, $j = 1..N$ are analytic in C_1 and $\{\gamma_j |\ j = 1..N\ \}$ are any non-zero real numbers. The poles positions, $\zeta_j(0)$, $j = 1..N$, are taken to be strictly inside C_1. (We are restricting consideration to the annular blob scenario. Extension of the present analysis to the two-bubble problem is straightforward). Note that the map is also known to be analytic everywhere in C_0. The following demonstration is modelled on arguments given by Tanveer and Vasconcelos [8] in their treatment of the related problem of a single bubble in an arbitrary straining flow.

Theorem 4.2.1

Given an initial conformal map of the form (4.28), and satisfying the conditions just stated, such a map retains the same singularity structure within C_1 for all times that the solution exists provided the singularities at $\zeta_j(t)$, $j = 1..N$ evolve according to

$$\frac{d}{dt}\bar{\zeta}_j^{-1} = -\bar{\zeta}_j^{-1} I(\bar{\zeta}_j^{-1}, t) \tag{4.29}$$

Proof: It has already been deduced that the analytic continuation of equation (4.2) into C_1 is given by (4.3). Substituting the decomposition (4.28) into (4.3) it is clear that $z(\zeta, t)$ retains the form

$$z(\zeta,t) = \sum_{j=1}^{N} \frac{E_j(\zeta,t)}{(\zeta - \zeta_j(t))^{\gamma_j}} + M(\zeta,t) \tag{4.30}$$

for all times that the solution exists where $M(\zeta,t)$ and $E_j(\zeta,t)$ are analytic in C_1 provided the following equations are satisfied

$$\dot{\zeta}_j = -q_1(\zeta_j,t) \tag{4.31}$$

$$M_t = q_1 M_\zeta + q_3 M + q_2 \tag{4.32}$$

$$E_{jt} = q_1 E_{j\zeta} + q_3 E_j + \frac{\gamma_j E_j}{(\zeta-\zeta_j)}\{q_1(\zeta,t) - q_1(\zeta_j,t)\} \tag{4.33}$$

where we define

$$q_2(\zeta,t) = 2\bar{G}(\zeta^{-1},t) \tag{4.34}$$

The crucial point to note is that the coefficient functions in the first order partial differential equations (4.32) and (4.33) are known *a priori* to be analytic in C_1. This therefore implies that if $M(\zeta,0)$ and $E_j(\zeta,0)$ are initially analytic in C_1, then $M(\zeta,t)$ and $E_j(\zeta,t)$ will remain analytic in C_1 for all times that the solution exists. This therefore implies that $z(\zeta,t)$ will retain the singularity structure given in (4.28) (provided, of course, that none of the singularities move out of C_1. Note however that, if this was to happen, the exact solutions would be deemed invalid thereafter anyway). Finally, using the result (4.12) of Lemma 4.1.2, it is a simple matter to show that equation (4.31) is equivalent to equation (4.29) (this latter equation being the evolution equation of the poles of the mapping as determined in chapter 2).

4.3 Loxodromic Function Solutions

Combining the above two important results, namely that a conformal map that initially satisfies the loxodromic property continues to satisfy the loxodromic property under Stokes evolution and the fact that the singularity structure of the above class of initial conformal maps in the union of the annuli C_0 and C_1 is conserved under evolution (recall that for an annular blob, the conformal map is necessarily analytic everywhere in C_0) then provides the result that, provided a solution exists, a conformal map that is initially a loxodromic function remains a loxodromic function under evolution. This is because the two properties that define a loxodromic function i.e. a function that is both *meromorphic* and satisfies the *loxodromic property* have been shown to be conserved under evolution. Note that, by the known loxodromic property of the solution and the known singularity structure in C_0 and C_1 for all times, it is well-known that this is enough to define the loxodromic function everywhere in the finite ζ-plane excluding the origin, i.e. the union of C_0 and C_1 represents a *fundamental annulus* for the loxodromic function. Thus the behaviour of the function $z(\zeta, t)$ in every other annulus is also known.

The above analysis provides an instructive alternative mathematical argument for the existence of the exact solutions detailed in previous chapters, although it still remains to show that the resulting finite system to which the problem has been reduced has a solution. This particular question is more clearly studied using the integral approach developed in previous chapters. We also remark that the two results just demonstrated can also be used to argue the existence of exact mathematical solutions for 4-bubbles with symmetry. This problem is treated in chapter 5.

An interesting question which arises from this section is whether *non-meromorphic* functions which satisfy the loxodromic property might represent exact solutions of the problem. For example, it is conceivable that a single-valued function possessing an *infinite* number of zeros and poles in the fundamental annulus (thus implying an essential singularity of the function in C_0) might constitute an exact solution. Intuitively, for the evolution of the infinity of poles or zeros to be describable in a finite form it would seem to be necessary that the poles and zeros behave in a way describable by some finite number of time-evolving quantities, with a corresponding set of conserved quantities. Perhaps the conservation of loxodromy might provide the underlying mathematical structure necessary for this to be so.

Chapter 5

Symmetric 4-Bubble Solutions

5.1 Conformal Mapping with Symmetry

Another natural question to ask, now that exact solutions have been identified in both the singly and doubly-connected scenarios, is whether exact solutions exist for fluid regions of connectivity greater than two.

A general answer to this question is not known, at least by the present author, at the time of writing. However, it is possible to demonstrate, at least in principle, the existence of a class of mathematical solutions to the Stokes equations corresponding to a 4-bubble scenario where a certain degree of symmetry is assumed. Physically, these mathematical solutions turn out to be somewhat artificial in that physical parameters which one would expect to be externally specifiable, are in fact determined by the exact solution itself. Nevertheless, it is important to document that such mathematical solutions exist, since they might point the way to a more physically relevant generalization. Their existence certainly does nothing to hinder a conviction that certain classes of exact solutions exist for fluid regions of connectivity greater than two.

Consider a conformal map from the upper-half annulus between ρ and 1 that maps to the region exterior to two half-bubbles centered on the real and imaginary axes in the first quadrant of the physical plane (see Figures 5.1 and 5.2). In the full physical domain, it is assumed that each bubble is mirror-symmetric about the axis on which it is located. It is also assumed that the entire geometric configuration is mirror-symmetric about both the x- and y-axes. See Figure 5.1. This symmetry assumption necessarily implies that the origin is a stagnation point of the flow. In addition, the velocity of the fluid *normal* to the real and imaginary axes (exterior to the bubbles)

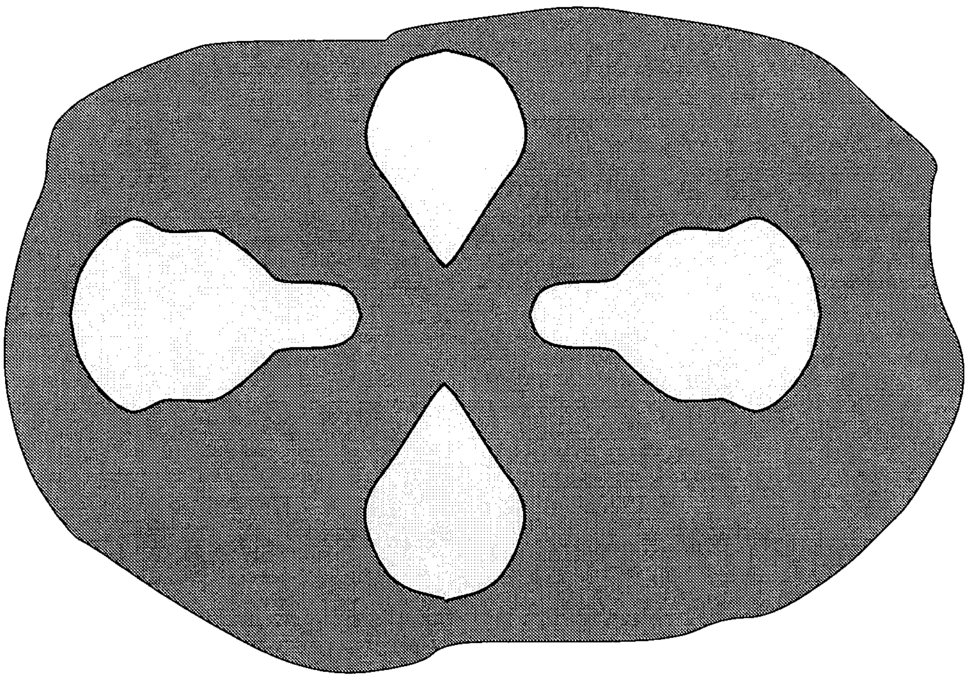

Figure 5.1: Four symmetric bubbles

is necessarily zero at any point on these axes. It is also necessary that the boundary conditions at infinity are such as to respect the flow symmetries. Thus, we assume a linear strain flow at infinity which is symmetric with respect to the four-quadrants of the physical plane. It must also be assumed that there is no uniform flow field at infinity. The velocity $\mathbf{u}$ therefore has the form

$$\mathbf{u} = (\delta_1(t)x,\ -\delta_1(t)y) + \frac{m(t)}{2\pi}\frac{\hat{\mathbf{x}}}{|\mathbf{x}|} \text{ as } |x| \to \infty \tag{5.1}$$

where $\delta_1(t)$, $m(t)$ are real time-dependent parameters.

Remark 5.1.1

It is noted that it is also possible to find another class of 4-bubble solutions by assuming a greater degree of symmetry. In this case, all four bubbles are assumed to be geometrically identical so that each bubble is a rotation about the physical origin through some integer multiple of $\frac{\pi}{2}$ of any other bubble. While the existence of this class of solutions is equally important, we choose to present the theory for the

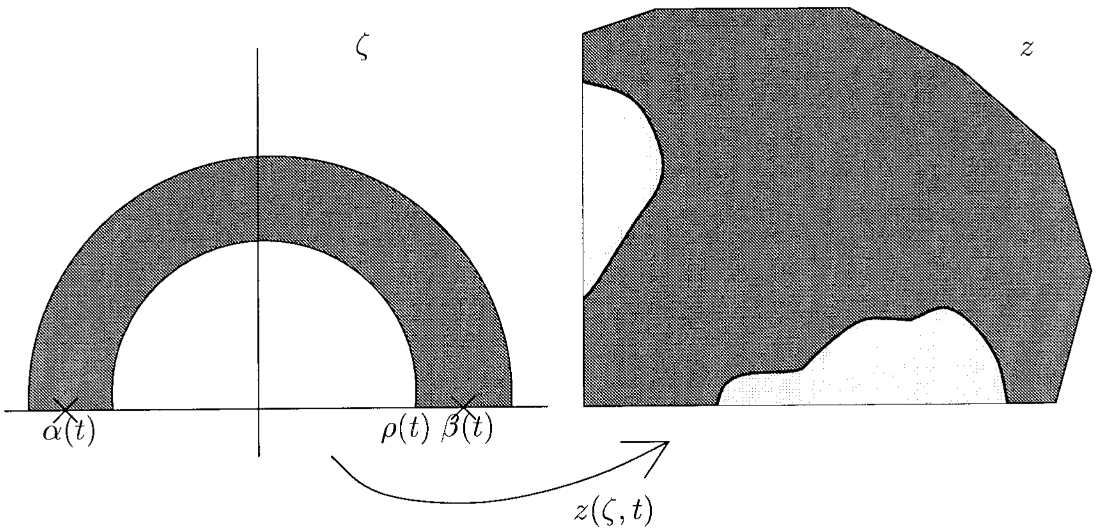

Figure 5.2: Conformal mapping domains

more general class where a lesser degree of symmetry is assumed – such solutions will presumably contain the more symmetric solutions as a subclass.

It is straightforward to show that the conformal map must take the following general form for all times

$$z(\zeta, t) = i\frac{(\zeta - \alpha(t))^{1/2}}{(\zeta - \beta(t))^{1/2}}\hat{h}(\zeta, t) \tag{5.2}$$

where the real point $\alpha(t)$ maps to the physical origin while the real point $\beta(t)$ maps to infinity. See Figure 5.2. The upper half of the circle $|\zeta| = 1$ is taken to map to bubble 1 centred on the imaginary axis and the upper half of circle $|\zeta| = \rho$ to bubble 2 centered on the real axis. To respect the assumed symmetries, the function $\hat{h}(\zeta, t)$ must be taken to be real on the real axis and analytic in the upper half of C_0 (with C_0 defined as in the previous chapter). By the Schwarz reflection principle, we immediately deduce that $\hat{h}(\zeta, t)$ is analytic everywhere in C_0. It will also be assumed in what follows that a suitable $\hat{h}(\zeta, 0)$ can be found such that the mapping is conformal and univalent in the upper half annulus.

Since $f(z)$ and $g'(z)$ are analytic functions of z at $z = 0$ and since the velocity field at $z = 0$ must vanish, it is necessary that

$$f(z) \sim F_0 z + O(z^2) \tag{5.3}$$

$$g'(z) \sim G_0 z + O(z^2) \tag{5.4}$$

as $z \to 0$ for some F_0, G_0. This can be seen from the expression (1.10) for the velocity field in terms of $f(z)$ and $g'(z)$. In addition, it is known by the boundary conditions at infinity that

$$f(z) \to F_\infty z + O(1) \tag{5.5}$$

$$g'(z) \to G_\infty z + O(1) \tag{5.6}$$

as $z \to \infty$ for some F_∞, G_∞. Thus, from these local analyses, it is clearly seen that $F(\zeta, t)$ and $G(\zeta, t)$ must also have decompositions of the form

$$F(\zeta, t) = i\frac{(\zeta - \alpha(t))^{1/2}}{(\zeta - \beta(t))^{1/2}}\hat{F}(\zeta, t) \tag{5.7}$$

$$G(\zeta, t) = i\frac{(\zeta - \alpha(t))^{1/2}}{(\zeta - \beta(t))^{1/2}}\hat{G}(\zeta, t) \tag{5.8}$$

where $\hat{F}(\zeta, t)$ and $\hat{G}(\zeta, t)$ must be analytic everywhere in the upper-half of C_0. From the expression (1.10) for the velocity field in terms of $f(z)$ and $g'(z)$, it can also be deduced that $\hat{F}(\zeta, t)$ and $\hat{G}(\zeta, t)$ must be *real* on the real axis. This will ensure that the velocity of the fluid anywhere on the real or imaginary axes (exterior to the bubbles) is *tangential* to the respective axes. This is a necessary condition due to the symmetry constraints. By the Schwarz reflection principle we then immediately deduce that $\hat{F}$ and $\hat{G}$ are also analytic everywhere in C_0.

The kinematic boundary conditions on the upper-arcs $\zeta = 1,\ \rho$ can be written

$$Re\left[\frac{\hat{h}_t(\zeta,t) - \left(\frac{\dot{\alpha}(t)}{2(\zeta-\alpha)} - \frac{\dot{\beta}(t)}{2(\zeta-\beta)}\right)\hat{h}(\zeta,t) + 2\hat{F}(\zeta,t)}{\zeta\left(\hat{h}_\zeta(\zeta,t) + \left(\frac{1}{2(\zeta-\alpha)} - \frac{1}{2(\zeta-\beta)}\right)\hat{h}(\zeta,t)\right)}\right] = \frac{1}{2z_\zeta^{1/2}(\zeta,t)\bar{z}_\zeta^{1/2}(\bar{\zeta},t)} \tag{5.9}$$

$$Re\left[\frac{\hat{h}_t(\zeta,t) - \left(\frac{\dot{\alpha}(t)}{2(\zeta-\alpha)} - \frac{\dot{\beta}(t)}{2(\zeta-\beta)}\right)\hat{h}(\zeta,t) + 2\hat{F}(\zeta,t)}{\zeta\left(\hat{h}_\zeta(\zeta,t) + \left(\frac{1}{2(\zeta-\alpha)} - \frac{1}{2(\zeta-\beta)}\right)\hat{h}(\zeta,t)\right)}\right] = -\frac{1}{2\rho z_\zeta^{1/2}(\zeta,t)\bar{z}_\zeta^{1/2}(\bar{\zeta},t)} \tag{5.10}$$

The function in square brackets on the left hand sides of (5.9) and (5.10) can be seen to be analytic everywhere in the upper-half annulus $\rho < |\zeta| < 1$ (including at $\zeta = \alpha$ and β where the square-root branch points of the numerator and denominator cancel out) and also real on the real axis for $\rho < |\zeta| < 1$. By the Schwarz reflection principle, the function in square brackets on the left hand side of (5.9) and (5.10) must therefore be analytic everywhere in C_0 and also satisfy (5.9) on the lower-arc $|\zeta| = 1$ and satisfy (5.10) on the lower-arc $|\zeta| = \rho$. By the Villat formula, we can therefore write

$$\frac{\hat{h}_t(\zeta,t) - \left(\frac{\dot{\alpha}(t)}{2(\zeta-\alpha)} - \frac{\dot{\beta}(t)}{2(\zeta-\beta)}\right)\hat{h}(\zeta,t) + 2\hat{F}(\zeta,t)}{\zeta\left(\hat{h}_\zeta(\zeta,t) + \left(\frac{1}{2(\zeta-\alpha)} - \frac{1}{2(\zeta-\beta)}\right)\hat{h}(\zeta,t)\right)} = I(\zeta,t) \tag{5.11}$$

which is valid everywhere inside and on C_0. In this equation, the function $I(\zeta,t)$ is the function defined in chapter 2. (We remark that, due to the symmetries in this case, there is no choice but to have $C_2(t) = 0$). Given the decomposition (5.2), (5.11) is equivalent to saying that the equation

$$z_t + 2F(\zeta,t) = \zeta I(\zeta,t) z_\zeta(\zeta,t) \tag{5.12}$$

holds everywhere in the annulus C_0. In addition, the stress condition on the upper-arc

$|\zeta| = 1$ is known to be given by

$$2\bar{G}(\bar{\zeta}, t) = -2F(\zeta, t) - 2z(\zeta, t)\frac{\bar{F}_\zeta(\bar{\zeta}, t)}{\bar{z}_\zeta(\bar{\zeta}, t)} + \frac{\zeta z_\zeta^{1/2}(\zeta, t)}{\bar{z}_\zeta^{1/2}(\bar{\zeta}, t)} \tag{5.13}$$

(where the constant of integration has been taken to be zero) thus, by analytic continuation this also holds everywhere else (including on the lower arc $|\zeta| = 1$). The same can be shown for the stress condition on $|\zeta| = \rho$ (again with this second constant of integration taken to be zero – resulting in a similar loss of generality of the solutions as was the case in previous chapters). We also remark that the evolution equation for the conformal modulus $\rho(t)$ is exactly as given in chapter 2.

The result of these arguments is that we have shown that, for the class of symmetric 4-bubble solutions under consideration, the equations for the relevant conformal map are the *same* equations, valid in the *same* regions of the parametric ζ-plane, as those that were treated in chapter 3 for the general two-bubble problem.

It is now possible to immediately import the general results established in chapter 4 for conformal maps satisfying the above equations, namely

(i) a conformal map that initially satisfies the loxodromic property continues to satisfy it under the evolution equations derived in chapter 3

(ii) certain classes of initial singularities of the conformal map in C_1 are preserved under evolution, with no spontaneous production of additional singularities.

5.2 Form of the Exact Solutions

Now we consider an initial map of the form

$$z(\zeta, 0) = \left[\frac{P(\zeta\alpha(0)^{-1})P(\zeta\alpha(0))}{P(\zeta\beta(0)^{-1})P(\zeta\beta(0))}\right]^{1/2} h(\zeta, 0) \tag{5.14}$$

where $h(\zeta, 0)$ is taken to be a loxodromic function with real poles and zeros. Note that $z(\zeta, 0)$ satisfies the loxodromic property even though it is **not** a loxodromic function of ζ (it is not meromorphic). Note also that $z(\zeta, 0)$ also has the required local (square root) behaviour at $\zeta = \alpha, \beta$ that is required from earlier considerations. As mentioned earlier, it is assumed that it is possible to find a suitable $h(\zeta, 0)$ such that the map has the necessary univalency and conformality properties in C_0.

The important observation is that an initial condition of the form (5.14) falls within the special class of initial conditions mentioned in (i) and (ii) above and so, combining results (i) and (ii), it is straightforward to show that, given that a solution exists, the *only possible way* for $z(\zeta, 0)$ to evolve in time is for it to remain of the form

$$z(\zeta, t) = \left[\frac{P(\zeta\alpha(t)^{-1})P(\zeta\alpha(t))}{P(\zeta\beta(t)^{-1})P(\zeta\beta(t))}\right]^{1/2} h(\zeta, t) \tag{5.15}$$

where $h(\zeta, t)$ remains a loxodromic function of the same order as $h(\zeta, 0)$. This is the unique solution (if it exists).

Remark 5.2.1

This deduction implicitly utilizes the fact (established from the general considerations in chapter 4) that the singularities in C_1 at points $\alpha(t)^{-1}$ and $\beta(t)^{-1}$ must evolve according to the following equations

$$\frac{d}{dt}\alpha(t)^{-1} = -\alpha^{-1} I(\alpha^{-1}, t) \tag{5.16}$$

$$\frac{d}{dt}\beta(t)^{-1} = -\beta^{-1} I(\beta^{-1}, t) \tag{5.17}$$

(this can be deduced from (4.31)) while it must also be the case that $\alpha(t)$ and $\beta(t)$ satisfy the equations

$$\frac{d}{dt}\alpha(t) = -\alpha I(\alpha, t) \tag{5.18}$$

$$\frac{d}{dt}\beta(t) = -\beta I(\beta, t) \tag{5.19}$$

(this can be deduced from equation (5.12)). Due to the result of Lemma 4.1.2, the two (seemingly different) evolution equations for $\alpha(t)$ and $\beta(t)$ are in fact identical and there is no inconsistency.

Thus provided the finite set of evolution equations for the finite set of parameters in $h(\zeta, t)$ can be found, (5.15) constitutes an **exact solution** for the symmetric 4-bubble problem.

We also observe that the natural domain of the function given in (5.15) is a highly complicated two-sheeted Riemann ζ-surface. One might visualize the two Riemann surfaces as being "boot-laced" together at an infinity of different points. The loxodromic property means that consideration can be confined to a *fundamental* region and on the fundamental annulus there will be two Riemann sheets associated with the four square-root branch points at $\alpha, \beta, \alpha^{-1}$ and β^{-1}. There will also be four equivalent branch points in each of the countable infinity of equivalent annuli. However, moving from one of these sheets to another involves no more than a flipping of the sign of $z(\zeta, t)$ (and also $F(\zeta, t)$ and $G(\zeta, t)$). This is clear from the decompositions of these functions as detailed above. Such a flipping of sign in z, F and G can be shown to have no effect on the equations (derived from the kinematic and stress boundary conditions) relating the functions z, F and G and their derivatives on each sheet of the ζ-surface. Therefore it is enough to solve the problem on just one sheet of the Riemann ζ-surface.

In summary, the assumed symmetry of the problem combined with the non-trivial fact that the loxodromic property of a mapping function can be shown to hold for all times if it holds initially, both conspire to allow reduction of the problem to just one sheet of a highly complicated two-sheeted Riemann surface. It is remarkable that the equations admit such a reduction.

Having justified, in principle, the existence of mathematical solutions of the form given in (5.15), it remains to investigate whether the equations for the evolution of the finite number of parameters appearing in (5.15) can indeed be solved to yield solutions. If not, exact solutions of the deduced form simply do not exist. The considerations above reduce this investigation to a "counting" problem. Further, if the counting problem is consistent, it remains to verify solvability of the resulting finite nonlinear system. Note that the purely *local* considerations of this (and the previous) chapter reveal no information about conserved quantities. Nor do they reveal, in a clear fashion, whether the counting problem to which the problem is reduced is solvable, even in principle. The easiest way to proceed with the question of consistency in the "counting" is to attempt an extension of the more global considerations expounded in previous chapters and to investigate whether the line integral approach can be extended to incorporate this 4-symmetric bubble scenario.

5.3 Line Integral Formulation

There are two important reasons for investigating whether the line integral approach can be extended to the present four bubble case: first, from a mathematical viewpoint, the line integral approach seems to display the underlying structure of the equations in a very transparent way (e.g. the evolution equations for the line integral quantities had an attractive "triangular" shape); second, it is of interest to see whether the line integral approach is "robust" i.e. that it is general enough to incorporate this new class of exact solutions.

We first review the general methodology of the line integral approach. In essence, the approach involves defining a countable infinity of "moments" (in a generalized sense) of the geometrical quantity $\bar{z}(\bar{\zeta}, t)z_\zeta(\zeta, t)$ about each and all of its initial singularities in the region corresponding to the fluid domain. In the cases studied so far, it was then shown that all but a finite set of the moments are conserved under the equations of Stokes evolution. This was true principally because the **only** initial singularities of the integrand in the region corresponding to the fluid domain (in the cases considered so far) was a finite distribution of multipoles, and these could be "cancelled off", one by one, by appropriate choices of the kernel function $K(\zeta, t)$, analytic in the fluid region. The finite set of non-trivially evolving "moments" then (implicitly) gave the evolution of the parameters in the conformal map.

One immediate concern in generalizing this approach to this case is the appearance of the square root branch points of z, F and G at points $\alpha(t)$ and $\beta(t)$ inside C_0. At first glance, it appears that it will be necessary to integrate around cuts within C_0 thereby destroying the structure on which the success of the line integral approach seems to depend. However, more careful inspection reveals that this is not necessary for the decomposition as given in (5.15) because the integrands of the line integral quantities (as well as the integrands of all the integrals appearing in the time evolution equations) are such that all square root singularities conveniently cancel out yielding integrands that have (at worst) multipole singularities inside C_0 and no branch points.

Indeed, it was precisely the desire to have the possible branch points of the integrand (inside C_0) disappear that led to the choice of the square root prefactor given in (5.15). Indeed, this is the **only** choice of prefactor that had the three necessary properties, namely:

1. the necessary square root behaviors at $\zeta = \alpha$ and β in C_0 (and **nowhere else** inside C_0)

2. the loxodromic property

3. invariance to the transformation $\zeta \mapsto \zeta^{-1}$ so that the branch points of the integrands in the line integrals disappear.

It is a remarkable fact that the **quadratic** nature (in z) of the integrand of the line integrals facilitates the cancelling out of the possible square root branch points, resulting in an integrand that is purely meromorphic in C_0 with no branch point singularities.

5.4 Evolution Equations

It still remains to find evolution equations for the zeros and poles of $h(\zeta, t)$. Assume that $h(\zeta, t)$ has the following form:

$$h(\zeta, t) = R(t) \frac{\prod_{k=1}^{M_0} P(\zeta \eta_k^{-1})}{\prod_{j=1}^{j=N} P(\zeta \zeta_j^{-1})^{\gamma_j}} \tag{5.20}$$

where γ_j is the order of the pole at the point ζ_j taken to be in the annulus C_1. Note that since $h(\zeta, t)$ must be real on the real axis, all parameters appearing in (5.20) must be real.

We shall now state (without proof) the modifications to this case of the line integral approach expounded earlier in this thesis. The initial singularities inside C_0 of $\bar{z}z_\zeta$ (for the initial form (5.14)) are clearly at $\zeta = \alpha(0),\ \beta(0),\ \zeta_j^{-1}(0)\ (j = 1..N)$. Therefore, we deduce from the line integral approach that provided

$$\dot{\alpha} = -\alpha I(\alpha, t),\ \dot{\beta} = -\beta I(\beta, t) \tag{5.21}$$

and provided the poles of $h(\zeta, t)$ evolve according to

$$\frac{d}{dt}\zeta_j^{-1} = -\zeta_j^{-1} I(\zeta_j^{-1}, t)\ j = 1..N \tag{5.22}$$

then, defining the following line integral quantities

$$J_{k_\infty}^{\infty}(t) = \oint_{C_0} K_\infty(\zeta, t; k_\infty)\bar{z}(\bar{\zeta}, t) z_\zeta(\zeta, t) d\zeta \tag{5.23}$$

where

$$K_\infty(\zeta, t; k_\infty) = P(\zeta\beta)^{k_\infty} \prod_{j=1}^{N} P(\zeta\zeta_j)^{\gamma_j},\ k_\infty = 0, 1, \ldots. \tag{5.24}$$

and

$$J_{k_0}^{0}(t) = \oint_{C_0} K_0(\zeta, t; k_0)\bar{z}(\bar{\zeta}, t) z_\zeta(\zeta, t) d\zeta \tag{5.25}$$

where

$$K_0(\zeta, t; k_0) = P(\zeta\alpha)^{k_0} P(\zeta\beta)^2 \prod_{j=1}^{N} P(\zeta\zeta_j)^{\gamma_j},\ k_0 = 0, 1, \ldots. \tag{5.26}$$

and finally,

$$J^j_{k_j}(t) = \oint_{C_0} K_j(\zeta, t; k_j)\bar{z}(\bar{\zeta}, t) z_\zeta(\zeta, t) d\zeta \tag{5.27}$$

where

$$K_j(\zeta, t; k_j) = P(\zeta\beta)^2 P(\zeta\zeta_j)^{k_j} \prod_{\substack{p=1 \\ p\neq j}}^{N} P(\zeta\zeta_p)^{\gamma_p}, \; k_j = 0, 1, ... \tag{5.28}$$

the only time-evolving line integrals are

$$J^\infty_{k_\infty}(t), \; k_\infty = 0, 1 \tag{5.29}$$

and

$$J^j_{k_j}(t), \; j = 0..\gamma_j - 1 \text{ for } j = 1..N \tag{5.30}$$

all other line integrals being initially zero and, more importantly, remaining zero under evolution. Note that $J^0_{k_0}(t) = 0$ for all $k_0 = 0, 1, ...$ provided $\alpha(t)$ satisfies (5.21).

In total, there will be $M_0 + 2$ non-trivially evolving line integrals. Note that the equations for the evolution of the two quantities in (5.29) will depend on both $\delta_1(t)$ and $m(t)$. For a physically meaningful solution, we expect that these should be externally specifiable parameters. Unfortunately, there are only M_0 unknown parameters in the conformal map whose evolution needs to be determined, namely the M_0 zeros of $h(\zeta, t)$ satisfying the constraint

$$\prod_{j=1}^{N} \zeta_j^{\gamma_j} = \prod_{p=1}^{M_0} \eta_p$$

as well as the parameter $R(t)$. This makes a total of M_0 parameters. Since there are $M_0 + 2$ time evolving line integrals whose evolution depends on 2 parameters, $\delta_1(t)$ and $m(t)$ (namely, strain rate and mass flux at infinity), the only way that this finite system has a solution at all is if $\delta_1(t)$ and $m(t)$ are **not** externally specified but are quantities determined by the exact solutions themselves. It is in this sense that the problem has exact *mathematical* solutions (i.e. the finite system to which the problem has been reduced can be solved) which, unfortunately, have a rather artificial physical significance.

5.5 Discussion

It is remarkable that the "counting problem" above conveniently works out so as to provide the existence of exact mathematical solutions (albeit solutions with a rather artificial physical significance). Had there been *more* non-trivially evolving line-integrals than undetermined parameters in the mapping, the finite nonlinear system to which the problem has been reduced would **not** be solvable and exact mathematical solutions of the form deduced in this chapter would quite simply not exist. Even given the artificiality of the solutions from a physical standpoint, the potential usefulness of exact mathematical solutions of the form deduced here as *qualitative* models of the true physical behaviour should not be underestimated. The usefulness of the exact solutions as a "check" on a numerical code designed to solve the full free boundary value problem is also clear.

We mention here that it is possible to extend the Theorem of Invariants to this case. For completeness, it is also necessary to state a conjecture, analogous to that stated in chapters 1–3, on the matter of whether the evolution of the above line integral quantities uniquely defines (at least locally) the evolution of the parameters appearing in the conformal map. By analogy with earlier chapters, although we do not explicitly verify it here, such a conjecture is again expected to be true (in general).

Despite the drawbacks of the solutions from a *physical* viewpoint, the existence of such *mathematical* solutions is of enormous theoretical interest. These solutions provide evidence for the case that a mathematical generalization of the above methods might be possible in order to find a more general class of exact solutions for 4-bubbles *without* any assumed symmetry.

PART II

Chapter 6

General Solutions to the Two-Dimensional Liouville Equations

6.1 Introduction

This chapter[1] presents the most general exact solutions of the quasi-linear partial differential equations

$$\psi_{xx} + \psi_{yy} = \tilde{c}e^{d\psi} \tag{6.1}$$

$$\psi_{xx} - \psi_{yy} = \tilde{c}e^{d\psi} \tag{6.2}$$

where $\tilde{c}$ and d are real non-zero constants. Equations (6.1) and (6.2) are both generally recognized as being forms of the two-dimensional Liouville equation [25], and throughout this paper they will be referred to as the elliptic and hyperbolic Liouville equations respectively. The importance of these equations in various areas of mathematical physics from plasma physics and field theoretical modelling to fluid dynamics has made them the topic of many investigations for solution. A variety of exact solutions has been reported in the literature, many derived using highly sophisticated mathematical techniques [18]-[22],[25]-[34], [37]. For example, most recently Popov [37] employed a geometrical method on a Lobachevskii plane to obtain some general solutions to (6.1) from solutions of the two-dimensional Laplace equation, while Bhutani, Moussa and Vijayakumar [18] recently reported a new general solution of

[1]This chapter is based on material from an article entitled "General Solutions to the 2D Liouville Equation" by Darren G. Crowdy, *International Journal of Engineering Science*, **35**, Issue 2, pp. 141–149, Copyright (1997). It is reproduced here with the kind permission of Elsevier Science Ltd., The Boulevard, Langford Lane, Kidlington, OX5 1GB, U.K.

(6.2), and retrieved all previously known general solutions, using a direct method based on the formalism devised by Clarkson and Kruskal [36]. Neither of these two solution methods provide the *most* general solutions to (6.1) and (6.2) and solutions of the generality presented in this chapter have not, to the best of the author's knowledge, been reported before. This chapter therefore serves as a unification of many disparate results spread throughout the literature, and also provides many previously unknown exact solutions.

The methods employed here are essentially elementary yet the solutions obtained are shown to be the most general. It is indicated how to retrieve currently known solutions as special cases of these most general solutions. The general solution to (6.1) is shown to depend on two arbitrary analytic functions and some constants while the general solution to (6.2) depends on four arbitrary real functions and some constants. For clarity, the development is presented as a series of theorems and proofs, but the exposition is non-rigorous. Accordingly, any deeper mathematical implications of the results will not be treated here.

6.2 The Elliptic Case

To illustrate the method of solution for the elliptic case we solve the elliptic Liouville equation in the (x, y)-plane given by

$$\psi_{xx} + \psi_{yy} = \tilde{c}\mathrm{e}^{d\psi} \tag{6.3}$$

where $\tilde{c}$, d are real constants which are assumed to be non-zero. By shifting to characteristic coordinates, $z = x + \mathrm{i}y$ and $\bar{z} = x - \mathrm{i}y$, we can equivalently solve

$$\psi_{z\bar{z}} = c\mathrm{e}^{d\psi} \tag{6.4}$$

for real solutions ψ where $c = \frac{\tilde{c}}{4}$. It is noted that by the linear change of dependent variable

$$\phi = d\psi + \log(|cd|) \tag{6.5}$$

(6.4) can be written in the canonical form

$$\phi_{z\bar{z}} = \text{sgn}[cd]\ \mathrm{e}^{\phi} \tag{6.6}$$

Theorem 6.2.1

Any function $\psi(z, \bar{z})$ that is twice differentiable with respect to z and $\bar{z}$ and is a solution to

$$\psi_{z\bar{z}} = c\ \mathrm{e}^{d\psi} \tag{6.7}$$

also satisfies

$$\psi_{\bar{z}\bar{z}} - \frac{d}{2}\psi_{\bar{z}}^2 = \bar{E}(\bar{z}) \tag{6.8}$$

where $\bar{E}$ is some analytic function of $\bar{z}$.

Proof: Integrating (6.7) with respect to z gives

$$\psi_{\bar{z}} = c\int_{z_0}^{z} \mathrm{e}^{d\psi} dz + \bar{F}(\bar{z})$$

for some arbitrary analytic function $\bar{F}(\bar{z})$. On differentiating with respect to $\bar{z}$ and using (6.7), we obtain

$$\psi_{\bar{z}\bar{z}} = \frac{d}{2}\psi_{\bar{z}}^2 + \bar{E}(\bar{z}) \tag{6.9}$$

where $\bar{E}(\bar{z}) = \bar{F}'(\bar{z})$. Hence Theorem 6.2.1 follows.

Theorem 6.2.2

Any real-valued solution $\psi(z, \bar{z})$ to (6.8) that is sufficiently differentiable with respect to $\bar{z}$ and z is also a solution to (6.7) for some real constant c.

Proof: A direct proof of this is possible – the general real solution of (6.8) can be found directly (see Theorem 6.2.3) and it can be checked by substitution that the resulting solutions satisfy (6.7) for some value of c. An alternative approach is to take the second derivative of (6.8) with respect to z giving

$$\psi_{\bar{z}\bar{z}zz} - d\psi_{\bar{z}}\psi_{\bar{z}zz} - d\psi_{\bar{z}z}^2 = 0 \tag{6.10}$$

Taking the complex conjugate of (1.10) gives

$$\psi_{zz\bar{z}\bar{z}} - d\psi_z\psi_{z\bar{z}\bar{z}} - d\psi_{z\bar{z}}^2 = 0 \tag{6.11}$$

We now define $\omega(z, \bar{z}) = \psi_{\bar{z}z}$. Subtracting (6.10) from (6.11) we obtain

$$\psi_{\bar{z}}\omega_z - \psi_z\omega_{\bar{z}} = 0 \tag{6.12}$$

It follows from (6.12) that $\omega = f(\psi)$ for some real-valued function f. Differentiating (6.8) once with respect to z yields

$$\psi_{\bar{z}\bar{z}z} - d\psi_{\bar{z}z}\psi_{\bar{z}} = 0 \tag{6.13}$$

Using $\omega(z, \bar{z}) = f(\psi)$ then implies

$$\psi_{\bar{z}}(f' - d\ f) = 0 \tag{6.14}$$

from which it is concluded that any non-trivial real-valued solution $\psi(z, \bar{z})$ of (6.8) satisfies (6.7) for some constant c.

Theorem 6.2.3

Every real valued solution to (6.7) is of the form

$$\psi = -\frac{2}{d}\log\Big[c_1 y_1(z)\bar{y}_1(\bar{z}) + c_4 y_2(z)\bar{y}_2(\bar{z}) + c_2 y_1(z)\bar{y}_2(\bar{z}) + \bar{c}_2\bar{y}_1(\bar{z})y_2(z)\Big] \tag{6.15}$$

where $\bar{y}_1(\bar{z})$ and $\bar{y}_2(\bar{z})$ are two independent solutions to

$$y_{\bar{z}\bar{z}} + \frac{d}{2}\bar{E}(\bar{z})y = 0 \tag{6.16}$$

for some analytic $\bar{E}(\bar{z})$ while c_1 and c_4 are real constants and c_2 is some complex constant.

Remark 6.2.1

Real solutions are only defined in regions of the (x, y)-plane where the argument of the logarithm in (6.15) is positive.

Remark 6.2.2

The conjugate function $\bar{f}(z)$ is defined as

$$\bar{f}(z) = \overline{f(\bar{z})}$$

Proof: From Theorem 6.2.1 it follows that a solution to (6.7) is also a solution to (6.8) for some $\bar{E}(\bar{z})$. Note that (6.8) is in the form of a Riccati equation and can be made into the linear second order differential equation (6.16) for $y = e^{-d\psi/2}$. Therefore, it follows that

$$y(z,\bar{z}) = E_1(z)\bar{y}_1(\bar{z}) + E_2(z)\bar{y}_2(\bar{z}) \tag{6.17}$$

for some functions $E_1(z)$ and $E_2(z)$. Since ψ (and therefore y) is real, by taking the complex conjugate of (6.17), it follows that

$$y(z, \bar{z}) = \bar{E}_1(\bar{z})y_1(z) + \bar{E}_2(\bar{z})y_2(z) \tag{6.18}$$

Now, since (6.18) is a solution to (6.16), it follows that

$$\bar{E}_1(\bar{z}) = \bar{c}_1\bar{y}_1(\bar{z}) + \bar{c}_2\bar{y}_2(\bar{z}) \tag{6.19}$$

$$\bar{E}_2(\bar{z}) = \bar{c}_3\bar{y}_1(\bar{z}) + \bar{c}_4\bar{y}_2(\bar{z}) \tag{6.20}$$

for some constants c_1, c_2, c_3 and c_4. On substituting (6.19) and (6.20) back into (6.17) and (6.18) and equating the two different expressions for y, we obtain the condition that c_1 and c_4 are each real and that $c_2 = \bar{c}_3$. Thus,

$$y(z, \bar{z}) = c_1y_1(z)\bar{y}_1(\bar{z}) + c_4y_2(z)\bar{y}_2(\bar{z}) + c_2y_1(z)\bar{y}_2(\bar{z}) + \bar{c}_2\bar{y}_1(\bar{z})y_2(z) \tag{6.21}$$

Thus, from the definition of y in terms of ψ, (6.15) follows.

Remark 6.2.3

Since $\bar{E}(\bar{z})$ is some *arbitrary* analytic function, the requirement that $\bar{y}_1(\bar{z})$ and $\bar{y}_2(\bar{z})$ are independent solutions to (6.16) can be replaced by choosing $\bar{y}_1$ to be an arbitrary analytic function of $\bar{z}$ while determining $\bar{y}_2(\bar{z})$ from the condition that the wronskian $\bar{w}(\bar{z}) \equiv \bar{y}_1(\bar{z})\bar{y}_2'(\bar{z}) - \bar{y}_1'(\bar{z})\bar{y}_2(\bar{z}) = 1$ (this can be done without any loss of generality). Clearly, once $\bar{y}_1(\bar{z})$ is chosen, an expression for $\bar{E}(\bar{z})$ follows from (6.16). This unwieldy method of determining $\bar{y}_2(\bar{z})$ from the wronskian can be avoided by use of the following theorem:

Theorem 6.2.4

Let $Y_1(z)$ and $Y_2(z)$ be two arbitrary but independent analytic functions of z. Denote their wronskian by $W(z) \equiv Y_1(z)Y_2'(z) - Y_1'(z)Y_2(z)$. Then $y_1(z) = Y_1(z)/\sqrt{W(z)}$ and

$y_2(z) = Y_2(z)/\sqrt{W(z)}$ *are two independent analytic functions with unit wronskian.*

Proof: Since Y_1 and Y_2 are independent, then $W(z)$ is not zero. The relation $c_1 y_1 + c_2 y_2 = 0$ clearly implies $c_1 Y_1 + c_2 Y_2 = 0$ in some open set; from the independence of Y_1 and Y_2, this implies $c_1 = 0$, $c_2 = 0$; i.e. y_1 and y_2 are independent. On substituting for y_1 and y_2 in terms of Y_1 and Y_2, it follows that the wronskian $w(z)$ of y_1 and y_2 is

$$w(z) = \frac{Y_1(z)Y_2'(z) - Y_2(z)Y_1'(z)}{W(z)} = 1 \tag{6.22}$$

Hence Theorem 6.2.4 is proved.

Theorem 6.2.5

Any real solution to (6.7) is of the form

$$\psi = -\frac{2}{d}\log\left[c_1 Y_1(z)\bar{Y}_1(\bar{z}) + c_4 Y_2(z)\bar{Y}_2(\bar{z}) + c_2 Y_1(z)\bar{Y}_2(\bar{z}) + \bar{c}_2\bar{Y}_1(\bar{z})Y_2(z)\right] + \frac{1}{d}\log\left[W(z)\bar{W}(\bar{z})\right] \tag{6.23}$$

for some independent analytic functions $Y_1(z)$ and $Y_2(z)$, where c_1 and c_4 are real constants and c_2 is a complex constant, while $W(z)$ is the wronskian of $Y_1(z)$ and $Y_2(z)$.

Proof: This follows by substituting for $y_1(z)$ and $y_2(z)$ in terms of functions $Y_1(z)$ and $Y_2(z)$ (and their wronskian $W(z)$) into (6.15).

Theorem 6.2.6

The most general real solution to (6.7) is given by (6.23), where $Y_1(z)$ and $Y_2(z)$ are any independent analytic functions of z, $W(z)$ is their wronskian, with real constants c_1 and c_4 and complex constant c_2 satisfying the constraint

$$cd = -2(c_1\, c_4 - |c_2|^2) \tag{6.24}$$

but which are otherwise arbitrary.

Proof: Since we know any solution of (6.7) is of the form (6.23), by directly substituting (6.23) into equation (6.7), it is found that (6.7) is satisfied if and only if the constraint (6.24) is satisfied.

According to Theorem 6.2.6 it should be possible to retrieve all known solutions of the elliptic Liouville equation as special cases of (6.23). The well-known general solution given by Liouville [8,16-18] is trivially retrieved as a special case of this most general solution. Liouville's solution of (6.7) when $cd < 0$, can be written

$$\mathrm{e}^{d\psi} = -\frac{2}{cd}\frac{({u_x}^2 + {u_y}^2)}{(u^2 + v^2 + 1)^2} \tag{6.25}$$

where u and v are arbitrary conjugate functions. This corresponds to the choice $Y_1(z) = f(z) = u + \mathrm{i}v$ where $f(z)$ is an arbitrary analytic function and $Y_2(z) = 1$ with $c_1 = c_4 = \sqrt{-\frac{cd}{2}}$ and $c_2 = 0$. The resulting solution (using (6.23)) is

$$\psi = -\frac{2}{d}\log\left[\sqrt{-\frac{cd}{2}}(f(z)\bar{f}(\bar{z}) + 1)\right] + \frac{1}{d}\log[f'(z)\bar{f}'(\bar{z})] \tag{6.26}$$

Observing that $f(z)\bar{f}(\bar{z}) = u^2 + v^2$ and $f'(z)\bar{f}'(\bar{z}) = {u_x}^2 + {u_y}^2$ we retrieve Liouville's solution (6.25) as a special case of (6.23). Stuart [34] lists a number of exact solutions of (6.7) including one that is similar to Liouville's solution for the case $cd > 0$ in the form

$$\mathrm{e}^{d\psi} = \frac{2}{cd}\frac{({u_x}^2 + {u_y}^2)}{(u^2 + v^2 - 1)^2} \tag{6.27}$$

This corresponds to $Y_1(z) = f(z) = u + \mathrm{i}v$ ($f(z)$ arbitrary) and $Y_2(z) = 1$ with $c_1 = \sqrt{\frac{cd}{2}}$, $c_4 = -\sqrt{\frac{cd}{2}}$ and $c_2 = 0$. Stuart [34] also reports a class of solutions (attributed to Varley) for the case $cd < 0$ in the form

$$\mathrm{e}^{-d\psi/2} = \alpha_1(z)\bar{\alpha}_1(\bar{z}) + \alpha_2(z)\bar{\alpha}_2(\bar{z}) \tag{6.28}$$

where $\alpha_1(z)$, $\alpha_2(z)$ are independent analytic functions of z satisfying the equation

$$f_{zz} - G(z)f = 0 \tag{6.29}$$

with $\alpha_1(z)\alpha_2'(z) - \alpha_2(z)\alpha_1'(z) = \lambda$ and $|\lambda|^2 = -cd/2$ and $G(z)$ is an arbitrary analytic function of z. In fact, using the theorems just developed, it can now be demonstrated that this general solution is equivalent to the *most* general solution for $cd < 0$. To see this, by combining and rewriting the various results of Theorems 6.2.1–6.2.6, it has now been established that the *most* general solution of (6.7) can be written

$$\psi = -\frac{2}{d}\log\left[c_1\left(y_1 + \frac{c_2}{c_1}y_2\right)\left(\bar{y}_1 + \frac{\bar{c}_2}{c_1}\bar{y}_2\right) - \frac{cd}{2c_1}y_2\bar{y}_2\right] \tag{6.30}$$

where y_1 and y_2 are independent solutions of (6.16) for some $E(z)$. If $cd < 0$ it is clear that in order for the argument of the logarithm in (6.30) to be positive then necessarily $c_1 > 0$. Identifying $\alpha_1(z) = \sqrt{c_1}\left(y_1 + \frac{c_2}{c_1}y_2\right)$ and $\alpha_2(z) = \sqrt{-\frac{cd}{2c_1}}y_2(z)$ it is seen that (6.28) is in fact equivalent to the *most* general solution for $cd < 0$. This very important and significant fact is not stated in Stuart [34], nor does it seem to have been acknowledged elsewhere in the literature. The three types of solution of $\nabla^2\psi = e^{\psi}$ (corresponding to $c = 1/4$, $d = 1$ in our notation) recently identified by Popov [37] using geometrical methods can be written

$$\psi = \log\left[\frac{2(v_x^2 + v_y^2)}{v^2}\right] \tag{6.31}$$

$$\psi = \log\left[\frac{2(v_x^2 + v_y^2)}{\sinh^2 v}\right] \tag{6.32}$$

$$\psi = \log\left[\frac{2(v_x^2 + v_y^2)}{\sin^2 v}\right] \tag{6.33}$$

where $v(x, y) = Re[f(z)]$ and $f(z)$ is a general analytic function of $z = x + iy$. To retrieve (6.31) take $c_1 = c_4 = 0$, $c_2 = \frac{1}{\sqrt{8}}$ with $Y_1(z) = f(z)$, $Y_2(z) = 1$ in (6.23).

Noting that $v_x^2 + v_y^2 = f'(z)\bar{f}'(\bar{z})$ we retrieve the required result. To obtain (6.32) we take $c_1 = c_4 = 0$, $c_2 = \frac{1}{\sqrt{8}}$ with $Y_1(z) = \sinh[f(z)/2]$, $Y_2(z) = \cosh[f(z)/2]$. Result (6.33) is obtained by taking $c_1 = c_4 = 0$, $c_2 = \frac{1}{\sqrt{8}}$ with $Y_1(z) = \sin[f(z)/2]$, $Y_2(z) = \cos[f(z)/2]$.

Note that from the canonical form (6.6) it is clear that there are essentially two distinct types of elliptic Liouville equation depending on sgn$[cd]$ which, by (6.24), is the same as the sign of the determinant-like quantity $|c_2|^2 - c_1c_4$. The solutions in each case have somewhat different behaviours. In particular it is known from more general analyses [38] [39] that when $cd > 0$ the elliptic Liouville equation possesses **no** solution valid in the entire plane, while for $cd < 0$ it does possess such solutions. These properties can now be demonstrated explicitly for the elliptic Liouville equation using the above general representation of the solutions. For example, we briefly sketch a direct proof of the fact that for $cd > 0$, (6.7) has *no* solutions valid in the entire complex z-plane. The proof is by contradiction. Suppose there exists a solution of (6.7) for $cd > 0$ valid in the entire plane. Then by Theorems 1–6, the solution *necessarily* has the form (6.30) where $y_1(z)$ and $y_2(z)$ are independent solutions of (6.16) for some $E(z)$ and, without loss of generality, $c_1 > 0$. Since the solution is valid everywhere, $y_1(z)$ and $y_2(z)$ must be entire functions. Also, in order that the argument of the logarithm in (6.30) is strictly positive, the following inequality must hold *everywhere* in the finite z-plane

$$\left|\sqrt{c_1}\left(y_1 + \frac{c_2}{c_1}y_2\right)\right| > \sqrt{\frac{cd}{2c_1}}|y_2| \tag{6.34}$$

In addition, $y_1 + \frac{c_2}{c_1}y_2$ can have no zeros in the finite z-plane because the argument of the logarithm in (6.30) would fail to be strictly positive at any zero of $y_1 + \frac{c_2}{c_1}y_2$. Equation (6.34) thus implies that

$$\left|\frac{y_2(z)}{y_1(z) + \frac{c_2}{c_1}y_2(z)}\right| < \sqrt{\frac{2c_1^2}{cd}} \tag{6.35}$$

However, since $y_1 + \frac{c_2}{c_1} y_2$ has no zeros and since $y_1(z)$ and $y_2(z)$ are entire then the function $\frac{y_2(z)}{y_1(z)+\frac{c_2}{c_1}y_2(z)}$ is also entire. But (6.35) states that it is a *bounded* entire function which implies (by the Liouville theorem) that it must be a constant function. Finally, this then implies that $y_1(z)$ and $y_2(z)$ are linearly *dependent*, which is the required contradiction. It is a nice feature that the Liouville *theorem* proves to be the result from analytic function theory needed to prove this result on solutions to the Liouville *equation.*

Finally, we remark that the Dirichlet boundary value problem in a bounded domain with finite boundary values always has a unique solution when $cd > 0$ (but not when $cd < 0$, e.g. [40]). Thus, as one example of the utility of the solutions presented here in solving real physical problems, it can be envisaged that the above representation of the most general solution, combined perhaps with conformal mapping techniques, might be used to solve such classical Dirichlet boundary value problems. The form given in (6.23) would seem to be the most convenient for such purposes.

6.3 The Hyperbolic Case

We now extend this analysis to the two-dimensional hyperbolic Liouville equation in the $(\tilde{x}, \tilde{y})$-plane given by

$$\psi_{\tilde{x}\tilde{x}} - \psi_{\tilde{y}\tilde{y}} = \tilde{c}e^{d\psi} \tag{6.36}$$

where $\tilde{c}, d$ are again real constants, assumed to be non-zero. By shifting to characteristic coordinates (x, t) where $x = \tilde{x} + \tilde{y}$ and $t = \tilde{x} - \tilde{y}$ we can equivalently solve

$$\psi_{xt} = ce^{d\psi} \tag{6.37}$$

where x and t are real coordinates, and $c = \frac{\tilde{c}}{4}$. Note that by the linear change of dependent variable (6.5) the canonical form for the hyperbolic Liouville equation

(6.37) can be written

$$\phi_{xt} = \text{sgn}[cd]\ \mathrm{e}^{\phi} \tag{6.38}$$

We now demonstrate that the general solution of the hyperbolic Liouville equation depends on four arbitrary real functions, in contrast to two arbitrary analytic functions as in the elliptic case in §2.

Theorem 6.3.1

Any function $\psi(x,t)$ that is twice differentiable with respect to both x and t and is a real solution to

$$\psi_{xt} = c\mathrm{e}^{d\psi} \tag{6.39}$$

simultaneously satisfies the two equations

$$\psi_{xx} - \frac{d}{2}\psi_x^2 = E(x) \tag{6.40}$$

$$\psi_{tt} - \frac{d}{2}\psi_t^2 = F(t) \tag{6.41}$$

for some choice of functions $E(x)$ and $F(t)$.

Proof: Integrating (6.39) with respect to t gives

$$\psi_x = c\int_{t_0}^{t} \mathrm{e}^{d\psi}\,dt + G(x)$$

for some arbitrary real function $G(x)$. Differentiating this equation with respect to x and using (6.39) gives

$$\psi_{xx} - \frac{d}{2}\psi_x^2 = E(x) \tag{6.42}$$

where $E(x) = G'(x)$. Hence ψ satisfies equation (6.40). By the symmetry of (6.39) in x and t, the same manipulations imply ψ also satisfies eqn (6.41) for some $F(t)$. Thus Theorem 6.3.1 follows.

Theorem 6.3.2

Any sufficiently differentiable solution of both (6.40) and (6.41) satisfies equation (6.39) for some choice of the constant c.

Proof: A direct proof of this is possible – general simultaneous solutions to equations (6.40) and (6.41) can be found directly (see Theorem 6.3.3), and it can be checked by substitution that these are solutions of (6.39). An alternative approach is to differentiate (6.40) twice with respect to t giving

$$\psi_{xxtt} - d\psi_{xtt}\psi_x - d{\psi_{xt}}^2 = 0 \tag{6.43}$$

Similarly, differentiating (6.41) twice with respect to x yields

$$\psi_{ttxx} - d\psi_{txx}\psi_t - d{\psi_{tx}}^2 = 0 \tag{6.44}$$

Subtracting (6.43) from (6.44) implies

$$\psi_{xtt}\psi_x - \psi_{txx}\psi_t = 0 \tag{6.45}$$

which implies

$$\psi_{xt} = f(\psi) \tag{6.46}$$

for some real function $f(\psi)$. Differentiating (6.40) once with respect to t and using (6.46) gives

$$\psi_x(f' - d\ f) = 0 \tag{6.47}$$

Thus implying that any non-trivial simultaneous solution of equations (6.40) and (6.41) satisfies (6.39) for some value of c.

Theorem 6.3.3

Every solution of (6.39) is of the form

$$\psi = -\frac{2}{d}\log\Big[c_1 y_1(t)w_1(x) + c_2 y_1(t)w_2(x) + c_3 y_2(t)w_1(x) + c_4 y_2(t)w_2(x)\Big] \tag{6.48}$$

There $y_1(t),\ y_2(t)$ are two independent solutions of

$$y_{tt} + \frac{d}{2}F(t)\ y = 0 \tag{6.49}$$

and $w_1(x),\ w_2(x)$ are two independent solutions of

$$w_{xx} + \frac{d}{2}E(x)\ w = 0 \tag{6.50}$$

and c_1, c_2, c_3, c_4 are real constants.

Remark 6.3.1

Real solutions are defined in regions of the (x,t)-plane where the argument of the logarithm in (6.48) is positive.

Proof: From Theorem 6.3.1, solutions of (6.39) simultaneously satisfy (6.40) and (6.41) for some $E(x)$ and $F(t)$. Note that (6.40) is of the Riccati form and can be made into a linear second order equation for $M(x,t) = e^{-d\psi/2}$. Using this transformation the resulting equation for $M(x,t)$ is (6.50) i.e.

$$M_{xx} + \frac{d}{2}E(x)\ M = 0 \tag{6.51}$$

Therefore,

$$M(x,t) = E_1(t)w_1(x) + E_2(t)w_2(x) \tag{6.52}$$

for some functions $E_1(t)$ and $E_2(t)$. Now since ψ is also a solution of (6.41) then $M(x,t)$ is also a solution to (6.49) and we deduce that

$$E_1(t) = c_1 y_1(t) + c_3 y_2(t) \tag{6.53}$$

and

$$E_2(t) = c_2 y_1(t) + c_4 y_2(t) \tag{6.54}$$

for some real constants c_1, c_2, c_3, c_4. Substituting (6.53) and (6.54) into (6.52) gives the result (6.48).

Theorem 6.3.4

Any real solution to (6.39) is of the form

$$\begin{aligned} \psi = -\frac{2}{d}\log\Big[c_1 Y_1(t) W_1(x) &+ c_2 Y_1(t) W_2(x) + c_3 Y_2(t) W_1(x) \\ &+ c_4 Y_2(t) W_2(x)\Big] + \frac{1}{d}\log\left[Y(t)W(x)\right] \end{aligned} \tag{6.55}$$

where $Y_1(t)$, $Y_2(t)$ are independent sufficiently differentiable functions with wronskian $Y(t)$, $W_1(x)$, $W_2(x)$ are independent sufficiently differentiable functions with wronskian $W(x)$ and c_1, c_2, c_3, c_4 are real constants.

Proof: Analogous to proof of Theorems 6.2.4 and 6.2.5.

Theorem 6.3.5

The most general real solution to (6.39) is given by (6.55), where $Y_1(t)$, $Y_2(t)$ are any independent functions with wronskian $Y(t)$, $W_1(x), W_2(x)$ are any independent functions of x with wronskian $W(x)$ and c_1, c_2, c_3, c_4 are real constants satisfying the constraint

$$cd = -2(c_1 c_4 - c_2 c_3) \tag{6.56}$$

but which are otherwise arbitrary.

Proof: Since we know any solution of (6.39) is of the form (6.55), by substituting (6.55) into equation (6.39), we find that (6.39) is satisfied if and only if constraint (6.56) is satisfied.

Theorem 6.3.5 implies that all known solutions of (6.39) should be retrievable as special cases of (6.55). For example, the choice

$$Y_1(t) = \sigma(t) + z_0,\ Y_2(t) = 1,\ W_1(x) = \theta(x),\ W_2(x) = 1 \tag{6.57}$$

with $c_1 = c_4 = 0,\ c_2 = c_3 = \frac{1}{\sqrt{2}}$ and with $\sigma(t)$ and $\theta(x)$ arbitrary real functions and z_0 a real constant, represents a well-known general solution to the hyperbolic Liouville equation (6.39) (with $c = d = 1$)

$$\psi(x,t) = \log\left[\frac{2\theta'(x)\sigma'(t)}{(\theta(x) + \sigma(t) + z_0)^2}\right] \tag{6.58}$$

which coincides with the one obtained by Ibragimov [21] using Backlund transformation techniques, by Tamizhmani and Lakshmanan [22] using a Painleve analysis, and by Bhutani, Moussa and Vijayakumar [18] using a direct method for finding similarity solutions following the Clarkson and Kruskal formalism [36]. It is also the general solution which normally appears in text-books [23] [24]. The choice

$$Y_1(t) = \cosh\left(\frac{-\sqrt{C}(\sigma(t) + z_0)}{2}\right),\ Y_2(t) = \sinh\left(\frac{-\sqrt{C}(\sigma(t) + z_0)}{2}\right)$$

$$W_1(x) = \sinh\left(\frac{-\sqrt{C}\theta(x)}{2}\right),\ W_2(x) = \cosh\left(\frac{-\sqrt{C}\theta(x)}{2}\right) \tag{6.59}$$

with $c_2 = c_3 = \frac{1}{\sqrt{2}}$ and $c_1 = c_4 = 0$ and where C is a real constant gives the solution

$$\psi = \log\left[-\left(\frac{C}{2}\right)\theta'(x)\sigma'(t)\text{sech}^2\left(\frac{-\sqrt{C}}{2}(\theta(x) + \sigma(t) + z_0)\right)\right] \tag{6.60}$$

which is the solution of (6.39) (for $c = d = 1$) discovered recently by Bhutani et al [18] using different methods and which can be related to that found in [19] using an isovector approach. The choice

$$Y_1(t) = \cos\left(\frac{\sqrt{C}(\sigma(t) + z_0)}{2}\right),\ Y_2(t) = -\sin\left(\frac{\sqrt{C}(\sigma(t) + z_0)}{2}\right)$$

$$W_1(x) = \sin\left(\frac{\sqrt{C}\theta(x)}{2}\right),\ W_2(x) = \cos\left(\frac{\sqrt{C}\theta(x)}{2}\right) \tag{6.61}$$

with $c_2 = c_3 = \frac{1}{\sqrt{2}}$ and $c_1 = c_4 = 0$ gives the solution

$$\psi = \log\left[\left(\frac{C}{2}\right)\theta'(x)\sigma'(t)\sec^2\left(\frac{\sqrt{C}}{2}(\theta(x) + \sigma(t) + z_0)\right)\right] \tag{6.62}$$

which corresponds to a solution of (6.39) (for $c = d = 1$) reported in Ibragimov [21] (when $C = 4$) and which was also retrieved by Bhutani et al [18].

PART III

Chapter 7

A Note on the Linear Stability of Burgers Vortex

7.1 Overview

In this chapter[1], a two-parameter family of analytical solutions of the linearized equations for axially-dependent disturbances to the three-dimensional base strain field associated with the well-known axisymmetric Burgers vortex is presented. The solutions are valid asymptotically at large axial distances from the stagnation point. By a formal perturbation analysis, perturbative solutions are also found for disturbances to the Burgers vortex for small Reynolds numbers. The solutions are believed to provide important insights into the nature of the as yet unsolved problem of the linear stability of Burgers vortex to axially-varying disturbances.

7.2 Introduction

The axisymmetric Burgers vortex represents one of the few known exact solutions to the full Navier-Stokes equations, however very little has been deduced about its stability properties since its discovery nearly fifty years ago [41]. Given the extensive use of the vortex as a model of the fine scale structure of turbulence, its stability properties are of great importance. The vortex consists of a pure swirl flow superposed on an irrotational base strain flow. The flow is incompressible. In cylindrical coordinates (in which lengths are non-dimensionalized with respect to the Burgers length-scale $\sqrt{\frac{\nu}{a}}$ and times with respect to a^{-1} where a is the strain rate of the background flow

[1]This chapter is based on an article entitled "A Note on the Linear Stability of Burgers Vortex" by Darren G. Crowdy, to appear in *Studies in Applied Mathematics.* It is reproduced here with the kind permission of The Editor (Professor David Benney), Studies in Applied Mathematics, M.I.T., 2-341, 77 Massachusetts Avenue, Cambridge, MA 02139.

field) the solution can be written

$$\mathbf{u}(r,\theta,z) = (-r,\ V_B(r),\ 2z) \tag{7.1}$$

where

$$V_B(r) = \frac{\Gamma}{2\pi r}(1 - e^{-r^2/2}) \equiv \frac{\Gamma}{2\pi}\tilde{V}_B(r) \tag{7.2}$$

Γ is the circulation. We define the Reynolds number to be

$$Re = \frac{\Gamma}{2\pi\nu} \tag{7.3}$$

where ν is the viscosity of the fluid.

Robinson and Saffman [42], and more recently, Prochazka and Pullin [43] have investigated the linear stability of the vortex to a general disturbance in the plane perpendicular to the axial straining direction and found it to be stable, at least for moderately high Reynolds numbers. Leibovich and Holmes [44] analysed the global stability of the vortex and showed it to be globally unstable for all Reynolds numbers. These results say nothing about the linear stability of the vortex to the important class of z-dependent disturbances, and no study of this, either analytical or numerical, seems to have been carried out before. This is probably due to the difficulty in even *formulating* the linear stability problem – the classical notion of "wave-number" typically associated with Fourier-mode eigenfunctions is not available owing to the lack of translational symmetries of the base strain flow on which the Burgers vortex is superposed. Rather than viewing this as a drawback, this note *exploits* the non-autonomous nature of the linearized disturbance equations to glean important analytical information on the large-z behaviour of solutions. We also note that the two-dimensional linear stability of the related Burgers vortex layer has recently received attention [45] but again, the important question of the its three-dimensional linear stability was not broached.

To elucidate our approach, consider the procedure for analysing the linear stability of a two-dimensional Blasius boundary layer [46]. Suppose x is the coordinate along the wall, and y is the coordinate perpendicular to the wall. In this case the linearized equations are also not autonomous in x, but by use of a *parallel-mean flow assumption* [46], the equations can be approximated by an autonomous set (especially at large Re – see [46]). The approximate equations then admit the following eigenfunctions for the streamfunction

$$\psi(x, y, t) = f(y)e^{ikx}e^{-i\omega t} \tag{7.4}$$

Fitting the boundary conditions on $f(y)$ (i.e. on the wall at $y = 0$ and at $y \to \infty$) then provides an eigenvalue relation between k and ω. For Burgers vortex, owing to the non-autonomous nature of the linearized partial differential equations, eigenfunctions analogous to the Fourier modes above are not available in general. However, in this chapter, we explicitly find a two-parameter family of self-consistent large-z asymptotic solutions of the linearized partial differential equations for small Reynolds numbers. For $Re = 0$ the solutions have an algebraic dependence on z as $z \to \infty$. Fitting the appropriate boundary conditions at $r = 0$ and $r \to \infty$ then provides the eigenvalue relation between the frequency and the exponent of z in the asymptotic solutions as expected by analogy with the Blasius boundary layer analysis.

The principal aim of this note is to present our analytical observations on the structure of a class of solutions of the linearized disturbance equations about the Burgers vortex for small Reynolds numbers. However, we go further and conjecture some possible implications of these observations. In the discussion section, we use the evidence of the explicit large-z solutions found here to put the case for a *spatial mode analysis* of the linear stability of the Burgers vortex to axially varying disturbances and conjecture the possible role played in such an analysis by the solutions found here. In particular, a spatial mode analysis would essentially involve causing a general oscillatory disturbance at some z-station near the stagnation point at the origin and observing whether the disturbances grow spatially as they are convected with the flow to $z \to \infty$. Clearly the possible behaviour of solutions as $z \to \infty$ is then of crucial interest and is fundamental to understanding the linear stability problem. We argue here that, for perturbations to the base strain field with no vortex (corresponding to $Re = 0$) there are two fundamental behaviours of solutions as $z \to \infty$: some solutions grow exponentially with z, while the remainder have milder (less-than-exponential) behaviour as $z \to \infty$. The existence of the latter class of solutions is demonstrated by explicit construction, and the subset of such solutions found here is shown to have an *algebraic* dependence on z as $z \to \infty$. It is argued that it is these solutions (and not the exponentially growing solutions) that are relevant to the linear stability analysis. By a formal perturbation procedure, similar explicit large-z asymptotic solutions can be found for perturbations to weak Burgers vortices (small Re). It does not seem to be possible to derive explicit analytic forms for the exponential solutions. Because the family of solutions presented here is not derived in any systematic way (so that there may well be other behaviours at infinity that we have not identified) it is not possible to make any *definite* statements on the linear (spatial) stability of the Burgers vortex for small Reynolds numbers, but some informed speculations on how to formulate a numerical treatment of the problem can now at least be made based on this analysis.

7.3 Large-z Solutions for $Re = 0$

It is clearly sufficient to consider the half-space $r \in [0, \infty)$, $z \in [0, \infty)$. The solution method is straightforward: an ansatz for large-z asymptotic solutions to the linearized disturbance equations is made. Assuming the ansatz, certain terms in the linearized equations are shown to be asymptotically negligible at large z. Solutions of the resulting asymptotic equations (satisfying appropriate boundary conditions at $r = 0$ and $r \to \infty$) are then explicitly found having the form assumed initially. Thus, such solutions are *consistent* large-z solutions of the original equations. This is a standard dominant balance argument [47]. The velocity field is written

$$\mathbf{u}(r, z, t) = (-r + u(r, z, t),\ V_B(r) + v(r, z, t),\ 2z + w(r, z, t)) \qquad (7.5)$$

and the pressure field,

$$P(r,z,t) = P_B(r,z) + p(r,z,t) \tag{7.6}$$

where $u(r,z,t), v(r,z,t), w(r,z,t)$ and $p(r,z,t)$ represent the perturbation quantities to be determined. $P_B(r,z)$ represents the pressure field associated with the steady Burgers vortex solution. For simplicity (and through lack of an analogue to Squire's theorem for this case) we simply assume that the solutions have no azimuthal dependence. This is permissible by the axisymmetry of the Burgers vortex and strain field. Substituting into the Navier-Stokes equations and linearizing, the non-dimensionalized evolution equations become,

$$\frac{\partial u}{\partial t} - u - r\frac{\partial u}{\partial r} + 2z\frac{\partial u}{\partial z} - Re\left(\frac{2\tilde{V}_B(r)}{r}\right)v = -\frac{\partial p}{\partial r} + \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} + \frac{\partial^2 u}{\partial z^2} \tag{7.7}$$

$$\frac{\partial v}{\partial t} - v - r\frac{\partial v}{\partial r} + 2z\frac{\partial v}{\partial z} + Re\left(\frac{\partial \tilde{V}_B}{\partial r} + \frac{\tilde{V}_B}{r}\right)u = \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2} + \frac{\partial^2 v}{\partial z^2} \tag{7.8}$$

$$\frac{\partial w}{\partial t} + 2w - r\frac{\partial w}{\partial r} + 2z\frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z} + \frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2} \tag{7.9}$$

$$\frac{1}{r}\frac{\partial(ru)}{\partial r} + \frac{\partial w}{\partial z} = 0 \tag{7.10}$$

Note that the solution structure is most clearly seen by working with the above equations. Thus we have deliberately avoided the alternative streamfunction-vorticity formulation. Any solutions for u, v, and w must be regular at $r = 0$. Since we are seeking perturbations to the Burgers vortex where the vorticity decays exponentially as $r \to \infty$, we shall require all components of the *perturbation vorticity* also to decay exponentially as $r \to \infty$. A *sufficient* (but not necessary) condition is that u, v and w decay exponentially. Thus, for our purposes, we shall impose the boundary condition on the perturbation velocities that they decay exponentially as $r \to \infty$.

We make the following ansatz for large-z asymptotic solutions:

$$\begin{aligned} u(r,z,t) &\sim \bar{u}(r)e^{-\mu t}\frac{1}{z^{\sigma+1}} \\ w(r,z,t) &\sim \bar{w}(r)e^{-\mu t}\frac{1}{z^{\sigma}} \\ v(r,z,t) &\sim \bar{v}(r)e^{-\mu t}\frac{1}{z^{\sigma+1}} \\ p(r,z,t) &\sim \bar{p}(r)e^{-\mu t}\frac{1}{z^{\sigma+1}} \end{aligned} \tag{7.11}$$

where μ, σ are some (generally complex) parameters. It is understood throughout that the *real part* of all functions should be taken to obtain a physical solution – by linearity this can always be done. *Assuming* the ansatz, it is clear that as $z \to \infty$ the $\frac{\partial^2}{\partial z^2}$ terms in equations (7.7)-(7.9) can be consistently neglected with respect to the other terms, as can the $\frac{\partial p}{\partial z}$ in equation (7.9). All neglected terms in each equation are $O(\frac{1}{z^2})$ (i.e. small for $z \gg 1$) compared to the terms retained. Note also that since z has been non-dimensionalized with respect to the Burgers length scale, the asymptotic solutions are valid for (dimensional) $z \gg \sqrt{\frac{\nu}{a}}$. Note that the alternative of balancing z-advection with z-diffusion in any of the momentum equations (7.7)-(7.9) e.g.

$$+2z\frac{\partial u}{\partial z} \sim +\frac{\partial^2 u}{\partial z^2} \tag{7.12}$$

is likely to lead to perturbation velocities *growing exponentially* with z as $z \to \infty$.

Substituting the above ansatz for $w(r,z,t)$ into the *asymptotic* version of (7.9) yields the following ordinary differential equation (o.d.e) for $\bar{w}(r)$

$$\frac{d^2\bar{w}}{dr^2} + \left(r + \frac{1}{r}\right)\frac{d\bar{w}}{dr} + (\mu - 2 + 2\sigma)\bar{w} = 0 \tag{7.13}$$

In principle, if $\bar{w}(r)$ (satisfying the boundary conditions) can be determined from (7.13), (7.10) must be solved for a $\bar{u}(r)$ which also satisfies the boundary conditions. If an appropriate $\bar{u}(r)$ can be found, the asymptotic version of (7.8) then provides an o.d.e for $\bar{v}(r)$. Finally, if a suitable $\bar{v}(r)$, satisfying the boundary conditions can be found, the asymptotic version of (7.7) can be directly integrated to give the corresponding $\bar{p}(r)$. It remains to see if appropriate solutions to the o.d.e's can be determined. (7.13) can be identified with a confluent hypergeometric equation and, in the notation of [17], the solution can be written

$$\bar{w}(r) = M\left[\frac{(\mu - 2 + 2\sigma)}{2};\ 1;\ -\frac{r^2}{2}\right] \tag{7.14}$$

where M is the confluent hypergeometric function, regular at the origin. The general asymptotic behaviour of this function as $r \to \infty$ is

$$M[a; b; -r^2/2] = \left(-\frac{r^2}{2}\right)^{-a} \frac{e^{ia\pi}\Gamma(b)}{\Gamma(b-a)} \left[\sum_{m=0}^{P-1} \frac{[a]_m[1-a-b]_m}{m!}\left(\frac{r^2}{2}\right)^{-m} + O\left(\frac{1}{r^{2P}}\right)\right]$$
$$+ e^{-r^2/2}\left(-\frac{r^2}{2}\right)^{a-b} \frac{\Gamma(b)}{\Gamma(a)} \left[\sum_{m=0}^{Q-1} \frac{[b-a]_m[1-a]_m}{m!}\left(-\frac{r^2}{2}\right)^{-m} + O\left(\frac{1}{r^{2Q}}\right)\right] \tag{7.15}$$

where $[a]_m = a(a+1)...(a+m-1)$. The requirement of *exponential* decay as $r \to \infty$ gives the eigenvalue condition $\Gamma(b-a)^{-1} = 0$, i.e. $(b-a) = -k$ where $k = 0, 1, 2....$ (using well-known properties of $\Gamma(z)$). Using the parameters in (7.14), the eigenvalue condition is

$$\mu = 2k - 2\sigma + 4 \tag{7.16}$$

Using Kummer's transformation to identify the solutions in terms of the generalized Laguerre polynomials $L_k^{(n)}(r^2/2)$, to within normalization,

$$\bar{w}(r) = e^{-r^2/2} L_k^{(0)}(r^2/2),\ k = 0, 1, 2... \tag{7.17}$$

with μ as in (7.16). The function $\bar{u}(r)$ must now be deduced from (7.10). It is easily shown that

$$\bar{u}(r) = \sigma U(r) \text{ where } U(r) \equiv \frac{1}{r}\int_0^r \tilde{r}\bar{w}(\tilde{r})d\tilde{r} \tag{7.18}$$

In general, for arbitrary choices of function $\bar{w}(r)$, regular at $r = 0$ and exponentially decaying as $r \to \infty$, the function obtained by integration as in (7.18) clearly cannot be expected to be exponentially decaying. However, we now illustrate that this is *not* the case for $\bar{w}(r)$ having the special form (7.17) provided $k \geq 1$. Substituting from (7.17) in (7.18) yields

$$U(r) = \frac{1}{r}\int_0^r \tilde{r}e^{-\tilde{r}^2/2}L_k^{(0)}(\tilde{r}^2/2)d\tilde{r} = \frac{1}{r}\sum_{j=0}^{k}\frac{b_j}{2^j}I_j(r) \tag{7.19}$$

where we denote the coefficients of the k-th order Laguerre polynomial $L_k^{(0)}(r^2/2)$ by $\{b_j | j = 0..k\}$ so that

$$L_k^{(0)}(r^2/2) = \sum_{j=0}^{k}\frac{b_j}{2^j}r^{2j} \tag{7.20}$$

and we define

$$I_j(r) = \int_0^r \tilde{r}^{2j+1}e^{-\tilde{r}^2/2}d\tilde{r} \tag{7.21}$$

It can be shown using integration by parts that

$$I_j(r) = f_j(r^2)e^{-r^2/2} + 2^j j! \ \forall j \geq 0 \tag{7.22}$$

for some polynomial $f_j(r^2)$. Thus

$$U(r) = \frac{1}{r}\sum_{j=0}^{k}\frac{b_j}{2^j}f_j(r^2)e^{-r^2/2} + \frac{1}{r}\sum_{j=0}^{k}b_j j! \tag{7.23}$$

from which, without further inspection, it might be concluded that $U(r) \sim \frac{1}{r}$ as

$r \to \infty$. But remarkably,

$$b_j = \frac{(-1)^j}{j!j!} \frac{k!}{(k-j)!} \tag{7.24}$$

which implies

$$\sum_{j=0}^{k} b_j j! = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \equiv 0 \ \forall \ k \geq 1 \tag{7.25}$$

which then implies that $U(r)$ is indeed exponentially decaying as $r \to \infty$ (for all $k \geq 1$) as required to satisfy the boundary conditions. Also, it is clear that $\bar{u}(r)$ is regular at $r = 0$. It still remains to establish that an appropriate $\bar{v}(r)$ can be found. Substituting the ansatz for $v(r,z,t)$ and $u(r,z,t)$ into the asymptotic form of (7.8) yields

$$\frac{d^2\bar{v}}{dr^2} + \left(r + \frac{1}{r}\right)\frac{d\bar{v}}{dr} + \left(\mu + 3 + 2\sigma - \frac{1}{r^2}\right)\bar{v} = (Re)e^{-r^2/2}\bar{u} \tag{7.26}$$

Even more remarkably, it can be shown that the spectrum of the self-adjoint linear differential operator (LDO) in (7.13) is a subset of the spectrum of the self-adjoint LDO on the left hand side of (7.26). Therefore, if (7.16) holds, the solution to the *homogeneous* equation (7.26) which satisfies the boundary condition at $r = 0$ and $r \to \infty$ is in fact given by

$$\bar{v}(r) = r \ M\left[\frac{\mu + 4 + 2\sigma}{2}; \ 2; \ -\frac{r^2}{2}\right] \tag{7.27}$$

or, again using Kummer's transformation (to within normalization)

$$\bar{v}(r) = r \ e^{-r^2/2} \ L_{k+2}^{(1)}(r^2/2) \tag{7.28}$$

Thus, in order for a $\bar{v}(r)$ (satisfying the boundary conditions) to exist, a Fredholm alternative compatibility condition will have to be satisfied by the inhomogeneous term of (7.26), namely,

$$\Big\langle (Re)e^{-r^2/2}\bar{u}(r),\ r\ e^{-r^2/2}L^{(1)}_{k+2}(r^2/2)\Big\rangle = 0 \tag{7.29}$$

where angle brackets denote the inner product defined by

$$\langle f(r),\ g(r)\rangle = \int_0^\infty f(r)g^*(r)e^{r^2/2}rdr \tag{7.30}$$

associated with the self-adjoint LDO on the left hand side of (7.26). In general, for $Re \neq 0$, (7.29) will not hold and there will be *no* solution for $\bar{v}(r)$ satisfying the boundary conditions and hence *no* solution having the form of the assumed ansatz. However, for $Re = 0$, the equation for $\bar{v}(r)$ can be solved and is given (to within normalization) by (7.28). The corresponding $\bar{p}(r)$ then follows immediately from integration of the asymptotic form of (7.7). We have therefore succeeded in finding a family of consistent large-z solutions (for $Re = 0$) parametrized by integers $k \geq 1$ and the complex parameter σ, with μ given by the eigenvalue relation (7.16), having the form originally hypothesized in the ansatz (7.11) and satisfying the required boundary conditions at $r = 0$ and $r \to \infty$.

7.4 Perturbation Theory for Small Re

Despite the string of fortuitous circumstances that led to the identification of the above two-parameter family of solutions, it is not expected that these represent isolated solutions which exist only for $Re = 0$. Indeed we expect to be able to find perturbative solutions about the $Re = 0$ results valid for small non-zero Re, although they will clearly not have the simple form given in (7.11), as already noted by the failure of such solutions to satisfy the secularity condition (7.29). The relevant perturbation analysis is outlined in detail in Appendix M. The analysis is not only interesting as an example of a tractable perturbation analysis on a system of linear partial differential equations (an analysis with some very interesting properties – in particular the eigenvalue relation for small Re can be determined to all orders and summed), but it also provides valuable insights into how the large-z solutions found for $Re = 0$ change when a weak Burgers vortex is superposed on the base strain field. To summarize the results obtained, the zeroth order solution is taken to be given by

$$\begin{aligned} w(r,z,t) &= \bar{w}(r)e^{-\mu_0 t}\frac{1}{z^{\sigma}} \\ v(r,z,t) &= \bar{v}(r)e^{-\mu_0 t}\frac{1}{z^{\sigma+1}} \\ u(r,z,t) &= \sigma U(r)e^{-\mu_0 t}\frac{1}{z^{\sigma+1}} \\ \mu_0 &= 2k - 2\sigma + 4 \end{aligned} \tag{7.31}$$

where we define

$$\begin{aligned} \bar{w}(r) &= e^{-r^2/2}L_k^{(0)}(r^2/2) \\ \bar{v}(r) &= r\ e^{-r^2/2}L_{k+2}^{(1)}(r^2/2) \\ U(r) &= \frac{1}{r}\int_0^r \tilde{r}e^{-\tilde{r}^2/2}L_k^{(0)}(\tilde{r}^2/2)d\tilde{r} \end{aligned} \tag{7.32}$$

The final perturbed solution for the velocity field can be written in the following form:

$$\begin{aligned} w(r,z,t) &= \bar{w}(r)e^{-\mu t}\frac{1}{z^{\tilde{\sigma}}} \\ v(r,z,t) &= \frac{e^{-\mu t}}{z^{\sigma+1}}\Bigg(\bar{v}(r) + Re\ \bar{v}_1(r) \\ &\quad + Re^2\ \left(v_2^{(2)}(r)(\log z)^2 + v_2^{(1)}(r)\log z + v_2^{(0)}(r)\right) + \ldots\Bigg) \\ u(r,z,t) &= \tilde{\sigma}U(r)e^{-\mu t}\frac{1}{z^{\tilde{\sigma}+1}} \\ \tilde{\sigma} &= \sigma\left(1 - \frac{Re\mu_1}{2\sigma + Re\mu_1}\right) \\ \mu &= 2k + 4 - 2\tilde{\sigma} \end{aligned} \tag{7.33}$$

with μ_1 given by

$$\mu_1 = \frac{\langle \sigma e^{-r^2/2}U(r),\ \bar{v}(r)\rangle}{\langle \bar{v}(r),\ \bar{v}(r)\rangle} \tag{7.34}$$

and where the functions $\{v_n^{(j)}(r) | 0 \leq j \leq n, n \geq 2\}$ can be found as eigenfunction expansions if needed. See the analysis in appendix M for full details.

Finally, note that as *Re* gets larger it is not expected that the large-z asymptotic assumptions (i.e. the "dominant balances") made to simplify the equations will continue to be valid and a numerical study of the full equations will probably be needed to see how these solutions continue for larger *Re*. However, for *small Re*, we have shown by explicit construction of self-consistent solutions that the asymptotic assumptions were the correct ones to make to find those solutions. The two important results of this section are to note that when $Re \neq 0$ the solutions become more complicated and do not take the simple separable form as given in (7.11), and also to note how the eigenvalue relation changes for small *Re* i.e. to first order

$$\mu = 2k + 4 - 2\sigma + Re\ \sigma\ \frac{\langle e^{-r^2/2}\bar{u}(r),\ \bar{v}(r)\rangle}{\langle \bar{v}(r),\ \bar{v}(r)\rangle} \tag{7.35}$$

7.5 Discussion

We now discuss the possible relevance of these solutions to the linear stability problem of Burgers vortex to axially-varying perturbations. Using the results of this note we now argue the case for a *spatial mode analysis* (see [48] and references therein). Such analyses are usually more appropriate than a *temporal mode analysis* in stability problems where there is an overall mean flow direction (the z-direction in this case). A suggested stability problem is to find the large-z asymptotic behaviour of disturbances forced by a general localized oscillatory perturbation near the stagnation point (cf. the oscillating Schubauer ribbon experiment in boundary-layer stability analysis [46]). In classical spatial mode analyses for flow problems allowing the usual Fourier-mode decomposition (cf. (7.4)), the eigenvalue relation is interpreted as a relation giving the (generally complex) wave-number k as a function of the real frequency ω, rather than a relation for the (generally complex) frequency ω as a function of the real wavenumber k (temporal mode analysis). The existence of spatially growing modes (e.g. a mode with Im$[k] > 0$ for some real ω in (7.4)) then implies spatial instability *provided* the group velocity of the spatially growing modes is such that the waves travel *downstream* of the excitation. We conjecture that the proposed exponentially growing modes suggested by the balance in (7.12) propagate *towards* the stagnation point *from* infinity and thus would be discounted physically using some generalized radiation condition. Note that it is clear that if a temporal mode analysis were being carried out, some form of boundary condition at $z \to \infty$ will be needed. In that case, it is not at all clear what form this boundary condition should take. We conjecture that the appropriate boundary condition should be to discount *exponentially* growing solutions, although the reason for this choice is more easily understood (if the radiation condition conjecture is correct) from a spatial mode perspective. Thus ruling out solutions which grow *exponentially* with z as physically irrelevant, then naively inverting the eigenvalue relation (2.12) for σ setting $\mu = i\omega$ for the class of large-z solutions found for $Re = 0$ yields

$$\sigma = k + 2 - \frac{i\omega}{2}, \quad k = 1, 2... \tag{7.36}$$

Since $\mathrm{Re}[\sigma] = k + 2 > 0$ for $k = 1, 2..$, implying algebraic *decay* as $z \to \infty$ of *all* the solutions of the form (7.11) forced by a purely oscillatory excitation of frequency ω (note that the analysis here made no *a priori* assumptions on the sign of $\mathrm{Re}[\sigma]$), this *suggests* spatial stability for $Re = 0$, but since we have not been able to systematically find all modes, no such comprehensive statement can be made. However, it can further be *speculated* that a possible solution (after transients) for $Re = 0$ to an initial value problem (IVP) with, say, no initial disturbance in $z \geq 0$ and forced by an *appropriate* excitation of single frequency ω at some z-station near the stagnation point could be written

$$w(r, z, t) \sim \mathrm{Re}\left[e^{-i\omega t} \sum_{k=1}^{\infty} A_k(\omega) \frac{e^{-r^2/2} L_k^{(0)}(r^2/2)}{z^{k+2-i\omega/2}}\right] \quad \text{as } z \to \infty \tag{7.37}$$

for some $\{A_k(\omega) | k = 1, 2..\}$, with similar expressions for u and v. By the term "appropriate" we mean a specially manufactured disturbance that will excite (at large

z) only those modes that we have explicitly found. Since we have not systematically found *all* modes, we cannot hope to write down the most general large-z asymptotic solution generated by a *general* oscillatory excitation of frequency ω. Note that for $Re \neq 0$, again formally inverting (7.35) for σ with $\mu = i\omega$ implies that

$$\sigma = \frac{2k + 4 - i\omega}{2 - Re\,\frac{\langle e^{-r^2/2}\bar{u}(r),\ \bar{v}(r)\rangle}{\langle \bar{v}(r),\ \bar{v}(r)\rangle}} \tag{7.38}$$

It is then seen from (7.38) (setting $\mu = i\omega$) that by introducing a weak Burgers vortex, the real part of the exponent of algebraic decay of the perturbation swirl velocity v is seen to increase or decrease according as the sign of the real quantity

$$\frac{\langle e^{-r^2/2}\bar{u}(r),\ \bar{v}(r)\rangle}{\langle \bar{v}(r),\ \bar{v}(r)\rangle} \tag{7.39}$$

is positive or negative, while that of u and w remains the same. Thus we might say that the particular $Re = 0$ solutions given in (M.1) become *more* or *less* spatially stable by the addition of a weak Burgers vortex according as the real quantity in (7.39) is positive or negative (note that the quantity in (7.39) depends implicitly on the integer k).

Even if we had systematically found all possible asymptotic behaviours, it would still be necessary to determine, using perhaps some generalized notion of group velocity, which modes propagate downstream of the excitation i.e. *towards* $z \to \infty$, and in particular that the proposed exponentially growing modes can be genuinely discounted for the physical reasons just conjectured. In general, given the complexity of the equations, a numerical solution of the full IVP will probably be needed to verify or disclaim these conjectures. This would constitute a somewhat formidable undertaking, especially if perturbations with azimuthal dependence are also included, and this is left for future study. In any event, it will be of great interest to see precisely what role the explicit asymptotic solutions found here play in the linear stability problem of Burgers vortex to axially-varying perturbations for small Reynolds numbers.

7.6 Summary

In summary, the analytical observations presented in this chapter throw light on the structure at large axial distances of a certain class of solutions of the linearized disturbance equations about the Burgers vortex. Given the complexity of the equations, it is remarkable that any such explicit analytical insights can be made at all. In this section it has further been argued that these observations are important for providing clues for the formulation of the linear stability problem for the vortex to general three-dimensional disturbances. At the very least, the results allow some definite mathematical questions to be asked which a future numerical treatment of the linear stability problem might attempt to answer. Certainly they suggest that allowance should be made in any numerical treatment for a *continuous* spectrum associated with the z direction and a *discrete* spectrum associated with the r direction (some collocation method using the complete set of Laguerre polynomials found above seems appropriate). The solutions found above might provide a useful check for a numerical code. Finally, we remark that the results also suggest the possible use of some form of Mellin transform technique as a tool in the numerical study of this problem.

Appendix A Proof of Theorem 1.4.2

Proof: First, assume that $J^0_{k_0}(t) = 0$ for $k_0 \geq M - M_0$. Then, from (1.39) and the definition of $J^0_{k_0}$ in (1.47), it follows that

$$\oint_C \zeta^j H(\zeta, t)\, d\zeta = 0 \text{ for } j = k_0 - M + M_0 \geq 0 \tag{A.1}$$

where

$$H(\zeta, t) = \zeta^M \bar{h}(1/\zeta, t)\, z_\zeta \tag{A.2}$$

since $H(\zeta, t)$ is known to be analytic on $|\zeta| = 1$, it must have a Laurent series convergent for $|\zeta| = 1$ (and locally in an enclosing annulus). Writing this as

$$H(\zeta, t) = \sum_{j=-\infty}^{\infty} H_n(t)\, \zeta^n \tag{A.3}$$

it is clear that (A.1) implies that $H_{-j-1} = 0$ for $j \geq 0$, i.e. all negative coefficients of the Laurent expansion for $H(\zeta, t)$ are zero. Thus $H(\zeta, t)$ is analytic in $|\zeta| \leq 1$. Since it is known that z_ζ is analytic and nonzero there, then it follows that $\zeta^M \bar{h}(\frac{1}{\zeta}, t)$ is also analytic in $|\zeta| \leq 1$. We conclude that $h(\zeta, t)$ must be polynomial of degree at most M.

Conversely, assume that $h(\zeta, t)$ is a polynomial of degree M or less. It follows that $\zeta^M \bar{h}(1/\zeta, t) z_\zeta$ is analytic for $|\zeta| \leq 1$. By Cauchy's theorem, we deduce $J^0_{k_0}(t) = 0$ for $k_0 \geq M - M_0$ and the proof of Theorem 1.4.2 is complete.

Appendix B Proof of Theorem 1.4.4

Proof: Note that since $M - M_0 \geq r_0$ by (1.34), then (1.53) gives the appropriate evolution equation for $J^0_{k_0}$ when $k_0 \geq M - M_0$. By inspection of (1.53), $J^0_{k_0}(t) = 0$ for $k_0 \geq M - M_0$ is clearly a solution of the initial value problem. However, this does not address the question of uniqueness. In order to show uniqueness, for $|\zeta| \leq 1$, it is convenient to express

$$I(\zeta, t) = \sum_{n=0}^{\infty} I_n \, \zeta^n \tag{B.1}$$

$$\sum_{p=1}^{N} \gamma_p \frac{\zeta I(\zeta, t) - \bar{\zeta}_p^{-1} I(\bar{\zeta}_p^{-1}, t)}{\zeta - \bar{\zeta}_p^{-1}} = \sum_{n=0}^{\infty} T_n \, \zeta^n = T(\zeta) \tag{B.2}$$

It is clear from (1.50) that

$$d_n = -k_0 I_n + T_n, \; n \geq 0 \tag{B.3}$$

Note that I_n and T_n are not dependent on k_0, unlike d_n defined in (1.50). Equation (1.53) can be then be rewritten as

$$\dot{J}^0_{k_0} = -\sum_{j=0}^{\infty} k_0 I_j J^0_{(k_0+j)} + \sum_{j=0}^{\infty} T_j J^0_{(k_0+j)} \tag{B.4}$$

It is convenient to extend the definition of I_j and T_j for $j < 0$ by setting them to zero. Then for $k_0 \geq M - M_0$,

$$\dot{J}^0_{k_0} = -\sum_{j=-\infty}^{\infty} k_0 I_j J^0_{(k_0+j)} + \sum_{j=-\infty}^{\infty} T_j J^0_{(k_0+j)} \tag{B.5}$$

We define new variables

$$U_k(t) = J_k^0(t) \text{ for } k \geq M - M_0 \tag{B.6}$$

Then for $k \geq M - M_0$

$$\dot{U}_k = - \sum_{j=-\infty}^{\infty} k \, I_j \, U_{(k+j)} + \sum_{j=-\infty}^{\infty} T_j \, U_{(k+j)} \tag{B.7}$$

We extend U_k to $k < M - M_0$ by requiring $U_k(0) = 0$ and demanding that it satisfies (B.7), even for $k < M - M_0$. If we now define

$$U(\zeta, t) = \sum_{k=-\infty}^{\infty} U_k(t) \zeta^k \tag{B.8}$$

$$\hat{I}(\zeta, t) = \sum_{n=-\infty}^{\infty} I_n \zeta^{-n} = I(\zeta^{-1}, t) \tag{B.9}$$

$$\hat{T}(\zeta, t) = \sum_{n=-\infty}^{\infty} T_n \zeta^{-n} = T(\zeta^{-1}, t) \tag{B.10}$$

Multiplying (B.7) by ζ^k and summing over k it is clear that $U(\zeta, t)$ satisfies

$$U_t + \zeta(\hat{I} \, U)_\zeta - \hat{T} \, U = 0 \tag{B.11}$$

We know that as long as $z_\zeta \neq 0$ in $|\zeta| \leq 1$, $I(\zeta, t)$ defined by (1.38) is analytic for $|\zeta| \leq 1$. This implies $\hat{I}(\zeta, t) = I(\zeta^{-1}, t)$ is analytic for $|\zeta| \geq 1$. Further, by inspection, it is clear that $\hat{T}(\zeta, t)$ is analytic in this domain as well. The initial conditions on $J_{k_0}^0$ for $k_0 \geq M - M_0$ imply $U_{k_0} = 0$ for all k_0 and hence $U(\zeta, 0) = 0$. From the well known theory of first order partial differential equations, whose coefficients are known *a priori* to be analytic over some domain, it follows from (B.11) that the unique solution is $U(\zeta, t) = 0$. This implies all $U_k(t)$ (and hence all $J_k^0(t)$) for $k \geq M - M_0$ are zero. Thus, Theorem 1.4.4 is proved.

Appendix C Proof of Theorem 1.5.1

Proof: We use Theorem 1.4.1 and the expression for K_j in (1.56) to conclude that

$$\begin{aligned}
\dot{J}^j_{k_j}(t) &= k_j \oint_C K_j(\zeta,t;k_j)\left[\frac{-\frac{d}{dt}(\bar{\zeta}_j^{-1}) - \zeta I(\zeta,t)}{\zeta-\bar{\zeta}_j^{-1}}\right]\bar{z}(\bar{\zeta},t)z_\zeta(\zeta,t)d\zeta \\
&+ \sum_{\substack{p=1\\p\neq j}}^{N}\gamma_p \oint_C K_j(\zeta,t;k_j)\left[\frac{-\frac{d}{dt}\bar{\zeta}_p^{-1} - \zeta I(\zeta,t)}{\zeta-\bar{\zeta}_p^{-1}}\right]\bar{z}(\bar{\zeta},t)z_\zeta(\zeta,t)d\zeta \\
&- (M-M_0)\oint_C K_j(\zeta,t;k_j)I(\zeta,t)\bar{z}(\bar{\zeta},t)z_\zeta(\zeta,t)d\zeta \\
&+ \oint_C K_j(\zeta,t;k_j)2G(\zeta,t)z_\zeta(\zeta,t)d\zeta
\end{aligned} \tag{C.1}$$

Using (1.51) the integrands in (C.1) are seen to be analytic for $|\zeta| \leq 1$, except possibly at $\zeta = \bar{\zeta}_j^{-1}$. We deform the contour and rewrite (C.1) as

$$\begin{aligned}
\dot{J}^j_{k_j}(t) &= k_j \oint_{|\zeta-\bar{\zeta}_j^{-1}|=\epsilon} K_j(\zeta,t;k_j)\left[\frac{-\frac{d}{dt}(\bar{\zeta}_j^{-1}) - \zeta I(\zeta,t)}{\zeta-\bar{\zeta}_j^{-1}}\right]\bar{z}(\bar{\zeta},t)z_\zeta(\zeta,t)d\zeta \\
&+ \sum_{\substack{p=1\\p\neq j}}^{N}\gamma_p \oint_{|\zeta-\bar{\zeta}_j^{-1}|=\epsilon} K_j(\zeta,t;k_j)\left[\frac{-\frac{d}{dt}(\bar{\zeta}_p^{-1}) - \zeta I(\zeta,t)}{\zeta-\bar{\zeta}_p^{-1}}\right]\bar{z}(\bar{\zeta},t)z_\zeta(\zeta,t)d\zeta \\
&- (M-M_0)\oint_{|\zeta-\bar{\zeta}_j^{-1}|=\epsilon} K_j(\zeta,t;k_j)I(\zeta,t)\bar{z}(\bar{\zeta},t)z_\zeta(\zeta,t)d\zeta \\
&+ \oint_{|\zeta|=1} K_j(\zeta,t;k_j)2G(\zeta,t)z_\zeta(\zeta,t)d\zeta
\end{aligned} \tag{C.2}$$

where ϵ is chosen small enough to ensure that the series in (1.57) is convergent for $|\zeta-\bar{\zeta}_j^{-1}| \leq \epsilon$. Using (1.57), and carrying out term by term integration (valid since the convergence is uniform), the result (1.58) immediately follows. Further, if $k_j \geq r_j$ it is clear that the integrand $K_j(\zeta,t;k_j)2G(\zeta,t)z_\zeta(\zeta,t)$ is analytic everywhere in $|\zeta| \leq 1$ and hence (1.59) follows. The proof of the Theorem 1.5.1 is then complete.

Appendix D Proof of Theorem 1.6.1

Proof: It is clear from the results of previous sections that the evolution of the blob is given by

$$z(\zeta, t) = \frac{h(\zeta, t)}{\prod_{j=1}^{N}(\zeta - \zeta_j(t))^{\gamma_j}} \tag{D.1}$$

where $h(\zeta, t)$ remains a polynomial of degree M, and the poles ζ_j, $j = 1..N$, evolve according to (1.51). Suppose there exists an index j such that $\gamma_j = 1$ with $r_j = 0$, *i.e.* $G(\zeta, t)$ is free of any singularity at $\zeta = \bar{\zeta}_j^{-1}$. Consider $k_j = 0$ in (1.58); it is clear from (1.58) that since $r_j = 0$,

$$\dot{J}_0^j = -\hat{d}_0^j J_0^j \tag{D.2}$$

Using (1.51) and (1.57), it follows that

$$\hat{d}_0^j = \sum_{\substack{p=1 \\ p \neq j}}^{N} \gamma_p \left[\frac{-\frac{d}{dt}(\bar{\zeta}_p^{-1}) + \frac{d}{dt}(\bar{\zeta}_j^{-1})}{\bar{\zeta}_j^{-1} - \bar{\zeta}_p^{-1}} \right] - (M - M_0) I(\bar{\zeta}_j^{-1}, t) \tag{D.3}$$

From (D.2) and (D.3),

$$\frac{d}{dt}\log(J_0^j(t)) = \sum_{\substack{p=1 \\ p \neq j}}^{N} \gamma_p \frac{d}{dt}\log(\bar{\zeta}_j^{-1} - \bar{\zeta}_p^{-1}) - (M - M_0)\frac{d}{dt}\log(\bar{\zeta}_j) \tag{D.4}$$

Integrating with respect to time yields,

$$B_j = \frac{J_0^j(t)\bar{\zeta}_j^{M-M_0}}{\prod_{\substack{p=1 \\ p \neq j}}^{N}(\bar{\zeta}_j^{-1} - \bar{\zeta}_p^{-1})^{\gamma_p}} \tag{D.5}$$

where the complex constants B_j are determined from initial conditions. Hence the theorem is proved.

Appendix E Dirichlet Formula for an Annulus

In this appendix we derive the Poisson integral formula for a general function $f(\zeta) = u(r,\phi) + iv(r,\phi)$ (where $\zeta = r\exp(i\phi)$) analytic in an annulus $\rho < |\zeta| < 1$ in terms of the values of u on the two boundaries $|\zeta| = 1$ and $|\zeta| = \rho$. Since $f(\zeta)$ is analytic in the annulus, its real part will be harmonic there. One way to derive the result is to use the well-known result that a general harmonic function in the annulus can be written

$$u(r,\phi) = a_0 + \sum_{n=1}^{\infty}\left[a_n r^n + \frac{c_n\rho^n}{r^n}\right]\cos(n\phi) + \sum_{n=1}^{\infty}\left[b_n r^n + \frac{d_n\rho^n}{r^n}\right]\sin(n\phi) \tag{E.1}$$

where

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} u(1,\phi')d\phi' = \frac{1}{2\pi}\int_0^{2\pi} u(\rho,\phi')d\phi' \tag{E.2}$$

(this might be termed an 'averaging condition') and for $n \geq 1$ it is easily shown that

$$a_n = \frac{1}{\pi(1-\rho^{2n})}\int_0^{2\pi}\left[u(1,\phi') - \rho^n u(\rho,\phi')\right]\cos(n\phi')d\phi' \tag{E.3}$$

$$b_n = \frac{1}{\pi(1-\rho^{2n})}\int_0^{2\pi}\left[u(1,\phi') - \rho^n u(\rho,\phi')\right]\sin(n\phi')d\phi' \tag{E.4}$$

$$c_n = \frac{1}{\pi(1-\rho^{2n})}\int_0^{2\pi}\left[u(\rho,\phi') - \rho^n u(1,\phi')\right]\cos(n\phi')d\phi' \tag{E.5}$$

$$d_n = \frac{1}{\pi(1-\rho^{2n})}\int_0^{2\pi}\left[u(\rho,\phi') - \rho^n u(1,\phi')\right]\sin(n\phi')d\phi' \tag{E.6}$$

Substituting these integral expressions for the coefficients into E.1 yields

$$u(r,\phi) = \frac{1}{2\pi}\int_0^{2\pi} u(1,\phi')d\phi' + \frac{1}{\pi}\int_0^{2\pi} d\phi' u(1,\phi')\left(\sum_{n=1}^{\infty}\left[r^n - \frac{\rho^{2n}}{r^n}\right]\frac{\cos(n(\phi-\phi'))}{(1-\rho^{2n})}\right)$$
$$+\frac{1}{\pi}\int_0^{2\pi} d\phi' u(\rho,\phi')\left(\sum_{n=1}^{\infty}\left[-r^n\rho^n + \frac{\rho^n}{r^n}\right]\frac{\cos(n(\phi-\phi'))}{(1-\rho^{2n})}\right) \quad \text{(E.7)}$$

We now make two important observations: first,

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{(1-\rho^{2n})}\left[r^n - \frac{\rho^{2n}}{r^n}\right]\cos(n(\phi-\phi'))\\
&= \sum_{n=1}^{\infty}\left[\frac{\rho^{2n}}{(1-\rho^{2n})}(r^n - \frac{1}{r^n}) + r^n\right]\cos(n(\phi-\phi')\\
&= Re\sum_{n=1}^{\infty}\left[\frac{\rho^{2n}}{(1-\rho^{2n})}\left[\left(\frac{\zeta}{\zeta'}\right)^n - \left(\frac{\zeta}{\zeta'}\right)^{-n}\right]\right.\\
&\left. + \sum_{n=1}^{\infty}\left(\frac{\zeta}{\zeta'}\right)^n\right]\end{aligned} \quad \text{(E.8)}$$

since on $|\zeta'| = 1$, $\zeta' = \exp(i\phi')$. next we note that,

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{(1-\rho^{2n})}\left[\frac{\rho^n}{r^n} - \rho^n r^n\right]\cos(n(\phi-\phi'))\\
&= \sum_{n=1}^{\infty}\left[\frac{\rho^{2n}}{(1-\rho^{2n})}\left[\left(\frac{\rho}{r}\right)^n - \left(\frac{r}{\rho}\right)^n\right] + \left(\frac{\rho}{r}\right)^n\right]\cos(n(\phi-\phi'))\\
&= Re\sum_{n=1}^{\infty}\left[\frac{\rho^{2n}}{(1-\rho^{2n})}\left[-\left(\frac{\zeta}{\zeta'}\right)^n + \left(\frac{\zeta}{\zeta'}\right)^{-n}\right]\right.\\
&\left. + \sum_{n=1}^{\infty}\left(\frac{\zeta'}{\zeta}\right)^n\right]\end{aligned} \quad \text{(E.9)}$$

Thus substituting these expressions (and summing the geometric series appearing in them) we can write

$$f(\zeta) = \frac{1}{2\pi}\int_0^{2\pi} d\phi' u(1,\phi')\left[1 + \frac{2\frac{\zeta}{\zeta'}}{(1-\frac{\zeta}{\zeta'})} + 2\sum_{n=1}^{\infty}\frac{\rho^{2n}}{(1-\rho^{2n})}\left[\left(\frac{\zeta}{\zeta'}\right)^n - \left(\frac{\zeta}{\zeta'}\right)^{-n}\right]\right]$$
$$+\frac{1}{2\pi}\int_0^{2\pi} d\phi' u(\rho,\phi')\left[\frac{2\frac{\zeta'}{\zeta}}{(1-\frac{\zeta'}{\zeta})} - 2\sum_{n=1}^{\infty}\frac{\rho^{2n}}{(1-\rho^{2n})}\left[\left(\frac{\zeta}{\zeta'}\right)^n - \left(\frac{\zeta}{\zeta'}\right)^{-n}\right]\right] + iC \quad \text{(E.10)}$$

where C is some purely real constant. This can be rewritten as

$$\begin{aligned}
f(\zeta) = \frac{1}{2\pi}\int_0^{2\pi} d\phi' u(1,\phi')\left[\frac{1+\frac{\zeta}{\zeta'}}{1-\frac{\zeta}{\zeta'}} + 2\sum_{n=1}^{\infty}\frac{\rho^{2n}}{(1-\rho^{2n})}\left[\left(\frac{\zeta}{\zeta'}\right)^n - \left(\frac{\zeta}{\zeta'}\right)^{-n}\right]\right] \\
- \frac{1}{2\pi}\int_0^{2\pi} d\phi' u(\rho,\phi')\left[\frac{1+\frac{\zeta}{\zeta'}}{1-\frac{\zeta}{\zeta'}} + 2\sum_{n=1}^{\infty}\frac{\rho^{2n}}{(1-\rho^{2n})}\left[\left(\frac{\zeta}{\zeta'}\right)^n - \left(\frac{\zeta}{\zeta'}\right)^{-n}\right]\right] \\
- \frac{1}{2\pi}\int_0^{2\pi} d\phi' u(\rho,\phi') + iC
\end{aligned} \tag{E.11}$$

Now note that the kernel function can be rewritten in a natural way in terms of the function $P(\zeta)$ which is defined in Appendix F. To see this, note that

$$\begin{aligned}
1 - 2\zeta\frac{P'(\zeta)}{P(\zeta)} &= 1 + 2\zeta\left(\frac{1}{1-\zeta} + \sum_{m=1}^{\infty}\frac{\rho^{2m}}{1-\rho^{2m}\zeta} - \sum_{m=1}^{\infty}\frac{\rho^{2m}}{\zeta^2(1-\frac{\rho^{2m}}{\zeta})}\right) \\
&= \frac{1+\zeta}{1-\zeta} + 2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\rho^{2mn}\left(\zeta^n - \frac{1}{\zeta^n}\right) \\
&= \frac{1+\zeta}{1-\zeta} + 2\sum_{n=1}^{\infty}\left(\zeta^n - \frac{1}{\zeta^n}\right)\frac{\rho^{2n}}{1-\rho^{2n}}
\end{aligned} \tag{E.12}$$

Using this result we clearly get (in obvious notation)

$$\begin{aligned}
f(\zeta) = \frac{1}{2\pi i}\int_{|\zeta'|=1}\frac{d\zeta'}{\zeta'}u(\zeta')\left(1 - 2\frac{\zeta}{\zeta'}\frac{P'(\frac{\zeta}{\zeta'})}{P(\frac{\zeta}{\zeta'})}\right) \\
- \frac{1}{2\pi i}\int_{|\zeta'|=\rho}\frac{d\zeta'}{\zeta'}u(\zeta')\left(1 - 2\frac{\zeta}{\zeta'}\frac{P'(\frac{\zeta}{\zeta'})}{P(\frac{\zeta}{\zeta'})}\right) \\
- \frac{1}{2\pi i}\int_{|\zeta'|=\rho}\frac{d\zeta'}{\zeta'}u(\zeta') + iC
\end{aligned} \tag{E.13}$$

Thus the result is an integral representation of the function $f(\zeta)$ in terms of its real parts on the two boundaries of the annulus.

Appendix F The Function $P(\zeta)$

The function $P(\zeta)$ is defined, for all $\zeta \neq 0$, by the following product representation

$$P(\zeta) = (1-\zeta) \prod_{m=1}^{\infty} (1-\rho^{2m}\zeta) \prod_{n=1}^{\infty} \left(1 - \frac{\rho^{2n}}{\zeta}\right) \tag{F.1}$$

where ρ is taken to be any positive real number, $0 < \rho < 1$. The function has an implicit dependence on the parameter ρ which is suppressed in the present notation.

We note the following important properties of this function:

$$P(\frac{1}{\zeta}) = -\frac{1}{\zeta} P(\zeta) \tag{F.2}$$

$$P(\rho^2\zeta) = -\frac{1}{\zeta} P(\zeta) \tag{F.3}$$

$P(\zeta)$ is related to the first theta function via

$$P(\zeta) = -iG^{-1}\rho^{-1/4} e^{i\pi u} \theta_1(\pi u, \rho) \tag{F.4}$$

where

$$\zeta = e^{2i\pi u} \text{ and } G = \prod_{n=1}^{\infty} (1-\rho^{2n}) \tag{F.5}$$

See Whittaker and Watson [13] for more details. $P(\zeta)$ has *simple* zeros at all points $\zeta = \rho^{2n}$ where n is any integer, and no zeros at any other points. The origin and the point at infinity are limit points of these zeros. It is also noted that $P(\zeta)$ is analytic everywhere in the finite ζ plane apart from the origin.

Further properties of the function $P(\zeta)$ can be found, for example, in [13].

Appendix G Proof of Theorem 2.4.1

Proof: First note that on $|\zeta| = 1$

$$\frac{d}{dt}z = -2F + \zeta I z_\zeta \tag{G.1}$$

while on $|\zeta| = \rho$

$$\frac{d}{dt}z = -2F + \zeta I z_\zeta + \frac{\dot{\rho}}{\rho}\zeta z_\zeta \tag{G.2}$$

where $\frac{d}{dt}z$ is defined to be the time derivative of z, keeping $\nu = Arg\ \zeta$ fixed. Note, this is **not** the same as $z_t(\zeta, t)$ on the inner boundary $\zeta = \rho e^{i\nu}$. Throughout this proof, all conjugate functions are understood to be functions of the conjugate variable $\bar{\zeta}$. Also $z_\zeta(\zeta, t)$ is understood to mean the partial derivative of z with respect to the first variable. Consider the time derivative of $J_K(t)$:

$$\frac{d}{dt}J_K(t) = \oint_{\partial C} \frac{dK}{dt}\bar{z}z_\zeta d\zeta + K\frac{d\bar{z}}{dt}z_\zeta d\zeta + K\bar{z}\frac{d}{dt}\left(z_\zeta d\zeta\right) \tag{G.3}$$

Using (G.1) and (G.2) this becomes (writing out the integrals on each boundary separately)

$$\begin{aligned}\frac{d}{dt}J_K(t) = &\oint_{|\zeta|=1} K_t\bar{z}z_\zeta + K\left(-2\bar{F} + \frac{1}{\zeta}\bar{I}\bar{z}_\zeta\right)z_\zeta + K\bar{z}\left(-2F_\zeta + \frac{\partial}{\partial\zeta}\left(\zeta I z_\zeta\right)\right)d\zeta \\ &- \oint_{|\zeta|=\rho}\left(K_t + \frac{\dot{\rho}}{\rho}\zeta K_\zeta\right)\bar{z}z_\zeta + Kz_\zeta\left(-2\bar{F} + \frac{\rho^2}{\zeta}\bar{I}\bar{z}_\zeta + \frac{\rho\dot{\rho}}{\zeta}\bar{z}_\zeta\right) \\ &+ K\bar{z}\left(-2F_\zeta + \frac{\partial}{\partial\zeta}\left(\zeta I z_\zeta\right) + \frac{\dot{\rho}}{\rho}\frac{\partial}{\partial\zeta}\left(\zeta z_\zeta\right)\right)d\zeta\end{aligned} \tag{G.4}$$

Now we use the stress conditions on the boundary circles of the annulus which takes the form on $|\zeta| = 1$,

$$-2\bar{F}z_\zeta - 2F_\zeta \bar{z} = 2Gz_\zeta - \frac{z_\zeta^{1/2}\bar{z}_\zeta^{1/2}}{\zeta} \tag{G.5}$$

while on $|\zeta| = \rho$,

$$-2\bar{F}z_\zeta - 2F_\zeta \bar{z} = 2Gz_\zeta + \frac{\rho z_\zeta^{1/2}\bar{z}_\zeta^{1/2}}{\zeta} \tag{G.6}$$

Consider first the integral around $|\zeta| = 1$ in (G.4). Using parts and the stress condition (G.5) this takes the form

$$\oint_{|\zeta|=1} (K_t - \zeta I K_\zeta)\,\bar{z}z_\zeta + K\left(2Gz_\zeta - \frac{z_\zeta^{1/2}\bar{z}_\zeta^{1/2}}{\zeta} + \left(I + \bar{I}\right)\frac{z_\zeta \bar{z}_\zeta}{\zeta}\right) \tag{G.7}$$

Now using the fact that on $|\zeta| = 1$

$$I + \bar{I} = 2\ Re\ I \ = \ \frac{1}{z_\zeta^{1/2}\bar{z}_\zeta^{1/2}} \tag{G.8}$$

which follows from (2.13) (with $A_O = 0$). This reduces to

$$\oint_{|\zeta|=1} (K_t - \zeta I K_\zeta)\,\bar{z}z_\zeta + 2KGz_\zeta d\zeta \tag{G.9}$$

Now consider the integral around $|\zeta| = \rho$ in (G.4). By parts and using the stress condition (G.6) we get

$$\oint_{|\zeta|=\rho} \left(K_t + \frac{\dot{\rho}}{\rho}\zeta K_\zeta - \zeta I K_\zeta\right)\bar{z}z_\zeta + K\left(2Gz_\zeta + \frac{\rho z_\zeta^{1/2}\bar{z}_\zeta^{1/2}}{\zeta} + \frac{\rho^2}{\zeta}\left(I + \bar{I}\right)z_\zeta\bar{z}_\zeta\right)$$
$$- \frac{\dot{\rho}}{\rho}\zeta K_\zeta \bar{z}z_\zeta + \frac{2\dot{\rho}\rho}{\zeta}K z_\zeta \bar{z}_\zeta\ d\zeta \tag{G.10}$$

Using the fact that on $|\zeta| = \rho$

$$I + \bar{I} = 2\ Re\ I \ = \ -\frac{1}{\rho z_\zeta^{1/2} \bar{z}_\zeta^{1/2}} - \frac{2\dot{\rho}}{\rho} \tag{G.11}$$

that follows from (2.13) (with $A_I = 0$), (G.10) becomes

$$\oint_{|\zeta|=\rho} (K_t - \zeta I K_\zeta)\, \bar{z} z_\zeta d\zeta + 2KGz_\zeta d\zeta \tag{G.12}$$

Subtracting (G.12) from (G.9) gives the required result.

Appendix H Proof of Theorem 2.4.2

Proof: First we observe that

$$\frac{\partial}{\partial t}K_0(\zeta,t;k_0) = K_0(\zeta,t;k_0)\left[\sum_{j=1}^{N}\gamma_j\zeta\frac{d\bar{\zeta}_j}{dt}\frac{P'(\zeta\bar{\zeta}_j)}{P(\zeta\bar{\zeta}_j)} + \gamma_j\dot{\rho}\frac{P_\rho(\zeta\bar{\zeta}_j)}{P(\zeta\bar{\zeta}_j)}\right] \tag{H.1}$$

where $P'(\zeta)$ denotes the usual differentiation with respect to the argument, and P_ρ denotes differentiation with respect to the parameter ρ. Note that our notation suppresses the dependence of $P(\zeta)$ on the parameter ρ. Also,

$$\frac{\partial}{\partial\zeta}K_0(\zeta,t;k_0) = K_0(\zeta,t;k_0)\left[\frac{k_0}{\zeta} + \sum_{j=1}^{N}\gamma_j\bar{\zeta}_j\frac{P'(\zeta\bar{\zeta}_j)}{P(\zeta\bar{\zeta}_j)}\right] \tag{H.2}$$

Applying the results of Theorem 2.4.1 we deduce

$$\dot{J}^0_{k_0}(t) = \oint_{\partial C}K_0 2Gz_\zeta d\zeta + \oint_{\partial C}K_0\left[-k_0 I + \sum_{j=1}^{N}\left(\gamma_j\zeta\frac{P'(\zeta\bar{\zeta}_j)}{P(\zeta\bar{\zeta}_j)}\left(\frac{d}{dt}\bar{\zeta}_j - \bar{\zeta}_j I(\zeta,t)\right) + \gamma_j\dot{\rho}\frac{P_\rho(\zeta\bar{\zeta}_j)}{P(\zeta\bar{\zeta}_j)}\right)\right]\bar{z}z_\zeta\, d\zeta \tag{H.3}$$

We now define the following function

$$T(\zeta,t) \equiv \sum_{j=1}^{N}\gamma_j\zeta\frac{P'(\zeta\bar{\zeta}_j)}{P(\zeta\bar{\zeta}_j)}\left(\frac{d\bar{\zeta}_j}{dt} - \bar{\zeta}_j I(\zeta,t)\right) + \gamma_j\dot{\rho}\frac{P_\rho(\zeta\bar{\zeta}_j)}{P(\zeta\bar{\zeta}_j)} \tag{H.4}$$

Note that

$$P_\rho(\zeta\bar{\zeta}_j) = -2P(\zeta\bar{\zeta}_j)\sum_{n=1}^{\infty} n\ \rho^{2n-1}\left(\frac{1}{\zeta\bar{\zeta}_j - \rho^{2n}} + \frac{1}{\frac{1}{\zeta\bar{\zeta}_j} - \rho^{2n}}\right) \tag{H.5}$$

Notice P_ρ/P is singular at $\zeta = \bar{\zeta}_j^{-1}\rho^{2n}$ for a *non-zero* integer n and therefore is analytic in C. The first term on the right hand side of (H.4) is also free of singularities in C provided $\zeta_j(t)$ is evolved according to $\frac{d\bar{\zeta}_j}{dt} = \bar{\zeta}_j I\left(\bar{\zeta}_j^{-1}, t\right)$, which can equivalently be written as

$$\frac{d\bar{\zeta}_j^{-1}}{dt} = -\bar{\zeta}_j^{-1} I\left(\bar{\zeta}_j^{-1}, t\right) \;\; j = 1..N \tag{H.6}$$

Since $T(\zeta, t)$ is then analytic in C it therefore has a Laurent series which we denote

$$T(\zeta, t) = \sum_{j=-\infty}^{\infty} T_j \zeta^j \tag{H.7}$$

This expansion is convergent everywhere in C. We also define the following Laurent series

$$I(\zeta, t) = \sum_{j=-\infty}^{\infty} I_j \zeta^j \tag{H.8}$$

which is also convergent in C. Using these expansions it is straightforward to see that equation (H.1) can be written

$$\dot{J}^0_{k_0}(t) = -\sum_{j=-\infty}^{\infty} k_0 I_j J^C_{k_0+j} + \sum_{j=-\infty}^{\infty} T_j J^0_{k_0+j} \tag{H.9}$$

Note that the first integral on the right hand side of (H.3) gives no contribution since $G(\zeta, t)$ is assumed to be analytic in C. We now define the function $J(\zeta, t)$ via

$$J(\zeta, t) \equiv \sum_{j=-\infty}^{\infty} J^0_j(t) \zeta^j \tag{H.10}$$

and also the following functions

$$\hat{I}(\zeta, t) = I(\zeta^{-1}, t) \tag{H.11}$$

$$\hat{T}(\zeta,t) = T(\zeta^{-1},t) \tag{H.12}$$

Multiplying (H.9) by ζ_0^k and summing over all integers k_0 yields the following partial differential equation for $J(\zeta,t)$

$$\frac{\partial J}{\partial t} + \zeta \frac{\partial}{\partial \zeta}(\ \hat{I} J\ ) - \hat{T} J = 0 \tag{H.13}$$

Note that in the annulus $1 < |\zeta| < \rho^{-1}$ the coefficient functions of the first order partial differential equation (H.13) are known *a priori* to be analytic. Thus if $J(\zeta,0) = 0$ in this domain, then by the well-known theory of first order linear partial differential equations whose coefficients are known *a priori* to be analytic over some domain, we deduce that the *unique* solution is $J(\zeta,t) = 0$ for all times that the solution exists. Hence the theorem is proved.

Appendix I Proof of Theorem 2.4.3

Proof: Define the Laurent series of z_ζ as follows

$$z_\zeta(\zeta,t) = \sum_{-\infty}^{\infty} Z_n \zeta^n \tag{I.1}$$

Since z_ζ is analytic in C then this series converges everywhere inside C and on the boundary ∂C. We also denote the Laurent expansion of $\bar{h}(\zeta,t)$ by

$$\bar{h}(\zeta,t) = \sum_{-\infty}^{\infty} H_n \zeta^n \tag{I.2}$$

which is also known to be convergent everywhere inside C and on the boundary ∂C. First note the following facts which result from the relation (2.27) for $z(\zeta,t)$ and the properties (F.2) and (F.3) (in Appendix F) for $P(\zeta)$

$$\bar{z}(\zeta^{-1},t) = R(t)\frac{\zeta^{M_0}\bar{h}(\zeta^{-1},t)}{\prod_{j=1}^{N}\left[P(\zeta\bar{\zeta}_j)\right]^{\gamma_j}} \tag{I.3}$$

$$\bar{z}(\rho^2\zeta^{-1},t) = \frac{\bar{h}(\rho^2\zeta^{-1},t)}{\prod_{j=1}^{N}\left[P(\zeta\bar{\zeta}_j)\right]^{\gamma_j}} \tag{I.4}$$

Now suppose that $J^0_{k_0}(t) = 0$ for all k_0. This implies that $\forall k_0$

$$R(t)\oint_{|\zeta|=1} \zeta^{k_0+M_0}\bar{h}(\zeta^{-1},t)z_\zeta(\zeta,t)\;d\zeta = \oint_{|\zeta|=\rho} \zeta^{k_0}\bar{h}(\rho^2\zeta^{-1},t)z_\zeta(\zeta,t)\;d\zeta \tag{I.5}$$

where each of the contour integrals in (I.5) is taken in the anticlockwise sense. Since $|\zeta^{-1}| = 1$ on $|\zeta| = 1$ and $|\rho^2\zeta^{-1}| = \rho$ on $|\zeta| = \rho$ then we can write

$$\bar{h}(\zeta^{-1},t) = \sum_{-\infty}^{\infty} H_n \zeta^{-n} \text{ on } |\zeta| = 1 \tag{I.6}$$

$$\bar{h}(\rho^2\zeta^{-1},t) = \sum_{-\infty}^{\infty} H_n \rho^{2n}\zeta^{-n} \text{ on } |\zeta| = \rho \tag{I.7}$$

Similarly expanding $z_\zeta(\zeta,t)$ as a Laurent series (valid on both boundaries of C) (I.5) becomes

$$R(t)\oint_{|\zeta|=1} \sum_{m=-\infty}^{\infty} \zeta^{k_0+m+M_0} \sum_{n=-\infty}^{\infty} H_n Z_{n+m} d\zeta = \oint_{|\zeta|=\rho} \sum_{m=-\infty}^{\infty} \zeta^{k_0+m} \sum_{n=-\infty}^{\infty} H_n \rho^{2n} Z_{n+m} d\zeta \tag{I.8}$$

for all k_0. Computing the integrals gives

$$R(t) \sum_{n=-\infty}^{\infty} H_n Z_{n-k_0-1-M_0} = \sum_{n=-\infty}^{\infty} H_n \rho^{2n} Z_{n-k_0-1} \; \forall k_0 \tag{I.9}$$

Now multiply equation (I.9) by ζ^{k_0+1} and sum over all k_0. Using k instead of k_0 we then get

$$R(t) \sum_{k=-\infty}^{\infty} \zeta^{k+1} \sum_{n=-\infty}^{\infty} H_n Z_{n-1-k-M_0} = \sum_{k=-\infty}^{\infty} \zeta^{k+1} \sum_{n=-\infty}^{\infty} H_n \rho^{2n} Z_{n-k-1} \tag{I.10}$$

Define the function $H(\zeta,t)$ by the Laurent series

$$H(\zeta,t) \equiv \sum_{n=-\infty}^{\infty} H_n \rho^{2n} \zeta^n \tag{I.11}$$

where the coefficients $\{H_n\}$ are the same as in (I.2). It is known that on $|\zeta| = \frac{1}{\rho}$ this series converges and equals $\bar{h}(\rho^2\zeta,t)$ there. Note also that the left hand side of (I.10) is equal to $R(t)\zeta^{-M_0}\bar{h}(\zeta,t)z_\zeta(\zeta^{-1},t)$ on $|\zeta| = 1$, where it is known to be analytic with a convergent Laurent expansion. This equality must hold anywhere the series converges. From consideration of the right hand side of (I.10), it is clear that it is equal to $H(\zeta,t)z_\zeta(\zeta^{-1},t)$ on $|\zeta| = \rho^{-1}$, where it is known to be analytic with a convergent Laurent expansion. Thus the series in (I.10) is convergent on $|\zeta| = \rho^{-1}$, as well. From the principle of analytic continuation, this implies

$$R(t)\zeta^{-M_0}\bar{h}(\zeta,t)z_\zeta(\zeta^{-1},t) = H(\zeta,t)z_\zeta(\zeta^{-1},t) \tag{I.12}$$

with each side of the equality (I.12) having the same convergent Laurent series. From the relation of H with $\bar{h}$, It follows that

$$R(t)\zeta^{-M_0}\bar{h}(\zeta,t) = \bar{h}(\rho^2\zeta,t) \tag{I.13}$$

Since we know that $\bar{h}(\zeta,t)$ is analytic (at least) in the entire annulus $\rho \leq |\zeta| \leq \rho^{-1}$, (I.13) furnishes the analytic continuation of $\bar{h}(\zeta,t)$ into the entire plane, excluding the points at 0 and ∞. This is the required result.

Conversely, if $\bar{h}(\zeta,t)$ satisfies condition (2.35) for all ζ, and is analytic everywhere except at 0 and ∞ then it is clear that $J^0_{k_0}(t)$ reduces to the integral around ∂C of the following function of ζ i.e.

$$R(t)\zeta^{k_0+M_0}\bar{h}(\zeta^{-1},t)z_\zeta(\zeta,t) \tag{I.14}$$

which is known to be analytic in C for all k_0, thus the result follows by Cauchy's theorem.

Appendix J Proof of Theorem 2.4.4

Proof: Suppose that $\bar{h}(\zeta, t)$ satisfies (2.36) and is analytic everywhere except possibly at zero and infinity. Consider the function $M(\zeta, t)$ defined by

$$M(\zeta, t) = \prod_{m=1}^{M_0} P\left(\frac{\zeta}{\beta_m}\right) \tag{J.1}$$

where $\{\beta_m(t)|m = 1..M_0\}$ are taken to satisfy

$$\prod_{m=1}^{M_0} [-\beta_m(t)] = R(t) \tag{J.2}$$

Note that $M(\zeta, t)$ satisfies the following functional equation which results from the properties of $P(\zeta)$ as described in Appendix F

$$M(\rho^2\zeta) = R(t)\zeta^{-M_0}M(\zeta, t) \tag{J.3}$$

Now define the function $N(\zeta, t)$ by

$$N(\zeta, t) \equiv \frac{\bar{h}(\zeta, t)}{M(\zeta, t)} \tag{J.4}$$

Then it is clear from the definitions of $N(\zeta, t)$ and $M(\zeta, t)$ and the known analyticity of $\bar{h}(\zeta, t)$ everywhere except at zero and infinity that $N(\zeta, t)$ is a meromorphic function everywhere (excluding 0 and ∞) with poles at the points $\{\beta_m(t)\}$ and all equivalent points $\{\rho^{2n}\beta_m(t)\}$, where n is an arbitrary integer. It is also easily seen that it satisfies the functional equation

$$N(\rho^2\zeta, t) = N(\zeta, t) \tag{J.5}$$

for all $\zeta \neq 0$. Thus, $N(\zeta,t)$ is a *loxodromic* function in the sense defined in Appendix L. By the representation theorem (Theorem 4 of Appendix L) for loxodromic functions we conclude that $N(\zeta,t)$ necessarily has a representation of the form

$$N(\zeta,t) = \bar{S}(t)\frac{\prod_{m=1}^{M_0} P(\frac{\zeta}{\bar{\eta}_m})}{\prod_{m=1}^{M_0} P(\frac{\zeta}{\beta_m})} \tag{J.6}$$

for some $\bar{S}(t)$ and some functions $\{\bar{\eta}_m(t)|m = 1..M_0\}$ satisfying the condition

$$\prod_{m=1}^{M_0}[-\bar{\eta}_m(t)] = \prod_{m=1}^{M_0}[-\beta_m(t)] = R(t) \tag{J.7}$$

Thus by comparison with the definition of $N(\zeta,t)$ we conclude that $\bar{h}(\zeta,t)$ can be written

$$\bar{h}(\zeta,t) = \bar{S}(t)\prod_{m=1}^{M_0} P\left(\frac{\zeta}{\bar{\eta}_m}\right) \tag{J.8}$$

for some $\bar{S}(t)$ with the $\{\bar{\eta}_m(t)\}$ satisfying (J.7). This is the required result.

The converse result is trivially established by using the properties F.2 and F.3 of the function $P(\zeta,t)$ in Appendix F.

Appendix K Proof of Theorem 2.6.1 (Theorem of Invariants)

Proof: Consider the time derivative of J_0^j. From (2.49), it follows that

$$\frac{dJ_0^j}{dt} = \oint_{\partial C} K_j \left[\sum_{\substack{p=1 \\ p\neq j}}^{N} \gamma_p \left(\frac{d\bar{\zeta}_p}{dt} - \bar{\zeta}_p I(\zeta, t) \right) \zeta \frac{P'(\zeta\bar{\zeta}_p)}{P(\zeta\bar{\zeta}_p)} + \gamma_p \dot{\rho} \frac{P_\rho(\zeta\bar{\zeta}_p)}{P(\zeta\bar{\zeta}_p)} \right] \bar{z} z_\zeta \, d\zeta \tag{K.1}$$

By inspection it can be seen that the only contribution to the integral in (K.1) comes from the simple pole in $\bar{z}$ at $\bar{\zeta}_j^{-1}$ since the rest of the integrand is analytic if the evolution of the poles is given by (2.40). Since the pole at $\bar{\zeta}_j^{-1}$ is *simple*, it is easily seen that (K.1) can be written as:

$$\frac{dJ_0^j}{dt} = \left[\sum_{\substack{p=1 \\ p\neq j}}^{N} \gamma_p \left(\frac{d\bar{\zeta}_p}{dt} - \bar{\zeta}_p I(\bar{\zeta}_j^{-1}, t) \right) \bar{\zeta}_j^{-1} \frac{P'(\bar{\zeta}_p\bar{\zeta}_j^{-1})}{P(\bar{\zeta}_p\bar{\zeta}_j^{-1})} + \gamma_p \dot{\rho} \frac{P_\rho(\bar{\zeta}_p\bar{\zeta}_j^{-1})}{P(\bar{\zeta}_p\bar{\zeta}_j^{-1})} \right] J_0^j \tag{K.2}$$

Using the fact that

$$I(\bar{\zeta}_j^{-1}, t) = -\frac{d\log(\bar{\zeta}_j^{-1})}{dt} \tag{K.3}$$

(K.2) can be written

$$\frac{d\log J_0^j(t)}{dt} = \sum_{\substack{p=1 \\ p\neq j}}^{N} \frac{d}{dt} \log\left(\left[P(\bar{\zeta}_p\bar{\zeta}_j^{-1}) \right]^{\gamma_p} \right) \tag{K.4}$$

which can clearly be directly integrated with respect to time to give the required result.

Appendix L Loxodromic Functions

Again it is assumed that the real parameter ρ is such that $0 < \rho < 1$. A loxodromic function $L(\zeta)$ is a meromorphic function satisfying the functional equation

$$L(\rho^2\zeta) = L(\zeta) \tag{L.1}$$

for all $\zeta \neq 0$. The *fundamental annulus* is defined as the annulus

$$\rho^2 < |\zeta| \leq 1 \tag{L.2}$$

Note that whenever zeros and poles are discussed in the following, each zero or pole is assumed to appear repeatedly according to its multiplicity. We now state, without proof, some of the important theorems concerning loxodromic functions that are used in the main body of this paper. The proofs can be found in the standard references (e.g. [12]–[14]).

Theorem 1

A loxodromic function $L(\zeta)$ has the same number of zeros and poles in the fundamental annulus.

Theorem 2

If a loxodromic function $L(\zeta)$ has zeros at points $\{\eta_j | j = 1..M\}$ and poles at points $\{\zeta_j | j = 1..M\}$ then

$$\frac{\prod_{j=1}^{M} \zeta_j}{\prod_{j=1}^{M} \eta_j} = \rho^{2\mu} \tag{L.3}$$

for some integer μ (positive, negative or zero).

Theorem 3

If $\zeta = \eta_1$ is a zero of a loxodromic function, it has countably many equivalent zeros at the points $\eta_1\rho^{2n}$ where n is any integer. Similarly if $\zeta = \zeta_1$ is a pole of a loxodromic *function it has countably many equivalent poles (of the same order) at points* $\zeta = \zeta_1 \rho^{2n}$.

Theorem 4

(Representation Theorem for Loxodromic Functions) Any loxodromic function with zeros at points $\{\eta_j | j = 1..M\}$ *and poles at points* $\{\zeta_j | j = 1..M\}$ *can be written*

$$L(\zeta) = R \frac{P(\frac{\zeta}{\eta_1})P(\frac{\zeta}{\eta_2})....P(\frac{\zeta}{\eta_M})}{P(\frac{\zeta}{\zeta_1})P(\frac{\zeta}{\zeta_2})....P(\frac{\zeta\rho^{2\mu}}{\zeta_M})} \tag{L.4}$$

for some integer μ.

Remark 1

Loxodromic functions and elliptic functions are intimately related. In particular, by use of the exponential map in Appendix F and Theorem 4, we see that a general loxodromic function $L(\zeta)$ can be written

$$L(\zeta) = \tilde{R} \frac{\theta_1[-i/2 \log(\frac{\zeta}{\eta_1}), \rho]...\theta_1[-i/2 \log(\frac{\zeta}{\eta_M}), \rho]}{\theta_1[-i/2 \log(\frac{\zeta}{\zeta_1}), \rho]...\theta_1[-i/2 \log(\frac{\zeta\rho^{2\mu}}{\zeta_M}), \rho]} \tag{L.5}$$

where μ is an integer, and $\tilde{R}$ is a constant. This is the well-known representation theorem for elliptic functions. In particular, defining $u = \log \zeta$ and $\hat{L}(u) \equiv L(\zeta)$ it is clear that

$$\hat{L}(u + 2\log\rho) = \hat{L}(u) \tag{L.6}$$

$$\hat{L}(u + 2\pi i) = \hat{L}(u) \tag{L.7}$$

i.e. $\hat{L}(u)$ is a meromorphic, doubly-periodic function of u, that is, an elliptic function. This identification is useful since the theory of elliptic functions is very well developed and many results can be imported to assist in the calculations carried out in this paper.

Remark 2

From the above representation theorem it is clear that a loxodromic function is completely determined once its zeros and poles in the fundamental annulus are known, as well as its value at one other point.

Theorem 5

A loxodromic function cannot have only one simple pole in the fundamental annulus.

Theorem 6

(Liouville Theorem for loxodromic functions) A loxodromic function with no poles is a constant function.

Further information on the theory of loxodromic and elliptic functions can be found in [12] [13] [14] [16] [17].

Appendix M Perturbation theory of small Re

The zeroth order solution is taken to be given by

$$\begin{aligned} w(r,z,t) &= \bar{w}(r)e^{-\mu_0 t}\frac{1}{z^{\sigma}} \\ v(r,z,t) &= \bar{v}(r)e^{-\mu_0 t}\frac{1}{z^{\sigma+1}} \\ u(r,z,t) &= \sigma U(r)e^{-\mu_0 t}\frac{1}{z^{\sigma+1}} \\ \mu_0 &= 2k-2\sigma+4 \end{aligned} \tag{M.1}$$

where we define

$$\begin{aligned} \bar{w}(r) &= e^{-r^2/2}L_k^{(0)}(r^2/2) \\ \bar{v}(r) &= r\ e^{-r^2/2}L_{k+2}^{(1)}(r^2/2) \\ U(r) &= \frac{1}{r}\int_0^r \tilde{r}e^{-\tilde{r}^2/2}L_k^{(0)}(\tilde{r}^2/2)d\tilde{r} \end{aligned} \tag{M.2}$$

It is taken to be understood that the solutions sought are *asymptotic* solutions valid at large z, although we shall use $=$ rather than $\sim$ throughout. We now seek to continue these solutions for small non-zero Re. In the following analysis, special care must be taken to ensure that we are always finding a perturbation to the above zeroth order solution for given k and σ, and that we do not add onto the perturbed solution any contributions from neighbouring solutions. This will ensure uniqueness of the perturbed solution.

We now attempt to solve the *same* large-z asymptotic equations as done in chapter 7 in the case of zero Reynolds number – in other words, the same dominant balance is expected to be good for solutions for small Re. First it is observed that the large-z asymptotic equation for $w(r,z,t)$ is independent of Reynolds number. Thus the perturbed solution for $w(r,z,t)$ must also have the form

$$w(r,z,t) = e^{-r^2/2} L_k^{(0)}(r^2/2) e^{-\mu t} \frac{1}{z^{\tilde{\sigma}}} \tag{M.3}$$

where we take

$$\mu = \mu_0 + Re\ \mu_1 + Re^2\ \mu_2 + ... \tag{M.4}$$

Note also that the eigenvalue condition continues to be

$$\mu = 2k - 2\tilde{\sigma} + 4,\ \ k \geq 1 \tag{M.5}$$

Thus we immediately conclude that

$$\tilde{\sigma} = \sigma - Re\ \frac{\mu_1}{2} - Re^2\ \frac{\mu_2}{2} - ... \tag{M.6}$$

It is seen that the perturbed expression for $w(r,z,t)$ has exactly the same functional form as the zeroth order solution but with perturbed parameters. By continuity, the same is true of the perturbed $u(r,z,t)$ which can be written

$$u(r,z,t) = \left(\frac{\tilde{\sigma}}{r} \int_0^r \tilde{r} e^{-\tilde{r}^2/2} L_k^{(0)}(\tilde{r}^2/2) d\tilde{r}\right) e^{-\mu t} \frac{1}{z^{\tilde{\sigma}+1}} \tag{M.7}$$

This is again fortuitous because it means that exactly the same arguments as used before ((7.19)-(7.25)) can be applied to the perturbed $u(r,z,t)$ to demonstrate that it is a function decaying exponentially as $r \to \infty$. It remains to determine $\mu_1,\ \mu_2...$ which are derived from solvability conditions for the perturbed $v(r,z,t)$. We note at this point that while solving for $v(r,z,t)$, care will be taken not to add into the solution any terms having the following form

$$\frac{CRe^j\ \log z\ \bar{v}(r)}{z^{\sigma+1}} e^{-\mu t} \tag{M.8}$$

for any integer j and any constant C. This is clearly the $O(Re^j)$ term in a small

Reynolds number expansion of

$$\frac{\bar{v}(r)}{z^{\hat{\sigma}+1}} e^{-\mu t} \text{ where } \hat{\sigma} = \sigma - C\ Re^{j} \tag{M.9}$$

It is straightforward to show that adding in any such solution would correspond to altering the parameter σ in the zeroth order solution, however it is assumed that a particular value of σ is specified *a priori*.

As a convenient shorthand we define the following linear operators:

$$\begin{aligned}
M(r,z) &\equiv \frac{\partial^2}{\partial r^2} + \left(r + \frac{1}{r}\right)\frac{\partial}{\partial r} + \left(\mu + 1 - \frac{1}{r^2}\right) - 2z\frac{\partial}{\partial z} \\
M_0(r,z) &\equiv \frac{\partial^2}{\partial r^2} + \left(r + \frac{1}{r}\right)\frac{\partial}{\partial r} + \left(\mu_0 + 1 - \frac{1}{r^2}\right) - 2z\frac{\partial}{\partial z} \\
M_0(r) &\equiv \frac{d^2}{dr^2} + \left(r + \frac{1}{r}\right)\frac{d}{dr} + \left(\mu_0 + 1 - \frac{1}{r^2}\right) + 2(\sigma + 1)
\end{aligned} \tag{M.10}$$

Note that the solution of the ordinary differential equation

$$M_0(r)\Omega(r) = 0 \tag{M.11}$$

is

$$\Omega(r) = A\bar{v}(r) \tag{M.12}$$

for some constant A.

Note also how the operator $M_0(r,z)$ acts on functions such as $\Omega(r)\frac{\log^p z}{z^{\sigma+1}}$ (p an integer):

$$M_0(r,z)\left(\Omega(r)\frac{\log^p z}{z^{\sigma+1}}\right) = \left(\frac{\log^p z}{z^{\sigma+1}}\right) M_0(r)\Omega(r) - 2p\left(\frac{\log^{(p-1)} z}{z^{\sigma+1}}\right)\Omega(r) \tag{M.13}$$

We now write

$$v(r,z,t) = \hat{v}(r,z)e^{-\mu t},\ \ u(r,z,t) = \hat{u}(r,z)e^{-\mu t} \tag{M.14}$$

where it is already known from (M.7) that

$$\hat{u}(r,z) = \tilde{\sigma}\frac{U(r)}{z^{\tilde{\sigma}+1}} \tag{M.15}$$

with $\tilde{\sigma}$ given by (M.6). In terms of this notation, the equation for $\hat{v}(r,z)$ can be written

$$M(r,z)\ \hat{v}(r,z) = (Re)\ e^{-r^2/2}\ \hat{u}(r,z) \tag{M.16}$$

This is a partial differential equation for $\hat{v}(r,z)$ with an $O(Re)$ forcing depending on $\hat{u}(r,z)$. Expanding (M.15) for small Re gives

$$\begin{aligned}\hat{u}(r,z) = \frac{\sigma U(r)}{z^{\sigma+1}}\Bigg(1 + Re\left(\frac{\mu_1 \log z}{2} - \frac{\mu_1}{2\sigma}\right) \\ + Re^2\left(\frac{{\mu_1}^2 \log^2 z}{8} + \log z\left(\frac{\mu_2}{2} - \frac{{\mu_1}^2}{4\sigma}\right) - \frac{\mu_2}{2\sigma}\right)\Bigg)\end{aligned} \tag{M.17}$$

We now write

$$\hat{v}(r,z) = v_0(r,z) + Re\ v_1(r,z) + Re^2\ v_2(r,z) + ... \tag{M.18}$$

Substituting the expansions for μ, $u(r,z,t)$ and $v(r,z,t)$ into (M.16) gives

$$\begin{aligned}\left(M_0(r,z) + Re\ \mu_1 + (Re^2)\mu_2 + ...\right)\left(v_0(r,z) + (Re)v_1(r,z) + (Re)^2 v_2(r,z) + ...\right) \\ = \sigma(Re)e^{-r^2/2}\left(\frac{U(r)}{z^{\sigma+1}}\right)\left(1 + (Re)\left(-\frac{\mu_1}{2\sigma} + \frac{\mu_1 \log z}{2}\right) + ...\right)\end{aligned} \tag{M.19}$$

The leading order equation obviously gives the zeroth order solution

$$v_0(r,z) = \frac{\bar{v}(r)}{z^{\sigma+1}} \tag{M.20}$$

At first order in Re we get

$$M_0(r,z)v_1(r,z) = -\frac{\mu_1 \bar{v}(r)}{z^{\sigma+1}} + \frac{\sigma e^{-r^2/2}U(r)}{z^{\sigma+1}} \tag{M.21}$$

To solve this, we try $v_1(r,z) = \frac{\bar{v}_1(r)}{z^{\sigma+1}}$ yielding

$$M_0(r)\bar{v}_1(r) = -\mu_1\bar{v}(r) + \sigma e^{-r^2/2}U(r) \tag{M.22}$$

This is an ordinary differential equation for the function $\bar{v}_1(r)$ and by the self-adjointness of the operator $M_0(r)$ a solution for $\bar{v}_1(r)$ satisfying the boundary conditions only exists provided a Fredholm alternative condition is satisfied by the inhomogeneous term in (M.22), namely,

$$\langle -\mu_1\bar{v}(r) + \sigma e^{-r^2/2}U(r),\ \bar{v}(r)\rangle = 0 \tag{M.23}$$

yielding the result

$$\mu_1 = \frac{\langle \sigma e^{-r^2/2}U(r),\ \bar{v}(r)\rangle}{\langle \bar{v}(r),\ \bar{v}(r)\rangle} \tag{M.24}$$

Given this solvability condition, $\bar{v}_1(r)$ can, in principle, be computed as an expansion in the complete set of eigenfunctions $\{rL_p^{(1)}(r^2/2)\exp(-r^2/2)|p = 0, 1..\}$ if needed. Observe that $\bar{v}_1(r)$ seems only to be determined to within an arbitrary multiple of $\bar{v}(r)$, but adding any amount of the function $\bar{v}(r)$ simply alters the normalization of the zeroth order solution which is assumed fixed *a priori.* At second order in Re we obtain

$$M_0(r,z)v_2(r,z) = -\frac{\mu_1\bar{v}_1(r)}{z^{\sigma+1}} - \frac{\mu_2\bar{v}(r)}{z^{\sigma+1}} - \frac{\mu_1 U(r)e^{-r^2/2}}{2z^{\sigma+1}} + \frac{\sigma\mu_1 U(r)e^{-r^2/2}\log z}{2z^{\sigma+1}} \tag{M.25}$$

We try

$$v_2(r,z) = \left(\frac{\log^2 z}{z^{\sigma+1}}\right)\bar{v}_2^{(2)}(r) + \left(\frac{\log z}{z^{\sigma+1}}\right)\bar{v}_2^{(1)}(r) + \left(\frac{1}{z^{\sigma+1}}\right)\bar{v}_2^{(0)}(r) \tag{M.26}$$

Substitution and use of (M.11)–(M.13) yields

$$\left(\frac{\log^2 z}{z^{\sigma+1}}\right) M_0(r)\bar{v}_2^{(2)}(r) + \left(\frac{\log z}{z^{\sigma+1}}\right) M_0(r)\bar{v}_2^{(1)}(r) + \left(\frac{1}{z^{\sigma+1}}\right) M_0(r)\bar{v}_2^{(0)}(r)$$
$$= \left(\frac{\log z}{z^{\sigma+1}}\right)\left(4\bar{v}_2^{(2)}(r) + \frac{\sigma\mu_1 U(r)e^{-r^2/2}}{2}\right) - \frac{\mu_1\bar{v}_1(r)}{z^{\sigma+1}} - \frac{\mu_2\bar{v}(r)}{z^{\sigma+1}}$$
$$- \frac{\mu_1 U(r)e^{-r^2/2}}{2z^{\sigma+1}} + \frac{2\bar{v}_2^{(1)}(r)}{z^{\sigma+1}} \quad \text{(M.27)}$$

Using linearity and equating coefficients of the three different functions of z,

$$M_0(r)\bar{v}_2^{(2)}(r) = 0 \quad \text{(M.28)}$$

from this we deduce that

$$\bar{v}_2^{(2)}(r) = A\bar{v}(r) \quad \text{(M.29)}$$

for some constant A. Also,

$$M_0(r)\bar{v}_2^{(1)}(r) = 4A\bar{v}(r) + \frac{\sigma\mu_1 U(r)e^{-r^2/2}}{2} \quad \text{(M.30)}$$

The Fredholm alternative condition that an appropriate $\bar{v}_2^{(1)}(r)$ should exist yields the value of the constant A. We can then solve for the function $\bar{v}_2^{(1)}(r)$ as an eigenfunction expansion to within an arbitrary multiple of $\bar{v}(r)$, the kernel function. As previously discussed, if we were to add any of the kernel function to $\bar{v}_2^{(1)}(r)$, we would be adding a term of the form (M.8) which we have disallowed for reasons discussed earlier. This requirement implies that $\bar{v}_2^{(1)}(r)$ is uniquely determined. Finally

$$M_0(r)\bar{v}_2^{(0)}(r) = -\mu_1\bar{v}_1(r) - \mu_2\bar{v}(r) - \frac{\mu_1 U(r)e^{-r^2/2}}{2} + 2\bar{v}_2^{(1)}(r) \quad \text{(M.31)}$$

By construction, the first and fourth terms on the right hand side are orthogonal to

$\bar{v}(r)$ but solvability for $\bar{v}_2^{(0)}(r)$ uniquely gives the value of μ_2 i.e.

$$\mu_2 = -\frac{\langle \mu_1 U(r) e^{-r^2/2},\ \bar{v}(r)\rangle}{2\langle \bar{v}(r),\ \bar{v}(r)\rangle} = -\frac{\mu_1^2}{2\sigma} \tag{M.32}$$

Again, $v_2^{(0)}(r)$ can be written as an eigenfunction expansion if required. The perturbation calculation was carried out to $O(Re^3)$ and it became clear that the procedure could in principle be carried out indefinitely. Indeed, the perturbation analysis reveals a particularly interesting structure which in fact allows the eigenvalue relation to be not only determined to all orders, but also summed. This is primarily a result of the fact that the form of the forcing in (M.16) is known at all orders in Re. It becomes clear that

$$\mu_n = -\frac{\mu_{n-1}}{2}\frac{\langle e^{-r^2/2}U(r),\ \bar{v}(r)\rangle}{\langle \bar{v}(r),\ \bar{v}(r)\rangle} = -\frac{\mu_1}{2\sigma}\mu_{n-1}\ \forall n \geq 2 \tag{M.33}$$

$$v_n(r, z) = \frac{1}{z^{\sigma+1}}\sum_{j=0}^{n}(\log z)^j v_n^{(j)}(r)\ \forall n \geq 2 \tag{M.34}$$

where $v_n^{(j)}(r)$ are some functions of r which can be determined as eigenfunction expansions from the perturbation analysis. The results (M.33) and (M.34) can be formally proved by induction. Using (M.33) in (M.6) it becomes clear that

$$\begin{aligned}\tilde{\sigma} &= \sigma - \frac{1}{2}(Re\ \mu_1 + Re^2\ \mu_2 + Re^3\ \mu_3 + ...)\\ &= \sigma - \frac{\mu_1\ Re}{2}\left(1 - Re\frac{\mu_1}{2\sigma} + Re^2\ \left(\frac{\mu_1}{2\sigma}\right)^2 + ...\right)\\ &= \sigma\left(1 - \frac{Re\mu_1}{2\sigma + Re\mu_1}\right)\end{aligned} \tag{M.35}$$

The final perturbed solution for the velocity field can therefore be written

$$
\begin{aligned}
w(r,z,t) &= \bar{w}(r)e^{-\mu t}\frac{1}{z^{\tilde{\sigma}}} \\
v(r,z,t) &= \frac{e^{-\mu t}}{z^{\sigma+1}}\Big(\bar{v}(r) + Re\ \bar{v}_1(r) \\
&\quad + Re^2\ \left(v_2^{(2)}(r)(\log z)^2 + v_2^{(1)}(r)\log z + v_2^{(0)}(r)\right) + \ldots\Big) \\
u(r,z,t) &= \tilde{\sigma}U(r)e^{-\mu t}\frac{1}{z^{\tilde{\sigma}+1}} \\
\tilde{\sigma} &= \sigma\left(1 - \frac{Re\mu_1}{2\sigma + Re\mu_1}\right) \\
\mu &= 2k + 4 - 2\tilde{\sigma}
\end{aligned} \tag{M.36}
$$

with μ_1 given in (M.24) and where the functions $\{v_n^{(j)}(r)|0 \leq j \leq n, n \geq 2\}$ can be found as eigenfunction expansions if needed.

Although the radius of convergence of the expansion in Re is not known, we do not anticipate any problems with convergence. The fact that the eigenvalue can be found to all orders and has a finite sum lends credence to this. It is straightforward to see that the corresponding perturbation pressure will have the large-z asymptotic form

$$
\begin{aligned}
p(r,z) = \frac{\bar{p}(r)}{z^{\sigma+1}} &+ Re\ \left[\frac{p_1^{(0)}(r)}{z^{\sigma+1}} + p_1^{(1)}(r)\frac{\log z}{z^{\sigma+1}}\right] \\
&+ Re^2\left[\frac{p_2^{(0)}(r)}{z^{\sigma+1}} + p_2^{(1)}(r)\frac{\log z}{z^{\sigma+1}} + p_2^{(2)}(r)\frac{\log^2 z}{z^{\sigma+1}}\right] + O(Re^3)
\end{aligned} \tag{M.37}
$$

where the functions of r appearing in (M.37) can be obtained by direct integration of the asymptotic form of (7.7).

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