Appendix C: Damping Due to Eddy Currents

Figure C.1 shows a cutaway view of the cylindrical sensor magnet moving inside an annular magnet. The relative motion of these magnets induces eddy currents in both the sensor and the annulus, with such currents being proportional to the material's conductivity. The goal of this appendix is to calculate the oscillator's damping rate $\gamma = 2/\tau$ due to eddy currents in the BOOMERANG prototype.

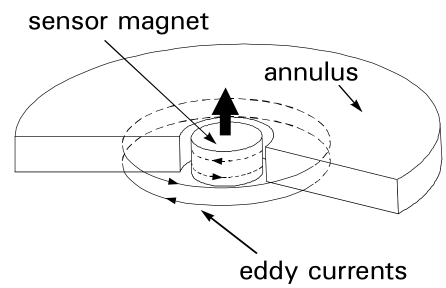

Figure C.1. Eddy currents due to relative motion of the magnets.

We start by considering the electric field produced by the sensor magnet moving in the rest frame of the annular magnet. If a stationary magnetized body produces a magnetic field $\mathbf{B}(\mathbf{r})$, then it produces an electric field

$$\mathbf{E}(\mathbf{r}) = -\mathbf{v} \times \mathbf{B}(\mathbf{r}) \tag{C.1}$$

in a coordinate frame in which its velocity is $\mathbf{v}$. The same result is obtained by considering the time derivative of the local vector potential. This electric field induces eddy currents in the conducting annular magnet that are strictly azimuthal due to the symmetry of the magnets, as we shall see. We will neglect eddy currents in other metal parts due to the fact that these parts are far more distant from the sensor than the annulus is and because, as we will see, the eddy current density falls off very rapidly with distance. Eddy currents in the sensor (by virtue of its motion in the field of the other magnets) can also be calculated in the rest frame of the sensor magnet using Equation (C.1).

Before we continue, an important point must be made. Our analysis will leave out the fact that the eddy currents themselves are time-varying. A given induced current element therefore gives rise to oscillating electromagnetic fields and secondary eddy currents in nearby conductors. A more rigorous analysis of the problem must therefore be cast in terms of field equations[1], which are further complicated by moving media[2]. We shall continue with our more or less rough estimate of the eddy currents nonetheless. It will turn out that while the skin depth in the mu-metal magnets (which can be said to quantify the importance of this "self-consistent-field" issue),

$$\delta = \sqrt{\frac{2}{\omega\sigma\mu}} = \sqrt{\frac{2}{(2\pi \cdot 500\,\mathrm{Hz})(1.72 \cdot 10^{6}\,\Omega^{-1}\mathrm{m}^{-1})(50000 \cdot 4\pi \cdot 10^{-7}\ \mathrm{TmA}^{-1})}} = 77\,\mu\mathrm{m}\,, \quad (C.2)$$

is far smaller than the size of the magnets, it is about the same size as the range over which eddy currents are strong. So, the more rigorous analysis will not differ wildly from our simpler theory at the prototype size scale, and agreement between the theories will become closer as size scales are reduced.

The system of sensor magnet and annulus is symmetric with respect to rotation about the vertical axis. Consequently, the azimuthal component of the magnetic field

$$\mathbf{B}(\rho, z) = B_\rho(\rho, z)\hat{\rho} + B_z(\rho, z)\hat{\mathbf{z}} \tag{C.3}$$

vanishes, and we may work in a gauge in which the magnetic vector potential $\mathbf{A}(\rho, z) = A_\phi(\rho, z)\hat{\phi}$ is strictly azimuthal. Here and in what follows we make use of a cylindrical coordinate frame $\{\hat{\rho}, \hat{\phi}, \hat{\mathbf{z}}\}$, with z along the symmetry axis. The velocity of the sensor magnet $\mathbf{v} = v\,\hat{\mathbf{z}}$ is also along the symmetry axis, and since the symmetry is therefore not broken, we may write Equation (C.1) as

$$\mathbf{E} = -v\,B_\rho\hat{\phi}\ . \tag{C.4}$$

Since the boundaries of the cylindrical magnets are parallel to this azimuthal electric field, and since the conductivity σ is isotropic, the induced currents $\mathbf{J}(\mathbf{r}) = \sigma\mathbf{E}(\mathbf{r})$ at every position $\mathbf{r}$ are also strictly azimuthal. We then find that the local dissipated power density is

$$W = \mathbf{J} \cdot \mathbf{E} = \sigma E^2 = \sigma v^2 B_\rho^2\ . \tag{C.5}$$

The velocity $v(t) = v_0 \cos \omega t$ is a function of time, and we may write the total instantaneous power dissipation as an integral over the volume V of the conductor in terms of the radial field B_ρ produced by the moving element,

$$P(t) = \int_V W dV = \sigma v(t)^2 \int_V B_\rho^2 dV\ . \tag{C.6}$$

The dissipated power is also the (negative) time derivative of the work done by the dissipative force $\mathbf{F} = -\alpha\mathbf{v}$:

$$P(t) = -\frac{\mathrm{d}}{\mathrm{d}t}U = -\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \mathbf{F}\cdot\mathbf{dl} = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \alpha\mathbf{v}(t')\cdot\mathbf{v}(t')dt' = \alpha v(t)^2 \,. \tag{C.7}$$

Setting Equations (C.6) and (C.7) equal, we find

$$\gamma = \frac{1}{m}\alpha = \frac{\sigma}{m}\int_V B_\rho^2 dV \,. \tag{C.8}$$

An important conclusion to be drawn from Equation (C.8) is that, since the conductivity σ and the field B are scale-invariant, and since m and V are both proportional to r^3, the damping rate γ is scale-invariant. To evaluate the integral in (C.8), we must first know B_ρ, which is itself the result of a volume integration over contributions from dipole elements in the moving magnet. It is convenient first to find the vector potential $\mathbf{A}(\mathbf{r})$, in terms of which we have

$$B_\rho = \hat{\rho}\cdot\nabla\times\mathbf{A} = -\frac{\partial A_\phi}{\partial z} \,. \tag{C.9}$$

If we assume that the magnetization is uniform, then the necessary volume integration,

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\iiint_{\text{magnet}} \frac{M\hat{\mathbf{z}}\times\hat{\mathbf{r}}}{r^2}dV \,, \tag{C.10}$$

is simplified by standard integral theorems. We obtain the well-known result[3] that the field outside a cylindrical magnet with uniform axial magnetization is the same as if the magnet were replaced by a solenoid of the same dimensions with the surface current density equal to the magnetization M:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 M}{4\pi} \iint_{\text{side}} \frac{\hat{\phi}}{r}\, ds\,. \tag{C.11}$$

The surface element in this integral, $ds = \rho\, d\phi\, dz$ includes the axial coordinate z, and so combination of Equations (C.9) and (C.11) is facilitated by the fundamental theorem of calculus, which offsets the axial derivative and integration. The result is

$$B_\rho(\mathbf{r}) = \frac{\mu_0 M}{4\pi} \left[\oint_{\substack{\text{top}\\ \text{edge}}} \frac{a\hat{\phi}}{r}\, d\phi - \oint_{\substack{\text{bottom}\\ \text{edge}}} \frac{a\hat{\phi}}{r}\, d\phi \right], \tag{C.12}$$

where a is the radius of the magnet. The integration paths for these integrals are as shown in Figure C.2.

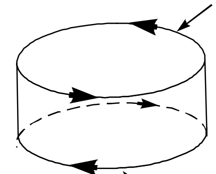

Figure C.2. Integration paths for Equation (C.8) when the sensor magnet is considered the source of electric fields (in the other magnets).

The line integrals in Equation (C.12) may be evaluated in terms of the complete elliptic integrals of the first and second kinds, $K(m)$ and $E(m)$. The result is

$$I(\rho, z) \equiv \oint_{\substack{\text{edge}\\ \text{ring}}} \frac{a\hat{\phi}}{r}\, d\phi = \sqrt{\frac{4a}{\rho}} \left(\frac{(2-m)K(m) - 2E(m)}{\sqrt{m}} \right) \hat{\phi}\,, \tag{C.13}$$

where $m = \frac{4a\rho}{z^2 + (\rho + a)^2}$, and where z is measured from the plane of the given edge ring (top or bottom).

The same analysis applies to calculation of eddy currents in the sensor magnet if one calculates with the other magnets moving with velocity **v** in the rest frame of the sensor magnet. (Effects due to acceleration of the magnets are negligible and ignored.) Equations (C.12) and (C.13) may be used to calculate the integrand B_ρ^2 in Equation (C.8). Figure C.3 shows a contour plot of this value (scaled by the Jacobian determinant for the integration, which is ρ). The picture is a detailed map of how the dissipated power density is distributed inside the magnets. The eddy currents are concentrated near the sharp edges of both the sensor magnet and annulus.

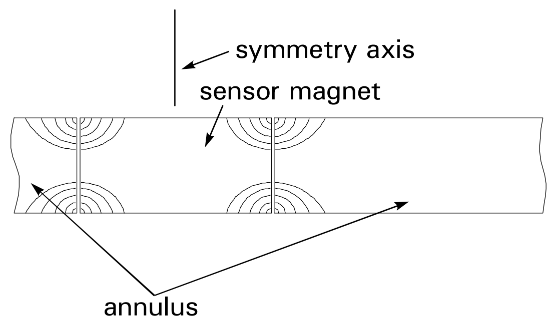

Figure C.3. Power dissipation in the magnets. Contours show where most of the power is dissipated. Each contour represents a factor of 2 decrease in power density.

Figure C.3 shows that the power density falls off approximately exponentially with distance from the sharp corners of the magnets. The calculated distance over which the power density decreases by *e* in this model is 110 μm, which is close to the skin depth (77 μm) calculated at 500 Hz for mu metal. This means that we are somewhat *over*-estimating the damping rate, which we calculate to be 0.93 Hz by numerically integrating (C.8). Were the skin depth much smaller (or the magnets larger), the conclusions drawn from our rough theory of eddy currents would have to be changed. In particular, the effective conductivity, and therefore the damping rate, would be reduced. The effective size of the conductor would also scale as an area rather than a volume, and so, above the size scale of the BOOMERANG prototype, the damping rate γ scales inversely with size (as r^{-1}).

Again, because of the skin effect, we are slightly overestimating the eddy current damping with our simplified model relative to a more exact calculation with field equations. However, the observed damping rate is still somewhat *larger* than we've calculated. Empirically we find that introducing radial slits to interrupt and redirect the eddy currents reduces the damping rate, as does increasing the gap or rounding the edges of the magnets[4]. So it is likely that our assumption of perfectly uniform magnetization is an oversimplification. Indeed, allowing the magnetization to have a nonvanishing radial component would increase B_ρ in Equation (C.8), and this would also increase our estimate of the damping rate.

References

1 W. R. Smythe, *Static and Dynamic Electricity*, 3rd ed. (McGraw-Hill, New York, 1968).

2 P. Penfield and H. A. Haus, *Electrodynamics of Moving Media* (M.I.T. Press, Cambridge, MA, 1967).

3 P. Lorrain, D. R. Corson, and F. Lorrain, *Electromagnetic Fields and Waves*, 3rd ed. (W. H. Freeman and Co., New York, 1988).

4 L. A. Madsen, Ph. D. Thesis, California Institute of Technology, 2002.