Irreducibility of the Lawrence-Krammer representation of the BMW algebra of type $\mathbf{A_{n-1}}$

Thesis by

Claire Levaillant

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To Jean-Charles Andre

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Abstract

Given two nonzero complex parameters l and m, we construct by the mean of knot theory a matrix representation of size $\frac{n(n-1)}{2}$ of the BMW algebra of type A_{n-1} with parameters l and m over the field $\mathbb{Q}(l, r)$, where $m = \frac{1}{r} - r$. As a representation of the Braid group on n strands, it is equivalent to the Lawrence-Krammer representation that was introduced by Lawrence and Krammer to show the linearity of the Braid groups. We prove that the Lawrence-Krammer representation is generically irreducible, but that for some values of the parameters l and r, it becomes reducible. In particular, we show that for these values of the parameters l and r, the BMW algebra is not semisimple. When the representation is reducible, the action on a proper invariant subspace of the Lawrence-Krammer space must be a Hecke algebra action. It allows us to describe the invariant subspaces when the representation is reducible.

Contents

Chapter 1

Introduction

The Problem and the Main Results

In this thesis, we build and examine a representation of degree $\frac{n(n-1)}{2}$ of the BMW algebra of type A_{n-1} over the field $\mathbb{Q}(l, m)$ that is equivalent to the Lawrence-Krammer representation introduced by Lawrence and Krammer in [7] and [9]. A result in [4] states that this representation is generically irreducible. Here we recover this result by a different method, and show further that when l and m are specified in the field of complex numbers, the Lawrence-Krammer representation is always irreducible except when l and m are related in a certain way. We let r and $-\frac{1}{r}$ be the roots of the quadratics $X^2 + mX - 1 = 0$. We define $\mathcal{H}_{F,r^2}(n)$ as the Iwahori-Hecke algebra of the symmetric group $Sym(n)$ over the field $F = \mathbb{Q}(l, r)$, with parameter r^2. Its generators $g_1, \dots, g_{n-1}$ satisfy the braid relations and the relation $g_i^2 + m\, g_i = 1$ for all i. This definition is the same as the one given in [10], after the generators have been rescaled by a factor $\frac{1}{r}$. We assume that $\mathcal{H}_{F,r^2}(n)$ is semisimple. It suffices to assume that $(r^2)^k \neq 1$ for every $k \in \{1, \dots, n\}$. We prove the following theorem:

Main Theorem.

Let m, l and r be three nonzero complex numbers with $m = \frac{1}{r} - r$. Let n be an integer with $n \geq 3$. Assume that $(r^2)^k \neq 1$ for every $k \in \{1, \dots, n\}$ and so $\mathcal{H}_{F,r^2}(n)$ is semisimple.

When $n \geq 4$, the Lawrence-Krammer representation of the BMW algebra $B(A_{n-1})$ with parameters l and $m = \frac{1}{r} - r$ over the field $\mathbb{Q}(l, r)$ is irreducible, except when $l \in \{r, -r^3, \frac{1}{r^{n-3}}, -\frac{1}{r^{n-3}}, \frac{1}{r^{2n-3}}\}$ when it is reducible.

When $n = 3$, the Lawrence-Krammer representation of the BMW algebra $B(A_2)$ with parameters l and $m = \frac{1}{r} - r$ over the field $\mathbb{Q}(l, r)$ is irreducible except when $l \in \{-r^3, \frac{1}{r^3}, 1, -1\}$ when it is reducible.

Moreover, for each of the values of l and r when the representation is reducible, we show that there exists a unique nonzero proper invariant subspace of the Lawrence-Krammer space, except when $l = -r^3$ and $r^{2n} = -1$ and we are able to give its dimension. When $l = -r^3$ and $r^{2n} = -1$, we prove that the Lawrence-Krammer representation contains a direct sum of a representation of degree one and an irreducible representation of degree $\frac{(n-1)(n-2)}{2}$ and that there are no other irreducibles inside the Lawrence-Krammer space. Suppose $\mathcal{V}^{(n)}$ is the Lawrence-Krammer vector space over $\mathbb{Q}(l, r)$ of dimension $\frac{n(n-1)}{2}$. We prove the following five theorems and part of Conjecture A below. In the Theorems, the integer n is taken to be greater than or equal to 3. When the lower bound is not 3 in the theorems below, it means these cases for those small integers are special and need to be formulated apart. In fact, we prove that Conjecture A below, that we believe to be true for a general $n \geq 3$, holds in these cases. In what follows we still assume that $\mathcal{H}_{F,r^2}(n)$ is semisimple, or which is equivalent $(r^2)^k \neq 1$ for every positive integer k with $1 \leq k \leq n$.

Theorem A. *Let n be an integer with $n \geq 4$. There exists a one-dimensional* *invariant subspace of* $\mathcal{V}^{(n)}$ *if and only if* $l = \frac{1}{r^{2n-3}}$. *If so it is unique.*

Theorem B**.** *Let* n *be an integer with* $n \geq 3$ *and* $n \neq 4$. *There exists an irreducible* $(n-1)$*-dimensional invariant subspace of* $\mathcal{V}^{(n)}$ *if and only if* $l \in \{\frac{1}{r^{n-3}}, -\frac{1}{r^{n-3}}\}$. *If so, it is unique.*

Theorem C**.** *Let* n *be an integer with* $n \geq 4$. *There exists an irreducible* $\frac{n(n-3)}{2}$*-dimensional invariant subspace of* $\mathcal{V}^{(n)}$ *if and only if* $l = r$. *If so, it is unique.*

Theorem D**.** *Let* n *be an integer with* $n \geq 5$. *There exists an irreducible* $\frac{(n-1)(n-2)}{2}$*-dimensional invariant subspace of* $\mathcal{V}^{(n)}$ *if and only if* $l = -r^3$.

Theorem E**.** *When the Lawrence-Krammer representation is reducible,* $\mathcal{V}^{(n)}$ *has a unique nontrivial proper invariant subspace except when* $l = -r^3$ *and* $r^{2n} = -1$ *when* $\mathcal{V}^{(n)}$ *contains a direct sum of a one-dimensional invariant subspace and an irreducible* $\frac{(n-1)(n-2)}{2}$*-dimensional one.*

Moreover, in the case $l = \frac{1}{r^{2n-3}}$, we can give a spanning vector for the one-dimensional invariant subspace and in the case $l \in \{\frac{1}{r^{n-3}}, -\frac{1}{r^{n-3}}\}$, we are also able to describe the two respective irreducible $(n-1)$-dimensional subspaces of $\mathcal{V}$. Also, when $l = r$ we can describe the irreducible $\frac{n(n-3)}{2}$-dimensional invariant subspace of $\mathcal{V}^{(n)}$.

Furthermore, the cases $n = 3$ and $n = 4$ are described in the following conjecture. The conjecture holds in these cases, by Theorems 4 and 5 when $n = 3$ and by Theorems 4 and 6 and Corollary 4 when $n = 4$. It is proven that the Conjecture is also true for $n = 5$, namely by Theorems $4, 5, 8$ and Result 1 in the thesis. Part of the arguments lies in the proof of these theorems when we show that the conjugate Specht modules $S^{(1^n)}$ (cases $n = 4, 5$ in the proof of Theorem 4), $S^{(2,1^{n-2})}$ (case $n \in \{3, 5\}$ in the proof of Theorem 5), $S^{(2,2,1^{n-4})}$ (case $n = 5$ in the proof of Result 1) cannot occur in the

L-K space $\mathcal{V}^{(n)}$. The results for these cases are respectively gathered in Appendices E, F and G at the end of the thesis. We will sometimes abbreviate L-K representation instead of Lawrence-Krammer representation and L-K space instead of Lawrence-Krammer space.

Conjecture A. *Let n be an integer with $n \geq 3$. Assume $\mathcal{H}_{F,r^2}(n)$ is semisimple and $r^{2n} \neq -1$ when $l = -r^3$.*
When the L-K representation is reducible, its unique nontrivial proper invariant subspace is isomorphic to one of the Specht modules

$$S^{(n)},\, S^{(n-1,1)},\, S^{(n-2,2)},\, S^{(n-1,1,1)},$$

which respectively arise when and only when

$$l = \frac{1}{r^{2n-3}},\, l \in \left\{\frac{1}{r^{n-3}}, -\frac{1}{r^{n-3}}\right\},\, l = r,\, l = -r^3$$

When $l = -r^3$ and $r^{2n} = -1$, there are exactly three proper nontrivial invariant subspaces of $\mathcal{V}^{(n)}$ and they are respectively isomorphic to $S^{(n)}$, $S^{(n-2,1,1)}$ and $S^{(n)} \oplus S^{(n-2,1,1)}$

The BMW algebra $B(A_{n-1})$ is generated by invertible elements g_i's, that are analogue to those from the Braid group and that we will call generators of the first type. There is a second set of $(n-1)$ elements in $B(A_{n-1})$ that generate an ideal, namely the e_i's (see section 2 entitled "Background and Notations" of the thesis). They are called generators of the second type. We will see in the thesis that the generators of the second type and some of their conjugates by products of generators of the first type play a critical role. We think that the following result is true. If so it will yield a different approach to solving Lemma 10 of the thesis, which is a key lemma on the road to Theorem D (and C as we did in the thesis without the use of the Branching Rule).

Conjecture B. *Let T denote the left action on the Lawrence-Krammer space of the sum of all the conjugates of the e_i's by products of the g_i's. Fix $r \in \mathbb{C}$. Then, $det\,T \in \mathbb{Q}(l)$, the degree of T is the degree $\frac{n(n-1)}{2}$ of the Lawrence-Krammer representation and*

$$det(T) = \frac{(l-r)^{\frac{n(n-3)}{2}}(l-r^3)^{\frac{(n-1)(n-2)}{2}}(l-\frac{1}{r^{n-3}})^{n-1}(l+\frac{1}{r^{n-3}})^{n-1}(l-\frac{1}{r^{2n-3}})}{\lambda_r\, l^{\frac{n(n-1)}{2}}}$$

where λ_r is a constant that depends on the value of the parameter r.

The conjecture is shown through Maple for $n \in \{3,4,5,6\}$ in the Appendix A where the procedure NOTIRR provides a necessary condition on l and r so that the L-K space $\mathcal{V}^{(n)}$ is reducible. When the representation is reducible, l must be a root of some polynomial of $\mathbb{Q}(r)[X]$, with multiplicity given in the output. Visibly, this multiplicity is the dimension of the corresponding invariant subspace.

The Approach

Our approach is to use knot theory as a tool to build the Lawrence-Krammer representation. We use a deformation of the Brauer centralizer algebra, namely the Kauffman tangle algebra. This algebra MT_n constructed by Morton, Traczyk and Kauffman was shown by Morton and Wassermann in 1989 to be generically isomorphic to the BMW algebra of type A_{n-1} (see [12]). One difference between the Brauer centralizer algebra and the Morton-Traczyk algebra is that the group algebra of the symmetric group has been replaced with the Iwahori-Hecke algebra of the symmetric group.

This introduces a new type of braids where over-crossings have to be distinguished from under-crossings. A tangle with an over-crossing is related to a tangle with an under-crossing by the Kauffman skein relation that uses the parameter of the Iwahori-Hecke algebra of the symmetric group. The geometric approach of the Kauffman tangle algebra leads us to visualize (and later on prove for the algebraic version of the representation) that if $\mathcal{W}$ is a proper invariant subspace of the Lawrence-Krammer vector space $\mathcal{V}$, then the action on $\mathcal{W}$ must be an Iwahori-Hecke action. This was incidentally shown by Arjeh Cohen, Die Gijsbers and David Wales in a more general setting when they also deal with representations of the BMW algebra of types D and E. By using the Lawrence-Krammer representation, we can find the two inequivalent irreducible matrix representations of $\mathcal{H}_{F,r^2}(n)$ of degree $(n-1)$. We then use those irreducible representations to show that if there exists an irreducible $(n-1)$-dimensional invariant subspace of $\mathcal{V}$, it forces $l \in \{\frac{1}{r^{n-3}}, -\frac{1}{r^{n-3}}\}$. Conversely, if l takes one of these fractional values in r, we show that there exists a unique irreducible $(n-1)$-dimensional invariant subspace of $\mathcal{V}$. For the small values $3, 4, 5$ of n we are able to find with Maple from the Lawrence-Krammer representation the inequivalent irreducible matrix representations of $\mathcal{H}_{F,r^2}(n)$; we then deduce from these matrix representations wether or not the Lawrence-Krammer representation can be reducible and if so for which values of the parameters l and r. The table in Appendix E, (resp F, G, H) lists which irreducible representations of $\mathcal{H}_{F,r^2}(n)$, $n = 3$ (resp $4, 5, 6$) can possibly occur in the Lawrence-Krammer representation and for which corresponding values of l and r. All the cases described in the tables are proven in the thesis. The case $n = 6$ could be completed by using the Branching Rule. In the table for n, each Ferrers diagram associated with a partition of the integer

n corresponds to a class of irreducible $\mathcal{H}_{F,r^2}(n)$-module. Only the irreducible representations of $\mathcal{H}_{F,r^2}(n)$ of degree less than the degree $\frac{n(n-1)}{2}$ of the Lawrence-Krammer representation are represented in the table. Let the α_i's for $i = 1, \ldots, n-1$ denote the $(n-1)$ simple roots of the root system of type A_{n-1} and let ϕ^+ denote the set of all the positive roots. There are $\frac{n(n-1)}{2}$ of these. The Lawrence-Krammer vector space $\mathcal{V}^{(n)}$ is the vector space over $\mathbb{Q}(l, r)$ with the generating set $\{x_\beta | \beta \in \phi^+\}$, indexed by all the positive roots. Thus it has dimension $\frac{n(n-1)}{2}$. If $\beta = \alpha_i + \cdots + \alpha_{j-1}$ is a positive root where $i < j$, then to simplify the notations, we let w_{ij} denote the vector $x_{\alpha_i + \cdots + \alpha_{j-1}}$. Thus, $\mathcal{V}^{(n)}$ is spanned over $\mathbb{Q}(l, r)$ by the $\frac{n(n-1)}{2}$ vectors w_{ij} with $1 \leq i < j \leq n$. These are the vectors that appear in the tables of the Appendices.

Some History and Recent Developments

The BMW algebras of type A were introduced by J. Murakami in [13] and separately by J.S. Birman and H. Wenzl in [2] in order to try to find faithful representations of the Braid group. Birman and Wenzl wanted to use these algebras to construct representations of the Braid group B_n over $n-1$ generators in order to investigate wether this group is linear. Linearity of a group means that there exists a faithful representation into $GL(m, \mathbb{R})$ for some positive integer m. Burau discovered an n-dimensional representation of B_n that was faithful for $n = 3$, but shown not to be faithful for $n \geq 9$ (Moody, [11]). For a long time, if it was known that B_n was linear for $n \in \{2, 3\}$, for $n \geq 4$, the problem remained open. Krammer later introduced a representation of B_n and showed it to be faithful for $n = 4$ (cf [7]). Since the same representation occurs in an earlier work of Lawrence, it is called the Lawrence-Krammer representation. In [1], using topology, Bigelow proved the Lawrence-Krammer representation to be faithful for all n, thus stating that all the Artin groups of type A are linear. Shortly after, Krammer also showed with specific real values of the parameters that his representation is a faithful representation of the Braid group, using algebraic methods this time (see [8]). In 2002, A.M. Cohen and D.B. Wales generalized the Lawrence-Krammer representation to Artin groups of finite types, as well as their proof of linearity, thus showing in [5] that every Artin group of finite type is linear. Following the observation that the Lawrence-Krammer representation of the Artin group of type A_{n-1} factors through the BMW algebra of the same type, they then generalized the definition of BMW algebras to other types D and E and built representations of these algebras that they showed to be generically irreducible.

As part of this work, we show that if the Lawrence-Krammer space has a proper invariant subspace $\mathcal{W}$, the g_i conjugates of the e_i's all annihilate $\mathcal{W}$. That is how the Main theorem implies that for these specific values of the parameters l and r, the BMW algebra B of type A_{n-1} with parameters l and $m = \frac{1}{r} - r$ is not semisimple. Hans Wenzl showed a similar result when he states in [15] that these algebras are semisimple except possibly if r is a root of unity or $l = r^k$ for some $k \in \mathbb{Z}$, where he also considers complex parameters. However, he does not prove or mention for which integers k or which roots of unity the algebras fail to be semisimple. Hebing Rui and Mei Si more recently obtained the same result as ours in showing that for the values of l and r that we found (and other values), B is not semisimple. They used the representation theory of cellular algebras.

Chapter 2

Background and Notations

We consider the BMW algebra $B(A_{n-1})$, with nonzero complex parameters m and l, as defined in [4]. In this thesis we build a representation of the algebra $B(A_{n-1})$ and we find necessary and sufficient conditions on l and m so that it is irreducible. Throughout the thesis we will use the change of parameter $m = 1/r - r$, so that the two relevant parameters will be l and r. We recall below the relations defining $B(A_{n-1})$, where we assumed $r^2 \neq 1$ and so $m \neq 0$:

$$\begin{aligned}
g_i g_j &= g_j g_i & \forall 1 \leq i, j \leq n-1,\ |i-j| \geq 2 \quad (1)\\
g_i g_{i+1} g_i &= g_{i+1} g_i g_{i+1} & \forall 1 \leq i \leq n-2 \quad (2)\\
e_i &= \frac{l}{m} * (g_i^2 + m\, g_i - 1) & \forall 1 \leq i \leq n-1 \quad (3)\\
g_i e_i &= l^{-1} e_i & \forall 1 \leq i \leq n-1 \quad (4)\\
e_i g_{i+1} e_i &= l e_i & \forall 1 \leq i \leq n-2 \quad (5)\\
e_i g_{i-1} e_i &= l e_i & \forall 2 \leq i \leq n-1 \quad (6)
\end{aligned}$$

as well as some important and direct consequences of these defining relations:

$$\begin{aligned}
e_i g_i &= l^{-1} e_i && \forall 1 \leq i \leq n-1 && (7)\\
g_i^2 &= 1 - m g_i + m l^{-1} e_i && \forall 1 \leq i \leq n-1 && (8)\\
g_i^{-1} &= g_i + m - m\, e_i && \forall 1 \leq i \leq n-1 && (9)
\end{aligned}$$

We will also use the "mixed Braid relations":

$$\begin{aligned}
g_i g_{i+1} e_i &= e_{i+1} e_i && \forall 1 \leq i \leq n-2 && (10)\\
g_i g_{i-1} e_i &= e_{i-1} e_i && \forall 2 \leq i \leq n-1 && (11)\\
g_i e_{i+1} e_i &= g_{i+1} e_i + m(e_i - e_{i+1} e_i) && \forall 1 \leq i \leq n-2 && (12)\\
g_i e_{i-1} e_i &= g_{i-1} e_i + m(e_i - e_{i-1} e_i) && \forall 2 \leq i \leq n-1 && (13)
\end{aligned}$$

Finally, for any node $i = 1, \ldots n-1$, we will make use of the idempotent relation $e_i^2 = x\, e_i$ where the parameter x is defined by $x = 1 - \frac{l-1/l}{1/r-r}$

We will work over the field $F = \mathbb{Q}(r, l)$. Let I_1 and I_2 be the two-sided ideals of $B(A_{n-1})$ respectively generated by the e_i's and the $e_i e_j$'s with $i \nsim j$ as in [4]. The representation will be built inside I_1/I_2. It is shown in [12] that the BMW algebra $B(A_{n-1})$ is isomorphic to the Kauffman algebra TM_n and from now on we will think of BMW elements in terms of tangles. The algebra $B(A_{n-1})$ acts by the left on the subspace of I_1/I_2 spanned by the elements with a fixed bottom horizontal line joining any two nodes, giving to this subspace a structure of $B(A_{n-1})$-module. We fix such a bottom horizontal line and we denote by V the corresponding subspace. Without loss of generality we take this horizontal line to be the one joining nodes 1 and

2 on the bottom line. From now on, all the elements that we consider are in V and so they are linear combinations over F of tangles all having their nodes 1 and 2 joined on the bottom line. When two tangles have the same top horizontal line, we will say that they are "similar" as in the definition below:

Definition 1.
Two tangles in V are "similar" if their respective top horizontal lines are the same, with this top horizontal line overcrossing the eventual vertical strings.

We now recall some basic facts about the Artin group of type A_{n-1} and its associated root system and we introduce a set of tangles in V that contains as many elements as there are positive roots and which is in correspondence with the set of positive roots. These tangles are obtained by picking any random top horizontal edge. Moreover, we agree that this top horizontal edge always overcrosses any of the vertical strings that it intersects.
Let $M = (m_{ij})$ be the Coxeter matrix of the Artin group A of type A_{n-1} with generators $s_1, ..., s_{n-1}$. Let $(\epsilon_i)_{i=1...n-1}$ be the canonical basis of $\mathbb{R}^{n-1}$ and let B_M be the canonical symmetric bilinear form over $\mathbb{R}^{n-1}$ associated to M.

$$B_M(\epsilon_i, \epsilon_j) = -cos(\pi/m_{ij}) \text{ where } m_{ij} = \begin{cases} 2 & \text{if } |i-j| > 1, \\ 3 & \text{if } |i-j| = 1 \\ 1 & \text{if } i = j \end{cases}$$

For $i = 1, ..., n-1$, let s_{H_i} denote the reflection with respect to the hyperplane $H_i := KerB_M(., \epsilon_i)$. By the theory in [3], we know that the Coxeter group $\mathcal{W}$ of the Artin group of type A_{n-1} is isomorphic to the reflection group spanned by the s_{H_i}'s. We have a root system and $\mathcal{W}$ is finite as it permutes the roots. Hence B_M is an inner product and throughout the paper we will denote it by (,). Finally, instead of $\epsilon_1, \epsilon_2, ..., \epsilon_{n-1}$, we denote the simple roots by $\alpha_1, ..., \alpha_{n-1}$ and the associated simple reflections $s_{H1}, ..., s_{H_{n-1}}$ by $r_1, ..., r_{n-1}$. We associate to each positive root β a tangle in V in the following way: α_1 is e_1. Each positive root is then built from α_1. For $\beta \in \phi^+$, where ϕ^+ is the set of positive roots, let $w_{\beta 1}$ be the unique element in the Weyl group of minimal length mapping α_1 to β, as in [4]. For instance, we have $w_{\alpha_2 1} = r_1 r_2$ and $w_{\alpha_2+\alpha_3+\alpha_4,1} = r_4 r_3 r_1 r_2$. Taking the same notations as in [4], there is a map:

$$\begin{pmatrix} \mathcal{W} & \xrightarrow{\psi} & A \\ r_{i_1}...r_{i_l} & \longmapsto & s_{i_1}...s_{i_l} \end{pmatrix}$$

where $r_{i_1}...r_{i_l}$ is a reduced decomposition. To have this map well defined, we need to show that if $r_{i_1}...r_{i_l}$ and $r_{j_1}...r_{j_l}$ are two reduced decompositions in W such that $r_{i_1}...r_{i_l} = r_{j_1}...r_{j_l}$ then $s_{i_1}...s_{i_l}$ can be transformed into $s_{j_1}...s_{j_l}$ by using the braid relations

$$\begin{aligned} s_i s_j &= s_j s_i & \forall 1 \leq i, j \leq n-1, |i-j| > 2 && (14) \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} & \forall 1 \leq i \leq n-2 && (15) \end{aligned}$$

This is Matsumoto's theorem and it is stated (and proved) in [10].

And there is a morphism of groups:

$$\begin{pmatrix} A & \xrightarrow{b} & B(A_{n-1})^\times \\ s_i & \mapsto & g_i \end{pmatrix}$$

where $B(A_{n-1})^\times$ is the group of units of $B(A_{n-1})$.

We denote by φ the composition of these two maps and we now associate to β the BMW element $\varphi(w_{\beta 1})e_1$. Thus α_2 is associated with $g_1 g_2 e_1$ and using the relations defining $B(A_{n-1})$, this element is in fact $e_2 e_1$. Similarly, $\alpha_2+\alpha_3+\alpha_4$ is associated with $g_4 g_3 g_1 g_2 e_1$, which can be rewritten to $g_4 g_3 e_2 e_1$ thanks to relation (10). We note that more generally, the α_i's are associated with the $e_i e_{i-1}...e_1$'s and are obtained from α_1 by shifting the top horizontal line to the right. Then from α_i, get a positive root $\alpha_i + ... + \alpha_{i+k}$ by acting by the left on $e_i...e_1$ with $g_{i+k}...g_{i+1}$, the top horizontal line crossing now k vertical lines. More generally the top horizontal line of the tangle representing a root of height h always crosses $h-1$ vertical lines. We denote by W the subspace of V spanned by these tangles and by $\mathcal{B}$ the basis of W composed of these tangles. By construction, the cardinality of $\mathcal{B}$ equals the number of positive roots.

We also introduce elements X_{ij} in TM_n which will be useful throughout the paper. By definition, X_{ij} is the element with two horizontal lines, one at the top and the other one at the bottom, both joining nodes i and j, all the other nodes being joined by straight vertical lines. Moreover, in this definition we will take the horizontal lines overcrossing the vertical lines. We note that there are as many elements X_{ij}'s as there are positive roots since for each positive root we built a unique tangle in V by taking α_1 to be e_1 and by letting the horizontal edge on the top line vary for the other ones, allowing all the possible combinations. And in fact,

$$\big|\{X_{ij}; 1 \leq i < j \leq n\}\big| = |\phi^+| = |\mathcal{B}| = \binom{n}{2}$$

The following lemma relies on the fact that the X_{ij}'s are conjugate to the e_i's:

Lemma 1. $\forall x, \left(x \in \bigcap_{1\leq i<j\leq n} Ker X_{ij} \Rightarrow \forall k, \forall l,\ g_l\, x \in Ker\ e_k\right)$

PROOF OF THE LEMMA: let x be in the intersection of the kernels of the X_{ij}'s. By definition, we have $e_k x = 0$. Then, by (7), it comes $e_k g_k x = 0$. So we get: $g_k x \in Ker\ e_k$. Next, from $X_{k\,k+2} x = 0$, we derive $g_{k+1} e_k g_{k+1}^{-1} x = 0$, which implies $e_k g_{k+1}^{-1} x = 0$, by multiplying by the left the last equality with g_{k+1}^{-1}. Hence, $g_{k+1}^{-1} x \in Ker\ e_k$. By using (9), it follows that $g_{k+1} x \in Ker\ e_k$. Also, from $X_{k-1k+1} x = 0$, we derive $g_{k-1}^{-1} e_k g_{k-1} x = 0$ which implies that $g_{k-1} x \in Ker\ e_k$. At this point we have for any two nodes k and l:

$$\text{if } k = l \text{ or if } k \sim l \text{ then } g_l x \in Ker\ e_k$$

Further, if $k \nsim l$, by using the braid relation (1) and the polynomial expression (3), e_k and g_l commute and it immediately follows that $g_l x \in Ker\ e_k$. Thus we have proved the lemma.

Corollary 1. $\forall x \in \bigcap_{1\leq i<j\leq n} Ker\ X_{ij},\ \forall 1 \leq l \leq n-1,\ g_l^{-1} x \in \bigcap_{1\leq k\leq n-1} Ker\ e_k$

PROOF: direct by using the lemma and relation (9).

We now prove the stronger lemma:

Lemma 2. $\forall x, \left(x \in \bigcap_{1\leq i<j\leq n} Ker X_{ij} \Rightarrow \forall l,\ g_l\, x \in \bigcap_{1\leq i<j\leq n} Ker X_{ij}\right)$

PROOF: Given an x in the intersection of the kernels of the X_{ij}'s and given $j \geq i+2$, we want to show that $g_l\, x \in Ker\, X_{ij}$ for any node l. The case $j = i+1$ has already been seen in Lemma 1. The idea is to consider different conjugation formulas for X_{ij}. And indeed, let's first write:

$$X_{ij} = g_{j-1} \ldots g_{i+1}\, e_i\, g_{i+1}^{-1} \ldots g_{j-1}^{-1}$$

From there we deduce two things:

- If $k \nsim \{i, \ldots, j-1\}$, then $g_k\, x \in Ker\, X_{ij}$ as g_k commutes with all the elements composing the word above.
- $g_{j-1}x \in Ker\, X_{ij}$. Indeed, we have in the case $j > i+2$:

$$\begin{aligned} X_{ij}g_{j-1}x &= (g_{j-1}\ldots g_{i+1}\, e_i\, g_{i+1}^{-1}\ldots g_{j-1}^{-1})g_{j-1}x \\ &= g_{j-1}(g_{j-2}\ldots g_{i+1}\, e_i\, g_{i+1}^{-1}\ldots g_{j-2}^{-1})x \\ &= 0 \qquad \text{since } x \in Ker\, X_{i\,j-1} \end{aligned}$$

If $j = i+2$, we have $X_{i\,i+2} = g_{i+1}\, e_i\, g_{i+1}^{-1}$ and it follows that $X_{i\,i+2}\, g_{i+1}x = g_{i+1}\, e_i\, x = 0$ since $x \in Ker\, e_i$.

We now deal with the nodes $i,\ i+1, \ldots j-2$. Instead of starting from the left with e_i like in the previous decomposition for X_{ij}, we start from the right with e_{j-1} and get the following expression:

$$X_{ij} = g_i^{-1}\ldots g_{j-2}^{-1}\, e_{j-1}\, g_{j-2}\ldots g_i \tag{16}$$

Again we deduce two things from this writing:

- Let k be any value in $\{2, \ldots, j-i-1\}$. We look at the action of X_{ij}

on $g_{j-k}\,x$. We get successively:

$$\begin{aligned}
X_{ij}\,g_{j-k}\,x &= g_i^{-1}\cdots g_{j-2}^{-1}\,e_{j-1}\,g_{j-2}\cdots g_i\,g_{j-k}\,x \\
&= g_i^{-1}\cdots g_{j-2}^{-1}\,e_{j-1}\,g_{j-2}\cdots g_{j-k}g_{j-k-1}g_{j-k}g_{j-k-2}\cdots g_i\,x \\
&= g_i^{-1}\cdots g_{j-2}^{-1}\,e_{j-1}\,g_{j-2}\cdots g_{j-k+1}g_{j-k-1}g_{j-k}g_{j-k-1}\cdots g_i x \\
&= g_i^{-1}\cdots g_{j-2}^{-1}g_{j-k-1}\,e_{j-1}\,g_{j-2}\cdots g_i\,x \\
&= g_i^{-1}\cdots g_{j-2}^{-1}g_{j-k-1}^{-1}\,e_{j-1}\,g_{j-2}\cdots g_i\,x - m\,X_{ij}\,x \\
&= g_i^{-1}\cdots g_{j-k-1}^{-1}g_{j-k}^{-1}g_{j-k-1}^{-1}\cdots g_{j-2}^{-1}e_{j-1}g_{j-2}\cdots g_i\,x \\
&= g_i^{-1}\cdots g_{j-k-2}^{-1}g_{j-k}^{-1}g_{j-k-1}^{-1}g_{j-k}^{-1}\cdots g_{j-2}^{-1}e_{j-1}g_{j-2}\cdots g_i\,x \\
&= g_{j-k}^{-1}\,g_i^{-1}\cdots g_{j-2}^{-1}e_{j-1}g_{j-2}\cdots g_i\,x \\
&= g_{j-k}^{-1}\,X_{ij}\,x \\
&= 0
\end{aligned}$$

To quickly manipulate the equalities above, it will be useful to notice that moving from the right to the left, the indices increase on the right of e_{j-1} and they decrease on the left of e_{j-1}. We have the first equality by definition of X_{ij}. The second equality is obtained by using the braid relation (1), commuting g_{j-k} till we reach g_{j-k-1} and the third one is obtained by using the braid relation (2) on the nodes $j-k$ and $j-k-1$: $g_{j-k}g_{j-k-1}g_{j-k} = g_{j-k-1}g_{j-k}g_{j-k-1}$. One of the factors g_{j-k-1} lies now next to g_{j-k+1} and we may again apply the braid relation (1) repeatedly to move it to the left of e_{j-1}. This is the object of the equality number four. Next since all the terms on the left of e_{j-1} are inverses, it is natural to express g_{j-1} in terms of its inverse by using (9). The fifth equality is obtained by using (9) and the fact that $e_{j-k-1}e_{j-1} = 0$ (since $j-k-1 \in \{i,\ldots,j-3\}$). When the factor

g_{j-k-1} "disappears" in (9), it yields $X_{ij}\, x$, which is zero by choice of x in the intersection of all the kernels. Thus, in the product, g_{j-k-1} may in fact be replaced by g_{j-k-1}^{-1} without modifying the word. Next, in the equality number 6, we commute g_{j-k-1}^{-1} to the left till we get stuck. Then we apply again the braid relation (2) on the inverses to get the equality number 7. We may now move g_{j-k}^{-1} to the extreme left by using again the commuting relation (1) on the inverses. It finally allows us to isolate X_{ij} which acting by the left on x yields 0. Thus, we have proved the following:

$$\forall k \in \{i+1, \dots j-2\},\; g_k\, x \in Ker\, X_{ij}$$

– The second thing that we derive from the expression (16) is direct: $g_i^{-1} \dots g_{j-2}^{-1}\, e_{j-1}\, g_{j-2} \dots g_{i-1} g_i\, g_i\, x = 0$ by expanding the square g_i^2 with (8) and using the fact that $X_{i-1\,j} x = X_{ij} x = e_i\, x = 0$. Thus, we have $g_i\, x \in Ker\, X_{ij}$.

It remains to show that $g_{i-1}\, x \in Ker\, X_{ij}$ and $g_j\, x \in Ker\, X_{ij}$ Let's write:

$$X_{ij} = g_j^{-1} X_{i\,j+1} g_j$$

The trick here is to go "one node too far" to make a g_j appear on the right extremity of the word. Then, using the formula $g_j^2 = 1 - m\, g_j + m l^{-1}\, e_j$ and the fact that $X_{i\,j+1}\, x = X_{ij}\, x = e_j\, x = 0$ by our hypothesis on x, it immediately follows that $X_{ij}\, g_j\, x = 0$. Hence we have $g_j\, x \in Ker\, X_{ij}$.
Finally we write:

$$X_{ij} = g_{i-1} X_{i-1\,j} g_{i-1}^{-1}$$

Here, the trick is to go one node backwards to make a "g_{i-1}" appear. It follows that $X_{ij}g_{i-1}\,x = g_{i-1}X_{i-1\,j}\,x = 0$ since $x \in Ker\,X_{i-1\,j}$. Hence $g_{i-1}\,x \in Ker\,X_{ij}$ and we have now proved that

$$\forall l,\; g_l\,x \in Ker\,X_{ij}$$

We have the immediate corollary:

Corollary 2. *$\forall n$, $\bigcap_{1 \leq i < j \leq n} Ker X_{ij}$ is a $B(A_{n-1})$-module*

Chapter 3

The Case n=3

In this section we explicitly build a representation of $B(A_2)$ in V.

There are three positive roots in A_2: α_1, α_2, $\alpha_1+\alpha_2$ and V is spanned by e_1, e_2e_1 and g_2e_1. Acting by g_1 on these elements yields:

$g_1e_1 = l^{-1}e_1$, $g_1e_2e_1 = g_2e_1 + me_1 - me_2e_1$ and $g_1g_2e_1 = e_2e_1$.

Thus, the matrix of the action of g_1 is given by:

$$\begin{array}{cc} & \begin{array}{ccc} e_1 & e_2e_1 & g_2e_1 \end{array} \\ \begin{array}{c} e_1 \\ e_2e_1 \\ g_2e_1 \end{array} & \begin{pmatrix} l^{-1} & m & 0 \\ 0 & -m & 1 \\ 0 & 1 & 0 \end{pmatrix} \end{array} =: G_1(3)$$

Similarly, we have $g_2e_2e_1 = l^{-1}e_2e_1$ and $g_2g_2e_1 = e_1 + ml^{-1}e_2e_1 - mg_2e_1$.

Thus, the matrix of the action of g_2 is:

$$\begin{array}{cc} & \begin{array}{ccc} e_1 & e_2e_1 & g_2e_1 \end{array} \\ \begin{array}{c} e_1 \\ e_2e_1 \\ g_2e_1 \end{array} & \begin{pmatrix} 0 & 0 & 1 \\ 0 & l^{-1} & ml^{-1} \\ 1 & 0 & -m \end{pmatrix} \end{array} =: G_2(3)$$

We verify that the braid relation $G_1G_2G_1 = G_2G_1G_2$ is satisfied. Next, for $i \in \{1,2\}$, we define E_i by the matrix relation:

$$E_i := l/m * (G_i^2 + mG_i - Id) \tag{17}$$

and we check that for $i \in \{1,2\}$ and $j \in \{1,2\}$ the matrix relations

$$\begin{aligned} G_iE_i &= l^{-1}E_i & (18) \\ E_iG_jE_i &= lE_i & (19) \end{aligned}$$

are satisfied. Hence we have a left representation: $B(A_2) \to End_F(V)$. Suppose now that V is not irreducible. Then it has a proper nonzero invariant subspace U. Let u be a nonzero element of U. Let's decompose u over the basis $\mathcal{B} = (e_1, e_2e_1, g_2e_1)$, say $u = \lambda_1\ e_1 + \lambda_2\ e_2e_1 + \lambda_3\ g_2e_1$ with $(\lambda_1, \lambda_2, \lambda_3) \neq (0,0,0)$. We look at the action of $X_{12} = e_1$. We have:

$$\left\{ \begin{array}{rcl} e_1e_1 = e_1^2 &=& xe_1 \\ e_1(e_2e_1) &=& e_1 \\ e_1(g_2e_1) &=& le_1 \end{array} \right.$$

Thus, we get $X_{12}u = (\lambda_1x + \lambda_2 + \lambda_3l)\ e_1$. Assume that $X_{12}u \neq 0$ i.e. $\lambda_1x + \lambda_2 + \lambda_3l \neq 0$. It follows that e_1 is in U. Then e_2e_1 and g_2e_1 are also in U as U is a $B(A_2)$-submodule of V. Then U would not be proper which is a contradiction. Hence we have $X_{12}u = 0$. This last equality being true for any nonzero element of U, it comes $X_{12}U = 0$. Similarly, acting by $X_{23} = e_2$ on an element u of U gives a multiple of e_2e_1 and acting by X_{13}

gives a multiple of g_2e_1. Since

$$\left\{\begin{array}{rcl} g_1^{-1}(e_2e_1) & = & g_2e_1 \\ e_1(e_2e_1) & = & e_1 \end{array}\right. \text{ and } \left\{\begin{array}{rcl} g_2^{-1}(g_2e_1) & = & e_1 \\ (le_2)(g_2e_1) & = & e_2e_1 \end{array}\right. ,$$

the first set of equalities implies that $X_{23}U = 0$ and the second set of equalities implies that $X_{13}U = 0$ (otherwise U would be the whole space V, which contradicts U proper). Let $S := X_{12} + X_{23} + X_{13}$. Thus, we have shown that if V is not irreducible and if U is a nontrivial proper invariant subspace of V then $SU = 0$. This equality implies in turn that $det(S) = 0$ as U is nonzero. Let's compute the matrix of S in the basis $\mathcal{B}$. The action of X_{12} gives the first row of the matrix, the action of X_{23} the second row of the matrix and the action of X_{13} the third row of the matrix as we make it appear on the matrix below:

$$Mat_{\mathcal{B}}S = \begin{array}{c} \\ X_{12} \\ X_{23} \\ X_{13} \end{array} \begin{array}{c} \begin{array}{ccc} e_1 & e_2e_1 & g_2e_1 \end{array} \\ \begin{pmatrix} x & 1 & l \\ 1 & x & \frac{1}{l} \\ \frac{1}{l} & l & x \end{pmatrix} \end{array}$$

Using the defining relation $x = 1 - \frac{l-1/l}{1/r-r}$ and solving the equation $det(S) = 0$ in Maple, we get: $det(S) = 0 \Leftrightarrow l \in \{-r^3, -1, 1, 1/r^3\}$. Thus if l does not belong to any of these values, then V is irreducible. Conversely, we show that if V is irreducible then $l \notin \{-r^3, -1, 1, 1/r^3\}$. Indeed, for each of these values of l we claim that $\cap_{1\leq i<j\leq 3}(Ker\ X_{ij} \cap V)$ is a nontrivial proper invariant subspace of V. It suffices to show that $\cap_{1\leq i<j\leq 3}Ker\ X_{ij}$ is a $B(A_2)$-module and that for each of the values of l above, there exists a non-zero element in V which is annihilated by all the X_{ij}'s. The first point

comes from corollary 2 applied with $n = 3$. To see the second point, we prove the following lemma:

Lemma 3.

$$\textit{If } l = -r^3, \quad y(3) := -re_1 - \frac{1}{r}e_2e_1 + g_2e_1 \in \bigcap_{1\leq i<j\leq 3} Ker\, X_{ij} \tag{20}$$

$$\textit{If } l \in \{1, -1\}, \quad z(3) := e_1 - e_2e_1 \in \bigcap_{1\leq i<j\leq 3} Ker\, X_{ij} \tag{21}$$

$$\textit{If } l = \frac{1}{r^3}, \quad t(3) := \frac{1}{r}e_1 + re_2e_1 + g_2e_1 \in \bigcap_{1\leq i<j\leq 3} Ker\, X_{ij} \tag{22}$$

PROOF: for $l = 1$ or $l = -1$, it is easy to see that $e_1 - e_2e_1 \in Ker\, e_1$ by using $x = 1$ and the relations $e_1^2 = xe_1$ and $e_1e_2e_1 = e_1$. Also, it is immediate that $e_1 - e_2e_1 \in Ker\, e_2$. It remains to show that $X_{13}(e_1 - e_2e_1) = 0$. On one hand we have $X_{13}e_2 = g_1^{-1}e_2g_1e_2 = g_1^{-1}le_2$ by (6) and so $X_{13}e_2e_1 = g_1^{-1}le_2e_1$. On the other hand we have $X_{13}e_1 = g_1^{-1}e_2g_1e_1 = g_1^{-1}e_2l^{-1}e_1$. Replacing l by its value yields the desired result. If $l = -r^3$, we compute $x = -r^2 - 1/r^2$. The left action by e_1 on $y(3)$ gives after the use of the classical relations $(-rx - 1/r + l)e_1$, which after simplification is zero. Similarly, we obtain $e_2.y(3) = (-r - x/r + 1/l)e_2e_1$ and the coefficient in the parenthesis is zero after replacing x and l by their respective values. Let's now compute $X_{13}.y(3)$. We have: $X_{13}g_2e_1 = (g_2e_1g_2^{-1})g_2e_1 = g_2e_1^2 = x\,g_2e_1$. Next, from $e_2e_1 = g_1g_2e_1$, we derive $g_1^{-1}e_2e_1 = g_2e_1$. Using the formulas above that give the actions of X_{13} on e_2e_1 and e_1 respectively and replacing now $g_1^{-1}e_2e_1$ with g_2e_1, we get: $X_{13}.y(3) = (-r/l - l/r + x)g_2e_1 = 0$. This finishes the proof in the case $l = -r^3$. Finally, (22) is obtained the same way as (20) and is left to the reader. We may now state the following theorem:

Theorem 1.

$Span_F(e_1,\, e_2e_1,\, g_2e_1)$ is an irreducible $B(A_2)$-module iff $l \notin \{-r^3, -1, 1, 1/r^3\}$

Chapter 4

The Case n=4

Here W is spanned over F by

$$e_1, e_2e_1, g_2e_1, e_3e_2e_1, g_3e_2e_1, g_3g_2e_1$$

and the six positive roots are (in the same order):

$$\alpha_1, \alpha_2, \alpha_1 + \alpha_2, \alpha_3, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3$$

Generally speaking, in $B(A_{n-1})$ we will always order the roots in the following way:

$$\alpha_1, \alpha_2, \alpha_2+\alpha_1, \alpha_3, \alpha_3+\alpha_2, \alpha_3+\alpha_2+\alpha_1, ..., \alpha_{n-1}, \alpha_{n-1}+\alpha_{n-2}, ..., \alpha_{n-1}+...+\alpha_1$$

Recall that one top horizontal line is enough to determine a positive root. Geometrically, moving from the left to the right on the top line, start with a node ≥ 2 and join it to its left neighbors, the closest nodes being considered first. We note and we recall that the height of a root increases with the number of crossings. The main interest of ordering the roots this way is that the

action of $g_1, g_2, ..., g_{n-2}$ on a positive root β whose support does not contain the node $n-1$ has already been computed in the case of $B(A_{n-2})$, as the tangle representing β in $B(A_{n-1})$ is obtained from the tangle representing β in $B(A_{n-2})$ by adding a vertical line on the right side, which is left invariant with the action of the elements $g_1, g_2, ..., g_{n-2}$ (such actions don't impact the last node). As we will see later on, this ordering of the basis will allow us to define the matrices of $g_1, ..., g_{n-2}$ by blocks, inductively on n.
We try the same method as in the case $n = 3$. The difference is now that there are two vertical braids instead of one and they can either cross or not. Only in this part, we will denote by W^c the subspace of V spanned by the tangles representing the positive roots as in W with the difference that their two vertical strings are overcrossing. $\mathcal{B}^c$ denotes the basis in W^c formed by these elements. We look at the action of $g_1,\ g_2,\ g_3$ on $\mathcal{B}$ and $\mathcal{B}^c$ respectively. We first deal with the non crossed tangles. When acting by the g_i's, crossings appear. However, we still use matrices as a way of representing the actions. We will indicate that there is a crossing by adding a c when the crossing is over and a c' when the crossing is under, as an exponent on the right hand side of the coefficient (those special coefficients are indicated by boxes in the matrices below). For instance, with our conventions,

$$g_1(g_3e_2e_1) = \mathbf{m^c e_1} + g_3g_2e_1 - mg_3e_2e_1$$

means

$$g_1(g_3e_2e_1) = \mathbf{m\, e_1 g_3} + g_3g_2e_1 - mg_3e_2e_1$$

and

$$g_3(g_3g_2e_1) = g_2e_1 - mg_3g_2e_1 + \mathbf{(ml^{-1})^{c'} e_3e_2e_1}$$

means

$$g_3(g_3g_2e_1) = g_2e_1 - mg_3g_2e_1 + \mathbf{ml^{-1}\, e_3e_2e_1g_3^{-1}}$$

We gather the results of the action of g_1 in the following matrix:

$$\begin{array}{c c}
 & \begin{array}{cccccc} e_1 & e_2e_1 & g_2e_1 & e_3e_2e_1 & g_3e_2e_1 & g_3g_2e_1 \end{array} \\
\begin{array}{c} e_1 \\ e_2e_1 \\ g_2e_1 \\ e_3e_2e_1 \\ g_3e_2e_1 \\ g_3g_2e_1 \end{array} &
\begin{pmatrix}
l^{-1} & m & 0 & 0 & \boxed{m^c} & 0 \\
0 & -m & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \boxed{1^c} & 0 & 0 \\
0 & 0 & 0 & 0 & -m & 1 \\
0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}
\end{array}$$

And similarly, we get the matrix corresponding to the action of g_2. For instance, we have:

$$\begin{aligned}
g_2(g_3g_2e_1) &= g_3g_2g_3e_1 \text{ by the braid relation (1)} \\
&= (g_3g_2e_1)g_3 \text{ by commutativity of } e_1 \text{ and } g_3,
\end{aligned}$$

which yields the coefficient of the bottom right hand side.

$$\begin{array}{c c}
 & \begin{array}{cccccc} e_1 & e_2e_1 & g_2e_1 & e_3e_2e_1 & g_3e_2e_1 & g_3g_2e_1 \end{array} \\
\begin{array}{c} e_1 \\ e_2e_1 \\ g_2e_1 \\ e_3e_2e_1 \\ g_3e_2e_1 \\ g_3g_2e_1 \end{array} &
\begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 \\
0 & l^{-1} & ml^{-1} & m & 0 & 0 \\
1 & 0 & -m & 0 & 0 & 0 \\
0 & 0 & 0 & -m & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \boxed{1^c}
\end{pmatrix}
\end{array}$$

Finally the matrix corresponding to the action of g_3 is:

$$
\begin{array}{c|c}
 & \begin{array}{cccccc} e_1 & e_2e_1 & g_2e_1 & e_3e_2e_1 & g_3e_2e_1 & g_3g_2e_1 \end{array} \\
\begin{array}{c} e_1 \\ e_2e_1 \\ g_2e_1 \\ e_3e_2e_1 \\ g_3e_2e_1 \\ g_3g_2e_1 \end{array} &
\begin{pmatrix}
\boxed{1^c} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & l^{-1} & ml^{-1} & \boxed{(ml^{-1})^{c'}} \\
0 & 1 & 0 & 0 & -m & 0 \\
0 & 0 & 1 & 0 & 0 & -m
\end{pmatrix}
\end{array}
$$

Note that these matrices can be obtained by using the classical relations of the $B(A_3)$ algebra looking at the action of the g_i's on the elements of the basis of W, or can be obtained by manipulating the tangles in TM_4. Both ways lead of course to the same result and it is to the reader to determine which way (or which combination of ways) he or she decides to use. While it is easy to see the action of g_1 on $e_1, e_2e_1, g_2e_1, g_3e_2e_1, g_3g_2e_1$ by using the rules of the first page, it is direct to see the overcrossing for the tangle $e_3e_2e_1$ after the left action by g_1. Because of the terms in the boxes, we see that W is not a $B(A_3)$-module. An idea is to look at the action of the g_i's on the crossed roots e_1g_3, $e_2e_1g_3$, $g_2e_1g_3$, $e_3e_2e_1g_3$, $g_3e_2e_1g_3$, $g_3g_2e_1g_3$ which span W^c to see in turn what we get and exhibit from there an invariant subspace. Obviously, the coefficients which are not in the boxes are unchanged since all we do is make the words bigger by adding to them a factor g_3 on the right (which makes the crossing appear) and the action by the g_i's on these words takes place on the left. Next, we have:

$$
\begin{aligned}
g_1(e_3e_2e_1g_3) &= e_3(g_1e_2e_1)g_3 \text{ by commutativity of } e_1 \text{ and } g_3 \\
&= e_3(g_2e_1 + m(e_1 - e_2e_1))g_3 \text{ by (12)} \\
&= e_3g_2e_1g_3 - m\, e_3e_2e_1g_3 \text{ since } e_3e_1 = 0 \text{ in } I_1/I_2 \\
&= e_3g_2g_3e_1 - m\, e_3e_2e_1g_3 \text{ by commutativity of } e_1 \text{ and } g_3 \\
&= \mathbf{e_3e_2e_1 - m\, e_3e_2e_1g_3} \qquad (23)
\end{aligned}
$$

The last equality is obtained by using (11) and the anti-involution on products of generators of $B(A_4)$:

$$g_{i_1}...g_{i_q} \longmapsto g_{i_q}...g_{i_q}$$

We also have:

$$g_1(g_3e_2e_1g_3) = \mathbf{m(e_1 - me_1g_3)} - mg_3e_2e_1g_3 + g_3g_2e_1g_3 \tag{24}$$

$$g_2(g_3g_2e_1g_3) = \mathbf{g_3g_2e_1 - mg_3g_2e_1g_3} \tag{25}$$

$$g_3(e_1g_3) = \mathbf{e_1 - me_1g_3} \tag{26}$$

We notice that:

1^c is replaced by "$noncrossed - m\,crossed$" as in (23), (25) and (26)

m^c is replaced by "$m(noncrossed - m\,crossed)$" as in (24)

And indeed, we look at the left action of the g_i's on words of the form wg_3. For such prefix w we have established that:

$$\boxed{g_iw = \lambda\, w' g_3 + s} \tag{27}$$

where i is adequately picked, with λ the appropriate coefficient and with s a sum of noncrossed terms (eventually zero). Now we have:

$$\begin{aligned} g_i(wg_3) &= (g_iw)g_3 && \text{by associativity} \\ &= \lambda\, w' g_3^2 \,+\, s\, g_3 && \text{by replacing with (27)} \\ &= \lambda(w' - m\, w' g_3 + ml^{-1}w' e_3) + s\, g_3 && \text{by application of } (8) \end{aligned}$$

But $w^{'}$ belongs to W and has its nodes 1 and 2 joined on the bottom line. Then, the tangle obtained by multiplying $w^{'}$ with e_3 has nodes 1 and 2, 3 and 4 respectively joined on the bottom line, hence is zero in I_1/I_2. So we finally get:

$$\boxed{g_i(wg_3) = \lambda(w^{'} - m\,w^{'}g_3) +\ s\,g_3} \tag{28}$$

Thus,

$$\lambda^c \text{ is replaced by } \lambda\,(noncrossed - m\ crossed)$$

It remains to compute

$$\begin{aligned} g_3(g_3g_2e_1g_3) &= g_2e_1g_3 - mg_3g_2e_1g_3 + ml^{-1}e_3g_2e_1g_3 \text{ by the rule } (8) \\ &= g_2e_1g_3 - mg_3g_2e_1g_3 + ml^{-1}\ e_3e_2e_1 \end{aligned}$$

to see how the last box is being modified. Here the last last term of the last equation is obtained from $e_3g_2e_1g_3$ by commuting e_1 and g_3, then applying the rule (11) with the anti-involution described above. Note that another way to see it is to notice that the term in the box is nothing else but $ml^{-1}e_3e_2e_1g_3^{-1}$, where g_3^{-1} makes the undercrossing appear. Then acting by g_3 on the left yields the non crossed term $ml^{-1}e_3e_2e_1$.

We summarize our results in the matrices below. Like in the non crossed case, these matrices are not mathematical objects, but are used as a convenient way of representing the actions of the g_i's.

For the action of g_1 on the elements of $\mathcal{B}^c$, we get:

$$\begin{array}{c c} & \begin{array}{cccccc} e_1g_3 & e_2e_1g_3 & g_2e_1g_3 & e_3e_2e_1g_3 & g_3e_2e_1g_3 & g_3g_2e_1g_3 \end{array} \\ \begin{array}{c} e_1g_3 \\ e_2e_1g_3 \\ g_2e_1g_3 \\ e_3e_2e_1g_3 \\ g_3e_2e_1g_3 \\ g_3g_2e_1g_3 \end{array} & \begin{pmatrix} l^{-1} & m & 0 & 0 & \boxed{m(1^{nc}-m)} & 0 \\ 0 & -m & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boxed{1^{nc}-m} & 0 & 0 \\ 0 & 0 & 0 & 0 & -m & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

And similarly we represent the action of g_2 on $\mathcal{B}^c$ by the matrix

$$\begin{array}{c c} & \begin{array}{cccccc} e_1g_3 & e_2e_1g_3 & g_2e_1g_3 & e_3e_2e_1g_3 & g_3e_2e_1g_3 & g_3g_2e_1g_3 \end{array} \\ \begin{array}{c} e_1g_3 \\ e_2e_1g_3 \\ g_2e_1g_3 \\ e_3e_2e_1g_3 \\ g_3e_2e_1g_3 \\ g_3g_2e_1g_3 \end{array} & \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & l^{-1} & ml^{-1} & m & 0 & 0 \\ 1 & 0 & -m & 0 & 0 & 0 \\ 0 & 0 & 0 & -m & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \boxed{1^{nc}-m} \end{pmatrix} \end{array}$$

and we represent the action of g_3 on $\mathcal{B}^c$ by the matrix

$$\begin{array}{c c} & \begin{array}{cccccc} e_1g_3 & e_2e_1g_3 & g_2e_1g_3 & e_3e_2e_1g_3 & g_3e_2e_1g_3 & g_3g_2e_1g_3 \end{array} \\ \begin{array}{c} e_1g_3 \\ e_2e_1g_3 \\ g_2e_1g_3 \\ e_3e_2e_1g_3 \\ g_3e_2e_1g_3 \\ g_3g_2e_1g_3 \end{array} & \begin{pmatrix} \boxed{1^{nc}-m} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & l^{-1} & ml^{-1} & \boxed{(ml^{-1})^{nc}} \\ 0 & 1 & 0 & 0 & -m & 0 \\ 0 & 0 & 1 & 0 & 0 & -m \end{pmatrix} \end{array}$$

Let's now consider the subspace W_r of W defined as follows:

$$W_r := \text{Span}_F(e_1 + r\, e_1g_3,\ e_2e_1 + re_2e_1g_3,\ g_2e_1 + r\, g_2e_1g_3,$$
$$e_3e_2e_1 + r\, e_3e_2e_1g_3,\ g_3e_2e_1 + r\, g_3e_2e_1g_3,\ g_3g_2e_1 + r\, g_3g_2e_1g_3)$$

W_r is spanned over F by linear combinations of non crossed tangles spanning W and their corresponding crossed tangles spanning W^c. The spanning elements above form a basis of W_r that we denote by $\mathcal{B}_r$. We will show that W_r is an invariant subspace. To that aim, let's consider the linear combination of non crossed tangle and crossed tangle $w + r\,wg_3$. Using (27) and (28) above, we compute:

$$\begin{aligned} g_i(w + rwg_3) &= \lambda(r\,w^{'} + (1 - mr)w^{'}g_3) + (s\, +\, r\,sg_3) \\ &= \lambda r(\underbrace{w^{'} + r\,w^{'}g_3}_{\in W_r}) + (\underbrace{s\, +\, r\,sg_3}_{\in W_r}) \end{aligned} \tag{29}$$

Further, we have:

$$\begin{aligned} g_3(g_3g_2e_1 + r\,g_3g_2e_1g_3) = (g_2e_1 + r\,g_2e_1g_3) - m(g_3g_2e_1 + r\,g_3g_2e_1g_3) \\ + ml^{-1}(e_3e_2e_1g_3^{-1} + re_3e_2e_1) \end{aligned} \tag{30}$$

But using the tangle formula:

$$T^{-} = T^{+} + m(T^{\circ} - T^{\infty}) \tag{31}$$

which is one of the defining relations of the Kauffman's tangle algebra (see [12]), we can express $e_3e_2e_1g_3^{-1}$ as a sum of two similar tangles as follows:

$$e_3e_2e_1g_3^{-1} = e_3e_2e_1g_3 + m\,e_3e_2e_1$$

T^{∞} being zero for this tangle as we work in I_1/I_2. It follows that (30) can be rewritten in the following way:

$$\begin{aligned} g_3(g_3g_2e_1 + r\,g_3g_2e_1g_3) = (g_2e_1 + r\,g_2e_1g_3) - m(g_3g_2e_1 + r\,g_3g_2e_1g_3) \\ + ml^{-1}((m + r)e_3e_2e_1 + e_3e_2e_1g_3) \end{aligned}$$

Recall that by definition, $m = r^{-1} - r$, so that $m + r = r^{-1}$. Hence, we finally get:

$$g_3(g_3g_2e_1 + r\, g_3g_2e_1g_3) = (\overbrace{g_2e_1 + r\, g_2e_1g_3}^{\in W_r}) - m(\overbrace{g_3g_2e_1 + r\, g_3g_2e_1g_3}^{\in W_r}) \\ + \frac{ml^{-1}}{r}(\underbrace{e_3e_2e_1 + r\, e_3e_2e_1g_3}_{\in W_r}) \tag{32}$$

Since the columns that don't contain any box in the matrices are the same in both cases crossed and non crossed, we conclude that W_r is an invariant subspace of V as announced. Let's give the matrices of the left actions by g_1, g_2 and g_3 in the basis $\mathcal{B}_r$. We denote them by $G_1(4)$, $G_2(4)$ and $G_3(4)$ respectively. We need to replace the coefficients in the boxes by appropriate new ones. And in fact, we see that

$$\text{by } (29),\ \boxed{\lambda^c \ \text{ must be replaced by } \ \lambda\, r} \tag{33}$$

and

$$\text{by } (32),\ \boxed{(ml^{-1})^{c'} \ \text{ must be replaced by } \ \frac{ml^{-1}}{r}} \tag{34}$$

We obtain the following matrices:

$$G_1(4) := \begin{pmatrix} l^{-1} & m & 0 & 0 & m\,r & 0 \\ 0 & -m & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r & 0 & 0 \\ 0 & 0 & 0 & 0 & -m & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$G_2(4) := \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & l^{-1} & ml^{-1} & m & 0 & 0 \\ 1 & 0 & -m & 0 & 0 & 0 \\ 0 & 0 & 0 & -m & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & r \end{pmatrix}$$

$$G_3(4) := \begin{pmatrix} r & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & l^{-1} & ml^{-1} & \frac{ml^{-1}}{r} \\ 0 & 1 & 0 & 0 & -m & 0 \\ 0 & 0 & 1 & 0 & 0 & -m \end{pmatrix}$$

We verify that the braid relations (1) and (2) of the front page are satisfied on these matrices and using these matrices, we define new matrices $E_i(4)$ for $i \in \{1,2,3\}$. They are given by the formula:

$$E_i(4) = \frac{l}{m}(G_i^2(4) + mG_i(4) - I_6) \qquad (35)$$

where I_6 is the identity matrix of size 6. We check in turn that the relations (4), (5) and (6) are satisfied on the $G_i(4)$'s and $E_i(4)$'s. Hence we get a representation of $B(A_3)$ in W_r which is defined on the generators e_1, e_2, e_3, g_1, g_2, g_3 of the BMW algebra $B(A_3)$ by the map:

$$\begin{array}{ccc} B(A_3) & \longrightarrow & \mathcal{M}(6,F) \\ g_1,\, g_2,\, g_3 & \longmapsto & G_1(4),\, G_2(4),\, G_3(4) \\ e_1,\, e_2, \epsilon_3 & \longmapsto & E_1(4),\, E_2(4),\, E_3(4) \end{array}$$

Suppose that W_r is not irreducible. Let U be a non trivial proper invariant subspace of W_r. We claim that the X_{ij}'s, $1 \leq i < j \leq 4$ act trivially on U. Indeed, suppose that there exists a non zero element u in U such that $X_{ij}u \neq 0$. Since the tangle resulting from the product $X_{ij}u$ is in W_r and is a linear combination of tangles having their nodes i and j joined at the top, with the edge (ij) overcrossing the eventual vertical lines that it intersects as in X_{ij}, and their nodes 1 and 2 joined at the bottom, it must be proportional to the element of $\mathcal{B}_r$ having the same horizontal lines. Let's rename the six elements of the basis $\mathcal{B}_r$:

$$\mathcal{B}_r = (x_{\alpha_1}, x_{\alpha_2}, x_{\alpha_2+\alpha_1}, x_{\alpha_3}, x_{\alpha_3+\alpha_2}, x_{\alpha_3+\alpha_2+\alpha_1})$$

So $X_{ij}u$ is proportional to an x_β where β is one of the six positive roots. Recall that by part 2, an expression for x_β is:

$$\begin{aligned} x_\beta &= \varphi(w_{\beta 1})e_1 + r\varphi(w_{\beta 1})e_1 g_3 \\ &= \varphi(w_{\beta 1})(e_1 + re_1 g_3) \\ &= \varphi(w_{\beta 1})x_{\alpha_1} \end{aligned} \tag{36}$$

We see with (36) that any x_β can be obtained from x_{α_1} by multiplying x_{α_1} by the left with an element in $B(A_3)$. Moreover, $\varphi(w_{\beta 1})$ is invertible as it is a word containing only g_i's by construction. We conclude that if x_β is in U, then x_{α_1} is in U as U is a $B(A_3)$-module and then all the other x_γ with $\gamma \in \phi^+ \setminus \{\alpha_1, \beta\}$ are also in U by above. Now if $X_{ij}u$ is non zero and proportional to x_β, then x_β is in U and by what preceeds all the elements of the basis $\mathcal{B}_r$ are also in U. It implies that $U = W_r$, which contradicts the fact that U is proper. Thus, $X_{ij}u = 0$ for all $u \in U$ and since the argument can be applied to any of the X_{ij}'s, we have actually proved that all the X_{ij}'s annihilate U. Their sum also annihilates U. Let's consider their sum:

$$S := X_{12} + X_{13} + X_{14} + X_{23} + X_{24} + X_{34}$$

It comes:

$$S\,U = 0$$

Since U is non zero, we must have $det(S) = 0$. We will compute the matrix of the left action of S in the basis $\mathcal{B}_r$ and then compute its determinant. Note that each row in this matrix corresponds to the action of one of the

X_{ij}'s. The matrix can be obtained by calculating the actions of the X_{ij}'s directly on the tangles x_{α_1}, x_{α_2}, $x_{\alpha_2+\alpha_1}$, x_{α_3}, $x_{\alpha_3+\alpha_2}$, $x_{\alpha_3+\alpha_2+\alpha_1}$ or by using Maple, the expressions of $G_1(4)$, $G_2(4)$, $G_3(4)$, the formula (35), the formula (9) and the conjugation formulas for $j > i+1$:

$$X_{ij} = g_{j-1} \dots g_{i+1}\, e_i\, g_{i+1}^{-1} \dots g_{j-1}^{-1}$$

We get the matrix:

$$Mat_{\mathcal{B}_r}(S) \;=\; \begin{array}{c} \\ X_{12} \\ X_{23} \\ X_{13} \\ X_{34} \\ X_{24} \\ X_{14} \end{array} \begin{array}{c} \begin{array}{cccccc} x_{\alpha_1} & x_{\alpha_2} & x_{\alpha_1+\alpha_2} & x_{\alpha_3} & x_{\alpha_3+\alpha_2} & x_{\alpha_3+\alpha_2+\alpha_1} \end{array} \\ \begin{pmatrix} x & 1 & l & 0 & r & lr \\ 1 & x & \frac{1}{l} & 1 & l & 0 \\ \frac{1}{l} & l & x & r & (\frac{1}{r}-r)(r-l) & l \\ 0 & 1 & \frac{1}{r} & x & \frac{1}{l} & \frac{1}{lr} \\ \frac{1}{r} & \frac{1}{l} & (r-\frac{1}{r})(\frac{1}{r}-\frac{1}{l}) & l & x & \frac{1}{l} \\ \frac{1}{lr} & 0 & \frac{1}{l} & lr & l & x \end{pmatrix} \end{array}$$

Using Maple to compute the determinant of this 6×6 matrix, we get that $det(S)$ is zero if and only if $l \in \{r, -r^3, \frac{1}{r}, -\frac{1}{r}, \frac{1}{r^5}\}$ and we conclude that if W_r is not irreducible then l must be one of these values. Conversely, we show that if l takes one of these values, then W_r is not irreducible. It suffices to exhibit a nontrivial proper invariant subspace of W_r for each of these values of l. In what follows, we will denote by T_4 the set $\{r, -r^3, \frac{1}{r}, -\frac{1}{r}, \frac{1}{r^5}\}$

Proposition 1. *If l belongs to the set T_4 then $\bigcap_{1\le i<j\le 4} (Ker\, X_{ij} \cap W_r)$ is a non trivial proper invariant subspace of W_r.*

PROOF: by corollary (2) applied with $n = 4$, we know that $\bigcap_{1\le i<j\le 4} Ker\, X_{ij}$ is a $B(A_3)$-module. Hence $\bigcap_{1\le i<j\le 4} (Ker\, X_{ij} \cap W_r)$ is an invariant subspace of W_r. It is obviously not W_r itself since for instance $e_1 + r\, e_1 g_3 \in W_r$, but

$e_2(e_1+r\,e_1g_3) =\neq 0$, so that $e_1+r\,e_1g_3 \notin\ Ker\,e_2$, which implies $e_1+r\,e_1g_3 \notin \bigcap_{1\leq i<j\leq 4}(Ker\,X_{ij}\cap W_r)$. It remains to show that $\bigcap_{1\leq i<j\leq 4}(Ker\,X_{ij}\cap W_r)$ is non trivial. To this aim, we show the following lemma that exhibits for each value of l in T_4 an element in W_r that is annihilated by all the X_{ij}'s:

Lemma 4.

Let $x(4) := r^2\,x_{\alpha_1+\alpha_2} + x_{\alpha_2+\alpha_3} - r\,x_{\alpha_1+\alpha_2+\alpha_3} - r\,x_{\alpha_2}$

and $y(4) := -r^2\,x_{\alpha_1} - \frac{1}{r^2}x_{\alpha_3} + x_{\alpha_1+\alpha_2+\alpha_3} - x_{\alpha_2}$

and $z(4) := x_{\alpha_1} + x_{\alpha_3} + x_{\alpha_1+\alpha_2+\alpha_3} - x_{\alpha_2}$

and $z^{'}(4) := x_{\alpha_1+\alpha_2+\alpha_3} - x_{\alpha_2}$

and $t(4) := \frac{1}{r^2}x_{\alpha_1} + r^2x_{\alpha_3} + \frac{1}{r}x_{\alpha_1+\alpha_2} + r\,x_{\alpha_2+\alpha_3} + x_{\alpha_1+\alpha_2+\alpha_3} + x_{\alpha_2}$

We have for the distinct values of l:

$$\textit{If } l = r \quad \textit{then } x(4) \in \bigcap_{1\leq i<j\leq 4} Ker\,X_{ij} \tag{37}$$

$$\textit{If } l = -r^3 \ \textit{then } y(4) \in \bigcap_{1\leq i<j\leq 4} Ker\,X_{ij} \tag{38}$$

$$\textit{If } l = -\frac{1}{r} \ \textit{then } z(4) \in \bigcap_{1\leq i<j\leq 4} Ker\,X_{ij} \tag{39}$$

$$\textit{If } l = \frac{1}{r} \quad \textit{then } z'(4) \in \bigcap_{1\leq i<j\leq 4} Ker\,X_{ij} \tag{40}$$

$$\textit{If } l = \frac{1}{r^5} \quad \textit{then } t(4) \in \bigcap_{1\leq i<j\leq 4} Ker\,X_{ij} \tag{41}$$

PROOF OF THE LEMMA: Since each row of the matrix, say $M(l)$, of the left action of S on the basis $\mathcal{B}_r$ corresponds to the action of exactly one of the X_{ij}'s, it suffices to verify that $M(r)X = 0$, $M(-r^3)Y = 0$, $M(-1/r)Z = 0$, $M(1/r)Z' = 0$ and $M(1/r^5)T = 0$ where X, Y, Z, Z', T are the column vectors respectively corresponding to $x(4)$, $y(4)$, $z(4)$, $z'(4)$, $t(4)$ in the basis $\mathcal{B}_r$.

$$\begin{pmatrix} 2 & 1 & r & 0 & r & r^2 \\ 1 & 2 & \frac{1}{r} & 1 & r & 0 \\ \frac{1}{r} & r & 2 & r & 0 & r \\ 0 & 1 & \frac{1}{r} & 2 & \frac{1}{r} & \frac{1}{r^2} \\ \frac{1}{r} & \frac{1}{r} & 0 & r & 2 & \frac{1}{r} \\ \frac{1}{r^2} & 0 & \frac{1}{r} & r^2 & r & 2 \end{pmatrix} \begin{pmatrix} 0 \\ -r \\ r^2 \\ 0 \\ 1 \\ -r \end{pmatrix} = 0$$

$$\begin{pmatrix} -\frac{1}{r^2} - r^2 & 1 & -r^3 & 0 & r & -r^4 \\ 1 & -\frac{1}{r^2} - r^2 & -\frac{1}{r^3} & 1 & -r^3 & 0 \\ -\frac{1}{r^3} & -r^3 & -\frac{1}{r^2} - r^2 & r & 1 - r^4 & -r^3 \\ 0 & 1 & \frac{1}{r} & -\frac{1}{r^2} - r^2 & -\frac{1}{r^3} & -\frac{1}{r^4} \\ \frac{1}{r} & -\frac{1}{r^3} & 1 - \frac{1}{r^4} & -r^3 & -\frac{1}{r^2} - r^2 & -\frac{1}{r^3} \\ -\frac{1}{r^4} & 0 & -\frac{1}{r^3} & -r^4 & -r^3 & -\frac{1}{r^2} - r^2 \end{pmatrix} \begin{pmatrix} -r^2 \\ -1 \\ 0 \\ -\frac{1}{r^2} \\ 0 \\ 1 \end{pmatrix} = 0$$

$$\begin{pmatrix} 2 & 1 & -\frac{1}{r} & 0 & r & -1 \\ 1 & 2 & -r & 1 & -\frac{1}{r} & 0 \\ -r & -\frac{1}{r} & 2 & r & \frac{1}{r^2} - r^2 & -\frac{1}{r} \\ 0 & 1 & \frac{1}{r} & 2 & -r & -1 \\ \frac{1}{r} & -r & r^2 - \frac{1}{r^2} & -\frac{1}{r} & 2 & -r \\ -1 & 0 & -r & -1 & -\frac{1}{r} & 2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} = 0$$

$$\begin{pmatrix} 0 & 1 & \frac{1}{r} & 0 & r & 1 \\ 1 & 0 & r & 1 & \frac{1}{r} & 0 \\ r & \frac{1}{r} & 0 & r & 2 - \frac{1}{r^2} - r^2 & \frac{1}{r} \\ 0 & 1 & \frac{1}{r} & 0 & r & 1 \\ \frac{1}{r} & r & 0 & \frac{1}{r} & 0 & r \\ 1 & 0 & r & 1 & \frac{1}{r} & 0 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = 0$$

$$\begin{pmatrix} z & 1 & \frac{1}{r^5} & 0 & r & \frac{1}{r^4} \\ 1 & z & r^5 & 1 & \frac{1}{r^5} & 0 \\ r^5 & \frac{1}{r^5} & z & r & 1-r^2+\frac{1}{r^4}-\frac{1}{r^6} & \frac{1}{r^5} \\ 0 & 1 & \frac{1}{r} & z & r^5 & r^4 \\ \frac{1}{r} & r^5 & 1-\frac{1}{r^2}+r^4-r^6 & \frac{1}{r^5} & z & r^5 \\ r^4 & 0 & r^5 & \frac{1}{r^4} & \frac{1}{r^5} & z \end{pmatrix} \begin{pmatrix} \frac{1}{r^2} \\ 1 \\ \frac{1}{r} \\ r^2 \\ r \\ 1 \end{pmatrix} = 0$$

where in the last matrix $z := -\left(r^4 + \frac{1}{r^4} + r^2 + \frac{1}{r^2}\right)$. These equalities end the proof of the lemma. We are now able to conclude. By proposition 1, if l is in T_4 then W_r is not irreducible as it contains a non trivial proper invariant subspace. Conversely, we have seen above that if W_r is not irreducible then l must belong to T_4. We summarize this result in a theorem:

Theorem 2. *$Span_F(x_{\alpha_1}, x_{\alpha_2}, x_{\alpha_2+\alpha_1}, x_{\alpha_3}, x_{\alpha_3+\alpha_2}, x_{\alpha_3+\alpha_2+\alpha_1})$ is an irreducible $B(A_3)$-module if and only if $l \notin \{r, -r^3, -\frac{1}{r}, \frac{1}{r}, \frac{1}{r^5}\}$*

Chapter 5

The Case n=5

There are ten positive roots:

$$\phi^+ = \{\alpha_1,\, \alpha_2, \alpha_2 + \alpha_1,\, \alpha_3,\, \alpha_3 + \alpha_2,\, \alpha_3 + \alpha_2 + \alpha_1,$$
$$\alpha_4,\, \alpha_4 + \alpha_3,\, \alpha_4 + \alpha_3 + \alpha_2,\, \alpha_4 + \alpha_3 + \alpha_2 + \alpha_1\}$$

corresponding to the following tangles in V:

$$e_1,\, e_2e_1, g_2e_1,\, e_3e_2e_1,\, g_3e_2e_1, g_3g_2e_1,$$
$$e_4e_3e_2e_1,\, g_4e_3e_2e_1,\, g_4g_3e_2e_1, g_4g_3g_2e_1$$

The first six tangles are obtained from the same six tangles in TM_4 by adding a straight vertical line on the right hand side. Hence the actions of g_1, g_2, g_3 (which affect only the first four nodes) are the same as in the case $n = 4$. We only need to compute the actions of g_1, g_2, g_3 on the four remaining tangles. But we need to do all the computations for the action of g_4. Crossings eventually appear between the three vertical braids. We represent them by means of permutations that are either transposes or cycles of lengths three. We will again denote those crossed tangles by their non crossed analogues, but we will specify the permutation corresponding to the vertical braids by an exponent on the right hand side of the algebra element. When the permutation is the identity, the exponent will be omitted. Moreover and unless otherwise mentioned, all the crossings are over. When there is one undercrossing (on the vertical braids), we will indicate it with a prime on the transpose and if there are two undercrossings, we will indicate them with a double prime on the 3-cycle. Let's compute the action of g_1 on the four tangles involving α_4:

$$\begin{aligned}
g_1(e_4e_3e_2e_1) &= e_4e_3g_1e_2e_1 && \text{by } (1) \\
&= e_4e_3g_2e_1 - me_4e_3e_2e_1 && \text{by } (12) \text{ and the fact that } e_3e_1 = 0 \\
&= e_4e_3e_2e_1^{(12)} && \text{by } (31) \qquad (42) \\
g_1(g_4e_3e_2e_1) &= g_4e_3e_2e_1^{(12)} && \qquad (43) \\
g_1(g_4g_3e_2e_1) &= g_4g_3g_1e_2e_1 && \text{by } (1) \\
&= g_4g_3g_2e_1 + m\, g_4g_3e_1 - m\, g_4g_3e_2e_1 && \text{by } (12) \\
&= g_4g_3g_2e_1 + m\, e_1^{(123)} - m\, g_4g_3e_2e_1 && \text{(rewriting)} \qquad (44) \\
g_1(g_4g_3g_2e_1) &= g_4g_3g_1g_2e_1 && \text{by } (1) \\
&= g_4g_3e_2e_1 && \text{by } (10) \qquad (45)
\end{aligned}$$

(43) was not detailed because it is obtained the same way as (42). Let's do the same with g_2:

$$
\begin{aligned}
g_2(e_4e_3e_2e_1) &= e_4g_2e_3e_2e_1 && \text{by } (1) \\
&= e_4g_3e_2e_1 - m\, e_4e_3e_2e_1 && \text{by } (12) \text{ and the fact that } e_4e_2 = 0 \\
&= e_4e_3e_2e_1^{(23)} && \text{by } (31) \qquad (46) \\
g_2(g_4e_3e_2e_1) &= g_4g_2e_3e_2e_1 && \text{by } (1) \\
&= g_4g_3e_2e_1 + m\, g_4e_2e_1 - m\, g_4e_3e_2e_1 && \text{by } (12) \\
&= g_4g_3e_2e_1 + m\, e_2e_1^{(23)} - m\, g_4e_3e_2e_1 && \text{(rewriting)} \qquad (47) \\
g_2(g_4g_3e_2e_1) &= g_4g_2g_3e_2e_1 && \text{by } (1) \\
&= g_4e_3e_2e_1 && \text{by } (10) \qquad (48) \\
g_2(g_4g_3g_2e_1) &= g_4g_2g_3g_2e_1 && \text{by } (1) \\
&= g_4g_3g_2g_3e_1 && \text{by } (2) \\
&= g_4g_3g_2e_1^{(12)} && \text{(rewriting)} \qquad (49)
\end{aligned}
$$

We have for g_3:

$$
\begin{aligned}
g_3(e_4e_3e_2e_1) &= g_4e_3e_2e_1 + m\, e_3e_2e_1 - m\, e_4e_3e_2e_1 && \text{by } (12) \qquad (50) \\
g_3(g_4e_3e_2e_1) &= e_4e_3e_2e_1 && \text{by } (10) \qquad (51) \\
g_3(g_4g_3e_2e_1) &= g_4g_3g_4e_2e_1 && \text{by } (2) \\
&= g_4g_3e_2e_1g_4 && \text{by } (1) \\
&= g_4g_3e_2e_1^{(23)} && \text{(rewriting)} \qquad (52) \\
g_3(g_4g_3g_2e_1) &= g_4g_3g_4g_2e_1 && \text{by } (2) \\
&= g_4g_3g_2e_1g_4 && \text{by } (1) \\
&= g_4g_3g_2e_1^{(23)} && \text{(rewriting)} \qquad (53)
\end{aligned}
$$

Let's now compute the action of g_4 on all the elements of the basis $\mathcal{B}$ of W.

We have:

$$
\begin{aligned}
g_4 e_1 &= e_1^{(23)} && \text{(rewriting)} && (54)\\
g_4(e_2e_1) &= e_2e_1^{(23)} && \text{(rewriting)} && (55)\\
g_4(g_2e_1) &= g_2e_1^{(23)} && \text{(rewriting)} && (56)\\
g_4(e_3e_2e_1) &= g_4e_3e_2e_1 && \text{(unchanged)} && (57)\\
g_4(g_3e_2e_1) &= g_4g_3e_2e_1 && \text{(unchanged)} && (58)\\
g_4(g_3g_2e_1) &= g_4g_3g_2e_1 && \text{(unchanged)} && (59)\\
g_4(e_4e_3e_2e_1) &= l^{-1}e_4e_3e_2e_1 \quad \text{by } (4) && && (60)\\
g_4(g_4e_3e_2e_1) &= e_3e_2e_1 - m\, g_4e_3e_2e_1 + ml^{-1}e_4e_3e_2e_1 \ \text{ by } (8) && && (61)\\
g_4(g_4g_3e_2e_1) &= g_3e_2e_1 - m\, g_4g_3e_2e_1 + ml^{-1}e_4g_3e_2e_1 \ \text{ by } (8) && &&\\
&= g_3e_2e_1 - m\, g_4g_3e_2e_1 + ml^{-1}e_4e_3e_2e_1^{(23)'} && && (62)\\
g_4(g_4g_3g_2e_1) &= g_3g_2e_1 - m\, g_4g_3g_2e_1 + ml^{-1}e_4g_3g_2e_1 \ \text{ by } (8) && &&\\
&= g_3g_2e_1 - m\, g_4g_3g_2e_1 + ml^{-1}e_4e_3e_2e_1^{(132)''} && && (63)
\end{aligned}
$$

In (62), there is one undercrossing indicated with a prime. In (63), there are two undercrossings that are indicated with a double prime. We summarize these actions in matrices: the action of g_1 is represented by the matrix:

$$
\begin{pmatrix}
l^{-1} & m & 0 & 0 & \boxed{m^{(12)}} & 0 & 0 & 0 & \boxed{m^{(123)}} & 0\\
0 & -m & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & \boxed{1^{(12)}} & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & -m & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & \boxed{1^{(12)}} & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1^{(12)}} & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -m & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}
$$

The matrix giving the action of g_2 is:

$$\begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & l^{-1} & ml^{-1} & m & 0 & 0 & 0 & \boxed{m^{(23)}} & 0 & 0\\
1 & 0 & -m & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -m & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \boxed{1^{(12)}} & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & \boxed{1^{(23)}} & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -m & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1^{(12)}}
\end{pmatrix}$$

And the one giving the action of g_3 is:

$$\begin{pmatrix}
\boxed{1^{(12)}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & l^{-1} & ml^{-1} & \boxed{(ml^{-1})^{(12)'}} & m & 0 & 0 & 0\\
0 & 1 & 0 & 0 & -m & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & -m & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & -m & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1^{(23)}} & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1^{(23)}}
\end{pmatrix}$$

As already mentioned above, in those three matrices respectively representing the actions of g_1, g_2 and g_3 on the tangles spanning W, the upper left 6×6 block is the same as in the case $n = 4$ where the c (resp c') indicating the over (resp under) crossing on the two first vertical braids has been replaced with our new notations by the transpose (12) (resp $(12)'$). Finally, let's give the matrix representing the action of g_4:

$$\begin{pmatrix}
\boxed{1^{(23)}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \boxed{1^{(23)}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \boxed{1^{(23)}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & l^{-1} & ml^{-1} & \boxed{(ml^{-1})^{(23)'}} & \boxed{(ml^{-1})^{(132)''}} \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & -m & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -m & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -m
\end{pmatrix}$$

In the case $n = 5$, it appears to be harder to explicit an invariant subspace of W like we did previously. In the case $n = 4$, we got a representation by multiplying the coefficients in the boxes by r when the crossing was over (as in (33)) and by dividing them by r when the crossing was under (as in (34)). Here we try the same method, the power of r increasing with the number of crossings: if there are two over crossings among the vertical strands, we multiply the corresponding coefficient in the box by r^2. If there are two under crossings among the vertical strands, we will divide the corresponding coefficient in the box by r^2. In other words, a transpose is replaced by a multiplication by r and a cycle of length three is replaced by a multiplication by r^2 when the vertical crossings are over. A transpose is replaced by a division by r and a cycle of length three is replaced by a division by r^2 when all the vertical crossings are under. It yields the matrices:

$$G_1(5) = \begin{pmatrix} l^{-1} & m & 0 & 0 & mr & 0 & 0 & 0 & mr^2 & 0 \\ 0 & -m & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -m & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & r & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -m & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$G_2(5) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & l^{-1} & ml^{-1} & m & 0 & 0 & 0 & mr & 0 & 0 \\ 1 & 0 & -m & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -m & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & r & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -m & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & r \end{pmatrix}$$

$$G_3(5) = \begin{pmatrix} r & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & l^{-1} & ml^{-1} & \frac{ml^{-1}}{r} & m & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -m & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -m & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -m & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & r & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & r \end{pmatrix}$$

$$G_4(5) = \begin{pmatrix} r & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & r & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & r & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & l^{-1} & ml^{-1} & \frac{ml^{-1}}{r} & \frac{ml^{-1}}{r^2} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -m & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -m & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -m \end{pmatrix}$$

All these matrices are invertible and have the same determinant $\frac{-r^3}{l}$. As in (17) and (35), we define for all $i \in \{1,\, 2,\, 3,\, 4\}$:

$$E_i(5) = \frac{l}{m}(G_i(5)^2 + mG_i(5) - I_{10})$$

where I_{10} is the identity matrix of size 10. We verify with Maple that the braid relations (1) and (2) are satisfied on the G_i's and that the relations (4), (5) and (6) combining the e_i's and the g_i's are also satisfied on the matrices E_i's and G_i's. We define a morphism of groups:

$$\begin{array}{rcl} B(A_4)^{\times} & \longrightarrow & GL_{10}(F) \\ g_i & \longmapsto & G_i(5) \end{array}$$

where $B(A_4)^{\times}$ is the group of units of $B(A_4)$ that we extend to a morphism of algebras defined on the generators g_i's and e_i's of $B(A_4)$:

$$\begin{array}{rcl} B(A_4) & \longrightarrow & \mathcal{M}(10, F) \\ g_i & \longmapsto & G_i(5) \\ e_i & \longmapsto & E_i(5) \end{array}$$

Thus, we have a representation Υ of $B(A_4)$ in F^{10}:

$$B(A_4) \xrightarrow{\Upsilon} End_F(F^{10})$$

defined by $\Upsilon(g_i)(X) = G_i(5)X$ for any $X \in F^{10}$ and any $i \in \{1,\, \dots,\, 4\}$. Let us denote by $\mathcal{E} = (\epsilon_1,\, \epsilon_2 \dots, \epsilon_{10})$ the canonical basis of F^{10}. These vectors have one on their ith coordinate and zeros elsewhere. $G_i(5)$ is the matrix of $\Upsilon(g_i)$ in the basis $\mathcal{E}$. Suppose that Υ is not irreducible. Then there exists a nonzero proper F-vector subspace H of F^{10} such that $\Upsilon(w)(H) \subseteq H$ for any word w in $B(A_4)$. In particular, we must have $\Upsilon(X_{ij})(H) \subseteq H$ for all $1 \leq i < j \leq 5$. Computations in Maple show that the matrices representing the X_{ij}'s each have exactly one nonzero row: the one corresponding to the positive root which has a top horizontal line joining the nodes i and j. For instance, the tangle associated to the positive root $\alpha_2 + \alpha_3$ has its top horizontal line joining the nodes 2 and 4 and is the fifth element of the basis $\mathcal{B}$. And the fifth row is the only nonzero row in the matrix of the endomorphism $\Upsilon(X_{24})$ in the basis $\mathcal{E}$. Let's give an explicit expression for the matrix S representing the sum of the X_{ij}'s. We indicated in front of each row the corresponding X_{ij}:

$$\begin{array}{c} X_{12} \\ X_{23} \\ X_{13} \\ X_{34} \\ X_{24} \\ X_{14} \\ X_{45} \\ X_{35} \\ X_{25} \\ X_{15} \end{array}
\begin{pmatrix}
x & 1 & l & 0 & r & lr & 0 & 0 & r^2 & lr^2 \\
1 & x & \frac{1}{l} & 1 & l & 0 & 0 & r & lr & 0 \\
\frac{1}{l} & l & x & r & (r-\frac{1}{r})(l-r) & l & 0 & r^2 & (r^2-1)(l-r) & lr \\
0 & 1 & \frac{1}{r} & x & \frac{1}{l} & \frac{1}{lr} & 1 & l & 0 & 0 \\
\frac{1}{r} & \frac{1}{l} & (\frac{1}{r}-r)(\frac{1}{l}-\frac{1}{r}) & l & x & \frac{1}{l} & r & (r-\frac{1}{r})(l-r) & l & 0 \\
\frac{1}{lr} & 0 & \frac{1}{l} & lr & l & x & r^2 & (r^2-1)(l-r) & (r-\frac{1}{r})(l-r) & l \\
0 & 0 & 0 & 1 & \frac{1}{r} & \frac{1}{r^2} & x & \frac{1}{l} & \frac{1}{lr} & \frac{1}{lr^2} \\
0 & \frac{1}{r} & \frac{1}{r^2} & \frac{1}{l} & (\frac{1}{r}-r)(\frac{1}{l}-\frac{1}{r}) & (\frac{1}{r^2}-1)(\frac{1}{l}-\frac{1}{r}) & l & x & \frac{1}{l} & \frac{1}{lr} \\
\frac{1}{r^2} & \frac{1}{lr} & (\frac{1}{r^2}-1)(\frac{1}{l}-\frac{1}{r}) & 0 & \frac{1}{l} & (\frac{1}{r}-r)(\frac{1}{l}-\frac{1}{r}) & lr & l & x & \frac{1}{l} \\
\frac{1}{lr^2} & 0 & \frac{1}{lr} & 0 & 0 & \frac{1}{l} & lr^2 & lr & l & x
\end{pmatrix} = S$$

Suppose now that $\Upsilon(X_{24})(H)$ is nonzero. Let h be a vector in H such that $\Upsilon(X_{24})(h) \neq 0$. $\Upsilon(X_{24})(h)$ is a vector in H whose only nonzero coordinate is the fifth one. It follows that ϵ_5 belongs to H. Next, we read on the fifth column of the matrix S:

$$\begin{array}{rcccl}\Upsilon(X_{12})(\epsilon_5) & = & r\,\epsilon_1 & \Rightarrow & \epsilon_1 \in H \\ \Upsilon(X_{34})(\epsilon_5) & = & \frac{1}{l}\,\epsilon_4 & \Rightarrow & \epsilon_4 \in H \\ \Upsilon(X_{45})(\epsilon_5) & = & \frac{1}{r}\,\epsilon_7 & \Rightarrow & \epsilon_7 \in H \\ \Upsilon(X_{25})(\epsilon_5) & = & \frac{1}{l}\,\epsilon_9 & \Rightarrow & \epsilon_9 \in H\end{array}$$

Then we read on the first column of the matrix S:

$$\begin{array}{rcccl}\Upsilon(X_{13})(\epsilon_1) & = & \frac{1}{l}\,\epsilon_3 & \Rightarrow & \epsilon_3 \in H \\ \Upsilon(X_{14})(\epsilon_1) & = & \frac{1}{lr}\,\epsilon_6 & \Rightarrow & \epsilon_6 \in H\end{array}$$

Finally, we read on the third column of S:

$$\begin{array}{rcccl}\Upsilon(X_{35})(\epsilon_3) & = & \frac{1}{r^2}\,\epsilon_8 & \Rightarrow & \epsilon_8 \in H \\ \Upsilon(X_{15})(\epsilon_3) & = & \frac{1}{lr}\,\epsilon_{10} & \Rightarrow & \epsilon_{10} \in H \\ \Upsilon(X_{23})(\epsilon_3) & = & \frac{1}{l}\,\epsilon_2 & \Rightarrow & \epsilon_2 \in H\end{array}$$

We conclude that the whole basis $\mathcal{E}$ is contained in H. Thus, $H = F^{10}$, which contradicts H proper. So $\Upsilon(X_{24})(H) = 0$. Similarly, the positive root having nodes i and j joined on the top line in the tangles is $\alpha_{j-1} + \cdots + \alpha_i$. The associated tangle is the $\{[1+2+\cdots+(j-2)]+(j-i)\}$-th vector of the basis $\mathcal{B}$ of W. Hence we have:

$$\Big(\Upsilon(X_{ij})(h) \neq 0, \text{ some } h \in H\Big) \Longrightarrow \Big(\epsilon_{\frac{(j-1)(j-2)}{2}+j-i} \in H\Big)$$

Next, we observe that in each column of S, there are at least six nonzero and non diagonal coefficients. In particular, there are at least six nonzero and non diagonal coefficients in the column k of S, where

$$k := \frac{(j-1)(j-2)}{2} + j - i$$

From there, reasoning like above, we deduce that there are six other elements of the basis $\mathcal{B}$, say $\epsilon_{i_2}, \ldots \epsilon_{i_7}$, that are in H. We need to verify that the three remaining ones, say ϵ_{i_8}, ϵ_{i_9} and $\epsilon_{i_{10}}$, also belong to H. We notice again that in each row of S, there are at least six nonzero and non diagonal coefficients. Let $s_{i_8 j_1}, \ldots, s_{i_8 j_6}$ be six non zero and non diagonal coefficients of the i_8-th row of S. Necessarily, there exists $s \in \{2, \ldots, 7\}$ and $t \in \{1, \ldots, 6\}$ such that $i_s = j_t$. It comes:

$$\Upsilon(X_{kl})(\epsilon_{i_s}) = s_{i_8 j_t} \epsilon_{i_8}$$

where X_{kl} is in front of the i_8-th row. It follows that ϵ_{i_8} belongs to H and the same method applies to show that ϵ_{i_9} and $\epsilon_{i_{10}}$ are in H. We conclude that $\Upsilon(X_{ij})(H) = 0$, since otherwise H wouldn't be proper. This equality holds for all the X_{ij}'s, hence considering their sum $\mathcal{S}$, we get $\Upsilon(\mathcal{S})(H) = 0$. In other words, we have $\forall z \in H,\ Sz = 0$. We conclude that if Υ is not irreducible, we must have $det(S) = 0$. Using Maple yields the equivalence $det(S) = 0 \iff l \in \{r, -r^3, -\frac{1}{r^2}, \frac{1}{r^2}, \frac{1}{r^7}\}$. Let's show conversely that for any of these values of l, the representation Υ is not irreducible. First, we show that

$$\Upsilon(g_k)\big(\bigcap_{1 \le i < j \le 5} Ker\, \Upsilon(X_{ij})\big) \subset \bigcap_{1 \le i < j \le 5} Ker\, \Upsilon(X_{ij})$$

for all k in $\{1, \ldots, 4\}$. Let h be a vector of F^{10} such that $\Upsilon(X_{ij})(h) = 0$ for all $1 \le i < j \le 5$. Given i and j such that $1 \le i < j \le 5$, using the conjugation formulas for X_{ij} and the algebra relations like in the proof of lemma 2, but this time on the matrices, we get $Mat_{\mathcal{E}}(\Upsilon(X_{ij}))G_k(5)h = 0$ for all $k \in \{1, \ldots, 4\}$. So we have $(\Upsilon(X_{ij}) \circ \Upsilon(g_k))(h) = 0$ *i.e.* $\Upsilon(g_k)(h) \in Ker\,\Upsilon(X_{ij})$ for all $k \in \{1, \ldots, 4\}$. Hence the inclusion above is satisfied and the F-vector subspace $\cap_{1\leq i<j\leq 5} Ker\,\Upsilon(X_{ij})$ of F^{10} is a $B(A_4)$-module. The next step is to show that for each of the values of l above, this space is nonzero. A vector v of F^{10} is in $\cap_{1\leq i<j\leq 5} Ker\,\Upsilon(X_{ij})$ if and only if it is in the kernel of the matrix S. We check that:

$$\text{If } l = r, \quad \text{then } (r^2, 0, -r, 1, -r, 0, 0, 0, 0, 0) \in Ker\,S \tag{64}$$

$$\text{If } l = -r^3, \text{ then } (0, -r, 0, -\frac{1}{r}, 1, 0, 0, 0, 0, 0) \in Ker\,S \tag{65}$$

$$\text{If } l = -\frac{1}{r^2}, \text{ then } (-r^2, r^2 + \frac{1}{r}, r, -\frac{1}{r}, 1, 0, 0, -1, r, 0) \in Ker\,S \tag{66}$$

$$\text{If } l = \frac{1}{r^2}, \quad \text{then } (r^2, -r^2 + \frac{1}{r}, -r, -\frac{1}{r}, 1, 0, 0, -1, r, 0) \in Ker\,S \tag{67}$$

$$\text{If } l = \frac{1}{r^7}, \quad \text{then } (\frac{1}{r^3}, \frac{1}{r}, \frac{1}{r^2}, r, 1, \frac{1}{r}, r^3, r^2, r, 1) \in Ker\,S \tag{68}$$

We conclude that if l is one of these values, then Υ is not irreducible. Hence we have the equivalence: Υ is irreducible *iff* $l \notin \{r, -r^3, -\frac{1}{r^2}, \frac{1}{r^2}, \frac{1}{r^7}\}$

Chapter 6

Generalization

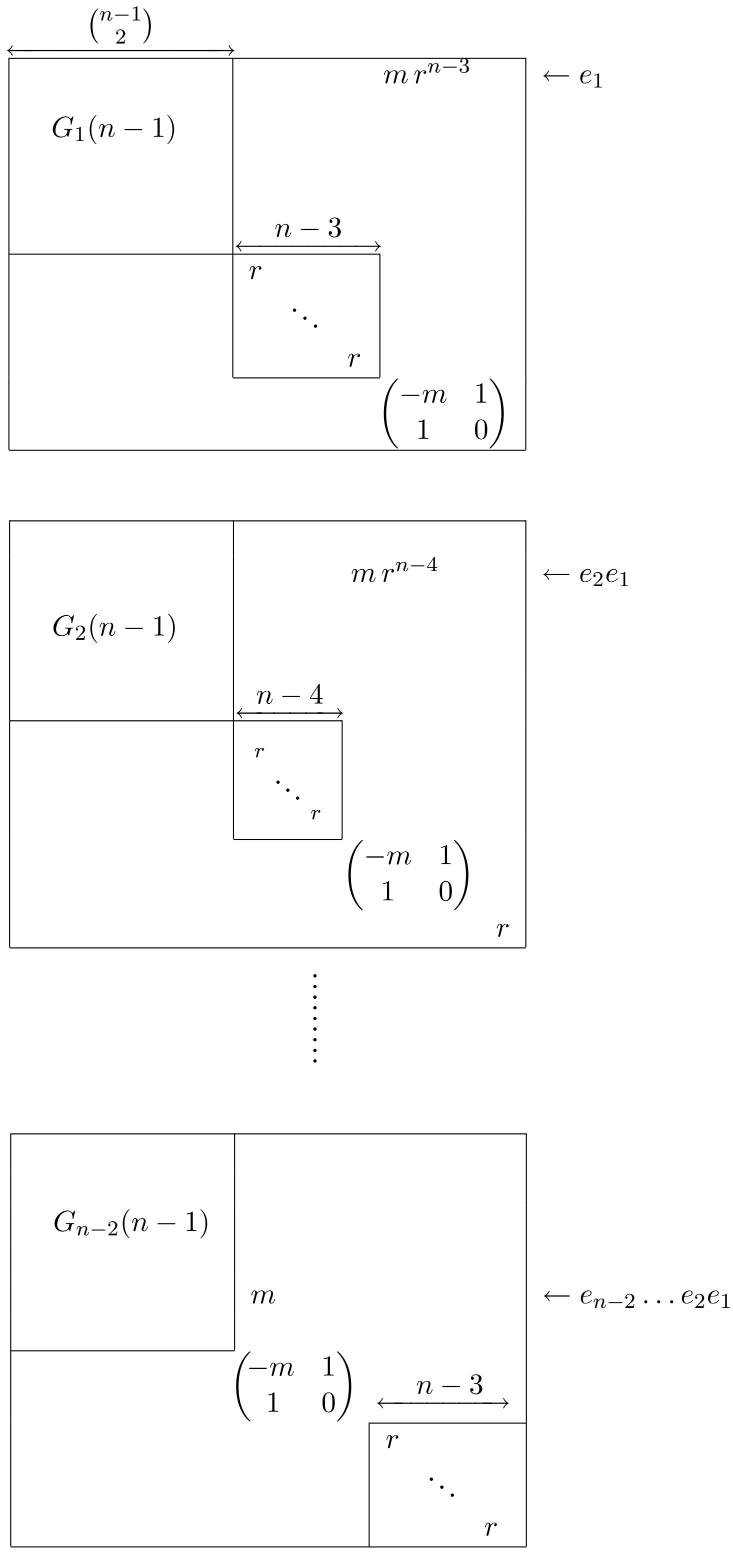
In this section, we extend the previous constructions to $B(A_{n-1})$, for any integer n. In A_{n-1}, there are $\binom{n}{2}$ positive roots. As already described in section 4, we order the roots in the following way: incrementing i, we start with α_i and list all the positive roots that have nodes less or equal to i in their support (there are i of them), in an increasing height order. This ordering allows us to "build A_1 up to A_{n-1}". Geometrically with the tangles, we join a node to its left neighbors starting with its adjacent neighbor and moving to the left; we shift the right extremity of the top horizontal line and start again. After step i, all the possible pairs of nodes for the top horizontal line appear exactly once on the first $i+1$ nodes. Hence we have listed $\binom{i+1}{2}$ tangles. The first $\binom{n-1}{2}$ positive roots correspond to the tangles spanning W in $B(A_{n-2})$ where a vertical string has been added to the right side. Hence, proceeding inductively, we already know the action of $g_1, \ldots, g_{n-2}$ on these tangles, the last vertical string on the right being left invariant with these actions. To obtain the matrices $G_1, G_2, \ldots, G_{n-2}$ inductively, it remains to compute the actions of these g_i's on the $n-1$ tangles

$$e_{n-1}e_{n-2}\cdots e_1,\ \ g_{n-1}e_{n-2}\cdots e_1,\ \ g_{n-1}g_{n-2}e_{n-3}\cdots e_1$$
$$,\ldots,\ \ g_{n-1}g_{n-2}\cdots g_2e_1$$

The action by g_1 on the first $n-3$ tangles (in the roots $\alpha_{n-1},\ \alpha_{n-1}+\alpha_{n-2},\ldots,$ $\alpha_{n-1}+\alpha_{n-2}+\cdots+\alpha_3$) is simply a crossing, hence a multiplication by r. Recall that the top horizontal line of the tangle associated with the positive root $\alpha_{n-1}+\cdots+\alpha_i$ joins the nodes i and n. As long as the top horizontal line does not begin with either node i or node $i+1$, as in $\alpha_{n-1},\ \ldots,\ \alpha_{n-1}+\cdots+\alpha_{i+2}$ (first $n-i-2$ tangles) or as in $\alpha_{n-1}+\cdots+\alpha_{i-1},\ \ldots,\ \alpha_{n-1}+\cdots+\alpha_1$ (last $i-1$ tangles), the action by g_i is a crossing and the coefficient in the matrix is an r on the diagonal. Suppose now that we are looking for the action of a g_i on the tangle whose top horizontal line starts at node i (and ends at node n). The left extremity of the top line of the tangle resulting from this action is now shifted to the right and starts at node $i+1$. In other words, the root $\alpha_{n-1}+\cdots+\alpha_i$ is sent to the root $\alpha_{n+1}+\cdots+\alpha_{i+1}$, which corresponds to a 1 above the diagonal in the matrix of G_i. It remains to look at the action of g_i on the tangle $(i+1,n)$. The node $i+1$ is sent onto the node i with an undercrossing. Using the tangle formula (31), we get:

$$g_i.(i+1,n)=(i,n)+m\,e_i.(i+1,n)-m\,(i+1,n) \tag{69}$$

where the permutation (k,l) denotes the tangle in W whose top line starts with node k and ends with node l. Acting by e_i transforms the top horizontal line of the tangle into a vertical line that still over crosses the same number of vertical strings, that is a total of $n-i-2$ vertical strings. Its top horizontal line now joins the nodes i and $i+1$ as in $e_ie_{i-1}\ldots e_1$. We read on the equation (69) that there are three coefficients in the column of the matrix. The first one is a 1 just below the diagonal, the second one is mr^{n-i-2} on the $\{1+2+\cdots+(i-1)+1\}$th row of the matrix, where the factor on the right corresponds to the $n-i-2$ over crossings. The third coefficient is a $-m$ on the diagonal. We get the following matrices for G_1, $G_2, \ldots, G_{n-2}$ (that we will call "matrices of the first kind"):

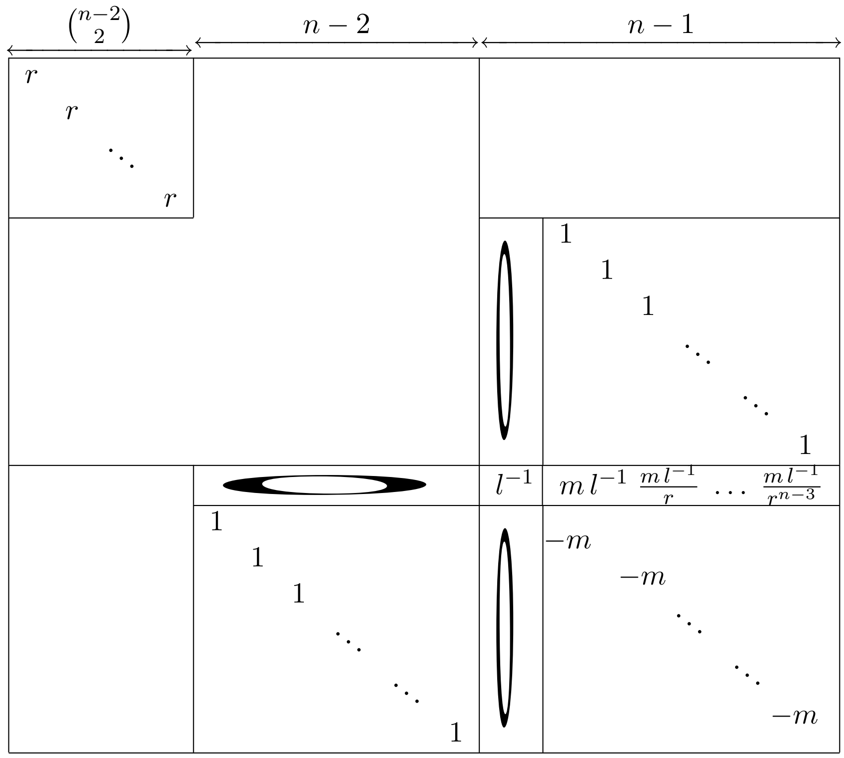


$\binom{n-1}{2}$
$G_1(n-1)$
$m\,r^{n-3}$
$\leftarrow e_1$
$n-3$
r
r
$\begin{pmatrix} -m & 1 \\ 1 & 0 \end{pmatrix}$
$G_2(n-1)$
$m\,r^{n-4}$
$\leftarrow e_2 e_1$
$n-4$
r
r
$\begin{pmatrix} -m & 1 \\ 1 & 0 \end{pmatrix}$
r
$G_{n-2}(n-1)$
m
$\leftarrow e_{n-2} \dots e_2 e_1$
$\begin{pmatrix} -m & 1 \\ 1 & 0 \end{pmatrix}$
$n-3$
r
r

We now compute the action of g_{n-1} on the spanning tangles of W. When $n-2$ and $n-1$ are not in the support of the corresponding positive roots, the tangle's top line does not end with node $n-1$ or node n. Thus the result of the action by g_{n-1} is a crossing of the last two vertical strings. Consequently, the upper left square block of size $\binom{n-2}{2}$ of the matrix G_{n-1} is a scalar matrix with r's on the diagonal. Now if $n-2$ is in the support of the positive root, the top line of the associated tangle ends with node $n-1$ and a left action by g_{n-1} shifts this end to node n. In other words, the positive root $\alpha_{n-2}+\cdots+\alpha_i$ is sent to the positive root $\alpha_{n-1}+\alpha_{n-2}+\cdots+\alpha_i$. Next, if $n-1$ is in the support of the positive root, the top line of the associated tangle ends with node n. We start with the easy case of the action of g_{n-1} on the positive root α_{n-1} associated with the tangle $e_{n-1}\dots e_1$. The result of this action is simply a multiplication by $1/l$ to remove the loop, hence a $1/l$ on the diagonal. If the top horizontal line crosses now 1 to $n-2$ vertical strings, as in $\alpha_{n-1}+\alpha_{n-2},\dots,\alpha_{n-1}+\cdots+\alpha_1$, the action by g_{n-1} shifts the right extremity of the top horizontal line to the left. The top line ends now in node $n-1$ instead of node n. But to be able to use the regular isotopy and more specifically Reidemeister's move II (see for instance [12]) that allows to separate two strings that intersect in two overcrossings or two undercrossings, we first need to transform the overcrossing of g_{n-1} into an undercrossing, using the tangle formula (31). It yields:

$$g_{n-1}.(i,n) = (i,n-1) + \; m\, e_{n-1}.(i,n) - m(i,n)$$

The action by e_{n-1} transforms the top horizontal line into a vertical string that overcrosses one less vertical lines as the last vertical line is now part of a loop. In the general picture, the $n-i-2$ crossings are under and to build a representation as before, we divide the corresponding coefficient by the power r^{n-i-2}. The top line now joins the two last nodes as in $e_{n-1}\dots e_1$. Thus, for $2 \le j \le n-1$, the $(1+2+\cdots+(n-2)+1, 1+2+\cdots+(n-2)+j)$ coefficient of the matrix G_{n-1} is $\frac{m}{lr^{j-2}}$. The matrix G_{n-1} has the following shape, where the blanks must be filled with zeros. We will call it a "matrix of the second kind":



We defined matrices $G_1, \dots, G_{n-1}$ that correspond to the actions of $g_1, \dots, g_{n-1}$ on the spanning elements of W. We now define matrices $E_1, \dots, E_{n-1}$ by setting

$$\forall 1 \le i \le n-1,\ E_i := \frac{l}{m}\Big(G_i^2 + m\,G_i - I_{\binom{n}{2}}\Big)$$

Our conjecture is that:

Conjecture 1.

<u>*Case n=3*</u>

$$\begin{array}{ccc} B(A_2) & \longrightarrow & \mathcal{M}(3,F) \\ g_1,\, g_2 & \longmapsto & G_1,\, G_2 \\ e_1,\, e_2 & \longmapsto & E_1,\, E_2 \end{array} \quad \textit{is a representation.}$$

It is irreducible iff $l \notin \{-r^3, -1, 1, \frac{1}{r^3}\}$.

<u>*General case*</u>

$$\begin{array}{ccc} B(A_{n-1}) & \longrightarrow & \mathcal{M}(\binom{n}{2}, F) \\ g_i & \longmapsto & G_i \\ e_i & \longmapsto & E_i \end{array} \quad \textit{is a representation.}$$

It is irreducible iff $l \notin \{r, -r^3, -\frac{1}{r^{n-3}}, \frac{1}{r^{n-3}}, \frac{1}{r^{2n-3}}\}$.

The conjecture is verified in the cases $n = 3, 4, 5$ (cf respective sections 3, 4, 5). Computations with Maple were done in the case $n = 6$. We wrote a program in Maple that defines the matrices $G_1, G_2, \ldots, G_{n-1}$ for a given n, computes the X_{ij}'s, forms their sum matrix S, then solves in the variable l the equation $det(S) = 0$. We ran the program for $n = 6$, obtaining a similar results (but without any proof) as in the cases $n = 4$ and $n = 5$ and leading us to the conjecture 1 for a general n. We will prove the conjecture in the general case, but first we need to give a more visible expression for the announced representation. This is the object of the next section.

Chapter 7

The Representation Itself

Let $\mathcal{V}$ denote the vector space over F with spanning vectors x_β indexed by $\beta \in \phi^+$, where ϕ^+ is the set of positive roots. In this section we give a formal definition of a representation

$$\begin{array}{ccc} B(A_{n-1}) & \longrightarrow & End_F(\mathcal{V}) \\ g_i & \longmapsto & \nu_i \end{array}$$

In the case $n = 4$, an element x_β can be viewed as an appropriate linear combination over F of the tangles associated with β (cf section 4). In what follows, $Supp(\beta)$ denotes the support of the positive root β *i.e.* the set of $k \in \{1 \dots n\}$ such that the coefficient of α_k in β is nonzero ; $ht(\beta)$ denotes the height of the positive root β: if $\beta = \sum_i n_i \alpha_i$, then $ht(\beta) := \sum_i n_i$. We read on the matrices of the previous section an expression for the representation in two special cases: $i = n-1$ on one hand (point 1. below) and $i \in \{1, ..., n-2\}$ and $n-1 \in Supp(\beta)$ on the other hand (point 2. below). We then deduce inductively a formula for the representation in the case $i \in \{1, \dots, n-2\}$ (point 3. below). We have:

1. If $i = n-1$, then we read on the matrix G_{n-1} of previous section:

$$\nu_i(x_\beta) = \begin{array}{r|lrcl} \text{(I)} & r \quad x_\beta & \text{if } (\beta|\alpha_i) & = & 0 \\ \text{(II)} & l^{-1} x_\beta & \text{if } (\beta|\alpha_i) & = & 1 \\ \text{(III)} & x_{\beta-\alpha_i} + \frac{m}{l\, r^{ht(\beta)-2}}\, x_{\alpha_i} - m x_\beta & \text{if } (\beta|\alpha_i) & = & \frac{1}{2} \\ \text{(IV)} & x_{\beta+\alpha_i} & \text{if } (\beta|\alpha_i) & = & -\frac{1}{2} \end{array}$$

2. If $i \in \{1, \ldots, n-2\}$ and if $n-1 \in Supp(\beta)$, then we read on the last $n-1$ columns of the matrices $G_1, \ldots, G_{n-2}$ of previous section:

$$\nu_i(x_\beta) = \begin{array}{r|lrcl} \text{(I)} & r\, x_\beta & \text{if } (\beta|\alpha_i) & = & 0 \\ \text{(III}'\text{)} & x_{\beta-\alpha_i} & \text{if } (\beta|\alpha_i) & = & \frac{1}{2} \\ \text{(IV}'\text{)} & x_{\beta+\alpha_i} + m\, r^{n-i-2} x_{\alpha_i} - m\, x_\beta & \text{if } (\beta|\alpha_i) & = & -\frac{1}{2} \end{array}$$

3. $i \in \{1, 2, \ldots, n-2\}$ and no restriction on β: we use induction with the expressions for the representation in points 1. and 2. above. More specifically, we will always use the induction in the following way: suppose that we want to evaluate $\nu_i(n)(x_{\alpha_l+\cdots+\alpha_k})$ where $l \geq k$ and $l \leq n-2$. By induction this value has already been computed in lower dimension (in $B(A_{n-2})$) and proceeding successively by induction, we may in fact decrement the integer n till we reach the integer $M := Max(i+1, l+1)$. On the matrices the picture is to keep moving backward in the upper left corner till you can either use the last columns of a matrix of the first kind (in the case $M = l+1$) or the matrix of the second kind (in the case $M = i+1$) in the suitable $B(A_{M-1})$. We note that if $i+1 < l+1$ then by definition $M = l+1$ and we have $i \leq l-1 = (l+1)-2$, so that the inductive steps are all justified and

the expression for $\nu_i(l+1)(x_{\alpha_l+\cdots+\alpha_k})$ may indeed be read on the last l columns of the matrix of the first kind $G_i(l+1)$. If on the contrary $i+1 \geq l+1$, then by definition $M = i+1$. The inductive steps are again justified by the fact that $i \leq (i+2)-2$ and $l \leq i$. The expression for $\nu_i(i+1)(x_{\alpha_l+\cdots+\alpha_k})$ is now obtained by using the matrix of the second kind $G_i(i+1)$ in $B(A_i)$. We will now apply those preliminary remarks, while dealing with the different values for the inner product $(\beta|\alpha_i)$.

- If $(\beta|\alpha_i) = -\frac{1}{2}$, there are two cases:
  - $\star$ $\beta = \alpha_{i-1} + \cdots + \alpha_k$ with $k \leq i-1$. We have:

$$\begin{aligned} \nu_i(n)(x_{\alpha_{i-1}+\cdots+\alpha_k}) = \ldots = & \ \nu_i(i+1)(x_{\alpha_{i-1}+\cdots+\alpha_k}) \\ = & \ x_{\beta+\alpha_i} \quad \text{by using} \quad \text{(IV)} \end{aligned}$$

  - $\star$ $\beta = \alpha_l + \cdots + \alpha_{i+1}$ with $l \geq i+1$. Then we have:

$$\begin{aligned} \nu_i(n)(x_{\alpha_l+\cdots+\alpha_{i+1}}) = \ldots = & \ \nu_i(l+1)(x_{\alpha_l+\cdots+\alpha_{i+1}}) \\ = & \ x_{\beta+\alpha_i} + m\, r^{l-i-1} x_{\alpha_i} - m\, x_\beta \end{aligned}$$

    The last equality is obtained by using (IV$'$) of point 2.

    For the last equality to hold, we must have $i \leq (l+1)-2$, *i.e.* $i \leq l-1$ which is true by our assumption in $\star$.

- If $(\beta|\alpha_i) = \frac{1}{2}$, there are again two cases:
  - $\star$ $\beta = \alpha_i + \cdots + \alpha_k$ with $k \leq i-1$. Then we have:

$$\begin{aligned} \nu_i(n)(x_{\alpha_i+\cdots+\alpha_k}) &= \nu_i(i+1)(x_{\alpha_i+\cdots+\alpha_k}) \\ &= x_{\beta-\alpha_i} + \frac{m}{l\, r^{ht(\beta)-2}}\, x_{\alpha_i} - m\, x_\beta \end{aligned}$$

The last equality is obtained by using the expression (III) of point 1.

⋆ $\beta = \alpha_l + \cdots + \alpha_i$ with $l \geq i+1$. Then we have:

$$\begin{aligned} \nu_i(n)(x_{\alpha_l+\cdots+\alpha_i}) &= \nu_i(l+1)(x_{\alpha_l+\cdots+\alpha_i}) \\ &= x_{\beta-\alpha_i} \quad \text{by} \quad (\text{III}') \end{aligned}$$

- If $(\beta|\alpha_i) = 0$, then there are again two possibilities:

⋆ $i-1,\, i,\, i+1 \notin Supp(\beta)$

Then $\nu_i(x_\beta) = r\, x_\beta$

⋆ $\{i-1,\, i,\, i+1\} \subseteq Supp(\beta)$

Then we can write $\beta = \alpha_l + \cdots + \alpha_{i+1} + \alpha_i + \alpha_{i-1} + \cdots + \alpha_k$, some $l \geq i+1$ and $k \leq i-1$. It comes:

$$\begin{aligned} &\nu_i(n)(x_{\alpha_l+\cdots+\alpha_{i+1}+\alpha_i+\alpha_{i-1}+\cdots+\alpha_k}) \\ &\qquad = \cdots = \nu_i(l+1)(x_{\alpha_l+\cdots+\alpha_{i-1}+\alpha_i+\alpha_{i+1}+\cdots+\alpha_k}) \end{aligned}$$

We have by assumption $i \leq l-1 = (l+1)-2$, so that we may apply point 2. Further, β has height at least $l-i+2$ and there are $(l+1)-i-2 = l-i-1$ r's on the diagonal of the first $l-i-1$ columns of the last l columns of the matrix $G_i(l+1)$. By previous section, skip two more columns and get again a pattern of r's on the diagonal, which now yields:

$$\nu_i(n)(x_{\alpha_l+\cdots+\alpha_{i+1}+\alpha_i+\alpha_{i-1}+\cdots+\alpha_k}) = r\, x_\beta$$

Thus, we see that in both cases we have $\nu_i(x_\beta) = r\, x_\beta$.

- If $(\beta|\alpha_i) = 1$, the only possibility is $\beta = \alpha_i$ and we have:

$$\begin{aligned} \nu_i(n)(x_{\alpha_i}) &= \nu_i(i+1)(x_{\alpha_i}) \\ &= \frac{1}{l}\, x_{\alpha_i} \quad \text{as in (II)} \end{aligned}$$

  We deduce from that last point and from the equation (II) of point 1. that if $(\beta|\alpha_i) = 1$, then $\nu_i(x_\beta) = \frac{1}{l}\, x_\beta$ for all i.

We now gather all these results to obtain the final definition of the endomorphisms ν_i's. In what follows, let $\prec$ be the total order on the positive roots, as used and described several times before:

$$\begin{aligned} &\alpha_1 \prec \alpha_2 \prec \alpha_2 + \alpha_1 \prec \alpha_3 \prec \alpha_3 + \alpha_2 \prec \alpha_3 + \alpha_2 + \alpha_1 \\ &\qquad \prec \ldots \prec \alpha_{n-1} \prec \alpha_{n-1} + \alpha_{n-2} \prec \ldots \prec \alpha_{n-1} + \alpha_{n-2} + \cdots + \alpha_1 \end{aligned}$$

It will also be convenient to read $\beta \succ \gamma$ for some positive roots β and γ. By $\beta \succ \gamma$ we will understand $\gamma \prec \beta$. We see with points 1., 2. and 3. that we always have $\nu_i(x_\beta) = r\, x_\beta$ when $(\beta|\alpha_i) = 0$. Similarly, by points 1. and 3. we always have $\nu_i(x_\beta) = l^{-1}\, x_\beta$ when $(\beta|\alpha_i) = 1$. Suppose now that $(\beta|\alpha_i) = \frac{1}{2}$. Then $\beta - \alpha_i$ is a positive root and there are two cases: either $\beta - \alpha_i \prec \alpha_i$ or $\beta - \alpha_i \succ \alpha_i$. The first situation occurs in point 1. since $\beta - \alpha_{n-1}$ starts with α_{n-2} and is thus ranked before α_{n-1} and in point 3. when $\beta = \alpha_i + \cdots + \alpha_k$ with $k \leq i-1$. In the latter case, $\beta - \alpha_i = \alpha_{i-1} + \cdots + \alpha_k \prec \alpha_i$. In both cases we have $\nu_i(x_\beta) = x_{\beta-\alpha_i} + \frac{m}{l\, r^{ht(\beta)-2}}\, x_{\alpha_i} - m\, x_\beta$. The second situation occurs in point 2. since then $\beta = \alpha_{n-1} + \cdots + \alpha_{i+1} + \alpha_i$ and so $\beta - \alpha_i = \alpha_{n-1} + \cdots + \alpha_{i+1} \succ \alpha_i$. It also occurs in point 3. when $\beta = \alpha_l + \cdots + \alpha_i$ with $l \geq i+1$, since then $\beta - \alpha_i = \alpha_l + \cdots + \alpha_{i+1} \succ \alpha_i$. Again, in both cases the result is the same and we have $\nu_i(x_\beta) = x_{\beta-\alpha_i}$. Finally the last case $(\beta|\alpha_i) = -\frac{1}{2}$ splits in turn into two cases: $\beta \succ \alpha_i$ on one hand and $\beta \prec \alpha_i$ on the other hand. Indeed, in point 1. and in point 3. when $\beta = \alpha_{i-1} + \cdots + \alpha_k$ with $k \leq i-1$, we have $\beta \prec \alpha_i$ and $\nu_i(x_\beta) = x_{\beta+\alpha_i}$. Whereas in point 2. and in point 3. when $\beta = \alpha_l + \cdots + \alpha_{i+1}$ with $l \geq i+1$, we have $\beta \succ \alpha_i$ and $\nu_i(x_\beta) = x_{\beta+\alpha_i} + m\, r^{ht(\beta)-1}\, x_{\alpha_i} - m\, x_\beta$.

Note that when $(\beta|\alpha_i) = \frac{1}{2}$, a direct comparison between β and α_i is not the criterion as in that case i lies in the support of β, which always yields β greater than α_i. In fact, if $(\beta|\alpha_i) = \frac{1}{2}$, i is contained in the support of β and moreover β ends with α_i or begins with α_i. Comparing $\beta - \alpha_i$ with α_i tells the relative position of α_i in the sum. If $(\beta|\alpha_i) = -\frac{1}{2}$, then β "stops just before α_i" (in which case $\beta \succ \alpha_i$) or "starts right after α_i" (in which case $\beta \prec \alpha_i$).

We are now able to give a complete expression for the endomorphism ν_i:

$$\nu_i(x_\beta) = \begin{cases} r \quad x_\beta & \text{if } (\beta|\alpha_i) = \quad 0 & (a) \\ l^{-1}\, x_\beta & \text{if } (\beta|\alpha_i) = \quad 1 & (b) \\ x_{\beta-\alpha_i} & \text{if } (\beta|\alpha_i) = \quad \frac{1}{2} \;\&\; \beta - \alpha_i \succ \alpha_i & (c) \\ x_{\beta-\alpha_i} + \frac{m}{l\, r^{ht(\beta)-2}} \quad x_{\alpha_i} - m\, x_\beta & \text{if } (\beta|\alpha_i) = \quad \frac{1}{2} \;\&\; \beta - \alpha_i \prec \alpha_i & (d) \\ x_{\beta+\alpha_i} + m\, r^{ht(\beta)-1}\, x_{\alpha_i} - m\, x_\beta & \text{if } (\beta|\alpha_i) = -\frac{1}{2} \;\&\; \quad \beta \quad \succ \alpha_i & (e) \\ x_{\beta+\alpha_i} & \text{if } (\beta|\alpha_i) = -\frac{1}{2} \;\&\; \quad \beta \quad \prec \alpha_i & (f) \end{cases}$$

We now define $\nu(e_i) := \frac{l}{m}(\nu_i^2 + m\, \nu_i - id_{\mathcal{V}})$ and compute the explicit expression of the endomorphism $\nu(e_i)$. For $(\beta|\alpha_i) = 0$, we use the defining relation $r^2 + m\,r - 1 = 0$. When $(\beta|\alpha_i) = 1$, we get:

$$\begin{aligned}
\nu(e_i)(x_\beta) &= \frac{l}{m}\Big(\frac{1}{l^2} + \frac{m}{l} - 1\Big)\,x_\beta \\
&= \Big(\frac{1}{ml} + 1 - \frac{l}{m}\Big)\,x_\beta \\
&= \Big(1 - \frac{l - \frac{1}{l}}{m}\Big)\,x_\beta \\
&= \Big(1 - \frac{l - \frac{1}{l}}{m}\Big)\,x_{\alpha_i} \quad \text{as in that case } \beta = \alpha_i
\end{aligned}$$

In the other cases, we notice that:

$$\begin{array}{llll}
\text{if } (\beta|\alpha_i) = \frac{1}{2}\,, & \text{then } \beta - \alpha_i & \text{is a root and} & (\beta - \alpha_i|\alpha_i) = \frac{1}{2} - 1 = -\frac{1}{2} \\
\text{if } (\beta|\alpha_i) = -\frac{1}{2}\,, & \text{then } \beta + \alpha_i & \text{is a root and} & (\beta + \alpha_i|\alpha_i) = -\frac{1}{2} + 1 = \frac{1}{2}
\end{array}$$

The computation of the square ν_i^2 uses the third relation together with the fifth relation and the fourth relation together with the sixth relation in a very pretty way. The idea is that a condition of application of (c) (resp (d)) with β yields a condition of application of (e) (resp (f)) with $\beta - \alpha_i$ and a condition of application of (e) (resp (f)) with β yields a condition of application of (c) (resp (d)) with $\beta + \alpha_i$. And indeed,
if $(\beta|\alpha_i) = \frac{1}{2}$ and $\beta - \alpha_i \succ \alpha_i$, then $(\beta - \alpha_i|\alpha_i) = -\frac{1}{2}$ and $\beta - \alpha_i \succ \alpha_i$.
It follows that:

$$\begin{aligned}
\nu_i^2(x_\beta) &= \nu_i(x_{\beta-\alpha_i}) && \text{by application of } (c) \text{ with } \beta \\
&= x_{(\beta-\alpha_i)+\alpha_i} + m\,r^{ht(\beta-\alpha_i)-1}\,x_{\alpha_i} - m\,x_{\beta-\alpha_i} && \text{by application of } (e) \text{ with } \beta - \alpha_i \\
&= x_\beta + m\,r^{ht(\beta)-2}\,x_{\alpha_i} - m\,x_{\beta-\alpha_i}
\end{aligned}$$

Computing $\nu(e_i)(x_\beta)$, the first term on the right hand side of the last equality above cancels with $-Id_{\mathcal{V}}(x_\beta) = -x_\beta$ and the third term cancels with $m\,\nu_i(x_\beta) = m\,x_{\beta-\alpha_i}$. It remains the term proportional to α_i which multiplied by the coefficient $\frac{l}{m}$ yields $l\,r^{ht(\beta)-2}\,x_{\alpha_i}$.

If $(\beta|\alpha_i) = \frac{1}{2}$ and $\beta - \alpha_i \prec \alpha_i$, $(\beta - \alpha_i|\alpha_i) = -\frac{1}{2}$ and $\beta - \alpha_i \prec \alpha_i$, so that:

$$\begin{aligned} \nu_i^2(x_\beta) &= \nu_i(x_{\beta-\alpha_i} + \frac{m}{l\,r^{ht(\beta)-2}}\,x_{\alpha_i} - m\,x_\beta) && \text{by application of } (d) \text{ with } \beta \\ &= x_\beta + \frac{m}{l^2\,r^{ht(\beta)-2}}\,x_{\alpha_i} - m\,\nu_i(x_\beta) && \text{by application of } (f) \text{ with } \beta - \alpha_i \text{ and by } (b) \end{aligned}$$

When computing $\nu(e_i)$, the two terms on the extremities of the last equality are canceled and it remains the only term $\frac{1}{l\,r^{ht(\beta)-2}}\,x_{\alpha_i}$ after multiplication by the factor $\frac{l}{m}$.

If $(\beta|\alpha_i) = -\frac{1}{2}$ and $\beta \succ \alpha_i$, then $(\beta + \alpha_i|\alpha_i) = \frac{1}{2}$ and $(\beta + \alpha_i) - \alpha_i \succ \alpha_i$. Hence,

$$\begin{aligned} \nu_i^2(x_\beta) &= \nu_i(x_{\beta+\alpha_i} + m\,r^{ht(\beta)-1}\,x_{\alpha_i} - m\,x_\beta && \text{by application of } (e) \text{ with } \beta \\ &= x_\beta + \frac{m\,r^{ht(\beta)-1}}{l}\,x_{\alpha_i} - m\,\nu_i(x_\beta) && \text{by application of } (c) \text{ with } \beta + \alpha_i \text{ and by } (b) \end{aligned}$$

As in the previous cases, the only remaining term in $\nu(e_i)(x_\beta)$ is the multiple of x_{α_i} and this time, the coefficient is just $r^{ht(\beta)-1}$.

Finally if $(\beta|\alpha_i) = -\frac{1}{2}$ and $\beta \prec \alpha_i$, then $(\beta+\alpha_i|\alpha_i) = \frac{1}{2}$ and $(\beta+\alpha_i)-\alpha_i \prec \alpha_i$.

It follows that:

$$\begin{aligned}
\nu_i^2(x_\beta) &= \nu_i(x_{\beta+\alpha_i}) && \text{by application of } (f) \text{ with } \beta \\
&= x_{(\beta+\alpha_i)-\alpha_i} + \frac{m}{l\,r^{ht(\beta+\alpha_i)-2}}\,x_{\alpha_i} - m\,x_{\beta+\alpha_i} && \begin{array}{l}\text{by application of } (d) \\ \text{with } \beta+\alpha_i\end{array}
\end{aligned}$$

After canceling the two terms in x_β and in $x_{\beta+\alpha_i}$, the final result is $\frac{1}{r^{ht(\beta)-1}}\,x_{\alpha_i}$. Thus, we have:

$$\nu(e_i)(x_\beta) = \begin{cases}
0 & \text{if } (\beta|\alpha_i) = 0 \\
\left(1 - \frac{l-\frac{1}{l}}{m}\right) x_{\alpha_i} & \text{if } (\beta|\alpha_i) = 1 \\
l\,r^{ht(\beta)-2} \quad x_{\alpha_i} & \text{if } (\beta|\alpha_i) = \frac{1}{2} \ \& \ \beta-\alpha_i \succ \alpha_i \\
\frac{1}{l\,r^{ht(\beta)-2}} \quad x_{\alpha_i} & \text{if } (\beta|\alpha_i) = \frac{1}{2} \ \& \ \beta-\alpha_i \prec \alpha_i \\
r^{ht(\beta)-1} \quad x_{\alpha_i} & \text{if } (\beta|\alpha_i) = -\frac{1}{2} \ \& \ \beta \succ \alpha_i \\
\frac{1}{r^{ht(\beta)-1}} \quad x_{\alpha_i} & \text{if } (\beta|\alpha_i) = -\frac{1}{2} \ \& \ \beta \prec \alpha_i
\end{cases}$$

By construction of ν_i, G_i is the matrix of the endomorphism ν_i in the basis $\mathcal{B}_\mathcal{V} := (x_{\alpha_1}, x_{\alpha_2}, x_{\alpha_2+\alpha_1}, \ldots, x_{\alpha_{n-1}}, x_{\alpha_{n-1}+\alpha_{n-2}}, x_{\alpha_{n-1}+\alpha_{n-2}+\cdots+\alpha_1})$ and by definition, E_i is the matrix of the endomorphism $\nu(e_i)$ in the basis $\mathcal{B}_\mathcal{V}$.

Theorem 3.

$$\nu^{(n)} : \begin{array}{ccc} B(A_{n-1}) & \longrightarrow & End_F(\mathcal{V}) \\ g_i & \longmapsto & \nu_i \\ e_i & \longmapsto & \nu(e_i) \end{array}$$

is a representation of the algebra $B(A_{n-1})$ in the F-vector space $\mathcal{V}$.

If we can show the theorem, then by the remark above, the maps on the generators in the conjecture 1. also define a representation of $B(A_2)$ (resp $B(A_{n-1})$).

PROOF OF THE THEOREM:

First we must show that for any two nodes i and j, if $i \nsim j$, then $\nu_i\nu_j = \nu_j\nu_i$ and if $i \sim j$ then $\nu_i\nu_j\nu_i = \nu_j\nu_i\nu_j$. Suppose first that $i \nsim j$. Then we have $(\alpha_i|\alpha_j) = 0$. We want to show that:

$$\begin{aligned} \nu_i\nu_j(x_{\alpha_j}) &= \nu_j\nu_i(x_{\alpha_j}) \\ \nu_i\nu_j(x_\beta) &= \nu_j\nu_i(x_\beta) \quad \text{for } \beta \notin \{\alpha_i, \alpha_j\} \end{aligned}$$

It is a direct consequence of (a) and (b) that the two members of the first equality are equal to $\frac{r}{l}x_{\alpha_j}$. As for the second equality, we check it by computing the common value depending on the inner products $(\beta|\alpha_i)$ and $(\beta|\alpha_j)$. We summarize the results in the following table:

case	$(\beta\|\alpha_i)$	$(\beta\|\alpha_j)$	$\nu_i\nu_j(x_\beta) = \nu_j\nu_i(x_\beta)$	rule
1	0	0	$r^2\, x_\beta$	(a)
2	0	$\frac{1}{2}$ & $\beta - \alpha_j \succ \alpha_j$	$r\, x_{\beta-\alpha_j}$	(a) & (c)
3	0	$\frac{1}{2}$ & $\beta - \alpha_j \prec \alpha_j$	$r\, x_{\beta-\alpha_j} + \frac{m}{l\, r^{ht(\beta)-1}}\, x_{\alpha_j} - mr\, x_\beta$	(a) & (d)
4	0	$-\frac{1}{2}$ & $\beta \succ \alpha_j$	$r\, x_{\beta+\alpha_j} + m\, r^{ht(\beta)} x_{\alpha_j} - mr\, x_\beta$	(a) & (e)
5	0	$-\frac{1}{2}$ & $\beta \prec \alpha_j$	$r\, x_{\beta+\alpha_j}$	(a) & (f)
6	$\frac{1}{2}$ & $\beta - \alpha_i \prec \alpha_i$	$\frac{1}{2}$ & $\beta - \alpha_j \succ \alpha_j$	$x_{\beta-\alpha_j-\alpha_i} + \frac{m}{l\, r^{ht(\beta)-3}}\, x_{\alpha_i} - m\, x_{\beta-\alpha_j}$	(a) & (c) & (d)
7	$-\frac{1}{2}$ & $\beta \prec \alpha_i$	$-\frac{1}{2}$ & $\beta \succ \alpha_j$	$x_{\beta+\alpha_i+\alpha_j} + m\, r^{ht(\beta)}\, x_{\alpha_j} - m\, x_{\beta+\alpha_i}$	(a) & (e) & (f)
8	$\frac{1}{2}$ & $\beta - \alpha_i \succ \alpha_i$	$-\frac{1}{2}$ & $\beta \prec \alpha_j$	$x_{\beta+\alpha_j-\alpha_i}$	(c) & (f)
9	$\frac{1}{2}$ & $\beta - \alpha_i \prec \alpha_i$	$-\frac{1}{2}$ & $\beta \succ \alpha_j$	$x_{\beta+\alpha_j-\alpha_i} - m\, x_{\beta+\alpha_j} - m\, x_{\beta-\alpha_i}$ $+mr^{ht\beta} x_{\alpha_j} + \frac{m}{lr^{ht(\beta)-3}}\, x_{\alpha_i} + m^2 x_\beta$	(a) & (d) & (e)

The cases 1 to 9 are the only possible cases when excluding $(\beta|\alpha_i) = 1$ or $(\beta|\alpha_j) = 1$. In the table above, (a) is often used with $(\alpha_i|\alpha_j) = 0$. In the first five cases, $(\beta|\alpha_i) = 0$. Since we assumed that $(\alpha_i|\alpha_j) = 0$, it follows that $(\beta - \alpha_j|\alpha_i) = 0$ (used in cases 2 and 3 to compute $\nu_i\nu_j(x_\beta)$) and $(\beta + \alpha_j|\alpha_i) = 0$ (used in cases 4 and 5 to compute $\nu_i\nu_j(x_\beta)$). Next, if both inner products are equal to $\frac{1}{2}$, then without loss of generality, β "starts" with

α_i and "ends with" α_j, *i.e.* $\beta = \alpha_i + \cdots + \alpha_j$ with $i > j$. Then $\beta - \alpha_i \prec \alpha_i$ and $\beta - \alpha_j \succ \alpha_j$. Moreover, $(\beta - \alpha_i|\alpha_j) = (\beta|\alpha_j) = \frac{1}{2}$ and we read on the expression giving β that $(\beta - \alpha_i) - \alpha_j \succ \alpha_j$. Then we may use (c) with $\beta - \alpha_i$ and α_j for computing $\nu_j\nu_i(x_\beta)$. Similarly, $(\beta - \alpha_j|\alpha_i) = \frac{1}{2}$ and $(\beta - \alpha_j) - \alpha_i \prec \alpha_i$, so we may use (d) with $\beta - \alpha_j$ and α_i for computing $\nu_i\nu_j(x_\beta)$. If both inner products are equal to $-\frac{1}{2}$, then without loss of generality, we may write $\beta = \alpha_{i-1} + \cdots + \alpha_{j+1}$. We have $\alpha_j \prec \beta \prec \alpha_i$ and also $\beta + \alpha_j \prec \alpha_i$ and $\beta + \alpha_i \succ \alpha_j$. This allows us to use (f) with $\beta + \alpha_j$ and α_i when computing $\nu_i\nu_j(x_\beta)$ on one hand and (e) with $\beta + \alpha_i$ and α_j when computing $\nu_j\nu_i(x_\beta)$ on the other hand. The same methods apply for the cases 8 and 9, with $\beta = \alpha_{j-1} + \cdots + \alpha_i$ in 8 and $\beta = \alpha_i + \cdots + \alpha_{j+1}$ in 9.

Suppose now that $i \sim j$. Then we have $(\alpha_i|\alpha_j) = -\frac{1}{2}$. Again by symmetry of the roles played by i and j, we need to show that:

$$\begin{aligned} \nu_i\nu_j\nu_i(x_{\alpha_j}) &= \nu_j\nu_i\nu_j(x_{\alpha_j}) \\ \nu_i\nu_j\nu_i(x_\beta) &= \nu_j\nu_i\nu_j(x_\beta) \quad \text{for } \beta \notin \{\alpha_i, \alpha_j\} \end{aligned}$$

The right hand side of the first equation is $l^{-1}\nu_j\nu_i(x_{\alpha_j})$. Next, we need to distinguish between two cases: $\alpha_j \prec \alpha_i$ and $\alpha_j \succ \alpha_i$. The first case is the short one as it uses the short expressions (c) and (f), which immediately yield $\nu_i\nu_j\nu_i(x_{\alpha_j}) = \nu_j\nu_i\nu_j(x_{\alpha_j}) = l^{-1}x_{\alpha_i}$. In the second case, we have, $\nu_i(x_{\alpha_j}) = x_{\alpha_i+\alpha_j} + m\,x_{\alpha_i} - mx_{\alpha_j}$. Since $(\alpha_i + \alpha_j|\alpha_j) = 1 - \frac{1}{2} = \frac{1}{2}$ and $(\alpha_i + \alpha_j) - \alpha_j = \alpha_i \prec \alpha_j$, we get $\nu_j\nu_i(x_{\alpha_j}) = x_{\alpha_i} + \frac{m}{l}x_{\alpha_j} - m\,x_{\alpha_i+\alpha_j} + m\,x_{\alpha_i+\alpha_j} - \frac{m}{l}x_{\alpha_j}$, *i.e.* $\nu_j\nu_i(x_{\alpha_j}) = x_{\alpha_i}$. In that case the final result is again $l^{-1}x_{\alpha_i}$ for both members of the first equality. The proof and results for the second equality are gathered in the following table:

case	$(\beta\|\alpha_i)$	$(\beta\|\alpha_j)$	$\nu_i\nu_j\nu_i(x_\beta) = \nu_j\nu_i\nu_j(x_\beta)$	rule
1	0	0	$r^3\, x_\beta$	(a)
2	0	$\frac{1}{2}$ & $\beta - \alpha_j \succ \alpha_j$	$r\, x_{\beta-\alpha_j-\alpha_i}$	(a) & (c)
3	0	$\frac{1}{2}$ & $\beta - \alpha_j \prec \alpha_j$	$r\, x_{\beta-\alpha_i-\alpha_j} - mr\, x_{\beta-\alpha_j} - mr^2 x_\beta$ $+ \frac{m}{lr^{ht\beta-3}}\, x_{\alpha_i+\alpha_j} + \frac{m}{lr^{ht\beta-2}}\, x_{\alpha_i} - \frac{m^2}{lr^{ht\beta-3}}\, x_{\alpha_j}$	all of them
4	0	$-\frac{1}{2}$ & $\beta \succ \alpha_j$	$r\, x_{\beta+\alpha_i+\alpha_j} - mr\, x_{\beta+\alpha_j} - mr^2 x_\beta$ $+ mr^{ht\beta} x_{\alpha_i+\alpha_j} + mr^{ht\beta-1} x_{\alpha_i} - m^2 r^{ht\beta} x_{\alpha_j}$	all of them
5	0	$-\frac{1}{2}$ & $\beta \prec \alpha_j$	$r\, x_{\beta+\alpha_i+\alpha_j}$	(a) & (f)
6	$\frac{1}{2}$ & $\beta - \alpha_i \prec \alpha_i$	$\frac{1}{2}$ & $\beta - \alpha_j \succ \alpha_j$	$\frac{1}{l}\, x_\beta + \frac{m}{l}\, x_{\alpha_j} - \frac{m}{l}\, x_{\beta-\alpha_j}$	all of them except (a)
7	$\frac{1}{2}$ & $\beta - \alpha_i \succ \alpha_i$	$-\frac{1}{2}$ & $\beta \succ \alpha_j$	$r\, x_\beta + mr^{ht\beta-1}\, x_{\alpha_i} - mr\, x_{\beta-\alpha_i}$	(a) & (c) & (e) & (f)
8	$\frac{1}{2}$ & $\beta - \alpha_i \prec \alpha_i$	$-\frac{1}{2}$ & $\beta \prec \alpha_j$	$r\, x_\beta + \frac{m}{lr^{ht\beta-2}}\, x_{\alpha_j} - mr\, x_{\beta+\alpha_j}$	(a) & (c) & (d) & (f)

The values for the inner products and the fact that i and j are adjacent nodes determine in each case the only possible expression for β. In case 2, we have $\beta = \alpha_l + \cdots + \alpha_{i+1} + \alpha_i + \alpha_j$ with $l \geq i+1$. Then $(\beta - \alpha_j|\alpha_i) = \frac{1}{2}$ and $\beta - \alpha_j - \alpha_i \succ \alpha_i$ and $(\beta - \alpha_j - \alpha_i|\alpha_j) = 0$. In case 3, we have $\beta = \alpha_j + \alpha_i + \alpha_{i-1} + \cdots + \alpha_k$ with $k \leq i-1$.
In case 4, we have $\beta = \alpha_l + \cdots + \alpha_{j+1}$ with $l \geq j+1$ and $\alpha_j \succ \alpha_i$.
In case 5, we have $\beta = \alpha_{j-1} + \cdots + \alpha_k$ with $k \leq j-1$ and $\alpha_i \succ \alpha_j$.
If $(\beta|\alpha_i) = (\beta|\alpha_j) = \frac{1}{2}$ (case 6), the only possibility is $\beta = \alpha_i + \alpha_j$ with without loss of generality $\alpha_i \succ \alpha_j$. We get:

$$\begin{aligned} \nu_i\nu_j\nu_i(x_\beta) &= \frac{1}{l}\, x_\beta + \frac{m}{lr^{ht\beta-2}}\, x_{\alpha_j} - \frac{m}{l}\, x_{\beta-\alpha_j} \\ \nu_j\nu_i\nu_j(x_\beta) &= \frac{1}{l}\, x_\beta + \frac{mr^{ht\beta-2}}{l}\, x_{\alpha_j} - \frac{m}{l}\, x_{\beta-\alpha_j} \end{aligned}$$

Since $ht\beta = 2$, the two results are the same.
The case $(\beta|\alpha_i) = (\beta|\alpha_j) = -\frac{1}{2}$ is impossible.
In case 7, we have $\beta = \alpha_l + \cdots + \alpha_i$, with $l > i$ and $i \sim j$, $i > j$. Finally in case 8, we have $\beta = \alpha_i + \cdots + \alpha_k$ with $k < i$ and $j \sim i$, $j > i$.

To get a representation, there are two more things to check: $\nu_i\nu(e_i) = l^{-1}\nu(e_i)$ and $\nu(e_i)\nu_j\nu(e_i) = l\nu(e_i)$ for any two adjacent nodes i and j. The first equality holds since $\nu_i(x_{\alpha_i}) = l^{-1}x_{\alpha_i}$ and $\nu(e_i)$ is always a multiple of x_{α_i}. Next, given two adjacent nodes i and j, their inner product is equal to $-\frac{1}{2}$. We want to show that $\nu(e_i)\nu_j(x_{\alpha_i}) = lx_{\alpha_i}$. Suppose first that $\alpha_i \prec \alpha_j$. Then $\nu_j(x_{\alpha_i}) = x_{\alpha_i+\alpha_j}$. Since $(\alpha_i + \alpha_j|\alpha_i) = 1 - \frac{1}{2} = \frac{1}{2}$ and since $\alpha_j \succ \alpha_i$, we get $\nu(e_i)(x_{\alpha_i+\alpha_j}) = l\, x_{\alpha_i}$. Next suppose that $\alpha_i \succ \alpha_j$. We have $\nu(e_i)(x_{\alpha_i+\alpha_j} + m\, x_{\alpha_j} - m\, x_{\alpha_i}) = (\frac{1}{l} + m(1-x))x_{\alpha_i} = l\, x_{\alpha_i}$. This ends the proof of the theorem 3.

Finally, it is direct to observe that $\nu(e_i e_j) = 0$ when $i \nsim j$. Thus, the so built representation factors through the quotient $B(A_{n-1})/I_2$.

Chapter 8

Proof of the Main Theorem

8.1 Properties of the Representation and the Case $\mathrm{n} = 6$

Suppose that the representation ν is not irreducible. Then there exists $\mathcal{U}$ an F-vector subspace of $\mathcal{V}$ such that $\mathcal{U} \neq 0$, $\mathcal{U} \neq \mathcal{V}$ and $\nu(w)(\mathcal{U}) \subseteq \mathcal{U}$ for any element w of the BMW algebra $B(A_{n-1})$. We will show that $\nu(X_{ij})(\mathcal{U}) = 0$ for any $1 \leq i < j \leq n$. And indeed, suppose that there exists $u \in \mathcal{U}$ such that $\nu(e_i)(u) \neq 0$. Then by the action of $\nu(e_i)$ described above, we get that x_{α_i} is in $\mathcal{U}$. From there we deduce that all the x_β's are in $\mathcal{U}$ and there are three steps. First we have $\nu_{i+1}(x_{\alpha_i}) \in \mathcal{U}$, *i.e.* $x_{\alpha_{i+1}+\alpha_i} \in \mathcal{U}$. By successive applications of (f), it follows that all the $x_{\alpha_l+\dots+\alpha_i}$ with $l \geq i$ are in $\mathcal{U}$. Next, starting again from $x_{\alpha_{i+1}+\alpha_i}$, we have $\nu_i(x_{\alpha_{i+1}+\alpha_i}) = x_{\alpha_{i+1}}$, so that $x_{\alpha_{i+1}} \in \mathcal{U}$ and proceeding inductively with step 1, all the x_{α_l} with $l \geq i$ are also in $\mathcal{U}$. Third, $\nu_{i-1}(x_{\alpha_i}) = x_{\alpha_i+\alpha_{i-1}} + m\, x_{\alpha_{i-1}}$ *modulo* $F x_{\alpha_i}$ and another application of ν_{i-1} yields $\nu_{i-1}(x_{\alpha_i+\alpha_{i-1}} + m\, x_{\alpha_{i-1}}) = x_{\alpha_i} + \frac{m}{l}\, x_{\alpha_{i-1}}$, from which we derive that $x_{\alpha_{i-1}}$ is in $\mathcal{U}$. Again by induction we also have $x_{\alpha_1}, \dots, x_{\alpha_{i-2}} \in \mathcal{U}$. We gather all these informations as follows: by the steps 2 and 3, all the x_{α_i}'s are in $\mathcal{U}$. Furthermore, by the first step applied to the x_{α_i}'s, all the x_β's are in $\mathcal{U}$. This contradicts $\mathcal{U} \neq \mathcal{V}$. We conclude that $\nu(e_i)(\mathcal{U}) = 0$ for all i. It remains to show that $\forall j \geq i+2$, $\nu(X_{ij})(\mathcal{U}) = 0$. Using the conjugation formulas, an expression for $\nu(X_{ij})$ is:

$$\nu(X_{ij}) = \nu_{j-1} \dots \nu_{i+1} \nu(e_i) \nu_{i+1}^{-1} \dots \nu_{j-1}^{-1}$$

For any positive root β, $\nu(X_{ij})(x_\beta)$ is a multiple of $\nu_{j-1} \dots \nu_{i+1}(x_{\alpha_i})$, hence a multiple of $x_{\alpha_i + \dots + \alpha_{j-1}}$. We will remember this fact:

Proposition 2.
$\nu(X_{ij})(x_\beta)$ is always a multiple of $x_{\alpha_i + \dots + \alpha_{j-1}}$.
In other words, the $\{(1+2+\dots+j-2)+(j-i)\}$th row is the only non zero row in the matrix $Mat_{\mathcal{B}_\mathcal{V}} \nu(X_{ij})$.

Thus, if $\nu(X_{ij})(u) \neq 0$, some $u \in \mathcal{U}$, it comes $x_{\alpha_i + \dots + \alpha_{j-1}} \in \mathcal{U}$. As expected, $\alpha_i + \dots + \alpha_{j-1}$ is the positive root having nodes i and j joined on its top horizontal line like in X_{ij}. Then, by successive applications of (c), we get $x_{\alpha_{j-1}} \in \mathcal{U}$. By the same arguments as above, we deduce from that fact that all the x_β's are in fact in $\mathcal{U}$, which is a contradiction. Again we conclude that $\nu(X_{ij})(\mathcal{U}) = 0$. Thus, we have:

$$\nu\left(\sum_{1 \leq i < j \leq n} X_{ij}\right)(\mathcal{U}) = 0$$

Since S is the matrix of the endomorphism $\nu(\sum_{1 \leq i < j \leq n} X_{ij})$ in the basis $\mathcal{B}_\mathcal{V}$, ν not irreducible implies $det(S) = 0$ (otherwise $\mathcal{U}$ would be trivial). A direct consequence of that fact is that the sufficient condition on l found with Maple so that the representation is irreducible holds in the case $n = 6$. Furthermore, we obtained with Maple that for each of the values of l, there exists a nonzero vector in the kernel of the matrix S. Such vectors belong to the intersection $\cap Ker\ \nu(X_{ij})$ by proposition 2. Since the F-vector subspace $\cap Ker\ \nu(X_{ij})$ is stable under the $\nu(w)'s,\ w \in B(A_5)$ by the same arguments as those exposed at the end of part 5, the Main Theorem is true in the case $n = 6$.

8.2 The Case $\mathbf{l = \frac{1}{r^{2n-3}}}$

In this section, we show that for $l = \frac{1}{r^{2n-3}}$ the representation is reducible and we give a necessary and sufficient condition on l so that there exists a one dimensional invariant subspace of $\mathcal{V}$. We introduce new notations. Recall that each of the positive roots is associated with a top line joining two nodes in a unique way. We will denote by w_{ij} $(i < j)$ the root whose associated tangle has nodes i and j joined on its top horizontal line. Note that w_{ij} is nothing else but $\alpha_i + \cdots + \alpha_{j-1}$. We will again denote by w_{ij} the element $x_{w_{ij}}$ of $\mathcal{V}$. Our result is the following:

Theorem 4. *Assume* $(r^2)^2 \neq 1$.

Suppose $\mathbf{n = 3}$. *There exists a one dimensional invariant subspace of* $\mathcal{V}$ *if and only if* $l = \frac{1}{r^3}$ *or* $l = -r^3$.

If those values are distinct then, if such a space exists, it is unique and

If $l = \frac{1}{r^3}$, *it is spanned by* $w_{12} + r\, w_{13} + r^2\, w_{23}$

If $l = -r^3$, *it is spanned by* $w_{12} - \frac{1}{r}\, w_{13} + \frac{1}{r^2}\, w_{23}$

Moreover, if $-r^3 = \frac{1}{r^3}$, i.e $r^6 = -1$, *then there exists exactly two one-dimensional* *invariant subspaces:*

$$Span_F(w_{12} + r\, w_{13} + r^2\, w_{23}) \;\; \textit{and} \;\; Span_F(w_{12} - \frac{1}{r}\, w_{13} + \frac{1}{r^2}\, w_{23})$$

In other words, the sum

$$Span_F(w_{12} + r\, w_{13} + r^2\, w_{23}) + Span_F(w_{12} - \frac{1}{r}\, w_{13} + \frac{1}{r^2}\, w_{23})$$

is direct.

Suppose $\mathbf{n} \geq \mathbf{4}$. *There exists a one dimensional invariant subspace of* $\mathcal{V}$ *if and only if* $l = \frac{1}{r^{2n-3}}$. *If so, it is spanned by* $\sum_{1\leq s<t\leq n} r^{s+t} w_{st}$

PROOF OF THE THEOREM: suppose that there exists a one-dimensional invariant subspace $\mathcal{U}$ of $\mathcal{V}$, say $\mathcal{U}$ is spanned by u. We have seen that

$$\mathcal{U} \subseteq \bigcap_{1\leq i<j\leq n} Ker\, \nu(X_{ij})$$

In particular, we must have $\nu(e_i)(\mathcal{U}) = 0$ for all i. By definition of $\nu(e_i)$, this implies $(\nu_i^2 + m\, \nu_i - id_{\mathcal{V}})(u) = 0$. For each i, let λ_i be the scalar so that $\nu_i(u) = \lambda_i u$. It comes $\lambda_i^2 + m\lambda_i - 1 = 0$, hence $\lambda_i \in \{r, -\frac{1}{r}\}$. Furthermore, by using the braid relation $\nu_i\nu_j\nu_i = \nu_j\nu_i\nu_j$, we see that all the λ_i's for $1 \leq i \leq n-1$ must take the same value r or $-\frac{1}{r}$. Let's denote this common value by λ. Thus all the λ_i's are determined by λ_1. We want to determine wether $\lambda_1 = r$ or $\lambda_1 = -\frac{1}{r}$. Let's write

$$u = \sum_{1\leq i<j\leq n} \mu_{ij} w_{ij}$$

We claim that the coefficients in this sum are related in a certain way. First we notice that an action of ν_i on $\mu_{i,k}\, w_{i,k}$ makes a term in $w_{i+1,k}$ appear with coefficient $\mu_{i,k}$ and an action of ν_i on $\mu_{i+1,k}\, w_{i+1,k}$ makes a term in $w_{i+1,k}$ appear with coefficient $-m\mu_{i+1,k}$. Moreover, $w_{i+1,k}$ cannot be obtained from other w_{st}'s by acting with ν_i. Thus, if $\nu_i(u) = \lambda u$, the coefficients $\mu_{i,k}$ and $\mu_{i+1,k}$ must be related by the relation:

$$\mu_{i,k} - m\, \mu_{i+1,k} = \lambda\, \mu_{i+1,k}$$

We note that if one of the coefficients $\mu_{i,k}$, $\mu_{i+1,k}$ is zero, then the other one is zero. In other words the two coefficients are either both zero or both non zero. They are related by

$$\mu_{i+1,k} = \lambda\, \mu_{i,k} \tag{70}$$

Similarly, by operating on $w_{l,i}$ and $w_{l,i+1}$ with ν_i, we get the equation:

$$\mu_{l,i+1} = \lambda\, \mu_{l,i} \tag{71}$$

Now the relations (70) and (71) show that all the coefficients μ_{ij}'s are nonzero as u itself is nonzero. Moreover, up to a multiplication by a scalar, u is of the form

$$u = \sum_{1 \le i < j \le n} \lambda^{i+j} w_{ij}$$

Suppose first $n = 3$. Then u can be written in the form

$$u = \lambda^3\, w_{12} + \lambda^4\, w_{13} + \lambda^5\, w_{23}$$

Let's compute $\nu_1(u)$ and $\nu_2(u)$:

$$\begin{array}{rclcccccc} \nu_1(u) & = & (\frac{\lambda^3}{l} + m\,\lambda^5) & w_{12} & + & \lambda^5\,w_{13} & + & \lambda^6 & w_{23} \\ \nu_2(u) & = & \lambda^4 & w_{12} & + & \lambda^5\,w_{13} & + & (\frac{\lambda^5}{l} + \frac{m\,\lambda^4}{l}) & w_{23} \end{array}$$

Since $\nu_1(u) = \lambda\,u$ and $\nu_2(u) = \lambda\,u$, we must have:

$$\left\{ \begin{array}{llllll} \frac{\lambda^3}{l} + m\,\lambda^5 = \lambda^4 & \textit{i.e.} \quad \frac{1}{l} = \lambda(1 - m\,\lambda) & \textit{i.e.} \quad l = \frac{1}{\lambda^3} & \text{as} & 1 - m\,\lambda = \lambda^2 \\ \frac{\lambda^5}{l} + \frac{m\,\lambda^4}{l} = \lambda^6 & \textit{i.e.} \quad l = \frac{1}{\lambda}(1 + \frac{m}{\lambda}) & \textit{i.e.} \quad l = \frac{1}{\lambda^3} & \text{as} & 1 + \frac{m}{\lambda} = \frac{1}{\lambda^2} \end{array} \right.$$

Thus, in the case $n = 3$, if there exists a one dimensional invariant subspace of $\mathcal{V}$ then l must take the values $\frac{1}{r^3}$ or $-r^3$. Conversely, let's consider the two vectors:

$$\begin{array}{rcccccc} u_r & = & w_{12} & + & r\,w_{13} & + & r^2\,w_{23} \\ u_{-\frac{1}{r}} & = & w_{12} & - & \frac{1}{r}\,w_{13} & + & \frac{1}{r^2}\,w_{23} \end{array}$$

We read on the equations giving the expressions for $\nu_1(u)$ and $\nu_2(u)$ that

$$\begin{array}{lccclcccc} \text{If} & l & = & \frac{1}{r^3} & \text{then} & \nu_1(u_r) & = & \nu_2(u_r) & = & r\,u_r \\ \text{If} & l & = & -r^3 & \text{then} & \nu_1(u_{-\frac{1}{r}}) & = & \nu_2(u_{-\frac{1}{r}}) & = & -\frac{1}{r}\,u_{-\frac{1}{r}} \end{array}$$

The theorem is thus proved in the case $n = 3$. Suppose now $n \geq 4$. By the above, the coefficient μ_{34} of u is nonzero. An action of ν_1 on w_{34} is a multiplication by r and an action of ν_1 on the other w_{ij}'s does not affect the coefficient of w_{34}. This last point forces $\lambda = r$ and u can be rewritten:

$$u = \sum_{1 \leq i < j \leq n} r^{i+j} w_{ij}$$

We will now see how this expression of u forces the value of l. It suffices for instance to look at the action of ν_1 on u and the resulting coefficient in w_{12}. A term in w_{12} appears when ν_1 acts on w_{12} and on the w_{2j}'s for $3 \leq j \leq n$.

In the first case the resulting coefficient is $\frac{r^3}{l}$ and in the second case the resulting coefficient is $m\, r^{j-3} \times r^{j+2}$. Hence we get the equation:

$$\frac{m}{r}\sum_{j=3}^{n}(r^2)^j + \frac{r^3}{l} = r^4 \tag{72}$$

from which we derive that $l = \frac{1}{r^{2n-3}}$. Conversely, for this value of l and letting $u = \sum_{1\le i<j\le n} r^{i+j} w_{ij}$, we check that $\nu_i(u) = r\,u$ for all integers $1 \le i \le n-1$. We start the proof by showing that the coefficient r^{2i+1} of $w_{i,i+1}$ is multiplied by r when we act by ν_i. We notice that the root α_i appears when we act by ν_i on:

1. α_i
2. β with $(\beta|\alpha_i) = \frac{1}{2}$ and $\beta - \alpha_i \prec \alpha_i$
3. β with $(\beta|\alpha_i) = -\frac{1}{2}$ and $\beta \succ \alpha_i$

In the first case, it yields the coefficient $r^{2n-3} \times r^{2i+1} = r^{2n+2i-2}$

In the second case, $\beta = w_{k,i+1}$ with $k = 1, \ldots, i-1$. It yields the coefficient:

$$\sum_{k=1}^{i-1} \frac{m\, r^{2n-3}}{r^{i-k-1}} \times r^{k+i+1} = m\, r^{2n-1}\sum_{k=1}^{i-1}(r^2)^k = r^{2n} - r^{2i+2n-2}$$

In the third case, $\beta = w_{i+1,s}$ with $s = i+2, \ldots, n$. It yields the coefficient:

$$\sum_{s=i+2}^{n} m\, r^{s-i-2} \times r^{s+i+1} = \frac{m}{r}\sum_{s=i+2}^{n}(r^2)^s = r^{2i+2} - r^{2n}$$

When summing the three coefficients, it only remains $r^{2i+2} = r \times r^{2i+1}$. Thus the coefficient r^{2i+1} of $w_{i,i+1}$ is multiplied by r.

Next, given a positive root β, if none of the nodes $i-1$, i, $i+1$ is in the support of β or if all three nodes $i-1$, i, $i+1$ are in the support of β, then it comes:

$$\nu_i(x_\beta) = r\, x_\beta$$

Thus, we only need to study the combined effect of ν_i on $w_{k,i}$, $w_{k,i+1}$, with $k \leq i-1$ on one hand and $w_{i,l}$, $w_{i+1,l}$, with $l \geq i+2$ on the other hand.
We have:

$$\begin{aligned} r^{k+i}\, \nu_i(w_{k,i}) &= r^{k+i}\, w_{k,i+1} && (73) \\ r^{k+i+1}\, \nu_i(w_{k,i+1}) &= r^{k+i+1}\, w_{k,i} - m\, r^{k+i+1}\, w_{k,i+1}\ modulo\, F\, x_{\alpha_i} && (74) \end{aligned}$$

So we get:

$$\nu_i\Big(r^{k+i}\, w_{k,i} + r^{k+i+1}\, w_{k,i+1}\Big) = r^{k+i+1}\, w_{k,i} + r^{k+i+2}\, w_{k,i+1}\ modulo\, F\, x_{\alpha_i}$$

Similarly, we have:

$$\begin{aligned} r^{l+i}\, \nu_i(w_{i,l}) &= r^{l+i}\, w_{i+1,l} && (75) \\ r^{l+i+1}\, \nu_i(w_{i+1,l}) &= r^{l+i+1}\, w_{i,l} - m\, r^{l+i+1}\, w_{i+1,l}\ modulo\, F\, x_{\alpha_i} && (76) \end{aligned}$$

so that:

$$\nu_i\Big(r^{l+i}\, w_{i,l} + r^{l+i+1}\, w_{i+1,l}\Big) = r^{l+i+1}\, w_{i,l} + r^{l+i+2}\, w_{i+1,l}\ modulo\, F\, x_{\alpha_i}$$

This ends the proof of Theorem 4.

8.3 The Cases $\mathbf{l = \frac{1}{r^{n-3}}}$ and $\mathbf{l = -\frac{1}{r^{n-3}}}$

In this section we show a necessary and sufficient condition on l and r so that there exists an irreducible $(n-1)$-dimensional invariant subspace of the

F-vector space $\mathcal{V}$, under the condition that the Iwahori-Hecke algebra of the symmetric group $Sym(n)$ with parameter r^2 over the field F is semisimple. This last condition is equivalent to $(r^2)^k \neq 1$ for all $k = 1, \ldots, n$. Explicitly we have the two theorems:

Theorem 5.

Let n be a positive integer with $n \geq 3$ and $n \neq 4$. Let's assume that the Iwahori-Hecke agebra of the symmetric group $Sym(n)$ with parameter r^2 over the field F is semisimple. Then, there exists an irreducible $(n-1)$-dimensional invariant subspace of $\mathcal{V}$ if and only if $l = \frac{1}{r^{n-3}}$ or $l = -\frac{1}{r^{n-3}}$.

If so, it is spanned by the v_i's, $1 \leq i \leq n-1$, where v_i is defined by the formula:

$$v_i := \Big(\frac{1}{r} - \frac{1}{l}\Big) w_{i,i+1} + \sum_{k=i+2}^{n} r^{k-i-2}(w_{i,k} - \frac{1}{r}\, w_{i+1,k}) + \epsilon_l \sum_{s=1}^{i-1} r^{n-i-2+s}(w_{s,i} - \frac{1}{r}\, w_{s,i+1})$$

$$\textit{with} \quad \begin{cases} \epsilon_{\frac{1}{r^{n-3}}} & = 1 \\ \epsilon_{-\frac{1}{r^{n-3}}} & = -1 \end{cases}$$

Theorem 6. *(Case $n = 4$)*
Let's assume that the Iwahori-Hecke algebra of the symmetric group $sym(4)$ with parameter r^2 over the field F is semisimple. Then:

there exists an irreducible 3-dimensional invariant subspace of $\mathcal{V}$ if and only if $l \in \{\frac{1}{r}, -\frac{1}{r}, -r^3\}$.

If $l \in \{-\frac{1}{r}, \frac{1}{r}\}$, it is spanned by the vectors:

$$\begin{array}{lllllll}
v_1 & = & (\frac{1}{r}-\frac{1}{l})w_{12} & + & (w_{13}-\frac{1}{r}\,w_{23}) & + & r\,(w_{14}-\frac{1}{r}\,w_{24}) \\
v_2 & = & (\frac{1}{r}-\frac{1}{l})w_{23} & + & (w_{24}-\frac{1}{r}\,w_{34}) & + & \epsilon_l\,r\,(w_{12}-\frac{1}{r}\,w_{13}) \\
v_3 & = & (\frac{1}{r}-\frac{1}{l})w_{34} & + & \epsilon_l\,(w_{13}-\frac{1}{r}\,w_{14}) & + & \epsilon_l(w_{23}-\frac{1}{r}\,w_{24})
\end{array}$$

$$\textit{where} \quad \begin{cases} \epsilon_{\frac{1}{r}} & = 1 \\ \epsilon_{-\frac{1}{r}} & = -1 \end{cases}$$

If $l = -r^3$, it is spanned by the vectors:

$$\begin{array}{lllllllllllllllll}
v_1 & = & & & & r & w_{23} & + & & w_{13} & + & (\frac{1}{r}+\frac{1}{r^3}) & w_{34} & - & & w_{24} & - & \frac{1}{r} & w_{14} \\
v_2 & = & -r & w_{12} & - & & & & r^2 & w_{13} & - & \frac{1}{r} & w_{34} & - & \frac{1}{r^2} & w_{24} & + & (r+\frac{1}{r}) & w_{14} \\
v_3 & = & (r+\frac{1}{r^3}) & w_{12} & + & \frac{1}{r} & w_{23} & - & & w_{13} & + & & & & & w_{24} & - & r & w_{14}
\end{array}$$

JOINT PROOF OF THE THEOREMS:

We first recall that if $\mathcal{U}$ is a proper invariant subspace of $\mathcal{V}$, it must be annihilated by all the algebra elements X_{ij}'s. In particular it is annihilated by all the e_i's. Thus, the action of the BMW algebra $B(A_{n-1})$ on the F-vector space $\mathcal{U}$ is an Iwahori-Hecke algebra action. Further, for $n \geq 4$ and $n \neq 6$, there are exactly two inequivalent irreducible (n-1)-dimensional representations of the Iwahori-Hecke algebra of the symmetric group $Sym(n)$ with parameter r^2 over the field F and they are respectively given by the matrices:

$$M_1 = \begin{bmatrix} -1/r & 1/r & & & \\ & r & & & \\ & & r & & \\ & & & \ddots & \\ & & & & r \end{bmatrix}, \; M_2 = \begin{bmatrix} r & & & & & \\ r & -1/r & 1/r & & & \\ & & r & & & \\ & & & r & & \\ & & & & \ddots & \\ & & & & & r \end{bmatrix},$$

$$M_3 = \begin{bmatrix} r & & & & & & \\ & r & & & & & \\ & r & -1/r & 1/r & & & \\ & & & r & & & \\ & & & & r & & \\ & & & & & \ddots & \\ & & & & & & r \end{bmatrix}, \ \ldots, \ M_{n-2} = \begin{bmatrix} r & & & & & & \\ & \ddots & & & & & \\ & & r & & & & \\ & & & r & & & \\ & & & r & -1/r & 1/r \\ & & & & & r \end{bmatrix},$$

$$M_{n-1} = \begin{bmatrix} r & & & & \\ & \ddots & & & \\ & & r & & \\ & & & r & \\ & & & r & -1/r \end{bmatrix}$$

and for the conjugate representation

$$N_1 = \begin{bmatrix} r & -r & & & \\ & -1/r & & & \\ & & -1/r & & \\ & & & \ddots & \\ & & & & -1/r \end{bmatrix},$$

$$N_2 = \begin{bmatrix} -1/r & & & & & \\ -1/r & r & -r & & & \\ & & -1/r & & & \\ & & & -1/r & & \\ & & & & \ddots & \\ & & & & & -1/r \end{bmatrix},$$

$$N_3 = \begin{bmatrix} -1/r & & & & & & \\ & -1/r & & & & & \\ & -1/r & r & -r & & & \\ & & & -1/r & & & \\ & & & & -1/r & & \\ & & & & & \ddots & \\ & & & & & & -1/r \end{bmatrix}, \ \ldots,$$

$$N_{n-2} = \begin{bmatrix} -1/r & & & & & \\ & \ddots & & & & \\ & & -1/r & & & \\ & & & -1/r & & \\ & & & -1/r & r & -r \\ & & & & & -1/r \end{bmatrix},$$

$$N_{n-1} = \begin{bmatrix} -1/r & & & & \\ & \ddots & & & \\ & & -1/r & & \\ & & & -1/r & \\ & & & -1/r & r \end{bmatrix}$$

where all the matrices are square matrices of size $n-1$ and where the blanks are filled with zeros. Note that those two representations are equivalent for $n = 3$. We now show that the latter representation cannot occur inside $\mathcal{V}$ when $n \geq 5$. Indeed, suppose that there exists $\mathcal{U}$ an irreducible $(n-1)$-dimensional invariant subspace of $\mathcal{V}$. Let $(v_1, \ldots, v_{n-1})$ be a basis of $\mathcal{U}$ in which the matrices of the left actions of the g_i's are the N_i's. We have the relations (when the indices make sense):

$$(\triangledown)\left|\begin{array}{rcll} \nu_t(v_i) & = & -1/r\ v_i, & \text{for } t \notin \{i-1,\, i,\, i+1\} \\ \nu_i(v_i) & = & r\ v_i & \\ \nu_{i+1}(v_i) & = & -1/r(v_i + v_{i+1}) & \\ \nu_{i-1}(v_i) & = & -1/r\ v_i - r\ v_{i-1} & \end{array}\right.$$

Lemma 5. *It is impossible to have such a set of relations.*

Throughout the proof of the lemma and the theorems, we will make an extensive use of the following equalities. We have for any node q:

$$\begin{array}{lcll} \forall k \geq q+2, \quad \nu_q(w_{q+1,k}) & = & w_{q,k} - m\, w_{q+1,k} \ \text{ modulo } F\, x_{\alpha_q} & (\star) \\ \forall s \leq q-1, \quad \nu_q(w_{s,q+1}) & = & w_{s,q} - m\, w_{s,q+1} \ \text{ modulo } F\, x_{\alpha_q} & (\star\star) \end{array}$$

We claim that $(\star)$ (resp $(\star\star)$) is the only way to get a term in $w_{q,k}$ for a given $k \geq q+2$ (resp $w_{s,q}$ for a given $s \leq q-1$) when acting by ν_q. Indeed, an action by ν_q results in a linear combination of the following operations:

(i) create a root α_q

(ii) add or subtract a root α_q to an existing positive root

(iii) leave an element of $\mathcal{V}$ unchanged

in the way described in the defining representation. With this remark, we see that $w_{q,k}$ (resp $w_{s,q}$) can only be obtained from $w_{q+1,k}$ or $w_{q,k}$ itself (resp from $w_{s,q+1}$ or $w_{s,q}$ itself). Since $\nu_q(w_{q,k}) = w_{q+1,k}$ (resp $\nu_q(w_{s,q}) = w_{s,q+1}$), $w_{q,k}$ (resp $w_{s,q}$) can only be obtained from $w_{q+1,k}$ (resp from $w_{s,q+1}$). Hence, if $\nu_q(v_i) = \lambda\, v_i$ with $\lambda \in \{r, -1/r\}$ and the $\mu_{s,t}$'s are the respective coefficients of the $w_{s,t}$'s in v_i, the relations $(\star)$ and $(\star\star)$ respectively imply that:

$$\forall k \geq q+2,\ \mu_{q+1,k} = \lambda\mu_{q,k} \tag{77}$$

$$\forall s \leq q-1,\ \mu_{s,q+1} = \lambda\mu_{s,q} \tag{78}$$

We will make an extensive use of these equalities. It will also be useful to note that for any node $q \in \{1, \ldots, n-1\}$, we have:

$$\begin{aligned} \forall k \geq q+2,\ \nu_q(w_{q,k}) &= w_{q+1,k} \\ \forall s \leq q-1,\ \nu_q(w_{s,q}) &= w_{s,q+1} \end{aligned}$$

as it has already been mentioned above. Finally, we will let the endomorphisms of $\mathcal{V}$ over F act on the right on $\mathcal{V}$ by

$$x_\beta.\nu_i = \nu_i(x_\beta)$$

Consider the first vector of the basis v_1 and recall that $n \geq 4$. The relation $v_1 . \nu_{n-1} = -\frac{1}{r} v_1$ implies that in v_1 there are no terms in $w_{s,t}$ for integers $s, t \in \{1, \ldots, n-2\}$ such that $s < t$. Hence we may write:

$$v_1 = \sum_{j=1}^{n-2} \mu_{j,n-1} w_{j,n-1} + \sum_{j=1}^{n-1} \mu_{j,n} w_{j,n} \tag{79}$$

We now use the relation $v1 . \nu_3 = -\frac{1}{r} v_1$ to get rid of more terms in this sum. Indeed, this relation implies that there are no terms in $w_{j,k}$ for $j \geq 5$ in v_1. Furthermore, the relations:

$$\begin{cases} w_{2,n-1} . \nu_1 &= w_{1,n-1} + mr^{n-4} w_{12} - m\, w_{2,n-1} \\ w_{2,n} . \nu_1 &= w_{1,n} + mr^{n-3} w_{12} - m\, w_{2,n} \end{cases}$$

imply in turn that:

$$mr^{n-4} \mu_{2,n-1} + mr^{n-3} \mu_{2,n} = 0$$

i.e

$$\mu_{2,n} = -\frac{1}{r}\, \mu_{2,n-1} \tag{80}$$

as there is no term in w_{12} in v_1 by the first point. Finally, an application of (77) with $q = 1$ and $\lambda = r$ also yields the relations between the coefficients:

$$\mu_{2,n-1} = r\, \mu_{1,n-1} \tag{81}$$

$$\mu_{2,n} = r\, \mu_{1,n} \tag{82}$$

We gather all these results to get, up to a multiplication by a scalar:

$$v_1 = w_{1,n-1} + r\, w_{2,n-1} - \frac{1}{r}\, w_{1,n} - w_{2,n} \\ + \mu_{3,n-1}\, w_{3,n-1} + \mu_{4,n-1}\, w_{4,n-1} + \mu_{3,n}\, w_{3,n} + \mu_{4,n}\, w_{4,n} \quad (83)$$

or

$$v_1 = \mu_{3,n-1}\, w_{3,n-1} + \mu_{4,n-1}\, w_{4,n-1} + \mu_{3,n}\, w_{3,n} + \mu_{4,n}\, w_{4,n} \quad (84)$$

1. **Suppose first that $\mathbf{n} \geq \mathbf{5}$**

   Then $v1.\, \nu_3 = -\frac{1}{r}\, v_1$ implies that $\mu_{2,n} = 0$. Then the expression for v_1 is given by (84). We will deal with the case $n = 5$ separately. Hence, assume first $n > 5$.

   (a) $\underline{n > 5}$

   We keep looking at the action of ν_3. By (77) applied with $\lambda = -\frac{1}{r}$ and $q = 3$, we have:

$$\mu_{4,n} = -\frac{1}{r}\, \mu_{3,n} \quad (85)$$

$$\mu_{4,n-1} = -\frac{1}{r}\, \mu_{3,n-1} \quad (86)$$

   Furthermore, by the set of equalities:

$$\begin{cases} w_{4,n-1}.\, \nu_3 = w_{3,n-1} + m r^{n-6}\, w_{34} - m\, w_{4,n-1} \\ w_{4,n}.\, \nu_3 = w_{3,n} + m r^{n-5}\, w_{34} - m\, w_{4,n} \end{cases}$$

   we get by the same computation as in (80) that:

$$\mu_{4,n} = -\frac{1}{r}\, \mu_{4,n-1} \quad (87)$$

as there is no term in w_{34} in v_1 (recall that $n > 5$). Next, we have $w_{3,n}.\,\nu_4 = r\,w_{3,n}$ and an action of ν_4 on the other terms of v_1 never makes any term in $w_{3,n}$ appear. Hence, $v1.\,\nu_4 = -\frac{1}{r}\,v_1$ forces $\mu_{3,n} = 0$. Then by (85), (87) and (86), all the coefficients of v_1 are zero. In other words $v_1 = 0$, which is impossible as v_1 is a basis vector.

(b) The case $n = 5$

In this case, there are only three terms in v_1:

$$v_1 = \mu_{34}\,w_{34} + \mu_{35}w_{35} + \mu_{45}w_{45}$$

Let's act by ν_3. We have:

$$w_{34}.\,\nu_3 = \frac{1}{l}\,w_{34} \tag{88}$$

$$w_{45}.\,\nu_3 = w_{35} + m\,w_{34} - m\,w_{45} \tag{89}$$

from which we derive by looking at the term in w_{34}:

$$\left(\frac{1}{l} + \frac{1}{r}\right)\mu_{34} = -m\,\mu_{45}$$

Moreover, by an equality of type (77) with $q = 3$ and $\lambda = -\frac{1}{r}$, we get:

$$\mu_{45} = -\frac{1}{r}\,\mu_{35}$$

At this point, we need to distinguish between two cases:

i. If $l = -r$, we get by the two previous equalities: $\mu_{45} = \mu_{35} = 0$. Then, $v_1 = \mu_{34}\,w_{34}$ and $\mu_{34} \neq 0$. Then we must have

$v_1.\,\nu_4 = \mu_{34}\,w_{35} = -\frac{1}{r}\,\mu_{34}\,w_{34}$ *i.e* $w_{35} = -\frac{1}{r}\,w_{34}$, which is a contradiction.

ii. If $l \neq -r$, we get $\mu_{34} = -\frac{m}{\frac{1}{r}+\frac{1}{l}}\,\mu_{45}$, so that v_1 is proportional to

$$w_{35} - \frac{1}{r}\,w_{45} + \frac{ml}{l+r}\,w_{34}$$

We now check that this expression for v_1 is compatible with $v_1.\,\nu_4 = -\frac{1}{r}\,v_1$ for a certain value of l. We have:

$$\begin{cases} w_{35}.\,\nu_4 &= & w_{34} + \frac{m}{l}\,w_{45} - m\,w_{35} \\ w_{45}.\,\nu_4 &= & \frac{1}{l}\,w_{45} \\ w_{34}.\,\nu_4 &= & w_{35} \end{cases}$$

By looking at the coefficient of the term in w_{45} in $v_1.\,\nu_4 = -\frac{1}{r}\,v_1$, we must have:
$\frac{1}{r^2} = \frac{m}{l} - \frac{1}{lr}$ *i.e* $l = -r^3$. We note that $-r^3 \neq -r$ as we have assumed that the parameter m is nonzero. Looking at the coefficients of the terms in w_{34} and w_{35} in $v_1.\,\nu_4 = -\frac{1}{r}\,v_1$ yields in turn:

$$-r = \frac{ml}{l+r}$$

With $l = -r^3$, this equality holds. We replace l by $-r^3$ in the expression giving v_1 to get:

$$v_1 = w_{35} - \frac{1}{r}\,w_{45} - r\,w_{34} \tag{90}$$

Next, we must have $v_1 . \nu_2 = -\frac{1}{r}(v_1 + v_2)$ and we have:

$$w_{35} . \nu_2 = w_{25} + mr\, w_{23} - m\, w_{35} \tag{91}$$

$$w_{45} . \nu_2 = r\, w_{45} \tag{92}$$

$$w_{34} . \nu_2 = w_{24} + m\, w_{23} - m\, w_{34} \tag{93}$$

We need to investigate the coefficients for v_2. The equality $v_2 . \nu_4 = -\frac{1}{r} v_2$ forces the coefficients of the w_{ij}'s to be zero for $1 \leq i < j \leq 3$. Hence we may write:

$$v_2 = \lambda_{14}\, w_{14} + \lambda_{24}\, w_{24} + \lambda_{34}\, w_{34} + \\ \lambda_{15}\, w_{15} + \lambda_{25}\, w_{25} + \lambda_{35}\, w_{35} + \lambda_{45}\, w_{45} \tag{94}$$

Moreover, since $v_2 . \nu_2 = r\, v_2$, we have by (77) applied with $\lambda = r$ and $q = 2$:

$$\lambda_{34} = r\, \lambda_{24}$$

$$\lambda_{35} = r\, \lambda_{25}$$

and with the equalities (91) and (93) and the fact that there are no terms in w_{23} in v_1 and v_2, we also deduce that:

$$\lambda_{35} = -\frac{1}{r}\, \lambda_{34}$$

Furthermore, since there must be a term in w_{25} in v_2, none of these coefficients is zero. Finally, by the equalities (91), (92), (93), the expression for v_1 in (90) and the fact that $v_1 . \nu_2 = -\frac{1}{r}(v_1 + v_2)$, we have: $\lambda_{14} = \lambda_{15} = 0$. Thus, v_2 must be proportional to the vector

$$w_{24} + r\,w_{34} - w_{35} - \frac{1}{r}\,w_{25} + \lambda_{45}\,w_{45}$$

Next, if v_1 is given by (90), by looking at the coefficient of w_{25} in $v_1.\,\nu_2 = -\frac{1}{r}\,(v_1 + v_2)$, the vector v_2 must be given by:

$$v_2 = r^2\,w_{24} + r^3\,w_{34} - r^2 w_{35} - r\,w_{25} + \lambda_{45}\,w_{45}$$

From there, the contradiction comes from the relation $v_2.\,\nu_1 = -\frac{1}{r}\,v_2 - r\,v_1$, as for instance the coefficient of w_{34} in $v_2.\,\nu_1$ is r^4 while the coefficient of w_{34} in $-\frac{1}{r}\,v_2 - r\,v_1$ is zero. We conclude that when $n = 5$ it is also impossible to have an irreducible 4- dimensional invariant subspace of $\mathcal{V}$ given by the matrices N_i's. It remains to study the case $n = 4$.

2. **The case $\mathbf{n} = \mathbf{4}$.**

By the expressions given in (83) and (84), we know that v_1 is proportional to $w_{13} + r\,w_{23} - \frac{1}{r}\,w_{14} - w_{24} + \mu_{34} w_{34}$ or to w_{34}. Assume first that v_1 is a multiple of w_{34}. The relation $v_1.\,\nu_3 = -\frac{1}{r}\,v_1$ forces $l = -r$. Suppose $l = -r$ and without loss of generality $v_1 = w_{34}$. We must have $v_1.\,\nu_2 = -\frac{1}{r}\,(v_1 + v_2)$, *i.e*

$$w_{24} + m\,w_{23} - m\,w_{34} = -\frac{1}{r}\,w_{34} - \frac{1}{r}\,v_2 \tag{95}$$

A general form for v_2 is:

$$v_2 = \lambda_{12}\,w_{12} + \lambda_{13}\,w_{13} + \lambda_{14}\,w_{14} + \lambda_{23}\,w_{23} + \lambda_{24}\,w_{24} + \lambda_{34}\,w_{34} \tag{96}$$

By (95), we have $\lambda_{12} = \lambda_{13} = \lambda_{14} = 0$. Next, by (77) applied with $q = 2$ and $\lambda = r$, we get $\lambda_{34} = r\,\lambda_{24}$. Also, from

$$\begin{aligned} w_{34}.\,\nu_2 &= w_{24} + m\,w_{23} - m\,w_{34} \\ w_{23}.\,\nu_2 &= -\frac{1}{r}\,w_{23} \end{aligned}$$

we deduce that

$$\lambda_{23} = \frac{m}{r+\frac{1}{r}}\,\lambda_{34}$$

Thus, the last three coefficients in v_2 are nonzero and v_2 is proportional to the vector

$$w_{24} + r\,w_{34} + \frac{mr}{r+\frac{1}{r}}\,w_{23},$$

and in fact (95) forces

$$v_2 = -r\left(w_{24} + r\,w_{34} + \frac{mr}{\frac{1}{r}+r}\,w_{23}\right)$$

to match the terms in w_{24} and w_{34}. Now the contradiction comes from the term in w_{23}: we must have:

$$m = \frac{mr}{r+\frac{1}{r}} \qquad i.e \qquad \frac{1}{r} = 0,$$

which is impossible.

Thus, we must have $v_1 = w_{13} + r\,w_{23} - \frac{1}{r}\,w_{14} - w_{24} + \mu_{34}w_{34}$.

We have the set of equations:

$$w_{13}.\,\nu_2 = w_{12} + \frac{m}{l}\,w_{23} - m\,w_{13} \tag{97}$$
$$w_{23}.\,\nu_2 = \frac{1}{l}\,w_{23} \tag{98}$$
$$w_{14}.\,\nu_2 = r\,w_{14} \tag{99}$$
$$w_{24}.\,\nu_2 = w_{34} \tag{100}$$
$$w_{34}.\,\nu_2 = w_{24} + m\,w_{23} - m\,w_{34} \tag{101}$$

By looking at the coefficient of w_{12} in the relation $v1.\,\nu_2 = -\frac{1}{r}\,v_1 - \frac{1}{r}\,v_2$, we get by using the equation (97) and the expression for v_1 above:

$$1 = -\frac{\lambda_{12}}{r} \qquad \textit{i.e.} \qquad \lambda_{12} = -r$$

Consequently,

$$\lambda_{13} = r\lambda_{12} = -r^2$$

by the relation $v_2.\,\nu_2 = r\,v_2$ and (78). By looking at the coefficient of w_{14} in the relation $v1.\,\nu_2 = -\frac{1}{r}\,v_1 - \frac{1}{r}\,v_2$ and by using (99) and the expression for v_1, we get:

$$-1 = -\frac{\lambda_{14}}{r} + \frac{1}{r^2} \qquad \textit{i.e.} \qquad \lambda_{14} = r + \frac{1}{r}$$

Similarly, by studying the coefficient of w_{24} in the same relation, we get:

$$\mu_{34} = \frac{1}{r} - \frac{\lambda_{24}}{r}$$

Next, we use the relation $v_2.\,\nu_1 = -\frac{1}{r}\,v_2 - r\,v_1$ together with the set of equations:

$$\begin{aligned}
w_{12}.\,\nu_1 &= \frac{1}{l}\, w_{12} \\
w_{23}.\,\nu_1 &= w_{13} + m\, w_{12} - m\, w_{23} \\
w_{13}.\,\nu_1 &= w_{23} \\
w_{34}.\,\nu_1 &= r\, w_{34} \\
w_{24}.\,\nu_1 &= w_{14} + mr\, w_{12} - m\, w_{24} \\
w_{14}.\,\nu_1 &= w_{24}
\end{aligned}$$

By looking at the coefficient in w_{13}, we get with $\lambda_{13} = -r^2$ (in v_2) and $\mu_{13} = 1$ (in v_1):

$$\lambda_{23} = r - r = 0$$

Also, by looking at the coefficient in w_{14}, we get

$$\lambda_{24} = 1 - \frac{\lambda_{14}}{r}$$

And since $\lambda_{14} = r + \frac{1}{r}$ from above, we actually get:

$$\lambda_{24} = -\frac{1}{r^2}$$

Then we also have

$$\lambda_{34} = r\, \lambda_{24} = -\frac{1}{r}$$

by $v_2.\,\nu_2 = r\, v_2$ and (77). From the expression for λ_{24}, it also follows that

$$\mu_{34} = \frac{1}{r} + \frac{1}{r^3}$$

To finish, we look at the coefficient of w_{12} in $v_2.\,\nu_1 = -\frac{1}{r}\, v_2 - r\, v_1$. By using the second set of equations with the adequate coefficients λ_{12}, λ_{23} and λ_{24} for v_2, we immediately get:

$$\frac{-r}{l} - \frac{m}{r} = 1$$

$$i.e \qquad \boxed{l = -r^3}$$

We conclude that $n = 4$ is the only case for which it is possible to have an irreducible representation of degree $n-1$ inside $\mathcal{V}$ that is equivalent to the matrix representation defined by the matrices N_i's. Moreover, we showed in that case that l must take the value $-r^3$. Furthermore, we have seen along the proof that such a 3-dimensional invariant subspace must be spanned over F by the vectors:

$$\begin{aligned} v_1 &= w_{13} + r\, w_{23} - \frac{1}{r}\, w_{14} - w_{24} + \big(\frac{1}{r} + \frac{1}{r^3}\big) w_{34} \\ v_2 &= -r\, w_{12} - r^2\, w_{13} - \frac{1}{r}\, w_{34} - \frac{1}{r^2}\, w_{24} + \big(r + \frac{1}{r}\big) w_{14} \end{aligned}$$

and a third linearly independent vector v_3 such that:

$$\left| \begin{array}{lcl} v_3 . \nu_3 &=& r\, v_3 \\ v_3 . \nu_2 &=& -\frac{1}{r}\, v_3 - r\, v_2 \\ v_2 . \nu_3 &=& -\frac{1}{r}(v_2 + v_3) \end{array} \right.$$

Only two of the above relations will be useful to force the value for the coefficients of v_3 in the basis of $\mathcal{V}$. More precisely, we use the action by g_3 as in the first and third relations, together with the following defining equations (where we took care to replace l by its specialization):

$$
\begin{aligned}
w_{12}.\,\nu_3 &= r\,w_{12} \\
w_{23}.\,\nu_3 &= w_{24} \\
w_{13}.\,\nu_3 &= w_{14} \\
w_{34}.\,\nu_3 &= -\frac{1}{r^3}\,w_{34} \\
w_{24}.\,\nu_3 &= w_{23} - \frac{m}{r^3}\,w_{34} - m\,w_{24} \\
w_{14}.\,\nu_3 &= w_{13} - \frac{m}{r^4}\,w_{34} - m\,w_{14}
\end{aligned}
$$

Let's write a general form for v_3:

$$v_3 := \gamma_{12}\,w_{12} + \gamma_{23}\,w_{23} + \gamma_{13}\,w_{13} + \gamma_{34}\,w_{34} + \gamma_{24}\,w_{24} + \gamma_{14}\,w_{14}$$

By looking at the coefficient of w_{23} in the third relation, we get $\lambda_{24} = -\frac{1}{r}\,\gamma_{23}$, from which we derive $\gamma_{23} = \frac{1}{r}$ by replacing λ_{24} by its value. Similarly by looking at the coefficient of w_{14}, still using the third relation, we see that:

$$\lambda_{14} = r - \frac{1}{r}\,\gamma_{13} \qquad \textit{i.e} \qquad r + \frac{1}{r} = r - \frac{1}{r}\,\gamma_{13} \qquad \textit{i.e} \qquad \gamma_{13} = -1$$

Since by the first relation and (78) we have

$$\begin{cases} \gamma_{14} &= r\,\gamma_{13} \\ \gamma_{24} &= r\,\gamma_{23} \end{cases},$$

we derive that $\gamma_{14} = -r$ on one hand and $\gamma_{24} = 1$ on the other hand. It remains to find the coefficients γ_{12} and γ_{34}. Still by the last relation and by looking this time at the coefficient in w_{12}, we get:

$$r\lambda_{12} = 1 - \frac{\gamma_{12}}{r}$$

Replacing λ_{12} by the value $-r$ yields:

$$\gamma_{12} = r + r^3$$

Finally, using the third relation for the last time and looking at the coefficient in w_{34} yields:

$$\Big(-\frac{1}{r^3}\Big)\Big(-\frac{1}{r}\Big) - \frac{m}{r^3}\Big(-\frac{1}{r^2}\Big) - \frac{m}{r^4}\Big(r + \frac{1}{r}\Big) = \frac{1}{r^2} - \frac{1}{r}\,\gamma_{34},$$

which leads to $\gamma_{34} = 0$.

When gathering all the coefficients for v_3, we obtain:

$$v_3 = (r + r^3)\,w_{12} + \frac{1}{r}\,w_{23} - w_{13} + w_{24} - r\,w_{14}$$

Thus, if there exists an irreducible 3-dimensional invariant subspace of $\mathcal{V}$ whose matrix representation is equivalent to the one defined by the matrices N_i's, then it is spanned by the vectors v_1, v_2, v_3 as we defined them above and l must take the value $-r^3$.

Conversely, we show that for the value $-r^3$ of l, the vectors

$$\left\{\begin{array}{lcc} v_1 & = & r\,w_{23} + w_{13} + \big(\frac{1}{r} + \frac{1}{r^3}\big)\,w_{34} - w_{24} - \frac{1}{r}\,w_{14} \\ v_2 & = & -r\,w_{12} - r^2\,w_{13} - \frac{1}{r}\,w_{34} - \frac{1}{r^2}\,w_{24} + \big(r + \frac{1}{r}\big)\,w_{14} \\ v_3 & = & (r + r^3)\,w_{12} + \frac{1}{r}\,w_{23} - w_{13} + w_{24} - r\,w_{14} \end{array}\right.$$

form a free family of vectors that satisfy to the relations $(\bigtriangledown)$. This will prove that $\text{Span}_F(v_1, v_2, v_3)$ is an irreducible 3-dimensional invariant subspace of $\mathcal{V}$. For the freedom of the family of vectors, we note that in v_1 (resp v_2, resp v_3) there is no term in w_{12} (resp w_{23}, resp w_{34}). Then, if λ_1, λ_2 and λ_3 are scalars such that:

$$\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = 0,$$

they must be related by:

$$\begin{cases} \lambda_2 &= (1+r^2)\,\lambda_3 \\ \lambda_2 &= (1+\frac{1}{r^2})\,\lambda_1 \\ \lambda_3 &= -r^2\,\lambda_1 \end{cases},$$

where we used the freedom of the w_{ij}'s. These equations imply that:

$$(1+r^2+r^4+r^6)\lambda_1 = 0$$

Since $r^2 \neq 1$ and $(r^2)^4 \neq 1$, we get $\lambda_1 = 0$. It follows that $\lambda_1 = \lambda_2 = \lambda_3 = 0$. Hence the family (v_1, v_2, v_3) is free. It remains to show that the relations $(\bigtriangledown)$ are satisfied on the vectors v_1, v_2, v_3. This is an easy verification that is left to the reader.

Let's go back to the proof of the theorems and suppose that there is an irreducible $(n-1)$-dimensional invariant subspace $\mathcal{U}$ of $\mathcal{V}$. When $n = 4$, assume that $l \neq -r^3$. By the preceding work, there exists a basis $(v_1, \ldots, v_{n-1})$ of $\mathcal{U}$ in which the matrices of the left actions of the g_i's are the matrices M_i's and we have the relations, when the indices make sense:

$$(\triangle)\left|\begin{array}{rcll} \nu_t(v_i) & = & r\, v_i, & \text{for } t \notin \{i-1,\, i,\, i+1\} \\ \nu_i(v_i) & = & -\frac{1}{r}\, v_i & \\ \nu_{i+1}(v_i) & = & r(v_i + v_{i+1}) & \\ \nu_{i-1}(v_i) & = & r\, v_i + \frac{1}{r}\, v_{i-1} & \end{array}\right.$$

We will show that these relations force the values $\frac{1}{r^{n-3}}$ or $-\frac{1}{r^{n-3}}$ for l.

First of all, the relation $v_i.\,\nu_i = -\frac{1}{r}\, v_i$ implies that in v_i there is no term in $w_{s,t}$ for

$$\begin{array}{lll} & t \leq i-1 & (a) \\ or & s \geq i+2 & (b) \\ or & \left\{\begin{array}{l} s \leq i-1 \\ t \geq i+2 \end{array}\right. & \begin{array}{l} (c) \\ (d) \end{array} \end{array}$$

since those terms are all multiplied by r when acting by ν_i and they cannot be obtained from other $w_{k,q}$'s with an action by ν_i by points (i), (ii) and (iii) above. By (b), either $s \leq i-1$ or $s = i$ or $s = i+1$. In the first case, by (d), the integer t must be less or equal to $i+1$. By (a) the only possibilities for t are $t = i$ or $t = i+1$. When $s = i$, we may have $t \geq i+1$ and when $s = i+1$, we may have $t \geq i+2$. Thus, the possible values for s and t for $w_{s,t}$ in v_i are:

$s \leq i-1$ and $t \in \{i,\, i+1\}$

or

$s = i$ and no restriction on t ($t \geq i+1$)

or

$s = i+1$ and no restriction on t ($t \geq i+2$)

Thus, a general form for v_i must be:

$$
\begin{aligned}
v_i = \mu_{i,i+1}\, w_{i,i+1} + \sum_{k=i+2}^{n} \mu_{i,k}\, w_{i,k} + \sum_{k=i+2}^{n} \mu_{i+1,k}\, w_{i+1,k} \\
+ \sum_{s=1}^{i-1} \mu_{s,i}\, w_{s,i} + \sum_{s=1}^{i-1} \mu_{s,i+1}\, w_{s,i+1}
\end{aligned} \tag{102}
$$

Next, from $v_i.\nu_i = -\frac{1}{r}\, v_i$, we have by (78) applied with $q = i$ and $\lambda = -\frac{1}{r}$ on one hand and (77) applied with $q = i$ and $\lambda = -\frac{1}{r}$ on the other hand:

$$
\begin{cases}
\mu_{s,i+1} = -\frac{1}{r}\, \mu_{s,i}, & \forall s \leq i-1 \\
\mu_{i+1,k} = -\frac{1}{r}\, \mu_{i,k}, & \forall k \geq i+2
\end{cases}
$$

Further, to get more relations between the coefficients, we use the relations

$$
v_i.\, \nu_q = r\, v_i \;\text{ for } q \notin \{i-1, i, i+1\}
$$

By (77) applied with each $q \notin \{i-1,\, i,\, i+1\}$ and $\lambda = r$, we get:

$$
\forall k \geq q+2, \;\; \mu_{q+1,k} = r\, \mu_{q,k} \tag{103}
$$

and by (78) also applied with each $q \notin \{i-1,\, i,\, i+1\}$ and $\lambda = r$, we get:

$$
\forall s \leq q-1, \;\; \mu_{s,q+1} = r\, \mu_{s,q} \tag{104}
$$

Let's apply (103) with $q = s \leq i-2$ and $k \in \{i,\, i+1\}$ $(i \geq s+2)$ on one hand and (104) with $q = k \geq i+2$ and $s \in \{i,\, i+1\}$ $(i+1 \leq k-1)$ on the other hand (where we used the same notations as in the sums of (102)) to get:

$$\forall\ 1 \leq s \leq i-2, \begin{cases} \mu_{s+1,i} & = & r\,\mu_{s,i} \\ \mu_{s+1,i+1} & = & r\,\mu_{s,i+1} \end{cases} \qquad \text{on one hand}$$

$$\forall\ i+2 \leq k \leq n-1, \begin{cases} \mu_{i,k+1} & = & r\,\mu_{i,k} \\ \mu_{i+1,k+1} & = & r\,\mu_{i+1,k} \end{cases} \qquad \text{on the other hand}$$

The formula (102) can now be rewritten as:

$$\begin{aligned} v_i = \mu^i\, w_{i,i+1} &+ \sum_{k=i+2}^{n} \delta^i\, r^{k-i-2}(w_{i,k} - \frac{1}{r}\, w_{i+1,k}) \\ &+ \sum_{s=1}^{i-1} \lambda^i\, r^{s-1}(w_{s,i} - \frac{1}{r}\, w_{s,i+1}) \end{aligned}$$

We will show that the δ^i 's for $i = 1,\ \ldots,\ n-2$ can be set to the value one. First, we notice that if $v_1,\ \ldots,\ v_{n-1}$ are vectors satisfying to the relations $(\triangle)$, then the vectors $\delta\, v_1,\ \ldots,\ \delta\, v_{n-1}$ also satisfy to the relations $(\triangle)$ for any nonzero scalar δ. Thus, without loss of generality, we may write:

$$v_1 = \mu^1\, w_{12} + \sum_{k=3}^{n} r^{k-3}(w_{1,k} - \frac{1}{r}\, w_{2,k})$$

where we set $\delta^1 = 1$. Next, we notice on the expression above that an action of ν_2 on v_1 never makes a term in w_{24} appear. From there, it suffices to look at the coefficient of w_{24} in the relation

$$v_1.\nu_2 = r\, v_1 + r\, v_2$$

to get:

$$0 = -r + r\,\delta^2$$

i.e.

$$\delta^2 = 1$$

Let's proceed by induction and suppose that $\delta^i = 1$ for a given i with $2 \leq i \leq n-3$. We notice that δ^{i+1} is the coefficient of $w_{i+1,i+3}$ in v_{i+1}. Since an action of ν_{i+1} on v_i never makes a term in $w_{i+1,i+3}$ appear, by looking at the coefficient of $w_{i+1,i+3}$ in the relation $\nu_{i+1}\,v_i = r\,v_i + r\,v_{i+1}$, we get:

$$0 = -r\,\delta^i + r\,\delta^{i+1}$$

i.e.

$$\delta^{i+1} = 1$$

Thus, we have shown that all the δ^i 's, for $1 \leq i \leq n-2$, can be set to the value one. Hence the formula giving the v_i's can be rewritten as follows:

$$\begin{aligned} v_i = \mu^i\,w_{i,i+1} + &\sum_{k=i+2}^{n} r^{k-i-2}(w_{i,k} - \frac{1}{r}\,w_{i+1,k}) \\ + &\sum_{s=1}^{i-1} \lambda^i\,r^{s-1}(w_{s,i} - \frac{1}{r}\,w_{s,i+1}) \end{aligned} \tag{105}$$

where the only two unknown coefficients are λ^i and μ^i. It remains to find the coefficients λ^i and μ^i. By looking at the coefficient of the term $w_{i,i+1}$ in the relation $v_i.\,\nu_{i+1} = r(v_i + v_{i+1})$, we get the equations:

$$\forall\, 1 \leq i \leq n-2, \quad \boxed{r\,\mu^i + r^i\,\lambda^{i+1} = 1} \tag{106}$$

Next, by looking at the coefficient of the same term $w_{i,i+1}$ in the relation

$v_i.\,\nu_{i-1} = r\,v_i + \frac{1}{r}\,v_{i-1}$ for $i \geq 2$, we get the equations:

$$\forall\, 2 \leq i \leq n-1, \quad -m\,\mu^i - \lambda^i\,r^{i-3} = r\,\mu^i - \frac{1}{r^2}$$

which can be rewritten after multiplication by a factor r^2:

$$\forall\, 2 \leq i \leq n-1, \quad \boxed{r\,\mu^i + r^{i-1}\,\lambda^i = 1} \tag{107}$$

Subtracting equalities (106) and (107) yields:

$$\forall\, 2 \leq i \leq n-2,\ \lambda^{i+1} = \frac{1}{r}\,\lambda^i \tag{108}$$

Next, we write equality (106) with $i = 1$ and equality (107) with $i = 2$:

$$\begin{aligned} r\,\mu^1 + r\,\lambda^2 &= 1 \qquad (109)\\ r\,\mu^2 + r\,\lambda^2 &= 1 \end{aligned}$$

from which we derive:

$$\mu^1 = \mu^2 \tag{110}$$

Further, by (107), we have:

$$\forall\, 2 \leq i \leq n-2, \quad r\,\mu^{i+1} + r^i\,\lambda^{i+1} = 1$$

And by using (108) we get:

$$\forall\, 2 \leq i \leq n-2, \quad r\,\mu^{i+1} + r^{i-1}\,\lambda^i = 1 \tag{111}$$

Now by (107) and (111), it comes:

$$\forall\, 2 \leq i \leq n-2,\;\; \mu^{i+1} = \mu^{i} \tag{112}$$

Gathering (110) and (112), we get:

$$\mu_1 = \mu_2 = \cdots = \mu_{n-1} \tag{113}$$

Thus, the coefficient μ^i of the term $w_{i,i+1}$ in the expression giving v_i does not depend on the integer i. We will denote it by μ.
By (108), all the λ^i 's are determined by λ^2 in the following way:

$$\forall\, 2 \leq i \leq n-1,\;\; \lambda^i = \lambda^2 \left(\frac{1}{r}\right)^{i-2} \tag{114}$$

By (109), we have $\lambda^2 = \frac{1}{r} - \mu$. Thus, by determining the coefficient μ, we will get a complete expression for all the vectors v_i's. Recall that μ is the coefficient of $w_{i,i+1}$ in v_i. We look at the coefficient of $w_{i,i+1}$ in $v_i.\, \nu_i = -\frac{1}{r}\, v_i$. We have:

$$\begin{aligned} \forall\, k \geq i+2,\;\; \Big[w_{i+1,k}.\, \nu_i\Big]_{w_{i,i+1}} &= m\, r^{k-i-2} \\ \forall\, s \leq i-1,\;\; \Big[w_{s,i+1}.\, \nu_i\Big]_{w_{i,i+1}} &= \frac{m}{l\, r^{i-s-1}} \end{aligned}$$

Hence we get the equation:

$$\frac{\mu}{l} - \sum_{k=i+2}^{n} r^{k-i-3}\, m\, r^{k-i-2} - \sum_{s=1}^{i-1} \lambda^i\, r^{s-2}\, \frac{m}{l\, r^{i-s-1}} = -\frac{\mu}{r}$$

i.e

$$\frac{\mu}{l} - \frac{1}{r^2} + (r^2)^{n-i-2} - \frac{\lambda^i}{l}\Big(\frac{1}{r^i} - r^{i-2}\Big) = -\frac{\mu}{r} \qquad (\star)_i$$

Let's write down $(\star)_2$ and $(\star)_3$:

$$\begin{array}{rcll} \mu\Big(\dfrac{1}{l}+\dfrac{1}{r}\Big) & = & \dfrac{\lambda^2}{l}\Big(\dfrac{1}{r^2}-1\Big)+\dfrac{1}{r^2}-(r^2)^{n-4} & \qquad (\star)_2 \\ \mu\Big(\dfrac{1}{l}+\dfrac{1}{r}\Big) & = & \dfrac{\lambda^2}{lr}\Big(\dfrac{1}{r^3}-r\Big)+\dfrac{1}{r^2}-(r^2)^{n-5} & \qquad (\star)_3 \end{array}$$

where λ^3 has been replaced by $\frac{\lambda^2}{r}$. Let's subtract these two equalities:

$$\frac{\lambda^2}{l}\Big(\frac{1}{r^2}-\frac{1}{r^4}\Big) = (r^2)^{n-4}\Big(1-\frac{1}{r^2}\Big) \qquad (\star)_2-(\star)_3$$

We multiply this last equality by $\frac{1}{r^2}$ and then divide it by $\frac{1}{r^2}-\frac{1}{r^4}$ (licit as $m \neq 0$) to get:

$$\lambda^2 = l\,(r^2)^{n-3}$$

We recall that $\mu = \frac{1}{r} - \lambda^2$. Hence we get a relation binding μ and l as follows:

$$\mu = \frac{1}{r} - l\,(r^2)^{n-3}$$

Next, by looking at the coefficient of w_{12} in $v_1.\,\nu_1 = -\frac{1}{r}\,v_1$, we get:

$$\mu\Big(\frac{1}{l}+\frac{1}{r}\Big) = \frac{1}{r^2}-(r^2)^{n-3} \tag{115}$$

as v_1 is only composed of one term and one sum as below:

$$v_1 = \mu\,w_{12} + \sum_{k=3}^{n} r^{k-3}(w_{1,k} - \frac{1}{r}\,w_{2,k})$$

Replacing μ by its value in equality (115) yields: $l^2 = \frac{1}{(r^2)^{n-3}}$. Hence there are two possible values for l:

$$\begin{array}{lllllll} \text{Either} & l = \frac{1}{r^{n-3}} & \text{and} & \mu = \frac{1}{r} - r^{n-3} & \text{and} & \lambda^2 = r^{n-3} \\ \text{Or} & l = -\frac{1}{r^{n-3}} & \text{and} & \mu = \frac{1}{r} + r^{n-3} & \text{and} & \lambda^2 = -r^{n-3} \end{array}$$

In both cases, we have $\mu = \frac{1}{r} - \frac{1}{l}$. In the first case, we get $\lambda^i = r^{n-i-1}$ and in the second case, we get $\lambda^i = -r^{n-i-1}$, where we used the expression given in (114).

At this point we have proven that if there exists an irreducible $(n-1)$-dimensional invariant subspace $\mathcal{U}$ of $\mathcal{V}$, then l must take the values $\frac{1}{r^{n-3}}$ or $-\frac{1}{r^{n-3}}$. Conversely, given $l \in \{\frac{1}{r^{n-3}}, -\frac{1}{r^{n-3}}\}$, we show that the vectors v_i's, $1 \leq i \leq n-1$, defined by:

$$v_i := \Big(\frac{1}{r} - \frac{1}{l}\Big) w_{i,i+1} + \sum_{k=i+2}^{n} r^{k-i-2}(w_{i,k} - \frac{1}{r}\ w_{i+1,k}) + \epsilon_l \sum_{s=1}^{i-1} r^{n-i-2+s}(w_{s,i} - \frac{1}{r}\ w_{s,i+1}) \tag{116}$$

$$\text{where} \quad \begin{cases} \epsilon_{\frac{1}{r^{n-3}}} & = 1 \\ \epsilon_{-\frac{1}{r^{n-3}}} & = -1 \end{cases}$$

form a free family of vectors and satisfy to the relations (when the indices make sense):

$$\left| \begin{array}{lcll} v_i.\, \nu_t & = & r\ v_i, & \text{for } t \notin \{i-1,\, i,\, i+1\} \\ v_i.\nu_i & = & -\frac{1}{r}\ v_i & \\ v_i.\, \nu_{i+1} & = & r(v_i + v_{i+1}) & \\ v_i.\, \nu_{i-1} & = & r\ v_i + \frac{1}{r}\ v_{i-1} & \end{array} \right.$$

Let's compute $v_i.\, \nu_t$ for $t < i-1$. All the terms in (116) are multiplied by a factor r when acting by ν_t, except possibly:

$$\begin{gathered} \epsilon_l\ r^{n-i-2+t}(w_{t,i} - \frac{1}{r}\ w_{t,i+1}) \\ \epsilon_l\ r^{n-i-2+t+1}(w_{t+1,i} - \frac{1}{r}\ w_{t+1,i+1}) \end{gathered} \tag{117}$$

By definition of ν_t, we have:

$$w_{t,i}.\,\nu_t = w_{t+1,i} \tag{118}$$

$$w_{t,i+1}.\,\nu_t = w_{t+1,i+1} \tag{119}$$

$$w_{t+1,i}.\,\nu_t = w_{t,i} + m\,r^{i-t-2}\,w_{t,t+1} - m\,w_{t+1,i} \tag{120}$$

$$w_{t+1,i+1}.\,\nu_t = w_{t,i+1} + m\,r^{i-t-1}\,w_{t,t+1} - m\,w_{t+1,i+1} \tag{121}$$

When we act by ν_t on v_i, all the terms $w_{s,t}$'s composing v_i are multiplied by r except the four mentioned above. Hence we read on the expressions (118), (119), (120) and (121) that $w_{t,i}$ is obtained from and only from $w_{t+1,i}$. And similarly, $w_{t,i+1}$ is obtained from and only from $w_{t+1,i+1}$. Now we read on (117) and (120) that the coefficient of $w_{t,i}$ is multiplied by r when acting by ν_t on v_i. Similarly, by (117) and (121), the coefficient of $w_{t,i+1}$ is multiplied by r when acting by ν_t on v_i. Similarly, we read on (117), (118) and (120) that after acting by ν_t on v_i, the coefficient of $w_{t+1,i}$ is $\epsilon_l\,r^{n-i-2+t}(1-mr)$, *i.e* $\epsilon_l\,r^{n-i+t} = r \times \epsilon_l\,r^{n-i-2+t+1}$. And we read on (117), (119) and (121) that the coefficient of $w_{t+1,i+1}$ is $-\frac{\epsilon_l}{r}\,r^{n-i-2+t}(1-mr)$, *i.e* $-\frac{\epsilon_l}{r}\,r^{n-i+t} = r \times \epsilon_l\,r^{n-i-2+t+1}\Big(-\frac{1}{r}\Big)$. We conclude that all the terms composing v_i are multiplied by a factor r when we act by ν_t on v_i. And we note that the terms in $w_{t,t+1}$ appearing in (120) and (121) cancel each other with the adequate coefficients 1 and $-\frac{1}{r}$ of $w_{t+1,i}$ and $w_{t+1,i+1}$. Thus, we have shown that:

$$\forall\, t \leq i-2,\ v_i.\,\nu_t = r\,v_i$$

Similarly, given $t \geq i+2$, all the terms in (116) are multiplied by a factor r

when acting by ν_t, except possibly:

$$\begin{aligned} & r^{t-i-2}(w_{i,t} - \frac{1}{r}\, w_{i+1,t}) \\ & r^{t-i-1}(w_{i,t+1} - \frac{1}{r}\, w_{i+1,t+1}) \end{aligned} \tag{122}$$

We compute the action of ν_t on the four terms $w_{i,t}$, $w_{i+1,t}$, $w_{i,t+1}$ and $w_{i+1,t+1}$:

$$w_{i,t}.\,\nu_t = w_{i,t+1} \tag{123}$$

$$w_{i+1,t}.\,\nu_t = w_{i+1,t+1} \tag{124}$$

$$w_{i,t+1}.\,\nu_t = w_{i,t} + \frac{m}{l\,r^{t-i-1}}\, w_{t,t+1} - m\, w_{i,t+1} \tag{125}$$

$$w_{i+1,t+1}.\,\nu_t = w_{i+1,t} + \frac{m}{l\,r^{t-i-2}}\, w_{t,t+1} - m\, w_{i+1,t+1} \tag{126}$$

We read on the equalities $(123) - (126)$ that when acting by ν_t, the term $w_{i,t}$ (resp $w_{i,t+1}$) can be obtained from and only from the term $w_{i,t+1}$ (resp $w_{i+1,t+1}$). By (122) and (125), the coefficient of $w_{i,t}$ is multiplied by a factor r when acting by ν_t on v_i and by (122) and (126), the one of $w_{i+1,t}$ is also multiplied by a factor r. Like above, we read on (122), (123) and (125) that the coefficient of $w_{i,t+1}$ is multiplied by r when acting by ν_t on v_i and we read on (122), (124) and (126) that the coefficient of $w_{i+1,t+1}$ is also multiplied by a factor r. Thus, all the terms $w_{s,t}$'s in (116) are in fact multiplied by r and we have:

$$\forall\, t \geq i+2,\; v_i.\,\nu_t = r\, v_i$$

We now show that $v_i.\,\nu_i = -\frac{1}{r}\, v_i$.

We have for all $k \geq i+2$ and all $s \leq i-1$:

$$\begin{cases} w_{i,i+1} \;.\; \nu_i = \frac{1}{l}\, w_{i,i+1} \\ w_{i,k} \;.\; \nu_i = w_{i+1,k} \\ w_{i+1,k} \;.\; \nu_i = w_{i,k} + m\, r^{k-i-2}\, w_{i,i+1} - m\, w_{i+1,k} \\ w_{s,i} \;.\; \nu_i = w_{s,i+1} \\ w_{s,i+1} \;.\; \nu_i = w_{s,i} + \frac{m}{l\, r^{i-s-1}}\, w_{i,i+1} - m\, w_{s,i+1} \end{cases}$$

We read on the equations above that the coefficient of $w_{i,i+1}$ after an action by ν_i on v_i is:

$$\left(\frac{1}{r} - \frac{1}{l}\right) \times \frac{1}{l} - \frac{m}{r} \sum_{k=i+2}^{n} (r^{k-i-2})^2 - \frac{m\epsilon_l}{lr}.\, r^{n+1}.\, \sum_{s=1}^{i-1} (r^{s-i-1})^2$$

i.e.

$$\left(\frac{1}{r} - \frac{1}{l}\right) \times \frac{1}{l} - \frac{1}{r^2}\Big[1 - r^{2n-2i-2} + \frac{\epsilon_l}{l}(r^{n-2i+1} - r^{n-1})\Big]$$

where we used the relation $\frac{m}{1-r^2} = \frac{1}{r}$. Next, we note that:

$$\forall\, l \in \left\{\frac{1}{r^{n-3}}, -\frac{1}{r^{n-3}}\right\},\ \frac{\epsilon_l}{l} = r^{n-3}$$

Hence, the term in the bracket is nothing else but

$$1 - r^{2n-4} = (1 - r^{n-2})(1 + r^{n-2}) = \left(1 - r\,\frac{\epsilon_l}{l}\right)\left(1 + r\,\frac{\epsilon_l}{l}\right)$$

Then, "distributing" the coefficient $\frac{1}{r^2}$ inside these two factors yields the product $\left(\frac{1}{r} - \frac{\epsilon_l}{l}\right)\left(\frac{1}{r} + \frac{\epsilon_l}{l}\right)$. Since $\epsilon_l \in \{-1, 1\}$, this product is $\left(\frac{1}{r} - \frac{1}{l}\right)\left(\frac{1}{r} + \frac{1}{l}\right)$. Thus, the coefficient of $w_{i,i+1}$ after an action by ν_i on v_i is:

$$\left(\frac{1}{r} - \frac{1}{l}\right)\frac{1}{l} - \left(\frac{1}{r} + \frac{1}{l}\right) = \boxed{-\frac{1}{r}\left(\frac{1}{r} - \frac{1}{l}\right)}$$

So the coefficient of $w_{i,i+1}$ after an action by ν_i on v_i is multiplied by a factor $-\frac{1}{r}$. Let's look at the other terms. For each $i+2 \leq k \leq n$, the coefficient of the term in $w_{i+1,k}$ is given by $r^{k-i-2}(1+\frac{m}{r}) = r^{k-i-4}$, hence is multiplied by $-\frac{1}{r}$. Similarly, for each $1 \leq s \leq i-1$, the coefficient of $w_{s,i+1}$ is, after an action by ν_i, $\epsilon_l r^{n-i-2+s}(1+\frac{m}{r}) = \epsilon_l\, r^{n-i+s-4}$, hence is multiplied by $-\frac{1}{r}$. Moreover, it can be read directly on the expression (116) and the third (resp the fifth) equation above that the coefficient of $w_{i,k}$ (resp $w_{s,i}$) gets multiplied by a factor $-\frac{1}{r}$. Thus, we have:

$$v_i.\,\nu_i = -\frac{1}{r}\,v_i$$

Let's now compute $v_i.\,\nu_{i+1}$. When acting by ν_{i+1}, most of the terms $w_{s,t}$'s in (116) are multiplied by r, with the exception of:

- $w_{i,i+1}$
- $w_{i,i+2}$, $w_{i+1,i+2}$ and $w_{i+1,k}$ for $i+3 \leq k \leq n$, in the first sum
- $w_{s,i+1}$ for $1 \leq s \leq i-1$, in the second sum

We compute the action of ν_{i+1} on these terms:

$$w_{i,i+1}.\,\nu_{i+1} = w_{i,i+2} \tag{127}$$

$$w_{i,i+2}.\,\nu_{i+1} = w_{i,i+1} + \frac{m}{l}\,w_{i+1,i+2} - m\,w_{i,i+2} \tag{128}$$

$$w_{i+1,i+2}.\,\nu_{i+1} = \frac{1}{l}\,w_{i+1,i+2} \tag{129}$$

$$w_{i+1,k}.\,\nu_{i+1} = w_{i+2,k} \qquad (i+3 \leq k \leq n) \tag{130}$$

$$w_{s,i+1}.\,\nu_{i+1} = w_{s,i+2} \qquad (1 \leq s \leq i-1) \tag{131}$$

From there,

$$\left.\begin{array}{l}\text{the coefficient of } w_{i,i+1} \text{ in } v_i.\,\nu_{i+1} \text{ is } 1,\\[2mm] \text{the coefficient of } w_{i,i+1} \text{ in } v_i \text{ is } \frac{1}{r}-\frac{1}{l},\\[2mm] \text{the coefficient of } w_{i,i+1} \text{ in } v_{i+1} \text{ is } \epsilon_l\, r^{n-3} = \epsilon_l\,\frac{\epsilon_l}{l} = \frac{1}{l}\end{array}\right\} 1 = r\Big(\frac{1}{r}-\frac{1}{l}\Big)+\frac{r}{l}$$

$$\left.\begin{array}{l}\text{the coefficient of } w_{i,i+2} \text{ in } v_i.\,\nu_{i+1} \text{ is } r-\frac{1}{l},\\[2mm] \text{the coefficient of } w_{i,i+2} \text{ in } v_i \text{ is } 1,\\[2mm] \text{the coefficient of } w_{i,i+2} \text{ in } v_{i+1} \text{ is } -\frac{1}{lr}\end{array}\right\} r-\frac{1}{l} = r - r\times\frac{1}{lr}$$

$$\left.\begin{array}{l}\text{the coefficient of } w_{i+,i+2} \text{ in } v_i.\,\nu_{i+1} \text{ is } \frac{m}{l}-\frac{1}{rl} = -\frac{r}{l},\\[2mm] \text{the coefficient of } w_{i+1,i+2} \text{ in } v_i \text{ is } -\frac{1}{r},\\[2mm] \text{the coefficient of } w_{i+1,i+2} \text{ in } v_{i+1} \text{ is } \frac{1}{r}-\frac{1}{l}\end{array}\right\} -\frac{r}{l} = -1 + r\Big(\frac{1}{r}-\frac{1}{l}\Big)$$

From now on we do not need to worry about these terms anymore. We study the action of ν_{i+1} on the first sum of v_i for indices k such that $i+3 \leq k \leq n$. By the equation (130), an action by ν_{i+1} on v_i makes a term in $w_{i+2,k}$ appear with coefficient the one of $w_{i+1,k}$ in v_i, that is $-\frac{r^{k-i-2}}{r}$. Hence, we get a term:

$$r\sum_{k=i+3}^{n}(r^{k-i-3}\,w_{i+1,k}-\frac{1}{r}\,w_{i+2,k}),$$

where we need to add

$$-\sum_{k=i+3}^{n} r^{k-i-2}\,w_{i+1,k}$$

The first expression is the first sum in v_{i+1}, multiplied by a factor r. The latter expression is a part of the first sum in $r\,v_i$ with indices k greater than $i+3$. Since all the terms $w_{i,k}$'s for $i+3 \leq k \leq n$ are multiplied by r when acting by ν_{i+1}, we get a sum:

$$r \sum_{k=i+3}^{n} r^{k-i-2}(w_{i,k} - \frac{1}{r}\, w_{i+1,k})$$

Next, we study the action of ν_{i+1} on the second sum of v_i. Acting by ν_{i+1} makes a term in $w_{s,i+2}$ appear by equation (131), with a coefficient $-\epsilon_l\, \frac{r^{n-i-2+s}}{r}$. Hence we get a sum:

$$r\epsilon_l \sum_{s=1}^{i-1} r^{n-i-3+s}(w_{s,i+1} - \frac{1}{r}\, w_{s,i+2}) \tag{132}$$

where we need to add

$$-\epsilon_l \sum_{s=1}^{i-1} r^{n-i-2+s}\, w_{s,i+1}$$

This latter expression is a part of the second sum in $r\, v_i$. And since all the terms $w_{s,i}$'s, for $1 \leq s \leq i-1$ get multiplied by r when acting by ν_{i+1}, we actually get the whole second sum

$$r\, \epsilon_l \sum_{s=1}^{i-1} r^{n-i-2+s}(w_{s,i} - \frac{1}{r}\, w_{s,i+1})$$

of $r\, v_i$.

The sum (132) is the second sum in $r\, v_{i+1}$, minus the term

$$\frac{r}{l}\,(w_{i,i+1} - \frac{1}{r}\, w_{i,i+2}),$$

which corresponds to $s = i$. Since those terms in $w_{i,i+1}$ and $w_{i,i+2}$ have already been processed above, we may now conclude that:

$$v_i.\, \nu_{i+1} = r(v_i + v_{i+1})$$

Let $2 \leq i \leq n-1$. It remains to show that

$$v_i.\nu_{i-1} = r\, v_i + \frac{1}{r}\, v_{i-1}$$

When we act by ν_{i-1} on v_i, all the terms $w_{s,t}$ in (116) get multiplied by r, except

- $w_{i,i+1}$
- $w_{i,k}$, each $i+2 \leq k \leq n$, in the first sum
- $w_{i-1,i}$, $w_{s,i}$ for $1 \leq s \leq i-2$ and $w_{i-1,i+1}$ in the second sum

Let's compute the action of ν_{i-1} on these terms.

$$w_{i,i+1}.\,\nu_{i-1} \quad = \quad w_{i-1,i+1} + m\, w_{i-1,i} - m\, w_{i,i+1} \tag{133}$$

$$\forall\, k \geq i+2, \quad w_{i,k}.\,\nu_{i-1} \quad = \quad w_{i-1,k} + m\, r^{k-i-1}\, w_{i-1,i} - m\, w_{i,k} \tag{134}$$

$$w_{i-1,i}.\,\nu_{i-1} \quad = \quad \frac{1}{l}\, w_{i-1,i} \tag{135}$$

$$\forall\, s \leq i-2, \quad w_{s,i}.\,\nu_{i-1} \quad = \quad w_{s,i-1} + \frac{m}{l\, r^{i-s-2}}\, w_{i-1,i} - m\, w_{s,i} \tag{136}$$

$$w_{i-1,i+1}.\,\nu_{i-1} \quad = \quad w_{i,i+1} \tag{137}$$

We see with the equations (133) and (137) that the coefficient of $w_{i,i+1}$ in $v_i.\,\nu_{i-1}$ is:

$$-m\Big(\frac{1}{r} - \frac{1}{l}\Big) - \epsilon_l.\frac{r^{n-3}}{r}$$

We recall from before that:

$$r^{n-3} = \frac{\epsilon_l}{l}$$

Thus,

$$\epsilon_l . r^{n-3} = \frac{\epsilon_l^2}{l} = \frac{1}{l}$$

We will use this equality extensively. Therefore, the coefficient of $w_{i,i+1}$ in $v_i . \nu_{i-1}$ is in fact:

$$\boxed{1 - \frac{r}{l} - \frac{1}{r^2}}$$

We now compute the coefficient of $w_{i-1,i}$ in $v_i . \nu_{i-1}$. There are several contributions coming from four different sources:

- $\star$ the term $w_{i,i+1}$ with coefficient $m\left(\frac{1}{r} - \frac{1}{l}\right)$
- $\star$ the terms $w_{i,k}$'s for $i + 2 \leq k \leq n$ with coefficient

$$\begin{aligned} \sum_{k=i+2}^{n} m\, r^{k-i-1} r^{k-i-2} &= m\, r^{-2i-3} \sum_{k=i+2}^{n} (r^2)^k \\ &= m\, r^{-2i-3}\, r^{2i+4}\, \frac{1-(r^2)^{n-i-1}}{1-r^2} \\ &= 1 - r^{2n-2i-2} \end{aligned}$$

- $\star$ the term $w_{i-1,i}$ with coefficient $\frac{1}{l^2}$
- $\star$ the terms $w_{s,i}$'s for $1 \leq s \leq i-2$ with coefficient

$$\begin{aligned} \epsilon_l \sum_{s=1}^{i-2} \frac{m}{l r^{i-s-2}}\, r^{n-i-2+s} &= r^{2n-3} r^{-2i}\, m \sum_{s=1}^{i-2} (r^2)^s \\ &= r^{2n-3} r^{-2i}\, r\, (1 - (r^2)^{i-2}) \\ &= r^{2n-2i-2} - (r^{n-3})^2 \\ &= r^{2n-2i-2} - \frac{1}{l^2} \end{aligned}$$

The sum of all these contributions is:

$$1+m\Big(\frac{1}{r}-\frac{1}{l}\Big)=\frac{1}{r^2}+\frac{r}{l}-\frac{1}{rl}=\boxed{\frac{1}{r}\Big(\frac{1}{r}-\frac{1}{l}\Big)+\frac{r}{l}}$$

Now we have:

$$\left.\begin{array}{l}\text{the coefficient of } w_{i,i+1} \text{ in } v_i.\,\nu_{i-1} \text{ is } 1-\frac{r}{l}-\frac{1}{r^2},\\ \text{the coefficient of } w_{i,i+1} \text{ in } v_i \text{ is } \Big(\frac{1}{r}-\frac{1}{l}\Big),\\ \text{the coefficient of } w_{i,i+1} \text{ in } v_{i-1} \text{ is } -\frac{1}{r}\end{array}\right\} 1-\frac{r}{l}-\frac{1}{r^2}=r\Big(\frac{1}{r}-\frac{1}{l}\Big)+\frac{1}{r}\Big(-\frac{1}{r}\Big)$$

$$\left.\begin{array}{l}\text{the coefficient of } w_{i-1,i} \text{ in } v_i.\,\nu_{i-1} \text{ is } \frac{1}{r}\Big(\frac{1}{r}-\frac{1}{l}\Big)+\frac{r}{l},\\ \text{the coefficient of } w_{i-1,i} \text{ in } v_i \text{ is } \frac{1}{l},\\ \text{the coefficient of } w_{i-1,i} \text{ in } v_{i-1} \text{ is } \frac{1}{r}-\frac{1}{l}\end{array}\right\} \frac{1}{r}\Big(\frac{1}{r}-\frac{1}{l}\Big)+\frac{r}{l}=r.\frac{1}{l}+\frac{1}{r}.\Big(\frac{1}{r}-\frac{1}{l}\Big)$$

$$\left.\begin{array}{l}\text{the coefficient of } w_{i-1,i+1} \text{ in } v_{i-1}.\,\nu_{i-1} \text{ is } \frac{1}{r}-\frac{1}{l},\\ \text{the coefficient of } w_{i-1,i+1} \text{ in } v_i \text{ is } -\frac{1}{lr},\\ \text{the coefficient of } w_{i-1,i+1} \text{ in } v_{i-1} \text{ is } 1\end{array}\right\} \frac{1}{r}-\frac{1}{l}=r.\big(-\frac{1}{lr}\big)+\frac{1}{r}\times 1$$

From now on, we do not worry about the terms in $w_{i,i+1}$, $w_{i-1,i}$ and $w_{i-1,i+1}$ anymore. We look at the equations (134) and (136). By (134), the term $w_{i,k}$ gets multiplied by $r-\frac{1}{r}$. Since $w_{i+1,k}.\,\nu_{i-1}=r\,w_{i+1,k}$, we get the whole first sum for v_i, multiplied by a factor r and a term

$$-\frac{1}{r}\sum_{k=i+2}^{n} r^{k-i-2}\,w_{i,k},$$

which is part of the global sum:

$$\frac{1}{r}\sum_{k=i+2}^{n} r^{k-i-1}(w_{i-1,k}-\frac{1}{r}\,w_{i,k})$$

of $\frac{1}{r}\,v_{i-1}$, where the terms in $w_{i-1,i+1}$ and $w_{i,i+1}$, corresponding to $k = i+1$, have been omitted. These terms were already processed above. But the action of ν_{i-1} on v_i also makes a term in $w_{i-1,k}$ appear (still by (134)), with coefficient r^{k-i-2}. Thus we get the whole first sum for v_{i-1}, multiplied by a factor $\frac{1}{r}$. Similarly, by (136), the term $w_{s,i}$ gets multiplied by $r - \frac{1}{r}$ for each $1 \leq s \leq i-2$. And since $w_{s,i+1}$ gets multiplied by r when acting by ν_{i-1}, we get the whole second sum

$$\epsilon_l \sum_{s=1}^{i-2} r^{n-i-2+s}(w_{s,i} - \frac{1}{r}\,w_{s,i+1})$$

of $r\,v_i$, minus the terms in $w_{i-1,i}$ and $w_{i-1,i+1}$ corresponding to the integer $s = i-1$. Those terms have already been processed above. Next, the part $-\frac{1}{r}\,w_{s,i}$ yields a sum:

$$\frac{\epsilon_l}{r} \sum_{s=1}^{i-2} r^{n-i-1+s}\big(-\frac{1}{r}\big) w_{s,i},$$

which is a part of the second sum:

$$\frac{\epsilon_l}{r} \sum_{s=1}^{i-2} r^{n-i-1+s}(w_{s,i-1} - \frac{1}{r}\,w_{s,i})$$

of $\frac{1}{r}\,v_{i-1}$. Since the action of ν_{i-1} on v_i also makes a term in $w_{s,i-1}$ appear with coefficient the one of $w_{s,i}$ in v_i by (136), that is $\frac{\epsilon_l}{r}\,r^{n-i-1+s}$, we get the whole second sum of $\frac{1}{r}\,v_{i-1}$.
All the preceding results lead to conclude that $v_i.\,\nu_{i-1} = r\,v_i + \frac{1}{r}\,v_{i-1}$. We have now shown that the v_i's satisfy to the announced relations.

Let's show that the family $(v_1,\,v_2,\,\ldots,\,v_{n-1})$ is free. In what follows, we will denote by $\mathcal{H}_{F,r^2}(n)$ the Iwahori-Hecke algebra of the symmetric group $Sym(n)$ with parameter r^2 over the field F. For large values of the integer

n, we will make use of a result of James on the dimensions of the irreducible representations of the symmetric group $Sym(n)$. For $n \geq 7$, James states in [6] that an irreducible $K\,Sym(n)$-module, where K is a field for characteristic zero, is either one of the Specht modules $S^{(n)}$, $S^{(1^n)}$, $S^{(n-1,1)}$, $S^{(2,1^{n-2})}$ or has dimension greater than $n-1$. Further for $n = 5$, if d is the degree of an irreducible representation of $Sym(5)$ over K, then $d \in \{1, 4, 5, 6\}$. Hence this fact also holds for $n = 5$. In characteristic zero, when the Iwahori-Hecke algebra of the symmetric group $Sym(n)$ is semisimple, the degrees of its irreducible representations are the same as the degrees of the irreducible representations of the symmetric group (See for instance [10]). Thus, for $n = 5$ or $n \geq 7$, if dim $\text{Span}_F(v_1, \ldots, v_{n-1}) = k < n-1$, then $\text{Span}_F(v_1, \ldots, v_{n-1})$ is either one dimensional or must contain a one dimensional invariant subspace. Then $l = \frac{1}{r^{2n-3}}$ by Theorem 4. But we have assumed that $l \in \{\frac{1}{r^{n-3}}, -\frac{1}{r^{n-3}}\}$. Hence we get a contradiction since $r^{2n-3} = \epsilon\, r^{n-3}$ with $\epsilon \in \{1, -1\}$ would imply $r^{2n} = 1$, which is forbidden when $\mathcal{H}_{F,r^2}(n)$ is semisimple. Similarly for $n = 3$, if v_1 and v_2 are linearly dependent, then the invariant vector space spanned by v_1 and v_2 over the field F has dimension one. This forces $l = -r^3$ or $l = \frac{1}{r^3}$ by Theorem 4. But we have assumed that $l \in \{1, -1\}$ in this case and the condition of semisimplicity $(r^2)^3 \neq 1$ prevents $r^3 = 1$ or $r^3 = -1$ from happening. Thus, in the case $n = 3$, the vectors v_1 and v_2 are linearly independent. Let's now deal with the case $n = 4$. We want to show that (v_1, v_2, v_3) is free. By definition, we have:

$$\begin{cases} v_1 &= (\frac{1}{r} - \frac{1}{l})\, w_{12} &+ \quad (w_{13} - \frac{1}{r}\, w_{23}) &+ \quad r(w_{14} - \frac{1}{r}\, w_{24}) \\ v_2 &= (\frac{1}{r} - \frac{1}{l})\, w_{23} &+ \quad (w_{24} - \frac{1}{r}\, w_{34}) &+ \quad r\,(w_{12} - \frac{1}{r}\, w_{13})\, \epsilon_l \\ v_3 &= (\frac{1}{r} - \frac{1}{l}) w_{34} &+ \quad (w_{13} - \frac{1}{r}\, w_{14})\, \epsilon_l &+ \quad r(w_{23} - \frac{1}{r}\, w_{24})\, \epsilon_l \end{cases}$$

Suppose λ_1, λ_2 and λ_3 are scalars such that:

$$\lambda_1\, v_1 + \lambda_2\, v_2 + \lambda_3\, v_3 = 0 \tag{138}$$

Since the w_{ij}'s are linearly independent over F, we derive the set of equations:

$$\begin{aligned} \lambda_1 - \lambda_2\, \epsilon_l + \lambda_3\, \epsilon_l &= 0 && (139)\\ r\, \lambda_1 - \frac{\epsilon_l}{r}\, \lambda_3 &= 0 && (140)\\ -\lambda_1 + \lambda_2 - \lambda_3\, \epsilon_l &= 0 && (141)\\ -\frac{\lambda_2}{r} + \lambda_3 \big(\frac{1}{r} - \frac{1}{l}\big) &= 0 && (142) \end{aligned}$$

(139), (140), (141) and (142) are respectively obtained by equaling to zero the respective coefficients of w_{13}, w_{14}, w_{24} and w_{34} in the relation (138). Let's first deal with $l = -\frac{1}{r}$ and $\epsilon_l = -1$. By (139) and (141), we have:

$$\lambda_3 - \lambda_2 = \lambda_2 + \lambda_3,$$

from which we derive $\lambda_2 = 0$ since our base field has characteristic zero. Then by (129), we get $\lambda_1 = \lambda_3$ and by (140), we get $(r + \frac{1}{r})\, \lambda_3 = 0$. Since we have assumed that $(r^2)^2 \neq 1$ by semisimplicity of the Iwahori-Hecke algebra $\mathcal{H}_{F,r^2}(4)$, we cannot have $r^2 = -1$. Hence we get $\lambda_3 = 0$ and so $\lambda_1 = \lambda_2 = \lambda_3 = 0$.

Suppose now $l = \frac{1}{r}$ and $\epsilon_l = 1$. By (140), we have $\lambda_1 = \frac{\lambda_3}{r^2}$ and by (139) or (141), we get: $\lambda_2 = (1 + \frac{1}{r^2})\, \lambda_3$. Then by (142), we get:

$$\Big(\frac{1}{r^3} + r\Big)\lambda_3 = 0$$

Since we have assumed that $(r^2)^4 \neq 1$, it is impossible to have $(r^2)^2 = -1$. It follows that $\lambda_3 = 0$, and by the above relations binding λ_1 and λ_3 on one hand and λ_2 and λ_3 on the other hand, we also get $\lambda_1 = \lambda_2 = 0$. This achieves the proof of the fact that the family of vectors (v_1, v_2, v_3) is free.

We are now able to conclude: the vector space $\mathrm{Span}_F(v_1, \ldots, v_{n-1}) \subset \mathcal{V}$ is $(n-1)$-dimensional, is invariant under the action of the g_i's and is a $\mathcal{H}_{F,r^2}(n)$-module since it is a proper nontrivial invariant subspace of $\mathcal{V}$. Then by the relations described above, it is also irreducible.

We note that for $n = 6$ the sufficient condition on l and r so that there exists an irreducible 5-dimensional invariant subspace of $\mathcal{V}$ holds in Theorem 5, since simple computations of the dimensions of the irreducible $F\,Sym(6)$-modules show that there is no irreducible representation of $\mathcal{H}_{F,r^2}(6)$ of degree d with $1 < d < 5$. This forces the family of vectors $(v_1, v_2, v_3, v_4, v_5)$ to be free by the same argument as in the cases $n = 5$ and $n \geq 7$. However, for the whole theorem to be true in the case $n = 6$, we need to investigate further about the irreducible representations of $\mathcal{H}_{F,r^2}(6)$ of degree 5 corresponding to the partition $(3, 3)$ and its conjugate partition $(2, 2, 2)$. An idea is to reduce somehow to the case $n = 5$. We will use the branching rule for the restriction of the Specht modules of the Iwahori-Hecke algebras that are semisimple, as it is described in Corollary 6.2 of [10]. Assuming that $\mathcal{H}_{F,r^2}(6)$ is semisimple, we have:

$$S^{(3,3)} \downarrow_{\mathcal{H}_{F,r^2}(5)} \simeq S^{(3,2)} \qquad\qquad S^{(2,2,2)} \downarrow_{\mathcal{H}_{F,r^2}(5)} \simeq S^{(2,2,1)}$$

Moreover, we have the following fact:

Fact 1. *Suppose $\mathcal{H}_{F,r^2}(5)$ is semisimple. Then, up to equivalence, the two irreducible matrix representations of degree 5 of $\mathcal{H}_{F,r^2}(5)$ are respectively defined by* *the matrices P_1, P_2, P_3, P_4 and Q_1, Q_2, Q_3, Q_4 given by:*

$$P_1 := \begin{bmatrix} r & & & & \\ & r & & & \\ & & r & & \\ 1 & & -r^2 & -\frac{1}{r} & \\ & 1 & & & -\frac{1}{r} \end{bmatrix}, \; P_2 := \begin{bmatrix} -\frac{1}{r} & & & 1 & \\ & -\frac{1}{r} & 1 & & 1 \\ & & r & & \\ & & & r & \\ & & & & r \end{bmatrix}$$

$$P_3 := \begin{bmatrix} r & & & & \\ & r & & & \\ & 1 & -\frac{1}{r} & & \\ 1 & & & -\frac{1}{r} & -r^2 \\ & & & & r \end{bmatrix}, \; P_4 := \begin{bmatrix} & 1 & -r & & \\ 1 & r-\frac{1}{r} & 1 & & \\ & & r & & \\ & & -r^2 & & 1 \\ & & r & 1 & r-\frac{1}{r} \end{bmatrix}$$

and for the conjugate representation:

$$Q_1 := \begin{bmatrix} -\frac{1}{r} & & & & \\ & -\frac{1}{r} & & & \\ & & -\frac{1}{r} & & \\ 1 & & -\frac{1}{r^2} & r & \\ & 1 & & & r \end{bmatrix}, \; Q_2 := \begin{bmatrix} r & & & 1 & \\ & r & 1 & & 1 \\ & & -\frac{1}{r} & & \\ & & & -\frac{1}{r} & \\ & & & & -\frac{1}{r} \end{bmatrix}$$

$$Q_3 := \begin{bmatrix} -\frac{1}{r} & & & & \\ & -\frac{1}{r} & & & \\ & 1 & r & & \\ 1 & & & r & -\frac{1}{r^2} \\ & & & & -\frac{1}{r} \end{bmatrix}, \; Q_4 := \begin{bmatrix} & 1 & \frac{1}{r} & & \\ 1 & r-\frac{1}{r} & 1 & & \\ & & -\frac{1}{r} & & \\ & & -\frac{1}{r^2} & & 1 \\ & & -\frac{1}{r} & 1 & r-\frac{1}{r} \end{bmatrix}$$

where the blanks are to be filled with zeros.

PROOF OF THE FACT: since the second matrix representation is the conjugate of the first one where we replaced r by $-\frac{1}{r}$, it suffices to show that the first set of matrices defines a representation of $\mathcal{H}_{F,r^2}(5)$ that is irreducible. It is a direct verification that that the so-defined matrices P_i's, $i = 1 \ldots 4$, satisfy to the braid relations and to the Hecke algebra relations $P_i^2 + m\, P_i = I_5$ where I_5 is the identity matrix of size 5. Let us now show that this Iwahori-Hecke algebra matrix representation is irreducible. Assuming $\mathcal{H}_{F,r^2}(5)$ is semisimple, a non irreducibility of the representation means that there exists a nonzero vector w in F^5 and scalars $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ in F such that for all $i = 1, \ldots, 4$ we have:

$$P_i\, w = \lambda_i\, w \qquad (\star_i)$$

Let w_1, w_2, w_3, w_4, w_5 be the coordinates of w in the canonical basis of F^5. We write all the equations provided by $(\star_1)$, $(\star_2)$, $(\star_3)$ and $(\star_4)$.

$$(\star_1)\left\{\begin{array}{ccll} r\, w_1 & = & \lambda_1\, w_1 & (E1)\\ r\, w_2 & = & \lambda_1\, w_2 & (E2)\\ r\, w_3 & = & \lambda_1\, w_3 & (E3)\\ w_1 - r^2\, w_3 - \frac{1}{r}\, w_4 & = & \lambda_1\, w_4 & (E4)\\ w_2 - \frac{1}{r}\, w_5 & = & \lambda_1\, w_5 & (E5) \end{array}\right.$$

$$(\star_2)\left\{\begin{array}{ccll} -\frac{1}{r}\, w_1 + w_4 & = & \lambda_2\, w_1 & (E6)\\ -\frac{1}{r}\, w_2 + w_3 + w_5 & = & \lambda_2\, w_2 & (E7)\\ r\, w_3 & = & \lambda_2\, w_3 & (E8)\\ r\, w_4 & = & \lambda_2\, w_4 & (E9)\\ r\, w_5 & = & \lambda_2\, w_5 & (E10) \end{array}\right.$$

$$(\star_3)\left\{\begin{array}{ccll} r\, w_1 & = & \lambda_3\, w_1 & (E11)\\ r\, w_2 & = & \lambda_3\, w_2 & (E12)\\ w_2 - \frac{1}{r}\, w_3 & = & \lambda_3\, w_3 & (E13)\\ w_1 - \frac{1}{r}\, w_4 - r^2\, w_5 & = & \lambda_3\, w_4 & (E14)\\ r\, w_5 & = & \lambda_3\, w_5 & (E15) \end{array}\right.$$

$$(\star_4)\begin{cases} w_2 - r\,w_3 & = \lambda_4\,w_1 \quad (E16) \\ w_1 + (r - \frac{1}{r})\,w_2 + w_3 & = \lambda_4\,w_2 \quad (E17) \\ r\,w_3 & = \lambda_4\,w_3 \quad (E18) \\ -r^2\,w_3 + w_5 & = \lambda_4\,w_4 \quad (E19) \\ r\,w_3 + w_4 + (r - \frac{1}{r})w_5 & = \lambda_4\,w_5 \quad (E20) \end{cases}$$

We will show that it is impossible to have such a set of relations. Our first step is to show that these relations force all the λ_i's to take the same value r. From there, we show that all the w_i's must then be zero, which is a contradiction. First, since all the λ_i's must be equal to r or all the λ_i's must be equal to $-\frac{1}{r}$, it suffices to show that $\lambda_1 = r$.

Now, if $\lambda_1 \neq r$, we get:

Equation used	Result
(E_1)	$w_1 = 0$
(E_2)	$w_2 = 0$
(E_3)	$w_3 = 0$
(E_6)	$w_4 = 0$
(E_7)	$w_5 = 0$

Since we get that all the coordinates of the vector w must be equal to zero, we see that λ_1 must equal r, so that all the λ_i's must in fact be equal to r. Now, from (E_4) and (E_{14}), we have:

$$\begin{aligned} \left(r + \frac{1}{r}\right) w_4 &= w_1 - r^2\,w_3 \\ \left(r + \frac{1}{r}\right) w_4 &= w_3 - r^2\,w_5 \end{aligned}$$

from which we derive that:

$$w_3 = w_5$$

Using this equality, we get from (E_{13}) that:

$$w_2 = \left(r + \frac{1}{r}\right) w_5$$

and from (E_7) that:

$$\left(r + \frac{1}{r}\right) w_2 = 2\, w_5$$

Gathering these two equalities, it yields:

$$\left(r + \frac{1}{r}\right)^2 w_5 = 2\, w_5$$

Since we have assumed that $\mathcal{H}_{F,r^2}(5)$ is semisimple, we have $(r^2)^4 \neq 1$. Thus, it is impossible to have $\left(r + \frac{1}{r}\right)^2 = 2$. Hence we get $w_5 = w_3 = 0$. Then it comes $w_2 = 0$ by (E_{13}) and $w_4 = 0$ by (E_{20}) and $w_1 = 0$ by (E_{17}). It is a contradiction to have $w = 0$, hence our so-defined matrix representations are irreducible. We use these two irreducible 5-dimensional matrix representations to show the following theorem:

Result 1.

Suppose $n = 5$. Then there exists an irreducible 5-dimensional invariant subspace of $\mathcal{V}$ if and only if $l = r$.

PROOF: suppose that there exists an irreducible 5-dimensional invariant subspace of $\mathcal{V}$, say $\mathcal{W}$. Then $\mathcal{W}$ is an irreducible $\mathcal{H}_{F,r^2}(5)$-module. Then there must exist a basis $(w_1, w_2, w_3, w_4, w_5)$ of $\mathcal{W}$ in which the matrices of the left action of the g_i's, $i = 1, \ldots, 4$, are either the P_i's or the Q_i's. We show that the first possibility leads to force the value r for l. Suppose such vectors exist. We read on the matrices P_1 and P_4 that:

$$\begin{aligned} g_1.w_4 &= -\frac{1}{r}\, w_4 \\ g_3.w_4 &= -\frac{1}{r}\, w_4 \end{aligned}$$

These two relations imply that in w_4 there are no terms in w_{34}, w_{45}, w_{35} and in w_{12}, w_{15}, w_{25}. In what follows, we will denote by $\mu_{ij}^{(k)}$ the coefficient of w_{ij} in w_k. Thus, we have:

$$w_4 = \mu_{23}^{(4)}\, w_{23} + \mu_{13}^{(4)}\, w_{13} + \mu_{24}^{(4)}\, w_{24} + \mu_{14}^{(4)}\, w_{14}$$

Furthermore, we read on the matrix P_2 that:

$$g_2.w_4 = r\, w_4 + w_1$$

By looking at the coefficient of w_{23} in this equality and by using the expression for w_4, we get:

$$\frac{1}{l}\, \mu_{23}^{(4)} + \frac{m}{l}\, \mu_{13}^{(4)} = r\, \mu_{23}^{(4)} + \mu_{23}^{(1)} \ (\star)$$

We will show that $\mu_{23}^{(1)} = 0$. To this aim, we look at the coefficient of w_{13} in

$$g_1.w_1 = r\, w_1 + w_4$$

to obtain:

$$\mu_{23}^{(1)} = r\, \mu_{13}^{(1)} + \mu_{13}^{(4)}$$

Further we look at the coefficient of w_{13} in $g_3.w_1 = r\, w_1 + w_4$ to get the equation $\mu_{14}^{(1)} = r\, \mu_{13}^{(1)} + \mu_{13}^{(4)}$. Since $g_2.w_1 = -\frac{1}{r}\, w_1$, in w_1 there is no term in

w_{14}. Hence we get:

$$\mu_{13}^{(4)} = -r\,\mu_{13}^{(1)}$$

Mixing the two equalities now yields

$$\mu_{23}^{(1)} = 0,$$

as desired. Now $(\star)$ can be rewritten as:

$$\frac{1}{l}\,\mu_{23}^{(4)} + \frac{m}{l}\,\mu_{13}^{(4)} = r\,\mu_{23}^{(4)}\ (\star\star)$$

Since $\mu_{14}^{(1)} = \mu_{23}^{(1)} = 0$, we derive from

$$g_1.w_1 = r\,w_1 + w_4$$

that

$$\mu_{24}^{(1)} = \mu_{14}^{(4)}$$

and we derive from

$$g_3.w_1 = r\,w_1 + w_4$$

that

$$\mu_{24}^{(1)} = \mu_{23}^{(4)}$$

Now it comes:

$$\mu_{23}^{(4)} = \mu_{14}^{(4)}$$

Moreover, since $g_3.w_4 = -\frac{1}{r}\,w_4$, we have by (78) applied with $s = 1$, $q = 3$ and $\lambda = -\frac{1}{r}$ that:

$$\mu_{14}^{(4)} = -\frac{1}{r}\,\mu_{13}^{(4)}$$

It follows that

$$\mu_{23}^{(4)} = -\frac{1}{r}\,\mu_{13}^{(4)}$$

Plugging this value for $\mu_{23}^{(4)}$ into $(\star\star)$ yields:

$$\left(\frac{m}{l} - \frac{1}{lr}\right)\mu_{13}^{(4)} = -\mu_{13}^{(4)} \quad (\star\star\star)$$

Furthermore, the coefficients in w_4 are related in a certain way that prevents $\mu_{13}^{(4)}$ from being zero. Indeed, as seen along the way, we have:

$$\left|\begin{array}{ll} \mu_{23}^{(4)} = & \mu_{14}^{(4)} \\ \mu_{14}^{(4)} = & -\frac{1}{r}\,\mu_{13}^{(4)} \end{array}\right.$$

Further, with $g_3.w_4 = -\frac{1}{r}\,w_4$, we add a new equation by (78) applied with $s = 2$, $q = 3$ and $\lambda = -\frac{1}{r}$:

$$\mu_{24}^{(4)} = -\frac{1}{r}\,\mu_{23}^{(4)}$$

It now appears clearly from these relations that if $\mu_{13}^{(4)} = 0$, then the other three coefficients in w_4 are also zero. This is naturally impossible as w_4 is a basis vector. Thus, $(\star\star\star)$ reduces to:

$$\frac{m}{l} - \frac{1}{lr} = -1,$$

i.e

$$\boxed{l = r}$$

Conversely, suppose $l = r$.

Let $w4 := w_{23} - \frac{1}{r}\,w_{24} + w_{14} - r\,w_{13}$

and set

$$\begin{aligned} w5 &:= g_4.w4 = r\, w_{23} - r^2\, w_{13} - \frac{1}{r}\, w_{25} + w_{15} \\ w1 &:= g_2.w4 - r\, w4 = -\frac{1}{r}\, w_{34} + w_{24} - r\, w_{12} + w_{13} \\ w2 &:= g_4.w1 = -\frac{1}{r}\, w_{35} + w_{25} - r^2\, w_{12} + r\, w_{13} \\ w3 &:= g_3.w2 - r\, w2 = -r^2\, w_{13} + r\, w_{14} - \frac{1}{r}\, w_{45} + w_{35} \end{aligned}$$

Claim 1. *If $l = r$, then the vectors $w1, w2, w3, w4, w5$ span an irreducible 5-dimensional invariant subspace of $\mathcal{V}$ over $\mathbb{Q}(r)$.*

PROOF: it is a direct verification that the family of vectors $w1, w2, w3, w4, w5$ are linearly independent over $\mathbb{Q}(r)$ and that they satisfy to the relations:

$$\left|\begin{array}{lcl} g_1.w1 & = & r\,w1 + w4 \\ g_2.w1 & = & -\frac{1}{r}\,w1 \\ g_3.w1 & = & r\,w1 + w4 \end{array}\right.$$

$$\left|\begin{array}{lcl} g_1.w2 & = & r\,w2 + w5 \\ g_2.w2 & = & -\frac{1}{r}\,w2 \\ g_4.w2 & = & (r-\frac{1}{r})\,w2 + w1 \end{array}\right.$$

$$\left|\begin{array}{lcl} g_1.w3 & = & r\,w3 - r^2\,w4 \\ g_2.w3 & = & r\,w3 + w2 \\ g_3.w3 & = & -\frac{1}{r}\,w3 \\ g_4.w3 & = & -r\,w1 + w2 + r\,w3 - r^2\,w4 + r\,w5 \end{array}\right.$$

$$\left|\begin{array}{lcl} g_1.w4 & = & -\frac{1}{r}\,w4 \\ g_3.w4 & = & -\frac{1}{r}\,w4 \end{array}\right.$$

$$\left|\begin{array}{lcl} g_1.w5 & = & -\frac{1}{r}\,w5 \\ g_2.w5 & = & r\,w5 + w2 \\ g_3.w5 & = & r\,w5 - r^2\,w4 \\ g_4.w5 & = & (r-\frac{1}{r})\,w5 + w4 \end{array}\right.$$

By Fact 1 the claim is hence proven. To finish the proof of Result 1, it suffices to show that the second irreducible representation of degree 5 described by the matrices Q_i's cannot occur. Suppose that there exists a basis $(v_1, v_2, v_3, v_4, v_5)$ of $\mathcal{W}$ in which the matrices of the left action of the g_i's, $i = 1, \ldots, 4$, are the Q_i's. In what follows, we will denote by $\lambda_{ij}^{(k)}$ the coefficient of w_{ij} in v_k. By the relations

$$\begin{aligned} g_3.v_4 &= r\,v_4 \\ g_1.v_4 &= r\,v_4 \end{aligned}$$

we get the relations between the coefficients in v_4:

$$\begin{aligned} \lambda_{45}^{(4)} &= r\,\lambda_{35}^{(4)} \\ \lambda_{14}^{(4)} &= r\,\lambda_{13}^{(4)} \\ \lambda_{24}^{(4)} &= r\,\lambda_{23}^{(4)} \\ \lambda_{25}^{(4)} &= r\,\lambda_{15}^{(4)} \\ \lambda_{24}^{(4)} &= r\,\lambda_{14}^{(4)} \\ \lambda_{23}^{(4)} &= r\,\lambda_{13}^{(4)} \end{aligned}$$

Next, since $g_4.v_4 = v_5$ and $g_3.v_5 = -\frac{1}{r}\,v_5 - \frac{1}{r^2}\,v_4$, we get

$$g_3\,g_4\,v_4 = -\frac{1}{r}\,g_4\,v_4 - \frac{1}{r^2}\,v_4$$

Looking at the term in w_{12} in this equation yields:

$$r^2\,\lambda_{12}^{(4)} = -\lambda_{12}^{(4)} - \frac{1}{r^2}\,\lambda_{12}^{(4)}$$

Since $(r^2)^3 \neq 0$, we have $r^2 + \frac{1}{r^2} + 1 \neq 0$. Thus we get $\lambda_{12}^{(4)} = 0$. By looking at the coefficient of w_{12} in $g_3.v_1 = -\frac{1}{r}\,v_1 + v_4$, we now get: $r\,\lambda_{12}^{(1)} = -\frac{1}{r}\,\lambda_{12}^{(1)}$. Thus we also have $\lambda_{12}^{(1)} = 0$. Further since $\lambda_{13}^{(1)} = r\,\lambda_{12}^{(1)}$ by the relation $g_2.v_1 = r\,v_1$, it follows that $\lambda_{13}^{(1)} = 0$. We then look at the coefficient of w_{13} in $g_2.v_4 = -\frac{1}{r}\,v_4 + v_1$ to get: $-m\lambda_{13}^{(4)} = -\frac{1}{r}\,\lambda_{13}^{(4)}$ where we used that $\lambda_{12}^{(4)} = 0$. It comes $\lambda_{13}^{(4)} = 0$. From there we derive from the set of relations above that:

$$\lambda_{13}^{(4)} = \lambda_{23}^{(4)} = \lambda_{24}^{(4)} = \lambda_{14}^{(4)} = 0$$

Thus we have:

$$v_4 = \lambda_{34}^{(4)}\, w_{34} + r\,\lambda_{35}^{(4)}\, w_{45} + \lambda_{35}^{(4)}\, w_{35} + r\,\lambda_{15}^{(4)}\, w_{25} + \lambda_{15}^{(4)}\, w_{15}$$

Let's look at the term in w_{25} in $g_3\, g_4\, v_4 = -\frac{1}{r}\, g_4\, v_4 - \frac{1}{r^2}\, v_4$. It comes:

$$-mr^2\,\lambda_{15}^{(4)} = -\frac{1}{r}\Big(- mr\,\lambda_{15}^{(4)}\Big) - \frac{1}{r^2}\, r\,\lambda_{15}^{(4)}$$

i.e

$$-r^3\,\lambda_{15}^{(4)} = 0$$

Hence $\lambda_{15}^{(4)} = 0$ and there are only three terms left in v_4:

$$v_4 = \lambda_{34}^{(4)}\, w_{34} + r\,\lambda_{35}^{(4)}\, w_{45} + \lambda_{35}^{(4)}\, w_{35}$$

Let's now look at the term in w_{34} in the relation $g_1.v_1 = -\frac{1}{r}\, v_1 + v_4$. We get:

$$r\lambda_{34}^{(1)} = -\frac{1}{r}\,\lambda_{34}^{(1)} + \lambda_{34}^{(4)}$$

i.e

$$\lambda_{34}^{(4)} = \Big(r + \frac{1}{r}\Big)\,\lambda_{34}^{(1)}$$

On the other hand, by looking at the coefficient of w_{34} in $g_2.\, v_4 = -\frac{1}{r}\, v_4 + v_1$, we get:

$$-m\,\lambda_{34}^{(4)} = -\frac{1}{r}\,\lambda_{34}^{(4)} + \lambda_{34}^{(1)}$$

i.e

$$\lambda_{34}^{(4)} = \frac{1}{r}\,\lambda_{34}^{(1)}$$

The two relations binding $\lambda_{34}^{(4)}$ and $\lambda_{34}^{(1)}$ now yield $\lambda_{34}^{(4)} = \lambda_{34}^{(1)} = 0$. Thus, there are only two terms left in v_4. Explicitly we have:

$$v_4 = r\,\lambda_{35}^{(4)}\,w_{45} + \lambda_{35}^{(4)}\,w_{35}$$

But by looking at the coefficient of w_{34} in $g_3.v_4 = r\,v_4$ we have:

$$m\,\lambda_{45}^{(4)} = 0$$

Then v_4 would be zero, a contradiction.

Let's go back to the proof of Theorem 5. When $n = 6$, suppose that there exists a 5-dimensional invariant subspace $\mathcal{W}$ of $\mathcal{V}$ with

$$\mathcal{W} \simeq S^{(3,3)} \text{ or } \mathcal{W} \simeq S^{(2,2,2)}$$

Then

$$\mathcal{W}\downarrow_{\mathcal{H}_{F,r^2}(5)} \simeq S^{(3,2)} \text{ or } \mathcal{W}\downarrow_{\mathcal{H}_{F,r^2}(5)} \simeq S^{(2,2,1)}$$

Then there must exist a basis of $\mathcal{W}$ in which the matrices of the left action by the g_i's are the P_i's or the Q_i's. We first place ourself in the first situation and adapt the proof of Result 1. Let $(w_1, w_2, w_3, w_4, w_5)$ be a basis of $\mathcal{W}$ in which the matrices of the left action by the g_i's are the P_i's. Since the relation $g_1.w_4 = -\frac{1}{r}\,w_4$ (resp $g_3.w_4 = -\frac{1}{r}\,w_4$) forces that in w_4 there is no term in w_{36}, w_{46}, w_{56} (resp in w_{26}, w_{16}), we see that the shape of w_4 is still the same. Then the presence of a sixth node does not modify the arguments of Result 1 and we get by the exact same arguments as in the proof of Result

1 that $l = r$. Furthermore $\mathcal{W}$ must be spanned by the linearly independent vectors w_1, w_2, w_3, w_4, w_5 provided in the proof of Result 1. But $\mathcal{W}$ is a submodule of $\mathcal{V}$. Hence $g_5.\mathcal{W} \subseteq \mathcal{W}$. This is not compatible with the spanning vectors above.

Second, still following the proof in Result 1, we show that it is impossible to have a basis $(v_1, v_2, v_3, v_4, v_5)$ of $\mathcal{W}$ in which the matrices of the left action by the g_i's are the Q_i's. Suppose such vectors can be found. Taking the same notations as in Result 1 and using the same arguments as in the proof of Result 1 with the addition of a sixth node, we must have:

$$v_4 = r\lambda_{35}^{(4)}\, w_{45} + \lambda_{35}^{(4)}\, w_{35} + \\ \lambda_{56}^{(4)}\, w_{56} + r\lambda_{36}^{(4)}\, w_{46} + \lambda_{36}^{(4)}\, w_{36} + r\lambda_{16}^{(4)}\, w_{26} + \lambda_{16}^{(4)}\, w_{16}$$

We simplify further the shape of v_4 by looking at the terms in w_{12} in $g_1.v_4 = r\, v_4$. Such terms appear only when g_1 acts on w_{26}, with coefficient $m\, r^3$. Since there is no term in w_{34} in v_4, we simply get:

$$m\, r^3 \lambda_{26}^{(4)} = 0$$

Hence, $\lambda_{26}^{(4)} = \lambda_{16}^{(4)} = 0$.

Further, by looking at the coefficient of w_{34} in $g_3.v_4 = r\, v_4$, we obtain:

$$m\, r\lambda_{35}^{(4)} + m\, r^2\, \lambda_{36}^{(4)} = 0$$

i.e

$$\lambda_{36}^{(4)} = -\frac{1}{r}\, \lambda_{35}^{(4)}$$

Furthermore, we claim that v_4 cannot be a multiple of w_{56}. Indeed, if so, in $-\frac{1}{r}\, g_4.v_4 - \frac{1}{r^2}\, v_4$ there is no term in w_{36}, but in $g_3g_4.v_4$, there is one. This would contradict

$$g_3g_4.v_4 = -\frac{1}{r}\, g_4.v_4 - \frac{1}{r^2}\, v_4$$

Thus, without loss of generality, we may set $\lambda^{(4)}_{35} = 1$ and we rewrite v_4 as follows:

$$v_4 = r\, w_{45} + w_{35} + \lambda^{(4)}_{56}\, w_{56} - w_{46} - \frac{1}{r}\, w_{36}$$

We now derive from $v_1 = g_2.v_4 + \frac{1}{r}\, v_4$ that:

$$\begin{aligned} v_1 = {} & (1+r^2)\, w_{45} + r\, w_{35} + w_{25} \\ & + \left(r + \frac{1}{r}\right) \lambda^{(4)}_{56}\, w_{56} - \left(r + \frac{1}{r}\right) w_{46} - w_{36} - \frac{1}{r}\, w_{26} \end{aligned}$$

If we look at the coefficient of w_{26} in $g_3.v_1 = -\frac{1}{r}\, v_1 + v_4$ we get

$$-1 = \frac{1}{r^2},$$

which is a contradiction as $(r^2)^2 \neq 1$.

Thus, we conclude that when $n = 6$ it is impossible to have an irreducible 5-dimensional submodule of $\mathcal{V}$ isomorphic to $S^{(3,3)}$ or isomorphic to $S^{(2,2,2)}$. Rather, it must be isomorphic to $S^{(5,1)}$ or $S^{(2,1^4)}$. Then, as seen before, one of these representations leads to $l \in \{\frac{1}{r^3}, -\frac{1}{r^3}\}$ while its conjugate representation cannot occur. Hence, the proof of Theorem 5 is now complete.

8.4 Proof of the Main Theorem

In this section we prove the theorem:

Theorem 7. *Assume $\mathcal{H}_{F,r^2}(n)$ is semisimple. Then,*

$\nu^{(3)}$ *is irreducible if and only if* $l \notin \{-r^3, 1, -1, \frac{1}{r^3}\}$

For $n \geq 4$, $\nu^{(n)}$ *is irreducible if and only if* $l \notin \{r, -r^3, \frac{1}{r^{n-3}}, -\frac{1}{r^{n-3}}, \frac{1}{r^{2n-3}}\}$

PROOF: we already proved the theorem for the small values $n = 3, 4, 5, 6$ in the previous sections. For larger values of n, we will use the following two statements of James which can be found in [6]:

Proposition 3. *Let* K *be a field of characteristic zero.*
For all $n \geq 7$*, every irreducible* $K\, Sym(n)$*-module is either isomorphic to one of the Specht modules* $S^{(n)}$, $S^{(1^n)}$, $S^{(n-1,1)}$, $S^{(2,1^{n-2})}$ *or has dimension greater than n-1.*

Proposition 4. *Let* K *be a field of characteristic zero.*
For all $n \geq 9$*, every irreducible* $K\, Sym(n)$*-module is either isomorphic to one of the Specht modules* $S^{(n)}$, $S^{(n-1,1)}$, $S^{(n-2,2)}$, $S^{(n-2,1,1)}$ *or their conjugates, or has dimension greater than* $\frac{(n-1)(n-2)}{2}$.

For a proof of these two facts, we refer the reader to [6]. We will use an immediate corollary:

Corollary 3. *Let* $\mathcal{D}$ *be an irreducible* $F\, Sym(n)$*-module with* $n = 7$ *or* $n \geq 9$*, where* F *is a field of characteristic zero. Then, there are two possibilities:*

$$\text{either} \quad dim\, \mathcal{D} \in \{1, n-1, \tfrac{n(n-3)}{2}, \tfrac{(n-1)(n-2)}{2}\}$$

$$\text{or} \qquad dim\, \mathcal{D} > \tfrac{(n-1)(n-2)}{2}$$

PROOF OF THE COROLLARY: the dimensions of the irreducible $F\, Sym(n)$-modules $S^{(n-2,1,1)}$, $S^{(n-2,2)}$ (and their respective conjugates) are given by the number of standard Young tableau of shapes $(n-2,1,1)$ and $(n-2,2)$. In a standard tableau, the numbers increase down each row and down each column. For the shape $(n-2,2)$, the two possible configurations are the following:

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & \ldots\ldots & \\ \hline & & \multicolumn{2}{c}{} \\ \cline{1-2}\end{array}$$

where you need to pick two integers out of the $(n-2)$ remaining ones, to fill in the two boxes on the second row. The other possible configuration is:

$$\begin{array}{|c|c|c|c|}\hline 1 & 3 & \ldots\ldots & \\ \hline 2 & & \multicolumn{2}{c}{} \\ \cline{1-2}\end{array}$$

And there are $(n-3)$ possible choices for the node $(2,2)$. Thus, the total number of standard Young tableau of shape $(n-2,2)$ is

$$\binom{n-2}{2} + (n-3) = \frac{n(n-3)}{2}$$

Hence we have $\dim S^{(n-2,2)} = \frac{n(n-3)}{2}$. And there are $\binom{n-1}{2}$ standard Young tableaux of shape $(n-2,1,1)$, so $\dim S^{(n-2,1,1)} = \frac{(n-1)(n-2)}{2}$.

Now for $n \geq 9$, the corollary is nothing else but a reformulation of Proposition 4 with $\dim S^{(n-2,2)} = \frac{n(n-3)}{2}$ and $\dim S^{(n-2,1,1)} = \frac{(n-1)(n-2)}{2}$. By Proposition 3, the irreducible $F\,Sym(7)$-modules have dimension $1, 6$ or dimension greater than 6. In the table below, we computed the dimensions of the irreducible $F\,Sym(7)$-modules that have dimensions greater than 6. We used the Hook formula. Here, the conjugates of the Specht modules mentioned in the table have been omitted since they have the same dimensions by the Hook formula. As illustrated by the table, the irreducible $sym(7)$-modules have dimensions either 1 or 6 or $\frac{7\times 4}{2} = 14$ or $\frac{6\times 5}{2} = 15$ or dimensions greater than 15. This achieves the proof of the corollary. Note that Proposition 4 fails in the case $n = 7$, as for instance $\dim S^{(4,3)} = 14$, while Corollary 3 holds in this case.

class of irreducible $Sym(7)$-module	hooklengths	dimension
$S^{(5,2)}$	$\begin{array}{ccccc} 6 & 5 & 3 & 2 & 1 \\ 2 & 1 & & & \end{array}$	$f^{(5,2)} = \frac{7!}{6\times5\times4\times3} = 14$
$S^{(5,1,1)}$	$\begin{array}{ccccc} 7 & 4 & 3 & 2 & 1 \\ 2 & & & & \\ 1 & & & & \end{array}$	$f^{(5,1,1)} = \frac{7!}{7\times4\times3\times2\times2} = 15$
$S^{(4,3)}$	$\begin{array}{cccc} 5 & 4 & 3 & 1 \\ 3 & 2 & 1 & \end{array}$	$f^{(4,3)} = \frac{7!}{5\times4\times3\times2\times3} = 14$
$S^{(4,2,1)}$	$\begin{array}{cccc} 6 & 4 & 2 & 1 \\ 3 & 1 & & \\ 1 & & & \end{array}$	$f^{(4,2,1)} = \frac{7!}{6\times4\times3\times2} = 35$
$S^{(4,1,1,1)}$	$\begin{array}{cccc} 7 & 3 & 2 & 1 \\ 3 & & & \\ 2 & & & \\ 1 & & & \end{array}$	$f^{(4,1,1,1)} = \frac{7!}{7\times6\times3\times2} = 20$
$S^{(3,3,1)}$	$\begin{array}{ccc} 5 & 3 & 2 \\ 4 & 2 & 1 \\ 1 & & \end{array}$	$f^{(3,3,1)} = \frac{7!}{5\times4\times3\times2\times2} = 21$

Finally, we note that Corollary 3 is not true for $n = 8$, since $\frac{8\times5}{2} = 20$ and $\dim S^{(4,4)} = 14$.

Let's go back to the proof of the theorem. There are two parts in the proof. First of all we need to show that for each of the values r, $-r^3$, $\frac{1}{r^{n-3}}$, $-\frac{1}{r^{n-3}}$, $\frac{1}{r^{2n-3}}$ of l, the representation $\nu^{(n)}$ is reducible. Second we need to show that if $\nu^{(n)}$ is reducible, then it forces one of the values r, $-r^3$, $\frac{1}{r^{n-3}}$, $-\frac{1}{r^{n-3}}$, $\frac{1}{r^{2n-3}}$ for the parameter l. The first point is achieved by exhibiting a non trivial proper invariant subspace inside $\mathcal{V}$ as it was already done in the case $l = \frac{1}{r^{2n-3}}$ (cf Theorem 4) and in the cases $l \in \{\frac{1}{r^{n-3}}, -\frac{1}{r^{n-3}}\}$ (cf Theorems 5 and 6). We leave the cases $l = r$ and $l = -r^3$ for later. For the second part of the proof, we will use our preliminary remarks and results of G.D. James. We proceed by induction on n to show the following property:

$$(\P_n) : \nu^{(n)} \text{ is reducible} \Rightarrow l \in \left\{r, -r^3, \frac{1}{r^{n-3}}, -\frac{1}{r^{n-3}}, \frac{1}{r^{2n-3}}\right\}$$

First we show that $(\P_7)$ and $(\P_8)$ hold. Suppose $\nu^{(7)}$ is reducible. Let $\mathcal{W}$ be a proper non trivial invariant subspace of $\mathcal{V}$, with $\mathcal{W}$ irreducible. Then $\mathcal{W}$ is an irreducible $\mathcal{H}_{F,r^2}(7)$-module. By Corollary 3, we have $\dim \mathcal{W} \in \{1, 6, 14, 15\}$ or $\dim \mathcal{W} > 15$. If $\dim \mathcal{W} = 1$, Theorem 4 implies that $l = \frac{1}{r^{11}}$. Also, if $\dim \mathcal{W} = 6$, Theorem 5 implies that $l \in \{\frac{1}{r^4}, -\frac{1}{r^4}\}$. Let $\mathcal{V}_0$ be the vector subspace of $\mathcal{V}$ over F, spanned by the w_{ik}'s for $1 \leq i < k \leq 6$. The vector space $\mathcal{V}_0$ is a $B(A_5)$-module with the action provided by the restriction of $\nu^{(7)}$ to $B(A_5)$, but is not a $B(A_6)$-module since for instance $g_6.w_{5,6} = w_{5,7} \notin \mathcal{V}_0$. In particular, $\mathcal{V}_0 \neq \mathcal{W}$. Further, if $\mathcal{V}_0 \subseteq \mathcal{W}$, then $x_{\alpha_1} \in \mathcal{W}$. By § **8.1** this would imply that $\mathcal{W} = \mathcal{V}$, which is impossible. Thus, considering the intersection of vector spaces $\mathcal{W} \cap \mathcal{V}_0$, we get:

$$0 \subseteq \mathcal{W} \cap \mathcal{V}_0 \subset \mathcal{V}_0$$

♣ Suppose $0 = \mathcal{W} \cap \mathcal{V}_0$. Since $\mathcal{W} \oplus \mathcal{V}_0 \subseteq \mathcal{V}$, we must have $\dim_F \mathcal{W} +$

$\dim_F \mathcal{V}_0 \leq \dim_F \mathcal{V}$,

$$i.e \qquad \dim_F \mathcal{W} \leq \binom{7}{2} - \binom{6}{2} = 6$$

In this situation, we get, as seen above $l \in \left\{\frac{1}{r^{11}}, -\frac{1}{r^4}, \frac{1}{r^4}\right\}$.

♣♣ Otherwise, we have $0 \subset \mathcal{W} \cap \mathcal{V}_0 \subset \mathcal{V}_0$.
Since $\mathcal{W}$ is also a $B(A_5)$-module with an action given by the restriction $\nu^{(7)} \downarrow_{B(A_5)}$, the vector space $\mathcal{W} \cap \mathcal{V}_0$ can be viewed as a $B(A_5)$-module. By forgetting the last vertical line in the tangles representing the spanning elements $w_{i,k}$'s, $1 \leq i < k \leq 6$ of $\mathcal{V}_0$, the inclusions above mean exactly that $\nu^{(6)}$ is reducible. Then, by the case $n = 6$, we get $l \in \{r, -r^3, \frac{1}{r^3}, -\frac{1}{r^3}, \frac{1}{r^9}\}$.

We now use the same technique to still get more informations, but this time with the F-vector space $\mathcal{V}_1 := \langle w_{i,k} | 1 \leq i < k \leq 5 \rangle_F$. With the action given by $\nu^{(7)} \downarrow_{B(A_4)}$ the vector subspace $\mathcal{V}_1$ of $\mathcal{V}$ is a $B(A_4)$-module. For the same reasons as above, it is impossible to have $\mathcal{V}_1 \subseteq \mathcal{W}$, hence we have:

$$0 \subseteq \mathcal{W} \cap \mathcal{V}_1 \subset \mathcal{V}_1$$

And there are again two cases:

♠ If $\mathcal{W} \cap \mathcal{V}_1 = \{0\}$, then $\dim \mathcal{W} \leq \binom{7}{2} - \binom{5}{2} = 11$. Then by the above, $\dim \mathcal{W} \in \{1, 6\}$ and $l \in \{\frac{1}{r^4}, -\frac{1}{r^4}, \frac{1}{r^{11}}\}$

♠♠ Otherwise, $0 \subset \mathcal{W} \cap \mathcal{V}_1 \subset \mathcal{V}_1$. By forgetting the last two vertical lines in the tangles representing the spanning elements $w'_{i,k}s$ $1 \leq i < k \leq 5$ of $\mathcal{V}_1$, we read on the inclusions above that $\nu^{(5)}$ is reducible. By the case $n = 5$, it yields $l \in \{r, -r^3, \frac{1}{r^2}, -\frac{1}{r^2}, \frac{1}{r^7}\}$.

Suppose now that $l \notin \{\frac{1}{r^4}, -\frac{1}{r^4}, \frac{1}{r^7}\}$. Then points ♣ and ♠ cannot happen. By points ♣♣ and ♠♠ we get:

$$\begin{cases} l \in \{r, -r^3, \frac{1}{r^3}, -\frac{1}{r^3}, \frac{1}{r^9}\} \\ \qquad\qquad \& \\ l \in \{r, -r^3, \frac{1}{r^2}, -\frac{1}{r^2}, \frac{1}{r^7}\} \end{cases}$$

As usual let ϵ take the value $+1$ or -1. We have the equivalences:

$$r^2 = \epsilon\, r^3 \quad \Leftrightarrow \quad r = \epsilon \tag{143}$$

$$r^2 = \epsilon\, r^9 \quad \Leftrightarrow \quad r^7 = \epsilon \tag{144}$$

$$r^7 = \epsilon\, r^3 \quad \Leftrightarrow \quad r^4 = \epsilon \tag{145}$$

$$r^7 = \quad r^9 \quad \Leftrightarrow \quad r^2 = 1 \tag{146}$$

Since $r^2 \neq 1$ as $m \neq 0$ and since $(r^2)^7 \neq 1$ and $(r^2)^4 \neq 1$ by semisimplicity of $\mathcal{H}_{F,r^2}(7)$, none of $(143) - (146)$ can occur. Thus, l must take the values r or $-r^3$.

We conclude that $l \in \{r, -r^3, \frac{1}{r^4}, -\frac{1}{r^4}, \frac{1}{r^{11}}\}$, as expected.

Let's now prove $(¶_8)$. Taking the same notations as before, suppose $\mathcal{V}$ is reducible and let $\mathcal{W}$ be an irreducible submodule of $\mathcal{V}$. By § **8.1** and the relations defining the BMW algebra, $\mathcal{W}$ is also an irreducible $\mathcal{H}_{F,r^2}(8)$-module. If dim$\mathcal{W} = 1$, then $l = \frac{1}{r^{13}}$ by Theorem 4 and if dim$\mathcal{W} = 7$, then $l \in \{\frac{1}{r^5}, -\frac{1}{r^5}\}$ by Theorem 5. Otherwise, dim$\mathcal{W} \geq 8$ by Proposition 3, a piece of the work of James in [6]. Still taking the same notations as before, we define the vector spaces $\mathcal{V}_0$ and $\mathcal{V}_1$:

$$\mathcal{V}_0 = \left\langle w_{i,k} | 1 \leq i < k \leq 7 \right\rangle_F$$
$$\mathcal{V}_1 = \left\langle w_{j,s} | 1 \leq j < s \leq 6 \right\rangle_F$$

If $\mathcal{W} \cap \mathcal{V}_0 = \{0\}$, then it forces $\dim \mathcal{W} \leq \binom{8}{2} - \binom{7}{2} = 7$, so that $l \in \{\frac{1}{r^{13}}, -\frac{1}{r^5}, \frac{1}{r^5}\}$ by the above. Otherwise, we have $0 \subset \mathcal{W} \cap \mathcal{V}_0 \subset \mathcal{V}_0$. If we forget about the last node (number eight), we see with these inclusions that $\nu^{(7)}$ is reducible. Then by ($\P_7$), it yields $l \in \{r, -r^3, \frac{1}{r^4}, -\frac{1}{r^4}, \frac{1}{r^{11}}\}$.
If $\mathcal{W} \cap \mathcal{V}_1 = \{0\}$, then it forces $\dim \mathcal{W} \leq \binom{8}{2} - \binom{6}{2} = 13$. At this point, we need to study wether or not there could be an irreducible $\mathcal{H}_{F,r^2}(8)$-module of dimension d with $8 \leq d \leq 13$. The answer to that question is no as shown by the table below. We referred to the Appendix table in [6]. James gives a polynomial lower bound in n for the dimension of an irreducible $K\, Sym(n)$-module depending on the shape $\bar{\mu} = (\mu_2, \mu_3, \dots)$ of the partition $\mu = (\mu_1, \mu_2, \dots) \vdash n$ of n and on the characteristic of the field. In characteristic zero, these bounds are reached for any n. We summarize the results in the case $n = 8$ and $char(K) = 0$ in the table below. In this case, a complete list of Specht modules of dimension greater or equal to 8 is:

$$S^{(6,2)},\ S^{(6,1,1)},\ S^{(5,3)},\ S^{(5,2,1)},\ S^{(5,1,1,1)},\ S^{(4,4)},\ S^{(4,3,1)},\ S^{(4,2,2)},\ S^{(4,2,1,1)},\ S^{(3,3,2)},$$

and their respective conjugates (when these are not self-conjugate), as illustrated in the table of our appendix. Hence we see that the different possibilities for $\bar{\mu}$ are:

$$(2),\ (1^2),\ (3),\ (2,1),\ (1^3),\ (4),\ (3,1),\ (2,1,1),\ (2,2),\ (3,2)$$

For each of these shapes corresponding to the rows of height ≥ 2 in the

Ferrers diagram, the dimension of the corresponding class of irreducible $K\,Sym(8)$-module is:

Specht module	$\bar{\mu}$	dimension
$S^{(6,2)}$	(2)	20
$S^{(6,1,1)}$	(1^2)	21
$S^{(5,3)}$	(3)	28
$S^{(5,2,1)}$	$(2,1)$	64
$S^{(5,1,1,1)}$	(1^3)	35
$S^{(4,4)}$	(4)	14
$S^{(4,3,1)}$	$(3,1)$	70
$S^{(4,2,1,1)}$	$(2,1^2)$	90
$S^{(4,2,2)}$	(2^2)	56
$S^{(3,3,2)}$	$(3,2)$	42

The last row of the array was obtained by the Hook formula as the appendix table of James gives results only for partitions $\bar{\mu} \vdash m$ with $m \leq 4$. From the table, we gather the information that the next smallest degree of an irreducible representation of the Iwahori-Hecke algebra $\mathcal{H}_{F,r^2}(8)$ is 14. Thus, dim$\mathcal{W} \leq 13$ implies in fact that dim$\mathcal{W} \in \{1, 7\}$, and this forces $l \in \{\frac{1}{r^{13}}, \frac{1}{r^5}, -\frac{1}{r^5}\}$ by Theorem 4 and Theorem 5. On the other hand, if $\mathcal{W} \cap \mathcal{V}_1 \neq \{0\}$, then it comes $0 \subset \mathcal{W} \cap \mathcal{V}_1 \subset \mathcal{V}_1$, the last inclusion holding for the same reasons as before. The F-vector space $\mathcal{W} \cap \mathcal{V}_1$ is a $B(A_5)$-module with the action given by $\nu^{(8)} \downarrow_{B(A_5)}$. When the BMW algebra $B(A_5)$ acts on the tangles representing the elements $w_{i,k}$'s, $1 \leq i \leq 6$ of $\mathcal{V}_1$, it leaves their last two vertical lines invariant. Hence, we may as well forget about them and we read on the inclusions above that $\nu^{(6)}$ is reducible. By the case $n = 6$, it follows that $l \in \{r, -r^3, \frac{1}{r^3}, -\frac{1}{r^3}, \frac{1}{r^9}\}$.

Suppose now that $l \notin \{\frac{1}{r^13}, \frac{1}{r^5}, -\frac{1}{r^5}\}$. Then, by the previous considerations, we get:

$$\begin{cases} l \in \{r, -r^3, \frac{1}{r^3}, -\frac{1}{r^3}, \frac{1}{r^9}\} \\ \qquad\qquad \& \\ l \in \{r, -r^3, \frac{1}{r^4}, -\frac{1}{r^4}, \frac{1}{r^{11}}\} \end{cases}$$

Again, we cannot have:

$$\begin{cases} r^3 &= \epsilon\, r^4 \\ r^9 &= r^{11} \end{cases}$$

because $r \notin \{1, -1\}$ as $m \neq 0$. Also, we cannot have:

$$\begin{cases} r^3 &= \epsilon\, r^{11} \\ r^9 &= \epsilon\, r^4 \end{cases}$$

since $(r^2)^4 \neq 1$ and $(r^2)^5 \neq 1$ by semisimplicity of $\mathcal{H}_{F,r^2}(8)$. Then, the only possibilities are: $l = r$ or $l = -r^3$. In summary, $l \in \{r, -r^3, \frac{1}{r^5}, -\frac{1}{r^5}, \frac{1}{r^{13}}\}$. This achieves the proof of $(¶_8)$.

We are now in a position to deal with the general case. Let n be any integer greater or equal to 9 and suppose that $(¶_{n-2})$ and $(¶_{n-1})$ hold. Suppose that $\nu^{(n)}$ is reducible and let $\mathcal{W}$ be a proper nontrivial invariant subspace of $\mathcal{V}$. The F-vector subspace $\mathcal{W}$ must satisfy to:

$$\begin{cases} 0 \subset \mathcal{W} \subset \mathcal{V} \\ \forall 1 \leq i \leq n-1,\ \nu_i(\mathcal{W}) \subseteq \mathcal{W} \end{cases} \qquad (\bigstar)$$

Without loss of generality, let's assume that $\mathcal{W}$ is irreducible. We have the lemma:

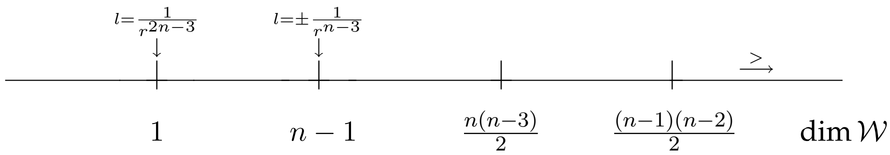
Lemma 6. *Let $\mathcal{G}$ be a F-vector subspace of $\mathcal{V}$. Then,*
$\mathcal{G}$ is a $B(A_{n-1})$-module if and only if $\mathcal{G}$ is a $\mathcal{H}_{F,r^2}(n)$-module.

PROOF OF THE LEMMA: Suppose first that $\mathcal{G}$ is a $B(A_{n-1})$-module. By § **8.1** and the defining relation (3) of the BMW algebra, the $B(A_{n-1})$-action on $\mathcal{G}$ is a Iwahori-Hecke algebra action by $\mathcal{H}_{F,r^2}(n)$ since $e_i.\mathcal{G} = 0$ for all $1 \leq i \leq n-1$. Conversely, if $\mathcal{G}$ is an $\mathcal{H}_{F,r^2}(n)$-module, it is obviously a $B(A_{n-1})$-module since the expression defining e_i is polynomial in g_i.

By the lemma, $\mathcal{W}$ is an irreducible $\mathcal{H}_{F,r^2}(n)$-module. Then by Corollary 3, we have $\dim \mathcal{W} \in \{1, n-1, \frac{n(n-3)}{2}, \frac{(n-1)(n-2)}{2}\}$ or $\dim \mathcal{W} > \frac{(n-1)(n-2)}{2}$.

$$\text{If } \dim \mathcal{W} = 1, \text{ then } l = \frac{1}{r^{2n-3}} \text{ by Theorem } 4.$$
$$\text{If } \dim \mathcal{W} = n-1, \text{ then } l \in \{\frac{1}{r^{n-3}}, -\frac{1}{r^{n-3}}\} \text{ by Theorem5.}$$

Otherwise, $\dim \mathcal{W} = \frac{n(n-3)}{2}$ or $\dim \mathcal{W} = \frac{(n-1)(n-2)}{2}$ or $\dim \mathcal{W} > \frac{(n-1)(n-2)}{2}$.



As for $n = 7$ and $n = 8$, let's define the two F-vector spaces:

$$\mathcal{V}_0 = \left\langle w_{i,k} | 1 \leq i < k \leq n-1 \right\rangle_F$$
$$\mathcal{V}_1 = \left\langle w_{j,s} | 1 \leq j < s \leq n-2 \right\rangle_F$$

of respective dimensions $\binom{n-1}{2}$ and $\binom{n-2}{2}$. If $\mathcal{W}$ contains $\mathcal{V}_0$ or $\mathcal{V}_1$, then $\mathcal{W}$ contains x_{α_1} and by §**8.1**, all the x_β's are in fact in $\mathcal{W}$. Then $\mathcal{W}$ is the whole space $\mathcal{V}$, which is impossible. Thus we have the inclusions of vector spaces:

$$0 \subseteq \mathcal{W} \cap \mathcal{V}_0 \subset \mathcal{V}_0$$
$$0 \subseteq \mathcal{W} \cap \mathcal{V}_0 \subset \mathcal{V}_1$$

If $\mathcal{W} \cap \mathcal{V}_0 = \{0\}$, then $\mathcal{W} \bigoplus \mathcal{V}_0$ and we have:

$$\begin{array}{lrcl} & \dim \mathcal{W} + \dim \mathcal{V}_0 & \leq & \dim \mathcal{V} \\ \textit{i.e} & \dim \mathcal{W} & \leq & \binom{n}{2} - \binom{n-1}{2} \\ \textit{i.e} & \dim \mathcal{W} & \leq & n-1 \end{array} \tag{147}$$

Similarly, if $\mathcal{W} \cap \mathcal{V}_1 = \{0\}$, then $\mathcal{W} \bigoplus \mathcal{V}_1$ and we get:

$$\begin{array}{lrcl} & \dim \mathcal{W} + \dim V_1 & \leq & \dim \mathcal{V} \\ \textit{i.e} & \dim \mathcal{W} & \leq & \binom{n}{2} - \binom{n-2}{2} \\ \textit{i.e} & \dim \mathcal{W} & \leq & 2n-3 \end{array} \tag{148}$$

Since the inequality $2n - 3 < \frac{n(n-3)}{2}$ holds as soon as $n \geq 7$, so in particular for any $n \geq 9$, we see in each case and as illustrated by the figure, that there are only two possibilities for the dimension of $\mathcal{W}$: either $\dim \mathcal{W} = 1$ or $\dim \mathcal{W} = n - 1$. Thus, each of the conditions (147) and (148) implies that $l \in \{\frac{1}{r^{n-3}}, -\frac{1}{r^{n-3}}, \frac{1}{r^{2n-3}}\}$. Suppose now that l does not take any of these values. Then we have $\mathcal{W} \cap \mathcal{V}_0 \neq \{0\}$ and $\mathcal{W} \cap \mathcal{V}_1 \neq \{0\}$, so that:

$$0 \subset \mathcal{W} \cap \mathcal{V}_0 \subset \mathcal{V}_0$$
$$0 \subset \mathcal{W} \cap \mathcal{V}_0 \subset \mathcal{V}_1$$

By definition of the representation, $\nu^{(n-1)}$ is a representation of $B(A_{n-2})$ in $\mathcal{V}_0$. Still by definition of the representation, for each $i = 1, \ldots, n-2$ and for all basis vector x_β in $\mathcal{V}_0$, the vectors $\nu^{(n-1)}(g_i)(x_\beta)$ and $\nu^{(n)}(g_i)(x_\beta)$ only depend on the inner product $(\beta|\alpha_i)$ and we have:

$$\nu^{(n-1)}(g_i)(x_\beta) = \nu^{(n)}(g_i)(x_\beta) = \nu_i(x_\beta)$$

Let $z \in \mathcal{W} \cap \mathcal{V}_0$. Then $\forall\, 1 \leq i \leq n-2,\ \nu^{(n-1)}(g_i)(z) = \nu^{(n)}(g_i)(z) \in \mathcal{W}$ since $\mathcal{W}$ is a $B(A_{n-1})$-submodule of $\mathcal{V}$. Thus,

$$\forall\ i = 1, \ldots, n-2,\ \nu^{(n-1)}(g_i)(\mathcal{W} \cap \mathcal{V}_0) \subseteq \mathcal{W} \cap \mathcal{V}_0$$

Then $\nu^{(n-1)}$ is reducible and $(\P_{n-1})$ implies that $l \in \{r, -r^3, \frac{1}{r^{n-4}}, -\frac{1}{r^{n-4}}, \frac{1}{r^{2n-5}}\}$. Similarly, $\nu^{(n-2)}$ is a representation of $B(A_{n-3})$ in $\mathcal{V}_1$ and we have:

$$\forall 1 \leq i \leq n-3,\ \forall z \in \mathcal{W} \cap \mathcal{V}_1,\ \nu^{(n-2)}(g_i)(z) = \nu_i(z) \in \mathcal{W} \text{ by } (\bigstar)$$

Thus, we have:

$$\forall i = 1, \ldots, n-3,\ \nu^{(n-2)}(g_i)(\mathcal{W} \cap \mathcal{V}_1) \subseteq \mathcal{W} \cap \mathcal{V}_1,$$

so that the representation $\nu^{(n-2)}$ is reducible. Then $(\P_{n-2})$ implies that

$$l \in \left\{r, -r^3, \frac{1}{r^{n-5}}, -\frac{1}{r^{n-5}}, \frac{1}{r^{2n-7}}\right\}$$

In summary, if $l \notin \{\frac{1}{r^{n-3}}, -\frac{1}{r^{n-3}}, \frac{1}{r^{2n-3}}\}$, then:

$$l \in \left\{r, -r^3, \frac{1}{r^{n-4}}, -\frac{1}{r^{n-4}}, \frac{1}{r^{2n-5}}\right\} \text{ and } l \in \left\{r, -r^3, \frac{1}{r^{n-5}}, -\frac{1}{r^{n-5}}, \frac{1}{r^{2n-7}}\right\}$$

At this stage, we recall that we made the assumption that $\mathcal{H}_{F,r^2}(n)$ is semisimple. By Corollary 3.44 in [10], the smallest integer e such that

$$1 + r^2 + \cdots + (r^2)^{e-1} = 0$$

($e = \infty$ if no such integer exists) must be greater than n. Since $m \neq 0$ implies $r^2 \neq 1$, this condition is in fact equivalent to: "the smallest integer

e, if it exists, such that $(r^2)^e = 1$ is greater than n". Thus, we must have:

$$(r^2)^k \neq 1, \, \forall k = 1, \ldots, n$$

We use this constraint on r to show that the two conditions on l above force $l \in \{r, -r^3\}$. Indeed, the condition

$$\begin{array}{c} l \in \{r, -r^3, \frac{1}{r^{n-4}}, -\frac{1}{r^{n-4}}, \frac{1}{r^{2n-5}}\} \\ \& \\ l \in \{r, -r^3, \frac{1}{r^{n-5}}, -\frac{1}{r^{n-5}}, \frac{1}{r^{2n-7}}\} \end{array}$$

and the impossibility to have any of the following equalities below

$$\begin{array}{ccccclccc} \frac{\epsilon}{r^{n-4}} & = & \frac{\epsilon'}{r^{n-5}} & \Leftrightarrow & r \in \{-1,1\} & \text{impossible as} & m & \neq & 0 \\ \frac{\epsilon}{r^{n-4}} & = & \frac{1}{r^{2n-7}} & \Leftrightarrow & r^{n-3} = \epsilon & \text{impossible as} & (r^2)^{n-3} & \neq & 1 \\ \frac{1}{r^{2n-5}} & = & \frac{\epsilon}{r^{n-5}} & \Leftrightarrow & r^n = \epsilon & \text{impossible as} & (r^2)^n & \neq & 1 \\ \frac{1}{r^{2n-5}} & = & \frac{1}{r^{2n-7}} & \Leftrightarrow & r^2 = 1 & \text{impossible as} & m & \neq & 0 \end{array}$$

imply that $l \in \{r, -r^3\}$. We conclude that if $\nu^{(n)}$ is reducible, then $l \in \{r, -r^3, \frac{1}{r^{n-3}}, -\frac{1}{r^{n-3}}, \frac{1}{r^{2n-3}}\}$. Hence,

$$(\P_{n-2})\&(\P_{n-1}) \Longrightarrow (\P_n)$$

Since $(\P_7)\&(\P_8)$ hold, $(\P_n)$ holds for all $n \geq 9$.

To complete the proof of the Theorem, it remains to show that for $l = r$ and $l = r^3$, the representation $\nu^{(n)}$ is reducible. We know from before that $\cap_{1 \leq i < j \leq n} Ker\, \nu(X_{ij})$ is a proper invariant subspace of $\mathcal{V}$. If we can show that for $l = r$ and for $l = r^3$, this subspace is non zero, then $\nu^{(n)}$ is reducible for each of these values of l. For $n = 4, 5$ and for each value of $l = r, -r^3$

we exhibited a non zero vector belonging to $\cap_{1\leq i<j\leq n} Ker\,\nu(X_{ij})$, which showed the reducibility of $\nu^{(n)}$ in these cases. For bigger values of n, we have the result:

Proposition 5. *Let n be an integer with $n \geq 5$.*

$$\begin{array}{lllrcl} \textit{If} & l & = & r, & \mathcal{X} := r^2\, w_{12} - r\, w_{13} + w_{34} - r\, w_{24} & \in \bigcap_{1\leq i<j\leq n} Ker\,\nu(X_{ij}) \\ \textit{If} & l & = & -r^3, & \mathcal{Y} := -r\, w_{23} - \frac{1}{r}\, w_{34} + w_{24} & \in \bigcap_{1\leq i<j\leq n} Ker\,\nu(X_{ij}) \end{array}$$

PROOF OF THE PROPOSITION: for $n = 5$, the vectors of the proposition are the ones annihilating the matrix S of § **5**. By construction, S is the matrix of the endomorphism $\sum_{1\leq i<j\leq n} \nu(X_{ij})$ in the basis $\mathcal{B}$. By Proposition 2 of § **8**.**1**, the matrix $Mat_{\mathcal{B}}\nu(X_{ij})$ is the matrix whose $\{(1+2+\cdots+j-2)+(j-i)\}$-th row is the one of $Mat_{\mathcal{B}}\sum_{1\leq i<j\leq n}\nu(X_{ij})$ and with zeros elsewhere. Hence Proposition 5 holds for $n = 5$. Moreover, to show in the general case that $\mathcal{X}, \mathcal{Y} \in \cap_{1\leq i<j\leq n} Ker\,\nu(X_{ij})$, it suffices to show that each row of the sum matrix annihilates $\mathcal{X}, \mathcal{Y}$. Further, since the last $\binom{n}{2} - 5$ coordinates of the vectors $\mathcal{X}$ and $\mathcal{Y}$ are zero, it suffices to check that the five first coordinates of each row of the sum matrix annihilate the column vectors $(r^2, 0, -r, 1, -r)$ and $(0, -r, 0, -\frac{1}{r}, 1)$. Furthermore, by the remark above, the first $\binom{5}{2}$ rows of the sum matrix correspond to the only nonzero row of the matrices $Mat_{\mathcal{B}}\nu(X_{i,j})$ for $1 \leq i < j \leq 5$. Since an expression for $\nu(X_{ij})$ is:

$$\nu(X_{ij}) = \nu_{j-1}\ldots\nu_{i+1}\nu(e_i)\nu_{i+1}^{-1}\ldots\nu_{j-1}^{-1},$$

and since $\nu_i^{-1}(x_\beta)$ only depends on the inner product $(\alpha_i|\beta)$ as in the expression of ν_i^{-1} below:

$$\nu_i^{-1}(x_\beta) = \begin{cases} \frac{1}{r}\, x_\beta & \text{if} \quad (\beta|\alpha_i) = 0 \\ l\, x_\beta & \text{if} \quad (\beta|\alpha_i) = 1 \\ x_{\beta-\alpha_i} - ml\, r^{ht(\beta)-2}\, x_{\alpha_i} + m\, x_\beta & \text{if} \quad (\beta|\alpha_i) = \frac{1}{2} \;\&\; \beta-\alpha_i \succ \alpha_i \\ x_{\beta-\alpha_i} & \text{if} \quad (\beta|\alpha_i) = \frac{1}{2} \;\&\; \beta-\alpha_i \prec \alpha_i \\ x_{\beta+\alpha_i} & \text{if} \quad (\beta|\alpha_i) = -\frac{1}{2} \;\&\; \beta \succ \alpha_i \\ x_{\beta+\alpha_i} - \frac{m}{r^{ht(\beta)-1}}\, x_{\alpha_i} + m\, x_\beta & \text{if} \quad (\beta|\alpha_i) = -\frac{1}{2} \;\&\; \beta \prec \alpha_i \end{cases},$$

the action of $\nu^{(n)}(X_{ij})$, $j \leq 5$ on $x_{\alpha_1}, x_{\alpha_2}, x_{\alpha_1+\alpha 2}, x_{\alpha_3}, x_{\alpha_3+\alpha_2}$ is the same as the action of $\nu^{(5)}(X_{ij})$, $j \leq 5$ on the same vectors. Thus, we only need to check that the five first coordinates of the last $\binom{n}{2} - \binom{5}{2}$ rows of the sum matrix annihilates the column vectors $(r^2, 0, -r, 1, -r)$ and $(0, -r, 0, -\frac{1}{r}, 1)$. In other words, we need to check that for all the BMW algebra elements X_{ij}'s with $1 \leq i < j \leq n$ and $j \geq 6$:

If $l = r$,

$$r^2\, [\nu(X_{ij})(x_{\alpha_1})]_{w_{ij}} - r\, [\nu(X_{ij})(x_{\alpha_1+\alpha_2})]_{w_{ij}} + [\nu(X_{ij})(x_{\alpha_3})]_{w_{ij}} - r\, [\nu(X_{ij})(x_{\alpha_3+\alpha_2})]_{w_{ij}} = 0 \quad (\divideontimes)$$

If $l = -r^3$,

$$-r\, [\nu(X_{ij})(x_{\alpha_2})]_{w_{ij}} - \frac{1}{r}\, [\nu(X_{ij})(x_{\alpha_3})]_{w_{ij}} + [\nu(X_{ij})(x_{\alpha_2+\alpha_3})]_{w_{ij}} = 0 \qquad (\divideontimes\divideontimes)$$

For a proof of these two equalities, we shall use the table of the appendix that gives a complete description of how the X_{ij}'s act on the w_{sk}'s or the straightforward computations below that use the conjugation formulas for the X_{ij}'s. First we have the lemma:

Lemma 7.

For all $1 \leq i < j-1 \leq n$ with $j \geq 6$ and $i \geq 5$, we have:

$$\forall \beta \in \{\alpha_1, \alpha_2, \alpha_2+\alpha_1, \alpha_3, \alpha_3+\alpha_2\},\ \nu(X_{ij})(x_\beta) = \nu_{j-1} \dots \nu_{i+1}\nu(e_i)\nu_{i+1}^{-1} \dots \nu_{j-1}^{-1}(x_\beta) = 0$$

For all $i \geq 5$, we have:

$$\forall \beta \in \{\alpha_1, \alpha_2, \alpha_2+\alpha_1, \alpha_3, \alpha_3+\alpha_2\},\ \nu(X_{i,i+1})(x_\beta) = \nu(e_i)(x_\beta) = 0$$

PROOF OF THE LEMMA: for $1 \leq i < j-1 \leq n$ and $j \geq 6$ and $i \geq 5$, it suffices to notice that:

$$\mathrm{Supp}(\beta) \cap \{i-1, i, i+1, \dots, j-1, j\} = \emptyset$$

Hence, the vector $\nu(e_i)\nu_{i+1}^{-1} \dots \nu_{j-1}^{-1}(x_\beta)$ is zero. As for the second equality, it simply comes from the fact that $(\beta|\alpha_i) = 0$. $\square$

In what follows, we shall denote by ϕ_1^+ the subset of ϕ^+ composed of the positive roots $\alpha_1, \alpha_2, \alpha_2+\alpha_1, \alpha_3, \alpha_3+\alpha_2$.

It remains to show $(\circledast)$ and $(\circledast\circledast)$ for $i \in \{1,2,3,4\}$ and $j \geq 6$. Let's start with $i = 4$. We have for all $\beta \in \phi_1^+$:

$$\begin{aligned}
\nu(X_{4j})(x_\beta) &= \nu_{j-1} \dots \nu_5 \nu(e_4)\nu_5^{-1} \dots \nu_{j-1}^{-1}(x_\beta) \\
&= \frac{1}{r^{j-5}}\, \nu_{j-1} \dots \nu_5 \nu(e_4)(x_\beta) \\
&= \begin{cases} 0 & \text{if } \beta \in \{\alpha_1, \alpha_2, \alpha_1+\alpha_2\} \\ \frac{1}{r^{j-5}}\nu_{j-1} \dots \nu_5(x_{\alpha_4}) & \text{if } \beta = \alpha_3 \\ \frac{1}{r^{j-5}}\nu_{j-1} \dots \nu_5(x_{\alpha_4}).\frac{1}{r} & \text{if } \beta = \alpha_3+\alpha_2 \end{cases}
\end{aligned}$$

By the game of the coefficients in $(\circledast)$ and $(\circledast\circledast)$, we conclude that $(\circledast)$ and $(\circledast\circledast)$ hold when $(i,j) = (4,j)$ with $j \geq 6$.

Next, we have for all $\beta \in \phi_1^+$ and all $j \geq 6$:

$$
\begin{aligned}
\nu(X_{3j})(x_\beta) &= \nu_{j-1}\dots\nu_4\nu(e_3)\nu_4^{-1}\dots\nu_{j-1}^{-1}(x_\beta) \\
&= \frac{1}{r^{j-5}}\,\nu_{j-1}\dots\nu_4\nu(e_3)\nu_4^{-1}(x_\beta) \\
&= \begin{cases}
0 & \text{if } \beta = \alpha_1 \\
\frac{1}{r^{j-4}} \quad \nu_{j-1}\dots\nu_4(x_{\alpha_3}) & \text{if } \beta = \alpha_2 \\
\frac{1}{r^{j-3}} \quad \nu_{j-1}\dots\nu_4(x_{\alpha_3}) & \text{if } \beta = \alpha_2+\alpha_1 \\
\frac{1}{r^{j-5}}\frac{1}{l} \quad \nu_{j-1}\dots\nu_4(x_{\alpha_3}) & \text{if } \beta = \alpha_3 \\
\frac{1}{r^{j-5}}\, m\left(\frac{1}{l}-\frac{1}{r}\right) \quad \nu_{j-1}\dots\nu_4(x_{\alpha_3}) & \text{if } \beta = \alpha_3+\alpha_2
\end{cases}
\end{aligned}
$$

When $l = r$, the last term above is zero. Thus, the only nonzero terms in $(\circledast)$ are $-r\,\frac{1}{r^{j-3}}\nu_{j-1}\dots\nu_4(x_{\alpha_3})$ and $\frac{1}{r^{j-5}}\frac{1}{r}\nu_{j-1}\dots\nu_4(x_{\alpha_3})$ and they cancel each other. As for $(\circledast\circledast)$ when $l = -r^3$, all the powers in $\frac{1}{r}$ cancel each other, as in the equation:

$$
-r\,\frac{1}{r^{j-4}} + \frac{1}{r}\,\frac{1}{r^{j-5}}\,\frac{1}{r^3} + \frac{1}{r^{j-5}}\Big(\frac{1}{r}-r\Big)\Big(-\frac{1}{r^3}-\frac{1}{r}\Big) = 0
$$

More computations show that for all $\beta \in \phi_1^+$ and for all $j \geq 6$,

$$
\begin{aligned}
\nu(X_{2j})(x_\beta) &= \nu_{j-1}\dots\nu_3\nu(e_2)\nu_3^{-1}\dots\nu_{j-1}^{-1}(x_\beta) \\
&= \frac{1}{r^{j-5}}\,\nu_{j-1}\dots\nu_3\nu(e_2)\nu_3^{-1}\nu_4^{-1}(x_\beta) \\
&= \begin{cases}
\frac{1}{r^{j-3}} \quad \nu_{j-1}\dots\nu_3(x_{\alpha_2}) & \text{if } \beta = \alpha_1 \\
\frac{1}{r^{j-4}}\cdot\frac{1}{l} \quad \nu_{j-1}\dots\nu_3(x_{\alpha_2}) & \text{if } \beta = \alpha_2 \\
\frac{1}{r^{j-4}}\, m\left(\frac{1}{l}-\frac{1}{r}\right) \quad \nu_{j-1}\dots\nu_3(x_{\alpha_2}) & \text{if } \beta = \alpha_2+\alpha_1 \\
0 & \text{if } \beta = \alpha_3 \\
\frac{1}{r^{j-5}}\cdot\frac{1}{l} \quad \nu_{j-1}\dots\nu_3(x_{\alpha_2}) & \text{if } \beta = \alpha_3+\alpha_2
\end{cases}
\end{aligned}
$$

The zero comes from the fact that:

$$\nu_3^{-1}\nu_4^{-1}(x_{\alpha_3}) = x_{\alpha_4} \tag{149}$$

Moreover, if $l = r$ the third row is also zero. Since

$$r^2\,\frac{1}{r^{j-3}} - r\,\frac{1}{r^{j-5}}\,\frac{1}{r} = 0,$$

we see that the first and the last term on the left hand side of the equality ($\divideontimes$) cancel each other. Also, the fact that:

$$r\,\frac{1}{r^{j-4}}\,\frac{1}{r^3} - \frac{1}{r^{j-5}}\,\frac{1}{r^3} = 0$$

and the computations above imply that ($\divideontimes\divideontimes$) holds.

Let's now study the action of $\nu(e_1)\nu_2^{-1}\nu_3^{-1}\nu_4^{-1}$ on an x_β with $\beta \in \phi_1^+$.

We have:

$$
\begin{aligned}
\nu(e_1)\nu_2^{-1}(x_{\alpha_1}) &= \nu(e_1)(x_{\alpha_2+\alpha_1} - m\,x_{\alpha_2} + m\,x_{\alpha_1}) \\
&= \frac{1}{l}\,x_{\alpha_1} \\
\nu(e_1)\nu_2^{-1}\nu_3^{-1}(x_{\alpha_2}) &= \nu(e_1)\nu_2^{-1}(x_{\alpha_2+\alpha_3} - m\,x_{\alpha_3} + m\,x_{\alpha_2}) \\
&= \nu(e_1)(x_{\alpha_3}) \\
&= 0 \\
\nu(e_1)\nu_2^{-1}\nu_3^{-1}(x_{\alpha_2+\alpha_1}) &= \nu(e_1)\nu_2^{-1}(x_{\alpha_3+\alpha_2+\alpha_1} - \frac{m}{r}\,x_{\alpha_3} + m\,x_{\alpha_2+\alpha_1}) \\
&= \nu(e_1)(\frac{1}{r}\,x_{\alpha_3+\alpha_2+\alpha_1} - \frac{m}{r}\,x_{\alpha_2+\alpha_3} + m\,x_{\alpha_1}) \\
&= \frac{1}{l}\,x_{\alpha_1} \\
\nu(e_1)\nu_2^{-1}\nu_3^{-1}\nu_4^{-1}(x_{\alpha_3}) &= \nu(e_1)\nu_2^{-1}(x_{\alpha_4}) \ \text{ by } (149) \\
&= \frac{1}{r}\,\nu(e_1)(x_{\alpha_4}) \\
&= 0 \\
\nu(e_1)\nu_2^{-1}\nu_3^{-1}\nu_4^{-1}(x_{\alpha_3+\alpha_2}) &= \nu(e_1)\nu_2^{-1}\nu_3^{-1}(x_{\alpha_4+\alpha_3+\alpha_2} - \frac{m}{r}\,x_{\alpha_4} + m\,x_{\alpha_3+\alpha_2}) \\
&= \nu(e_1)\nu_2^{-1}(\frac{1}{r}\,x_{\alpha_4+\alpha_3+\alpha_2} - \frac{m}{r}x_{\alpha_3+\alpha_4} + m\,x_{\alpha_2}) \\
&= \frac{1}{r}\,\nu(e_1)(x_{\alpha_3+\alpha_4}) \\
&= 0
\end{aligned}
$$

Therefore, we have for all $\beta \in \phi_1^+$ and all $j \geq 6$:

$$
\begin{array}{rcl}
\nu(X_{1j})(x_\beta) & = & \nu_{j-1}\dots\nu_2\nu(e_1)\nu_2^{-1}\dots\nu_{j-1}^{-1}(x_\beta) \\
 & = & \dfrac{1}{r^{j-5}}\,\nu_{j-1}\dots\nu_2\nu(e_1)\nu_2^{-1}\nu_3^{-1}\nu_4^{-1}(x_\beta) \\
 & = & \left\{\begin{array}{llrcl}
\frac{1}{r^{j-5}}\,\frac{1}{r^2}\,\frac{1}{l} & \nu_{j-1}\dots\nu_2(x_{\alpha_1}) & \text{if } \beta & = & \alpha_1 \\
0 & & \text{if } \beta & = & \alpha_2 \\
\frac{1}{r^{j-5}}\,\frac{1}{r}\,\frac{1}{l} & \nu_{j-1}\dots\nu_2(x_{\alpha_1}) & \text{if } \beta & = & \alpha_2+\alpha_1 \\
0 & & \text{if } \beta & = & \alpha_3 \\
0 & & \text{if } \beta & = & \alpha_3+\alpha_2
\end{array}\right.
\end{array}
$$

Then, we see that all the terms of the left hand side of equality (⁂⁂) are zero. And we read on the first expression and on the third expression above that

$$r^2\left[\nu(X_{1,j})(x_{\alpha_1})\right]_{w_{1j}} - r\left[\nu(X_{1,j})(x_{\alpha_1+\alpha_2})\right]_{w_{1j}} = 0$$

Then (⁂) also holds. This achieves the proof of Proposition 5 and the conjecture is thus proven when the Iwahori-Hecke algebra $\mathcal{H}_{F,r^2}(n)$ is semisimple.

Chapter 9

More Properties of the Representation when $\mathbf{l \in \{r, -r^3\}}$

In this part we still assume that the Iwahori-Hecke algebra $\mathcal{H}_{F,r^2}(n)$ for the adequate integer n is semisimple and we show more properties of the representation $\nu^{(n)}$. We focus on the cases when the representation is reducible with $l \in \{r, -r^3\}$. We introduce new notations: we will denote by $K(n)$ the intersection module

$$K(n) := \bigcap_{1 \le i < j \le n} Ker\, \nu^{(n)}(X_{ij}),$$

and by $k(n)$ its dimension as a vector space over F. We first study some properties of reducibility or irreducibility of $K(n)$ depending on the values of l and r.

9.1 Proprieties of Reducibility/Irreducibility of the $\mathrm{K}(\mathrm{n})$'s

We have the Proposition:

Proposition 6.

Let n be an integer with $n \geq 4$.

When $l = r$, $K(n)$ is always irreducible.

Let n be an integer with $n \geq 3$.

When $l = -r^3$ and $n \neq 8$, there are two cases:

$$\begin{cases} r^{2n} \neq -1 & \textit{and } K(n) \textit{ is irreducible} \\ r^{2n} = -1 & K(n) \textit{ is reducible and } k(n) \geq \frac{(n-1)(n-2)}{2} \end{cases}$$

(Case n=8) When $l = -r^3$, there are two cases:

$$\begin{cases} r^{16} \neq -1 & \textit{and } K(8) \textit{ is irreducible} \\ r^{16} = -1 & K(8) \textit{ is reducible and } k(8) \in \{15, 21, 22\} \end{cases}$$

PROOF:

- $\star$ Let's first assume that $n \geq 5$ and $n \neq 8$. Suppose $l = r$. The case $l = -r^3$ will be treated separately later on. We know that for $n \geq 5$ and $n \neq 8$, the irreducible $\mathcal{H}_{F,r^2}(n)$-modules have dimension 1, $n-1$, $\frac{n(n-3)}{2}$, $\frac{(n-1)(n-2)}{2}$ or dimension greater than $\frac{(n-1)(n-2)}{2}$. Suppose $K(n)$ is reducible. By semisimplicity of $\mathcal{H}_{F,r^2}(n)$, the $\mathcal{H}_{F,r^2}(n)-$ module $K(n)$ would decompose as a direct sum $K_1(n) \bigoplus K_2(n)$ with $K_1(n)$ irreducible. With the same notations as before, let's write $k_1(n)$ for the dimension of $K_1(n)$ over F and $k_2(n)$ for the dimension of

$K_2(n)$ over F. Suppose now $k_1(n) \geq \frac{(n-1)(n-2)}{2}$. Then it comes:

$$k_2(n) < \frac{n(n-1)}{2} - \frac{(n-1)(n-2)}{2} = n-1$$

By Theorem 4, this would force $l = \frac{1}{r^{2n-3}}$, which contradicts our assumption $l = r$. Furthermore by theorem 5, it is impossible to have $k_1(n) = n-1$ and by theorem 4, it is also impossible to have $k_1(n) = 1$. Then, the only remaining possibility for $k_1(n)$ is $k_1(n) = \frac{n(n-3)}{2}$. It follows that:

$$k_2(n) + \frac{n(n-3)}{2} < \frac{n(n-1)}{2},$$

$$i.e \qquad\qquad k_2(n) < n \leq \frac{n(n-3)}{2}$$

This last set of inequalities now forces $k_2(n) \in \{1, n-1\}$, which is again a contradiction. Hence $K(n)$ is irreducible in this case.

⋆ Let's now deal with the case $n = 3$. Here we assume that $l = -r^3$. We have $k(3) \in \{1, 2\}$.

  * If $r^6 \neq -1$.

    Suppose $k(3) = 2$. Since $l = -r^3$, we know that there exists a one-dimensional invariant subspace of $\mathcal{V}$. Theorem 4 also states that this one-dimensional invariant subspace is unique with our assumption on r. Moreover, it is contained in $K(3)$. Then it must have a one-dimensional summand, which is impossible by uniqueness of the one-dimensional invariant subspace in that case. Hence we must have $k(3) = 1$. Then $K(3)$ is irreducible and as a matter of fact, $K(3)$ is the one-dimensional invariant subspace of $\mathcal{V}$.

* If $r^6 = -1$.

  Then, by Theorem 4, there exists two one-dimensional invariant subspaces of $\mathcal{V}$ whose sum is direct. Since these spaces are distinct, $k(3)$ cannot equal 1. Hence $k(3) = 2$, $K(3)$ is reducible and $K(3)$ is a direct sum of these two one-dimensional invariant subspaces.

⋆ The case $n = 4$.

First we prove the following Theorem:

Result 2.

Suppose $\nu^{(4)}$ is reducible and let $\mathcal{W}$ be an irreducible submodule of $\mathcal{V}$. If dim $\mathcal{W} = 2$, then $l = r$ and $\mathcal{W}$ is spanned over F by the vectors:

$$(w_{13} - \tfrac{1}{r}\, w_{23}) - \tfrac{1}{r}\,(w_{14} - \tfrac{1}{r}\, w_{24})$$
$$(w_{12} - \tfrac{1}{r}\, w_{13}) - \tfrac{1}{r}\,(w_{24} - \tfrac{1}{r}\, w_{34})$$

Conversely, if $l = r$, the two linearly independent vectors above are stable under the action of g_1, g_2, g_3.

PROOF: By lemma 6, $\mathcal{W}$ is an irreducible 2-dimensional $\mathcal{H}_{F,r^2}(4)$-module. Let's set H_1, H_2 and H_3 to be the matrices:

$$H_1 := \begin{bmatrix} -\frac{1}{r} & 1 \\ 0 & r \end{bmatrix},\ H_2 := \begin{bmatrix} r & 0 \\ 1 & -\frac{1}{r} \end{bmatrix},\ H_3 := \begin{bmatrix} -\frac{1}{r} & 1 \\ 0 & r \end{bmatrix}$$

By the symmetry of the roles played by the scalars r and $-\frac{1}{r}$ and the choice $H_3 = H_1$, it suffices to check that:

$$\begin{array}{l} H_1^2 + m\,H_1 = I \\ H_1 H_2 H_1 = H_2 H_1 H_2 \end{array},$$

(where I is the identity matrix of size 2) to see that these matrices define a matrix representation of degree 2 of the Iwahori-Hecke algebra $\mathcal{H}_{F,r^2}(4)$. Furthermore, this representation is irreducible. And indeed, if the representation were reducible, there would exist a nonzero vector $v = (v_1, v_2)$ of F^2 and scalars λ_1 and λ_2 such that:

$$\left\{ \begin{array}{lcl} H_1.v & = & \lambda_1\,v \\ H_2.v & = & \lambda_2\,v \end{array} \right.$$

By using the definitions for H_1, we get from the first equation:

$$\left\{ \begin{array}{ccc} -\frac{1}{r}\,v_1 + v_2 & = & \lambda_1\,v_1 \\ r\,v_2 & = & \lambda_1\,v_2 \end{array} \right.,$$

from which we derive the new set of equations:

$$\left\{ \begin{array}{ccc} (r + \frac{1}{r})\,v_1 & = & v_2 \\ \lambda_1 & = & r \end{array} \right. \quad \text{or} \quad \left\{ \begin{array}{ccc} v_2 & = & 0 \\ \lambda_1 & = & -\frac{1}{r} \end{array} \right.$$

Since it is visible on the matrices that in $H_2.v = \lambda_2\,v$, the scalar v_1 has been replaced by the scalar v_2, we also get the set of equations:

$$\left\{ \begin{array}{ccc} (r + \frac{1}{r})\,v_2 & = & v_1 \\ \lambda_2 & = & r \end{array} \right. \quad \text{or} \quad \left\{ \begin{array}{ccc} v_1 & = & 0 \\ \lambda_2 & = & -\frac{1}{r} \end{array} \right.$$

where v_1 has been replaced by v_2 and λ_1 by λ_2.

From there, we see that it is impossible to have $v_2 = 0$ or $v_1 = 0$ because since $(r^2)^2 \neq 1$, this would yield $v_1 = v_2 = 0$. Hence the braces on the right hand side are to be excluded. Now we get:

$$v_1 = \left(r + \frac{1}{r}\right)^2 v_1 \text{ and } v_1 \neq 0,$$

so that $\left(r + \frac{1}{r}\right)^2 = 1$,

$$\textit{i.e} \qquad r + \frac{1}{r} = 1 \text{ or } r + \frac{1}{r} = -1$$

Let's solve the quadratics $r^2 - r + 1 = 0$ and $r^2 + r + 1 = 0$. Both have discriminant $(i\sqrt{3})^2$. Thus we get the solutions:

$$r \in \left\{ \frac{1}{2} + i\frac{\sqrt{3}}{2}, \frac{1}{2} - i\frac{\sqrt{3}}{2}, -\frac{1}{2} + i\frac{\sqrt{3}}{2}, -\frac{1}{2} - i\frac{\sqrt{3}}{2} \right\}$$

Equivalently,

$$r \in \{e^{i\frac{\pi}{3}}, e^{-i\frac{\pi}{3}}, e^{2i\frac{\pi}{3}}, e^{4i\frac{\pi}{3}}\}$$

Then it comes:

$$r^2 \in \{e^{2i\frac{\pi}{3}}, e^{-2i\frac{\pi}{3}}, e^{4i\frac{\pi}{3}}, e^{8i\frac{\pi}{3}}\}$$

Now it is visible that $(r^2)^3$=1, which is forbidden. Thus, the so given representation of $\mathcal{H}_{F,r^2}(4)$ of degree 2 is irreducible. Then there exists a basis (v_1, v_2) in $\mathcal{W}$ such that:

$$\begin{cases} \nu_1\, v_1 = -\frac{1}{r}\, v_1 & (\circledast)_1 \\ \nu_1\, v_2 = v_1 + r\, v_2 & (\circledast)_2 \end{cases} , \quad \begin{cases} \nu_2\, v_1 = r\, v_1 + v_2 & (\circledast)_3 \\ \nu_2\, v_2 = -\frac{1}{r}\, v_2 & (\circledast)_4 \end{cases} , \quad \begin{cases} \nu_3\, v_1 = -\frac{1}{r}\, v_1 & (\circledast)_5 \\ \nu_3\, v_2 = v_1 + r\, v_2 & (\circledast)_6 \end{cases}$$

By $(\circledast)_1$ (resp $(\circledast)_5$), there is no term in w_{34} (resp w_{12}) in v_1. Similarly, by $(\circledast)_4$, there is no term in w_{14} in v_2. Next, let the λ_{ij}'s be the coefficients of the w_{ij}'s in v_1 and let the μ_{ij}'s be the coefficients of the w_{ij}'s in v_2. By $(\circledast)_1$ and (77) applied with $q = 1$ and $\lambda = -\frac{1}{r}$, we get:

$$\lambda_{23} = -\frac{1}{r}\,\lambda_{13} \;\;\&\;\; \lambda_{24} = -\frac{1}{r}\,\lambda_{14}$$

By $(\circledast)_5$ and (78) applied with $q = 3$ and $\lambda = -\frac{1}{r}$, we get:

$$\lambda_{24} = -\frac{1}{r}\,\lambda_{23} \;\;\&\;\; \lambda_{14} = -\frac{1}{r}\,\lambda_{13}$$

By $(\circledast)_4$ and respectively (77) and (78) applied with $q = 2$ and $\lambda = -\frac{1}{r}$, we get:

$$\mu_{34} = -\frac{1}{r}\,\mu_{24} \;\;\&\;\; \mu_{13} = -\frac{1}{r}\,\mu_{12}$$

Gathering these relations between the coefficients, we obtain:

$$\begin{array}{lcl} v_1 & = & w_{13} - \frac{1}{r}\,w_{23} + \frac{1}{r^2}\,w_{24} - \frac{1}{r}\,w_{14} \\ v_2 & \alpha & w_{12} - \frac{1}{r}\,w_{13} + \mu\,(w_{24} - \frac{1}{r}\,w_{34}) + \mu^{'}\,w_{23} \end{array}$$

where μ and $\mu^{'}$ are scalars to determine. For that we use the mixed relations $(\circledast)_2$, $(\circledast)_3$, $(\circledast)_6$. First and foremost, $(\circledast)_3$ sets the coefficient of proportionality of v_2 to be one. Also, by looking at the coefficient of w_{24} in $(\circledast)_3$, we get $\mu = -\frac{1}{r}$. Further, by looking at the coefficient of w_{23} in $(\circledast)_6$ and replacing μ by its value, we get:

$$-\frac{1}{r} = -\frac{1}{r} + r\,\mu^{'}$$

It follows that $\mu^{'} = 0$. Thus, if $\mathcal{W}$ is an irreducible two-dimensional invariant subspace of $\mathcal{V}$, then it is spanned by the vectors

$$v_1 = w_{13} - \frac{1}{r}\,w_{23} + \frac{1}{r^2}\,w_{24} - \frac{1}{r}\,w_{14} \qquad (150)$$

$$v_2 = w_{12} - \frac{1}{r}\,w_{13} - \frac{1}{r}\,w_{24} + \frac{1}{r^2}\,w_{34} \qquad (151)$$

Furthermore, looking at the coefficient of w_{12} in $(\circledast)_2$ yields:

$$\frac{1}{l} - m = r,$$

from which we derive $\quad \frac{1}{l} = \frac{1}{r} \quad$ *i.e* $\quad \boxed{l = r}$

Conversely, if $l = r$, it is a direct verification that the vectors v_1 and v_2 defined by (150) and (151) satisfy to all the relations $(\circledast)_i$ with $i = 1, \dots, 6$. Thus, those linearly independent vectors span an irreducible two-dimensional invariant subspace of $\mathcal{V}$.
We have the immediate corollary:

Corollary 4. *Let $n = 4$. Then, there exists an irreducible 2-dimensional invariant subspace of $\mathcal{V}$ if and only if $l = r$.*

PROOF: contained in the above.

Aparte: we note at this stage that for $l \in \{\frac{1}{r}, -\frac{1}{r}\}$, the freedom of the family of vectors (v_1, v_2, v_3) of Theorem 6 is a direct consequence of Theorem 4 and Corollary 4. Indeed, if the family (v_1, v_2, v_3) is not free, then $\dim Span_F(v_1, v_2, v_3) = 1$ or $\dim Span_F(v_1, v_2, v_3) = 2$. In the first case, we get $l = \frac{1}{r^5}$ by Theorem 4, which is impossible. In the second case, the existence of an irreducible 2-dimensional invariant subspace would force $l = r$, also impossible. We note that the same argument does not hold for the spanning vectors of an irreducible 3-

dimensional invariant subspace when $l = -r^3$ since we could have $-r^3 = \frac{1}{r^5}$. Freedom needs to be shown by hands in that case.

We have a corollary of Corollary 4:

Corollary 5. *When $l = r$, the intersection module $K(4)$ is the irreducible 2-dimensional invariant subspace of $\mathcal{V}$.*

PROOF: by Result 2, an irreducible 2-dimensional invariant subspace of $\mathcal{V}$ when it exists must be unique. Also, as a matter of fact, it must be contained in $K(4)$. Suppose $l = r$ and $k(4) > 2$. Then $k(4) \in \{3, 4, 5\}$. Assume first $k(4) = 3$. If $K(4)$ were not irreducible, it would contain a 2-dimensional submodule that has a one-dimensional summand. Then $l = \frac{1}{r^5} = r$, i.e $(r^2)^3 = 1$: absurd. So $K(4)$ is irreducible, 3-dimensional and $l \in \{\frac{1}{r}, -\frac{1}{r}, -r^3\}$ by Theorem 6. None of these possibilities is compatible with $l = r$. Hence $k(4) \neq 3$. If $k(4) = 4$, then $K(4)$ is not irreducible as its dimension is not 1, 2 or 3. Then by uniqueness in Result 2 it must be a direct sum of a 1-dimensional submodule and an irreducible 3-dimensional one. This is again impossible. The other remaining possibility is $k(4) = 5$. Then the only possibility for $K(4)$ is to decompose as a direct sum of an irreducible 3-dimensional invariant subspace and an irreducible two-dimensional one, which for the same reasons as above has to be excluded. We must conclude that $k(4) = 2$, which achieves the proof of the corollary.

Let's go back to the proof of Proposition 6 for $n = 4$. First, by Corollary 5, half of the work is done in the case $n = 4$. Indeed, the corollary says that when $l = r$, the submodule $K(4)$ of $\mathcal{V}$ is irreducible. When $l = -r^3$, we know from Theorem 6 of the existence of a (unique) irreducible 3-dimensional invariant subspace of $\mathcal{V}$. Then $k(4) \geq 3$. We attempt to exclude turn by turn the two possibilities $k(4) = 4$ and $k(4) = 5$ and summarize the usual arguments in the following table:

k(4)	K(4)	value for l	Contradiction
4	$3 \bigoplus 1$	$l = \frac{1}{r^5} = -r^3$	NONE, apparently
5	$3 \bigoplus 2$	$l = r = -r^3$	$(r^2)^2 \neq 1$

We need to investigate further in the case $k(4) = 4$. If $r^8 \neq -1$, then it is impossible to have $k(4) = 4$. Since it is also impossible to have $k(4) = 5$, it comes $k(4) = 3$ and $K(4)$ is irreducible. If on the contrary $r^8 = -1$, then we have $l = -r^3 = \frac{1}{r^5}$. The fact that $l = \frac{1}{r^5}$ forces the existence of a one-dimensional invariant subspace of $\mathcal{V}$. Since we have seen that $k(4) \geq 3$, this one-dimensional invariant subspace is not $K(4)$, but is contained in $K(4)$. Then by semisimplicity of $\mathcal{H}_{F,r^2}(4)$, it has a summand in $K(4)$. This summand cannot be two-dimensional (otherwise $l = r$, impossible with $l = -r^3$). Since by the table above, $k(4) < 5$, this summand must in fact be 3-dimensional. It forces $k(4) = 4$ and $K(4)$ is a direct sum of the unique one-dimensional invariant subspace of $\mathcal{V}$ and of the unique irreducible 3-dimensional invariant subspace of $\mathcal{V}$. This terminates the case $n = 4$.

$\star$ Let's go back to the general case with $n \geq 5$ and $n \neq 8$ when $l = -r^3$. First if $r^{2n} \neq -1$ then it is impossible to have $l = -r^3 = \frac{1}{r^{2n-3}}$. Thus, by the same arguments as in the case $l = r$, $K(n)$ is irreducible. If now $r^{2n} = -1$, we have $l = -r^3 = \frac{1}{r^{2n-3}}$. By Theorem 4, there exists a one-dimensional invariant subspace in $\mathcal{V}$. Since by Proposition 5, the intersection $K(n) \cap \mathcal{V}_0$ is non-trivial as the vector $\mathcal{Y}$ is in $K(n)$ for every $n \geq 5$ and since by choice of l and r and the previous study, the module $K(n-1)$ is irreducible, we have the inclusions:

$$0 \subset K(n) \cap \mathcal{V}_0 \subseteq K(n)$$
$$0 \subset K(n) \cap \mathcal{V}_0 = K(n-1)$$

If $k(n) = 1$, then we must have $K(n) \cap \mathcal{V}_0 = K(n) = K(n-1)$. But still by choice of l and r, $K(n-1)$ cannot be one-dimensional, hence a contradiction. Thus it is impossible to have $k(n) = 1$ and the one-dimensional invariant subspace of $\mathcal{V}$, say $\mathcal{V}_{1,\frac{1}{r^{2n-3}}}$ must be strictly contained in $K(n)$. This proves that $K(n)$ is reducible. Moreover, by semisimplicity of $\mathcal{H}_{F,r^2}(n)$, the module $\mathcal{V}_{1,\frac{1}{r^{2n-3}}}$ has a summand in $K(n)$. This summand cannot be or contain an irreducible $(n-1)$-dimensional invariant subspace (otherwise $l \in \{\frac{1}{r^{n-3}}, -\frac{1}{r^{n-3}}\}$ by Theorem 5, a contradiction with $l = -r^3$). Since

$$\frac{n(n-1)}{2} - \frac{(n-1)(n-2)}{2} = n-1,$$

this summand must be irreducible, of dimension $\frac{n(n-3)}{2}$, $\frac{(n-1)(n-2)}{2}$ or dimension greater than $\frac{(n-1)(n-2)}{2}$. Hence $k(n) \geq \frac{(n-1)(n-2)}{2}$.

⋆ The case $n = 8$.

If $\mathcal{W}$ is an irreducible submodule of $\mathcal{V}$, then

$$\dim \mathcal{W} \in \{1, 7, 14, 20, 21\}$$

* <u>Assume $l = r$</u>

The argument is the same as in the first general case, except that there is one more possibility for a dimension between 7 and 21. Explicitly, suppose $K(8)$ is reducible. Since we assumed that $\mathcal{H}_{F,r^2}(8)$ is semisimple, there exists $K_1(8)$ and $K_2(8)$ submodules of $K(8)$ such that $K(8) = K_1(8) \bigoplus K_2(8)$. Without loss of generality, $K_1(8)$ is irreducible. Like in the first case, if $k_1(8) = 21$, then $k_2(8) < 28 - 21 = 7$. Then $K_2(8)$ is irreducible and one-dimensional. It follows that $l = \frac{1}{r^{13}}$ by Theorem 4. Since $(r^2)^7 \neq 1$, we get a contradiction. Also, since $(r^2)^6 \neq 1$ and $(r^2)^7 \neq 1$, we cannot have $k_1(8) \in \{1, 7\}$. Hence $k_1(8) \in \{14, 20\}$. Suppose first $k_1(8) = 14$. Then $k_2(8) < 28 - 14 = 14$. This is impossible by the same arguments as before. On the other hand, if $k_1(8) = 20$, then $k_2(8) < 28 - 20 = 8$. Then $K_2(8)$ is irreducible and has dimension 1 or 7, which is impossible. We conclude that $K(8)$ is irreducible when $l = r$.

* <u>Assume $l = -r^3$</u>

  a) If $r^{16} \neq -1$ then $\frac{1}{r^{13}} \neq -r^3$. Since we also have $(r^2)^8 \neq 1$, the same arguments as for $l = r$ yield the irreducibility of $K(8)$ in that case.

  b) If $r^{16} = -1$, then $l = -r^3 = \frac{1}{r^{13}}$. By Theorem 4, there exists a unique one-dimensional invariant subspace of $\mathcal{V}$. If it was $K(8)$, we would have $K(8) = K(8) \cap \mathcal{V}_0 = K(7)$, the last equality holding by irreducibility of $K(7)$ for these values of l and r. Then $K(7)$ is also one-dimensional which forces $l = \frac{1}{r^{11}}$: impossible. We conclude that $K(8)$ is reducible and contains a one-dimensional invariant subspace that has a summand in $K(8)$. This summand cannot be one-dimensional or contain any one-dimensional submodule by uniqueness in Theorem 4. This summand may also not be 7-dimensional since it would then be irreducible and we would have $l \in \{\frac{1}{r^5}, -\frac{1}{r^5}\}$. This is not compatible with $l = -r^3$ and $(r^2)^8 \neq 1$. Neither can it contain an irreducible 7-dimensional invariant subspace. Since it has dimension less than 28, it must be 14, 20 or 21-dimensional. This ends the proof of Proposition 6.

We deduce from the Proposition some properties of inclusions of the $K(n)$'s. Let us first introduce a few more notations. We extend the definitions of $\mathcal{V}_0$ and $\mathcal{V}_1$ and define $\mathcal{V}_{n-s}$ to be the F-vector space spanned by the w_{ij}'s with $1 \leq i < j \leq s-1$. Precisely,

Definition 2.

$$\mathcal{V}_{n-s} := \mathit{Span}_F(w_{ij} | 1 \leq i < j \leq s-1) \qquad \forall s = 4, \dots, n \tag{152}$$

And for better clarity in the notations, we set:

$$\begin{cases} \mathcal{V}^{(s)} & := & \mathcal{V}_{n-s-1} \quad \forall s = 3, \dots, n-1 \\ \mathcal{V}^{(n)} & := & \mathcal{V} \end{cases}$$

We will use either of these definitions depending on the context. The second definition is somehow nicer as it has a meaning independently of the choice of integer n. We have the Propositions:

Proposition 7. *Suppose $l = r$. Assume $\mathcal{H}_{F,r^2}(n)$ is semisimple with $n \geq 4$. Then*

$$K(n) \supset K(n-1)$$

PROOF OF PROPOSITION 7

The inclusion uses the irreducibility of the $K(n)$'s for $n \geq 4$, when $l = r$ and the fact that they are nontrivial. To show that the inclusion is strict uses the result of Proposition 5.

The proof itself: suppose $l = r$ and let n be an integer with $n \geq 5$. We consider the intersection $K(n) \cap \mathcal{V}^{(n-1)}$. If this intersection was trivial, we would have $k(n) \leq n-1$, which is impossible. Since $K(n) \cap \mathcal{V}^{(n-1)} \subset \mathcal{V}^{(n-1)}$ (otherwise $K(n) = \mathcal{V}$ by §8), $K(n) \cap \mathcal{V}^{(n-1)}$ is a proper non-trivial invariant subspace of $\mathcal{V}^{(n-1)}$ and we must have:

$$0 \subset K(n) \cap \mathcal{V}^{(n-1)} \subseteq K(n-1) \tag{153}$$

By irreducibility of $K(n)$ for every $n \geq 4$ when $l = r$, we now get:

$$K(n) \cap \mathcal{V}^{(n-1)} = K(n-1) \tag{154}$$

Thus, we read:

$$K(n-1) \subseteq K(n)$$

Further, if the equality holds between the two sets, then,

$$K(n) \subset \mathcal{V}^{(n-1)} \tag{155}$$

We show that (155) is a contradiction. We recall from Proposition 5 that the vector $\mathcal{X}$ defined by:

$$\mathcal{X} = r^2\, w_{12} - r\, w_{13} + w_{34} - r\, w_{24}$$

belongs to $K(n)$. Since $K(n)$ is a $B(A_{n-1})$-module, the vector

$$(g_{n-1}\dots g_5 g_4)\,.\,\mathcal{X} = r^{n-2}\,w_{12} - r^{n-3}\,w_{13} + w_{3n} - r\,w_{2n}$$

also belongs to $K(n)$. As a matter of fact, this vector is in $K(n)$, but is not in $\mathcal{V}^{(n-1)}$, hence the contradiction in (155). We conclude that $K(n) \supset K(n-1)$. It remains to show that $K(3) \subset K(4)$. But by Theorem 7, the representation $\nu^{(3)}$ is irreducible when $l = r$. Moreover, $K(3)$ is a submodule of $\mathcal{V}$ and is not $\mathcal{V}$ itself. Thus $K(3)$ must be trivial. Finally, $K(4) \neq 0$ as it contains for instance the vector $x(4) = r^2\,w_{13} + w_{24} - r\,w_{14} - r\,w_{23}$. Thus, when $l = r$, we have:

$$0 = K(3) \subset K(4)$$

This ends the proof of Proposition 7.

Proposition 8.
In Proposition 7 it suffices to assume that $\mathcal{H}_{F,r^2}(n-1)$ is semisimple.

Indeed, in the proof above, we considered the intersection module $K(n) \cap \mathcal{V}^{(n-1)}$. Consider instead the intersection $K(n) \cap K(n-1)$ and use Proposition 5 to claim that this intersection is non trivial for $n \geq 6$. When $n = 5$, use for instance that $x(4) \in K(4) \cap K(5)$. Then by irreducibility of $K(n-1)$, we have $K(n-1) \cap K(n) = K(n-1)$, from which we derive that $K(n) \supseteq K(n-1)$. Since $(g_{n-1}\dots g_4).\mathcal{X} \in K(n) \setminus K(n-1)$, we have $K(n) \supset K(n-1)$.

We now study some properties of inclusions of the $K(n)$'s when $l = -r^3$.

Proposition 9. *Let n be any integer with $n \geq 4$.* ***Suppose*** *$l = -r^3$* ***and suppose*** *$\mathcal{H}_{F,r^2}(n-1)$* ***is semisimple.***

1) *If* $r^{2(n-1)} \neq -1$, *then* $K(n-1) \subset K(n)$.

2) $(n=4)$ *If* $r^6 = -1$, *then we have:*

$$K(3) \not\subseteq K(4) \subset K(5) \subset K(6)$$

3) *If* $r^{2(n-1)} = -1$ *for some* $n \geq 5$ *then we have the towers of inclusion:*

$$K(3) \subset \cdots \subset K(n-1) \not\subseteq K(n) \subset \cdots \subset K(2(n-1))$$

PROOF: let's first prove 1). Following the same scheme as in the proof of Proposition 8, we show the inclusion $K(n-1) \subset K(n)$. For $n \geq 6$, the intersection $K(n) \cap K(n-1)$ is nontrivial as $\mathcal{Y}$ belongs to all the $K(s),\ s \geq 5$ by Proposition 5. When $n=5$, $y(4) \in K(5) \cap K(4)$. Also, when $n=4$, $-r\, w_{12} - \frac{1}{r}\, w_{23} + w_{13} \in K(4) \cap K(3)$. Thus, $0 \subset K(n) \cap K(n-1) \subseteq K(n-1)$. By irreducibility of $K(n-1)$ when $l = -r^3$, $\mathcal{H}_{F,r^2}(n-1)$ is semisimple and $r^{2(n-1)} \neq -1$, it follows that $K(n-1) \subseteq K(n)$. Since

$$(g_{n-1} \ldots g_4).\mathcal{Y} = -r\, w_{23} - \frac{1}{r}\, w_{3n} + w_{2n} \in K(n) \setminus K(n-1),$$

we see that $K(n-1) \subset K(n)$. This ends the proof of point 1). Let's prove points 2) and 3).

Claim 2. *Suppose* $r^{2n} = -1$ *for some* $n \geq 3$ *and assume that* $\mathcal{H}_{F,r^2}(n)$ *is semisimple. Then* $r^{2k} \neq -1,\ \forall k \notin (2\mathbb{N}+1)\, n$.

PROOF: Suppose $r^{2k} = -1$, some $k \neq n$. The proof is in two steps.

- First $k \leq 2n$: we have

$$-1 = r^{2n} = r^{2k} r^{2(n-k)}$$

If $r^{2k} = -1$, it comes $r^{2(n-k)} = 1$, a contradiction with $\mathcal{H}_{F,r^2}(n)$ semisimple as $k - n \leq n$.

- If $k \geq 2n$, let's divide k by $2n$:

$$k = t \times 2n + s \text{ with } 0 \leq s \leq 2n - 1$$

It comes $-1 = r^{2k} = r^{2s}(r^{2n})^{2t}$. Then $r^{2s} = -1$ with $0 \leq s \leq 2n - 1$. It forces $s = n$ by the first point. Then $k = (1 + 2t)n$. Hence $k \in (2\mathbb{N} + 1)\, n$.

We deduce from the claim a lemma:

Lemma 8. *If $r^{2n} = -1$ and $\mathcal{H}_{F,r^2}(n)$ is semisimple, then $\mathcal{H}_{F,r^2}(2n-1)$ is semisimple (but $\mathcal{H}_{F,r^2}(2n)$ is not).*

PROOF OF THE LEMMA: first if $r^{2n} = -1$, then $(r^2)^{2n} = 1$, so $\mathcal{H}_{F,r^2}(2n)$ is not semisimple. What the lemma says is that $2n$ is the first positive integer k for which $r^{2k} = 1$. In other words, $r^2 \neq 1, \ldots, (r^2)^{2n-1} \neq 1$, so that $\mathcal{H}_{F,r^2}(2n-1)$ is semisimple. Suppose that there exists an integer k_0 such that $0 < k_0 < 2n$ and $r^{2k_0} = 1$. Then it comes:

$$-1 = r^{2k_0} r^{2n-2k_0} = r^{2(n-k_0)}$$

so that

$$r^{2(n-k_0)} = -1$$

Now $0 < |n - k_0| < n$ implies that $|n - k_0| \notin (2\mathbb{N} + 1)n$. Then by the claim we have $r^{2|n-k_0|} \neq -1$, which yields a contradiction. Thus, $\forall 0 < k \leq 2n - 1,\ r^{2k} \neq 1$. This ends the proof of the lemma.

Suppose $n \geq 5$ and let's go back to the proof of Proposition 9: by the lemma $\mathcal{H}_{F,r^2}(2n-3)$ is semisimple. Also, by the claim, $r^{2k} \neq -1,\ \forall k \notin (2\mathbb{N}+1)(n-1)$. In particular, for all positive integer k such that $k \leq 2n-3$ and $k \neq n-1$, we have $r^{2k} \neq -1$. By applying point $1)$ of Proposition 9 to all these admissible k's, we get:

$$K(3) \subset \cdots \subset K(n-1)$$
$$K(n) \subset \cdots \subset K(2n-2)$$

It remains to show that $K(n-1) \not\subseteq K(n)$. When $l = -r^3$ and $r^{2(n-1)} = -1$, there exists a one-dimensional invariant subspace of $\mathcal{V}^{(n-1)}$, say $\mathcal{V}_{1,n-1}$. Moreover, by Theorem 4,

$$\mathcal{V}_{1,n-1} = Span_F(\sum_{1\leq s<t\leq n-1} r^{s+t}\, w_{s,t})$$

We have:

$$\begin{aligned}
e_{n-1}.\sum_{1\leq s<t\leq n-1} r^{s+t}\, w_{s,t} &= \sum_{i=1}^{n-2} r^{i+n-1}\, e_{n-1}.\, w_{i,n-1} \\
&= \sum_{i=1}^{n-2} \frac{r^{i+n-1}}{r^{n-i-2}}\, w_{n-1,n} \\
&= r\sum_{i=1}^{n-2} (r^2)^i\, w_{n-1,n} \\
&= r\,\frac{1-(r^2)^{n-2}}{1-r^2}\, w_{n-1,n}
\end{aligned}$$

Since $(r^2)^{n-2} \neq 1$, we see that $e_{n-1}.\sum_{1\leq s<t\leq n-1} r^{s+t}\, w_{s,t} \neq 0$. Hence $\mathcal{V}_{1,n-1} \not\subseteq K(n)$ and a fortiori, $K(n-1) \not\subseteq K(n)$. This ends the proof of

point 3). Let's prove 2). If $l = -r^3$ and $r^6 = -1$, we know from before that

$$K(3) = Span_F(w_{12} + r\, w_{13} + r^2\, w_{23}) \bigoplus Span_F(w_{12} - \frac{1}{r}\, w_{13} + \frac{1}{r^2}\, w_{23})$$

Since $e_3.(w_{12} + r\, w_{13} + r^2\, w_{23}) = (1 + r^2)\, w_{34} \neq 0$, we see that

$$K(3) \not\subseteq K(4)$$

Let's show that $K(4) \subset K(5)$. When $l = -r^3$, $y(4) \in K(4) \cap K(5)$, hence $K(4) \cap K(5) \neq 0$. Moreover $K(4)$ is irreducible as $r^8 \neq -1$ (otherwise $r^2 = 1$) and $r^8 \neq 1$ (otherwise $r^2 = -1$). Thus $K(4) \cap K(5) = K(4)$, which implies $K(4) \subseteq K(5)$. From $y(4) \in K(4)$ we derive $g_4.y(4) \in K(5)$ id est $-r^3\, w_{12} - \frac{1}{r^2} w_{35} + w_{15} - r\, w_{23} \in K(5)$. This element is not in $K(4)$. Hence we have $K(4) \subset K(5)$. Finally, by Proposition 5, $K(5) \cap K(6) \neq 0$ and since $r^{10} \neq -1$ (as otherwise $(r^2)^2 = 1$) and $r^{10} \neq 1$ (as otherwise $(r^2)^2 = 1$), we know from Proposition 6 that $K(5)$ is irreducible. Thus, $K(5) \cap K(6) = K(5)$ and $K(5) \subseteq K(6)$. Further, by the usual argument,

$$g_5.\,\mathcal{Y} \in K(6) \setminus K(5)$$

Thus $K(5) \subset K(6)$. In summary,

When $l = -r^3$, $(r^2)^2 \neq 1$ and $r^6 = -1$, we have:
$K(3) \not\subseteq K(4) \subset K(5) \subset K(6)$

This achieves the proof of Proposition 9. In the next section, we prove two more theorems that finish characterizing the dimensions of the invariant subspaces of $\mathcal{V}$ when the representation is reducible.

9.2 Properties of Dimension of the $\mathrm{K(n)}$'s

Our first main result is the following:

Proposition 10. *Let $n \geq 4$. When $l = r$, $k(n) = \frac{n(n-3)}{2}$*

PROOF: first we show that when $l = r$, we have $k(n) \geq \frac{n(n-3)}{2}$. We will deal with the case $n = 8$ separately. By Corollary 3, for $n = 7$ or $n \geq 9$, the irreducible $\mathcal{H}_{F,r^2}(n)$-modules have dimensions $1, n-1, \frac{n(n-3)}{2}, \frac{(n-1)(n-2)}{2}$ or dimension greater than $\frac{(n-1)(n-2)}{2}$. Hence in the case $n = 7$ or $n \geq 9$, if $k(n) < \frac{n(n-3)}{2}$, then there exists an irreducible $(n-1)$-dimensional invariant subspace of $\mathcal{V}$ or there exists a one-dimensional invariant subspace of $\mathcal{V}$, which forces $l \in \{\frac{1}{r^{2n-3}}, \frac{1}{r^{n-3}}, -\frac{1}{r^{n-3}}\}$. None of this value is compatible with $l = r$. Thus, $k(n) \geq \frac{n(n-3)}{2}$ in these cases. When $n \in \{5, 6\}$, the story is similar as Corollary 3 still holds in these cases. As for $n = 4$, $k(4) = 2$ by Corollary 5. Suppose now $n = 8$. The irreducible $\mathcal{H}_{F,r^2}(8)$-modules have dimensions $1, 7, 14, 20, 21$ or dimension greater or equal to $28 = \dim \mathcal{V}^{(8)}$. By Proposition 7, we know that $K(8) \supset K(7)$. By the above, we have $k(7) \geq 14$. Hence it comes $k(8) > 14$. Moreover, by Proposition 6, $K(8)$ is irreducible when $l = r$. Then $k(8) \geq 20$ and we are done with all the cases. We will prove the other inequality geometrically, by using the tangles. Let's introduce a few notations. Let $T(n)$ denote the matrix of the sum endomorphism $\nu(n) := \sum_{1 \leq i < j \leq n} \nu^{(n)}(X_{ij})$ as in the definition below:

Definition 3.

$$\nu(n) := \sum_{1 \leq i < j \leq n} \nu^{(n)}(X_{ij})$$
$$T(n) := Mat_{\mathcal{B}_{\mathcal{V}^{(n)}}} \nu(n)$$

Property 1. $K(n) = Ker\, \nu(n)$

PROOF: by Proposition 2, $Mat_{\mathcal{B}_{\mathcal{V}^{(n)}}} \nu^{(n)}(X_{ij})$ is the matrix whose $[\binom{j-1}{2} +$

$(j-i)$]-th row is $T(n)$'s and with zeros elsewhere. It follows that

$$\bigcap_{1\leq i<j\leq n} \mathrm{Ker}\,\nu^{(n)}(X_{ij}) = \mathrm{Ker}\Big(\sum_{1\leq i<j\leq n} \nu^{(n)}(X_{ij})\Big)$$

With our notations,

$$K(n) = \mathrm{Ker}\,(\nu(n))$$

Since $rk(\nu(n)) = rk(T(n))$, we get the equality on the dimensions:

$$k(n) + rk(T(n)) = \frac{n(n-1)}{2}$$

Thus, to show that $k(n) \leq \frac{n(n-3)}{2}$, it suffices to show that $rk(T(n)) \geq n$.

Property 2. *When $l = r$ and $n \geq 4$, $rk(T(n)) \geq n$*

PROOF: when $n = 4$, since $k(4) = 2$, $rk(T(4)) = 6 - 2 = 4$. When $n = 5$ we recall from Result 1 that there exists an irreducible 5-dimensional invariant subspace. Since $K(5)$ is irreducible by Proposition 6, we get $k(5) = 5$, hence $rk(T(5)) = 10 - 5 = 5$. Since the square submatrix of $T(6)$ composed of the first ten rows and first ten columns is $T(5)$, to show that $rk(T(6)) \geq 6$, an idea consists of extracting from $T(5)$ a square submatrix of size 5 (such a matrix exists as $rk(T(5)) = 5$) and try and build from it an invertible square submatrix of $T(6)$ of size 6 by adding a sixth subrow and a sixth subcolumn. We notice that in each last five rows of $T(6)$, there are six zeros amongst the first ten coefficients. These are indicated in bold below:

$$
T(6) = \left(\begin{array}{cccccccccc|ccccc}
\multicolumn{10}{c|}{T(5)} & \multicolumn{5}{c}{*} \\
\hline
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 1 & \frac{1}{r} & \frac{1}{r^2} & \frac{1}{r^3} & 2 & \frac{1}{r} & \frac{1}{r^2} & \frac{1}{r^3} & \frac{1}{r^4} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{1}{r} & \frac{1}{r^2} & \frac{1}{r^3} & \frac{1}{r} & \mathbf{0} & \mathbf{0} & \mathbf{0} & r & 2 & \frac{1}{r} & \frac{1}{r^2} & \frac{1}{r^3} \\
\mathbf{0} & \frac{1}{r^2} & \frac{1}{r^3} & \frac{1}{r^2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{1}{r} & \mathbf{0} & \mathbf{0} & r^2 & r & 2 & \frac{1}{r} & \frac{1}{r^2} \\
\frac{1}{r^3} & \frac{1}{r^3} & \mathbf{0} & \mathbf{0} & \frac{1}{r^2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{1}{r} & \mathbf{0} & r^3 & r^2 & r & 2 & \frac{1}{r} \\
\frac{1}{r^4} & \mathbf{0} & \frac{1}{r^3} & \mathbf{0} & \mathbf{0} & \frac{1}{r^2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \frac{1}{r} & r^4 & r^3 & r^2 & r & 2
\end{array}\right)
$$

Since the coefficients in the lower right part of the matrix are all non-zero, life would be wonderful if there existed five columns in $T(5)$ aligned on one of these rows of zeros that made a square submatrix of $T(5)$ invertible. Then, adding this subrow of five zeros and any subcolumn amongst the last five columns of $T(6)$ would still make the determinant of the resulting extended submatrix non-zero. Unfortunately things don't happen this way. We wrote a program in Maple that computes the determinant of all the square submatrices of $T(5)$ (See appendix D). The first five numbers that are printed in the output correspond to the columns and the five ones that follow, to the rows. We fix an admissible column and see if for one of the rows we get a nonzero determinant. To do that we scroll the bar down onto an admissible column (the first one being $[1, 2, 3, 4, 5]$ and the last one being $[5, 6, 7, 9, 10]$ and there are 30 of these) and look for a row that would give a nonzero coefficient at the end. The result is negative. The mathematical explanation for that is unknown to the author. We need to ask for slightly less and content ourselves with five columns aligned on only four zeros. Let's pick the first four zeros of row 11 and notice that it could be nice to have a one as the last non-zero coefficient. Thus, we investigate about the determinants of the submatrices of $T(5)$ with the pattern of columns $\mathcal{C} := [1, 2, 3, 4, 7]$. The first (in the lexicographic order) admissible 5-tuple of rows is actually $\mathcal{R} := [1, 2, 3, 4, 7](= \mathcal{C})$. And in fact picking $\mathcal{C}$ and $\mathcal{R}$ is also a natural choice as, by our program, it is for these subcolumns and subrows that the first nonzero determinant arises in lexicographic order (when ordering the columns first).

A few notations:

- Given a matrix A, submatrix$(A, [i_1, \ldots, i_s], [j_1, \ldots, j_s])$ denotes the square submatrix of size s of A with subrows $i_1, \ldots, i_s$ and subcolumns $j_1, \ldots, j_s$, where we followed Maple's notations.
- Given $\mathcal{R}$ a subset of rows, $c_{i_k,\mathcal{R}}$ denotes the extracted i_k-th column with rows $\mathcal{R}$.

Let $M :=$ submatrix$(T(5), [1, 2, 3, 4, 7], [1, 2, 3, 4, 7])$.
We have $det(M) = \frac{(r^2+1)^2}{r^2} \neq 0$, as $(r^2)^2 \neq 1$.

Hence the family of columns $(c_{1,\mathcal{R}}, c_{2,\mathcal{R}}, c_{3,\mathcal{R}}, c_{4,\mathcal{R}}, c_{7,\mathcal{R}})$ is free. A fortiori, the subfamily $(c_{1,\mathcal{R}}, c_{2,\mathcal{R}}, c_{3,\mathcal{R}}, c_{4,\mathcal{R}})$ is free.

Let $M' :=$ submatrix$(T(6), [1, 2, 3, 4, 7], [1, 2, 3, 4, 12])$
$det(M') = 0$ (see Appendix D)

Then $(c_{1,\mathcal{R}}, c_{2,\mathcal{R}}, c_{3,\mathcal{R}}, c_{4,\mathcal{R}}, c_{12,\mathcal{R}})$ is not free, but $(c_{1,\mathcal{R}}, c_{2,\mathcal{R}}, c_{3,\mathcal{R}}, c_{4,\mathcal{R}})$ is. Hence, the column $c_{12,\mathcal{R}}$ is a linear combination of columns $c_{1,\mathcal{R}}, c_{2,\mathcal{R}}, c_{3,\mathcal{R}}, c_{4,\mathcal{R}}$.

Thus, we don't modify the determinant of M by adding to $c_{7,\mathcal{R}}$ a multiple of $c_{12,\mathcal{R}}$. The 12th column of $T(6)$ is the vector $\nu(6)(w_{46})$. The coefficient on the 11th row is $[\nu(6)(w_{46})]_{x_{\alpha_5}} = [\nu^{(6)}(X_{56})(w_{46})]_{x_{\alpha_5}} = \frac{1}{r}$ by $(TL)_1$ of Appendix C with $l = r$. We recall that the coefficient of the 7th column and the 11th row is a one, while the other coefficients of the extracted matrix on the same row are 0's. We now do the operation on the columns

$$c_{7,\mathcal{R}@[\binom{5}{2}+1]} \leftarrow c_{7,\mathcal{R}@[\binom{5}{2}+1]} - r\, c_{12,\mathcal{R}@[\binom{5}{2}+1]}$$

in order to make a fifth zero appear, where we used the symbol @ for concatenation of lists. By doing so the determinant of M is unchanged. Let's consider the matrix

$$M^{''} := \text{submatrix}(T(6), [1,2,3,4,7,11], [1,2,3,4,7,12])$$

$M^{''}$ is a square submatrix of $T(6)$ of size 6 and

$$\det(M^{''}) = \underbrace{\det(M)}_{\neq 0} \times \frac{1}{r} \neq 0$$

Thus, we have exhibited an invertible submatrix of $T(6)$ of size 6. This shows that $rk(T(6)) \geq 6$. Furthermore, we have seen that when $l = r$, we have for all $n \geq 4$ that $k(n) \geq \frac{n(n-3)}{2}$. This is equivalent to $rk(T(n)) \leq n$. Thus, we have $rk(T(6)) \leq 6$. Hence $rk(T(6)) = 6$. Next, for $n \geq 7$ we inductively build invertible submatrices $S(n)$ of $T(n)$ of size n in the following way:

$$S(5) := \text{submatrix}(T(5), \mathcal{R}_5, \mathcal{C}_5) \text{ with } \mathcal{C}_5 = \mathcal{R}_5 := [1, 2, 3, 4, 7](= \mathcal{R})$$

$$S(n) := \text{submatrix}(T(n), \mathcal{R}_{n-1}@[\tbinom{n-1}{2} + (n-5)], \mathcal{C}_{n-1}@[\tbinom{n-1}{2} + (n-4)])$$

$S(n)$ is built from $S(n-1)$ by adding the extracted $[\binom{n-1}{2} + (n-5)]$-th subrow and the extracted $[\binom{n-1}{2} + (n-4)]$-th subcolumn. In other words, $S(n)$ is built from $S(n-1)$ by adding the subrow corresponding to X_{5n} and the subcolumn corresponding to w_{4n} as shown on Figure 1. In the table below, we gathered the results of the actions of X_{12}, X_{23}, X_{13}, X_{34} and X_{45} on the vectors $w_{4,n}$'s for $n \geq 6$. To calculate the coefficient of the action of X_{34} on w_{4k}, we used $(SR)_{k-4}$ with $i = 3$ and $j = 4$. It yields the coefficient r^{k-5}; to calculate the coefficient of the action of X_{45} on w_{4k}, we used $(TR)_{k-5}$ with $l = r$. It also yields the coefficient r^{k-5}.

	w_{46}	w_{47}	$\dots$	w_{4n}	$\dots$
X_{12}	0	0		0	
X_{23}	0	0		0	
X_{13}	0	0	$\dots$	0	$\dots$
X_{34}	r	r^2		r^{n-5}	
X_{45}	r	r^2		r^{n-5}	

It appears clearly on the table that $c_{\binom{n-1}{2}+(n-4),\mathcal{R}}$ is a multiple of $c_{12,\mathcal{R}}$. Since, as seen above, $c_{12,\mathcal{R}}$ is a linear combination of the extracted columns $c_{1,\mathcal{R}}, c_{2,\mathcal{R}}, c_{3,\mathcal{R}}, c_{4,\mathcal{R}}$ of $T(5)$, each extracted column $c_{\binom{n-1}{2}+(n-4),\mathcal{R}}$ of $T(n)$, $n \geq 7$, is also a linear combination of columns $c_{1,\mathcal{R}}, c_{2,\mathcal{R}}, c_{3,\mathcal{R}}, c_{4,\mathcal{R}}$. Thus, for any $n \geq 7$, we won't modify the determinant of $S(5)(= M)$ by adding to its 7-th column $c_{7,\mathcal{R}}$ a multiple of $c_{\binom{n-1}{2}+(n-4),\mathcal{R}}$. As it appears on the Figure below,

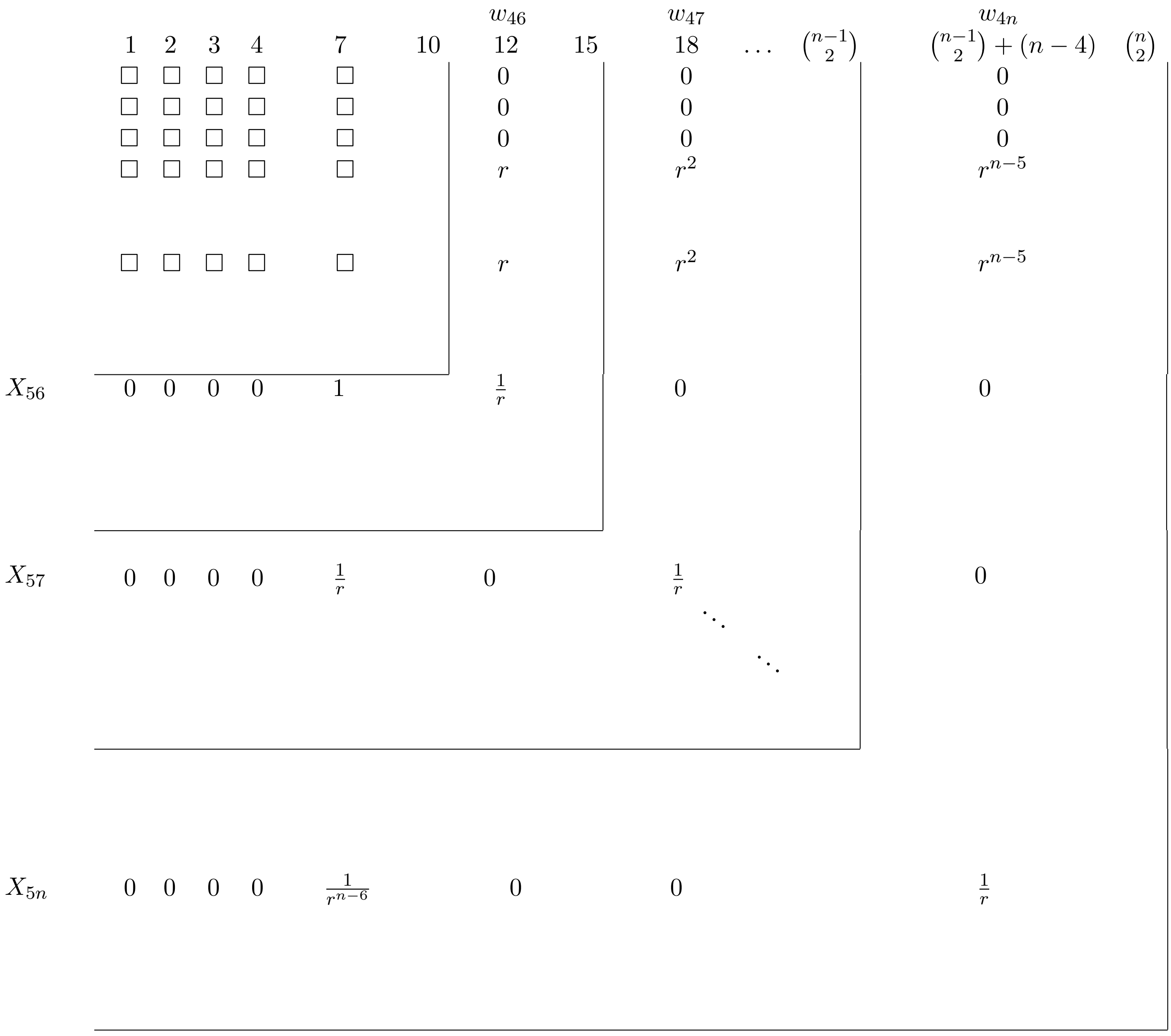


Figure 1

we have:

$$\begin{array}{ll}
\forall n \geq 7,\ X_{5n}.w_{4n} = \frac{1}{r}\, w_{5n} & \text{(Rule } (TL)_1 \text{ with } l = r) \\
\forall n \geq 7,\ X_{5n}.w_{45} = \frac{1}{r^{n-6}}\, w_{5n} & \text{(Rule } (SL)_1 \text{ with } i = 5 \ \&\ j = n) \\
\multicolumn{2}{l}{\forall n \geq 7,\ X_{5n}.w_{4j} = 0 \text{ for all } 6 \leq j \leq n-1 \text{ BECAUSE } l = r \text{ (cf Rule } (CL))} \\
\multicolumn{2}{l}{\forall n \geq 7,\ X_{5n}.w_{s,t} = 0 \text{ for all } (s,t) \in \{(1,2),(2,3),(1,3),(3,4)\}} \\
\multicolumn{2}{l}{\forall n \geq 7,\ X_{5k}.w_{4j} = 0,\ \forall 6 \leq k \leq n-1,\ \forall k+1 \leq j \leq n}
\end{array}$$

Hence, doing the following operation $\mathcal{O}_n$ on $S(n)$ with:

$$\mathcal{O}_s : \mathcal{C}_{7,\mathcal{R}_s} \leftarrow \mathcal{C}_{7,\mathcal{R}_s} - \sum_{k=0}^{n-6} \frac{1}{r^{k-1}}\, \mathcal{C}_{\binom{k+5}{2}+(k+2),\mathcal{R}_s}$$

transforms $S(n)$ into a matrix by blocks with 0's in the lower left quadrant, a square matrix $M_{\mathcal{O}_5} = S(5)_{\mathcal{O}_5}$ of size 5 obtained from M by doing the operation $\mathcal{O}_5$ and whose determinant equals the one of M in the upper left quadrant and an upper triangular square matrix of size $n-5$ with $\frac{1}{r}$'s over the diagonal in the lower right quadrant:

$$S(n)_{\mathcal{O}_n} = \left[\begin{array}{c|ccc}
S(5)_{\mathcal{O}_5} & & * & \\
\hline
 & \frac{1}{r} & & \\
0 & & \ddots & * \\
 & 0 & & \ddots \\
 & & & \frac{1}{r}
\end{array}\right]$$

$$\underset{5}{\longleftrightarrow}\ \underset{n-5}{\longleftrightarrow}$$

Then it comes:

$$\begin{aligned}
\det(S(n)) = \det(S(n)_{\mathcal{O}_n}) &= \det(S(5)_{\mathcal{O}_5}).\left(\frac{1}{r}\right)^{n-5}\\
&= \det(M_{\mathcal{O}_5}).\left(\frac{1}{r}\right)^{n-5}\\
&= \det(M).\left(\frac{1}{r}\right)^{n-5}\\
&\neq 0
\end{aligned}$$

Hence, $S(n)$ is a square submatrix of size n of $T(n)$ that is invertible. This shows that $rk(T(n)) \geq n$ for all $n \geq 7$. And since we know from before that for all $n \geq 4$, $rk(T(n)) \leq n$ when $l = r$, we have in fact: When $l = r,\ rk(T(n)) = n$ for all $n \geq 7$. This is the same as saying that When $l = r,\ k(n) = \frac{n(n-3)}{2}$ for all $n \geq 7$. Since as we have seen along the proof, we also have $rk(T(4)) = 4$ (*i.e* $k(4) = 2 = \frac{4.1}{2}$), $rk(T(5)) = 5$ (*i.e* $k(5) = 5 = \frac{5.2}{2}$) and $rk(T(6)) = 6$ (*i.e* $k(6) = 9 = \frac{6.3}{2}$) , this ends the proof of Proposition 10. Moreover we will remember the following fact, equivalent to Proposition 10:

Fact 2. *When $l = r$, the rank of $T(n)$ is n for all $n \geq 4$.*

We have two direct consequences of Propostion 10. Unless otherwise mentioned, $\mathcal{H}_{F,r^2}(n)$ is assumed to be semisimple.

Corollary 6. *Let n be an integer with $n \geq 4$. If $l = r$, then there exists an irreducible $\frac{n(n-3)}{2}$-dimensional invariant subspace of $\mathcal{V}$.*

PROOF: by Proposition 6, $K(n)$ is irreducible. By Proposition 10, $K(n)$ is $\frac{n(n-3)}{2}$-dimensional. This ends the proof.

Corollary 7. *Let n be an integer with $n \geq 5$. If there exists an irreducible $\frac{(n-1)(n-2)}{2}$-dimensional invariant subspace of $\mathcal{V}$, then $l = -r^3$.*

PROOF: Suppose $n \geq 6$. Let $\mathcal{W}$ be an irreducible $\frac{(n-1)(n-2)}{2}$-dimensional invariant subspace. Since for $n \geq 6$, we have $2n-3 \leq \frac{n(n-3)}{2} < \frac{n(n-3)}{2} + 1 = \frac{(n-1)(n-2)}{2}$, it comes $\dim \mathcal{W} > 2n-3$, hence $\mathcal{W} \cap \mathcal{V}_1 \neq \{0\}$. Also, $n-1 < \frac{(n-1)(n-2)}{2}$ implies that $\mathcal{W} \cap \mathcal{V}_0 \neq \{0\}$. Thus, by theorem 7 we get: $l \in \{r, -r^3, \frac{1}{r^{2(n-2)-3}}, \frac{1}{r^{n-5}}, -\frac{1}{r^{n-5}}\}$ and $l \in \{r, -r^3, \frac{1}{r^{2(n-1)-3}}, \frac{1}{r^{n-4}}, -\frac{1}{r^{n-4}}\}$. This only leaves the possibility $l \in \{r, -r^3\}$ by semisimplicity of $\mathcal{H}_{F,r^2}(n)$. Now $\dim \mathcal{W} = \frac{(n-1)(n-2)}{2}$ implies that $k(n) \geq \frac{n(n-3)}{2} + 1$ and when $l = r$ we have $k(n) = \frac{n(n-3)}{2}$ by Proposition 10. Hence the only possibility for l is in fact $l = -r^3$.

The case $n = 5$

Suppose that there exists an irreducible 6-dimensional invariant subspace of $\mathcal{V}$. Then we claim that there cannot exist an irreducible 5-dimensional invariant subspace of $\mathcal{V}$. Indeed let's name them $\mathcal{V}_5$ and $\mathcal{V}_6$. We have $\mathcal{V}_5 \subseteq K(5)$ and $\mathcal{V}_6 \subseteq K(5)$. The existence of an irreducible 5-dimensional invariant subspace of $\mathcal{V}$ implies $l = r$ by Result 1; by Proposition 6, $K(5)$ is irreducible as $l = r$. Then $K(5) = \mathcal{V}_5 = \mathcal{V}_6$, contradiction. Hence there cannot exist an irreducible 5-dimensional invariant subspace. Hence $l \neq r$. But $\nu^{(5)}$ is reducible, hence $l \in \{-r^3, \frac{1}{r^2}, -\frac{1}{r^2}, \frac{1}{r^7}\}$. Since $6 > 4$, $\mathcal{V}_6 \cap \mathcal{V}^{(4)} \neq 0$. Hence, $\nu^{(4)}$ is reducible and $l \in \{-r^3, \frac{1}{r}, -\frac{1}{r}, \frac{1}{r^5}\}$. Then l must take one of the two values $-r^3$ or $\frac{1}{r^7}$. Since $l \neq r$, we have $k(5) \in \{6, 7\}$. If $l = \frac{1}{r^7}$ then there exists a one-dimensional invariant subspace $\mathcal{V}_1$ of $\mathcal{V}^{(5)}$ which is in direct sum with $\mathcal{V}_6$. Hence $k(5) = 7$ and $K(5) = \mathcal{V}_1 \oplus \mathcal{V}_6$. If $l \neq \frac{1}{r^7}$, then $l = -r^3$ and $k(5) = 6$. Then $K(5)$ is irreducible and $K(5) = \mathcal{V}_6$. Let's push the study a little bit more. We have $\dim(\mathcal{V}_6 \cap \mathcal{V}^{(4)}) = 6 + 6 - \dim(\mathcal{V}_6 + \mathcal{V}^{(4)}) \geq 2$ and $\mathcal{V}_6 \cap \mathcal{V}^{(4)} \subseteq K(4)$. Hence, $k(4) \geq 2$. Moreover, since $l \neq r$, $k(4) \in \{3, 4\}$. Suppose $k(4) = 4$. Then $K(4)$ is a direct sum of an irreducible 3-dimensional invariant subspace of $\mathcal{V}^{(4)}$ and of a one-dimensional one. Then $l = \frac{1}{r^5} = -r^3$. So $r^8 = -1$. Then $r^{10} \neq -1$ and $l = -r^3$. Hence $K(5)$ is irreducible. Moreover $l \neq \frac{1}{r^7}$. Thus, if $l = \frac{1}{r^7}$, then $k(4) = 3$ and $K(4)$ is irreducible as $l \neq r$. Also it follows that $l \in \{-\frac{1}{r}, -r^3\}$, hence either $r^6 = -1$ or $r^{10} = -1$. We conclude that Corollary 7 holds for $n = 5$ in the case $r^6 \neq -1$. Also, we deduce from this discussion that if $K(4)$ is reducible, then $l \neq \frac{1}{r^7}$. Hence $l = -r^3$ and $K(5)$ is irreducible. Thus, $K(4)$ and $K(5)$ cannot be simultaneously reducible. Suppose now that $r^6 = -1$. We will show that $l = -r^3$, and this will prove that Corollary 7 holds in fact in any case. We have $r^{10} \neq -1$. Thus, $\frac{1}{r^7} \neq -r^3$. Suppose $l \neq -r^3$. Then $l = \frac{1}{r^7}$. Then as seen above, $K(5) = \mathcal{V}_1 \oplus \mathcal{V}_6$. Then $k(5) = 7 > 4$ and $K(4)$ is irreducible as for instance $K(5)$ is reducible. It follows that $K(5) \cap \mathcal{V}^{(4)} = K(4)$. In particular, we have $K(4) \subseteq K(5)$. Also, by the above, when $l = \frac{1}{r^7}$, we have $k(4) = 3$. Thus, $K(4)$ is irreducible, 3-dimensional. Since $l = \frac{1}{r^7} = -\frac{1}{r}$, we know from Theorem 6 that $K(4)$ is spanned over F by the vectors:

$$\begin{aligned}
v_1 &= \left(\frac{1}{r} + r\right) w_{12} + (w_{13} - \frac{1}{r}\, w_{23}) + r\,(w_{14} - \frac{1}{r}\, w_{24}) \\
v_2 &= \left(\frac{1}{r} + r\right) w_{23} + (w_{24} - \frac{1}{r}\, w_{34}) - r(w_{12} - \frac{1}{r}\, w_{13}) \\
v_3 &= \left(\frac{1}{r} + r\right) w_{34} - (w_{13} - \frac{1}{r}\, w_{14}) - (w_{23} - \frac{1}{r}\, w_{24})
\end{aligned}$$

Let's compute the action of X_{35} on v_2. By using the table in Appendix C, where we replaced l by $-\frac{1}{r}$, we have the equalities:

$$\begin{array}{lclll} X_{35}.w_{34} & = & -r & x_{\alpha_3+\alpha_4} & \text{by (ML)}_1 \\ X_{35}.w_{13} & = & \frac{1}{r^2} & x_{\alpha_3+\alpha_4} & \text{by (SL)}_2 \\ X_{35}.w_{23} & = & \frac{1}{r} & x_{\alpha_3+\alpha_4} & \text{by (SL)}_1 \\ X_{35}.w_{24} & = & (\frac{1}{r}-r)(-\frac{1}{r}-r) & x_{\alpha_3+\alpha_4} & \text{by (CL)}_{(1,1)} \end{array}$$

Then it comes:

$$X_{35}.v_2 = \left(r + \frac{1}{r}\right)^2 x_{\alpha_3+\alpha_4} \tag{156}$$

Since $K(4) \subseteq K(5)$, we must have $\left(r + \frac{1}{r}\right)^2 = 0$, a contradiction. Hence it is impossible to have $l = \frac{1}{r^7}$ when $r^6 = -1$. Then l must take the value $-r^3$, which ends all the cases. Conversely, if $l = -r^3$, we claim that there exists an irreducible 6-dimensional invariant subspace inside $\mathcal{V}^{(5)}$. Indeed, by Theorem 7, $\nu^{(5)}$ is reducible, hence there exists an irreducible $B(A_4)$-submodule of $\mathcal{V}$. It cannot be 5-dimensional by Result 1 ($l \neq r$); nor can it be 4-dimensional ($l \notin \{\frac{1}{r^2}, -\frac{1}{r^2}\}$). If $r^{10} \neq -1$, it cannot be one-dimensional either. Then it must be 6-dimensional. If $r^{10} = -1$, then we recall from Proposition 6 that $k(5) \geq 6$. Then there must exist an irreducible 6-dimensional submodule as well. Hence the Theorem:

Theorem 8.

Suppose $n = 5$. There exists an irreducible 6-dimensional invariant subspace of $\mathcal{V}$ if and only if $l = -r^3$.

Finally we note that Corollary 7 does not hold for $n = 4$. Indeed, since $(r^2)^4 \neq 1$, we have $\frac{1}{r} \neq -r^3$ and $-\frac{1}{r} \neq -r^3$. And when $l \in \{\frac{1}{r}, -\frac{1}{r}\}$, there exists an irreducible $\frac{3.2}{2}$-dimensional invariant subspace of $\mathcal{V}^{(4)}$ by Theorem 6.

We have a corollary of Corollary 7:

Corollary 8.

Let n be an integer with $n \geq 4$.
$i)$ If there exists an irreducible $\frac{n(n-3)}{2}$-dimensional invariant subspace of $\mathcal{V}$, then there does not exist any $\frac{(n-1)(n-2)}{2}$-dimensional invariant subspace of $\mathcal{V}$.
$ii)$ If there exists an irreducible $\frac{(n-1)(n-2)}{2}$-dimensional invariant subspace of $\mathcal{V}$, then there does not exist any $\frac{n(n-3)}{2}$-dimensional invariant subspace of $\mathcal{V}$.

PROOF: we prove both points at the same time. Let's first assume $n \geq 5$. Suppose that there exists both an irreducible $\frac{n(n-3)}{2}$-dimensional invariant subspace $\mathcal{V}_{\frac{n(n-3)}{2}}$ of $\mathcal{V}$ and an irreducible $\frac{(n-1)(n-2)}{2}$ invariant subspace $\mathcal{V}_{\frac{(n-1)(n-2)}{2}}$ of $\mathcal{V}$. By Corollary 7, we have $l = -r^3$. If $r^{2n} \neq -1$, $K(n)$ is irreducible by Proposition 6, hence a contradiction. If $r^{2n} = -1$, then on one hand there exists a one-dimensional invariant subspace $\mathcal{V}_1$ of $\mathcal{V}$, which is in direct sum with $\mathcal{V}_{\frac{(n-1)(n-2)}{2}}$. Hence we have $\mathcal{V}_1 \oplus \mathcal{V}_{\frac{(n-1)(n-2)}{2}} \subseteq K(n)$. Then $k(n) \geq \frac{(n-1)(n-2)}{2} + 1$. On the other hand, $\mathcal{V}_{\frac{n(n-3)}{2}}$ has a summand, say S in $K(n)$. Let's denote by s its dimension. Since $k(n) \geq \frac{n(n-3)}{2} + 2$, we must have $s \geq 2$. Since $l = -r^3$, there does not exist any $(n-1)$-dimensional invariant subspace inside $\mathcal{V}$. Hence $s \geq 2$ implies in fact $s > n-1$. Now if $s \geq n$, we have $\dim(\mathcal{V}_{\frac{n(n-3)}{2}} \oplus S) \geq n + \frac{n(n-3)}{2} = \frac{n(n-1)}{2}$: a contradiction since $k(n) < \frac{n(n-1)}{2}$. This ends the proof in the case $n \geq 5$. It remains to do the case $n = 4$. In this case, if there exists an irreducible 2-dimensional invariant subspace of $\mathcal{V}^{(4)}$, then $l = r$ by Result 2; if there exists an irreducible 3-dimensional invariant subspace of $\mathcal{V}^{(4)}$, then $l \in \{-r^3, \frac{1}{r}, -\frac{1}{r}\}$ by Theorem 6. Then it is impossible to have an irreducible 3-dimensional invariant subspace and an irreducible 2-dimensional one.

When $l = -r^3$, it appears that it is not as easy to show properties on the rank of the matrix $T(n)$. However, we have the nice result:

Proposition 11. *Suppose $\mathcal{H}_{F,r^2}(n-1)$ is semisimple, $l = -r^3$ and $r^{2(n-1)} \neq -1$. Then,*

$$k(n) \geq k(n-1) + (n-2)$$

PROOF OF THE PROPOSITION: by proposition 9, point 1), we know under these assumptions that $K(n-1) \subset K(n)$. Thus, any vector that annihilates the matrix $T(n-1)$ annihilates the matrix $T(n)$. Let $v_1, v_2, \dots, v_{k(n-1)}$ be a basis of $K(n-1)$. Define $(n-2)$ linearly independent vectors by:

$$V_k = w_{k+1,n} - r\, w_{k,n} + r^{n-k}\, w_{k,k+1}, \quad k = 1 \dots n-2$$

Claim 3.
$v_1, v_2, \dots, v_{k(n-1)}, V_1, \dots, V_{n-2}$ are $k(n-1) + (n-2)$ linearly independent vectors of $K(n)$

PROOF OF THE CLAIM: we want to show that the X_{ij}'s annihilate the V_k's for $k = 1, \dots, n-2$. First, let's show that the last $(n-1)$ rows of the matrix $T(n)$ annihilate these vectors. To that aim, we compute:

$$\forall\, 1 \leq j, k \leq n-1, \quad [X_{j,n}.w_{k,n}]_{w_{j,n}} = \begin{cases} -r^{k-j-2} & \text{if } k < j \\ -r^{k-j+2} & \text{if } k > j \\ -\frac{1}{r^2} - r^2 & \text{if } k = j \end{cases}$$

Let's fix a row of the matrix $T(n)$ that corresponds to the action of a X_{jn} with $1 \leq j \leq n-1$. We want to multiply this row by the vector V_k. First, we let $X_{j,n}$ act on the vectors $w_{k+1,n}$ and $w_{k,n}$ and multiply the first resulting coefficient by 1 and the second resulting coefficient by $-r$, then add the two coefficients.

* If $k > j$, we get $-r^{k-j+3} + r^{k-j+3} = 0$
* If $k < j-1$, we get $-r^{k-j-1} + r^{k-j-1} = 0$
* If $k = j-1$, we get $-\frac{1}{r^2} - r^2 + \frac{1}{r^2} = -r^2$
* If $k = j$, we get $-r^3 + \frac{1}{r} + r^3 = \frac{1}{r}$

Now fix a $k \in \{1, \dots, n-2\}$. We have:

$$X_{jn}.(w_{k+1,n} - r\, w_{k,n}) = 0 \text{ except for } j \in \{k, k+1\}$$

Also, we have:

$$\begin{array}{rcll} [X_{k+1,n}.w_{k,k+1}]_{w_{k+1,n}} & = & \frac{1}{r^{n-k-2}} & \\ \Big[X_{k,n}.w_{k,k+1}\Big]_{w_{k,n}} & = & -\frac{1}{r^{n-k+1}} & \\ \Big[X_{j,n}.w_{k,k+1}\Big]_{w_{j,n}} & = & 0 & \text{if } j \notin \{k, k+1\} \end{array}$$

Thus, we get:

$$\forall\, 1 \le j \le n-1,\ \forall\, 1 \le k \le n-2,\ X_{jn}.(w_{k+1,n} - r\, w_{k,n} + r^{n-k}\, w_{k,k+1}) = 0$$

This shows that the last $(n-1)$ rows of the matrix $T(n)$ annihilate the vectors V_k's, $k = 1, \dots, n-2$. We will now show that the whole matrix $T(n)$ annihilates in fact these vectors. Given two positive integers s and t such that $1 \le s < t \le n-1$, we have:

$$\left[X_{s,t}.\, w_{k,n}\right]_{w_{s,t}} = \begin{cases} -r^{n-t+2} & \text{if } s = k \\ r^{n-s-2} & \text{if } t = k \\ (-r^3 - r)(r^{k-s+n-t-1} - r^{k-s+n-t-3}) & \text{if } s < k \text{ and } t > k \\ 0 & \text{otherwise} \end{cases}$$

$$\left[X_{s,t}.\, w_{k+1,n}\right]_{w_{s,t}} = \begin{cases} -r^{n-t+2} & \text{if } s = k+1 \\ r^{n-s-2} & \text{if } t = k+1 \\ (-r^3 - r)(r^{k-s+n-t} - r^{k-s+n-t-2}) & \text{if } s < k+1 \text{ and } t > k+1 \\ -r^2 - \frac{1}{r^2} & \text{if } (s,t) = (k, k+1) \\ 0 & \text{otherwise} \end{cases}$$

$$\left[X_{s,t}.\, w_{k,k+1}\right]_{w_{s,t}} = \begin{cases} -\frac{1}{r^{t-k+1}} & \text{if } s = k \text{ and } t > k+1 \\ \frac{1}{r^{t-k-2}} & \text{if } s = k+1 \text{ and } t > k+1 \\ r^{k-s-1} & \text{if } t = k \text{ and } s < k \\ -r^{k-s+2} & \text{if } t = k+1 \text{ and } s < k \\ 0 & \text{otherwise} \end{cases}$$

From these equalities, we derive:

$$\begin{aligned}
\forall\, t > k+1,\ X_{k,t}.V_k &= (-r^3 - r)(r^{n-t} - r^{n-t-2}) + r\, r^{n-t+2} - r^{n-t-1} \\
&= 0 \qquad (157) \\
\forall\, s < k,\ X_{s,k}.V_k &= -r\, r^{n-s-2} + r^{n-k}\, r^{k-s-1} \\
&= 0 \qquad (158) \\
\forall\, s < k,\ X_{s,k+1}.V_k &= r^{n-s-2} + (r^4 + r^2)(r^{n-s-2} - r^{n-s-4}) - r^{n-k}\, r^{k-s+2} \\
&= 0 \qquad (159) \\
\forall\, t > k+1,\ X_{k+1,t}.V_k &= -r^{n-t-2} + r^{n-k}\, r^{k-t+2} \\
&= 0 \qquad (160)
\end{aligned}$$

Equalities (157), (158), (159) and (160) show that all the X_{ij}'s with $1 \leq i < j \leq n-1$ annihilate the vectors V_k's, $1 \leq k \leq n-2$. Thus, we have shown that the first $\binom{n-1}{2}$ rows of the matrix $T(n)$ annihilate the vectors $V_1, \ldots, V_{n-2}$. And with the work from before, all the rows of $T(n)$ annihilate in fact the vectors $V_1, \ldots, V_{n-2}$. Thus, these vectors belong to $K(n)$. Since we picked the vectors $v_1, v_2, \ldots, v_{k(n-1)}$ to form a basis of $K(n-1)$ and since $K(n-1)$ is contained in $K(n)$, the latter vectors also belong to $K(n)$. Finally, it is visible that all the participating vectors in Claim 3 are linearly independent. thus, we conclude that the claim holds and Proposition 11 as well.

Corollary 9. *Let $n \geq 4$. Assume that $r^{n+1} \neq -1,\ r^{n+2} \neq -1,\ \ldots,\ r^{2n} \neq -1$. If there exists an irreducible $\frac{n(n-3)}{2}$-dimensional invariant subspace of $\mathcal{V}$, then $l = r$.*

PROOF: we proceed by induction on n. When $n = 4$, if there exists an irreducible 2-dimensional invariant subspace of $\mathcal{V}$, then $l = r$ without any further assumption on r by Result 2. When $n = 5$, if there exists an irreducible 5-dimensional invariant subspace of $\mathcal{V}$ then $l = r$ still without any further assumption on r by Result 1. Let n be an integer with $n \geq 6$. Let's name W the irreducible $\frac{n(n-3)}{2}$-dimensional invariant subspace of $\mathcal{V}$. Consider the intersection $W \cap \mathcal{V}^{(n-1)}$. Since $\dim W = \frac{n(n-3)}{2} > n-1$ for any n greater or equal to 6, this intersection is nontrivial. Hence we get $l \in \{r, -r^3, \frac{1}{r^{n-4}}, -\frac{1}{r^{n-4}}, \frac{1}{r^{2n-5}}\}$ as $\nu^{(n-1)}$ is then reducible. Also, from $W \cap \mathcal{V}^{(n-1)} \subseteq K(n-1)$, we derive the inequality on the dimensions over F:

$$k(n-1) \geq \frac{n(n-3)}{2} - (n-1) = \frac{n^2-5n}{2} + 1 = \frac{(n-1)(n-4)}{2} - 1 \qquad (\mathcal{I})_n$$

We note that $\frac{n^2-5n}{2} + 1 > n-2$ is equivalent to $(n-1)(n-6) > 0$, which is itself equivalent to $\frac{n(n-3)}{2} > 2n-3$. Let's first deal with the case $n \geq 7$, assuming that Corollary 9 holds for $n = 6$. Then $l \in \{r, -r^3, -\frac{1}{r^{n-5}}, \frac{1}{r^{n-5}}, \frac{1}{r^{2n-7}}\}$, as $\nu^{(n-2)}$ is reducible ($\dim \mathcal{W} > 2n-3$). Then $l \in \{r, -r^3\}$. We want to show that $l = r$. Suppose $l = -r^3$. Since $r^{2n} \neq -1$, we have seen in Proposition 6 that $K(n)$ is irreducible. Then $K(n) = W$ and $k(n) = \frac{n(n-3)}{2}$. If $r^{2(n-1)} \neq -1$, then an application of Proposition 11 with $k(n) = \frac{n(n-3)}{2}$ yields:

$$k(n-1) \leq \frac{n(n-3)}{2} - (n-2) = \frac{n^2-5n}{2} + 2 = \frac{(n-1)(n-4)}{2}$$

Also, if $r^{2(n-1)} \neq -1$ and $l = -r^3$, then by Proposition 6, $K(n-1)$ is irreducible. Suppose first $n \neq 9$. Then $n-1 \neq 8$, hence,

$$n-2 < \frac{(n-1)(n-4)}{2} - 1 \leq k(n-1) \leq \frac{(n-1)(n-4)}{2}$$

forces in fact $k(n-1) = \frac{(n-1)(n-4)}{2}$, as there is no irreducible invariant subspace of $\mathcal{V}^{(n-1)}$ of dimension strictly between $(n-2)$ and $\frac{(n-1)(n-4)}{2}$. Also, if $n = 9$ *i.e* $n - 1 = 8$, then the inequality above reads $7 < 19 \leq k(8) \leq 20$. Since the degrees of the irreducible representations of $\mathcal{H}_{F,r^2}(8)$ are $1, 7, 14, 20, 21$ and degrees higher or equal to 28, the case $n = 9$ is not different. Thus we get $k(n-1) = \frac{(n-1)(n-4)}{2}$. Then by induction hypothesis, the existence of an irreducible $\frac{(n-1)(n-4)}{2}$-dimensional invariant subspace of $\mathcal{V}^{(n-1)}$ implies $l = r$. This is a contradiction with $l = -r^3$. Hence our hypothesis $l = -r^3$ was absurd and $l = r$ as announced. It remains to study the case $n = 6$. When $n = 6$, we find another argument to claim that the intersection $W \cap \mathcal{V}^{(4)}$ is non trivial. Indeed, it suffices to notice that $\dim W + \dim \mathcal{V}^{(4)} = 9 + 6 = 15 = \dim \mathcal{V}^{(6)}$. Then, if the sum $W + \mathcal{V}^{(4)}$ is direct, it comes $W \oplus \mathcal{V}^{(4)} = \mathcal{V}^{(6)}$. Acting with e_5 on both sides yields $e_5.\mathcal{V}^{(6)} = 0$, which is a contradiction. Hence $W \cap \mathcal{V}^{(4)} \neq 0$ and again in the case $n = 6$ we must have $l \in \{r, -r^3\}$. In particular, since $r^{10} \neq -1$, this implies that $K(5)$ is irreducible. Further, the inequality $(\mathcal{I})_6$ reads $k(5) \geq 4$. Thus, we have $k(5) \in \{4, 5, 6\}$. Now $k(5) = 4$ is to eliminate as the existence of an irreducible 4-dimensional invariant subspace of $\mathcal{V}^{(5)}$ would force $l \in \{\frac{1}{r^2}, -\frac{1}{r^2}\}$. Hence $k(5) \in \{5, 6\}$. We show that in any case this forces $l = r$. And indeed if $k(5) = 5$, it is automatic with Result 1. Suppose now $k(5) = 6$. We show that it is not possible to have $l = -r^3$. If $l = -r^3$, since $r^{10} \neq -1$, an application of Proposition 11 with $n = 6$ yields:

$$k(6) \geq k(5) + 4$$

Hence it comes $k(6) \geq 6 + 4 = 10$. But since $l = -r^3$ and $r^{12} \neq -1$, $K(6)$ is irreducible by Proposition 6. Hence $K(6) = W$ and $k(6) = 9$: contradiction.

Thus, we must have $l = r$. This ends the proof of Corollary 9. The next Corollary specifies the cases $r^{2n} = -1$ and uses the result of Corollary 9.

Corollary 10. *Let n be an integer with $n \geq 4$. If $r^{2n} = -1$ and there exists an irreducible $\frac{n(n-3)}{2}$-dimensional invariant subspace of $\mathcal{V}$, then*

$$\begin{array}{lllll} \textit{Either} & k(n) = \frac{n(n-3)}{2}, & k(n-1) = \frac{(n-1)(n-4)}{2} & \textit{and} & l = r \\ \textit{Or} & k(n) = \frac{(n-1)(n-2)}{2}, & k(n-1) = \frac{(n-2)(n-3)}{2} & \textit{and} & l = -r^3 \end{array}$$

PROOF: Let's first assume $n \geq 6$. Following the proof of Corollary 9, the existence of an irreducible $\frac{n(n-3)}{2}$-dimensional invariant subspace of $\mathcal{V}$ forces $l \in \{r, -r^3, \frac{1}{r^{n-4}}, -\frac{1}{r^{n-4}}, \frac{1}{r^{2n-5}}\}$ and $l \in \{r, -r^3, \frac{1}{r^{n-5}}, -\frac{1}{r^{n-5}}, \frac{1}{r^{2n-7}}\}$, which forces in turn $l \in \{r, -r^3\}$. If $l = r$, we already know from Proposition 10 that $k(n-1) = \frac{(n-1)(n-4)}{2}$ and $k(n) = \frac{n(n-3)}{2}$. If $l = -r^3$, since $r^{2n} = -1$, there exists a one-dimensional invariant subspace of $\mathcal{V}$, say $\mathcal{V}_1$. By hypothesis, there also exists an irreducible $\frac{n(n-3)}{2}$-dimensional invariant subspace, say $\mathcal{V}_{\frac{n(n-3)}{2}}$, of $\mathcal{V}$. The two vector spaces must be in direct sum as $\mathcal{V}_{\frac{n(n-3)}{2}}$ is irreducible. Hence we have $\mathcal{V}_1 \oplus \mathcal{V}_{\frac{n(n-3)}{2}} \subseteq K(n)$. If this inclusion is strict, then $k(n) > \frac{(n-1)(n-2)}{2}$ and $\mathcal{V}_1 \oplus \mathcal{V}_{\frac{n(n-3)}{2}}$ has a summand in $K(n)$ whose dimension is strictly less than $n-1$ (recall that $\frac{n(n-1)}{2} - \frac{(n-1)(n-2)}{2} = n-1$). This is a contradiction. Hence $K(n) = \mathcal{V}_1 \oplus \mathcal{V}_{\frac{n(n-3)}{2}}$ and $k(n) = \frac{(n-1)(n-2)}{2}$. To complete the proof, it remains to show that $k(n-1) = \frac{(n-2)(n-3)}{2}$. Since $\frac{(n-1)(n-2)}{2} > n-1$, we have $K(n) \cap \mathcal{V}^{(n-1)} \neq 0$. Hence $K(n) \cap \mathcal{V}^{(n-1)} \subseteq K(n-1)$. It follows on the dimensions over F that:

$$k(n-1) \geq \frac{(n-1)(n-2)}{2} - (n-1) = \frac{(n-1)(n-4)}{2}$$

Here comes the use of Corollary 9: since $r^{2n} = -1$, we have $r^n \neq -1$, $r^{n+1} \neq$

$-1, \ldots, r^{2(n-1)} \neq -1$. In particular $r^{2(n-1)} \neq -1$ implies that $K(n-1)$ is irreducible. If $k(n-1) = \frac{(n-1)(n-4)}{2}$, a licit application of Corollary 9 yields $l = r$, a contradiction with $l = -r^3$. Thus, the inequality above can be bettered:

$$k(n-1) \geq \frac{(n-2)(n-3)}{2}$$

As for the other way, again since $r^{2n} = -1$, we have $r^{2(n-1)} \neq -1$. Hence we may apply Proposition 11. It provides us with the inequality: $\frac{(n-1)(n-2)}{2} \geq k(n-1) + (n-2)$, *i.e*

$$k(n-1) \leq \frac{(n-2)(n-3)}{2}$$

Gathering the two inequalities finally yields:

$$k(n-1) = \frac{(n-2)(n-3)}{2}$$

Finally the Corollary is true when $n = 5$ and in that case we know from before that $l = r$ and $k(5) = 5$ and $k(4) = 2$. As for $n = 4$, we also have $l = r$ and $k(4) = 2$. Moreover, it is true that $k(3) = 0$ since for $l = r$ the representation $\nu^{(3)}$ is irreducible by Theorem 7.

Joining the results of Proposition 6 and Corollary 10 adds a bit of information to Proposition 6:

Corollary 11. *Let n be an integer with $n \geq 4$. If $l = -r^3$ and $r^{2n} = -1$, then either there exists an irreducible $\frac{n(n-3)}{2}$-dimensional invariant subspace of $\mathcal{V}$ and $k(n) = \frac{(n-1)(n-2)}{2}$ or $k(n) > \frac{(n-1)(n-2)}{2}$*

PROOF: first the corollary holds for $n = 4$ and $n = 5$. Indeed, for $n = 4$, if $l = -r^3$, there does not exist any irreducible 2-dimensional invariant subspace of $\mathcal{V}$ (otherwise $l = r$ by Result 2). On the contrary, there exists an irreducible 3-dimensional invariant subspace of $\mathcal{V}$ by Theorem 6 and there exists a one-dimensional invariant subspace of $\mathcal{V}$ since $l = -r^3 = \frac{1}{r^5}$ ($r^8 = -1$ by hypothesis). Those two invariant subspaces are in direct sum. Hence $k(4) > 3$. When $n = 5$, there does not exist any irreducible 5-dimensional invariant subspace of $\mathcal{V}$ (otherwise $l = r$ by Result 1). By Theorem 8 there exists an irreducible 6-dimensional invariant subspace of $\mathcal{V}$. This subspace must be in direct sum with the existing one-dimensional one. Hence $k(5) > 6$. Suppose now $n \geq 6$. If there exists an irreducible $\frac{n(n-3)}{2}$-dimensional invariant subspace of $\mathcal{V}$, then by Corollary 10, case $l = -r^3$, we have $k(n) = \frac{(n-1)(n-2)}{2}$. Suppose that there does not exist any irreducible $\frac{n(n-3)}{2}$-dimensional invariant subspace of $\mathcal{V}$. Since $l = -r^3$ and $r^{2n} = -1$, we know from Proposition 6 that $K(n)$ is reducible and $k(n) \geq \frac{(n-1)(n-2)}{2}$. Since with our assumptions there must exist a one-dimensional invariant subspace $\mathcal{V}_1$ of $\mathcal{V}$, if we had $k(n) = \frac{(n-1)(n-2)}{2}$, then this one-dimensional invariant subspace of $\mathcal{V}$ would have an irreducible $\frac{n(n-3)}{2}$-dimensional summand in $K(n)$, impossible by hypothesis. Thus, $k(n) \geq \frac{(n-1)(n-2)}{2} + 1$.

Corollary 12. *Let n be an integer with $n \geq 4$. If $l = -r^3$ and $r^{2n} = -1$, then either there exists an irreducible $\frac{n(n-3)}{2}$-dimensional invariant subspace of $\mathcal{V}$ and $k(n) = \frac{(n-1)(n-2)}{2}$ or there exists an irreducible $\frac{(n-1)(n-2)}{2}$-dimensional invariant subspace of $\mathcal{V}$ and $k(n) = 1 + \frac{(n-1)(n-2)}{2}$.*

PROOF: by Corollary 11, it suffices to prove that if there does not exist an irreducible $\frac{n(n-3)}{2}$-dimensional invariant subspace of $\mathcal{V}$, then there exists an irreducible $\frac{(n-1)(n-2)}{2}$-dimensional invariant subspace of $\mathcal{V}$ and $k(n) = 1 + \frac{(n-1)(n-2)}{2}$.

Let's first deal with the case $n = 4$. In this case, there does not exist any irreducible 2-dimensional invariant subspace of $\mathcal{V}$ (otherwise $l = r$, impossible with $l = -r^3$). Further, by choice of l and r, there exists an irreducible 3-dimensional invariant subspace of $\mathcal{V}$ and a one-dimensional invariant subspace of $\mathcal{V}$ and their sum is direct. Then by uniqueness of the one-dimensional invariant subspace of $\mathcal{V}$, it is impossible to have $k(4) = 5$. Hence $k(4) = 4$. This ends the case $n = 4$.

When $n = 5$, by choice of l and r, we know that there exists an irreducible 6-dimensional invariant subspace of $\mathcal{V}$ (cf Theorem 8) and there exists a one-dimensional invariant subspace of $\mathcal{V}$. Their sum is direct. It forbids $k(5) = 8$ or $k(5) = 9$. Hence $k(5) = 7$. Also, since $l \neq r$, there does not exist any irreducible 5-dimensional invariant subspace of $\mathcal{V}$ by Result 1, hence we are done with the case $n = 5$.

Suppose now $n \geq 6$ and suppose that there does not exist any irreducible $\frac{n(n-3)}{2}$-dimensional invariant subspace of $\mathcal{V}$. By Corollary 11, we have $k(n) \geq 1 + \frac{(n-1)(n-2)}{2}$. In particular $k(n) > n - 1$, hence the intersection $K(n) \cap \mathcal{V}^{(n-1)}$ is non-trivial. Further, since $r^{2n} = -1$ by hypothesis, we have $r^{2(n-1)} \neq -1$. Hence by Proposition 6, $K(n-1)$ is irreducible. Thus we get $K(n) \cap \mathcal{V}^{(n-1)} = K(n-1)$. From there, we have:

$$k(n-1) = k(n) + \frac{(n-1)(n-2)}{2} - \dim(K(n) + \mathcal{V}^{(n-1)})$$

Hence, we get:

$$k(n-1) \geq k(n) - (n-1)$$

that we rewrite:

$$k(n) \leq k(n-1) + (n-1)$$

Since we already know that $k(n) \geq 1 + \frac{(n-1)(n-2)}{2}$, it suffices to show that $k(n-1) = \frac{(n-2)(n-3)}{2}$ to get the desired result. To that aim, we show a lemma:

Lemma 9. *Let* $n \geq 3$. *If* $r^{n+1} \neq -1, r^{n+2} \neq -1, \ldots, r^{2n} \neq -1$ *and* $l = -r^3$, *then* $k(n) = \frac{(n-1)(n-2)}{2}$.

PROOF OF THE LEMMA: when $n = 3$ and $l = -r^3$ and $r^6 \neq -1$, there exists a unique one-dimensional invariant subspace of $\mathcal{V}$. Moreover, there does not exists any irreducible 2-dimensional invariant subspace as $l \notin \{-1, 1\}$. Thus we have $k(3) = 1$. When $n = 4$ and $l = -r^3$, there exists an irreducible 3-dimensional invariant subspace of $\mathcal{V}$. Moreover, since $l \neq r$, there does not exist any irreducible 2-dimensional invariant subspace of $\mathcal{V}$. Hence $k(4) \neq 5$. Also since $r^8 \neq -1$, there does not exist any one-dimensional invariant subspace of $\mathcal{V}$, hence $k(4) \neq 4$. Thus, we have $k(4) = 3$. Let's also do the case $n = 5$. By Theorem 8, there exists an irreducible 6-dimensional invariant subspace of $\mathcal{V}$. Hence $k(5) \geq 6$. Since there does not exist any one-dimensional invariant subspace of $\mathcal{V}$, $k(5)$ cannot equal $7, 8$ or 9. Thus we have $k(5) = 6$. Let $n \geq 6$. Under the assumptions on l and r, we know that there does not exist any one-dimensional invariant subspace of $\mathcal{V}$ by Theorem 4, there does not exist any irreducible $(n-1)$-dimensional invariant subspace of $\mathcal{V}$ by Theorem 5 and there does not exist any irreducible $\frac{n(n-3)}{2}$-dimensional invariant subspace of $\mathcal{V}$ by Corollary 9. Moreover, by Proposition 6 when $l = -r^3$ and $r^{2n} \neq -1$, we know that $K(n)$ is irreducible. We recall that when $n = 6, 7$ or $n \geq 9$, the irreducible representations of $\mathcal{H}_{F,r^2}(n)$ have degree 1, $n-1$, $\frac{n(n-3)}{2}$, $\frac{(n-1)(n-2)}{2}$ or a degree greater than $\frac{(n-1)(n-2)}{2}$. Hence, for these values of n, we must have $k(n) \geq \frac{(n-1)(n-2)}{2}$. When $n = 8$, we must have $k(8) \in \{14, 20, 21\}$. Since $k(7) \geq 15$ and since $K(7) \subset K(8)$ by Proposition 9 (as $r^{14} \neq -1$), it is impossible to have $k(8) = 14$. Hence, the case $n = 8$ is not exceptional and we must have $k(8) = 21$. In any case, we have $k(n) \geq \frac{(n-1)(n-2)}{2}$. We will show conversely that $k(n) \leq \frac{(n-1)(n-2)}{2}$. It is equivalent to show that the rank of the matrix $T(n)$ is greater or equal to $n-1$. We have the lemma:

Lemma 10. *Let n be an integer with $n \geq 5$. Suppose $r^{2n} \neq -1$. Then,*

$$l = -r^3 \Longrightarrow rk(T(n)) \geq n-1$$

PROOF OF THE LEMMA: we computed with Maple the determinant of the submatrix of $T(5)$, composed of subcolumns $1,3,4,7$ and subrows $1,3,4,7$. We define:

$$S(5) := \text{submatrix}(T(5), [1,3,4,7], [1,3,4,7])$$

in Maple notations. The value of the determinant of $S(5)$ is $\frac{1+r^4+r^8+r^{12}+r^{16}}{r^8}$. For future reference, define $\mathcal{R}_5 = \mathcal{C}_5 := [1,3,4,7]$. Next, given $n \geq 6$, we inductively build from $S(n-1)$ a submatrix $S(n)$ of $T(n)$ by adding the subrow correponding to the action of $X_{n-1,n}$ and the subcolumn corresponding to the vector $w_{4,n}$. Explicitly, we have:

$$S(n) := \text{submatrix}(T(n), \mathcal{R}_{n-1}@[\binom{n-1}{2}+1]), \mathcal{C}_{n-1}@[\binom{n-1}{2}+(n-4)])$$

And on Figure 2 is how the matrix looks like. We will show that when $r^{2n} \neq -1$, the determinant of this matrix is nonzero. In fact we have:

Proposition 12.

$$det(S(n)) = (-1)^{n+1}\left[\frac{1+r^4+\cdots+r^{4(n-1)}}{r^{8+\frac{n(n-5)}{2}}}\right]$$

PROOF OF THE PROPOSITION: by construction, there are only two nonzero coefficients on each of the $(\binom{k}{2}+1)$-th rows of the matrix $S(n)$ for $k =$

$5, \ldots, n-1$ and they are respectively given by:

$$[X_{k,k+1}.w_{4,k}]_{w_{k,k+1}} = \frac{1}{r^{k-5}} \qquad \text{by } (SL)_{k-4} \tag{161}$$

$$[X_{k,k+1}.w_{4,k+1}]_{w_{k,k+1}} = -\frac{1}{r^3.r^{k-5}} \qquad \text{by } (TL)_{k-4} \tag{162}$$

These coefficients are the ones corresponding to the columns $\mathcal{C}_{\binom{k-1}{2}+(k-4)}$ and $\mathcal{C}_{\binom{k}{2}+(k-3)}$. In particular, doing the operation

$$\mathcal{C}_{\binom{n-2}{2}+(n-5)} \leftarrow \mathcal{C}_{\binom{n-2}{2}+(n-5)} + r^3\,\mathcal{C}_{\binom{n-1}{2}+(n-4)}$$

on the columns makes a zero appear on the last row, making all the coefficients of the last row zero except $-\frac{1}{r^3.r^{n-6}}$. Hence the determinant of $S(n)$ is:

$$\det(S(n)) = -\frac{1}{r^3}\,\frac{1}{r^{n-6}}\,\det(\tilde{S}(n-1))$$

where $\tilde{S}(n-1)$ is obtained from $S(n-1)$ by replacing $\mathcal{C}_{\binom{n-2}{2}+(n-5)}(\mathcal{R}_{n-1})$ with $\mathcal{C}_{\binom{n-2}{2}+(n-5)}(\mathcal{R}_{n-1}) + r^3\,\mathcal{C}_{\binom{n-1}{2}+(n-4)}(\mathcal{R}_{n-1})$.
It comes:

$$\det(\tilde{S}(n-1)) = \det(S(n-1))$$
$$+r^3\det(\mathcal{C}_1(\mathcal{R}_{n-1}), \ldots, \mathcal{C}_{\binom{n-3}{2}+(n-6)}(\mathcal{R}_{n-1}), \widehat{\mathcal{C}_{\binom{n-2}{2}+(n-5)}}(\mathcal{R}_{n-1}), \mathcal{C}_{\binom{n-1}{2}+(n-4)}(\mathcal{R}_{n-1}))$$

The second determinant in the sum above is:

$$(-1)(-\frac{1}{r})(-\frac{1}{r^2})\ldots(-\frac{1}{r^{n-7}})\det([1,3,4,7],[1,3,4,\binom{n-1}{2}+(n-4)])$$

We computed with Maple $\det([1,3,4,7],[1,3,4,12])$ and found the value r^9.

Then,

$$\det([1,3,4,7],[1,3,4,\binom{n-1}{2}+(n-4)]) = r^{n+3}$$

Thus, we get:

$$\det(S(n)) = -\frac{1}{r^3}\,\frac{1}{r^{n-6}}\det S(n-1) - \frac{1}{r^{n-6}}\,(-1)^n\,\frac{r^{n+3}}{r^{\frac{(n-6)(n-7)}{2}}}$$

Let's proceed by induction on n and assume that Proposition 12 holds for $S(n-1)$. Then, replacing $\det(S(n-1))$ by its value yields the new equality:

$$\det(S(n)) = (-1)^{n+1}\,\frac{1}{r^{n-6}}\left[\frac{1+r^4+\cdots+r^{4(n-2)}}{r^{11+\frac{(n-1)(n-6)}{2}}} + \frac{r^{n+3}}{r^{\frac{(n-6)(n-7)}{2}}}\right]$$

And by reducing to the same denominator, we get:

$$\det(S(n)) = (-1)^{n+1}\left[\frac{1+r^4+\cdots+r^{4(n-1)}}{r^{8+\frac{n(n-5)}{2}}}\right]$$

As mentioned at the beginning of the proof of Lemma 10, the formula in Proposition 12 holds for $n=5$. Furthermore, we deduce from it the value for $\det(S(6))$. Indeed, by adding to the 7-th column r^3 times the 12-th column, we see that:

$$\begin{aligned}
\det(S(6)) &= -\frac{1}{r^3}\times\Big(\det(S(5)) + r^3\det([1,3,4,7],[1,3,4,12])\Big)\\
&= -\frac{1}{r^3}\times\Big(\frac{1+r^4+r^8+r^{12}+r^{16}}{r^8} + r^3.\,r^9\Big)\\
&= -\,\frac{1+r^4+r^8+r^{12}+r^{16}+r^{20}}{r^{11}}
\end{aligned}$$

So Proposition 12 also holds for $n=6$. Then, by induction, Proposition 12 holds for every $n \geq 5$.

$$
\begin{array}{c|cccccccccccc}
 & & & & & & & w_{46} & & w_{47} & & & w_{4n} & \\
 & 1 & 2 & 3 & 4 & 7 & 10 & 12 & 15 & 18 & \dots & \binom{n-1}{2} & \binom{n-1}{2}+(n-4) & \binom{n}{2} \\
\hline
 & \square & & \square & \square & \square & & 0 & & 0 & & & 0 & \\
 & \square & & \square & \square & \square & & 0 & & 0 & & & 0 & \\
 & \square & & \square & \square & \square & & r & & r^2 & & & r^{n-5} & \\
 & \square & & \square & \square & \square & & r & & r^2 & & & r^{n-5} & \\
X_{56} & 0 & & 0 & 0 & 1 & & -\frac{1}{r^3} & & 0 & & & 0 & \\
X_{67} & 0 & & 0 & 0 & 0 & & \frac{1}{r} & & -\frac{1}{r^3}\ \frac{1}{r} & & & 0 & \\
 & & & & & & & & & & \ddots & & & \\
X_{n-1,n} & 0 & & 0 & 0 & 0 & & 0 & & 0 & & & -\frac{1}{r^3}\ \frac{1}{r^{n-6}} & \\
\end{array}
$$

Figure 2

This achieves the proof of Lemma 9. Let's go back to the proof of the Corollary. Since when $r^{2n} = -1$, we have

$$r^n \neq -1,\ r^{n+1} \neq -1,\ \dots,\ r^{2(n-1)} \neq -1,$$

by Lemma 9, we get $k(n-1) = \frac{(n-2)(n-3)}{2}$. As already explained above, it follows that $k(n) = 1 + \frac{(n-1)(n-2)}{2}$. Then there exists an irreducible

$\frac{(n-1)(n-2)}{2}$-dimensional invariant subspace of $\mathcal{V}$. This ends the proof of Corollary 12. Thus, Proposition 6 can be slightly bettered and rewritten in the following way:

Proposition 13.

Let n be an integer with $n \geq 4$.
When $l = r$, $K(n)$ is always irreducible.

Let n be an integer with $n \geq 3$.
When $l = -r^3$, there are two cases:

1) $r^{2n} \neq -1$ *and* $K(n)$ *is irreducible*
2) $r^{2n} = -1$ *and* $K(n)$ *is reducible. Moreover, when $n \geq 4$,*

   $a)$ *Either there exists an irreducible $\frac{n(n-3)}{2}$-dimensional invariant subspace, $K(n)$ is the direct sum of an irreducible $\frac{n(n-3)}{2}$ dimensional invariant subspace and of the unique one-dimensional invariant subspace and*
   $k(n) = \frac{(n-1)(n-2)}{2}$.

   $b)$ *Or there exists an irreducible $\frac{(n-1)(n-2)}{2}$-dimensional invariant subspace $K(n)$ is the direct sum of an irreducible $\frac{(n-1)(n-2)}{2}$-dimensional invariant subspace and of the unique one-dimensional invariant subspace and*
   $k(n) = 1 + \frac{(n-1)(n-2)}{2}$.

(Case $n = 3$) There exists exactly two one-dimensional invariant subspaces of $\mathcal{V}$ and $K(3)$ is the direct sum of these two one-dimensional invariant subspaces.

In the case $n = 5$, we have a Corollary of this Proposition:

Corollary 13. *If $l = -r^3$ and if $r^{10} = -1$, then:*

$$K(5) + \mathcal{V}^{(4)} = \mathcal{V}^{(5)}$$

PROOF: Suppose that $r^{10} = -1$. Then $r^8 \neq -1$, so $K(4)$ is irreducible. If $K(5)+\mathcal{V}^{(4)}$ is strictly contained in $\mathcal{V}^{(5)}$, then its dimension as a vector space over F is less than 10. Then it comes:

$$\dim(K(5) \cap \mathcal{V}^{(4)}) = k(5) + 6 - \dim(K(5) + \mathcal{V}^{(4)}) \geq k(5) + 6 - 9 = k(5) - 3$$

Since by Proposition 13, we know that $k(5) \in \{6, 7\}$, the fact that $k(5) > 4$ implies that the intersection $K(5) \cap \mathcal{V}^{(4)}$ is nonzero. Thus by irreducibility of $K(4)$, we have $K(5) \cap \mathcal{V}^{(4)} = K(4)$ and the inequality above reads:

$$k(4) \geq k(5) - 3$$

Moreover, when $l = -r^3$, there exists an irreducible 3-dimensional invariant subspace of $\mathcal{V}^{(4)}$. Since $K(4)$ is irreducible, it must be $K(4)$. Hence $k(4) = 3$. Thus, the inequality above becomes:

$$k(5) \leq 6$$

Then $k(5) = 6$. Then by point $2)a)$ of Proposition 13, there must exist an irreducible 5-dimensional invariant subspace of $\mathcal{V}^{(5)}$. But this forces $l = r$ by Result 1: a contradiction. Thus, the vector spaces $K(5) + \mathcal{V}^{(4)}$ and $\mathcal{V}^{(5)}$ have the same dimension and they are actually equal.

In fact Corollary 13 generalizes to each n by noticing that the linearly independent set of vectors

$$\boxed{S := \mathcal{V}^{(n-1)} \cup \{V_1, \ldots, V_{n-2}\}}$$

of $K(n) + \mathcal{V}^{(n-1)}$ of cardinality $\frac{n(n-1)}{2} - 1$ does not span the vector space $K(n) + \mathcal{V}^{(n-1)}$ when $r^{2n} = -1$. Explicitly, we will prove the following Proposition:

Proposition 14. *Let n be an integer with $n \geq 5$. Suppose $l = -r^3$ and $r^{2n} = -1$. Then,*

$$K(n) + \mathcal{V}^{(n-1)} = \mathcal{V}^{(n)}$$

PROOF. Suppose that $K(n) + \mathcal{V}^{(n-1)} = \mathrm{Span}_F(V_1, \ldots, V_{n-2}) \oplus \mathcal{V}^{(n-1)}$. If $l = -r^3$ and $r^{2n} = -1$, then $l = \frac{1}{r^{2n-3}}$. Hence, there exists an irreducible 1-dimensional invariant subspace of $\mathcal{V}^{(n)}$. Moreover, by Theorem 4, it is spanned by

$$\sum_{1 \leq s < t \leq n} r^{s+t}\, w_{st}$$

Thus, if the equality above holds, the vector

$$w_{1,n} + r\, w_{2,n} + r^2\, w_{3,n} + \cdots + r^{n-2}\, w_{n-1,n}$$

must be a linear combination with coefficients in F of the $(n-2)$ vectors $w_{k+1,n} - r\, w_{k,n}$ where $k = 1, \ldots, n-2$. A direct verification shows right away that this is impossible. Then the set S is a linearly independent set of $K(n) + \mathcal{V}^{(n-1)}$ of cardinality $\frac{n(n-1)}{2} - 1$, which does not span $K(n) + \mathcal{V}^{(n-1)}$. This shows that the dimension of $K(n) + \mathcal{V}^{(n-1)}$ must be greater than $\frac{n(n-1)}{2} - 1$, hence must in fact equal $\frac{n(n-1)}{2}$, the dimension of $\mathcal{V}^{(n)}$. Thus, $K(n) + \mathcal{V}^{(n-1)} = \mathcal{V}^{(n)}$, as announced.

Proposition 14 now allows us to give a more accurate version of Proposition 13, point $2)$, as it shows that point $a)$ cannot occur.

Proposition 15.

Let n be an integer with $n \geq 4$.
When $l = r$, $K(n)$ is always irreducible.

Let n be an integer with $n \geq 3$.
When $l = -r^3$, there are two cases:

$1)$ $r^{2n} \neq -1$ *and* $K(n)$ *is irreducible*
$2)$ $r^{2n} = -1$ *and* $K(n)$ *is reducible.*

> *Moreover, when $n \geq 4$, $K(n)$ is the direct sum of an irreducible $\frac{(n-1)(n-2)}{2}$-dimensional invariant subspace and of the unique one-dimensional invariant subspace and $k(n) = 1 + \frac{(n-1)(n-2)}{2}$.*
>
> *(Case $n = 3$) There exists exactly two one-dimensional invariant subspaces of $\mathcal{V}$ and $K(3)$ is the direct sum of these two one-dimensional invariant subspaces.*

PROOF: Point $1)$ was already proven in the proof of Proposition 13 or its former version, and there is nothing more to add. Hence, let's prove the new version of point $2)$. Assume first $n \geq 5$. Suppose that $l = -r^3$ and $r^{2n} = -1$. By Proposition 13, point $1)$, $K(n-1)$ is irreducible, as under these assumptions on l and r, we have $l = -r^3$ and $r^{2(n-1)} \neq -1$. Moreover, by Proposition 13 point $2)$ this time, $k(n)$ is "big enough" so that $K(n) \cap \mathcal{V}^{(n-1)}$

cannot be trivial. Explicitly we have $\frac{(n-1)(n-2)}{2} > n-1$ for all $n \geq 5$.

$$\begin{array}{rrcl} \text{Hence we have:} & K(n) \cap \mathcal{V}^{(n-1)} & = & K(n-1) \\ \text{By Proposition } 14\text{, we also have:} & K(n) + \mathcal{V}^{(n-1)} & = & \mathcal{V}^{(n)} \end{array}$$

Both equalities yields the equality on the dimensions:

$$\begin{aligned} k(n-1) &= k(n) + \frac{(n-1)(n-2)}{2} - \frac{n(n-1)}{2} \\ &= k(n) - (n-1) \end{aligned}$$

Thus, we get:

$$k(n) = k(n-1) + (n-1) \tag{163}$$

Following the result of Proposition 13, only two cases are possible: $2)a)$ or $2)b)$. Suppose $2)a)$ holds. Then there exists an irreducible $\frac{n(n-3)}{2}$-dimensional invariant subspace of $\mathcal{V}^{(n)}$. By Corollary 10 we must have

$$k(n) = \frac{(n-1)(n-2)}{2} \qquad \& \qquad k(n-1) = \frac{(n-2)(n-3)}{2},$$

a contradiction with (163). Hence the hypothesis $2)a)$ was absurd and $2)b)$ holds: there exists an irreducible $\frac{(n-1)(n-2)}{2}$-dimensional invariant subspace of $\mathcal{V}^{(n)}$ and $k(n) = 1 + \frac{(n-1)(n-2)}{2}$. To achieve the proof, it remains to deal with the case $n = 4$. In that case, there exists a one-dimensional invariant subspace and an irreducible 3-dimensional one, and they must be in direct sum. Hence, again, situation $2)b)$ holds.
In turn, Corollary 10 can be rewritten:

Corollary 14. *Let n be an integer with $n \geq 4$. Suppose $r^{2n} = -1$. If there exists an irreducible $\frac{n(n-3)}{2}$-dimensional invariant subspace of $\mathcal{V}$, then $l = r$.*

9.3 A Proof of Theorems C and D

We are now in a position to give a complete characterization for the dimension of the irreducible representations. We gather our two main results in the following theorems:

Theorem 9. *Let n be an integer with $n \geq 4$. There exists an irreducible $\frac{n(n-3)}{2}$-dimensional invariant subspace of $\mathcal{V}$ if and only if $l = r$.*

Theorem 10. *Let n be an integer with $n \geq 5$. There exists an irreducible $\frac{(n-1)(n-2)}{2}$-dimensional invariant subspace of $\mathcal{V}$ if and only if $l = -r^3$.*

Before we start the joint proof of these two theorems, let's gather some known facts from earlier. In chronological order, we have the following results:

- For $n \geq 4$: if $l = r$, then there exists an irreducible $\frac{n(n-3)}{2}$-dimensional invariant subspace of $\mathcal{V}$. *(Corollary* 6*)*
- For $n \geq 5$: if there exists an irreducible $\frac{(n-1)(n-2)}{2}$-dimensional invariant subspace of $\mathcal{V}$, then $l = -r^3$. *(Corollary*7*)*
- For $n \geq 4$: $\mathcal{V}$ cannot contain both an irreducible $\frac{n(n-3)}{2}$-dimensional invariant subspace and an irreducible $\frac{(n-1)(n-2)}{2}$-dimensional one. *(Corollary* 8*)*
- For $n \geq 6$ and $r^{2(n-1)} \neq -1$: if $l = -r^3$, then $k(n) \geq k(n-1) + (n-2)$. *(Proposition* 11*)*
- For $n \geq 4$ and $r^{n+1} \neq -1$, $r^{n+2} \neq -1$, $\ldots$, $r^{2n} \neq -1$: if there exists an irreducible $\frac{n(n-3)}{2}$-dimensional invariant subspace of $\mathcal{V}$ then $l = r$. *(Corollary* 9*)*

- For $n \geq 5$ and $r^{2n} \neq -1$: if $l = -r^3$, then $k(n) \leq \frac{(n-1)(n-2)}{2}$.
  (Lemma 10*)*

- For $n \geq 3$ and $r^{2n} \neq -1$: if $l = -r^3$, then $K(n)$ is irreducible
  (Proposition 15*)*

- For $n \geq 4$ and $r^{2n} = -1$: if $l = -r^3$, then there exists an irreducible $\frac{(n-1)(n-2)}{2}$-invariant subspace of $\mathcal{V}$. Moreover, $k(n) = \frac{(n-1)(n-2)}{2} + 1$.
  (Proposition 15*)*

- For $n \geq 4$ and $r^{2n} = -1$: if there exists an irreducible $\frac{n(n-3)}{2}$-dimensional invariant subspace of $\mathcal{V}$, then $l = r$. *(Corollary* 14*)*

THE PROOF ITSELF: Given $n \geq 5$, it remains to show that if $r^{2n} \neq -1$ and $r^{2s} = -1$ for some integer $\frac{n+1}{2} \leq s \leq n-1$, then the following two statements hold:

statement 1: if there exists an irreducible $\frac{n(n-3)}{2}$-dimensional invariant subspace of $\mathcal{V}$, then $l = r$. $(S1)$
statement 2: if $l = -r^3$, then there exists an irreducible $\frac{(n-1)(n-2)}{2}$-dimensional invariant subspace of $\mathcal{V}$. $(S2)$

We will prove by induction that:

$$\forall n \geq 5, (\mathcal{P}_n)$$

where

$$(\mathcal{P}_n) : \text{if } r^{2n} \neq -1 \text{ and } r^{2s} = -1, \text{ some integer } \frac{n+1}{2} \leq s \leq n-1, \text{ then } (S1) \text{ and } (S2)$$

First, $(\mathcal{P}_5)$ holds: $(S1)$ is true by Result 1; $(S2)$ is true by Theorem 8. Let

n be an integer with $n \geq 6$ and suppose that $(\mathcal{P}_k)$ holds for all $5 \leq k \leq n-1$. Let's first deal with the case $r^{2s} = -1$ for some integer $\frac{n+1}{2} \leq s < n-1$. In particular, we have $r^{2(s+1)} \neq -1$ and $r^{2s} = -1$ with $s+1 < n$. By induction hypothesis, $(\mathcal{P}_{s+1})$ then holds. Since we assume $l = -r^3$ in $(S2)$, there exists an irreducible $\frac{s(s-1)}{2}$-dimensional invariant subspace of $\mathcal{V}^{(s+1)}$. Moreover, since for $l = -r^3$ and $r^{2(s+1)} \neq -1$, $K(s+1)$ is irreducible, we get $k(s+1) = \frac{s(s-1)}{2}$. Then we show by induction on l that:

$$\forall s+2 \leq l \leq n,\ k(l) = \frac{(l-1)(l-2)}{2} \tag{164}$$

For $l = s+2$, since $r^{s+1} \neq -1$ and $s \geq 4$, we may apply point number 4 with $n = s+2$. It yields:

$$k(s+2) \geq k(s+1) + s = \frac{s(s-1)}{2} + s = \frac{s(s+1)}{2}$$

Moreover, by point number 6 above, with $n = s+2$, we also have:

$$k(s+2) \leq \frac{s(s+1)}{2}$$

It follows that $k(s+2) = \frac{s(s+1)}{2}$. Let l be an integer with $s+3 \leq l \leq n$ and suppose equation (164) holds for the integer $l-1$. Again, since $r^{2(l-1)} \neq -1$, point number 4 forces:

$$k(l) \geq k(l-1) + (l-2) = \frac{(l-2)(l-3)}{2} + (l-2) = \frac{(l-1)(l-2)}{2}$$

And since $r^{2l} \neq -1$, point 6 again forces:

$$k(l) \leq \frac{(l-1)(l-2)}{2}$$

Thus, equation (164) holds for each $s+1 \leq l \leq n$ and in particular holds for n. Then $K(n)$ is irreducible and $\frac{(n-1)(n-2)}{2}$-dimensional, so that $(S2)$ holds. Now, if there exists an irreducible $\frac{n(n-3)}{2}$-dimensional invariant subspace of $\mathcal{V}$, then l cannot equal $-r^3$ by $(S2)$ and point number 3. Also, if $k(n) \geq \frac{n(n-3)}{2}$, then $K(n) \cap \mathcal{V}^{(n-1)} \neq \{0\}$, otherwise $k(n) \leq n-1$, but $\frac{n(n-3)}{2} \geq n$ for every $n \geq 5$. Assume first $n \geq 7$. Then, if $k(n) \geq \frac{n(n-3)}{2}$, then $K(n) \cap \mathcal{V}^{(n-2)} \neq \{0\}$, otherwise $k(n) \leq 2n-3$, but $\frac{n(n-3)}{2} > 2n-3$ as soon as $n \geq 7$. Then it comes:

$$l \in \left\{r, \frac{1}{r^{2n-5}}, \frac{1}{r^{n-4}}, -\frac{1}{r^{n-4}}\right\} \qquad \& \qquad l \in \left\{r, \frac{1}{r^{2n-7}}, \frac{1}{r^{n-5}}, -\frac{1}{r^{n-5}}\right\}$$

This only leaves the possibility $l = r$. It remains to deal with the case $n = 6$. When $n = 6$, s must be 4 and the only possibilities for l are $l = r$ or $l = \frac{1}{r^9}$. We will show that the second possibility for l is to be excluded. And indeed, if there exists a one-dimensional invariant subspace of $\mathcal{V}^{(6)}$, and an irreducible 9-dimensional one, these must be in direct sum, which forces $k(6) \geq 10$ and in fact $k(6) = 10$. Since $\frac{1}{r^9}$ must equal $\frac{1}{r^2}$ or $-\frac{1}{r^2}$, there must exist a unique irreducible 4-dimensional submodule of $\mathcal{V}^{(5)}$ that is the only submodule of $\mathcal{V}^{(5)}$. Thus, $k(5) = 4$. Now, from the inclusion

$$K(6) \cap \mathcal{V}^{(5)} \subseteq K(5),$$

we derive on the dimensions:

$$k(5) \geq k(6) - 5,$$

so that

$$k(6) \leq k(5) + 5 = 4 + 5 = 9,$$

a contradiction with $k(6) = 10$ as mentioned above. Hence the case $n = 6$ is no exception and $(S1)$ also holds in that case.

Suppose now $r^{2n} \neq -1$ and $r^{2(n-1)} = -1$ and let's show $(S1)$ and $(S2)$ under these assumptions. Let's first do it for $n = 6$. We assume that $r^{12} \neq -1$ and $r^{10} = -1$ and try and show $(S1)$. Suppose there exists an irreducible 9-dimensional invariant subspace of $\mathcal{V}^{(6)}$ and suppose $l = -r^3$. Let's first determine $k(5)$ and $k(6)$. Since $l = -r^3$ and $r^{10} = -1$, there exists a one-dimensional invariant subspace of $\mathcal{V}^{(5)}$. Moreover, since $l = -r^3$, there also exists an irreducible 6-dimensional invariant subspace of $\mathcal{V}^{(5)}$. Moreover, there cannot exist any irreducible 4-dimensional invariant subspace of $\mathcal{V}^{(5)}$ when $l = -r^3$ and there cannot exist any irreducible 5-dimensional invariant subspace as well. Hence we have $k(5) = 7$. Since $r^{12} \neq -1$ and $l = -r^3$, we also know from points 6 and 7 that $k(6) \in \{9, 10\}$. Since there exists an irreducible 9-dimensional invariant subspace of $\mathcal{V}^{(6)}$ by hypothesis, if $k(6)$ were equal to 10, there would also exist a one-dimensional submodule of $\mathcal{V}^{(6)}$, which would force $l = \frac{1}{r^9}$. But when $l = -r^3$ and $r^{12} \neq -1$, this is impossible. Hence we have $k(6) = 9$. Consider now the intersection $K(6) \cap \mathcal{V}^{(5)}$. The $\mathcal{H}_{F,r^2}(5)$-module $K(6) \cap \mathcal{V}^{(5)}$ is contained in $K(6) \downarrow_{\mathcal{H}_{F,r^2}(5)}$. By semisimplicity of $\mathcal{H}_{F,r^2}(5)$, there exists an $\mathcal{H}_{F,r^2}(5)$-submodule S of $K(6) \downarrow_{\mathcal{H}_{F,r^2}(5)}$ which is a summand for $K(6) \cap \mathcal{V}^{(5)}$:

$$K(6) \cap \mathcal{V}^{(5)} \oplus S = K(6) \downarrow_{\mathcal{H}_{F,r^2}(5)} \tag{165}$$

Let's study the dimension of $K(6) \cap \mathcal{V}^{(5)}$. First, we have the inequality:

$$\dim(K(6) \cap \mathcal{V}^{(5)}) \geq 9 + 10 - 15 = 4$$

If the dimension of $K(6)\cap\mathcal{V}^{(5)}$ were 4, $K(6)\cap\mathcal{V}^{(5)}$ would have a 3-dimensional summand in $K(5)$. This is impossible; if the dimension of $K(6)\cap\mathcal{V}^{(5)}$ were 5, $K(6)\cap\mathcal{V}^{(5)}$ would have a 2-dimensional summand in $K(5)$. This is impossible by uniqueness of the one-dimensional invariant subpsace of $\mathcal{V}^{(5)}$. If now $\dim(K(5)\cap\mathcal{V}^{(5)})=6$, then by (165) the dimension of S would be 3. We show that this is impossible.

Lemma 11. *Under the hypothesis $l=-r^3$ and $r^{10}=-1$, it is impossible to have:*

$$\begin{cases} S\subseteq K(6) \\ S \textit{ is a } \mathcal{H}_{F,r^2}(5)\textit{-module} \\ \textit{dim}\, S=3 \end{cases}$$

PROOF OF THE LEMMA: it suffices to show it when $\dim S=1$. We leave the proof for later in a more general setting (cf Lemma 13).

Assuming the lemma holds, the dimension of $K(6)\cap\mathcal{V}^{(5)}$ must hence be 7. But then

$$K(6)\cap\mathcal{V}^{(5)}=K(5),$$

which implies in particular:

$$K(5)\subseteq K(6)$$

But if $K(5)\subseteq K(6)$, the inequality of point 4 above becomes true, although $r^{10}=-1$ and we must have:

$$k(6)\geq k(5)+4$$

With $k(6)=9$ and $k(5)=7$, this inequality yields a contradiction.

Partial conclusion: we have proven that if there exists an irreducible 9-dimensional invariant subspace of $\mathcal{V}^{(6)}$, then l cannot equal $-r^3$. Then $l \in \{\frac{1}{r^9}, r\}$.

We have the general lemma:

Lemma 12. *Let n be an integer with $n \geq 5$. If $r^{2(n-1)} = -1$ and there exists an irreducible $\frac{n(n-3)}{2}$-dimensional invariant subspace of $\mathcal{V}$, then $l \in \{r, -r^3\}$.*

PROOF OF THE LEMMA: Let's denote by W the irreducible $\frac{n(n-3)}{2}$-dimensional invariant subspace of $\mathcal{V}$. For $n \geq 5$, we have $\frac{n(n-3)}{2} > n-1$, hence $W \cap \mathcal{V}^{(n-1)}$ is not trivial and so $l \in \{r, -r^3, \frac{1}{r^{2n-5}}, \frac{1}{r^{n-4}}, -\frac{1}{r^{n-4}}\}$. Also since $\nu^{(n)}$ is reducible we have: $l \in \{r, -r^3, \frac{1}{2n-3}, \frac{1}{r^{n-3}}, -\frac{1}{r^{n-3}}\}$. Thanks to the hypothesis $r^{2(n-1)} = -1$, it is impossible to have

$$\frac{1}{r^{2n-3}} = \frac{\epsilon}{r^{n-4}}$$

as otherwise $r^{2(n+1)} = 1$ and $r^{2(n-1)} = -1$ would force $r^2 = -1$, which is excluded. Since it is also impossible to have $\frac{1}{r^{n-3}} = \frac{\epsilon}{r^{2n-5}}$ and $\frac{1}{r^{n-3}} = \frac{\epsilon}{r^{n-4}}$, we see that $l \in \{r, -r^3\}$.

In fact, we have the more general proposition:

Proposition 16. *Let n be an integer with $n \geq 5$. Lemma 12 holds even without the assumption $r^{2(n-1)} = -1$: if there exists an irreducible $\frac{n(n-3)}{2}$-dimensional invariant subspace of $\mathcal{V}$, then $l \in \{r, -r^3\}$.*

PROOF OF THE PROPOSITION: by an argument repeated many times in the past, the existence of an irreducible $\frac{n(n-3)}{2}$-dimensional invariant subspace of $\mathcal{V}$ implies that $l \in \{r, -r^3, \frac{1}{r^{2n-3}}\}$. If $l = \frac{1}{r^{2n-3}}$, then there must exist a one-dimensional invariant subspace of $\mathcal{V}$. Denote it by $\mathcal{V}_1$ and denote the irreducible $\frac{n(n-3)}{2}$-dimensional invariant subspace of $\mathcal{V}$ by $\mathcal{V}_{\frac{n(n-3)}{2}}$.

$$\text{Let } \tilde{\mathcal{V}} := \mathcal{V}_1 \oplus \mathcal{V}_{\frac{n(n-3)}{2}}$$

$\tilde{\mathcal{V}}$ is $\frac{(n-1)(n-2)}{2}$-dimensional. The equation

$$\frac{(n-1)(n-2)}{2} > 2n-3$$

is equivalent to

$$n^2 - 7n + 8 > 0,$$

thus holds for every n. It follows that $\tilde{\mathcal{V}} \cap \mathcal{V}^{(n-2)}$ is not trivial, so that $l \in \{r, -r^3, \frac{1}{r^{2n-7}}, \frac{1}{r^{n-5}}, -\frac{1}{r^{n-5}}\}$ and $l \in \{r, -r^3, \frac{1}{r^{2n-5}}, \frac{1}{r^{n-4}}, -\frac{1}{r^{n-4}}\}$, which leaves the only possibilities $l \in \{r, -r^3\}$ for l.

Let's go back to our case $n = 6$. Applying the proposition with $n = 6$ excludes the possibility $l = \frac{1}{r^9}$. Thus, we have shown $(S1)$ when $n = 6$ and $r^{10} = -1$ but $r^{12} \neq -1$. From there we easily deduce $(S2)$ under the same assumptions. Indeed, if $l = -r^3$, the representation $\nu^{(6)}$ is reducible, hence $\mathcal{V}^{(6)}$ must have an irreducible submodule, which is also an $\mathcal{H}_{F,r^2}(6)$-module by the old lemma 6. The irreducible representations of $\mathcal{H}_{F,r^2}(6)$ have degrees $1, 5, 9$ or 10. The hypothesis $r^{12} \neq -1$ forbids to have $l = -r^3 = \frac{1}{r^9}$. So there cannot exist any one-dimensional invariant subspace of $\mathcal{V}^{(6)}$. Since it is also not possible to have $-r^3 \in \{\frac{1}{r^3}, -\frac{1}{r^3}\}$, there does not exist any irreducible 5-dimensional invariant subspace of $\mathcal{V}^{(6)}$ by Theorem 5. Moreover, the existence of an irreducible 9-dimensional submodule of $\mathcal{V}^{(6)}$ would force $l = r$ as we just saw in the proof of $(S1)$. Then the only remaining possibility is that there exists an irreducible 10-dimensional invariant subspace of $\mathcal{V}^{(6)}$. Hence $(S2)$ holds for $n = 6$, $r^{10} = -1$ and $r^{12} \neq -1$. We note that by showing the statements $(S1)$ and $(S2)$ under these conditions and together with all the previous considerations, we have actually shown the theorems 9 and 10 for $n = 6$.

To finish the proof of $(\mathcal{P}_n)$ by induction, let's go back to the general case under the assumptions $r^{2n} \neq -1$ and $r^{2(n-1)} = -1$. First let's deal with the proof of $(S1)$. Suppose there exists an irreducible $\frac{n(n-3)}{2}$-dimensional invariant subspace of $\mathcal{V}^{(n)}$. By Proposition 16, we know that $l \in \{r, -r^3\}$ and we want to show that $l = r$. We will follow the same path as in the case $n = 6$. Suppose $l = -r^3$. Since $l = -r^3$ and $r^{2(n-1)} = -1$, we have by point number 8 that $k(n-1) = 1 + \frac{(n-2)(n-3)}{2}$. With $l = -r^3$ and $r^{2n} \neq -1$, we also know from point 6 and 7 that $k(n) \in \{\frac{n(n-3)}{2}, \frac{(n-1)(n-2)}{2}\}$ (we will see later on in the proof that this is still true when $n = 8$ and admit it for now). Since there exists an irreducible $\frac{n(n-3)}{2}$-dimensional invariant subspace and there does not exist any one-dimensional invariant subspace, we must have $k(n) = \frac{n(n-3)}{2}$. Let S be a summand for $K(n) \cap \mathcal{V}^{(n-1)}$ in $K(n) \downarrow_{\mathcal{H}_{F,r^2}(n-1)}$:

$$K(n) \cap \mathcal{V}^{(n-1)} \oplus S = K(n) \downarrow_{\mathcal{H}_{F,r^2}(n-1)} \tag{166}$$

We have

$$\dim(K(n) \cap \mathcal{V}^{(n-1)}) \geq \frac{(n-1)(n-4)}{2} - 1$$

Then we must have

$$\dim(K(n) \cap \mathcal{V}^{(n-1)}) \geq \frac{(n-2)(n-3)}{2}$$

Moreover,

$$K(n) \cap \mathcal{V}^{(n-1)} \subseteq K(n-1)$$

implies that

$$\dim(K(n)\cap\mathcal{V}^{(n-1)}) \leq 1+\frac{(n-2)(n-3)}{2}$$

If $\dim(K(n)\cap\mathcal{V}^{(n-1)}) = 1+\frac{(n-2)(n-3)}{2}$, we get $K(n)\cap\mathcal{V}^{(n-1)} = K(n-1)$ which would imply that $K(n-1)\subseteq K(n)$. Then it comes:

$$k(n)\geq k(n-1)+(n-2) = \frac{(n-2)(n-3)}{2}+(n-2)+1 = \frac{(n-2)(n-1)}{2}+1,$$

a contradiction. Hence we must have $\dim(K(n)\cap\mathcal{V}^{(n-1)}) = \frac{(n-2)(n-3)}{2}$. But then, from equation (166), the $\mathcal{H}_{F,r^2}(n-1)$-module S would have dimension $n-3$. By James'result in Proposition 3 of the thesis, S must then contain a one-dimensional $\mathcal{H}_{F,r^2}(n-1)$-submodule, say U. When $n=7$, we also use the fact that there is no irreducible representation of $\mathcal{H}_{F,r^2}(6)$ of degree between 1 and 5.

Lemma 13. *Let n be an integer with $n\geq 5$. Suppose $l=-r^3$ and $r^{2(n-1)}=-1$. In $K(n)$, there does not exist any one-dimensional $\mathcal{H}_{F,r^2}(n-1)$-module.*

PROOF OF THE LEMMA: suppose such a module U exists and let

$$u=\sum_{1\leq i<j\leq n}\mu_{ij}\,w_{ij}$$

be a spanning vector of U over F. By the same arguments as in the proof of Theorem 4, we must have:

$$\forall 1\leq i\leq n-2,\ \nu_i(u)=\lambda\,u \ \text{ where } \ \lambda\in\{r,-\frac{1}{r}\}$$

It follows from these relations that for every node i with $1\leq i\leq n-2$, we have:

$$\begin{aligned}\forall k \geq i+2,\ \mu_{i+1,k} &= \lambda\, \mu_{i,k} &(167)\\ \forall l \leq i-1,\ \mu_{l,i+1} &= \lambda\, \mu_{l,i} &(168)\end{aligned}$$

From there, we see that if one of the coefficients μ_{st}, some $1 \leq s < t \leq n-1$, is zero then all of the coefficients μ_{ij} for $1 \leq i < j \leq n-1$ are zero. Suppose we are in this situation. Then u reduces to:

$$u = \sum_{i=1}^{n-1} \mu_{in}\, w_{in}$$

But we have:

$$\begin{aligned} e_1.\, w_{1n} &= -r^n\, w_{12} \\ e_1.\, w_{2n} &= r^{n-3}\, w_{12} \\ e_1.\, w_{jn} &= 0 \qquad\qquad \forall 3 \leq j \leq n-1 \end{aligned}$$

Then u would not be annihilated by e_1, which is impossible. If none of the coefficients for the w_{ij}'s with $1 \leq i < j \leq n-1$ are zero, in particular μ_{34} is nonzero. Then by the same argument as in the proof of Theorem 4, case $n \geq 4$, the coefficient λ must be r and not $-\frac{1}{r}$. Thus, u can be written as:

$$u = \sum_{1 \leq i < j \leq n-1} r^{i+j}\, w_{ij} + \mu \sum_{i=1}^{n-1} r^{i+n}\, w_{in}$$

Like we did in the proof of Theorem 4 (cf equation (72)), let's look at the action of ν_1 on u and the resulting coefficient in w_{12}. This time we get:

$$\frac{m}{r}\sum_{j=3}^{n-1}(r^2)^j + \mu\,\frac{m}{r}\,r^{2n} - 1 = r^4 \tag{169}$$

After simplifying this expression and replacing $r^{2(n-1)}$ by its value -1, we obtain:

$$\mu\,\frac{m}{r}\,r^{2n} = 0$$

Then it comes $\mu = 0$, so that U is in fact spanned by

$$u = \sum_{1\leq i<j\leq n-1} r^{i+j}\,w_{ij}$$

We note that we recover the fact that $l = \frac{1}{r^{2n-5}}$ (when $r^{2(n-1)} = -1$, we can check that $\frac{1}{r^{2n-5}} = -r^3$). To conclude, it suffices now to look at the action of e_{n-1} on u. We have for every i with $1 \leq i \leq n-1$:

$$e_{n-1}.\,w_{i,n-1} = \frac{1}{r^{n-i-2}}\,w_{n-1,n}$$

Then, it comes:

$$e_{n-1}.u = r\sum_{i=1}^{n-2}(r^2)^i\,w_{n-1,n} = r^3\,\frac{1-(r^2)^{n-2}}{1-r^2}\,w_{n-1,n}$$

Then, $e_{n-1}.u$ is nonzero as $r^{2(n-2)} \neq 1$. Then U is not contained in $K(n)$: a contradiction. On the way, we recovered the result from Proposition 9 that when $n \geq 5$, $l = -r^3$ and $r^{2(n-1)} = -1$, $K(n-1) \not\subseteq K(n)$, where we used the same action on the same vector.

Now Lemma 13 holds and we have hence proven that it is impossible to have $l = -r^3$. Thus, $l = r$ and $(S1)$ holds. Let's finally prove $(S2)$. If

$l = -r^3$, then $\nu^{(n)}$ is reducible. Moreover, if $r^{2n} \neq -1$, we know that $K(n)$ is irreducible by point 7 and $k(n) \leq \frac{(n-1)(n-2)}{2}$ by point 6. Suppose first $n = 7$ or $n \geq 9$. Then an irreducible $\mathcal{H}_{F,r^2}(n)$-module has dimension $1, n-1, \frac{n(n-3)}{2}, \frac{(n-1)(n-2)}{2}$ or dimension greater than $\frac{(n-1)(n-2)}{2}$. Since $l = -r^3$ and $r^{2n} \neq -1$, $K(n)$ cannot have dimension 1. Neither can it have dimension $n-1$ (as otherwise $l \in \{\frac{1}{r^{n-3}}, -\frac{1}{r^{n-3}}\}$). Then it must have dimension $\frac{n(n-3)}{2}$ or $\frac{(n-1)(n-2)}{2}$. If it had dimension $\frac{n(n-3)}{2}$, then we would have $l = r$ by $(S1)$. So we see that $K(n)$ must have dimension $\frac{(n-1)(n-2)}{2}$. Thus $(S2)$ is proven for $n = 7$ or $n \geq 9$. The case $n = 8$ is in fact not different, but needs to be slightly adapted. Recall that the irreducible representations of $\mathcal{H}_{F,r^2}(8)$ have dimensions $1, 7, 14, 20, 21$ or dimensions greater than or equal to 28. Still by points 6 and 7, we have $K(8)$ is irreducible with $k(8) \leq 21$. Then $k(8) \in \{14, 21\}$ by the same arguments as in the cases $n = 7$ or $n \geq 9$. If $k(8) = 14$, then $\dim(K(8) \cap \mathcal{V}^{(7)}) \geq 7$. By point number 9, the existence of an irreducible 14-dimensional invariant subspace of $\mathcal{V}^{(7)}$ must be excluded. Also, by semisimplicity of $\mathcal{H}_{F,r^2}(8)$, it is impossible to have $-r^3 \in \{\frac{1}{r^4}, -\frac{1}{r^4}\}$. Then $K(8) \cap \mathcal{V}^{(7)}$ must have dimension $1 + 15 = 16$, which is also the dimension of $K(7)$ when $l = -r^3$ and $r^{14} = -1$. Then we get $K(8) \cap \mathcal{V}^{(7)} = K(7)$ and in particular it comes $K(7) \subseteq K(8)$. This leads to a contradiction as any spanning vector for the one-dimensional invariant subspace of $\mathcal{V}^{(7)}$ does not belong to $K(8)$. So we have shown that it is impossible to have $k(8) = 14$ and the only remaining possibility is thus to have $k(8) = 21$. This shows $(S2)$ in the case $n = 8$ and $r^{16} \neq -1$ and $r^{14} = -1$.

To summarize, we have proven that $(S1)$ and $(S2)$ hold when $r^{2n} \neq -1$ and $r^{2(n-1)} = -1$. We have also proven that $(S1)$ and $(S2)$ hold when

$r^{2n} \neq -1$ and $r^{2s} = 1$ for some $\frac{n+1}{2} \leq s < n-1$ (and that is where we used induction). Then $(\mathcal{P}_n)$ holds for every $n \geq 5$ and Theorem 9 and 10 are thus entirely proven as soon as the Iwahori-Hecke algebra $\mathcal{H}_{F,r^2}(n)$ is assumed to be semisimple.

We have the immediate Corollary:

Corollary 15. *Let n be an integer with $n \geq 3$. Suppose $l = -r^3$.*

If $r^{2n} \neq -1$, then $k(n) = \frac{(n-1)(n-2)}{2}$

If $r^{2n} = -1$, then $k(n) = 1 + \frac{(n-1)(n-2)}{2}$

PROOF: the second point is point 2) of Proposition 15 when $n \geq 4$. When $n = 3$, we know from Theorem 4 that there exists exactly two one-dimensional invariant subspaces of $\mathcal{V}$ and that their sum is direct. Then $k(3) \geq 2$ and since $k(3) \neq 3$, in fact $k(3) = 2$. As for the first point, Lemma 10 for $n \geq 5$ yields the inequality $k(n) \leq \frac{(n-1)(n-2)}{2}$. Since when $l = -r^3$ and $n \geq 5$ there exists an irreducible $\frac{(n-1)(n-2)}{2}$-invariant subspace by Theorem 10, we also have $k(n) \geq \frac{(n-1)(n-2)}{2}$, so that $k(n) = \frac{(n-1)(n-2)}{2}$. It remains to deal with the cases $n = 3$ and $n = 4$. When $n = 3$, $l = -r^3$ and $l \neq \frac{1}{r^3}$, there exists a unique one-dimensional invariant subspace of $\mathcal{V}^{(3)}$. The uniqueness forces $k(3) = 1$. When $n = 4$ and $l = -r^3$, there exists an irreducible 3-dimensional invariant subspace of $\mathcal{V}^{(4)}$ by Theorem 6 and there does not exist any one-dimensional invariant subspace of $\mathcal{V}^{(4)}$ as $r^8 \neq -1$. Then $k(4) \in \{3, 5\}$. Moreover, since $l \neq r$ it is impossible to have $k(4) = 5$. Hence $k(4) = 3$, as announced.

Chapter 10

The Uniqueness Theorem and a Complete Description of the Invariant Subspace of $\mathcal{V}^{(\mathbf{n})}$ when $\mathrm{l} = \mathrm{r}$

In this part, we prove a theorem of uniqueness and describe the irreducible $\frac{n(n-3)}{2}$-dimensional invariant subspace of $\mathcal{V}^{(n)}$ when $l = r$.

Theorem 11. ***(Uniqueness)***
Let n be an integer with $n \geq 3$. We assume that $\nu^{(n)}$ is reducible and exclude the case when $l = -r^3$ and $r^{2n} = -1$. Then, there exists a unique non-trivial proper invariant subspace in $\mathcal{V}^{(n)}$.

PROOF: Assume first $n \geq 5$. Since $\nu^{(n)}$ is reducible, l must take one of the values $\frac{1}{r^{n-3}}, -\frac{1}{r^{n-3}}, \frac{1}{r^{2n-3}}, r$ or $-r^3$ by the Main Theorem. Moreover, with the assumptions $r^{2n} \neq -1$ and $\mathcal{H}_{F,r^2}(n)$ is semisimple, all these values are distinct. When $l = r$ or $l = -r^3$ and $r^{2n} \neq -1$, one of the results of

Proposition 15 is that $K(n)$ is irreducible. Since any proper invariant subspace of $\mathcal{V}^{(n)}$ must be contained in $K(n)$ and $K(n)$ is non-trivial for these values of l and r by Proposition 5, the vector space $K(n)$ must in fact be the only non-trivial proper invariant subspace of $\mathcal{V}^{(n)}$. Moreover, when $l = r$, $K(n)$ is $\frac{n(n-3)}{2}$-dimensional by Proposition 10 and when $l = -r^3$ and $r^{2n} \neq -1$, $K(n)$ is $\frac{(n-1)(n-2)}{2}$-dimensional as we just saw in Corollary 15. Suppose now that $l = \frac{1}{r^{2n-3}}$. We know by Theorem 4 that there exists a one-dimensional invariant subspace of $\mathcal{V}^{(n)}$. Further, by Theorem 5, (resp 9, resp 10), there cannot exist any irreducible $(n-1)$ (resp $\frac{n(n-3)}{2}$, resp $\frac{(n-1)(n-2)}{2}$) dimensional invariant subspace of $\mathcal{V}^{(n)}$. Then in the cases $n = 5$ and $n = 6$, this unique one-dimensional invariant subspace of $\mathcal{V}^{(n)}$ must be the only invariant subspace of $\mathcal{V}^{(n)}$. Let's now consider an integer n with $n = 7$ or $n \geq 9$. Then, if $K(n)$ is not one-dimensional, its dimension must be greater than or equal to $2 + \frac{(n-1)(n-2)}{2}$. It forces both $K(n) \cap \mathcal{V}^{(n-1)} \neq 0$ and $K(n) \cap \mathcal{V}^{(n-2)} \neq 0$. Then, $l \in \{r, -r^3\}$, which is impossible. Again, $K(n)$ is the unique invariant subspace of $\mathcal{V}^{(n)}$ in that case. When $n = 8$, if $k(8) \geq 15$, then again, $k(8) > 13$ and $k(8) > 7$, so that $l \in \{r, -r^3\}$, a contradiction. Hence we conclude again that $k(8) = 1$ in that case. Finally, suppose that $l \in \{\frac{1}{r^{n-3}}, -\frac{1}{r^{n-3}}\}$. In these cases, we know that there exists a unique $(n-1)$-dimensional invariant subspace of $\mathcal{V}^{(n)}$ and there does not exist any irreducible 1 or $\frac{n(n-3)}{2}$ or $\frac{(n-1)(n-2)}{2}$-dimensional invariant subspace of $\mathcal{V}^{(n)}$. Again, we will deal with the case $n = 8$ apart. When $n \in \{5, 6\}$, as before, we may conclude immediately. If now $n = 7$ or $n \geq 9$, suppose that there exists an irreducible s-dimensional invariant subspace of $\mathcal{V}^{(n)}$ with $s \geq 1 + \frac{(n-1)(n-2)}{2}$. That would force $l \in \{r, -r^3\}$, which is impossible. Thus, the unique $(n-1)$-dimensional invariant subspace of $\mathcal{V}^{(n)}$ is the only invariant subspace of $\mathcal{V}^{(n)}$. Finally, in the case $n = 8$, there cannot exist any irreducible 14-dimensional invariant subspace of $\mathcal{V}^{(8)}$ for the same reasons as before. Hence the unique 7-dimensional invariant subspace of $\mathcal{V}^{(8)}$ is the only invariant subspace of $\mathcal{V}^{(8)}$. This ends the proof of Theorem 11. Let's now describe the unique invariant subspace of $\mathcal{V}^{(n)}$ when $l = r$. We have the theorem:

Theorem 12.

<u>*Assume $l = r$.*</u>

- *When $n = 4$, the unique invariant subspace $K(4)$ of $\mathcal{V}^{(4)}$ is spanned by the two linearly independent vectors:*

$$\begin{aligned} w_1^{(4)} &:= (w_{14} - \tfrac{1}{r}\, w_{24}) + (w_{23} - r\, w_{13}) \\ w_2^{(4)} &:= (w_{24} - \tfrac{1}{r}\, w_{34}) + (w_{13} - r\, w_{12}) \end{aligned}$$

- *When $n \geq 5$, the unique invariant subspace $K(n)$ of $\mathcal{V}^{(n)}$ is built inductively as a direct sum of the unique invariant subspace $K(n-1)$ of $\mathcal{V}^{(n-1)}$ and of an $(n-2)$-dimensional vector space spanned by the vectors:*

$$\begin{aligned} w_1^{(n)} &:= w_{1,n} - \tfrac{1}{r}\, w_{2,n} + r^{n-4}\,(w_{23} - r\, w_{13}) \\ w_k^{(n)} &:= w_{k,n} - \tfrac{1}{r}\, w_{k+1,n} + r^{n-4}\,(w_{1,k+1} - r\, w_{1,k}), \quad 2 \leq k \leq n-2 \end{aligned}$$

PROOF OF THE THEOREM: the case $n = 4$ is contained in Result 2. Let's deal with $n \geq 5$. When $n = 5$, Claim 1 of Part 8.3 provides us with a spanning set of vectors for the unique invariant subspace of $\mathcal{V}^{(5)}$. Up to a reordering, we read that these vectors are $w_1^{(4)}$, $w_2^{(4)}$, the spanning vectors of the unique invariant subspace of $\mathcal{V}^{(4)}$ and the three vectors $w_1^{(5)}$, $w_2^{(5)}$ and $w_3^{(5)}$. Hence the Theorem holds in that case. We will proceed by induction. Given an integer n with $n \geq 6$, let's assume that the Theorem holds for the integer $n-1$. So $K(n-1)$ is spanned by all the vectors $w_s^{(t)}$'s of the Theorem with $4 \leq t \leq n-1$ and $1 \leq s \leq t-2$. When $l = r$, we know that $K(n)$ is the unique invariant subspace of $\mathcal{V}^{(n)}$ and that it contains $K(n-1)$, the unique invariant subspace of $\mathcal{V}^{(n-1)}$. Moreover, it appears that the $(n-2)$ vectors $w_i^{(n)},\ \ i = 1, \ldots, n-2$, span an $(n-2)$-dimensional subspace of $\mathcal{V}^{(n)}$ that is in direct sum with $K(n-1)$. Since we notice that

$$k(n) = \frac{n(n-3)}{2} = \frac{(n-1)(n-4)}{2} + (n-2) = k(n-1) + (n-2),$$

it will suffice to show that the $(n-2)$ vectors $w_i^{(n)}$, $i = 1, \ldots, n-2$, belong to $K(n)$. To do that, we follow the same steps as in the proof of Claim 3. First, let j be an integer with $1 \leq j \leq n-1$. We compute with the tables of Appendix C used with $l = r$:

$$\forall 1 \leq k \leq n-1,\ [X_{j,n}.w_{k,n}]_{w_{j,n}} = \begin{cases} r^{k-j} & \text{if } k \neq j \\ 2 & \text{if } k = j \end{cases}$$

It follows immediately that:

$$\begin{array}{rcll} \left[X_{j,n}.(w_{k,n} - \frac{1}{r}\, w_{k+1,n})\right]_{w_{j,n}} & = & 0 & \text{if } j \notin \{k, k+1\} \\ \left[X_{k,n}.(w_{k,n} - \frac{1}{r}\, w_{k+1,n})\right]_{w_{k,n}} & = & 1 & \\ \left[X_{k+1,n}.(w_{k,n} - \frac{1}{r}\, w_{k+1,n})\right]_{w_{k+1,n}} & = & -\frac{1}{r} & \end{array}$$

When $l = r$, the actions $(CL)_{(s,t)}$ of Appendix C are all trivial. Then we have for any integer k with $2 \leq k \leq n-2$:

$$\begin{aligned}
\left[X_{1,n}.(w_{1,k+1} - r\, w_{1,k})\right]_{w_{1,n}} &= \frac{1}{r.r^{n-k-2}} - \frac{r}{r.r^{n-k-1}} = 0 \\
\left[X_{j,n}.(w_{1,k+1} - r\, w_{1,k})\right]_{w_{j,n}} &= 0, \quad \text{for all } j \text{ such that } \begin{cases} 2 \leq j \leq n-1 \\ j \notin \{k, k+1\} \end{cases} \\
\left[X_{k,n}.(w_{1,k+1} - r\, w_{1,k})\right]_{w_{k,n}} &= -\frac{1}{r^{n-4}} \\
\left[X_{k+1,n}.(w_{1,k+1} - r\, w_{1,k})\right]_{w_{k+1,n}} &= \frac{1}{r^{n-3}}
\end{aligned}$$

Also, we have,

$$\begin{aligned}
\left[X_{j,n}.(w_{23} - r\, w_{13})\right]_{w_{j,n}} &= 0, \text{ forall } j \notin \{1,2,3\} \\
\left[X_{1,n}.(w_{23} - r\, w_{13})\right]_{w_{1,n}} &= -\frac{1}{r^{n-4}} \\
\left[X_{2,n}.(w_{23} - r\, w_{13})\right]_{w_{2,n}} &= \frac{1}{r^{n-3}} \\
\left[X_{3,n}.(w_{23} - r\, w_{13})\right]_{w_{3,n}} &= \frac{1}{r^{n-4}} - \frac{r}{r^{n-3}} = 0
\end{aligned}$$

Now it appears that the last $(n-1)$ rows of the matrix $T(n)$ annihilate the $(n-2)$ vectors $w_i^{(n)}$, $i = 1, \ldots, n-2$. To complete the proof, we want to show that the $X_{s,t}$'s with $1 \leq s < t \leq n-1$ all annihilate these vectors. This verification is left to the reader.

Chapter 11

A Proof of the Main Theorem without Maple

In this part, we proceed without using the results from previous part. From part 8, it suffices to show that the Main Theorem holds for the small values $n \in \{3,4,5,6\}$. In the case $n = 3$, the Main Theorem is proven with Theorem 4 and Theorem 5. Indeed, by Theorem 4, there exists a one-dimensional invariant subspace of $\mathcal{V}$ if and only if $l \in \{\frac{1}{r^3}, -r^3\}$. By Theorem 5, there exists an irreducible two-dimensional invariant subspace of $\mathcal{V}$ if and only if $l \in \{1,-1\}$. Similarly in the case $n = 4$, we know by Theorem 4 (resp Theorem 5, resp Corollary 4) that there exists an irreducible one (resp 3, resp 2)-dimensional invariant subspace of $\mathcal{V}$ if and only if $l = \frac{1}{r^5}$ (resp $l \in \{\frac{1}{r}, -\frac{1}{r}, -r^3\}$, resp $l = r$). Since the degrees of the irreducible representations of $\mathcal{H}_{F,r^2}(4)$ are 1, 2 or 3, this proves the Main Theorem in this case. Let's deal with the case $n = 5$. Suppose $\nu^{(5)}$ is reducible and let $\mathcal{W}$ be an irreducible invariant subspace of $\mathcal{V}$. Consider the F-vector space $\mathcal{W} \cap \mathcal{V}_0$, with the same notations as before. If this intersection is trivial, then the sum $\mathcal{W} + \mathcal{V}_0$ is direct and we must have $\dim \mathcal{W} + \dim\mathcal{V}_0 \leq \dim\mathcal{V}$, id est,

$\dim \mathcal{W} \leq 4$. Then $\dim \mathcal{W} \in \{1, 4\}$ as the degrees of the irreducible representations of $\mathcal{H}_{F,r^2}(5)$ are $1, 4, 5, 6$. By Theorem 4 and Theorem 5, this forces $l \in \{\frac{1}{r^7}, \frac{1}{r^2}, -\frac{1}{r^2}\}$. Suppose $l \notin \{\frac{1}{r^7}, \frac{1}{r^2}, -\frac{1}{r^2}\}$. Then we have $0 \subset \mathcal{W} \cap \mathcal{V}_0 \subset \mathcal{V}_0$. By the case $n = 4$ it follows that $l \in \{r, -r^3, \frac{1}{r^5}, -\frac{1}{r}, \frac{1}{r}\}$. Moreover, we have:

$$\begin{aligned}\dim(\mathcal{W} \cap \mathcal{V}_0) &= \dim \mathcal{W} + \dim \mathcal{V}_0 - \dim(\mathcal{W} + \mathcal{V}_0) \\ &\geq \dim \mathcal{W} + \dim \mathcal{V}_0 - \dim \mathcal{V} = \dim \mathcal{W} - 4\end{aligned}$$

Furthermore, by our assumption on l, we have $\dim \mathcal{W} \in \{5, 6\}$. If $\dim \mathcal{W} = 5$, then by the Result 1, we know that it forces $l = r$. From now on we suppose that $l \neq r$. So $\dim \mathcal{W} = 6$ and $\dim(\mathcal{W} \cap \mathcal{V}_0) \geq 2$. Our assumption on l is now $l \notin \{r, \frac{1}{r^7}, \frac{1}{r^2}, -\frac{1}{r^2}\}$ and $l \in \{-r^3, \frac{1}{r^5}, \frac{1}{r}, -\frac{1}{r}\}$. Our goal is to show that $l = -r^3$. Since $\mathcal{W} \cap \mathcal{V}_0$ is a proper invariant subspace of $\mathcal{V}_0$, it must be contained in $K(4)$. Therefore, we also have the inequality $\dim(\mathcal{W} \cap \mathcal{V}_0) \leq k(4)$, where we used the notations of previous section. We will show that it is impossible to have $l = \frac{1}{r^5}$ (unless $\frac{1}{r^5} = -r^3$) or $l = \frac{1}{r}$ or $l = -\frac{1}{r}$. If $l = \frac{1}{r^5}$, then by Theorem 4, there exists a unique one-dimensional invariant subspace inside $\mathcal{V}_0$. In particular $k(4) \neq 0$. Hence $k(4) \in \{1, 2, 3, 4, 5\}$. At this stage, we recall our assumption of semisimplicity for $\mathcal{H}_{F,r^2}(5)$. When $\mathcal{H}_{F,r^2}(5)$ is semisimple, a fortiori $\mathcal{H}_{F,r^2}(4)$ is semisimple. Suppose $k(4) = 2$. If $K(4)$ were not irreducible, it would contain a one-dimensional submodule that has a one-dimensional summand by semisimplicity of $\mathcal{H}_{F,r^2}(4)$. This is impossible by uniqueness of the one-dimensional invariant subspace of $\mathcal{V}_0$. Then $K(4)$ is an irreducible two dimensional invariant subspace of $\mathcal{V}_0$. By Corollary 4, the existence of an irreducible two-dimensional invariant subspace of $\mathcal{V}_0$ implies $l = r$, a contradiction with $(r^2)^3 \neq 1$. If $k(4) = 3$, there are three possibilities. Either $K(4)$ is irreducible and $l \in \{-r^3, \frac{1}{r}, -\frac{1}{r}\}$ by Theorem 6. Or $K(4)$ contains a one-dimensional submodule that has a two dimensional summand by semisimplicity of $\mathcal{H}_{F,r^2}(4)$. Then $l = r$ by uniqueness of the one-dimensional invariant subspace of $\mathcal{V}_0$ and Corollary 4. Or $K(4)$ contains a two-dimensional submodule that has a one-dimensional summand. Again, this forces $l = r$ by uniqueness of the one-dimensional invariant subspace of $\mathcal{V}_0$ and Corollary 4. Gathering these results, the only possibility for l that is compatible with $l = \frac{1}{r^5}$ and our assumption of semisimplicity for $\mathcal{H}_{F,r^2}(5)$ is to have $l = -r^3$. Suppose now $k(4) = 4$. Then $K(4)$ is reducible. If $K(4)$ contains a two- dimensional invariant subspace, then this two-dimensional subspace has a two-dimensional summand in $K(4)$. This is impossible by uniqueness of any one-dimensional (resp two-dimensional) invariant subspace inside $\mathcal{V}_0$. On the other hand, if $K(4)$ is a direct sum of an irreducible 3-dimensional submodule and a one-dimensional submodule, we must have $l \in \{-r^3, \frac{1}{r}, -\frac{1}{r}\}$ and $l = \frac{1}{r^5}$, which leaves the only possibility of $l = -r^3 = \frac{1}{r^5}$ for the parameters l and r. Finally suppose that $k(4) = 5$. The story is similar. $K(4)$ is reducible and is either a direct sum of a 4-dimensional submodule and a one-dimensional submodule or a direct sum of a 3-dimensional submodule and a 2-dimensional submodule, with no further decompositions allowed by uniqueness of an irreducible two-dimensional submodule of $\mathcal{V}_0$ or a one-dimensional submodule of $\mathcal{V}_0$ when these ones exist. Since there does not exist any irreducible representations of $\mathcal{H}_{F,r^2}(4)$ of degree 4, the first decomposition should still break, which is impossible. As for the second decomposition it forces $l = r$, which contradicts our assumption $l = \frac{1}{r^5}$. In summary, either $k(4) = 1$ or $l = -r^3$. To reach the goal it suffices to show that it is impossible to have $k(4) = 1$. When $k(4) = 1$, the two inequalities above read:

$$\begin{array}{rcl} \dim(\mathcal{W}\cap\mathcal{V}_0) & \geq & 2 \\ \dim(\mathcal{W}\cap\mathcal{V}_0) & \leq & 1 \end{array},$$

a contradiction. To achieve our goal it remains to show that it is impossible to have $l \in \{\frac{1}{r}, -\frac{1}{r}\}$. First we show that for these values of l we have $k(4) = 3$. The scheme of the proof is the same as in the case $l = \frac{1}{r^5}$. The existence of a 3-dimensional invariant subspace of $\mathcal{V}_0$ shows that $k(4) \neq 0$. Then, we eliminate turn by turn the possibilities $k(4) \in \{1, 2, 4, 5\}$. Immediately, if $k(4) = 1$, then $l = \frac{1}{r^5}$ by Theorem 4, in contradiction with $l \in \{\frac{1}{r}, -\frac{1}{r}\}$. Next, if $k(4) = 2$, then $K(4)$ is irreducible by Theorem 4 and $l = r$ by Corollary 4. This again contradicts $l \in \{\frac{1}{r}, -\frac{1}{r}\}$ by semisimplicity of $\mathcal{H}_{F,r^2}(4)$. And in fact those two cases could right away be excluded by a simple use of Theorem 6. Indeed, we have seen in Theorem 6 that when $l \in \{\frac{1}{r}, -\frac{1}{r}\}$, there exists a 3-dimensional invariant subspace inside $\mathcal{V}_0$. This implies that $k(4) \geq 3$. If $k(4) = 4$, by uniqueness in Theorem 4 and in Result 2, $K(4)$ must be a direct sum of a 3-dimensional submodule and a one-dimensional submodule with no further decomposition. This forces $l \in \{-r^3, \frac{1}{r}, -\frac{1}{r}\}$ by Theorem 6 and $l = \frac{1}{r^5}$ by Theorem 4. Since it is impossible to have $l = \frac{1}{r^5}$ and $l \in \{\frac{1}{r}, -\frac{1}{r}\}$, we are led to conclude that $k(4) \neq 4$ when $l \in \{\frac{1}{r}, -\frac{1}{r}\}$. Finally, if $k(4) = 5$, we get $l = r$ by the same arguments as those already described further above. This is again impossible with $l \in \{\frac{1}{r}, -\frac{1}{r}\}$. Thus, we have shown that when $l \in \{\frac{1}{r}, -\frac{1}{r}\}$, we have $k(4) = 3$. An immediate consequence is that $K(4)$ is irreducible. We also derive from this fact that $\dim(\mathcal{W} \cap \mathcal{V}_0) \leq 3$ by one of the two inequalities of the beginning. Then, for these two values of l, we have $2 \leq \dim(\mathcal{W} \cap \mathcal{V}_0) \leq 3$. Furthermore, $l \in \{\frac{1}{r}, -\frac{1}{r}\}$ forces $\dim(\mathcal{W} \cap \mathcal{V}_0) = 3$. By equality on the dimensions, we now get that $\mathcal{W} \cap \mathcal{V}_0 = K(4)$ and we note on the way that $\mathcal{W} \cap \mathcal{V}_0$ is irreducible.

Suppose first that $l = \frac{1}{r}$. Following Lemma 4, equation (40), it is an easy verification using the table of the appendix or directly by hands that $w_{14} - w_{23} \in K(4)$. Then $w_{14} - w_{23}$ also belongs to $\mathcal{W} \cap \mathcal{V}_0$ and in particular to $\mathcal{W}$. Since $\mathcal{W}$ is a $B(A_4)$-module, $e_4.(w_{14} - w_{23})$ must also belong to $\mathcal{W}$. But,

$$e_4.(w_{14} - w_{23}) = \frac{1}{r^2}\, x_{\alpha_4}$$

Then it comes $\mathcal{W} = \mathcal{V}$: contradiction.
Similarly for $l = -\frac{1}{r}$, we have:

$$x_{\alpha_1} + x_{\alpha_3} + x_{\alpha_1+\alpha_2+\alpha_3} - x_{\alpha_2} \in K(4) = \mathcal{W} \cap \mathcal{V}_0$$

It follows that:

$$e_4.(x_{\alpha_1} + x_{\alpha_3} + x_{\alpha_1+\alpha_2+\alpha_3} - x_{\alpha_2}) = \Big(1 + \frac{1}{r^2}\Big)\, x_{\alpha_4} \in \mathcal{W}$$

Since $(r^2)^2 \neq 1$ by semisimplicity of $\mathcal{H}_{F,r^2}(5)$, this implies that $x_{\alpha_4} \in \mathcal{W}$. But again $\mathcal{W}$ would then be the whole space $\mathcal{V}$ by the arguments of § 8.1. We conclude that it is impossible to have $l \in \{\frac{1}{r}, -\frac{1}{r}\}$. Thus, we have shown that if $\nu^{(5)}$ is reducible and $l \notin \{r, \frac{1}{r^7}, \frac{1}{r^2}, -\frac{1}{r^2}\}$, then $l = -r^3$. This says exactly that if $\nu^{(5)}$ is reducible, then $l \in \{r, -r^3, \frac{1}{r^7}, \frac{1}{r^2}, -\frac{1}{r^2}\}$. Conversely, if $l = \frac{1}{r^7}$, there exists a one-dimensional invariant subspace of $\mathcal{V}$ by Theorem 4, hence the representation is reducible; for $l \in \{\frac{1}{r^2}, -\frac{1}{r^2}\}$, there exists an irreducible 4-dimensional invariant subspace of $\mathcal{V}$ by Theorem 5 and so the representation is also reducible in that case. As for $l = r$ (resp $l = -r^3$), it is a direct verification that the vector $\mathcal{X}$ (resp $\mathcal{Y}$) of proposition 5 of § 8.4 belongs to the proper submodule $K(5)$ of $\mathcal{V}$. This achieves the proof of the Main Theorem in the case $n = 5$. It remains to do the case $n = 6$.

The degrees of the irreducible representations of $\mathcal{H}_{F,r^2}(6)$ are $1, 5, 9, 10, 16$. The vector space $\mathcal{V}$ is 15-dimensional. Hence, if $\mathcal{W}$ is an irreducible submodule, then $\dim \mathcal{W} \in \{1, 5, 9, 10\}$. If $\dim \mathcal{W} = 1$ then $l = \frac{1}{r^9}$ by Theorem 4. Also, if $\dim \mathcal{W} = 5$ then $l \in \{\frac{1}{r^3}, -\frac{1}{r^3}\}$ by Theorem 5. Suppose now $l \notin \{\frac{1}{r^9}, \frac{1}{r^3}, -\frac{1}{r^3}\}$. Then $\mathcal{W} \cap \mathcal{V}_0 \neq 0$. If $\mathcal{W} \cap \mathcal{V}_1 = 0$, then $\dim \mathcal{W} \leq 9$, which forces, with the condition on l above, $\dim \mathcal{W} = 9$. Also we note that $\mathcal{W} \bigoplus \mathcal{V}_1 = \mathcal{V}$ as $\dim \mathcal{W} + \dim \mathcal{V}_1 = 9 + 6 = 15 = \dim \mathcal{V}$. Since $\mathcal{W} \subseteq K(6)$, we must have $e_5.\mathcal{W} = 0$. Also, by definition of $\mathcal{V}_1$, we have $e_5.\mathcal{V}_1 = 0$. Then, it follows that $e_5.\mathcal{V} = 0$, which is a contradiction. Thus, if $l \notin \{\frac{1}{r^9}, \frac{1}{r^3}, -\frac{1}{r^3}\}$, then both $\mathcal{W} \cap \mathcal{V}_0$ and $\mathcal{W} \cap \mathcal{V}_1$ are nonzero. By the case $n = 4$ and the case $n = 5$ we now get:

$$\begin{cases} l \in \{r, -r^3, \frac{1}{r^5}, \frac{1}{r}, -\frac{1}{r}\} \\ \qquad\qquad \& \\ l \in \{r, -r^3, \frac{1}{r^7}, \frac{1}{r^2}, -\frac{1}{r^2}\} \end{cases}$$

which implies $l \in \{r, -r^3\}$. We conclude that if $\nu^{(6)}$ is reducible and $l \notin \{\frac{1}{r^9}, \frac{1}{r^3}, -\frac{1}{r^3}\}$, necessarily $l \in \{r, -r^3\}$. In other words, if $\nu^{(6)}$ is reducible then $l \in \{r, -r^3, \frac{1}{r^9}, \frac{1}{r^3}, -\frac{1}{r^3}\}$. Conversely, for $l = \frac{1}{r^9}$ (resp $l = \frac{1}{r^3}$ or $l = -\frac{1}{r^3}$), there exists a one-dimensional (resp 5-dimensional) invariant subspace of $\mathcal{V}$ by Theorem 4 (resp Theorem 5). Thus, the representation is reducible in these cases. As for $l = r$ (resp $l = -r^3$), we already know from the case $n = 5$ that $\mathcal{X}$ (resp $\mathcal{Y}$) belongs to $K(5)$. And as has been seen in the proof of Proposition 5, we also have $e_6.\mathcal{X} = X_{46}.\mathcal{X} = X_{36}.\mathcal{X} = X_{26}.\mathcal{X} = X_{16}.\mathcal{X} = 0$, with the same equalities holding for $\mathcal{Y}$ when $l = -r^3$, so that $\mathcal{X}$ (resp $\mathcal{Y}$) belongs to $K(6)$. Which proves the reducibility of the representation in these cases as well. The Main Theorem is thus proven in the case $n = 6$. And we have seen in Part 8 that if $(¶)_5$ and $(¶)_6$ hold, then $(¶)_n$ holds for all $n \geq 7$. Conversely, for each of the values of l in the set $\{\frac{1}{r^{2n-3}}, \frac{1}{r^{n-3}}, -\frac{1}{r^{n-3}}, r, -r^3\}$, the representation $\nu^{(n)}$ is reducible by Theorem 4, Theorem 5 and the proof of Proposition 5.

The program

```
SIZE := proc(n) binomial(n, 2) end proc


  G := proc(k, n)
local i, j, t, g;
    g;
    (l, s) -> array(1 .. SIZE(s), 1 .. SIZE(s));
    if n < 3 then ERROR(‘you have entered an invalid value‘)
    elif n = 3 then
        if k = 1 then
            g(k, n)[1, 1] := 1/l;
            g(k, n)[1, 2] := 1/r - r;
            g(k, n)[1, 3] := 0;
            g(k, n)[2, 1] := 0;
            g(k, n)[2, 2] := r - 1/r;
            g(k, n)[2, 3] := 1;
            g(k, n)[3, 1] := 0;
            g(k, n)[3, 2] := 1;
            g(k, n)[3, 3] := 0
        elif k = 2 then
            g(k, n)[1, 1] := 0;
            g(k, n)[1, 2] := 0;
            g(k, n)[1, 3] := 1;
            g(k, n)[2, 1] := 0;
            g(k, n)[2, 2] := 1/l;
            g(k, n)[2, 3] := (1/r - r)/l;
            g(k, n)[3, 1] := 1;
            g(k, n)[3, 2] := 0;
            g(k, n)[3, 3] := r - 1/r
        else ERROR(‘you entered an invalid first coordinate‘)
        end if
    else
        if k = n - 1 then
            for i to binomial(n - 2, 2) do for j to
                binomial(n, 2) do
                    if j <> i then g(k, n)[i, j] := 0
                    else g(k, n)[i, j] := r
                    end if
                end do
            end do;
            for i from 1 + binomial(n - 2, 2) to
            binomial(n - 1, 2) do for j to binomial(n, 2) do
```

```
                if j <> i + n - 1 then g(k, n)[i, j] := 0
                else g(k, n)[i, j] := 1
                end if
            end do
        end do;
        for i from 2 + binomial(n - 1, 2) to
        binomial(n, 2) do
            for j from 1 + binomial(n - 1, 2) to
            binomial(n, 2) do
                if j <> i then g(k, n)[i, j] := 0
                else g(k, n)[i, j] := r - 1/r
                end if
            end do;
            for j from 1 + binomial(n - 2, 2) to
            binomial(n - 1, 2) do
                if j <> i - n + 1 then g(k, n)[i, j] := 0
                else g(k, n)[i, j] := 1
                end if
            end do
        end do;
        for j from 2 + binomial(n - 1, 2) to
        binomial(n, 2) do
            t := j - binomial(n - 1, 2) - 2;
            g(k, n)[1 + binomial(n - 1, 2), j] :=
                (1/r - r)/(l*r^t)
        end do;
        g(k, n)[1 + binomial(n - 1, 2),
            1 + binomial(n - 1, 2)] := 1/l
    elif k < n - 1 then
        for i to binomial(n - 1, 2) do
            t := n - k - 2;
            for j from 1 + binomial(n - 1, 2) to
            binomial(n, 2) do
                if j <> binomial(n - 1, 2) + n - k - 1
                then g(k, n)[i, j] := 0
                else
                    if i = binomial(k, 2) + 1 then
                        g(k, n)[i, j] :=
                        r^(t - 1) - r^(t + 1)
                    else g(k, n)[i, j] := 0
                    end if
                end if
            end do
        end do;
```

```
            for i from 1 + binomial(n - 1, 2) to
            binomial(n, 2) do for j to binomial(n, 2) do
                    if
                    j = i and j <> binomial(n - 1, 2) + n - k
                     and j <> binomial(n - 1, 2) + n - k - 1
                    then g(k, n)[i, j] := r
                    elif i = binomial(n - 1, 2) + n - k - 1
                     and j = binomial(n - 1, 2) + n - k - 1
                    then g(k, n)[i, j] := r - 1/r
                    elif i = binomial(n - 1, 2) + n - k - 1
                     and j = binomial(n - 1, 2) + n - k then
                        g(k, n)[i, j] := 1
                    elif i = binomial(n - 1, 2) + n - k and
                    j = binomial(n - 1, 2) + n - k - 1 then
                        g(k, n)[i, j] := 1
                    else g(k, n)[i, j] := 0
                    end if
                end do
            end do;
            for i to binomial(n - 1, 2) do for j to
                binomial(n - 1, 2) do
                    g(k, n)[i, j] := G(k, n - 1)[i, j]
                end do
            end do
        else ERROR(`this element does not exist`)
        end if
    end if;
    Matrix(SIZE(n), g(k, n))
end proc


  IDENTITY := proc(n)
local identity, i;
    identity := Matrix(SIZE(n), SIZE(n));
    for i to SIZE(n) do identity[i, i] := 1 end do;
    Matrix(identity)
end proc


  e := proc(k, n)
    simplify(evalm(l*(
    `&*`(G(k, n), G(k, n)) + (1/r - r)*G(k, n) - IDENTITY(n))
    /(1/r - r)))
end proc
```

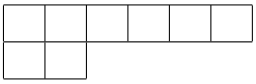
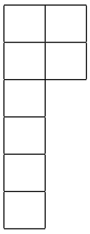
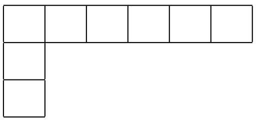
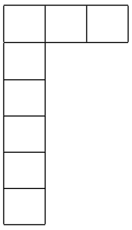
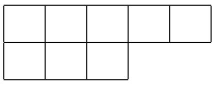
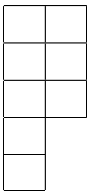
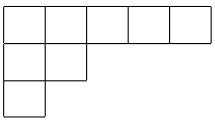
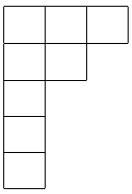
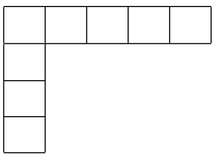
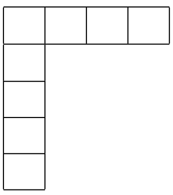
```
E := proc(k, n) Matrix(e(k, n)) end proc

  ginv := proc(k, n)
    simplify(evalm(
    G(k, n) + (1/r - r)*IDENTITY(n) + (r - 1/r)*E(k, n)))
end proc

GINV := proc(k, n) Matrix(ginv(k, n)) end proc

  T := proc(n)
local X, k, i, j;
    X;
    (s, t) -> Matrix(SIZE(n), SIZE(n));
    for i to n - 1 do for j from i + 1 to n do
            if j = i + 1 then X(i, j) := E(i, n)
            else X(i, j) := Matrix(simplify(evalm(`&*`(
                G(j - 1, n), X(i, j - 1), GINV(j - 1, n)))))
            end if
        end do
    end do;
    evalm(sum('sum('X(i, j)', 'j' = i + 1 .. n)',
        'i' = 1 .. n - 1))
end proc

d := proc(n) LinearAlgebra:-Determinant(Matrix(T(n))) end proc

NOTIRR := proc(n) solve({d(n) = 0}, {l}) end proc
```

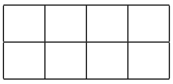
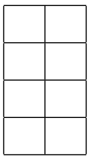
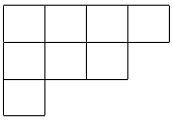
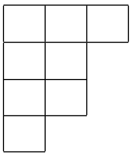
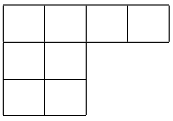
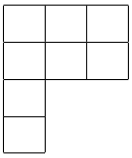
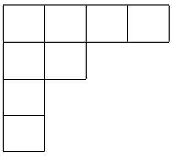
And running the procedure NOTIRR for $n = 3, 4, 5, 6$ yields:

```
> NOTIRR(3);

                                                         3          1
   {l = -1}, {l = -1}, {l = 1}, {l = 1}, {l = -r }, {l = ----}
                                                                    3
                                                                   r

> NOTIRR(4);

  {l = r}, {l = r}, {l = 1/r}, {l = 1/r}, {l = 1/r}, {l = - 1/r},

                                              1             3           3
        {l = - 1/r}, {l = - 1/r}, {l = ----}, {l = -r }, {l = -r },
                                                5
                                               r

                 3
        {l = -r }
```

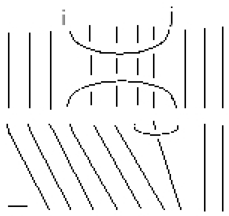
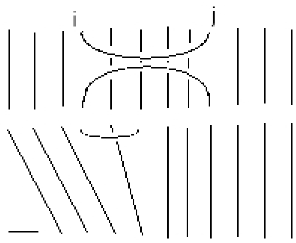
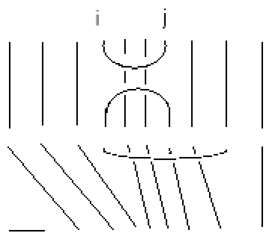
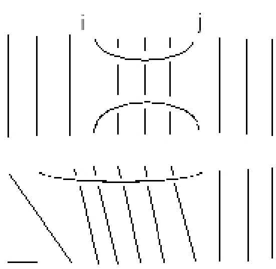
```
> NOTIRR(5);

                                                            1
  {l = r}, {l = r}, {l = r}, {l = r}, {l = r}, {l = ----},
                                                             7
                                                            r

             1              1              1              1
  {l = - ----}, {l = - ----}, {l = - ----}, {l = - ----},
              2              2              2              2
             r              r              r              r

           1           1           1           1           3
  {l = ----}, {l = ----}, {l = ----}, {l = ----}, {l = -r },
            2           2           2           2
           r           r           r           r

          3          3          3          3          3
  {l = -r }, {l = -r }, {l = -r }, {l = -r }, {l = -r }

> NOTIRR(6);

  {l = r}, {l = r}, {l = r}, {l = r}, {l = r}, {l = r}, {l = r},

                               1              1              1
  {l = r}, {l = r}, {l = ----}, {l = - ----}, {l = - ----},
                                9              3              3
                               r              r              r

             1              1              1            1
  {l = - ----}, {l = - ----}, {l = - ----}, {l = ----},
              3              3              3            3
             r              r              r            r

           1           1           1           1           3
  {l = ----}, {l = ----}, {l = ----}, {l = ----}, {l = -r },
            3           3           3           3
           r           r           r           r

          3          3          3          3          3
  {l = -r }, {l = -r }, {l = -r }, {l = -r }, {l = -r },

          3          3          3          3
  {l = -r }, {l = -r }, {l = -r }, {l = -r }
```

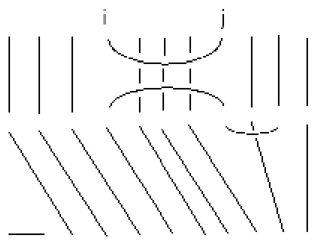
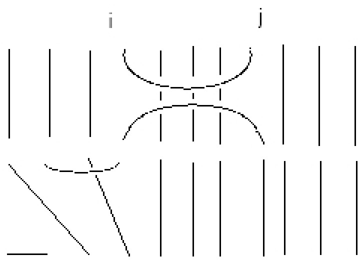
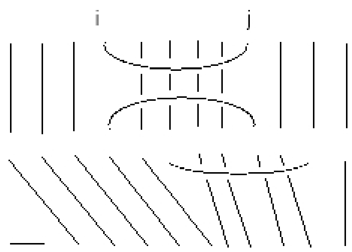
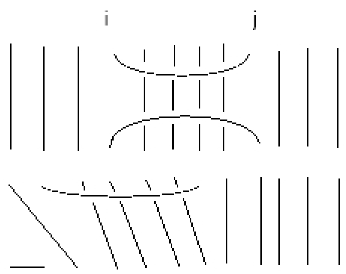
Table for $Sym(8)$

Specht module S^{λ}	Ferrers diagram	conjugate Ferrers diagram	conjugate Specht module $S^{\lambda'}$	dimension
$S^{(6,2)}$			$S^{(2,2,1,1,1,1)}$	20
$S^{(6,1,1)}$			$S^{(3,1,1,1,1,1)}$	21
$S^{(5,3)}$			$S^{(2,2,2,1,1)}$	28
$S^{(5,2,1)}$			$S^{(3,2,1,1,1)}$	64
$S^{(5,1,1,1)}$			$S^{(4,1,1,1,1)}$	35

$S^{(4,4)}$			$S^{(2,2,2,2)}$	14
$S^{(4,3,1)}$			$S^{(3,2,2,1)}$	70
$S^{(4,2,2)}$			$S^{(3,3,1,1)}$	56
$S^{(4,2,1,1)}$		self-conjugate	self-conjugate	90
$S^{(3,3,2)}$	$\begin{array}{\|c\|c\|c\|} \hline 5 & 4 & 2 \\ \hline 4 & 3 & 1 \\ \hline 2 & 1 & \\ \cline{1-2} \end{array}$	self-conjugate	self-conjugate	$\frac{8!}{5\times4\times3\times2\times4\times2} = 42$

How the X_{ij}'s act

Reference	Multiplication on the tangles	index range	algebraic result
$(\mathrm{MR})_k$		$k = 1, \dots, j-i-1$	$X_{ij}.\, w_{i+k,j} = l\, r^{k-1}\, w_{ij}$
$(\mathrm{ML})_k$		$k = 1, \dots, j-i-1$	$X_{ij}.\, w_{i,j-k} = \frac{1}{l\, r^{k-1}}\, w_{ij}$
$(\mathrm{TR})_k$		$k = 1, \dots, n-j$	$X_{ij}.\, w_{i,j+k} = l\, r^{k-1}\, w_{ij}$
$(\mathrm{TL})_k$		$k = 1, \dots, i-1$	$X_{ij}.\, w_{i-k,j} = \frac{1}{l\, r^{k-1}}\, w_{ij}$

Ref	Multiplication on the tangles	indices range	algebraic result
(SR)$_k$		$k = 1, \ldots, n-j$	$X_{ij}.\, w_{j,j+k} = r^{(k-1)+(j-i-1)}\, w_{ij}$
(SL)$_k$		$k = 1, \ldots, i-1$	$X_{ij}.\, w_{i-k,i} = \frac{1}{r^{(k-1)+(j-i-1)}}\, w_{ij}$
(CR)$_{(s,t)}$		$s = 1, \ldots, j-i-1$ $t = 1, \ldots, n-j$	$X_{ij}.\, w_{i+s,j+t} = (r^{t+s-1} - r^{t+s-3})(l-r)\, w_{ij}$
(CL)$_{(s,t)}$		$s = 1, \ldots, i-1$ $t = 1, \ldots, j-i-1$	$X_{ij}.\, w_{i-s,j-t} = \left(\frac{1}{r^{t+s-1}} - \frac{1}{r^{t+s-3}}\right)\left(\frac{1}{l} - \frac{1}{r}\right) w_{ij}$

Determinants of square submatrices of size 5 of $T(5)$

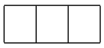
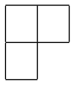
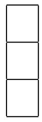
```
l:=r:N:=Matrix(T(5));

g:=proc () local M, d, i1, i2, i3, i4, i5, j1, j2, j3, j4, j5;
    for j1 to 10 do for j2 from j1+1 to 10 do for j3 from j2+1 to 10 do
         for j4 from j3+1 to 10 do for j5 from j4+1 to 10 do
             for i1 to 10 do for i2 from i1+1 to 10 do
                for i3 from i2+1 to 10 do for i4 from i3+1 to 10 do
                    for i5 from i4+1 to 10 do
                         with(linalg);
                         M := submatrix(N,[i1,i2, i3, i4, i5],
                             [j1, j2, j3, j4, j5]);
                         d := LinearAlgebra:-Determinant(
                              Matrix(M));
                         print(j1,j2,j3,j4,j5,i1,i2,i3,i4,i5,d)
                    end do
                  end do
                end do
              end do
            end do
          end do
        end do
      end do
    end do
  end do
end proc
```

And some fragments of output (columns, then rows, then determinant):

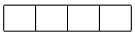
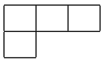
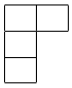
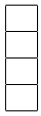
first line of output	$1,\ 2,\ 3,\ 4,\ 5,\ 1,\ 2,\ 3,\ 4,\ 5,\ \mathbf{0}$	
	$1,\ 2,\ 3,\ 4,\ 5,\ 1,\ 2,\ 3,\ 4,\ 6,\ \mathbf{0}$	
	$\vdots$	} determinants are all zeros
first	$1,\ 2,\ 3,\ 4,\ 7,\ 1,\ 2,\ 3,\ 4,\ 5,\ \mathbf{0}$	
5**-tuple**	$1,\ 2,\ 3,\ 4,\ 7,\ 1,\ 2,\ 3,\ 4,\ 6,\ \mathbf{0}$	
$[1,2,3,4,7]$	$1,\ 2,\ 3,\ 4,\ 7,\ 1,\ 2,\ 3,\ 4,\ 7,\ \mathbf{\frac{2\,r^2+r^4+1}{r^2}}$	$\leftarrow$ **first nonzero determinant**
is first	$1,\ 2,\ 3,\ 4,\ 7,\ 1,\ 2,\ 3,\ 4,\ 8,\ \frac{2\,r^2+r^4+1}{r}$	
"non-admissible"	$1,\ 2,\ 3,\ 4,\ 7,\ 1,\ 2,\ 3,\ 4,\ 9,\ 2\,r^2+r^4+1$	
pattern	$1,\ 2,\ 3,\ 4,\ 7,\ 1,\ 2,\ 3,\ 4,\ 10,\ r(2\,r^2+r^4+1)$	
of columns	$1,\ 2,\ 3,\ 4,\ 7,\ 1,\ 2,\ 3,\ 5,\ 6,\ 0$	
	$\vdots$	} whatever determinant
	$5,\ 6,\ 7,\ 8,\ 10,\ 5,\ 6,\ 7,\ 9,\ 10,\ 0$	
	$5,\ 6,\ 7,\ 8,\ 10,\ 5,\ 6,\ 8,\ 9,\ 10,\ 0$	
	$5,\ 6,\ 7,\ 8,\ 10,\ 5,\ 7,\ 8,\ 9,\ 10,\ -\frac{r^4+2\,r^2+1}{r^4}$	
	$5,\ 6,\ 7,\ 8,\ 10,\ 6,\ 7,\ 8,\ 9,\ 10,\ -\frac{r^4+2\,r^2+1}{r^3}$	
last	$5,\ 6,\ 7,\ 9,\ 10,\ 1,\ 2,\ 3,\ 4,\ 5,\ \mathbf{0}$	
"admissible"	$5,\ 6,\ 7,\ 9,\ 10,\ 1,\ 2,\ 3,\ 4,\ 6,\ \mathbf{0}$	
5**-tuple**	$\vdots$	} **determinants are all zeros**
of columns	$5,\ 6,\ 7,\ 9,\ 10,\ 6,\ 7,\ 8,\ 9,\ 10,\ \mathbf{0}$	
	$\vdots$	} whatever determinant
last line of output	$6,\ 7,\ 8,\ 9,\ 10,\ 6,\ 7,\ 8,\ 9,\ 10,\ \frac{r^4+2\,r^2+1}{r^2}$	

```
> with(linalg):l:=r:Determinant(Matrix(
                    submatrix(T(6),[1,2,3,4,7],[1,2,3,4,12])));

                                0
```

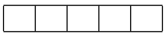
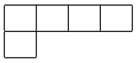
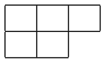
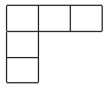
Reducibility of the L-K representation for $n = 3$

Ferrers diagrams for $n = 3$			
Reducibility of the L-K representation of degree 3	$l = \frac{1}{r^3}$	$l \in \{1, -1\}$	$l = -r^3$
Dimension of the corresponding invariant subspace of $\mathcal{V}^{(3)}$	1	2	1
Spanning vectors	$w_{12} + r\, w_{13} + r^2\, w_{23}$	$l = 1$: $\left\{ \begin{array}{l} \left(\frac{1}{r} - 1\right) w_{12} + \left(w_{13} - \frac{1}{r}\, w_{23}\right) \\ \left(\frac{1}{r} - 1\right) w_{23} + \left(w_{12} - \frac{1}{r}\, w_{13}\right) \end{array} \right.$ $l = -1$: $\left\{ \begin{array}{l} \left(\frac{1}{r} + 1\right) w_{12} + \left(w_{13} - \frac{1}{r}\, w_{23}\right) \\ \left(\frac{1}{r} + 1\right) w_{23} - \left(w_{13} - \frac{1}{r}\, w_{13}\right) \end{array} \right.$	$w_{12} - \frac{1}{r}\, w_{13} + \frac{1}{r^2}\, w_{23}$

Reducibility of the L-K representation for $n = 4$

F.D. for $n=4$					
Red. of the L-K rep. of deg 6	$l = \frac{1}{r^5}$	$l \in \{\frac{1}{r}, -\frac{1}{r}\}$	$l = r$	$l = -r^3$	cannot occur
Dim. of the inv. subspace of $\mathcal{V}^{(4)}$	1	3	2	3	
Spann. vect.	$\sum_{1\le i<j\le 4} r^{i+j}\, w_{ij}$	• $(\frac{1}{r} - \epsilon_l\, r) w_{12}$ $+(w_{13} - \frac{1}{r}\, w_{23})$ $+\epsilon_l\, r(w_{14} - \frac{1}{r}\, w_{24})$ • $(\frac{1}{r} - \epsilon_l\, r)\, w_{23}$ $+(w_{24} - \epsilon_l \frac{1}{r}\, w_{34})$ $+\epsilon_l\, r(w_{12} - \frac{1}{r}\, w_{13})$ • $(\frac{1}{r} - \epsilon_l\, r)\, w_{34}$ $+(w_{13} - \frac{1}{r}\, w_{14})$ $+\epsilon_l\, r(w_{23} - \frac{1}{r}\, w_{24})$ $\epsilon_l = \begin{cases} 1 & \text{if } l = \frac{1}{r}, \\ -1 & \text{if } l = -\frac{1}{r} \end{cases}$	• $(w_{13} - \frac{1}{r}\, w_{23})$ $-\frac{1}{r}(w_{14} - \frac{1}{r}\, w_{24})$ • $(w_{12} - \frac{1}{r}\, w_{13})$ $-\frac{1}{r}(w_{24} - \frac{1}{r}\, w_{34})$	• $(\frac{1}{r} + \frac{1}{r^3})\, w_{34}$ $+(w_{13} - \frac{1}{r}\, w_{14})$ $+r(w_{23} - \frac{1}{r}\, w_{24})$ • $(r + \frac{1}{r})\, w_{14}$ $-r\, w_{12} - r^2\, w_{13}$ $-\frac{1}{r}\, w_{34} - \frac{1}{r^2}\, w_{24}$ • $(r + \frac{1}{r^3})\, w_{12}$ $+w_{24} + \frac{1}{r}\, w_{23}$ $-w_{13} - r\, w_{14}$	

Reducibility of the L-K representation for $n = 5$

F.D. for $n = 5$	$\circledcirc$ 	$\circledcirc$ 	$\circledcirc$ 	
Red. of the L-K rep. of deg 10	$l = \frac{1}{r^7}$	$l \in \{\frac{1}{r^2}, -\frac{1}{r^2}\}$	$l = r$	$l = -r^3$
Dim. of the inv. subs. of $\mathcal{V}^{(5)}$	1	4	5	6
Spann. vect.	$\sum_{1 \leq i < j \leq 5} r^{i+j}\, w_{ij}$	For $i = 1, \ldots, 4$ $(\frac{1}{r} - \epsilon_l\, r^2)\, w_{i,i+1}$ $+ \sum_{k=i+2}^{5} r^{k-i-2}(w_{i,k} - \frac{1}{r}\, w_{i+1,k})$ $+ \epsilon_l \sum_{s=1}^{i-1} r^{s-i+3}\,(w_{s,i} - \frac{1}{r}\, w_{s,i+1})$ $\epsilon_l = \begin{cases} 1 & \text{if } l = \frac{1}{r^2}, \\ -1 & \text{if } l = -\frac{1}{r^2} \end{cases}$	• $w_1^{(4)},\, w_2^{(4)}$ • $w_1^{(5)},\, w_2^{(5)},\, w_3^{(5)}$	When $r^{10} \neq -1$: • $v_1^{(4)}, v_2^{(4)}, v_3^{(4)}$ • $v_1^{(5)}, v_2^{(5)}, v_3^{(5)}$

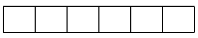
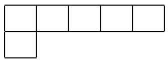
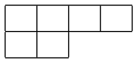
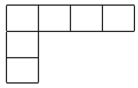
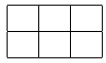
- A circled circ $\circledcirc$ on top of the Ferrers diagram indicates that the conjugate irreducible representation cannot occur in in the Lawrence-Krammer representation.
- The vectors $w_1^{(4)}$, $w_2^{(4)}$ are the spanning vectors of the unique irreducible 2-dimensional invariant subspace of $\mathcal{V}^{(4)}$.

The vectors $w_1^{(5)}$, $w_2^{(5)}$ and $w_3^{(5)}$ are given by: $\begin{cases} w_1^{(5)} &:= & w_{15} - \frac{1}{r}\, w_{25} + r\,(w_{23} - r\, w_{13}) \\ w_2^{(5)} &:= & w_{25} - \frac{1}{r}\, w_{35} + r\,(w_{13} - r\, w_{12}) \\ w_3^{(5)} &:= & w_{35} - \frac{1}{r}\, w_{45} + r\,(w_{14} - r\, w_{13}) \end{cases}$

- The vectors $v_1^{(4)}, v_2^{(4)}, v_3^{(4)}$ are the spanning vectors of the irreducible 3-dimensional invariant subspace of $\mathcal{V}^{(4)}$ of the previous table.

The vectors $v_1^{(5)}, v_2^{(5)}, v_3^{(5)}$ are defined by:
$v_k^{(5)} = w_{k+1,5} - r\, w_{k,5} + r^{5-k}\, w_{k,k+1}, \;\; k = 1, \ldots, 3$

Reducibility of the L-K representation for $n = 6$

F.D. for $n=6$	$\circledcirc$ 	$\circledcirc$ 	$\circledcirc$ 	$\circledcirc$ 	$\circledcirc$ 
Red. of the L-K rep. of deg 15	$l = \frac{1}{r^9}$	$l \in \{\frac{1}{r^3}, -\frac{1}{r^3}\}$	$l = r$	$l = -r^3$	cannot occur
Dim. of the inv. subs. of $\mathcal{V}^{(6)}$	1	5	9	10	
Spann. vect.	$\sum_{1\leq i<j\leq 6} r^{i+j}\, w_{ij}$	For $i = 1, \ldots, 5$ $(\frac{1}{r} - \epsilon_l\, r^3)\, w_{i,i+1}$ $+\sum_{k=i+2}^{6} r^{k-i-2}(w_{i,k} - \frac{1}{r}\, w_{i+1,k})$ $+\epsilon_l \sum_{s=1}^{i-1} r^{s-i+4}\,(w_{s,i} - \frac{1}{r}\, w_{s,i+1})$ $\epsilon_l = \begin{cases} 1 & \text{if } l = \frac{1}{r^3}, \\ -1 & \text{if } l = -\frac{1}{r^3} \end{cases}$	$\cdot\ \mathcal{W}^{(5)}$ $\cdot\ w_1^{(6)}$ $\cdot\ w_2^{(6)}$ $\cdot\ w_3^{(6)}$ $\cdot\ w_4^{(6)}$	When $r^{10}, r^{12} \neq -1$: $\cdot\ v_s^{(t)}, \begin{cases} t \in \{4,5\} \\ 1 \leq s \leq 3 \end{cases}$ $\cdot\ v_1^{(6)}$ $\cdot\ v_2^{(6)}$ $\cdot\ v_3^{(6)}$ $\cdot\ v_4^{(6)}$	

- A circled circ $\circledcirc$ on top of the Ferrers diagram indicates that the conjugate irreducible representation cannot occur in in the Lawrence-Krammer representation.
- $\mathcal{W}^{(5)} = \{w_1^{(4)}, w_2^{(4)}, w_1^{(5)}, w_2^{(5)}, w_3^{(5)}\}$, all the vectors defined in Appendices F and G.
- $$\begin{array}{rcll} w_1^{(6)} & = & w_{16} - \frac{1}{r}\, w_{26} & + \quad r^2\,(w_{23} - r\, w_{13}) \\ \forall\, 2 \leq k \leq 4, \quad w_k^{(6)} & = & w_{k,n} - \frac{1}{r}\, w_{k+1,n} & + \quad r^2\,(w_{1,k+1} - r\, w_{1,k}) \end{array}$$
- The vectors $v_s^{(t)}$ with $t \in \{4,5\}$ and $s \in \{1,2,3\}$ are defined in Appendices F and G.
- The $v_k^{(6)}$, $k = 1, \ldots, 4$ are given by:
  $v_k^{(6)} := w_{k+1,6} - r\, w_{k,6} + r^{6-k}\, w_{k,k+1}$

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