Fluid Locomotion and Trajectory Planning for Shape-Changing Robots

Thesis by

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Abstract

Motivated by considerations of shape changing propulsion of underwater robotic vehicles, I analyze the mechanics of deformable bodies operating in an ideal fluid. I give particular attention to fishlike robots which may be considered as one or more flexing or oscillating hydrofoils. I then describe methods of planning trajectories for a fishlike robot or any other sort of robot whose locomotion has a periodic or quasi-periodic nature.

This doctoral work was performed by Richard Mason at the California Institute of Technology. My thesis advisor was Professor Joel W. Burdick.

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Chapter 1 Introduction

1.1 Motivation

In this dissertation, I attack several problems relating to the general theme of locomotion in a fluid by a shape-changing body. The underlying motivation is to better understand the propulsion and control of biomimetic underwater vehicles.

Not surprisingly, many aquatic creatures are impressive swimmers, and observers have argued that the locomotion of various biological swimmers, from fish to cetaceans to penguins, must be exceptionally efficient. Such proposals date at least back to 1936 when Gray estimated the muscle power available to a dolphin and how much power should be required to propel a dolphin-shaped object through the water at delphinine speeds [Gra36]. "Gray's Paradox" is that the latter estimate was seven times higher than the former estimate, implying that dolphins should not be able to swim as fast as they do. Over the decades various flaws have been exposed in Gray's calculations, but the idea persists that dolphins and other biological swimmers may be unexpectedly efficient, having evolved to exploit some unusual fluid mechanics.

More recently, Triantafyllou et al. have proposed that fast-swimming fish use tail motion as a means of vorticity control, minimizing energy lost in the wake and recovering energy from vortices originating ahead of the tail. A flexible body utilizing these energy recovery methods could have an apparent drag less than that of a rigid body of the same size and baseline shape. There is experimental support for this theory: notably, the MIT "RoboTuna" [Bar96, BGT96] was able to achieve significant drag reduction with a flexible oscillating body, suggesting not only that fish-like propulsion can be efficient but that the effect can be reproduced by macroscopic motions of a mechanical device. Since non-nuclear underwater vehicles are often significantly constrained by the battery energy available to them, any mechanism for increasing their efficiency which could be copied from the biological world would be a significant benefit.

A variation on the idea of wake control is that if a biomimetic robot swimmer recaptures energy from its wake, it will make less noise in the water. Also, whatever noise it does make will presumably sound more like a fish (or cetacean, et cetera) and not like an ordinary vehicle. So fish-like robots may be useful in stealthy approach or surveillance applications.

Biological swimmers are also impressive in being notably more maneuverable than ordinary watercraft. Water vehicles with conventional propulsors and control surfaces typically require several body lengths to make a turn and have low maneuverability at low speeds. The addition of multiple propulsors along lateral axes can improve low-speed maneuverability but at a serious cost in weight, volume, and complexity. Meanwhile, some fish can make a 180-degree turn in a single body length, while predatory fish are capable of accelerations on the order of $100\mathrm{m}^2/\mathrm{s}^2$ from a dead stop. There may be a role for high-manueverability vehicles which work in the same way.

For all these reasons, in the last decade there has been wide interest in biomimetic swimming robots, which propel themselves through a fluid not by conventional thrusters, but by undulations or quasi-periodic changes in shape. Experimental efforts have included attempts to imitate the action of a "disembodied" fish tailfin [AHB$^+$91, ACS$^+$97, SLD00] and tow tank models of entire fish such as the MIT RoboTuna [Bar96, BGT96]. Free-swimming biomimetic robots include the Vorticity Control Unmanned Undersea Vehicle built at the Charles Stark Draper Laboratory [AK99]; the MIT RoboPike project; an eel-like robot at the University of Pennsylvania [CMOM01]; a fish robot at the Ship Research Institute in Tokyo [HTT00]; a dolphin robot at the Tokyo Institute of Technology [NKO00]; an underwater vehicle with paired mechanical pectoral fins at Tokai University [KWS00]; and ongoing efforts to develop robot lampreys [AWO00] and robot penguins [Ban00].

In this thesis I will suggest some modelling and planning techniques relevant to these and future biomimetic vehicles, and also describe an experimental platform constructed at Caltech.

1.2 Summary and Relation to Previous Work

In **Chapter 2**, I examine the self-propulsion of a *smooth body changing shape* in a quiescent irrotational potential flow, a problem previously studied by Saffman [Saf67] and by Miloh and Galper [Mil91, MG93, GM94]. I specialize to a body with a finite number of modes of deformation and derive the equations of motion emphasizing the "connection" from geometric mechanics: this may be considered an extension of Ostrowksi and Burdick's work on undulatory locomotion [OB98, Ost95] to fluid mechanical systems. I then focus on the particular example of the *squirming circle*, which was introduced in the case of one-dimensional motion by Kelly and Murray [KM96], and which bears comparison to the experimental *amoebot* built by Chen [CLC98]. I extend the example to motion in the plane and provide optimal control results for following a kinematic trajectory in fixed time while minimizing control effort.

In **Chapter 3**, I present general closed-form expressions for the force and moment on a *deformable Joukowski foil* executing arbitrary translation and rotation while also undergoing an arbitrary time-dependent change in its shape parameters. My motivation is to model the forces on a bending flexible fin or flexible streamlined body: however, I leave open the possibility that the foil has not only time-varying camber but also time-varying thickness, chord length, and/or area. I assume inviscid flow with a finite number of point vortices, including wake vortices shed from the trailing edge of the foil over time. These results may be considered a generalization of the closed-form expressions found by Streitlien [Str94, ST95] for a rigid Joukowski foil undergoing arbitrary rigid-body motion.

Of course, my results include the rigid-body case as a subset. In particular, I provide closed-form expressions for the added inertia coefficients of a rigid Joukowski foil. Previously published closed-form expressions for the added inertia [Str94, ST95, Sed65] suffer from typographical and other elementary errors such as dropped terms and dropped factors of two. This thesis corrects the historical record, while extending results to the deformable case.

In **Chapter 4**, I describe an *experimental three-link robot fish*, built by myself and others, which propels both itself and an attached gantry suspended above the water. The differences between this experimental platform and previous robot fish efforts are as follows:

- Our platform is fully mobile in the plane and genuinely self-propelled, as opposed to being statically mounted or towed through the water as in [Bar96, BGT96, AHB$^+$91, ACS$^+$97]. This makes it a better model of free swimming. In particular, the fact that our fish body is free to sideslip or rotate in the plane has a dramatic impact on swimming motions. Useful thrust provided by a tail-stroke can be much greater when the body is fixed, towed, or constrained to one dimension than when it yaws freely. Also, we were able to consider problems of turning and yaw stability.

- On the other hand, the restriction to planar motions simplifies analysis of the system to a planar problem. The attached gantry enables us to have much more precise telemetry of the fish's motion than is feasible for a truly free-swimming platform such as [AK99], enabling us to make more precise comparisons between theory and experiment. This has also enabled feedback control experiments.

I compare the experimental results from the platform to the results from theory and computer simulation. It turns out that a relatively simple model incorporating quasi-static lift and added-mass effects is good enough to make qualitative predictions about the platform's swimming behavior. I show that for swimmers executing certain types of maneuvers, lift forces dominate added mass forces, and vice versa.

In **Chapter 5**, I consider the equations of motion for the three-link swimmer in the *limit where added mass forces dominate,* and how to find solutions to these equations which are optimal according to a plausible performance index. Once found, these optimal solutions to short-time-horizon planning problems can be pieced together to form larger trajectories.

In **Chapter 6**, I examine the problem of *trajectory planning for a mobile robot restricted to a set of finite motions*, which can be iterated and concatenated to form larger trajectories. This is a reasonable model for shape-changing robots, which (a) locomote through repeated periodic undulations; (b) will likely have a finite number of gaits or periodic motions which are understood or deemed optimal; and (c) may not be small-time locally controllable.

I suggest three planning methods based on the density of points in configuration space reachable by the robot in a certain number of steps. The first algorithm proposed is directly inspired by the density-based Ebert-Uphoff algorithm for inverse kinematics of a discrete manipulator, as outlined by Chirikijian and Kyatkin [CK01]. I adapt the Ebert-Uphoff algorithm to the domain of mobile robots, notionally replacing the links of a discrete manipulator with segments of a path. One important change is that unlike the number of links in an existing manipulator, the number of steps in a mobile robot's path can be freely increased or decreased as necessary on a task-by-task basis. I discuss how the mobile robot can make task-by-task trade-offs between path length and the precision with which the goal is reached, and illustrate with an example using a model system. I also enhance the Ebert-Uphoff algorithm by evaluating each density function on several length scales instead of just one, since different length scales are appropriate depending on whether solutions near the desired goal are dense or sparse.

The second density-based planning algorithm uses a divide-and-conquer strategy instead of the essentially linear method of the Ebert-Uphoff algorithm. The divide-and-conquer algorithm requires exponentially less memory than the Ebert-Uphoff algorithm, and is exponentially faster during the expensive mapping/pre-computation phase. At run time, when a goal is provided and a trajectory is planned using the pre-computed map, the divide-and-conquer strategy may be either faster or slower than the linear strategy, depending on how well-behaved the mobile system's density functions are, and whether the divide-and-conquer algorithm can take advantage of parallel processing.

Finally, I suggest a way to use density functions to help search for a path through a field of static obstacles.

When compared to existing trajectory-planning methods based on Dijkstra's algorithm [Lat90], the density-based methods have the following advantages:

- Path length is a fully controllable parameter. The method can be used to find not just the shortest path, but a path with any specified acceptable length. This facilitates trade-offs between path length and goal accuracy, as mentioned earlier, and could also be valuable in rendezvous problems.

- Because the density function represents all feasible paths, instead of recording only the "best" path to each location like Dijkstra-based methods, the density-based methods are more robust in some ways. If the "best" path is unexpectedly unusable, the density-based methods are able to plan an alternative trajectory without starting from scratch.

- It is possible for Dijkstra-based methods to find a suboptimal solution, or no solution at all, because the correct solution was lost to "round-off error." This problem may be especially acute for robots, such as fish robots, which are not small-time locally controllable. The density-based planning methods can also lose solutions to round-off error, but because the density functions can be found by convolution of sets of large motions, instead of by incrementally adding small motions, the problem is less severe. This makes the density-based methods better suited for problems involving either long trajectories or a coarsely discretized configuration space.

Chapter 2 Smooth Deformable Bodies in an Ideal Fluid

2.1 Potential Flow Around a Smooth Deformable Body

Real amoebae are microscopically small. They operate at a very low Reynolds number and the relevant fluid equations are those of creeping flow [SW89]. However, if we were to build a macroscopic "robot amoeba" and expect it to swim through water, the Reynolds number of its ambient flow would be much higher and inertial forces will dominate instead of viscous ones. I thus make the reasonable idealization that the robot amoeba is a connected deformable body swimming through an inviscid and incompressible fluid. I also assume that the fluid is irrotational everywhere, and that the amoeba cannot generate vorticity in the fluid. Unless the amoeba grows sharp fins, this too is a reasonable assumption.

Any "amoeba" robot which we might actually construct would have a finite number of actuators. Indeed, we would like to use as few actuators as possible. Therefore, rather than allow the boundary of the amoeba to be infinitely variable, I assume that its shape can be described by a finite number, n_s, of *shape* variables, s. The space of all possible shapes, denoted by $\mathcal{S}$, is a finite-dimensional manifold.

I fix a frame, $\mathcal{F}_B$, to the body of the swimmer and let $\mathcal{F}_W$ denote a fixed reference frame. (See Figure 2.1.) The location of the $\mathcal{F}_B$ is given by $g(t) \in \mathrm{SE}(d)$, $d = 2, 3$. In coordinates, elements of $\mathrm{SE}(d)$ can be represented by homogeneous matrices, g.

$$g = \begin{pmatrix} R & \vec{p} \\ \vec{0}^T & 1 \end{pmatrix} \tag{2.1}$$

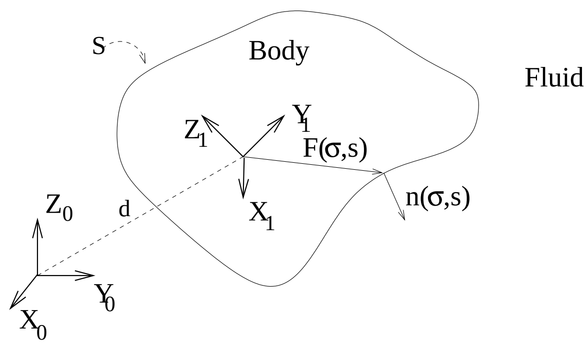


Figure 2.1: Schematic of deformable swimming body

The matrix $R \in \mathrm{SO}(d)$ describes the orientation of $\mathcal{F}_B$ with respect to $\mathcal{F}_W$, while $\vec{p} \in \mathbb{R}^d$ is the position of $\mathcal{F}_B$'s origin. The velocity of the moving reference frame, as seen by an observer in $\mathcal{F}_B$, is $g^{-1}\dot{g}$:

$$g^{-1}\dot{g} = \begin{pmatrix} \hat{\omega} & \vec{\xi} \\ \vec{0}^T & 0 \end{pmatrix} \tag{2.2}$$

where $\hat{\omega}$ is a $d \times d$ skew symmetric matrix and $\vec{\xi} \in \mathbb{R}^d$. The quantity $g^{-1}\dot{g}$ is an element of the Lie algebra of $\mathrm{SE}(d)$, $\mathfrak{se}(d)$. I shall denote by "$\vee$" the identification of $\mathfrak{se}(d)$ with $\mathbb{R}^{\frac{d(d+1)}{2}}$: $(g^{-1}\dot{g})^\vee = [\xi^T \ \omega^T]^T$, where ξ and ω are the linear and angular body velocities.

The swimmer's smooth surface, Σ, is parameterized by a coordinate chart that is a function of $(d-1)$ parameters $\sigma_1, \ldots, \sigma_{(d-1)}$, or by an atlas of $(d-1)$-dimensional charts. The surface parameters themselves are functions of the shape variables, $s_1, \cdots, s_{n_s}$.

Given the assumptions described above, the fluid motion around the swimmer is described by potential flow, and its domain, $\mathcal{D}$, is assumed to be unbounded. In the most general case, the ambient fluid undergoes non-uniform motion. I will assume that the ambient flow is quiescent. The Kirchoff principle for potential flow around a rigid body [VVP73] can be extended to show that the general fluid potential, ϕ, for an irrotational fluid surrounding a deformable body will take the form:

$$\phi = \sum_{i=1}^{3}(\xi_i \phi_i^g + \omega_i \phi_{i+3}^g) + \sum_{j=1}^{n_s} \phi_j^s \dot{s}_j. \tag{2.3}$$

The terms $\{\phi_i^g\}$ are the standard Kirchoff potentials for a rigid body—in this case, for the deformable body at a fixed shape, s. The term ϕ^s terms represent the contribution to the total potential due to body deformations.

Let $F(\vec{\sigma}, s)$ denote the location of a surface point with respect to a body fixed frame. The normal vector to the surface at that point is denoted $n(\vec{\sigma}, s)$. Then the instantaneous velocity, in the body frame, of the surface point parameterized by $\vec{\sigma}$ is $\vec{\xi} + \vec{\omega} \times F(\vec{\sigma}, s) + \frac{\partial}{\partial s_i} F(\vec{\sigma}, s)\dot{s}_i$. At the body surface, the fluid velocity and surface velocity in the normal direction must match, leading to the following boundary conditions.

$$\begin{aligned} \nabla\phi_i^g \cdot \vec{n}(\vec{\sigma}, s) &= n_i(\vec{\sigma}, s) \qquad\qquad i = 1, 2, 3 \\ \nabla\phi_i^g \cdot \vec{n}(\vec{\sigma}, s) &= (F(\vec{\sigma}, s) \times \vec{n}(\vec{\sigma}, s))_i \qquad i = 4, 5, 6 \\ -\nabla\phi_i^s \cdot n(\vec{\sigma}, s) &= \frac{\partial}{\partial s_i} F(\vec{\sigma}, s) \cdot n(\vec{\sigma}, s)\ \forall s_i \end{aligned} \tag{2.4}$$

These form separate Neumann problems for the Laplace equation ($\nabla^2\phi = 0$) for each term ϕ_i^g or ϕ_i^s. A unique solution (up to a constant) exists for each term [NS82]. [1]

The total kinetic energy of the constant density fluid and deformable body system is

$$\begin{aligned} T_{total} = T_{body} \ + \ T_{fluid} &= \tfrac{1}{2}\dot{q}^T \Lambda(q)\dot{q} \ + \ \frac{\rho_0}{2}\int_{\mathcal{D}} ||\nabla\phi||^2 \, dV \\ &= \tfrac{1}{2}(\dot{g}^T \ \dot{s}^T) \begin{pmatrix} \Lambda^{gg}(q) & \Lambda^{gs}(q) \\ (\Lambda^{gs})^T(q) & \Lambda^{ss}(q) \end{pmatrix} \begin{pmatrix} \dot{g} \\ \dot{s} \end{pmatrix} \ - \ \frac{\rho_0}{2}\int_{\Sigma} \phi(\nabla\phi \cdot n) \, d\Sigma, \end{aligned} \tag{2.5}$$

where ρ_0 is the fluid density, and $\Lambda(q)$ is the kinetic energy metric (or mass matrix) of the deformable body (in the absence of surrounding fluid). Because the potentials

[1]If the fluid external to the swimmer is considered to be an unbounded domain, $\mathcal{D}$, then a unique solution (up to an overall constant) is only guaranteed if the velocity of the fluid drops off sufficiently quickly (at least as fast as r^{-2}) at infinity–but this requirement is physically reasonable. Alternatively, one could adopt the "common-sense" position that the fluid really occupies a large but bounded domain.

take the form of Equation (2.3), the total kinetic energy can be put in the form

$$T_{total} = \tfrac{1}{2}\dot{q}^T M(q)\dot{q}.$$

For the moment I will ignore any additional potential forces acting on the deformable body, and hence the system's Lagrangian is equivalent to T_{total}. One could next derive the governing mechanics from the Euler-Lagrange equations. Instead I take a more abstract approach.

2.2 Ideas from Geometric Mechanics

New insight can be obtained by applying methods of geometric mechanics to the system described in the previous section. In particular I wish to find symmetries that lead to reduction. This section provides a brief summary of relevant ideas. More extensive background can be found in [MR94, MS93a].

2.2.1 Principal Fiber Bundles

Let Q denote the configuration space of the deformable swimmer, which consists of its position, $g \in \mathrm{SE}(d)$, and its shape, $s \in \mathcal{S}$. Hence, the configuration space is $\mathrm{SE}(d) \times S$, and a configuration $q \in Q$ can be given local coordinates $q = (g, s)$. This configuration space has a surprisingly rich structure due to the fact that $\mathrm{SE}(d)$ is a Lie group. Recall that every Lie group, G, has an associated Lie algebra, denoted $\mathfrak{g}$. In our context, the Lie algebra of $\mathrm{SE}(d)$, denoted by $\mathfrak{se}(d)$, consists of the velocities of $\mathcal{F}_B$ relative to $\mathcal{F}_W$, as seen by a body fixed observer. Elements of $\mathfrak{se}(d)$ can be represented by matrices of the form $g^{-1}\dot{g}$, as seen in Equation (2.2).

If the swimmer's initial body fixed frame position is $h \in G$, and it is displaced by an amount g, then its final position is gh. This *left translation* can be thought of as a map $L_g : G \to G$ given by $L_g(h) = gh$. The left translation induces a *left action* of G on Q. A left action is a smooth mapping $\Phi : G \times Q \to Q$ such that: (1) $\Phi_e(q) = q$ for all $q \in Q$; and (2) $\Phi_g(\Phi_h(q)) = \Phi_{L_g h}(q)$ for all $g, h \in G$ and $q \in Q$. The configuration space Q endowed with such an action is a *principal fiber bundle*. Q is called the *total space*, $\mathcal{S}$ the *base space* (or *shape space*), and G the *structure group*. The *canonical projection* $\pi : Q \to S = Q/G$ is a differentiable projection onto the second coordinate factor: $\pi(q) = \pi(g, s) = s$. The sets $\pi^{-1}(s) \subset Q$ for $s \in \mathcal{S}$ are the *fibers*, and Q is the union over $\mathcal{S}$ of its fibers. The usefulness of this structure will become more apparant in Section 2.2.3.

2.2.2 Symmetries

In Lagrangian mechanics, symmetries result in conservation laws. By a symmetry, we mean an invariance of the Lagrangian with respect to some operation.

Definition 1. The *lifted action* is the map $T\Phi_g : T_qQ \to T_{\Phi(q)}Q : (q, v) \mapsto (\Phi_g(q), T_q\Phi_g(v))$ for all $g \in G$ and $q \in Q$. I.e., $T_q\Phi_g$ is the Jacobian of the group action. For left translation on G, $T\Phi_g$ has the coordinate form:

$$T_q\Phi_h\dot{q} = \begin{pmatrix} T_gL_h\dot{g} \\ \dot{s} \end{pmatrix} = \begin{pmatrix} h\dot{g} \\ \dot{s} \end{pmatrix}. \tag{2.6}$$

Based on the group action and lifted action, we can introduce the following notions of symmetry, or *invariance*.

Definition 2. A Lagrangian function, $L \colon TQ \to I\!\!R$, is said to be G-*invariant* if it is invariant with respect to the lifted action, i.e., if

$$L(q, v_q) \quad = \quad L(\Phi_h(q), T_q\Phi_h v_q)$$

for all $h \in G$ and all $v_q \in T_qQ$.

2.2.3 Connections

To analyze and control propulsion of a deformable body, one would like to systematically derive an expression that answers the question: "If I wiggle the body's surface, how does the body move?" The relationship between shape and position changes can be formalized in terms of a *connection*, an intrinsic mathematical structure that is associated with a principal fiber bundle. I begin with some necessary technical definitions.

Definition 3. If $\xi \in \mathfrak{g}$, the vector field on Q denoted by ξ_Q and given by

$$\xi_Q(q) = \frac{d}{dt}\Phi_{\exp(t\xi)}(q)\big|_{t=0} \tag{2.7}$$

is called the *infinitesimal generator* of the action corresponding to ξ. The vertical space, $V_qQ \subset T_qQ$, is defined as

$$V_qQ = ker(T_q\pi) = \{v|\ v = \xi_Q(q)\ \ \forall\ \xi \in \mathfrak{g}\}.$$

Definition 4. The *connection one-form*, $\Gamma(q)\colon T_qQ \longrightarrow \mathfrak{g}$, is a Lie-algebra valued one-form having the following properties:

(i) $\Gamma(q)$ is linear in its action on T_qQ.

(ii) $\Gamma(q)\xi_Q = \xi$ for $\xi \in \mathfrak{g}$.

(iii) $\Gamma(q)\dot{q}$ is equivariant, i.e., it transforms as $\Gamma(\Phi_h(q))T_q\Phi_h(q)\dot{q} = \mathrm{Ad}_h\,\Gamma(q)\dot{q}$, where the adjoint action $\mathrm{Ad}_h : \mathfrak{g} \longrightarrow \mathfrak{g}$ is defined as $\mathrm{Ad}_h\,\xi = T_{h^{-1}}L_h(T_eR_{h^{-1}}\xi)$ for $\xi \in \mathfrak{g}$.

The infinitesimal generator of the adjoint representation is given by

$$\mathrm{ad}_\xi\,\eta = \xi_{\mathfrak{g}}\eta = [\xi, \eta] \quad \text{for } \eta, \xi \in \mathfrak{g}. \tag{2.8}$$

The *horizontal space* is the kernel of the connection one-form, $H_qQ = \{z \mid \Gamma(q)z = 0\}$, and is complementary to V_qQ. It can be shown that the connection one-form can be expressed in local coordinates $q = (g, s)$ as follows:

$$\Gamma(q)\dot{q} = Ad_g(A(s)\dot{s} + g^{-1}\dot{g}), \tag{2.9}$$

where $A\colon T\mathcal{S} \longrightarrow \mathfrak{g}$ is termed *the "local" form of the connection.* Hence, any $\dot{q} = (\dot{g}, \dot{s})$ which lies in H_qQ must satisfy a constraint of the following form:

$$g^{-1}\dot{g} = -A(s)\dot{s}. \tag{2.10}$$

Note that *the local connection plays a central role in the ensuing analysis.*

For Lagrangian systems with symmetries, such as the one studied here, the conservation laws that arise from symmetries constrain the overall system motion. This constraint can be expressed as a connection, called the *mechanical connection* [MR94]. It can be shown that the *mechanical connection* is given by the expression

$$\Gamma(q)v_q = I^{-1}(q)\mathrm{J}(v_q), \tag{2.11}$$

where the *momentum map*, $\mathrm{J} : TQ \to \mathfrak{g}^*$, satisfies:

$$\langle \mathrm{J}(v_q); \xi \rangle = \langle\langle v_q, \xi_Q \rangle\rangle,$$

for all $\xi \in \mathfrak{g}$ and $v_q \in T_qQ$. The expression $\langle\langle \cdot, \cdot \rangle\rangle$ denotes inner produce with respect to the kinetic energy metric. The *locked inertia tensor* is the map, $I(q) : \mathfrak{g} \to \mathfrak{g}^*$ which satisfies

$$\langle I(q)\xi; \eta \rangle = \langle\langle \xi_Q, \eta_Q \rangle\rangle \text{ for all } \xi, \eta \in \mathfrak{g}.$$

2.3 The Fluid Mechanical Connection

I now revisit the mechanics of the deformable swimmer in light of the ideas presented in the last section. I first show that a Lagrangian that is invariant with respect to a group action Φ_h induces a *reduced Lagrangian.* This is a general result that is independent of the particular fluid mechanical model that is studied in this paper.

Proposition 2.3.1. *[Ost95] If $L(q, \dot{q})$ is a G-invariant Lagrangian, i.e., it satisfies Definition 2, then the* reduced Lagrangian*, $l \colon TQ/G \to I\!R$, can be expressed as*

$$l(s, \dot{s}, \xi) = \begin{pmatrix} \xi^T & \dot{s}^T \end{pmatrix} \begin{pmatrix} I(s) & IA(s) \\ A^T I(s) & m(s) \end{pmatrix} \begin{pmatrix} \xi \\ \dot{s} \end{pmatrix} - V(s), \tag{2.12}$$

where $\xi = g^{-1}\dot{g} \in \mathfrak{g}$, $\dot{s} \in T_r\mathcal{S}$, $I(s)$ is the locked inertia tensor, and $A(s)$ is the local form of the mechanical connection.

Proof. The Lagrangian of a mechanical system can be written as

$$L(g,s,\dot g,\dot s) = \frac{1}{2}(\dot g^T, \dot s^T)\begin{pmatrix}\mathcal{G}_{11} & \mathcal{G}_{12}\\ \mathcal{G}_{12}^T & \mathcal{G}_{22}\end{pmatrix}\begin{pmatrix}\dot g\\ \dot s\end{pmatrix} - V(g,s).$$

Making use of (2.9) and (2.11), we see that

$$\begin{aligned}\dot g^T \mathcal{G}_{11}\dot g = \langle\langle(\dot g,0),(\dot g,0)\rangle\rangle = \langle \mathrm{J}(\dot g,0);\mathrm{Ad}_g\,\xi\rangle &= \langle I\Gamma(\dot g,0);\mathrm{Ad}_g\,\xi\rangle\\ &= \langle I\,\mathrm{Ad}_g\,\xi;\mathrm{Ad}_g\,\xi\rangle = \langle \mathrm{Ad}_g^*\,I\,\mathrm{Ad}_g\,\xi;\xi\rangle\end{aligned}$$

$$\begin{aligned}\dot s^T \mathcal{G}_{12}^T\dot g = \langle\langle(0,\dot s),(\dot g,0)\rangle\rangle = \langle \mathrm{J}(0,\dot s);\mathrm{Ad}_g\,\xi\rangle &= \langle I\Gamma(0,\dot s);\mathrm{Ad}_g\,\xi\rangle\\ &= \langle I\,\mathrm{Ad}_g\,A(s)\dot s;\mathrm{Ad}_g\,\xi\rangle = \langle \mathrm{Ad}_g^*\,I\,\mathrm{Ad}_g\,A(s)\dot s;\xi\rangle.\end{aligned}$$

Because of the G-invariance of L we can define $m(s) = \mathcal{G}_{22}(e,s)$ and $V(s) = V(q)$. If we also define $I(s) = \mathrm{Ad}_g^*\,I\,\mathrm{Ad}_g$ then the truth of the proposition follows. □

To apply Proposition 2.3.1 to our particular fluid mechanical problem, I must show that the fluid mechanical Lagrangian is invariant with respect to a group action. As seen in Section 2.1, the Lagrangian is a function of ϕ and $\nabla\phi$. Hence, invariance is based on the invariance of ϕ and $\nabla\phi$. Note that the potential, ϕ, is defined with respect to $\mathcal{F}_B$. Straightforward calculations based on this fact and an analysis of the boundary conditions can be used to prove the following fact.

Proposition 2.3.2. *In the case of quiescent ambient flow, the potential, ϕ, defined by Equation (2.3) and subject to boundary conditions (2.4) is* $\mathrm{SE}(d)$*-invariant. Similarly,* $\nabla\phi$ *is* $\mathrm{SE}(d)$*-invariant.*

This proposition cannot in general be extended to the case of a non-quiescent ambient flow. The following is a direct consequence of Proposition 2.3.2.

Proposition 2.3.3. *The fluid's kinetic energy, considered as the Lagrangian $T_{fluid}\colon TQ \to$*

$I\!R$ and given by

$$T_{fluid} = \frac{\rho_0}{2} \int_{\mathcal{D}} ||\nabla \phi||^2 \, dV = -\frac{\rho_0}{2} \int_{\Sigma} \phi (\nabla \phi \cdot n) \, dS,$$

is invariant with respect to a left SE(d)*-action.*

As a corollary to Prop. 2.3.1 and Prop. 2.3.3, I can state that the kinetic energy of a deformable swimmer in an inviscid irrotational fluid takes the following special form.

Proposition 2.3.4. *The kinetic energy of the deformable swimmer, Equation (2.5), assumes the reduced form:*

$$T_{total}(s, \dot{s}, \xi) = \frac{1}{2}(\xi^T, \dot{s}^T) \begin{pmatrix} I_d & I_d A_d \\ A_d^T I_d & m_d \end{pmatrix} \begin{pmatrix} \xi \\ \dot{s} \end{pmatrix}, \tag{2.13}$$

where I_d is the 6×6 "locked added inertia" tensor, with entries:

$$(I_d)_{ij} = \Lambda_{ij}^{gg}(s) - \frac{\rho_0}{2} \int_{\Sigma} (\phi_i^g (\nabla \phi_j^g \cdot n) + \phi_j^g (\nabla \phi_i^g \cdot n)) \, dS, \tag{2.14}$$

where $\Lambda^{gg}(s)$ is the locked inertia tensor of the deformable body system (considered in the absence of the fluid) and the second term is the classical added fluid mass/inertia. Meanwhile, $I_d A_d$ is a $(6 \times n_r)$ matrix with entries:

$$(I_d A_d)_{ij} = \Lambda_{ij}^{gs}(s) - \frac{\rho_0}{2} \int_{\Sigma} (\phi_i^g (\nabla \phi_j^s \cdot n) + \phi_j^s (\nabla \phi_i^g \cdot n)) \, dS. \tag{2.15}$$

Hence, the local form of the fluid mechanical connection*, A_d, can be computed as*

$$A_d(s) = I_d^{-1}(s)(I_d A_d)(s). \tag{2.16}$$

When the symmetry principles are taken into account, the governing equations of motion that one derives from the Euler-Lagrange mechanical equations *reduce* to this form:

$$g^{-1}\dot{g} = -A_d(s)\dot{s} + I_d^{-1}(s)\mu \tag{2.17}$$

$$\dot{\mu} = \mathrm{ad}^*_{\xi}\,\mu \tag{2.18}$$

$$M(s)\ddot{s} = T(s)\tau - B(s,\dot{s}) - C(s), \tag{2.19}$$

where μ is the system's momentum, in body coordinates. The variable τ represents the "shape forces." The first equation is the connection, modified to include the possibility that the swimmer starts with nonzero momentum. The second equation describes the evolution of the momentum, as seen in body coordinates. In spatial coordinates, this momentum is constant since it is a conserved quantity. We will hereafter assume that the swimmer starts with zero momentum, thereby eliminating the second equation and simplifying the first to the form of Equation (2.10). The third equation is known as the "shape" dynamics and is only a function of the shape varibles. *For the purposes of control analysis and trajectory generation, I need only focus on the connection.*

So, the infinitesimal relationship between shape changes and body velocity is described by the local form of the connection:

$$\dot{g} = -gA(s)\dot{s} = -gA_i(s)\dot{s}^i, \tag{2.20}$$

where the index i implies summation. I would like to find a solution for this equation that will aid in designing or evaluating motions that arise from shape variations. Because SE(d) is a Lie group, the solution to Equation (2.20) will generally have the form

$$g(t) = g(0)e^{z(t)},$$

where $z \in \mathfrak{se}(d)$. An expansion for the Lie algebra valued function $z(t)$ has been given by Magnus [Mag54].

$$z = \overline{A} + \frac{1}{2}\overline{[\overline{A}, A]} + \frac{1}{3}\overline{[[\overline{\overline{A}, A}], A]} + \frac{1}{12}[\overline{A}, \overline{[\overline{A}, A]}] + \cdots$$

$$\overline{A}(t) \equiv \int_0^t A(\tau)\dot{s}(\tau)d\tau,$$

where $[\cdot,\cdot]$ is the Jacobi-Lie bracket on $\mathfrak{g}$.

To obtain useful results, examine the group displacement resulting from a periodic path $\alpha\colon [0,T] \to M$ such that $\alpha(0) = \alpha(T)$. Taylor expand A_i about $\alpha(0)$ and then judiciously regroup, simplify, apply integration by parts, and use the fact that the path is cyclic [RB98].

$$\begin{aligned} z(\alpha) &= -\frac{1}{2}F_{ij}(0)\int_\alpha ds^i\, ds^j \\ &+ \frac{1}{3}(F_{ij,k} - [A_i, F_{jk}])(0)\int_\alpha ds^i\, ds^j\, ds^k + \cdots , \end{aligned} \tag{2.21}$$

where

$$F_{ij} \equiv A_{j,i} - A_{i,j} - [A_i, A_j] \tag{2.22}$$

is termed the *curvature* of the connection, the notation ",j" indicates differentiation with respect to s_j, and the following shorthand notation is used

$$\int_\alpha ds^i ds^j ds^k \equiv \int_0^T \int_0^{t_k} \int_0^{t_j} \dot{s}^i(t_i)dt_i\ \dot{s}^j(t_j)dt_j\ \dot{s}^k(t_k)dt_k.$$

Summation over indices is implied. The connection, A, and its curvature, F, are evaluated at $\alpha(0)$ so that the coefficients of the integrals are constants. For a complete discussion of this series, see [RB98].

Thus, our geometric analysis shows that for small boundary deformations, the displacement of the body over one period of shape deformation is proportional to the curvature of the connection. *The curvature is an excellent measure of the effectiveness of the swimmer.* This result further bolsters the central role of the fluid mechanical connection in the analysis of deformable swimmers.

For proportionally small deformations, the displacement experienced during one

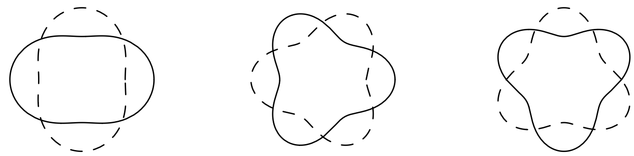


Figure 2.2: The variables $s_1(t), s_2(t), s_3(t)$ correspond to 3 deformation modes. The 1^{st} and 2^{nd} modes together yield motion in the x-direction. The 1^{st} and 1^{st} modes together yield motion in the y-direction.

deformation cycle is

$$g_{disp} = e^{z(\alpha)} \approx \exp\left(-\tfrac{1}{2}F_{ij}\int_{\alpha} ds^i\, ds^j\right). \tag{2.23}$$

The term $-\frac{1}{2}\sum_{i,j} F_{ij}\int_{\alpha} ds^i\, ds^j$ is Lie-algebra valued, and therefore it will take the form of $g^{-1}\dot{g}$ in Equation (2.2). The exponential of such a matrix is

$$\begin{bmatrix} e^{\hat{\omega}} & (I - e^{\hat{\omega}})(\hat{\omega}\xi) + (\vec{\omega}\cdot\xi)\vec{\omega} \\ \vec{0}^T & 1 \end{bmatrix}. \tag{2.24}$$

2.4 Planar Amoeba Example

As an example, I consider the propulsive movements of a roughly circular device whose boundary shape is modulated by a "small" amount. Bearing in mind that a practical robot amoeba should have as few actuators as possible, I restrict the possible deformations of its boundary to a set parameterized by three variables $s_1(t), s_2(t), s_3(t)$ as follows. Fix a frame in the body of the amoeba, and let the shape of the amoeba be described in polar coordinates in the body frame by the equation (see Figure 2.2):

$$F(\sigma, s) = r_0[1 + \epsilon(s_1\cos(2\sigma) + s_2\cos(3\sigma) + s_3\sin(3\sigma))].$$

(Note that the centroid C of the deformed amoeba is not, in general, located at the origin of the body frame; I will return to this point later.) Under this time-varying deformation the area A of the amoeba is

$$A(t) = \pi r_0^2[1 + \frac{1}{2}\epsilon^2(s_1(t)^2 + s_2(t)^2 + s_3(t)^2)].$$

The perfectly irrotational fluid surrounding the amoeba has density ρ. The potential ϕ is determined by the surface boundary conditions, by the requirement that $u = \nabla\phi$ go to zero as r approaches infinity, and by the requirement that there be no circulation around the amoeba. The surface boundary condition in polar coordinates is

$$\begin{aligned}(\nabla\phi \cdot \vec{n})|_\Sigma = &\begin{bmatrix}\dot{x}\cos\sigma + \dot{y}\sin\sigma & -\dot{x}\sin\sigma + \dot{y}\cos\sigma\end{bmatrix}^T \cdot \vec{n} \\ &+ \epsilon r_0 \begin{bmatrix}\dot{s}_1\cos(2\sigma) + \dot{s}_2\cos(3\sigma) + \dot{s}_3\sin(3\sigma) & 0\end{bmatrix}^T \cdot \vec{n}.\end{aligned}$$

The unit normal to the surface at any point is

$$n(s) = \begin{bmatrix} 1 - \frac{1}{2}\epsilon^2\beta^2 \\ \beta\ (\epsilon - (s_1\ \cos(2\,\sigma) + s_2\ \cos(3\sigma) + s_3\ \sin(3\sigma))\ \epsilon^2)\end{bmatrix} + \mathcal{O}(\epsilon^3),$$

where $\beta = -3\,s_3\ \cos(3\sigma) + 2\,s_1\ \sin(2\sigma) + 3\,s_2\ \sin(3\sigma)$. Solving Laplace's equation by separation of variables, I find

$$\phi = \phi_1\dot{x} + \phi_2\dot{y} + \phi_3\omega + \phi_1^s\dot{s_1} + \phi_2^s\dot{s_2} + \phi_3^s\dot{s_3} + \mathcal{O}(\epsilon^3),$$

where, using the notation $c(\sigma) = \cos(\sigma)$, $s(\sigma) = \sin(\sigma)$,

$$\begin{aligned}\phi_1 = &-\frac{r_0^2}{r}c(\sigma) + \epsilon\left(\frac{r_0^2}{r}s_1c(\sigma) + \frac{r_0^3}{r^2}s_2c(2\sigma) - \frac{r_0^4}{r^3}s_1c(3\sigma) - \frac{r_0^5}{r^4}s_2c(4\sigma)\right. \\ &\left.+\frac{r_0^3}{r^2}s_3s(2\sigma) - \frac{r_0^5}{r^4}s_3s(4\sigma)\right) + \epsilon^2\left(-\frac{r_0^2}{2r}(3s_1^2 + 5s_2^2 + 5s_3^2)c(\sigma)\right. \\ &-2\frac{r_0^3}{r^2}s_1s_2c(2\sigma) + \frac{r_0^4}{4r^3}s_1^2c(3\sigma) + \frac{r_0^5}{r^4}s_1s_2c(4\sigma) - \frac{r_0^6}{4r^5}(5s_1^2 - 3s_2^2 + 3s_3^2)c(5\sigma) \\ &-3\frac{r_0^7}{r^6}s_1s_2c(6\sigma) - \frac{7r_0^8}{4r^7}(s_2^2 - s_3^2)c(7\sigma) - \frac{2r_0^3}{r^2}s_1s_3s(2\sigma) + \frac{r_0^5}{r^4}s_1s_3s(4\sigma) \\ &\left.+\frac{3r_0^6}{2r^5}s_2s_3s(5\sigma) - \frac{3r_0^7}{r^6}s_1s_3s(6\sigma) - \frac{7r_0^8}{2r^7}s_2s_3s(7\sigma)\right)\end{aligned}$$

$$\phi_2 = -\frac{r_0^2}{r}s(\sigma) + \epsilon\left(\frac{r_0^3}{r^2}s_3c(2\sigma) + \frac{r_0^5}{r^4}s_3c(4\sigma) - \frac{r_0^2}{r}s_1s(\sigma) - \frac{r_0^3}{r^2}s_2s(2\sigma) - \frac{r_0^5}{r^3}s_1s(3\sigma)\right.$$
$$\left.-\frac{r_0^5}{r^4}s_2s(4\sigma)\right) + \epsilon^2\left(\frac{2r_0^3}{r^2}s_1s_3c(2\sigma) + \frac{r_0^5}{r^4}s_1s_3c(4\sigma) + \frac{3r_0^6}{2r^5}s_2s_3c(5\sigma) + \frac{3r_0^7}{r^6}s_1s_3c(6\sigma)\right.$$
$$+\frac{7r_0^8}{2r^7}s_2s_3c(7\sigma) - \frac{r_0^2}{2r}(3s_1^2 + 5s_2^2 + 5s_3^2)s(\sigma) - \frac{2r_0^3}{r^2}s_1s_2s(2\sigma) - \frac{r_0^4}{4r^3}s_1^2s(3\sigma)$$
$$\left.-\frac{r_0^5}{r^4}s_1s_2s(4\sigma) - \frac{r_0^6}{4r^5}(5s_1^2 + 3s_2^2 - 3s_3^2)s(5\sigma) - \frac{3r_0^7}{r^6}s_1s_2s(6\sigma) - \frac{7r_0^8}{4r^7}(s_2^2 - s_3^2)s(7\sigma)\right)$$

$$\phi_3 = \epsilon\left(\frac{r_0^5}{r^3}s_3c(3\sigma) - \frac{r_0^4}{r^2}s_1s(2\sigma) - \frac{r_0^5}{r^3}s_2s(3\sigma)\right)$$
$$+\epsilon^2\left(\frac{3r_0^3}{r}s_1s_3c(\sigma) + \frac{3r_0^7}{r^5}s_1s_3c(5\sigma) + \frac{7r_0^8}{2r^6}s_2s_3c(6\sigma) - \frac{3r_0^3}{r}s_1s_2s(\sigma)\right.$$
$$\left.-\frac{5r_0^6}{4r^4}s_1^2s(4\sigma) - \frac{3r_0^7}{r^5}s_1s_2s(5\sigma) - \frac{7r_0^8}{4r^6}(s_2^2 - s_3^2)s(6\sigma)\right)$$

$$\phi_1^s = \epsilon\left(-\frac{r_0^4}{2r^2}c(2\sigma)\right) + \epsilon^2\left(\frac{r_0^2}{2}s_1\ln(r) - \frac{5r_0^6}{8r^4}s_1c(4\sigma) - \frac{3r_0^7}{5r^5}s_2c(5\sigma) - \frac{3r_0^7}{5r^5}s_3s(5\sigma)\right)$$

$$\phi_2^s = \epsilon\left(-\frac{r_0^5}{3r^3}c(3\sigma)\right) + \epsilon^2\left(\frac{r_0^2}{2}s_2\ln(r) - \frac{r_0^3}{r}s_1c(\sigma) - \frac{3r_0^7}{5r^5}s_1c(5\sigma)\right.$$
$$\left.-\frac{7r_0^8}{12r^6}s_2c(6\sigma) - \frac{7r_0^8}{12r^6}s_3s(6\sigma)\right)$$

$$\phi_3^s = \epsilon\left(-\frac{r_0^5}{3r^3}s(3\sigma)\right) + \epsilon^2\left(\frac{r_0^2}{2}s_3\ln(r) + \frac{7r_0^8}{12r^6}s_3c(6\sigma) - \frac{r_0^3}{r}s_1s(\sigma)\right.$$
$$\left.-\frac{3r_0^7}{5r^5}s_1s(5\sigma) - \frac{7r_0^8}{12r^6}s_2s(6\sigma)\right)$$

From ϕ I can readily find the rightmost terms of equations (2.14) and (2.15); it remains to find Λ^{gg} and Λ^{gs}.

To compute Λ^{gg} and Λ^{gs}, I must make some assumptions regarding the amoeba's internal structure. For this example I assume that the amoeba is homogeneous (which implies that the center of mass is located at the centroid) and that it has the kinetic energy of an instantaneously rigid body of mass M with the same center-of-mass velocity and angular velocity. The velocity of the centroid is given by

$$\dot{C} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} + \epsilon^2 r_0 \begin{bmatrix} s_1 \dot{s_2} + \dot{s_1} s_2 - s_1 s_3 \omega \\ s_1 \dot{s_3} + \dot{s_1} s_3 + s_1 s_2 \omega \end{bmatrix} + \mathcal{O}(\epsilon^3),$$

while the moment of inertia around the centroid is

$$\begin{aligned} \frac{M}{A(t)} \int_0^{2\pi} \int_0^{r(t,\theta)} & \left| \left| \begin{bmatrix} r' \cos(\theta) - \epsilon^2 r_0 s_1(t) s_2(t) \\ r' \sin(\theta) - \epsilon^2 r_0 s_1(t) s_3(t) \end{bmatrix} \right| \right|^2 r' \, dr' \, d\theta \\ &= M \frac{\frac{1}{2}\pi r_0^4 (1 + 3\epsilon^2 (s_1^2 + s_2^2 + s_3^2)) + \mathcal{O}(\epsilon^3)}{\pi r_0^2 (1 + \frac{1}{2}\epsilon^2 (s_1(t)^2 + s_2(t)^2 + s_3(t)^2))} \\ &= M r_0^2 \left(\frac{1}{2} + \frac{5}{4} \epsilon^2 (s_1^2 + s_2^2 + s_3^2) \right) + \mathcal{O}(\epsilon^3), \end{aligned}$$

so the matrices Λ^{gg} and Λ^{gs} are given to order ϵ^2 by

$$\Lambda^{gg} \approx M \begin{bmatrix} 1 & 0 & -\epsilon^2 r_0 s_1 s_3 \\ 0 & 1 & \epsilon^2 r_0 s_1 s_2 \\ -\epsilon^2 r_0 s_1 s_3 & \epsilon^2 r_0 s_1 s_2 & r_0^2 (\frac{1}{2} + \frac{5}{4}\epsilon^2 (\sum_{i=1}^3 s_i^2)) \end{bmatrix}$$

$$\Lambda^{gs} = M \begin{bmatrix} \epsilon^2 r_0 s_2 & \epsilon^2 r_0 s_1 & 0 \\ \epsilon^2 r_0 s_3 & 0 & \epsilon^2 r_0 s_1 \\ 0 & 0 & 0 \end{bmatrix} + \mathcal{O}(\epsilon^3).$$

Using formulas (2.14) and (2.15), to find I_d and $I_d A_d$, I end with a local connection form $A_d = (I_d)^{-1}(I_d A_d)$.

$$A_d = \epsilon^2 \begin{bmatrix} r_0 (1-\mu) s_2 & r_0 \, s_1 & 0 \\ r_0 (1-\mu) s_3 & 0 & r_0 \, s_1 \\ 0 & \frac{-2\,\pi\,{r_0}^2\,\rho\,s_3}{M} & \frac{2\,\pi\,{r_0}^2\,\rho\,s_2}{M} \end{bmatrix} + \mathcal{O}(\epsilon^3), \tag{2.25}$$

where $\mu = (2\pi r_0^2 \rho)/(M + \pi r_0^2 \rho)$. After using the connection to derive the motion of

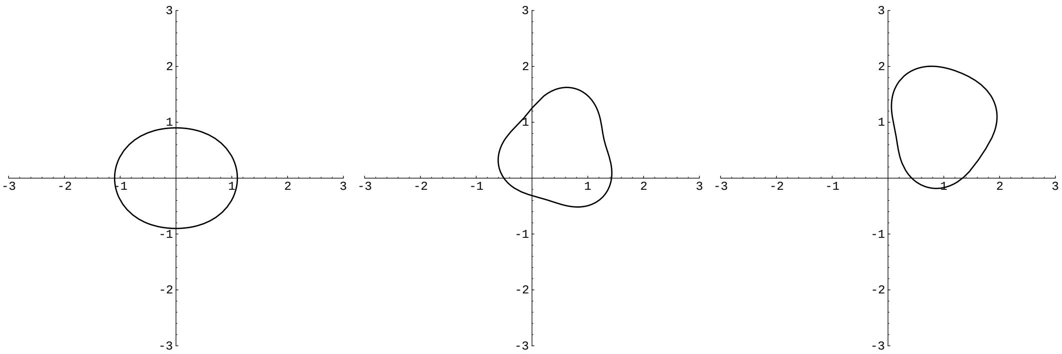


Figure 2.3: Computer animation of a planar amoeba whose density equals that of the surrounding fluid. There are roughly 15.3 oscillations between snapshots.

the frame, I may easily derive the velocity of the centroid:

$$\dot{C} = \epsilon^2 r_0 \left(\mu \begin{bmatrix} \dot{s}_1 s_2 \\ \dot{s}_1 s_3 \end{bmatrix} + \frac{2\rho\pi r_0^3}{M} s_1 \begin{bmatrix} \dot{s}_2 s_3^2 - \dot{s}_3 s_2 s_3 \\ \dot{s}_3 s_2^2 - \dot{s}_2 s_2 s_3 \end{bmatrix} \right) + \mathcal{O}(\epsilon^3).$$

As seen above, the abstract approach espoused here leads to a surprisingly succinct description of the essential governing equations.

2.4.1 Displacement by Periodic Motion

Example (*continued*)**:** Consider the case of our idealized planar amoeba. Since the curvature F of the connection is a skew symmetric quantity, there are only three independent nonzero curvature terms, F_{12}, F_{13}, and F_{23}. Let us assume that the first deformation mode is forced periodically by input $s_1 = \cos(\Omega t)$, while the second mode is periodically forced by input $s_2 = \sin(\Omega t)$. For the chosen input forcing functions, the integral terms associated with the F_{13} and F_{23} terms are zero. From Equations

(2.25) and (2.22) we see that

$$\hat{A}_1 = \epsilon^2 r_0 \frac{M - \pi r_0^2 \rho}{M + \pi r_0^2 \rho} \begin{bmatrix} 0 & 0 & s_2 \\ 0 & 0 & s_3 \\ 0 & 0 & 0 \end{bmatrix} + \mathcal{O}(\epsilon^3) \quad (2.26)$$

$$\hat{A}_2 = \epsilon^2 r_0 \begin{bmatrix} 0 & \frac{2\pi r_0 \rho}{M} s_3 & s_1 \\ -\frac{2\pi r_0 \rho}{M} s_3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \mathcal{O}(\epsilon^3) \quad (2.27)$$

$$F_{12} = \frac{2\pi r_0^3 \rho}{M + \pi r_0^2 \rho} \epsilon^2 \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \mathcal{O}(\epsilon^3). \quad (2.28)$$

Therefore, the displacement over one period of forcing is

$$\begin{aligned} g_{\text{disp}} &\approx \exp\left(-\tfrac{1}{2} F_{12}(0) \int_0^{\frac{2\pi}{\Omega}} \left(\int_0^t \dot{s}_1(\tau) d\tau \right) \dot{s}_2(t) dt \right. \\ &\quad \left. -\tfrac{1}{2} F_{21}(0) \int_0^{\frac{2\pi}{\Omega}} \left(\int_0^t \dot{s}_2(\tau) d\tau \right) \dot{s}_1(t) dt \right) \\ &= \exp(-\pi F_{12}(0)). \end{aligned} \quad (2.29)$$

If I discard the high-order terms in F_{12} and exponentiate only the curvature proportional to ϵ^2, we find

$$g_{\text{disp}} \approx \begin{bmatrix} 1 & 0 & \frac{-2\epsilon^2 \pi^2 r_0^3 \rho}{M + \pi r_0^2 \rho} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (2.30)$$

Thus, to $\mathcal{O}(\epsilon^2)$, each oscillation results in a displacement of $(-2\pi\epsilon^2 r_0 \frac{\pi r_0^2 \rho}{M + \pi r_0^2 \rho})$ along the x-axis. The simplicity with which this result can be obtained is a direct result of the geometric approach outlined in Sections 2.2 and 2.3.

2.4.2 Rectilinear Motion Planning

The series expansion outlined in Section 2.4.1 can be used as a basis for developing motion planning algorithms, as it directly relates control inputs to net displacement. Interested readers are referred to [RB98] for more details. Here I take a simplified approach to our example.

Consider the two-degree-of-freedom problem where we do not wish the amoeba to rotate, but wish the centroid to follow a path in the plane. Suppose that I choose sinusoidal inputs with time-varying amplitude, as follows: $s_1(t) = \cos(\Omega t)$, $s_2(t) = -a(t)\sin(\Omega t)$, $s_3(t) = -b(t)\sin(\Omega t)$. Then

$$\dot{C} = \epsilon^2 r_0 \frac{2\rho\pi r_0^2}{\rho\pi r_0^2 + M} \begin{bmatrix} a(t)\Omega\sin^2(\Omega t) \\ b(t)\Omega\sin^2(\Omega t) \end{bmatrix} + \mathcal{O}(\epsilon^3).$$

Thus, moving the centroid of the amoeba along a given curve in the plane is remarkably easy. (By contrast, moving the body frame origin along a given curve, using this form of input, would be more complicated, since the velocity of the body frame origin depends on the derivatives $\dot{a}(t)$ and $\dot{b}(t)$ to leading order.) At any point along the curve, I make the velocity of the centroid tangent to the curve by appropriate choice of a and b (b/a is the slope of the curve). As long as the curve is smooth, a and b are smooth functions of time.

This method steers the centroid along a path in the plane, but not necessarily at a desired speed at any given instant. In particular, the velocity of the centroid will periodically vanish (when $\sin^2(\Omega t)$ vanishes). Figure 2.3 shows snapshots of a computer simulation of this model as it tracks a straight line with unit slope.

I must also note that the average velocity of the centroid is disappointingly low, on the order of $\epsilon^2 r_0 \Omega$ (which is the norm of the connection's curvature!). Each shape change cycle moves the amoeba a distance on the order of $\epsilon^2 r_0$. Thus if $\epsilon = 0.1$, then 100 oscillations are required to move the amoeba one body radius. Mechanically feasible amoeba will be relatively slow swimmers.

2.4.3 Optimal Control Analysis for Amoeba

I now demonstrate that the sinusoidal inputs used in the last two sections are "optimal" inputs, according to one natural measure of performance. First I restrict the problem to motion along the x-axis. Therefore I have one base variable C_x and two shape variables, s_1 and s_2 (I can neglect s_3). Assume that the control inputs are $u_1 = \dot{s_1}$ and $u_2 = \dot{s_2}$, so $\dot{C_x} = \epsilon^2 r_0 \mu u_1 s_2$. Suppose I choose a minimum-control-effort performance index:

$$J(0) = \frac{1}{2} \int_0^T u^T u \, dt = \frac{1}{2} \int_0^T (u_1^2 + u_2^2) \, dt \tag{2.31}$$

Assume that $C_x(0) = 0$ and for simplicity $s_2(0) = 0$. At time T I require a final state $s_1(T) = s_1(0)$, $s_2(T) = s_2(0)$, and $C_x(0) = d$. The Hamiltonian and costate equations for this optimal control problem are

$$H = \frac{1}{2}(u_1^2 + u_2^2) + \lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 \epsilon^2 r_0 \mu u_1 s_2$$

$$-(\dot{\lambda_1}, \dot{\lambda_2}, \dot{\lambda_3}) = (0, \lambda_3 \epsilon^2 r_0 \mu u_1, 0)$$

From this we see that λ_1 and λ_3 are constants, while

$$\lambda_2(t) = \lambda_2(T) + \lambda_3 \epsilon^2 r_0 \mu \int_t^T u_1(t') \, dt'$$

The stationarity condition is

$$\begin{bmatrix} u_1 + \lambda_1 + \lambda_3 \epsilon^2 r_0 \mu s_2 \\ u_2 + \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

A solution to these equations is

$$\begin{aligned}
s_1 &= a\cos(N\pi t/T) \\
s_2 &= a\sin(N\pi t/T) \\
C_x &= -\epsilon^2 r_0 \mu a^2 \frac{N\pi}{T}[\frac{t}{2} - \frac{\sin((2N\pi/T)t)}{(4N\pi/T)}]
\end{aligned}$$

where $a = s_1(0)$ and $N = -\frac{2d}{\epsilon^2 a^2 r_0 \mu \pi}$.

To summarize, optimal inputs for $s_1(t)$ and $s_2(t)$ are sinusoidal functions of time, 90 degrees out of phase. "Optimal" inputs are those which cause the amoeba to swim a given distance along the x-axis in a given time, while minimizing control effort as defined in (2.31). By symmetry, sinusoids 90 degrees out of phase in $s_1(t)$ and $s_3(t)$ will cause optimal motion along the y-axis. Further, by making $s_2(t)$ and $s_3(t)$ each oscillate 90 degrees out of phase with $s_1(t)$, I can drive the amoeba in any direction in the plane, as seen in Section 2.4.2.

Chapter 3 Deformable Joukowski Foils

I now consider another deformable body: a somewhat "fishlike" swimmer shaped like a Joukowski foil with changeable shape parameters. The boundary Σ of a Joukowski foil is the image in the physical z-plane of a circle C, centered on the origin in the ζ plane, under the transformation

$$z = F(\zeta) = \zeta + \zeta_c + \frac{a^2}{\zeta + \zeta_c}. \tag{3.1}$$

The size and shape of the foil are determined by the parameters a and ζ_c, and the radius of C is $r_c = \|a - \zeta_c\|$. The real number a always has the opposite sign of the real part of ζ_c.

The inverse of the Joukowski map is given by:[1]

$$\zeta = F^{-1}(z) = \frac{1}{2}\left[z + \sqrt{z^2 - 4a^2}\right] - \zeta_c \tag{3.2}$$

[1] The square root in Equation (3.2) is a multivalued function and it is important to choose the correct branch, i.e., assuming z lies on or outside the foil, the sign of the square root should be chosen so that $\|\zeta\| \geq r_c$. Streitlien [Str94] suggests the expanded form $\zeta = \frac{1}{2}\left[z + \sqrt{z - 2a}\sqrt{z + 2a}\right] - \zeta_c$.This will yield the correct answer for most points in the z-plane but can still give the wrong branch for points close to the concave surface of a cambered foil.

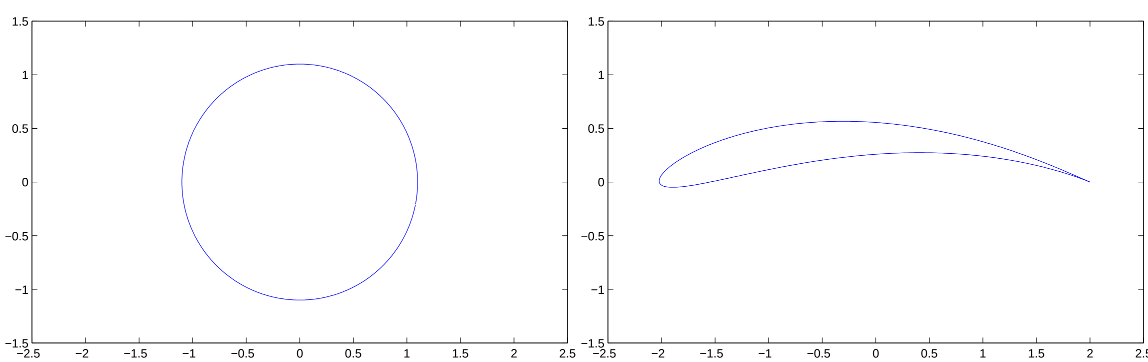


Figure 3.1: The Joukowski map $z = \zeta + \zeta_c + a^2/(\zeta + \zeta_c)$ sends a circle to an airfoil contour. Here $a = 1$, $r_c = 1.1$, and $\zeta_c = (-0.0781 + 0.2185i)$.

The area of the foil is given by

$$\begin{aligned} A &= \frac{1}{2i}\int_\Sigma \overline{z}\,dz \\ &= \frac{1}{2i}\int_C \left(\overline{\zeta}+\overline{\zeta_c}+\frac{a^2}{\overline{\zeta}+\overline{\zeta_c}}\right)\left(1-\frac{a^2}{(\zeta+\zeta_c)^2}\right)d\zeta \\ &= \frac{1}{2i}\int_C \left(\frac{r_c^2}{\zeta}+\overline{\zeta_c}+\frac{a^2\zeta}{r_c^2+\overline{\zeta_c}\zeta}\right)\left(1-\frac{a^2}{(\zeta+\zeta_c)^2}\right)d\zeta \\ &= \frac{1}{2i}\int_C \left(\frac{r_c^2}{\zeta}+\overline{\zeta_c}+\frac{a^2\zeta}{r_c^2+\overline{\zeta_c}\zeta}-\frac{r_c^2a^2}{\zeta(\zeta+\zeta_c)^2}-\frac{a^2\overline{\zeta_c}}{(\zeta+\zeta_c)^2}-\frac{a^4\zeta}{(r_c^2+\overline{\zeta_c}\zeta)(\zeta+\zeta_c)^2}\right)d\zeta \\ &= \pi\left(r_c^2-\frac{a^4r_c^2}{(r_c^2-\delta^2)^2}\right)=\pi r_c^2\left(1-\frac{a^4}{(r_c^2-\delta^2)^2}\right) \end{aligned}$$

where $\delta = \|\zeta_c\|$ and where I have taken advantage of the fact that $\overline{\zeta} = r_c^2/\zeta$ on the circle C.

The geometric center z_c of the foil is given by [Str94]:

$$z_cA = \frac{-1}{4i}\int_\Sigma z^2\,\overline{dz} \tag{3.3}$$

$$= \frac{-1}{4\pi}\int\left(\zeta+\zeta_c+\frac{a^2}{\zeta+\zeta_c}\right)^2\left(-\left(\frac{{r_c}^2}{\zeta^2}\right)+\frac{a^2\,{r_c}^2}{\zeta^2\left(\frac{{r_c}^2}{\zeta}+\overline{\zeta_c}\right)^2}\right)d\zeta \tag{3.4}$$

$$= \pi r_c^2\left[\zeta_c+\frac{a^6\overline{\zeta_c}}{(r_c^2-\delta^2)^3}\right] \tag{3.5}$$

$$z_c = \frac{a^6\overline{\zeta_c}+\zeta_c(r_c^2-\delta^2)^3}{(r_c^2-\delta^2)\left((r_c^2-\delta^2)^2-a^4\right)} \tag{3.6}$$

3.1 Allowed Deformations of the Foil

Suppose the Joukowski foil is translating with velocities U and V in the x and y directions and rotating about the origin at rate Ω, and furthermore the foil shape parameter ζ_c changes at a rate $\dot{\zeta_c} = \dot{\zeta_x} + i\dot{\zeta_y}$. Since ζ_c is varying in time, either r_c or a or both must also be time-varying so the relation $r_c = \|a - \zeta_c\|$ is satisfied. I can write:

$$r_c^2 = (a-\zeta_x)^2 + \zeta_y^2 \tag{3.7}$$

$$\dot{r_c} = \frac{1}{r_c}\left[(a-\zeta_x)\dot{a} - (a-\zeta_x)\dot{\zeta_x} + \zeta_y\dot{\zeta_y}\right] \tag{3.8}$$

I will choose to regard ζ_x, ζ_y, and a as the controllable shape parameters, and treat r_c as a dependent quantity, although in principle any three of the four variables could be taken as independent.

The velocity of the center of area z_c is

$$\begin{aligned}
\dot{z_c} &= U + iV + i\Omega z_c \\
&+ \left(\frac{a^6}{\left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)\,\left(-a^4 + \left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)^2\right)} - \frac{a^6\,\overline{\zeta_c}\,\left(\zeta_c + \overline{\zeta_c}\right)}{\left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)^2\,\left(a^4 - \left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)^2\right)} \right. \\
&\left. + \frac{2\,a^6\,\overline{\zeta_c}\,\left(\zeta_c + \overline{\zeta_c}\right) + \left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)^4 - a^4\,\left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)\,\left({r_c}^2 - \zeta_c\,\left(2\,\zeta_c + 3\,\overline{\zeta_c}\right)\right)}{\left(a^4 - \left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)^2\right)^2} \right) \dot{\zeta_x} \\
&+ \left(\frac{i\,a^6\,\left(\zeta_c - \overline{\zeta_c}\right)\,\overline{\zeta_c}}{\left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)^2\,\left(a^4 - \left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)^2\right)} + \frac{i\,a^6}{\left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)\,\left(a^4 - \left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)^2\right)} \right. \\
&\left. - \frac{i\,\left(2\,a^6\,\left(\zeta_c - \overline{\zeta_c}\right)\,\overline{\zeta_c} + a^4\,\left({r_c}^2 + \zeta_c\,\left(2\,\zeta_c - 3\,\overline{\zeta_c}\right)\right)\,\left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right) - \left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)^4\right)}{\left(a^4 - \left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)^2\right)^2} \right) \dot{\zeta_y} \\
&+ \left(\frac{2\,a^4\,r_c\,\left(a^6\,\overline{\zeta_c} - \left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)^2\,\left(3\,a^2\,\overline{\zeta_c} + 2\,\zeta_c\,\left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)\right)\right)}{\left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)^2\left(a^4 - \left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)^2\right)^2} \right) \dot{r_c} \\
&+ \left(\frac{6\,a^5\,\overline{\zeta_c}}{\left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)\,\left(-a^4 + \left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)^2\right)} + \frac{4\,a^3\,\left(a^6\,\overline{\zeta_c} + \zeta_c\left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)^3\right)}{\left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)\,\left(-a^4 + \left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)^2\right)^2} \right) \dot{a}
\end{aligned} \tag{3.9}$$

The acceleration of the center of area z_c is

$$\ddot{z}_c = \dot{U} + i\dot{V} + i\dot{\Omega} z_c + i\Omega \dot{z}_c$$
$$+ \frac{\partial z_c}{\partial \zeta_x}\ddot{\zeta}_x + \frac{\partial z_c}{\partial \zeta_y}\ddot{\zeta}_y + \frac{\partial z_c}{\partial a}\ddot{a} + \frac{\partial z_c}{\partial r_c}\ddot{r}_c$$
$$+ \frac{\partial^2 z_c}{\partial \zeta_x^2}\dot{\zeta}_x^{\,2} + \frac{\partial^2 z_c}{\partial \zeta_y^2}\dot{\zeta}_y^{\,2} + \frac{\partial^2 z_c}{\partial a^2}\dot{a}^2 + \frac{\partial^2 z_c}{\partial r_c^2}\dot{r}_c^{\,2}$$
$$+ 2\frac{\partial^2 z_c}{\partial \zeta_x \partial \zeta_y}\dot{\zeta}_x\dot{\zeta}_y + 2\frac{\partial^2 z_c}{\partial \zeta_x \partial a}\dot{\zeta}_x\dot{a} + 2\frac{\partial^2 z_c}{\partial \zeta_y \partial a}\dot{\zeta}_y\dot{a} + 2\frac{\partial^2 z_c}{\partial \zeta_x \partial r_c}\dot{\zeta}_x\dot{r}_c + 2\frac{\partial^2 z_c}{\partial \zeta_y \partial r_c}\dot{\zeta}_y\dot{r}_c + 2\frac{\partial^2 z_c}{\partial a \partial r_c}\dot{a}\dot{r}_c \tag{3.10}$$

A closed-form expression can be obtained by taking the appropriate partial derivatives of Equation (3.6). The derivatives with respect to ζ_x and ζ_y are facilitated by considering that if ζ_c and $\overline{\zeta_c}$ are regarded as independent variables, then:

$$\frac{\partial}{\partial \zeta_x} = \frac{\partial}{\partial \zeta_c} + \frac{\partial}{\partial \overline{\zeta_c}} \tag{3.11}$$
$$\frac{\partial}{\partial \zeta_y} = i\frac{\partial}{\partial \zeta_c} - i\frac{\partial}{\partial \overline{\zeta_c}} \tag{3.12}$$

For the time being I will not place any other constraints on what deformations of the foil are allowed. But if the deformations were required to preserve the area of the foil, for example, then since the area $A = \pi r_c^2(1 - a^4/(r_c^2 - \zeta_x^2 - \zeta_y^2)^2)$, I would require

$$\left((r_c^2 - \delta^2)^3 + a^4(r_c^2 + \delta^2)\right)\dot{r}_c - 2a^3 r_c(r_c^2 - \delta^2)\dot{a} - 2a^4 r_c \zeta_x \dot{\zeta}_x - 2a^4 r_c \zeta_y \dot{\zeta}_y = 0 \tag{3.13}$$

which, combined with Equation (3.8), yields

$$\dot{a}\left[(a - \zeta_x)\left((r_c^2 - \delta^2)^3 + a^4(r_c^2 + \delta^2)\right) - 2a^3 r_c^2(r_c^2 - \delta^2)\right]$$
$$- \dot{\zeta}_x\left[(a - \zeta_x)(r_c^2 - \delta^2)^3 + a^5(r_c^2 + \delta^2) + a^4\zeta_x(r_c^2 - \delta^2)\right]$$
$$+ \dot{\zeta}_y\left[(r_c^2 - \delta^2)\left((r_c^2 - \delta^2)^2 - a^4\right)\zeta_y\right] = 0 \tag{3.14}$$

If the area A is held constant then the velocity of the center of area simplifies to

$$\begin{aligned}\dot{z}_c &= U + iV + i\Omega z_c \\ &+ \left(\frac{\pi\, {r_c}^2 \, \left(\left({r_c}^2 - \zeta_c \,\overline{\zeta_c}\right)^4 + a^6 \, \left({r_c}^2 + \overline{\zeta_c}\, \left(2\,\zeta_c + 3\,\overline{\zeta_c}\right)\right)\right)}{A\left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)^4} \right) \dot{\zeta}_x \\ &+ \left(\frac{i\,\pi\, {r_c}^2 \, \left(\left({r_c}^2 - \zeta_c \,\overline{\zeta_c}\right)^4 - a^6 \, \left({r_c}^2 + \left(2\,\zeta_c - 3\,\overline{\zeta_c}\right)\,\overline{\zeta_c}\right)\right)}{A\left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)^4} \right) \dot{\zeta}_y \\ &+ \left(\frac{2\,\pi\, r_c \, \left({r_c}^8\,\zeta_c - 4\,{r_c}^6\,{\zeta_c}^2\,\overline{\zeta_c} - a^6\,\zeta_c\,\overline{\zeta_c}^2 + 6\,{r_c}^4\,{\zeta_c}^3\,\overline{\zeta_c}^2 + {\zeta_c}^5\,\overline{\zeta_c}^4 - 2\,{r_c}^2\,\overline{\zeta_c}\,\left(a^6 + 2\,{\zeta_c}^4\,\overline{\zeta_c}^2\right)\right)}{A\left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)^4} \right) \dot{r_c} \\ &+ \left(\frac{6\,a^5\,\pi\,{r_c}^2\,\overline{\zeta_c}}{A\left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)^3} \right) \dot{a} \qquad (3.15)\end{aligned}$$

A requirement of constant area is not the only reasonable choice of constraint. One alternative, non-area-preserving approach would be to allow ζ_c to vary while keeping r_c and a fixed. This would be equivalent to varying the camber of the foil while keeping its thickness and chord length approximately constant.

3.2 The Potential Function

Suppose the fluid around the foil is at rest at infinity, and allow for the presence of a central vortex with strength γ_c at the origin and a certain number of free vortices ($k = 1, \ldots, n_v$) with strengths γ_k at locations $z_k = F(\zeta_k)$ in the flow. Then the fluid flow in the z plane is described by the complex potential $w = \phi + i\psi$ made up of generalized Kirchoff potentials:

$$w = Uw_1(\zeta) + Vw_2(\zeta) + \Omega w_3(\zeta) + \gamma_c w_4(\zeta) + \sum_k \gamma_k w_5^k(\zeta;\zeta_k) + \dot{\zeta}_x w_1^s(\zeta) + \dot{\zeta}_y w_2^s(\zeta) + \dot{a} w_3^s(\zeta) \qquad (3.16)$$

where w_5^k is the portion of the potential associated with the kth wake vortex, and w_1^s, w_2^s, w_3^s are the parts of the potential associated with deformations of the shape of the foil.

First I find w_1, w_2, w_3 by considering motion of the rigid foil in the absence of deformation. As in Equation (2.4), at the surface of the foil Σ undergoing rigid motion, the flow must match the boundary condition:

$$(\nabla\phi \cdot n)_\Sigma = \frac{\partial\psi}{\partial\sigma}|_\Sigma = (\frac{\partial\psi}{\partial x}\frac{dx}{ds} + \frac{\partial\psi}{\partial y}\frac{dy}{ds})|_\Sigma = -(V + x\Omega)\frac{dx}{ds} + (U - y\Omega)\frac{dy}{ds} \quad (3.17)$$

where I use the same notation as above: n is a unit vector normal to the foil boundary, and σ is a coordinate parameterizing the surface. I use the subscript $_\Sigma$ to mean "at the surface Σ." From this one can conclude that

$$\frac{\partial\psi}{\partial x}|_\Sigma = -V - x\Omega \quad (3.18)$$

$$\frac{\partial\psi}{\partial y}|_\Sigma = U - y\Omega \quad (3.19)$$

So a sufficient condition to satisfy the boundary condition at the surface of the foil is

$$\begin{aligned} \mathsf{Im}\{w\}_\Sigma &= \psi|_\Sigma \\ &= \left(Uy - Vx - \frac{1}{2}\Omega(x^2 + y^2)\right)_\Sigma \\ &= \mathsf{Im}\left\{(U - iV)z - \frac{i}{2}\Omega z\bar{z}\right\}_\Sigma \quad (3.20) \end{aligned}$$

By substituting Equation (3.16) into the left-hand side of Equation (3.20), we see that the boundary condition for the w_1 component of the potential is

$$\begin{aligned} \mathsf{Im}\{w_1\}_\Sigma &= \mathsf{Im}\{z\}_\Sigma \\ &= \mathsf{Im}\left\{\zeta + \zeta_c + \frac{a^2}{\zeta + \zeta_c}\right\}_C \\ &= \mathsf{Im}\left\{\frac{r_c^2}{\bar{\zeta}} + \zeta_c + \frac{a^2}{\zeta + \zeta_c}\right\}_C \\ &= \mathsf{Im}\left\{-\frac{r_c^2}{\zeta} + \zeta_c + \frac{a^2}{\zeta + \zeta_c}\right\}_C \quad (3.21) \end{aligned}$$

(recalling that $\zeta = r_c^2/\overline{\zeta}$ on C) and so an acceptable solution for w_1 is:

$$w_1 = -\frac{r_c^2}{\zeta} + \zeta_c + \frac{a^2}{\zeta + \zeta_c} \tag{3.22}$$

Note that this solution for w_1 satisfies not only the boundary condition at the surface of the foil, but also the requirement that the fluid be at rest at infinity. The quantity $\zeta + \zeta_c + \frac{a^2}{\zeta+\zeta_c}$, for example, would have satisfied the boundary condition at the foil, but not at infinity—this is why the modification of Equation (3.21) was necessary.

Similarly:

$$\begin{aligned}
\mathsf{Im}\left\{w_2\right\}_\Sigma &= \mathsf{Im}\left\{-iz\right\}_\Sigma \\
&= \mathsf{Im}\left\{-i\zeta - i\zeta_c - i\frac{a^2}{\zeta + \zeta_c}\right\}_C \\
&= \mathsf{Im}\left\{-i\frac{r_c^2}{\overline{\overline{\zeta}}} - i\zeta_c - i\frac{a^2}{\zeta + \zeta_c}\right\}_C \\
&= \mathsf{Im}\left\{-i\frac{r_c^2}{\zeta} - i\zeta_c - i\frac{a^2}{\zeta + \zeta_c}\right\}_C
\end{aligned} \tag{3.23}$$

and an acceptable solution for w_2 is

$$w_2 = -i\frac{r_c^2}{\zeta} - i\zeta_c - i\frac{a^2}{\zeta + \zeta_c} \tag{3.24}$$

Meanwhile:

$$
\begin{aligned}
\mathsf{Im}\,\{w_3\}_\Sigma &= \mathsf{Im}\left\{\frac{-i}{2}z\bar{z}\right\}_\Sigma \\
&= \mathsf{Im}\left\{\frac{-i}{2}(\zeta+\zeta_c+\frac{a^2}{\zeta+\zeta_c})(\overline{\zeta}+\overline{\zeta_c}+\frac{a^2}{\overline{\zeta}+\overline{\zeta_c}})\right\}_C \\
&= \mathsf{Im}\left\{\frac{-i}{2}\left[\zeta\overline{\zeta}+\overline{\zeta_c}\zeta+\zeta_c\overline{\zeta}+\delta^2+a^2(\frac{\overline{\zeta}+\overline{\zeta_c}}{\zeta+\zeta_c}+\frac{\zeta+\zeta_c}{\overline{\zeta}+\overline{\zeta_c}})+\frac{a^4}{(\zeta+\zeta_c)(\overline{\zeta}+\overline{\zeta_c})}\right]\right\}_C \\
&= \mathsf{Im}\left\{\frac{-i}{2}\left[\zeta\overline{\zeta}+\overline{\zeta_c}\zeta+\zeta_c\overline{\zeta}+\delta^2+2a^2(\frac{\overline{\zeta}+\overline{\zeta_c}}{\zeta+\zeta_c})+\frac{a^4}{(\zeta+\zeta_c)(\overline{\zeta}+\overline{\zeta_c})}\right]\right\}_C \\
&= \mathsf{Im}\left\{\frac{-i}{2}\left[r_c^2+2\zeta_c\overline{\zeta}+\delta^2+2a^2(\frac{\overline{\zeta}+\overline{\zeta_c}}{\zeta+\zeta_c})+\frac{a^4}{(\zeta+\zeta_c)(\overline{\zeta}+\overline{\zeta_c})}\right]\right\}_C \\
&= \mathsf{Im}\left\{\frac{-i}{2}\left[r_c^2+2\zeta_c\frac{r_c^2}{\zeta}+\delta^2+2a^2(\frac{r_c^2/\zeta+\overline{\zeta_c}}{\zeta+\zeta_c})+\frac{a^4}{(\zeta+\zeta_c)(r_c^2/\zeta+\overline{\zeta_c})}\right]\right\}_C \\
&= \mathsf{Im}\left\{\frac{-i}{2}\left[r_c^2+2\zeta_c\frac{r_c^2}{\zeta}+\delta^2+2a^2(\frac{r_c^2/\zeta+\overline{\zeta_c}}{\zeta+\zeta_c})+\frac{a^4\zeta}{\overline{\zeta_c}(\zeta+\zeta_c)(\zeta+r_c^2/\overline{\zeta_c})}\right]\right\}_C
\end{aligned}
\tag{3.25}
$$

The expression inside the curly brackets satisfies the boundary conditions at the foil and at infinity, but the last term has a singularity at $\zeta = -r_c^2/\overline{\zeta_c}$, which is outside the boundary of C. To cancel this singularity, I add another singularity of opposite strength at $-r_c^2/\overline{\zeta_c}$, and use the Milne-Thomson circle theorem [MT68] to preserve the surface boundary condition. The term to be added is

$$
\frac{i}{2}\frac{a^4r_c^2}{\overline{\zeta_c}(r_c^2-\delta^2)(\zeta+r_c^2/\overline{\zeta_c})}-\frac{i}{2}\frac{a^4r_c^2}{\zeta_c(r_c^2-\delta^2)(r_c^2/\zeta+r_c^2/\zeta_c)}
$$

By inspection this corrective term is purely real on the circle C, so:

$$
\begin{aligned}
\mathsf{Im}\,\{w_3\}_\Sigma &= \mathsf{Im}\left\{\frac{-i}{2}\left[r_c^2+2\zeta_c\frac{r_c^2}{\zeta}+\delta^2+2a^2(\frac{r_c^2/\zeta+\overline{\zeta_c}}{\zeta+\zeta_c})+\frac{a^4\zeta}{\overline{\zeta_c}(\zeta+\zeta_c)(\zeta+r_c^2/\overline{\zeta_c})}\right.\right. \\
&\qquad \left.\left.-\frac{a^4r_c^2}{\overline{\zeta_c}(r_c^2-\delta^2)(\zeta+r_c^2/\overline{\zeta_c})}+\frac{a^4r_c^2}{\zeta_c(r_c^2-\delta^2)(r_c^2/\zeta+r_c^2/\zeta_c)}\right]\right\}_C \\
&= \mathsf{Im}\left\{\frac{-i}{2}\left[r_c^2+2\zeta_c\frac{r_c^2}{\zeta}+\delta^2+2a^2(\frac{r_c^2/\zeta+\overline{\zeta_c}}{\zeta+\zeta_c})+\frac{a^4(\zeta-\zeta_c)}{(\zeta+\zeta_c)(r_c^2-\delta^2)}\right]\right\}_C
\end{aligned}
\tag{3.26}
$$

The quantity inside the curly brackets now has no singularities outside the circle

C, and still approaches a constant at infinity. So we have a solution for w_3 which satisfies all boundary conditions:[2]

$$w_3 = \frac{-i}{2}\left[r_c^2 + 2\zeta_c\frac{r_c^2}{\zeta} + \delta^2 + 2a^2(\frac{r_c^2/\zeta + \overline{\zeta_c}}{\zeta+\zeta_c}) + \frac{a^4(\zeta-\zeta_c)}{(\zeta+\zeta_c)(r_c^2-\delta^2)}\right]. \tag{3.27}$$

The potential components due to vortices are straightforward:

$$w_4 = i\log(\frac{\zeta}{r_c}) \tag{3.28}$$

$$w_5^k = i\log\left(-\frac{r_c}{\zeta_k}\frac{\zeta-\zeta_k}{\zeta - r_c^2/\overline{\zeta_k}}\right). \tag{3.29}$$

Notice that w_5^k, the unit potential for a free vortex located at $z_k = F(\zeta_k)$, represents both the vortex itself and its image at the inverse point. The w_5^k terms do not create any net circulation around a large contour encircling the foil and all vortices; any such circulation is presumed to be folded into the w_4 central vortex term.

To find the potential component associated with changes of the foil shape parameters ζ_x, recall Equation (2.4) and observe that the boundary condition at the surface of the deforming foil is

$$\begin{aligned}
\mathsf{Re}\left\{-i\frac{dw_1^s}{d\zeta}\,d\zeta\right\}_C &= \mathsf{Re}\left\{\overline{\frac{\partial F(\zeta;\zeta_c)}{\partial\zeta_c}}(-i\,dz)\right\}_\Sigma \\
&= \mathsf{Re}\left\{-i\left(1-\frac{a^2}{(\overline{\zeta}+\overline{\zeta_c})^2}\right)\left(1-\frac{a^2}{(\zeta+\zeta_c)^2}\right)\,d\zeta\right\}_C \\
&= \mathsf{Re}\left\{-i\left(1-\frac{a^2}{(r_c^2/\zeta+\overline{\zeta_c})^2}\right)\left(1-\frac{a^2}{(\zeta+\zeta_c)^2}\right)\,d\zeta\right\}_C
\end{aligned} \tag{3.30}$$

[2] Streitlien [Str94, ST95] prescribes this same method of deriving the potential but, apparently in error, arrives at

$$w_3 = -i\left[\frac{\zeta_c r_c^2}{\zeta} + a^2\frac{r_c^2/\zeta+\overline{\zeta_c}}{\zeta+\zeta_c} + \frac{a^4\zeta_c}{(\zeta+\zeta_c)(r_c^2+\delta^2)} + \frac{a^4+r_c^4-\delta^4}{2(r_c^2-\delta^2)}\right].$$

This is not equivalent to the correct answer and does not satisfy the boundary condition at the surface of the foil.

Integrating both sides with respect to ζ, we find

$$\begin{aligned}\mathsf{Re}\,\{-iw_1^s\}_C \;=\; \mathsf{Re}\Bigg\{(-i)\Bigg(&\zeta - \frac{a^2\zeta}{\overline{\zeta_c}^2} - \frac{a^2r_c^4(-r_c^4+2r_c^2\zeta_c\overline{\zeta_c}+(a^2-\zeta_c^2)\overline{\zeta_c}^2)}{\overline{\zeta_c}^3(r_c^2+\zeta\overline{\zeta_c})(r_c^2-\zeta_c\overline{\zeta_c})^2}\\ &-\frac{a^2(-r_c^4+2r_c^2\zeta_c\overline{\zeta_c}+\zeta_c^2(a^2-\overline{\zeta_c}^2))}{(\zeta+\zeta_c)(r_c^2-\zeta_c\overline{\zeta_c})^2} - \frac{2a^4r_c^2\zeta_c\log((\zeta+\zeta_c)/r_c)}{(r_c^2-\zeta_c\overline{\zeta_c})^3}\\ &+2a^2r_c^2(r_c^6-3r_c^4\zeta_c\overline{\zeta_c}+3r_c^2\zeta_c^2\overline{\zeta_c}^2+\zeta_c(a^2-\zeta_c^2)\overline{\zeta_c}^3)\frac{\log((r_c^2+\zeta\overline{\zeta_c})/r_c^2)}{\overline{\zeta_c}^3(r_c^2-\zeta_c\overline{\zeta_c})^3}\Bigg)\Bigg\}_C\end{aligned} \tag{3.31}$$

where I have chosen the constant of integration in order to make the arguments of the logarithms dimensionless.

$$\begin{aligned}\mathsf{Im}\,\{w_1^s\}_C \;=\; \mathsf{Im}\Bigg\{&\zeta - \frac{a^2\zeta}{\overline{\zeta_c}^2} + \frac{a^2r_c^4(r_c^4-2r_c^2\delta^2-a^2\overline{\zeta_c}^2+\delta^4)}{\overline{\zeta_c}^3(r_c^2+\zeta\overline{\zeta_c})(r_c^2-\delta^2)^2}\\ &+\frac{a^2(r_c^4-2r_c^2\delta^2-\zeta_c^2a^2+\delta^4)}{(\zeta+\zeta_c)(r_c^2-\delta^2)^2} - \frac{2a^4r_c^2\zeta_c\log((\zeta+\zeta_c)/r_c)}{(r_c^2-\delta^2)^3}\\ &+2a^2r_c^2(r_c^6-3r_c^4\delta^2+3r_c^2\delta^4+\zeta_ca^2\overline{\zeta_c}^3-\delta^6)\frac{\log((r_c^2+\zeta\overline{\zeta_c})/r_c^2)}{\overline{\zeta_c}^3(r_c^2-\delta^2)^3}\Bigg\}_C\end{aligned} \tag{3.32}$$

$$\begin{aligned}\mathsf{Im}\,\{w_1^s\}_C \;=\; \mathsf{Im}\Bigg\{&\frac{r_c^2}{\overline{\zeta}} - \frac{a^2r_c^2}{\overline{\zeta_c}^2\overline{\zeta}} + \frac{a^2r_c^4((r_c^2-\delta^2)^2-a^2\overline{\zeta_c}^2)}{\overline{\zeta_c}^3(r_c^2+\zeta\overline{\zeta_c})(r_c^2-\delta^2)^2}\\ &+\frac{a^2((r_c^2-\delta^2)^2-\zeta_c^2a^2)}{(\zeta+\zeta_c)(r_c^2-\delta^2)^2} - \frac{2a^4r_c^2\zeta_c\log((\zeta+\zeta_c)/r_c)}{(r_c^2-\delta^2)^3}\\ &+2a^2r_c^2((r_c^2-\delta^2)^3+\zeta_ca^2\overline{\zeta_c}^3)\frac{\log((r_c^2+\zeta\overline{\zeta_c})/r_c^2)}{\overline{\zeta_c}^3(r_c^2-\delta^2)^3}\Bigg\}_C\end{aligned} \tag{3.33}$$

$$\begin{aligned}\mathsf{Im}\,\{w_1^s\}_C \;=\; \mathsf{Im}\Bigg\{&-\frac{r_c^2}{\zeta} + \frac{a^2r_c^2}{\zeta\zeta_c^2} + \frac{a^2r_c^4}{\overline{\zeta_c}^3(r_c^2+\zeta\overline{\zeta_c})} - \frac{a^4r_c^4}{\overline{\zeta_c}(r_c^2+\zeta\overline{\zeta_c})(r_c^2-\delta^2)^2}\\ &+\frac{a^2}{(\zeta+\zeta_c)} - \frac{a^4\zeta_c^2}{(\zeta+\zeta_c)(r_c^2-\delta^2)^2} - \frac{2a^4r_c^2\zeta_c\log((\zeta+\zeta_c)/r_c)}{(r_c^2-\delta^2)^3}\\ &+2a^2r_c^2\left(\frac{1}{\overline{\zeta_c}^3}+\frac{\zeta_ca^2}{(r_c^2-\delta^2)^3}\right)\log((r_c^2+\zeta\overline{\zeta_c})/r_c^2)\Bigg\}_C\end{aligned} \tag{3.34}$$

The quantity inside the curly brackets is prevented from being a valid solution for w_1^s by the singular behavior at $\zeta = -r_c^2/\overline{\zeta_c}$. As before, I will construct a corrective term which removes the singularity while preserving the other boundary conditions via the Milne-Thomson circle theorem. The corrective term is

$$- \left(\frac{a^2 r_c^4}{\overline{\zeta_c}^3} - \frac{a^4 r_c^4}{\overline{\zeta_c}(r_c^2 - \delta^2)^2} \right) \frac{1}{r_c^2 + \zeta\overline{\zeta_c}} - 2a^2 r_c^2 \left(\frac{1}{\overline{\zeta_c}^3} + \frac{\zeta_c a^2}{(r_c^2 - \delta^2)^3} \right) \log((r_c^2 + \zeta\overline{\zeta_c})/r_c^2)$$
$$- \left(\frac{a^2 r_c^4}{\zeta_c^3} - \frac{a^4 r_c^4}{\zeta_c(r_c^2 - \delta^2)^2} \right) \frac{1}{r_c^2 + (r_c^2/\zeta)\zeta_c} - 2a^2 r_c^2 \left(\frac{1}{\zeta_c^3} + \frac{\overline{\zeta_c} a^2}{(r_c^2 - \delta^2)^3} \right) \log((r_c^2 + (r_c^2/\zeta)\zeta_c)/r_c^2)$$

Since this expression is purely real on C, it follows that

$$\begin{aligned} \mathsf{Im}\,\{w_1^s\}_C \;=\; \mathsf{Im} \Big\{ & -\frac{r_c^2}{\zeta} + \frac{a^2 r_c^2}{\zeta\zeta_c^2} + \frac{a^2}{(\zeta + \zeta_c)} - \frac{a^4 \zeta_c^2}{(\zeta + \zeta_c)(r_c^2 - \delta^2)^2} \\ & - \frac{2a^4 r_c^2 \zeta_c \log((\zeta + \zeta_c)/r_c)}{(r_c^2 - \delta^2)^3} - \left(\frac{a^2 r_c^4}{\zeta_c^3} - \frac{a^4 r_c^4}{\zeta_c(r_c^2 - \delta^2)^2} \right) \frac{1}{r_c^2 + (r_c^2/\zeta)\zeta_c} \\ & - 2a^2 r_c^2 \left(\frac{1}{\zeta_c^3} + \frac{\overline{\zeta_c} a^2}{(r_c^2 - \delta^2)^3} \right) \log((r_c^2 + (r_c^2/\zeta)\zeta_c)/r_c^2) \Big\}_C \end{aligned} \tag{3.35}$$

$$\begin{aligned} \mathsf{Im}\,\{w_1^s\}_C \;=\; \mathsf{Im} \Big\{ & -\frac{r_c^2}{\zeta} + \frac{a^2 r_c^2}{\zeta\zeta_c^2} + \frac{a^2}{(\zeta + \zeta_c)} - \frac{a^4 \zeta_c^2}{(\zeta + \zeta_c)(r_c^2 - \delta^2)^2} \\ & - \frac{2a^4 r_c^2 \zeta_c}{(r_c^2 - \delta^2)^3} \log((\zeta + \zeta_c)/r_c) - \left(\frac{a^2 r_c^2}{\zeta_c^3} - \frac{a^4 r_c^2}{\zeta_c(r_c^2 - \delta^2)^2} \right) \frac{\zeta}{(\zeta + \zeta_c)} \\ & - 2a^2 r_c^2 \left(\frac{1}{\zeta_c^3} + \frac{\overline{\zeta_c} a^2}{(r_c^2 - \delta^2)^3} \right) \log(\frac{\zeta + \zeta_c}{\zeta}) \Big\}_C \end{aligned} \tag{3.36}$$

If I were now to use the quantity inside the curly brackets as a trial solution for w_1^s, we would find that it resulted in a nonzero amount of circulation around a closed contour surrounding the foil, which would vary with $\dot{\zeta_x}$. But Kelvin's circulation theorem states that the circulation around a closed material curve lying entirely in the fluid is an invariant of the motion, even if the curve is non-reducible [Saf92]. To satisfy this theorem and avoid producing changing circulation around a curve enclosing both the foil and any shed vortices, I add a central vortex term:

$$\left(\frac{2a^4r_c^2(i\zeta_y)}{(r_c^2-\delta^2)^3}\log(\frac{\zeta}{r_c})\right)$$

This is plainly real on C so that

$$\begin{aligned}\mathsf{Im}\,\{w_1^s\}_C &= \mathsf{Im}\Bigg\{-\frac{r_c^2}{\zeta}+\frac{a^2r_c^2}{\zeta\zeta_c^2}+\frac{a^2}{(\zeta+\zeta_c)}-\frac{a^4\zeta_c^2}{(\zeta+\zeta_c)(r_c^2-\delta^2)^2}\\ &\quad -\frac{2a^4r_c^2\zeta_c}{(r_c^2-\delta^2)^3}\log((\zeta+\zeta_c)/r_c)-\left(\frac{a^2r_c^2}{\zeta_c^3}-\frac{a^4r_c^2}{\zeta_c(r_c^2-\delta^2)^2}\right)\frac{\zeta}{(\zeta+\zeta_c)}\\ &\quad -2a^2r_c^2\left(\frac{1}{\zeta_c^3}+\frac{\overline{\zeta_c}a^2}{(r_c^2-\delta^2)^3}\right)\log(\frac{\zeta+\zeta_c}{\zeta})+\frac{2a^4r_c^2(i\zeta_y)}{(r_c^2-\delta^2)^3}\log(\frac{\zeta}{r_c})\Bigg\}_C\end{aligned} \quad (3.37)$$

So a valid solution for w_1^s is:

$$\begin{aligned}w_1^s &= -\frac{r_c^2}{\zeta}+\frac{a^2r_c^2}{\zeta\zeta_c^2}+\frac{a^2}{(\zeta+\zeta_c)}-\frac{a^4\zeta_c^2}{(\zeta+\zeta_c)(r_c^2-\delta^2)^2}\\ &\quad -\frac{2a^4r_c^2\zeta_c}{(r_c^2-\delta^2)^3}\log(\frac{\zeta+\zeta_c}{r_c})-\left(\frac{a^2r_c^2}{\zeta_c^3}-\frac{a^4r_c^2}{\zeta_c(r_c^2-\delta^2)^2}\right)\frac{\zeta}{(\zeta+\zeta_c)}\\ &\quad -2a^2r_c^2\left(\frac{1}{\zeta_c^3}+\frac{\overline{\zeta_c}a^2}{(r_c^2-\delta^2)^3}\right)\log(\frac{\zeta+\zeta_c}{\zeta})+\frac{2a^4r_c^2(i\zeta_y)}{(r_c^2-\delta^2)^3}\log(\frac{\zeta}{r_c})\end{aligned} \quad (3.38)$$

By a similar development we find that

$$\begin{aligned}\mathsf{Im}\,\{w_2^s\}_C &= \mathsf{Im}\Bigg\{(-i)\Bigg(\frac{r_c^2}{\zeta}-\frac{a^2r_c^2}{\zeta\zeta_c^2}+\frac{a^2r_c^4}{\overline{\zeta_c}^3(r_c^2+\zeta\overline{\zeta_c})}-\frac{a^4r_c^4}{\overline{\zeta_c}(r_c^2+\zeta\overline{\zeta_c})(r_c^2-\delta^2)^2}\\ &\quad +\frac{a^2}{(\zeta+\zeta_c)}-\frac{a^4\zeta_c^2}{(\zeta+\zeta_c)(r_c^2-\delta^2)^2}-\frac{2a^4r_c^2\zeta_c\log((\zeta+\zeta_c)/r_c)}{(r_c^2-\delta^2)^3}\\ &\quad +2a^2r_c^2\left(\frac{1}{\overline{\zeta_c}^3}+\frac{\zeta_ca^2}{(r_c^2-\delta^2)^3}\right)\log((r_c^2+\zeta\overline{\zeta_c})/r_c^2)\Bigg)\Bigg\}_C\end{aligned} \quad (3.39)$$

To which I add this corrective term, real on C, to remove singular behavior outside the foil boundary:

$$\left(\frac{a^2r_c^4}{\overline{\zeta_c}^3} - \frac{a^4r_c^4}{\overline{\zeta_c}(r_c^2-\delta^2)^2}\right)\frac{i}{r_c^2+\zeta\overline{\zeta_c}} + 2ia^2r_c^2\left(\frac{1}{\overline{\zeta_c}^3} + \frac{\zeta_c a^2}{(r_c^2-\delta^2)^3}\right)\log((r_c^2+\zeta\overline{\zeta_c})/r_c^2)$$
$$-\left(\frac{a^2r_c^4}{\zeta_c^3} - \frac{a^4r_c^4}{\zeta_c(r_c^2-\delta^2)^2}\right)\frac{i}{r_c^2+(r_c^2/\zeta)\zeta_c} - 2ia^2r_c^2\left(\frac{1}{\zeta_c^3} + \frac{\overline{\zeta_c}a^2}{(r_c^2-\delta^2)^3}\right)\log((r_c^2+(r_c^2/\zeta)\zeta_c)/r_c^2)$$

And another term, also real on C, to cancel any circulation produced around the foil,

$$(-i)\frac{2a^4r_c^2\zeta_x}{(r_c^2-\delta^2)^3}\log(\frac{\zeta}{r_c})$$

to find

$$\begin{aligned}\mathsf{Im}\left\{w_2^s\right\}_C \;=\; \mathsf{Im}\Bigg\{(-i)\Bigg(&\frac{r_c^2}{\zeta} - \frac{a^2r_c^2}{\zeta\zeta_c^2} + \frac{a^2}{(\zeta+\zeta_c)} - \frac{a^4\zeta_c^2}{(\zeta+\zeta_c)(r_c^2-\delta^2)^2}\\ &-\frac{2a^4r_c^2\zeta_c\log((\zeta+\zeta_c)/r_c)}{(r_c^2-\delta^2)^3} + \left(\frac{a^2r_c^2}{\zeta_c^3} - \frac{a^4r_c^2}{\zeta_c(r_c^2-\delta^2)^2}\right)\frac{\zeta}{(\zeta+\zeta_c)}\\ &+2a^2r_c^2\left(\frac{1}{\zeta_c^3} + \frac{\overline{\zeta_c}a^2}{(r_c^2-\delta^2)^3}\right)\log(\frac{\zeta+\zeta_c}{\zeta}) + \frac{2a^4r_c^2\zeta_x}{(r_c^2-\delta^2)^3}\log(\frac{\zeta}{r_c})\Bigg)\Bigg\}_C\end{aligned} \quad (3.40)$$

Thus:

$$\begin{aligned}w_2^s \;=\; (-i)\Bigg(&\frac{r_c^2}{\zeta} - \frac{a^2r_c^2}{\zeta\zeta_c^2} + \frac{a^2}{(\zeta+\zeta_c)} - \frac{a^4\zeta_c^2}{(\zeta+\zeta_c)(r_c^2-\delta^2)^2}\\ &-\frac{2a^4r_c^2\zeta_c\log((\zeta+\zeta_c)/r_c)}{(r_c^2-\delta^2)^3} + \left(\frac{a^2r_c^2}{\zeta_c^3} - \frac{a^4r_c^2}{\zeta_c(r_c^2-\delta^2)^2}\right)\frac{\zeta}{(\zeta+\zeta_c)}\\ &+2a^2r_c^2\left(\frac{1}{\zeta_c^3} + \frac{\overline{\zeta_c}a^2}{(r_c^2-\delta^2)^3}\right)\log(\frac{\zeta+\zeta_c}{\zeta}) + \frac{2a^4r_c^2\zeta_x}{(r_c^2-\delta^2)^3}\log(\frac{\zeta}{r_c})\Bigg)\end{aligned} \quad (3.41)$$

Finally,

$$\begin{aligned}\mathsf{Re}\left\{-i\frac{dw_3^s}{d\zeta}\,d\zeta\right\}_C &= \mathsf{Re}\left\{\overline{\frac{\partial F(\zeta;\zeta_c)}{\partial a}}(-i\,dz)\right\}_\Sigma\\ &= \mathsf{Re}\left\{-i\left(\frac{2a}{\overline{\overline{\zeta}+\overline{\zeta_c}}}\right)\left(1-\frac{a^2}{(\zeta+\zeta_c)^2}\right)d\zeta\right\}_C\\ &= \mathsf{Re}\left\{-i\left(\frac{2a}{r_c^2/\zeta+\overline{\zeta_c}}\right)\left(1-\frac{a^2}{(\zeta+\zeta_c)^2}\right)d\zeta\right\}_C\end{aligned} \quad (3.42)$$

Integrating both sides with respect to ζ, we find

$$\mathsf{Re}\,\{-iw_3^s\}_C = \mathsf{Re}\left\{(-i)2a\left(\frac{\zeta}{\overline{\zeta_c}} - \frac{a^2\zeta_c}{(\zeta+\zeta_c)(r_c^2-\delta^2)} - \frac{a^2r_c^2\log((\zeta+\zeta_c)/r_c)}{(r_c^2-\delta^2)^2} - \frac{r_c^2((r_c^2-\delta^2)^2 - a^2\overline{\zeta_c}^2)\log((r_c^2+\zeta\overline{\zeta_c})/r_c^2)}{\overline{\zeta_c}^2(r_c^2-\delta^2)^2}\right)\right\}_C \tag{3.43}$$

$$\mathsf{Im}\,\{w_3^s\}_C = \mathsf{Im}\left\{2a\left(\frac{\zeta}{\overline{\zeta_c}} - \frac{a^2\zeta_c}{(\zeta+\zeta_c)(r_c^2-\delta^2)} - \frac{a^2r_c^2\log((\zeta+\zeta_c)/r_c)}{(r_c^2-\delta^2)^2} - \frac{r_c^2((r_c^2-\delta^2)^2 - a^2\overline{\zeta_c}^2)}{\overline{\zeta_c}^2(r_c^2-\delta^2)^2}\log((r_c^2+\zeta\overline{\zeta_c})/r_c^2)\right)\right\}_C \tag{3.44}$$

$$\mathsf{Im}\,\{w_3^s\}_C = \mathsf{Im}\left\{2a\left(-\frac{r_c^2}{\zeta\zeta_c} - \frac{a^2\zeta_c}{(\zeta+\zeta_c)(r_c^2-\delta^2)} - \frac{a^2r_c^2\log((\zeta+\zeta_c)/r_c)}{(r_c^2-\delta^2)^2} - \frac{r_c^2((r_c^2-\delta^2)^2 - a^2\overline{\zeta_c}^2)}{\overline{\zeta_c}^2(r_c^2-\delta^2)^2}\log((r_c^2+\zeta\overline{\zeta_c})/r_c^2)\right)\right\}_C \tag{3.45}$$

The corrective term to remove the logarithmic singularity at $\zeta = -r_c^2/\overline{\zeta_c}$ is

$$\frac{r_c^2((r_c^2-\delta^2)^2 - a^2\overline{\zeta_c}^2)}{\overline{\zeta_c}^2(r_c^2-\delta^2)^2}\log((r_c^2+\zeta\overline{\zeta_c})/r_c^2) + \frac{r_c^2((r_c^2-\delta^2)^2 - a^2\zeta_c^2)}{\zeta_c^2(r_c^2-\delta^2)^2}\log((r_c^2+(r_c^2/\zeta)\zeta_c)/r_c^2)$$

So we find

$$\mathsf{Im}\,\{w_3^s\}_C = \mathsf{Im}\left\{2a\left(-\frac{r_c^2}{\zeta\zeta_c} - \frac{a^2\zeta_c}{(\zeta+\zeta_c)(r_c^2-\delta^2)} - \frac{a^2r_c^2\log((\zeta+\zeta_c)/r_c)}{(r_c^2-\delta^2)^2} + \frac{r_c^2((r_c^2-\delta^2)^2 - a^2\zeta_c^2)}{\zeta_c^2(r_c^2-\delta^2)^2}\log(\frac{(\zeta+\zeta_c)}{\zeta})\right)\right\}_C \tag{3.46}$$

And a valid solution for w_3^s is

$$w_3^s = 2a\left(-\frac{r_c^2}{\zeta\zeta_c} - \frac{a^2\zeta_c}{(\zeta+\zeta_c)(r_c^2-\delta^2)} - \frac{a^2r_c^2\log((\zeta+\zeta_c)/r_c)}{(r_c^2-\delta^2)^2} + \frac{r_c^2((r_c^2-\delta^2)^2 - a^2\zeta_c^2)}{\zeta_c^2(r_c^2-\delta^2)^2}\log(\frac{(\zeta+\zeta_c)}{\zeta})\right) \tag{3.47}$$

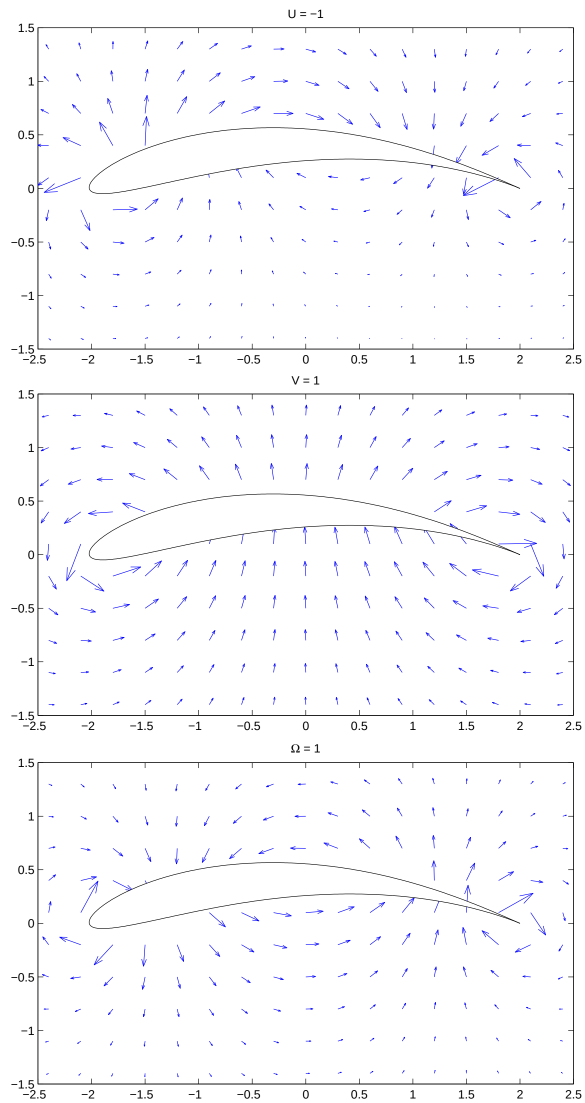


Figure 3.2: Fluid velocity resulting from the unit potentials w_1, w_2, and w_3 (top to bottom.) Here $a = 1$, $r_c = 1.1$, and $\zeta_c = (-0.0781 + 0.2185i)$. Note that I took U to be negative in the first figure. Negative U, or velocity in the negative x-direction, corresponds to "forward" motion when ζ_x is negative, as here.

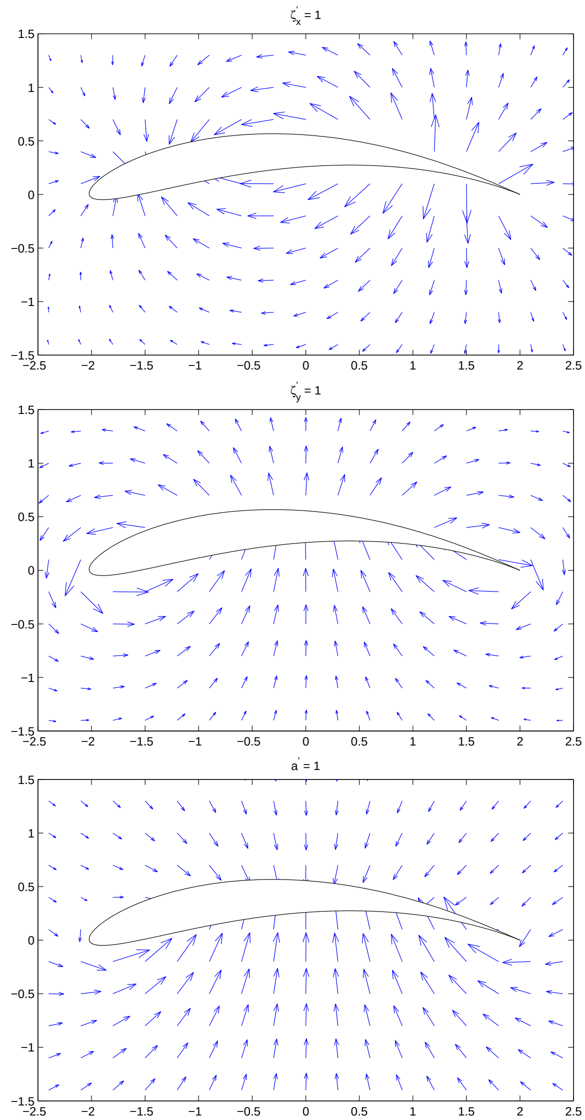


Figure 3.3: Fluid velocity resulting from the shape deformation unit potentials w_1^s, w_2^s, and w_3^s (top to bottom.) Here $a = 1$, $r_c = 1.1$, and $\zeta_c = (-0.0781 + 0.2185i)$.

$$
\begin{aligned}
w &= Uw_1(\zeta) + Vw_2(\zeta) + \Omega w_3(\zeta) + \gamma_c w_4(\zeta) + \sum_k \gamma_k w_5^k(\zeta;\zeta_k) + \dot{\zeta}_x w_1^s(\zeta) + \dot{\zeta}_y w_2^s(\zeta) + \dot{a} w_3^s(\zeta) \\
w_1 &= -\frac{r_c^2}{\zeta} + \zeta_c + \frac{a^2}{\zeta+\zeta_c} \\
w_2 &= -i\frac{r_c^2}{\zeta} - i\zeta_c - i\frac{a^2}{\zeta+\zeta_c} \\
w_3 &= \frac{-i}{2}\left[r_c^2 + 2\zeta_c\frac{r_c^2}{\zeta} + \delta^2 + 2a^2(\frac{r_c^2/\zeta + \overline{\zeta_c}}{\zeta+\zeta_c}) + \frac{a^4(\zeta-\zeta_c)}{(\zeta+\zeta_c)(r_c^2-\delta^2)}\right] \\
w_4 &= i\log(\frac{\zeta}{r_c}) \\
w_5^k &= i\log\left(-\frac{r_c}{\zeta_k}\frac{\zeta-\zeta_k}{\zeta - r_c^2/\overline{\zeta_k}}\right) \\
w_1^s &= \left[-\frac{r_c^2}{\zeta} + \frac{a^2 r_c^2}{\zeta\zeta_c^2} + \frac{a^2}{(\zeta+\zeta_c)} - \frac{a^4\zeta_c^2}{(\zeta+\zeta_c)(r_c^2-\delta^2)^2}\right. \\
&\quad -\frac{2a^4 r_c^2\zeta_c}{(r_c^2-\delta^2)^3}\log(\frac{\zeta+\zeta_c}{r_c}) - \left(\frac{a^2 r_c^2}{\zeta_c^3} - \frac{a^4 r_c^2}{\zeta_c(r_c^2-\delta^2)^2}\right)\frac{\zeta}{(\zeta+\zeta_c)} \\
&\quad \left. -2a^2 r_c^2\left(\frac{1}{\zeta_c^3} + \frac{\overline{\zeta_c}a^2}{(r_c^2-\delta^2)^3}\right)\log(\frac{\zeta+\zeta_c}{\zeta}) + \frac{2a^4 r_c^2(i\zeta_y)}{(r_c^2-\delta^2)^3}\log(\frac{\zeta}{r_c})\right] \\
w_2^s &= (-i)\left[\frac{r_c^2}{\zeta} - \frac{a^2 r_c^2}{\zeta\zeta_c^2} + \frac{a^2}{(\zeta+\zeta_c)} - \frac{a^4\zeta_c^2}{(\zeta+\zeta_c)(r_c^2-\delta^2)^2}\right. \\
&\quad -\frac{2a^4 r_c^2\zeta_c}{(r_c^2-\delta^2)^3}\log(\frac{\zeta+\zeta_c}{r_c}) + \left(\frac{a^2 r_c^2}{\zeta_c^3} - \frac{a^4 r_c^2}{\zeta_c(r_c^2-\delta^2)^2}\right)\frac{\zeta}{(\zeta+\zeta_c)} \\
&\quad \left. +2a^2 r_c^2\left(\frac{1}{\zeta_c^3} + \frac{\overline{\zeta_c}a^2}{(r_c^2-\delta^2)^3}\right)\log(\frac{\zeta+\zeta_c}{\zeta}) + \frac{2a^4 r_c^2\zeta_x}{(r_c^2-\delta^2)^3}\log(\frac{\zeta}{r_c})\right] \\
w_3^s &= 2a\left(-\frac{r_c^2}{\zeta\zeta_c} - \frac{a^2\zeta_c}{(\zeta+\zeta_c)(r_c^2-\delta^2)} - \frac{a^2 r_c^2}{(r_c^2-\delta^2)^2}\log(\frac{\zeta+\zeta_c}{r_c})\right. \\
&\quad \left. +\frac{r_c^2((r_c^2-\delta^2)^2 - a^2\zeta_c^2)}{\zeta_c^2(r_c^2-\delta^2)^2}\log(\frac{(\zeta+\zeta_c)}{\zeta})\right)
\end{aligned}
$$

Table 3.1: Solution for the deformable Joukowski foil potential function.

$$
\begin{aligned}
\frac{dw_1}{d\zeta} &= \frac{r_c^2}{\zeta^2} - \frac{a^2}{(\zeta+\zeta_c)^2} \\
\frac{dw_2}{d\zeta} &= i\frac{r_c^2}{\zeta^2} + i\frac{a^2}{(\zeta+\zeta_c)^2} \\
\frac{dw_3}{d\zeta} &= i\left[\frac{\zeta_c r_c^2}{\zeta^2} + \frac{a^2 r_c^2}{\zeta^2(\zeta+\zeta_c)} + \frac{a^2 r_c^2}{\zeta(\zeta+\zeta_c)^2} + \frac{a^2\overline{\zeta_c}}{(\zeta+\zeta_c)^2} - \frac{a^4\zeta_c}{(\zeta+\zeta_c)^2(r_c^2-\delta^2)}\right] \\
\frac{dw_4}{d\zeta} &= \frac{i}{\zeta} \\
\frac{dw_5^k}{d\zeta} &= i\left(\frac{1}{\zeta-\zeta_k} - \frac{1}{\zeta - r_c^2/\overline{\zeta_k}}\right) \\
\frac{dw_1^s}{d\zeta} &= \left[\frac{r_c^2}{\zeta^2} - \frac{a^2 r_c^2}{\zeta^2\zeta_c^2} - \frac{a^2}{(\zeta+\zeta_c)^2} + \frac{a^4\zeta_c^2}{(\zeta+\zeta_c)^2(r_c^2-\delta^2)^2} - \frac{2a^4 r_c^2\zeta_c}{(r_c^2-\delta^2)^3}\frac{1}{\zeta+\zeta_c} + \frac{2a^4 r_c^2 i\zeta_y}{(r_c^2-\delta^2)^3}\frac{1}{\zeta}\right. \\
&\quad \left. - \left(\frac{a^2 r_c^2}{\zeta_c^3} - \frac{a^4 r_c^2}{\zeta_c(r_c^2-\delta^2)^2}\right)\frac{\zeta_c}{(\zeta+\zeta_c)^2} + 2a^2 r_c^2\left(\frac{1}{\zeta_c^3} + \frac{\overline{\zeta_c}a^2}{(r_c^2-\delta^2)^3}\right)\frac{\zeta_c}{(\zeta+\zeta_c)(\zeta)}\right] \\
\frac{dw_2^s}{d\zeta} &= (-i)\left[-\frac{r_c^2}{\zeta^2} + \frac{a^2 r_c^2}{\zeta^2\zeta_c^2} - \frac{a^2}{(\zeta+\zeta_c)^2} + \frac{a^4\zeta_c^2}{(\zeta+\zeta_c)^2(r_c^2-\delta^2)^2} - \frac{2a^4 r_c^2\zeta_c}{(r_c^2-\delta^2)^3}\frac{1}{\zeta+\zeta_c} + \frac{2a^4 r_c^2\zeta_x}{(r_c^2-\delta^2)^3}\frac{1}{\zeta}\right. \\
&\quad \left. + \left(\frac{a^2 r_c^2}{\zeta_c^3} - \frac{a^4 r_c^2}{\zeta_c(r_c^2-\delta^2)^2}\right)\frac{\zeta_c}{(\zeta+\zeta_c)^2} - 2a^2 r_c^2\left(\frac{1}{\zeta_c^3} + \frac{\overline{\zeta_c}a^2}{(r_c^2-\delta^2)^3}\right)\frac{\zeta_c}{(\zeta+\zeta_c)(\zeta)}\right] \\
\frac{dw_3^s}{d\zeta} &= 2a\left[\frac{r_c^2}{\zeta^2\zeta_c} + \frac{a^2\zeta_c}{(\zeta+\zeta_c)^2(r_c^2-\delta^2)} - \frac{a^2 r_c^2}{(\zeta+\zeta_c)(r_c^2-\delta^2)^2} - \frac{r_c^2((r_c^2-\delta^2)^2 - a^2\zeta_c^2)}{\zeta_c^2(r_c^2-\delta^2)^2}\frac{\zeta_c}{\zeta(\zeta+\zeta_c)}\right]
\end{aligned}
$$

Table 3.2: Derivatives with respect to ζ of the deformable Joukowski foil potentials.

3.3 The Locked Added Inertia

The virtual inertia tensor of a rigid Joukowski foil ought to be well known. However, while the necessary integrals are conceptually straightforward, they are somewhat tedious, and perhaps for this reason, the existing literature [Str94, ST95, Sed65] is plagued with typographical and other elementary errors. So as I develop the equations of motion for a deformable foil, I will also have the incidental goal of producing an error-free treatment of the rigid foil's added inertia.

As I said in Proposition (2.3.4), the kinetic energy of the system is

$$T = \frac{1}{2}\dot{q}^T \Lambda \dot{q} + T_f \tag{3.48}$$

where the first term is the kinetic energy of the foil itself, and T_f is the kinetic energy of the fluid:

$$T_f = -\frac{\rho}{2}\int_\Sigma \phi(\nabla\phi \cdot n)\, d\sigma \tag{3.49}$$

Since

$$\phi = \mathsf{Re}\{w\} \tag{3.50}$$

$$(\nabla\phi \cdot n)\, d\sigma = \mathsf{Re}\{-i\tfrac{dw}{dz}dz\} \tag{3.51}$$

it follows that

$$\begin{aligned} T_f &= -\frac{\rho}{2}\int_\Sigma \mathsf{Re}\{w\}\mathsf{Re}\{-i\tfrac{dw}{dz}dz\} \\ &= -\frac{\rho}{2}\int_C \mathsf{Re}\{w\}\mathsf{Re}\left\{-i\tfrac{dw}{d\zeta}\, d\zeta\right\} \\ &= -\frac{\rho}{2}\mathsf{Re}\left\{\textstyle\int_C w\mathsf{Re}\left\{-i\tfrac{dw}{d\zeta}\, d\zeta\right\}\right\} \\ &= -\frac{\rho}{4}\mathsf{Re}\left\{\textstyle\int_C w(-i\tfrac{dw}{d\zeta}\, d\zeta + i\overline{\tfrac{dw}{d\zeta}}\,\overline{d\zeta})\right\} \\ &= -\frac{\rho}{4}\mathsf{Re}\left\{-i\textstyle\int_C w\tfrac{dw}{d\zeta}\, d\zeta + i\int_C w\overline{\tfrac{dw}{d\zeta}}\,\overline{d\zeta}\right\} \\ &= \frac{\rho}{4}\mathsf{Re}\left\{i\textstyle\int_C w\tfrac{dw}{d\zeta}\, d\zeta - i\int_C w\overline{\tfrac{dw}{d\zeta}}\,\overline{d\zeta}\right\} \end{aligned} \tag{3.52}$$

Considering the foil to be rigid, and disregarding any effects of vorticity, we see that T_f will be a quadratic function of body velocity $\dot{\xi}$.

$$T_f = \frac{1}{2}\begin{bmatrix} U & V & \Omega \end{bmatrix} I \begin{bmatrix} U \\ V \\ \Omega \end{bmatrix} \tag{3.53}$$

where I is the locked added inertia matrix and its components are

$$\begin{aligned}
I_{11} &= \frac{\rho}{2}\mathsf{Re}\Big\{ i\int_C w_1 \tfrac{dw_1}{d\zeta}\, d\zeta - i\int_C w_1 \overline{\tfrac{dw_1}{d\zeta}}\, \overline{d\zeta} \Big\} \\
&= \frac{\rho}{2}\mathsf{Re}\Big\{ -i\int_C \tfrac{-r_c^4}{\zeta\bar{\zeta}^2} + \tfrac{a^2 r_c^2}{(\zeta+\zeta_c)\bar{\zeta}^2} + \tfrac{r_c^2 a^2}{\zeta(\bar{\zeta}+\bar{\zeta_c})^2} - \tfrac{a^4}{(\zeta+\zeta_c)(\bar{\zeta}+\bar{\zeta_c})^2}\, \overline{d\zeta} \Big\} \\
&= \frac{\rho}{2}\mathsf{Re}\Big\{ (-i)\left(r_c^4 (2\pi i)\tfrac{1}{r_c^2} + \tfrac{a^2 r_c^2}{r_c^2}(-2\pi i) + \tfrac{r_c^2 a^2}{r_c^2}(-2\pi i) + (2\pi i) a^4 \tfrac{r_c^2}{(r_c^2-\delta^2)^2} \right) \Big\} \\
&= \pi\rho \left(r_c^2 - 2a^2 + a^4 \frac{r_c^2}{(r_c^2-\delta^2)^2} \right)
\end{aligned} \tag{3.54}$$

$$\begin{aligned}
I_{12} &= \frac{\rho}{4}\mathsf{Re}\Big\{ i\int_C w_1 \tfrac{dw_2}{d\zeta} + w_2 \tfrac{dw_1}{d\zeta}\, d\zeta - i\int_C w_1 \overline{\tfrac{dw_2}{d\zeta}} + w_2 \overline{\tfrac{dw_1}{d\zeta}}\, \overline{d\zeta} \Big\} \\
&= \frac{\rho}{4}\mathsf{Re}\Big\{ (-i)\int_C \tfrac{2ir_c^2 a^2}{\zeta(\bar{\zeta}+\bar{\zeta_c})^2} - \tfrac{2ir_c^2 a^2}{(\zeta+\zeta_c)\bar{\zeta}^2}\, \overline{d\zeta} \Big\} \\
&= \frac{\rho}{4}\mathsf{Re}\Big\{ (-i)\left(2ir_c^2 a^2 (\tfrac{-2\pi i}{r_c^2}) - 2ir_c^2 a^2 (\tfrac{-2\pi i}{r_c^2}) \right) \Big\} \\
&= 0
\end{aligned} \tag{3.55}$$

$$\begin{aligned}
I_{22} &= \frac{\rho}{2}\mathsf{Re}\Big\{ i\int_C w_2 \tfrac{dw_2}{d\zeta}\, d\zeta - i\int_C w_2 \overline{\tfrac{dw_2}{d\zeta}}\, \overline{d\zeta} \Big\} \\
&= \frac{\rho}{4}\mathsf{Re}\Big\{ i\int_C \tfrac{r_c^4}{\zeta\bar{\zeta}^2} + \tfrac{r_c^2 a^2}{\zeta(\bar{\zeta}+\bar{\zeta_c})^2} + \tfrac{a^2 r_c^2}{(\zeta+\zeta_c)\bar{\zeta}^2} + \tfrac{a^4}{(\zeta+\zeta_c)(\bar{\zeta}+\bar{\zeta_c})^2}\, d\bar{\zeta} \Big\} \\
&= \frac{\rho}{2}\mathsf{Re}\Big\{ 2\pi r_c^2 + 2\pi a^2 + 2\pi a^2 + 2\pi a^4 \left(\tfrac{r_c}{r_c^2-\delta^2}\right)^2 \Big\} \\
&= \pi\rho \left(r_c^2 + 2a^2 + a^4 \frac{r_c^2}{(r_c^2-\delta^2)^2} \right)
\end{aligned} \tag{3.56}$$

$$\begin{aligned}
I_{13} &= \frac{\rho}{4}\mathsf{Re}\Big\{-i\int_C w_3\overline{\tfrac{dw_1}{d\zeta}}\,\overline{d\zeta}-i\int_C w_1\overline{\tfrac{dw_3}{d\zeta}}\,\overline{d\zeta}\Big\}\\
&= \frac{\rho}{4}\mathsf{Re}\Bigg\{-\int_C\Big(\frac{\zeta_c r_c^2}{\zeta}+\frac{a^2r_c^2}{\zeta(\zeta+\zeta_c)}+\frac{a^2\overline{\zeta_c}}{(\zeta+\zeta_c)}+\frac{a^4(\zeta-\zeta_c)}{2(\zeta+\zeta_c)(r_c^2-\delta^2)}\Big)\Big(\frac{r_c^2}{\overline{\zeta}^2}-\frac{a^2}{(\overline{\zeta}+\overline{\zeta_c})^2}\Big)\overline{d\zeta}\\
&\quad+\int_C\Big(\frac{r_c^2}{\zeta}-\frac{a^2}{(\zeta+\zeta_c)}\Big)\Big(\frac{\overline{\zeta_c}r_c^2}{\overline{\zeta}^2}+\frac{a^2r_c^2}{\overline{\zeta}^2(\overline{\zeta}+\overline{\zeta_c})}+\frac{a^2r_c^2}{\overline{\zeta}(\overline{\zeta}+\overline{\zeta_c})^2}+\frac{a^2\zeta_c}{(\overline{\zeta}+\overline{\zeta_c})^2}-\frac{a^4\overline{\zeta_c}}{(\overline{\zeta}+\overline{\zeta_c})^2(r_c^2-\delta^2)}\Big)\overline{d\zeta}\Bigg\}\\
&= \frac{\rho}{4}\mathsf{Re}\Bigg\{2\pi i\left(r_c^2\zeta_c+a^2\overline{\zeta_c}-\frac{a^4\zeta_c}{(r_c^2-\delta^2)}-\zeta_c a^2-\frac{a^4}{\zeta_c}+\frac{a^4r_c^4}{\zeta_c(r_c^2-\delta^2)^2}-\frac{a^4r_c^2\overline{\zeta_c}}{(r_c^2-\delta^2)^2}+\frac{a^6r_c^2\zeta_c}{(r_c^2-\delta^2)^3}\right)\\
&\quad+\int_C\Big(\frac{r_c^2}{\zeta}-\frac{a^2}{(\zeta+\zeta_c)}\Big)\Big(\frac{\overline{\zeta_c}r_c^2}{\overline{\zeta}^2}+\frac{a^2r_c^2}{\overline{\zeta}^2(\overline{\zeta}+\overline{\zeta_c})}+\frac{a^2r_c^2}{\overline{\zeta}(\overline{\zeta}+\overline{\zeta_c})^2}+\frac{a^2\zeta_c}{(\overline{\zeta}+\overline{\zeta_c})^2}-\frac{a^4\overline{\zeta_c}}{(\overline{\zeta}+\overline{\zeta_c})^2(r_c^2-\delta^2)}\Big)\overline{d\zeta}\Bigg\}\\
&= \frac{\rho}{4}\mathsf{Re}\Bigg\{2\pi i\Bigg(r_c^2\zeta_c-\frac{a^4\zeta_c}{r_c^2-\delta^2}+a^2(\overline{\zeta_c}-\zeta_c)+\frac{a^4\delta^2}{\zeta_c(r_c^2-\delta^2)}+\frac{a^6r_c^2\zeta_c}{(r_c^2-\delta^2)^3}\\
&\quad-r_c^2\overline{\zeta_c}+a^2\overline{\zeta_c}-\frac{a^4\zeta_c}{r_c^2-\delta^2}-\frac{a^4r_c^2\zeta_c}{(r_c^2-\delta^2)^2}-a^2\zeta_c+\frac{a^4r_c^2\zeta_c}{(r_c^2-\delta^2)^2}+\frac{a^4\overline{\zeta_c}}{r_c^2-\delta^2}-\frac{a^6\overline{\zeta_c}r_c^2}{(r_c^2-\delta^2)^3}\Bigg)\Bigg\}\\
&= \frac{\pi\rho}{2}\mathsf{Re}\left\{i\left(r_c^2(\zeta_c-\overline{\zeta_c})-\frac{2a^4(\zeta_c-\overline{\zeta_c})}{r_c^2-\delta^2}-2a^2(\zeta_c-\overline{\zeta_c})+\frac{a^6r_c^2}{(r_c^2-\delta^2)^3}(\zeta_c-\overline{\zeta_c})\right)\right\}\\
&= \pi\rho\mathsf{Re}\left\{i\left(r_c^2\zeta_c-\frac{2a^4\zeta_c}{r_c^2-\delta^2}-2a^2\zeta_c+\frac{a^6r_c^2\zeta_c}{(r_c^2-\delta^2)^3}\right)\right\} \qquad (3.57)
\end{aligned}$$

where in the final step I have made use of the fact that $\mathsf{Re}\{i\zeta_c\}=-\mathsf{Re}\{i\overline{\zeta_c}\}$.

$$
\begin{aligned}
I_{23} &= \frac{\rho}{4}\mathsf{Re}\Big\{-i\int_C w_3\overline{\tfrac{dw_2}{d\zeta}}\,\overline{d\zeta} - i\int_C w_2\overline{\tfrac{dw_3}{d\zeta}}\,\overline{d\zeta}\Big\} \\
&= \frac{\rho}{4}\mathsf{Re}\left\{ i\int_C \left(\frac{\zeta_c r_c^2}{\zeta} + \frac{a^2 r_c^2}{\zeta(\zeta+\zeta_c)} + \frac{a^2\overline{\zeta_c}}{(\zeta+\zeta_c)} + \frac{a^4(\zeta-\zeta_c)}{2(\zeta+\zeta_c)(r_c^2-\delta^2)}\right)\left(\frac{r_c^2}{\overline{\zeta}^2} + \frac{a^2}{(\overline{\zeta}+\overline{\zeta_c})^2}\right)\overline{d\zeta}\right. \\
&\quad \left. + i\int_C \Big(\frac{r_c^2}{\zeta} + \frac{a^2}{(\zeta+\zeta_c)}\Big)\Big(\frac{\overline{\zeta_c}r_c^2}{\overline{\zeta}^2} + \frac{a^2r_c^2}{\overline{\zeta}^2(\overline{\zeta}+\overline{\zeta_c})} + \frac{a^2r_c^2}{\overline{\zeta}(\overline{\zeta}+\overline{\zeta_c})^2} + \frac{a^2\zeta_c}{(\overline{\zeta}+\overline{\zeta_c})^2} - \frac{a^4\overline{\zeta_c}}{(\overline{\zeta}+\overline{\zeta_c})^2(r_c^2-\delta^2)}\Big)\overline{d\zeta}\right\} \\
&= \frac{\rho}{4}\mathsf{Re}\left\{2\pi\left(\zeta_c r_c^2 + \zeta_c a^2 + \frac{a^4}{\zeta_c} - \frac{a^4 r_c^4}{\zeta_c(r_c^2-\delta^2)^2} + a^2\overline{\zeta_c} + \frac{a^4\overline{\zeta_c}r_c^2}{(r_c^2-\delta^2)^2} - \frac{a^4\zeta_c}{(r_c^2-\delta^2)} - \frac{a^6 r_c^2\zeta_c}{(r_c^2-\delta^2)^3}\right.\right. \\
&\quad \left.\left. + r_c^2\overline{\zeta_c} + a^2\overline{\zeta_c} - \frac{a^4\zeta_c}{r_c^2-\delta^2} - \frac{a^4\zeta_c r_c^2}{(r_c^2-\delta^2)^2} + a^2\zeta_c + \frac{a^4\zeta_c r_c^2}{(r_c^2-\delta^2)^2} - \frac{a^4\overline{\zeta_c}}{r_c^2-\delta^2} - \frac{a^6\overline{\zeta_c}r_c^2}{(r_c^2-\delta^2)^3}\right)\right\} \\
&= \frac{\pi\rho}{2}\mathsf{Re}\left\{(r_c^2+2a^2)(\zeta_c+\overline{\zeta_c}) - \frac{2a^4(\zeta_c+\overline{\zeta_c})}{r_c^2-\delta^2} - \frac{a^6 r_c^2(\zeta_c+\overline{\zeta_c})}{(r_c^2-\delta^2)^3}\right\} \\
&= \pi\rho\mathsf{Re}\left\{r_c^2\zeta_c + 2a^2\zeta_c - \frac{2a^4\zeta_c}{r_c^2-\delta^2} - \frac{a^6r_c^2\zeta_c}{(r_c^2-\delta^2)^3}\right\} \qquad (3.58)
\end{aligned}
$$

These expressions for the locked inertia terms are in agreement with the results of Streitlien [Str94, ST95] and with those of Sedov [Sed65], after correcting some typographical and other elementary errors.[3]

[3] In Sedov [Sed65], his expression λ_{xy} for the arbitrary Joukowski foil in Table 1 should read:

$$\lambda_{xy} = \frac{\rho\pi a^2}{2}\sin(2\alpha)$$

The expression λ_ω in the same table should read:

$$\lambda_\omega = \frac{\rho\pi a^4}{8} r^2R^2\left(8r^2R^2\cos^4(\alpha) - 2rR\sin^2(2\alpha) + \cos(4\alpha)\right)$$

The transformation formula in Sedov's equation (4.11) should read in part:

$$\lambda_{y'\omega'} = -(\lambda_x\eta - \lambda_{xy}\xi + \lambda_{x\omega})\sin(\beta) + (\lambda_{xy}\eta - \lambda_y\xi + \lambda_{y\omega})\cos(\beta)$$

In Streitlien, equation (41) in [ST95] and equation (49) in [Str94] should read:

$$\int_S w_3 z\,d\overline{z} = 2\pi\left[a^4 + a^2\zeta_c^2 + a^4\frac{\delta^4 - 2r_c^2\delta^2}{(r_c^2-\delta^2)^2} + a^6\frac{\overline{\zeta_c}^2}{(r_c^2-\delta^2)^2} + \frac{2a^8r_c^2\delta^2}{(r_c^2-\delta^2)^4} - r_c^2\frac{a^4+r_c^4-\delta^4}{2(r_c^2-\delta^2)} + a^4r_c^2\frac{a^4+r_c^4-\delta^4}{2(r_c^2-\delta^2)^3}\right]$$

Finally, the pure-rotational added inertia, m_{66} in Streitlien's notation, should consist not only of the real part of the above integral, but also of an added term equal to twice the real part of the foil's polar moment of inertia. So Streitlien's equation (48) in [ST95] and equation (59) in [Str94] should

$$
\begin{aligned}
I_{33} &= \frac{\rho}{2}\mathsf{Re}\Big\{ i\int_C w_3 \tfrac{dw_3}{d\zeta}\, d\zeta - i\int_C w_3 \overline{\tfrac{dw_3}{d\zeta}}\,\overline{d\zeta}\Big\} \\
&= \frac{\rho}{2}\mathsf{Re}\left\{ i\int_C \left[\frac{\zeta_c r_c^2}{\zeta} + \frac{a^2 r_c^2}{\zeta(\zeta+\zeta_c)} + \frac{a^2\overline{\zeta_c}}{(\zeta+\zeta_c)} + \frac{a^4(\zeta-\zeta_c)}{2(\zeta+\zeta_c)(r_c^2-\delta^2)}\right]\right. \\
&\quad \times \left.\left[\frac{\overline{\zeta_c} r_c^2}{\overline{\zeta}^2} + \frac{a^2 r_c^2}{\overline{\zeta}^2(\overline{\zeta}+\overline{\zeta_c})} + \frac{a^2 r_c^2}{\overline{\zeta}(\overline{\zeta}+\overline{\zeta_c})^2} + \frac{a^2\zeta_c}{(\overline{\zeta}+\overline{\zeta_c})^2} - \frac{a^4\overline{\zeta_c}}{(\overline{\zeta}+\overline{\zeta_c})^2(r_c^2-\delta^2)}\right]\overline{d\zeta}\right\} \\
&= \frac{\rho}{2}\mathsf{Re}\left\{ 2\pi\left(\delta^2 r_c^2 + a^2\overline{\zeta_c}^2 - a^4\frac{\delta^2}{r_c^2-\delta^2} + a^4\frac{r_c^2}{r_c^2-\delta^2} - a^4\frac{\delta^2}{r_c^2-\delta^2} + a^6\frac{\zeta_c^2}{(r_c^2-\delta^2)^2}\right.\right. \\
&\quad + a^4\frac{r_c^4}{(r_c^2-\delta^2)^2} - a^4\frac{r_c^2\delta^2}{(r_c^2-\delta^2)^2} + a^6\frac{r_c^2\zeta_c^2}{(r_c^2-\delta^2)^3} \\
&\quad \left. + a^2\zeta_c^2 + a^4 - a^4\frac{r_c^4}{(r_c^2-\delta^2)^2} + a^4\frac{\delta^2 r_c^2}{(r_c^2-\delta^2)^2} - a^6\frac{r_c^2\zeta_c^2}{(r_c^2-\delta^2)^3}\right) \\
&\quad \left. - i\int_C \left[\frac{\zeta_c r_c^2}{\zeta} + \frac{a^2 r_c^2}{\zeta(\zeta+\zeta_c)} + \frac{a^2\overline{\zeta_c}}{(\zeta+\zeta_c)} + \frac{a^4(\zeta-\zeta_c)}{2(\zeta+\zeta_c)(r_c^2-\delta^2)}\right]\frac{a^4\overline{\zeta_c}}{(\overline{\zeta}+\overline{\zeta_c})^2(r_c^2-\delta^2)}\overline{d\zeta}\right\} \\
&= \frac{\rho}{2}\mathsf{Re}\left\{ 2\pi\left(r_c^2\delta^2 + a^2(\zeta_c^2+\overline{\zeta_c}^2) + a^4\Big(2 - \frac{\delta^2}{(r_c^2-\delta^2)}\Big) + a^6\frac{\zeta_c^2}{(r_c^2-\delta^2)^2}\right.\right. \\
&\quad \left.\left. - a^4\frac{\delta^2}{(r_c^2-\delta^2)} - a^6\frac{\overline{\zeta_c}}{\zeta_c(r_c^2-\delta^2)} + a^6\frac{r_c^4\overline{\zeta_c}}{\zeta_c(r_c^2-\delta^2)^3} - a^6\frac{r_c^2\overline{\zeta_c}^2}{(r_c^2-\delta^2)^3} + a^8\frac{r_c^2\delta^2}{(r_c^2-\delta^2)^4}\right)\right\} \\
&= \pi\rho\mathsf{Re}\left\{ 2a^4 + r_c^2\delta^2 + a^2(\zeta_c^2+\overline{\zeta_c}^2) - 2a^4\frac{\delta^2}{(r_c^2-\delta^2)}\right. \\
&\quad \left. + a^6\left(\frac{\zeta_c^2}{(r_c^2-\delta^2)^2} - \frac{\overline{\zeta_c}^2}{\delta^2(r_c^2-\delta^2)} + \frac{r_c^4\overline{\zeta_c}^2}{\delta^2(r_c^2-\delta^2)^3} - \frac{r_c^2\overline{\zeta_c}^2}{(r_c^2-\delta^2)^3}\right) + a^8\frac{r_c^2\delta^2}{(r_c^2-\delta^2)^4}\right\} \\
&= \pi\rho\mathsf{Re}\left\{ 2a^4 + r_c^2\delta^2 + a^2(\zeta_c^2+\overline{\zeta_c}^2) - 2a^4\frac{\delta^2}{(r_c^2-\delta^2)} + a^6\frac{\zeta_c^2+\overline{\zeta_c}^2}{(r_c^2-\delta^2)^2} + a^8\frac{r_c^2\delta^2}{(r_c^2-\delta^2)^4}\right\} \\
&= \pi\rho\mathsf{Re}\left\{ 2a^4 + r_c^2\delta^2 + 2a^2\zeta_c^2 - 2a^4\frac{\delta^2}{(r_c^2-\delta^2)} + 2a^6\frac{\zeta_c^2}{(r_c^2-\delta^2)^2} + a^8\frac{r_c^2\delta^2}{(r_c^2-\delta^2)^4}\right\} \quad (3.59)
\end{aligned}
$$

In Figures 3.4–3.5, I plot the added mass components for a variety of foil shapes

---

read:

$$
m_{66} = 2\pi\mathsf{Re}\left[a^4 + a^2\zeta_c^2 + a^4\frac{\delta^4 - 2r_c^2\delta^2}{(r_c^2-\delta^2)^2} + a^6\frac{\overline{\zeta_c}^2}{(r_c^2-\delta^2)^2} + \frac{2a^8 r_c^2\delta^2}{(r_c^2-\delta^2)^4} - r_c^2\frac{a^4 + r_c^4 - \delta^4}{2(r_c^2-\delta^2)}\right.
$$
$$
\left. + a^4 r_c^2\frac{a^4 + r_c^4 - \delta^4}{2(r_c^2-\delta^2)^3} + \frac{r_c^2}{2}\left(r_c^2 + 2\delta^2 - a^8\frac{r_c^2 + 2\delta^2}{(r_c^2-\delta^2)^4}\right)\right]
$$

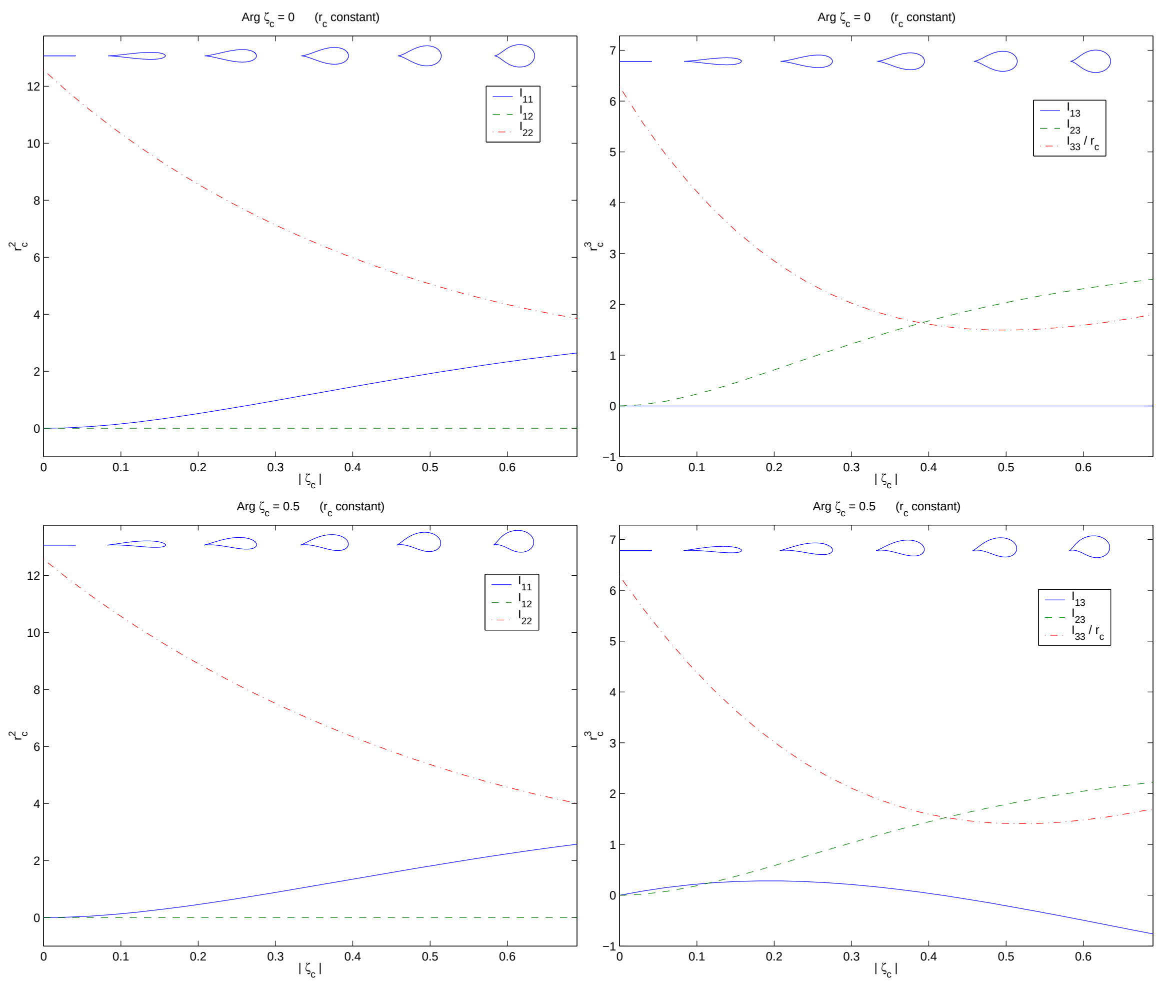


Figure 3.4: Plots of locked added inertia coefficients for a variety of Joukowski foils. In each plot, r_c and $\arg(\zeta_c)$ are held constant while $\|\zeta_c\|$ is varied. The coefficients I_{11}, I_{12}, I_{13} are given in units of r_c^2, while I_{13}, I_{23} are given in units of r_c^3 and I_{33} has units of r_c^4.

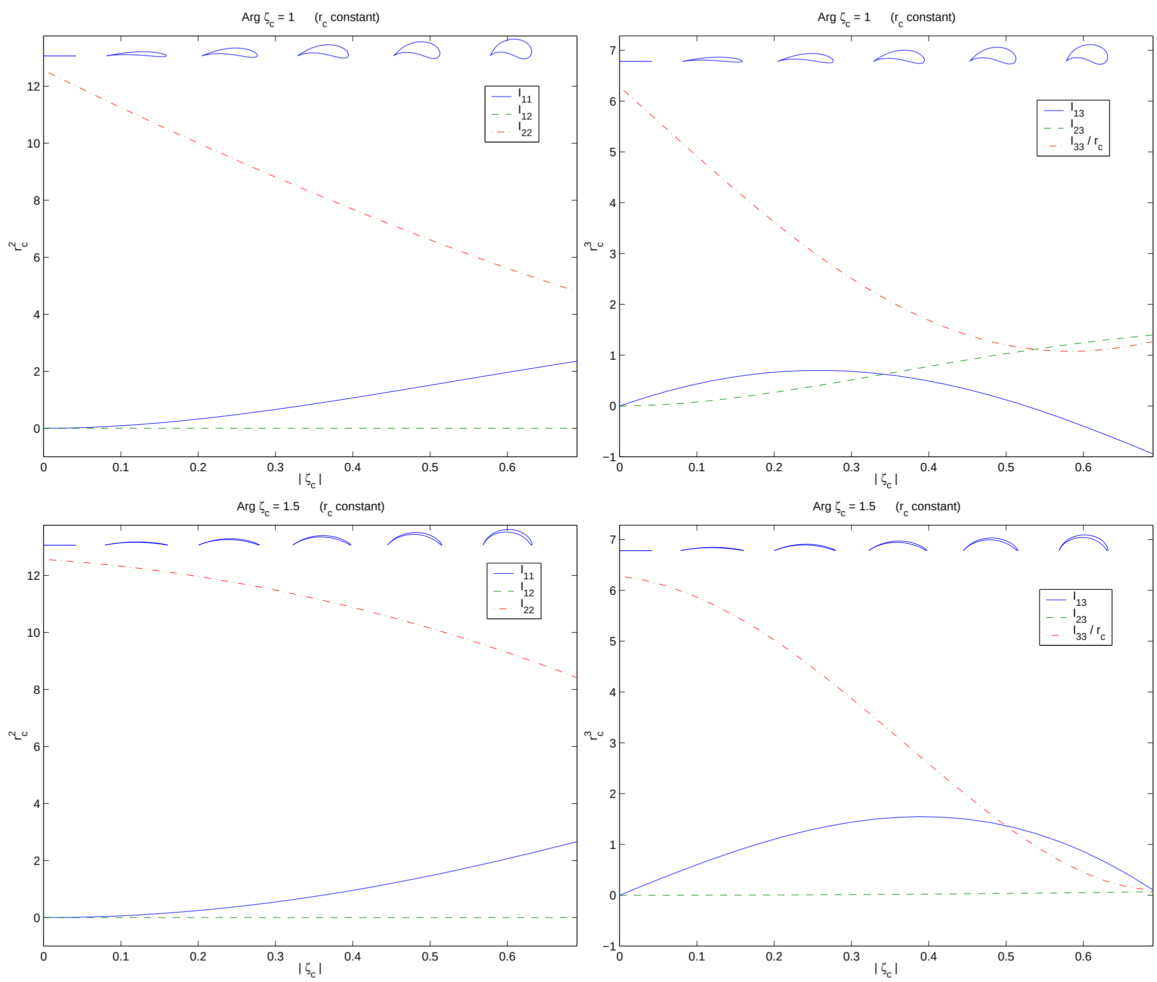


Figure 3.5: More plots of locked added inertia coefficients for a variety of Joukowski foils. In each plot, r_c and $\arg(\zeta_c)$ are held constant while $\|\zeta_c\|$ is varied. The coefficients I_{11}, I_{12}, I_{13} are given in units of r_c^2, while I_{13}, I_{23} are given in units of r_c^3 and I_{33} has units of r_c^4.

sampled from the (ζ_c, a) parameter space. Specifically, I let ζ_c vary along each of several different rays emanating from the origin, while always choosing a so as to keep $r_c = 1$. In other words, in each plot, $\|\zeta_c\|$ is the independent variable, while $\arg(\zeta_c)$ is constant (but a different constant in each pair of figures.) The closed form results for the added inertia were compared against results from numerical integration of the potential for these particular foil shapes, and found to be in agreement.

$$
\begin{aligned}
I_{11} &= \pi\rho\left(r_c^2 - 2a^2 + a^4\frac{r_c^2}{(r_c^2-\delta^2)^2}\right) \\
I_{12} &= 0 \\
I_{22} &= \pi\rho\left(r_c^2 + 2a^2 + a^4\frac{r_c^2}{(r_c^2-\delta^2)^2}\right) \\
I_{13} &= \pi\rho\mathsf{Re}\left\{i\left(r_c^2\zeta_c - 2a^2\zeta_c - \frac{2a^4\zeta_c}{r_c^2-\delta^2} + \frac{a^6 r_c^2\zeta_c}{(r_c^2-\delta^2)^3}\right)\right\} \\
&= \pi\rho\left(-r_c^2\zeta_y + 2a^2\zeta_y + \frac{2a^4\zeta_y}{r_c^2-\delta^2} - \frac{a^6 r_c^2\zeta_y}{(r_c^2-\delta^2)^3}\right) \\
I_{23} &= \pi\rho\mathsf{Re}\left\{r_c^2\zeta_c + 2a^2\zeta_c - \frac{2a^4\zeta_c}{r_c^2-\delta^2} - \frac{a^6 r_c^2\zeta_c}{(r_c^2-\delta^2)^3}\right\} \\
&= \pi\rho\left(r_c^2\zeta_x + 2a^2\zeta_x - \frac{2a^4\zeta_x}{r_c^2-\delta^2} - \frac{a^6 r_c^2\zeta_x}{(r_c^2-\delta^2)^3}\right) \\
I_{33} &= \pi\rho\mathsf{Re}\left\{2a^4 + r_c^2\delta^2 + 2a^2\zeta_c^2 - 2a^4\frac{\delta^2}{(r_c^2-\delta^2)} + 2a^6\frac{\zeta_c^2}{(r_c^2-\delta^2)^2} + a^8\frac{r_c^2\delta^2}{(r_c^2-\delta^2)^4}\right\} \\
&= \pi\rho\left(2a^4 + r_c^2\delta^2 + 2a^2(\zeta_x^2-\zeta_y^2) - 2a^4\frac{\delta^2}{(r_c^2-\delta^2)} + 2a^6\frac{(\zeta_x^2-\zeta_y^2)}{(r_c^2-\delta^2)^2} + a^8\frac{r_c^2\delta^2}{(r_c^2-\delta^2)^4}\right)
\end{aligned}
$$

Table 3.3: Components of the locked added inertia for the Joukowski foil.

3.4 Vortex Shedding

The wake from an airfoil moving in a potential flow may be computationally modelled as a trail of discrete point vortices. At each of a series of time steps, a new point vortex is shed from the airfoil, with strength chosen so as to satisfy the Kutta condition at that time step [Str94, ST95, Gie68]. The Kutta condition requires that the fluid velocities on the trailing-edge upper surface and lower surface be equal in magnitude but opposite in tangential direction [Gie68]. To satisfy this requirement, the component of $\overline{dw/d\zeta}$ tangential to the surface C must vanish at $\zeta = (a - \zeta_c)$, the point corresponding to the trailing edge. Since the shed vortex generates a fluid velocity tangential to C on C, it is always possible to choose a strength γ_k for the vortex to satisfy the Kutta condition. (In fact, we must have $\overline{dw/d\zeta} = 0$ at $\zeta = a - \zeta_c$ in order for the physical velocity $\overline{dw/dz}$ at the trailing edge to be bounded. But the normal component of $\overline{dw/d\zeta}$ must vanish naturally from the form of the potential: it cannot be made to vanish by choosing the strength of the shed vortex.)

Before solving for the strength γ_k, the vortex must be placed at some location near the trailing edge of the airfoil. There is no obvious unique way to choose the exact starting location of the vortex. I adapt a scheme from Streitlien [Str94], interpolating between the trailing edge and the last vortex shed. The idea, illustrated in Figure 3.6, is that each discrete vortex represents a segment of a continuous vortex sheet which departs the trailing edge parallel to it. Close to the trailing edge the sheet can be approximated by a circular arc (or in the degenerate case, a straight line) tangent to the trailing edge and intersecting the discrete vortex shed in the previous time step. The trailing edge is located at $z = 2a$. If $a > 0$, the angle of the cusped trailing edge is $\chi = -2\beta$ where $\beta = \arctan((\zeta_y)/(a - \zeta_x)$. If $a < 0$, the angle of the cusped trailing edge is $\chi = -2\beta + \pi$.

I place the new discrete vortex one-third of the way along this arc, so that each vortex is approximately at the midpoint of the wake segment that it represents.[4]

[4] Streitlien [Str94] also seeks to place the vortex at the midpoint of its wake segment, but uses $\frac{\theta}{4}$ instead of $\frac{\theta}{3}$! This seems to be a miscalculation. However, for small Δt the behavior of the simulation should not strongly depend on this detail.

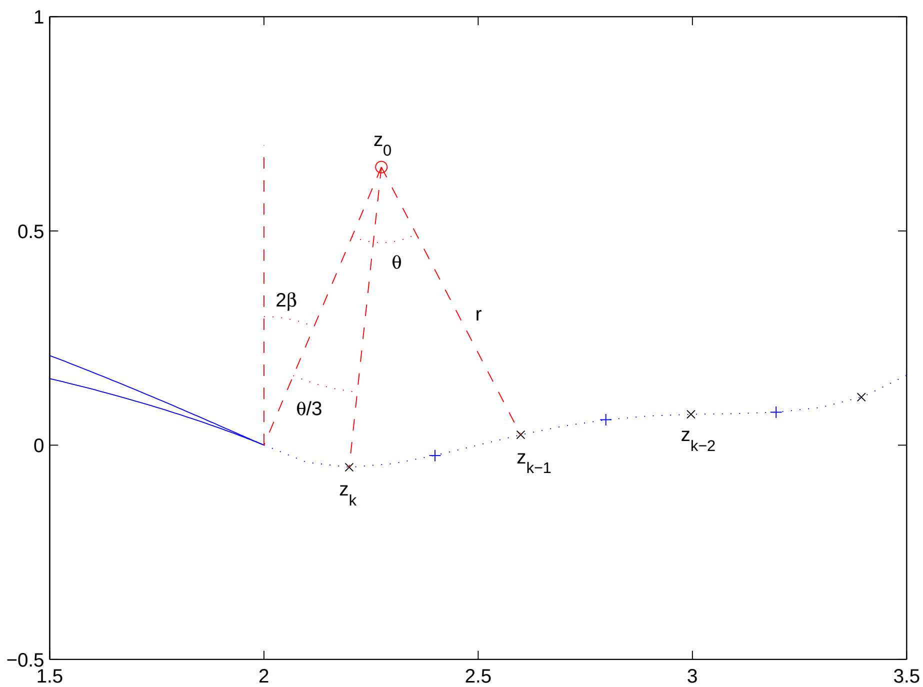


Figure 3.6: At each time step a new vortex is placed at the one-third point on a circular arc tangent to the trailing edge and intersecting the last vortex shed. The '$\times$' marks are the discrete vortices while the '+' marks delimit the wake segments which the vortices represent.

From examination of Figure 3.6, I have the following relations:

$$\begin{aligned} z_0 - z_{k-1} &= re^{i(\pi/2+\chi+\theta)} && (3.60) \\ z_0 - 2a &= re^{i(\pi/2+\chi)} && (3.61) \\ z_0 - z_k &= re^{i(\pi/2+\chi+\theta/3)} && (3.62) \end{aligned}$$

from which it follows that

$$\begin{aligned} z_{k-1} - 2a &= re^{i(\pi/2+\chi)}(1 - e^{i\theta}) && (3.63) \\ z_k - 2a &= re^{i(\pi/2+\chi)}(1 - e^{i\theta/3}) && (3.64) \end{aligned}$$

From the known location of the last shed vortex z_{k-1}, I can solve for θ:

$$e^{i\theta} = e^{-i2\chi}\frac{(z_{k-1} - 2a)^2}{\|z_{k-1} - 2a\|^2} \tag{3.65}$$

and then for z_k:

$$z_k - 2a = (z_{k-1} - 2a)\frac{1 - e^{i\theta/3}}{1 - e^{i\theta}} = \frac{(z_{k-1} - 2a)}{1 + e^{i\theta/3} + e^{i2\theta/3}} \tag{3.66}$$

In general, there are *two* circular arcs which intersect z_{k-1} and are tangent to the trailing edge of the foil. (Together they make up the circle which intersects z_{k-1} and is tangent to the trailing edge.) Note that $\theta \pm 2\pi$ is also a solution to Equation (3.65). I must make sure to pick the correct arc, i.e., the one that departs the trailing edge in the correct direction (not necessarily the shorter of the two arcs.) What I want is to choose the branch of θ (i.e., add or subtract 2π from θ as necessary) so that θ is positive if $\arg(z_{k-1} - 2a) \in (\chi, \chi + \pi)$, and θ is negative if $\arg(z_{k-1} - 2a) \in (\chi - \pi, \chi)$. Importantly, these equations give sensible results in the degenerate case, when the arc is a straight line and $\theta = 0$.

As a consequence of Kelvin's theorem, each one of the shed vortices subsequently moves with the same velocity as a fluid particle at the same location $z_k = F(\zeta_k)$.

That is, the vortices convect according to Routh's rule:

$$\begin{aligned}
\overline{\frac{dz_k}{dt}} &= \frac{d}{dz}\left[w - i\gamma_k \log(z - z_k)\right]_{z=z_k} \\
&= \left\{\frac{d\zeta}{dz}\frac{d}{d\zeta}\left[w - i\gamma_k \log(\zeta - \zeta_k)\right]\right\}_{\zeta=\zeta_k} + i\gamma_k \left\{\frac{d\zeta}{dz}\frac{d}{d\zeta}\left[\log(\zeta - \zeta_k) - \log(z - z_k)\right]\right\}_{\zeta=\zeta_k} \\
&= \left\{\frac{d\zeta}{dz}\frac{d}{d\zeta}\left[w - i\gamma_k \log(\zeta - \zeta_k)\right]\right\}_{\zeta=\zeta_k} + i\gamma_k \lim_{\zeta\to\zeta_k}\left\{\frac{d\zeta}{dz}\left[\frac{1}{\zeta - \zeta_k} - \frac{dz/d\zeta}{z - z_k}\right]\right\} \\
&= \left\{\frac{d\zeta}{dz}\frac{d}{d\zeta}\left[w - i\gamma_k \log(\zeta - \zeta_k)\right]\right\}_{\zeta=\zeta_k} + i\gamma_k \left\{\left[\frac{-(d^2z/d\zeta^2)}{2(dz/d\zeta)^2}\right]\right\}_{\zeta=\zeta_k} \\
&= \left\{\frac{d\zeta}{dz}\frac{d}{d\zeta}\left[w - i\gamma_k \log(\zeta - \zeta_k)\right]\right\}_{\zeta=\zeta_k} - i\gamma_k \left(\frac{a^2\ (\zeta_c + \zeta_k)}{\left(a^2 - (\zeta_c + \zeta_k)^2\right)^2}\right). \qquad (3.67)
\end{aligned}$$

The velocities $\dot{\zeta}_k$ can be recovered using

$$\frac{dz_k}{dt} = \frac{d\zeta_k}{dt} + \frac{d\zeta_c}{dt} - \frac{a^2}{(\zeta_k + \zeta_c)^2}\left(\frac{d\zeta_k}{dt} + \frac{d\zeta_c}{dt}\right) \qquad (3.68)$$

$$\frac{d\zeta_k}{dt} = \left(1 - \frac{a^2}{(\zeta_k + \zeta_c)^2}\right)^{-1}\frac{dz_k}{dt} - \frac{d\zeta_c}{dt}. \qquad (3.69)$$

In each time step the new vortex positions and velocities are found using a fourth- and fifth-order Runge-Kutta integration method. Since the location of each vortex exerts an influence in determining the velocity of every other vortex, the difficulty of integrating these differential equations forward in time grows as $\mathcal{O}\left(n_v^2\right)$ where n_v is the number of vortices. One way to keep the computational burden from growing too large is to coalesce clusters of vortices which are more than a few chord lengths from the foil, and replace each cluster by a single point vortex at the cluster's center of vorticity. Sarpkaya [Sar75] employed this procedure in simulating vortex shedding from an inclined flat plate, and found that vortex clusters more than eight chord lengths from the plate could be coalesced without a material effect on the simulation. Carrying this process to its logical conclusion, a vortex which is sufficiently far away may be regarded as infinitely distant and removed from the set of wake vortices, while the strength $(-\gamma_k)$ of its image is added to the central vortex strength γ_c.

The size of the time step Δt between shed vortices should obviously be chosen sufficiently small that the system's behavior is no longer sensitive to the size of Δt. For the flat plate, Sarpkaya [Sar75] found that a time step of $\Delta t = 0.04a/U$ was sufficiently small (with the results for $\Delta t = 0.02a/U$ and $\Delta t = 0.08a/U$ not being materially different, but with $\Delta t = 0.16a/U$ being too large.) For various motions of the deformable foil with U/a about unity and with the critical frequency of the deformations also being of order unity, I found time steps as large as $\Delta t = 0.12$ were adequate to qualitatively capture the wake structure and the forces on the foil, but not to reliably estimate the magnitude of the forces, for which a time step as small as $\Delta t = 0.04$ was required. Figure 3.7 shows a particular deformable foil simulation performed with a range of choices of Δt.

3.5 Forces on the Foil

In an inertial frame instantaneously aligned with the z coordinate frame, in which the fluid at infinity is at rest, the pressure p at each point in the fluid is given by [MT68]:

$$\begin{aligned} p &= -\frac{\partial \phi}{\partial t} - \frac{1}{2}(u^2 + v^2) \\ &= -\left(\frac{\partial}{\partial t}\frac{w + \overline{w}}{2}\right) - \frac{1}{2}\left(\frac{dw}{dz}\overline{\frac{dw}{dz}}\right) \end{aligned} \tag{3.70}$$

Using X and Y to represent the forces on the foil along the x- and y-axes respectively, we see that

$$X = \int_\Sigma p\,(-dy) \tag{3.71}$$

$$Y = \int_\Sigma p\,dx \tag{3.72}$$

or equivalently

$$X + iY = i\int_\Sigma p\,dz \tag{3.73}$$

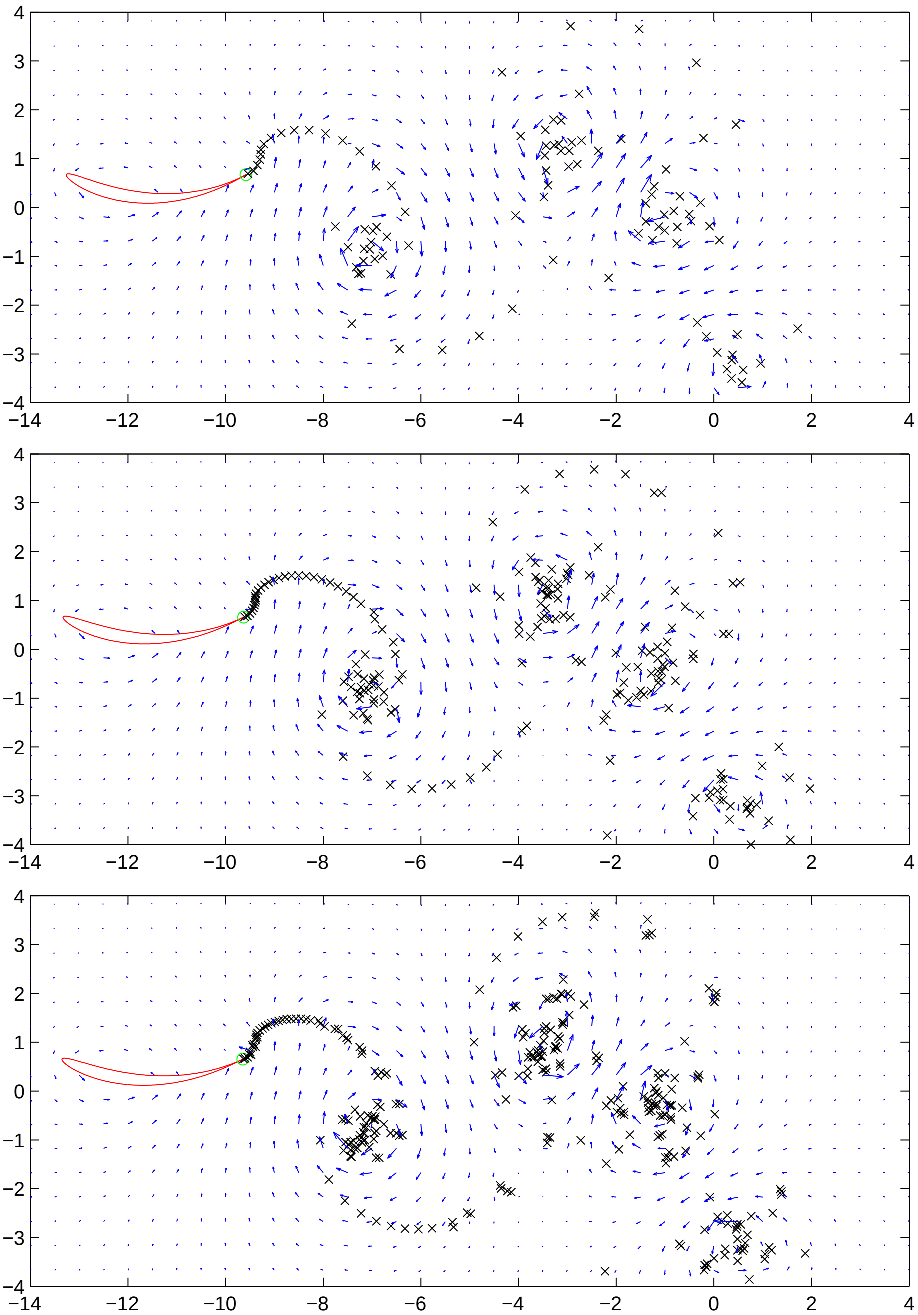


Figure 3.7: Wake from a heaving and flexing deformable Joukowski foil. The foil began its motion with its centroid at the origin. The centroid then translated with uniform velocity $U_{\rm cm} = -1$ in the x-direction while heaving in the y-direction with velocity $V_{\rm cm} = (-0.4 \sin t)$. Meanwhile the foil shape underwent a sinusoidal flexing motion with $(\zeta_x, \zeta_y) = (-0.05, 0.4 \sin t)$. The modelled wake behavior at $t = 12$ is shown with the time step Δt between shed vortex points being (top to bottom) $\Delta t = 0.12$, $\Delta t = 0.06$, $\Delta t = 0.04$.

so:

$$X + iY = -\frac{i}{2}\int_\Sigma \left(\frac{\partial w}{\partial t} + \frac{\partial \overline{w}}{\partial t} + \frac{dw}{dz}\overline{\frac{dw}{dz}}\right) dz. \tag{3.74}$$

This expression captures all the forces on the foil, including the added mass forces. Therefore in evaluating this integral I will, among other things, rediscover the locked inertia coefficients already derived in Section 3.3.

3.5.1 Unsteady Flow Force Terms

First I consider the part of integral (3.74) which is due to the time-varying potential.

$$\begin{aligned}\int_\Sigma \frac{\partial}{\partial t}(w+\overline{w})\, dz = \dot{U}\left(\int_\Sigma (w_1 + \overline{w_1})\, dz\right) + \dot{V}\left(\int_\Sigma (w_2 + \overline{w_2})\, dz\right) + \dot{\Omega}\left(\int_\Sigma (w_3 + \overline{w_3})\, dz\right) \\ + \dot{\gamma_c}\left(\int_\Sigma (w_4 + \overline{w_4}\, dz\right) + \sum_k \dot{\gamma_k}\left(2\int_\Sigma w_5^k\, dz\right) \\ + \ddot{\zeta_x}\left(\int_\Sigma (w_1^s + \overline{w_1^s}\, dz\right) + \ddot{\zeta_y}\left(\int_\Sigma (w_2^s + \overline{w_2^s}\, dz\right) + \ddot{a}\left(\int_\Sigma (w_3^s + \overline{w_3^s}\, dz\right) \\ + U\left(\int_\Sigma (\frac{\partial w_1}{\partial t} + \frac{\partial \overline{w_1}}{\partial t}\right) + V\left(\int_\Sigma (\frac{\partial w_2}{\partial t} + \frac{\partial \overline{w_2}}{\partial t}\right) + \Omega\left(\int_\Sigma (\frac{\partial w_3}{\partial t} + \frac{\partial \overline{w_3}}{\partial t}\right) \\ + \gamma_c\left(2\int_\Sigma \frac{\partial w_4}{\partial t}\right) + \sum_k \gamma_k\left(2\int_\Sigma \frac{\partial w_5^k}{\partial t}\, dz\right) \\ + \dot{\zeta_x}\left(\int_\Sigma (\frac{\partial w_1^s}{\partial t} + \frac{\partial \overline{w_1^s}}{\partial t}\right) + \dot{\zeta_y}\left(\int_\Sigma (\frac{\partial w_2^s}{\partial t} + \frac{\partial \overline{w_2^s}}{\partial t}\right) + \dot{a}\left(\int_\Sigma (\frac{\partial w_3^s}{\partial t} + \frac{\partial \overline{w_3^s}}{\partial t}\right)\end{aligned} \tag{3.75}$$

where I have used the fact that w_4 and w_5^k are real on Σ by construction.

Some elements of the integral can be evaluated using Cauchy's integral formula and the fact that $\overline{\zeta} = r_c^2/\zeta$ on C. The results are presented below with more details in Appendix A.

$$\int_\Sigma w_1\,dz \;=\; 2\pi i(a^2 - r_c^2) \tag{3.76}$$

$$\int_\Sigma \overline{w_1}\,dz \;=\; 2\pi i(a^2 - \frac{a^4 r_c^2}{(r_c^2-\delta^2)^2}) \tag{3.77}$$

$$\int_\Sigma w_2\,dz \;=\; 2\pi(a^2 + r_c^2) \tag{3.78}$$

$$\int_\Sigma \overline{w_2}\,dz \;=\; 2\pi(a^2 + \frac{a^4 r_c^2}{(r_c^2-\delta^2)^2}) \tag{3.79}$$

$$\int_\Sigma w_3\,dz \;=\; 2\pi\left(r_c^2\zeta_c + a^2\overline{\zeta_c} - \frac{a^4\zeta_c}{(r_c^2-\delta^2)}\right) \tag{3.80}$$

$$\int_\Sigma \overline{w_3}\,dz \;=\; 2\pi\left(\overline{\zeta_c}a^2 - \frac{a^4\zeta_c}{(r_c^2-\delta^2)} - \frac{a^6\overline{\zeta_c}r_c^2}{(r_c^2-\delta^2)^3}\right) \tag{3.81}$$

In order to perform the integrals involving logarithmic terms in w_4 and w_1^s, w_2^s, w_3^s, I do integration by parts, in each case choosing the branch cut associated with the logarithmic potential to pass through the point $z_{\mathrm{cut}} = 2a$, i.e., the trailing edge of the foil. More details are given in Appendix A.

$$\int_\Sigma w_4\,dz \;=\; -2\pi(2a - \zeta_c) \tag{3.82}$$

$$\int_\Sigma w_1^s\,dz = 2\pi i\left[-r_c^2 + a^2 + \frac{2a^2r_c^2}{\zeta_c^2} - \frac{a^4(\zeta_c^2+r_c^2)}{(r_c^2-\delta^2)^2} - \frac{4a^5r_c^2\zeta_c}{(r_c^2-\delta^2)^3} + \frac{2a^4r_c^2(i\zeta_y)}{(r_c^2-\delta^2)^3}(2a-\zeta_c) - 2a^2r_c^2\left(\frac{1}{\zeta_c^2} + \frac{\delta^2a^2}{(r_c^2-\delta^2)^3}\right)\right] \quad (3.83)$$

$$\int_\Sigma \overline{w_1^s}\,dz = 2\pi i\left[a^2 - \frac{a^4}{\overline{\zeta_c}^2} - \frac{a^4r_c^2}{(r_c^2-\delta^2)^2} + \frac{a^6r_c^2\overline{\zeta_c}^2}{(r_c^2-\delta^2)^4} - \frac{a^4r_c^4}{\overline{\zeta_c}^2(r_c^2-\delta^2)^2} + \frac{a^6r_c^4}{(r_c^2-\delta^2)^4} + \frac{2a^4r_c^2\overline{\zeta_c}}{(r_c^2-\delta^2)^3}\left(2a-\zeta_c+\frac{a^2\overline{\zeta_c}}{(r_c^2-\delta^2)}\right) + \frac{2a^4r_c^2(i\zeta_y)}{(r_c^2-\delta^2)^3}(2a-\zeta_c) + 2a^4r_c^2\left(\frac{1}{\overline{\zeta_c}^3} + \frac{\zeta_ca^2}{(r_c^2-\delta^2)^3}\right)\left(\frac{\overline{\zeta_c}}{(r_c^2-\delta^2)}\right)\right] \quad (3.84)$$

$$\int_\Sigma w_2^s\,dz = 2\pi\left[r_c^2 + a^2 - \frac{2a^2r_c^2}{\zeta_c^2} - \frac{a^4(\zeta_c^2-r_c^2)}{(r_c^2-\delta^2)^2} - \frac{4a^5r_c^2\zeta_c}{(r_c^2-\delta^2)^3} + \frac{2a^4r_c^2\zeta_x}{(r_c^2-\delta^2)^3}(2a-\zeta_c) + 2a^2r_c^2\left(\frac{1}{\zeta_c^2} + \frac{\delta^2a^2}{(r_c^2-\delta^2)^3}\right)\right] \quad (3.85)$$

$$\int_\Sigma \overline{w_2^s}\,dz = 2\pi\left[a^2 - \frac{a^4}{\overline{\zeta_c}^2} + \frac{a^4r_c^2}{(r_c^2-\delta^2)^2} - \frac{a^6r_c^2\overline{\zeta_c}^2}{(r_c^2-\delta^2)^4} - \frac{a^4r_c^4}{\overline{\zeta_c}^2(r_c^2-\delta^2)^2} + \frac{a^6r_c^4}{(r_c^2-\delta^2)^4} - \frac{2a^4r_c^2\overline{\zeta_c}}{(r_c^2-\delta^2)^3}\left(2a-\zeta_c+\frac{a^2\overline{\zeta_c}}{(r_c^2-\delta^2)}\right) + \frac{2a^4r_c^2\zeta_x}{(r_c^2-\delta^2)^3}(2a-\zeta_c) + 2a^4r_c^2\left(\frac{1}{\overline{\zeta_c}^3} + \frac{\zeta_ca^2}{(r_c^2-\delta^2)^3}\right)\left(\frac{\overline{\zeta_c}}{(r_c^2-\delta^2)}\right)\right] \quad (3.86)$$

$$\int_\Sigma w_3^s\,dz = 4\pi ia\left(-\frac{r_c^2}{\zeta_c} - \frac{a^2\zeta_c}{(r_c^2-\delta^2)} - \frac{2a^3r_c^2}{(r_c^2-\delta^2)^2} + \frac{r_c^2((r_c^2-\delta^2)^2-a^2\zeta_c^2)}{\zeta_c(r_c^2-\delta^2)^2}\right) \quad (3.87)$$

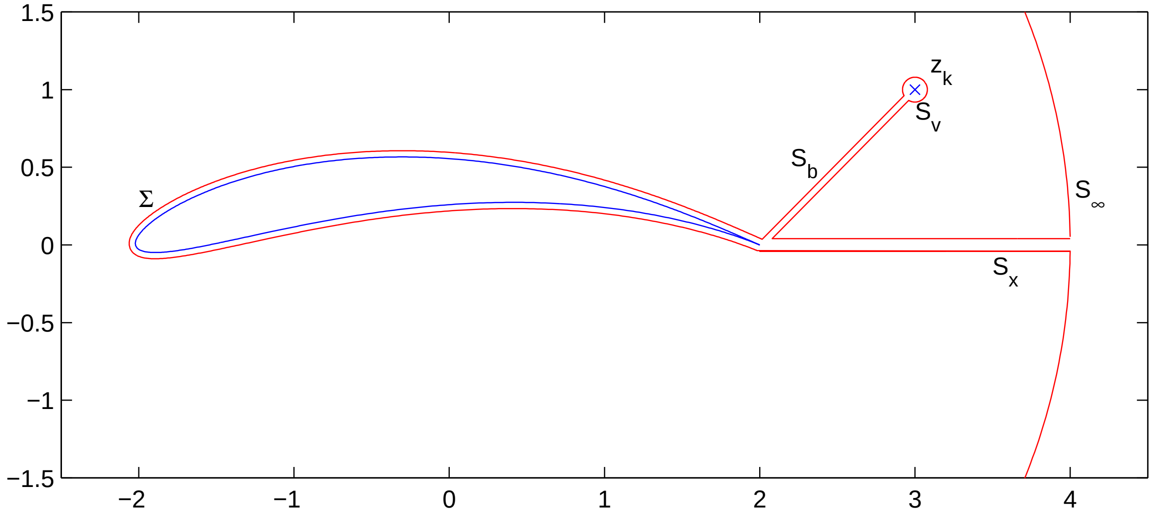


Figure 3.8: If the direction of integration is counterclockwise in each case, then $\int_\Sigma = \int_{S_\infty} - \int_{S_x} - \int_{S_v} - \int_{S_b}$.

$$\begin{aligned}\int_\Sigma \overline{w_3^s}\, dz &= 4\pi a i\left(\frac{a^2}{\overline{\zeta_c}} + \frac{a^4 r_c^2 \overline{\zeta_c}}{(r_c^2-\delta^2)^3}\right) \\ &\quad +4\pi i \frac{a^3 r_c^2}{(r_c^2-\delta^2)^2}\left(2a - \zeta_c + \frac{a^2\overline{\zeta_c}}{(r_c^2-\delta^2)}\right) \\ &\quad -4\pi i a^3 \frac{r_c^2((r_c^2-\delta^2)^2 - a^2\overline{\zeta_c}^{-2})}{\overline{\zeta_c}(r_c^2-\delta^2)^3} \qquad (3.88)\end{aligned}$$

Instead of evaluating w_5^k directly on Σ, we consider another contour of integration illustrated in Figure 3.8. The contour segment S_v encircles the vortex at $z = z_k$. The branch cut between that vortex and its image vortex inside the foil is chosen to pass through the trailing edge of the foil at $z = 2a$. The contour segment S_b encloses the branch cut between the wake vortex and the trailing edge. Since the contour $S_\infty - S_x - \Sigma - S_b - S_v$ encloses a space with no singularities or branch cuts of w_5^k, the integral of w_5^k over that contour must be zero.

$$\int_\Sigma w_5^k\, dz = \int_{S_\infty} w_5^k\, dz - \int_{S_x} w_5^k\, dz - \int_{S_v} w_5^k\, dz - \int_{S_b} w_5^k\, dz \qquad (3.89)$$

Since w_5^k is continuous in the vicinity of S_x, that integral vanishes. The integral around S_v also vanishes in the limit as the radius of S_v approaches zero. The value of w_5^k is 2π higher on the lower side of S_b than on the upper side, so

$$\begin{aligned}\int_{S_b} w_5^k\,dz &= \int_{z=2a}^{z=z_k} 2\pi\,dz \\ &= 2\pi(z_k - 2a)\end{aligned} \tag{3.90}$$

Finally, in the limit of large ζ,

$$\begin{aligned}w_5^k &= i\left[\log(\frac{-r_c}{\zeta_k}) + \log(\zeta-\zeta_k) - \log(\zeta - \frac{r_c^2}{\overline{\zeta_k}})\right] \\ &= i\log(\frac{-r_c}{\zeta_k}) + i\log(\zeta) - i\sum_{n=1}^{\infty}\zeta_k^n\frac{1}{n\zeta^n} - \left(i\log(\zeta) - i\sum_{n=1}^{\infty}\left(\frac{r_c^2}{\overline{\zeta_k}}\right)^n\frac{1}{n\zeta^n}\right)\end{aligned} \tag{3.91}$$

Identifying the terms that go as $1/\zeta$, we see that

$$\begin{aligned}\int_{S_\infty} w_5^k\,dz &= \int_{S_\infty} w_5^k\left(1 - \frac{a^2}{(\zeta+\zeta_c)^2}\right)d\zeta \\ &= 2\pi\left(\zeta_k - \frac{r_c^2}{\overline{\zeta_k}}\right)\end{aligned} \tag{3.92}$$

So,

$$\int_\Sigma w_5^k\,dz = 2\pi\left(\zeta_k - \frac{r_c^2}{\overline{\zeta_k}} - z_k + 2a\right) \tag{3.93}$$

The integrals involving time derivatives of the Kirchoff potentials can all be evaluated using Cauchy's integral formula.

$$\int_\Sigma \frac{\partial w_1}{\partial t}\,dz = 4\pi i\,(\,a\,\dot{a} - \,r_c\,\dot{r_c}) = 2\pi i\frac{d}{dt}\left(a^2 - r_c^2\right) \tag{3.94}$$

$$\int_\Sigma \frac{\partial\overline{w_1}}{\partial t}\,dz = \frac{4\,a^2\,i\,\pi\,\left(\dot{r_c}\,({r_c}^2-\delta^2)^3 + a\,\dot{a}\,(-{r_c}^5 + {r_c}^3\,\delta^2) + a^2\,{r_c}^3\,\zeta_c\,\left(-\dot{\zeta_x} + i\,\dot{\zeta_y}\right)\right)}{r_c\,({r_c}^2-\delta^2)^3} \tag{3.95}$$

$$\int_\Sigma \frac{\partial w_2}{\partial t}\,dz = 4\pi\,(\,a\,\dot{a} + \,r_c\,\dot{r_c}) = 2\pi\frac{d}{dt}\left(a^2 + {r_c}^2\right) \tag{3.96}$$

$$\int_\Sigma \frac{\partial \overline{w_2}}{\partial t}\,dz = \frac{4a^2\pi\left(a\dot{a}{r_c}^3\left({r_c}^2-\delta^2\right) + \dot{r_c}({r_c}^2-\delta^2)^3 + a^2{r_c}^3\zeta_c\left(\dot{\zeta}_x - i\dot{\zeta}_y\right)\right)}{r_c({r_c}^2-\delta^2)^3} \tag{3.97}$$

$$\begin{aligned}\int_\Sigma \frac{\partial w_3}{\partial t}\,dz &= 2\pi\Bigg[2r_c\dot{r_c}\zeta_c + 2a\dot{a}\overline{\zeta_c} + \frac{2a^4r_c\dot{r_c}\zeta_c}{({r_c}^2-\delta^2)^2} - \frac{4a^3\dot{a}\zeta_c}{{r_c}^2-\delta^2} + {r_c}^2\dot{\zeta}_x \\ &\quad -\frac{a^4\delta^2\dot{\zeta}_x}{({r_c}^2-\delta^2)^2} - \frac{a^4\dot{\zeta}_x}{{r_c}^2-\delta^2} + i{r_c}^2\dot{\zeta}_y - \frac{a^4 i\delta^2\dot{\zeta}_y}{({r_c}^2-\delta^2)^2} - \frac{a^4 i\dot{\zeta}_y}{{r_c}^2-\delta^2}\Bigg]\end{aligned} \tag{3.98}$$

$$\begin{aligned}\int_\Sigma \frac{\partial \overline{w_3}}{\partial t}\,dz &= \frac{2\,a^2\,\pi}{r_c\,({r_c}^2-\delta^2)^4}\,\Big(-4\,a^3\,\dot{a}\,{r_c}^3\,\overline{\zeta_c}\,\left({r_c}^2-\delta^2\right) - 2\,a\,\dot{a}\,r_c\,\zeta_c\left({r_c}^2-\delta^2\right)^3 \\ &\quad +\left({r_c}^2-\delta^2\right)^4\,\left(2\,\dot{r_c}\,\overline{\zeta_c} + r_c\,\left(\dot{\zeta}_x - i\,\dot{\zeta}_y\right)\right) \\ &\quad -a^2\,\zeta_c\left({r_c}^2-\delta^2\right)^2\,\left(4\,{r_c}^2\,\dot{r_c} - 2\,\dot{r_c}\,\delta^2 + r_c\,\zeta_c\,\left(\dot{\zeta}_x - i\,\dot{\zeta}_y\right)\right) \\ &\quad +a^4\,{r_c}^3\,\left(2\,r_c\,\dot{r_c}\,\overline{\zeta_c} - 2\,\delta^2\,\left(\dot{\zeta}_x - i\,\dot{\zeta}_y\right) + {r_c}^2\,\left(-\dot{\zeta}_x + i\,\dot{\zeta}_y\right)\right)\Big)\end{aligned} \tag{3.99}$$

$$\int_\Sigma \frac{\partial w_4}{\partial t}\,dz = 0 \tag{3.100}$$

$$\int_\Sigma \frac{\partial w_5^k}{\partial t}\,dz = 2\,\pi\,\left(\frac{r_c\,\left(-2\,\dot{r_c}\,\overline{\zeta_k} + r_c\,\dot{\overline{\zeta_k}}\right)}{\overline{\zeta_k}^2} + \frac{a^2\,\dot{\zeta}_k}{(\zeta_c+\zeta_k)^2}\right) \tag{3.101}$$

$$\int_\Sigma \frac{\partial w_1^s}{\partial t}\,dz = \dot{\zeta}_c\int_\Sigma \frac{\partial w_1^s}{\partial \zeta_c}\,dz + \dot{a}\int_\Sigma \frac{\partial w_1^s}{\partial a}\,dz + \dot{r_c}\int_\Sigma \frac{\partial w_1^s}{\partial r_c}\,dz \tag{3.102}$$

$$\int_{\Sigma} \frac{\partial w_1^s}{\partial \zeta_c}\, dz = \Bigg[4a^4\pi \Bigg(\frac{-2\,a\,i\,{r_c}^4}{({r_c}^2-\delta^2)^4} - \frac{2\,i\,{r_c}^4\,\zeta_c}{({r_c}^2-\delta^2)^4} - \frac{2\,i\,{r_c}^4\,\overline{\zeta_c}}{({r_c}^2-\delta^2)^4} - \frac{4\,a\,i\,{r_c}^2\,\delta^2}{({r_c}^2-\delta^2)^4} + \frac{2\,i\,{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}}{({r_c}^2-\delta^2)^4} - \frac{i\,{r_c}^2\,\zeta_c\,\overline{\zeta_c}^2}{({r_c}^2-\delta^2)^4} - \frac{6\,a\,{r_c}^2\,\overline{\zeta_c}\,\zeta_y}{({r_c}^2-\delta^2)^4} + \frac{3\,{r_c}^2\,\delta^2\,\zeta_y}{({r_c}^2-\delta^2)^4}\Bigg)\Bigg] \tag{3.103}$$

$$\int_{\Sigma} \frac{\partial w_1^s}{\partial a}\, dz = \Bigg[4\pi \Bigg(a\,i - \frac{8\,a^4\,i\,{r_c}^2\,\zeta_c}{({r_c}^2-\delta^2)^3} - \frac{4\,a^3\,i\,{r_c}^2\,\delta^2}{({r_c}^2-\delta^2)^3} - \frac{2\,a^3\,i\,{r_c}^2}{({r_c}^2-\delta^2)^2} - \frac{2\,a^3\,i\,{\zeta_c}^2}{({r_c}^2-\delta^2)^2} - \frac{8\,a^4\,{r_c}^2\,\zeta_y}{({r_c}^2-\delta^2)^3} + \frac{4\,a^3\,{r_c}^2\,\zeta_c\,\zeta_y}{({r_c}^2-\delta^2)^3}\Bigg)\Bigg] \tag{3.104}$$

$$\int_{\Sigma} \frac{\partial w_1^s}{\partial r_c}\, dz = \Bigg[4\pi \Bigg(-(i\,r_c) + \frac{2\,a^2\,i\,r_c}{{\zeta_c}^2} - \frac{2\,a^2\,i\,{r_c}^9}{{\zeta_c}^2\,({r_c}^2-\delta^2)^4} + \frac{8\,a^5\,i\,{r_c}^3\,\zeta_c}{({r_c}^2-\delta^2)^4} + \frac{8\,a^2\,i\,{r_c}^7\,\overline{\zeta_c}}{\zeta_c\,({r_c}^2-\delta^2)^4} + \frac{4\,a^4\,i\,{r_c}^3\,\delta^2}{({r_c}^2-\delta^2)^4} + \frac{4\,a^5\,i\,r_c\,{\zeta_c}^2\,\overline{\zeta_c}}{({r_c}^2-\delta^2)^4} - \frac{12\,a^2\,i\,{r_c}^5\,\overline{\zeta_c}^2}{({r_c}^2-\delta^2)^4} + \frac{2\,a^4\,i\,r_c\,{\zeta_c}^2\,\overline{\zeta_c}^2}{({r_c}^2-\delta^2)^4} + \frac{8\,a^2\,i\,{r_c}^3\,\zeta_c\,\overline{\zeta_c}^3}{({r_c}^2-\delta^2)^4} - \frac{2\,a^2\,i\,r_c\,{\zeta_c}^2\,\overline{\zeta_c}^4}{({r_c}^2-\delta^2)^4} + \frac{a^4\,i\,{r_c}^3}{({r_c}^2-\delta^2)^3} + \frac{2\,a^4\,i\,r_c\,{\zeta_c}^2}{({r_c}^2-\delta^2)^3} + \frac{a^4\,i\,r_c\,\delta^2}{({r_c}^2-\delta^2)^3} + \frac{8\,a^5\,{r_c}^3\,\zeta_y}{({r_c}^2-\delta^2)^4} - \frac{4\,a^4\,{r_c}^3\,\zeta_c\,\zeta_y}{({r_c}^2-\delta^2)^4} + \frac{4\,a^5\,r_c\,\delta^2\,\zeta_y}{({r_c}^2-\delta^2)^4} - \frac{2\,a^4\,r_c\,{\zeta_c}^2\,\overline{\zeta_c}\,\zeta_y}{({r_c}^2-\delta^2)^4}\Bigg)\Bigg] \tag{3.105}$$

$$\int_\Sigma \overline{\frac{\partial w_1^s}{\partial \zeta_c}}\,dz = \frac{4a^4\pi}{(r_c^2-\delta^2)^5}\left[-2\,a\,{r_c}^6 + 3\,a^2\,i\,{r_c}^4\,\zeta_c + {r_c}^6\,\zeta_c - i\,{r_c}^6\,\zeta_c \right.$$
$$-i\,{r_c}^4\,{\zeta_c}^3 - a^2\,{r_c}^4\,\overline{\zeta_c} + 2\,a^2\,i\,{r_c}^4\,\overline{\zeta_c} - 2\,a\,{r_c}^4\,\delta^2 + 3\,a^2\,i\,{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}$$
$$+{r_c}^4\,{\zeta_c}^2\,\overline{\zeta_c} + 2\,i\,{r_c}^4\,{\zeta_c}^2\,\overline{\zeta_c}$$
$$+2\,i\,{r_c}^2\,{\zeta_c}^4\,\overline{\zeta_c} - 2\,a^2\,{r_c}^2\,\zeta_c\,\overline{\zeta_c}^2 + a^2\,i\,{r_c}^2\,\zeta_c\,\overline{\zeta_c}^2 + 4\,a\,{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}^2$$
$$-2\,{r_c}^2\,{\zeta_c}^3\,\overline{\zeta_c}^2 - i\,{r_c}^2\,{\zeta_c}^3\,\overline{\zeta_c}^2 - i\,{\zeta_c}^5\,\overline{\zeta_c}^2 - 6\,a\,{r_c}^4\,\zeta_c\,\zeta_y$$
$$\left.+3\,{r_c}^4\,{\zeta_c}^2\,\zeta_y + 6\,a\,{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}\,\zeta_y - 3\,{r_c}^2\,{\zeta_c}^3\,\overline{\zeta_c}\,\zeta_y\right] \tag{3.106}$$

$$\int_\Sigma \overline{\frac{\partial w_1^s}{\partial a}}\,dz = \frac{-4a^3\pi}{(r_c^2-\delta^2)^4}\left[-2\,a^2\,{r_c}^2\,\left(-2\,\overline{\zeta_c}^2 + i\,\left({r_c}^2 + 2\,\delta^2 + \overline{\zeta_c}^2\right)\right) + 8\,a\,{r_c}^2\,\left({r_c}^2-\delta^2\right)\,\left(\overline{\zeta_c}+\zeta_y\right)\right.$$
$$\left.+\left({r_c}^2-\delta^2\right)\,\left(i\,\left({r_c}^2+{\zeta_c}^2\right)\,\left({r_c}^2-\delta^2\right) - 4\,{r_c}^2\,\zeta_c\,\left(\overline{\zeta_c}+\zeta_y\right)\right)\right] \tag{3.107}$$

$$\int_\Sigma \overline{\frac{\partial w_1^s}{\partial r_c}}\,dz = \frac{-4a^2\pi}{r_c(r_c^2-\delta^2)^5}\left[a^4\,i\,{r_c}^6 - i\,{r_c}^{10} + a^2\,i\,{r_c}^6\,{\zeta_c}^2 - 8\,a^3\,{r_c}^6\,\overline{\zeta_c} + 5\,a^4\,i\,{r_c}^4\,\delta^2\right.$$
$$+4\,a^2\,{r_c}^6\,\delta^2 + 5\,i\,{r_c}^8\,\delta^2 - 3\,a^2\,i\,{r_c}^4\,{\zeta_c}^3\,\overline{\zeta_c} - 4\,a^4\,{r_c}^4\,\overline{\zeta_c}^2 + 2\,a^4\,i\,{r_c}^4\,\overline{\zeta_c}^2$$
$$+4\,a^3\,{r_c}^4\,\zeta_c\,\overline{\zeta_c}^2 + 2\,a^4\,i\,{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}^2 - 2\,a^2\,{r_c}^4\,{\zeta_c}^2\,\overline{\zeta_c}^2 - 10\,i\,{r_c}^6\,{\zeta_c}^2\,\overline{\zeta_c}^2 + 3\,a^2\,i\,{r_c}^2\,{\zeta_c}^4\,\overline{\zeta_c}^2$$
$$-2\,a^4\,{r_c}^2\,\zeta_c\,\overline{\zeta_c}^3 + 4\,a^3\,{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}^3 - 2\,a^2\,{r_c}^2\,{\zeta_c}^3\,\overline{\zeta_c}^3$$
$$+10\,i\,{r_c}^4\,{\zeta_c}^3\,\overline{\zeta_c}^3 - a^2\,i\,{\zeta_c}^5\,\overline{\zeta_c}^3 - 5\,i\,{r_c}^2\,{\zeta_c}^4\,\overline{\zeta_c}^4 + i\,{\zeta_c}^5\,\overline{\zeta_c}^5 - 8\,a^3\,{r_c}^6\,\zeta_y + 4\,a^2\,{r_c}^6\,\zeta_c\,\zeta_y$$
$$\left.+4\,a^3\,{r_c}^4\,\delta^2\,\zeta_y - 2\,a^2\,{r_c}^4\,{\zeta_c}^2\,\overline{\zeta_c}\,\zeta_y + 4\,a^3\,{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}^2\,\zeta_y - 2\,a^2\,{r_c}^2\,{\zeta_c}^3\,\overline{\zeta_c}^2\,\zeta_y\right] \tag{3.108}$$

$$\int_\Sigma \frac{\partial w_2^s}{\partial \zeta_c}\,dz = \frac{4a^4\pi r_c^2}{(r_c^2-\delta^2)^4}\left[-2\,{r_c}^2\,\left(\zeta_c-\overline{\zeta_c}\right) + \delta^2\,\left(2\,\zeta_c+\overline{\zeta_c}-3\,i\,\zeta_y\right) - 2\,a\,\left({r_c}^2+2\,\delta^2-3\,i\,\overline{\zeta_c}\,\zeta_y\right)\right] \tag{3.109}$$

$$\int_\Sigma \frac{\partial w_2^s}{\partial a}\,dz \;=\; 4\,a\,\pi - \frac{32\,a^4\,\pi\,{r_c}^2\,\zeta_c}{\left({r_c}^2-\delta^2\right)^3} + \frac{16\,a^3\,\pi\,{r_c}^2\,\delta^2}{\left({r_c}^2-\delta^2\right)^3} + \frac{8\,a^3\,\pi\,{r_c}^2}{\left({r_c}^2-\delta^2\right)^2} - \frac{8\,a^3\,\pi\,{\zeta_c}^2}{\left({r_c}^2-\delta^2\right)^2} + \frac{16\,a^3\,i\,\pi\,{r_c}^2\,\left(2\,a-\zeta_c\right)\,\zeta_y}{\left({r_c}^2-\delta^2\right)^3} \quad (3.110)$$

$$\int_\Sigma \frac{\partial w_2^s}{\partial r_c}\,dz \;=\; 4\pi r_c\left[1 + \frac{12\,a^5\,{r_c}^2\,\zeta_c}{\left({r_c}^2-\delta^2\right)^4} - \frac{6\,a^4\,{r_c}^2\,\delta^2}{\left({r_c}^2-\delta^2\right)^4} - \frac{2\,a^4\,{r_c}^2}{\left({r_c}^2-\delta^2\right)^3} - \frac{4\,a^5\,\zeta_c}{\left({r_c}^2-\delta^2\right)^3} + \frac{2\,a^4\,{\zeta_c}^2}{\left({r_c}^2-\delta^2\right)^3} + \frac{2\,a^4\,\delta^2}{\left({r_c}^2-\delta^2\right)^3} + \frac{a^4}{\left({r_c}^2-\delta^2\right)^2} - \frac{12\,a^5\,i\,{r_c}^2\,\zeta_y}{\left({r_c}^2-\delta^2\right)^4} + \frac{6\,a^4\,i\,{r_c}^2\,\zeta_c\,\zeta_y}{\left({r_c}^2-\delta^2\right)^4} + \frac{4\,a^5\,i\,\zeta_y}{\left({r_c}^2-\delta^2\right)^3} - \frac{2\,a^4\,i\,\zeta_c\,\zeta_y}{\left({r_c}^2-\delta^2\right)^3}\right] \quad (3.111)$$

$$\int_\Sigma \overline{\frac{\partial w_2^s}{\partial \zeta_c}}\,dz \;=\; \frac{-4a^4\pi}{(r_c^2-\delta^2)^5}\left[a^2\,{r_c}^2\,\left(\delta^2\,\left(-3\,\zeta_c+\overline{\zeta_c}+2\,i\,\overline{\zeta_c}\right)+{r_c}^2\,\left(-3\,\zeta_c+(2+i)\,\overline{\zeta_c}\right)\right) + 2\,a\,i\,{r_c}^2\,\left({r_c}^2-\delta^2\right)\,\left({r_c}^2+2\,\delta^2+3\,\zeta_c\,\zeta_y\right) - \zeta_c\,\left({r_c}^2-\delta^2\right)\,\left((1+i)\,{r_c}^4+{\zeta_c}^3\,\overline{\zeta_c}-{r_c}^2\,\zeta_c\,\left(\zeta_c+\overline{\zeta_c}-2\,i\,\overline{\zeta_c}-3\,i\,\zeta_y\right)\right)\right] \quad (3.112)$$

$$\int_\Sigma \overline{\frac{\partial w_2^s}{\partial a}}\,dz \;=\; \frac{4a^3\pi}{(r_c^2-\delta^2)^4}\left[2\,a^2\,{r_c}^2\,\left({r_c}^2-\overline{\zeta_c}\,\left(-2\,\zeta_c+\overline{\zeta_c}+2\,i\,\overline{\zeta_c}\right)\right) - 8\,a\,i\,{r_c}^2\,\left({r_c}^2-\delta^2\right)\,\left(\overline{\zeta_c}+\zeta_y\right) + \left({r_c}^2-\delta^2\right)\,\left({r_c}^4+{\zeta_c}^3\,\overline{\zeta_c}-{r_c}^2\,\zeta_c\,\left(\zeta_c+\overline{\zeta_c}-4\,i\,\overline{\zeta_c}-4\,i\,\zeta_y\right)\right)\right] \quad (3.113)$$

$$\int_\Sigma \overline{\frac{\partial w_2^s}{\partial r_c}}\, dz = 4a^2\pi \Bigg[\frac{1}{r_c} - \frac{a^2}{r_c\,\overline{\zeta_c}^2} - \frac{a^4\,{r_c}^5}{\left({r_c}^2-\delta^2\right)^5} + \frac{12\,a^2\,{r_c}^5\,{\zeta_c}^2}{\left({r_c}^2-\delta^2\right)^5}$$
$$+\frac{2\,a^2\,{r_c}^9}{\overline{\zeta_c}^2\left({r_c}^2-\delta^2\right)^5} - \frac{8\,a^2\,{r_c}^7\,\zeta_c}{\overline{\zeta_c}\left({r_c}^2-\delta^2\right)^5} - \frac{5\,a^4\,{r_c}^3\,\delta^2}{\left({r_c}^2-\delta^2\right)^5}$$
$$-\frac{8\,a^2\,{r_c}^3\,{\zeta_c}^3\,\overline{\zeta_c}}{\left({r_c}^2-\delta^2\right)^5} + \frac{4\,a^4\,i\,{r_c}^3\,\overline{\zeta_c}^2}{\left({r_c}^2-\delta^2\right)^5} - \frac{2\,a^4\,r_c\,{\zeta_c}^2\,\overline{\zeta_c}^2}{\left({r_c}^2-\delta^2\right)^5}$$
$$+\frac{2\,a^2\,r_c\,{\zeta_c}^4\,\overline{\zeta_c}^2}{\left({r_c}^2-\delta^2\right)^5} + \frac{2\,a^4\,i\,r_c\,\zeta_c\,\overline{\zeta_c}^3}{\left({r_c}^2-\delta^2\right)^5} + \frac{8\,a^3\,i\,{r_c}^3\,\overline{\zeta_c}}{\left({r_c}^2-\delta^2\right)^4}$$
$$-\frac{4\,a^2\,i\,{r_c}^3\,\delta^2}{\left({r_c}^2-\delta^2\right)^4} + \frac{4\,a^3\,i\,r_c\,\zeta_c\,\overline{\zeta_c}^2}{\left({r_c}^2-\delta^2\right)^4} - \frac{2\,a^2\,i\,r_c\,{\zeta_c}^2\,\overline{\zeta_c}^2}{\left({r_c}^2-\delta^2\right)^4}$$
$$-\frac{a^2\,{r_c}^3}{\overline{\zeta_c}^2\left({r_c}^2-\delta^2\right)^2} - \frac{2\,a^4\,{r_c}^3\,\overline{\zeta_c}^2}{\left({r_c}^2-\delta^2\right)^2\left(-{r_c}^2+\delta^2\right)^3} + \frac{8\,a^3\,i\,{r_c}^3\,\zeta_y}{\left({r_c}^2-\delta^2\right)^4}$$
$$-\frac{4\,a^2\,i\,{r_c}^3\,\zeta_c\,\zeta_y}{\left({r_c}^2-\delta^2\right)^4} + \frac{4\,a^3\,i\,r_c\,\delta^2\,\zeta_y}{\left({r_c}^2-\delta^2\right)^4} - \frac{2\,a^2\,i\,r_c\,{\zeta_c}^2\,\overline{\zeta_c}\,\zeta_y}{\left({r_c}^2-\delta^2\right)^4}\Bigg] \tag{3.114}$$

$$\int_\Sigma \frac{\partial w_3^s}{\partial \zeta_c}\, dz = \frac{-4\,a^3\,i\,\pi\,{r_c}^2\,\left(3\,{r_c}^2+4\,a\,\overline{\zeta_c}-\zeta_c\,\overline{\zeta_c}\right)}{\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^3} \tag{3.115}$$

$$\int_\Sigma \frac{\partial w_3^s}{\partial a}\, dz = \frac{-12\,a^2\,i\,\pi\,\left(2\,a\,{r_c}^2+\zeta_c\,\left(2\,{r_c}^2-\zeta_c\,\overline{\zeta_c}\right)\right)}{\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^2} \tag{3.116}$$

$$\int_\Sigma \frac{\partial w_3^s}{\partial r_c}\, dz = \frac{16\,a^3\,i\,\pi\,r_c\,\left({r_c}^2\,\zeta_c+a\,\left({r_c}^2+\zeta_c\,\overline{\zeta_c}\right)\right)}{\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^3} \tag{3.117}$$

$$\int_\Sigma \overline{\frac{\partial w_3^s}{\partial \zeta_c}}\, dz = \frac{4a^3\pi}{(r_c^2-\delta^2)^4}\Big[-4\,a\,{r_c}^2\,\overline{\zeta_c}\,\left({r_c}^2-\delta^2\right)+\overline{\zeta_c}^2\,\left({r_c}^2-\delta^2\right)\,\left((2+i)\,{r_c}^2-i\,\delta^2\right)$$
$$-a^2\,{r_c}^2\,\left(3\,i\,{r_c}^2+2\,\delta^2+4\,i\,\delta^2\right)\Big] \tag{3.118}$$

$$\int_\Sigma \overline{\frac{\partial w_3^s}{\partial a}}\, dz = \frac{4a^2\pi}{(r_c^2-\delta^2)^3}\left[-6\,a\,{r_c}^4 + 3\,{r_c}^4\,\zeta_c - i\,{r_c}^4\,\zeta_c - 3\,a^2\,{r_c}^2\,\overline{\zeta_c} + 6\,a^2\,i\,{r_c}^2\,\overline{\zeta_c} \right.$$
$$\left. +6\,a\,{r_c}^2\,\delta^2 - 3\,{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c} + 2\,i\,{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c} - i\,{\zeta_c}^3\,\overline{\zeta_c}^2\right] \quad (3.119)$$

$$\int_\Sigma \overline{\frac{\partial w_3^s}{\partial r_c}}\, dz = \frac{-8a^3\pi}{r_c(r_c^2-\delta^2)^4}\left[a^2\,{r_c}^2\,\overline{\zeta_c}\,\left((-1+2\,i)\;{r_c}^2 + (-1+i)\;\delta^2\right)\right.$$
$$+\zeta_c\,\left({r_c}^2-\delta^2\right)\,\left((1+i)\;{r_c}^4 + (1-2\,i)\;{r_c}^2\,\delta^2 + i\,{\zeta_c}^2\,\overline{\zeta_c}^2\right)$$
$$\left. -2\,a\,\left({r_c}^6 - {r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}^2\right)\right] \quad (3.120)$$

3.5.2 Bernoulli Effect Force Terms

Now I turn to the last term in Equation (3.74), involving cross-terms between the Kirchoff potentials. In order to evaluate this integral, I first consider its complex conjugate.

$$\int_\Sigma \frac{dw}{dz}\overline{\frac{dw}{dz}}\, dz = \overline{\int_\Sigma \frac{dw}{dz}\overline{\frac{dw}{dz}}\,\overline{dz}} \quad (3.121)$$

$$\begin{aligned}\int_\Sigma \frac{dw}{dz}\overline{\frac{dw}{dz}}\,\overline{dz} &= \int_\Sigma \frac{dw}{d\zeta}\frac{d\zeta}{dz}\overline{\frac{dw}{d\zeta}\frac{d\zeta}{dz}}\,\overline{dz} \\ &= \int_C \frac{dw}{d\zeta}\overline{\frac{dw}{d\zeta}}\frac{d\zeta}{dz}\,\overline{d\zeta} \\ &= \int_C \frac{dw}{d\zeta}\overline{\frac{dw}{d\zeta}}\left(\frac{(\zeta+\zeta_c)^2}{(\zeta+\zeta_c)^2-a^2}\right)\left(\frac{-r_c^2}{\zeta^2}\right)d\zeta \\ &= \int_C \frac{dw}{d\zeta}\overline{\frac{dw}{d\zeta}}\left(\frac{(\zeta+\zeta_c)^2}{(\zeta+a+\zeta_c)(\zeta-a+\zeta_c)}\right)\left(\frac{-r_c^2}{\zeta^2}\right)d\zeta \quad (3.122)\end{aligned}$$

The factors apart from $dw/d\zeta$ and $\overline{dw/d\zeta}$ introduce poles at $\zeta = 0$; at $\zeta = (-a-\zeta_c)$; and at $\zeta = (a-\zeta_c)$. The former two poles lie inside C, and the third lies exactly on C.

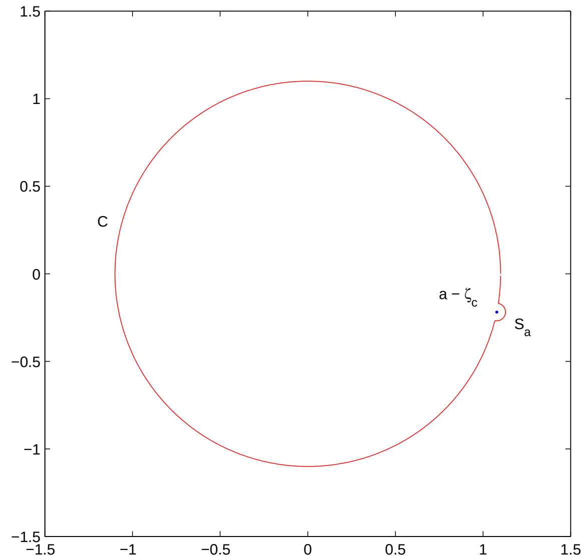


Figure 3.9: I consider a small excursion around the pole at $\zeta = a - \zeta_c$.

Now consider the contour $C_+ = C + S_a$ illustrated in Figure 3.9. The integral around the small semicircle S_a will be equal to the residue of the integrand at $\zeta = a - \zeta_c$ times πi. Therefore:

$$\begin{aligned}
\int_\Sigma \frac{dw}{dz}\overline{\frac{dw}{dz}}\,\overline{dz} &= \int_{C_+} \frac{dw}{d\zeta}\overline{\frac{dw}{d\zeta}}\left(\frac{(\zeta+\zeta_c)^2}{(\zeta+a+\zeta_c)(\zeta-a+\zeta_c)}\right)\left(\frac{-r_c^2}{\zeta^2}\right)d\zeta \\
&\quad - \int_{S_a} \frac{dw}{d\zeta}\overline{\frac{dw}{d\zeta}}\left(\frac{(\zeta+\zeta_c)^2}{(\zeta+a+\zeta_c)(\zeta-a+\zeta_c)}\right)\left(\frac{-r_c^2}{\zeta^2}\right)d\zeta \\
&= \int_{C_+} \frac{dw}{d\zeta}\overline{\frac{dw}{d\zeta}}\left(\frac{(\zeta+\zeta_c)^2}{(\zeta+a+\zeta_c)(\zeta-a+\zeta_c)}\right)\left(\frac{-r_c^2}{\zeta^2}\right)d\zeta \\
&\quad - i\pi\left[\frac{dw}{d\zeta}\overline{\frac{dw}{d\zeta}}\left(\frac{(\zeta+\zeta_c)^2}{(\zeta+a+\zeta_c)}\right)\left(\frac{-r_c^2}{\zeta^2}\right)\right]_{\zeta=a-\zeta_c} \\
&= \int_{C_+} \frac{dw}{d\zeta}\overline{\frac{dw}{d\zeta}}\left(\frac{(\zeta+\zeta_c)^2}{(\zeta+a+\zeta_c)(\zeta-a+\zeta_c)}\right)\left(\frac{-r_c^2}{\zeta^2}\right)d\zeta \\
&\quad + i\pi\frac{ar_c^2}{2(a-\zeta_c)^2}\left[\frac{dw}{d\zeta}\overline{\frac{dw}{d\zeta}}\right]_{\zeta=a-\zeta_c} \qquad (3.123)
\end{aligned}$$

As long as the Kutta condition is satisfied, $dw/d\zeta$ will equal zero at $\zeta = a - \zeta_c$, and the final term in Equation (3.123) vanishes.

Through the substitution $\overline{\zeta} \to r_c^2/\zeta$, I can replace $\overline{dw/d\zeta}$ with a function of ζ which takes on the same values on C_+. Then we can evaluate all terms of the integral on C_+ by the theory of residues. The results are given in Appendix B.

3.6 Moments on the Foil

The moment about the point $z = 0$ is given by

$$\begin{aligned} M &= \int_\Sigma x\,p\,dx + y\,p\,dy \\ &= \int_\Sigma p\,\mathsf{Re}\{z\,\overline{dz}\} \\ &= \mathsf{Re}\left\{\int_\Sigma p\,z\,\overline{dz}\right\} \end{aligned} \tag{3.124}$$

So employing Equation (3.70), we have

$$M = -\frac{1}{2}\mathsf{Re}\left\{\int_\Sigma \frac{\partial w}{\partial t} z\,\overline{dz} + \int_\Sigma \overline{\frac{\partial w}{\partial t}} z\,\overline{dz} + \int_\Sigma \frac{dw}{dz}\overline{\frac{dw}{dz}} z\,\overline{dz}\right\} \tag{3.125}$$

I have to make the replacements $z = F(\zeta)$ and $\overline{dz} = -(r_c^2/\zeta^2)(1 - a^2/((r_c^2/\zeta) + \overline{\zeta_c})^2)$. Then the following integrals can be readily evaluated in the ζ-plane using the theory of residues:

$$\begin{aligned} \int_\Sigma w_1 z\,\overline{dz} = -2\,a^2\,i\,\pi\,\Big(&{r_c}^6\,\zeta_c - {r_c}^4\,\left(2\,a^2 + 3\,{\zeta_c}^2\right)\overline{\zeta_c} - {\zeta_c}^2\,\left(a^2 + {\zeta_c}^2\right)\overline{\zeta_c}^3 \\ &+ {r_c}^2\,\overline{\zeta_c}\,\left(2\,a^4 + 3\,\zeta_c\,\left(a^2 + {\zeta_c}^2\right)\overline{\zeta_c}\right)\Big)/\left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)^3 \end{aligned} \tag{3.126}$$

$$\begin{aligned} \int_\Sigma \overline{w_1} z\,\overline{dz} &= \overline{\int_\Sigma w_1 \overline{z}\,dz} \\ &= -2\,i\,\pi\,\Big(a^6\,{r_c}^2\,\zeta_c + a^2\,\zeta_c\left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)^3 - {r_c}^2\,\zeta_c\left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)^3 \\ &\quad - a^4\,\left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)\left(-\left(\zeta_c\,\overline{\zeta_c}^2\right) + {r_c}^2\,\left(\zeta_c + \overline{\zeta_c}\right)\right)\Big)/\left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)^3 \end{aligned} \tag{3.127}$$

$$\int_\Sigma w_2 z\,\overline{dz} = -2\,a^2\,\pi\,\Big(-\left({r_c}^6\,\zeta_c\right) + {r_c}^4\,\left(2\,a^2 + 3\,{\zeta_c}^2\right)\,\overline{\zeta_c} + {\zeta_c}^2\,\left(a^2 + {\zeta_c}^2\right)\,\overline{\zeta_c}^3$$
$$+{r_c}^2\,\overline{\zeta_c}\,\left(2\,a^4 - 3\,\zeta_c\,\left(a^2 + {\zeta_c}^2\right)\,\overline{\zeta_c}\right)\Big)\,/\left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)^3 \quad (3.128)$$

$$\int_\Sigma \overline{w_2} z\,\overline{dz} = \overline{\int_\Sigma w_2\overline{z}\,dz}$$
$$= 2\,\pi\,\Big(a^6\,{r_c}^2\,\zeta_c + a^2\,\zeta_c\left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)^3 + {r_c}^2\,\zeta_c\left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)^3$$
$$-a^4\,\left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)\,\left({r_c}^2\,\overline{\zeta_c} - \zeta_c\,\left({r_c}^2 + \overline{\zeta_c}^2\right)\right)\Big)\,/\left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)^3 \quad (3.129)$$

$$\int_\Sigma w_3 z\,\overline{dz} = a^2\,\pi\,\Big(2\,a^4\,\overline{\zeta_c}^2\,\left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)^2 + 2\,{\zeta_c}^2\left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)^4 + a^6\,\left({r_c}^4 + 3\,{r_c}^2\,\zeta_c\,\overline{\zeta_c}\right)$$
$$+a^2\left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)^2\,\left({r_c}^4 - 7\,{r_c}^2\,\zeta_c\,\overline{\zeta_c} + 4\,{\zeta_c}^2\,\overline{\zeta_c}^2\right)\Big)\,/\left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)^4 \quad (3.130)$$

$$\int_\Sigma \overline{w_3} z\,\overline{dz} = \overline{\int_\Sigma w_3\overline{z}\,dz}$$
$$= \pi\,\Big(2\,a^6\,\overline{\zeta_c}^2\left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)^2 + 2\,a^2\,{\zeta_c}^2\left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)^4 + 2\,{r_c}^2\,\zeta_c\,\overline{\zeta_c}\left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)^4 - a^8\,\left({r_c}^4 + {r_c}^2\,\zeta_c\,\overline{\zeta_c}\right)$$
$$+a^4\left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)^2\,\left(3\,{r_c}^4 - 5\,{r_c}^2\,\zeta_c\,\overline{\zeta_c} + 4\,{\zeta_c}^2\,\overline{\zeta_c}^2\right)\,\Big)\,/\left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)^4 \quad (3.131)$$

In order to evaluate the terms involving $\log(\zeta/r_c)$, I make the substitution $\zeta = r_c e^{i\theta}$ (and $d\zeta = ir_c e^{i\theta}\,d\theta$) and integrate with respect to θ. The integration with respect to θ must respect the branch cut at the trailing edge of the foil and therefore the limits of integration must be $\theta = [-\beta, 2\pi - \beta]$ where $\beta = \arctan(\zeta_y/(a - \zeta_x))$.

$$
\begin{aligned}
\int_\Sigma \log(\frac{\zeta}{r_c})\, z\, \overline{dz} &= \int_C \log(\frac{\zeta}{r_c}) \left(\zeta + \zeta_c + \frac{a^2}{(\zeta + \zeta_c)}\right) \left(-\left(\frac{{r_c}^2}{\zeta^2}\right) + \frac{a^2\, {r_c}^2}{\left({r_c}^2 + \zeta\, \overline{\zeta_c}\right)^2}\right) d\zeta \\
= \frac{2\,\pi\, \left(2\, i\, a^3 - \beta\, {r_c}^2\, \zeta_c + \pi\, {r_c}^2\, \zeta_c + 2\, i\, a\, {\zeta_c}^2\right)}{\zeta_c} &- \frac{2\, (-i\, a^2\, \pi\, {r_c}^2 - a^2\, \beta\, \pi\, {r_c}^2 + a^2\, \pi^2\, {r_c}^2)}{\overline{\zeta_c}^2} \\
- \frac{2\, i\, (a\, \pi\, {r_c}^2 + a^2\, \pi\, \zeta_c)}{\overline{\zeta_c}} &- \frac{2\, i\, \pi\, \left(a^2 + {\zeta_c}^2\right)\, \overline{\zeta_c}}{\zeta_c} - \frac{2\, i\, (-(a^3\, \pi\, {r_c}^2) + a^4\, \pi\, \zeta_c)}{\zeta_c\, \left(-{r_c}^2 + \zeta_c\, \overline{\zeta_c}\right)} \\
&+ 2\, i\, a^2\, \pi\, {r_c}^2 \left(\overline{\zeta_c}^{-2} - \frac{a^2}{\left({r_c}^2 - \zeta_c\, \overline{\zeta_c}\right)^2}\right) \log(\frac{a}{\overline{\zeta_c}}) \\
&- 2\, i\, a^2\, \pi\, {r_c}^2 \left({\zeta_c}^{-2} - \frac{a^2}{\left({r_c}^2 - \zeta_c\, \overline{\zeta_c}\right)^2}\right) \log(\frac{{r_c}^2 + a\, \zeta_c - \zeta_c\, \overline{\zeta_c}}{{r_c}^2}) \quad (3.132)
\end{aligned}
$$

$$
\begin{aligned}
\int_\Sigma \log(\frac{\zeta}{r_c}) \overline{z}\, dz &= \int_C \log(\frac{\zeta}{r_c}) \left(\frac{{r_c}^2}{\zeta} + \overline{\zeta_c} + \frac{a^2\, \zeta}{{r_c}^2 + \zeta\, \overline{\zeta_c}}\right) \left(1 - \frac{a^2}{(\zeta + \zeta_c)^2}\right) d\zeta \\
= \frac{-2\, \pi\, {r_c}^2\, (-a + \zeta_c)\, (i\, a - a\, \beta + a\, \pi - \beta\, \zeta_c + \pi\, \zeta_c)}{{\zeta_c}^2} &+ \frac{2\, i\, a^3\, \pi\, {r_c}^2}{({r_c}^2 + a\, \zeta_c)\, \overline{\zeta_c}} - \frac{2\, i\, a^3\, \pi\, {r_c}^2}{\zeta_c\, \left(-{r_c}^2 + \zeta_c\, \overline{\zeta_c}\right)} \\
- \frac{2\, i\, \left(-2\, a^3\, \pi\, {r_c}^4 + a\, \pi\, {r_c}^6 - 2\, a^4\, \pi\, {r_c}^2\, \zeta_c + 2\, a^2\, \pi\, {r_c}^4\, \zeta_c + 2\, a^3\, \pi\, {r_c}^2\, {\zeta_c}^2\right)}{\zeta_c\, ({r_c}^2 + a\, \zeta_c)\, \left(-{r_c}^2 - a\, \zeta_c + \zeta_c\, \overline{\zeta_c}\right)} \\
&+ 2\, i\, a^2\, \pi\, {r_c}^2 \left({\zeta_c}^{-2} - \frac{a^2}{\left({r_c}^2 - \zeta_c\, \overline{\zeta_c}\right)^2}\right) \log(\frac{a}{\zeta_c}) \\
&- 2\, i\, a^2\, \pi\, {r_c}^2 \left(\overline{\zeta_c}^{-2} - \frac{a^2}{\left({r_c}^2 - \zeta_c\, \overline{\zeta_c}\right)^2}\right) \log(\frac{{r_c}^2 + a\, \overline{\zeta_c} - \zeta_c\, \overline{\zeta_c}}{{r_c}^2}) \quad (3.133)
\end{aligned}
$$

The integral of $\log((\zeta + \zeta_c)/\zeta)\, z\, \overline{dz}$, which occurs in the shape deformation potentials, can be best evaluated using the contour in Figure 3.10. I must account for the pole at $\zeta = -r_c^2/\overline{\zeta_c}$, which occurs in the factor replacing $\overline{dz}$. The integrand drops off as $\mathcal{O}\left(1/\zeta^2\right)$ for large ζ, so there is no contribution from C_∞, and the integrand is continuous in the vicinity of C_{b2} and C_x, so there is no contribution there. The only contribution comes from C_p, which can be evaluated by taking the residue of the

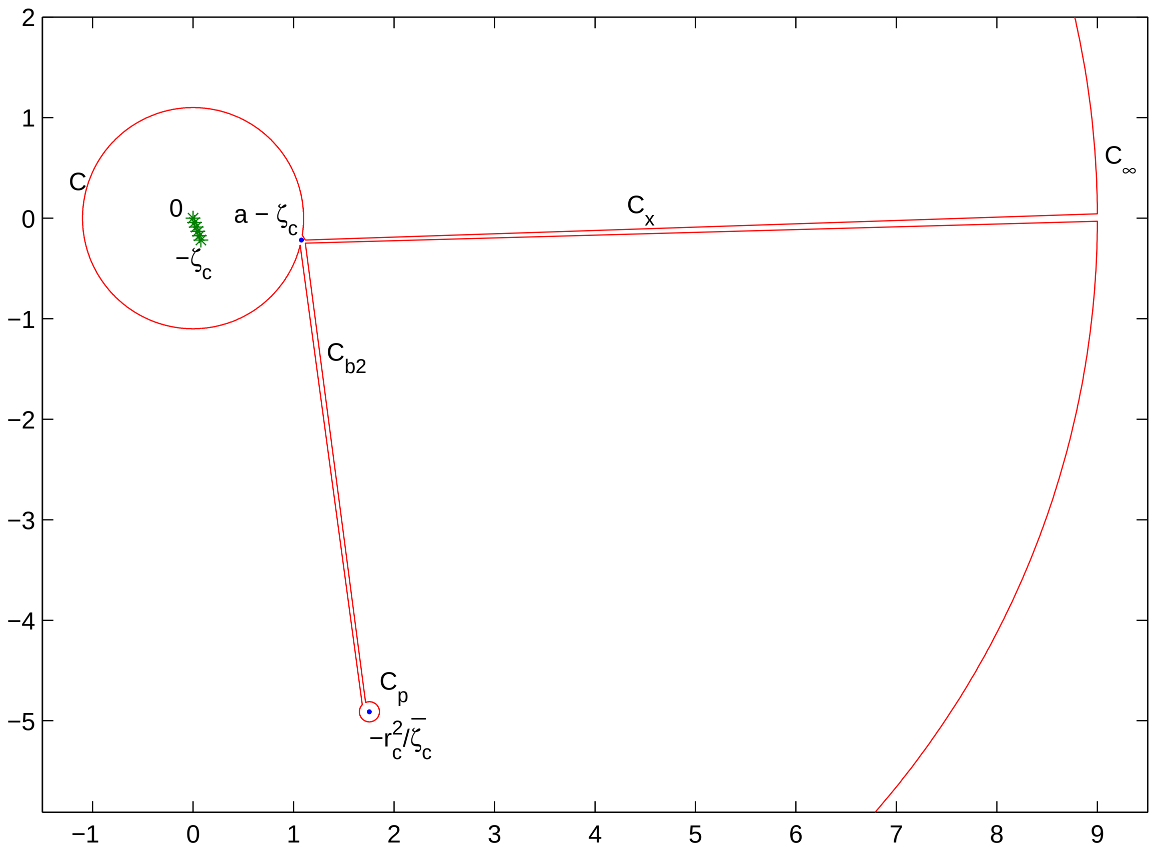


Figure 3.10: The contour C_p encloses a pole at $\zeta = -r_c^2/\overline{\zeta_c}$. There is a branch cut between the origin and $\zeta = -\zeta_c$. If integration is counterclockwise in each case, then $\int_C = \int_{C_\infty} - \int_{C_x} - \int_{C_p} - \int_{C_{b2}}$.

integrand at $\zeta = -r_c^2/\overline{\zeta_c}$.

$$\int_\Sigma \log(\frac{\zeta+\zeta_c}{\zeta})\, z\,\overline{dz} = \int_C \log(\frac{\zeta+\zeta_c}{\zeta})\left(\zeta+\zeta_c+\frac{a^2}{(\zeta+\zeta_c)}\right)\left(-\left(\frac{{r_c}^2}{\zeta^2}\right)+\frac{a^2\,{r_c}^2}{\zeta^2\left(\frac{{r_c}^2}{\zeta}+\overline{\zeta_c}\right)^2}\right)d\zeta$$

$$= -\int_{C_p} \log(\frac{\zeta+\zeta_c}{\zeta})\left(\zeta+\zeta_c+\frac{a^2}{(\zeta+\zeta_c)}\right)\left(-\left(\frac{{r_c}^2}{\zeta^2}\right)+\frac{a^2\,{r_c}^2}{\zeta^2\left(\frac{{r_c}^2}{\zeta}+\overline{\zeta_c}\right)^2}\right)d\zeta$$

$$= \frac{-2\,a^2\,i\,\pi\,\zeta_c\left({r_c}^4-2\,{r_c}^2\,\zeta_c\,\overline{\zeta_c}+a^2\,\overline{\zeta_c}^2+{\zeta_c}^2\,\overline{\zeta_c}^2\right)}{\overline{\zeta_c}\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^2}$$

$$+\frac{2\,a^2\,i\,\pi\,{r_c}^2\,\left({r_c}^2+a\,\overline{\zeta_c}-\zeta_c\,\overline{\zeta_c}\right)\left(-{r_c}^2+a\,\overline{\zeta_c}+\zeta_c\,\overline{\zeta_c}\right)\log\left(({r_c}^2-\zeta_c\,\overline{\zeta_c})/r_c^2\right)}{\overline{\zeta_c}^2\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^2} \quad (3.134)$$

The following integral does have a nonvanishing contribution from C_∞ as well as C_p.

$$\int_\Sigma \log(\frac{\zeta+\zeta_c}{\zeta})\,\overline{z}\,dz = \int_C \log(\frac{\zeta+\zeta_c}{\zeta})\left(1-\frac{a^2}{(\zeta+\zeta_c)^2}\right)\left(\frac{{r_c}^2}{\zeta}+\overline{\zeta_c}+\frac{a^2\,\zeta}{{r_c}^2+\zeta\,\overline{\overline{\zeta_c}}}\right)d\zeta$$

$$= \int_{C_\infty} \log(\frac{\zeta+\zeta_c}{\zeta})\left(1-\frac{a^2}{(\zeta+\zeta_c)^2}\right)\left(\frac{{r_c}^2}{\zeta}+\overline{\zeta_c}+\frac{a^2\,\zeta}{{r_c}^2+\zeta\,\overline{\overline{\zeta_c}}}\right)d\zeta$$

$$- \int_{C_p} \log(\frac{\zeta+\zeta_c}{\zeta})\left(1-\frac{a^2}{(\zeta+\zeta_c)^2}\right)\left(\frac{{r_c}^2}{\zeta}+\overline{\zeta_c}+\frac{a^2\,\zeta}{{r_c}^2+\zeta\,\overline{\overline{\zeta_c}}}\right)d\zeta$$

$$= 2\,i\,\pi\,\zeta_c\left(\frac{a^2}{\overline{\overline{\zeta_c}}}+\overline{\zeta_c}\right)-\frac{2\,i\,a^2\,\pi\,{r_c}^2\left(-{r_c}^4+2\,{r_c}^2\,\zeta_c\,\overline{\zeta_c}+a^2\,\overline{\zeta_c}^2-{\zeta_c}^2\,\overline{\zeta_c}^2\right)\log\left((r_c^2-\zeta_c\,\overline{\zeta_c})/r_c^2\right)}{\overline{\zeta_c}^2\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^2} \quad (3.135)$$

Finally, I can conclude that

$$\int_\Sigma \log(\frac{\zeta+\zeta_c}{r_c})\, z\,\overline{dz} = \int_\Sigma \left[\log(\frac{\zeta+\zeta_c}{\zeta})+\log(\frac{\zeta}{r_c})\right] z\,\overline{dz} \quad (3.136)$$

$$\int_\Sigma \log(\frac{\zeta+\zeta_c}{r_c})\,\overline{z}\,dz = \int_\Sigma \left[\log(\frac{\zeta+\zeta_c}{\zeta})+\log(\frac{\zeta}{r_c})\right]\overline{z}\,dz \quad (3.137)$$

I now have the tools to evaluate the moments arising from the shape deformations and from the central vortex potential. It immediately follows that

$$\int_\Sigma w_4 z\,\overline{dz} = \frac{2\,i\,\pi\,\left(2\,i\,a^3-\beta\,{r_c}^2\,\zeta_c+\pi\,{r_c}^2\,\zeta_c+2\,i\,a\,{\zeta_c}^2\right)}{\zeta_c} - \frac{2\,i\,\left(-i\,a^2\,\pi\,{r_c}^2-a^2\,\beta\,\pi\,{r_c}^2+a^2\,\pi^2\,{r_c}^2\right)}{\overline{\zeta_c}^2}$$
$$+\frac{2\,\left(a\,\pi\,{r_c}^2+a^2\,\pi\,\zeta_c\right)}{\overline{\zeta_c}}+\frac{2\,\pi\,\left(a^2+{\zeta_c}^2\right)\,\overline{\zeta_c}}{\zeta_c}+\frac{2\,\left(-\left(a^3\,\pi\,{r_c}^2\right)+a^4\,\pi\,\zeta_c\right)}{\zeta_c\,\left(-{r_c}^2+\zeta_c\,\overline{\zeta_c}\right)}$$
$$+2\,a^2\,\pi\,{r_c}^2\,\left(-\overline{\zeta_c}^{-2}+\frac{a^2}{\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^2}\right)\,\log(\frac{a}{\overline{\zeta_c}})$$
$$+2\,a^2\,\pi\,{r_c}^2\,\left({\zeta_c}^{-2}-\frac{a^2}{\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^2}\right)\,\log(\frac{{r_c}^2+a\,\zeta_c-\zeta_c\,\overline{\zeta_c}}{{r_c}^2}) \quad (3.138)$$

$$\int_\Sigma w_1^s\,z\,\overline{dz} = 2\,a^2\,\pi\,\Big(-i\,\left(r_c-\zeta_c\right)\,\left(r_c+\zeta_c\right)\,\left({r_c}^2+{\zeta_c}^2\right)\,\overline{\zeta_c}^2\,\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^5$$
$$+a^4\,\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^2\,\Big(i\,{r_c}^6\,\overline{\zeta_c}^2+i\,{\zeta_c}^3\,\overline{\zeta_c}^5+{r_c}^4\,\zeta_c\,\left(i\,\overline{\zeta_c}^3+2\,\left(i+\beta-\pi\right)\,{\zeta_c}^2\,\left(\zeta_c-i\,\zeta_y\right)\right)$$
$$-i\,{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}\,\left(4\,{\zeta_c}^3+2\,{\zeta_c}^2\,\left(\overline{\zeta_c}-i\,\zeta_y\right)+\overline{\zeta_c}^2\,\left(\overline{\zeta_c}-2\,i\,\zeta_y\right)\right)\Big)$$
$$-i\,a^6\,{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}^2\,\left({r_c}^4+2\,{\zeta_c}^2\,\overline{\zeta_c}\,\left(3\,\zeta_c+\overline{\zeta_c}-i\,\zeta_y\right)+{r_c}^2\,\zeta_c\,\left(-2\,\zeta_c+\overline{\zeta_c}+2\,i\,\zeta_y\right)\right)$$
$$+2\,a^5\,{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}^2\,\left({r_c}^2-2\,\zeta_c\,\overline{\zeta_c}\right)\,\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)\,\left(i\,\zeta_c+\zeta_y\right)$$
$$+2\,a^3\,{\zeta_c}^3\,\overline{\zeta_c}\,\left(-{r_c}^2+2\,\zeta_c\,\overline{\zeta_c}\right)\,\left({r_c}^3-r_c\,\zeta_c\,\overline{\zeta_c}\right)^2\,\left(i\,\zeta_c+\zeta_y\right)$$
$$+2\,a^2\,{r_c}^2\,\zeta_c\,\overline{\zeta_c}\,\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^2\,\Big(-i\,\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^3+{\zeta_c}^3\,\overline{\zeta_c}\,\left(\left(-\beta+\pi\right)\,{r_c}^2-i\,\zeta_c\,\overline{\zeta_c}\right)$$
$$+i\,{\zeta_c}^2\,\overline{\zeta_c}\,\left(\left(\beta-\pi\right)\,{r_c}^2+i\,\zeta_c\,\overline{\zeta_c}\right)\,\zeta_y\Big)$$
$$-2\,i\,a^2\,{r_c}^4\,\left({r_c}^2+\left(a-\zeta_c\right)\,\overline{\zeta_c}\right)\,\left({r_c}^2-\left(a+\zeta_c\right)\,\overline{\zeta_c}\right)\,\Big({r_c}^6-3\,{r_c}^4\,\zeta_c\,\overline{\zeta_c}+3\,{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}^2$$
$$+{\zeta_c}^3\,\left(-\overline{\zeta_c}^3+a^2\,\left(\zeta_c+\overline{\zeta_c}\right)\right)\Big)\,\log(\frac{{r_c}^2-\zeta_c\,\overline{\zeta_c}}{{r_c}^2})$$
$$-2\,i\,a^4\,{r_c}^4\,\zeta_c\,\left(\zeta_c-i\,\zeta_y\right)\,\Big(-\Big({\zeta_c}^2\,\left({r_c}^2+\left(a-\zeta_c\right)\,\overline{\zeta_c}\right)\,\left({r_c}^2-\left(a+\zeta_c\right)\,\overline{\zeta_c}\right)\,\log(\frac{a}{\overline{\zeta_c}})\Big)$$
$$+\left({r_c}^2+\zeta_c\,\left(a-\overline{\zeta_c}\right)\right)\,\overline{\zeta_c}^2\,\left({r_c}^2-\zeta_c\,\left(a+\overline{\zeta_c}\right)\right)\,\log(\frac{{r_c}^2+a\,\zeta_c-\zeta_c\,\overline{\zeta_c}}{{r_c}^2})\Big)\Big)$$
$$/\left({\zeta_c}^3\,\overline{\zeta_c}^2\,\left(-{r_c}^2+\zeta_c\,\overline{\zeta_c}\right)^5\right) \quad (3.139)$$

$$
\begin{aligned}
\int_\Sigma \overline{w_1^s}\, z\, \overline{dz} = 2\,\pi\, \Big(\zeta_c\, \Big(i\, a\, {r_c}^2\, {\zeta_c}^2\, \overline{\zeta_c}^4\, ({r_c}^2 - \zeta_c\, \overline{\zeta_c})^5 + i\, {r_c}^2\, {\zeta_c}^2\, \overline{\zeta_c}^3\, ({r_c}^2 - \zeta_c\, \overline{\zeta_c})^6 \\
+ i\, a^3\, \zeta_c\, \overline{\zeta_c}\, ({r_c}^2 - \zeta_c\, \overline{\zeta_c})^5\, \Big({r_c}^4 - \zeta_c\, \overline{\zeta_c}^3\Big) + i\, a^2\, \zeta_c\, ({r_c}^2 - \zeta_c\, \overline{\zeta_c})^6\, \Big({r_c}^4 - \zeta_c\, \overline{\zeta_c}^3\Big) \\
+ i\, a^9\, {r_c}^2\, \zeta_c\, \overline{\zeta_c}^3\, \Big({r_c}^4 + \zeta_c\, \overline{\zeta_c}^3\Big) + a^6\, ({r_c}^2 - \zeta_c\, \overline{\zeta_c})^3\, \Big(-i\, {r_c}^6\, \zeta_c - i\, \zeta_c\, \overline{\zeta_c}^6 \\
+ i\, {r_c}^2\, \overline{\zeta_c}^2\, \Big({\zeta_c}^3 - {\zeta_c}^2\, \overline{\zeta_c} + \zeta_c\, \overline{\zeta_c}^2 + 2\, \overline{\zeta_c}^3\Big) \\
+ {r_c}^4\, \zeta_c\, \overline{\zeta_c}\, (-i\, \zeta_c + 2\, (i + \beta - \pi)\, (\overline{\zeta_c} + i\, \zeta_y))\,) \\
+ a^7\, \overline{\zeta_c}\, ({r_c}^2 - \zeta_c\, \overline{\zeta_c}\,)^2\, \Big(-i\, {r_c}^6\, \zeta_c - i\, \zeta_c\, \overline{\zeta_c}^6 + i\, {r_c}^2\, \overline{\zeta_c}^2\, \Big({\zeta_c}^3 - {\zeta_c}^2\, \overline{\zeta_c} + \zeta_c\, \overline{\zeta_c}^2 + 2\, \overline{\zeta_c}^3\Big) \\
+ {r_c}^4\, \overline{\zeta_c}\, (-i\, {\zeta_c}^2 + 2\, (\beta - \pi)\, \zeta_c\, (\overline{\zeta_c} + i\, \zeta_y) + 2\, i\, \overline{\zeta_c}\, (\overline{\zeta_c} + i\, \zeta_y))\,) \\
+ i\, a^8\, {r_c}^2\, \zeta_c\, \overline{\zeta_c}^2\, ({r_c}^2 - \zeta_c\, \overline{\zeta_c})\, \Big({r_c}^4 + \zeta_c\, \overline{\zeta_c}^3 + 2\, {r_c}^2\, \overline{\zeta_c}\, (\overline{\zeta_c} + i\, \zeta_y)\Big) \\
+ a^4\, {r_c}^2\, \overline{\zeta_c}\, ({r_c}^2 - \zeta_c\, \overline{\zeta_c})^3\, \Big(2\, (-\beta + \pi)\, {r_c}^2\, \zeta_c\, \overline{\zeta_c}^3 \\
+ i\, \Big(2\, {r_c}^6 - 6\, {r_c}^4\, \zeta_c\, \overline{\zeta_c} + {\zeta_c}^2\, \overline{\zeta_c}^3\, (-\zeta_c + \overline{\zeta_c}) + {r_c}^2\, \zeta_c\, \overline{\zeta_c}^2\, (7\, \zeta_c + \overline{\zeta_c} - 2\, \beta\, \zeta_y + 2\, \pi\, \zeta_y)\Big)\Big) \\
+ a^5\, {r_c}^2\, \overline{\zeta_c}^2\, ({r_c}^2 - \zeta_c\, \overline{\zeta_c})^2\, \Big(-2\, {r_c}^2\, \zeta_c\, \overline{\zeta_c}\, \Big((\beta - \pi)\, \overline{\zeta_c}^2 + \zeta_c\, \zeta_y\Big) \\
+ i\, \Big(2\, {r_c}^6 - 6\, {r_c}^4\, \zeta_c\, \overline{\zeta_c} + {\zeta_c}^2\, \overline{\zeta_c}^3\, (-\zeta_c + \overline{\zeta_c}) + {r_c}^2\, \zeta_c\, \overline{\zeta_c}^2\, (9\, \zeta_c + \overline{\zeta_c} - 2\, \beta\, \zeta_y + 2\, \pi\, \zeta_y)\Big)\Big)\Big) \\
+ 2\, a^6\, {r_c}^4\, {\zeta_c}^2\, \overline{\zeta_c}\, ({r_c}^2 + (a - \zeta_c)\, \overline{\zeta_c}\,)^2\, (-{r_c}^2 + (a + \zeta_c)\, \overline{\zeta_c})\, (-i\, \overline{\zeta_c} + \zeta_y)\, \log(\frac{a}{\overline{\zeta_c}}) \\
+ 2\, a^4\, {r_c}^4\, ({r_c}^2 + \zeta_c\, (a - \overline{\zeta_c}))\, ({r_c}^2 + (a - \zeta_c)\, \overline{\zeta_c})\, ({r_c}^2 - \zeta_c\, (a + \overline{\zeta_c}))\, \Big(i\, \Big({r_c}^6 - 3\, {r_c}^4\, \zeta_c\, \overline{\zeta_c} \\
+ 3\, {r_c}^2\, {\zeta_c}^2\, \overline{\zeta_c}^2 + \overline{\zeta_c}^3\, (-{\zeta_c}^3 + a^2\, (\zeta_c + \overline{\zeta_c}))\Big)\, \log(\frac{{r_c}^2 - \zeta_c\, \overline{\zeta_c}}{{r_c}^2}) \\
+ a^2\, \overline{\zeta_c}^3\, (-i\, \overline{\zeta_c} + \zeta_y)\, \log(\frac{{r_c}^2 + a\, \zeta_c - \zeta_c\, \overline{\zeta_c}}{{r_c}^2})\Big)\Big) \\
/ \Big({\zeta_c}^2\, \overline{\zeta_c}^3\, ({r_c}^2 + (a - \zeta_c)\, \overline{\zeta_c})\, ({r_c}^2 - \zeta_c\, \overline{\zeta_c})^5\Big) \quad (3.140)
\end{aligned}
$$

The other shape deformation integrals can be evaluated similarly.

For the integrals involving the wake vortices:

$$\int_\Sigma w_5^k z\, \overline{dz} = \int_C w_5^k \left(\zeta + \zeta_c + \frac{a^2}{(\zeta + \zeta_c)}\right) \left(\frac{-r_c^2}{\zeta^2} + \frac{a^2 r_c^2}{(r_c^2 + \zeta\overline{\zeta_c})^2}\right) d\zeta \qquad (3.141)$$

I must consider the contour illustrated in Figure 3.11. Not only do I have to account

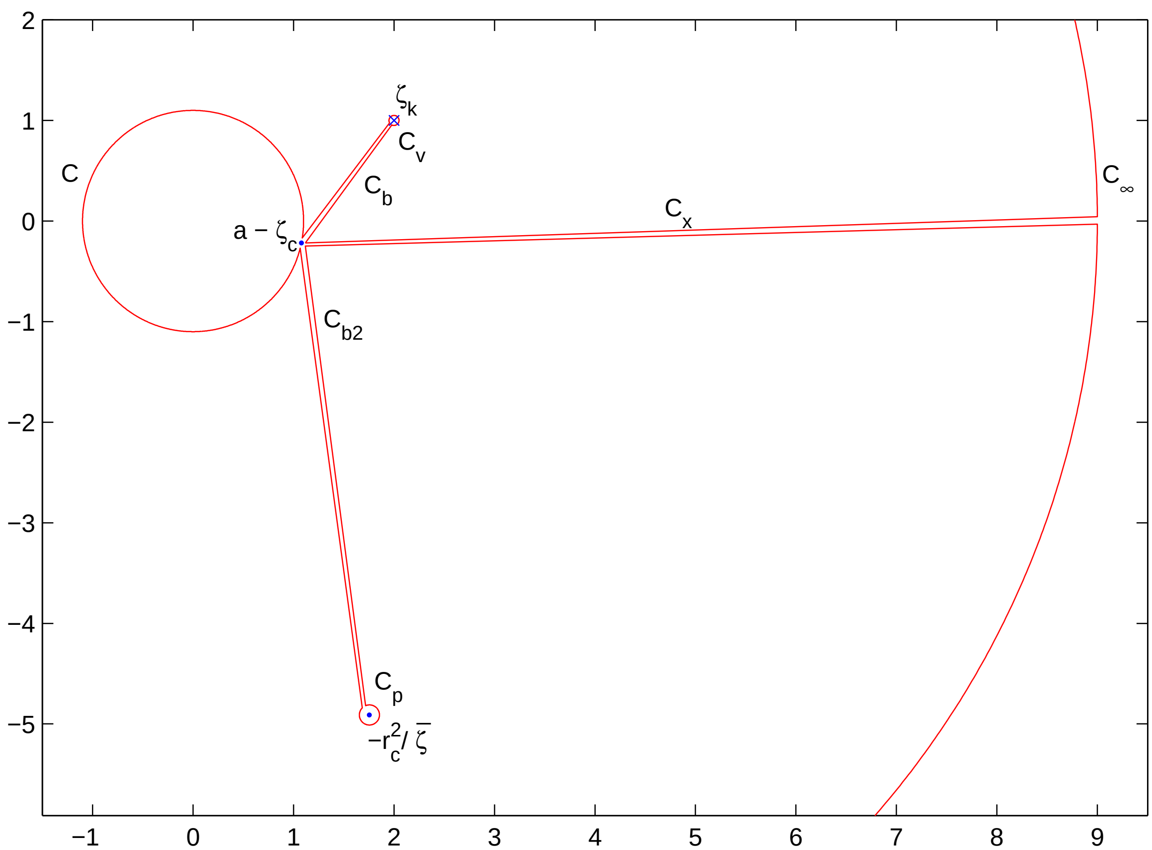


Figure 3.11: The contour C_p encloses a pole at $\zeta = -r_c^2/\overline{\zeta_c}$. If the direction of integration is counterclockwise in each case, then $\int_C = \int_{C_\infty} - \int_{C_x} - \int_{C_p} - \int_{C_v} - \int_{C_b}$.

for the pole at $\zeta = -r_c^2/\overline{\zeta_c}$, which occurs in the factor replacing $\overline{dz}$, but I must also allow for the branch cut between the wake vortex and its image.

Since the integral around the entire contour in Figure 3.11 must be zero, I have

$$\int_C w_5^k \left(\zeta + \zeta_c + \frac{a^2}{(\zeta+\zeta_c)}\right)\left(\frac{-r_c^2}{\zeta^2} + \frac{a^2 r_c^2}{(r_c^2 + \zeta\overline{\zeta_c})^2}\right) d\zeta$$
$$= \left(\int_{C_\infty} - \int_{C_x} - \int_{C_p} - \int_{C_v} - \int_{C_b}\right) w_5^k \left(\zeta + \zeta_c + \frac{a^2}{(\zeta+\zeta_c)}\right)\left(\frac{-r_c^2}{\zeta^2} + \frac{a^2 r_c^2}{(r_c^2 + \zeta\overline{\zeta_c})^2}\right) d\zeta \tag{3.142}$$

Recalling the series expansion for w_5^k in Equation (3.91),

$$w_5^k = i\log(\frac{-r_c}{\zeta_k}) - i\sum_{n=1}^{\infty} \zeta_k^n \frac{1}{n\zeta^n} + \left(i\sum_{n=1}^{\infty}\left(\frac{r_c^2}{\overline{\overline{\zeta_k}}}\right)^n \frac{1}{n\zeta^n}\right) \tag{3.143}$$

I isolate the part of the integrand of Equation (3.141) that goes as $1/\zeta$ and conclude

$$\int_{C_\infty} w_5^k \left(\zeta + \zeta_c + \frac{a^2}{(\zeta+\zeta_c)}\right)\left(\frac{-r_c^2}{\zeta^2} + \frac{a^2 r_c^2}{(r_c^2 + \zeta\overline{\zeta_c})^2}\right) d\zeta$$
$$= 2\pi\log(-\frac{r}{\zeta_k})\left(r_c^2 - \frac{a^2 r_c^2}{\overline{\zeta_c}^2}\right) \tag{3.144}$$

The integrand is continuous on C_x and C_b, so there is no contribution from those integrals. Since C_v encloses only a logarithmic singularity, it gives no contribution as the radius of C_v approaches zero. On C_b, w_5^k is larger by the amount 2π on the lower side than on the upper side. So,

$$\int_{C_b} w_5^k \left(\zeta + \zeta_c + \frac{a^2}{(\zeta+\zeta_c)}\right)\left(\frac{-r_c^2}{\zeta^2} + \frac{a^2 r_c^2}{(r_c^2 + \zeta\overline{\zeta_c})^2}\right) d\zeta$$
$$= 2\pi\int_{a-\zeta_c}^{\zeta_k} \left(\zeta + \zeta_c + \frac{a^2}{(\zeta+\zeta_c)}\right)\left(\frac{-r_c^2}{\zeta^2} + \frac{a^2 r_c^2}{(r_c^2 + \zeta\overline{\zeta_c})^2}\right) d\zeta \tag{3.145}$$

This is an elementary line integral in ζ which equals

$$\int_{C_b} w_5^k \left(\zeta + \zeta_c + \frac{a^2}{(\zeta+\zeta_c)}\right)\left(\frac{-r_c^2}{\zeta^2} + \frac{a^2 r_c^2}{(r_c^2 + \zeta\overline{\zeta_c})^2}\right) d\zeta$$
$$= -2\pi \left[{r_c}^2 \left(\frac{-a^2 - {\zeta_c}^2}{\zeta\,\zeta_c} + \frac{a^2\left({r_c}^4 - 2\,{r_c}^2\,\zeta_c\,\overline{\zeta_c} + a^2\,\overline{\zeta_c}^2 + {\zeta_c}^2\,\overline{\zeta_c}^2\right)}{\overline{\zeta_c}^2\left({r_c}^2 + \zeta\,\overline{\zeta_c}\right)\left(-{r_c}^2 + \zeta_c\,\overline{\zeta_c}\right)} + \frac{\left(-a^2 + {\zeta_c}^2\right)\log(\zeta)}{{\zeta_c}^2} \right.\right.$$
$$+ \frac{\left(a^2\,{r_c}^4 - a^4\,{\zeta_c}^2 - 2\,a^2\,{r_c}^2\,\zeta_c\,\overline{\zeta_c} + a^2\,{\zeta_c}^2\,\overline{\zeta_c}^2\right)\log(\zeta + \zeta_c)}{{\zeta_c}^2\left(-{r_c}^2 + \zeta_c\,\overline{\zeta_c}\right)^2}$$
$$\left.\left. + \frac{\left(-\left(a^2\,{r_c}^4\right) + 2\,a^2\,{r_c}^2\,\zeta_c\,\overline{\zeta_c} + a^4\,\overline{\zeta_c}^2 - a^2\,{\zeta_c}^2\,\overline{\zeta_c}^2\right)\log({r_c}^2 + \zeta\,\overline{\zeta_c})}{\overline{\zeta_c}^2\left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)^2} \right)\right]_{\zeta=(a-\zeta_c)}^{\zeta=\zeta_k} \quad (3.146)$$

Finally, the contribution from C_p is just $2\pi i$ times the residue of the integrand at $\zeta = -r_c^2/\overline{\zeta_c}$.

$$\int_{C_p} w_5^k \left(\zeta + \zeta_c + \frac{a^2}{(\zeta+\zeta_c)}\right)\left(\frac{-r_c^2}{\zeta^2} + \frac{a^2 r_c^2}{(r_c^2 + \zeta\overline{\zeta_c})^2}\right) d\zeta$$
$$= 2\,a^2\,\pi\,\left(\overline{\zeta_c}\left(-{r_c}^2 + \zeta_c\,\overline{\zeta_c}\right)\left({r_c}^4 - 2\,{r_c}^2\,\zeta_c\,\overline{\zeta_c} + \left(a^2 + {\zeta_c}^2\right)\overline{\zeta_c}^2\right)\left({r_c}^2 - \zeta_k\,\overline{\zeta_k}\right)\right.$$
$$\left. -{r_c}^2\left({r_c}^2 + (a - \zeta_c)\,\overline{\zeta_c}\right)\left({r_c}^2 - (a + \zeta_c)\,\overline{\zeta_c}\right)\left({r_c}^2 + \overline{\zeta_c}\,\zeta_k\right)\left(\overline{\zeta_c} + \overline{\zeta_k}\right)\log\left(-\frac{\left({r_c}^2 + \overline{\zeta_c}\,\zeta_k\right)\overline{\zeta_k}}{r_c\,\zeta_k\,\left(\overline{\zeta_c} + \overline{\zeta_k}\right)}\right)\right)$$
$$/\left(\overline{\zeta_c}^2\left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)^2\left({r_c}^2 + \overline{\zeta_c}\,\zeta_k\right)\left(\overline{\zeta_c} + \overline{\zeta_k}\right)\right) \quad (3.147)$$

Collecting all the terms, I find

$$\int_{\Sigma} w_5^k\, z\, \overline{dz} =$$
$$2\pi\left[\mathcal{A} + \mathcal{B}\log(-\frac{r_c}{\zeta_k}) + \overline{\mathcal{B}}\log(\frac{\zeta_k}{a - \zeta_c}) + \mathcal{C}\log(-\frac{(r_c^2 + a\overline{\zeta_c} - \delta^2)\overline{\zeta_k}}{r_c\zeta_k(\overline{\zeta_c} + \overline{\zeta_k})}) + \overline{\mathcal{C}}\log(\frac{\zeta_c + \zeta_k}{a})\right] \quad (3.148)$$

where

$$\mathcal{A} = -\left(r_c{}^2\left(\frac{a^2+\zeta_c{}^2}{-(a\,\zeta_c)+\zeta_c{}^2}+\frac{a^2\left(r_c{}^4-2\,r_c{}^2\,\zeta_c\,\overline{\zeta_c}+\left(a^2+\zeta_c{}^2\right)\,\overline{\zeta_c}^2\right)}{\overline{\zeta_c}^2\,\left(r_c{}^2+(a-\zeta_c)\,\overline{\zeta_c}\right)\,\left(-r_c{}^2+\zeta_c\,\overline{\zeta_c}\right)}\right)\right)$$
$$+r_c{}^2\left(-\left(\frac{a^2+\zeta_c{}^2}{\zeta_c\,\zeta_k}\right)+\frac{a^2\left(r_c{}^4-2\,r_c{}^2\,\zeta_c\,\overline{\zeta_c}+\left(a^2+\zeta_c{}^2\right)\,\overline{\zeta_c}^2\right)}{\overline{\zeta_c}^2\,\left(-r_c{}^2+\zeta_c\,\overline{\zeta_c}\right)\,\left(r_c{}^2+\overline{\zeta_c}\,\zeta_k\right)}\right)$$
$$-\frac{a^2\left(r_c{}^4-2\,r_c{}^2\,\zeta_c\,\overline{\zeta_c}+\left(a^2+\zeta_c{}^2\right)\,\overline{\zeta_c}^2\right)\,\left(r_c{}^2-\zeta_k\,\overline{\zeta_k}\right)}{\overline{\zeta_c}\,\left(-r_c{}^2+\zeta_c\,\overline{\zeta_c}\right)\,\left(r_c{}^2+\overline{\zeta_c}\,\zeta_k\right)\,\left(\overline{\zeta_c}+\overline{\zeta_k}\right)}$$
$$\mathcal{B} = \left(r_c^2-\frac{a^2r_c^2}{\overline{\zeta_c}^2}\right)$$
$$\mathcal{C} = \frac{a^2\,r_c{}^2\,\left(r_c{}^4-2\,r_c{}^2\,\zeta_c\,\overline{\zeta_c}+\left(-a^2+\zeta_c{}^2\right)\,\overline{\zeta_c}^2\right)}{\overline{\zeta_c}^2\,\left(r_c{}^2-\zeta_c\,\overline{\zeta_c}\right)^2}$$

Only the leading term in Equation (3.148) will actually appear after I take the real part to find the moment in Equation (3.125). To see this, I evaluate this integral on the contour of Figure 3.11:

$$\int_\Sigma w_5^k\,z\,\overline{dz} = \int_\Sigma \overline{w_5^k}\,z\,\overline{dz} = \overline{\int_\Sigma w_5^k\,\overline{z}\,dz} \tag{3.149}$$

$$\int_\Sigma w_5^k\overline{z}\,dz = \int_C w_5^k\left(1-\frac{a^2}{(\zeta+\zeta_c)^2}\right)\left(\frac{r_c{}^2}{\zeta}+\overline{\zeta_c}+\frac{a^2\,\zeta}{r_c{}^2+\zeta\,\overline{\zeta_c}}\right)\,d\zeta$$
$$= \int_C w_5^k\left(\frac{r_c{}^2}{\zeta}-\frac{a^2\,r_c{}^2}{\zeta\,(\zeta+\zeta_c)^2}+\overline{\zeta_c}-\frac{a^2\,\overline{\zeta_c}}{(\zeta+\zeta_c)^2}+\frac{a^2\,\zeta}{r_c{}^2+\zeta\,\overline{\zeta_c}}-\frac{a^4\,\zeta}{(\zeta+\zeta_c)^2\,\left(r_c{}^2+\zeta\,\overline{\zeta_c}\right)}\right)\,d\zeta \tag{3.150}$$

Since $w_5^k \approx i\log(-r_c/\zeta_k) - i\zeta_k/\zeta + ir_c^2/(\overline{\zeta_k}\zeta)$ for large ζ, I can identify the terms of the integrand that go as $1/\zeta$ and find

$$\int_{C_\infty} w_5^k \left(\frac{{r_c}^2}{\zeta} - \frac{a^2\,{r_c}^2}{\zeta\,(\zeta+\zeta_c)^2} + \overline{\zeta_c} - \frac{a^2\,\overline{\zeta_c}}{(\zeta+\zeta_c)^2} + \frac{a^2\,\zeta}{{r_c}^2 + \zeta\,\overline{\overline{\zeta_c}}} - \frac{a^4\,\zeta}{(\zeta+\zeta_c)^2\,\left({r_c}^2+\zeta\,\overline{\zeta_c}\right)} \right) d\zeta$$
$$= -2\,\pi \left(\left(\frac{a^2}{\overline{\overline{\zeta_c}}} + \overline{\zeta_c} \right) \left(-\zeta_k + \frac{{r_c}^2}{\overline{\zeta_k}} \right) + \left({r_c}^2 - \frac{a^2\,{r_c}^2}{\overline{\zeta_c}^2} \right) \log\left(\frac{-r_c}{\zeta_k} \right) \right) \quad (3.151)$$

The integral cancels around all the linear parts of the contour except for C_b.

$$\int_{C_b} w_5^k \left(1 - \frac{a^2}{(\zeta+\zeta_c)^2} \right) \left(\frac{{r_c}^2}{\zeta} + \overline{\zeta_c} + \frac{a^2\,\zeta}{{r_c}^2+\zeta\,\overline{\overline{\zeta_c}}} \right) d\zeta$$
$$= 2\pi \int_{a-\zeta_c}^{\zeta_k} \left(1 - \frac{a^2}{(\zeta+\zeta_c)^2} \right) \left(\frac{{r_c}^2}{\zeta} + \overline{\zeta_c} + \frac{a^2\,\zeta}{{r_c}^2+\zeta\,\overline{\overline{\zeta_c}}} \right) d\zeta$$
$$= 2\pi \Bigg[\frac{\zeta\,\left(a^2+\overline{\zeta_c}^2\right)}{\overline{\zeta_c}} + \frac{a^2\,\left({r_c}^4 + a^2\,{\zeta_c}^2 - 2\,{r_c}^2\,\zeta_c\,\overline{\zeta_c} + {\zeta_c}^2\,\overline{\zeta_c}^2\right)}{\zeta_c\,(\zeta+\zeta_c)\,\left(-{r_c}^2+\zeta_c\,\overline{\zeta_c}\right)} + \frac{\left(-\left(a^2\,{r_c}^2\right) + {r_c}^2\,{\zeta_c}^2\right)\,\log(\zeta)}{{\zeta_c}^2}$$
$$+ \frac{\left(a^2\,{r_c}^6 - a^4\,{r_c}^2\,{\zeta_c}^2 - 2\,a^2\,{r_c}^4\,\zeta_c\,\overline{\zeta_c} + a^2\,{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}^2\right)\,\log(\zeta+\zeta_c)}{{\zeta_c}^2\,\left(-{r_c}^2+\zeta_c\,\overline{\zeta_c}\right)^2}$$
$$+ \frac{\left(-\left(a^2\,{r_c}^6\right) + 2\,a^2\,{r_c}^4\,\zeta_c\,\overline{\zeta_c} + a^4\,{r_c}^2\,\overline{\zeta_c}^2 - a^2\,{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}^2\right)\,\log({r_c}^2+\zeta\,\overline{\zeta_c})}{\overline{\zeta_c}^2\,\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^2} \Bigg]_{\zeta=a-\zeta_c}^{\zeta=\zeta_k}$$
$$(3.152)$$

Finally, the contribution from C_p is found by taking the residue at $\zeta = -r_c^2/\overline{\zeta_c}$.

$$\int_{C_p} w_5^k \left(1 - \frac{a^2}{(\zeta+\zeta_c)^2} \right) \left(\frac{{r_c}^2}{\zeta} + \overline{\zeta_c} + \frac{a^2\,\zeta}{{r_c}^2+\zeta\,\overline{\overline{\zeta_c}}} \right) d\zeta$$
$$= -2\,a^2\,\pi\,{r_c}^2\,\left(-{r_c}^4 + 2\,{r_c}^2\,\zeta_c\,\overline{\zeta_c} + (a-\zeta_c)\,(a+\zeta_c)\,\overline{\zeta_c}^2 \right)$$
$$\times \log\left(-\frac{\left({r_c}^2+\overline{\zeta_c}\,\zeta_k\right)\,\overline{\zeta_k}}{r_c\,\zeta_k\,\left(\overline{\zeta_c}+\overline{\zeta_k}\right)} \right) / \left(\overline{\zeta_c}^2\,\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^2 \right) \quad (3.153)$$

So collecting all terms and taking the complex conjugate, I find

$$\int_\Sigma w_5^k \, z \, \overline{dz} = \int_\Sigma \overline{w_5^k} z \, \overline{dz} =$$
$$2\pi \left[\mathcal{A}^* - \mathcal{B} \log(\frac{\overline{\zeta_k}}{a - \overline{\overline{\zeta_c}}}) - \overline{\mathcal{B}} \log(-\frac{r_c}{\overline{\overline{\zeta_k}}}) - \mathcal{C} \log(\frac{\overline{\zeta_c} + \overline{\zeta_k}}{a}) - \overline{\mathcal{C}} \log(-\frac{(r_c^2 + a\zeta_c - \delta^2)\zeta_k}{r_c \overline{\zeta_k}(\zeta_c + \zeta_k)}) \right] \tag{3.154}$$

where

$$\begin{aligned} \mathcal{A}^* \;=\; & a \left(2\,\zeta_c - \frac{{r_c}^2}{\overline{\overline{\zeta_c}}} \right) + a^3 \left(\frac{1}{\zeta_c} + \frac{\overline{\zeta_c}}{-{r_c}^2 + \zeta_c \, \overline{\overline{\zeta_c}}} \right) \\ & - \frac{\zeta_c \, \left({r_c}^2 + \overline{\zeta_c} \, \zeta_k \right)}{\zeta_k} - \frac{a^4 \, \overline{\zeta_c}}{\left(-{r_c}^2 + \zeta_c \, \overline{\zeta_c} \right) \, \left(\overline{\zeta_c} + \overline{\zeta_k} \right)} \\ & - \frac{a^2 \, \left({r_c}^2 \, \left(\overline{\zeta_c}^2 - \zeta_c \, \zeta_k + \overline{\zeta_c} \, \overline{\zeta_k} \right) + \overline{\zeta_c} \, \zeta_k \, \left({\zeta_c}^2 + \overline{\zeta_c} \, \left(\overline{\zeta_c} + \overline{\zeta_k} \right) \right) \right)}{\zeta_c \, \overline{\zeta_c} \, \zeta_k \, \left(\overline{\zeta_c} + \overline{\zeta_k} \right)} \end{aligned}$$

The log terms in Equation (3.154) are negative complex conjugates of the log terms in Equation (3.148). This shows that all the log terms taken together are pure imaginary and will be discarded in Equation (3.125). So only the leading terms $\mathcal{A} + \mathcal{A}^*$ are actually significant in determining the moment on the foil.

The Bernoulli pressure terms can again be dealt with by using the contours illustrated in Figure 3.9.

$$
\begin{aligned}
\int_\Sigma & \frac{dw}{dz}\overline{\frac{dw}{dz}} z\,\overline{dz} \\
&= \int_C \frac{dw}{d\zeta}\overline{\frac{dw}{d\zeta}} \left(\frac{(\zeta+\zeta_c)^2}{(\zeta+\zeta_c)^2-a^2}\right)\left(\zeta+\zeta_c+\frac{a^2}{\zeta+\zeta_c}\right)\left(-\frac{r_c^2}{\zeta^2}\right)\,d\zeta \\
&= \int_{C_+} \frac{dw}{d\zeta}\overline{\frac{dw}{d\zeta}} \left(\frac{(\zeta+\zeta_c)^2}{(\zeta+\zeta_c)^2-a^2}\right)\left(\zeta+\zeta_c+\frac{a^2}{\zeta+\zeta_c}\right)\left(-\frac{r_c^2}{\zeta^2}\right)\,d\zeta \\
&- \int_{S_a} \frac{dw}{d\zeta}\overline{\frac{dw}{d\zeta}} \left(\frac{(\zeta+\zeta_c)^2}{(\zeta+\zeta_c)^2-a^2}\right)\left(\zeta+\zeta_c+\frac{a^2}{\zeta+\zeta_c}\right)\left(-\frac{r_c^2}{\zeta^2}\right)\,d\zeta \\
&= \int_{C_+} \frac{dw}{d\zeta}\overline{\frac{dw}{d\zeta}} \left(\frac{(\zeta+\zeta_c)^2}{(\zeta+\zeta_c)^2-a^2}\right)\left(\zeta+\zeta_c+\frac{a^2}{\zeta+\zeta_c}\right)\left(-\frac{r_c^2}{\zeta^2}\right)\,d\zeta \\
&- i\pi\left[\frac{dw}{d\zeta}\overline{\frac{dw}{d\zeta}} \left(\frac{(\zeta+\zeta_c)^2}{(\zeta+\zeta_c+a)}\right)\left(\zeta+\zeta_c+\frac{a^2}{\zeta+\zeta_c}\right)\left(-\frac{r_c^2}{\zeta^2}\right)\right]_{\zeta=a-\zeta_c} \\
&= \int_{C_+} \frac{dw}{d\zeta}\overline{\frac{dw}{d\zeta}} \left(\frac{(\zeta+\zeta_c)^2}{(\zeta+\zeta_c)^2-a^2}\right)\left(\zeta+\zeta_c+\frac{a^2}{\zeta+\zeta_c}\right)\left(-\frac{r_c^2}{\zeta^2}\right)\,d\zeta \\
&\qquad + i\pi\frac{a^2 r_c^2}{(a-\zeta_c)^2}\left[\frac{dw}{d\zeta}\overline{\frac{dw}{d\zeta}}\right]_{\zeta=a-\zeta_c}
\end{aligned}
\tag{3.155}
$$

where the last term vanishes due to the Kutta condition.

3.7 Equations of Motion

Collecting terms, we see that the force on the foil is

$$
\begin{aligned}
X + iY = & -\dot{U}\,\mathcal{A}_1 - \dot{V}\,\mathcal{A}_2 - \dot{\Omega}\,\mathcal{A}_3 + \ddot{\zeta}_x\,\mathcal{B}_1 + \ddot{\zeta}_y\,\mathcal{B}_2 + \ddot{a}\,\mathcal{B}_3 \\
& + U^2\,\mathcal{C}_1 + UV\,\mathcal{C}_2 + V^2\,\mathcal{C}_3 + U\Omega\,\mathcal{C}_4 + V\Omega\,\mathcal{C}_5 + \Omega^2\,\mathcal{C}_6 + U\dot{\zeta}_x\,\mathcal{D}_1 + U\dot{\zeta}_y\,\mathcal{D}_2 + U\dot{a}\,\mathcal{D}_3 + U\dot{r_c}\,\mathcal{D}_4 \\
& + V\dot{\zeta}_x\,\mathcal{D}_5 + V\dot{\zeta}_y\,\mathcal{D}_6 + V\dot{a}\,\mathcal{D}_7 + V\dot{r_c}\,\mathcal{D}_8 + \Omega\dot{\zeta}_x\,\mathcal{D}_9 + \Omega\dot{\zeta}_y\,\mathcal{D}_{10} + \Omega\dot{a}\,\mathcal{D}_{11} + \Omega\dot{r_c}\,\mathcal{D}_{12} \\
& + \dot{\zeta}_x{}^2\,\mathcal{E}_1 + \dot{\zeta}_x\dot{\zeta}_y\,\mathcal{E}_2 + \dot{\zeta}_y{}^2\,\mathcal{E}_3 + \dot{\zeta}_x\dot{a}\,\mathcal{E}_4 + \dot{\zeta}_y\dot{a}\,\mathcal{E}_5 + \dot{a}^2\,\mathcal{E}_6 + \dot{\zeta}_x\dot{r_c}\,\mathcal{E}_7 + \dot{\zeta}_y\dot{r_c}\,\mathcal{E}_8 + \dot{a}\dot{r_c}\,\mathcal{E}_9 \\
& + U\gamma_c\,\mathcal{F}_1 + V\gamma_c\,\mathcal{F}_2 + \Omega\gamma_c\,\mathcal{F}_3 + \dot{\zeta}_x\gamma_c\,\mathcal{F}_4 + \dot{\zeta}_y\gamma_c\,\mathcal{F}_5 + \dot{a}\gamma_c\,\mathcal{F}_6 \\
& + \sum_k \gamma_k \left(U\,\mathcal{G}_1^k + V\,\mathcal{G}_2^k + \Omega\,\mathcal{G}_3^k + \dot{\zeta}_x\,\mathcal{G}_4^k + \dot{\zeta}_y\,\mathcal{G}_5^k + \dot{a}\,\mathcal{G}_6^k + \dot{r_c}\,\mathcal{G}_7^k + \dot{\zeta}_k\,\mathcal{G}_8^k + \overline{\dot{\zeta}_k}\,\mathcal{G}_9^k + \gamma_c\,\mathcal{G}_{10}^k \right) \\
& + \sum_k \sum_{j \neq k} \gamma_k\gamma_j\,\mathcal{G}_{11}^k + \sum_k \gamma_k^2\,\mathcal{G}_{12}^k + \dot{\gamma}_c\,\mathcal{H}_1 + \sum_k \dot{\gamma}_k\,\mathcal{H}_2^k + \frac{\pi a r_c^2}{4(a-\zeta_c)^2} \left\| \frac{dw}{ds} \right\|^2_{\zeta = a - \zeta_c}
\end{aligned}
\tag{3.156}
$$

where the final term vanishes as long as the Kutta condition is satisfied, and where

$$
\begin{aligned}
\mathcal{A}_1 &= \pi \left(-2\,a^2 + {r_c}^2 + \frac{a^4\,{r_c}^2}{\left({r_c}^2 - \delta^2\right)^2} \right) \\
\mathcal{A}_2 &= i\,\pi \left(2\,a^2 + {r_c}^2 + \frac{a^4\,{r_c}^2}{\left({r_c}^2 - \delta^2\right)^2} \right) \\
\mathcal{A}_3 &= -\left(\frac{i\,\pi \left(a^6\,{r_c}^2\,\overline{\zeta_c} + 2\,a^4\,\zeta_c \left({r_c}^2 - \delta^2\right)^2 - {r_c}^2\,\zeta_c \left({r_c}^2 - \delta^2\right)^3 + 2\,a^2\,\overline{\zeta_c} \left(-{r_c}^2 + \delta^2\right)^3 \right)}{\left({r_c}^2 - \delta^2\right)^3} \right)
\end{aligned}
$$

I can relate some of the coefficients in Equation (3.156) to the locked inertia coefficients found in Section 3.3 as follows:

$$\mathcal{A}_1 = I_{11} + iI_{12} \tag{3.157}$$

$$\mathcal{A}_2 = I_{12} + iI_{22} \tag{3.158}$$

$$\mathcal{A}_3 = I_{13} + iI_{23} \tag{3.159}$$

The other coefficients are given in Appendix C.

After creation, the wake vortices are assumed to have constant strength γ_k. The central vortex strength γ_c is also normally assumed to be constant. Nevertheless, the terms in Equation (3.156) that are proportional to $\dot{\gamma_c}$ and $\dot{\gamma_k}$ can be significant whenever vortices are introduced to or removed from the flow.

Whenever the k^{th} vortex is shed from the foil into the wake, its strength may be considered to rise from zero to γ_k in the time step of creation, and the $\mathcal{H}_2^k$ term will exert some force on the foil (with $\dot{\gamma_k} = \gamma_k/(\Delta t)$). Streitlien [Str94] noted that if this effect is neglected in simulation of a rigid foil, the force calculations will still converge to the correct result as the time step Δt approaches zero, but only at a rate $\sqrt{\Delta t}$.

On the other hand, in order to reduce the computational burden and avoid tracking the location of all shed wake vortices indefinitely, we may wish to remove some vortices distant from the foil, and either combine them into a single vortex at their center of vorticity, or stop tracking them altogether and assume they have moved permanently to infinity. In the latter case, the images of the "infinitely distant" vortices move to the origin and the central vortex strength γ_c is changed by an amount $(-\gamma_k)$, the strength of the images. During the time step in which such simplification of the wake occurs, we should have force contributions from the $\mathcal{H}_2^k$ and possibly the $\mathcal{H}_1$ terms, which should substantially cancel if the wake simplification is justified.

The moment on the foil is given by

$$
\begin{aligned}
M = &-\dot{U}\,\mathcal{J}_1 - \dot{V}\,\mathcal{J}_2 - \dot{\Omega}\,\mathcal{J}_3 + \ddot{\zeta}_x\,\mathcal{K}_1 + \ddot{\zeta}_y\,\mathcal{K}_2 + \ddot{a}\,\mathcal{K}_3 \\
&+U^2\,\mathcal{L}_1 + UV\,\mathcal{L}_2 + V^2\,\mathcal{L}_3 + U\Omega\,\mathcal{L}_4 + V\Omega\,\mathcal{L}_5 + \Omega^2\,\mathcal{L}_6 + U\dot{\zeta}_x\,\mathcal{M}_1 + U\dot{\zeta}_y\,\mathcal{M}_2 + U\dot{a}\,\mathcal{M}_3 + U\dot{r}_c\,\mathcal{M}_4 \\
&+ V\dot{\zeta}_x\,\mathcal{M}_5 + V\dot{\zeta}_y\,\mathcal{M}_6 + V\dot{a}\,\mathcal{M}_7 + V\dot{r}_c\,\mathcal{M}_8 + \Omega\dot{\zeta}_x\,\mathcal{M}_9 + \Omega\dot{\zeta}_y\,\mathcal{M}_{10} + \Omega\dot{a}\,\mathcal{M}_{11} + \Omega\dot{r}_c\,\mathcal{M}_{12} \\
&+ \dot{\zeta}_x^{\,2}\,\mathcal{N}_1 + \dot{\zeta}_x\dot{\zeta}_y\,\mathcal{N}_2 + \dot{\zeta}_y^{\,2}\,\mathcal{N}_3 + \dot{\zeta}_x\dot{a}\,\mathcal{N}_4 + \dot{\zeta}_y\dot{a}\,\mathcal{N}_5 + \dot{a}^2\,\mathcal{N}_6 + \dot{\zeta}_x\dot{r}_c\,\mathcal{N}_7 + \dot{\zeta}_y\dot{r}_c\,\mathcal{N}_8 + \dot{a}\dot{r}_c\,\mathcal{N}_9 \\
&+ U\gamma_c\,\mathcal{P}_1 + V\gamma_c\,\mathcal{P}_2 + \Omega\gamma_c\,\mathcal{P}_3 + \dot{\zeta}_x\gamma_c\,\mathcal{P}_4 + \dot{\zeta}_y\gamma_c\,\mathcal{P}_5 + \dot{a}\gamma_c\,\mathcal{P}_6 + \dot{r}_c\gamma_c\,\mathcal{P}_7 \\
&+\sum_k \gamma_k \left(U\,\mathcal{Q}_1^k + V\,\mathcal{Q}_2^k + \Omega\,\mathcal{Q}_3^k + \dot{\zeta}_x\,\mathcal{Q}_4^k + \dot{\zeta}_y\,\mathcal{Q}_5^k + \dot{a}\,\mathcal{Q}_6^k + \dot{r}_c\,\mathcal{Q}_7^k + \dot{\zeta}_k\,\mathcal{Q}_8^k + \dot{\overline{\zeta_k}}\,\mathcal{Q}_9^k + \gamma_c\,\mathcal{Q}_{10}^k \right) \\
&+ \sum_k \sum_{j\neq k} \gamma_k\gamma_j\,\mathcal{Q}_{11}^k + \sum_k \gamma_k^2\,\mathcal{Q}_{12}^k + \dot{\gamma}_c\,\mathcal{R}_1 + \sum_k \dot{\gamma}_k\,\mathcal{R}_2^k + \frac{\pi a^2 \zeta_y (a-\zeta_x)}{r_c^2} \left\| \frac{dw}{ds} \right\|^2_{\zeta = a - \zeta_c}
\end{aligned}
\tag{3.160}
$$

where the final term vanishes as long as the Kutta condition is satisfied. The coefficients are given in Appendix D.

As an example, I calculate the forces and moments on a foil executing the following particular motion: the centroid of the foil moves with uniform velocity $(U_{\mathrm{cm}}, V_{\mathrm{cm}}) = (1, 0)$ while the foil pitches with $\Omega = 0.3\cos(t)$ and deforms with $(\zeta_x, \zeta_y) = (0.15, 0.3\sin(t))$, while $r_c = 1$. This trajectory, illustrated in Figures 3.12-3.13, is intended to be suggestive of a swimming fish, since the pitching and flexing deformation are countervailing, leaving the “nose” of the foil pointed in approximately the same direction while the rear part flaps back and forth. The forces and moments experienced by the foil are plotted in Figure 3.14. The motion generates a positive forward thrust and also a positive average moment, while the lateral forces settle into a symmetrical oscillation with zero mean. There is a transient spike in the lateral force in the first few time steps but otherwise the system displays periodic behavior immediately, i.e., there is little noticeable secular change in force on the foil as the wake evolves in time.

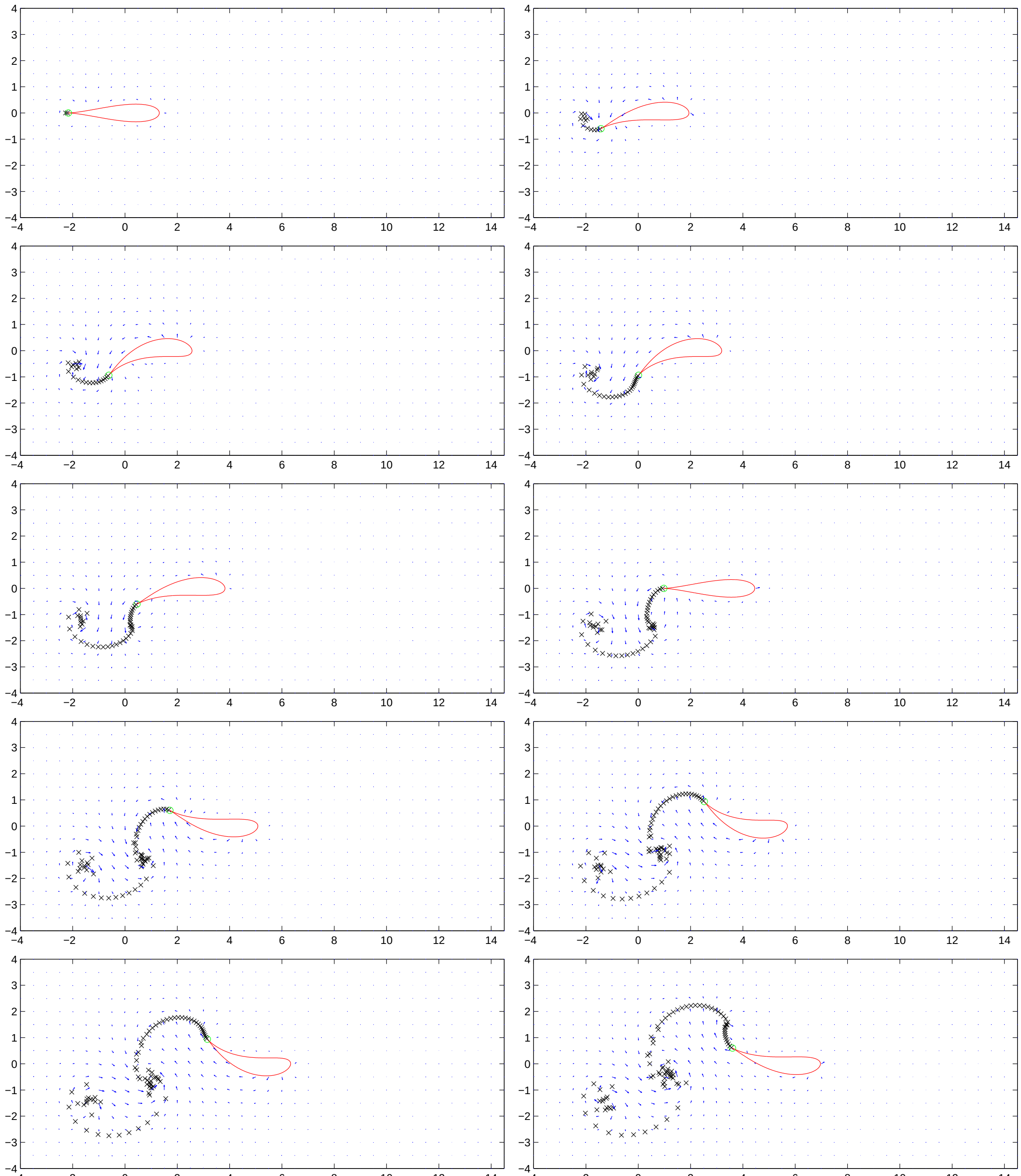


Figure 3.12: Flapping motion of a foil with $(U_{\rm cm}, V_{\rm cm}) = (1, 0)$, $\Omega = 0.3\cos(t)$, $(\zeta_x, \zeta_y) = (0.15, 0.3\sin(t))$, $r_c = 1$. The point vortices are being shed at intervals of $\Delta t = 2\pi/100 = 0.0628$. The snapshots shown are at intervals of $10\,\Delta t = 2\pi/10$.

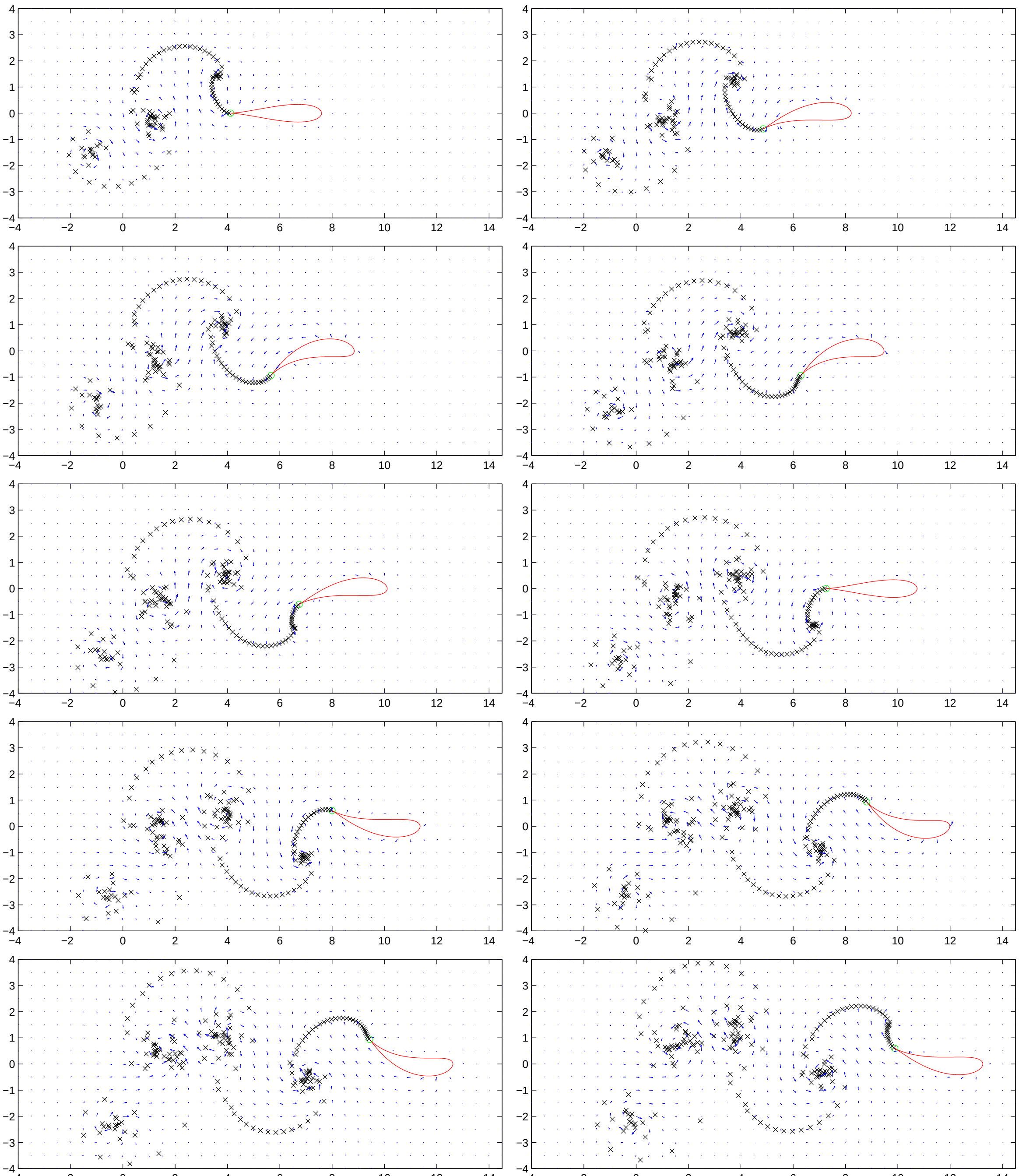


Figure 3.13: Continuation of Figure 3.12. Flapping motion of a foil with $(U_{\rm cm}, V_{\rm cm}) = (1, 0)$, $\Omega = 0.3\cos(t)$, $(\zeta_x, \zeta_y) = (0.15, 0.3\sin(t))$, $r_c = 1$. The point vortices are being shed at intervals of $\Delta t = 2\pi/100 = 0.0628$. The snapshots are at intervals of $10\Delta t = 2\pi/10$.

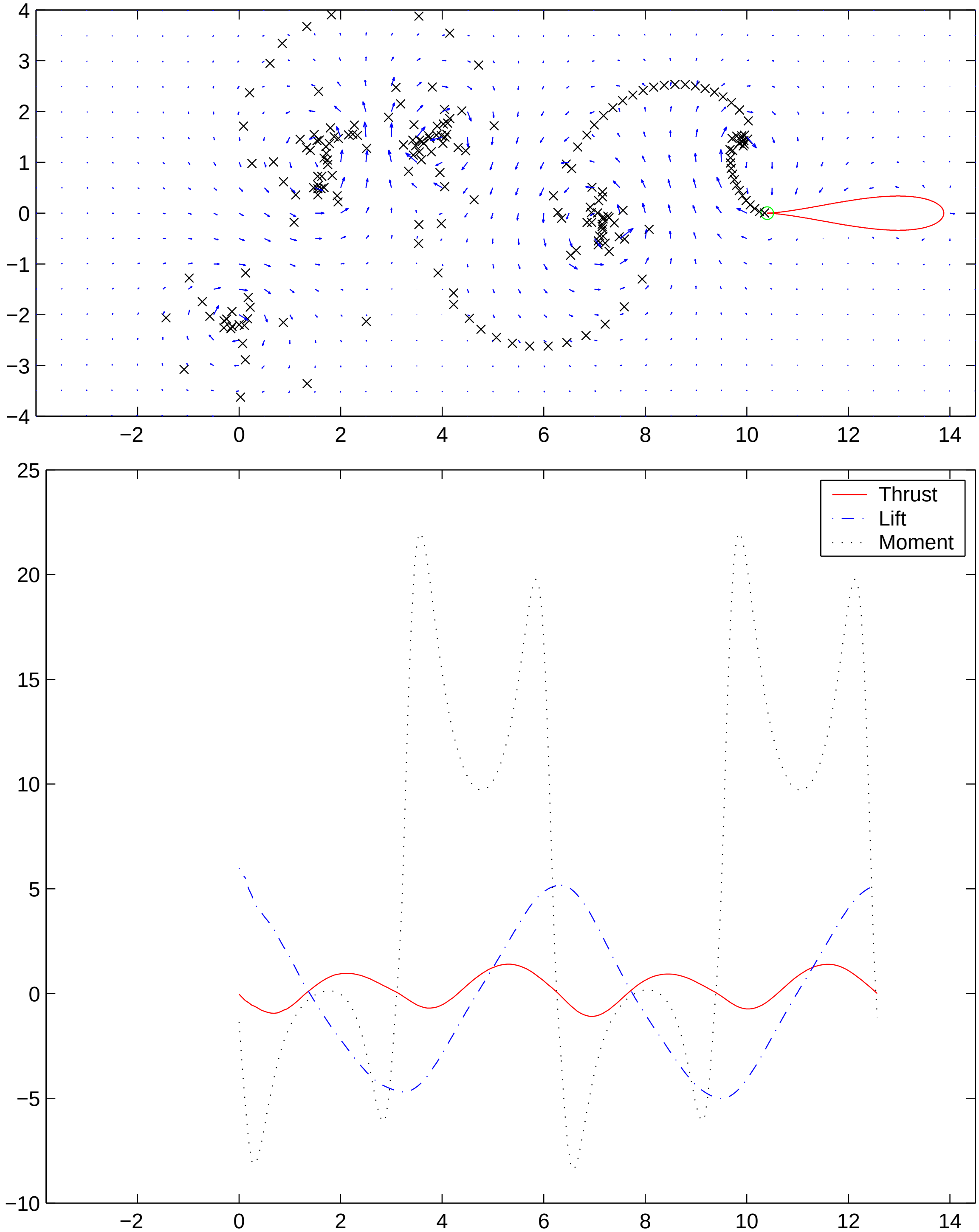


Figure 3.14: Forces and moments on the flapping foil of Figures 3.12-3.13, in a stationary frame oriented with the reader. While the oscillating force in the y-direction has an average approaching zero, the oscillating thrust in the x-direction has a positive mean value of 0.23. The moment also has a significantly positive mean value.

3.8 Self-Propulsion of a Deformable Foil

Equations (3.156) and (3.160) can be used to find the force and moment on a deformable foil undergoing any prescribed motion and deformation. They are also very broadly applicable without requiring any assumptions about the internal structure of the foil. A prescribed trajectory might arise in a practical sense if the deformable foil is attached to an external body which can move the foil through a given trajectory independent of the fluid forces experienced by the foil. In particular, if the flapping foil of Figure 3.14 were attached as a tailfin propulsor to another body, we can see how the tailfin could provide forward thrust to the body while inertial or internal forces provided by the body "absorbed" the lateral forces. By using stabilizing forward fins, or by using paired propulsors, the body could manage the moments generated by the tailfin and maintain the desired orientation.

Suppose, on the other hand, that instead of attaching the deformable foil as an appendage to another body, we are interested in its ability to swim as an isolated body in its own right. In order to find which motions (if any) generate self-propulsion, I have to make some assumption about the mass distribution $\rho(z)$.

The linear momentum of the foil, expressed as a complex number, is

$$P_x + iP_y = \int\int \rho(z) \left(U + iV + i\Omega z + \frac{\partial F}{\partial \zeta_c}\dot{\zeta}_c + \frac{\partial F}{\partial a}\dot{a} \right) dA, \tag{3.161}$$

while the angular momentum is

$$L = \mathsf{Re}\Big\{ -i \int\int \rho(z) \left(U + iV + i\Omega z + \tfrac{\partial F}{\partial \zeta_c}\dot{\zeta}_c + \tfrac{\partial F}{\partial a}\dot{a} \right) \overline{z}\, dA \Big\}. \tag{3.162}$$

As a consequence of Stokes's theorem, for an arbitrary function $f(z, \overline{z})$ on a region R bounded by ∂R:

$$\int_{\partial R} f\, \overline{dz} = -2i \int\int_R \frac{\partial f}{\partial z}\, dA \tag{3.163}$$

So if the function $\rho(z)$ is sufficiently congenial (for example, a polynomial of z and $\overline{z}$) I can transform the momentum expressions into contour integrals around the foil boundary, then make the usual transformations and evaluate the integrals using the theory of residues in the ζ-plane.

3.8.1 Uniform Foil Density

As an example, I assume that the foil is homogeneous with uniform density $\rho(z) = \rho$. (The density may, however, be time-dependent if the foil undergoes area-changing deformations.) Then:

$$P_x + iP_y = \frac{-1}{2i}\int_\Sigma \rho\left(Uz + iVz + i\Omega z^2/2 + \dot{\zeta}_c\frac{\partial}{\partial\zeta_c}F^2/2 + \dot{a}\frac{\partial}{\partial a}F^2/2\right)\overline{dz} \quad (3.164)$$

or

$$\begin{aligned} P_x + iP_y = \int_C \frac{-ir_c^2\rho}{2\zeta^2}\left(1 - \frac{a^2}{\left(\frac{r_c{}^2}{\zeta} + \overline{\zeta_c}\right)^2}\right)\Bigg(&\frac{2\,a\,\dot{a}\left(\zeta+\zeta_c+\frac{a^2}{\zeta+\zeta_c}\right)}{\zeta+\zeta_c} \\ &+\frac{i}{2}\,\Omega\left(\zeta+\zeta_c+\frac{a^2}{\zeta+\zeta_c}\right)^2 + \left(1 - \frac{a^2}{(\zeta+\zeta_c)^2}\right)\left(\zeta+\zeta_c+\frac{a^2}{\zeta+\zeta_c}\right)\dot{\zeta}_c \\ &+\left(\zeta+\zeta_c+\frac{a^2}{\zeta+\zeta_c}\right)U + i\left(\zeta+\zeta_c+\frac{a^2}{\zeta+\zeta_c}\right)V\Bigg)\,d\zeta \end{aligned} \quad (3.165)$$

The integral can be evaluated by finding the residues of the integrand at $\zeta = 0$ and $\zeta = -\zeta_c$, yielding

$$\begin{aligned} P_x + iP_y = \pi\,r_c{}^2\,\rho\Bigg[&\dot{a}\left(\frac{-4\,a^5\,r_c{}^2}{\zeta_c\left(-r_c{}^2+\zeta_c\,\overline{\zeta_c}\right)^3} - \frac{4\,a^5}{\zeta_c\left(-r_c{}^2+\zeta_c\,\overline{\zeta_c}\right)^2}\right) \\ &+\Omega\left(i\,\zeta_c - \frac{i\,a^6\,r_c{}^2}{\zeta_c\left(-r_c{}^2+\zeta_c\,\overline{\zeta_c}\right)^3} - \frac{i\,a^6}{\zeta_c\left(-r_c{}^2+\zeta_c\,\overline{\zeta_c}\right)^2}\right) \\ &+\left(1 + \frac{3\,a^6\,r_c{}^4}{\zeta_c{}^2\left(-r_c{}^2+\zeta_c\,\overline{\zeta_c}\right)^4} + \frac{6\,a^6\,r_c{}^2}{\zeta_c{}^2\left(-r_c{}^2+\zeta_c\,\overline{\zeta_c}\right)^3} + \frac{3\,a^6}{\zeta_c{}^2\left(-r_c{}^2+\zeta_c\,\overline{\zeta_c}\right)^2}\right)\dot{\zeta}_c \\ &+\left(1 - \frac{a^4}{\left(-r_c{}^2+\zeta_c\,\overline{\zeta_c}\right)^2}\right)U + \left(i - \frac{i\,a^4}{\left(-r_c{}^2+\zeta_c\,\overline{\zeta_c}\right)^2}\right)V\Bigg] \end{aligned} \quad (3.166)$$

If the total mass $m = \rho A$ of the body is conserved, then:

$$P_x + iP_y = m\left(U + i\,V + \frac{4\,a^5\,\dot{a}\,\overline{\zeta_c}}{\left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)\,\left(-a^4 + \left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)^2\right)} + \frac{i\,\Omega\,\left(a^6\,\overline{\zeta_c} + \zeta_c\left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)^3\right)}{\left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)\,\left(-a^4 + \left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)^2\right)} + \frac{\left({r_c}^8 - 4\,{r_c}^6\,\zeta_c\,\overline{\zeta_c} + 3\,\left(a^6 + 2\,{r_c}^4\,{\zeta_c}^2\right)\,\overline{\zeta_c}^2 - 4\,{r_c}^2\,{\zeta_c}^3\,\overline{\zeta_c}^3 + {\zeta_c}^4\,\overline{\zeta_c}^4\right)\,\dot{\zeta_c}}{\left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)^2\,\left(-a^4 + \left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)^2\right)}\right). \quad (3.167)$$

Meanwhile the angular momentum equals

$$L = \frac{1}{2}\mathsf{Re}\Big\{\int_\Sigma \rho\,\left(Uz + iVz + i\Omega z^2/2 + \dot{\zeta}_c\tfrac{\partial}{\partial\zeta_c}F^2/2 + \dot{a}\tfrac{\partial}{\partial a}F^2/2\right)\,\overline{z}\,\overline{dz}\Big\}. \quad (3.168)$$

Evaluating the integral by the method of residues I find that

$$L = m\mathsf{Re}\left(\frac{-2\,i\,a^7\,\dot{a}\,\left({r_c}^2 + 2\,\zeta_c\,\overline{\zeta_c}\right)}{\left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)^2\,\left(a^2 + {r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)\,\left(a^2 - {r_c}^2 + \zeta_c\,\overline{\zeta_c}\right)} + \frac{\Omega\,\left({r_c}^2 + 2\,\zeta_c\,\overline{\zeta_c}\right)\,\left(a^4 + \left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)^2\right)}{2\left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)^2} + \frac{i\,\overline{\zeta_c}\,\left(-\left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)^5 + 3\,a^8\,\left({r_c}^2 + \zeta_c\,\overline{\zeta_c}\right)\right)\,\dot{\zeta_c}}{\left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)^3\,\left(-a^4 + \left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)^2\right)} - \frac{i\,\left(a^6\,\zeta_c + \overline{\zeta_c}\left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)^3\right)\,U}{\left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)\,\left(-a^2 + {r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)\,\left(a^2 + {r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)} + \frac{\left(a^6\,\zeta_c + \overline{\zeta_c}\left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)^3\right)\,V}{\left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)\,\left(-a^2 + {r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)\,\left(a^2 + {r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)}\right). \quad (3.169)$$

Taking the real part, I find that

$$L = m \left(\frac{\Omega\,\left(r_c{}^2 + 2\,\zeta_c\,\overline{\zeta_c}\right)\,\left(a^4 + \left(-r_c{}^2 + \zeta_c\,\overline{\zeta_c}\right)^2\right)}{2\,\left(-r_c{}^2 + \zeta_c\,\overline{\zeta_c}\right)^2} \right.$$
$$- \frac{\left(-\left(r_c{}^2 - \zeta_c\,\overline{\zeta_c}\right)^5 + 3\,a^8\,\left(r_c{}^2 + \zeta_c\,\overline{\zeta_c}\right)\right)\,\dot{\zeta}_y\,\zeta_x}{\left(r_c{}^2 - \zeta_c\,\overline{\zeta_c}\right)^3\,\left(-a^4 + \left(-r_c{}^2 + \zeta_c\,\overline{\zeta_c}\right)^2\right)}$$
$$+ \frac{\left(-\left(r_c{}^2 - \zeta_c\,\overline{\zeta_c}\right)^5 + 3\,a^8\,\left(r_c{}^2 + \zeta_c\,\overline{\zeta_c}\right)\right)\,\dot{\zeta}_x\,\zeta_y}{\left(r_c{}^2 - \zeta_c\,\overline{\zeta_c}\right)^3\,\left(-a^4 + \left(-r_c{}^2 + \zeta_c\,\overline{\zeta_c}\right)^2\right)}$$
$$+ \left(-1 + a^2\,\left(\frac{1}{a^2 + r_c{}^2 - \zeta_c\,\overline{\overline{\zeta_c}}} + \frac{1}{-r_c{}^2 + \zeta_c\,\overline{\overline{\zeta_c}}}\right)\right)\,\zeta_y\,U$$
$$\left. + \left(1 + a^2\,\left(\frac{1}{-r_c{}^2 + \zeta_c\,\overline{\overline{\zeta_c}}} - \frac{1}{a^2 - r_c{}^2 + \zeta_c\,\overline{\overline{\zeta_c}}}\right)\right)\,\zeta_x\,V \right). \quad (3.170)$$

By setting

$$\frac{d}{dt}(P_x + iP_y) = X + iY \quad (3.171)$$
$$\frac{d}{dt}L = M \quad (3.172)$$

and using Equations (3.156) and (3.160) and (3.167) and (3.170), I can solve for $(\dot{U}, \dot{V}, \dot{\Omega})$ in terms of (U, V, Ω) and known quantities such as the prescribed deformation of the foil and the state of the wake vortices. I can then find the position (x, y, θ) of the z-frame attached to the foil, with respect to a stationary world frame, by observing that

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} U \\ V \\ \Omega \end{pmatrix} \quad (3.173)$$

$$\begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\theta} \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \dot{U} \\ \dot{V} \\ \dot{\Omega} \end{pmatrix} - \Omega \begin{pmatrix} \sin(\theta) & \cos(\theta) & 0 \\ -\cos(\theta) & \sin(\theta) & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} U \\ V \\ \Omega \end{pmatrix}$$

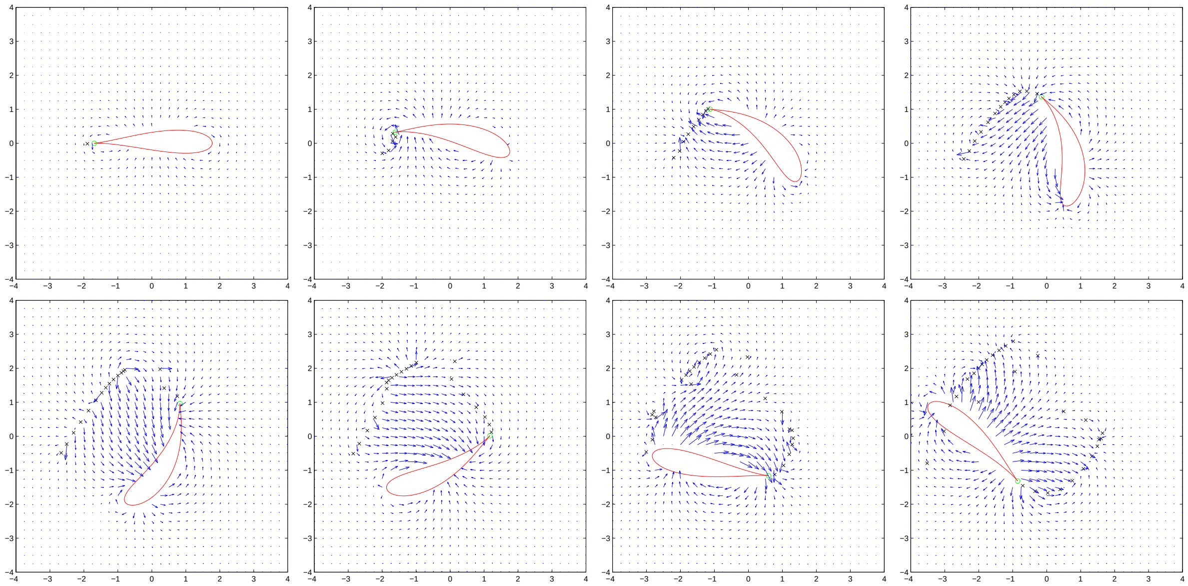


Figure 3.15: A foil starting from rest executes a turn by making the deformation $(\zeta_x, \zeta_y) = (0.15, 0.3\sin(t))$.

and integrating as necessary. It is important to note that our force and moment equations were derived in an inertial frame instantaneously aligned with the rotating z-coordinate frame, not in the rotating frame itself.

Efforts to find useful self-propulsion inputs for an isolated deformable foil are ongoing, but I can make some preliminary observations. For "typical" motions of the deformable foil, the angular accelerations that are generated by oscillating shape deformations appear large relative to the linear accelerations, so it is easy to discover inputs which yield large turns with a small amount of displacement[5], such as the turn illustrated in Figure 3.15. It is not so easy to generate the converse: steady swimming in a particular direction while keeping the orientation within tight bounds.

In further study of the issue, it may be useful to consider nonuniform mass distributions $\rho(z, \overline{z})$. The classic vertical silhouette of a fish with wide body, narrow peduncle, and wide tail means that when the planar silhouette is considered, the forward body and the tail have more inertia (both in actual mass and in added mass) that may be fairly represented by the uniform treatment I have tried in this section.

[5] Of course, the possibility of such maneuvers is part of the motivation for studying fishlike swimming [ACS+97, Wei72].

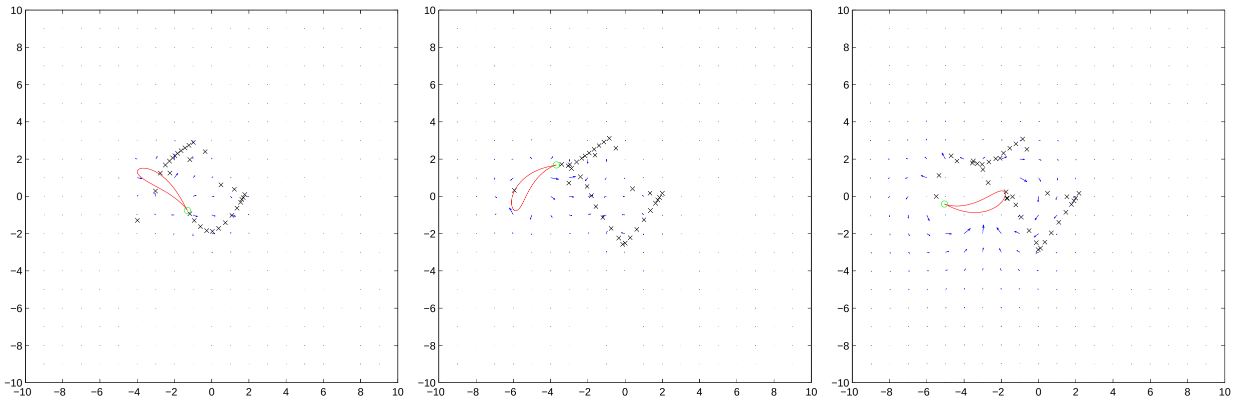


Figure 3.16: The continued progress of the oscillating foil from Figure 3.15. When the foil drives through its own wake, I assume that any vortices which come too close to the foil are annihilated.

To further stabilize the orientation, it may help to appeal to empirical dissipative or damping forces which do not arise naturally in the ideal fluid model.

When the foil executes general maneuvers in the fluid, it is possible for previously shed point vortices to come near the surface of the foil, as can be seen in Figure 3.15. In principle, if the vortex comes too close to the surface, it can attain very high velocities (because it is under powerful influence from its own counter-rotating image vortex), causing numerical difficulties in the integration of the simulation equations, but in reality such a vortex would probably be destroyed by the action of viscous effects [Sar75]. In simulations of flow over an inclined flat plate, Sarpkaya simply removed from the calculation any vortices which came closer than $0.1a$ to the surface of the plate, justifying the distance $0.1a$ in part by reference to the likely radius of the vortices in a real flow. I established a similar zone around the deformable foil when carrying out the simulation illustrated in Figure 3.16.

Chapter 4 Robot Carangiform Fish

4.1 Review of Carangiform Swimming

There are a wide variety of fish morphologies and at least a few different types of fish locomotion. This chapter focuses on a robot model of *carangiform*[1] fishes, fast-swimming fishes which resemble tuna and mackerel. Carangiform fishes typically have large, high-aspect-ratio tails, and they swim using primarily motions of the rear and tail, while the forward part of the body remains relatively immobile.[2]

To leading order, the geometry of carangiform swimming and the forces related to propulsion can be described as follows. First, I can roughly idealize the main body of the fish as a rigid body. The body is connected to the tail by a *peduncle*—a slender region of generally negligible hydrodynamic influence. Though three-dimensional effects may be important for some fish maneuvers, for purposes of modelling the gross thrust generation process, I simplify the geometry to the two dimensions of the horizontal plane, and our robot model moves in the plane.

As the tail moves, nonzero circulation is generated in the surrounding fluid so that the Kutta-Joukowski condition is satisfied. This vorticity is shed into the fish's wake, and the pattern of vortices left behind by the passing of the fish is roughly a reverse Karman vortex street (Figure 4.1). This shed vorticity is a source of energy loss [Lig75]. Biological studies [AHB$^+$91] suggest that flapping motions of the fish tails are optimized to recapture some of the energy lost in the wake. A vortex is shed near the extremum of the tail's sideways motion. The tail motion as it reverses direction can then be roughly interpreted as "pushing off" of the shed vortex. As a

[1] Latin for "mackerel-shaped."

[2] Some researchers use the additional category *thunniform* (Latin for "tuna-shaped") to refer to fish in which propulsive force is even more concentrated in a high-aspect-ratio tail, which is often lunate in shape. In their terminology our robot might resemble a thunniform rather than carangiform fish. However, I will use the more general term, since the distinction is primarily one of degree, and our experimental tailfin was typically a rectangular flat plate rather than lunate.

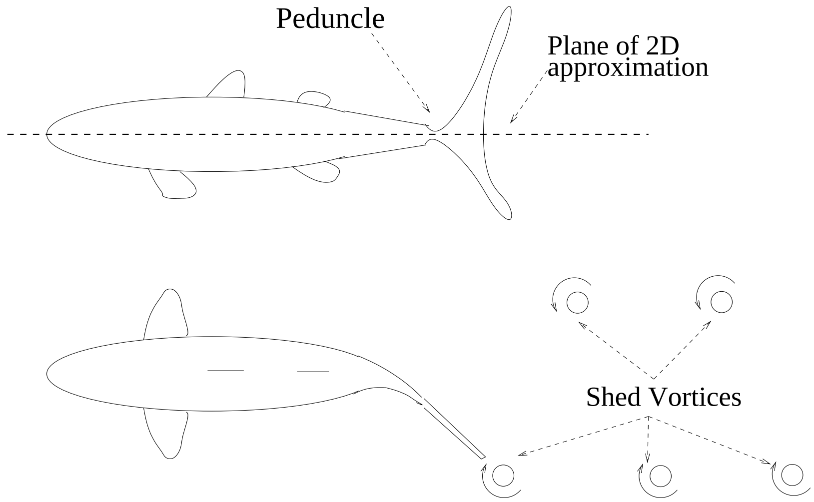


Figure 4.1: Schematic (side and top view) of Carangiform fish propulsion.

benefit, the counterrotating vortex generated by this secondary stroke helps to cancel the primary shed vortex, making the wake less detectable by predators. The shed vorticity may also influence the stability of the fish's motion, though there have not yet been significant studies of this effect. Finally, a fluid boundary layer along the fish's body induces drag, and sheds vorticity into the fish's wake. There is some evidence that a fish's geometry and tails motions may be adapted to recapture some of this lost energy, thereby improving efficiency.

In general, fish tails have a degree of flexibility,[3] but the caudal fins of carangiform swimmers are quite stiff and therefore it is a reasonable simplification to treat the tail as a rigid lifting surface.

Figure 4.2 gives an idea of how carangiform swimming can generate thrust without the benefit of a flexible tailfin. The tail of a fish is shown pitching and heaving up and down as the fish moves from right to left. The basic idea is that the tail maintains a negative angle of attack on the upstroke, and a positive angle of attack on the downstroke, with the result that the lift force on the hydrofoil/tail is always oriented so as to propel the fish forward [Lig75].

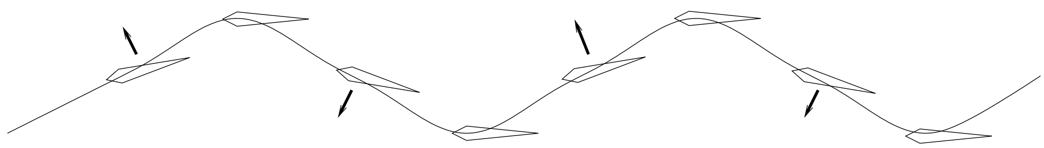


Figure 4.2: A "cartoon" of carangiform swimming consisting of snapshots of the tail as the fish swims from left to right. The arrows indicate lift forces acting on the tail. The fish body is removed for clarity.

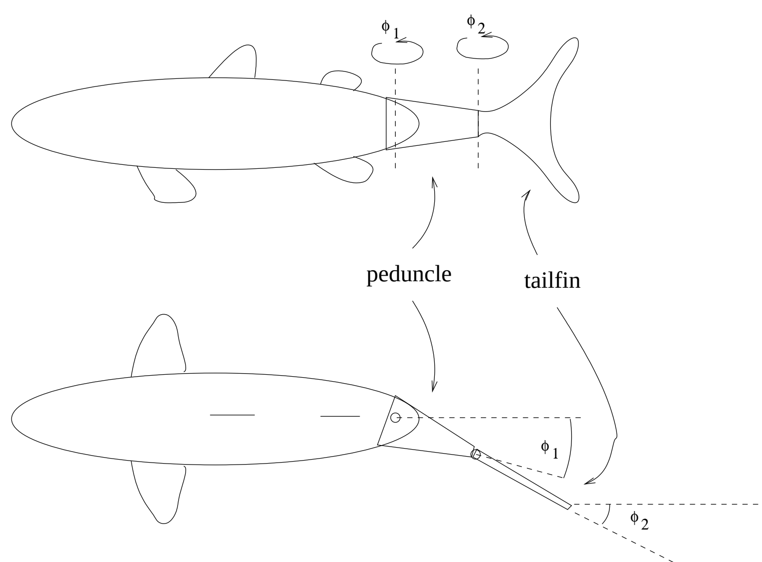


Figure 4.3: A two-jointed fish model. Our experiment replaces the three links with flat plates.

4.2 Description of the Experiment

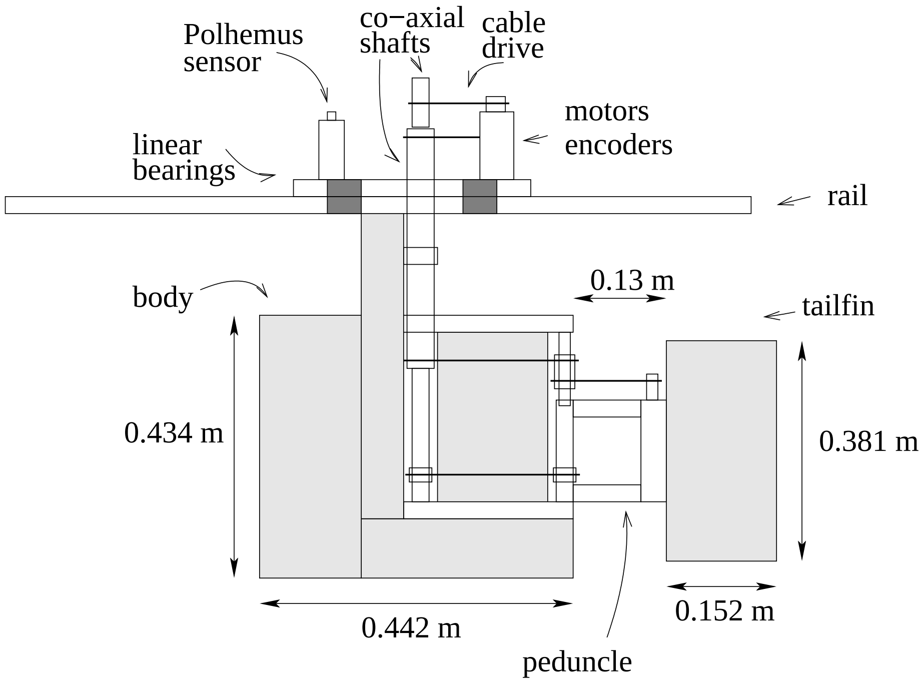
My colleagues and I built an experimental "fish" as a test-bed for our models. The experiment is intended to resemble an idealized carangiform fish that consists of only three links: a rigid body in front, a large wing-like tail at the rear, and a slender stem, or peduncle, which connects the two. The three rigid links are connected by rotational joints with joint angles θ_1 and θ_2. See Figure 4.3. I continue to idealize the model by supposing that I can neglect three-dimensional effects and regard the problem as essentially planar.

Figures 4.4 and 4.5 show schematic diagrams and photographs of the side and top views of the apparatus, while Figure 4.6 shows a photograph from beneath of an earlier prototype. The tailfin is a thin flat plate with a chord of 15.2 cm and a typical depth in the water of 38 cm. The peduncle is a thin supporting arm, 13 cm in length, which I believe experiences only negligible hydrodynamic forces. The body is a thick flat plate, intended to provide a degree of rotational inertia and rotational damping to help stabilize the fish during planar swimming.

The entire "fish" is suspended in a 4-foot-wide by 4-foot-deep by 36-foot-long water tank from a passive gantry-like multi-degree-of-freedom carriage. The supporting infrastructure consists of two orthogonal sets of rails and a rotating platform, all supported on low friction bearings. By flapping its tail, this mechanism allows the fish to propel itself and its supporting carriage around the tank. The frictional drag on the rails is sufficiently low that this carriage system is a reasonable approximation to untethered swimming. The system can move with three degrees of freedom in the plane: forward, sideways, and rotationally. Furthermore, the gantry suspension allows buoyancy effects to be ignored, thereby keeping the experiment focus on thrust generation and maneuvering. The carriage also simplifies the experiment by keeping the motors, electronics, etc. out of the water.

The tail and peduncle degrees of freedom are independently driven by two DC motors (Escap model 35 NT2 R82), each of which is capable of 75 W of power and

[3]Flexible tails might be considered deformable foils, as in Chapter 3.



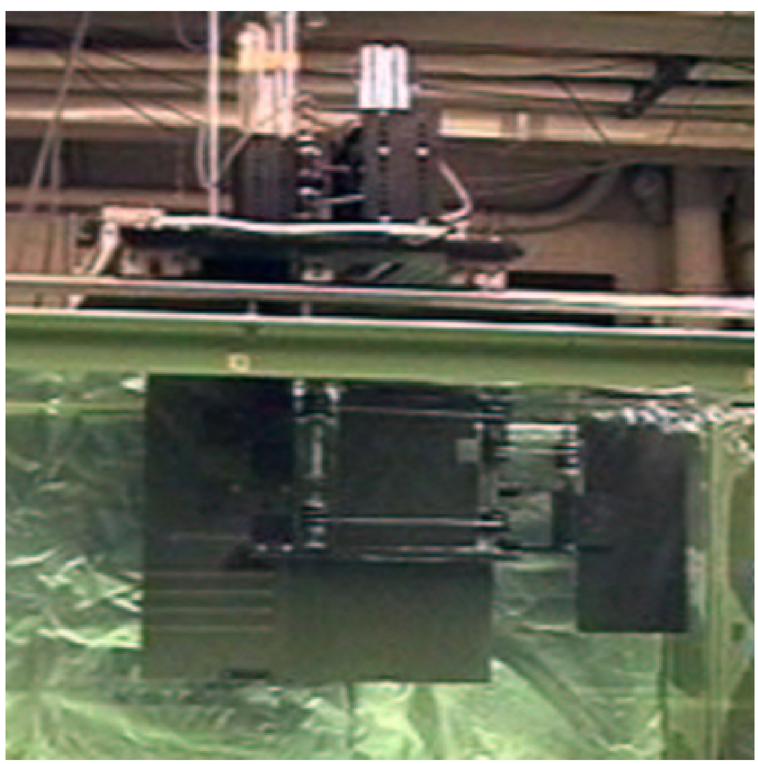
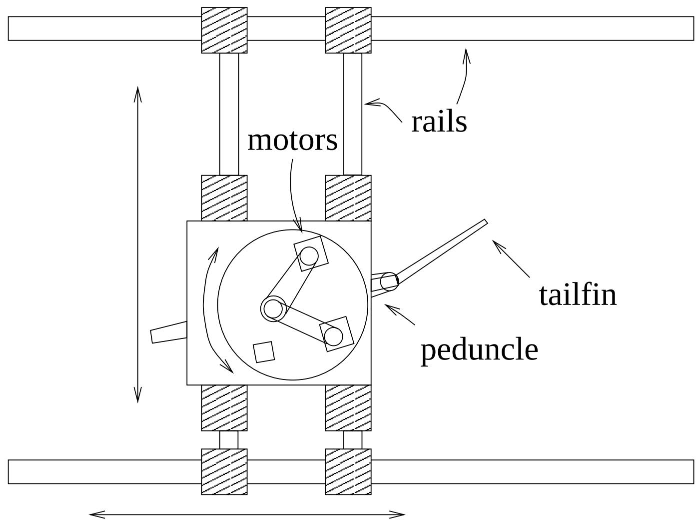


Figure 4.4: Side view schematic and photograph of experiment.



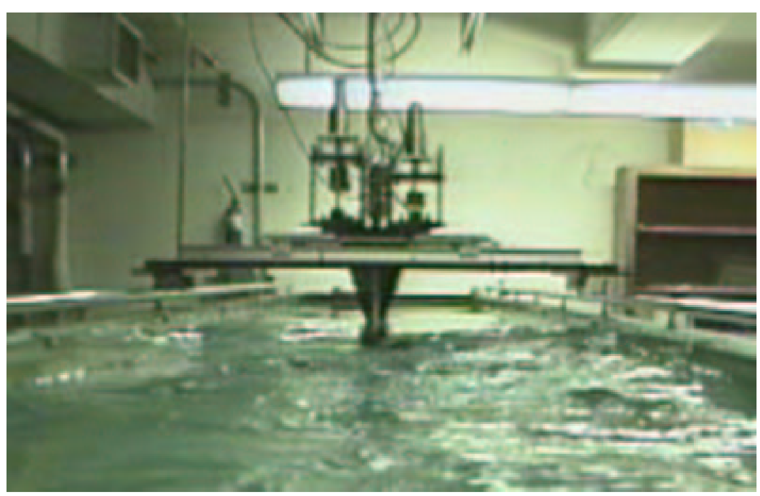


Figure 4.5: Top view schematic and rear top photo of experiment.

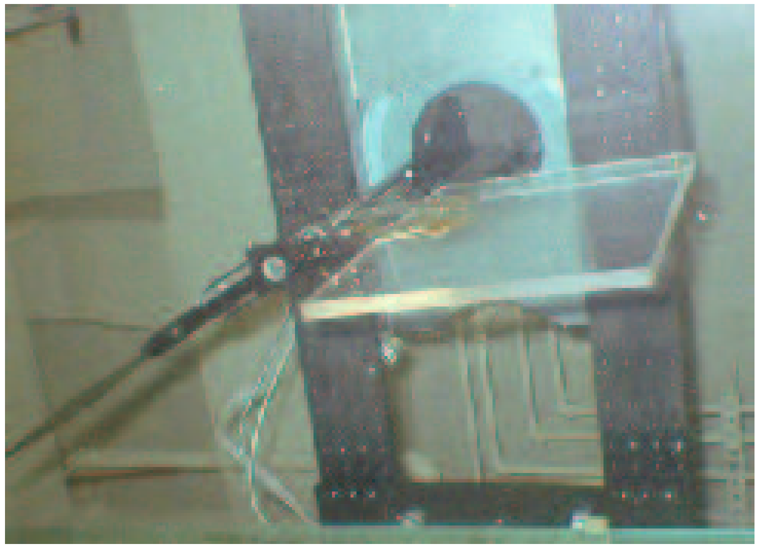


Figure 4.6: Bottom view of an earlier version of experiment.

110 mNm of torque. Timing belts and two coaxial shafts transmit power from the motors to the submerged joints of the fish. Also mounted on the carriage are optical shaft encoders (Hewlett-Packard HEDS-5500), which record the fish's joint angles at each instant and enable feedback control of the tail. The encoders are accurate to within about forty minutes of arc. Finally, a Polhemus position/orientation sensor is rigidly attached to the last stage of the carriage. In this way, the absolute position and orientation of the fish can be determined. Since every aspect of the fish's movement is instrumented, I may accurately compare experimental results with theory for the purposes of assessing the validity of simplified models.

For purposes of determining the center of mass or center of rotation of the system, the gantry carriage elements, which do not rotate with the submerged links but are connected to the three-link fish by a rotational bearing, can be regarded as a point mass located at the central axis of the rotational bearing. Since this mass is a significant fraction of the actual mass (if not the added mass) of the system, the location of the rotational bearing has a substantial impact on the location of the system's center of rotation. In an earlier version of the apparatus, the rotational bearing axis was located relatively far back on the body link and close to the peduncle. This setup could have been taken as equivalent to a free-floating swimmer with a short body or unusually massive hindquarters. This configuration had poor yaw stability.

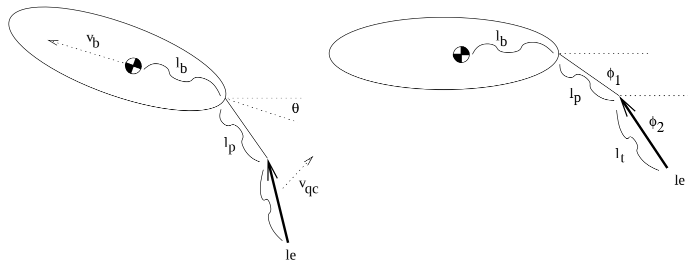


Figure 4.7: The idealized model.

Subsequently the experiment was modified to move the rotational axis forward relative to the body link. This resulted in dynamics more nearly comparable to those of a biological fish or a plausible free-swimming robot, and also improved the passive "weathervane" stability of the system with respect to yaw.

The mass of the entire robot and gantry system is $m = 35\,\mathrm{kg}$, and the moment of inertia of the body with the tail fully extended is $I_\Theta = 0.5038\,\mathrm{kg}\cdot\mathrm{m}^2$.

4.3 Model with Added Mass and Quasi-Steady Lift

In an effort to model the three-link experimental system in a relatively simple way, I treated the first link (body) as a source of drag, to be calculated empirically from measurements; the second link (peduncle) as hydrodynamically negligible; and the third link (tailfin) as a flat plate, generating lift and drag according to quasi-static two-dimensional wing theory. Figure 4.7 shows the geometry obtained by further idealizing the system in Figure 4.1. Let l_b be the distance between the body's center of rotation and the location of the body/peduncle connector. Let the total length of the body be l. The peduncle has length l_p. The tailfin has chord l_f and span d. Let $\vec{l_e}$ be a unit vector pointing in the direction of the leading edge of the tailfin hydrofoil.

$$\vec{l_e} = -\begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\phi_2) \\ \sin(\phi_2) \\ 0 \end{pmatrix}, \tag{4.1}$$

where θ is the orientation of the body in an inertial frame and ϕ_1, ϕ_2 are joint angles with respect to the body's longitudinal axis. Using the Kutta-Joukowski theorem and assuming that the tail hydrofoil is in a quasi-steady uniform flow with the velocity implied by the instantaneous velocity of the foil's quarter-chord point, I arrive at the following lift force on the hydrofoil:

$$L = \pi \rho_f \, l_f \, d(\vec{v}_{qc} \times \vec{l_e}) \times \vec{v}_{qc}, \tag{4.2}$$

where ρ_f is the density of the fluid, l_f is the chord of the tailfin plate, d is depth or span, and $\vec{v}_{qc}$ is the velocity of the *quarter-chord point* of the tailfin.[4]

From Lanchester-Prandtl wing theory, I estimate the drag on the tailfin at [AD59]:

$$D_f = -2\pi \rho_f \, l_f^2 \, \vec{v}_{qc} \, \frac{\|v_{qc}\|^2 - (\vec{v}_{qc} \cdot \vec{l_e})^2}{\|v_{qc}\|} \tag{4.3}$$

Meanwhile the quasi-static torque generated around the midpoint of the tail is:

$$\tau_f = -\pi \rho_f d \frac{l_f^2}{4} \left(\dot{x}_m \dot{y}_m \cos(2\theta + 2\phi_2) + \frac{(\dot{y}_m^2 - \dot{x}_m^2)}{2} \sin(2\theta + 2\phi_2) \right), \tag{4.4}$$

where $(\dot{x}_m, \dot{y}_m)$ is the velocity of the tail's midpoint.

The body of the fish has instantaneous translational velocity $\vec{v_b} = (\dot{x}, \dot{y}, 0)$; it also has instantaneous rotational velocity $\dot{\theta}$. The velocity of the hydrofoil's quarter-chord point and midpoint are respectively:

$$\begin{aligned}
\vec{v}_{qc} &= \begin{pmatrix} \dot{x} - l_b \sin(\theta)\dot{\theta} - l_p(\dot{\theta} + \dot{\phi}_1)\sin(\theta + \phi_1) - (l_f/4)(\dot{\theta} + \dot{\phi}_2)\sin(\theta + \phi_2) \\ \dot{y} + l_b \cos(\theta)\dot{\theta} + l_p(\dot{\theta} + \dot{\phi}_1)\cos(\theta + \phi_1) + (l_f/4)(\dot{\theta} + \dot{\phi}_2)\cos(\theta + \phi_2) \\ 0 \end{pmatrix} \\
\begin{pmatrix} \dot{x}_m \\ \dot{y}_m \\ 0 \end{pmatrix} &= \begin{pmatrix} \dot{x} - l_b \sin(\theta)\dot{\theta} - l_p(\dot{\theta} + \dot{\phi}_1)\sin(\theta + \phi_1) - (l_f/2)(\dot{\theta} + \dot{\phi}_2)\sin(\theta + \phi_2) \\ \dot{y} + l_b \cos(\theta)\dot{\theta} + l_p(\dot{\theta} + \dot{\phi}_1)\cos(\theta + \phi_1) + (l_f/2)(\dot{\theta} + \dot{\phi}_2)\cos(\theta + \phi_2) \\ 0 \end{pmatrix}.
\end{aligned}$$

The drag acting on the body of the fish was empirically measured by towing

[4] This implicitly assigns the wing a coefficient of lift $C_L = 2\pi \sin(\alpha)$. Then the drag estimate follows from the relationship $C_D = \frac{C_L^2}{\pi(d/l_f)}$, where d/l_f is the aspect ratio of the tailfin.

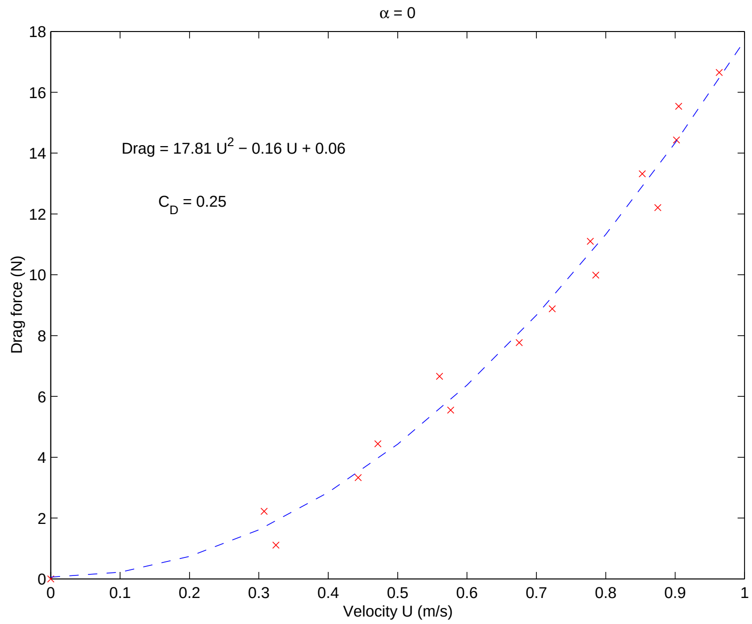


Figure 4.8: Drag force acting on the fish in forward motion.

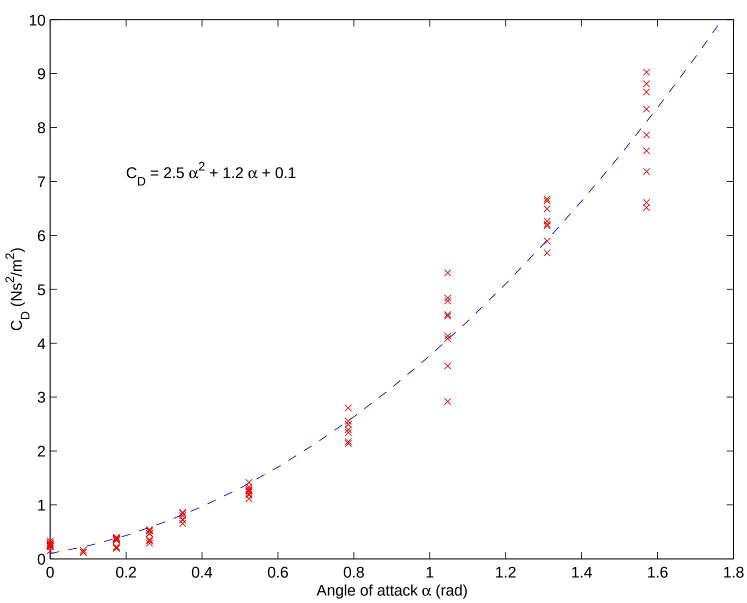


Figure 4.9: Drag force acting on the fish at varied angles of attack.

the apparatus through the water with a variety of falling weights and measuring the terminal velocity reached for a given amount of applied force. The drag force was found to be approximately quadratic with velocity. Figure 4.8 shows the drag force on the body moving straight forward at zero angle of attack. The drag, about $17.8\,\mathrm{Ns^2/m^2}$, can be normalized by the water density and the area of the body:

$$D = \frac{1}{2} C_D \, \rho_f \, l \, d \, U^2, \tag{4.5}$$

where $\rho_f = 1000\,\mathrm{kg/m^3}$, $l = 46\,\mathrm{cm}$, and $d = 31\,\mathrm{cm}$, to yield a normalized coefficient of drag $C_D = 0.25$ at $\alpha = 0$. However, the measured drag includes not only the various forms of hydrodynamical drag, but also any resistance due to the rail bearings. Figure 4.9 shows the experimentally measured coefficient of drag C_D for the fish body at a variety of different angles of attack α.

Dissipative forces acting against rotation were not measured explicitly, but they can be estimated if I suppose that the measured coefficients of drag C_D can be applied to incremental elements of the plate, each of which moves with a different velocity as the body rotates.

The total drag force acting on the body is

$$\begin{aligned} D_b &= -\frac{1}{2} \rho_f \, d \int_{l_b - l}^{l_b} C_D \|v(s)\|^2 \frac{v(s)}{\|v(s)\|} \, ds \\ &= -\frac{1}{2} \rho_f \, d \int_{l_b - l}^{l_b} C_D \|v(s)\| v(s) \, ds, \end{aligned} \tag{4.6}$$

where

$$v(s) = \begin{pmatrix} \dot{x} - \sin(\theta) \, s \, \dot{\theta} \\ \dot{y} + \cos(\theta) \, s \, \dot{\theta} \end{pmatrix}. \tag{4.7}$$

The total drag moment acting on the body is

$$\begin{aligned}\tau_b &= -\frac{1}{2}\rho_f\, d \int_{l_b-l}^{l_b} C_D \|v(s)\|^2 \frac{(s\cos(\theta), s\sin(\theta)) \times v(s)}{\|v(s)\|}\, ds \\ &= -\frac{1}{2}\rho_f\, d \int_{l_b-l}^{l_b} C_D \|v(s)\| (\cos(\theta)\dot{y} - \sin(\theta)\dot{x} + s\dot{\theta})\, s\, ds. \end{aligned} \tag{4.8}$$

Finally, I consider the added inertia from the fluid surrounding the body. The planar added mass coefficients for a flat plate of length l in a frame aligned with the plate at its midpoint are[5]

$$I_{11} = 0 \tag{4.9}$$

$$I_{12} = 0 \tag{4.10}$$

$$I_{13} = 0 \tag{4.11}$$

$$I_{22} = \pi\rho_f d(l/2)^2 \tag{4.12}$$

$$I_{23} = 0 \tag{4.13}$$

$$I_{33} = 2\pi\rho_f d(l/4)^4. \tag{4.14}$$

In another frame whose origin is located at (ξ, η) in the plate frame, and rotated by angle β, the transformed added mass coefficients are [Sed65]

$$\begin{aligned} I'_{11} &= I_{11}\cos^2(\beta) + I_{22}\sin^2(\beta) + I_{12}\sin(2\beta) \\ I'_{12} &= \frac{1}{2}(I_{22} - I_{11})\sin(2\beta) + I_{12}\cos(2\beta) \\ I'_{13} &= (I_{11}\eta - I_{12}\xi + I_{13})\cos(\beta) + (I_{12}\eta - I_{22}\xi + I_{23})\sin(\beta) \\ I'_{22} &= I_{11}\sin^2(\beta) + I_{22}\cos^2(\beta) - I_{12}\sin(2\beta) \\ I'_{23} &= -(I_{11}\eta - I_{12}\xi + I_{13})\sin(\beta) + (I_{12}\eta - I_{22}\xi + I_{23})\cos(\beta) \\ I'_{33} &= I_{11}\eta^2 + I_{22}\xi^2 - 2I_{12}\xi\eta + 2(I_{13}\eta - I_{23}\xi) + I_{33}. \end{aligned} \tag{4.15}$$

[5] These can be obtained from the results for the locked added inertia of a Joukowski foil, presented in Section 3.3, with the substitutions $\zeta_c = \delta = 0$ and $r_c = a = (l/4)$.

So at the center of rotation of the body, the added inertia due to the body plate is

$$\begin{aligned}
I_{11}^{\text{body}} &= \pi\rho_f d(l/2)^2 \sin^2(\theta) \\
I_{12}^{\text{body}} &= -\frac{1}{2}\pi\rho_f d(l/2)^2 \sin(2\theta) \\
I_{13}^{\text{body}} &= 0 \\
I_{22}^{\text{body}} &= \pi\rho_f d(l/2)^2 \cos^2(\theta) \\
I_{23}^{\text{body}} &= 0 \\
I_{33}^{\text{body}} &= 2\pi\rho_f d(l/4)^4,
\end{aligned} \tag{4.16}$$

while the added inertia due to the tailfin plate is

$$\begin{aligned}
I_{11}^{\text{tail}} &= \pi\rho_f d(l_f/2)^2 \sin^2(\theta+\phi_2) \\
I_{12}^{\text{tail}} &= -\frac{1}{2}\pi\rho_f d(l_f/2)^2 \sin(2\theta+2\phi_2) \\
I_{13}^{\text{tail}} &= \pi\rho_f d(l_f/2)^2 \sin(\theta+\phi_2)\left[(x-x_m)\cos(\theta+\phi_2)+(y-y_m)\sin(\theta+\phi_2)\right] \\
I_{22}^{\text{tail}} &= \pi\rho_f d(l_f/2)^2 \cos^2(\theta+\phi_2) \\
I_{23}^{\text{tail}} &= -\pi\rho_f d(l_f/2)^2 \cos(\theta+\phi_2)\left[(x-x_m)\cos(\theta+\phi_2)+(y-y_m)\sin(\theta+\phi_2)\right] \\
I_{33}^{\text{tail}} &= \pi\rho_f d(l_f/2)^2 \left[(x-x_m)^2\cos^2(\theta+\phi_2)+(y-y_m)^2\sin^2(\theta+\phi_2)\right. \\
&\quad \left.+(x-x_m)(y-y_m)\sin(2\theta+2\phi_2)\right] + 2\pi\rho_f d(l_f/4)^4,
\end{aligned} \tag{4.17}$$

where the tailfin midpoint (x_m, y_m) is

$$\begin{pmatrix} x_m \\ y_m \end{pmatrix} = \begin{pmatrix} x + l_b\cos(\theta) + l_p\cos(\theta+\phi_1) + (l_f/2)\cos(\theta+\phi_2) \\ y + l_b\sin(\theta) + l_p\sin(\theta+\phi_1) + (l_f/2)\sin(\theta+\phi_2) \end{pmatrix}. \tag{4.18}$$

These virtual inertias are added to the ordinary inertia of the system.

$$I = \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & I_\theta \end{pmatrix} \tag{4.19}$$

$$I^{\text{total}} = I + I^{\text{body}} + I^{\text{tail}} \tag{4.20}$$

Finally, the tailfin sees an additional added mass force and moment f_{am} and τ_{am} due to the acceleration component of the tailfin which is independent of $(\ddot{x}, \ddot{y}, \ddot{\theta})$.

$$f_{am} = -\pi\rho_f d\left(\frac{l_f}{2}\right)^2 \begin{pmatrix} \sin^2(\theta+\phi_2) & -\frac{1}{2}\sin(2\theta+2\phi_2) \\ -\frac{1}{2}\sin(2\theta+2\phi_2) & \cos^2(\theta+\phi_2) \end{pmatrix} \begin{pmatrix} \ddot{x}'_m \\ \ddot{y}'_m \end{pmatrix} \tag{4.21}$$

$$\begin{aligned} \ddot{x}'_m &= (-l_b\cos(\theta) - l_p\cos(\theta+\phi_1) - (l_f/2)\cos(\theta+\phi_2))\dot{\theta}^2 - l_p\cos(\theta+\phi_1)\dot{\phi_1}^2 \\ &\quad -(l_f/2)\cos(\theta+\phi_2)\dot{\phi_2}^2 - l_p\sin(\theta+\phi_1)\ddot{\phi_1} - (l_f/2)\sin(\theta+\phi_2)\ddot{\phi_2} \end{aligned} \tag{4.22}$$

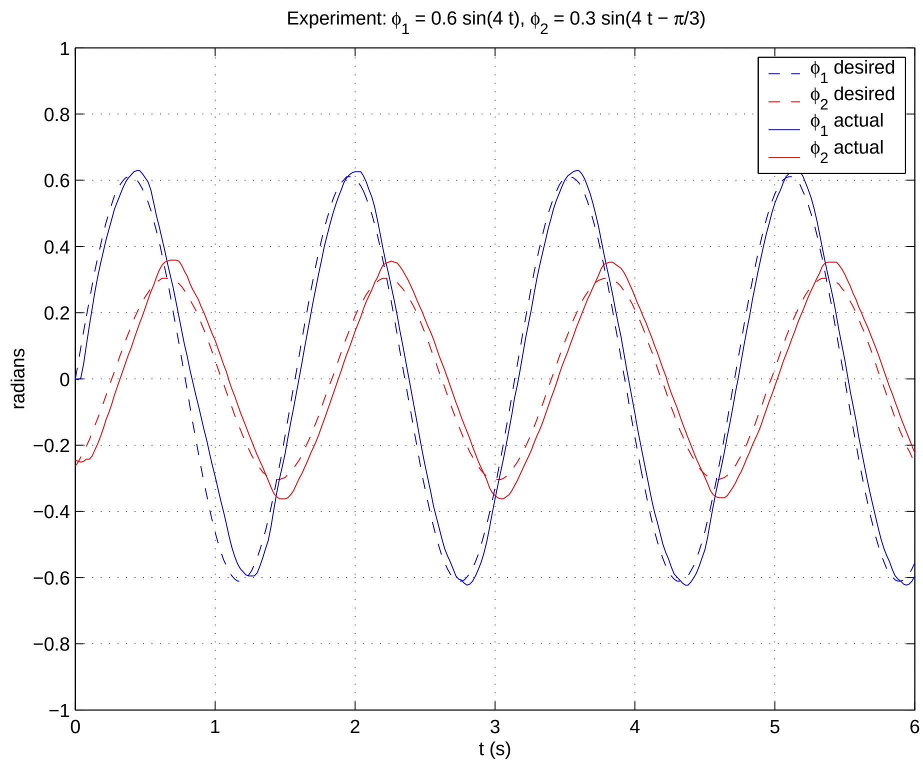
$$\begin{aligned} \ddot{y}'_m &= (-l_b\sin(\theta) - l_p\sin(\theta+\phi_1) - (l_f/2)\sin(\theta+\phi_2))\dot{\theta}^2 - l_p\sin(\theta+\phi_1)\dot{\phi_1}^2 \\ &\quad -(l_f/2)\sin(\theta+\phi_2)\dot{\phi_2}^2 + l_p\cos(\theta+\phi_1)\ddot{\phi_1} + (l_f/2)\cos(\theta+\phi_2)\ddot{\phi_2} \end{aligned} \tag{4.23}$$

$$\tau_{am} = -2\pi\rho_f d(l_f/4)^4\ddot{\phi_2} \tag{4.24}$$

At last, then, the equation of motion for the system is

$$I^{\text{total}} \begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\theta} \end{pmatrix} = \begin{pmatrix} L + D_f + D_b + f_{am} \\ \tau_f + \tau_b + \tau_{am} + (x_m - x, y_m - y) \times (L + f_{am}) \end{pmatrix} \tag{4.25}$$

Some things neglected in this model: By taking a quasi-static approximation to the lift force, I am disregarding any special spatial structure of the wake—roughly speaking, I am treating any vorticity shed from the tailfin as if it were swept away and immediately became very distant. The actual, as opposed to virtual, mass of the tailfin is considered small enough that I neglect any changes in I_θ due to changes in joint angle. I treat the bodies in isolation and ignore any hydrodynamical interactions between them, as well as any forces whatsoever acting on the peduncle. I also neglect some mundane details of the experimental apparatus: I ignore the presence of the tank walls, surface wave effects, and any stiction or resistance from the gantry bearings, except insofar as they may have influenced our empirical measurement of drag forces.



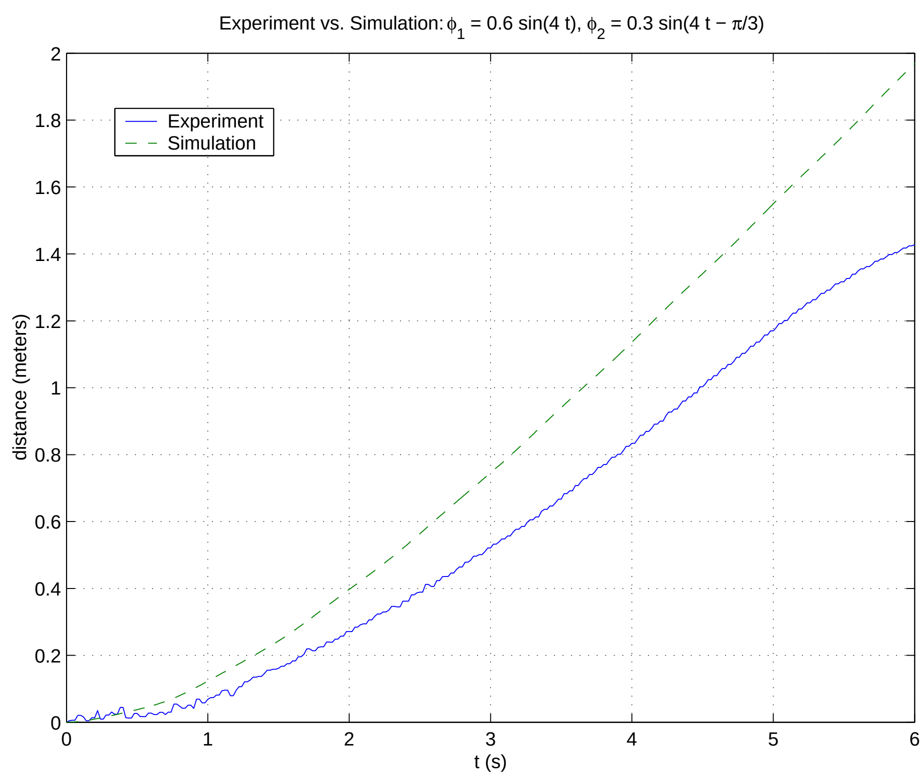


Figure 4.10: Experiment and simulation for the forward gait with $\phi_1 = 0.6\sin(4.0t)$, $\phi_2 = 0.3\sin(4.0t + \pi/3)$. A PD controller drives the joints to follow the desired trajectory. The simulation slightly overpredicts thrust (or underpredicts drag.)

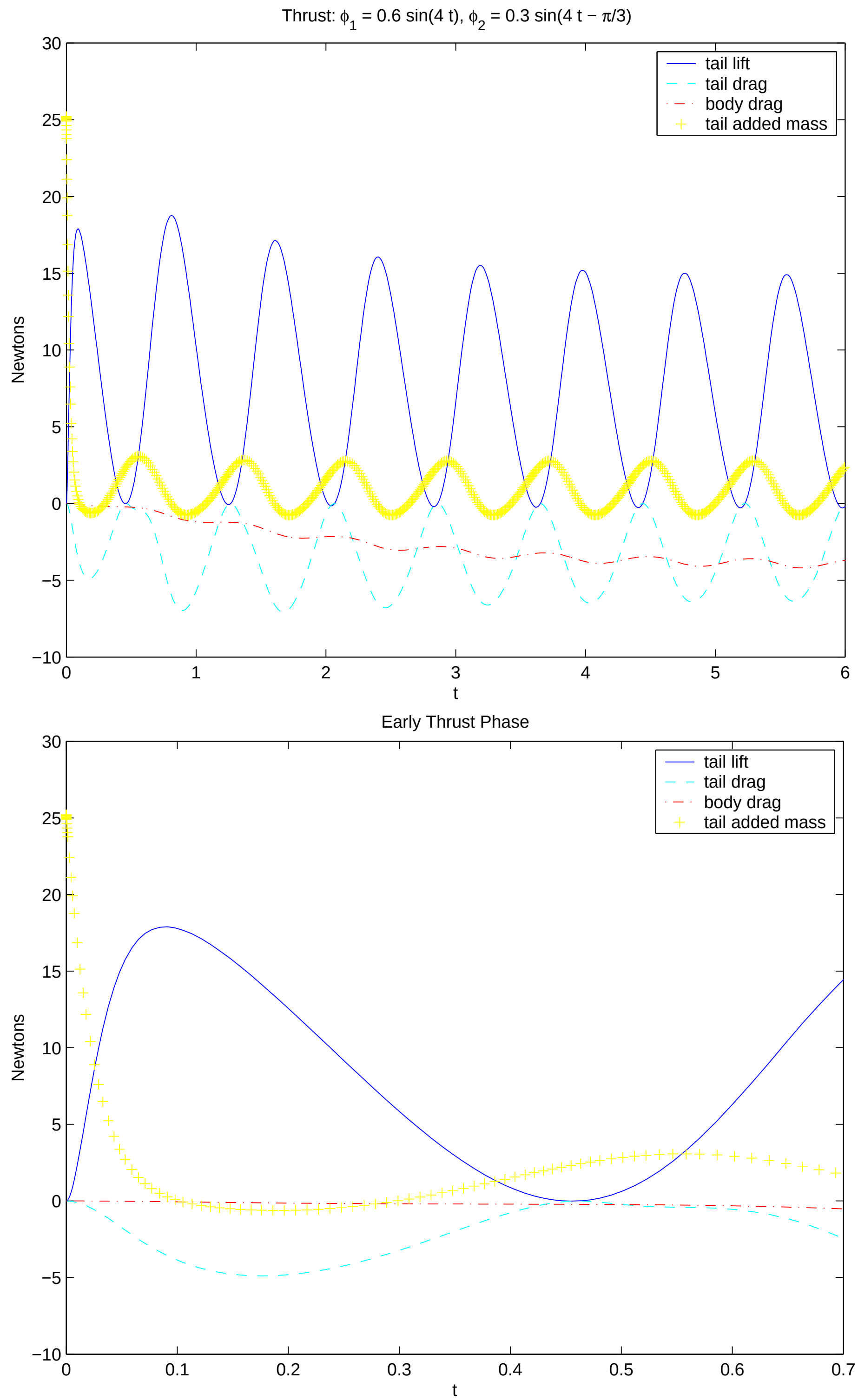


Figure 4.11: Forward thrust generated by the gait $\phi_1 = 0.6\sin(4.0t)$, $\phi_2 = 0.3\sin(4.0t + \pi/3)$, from Figure 4.10. Almost immediately the fish enters a periodic cycle where lift is the dominant source of thrust, but added mass forces on the tailfin are significant at the very start of thrust.

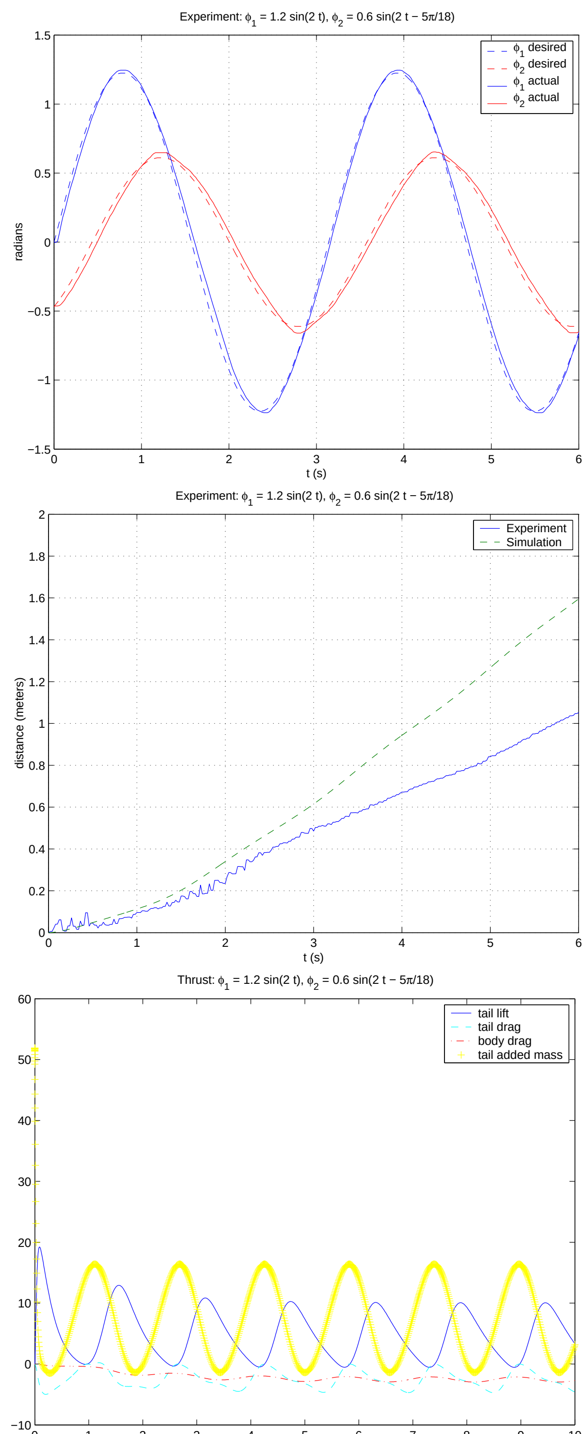


Figure 4.12: Experiment and simulation for the gait $\phi_1 = 1.2\sin(2.0t)$, $\phi_2 = 0.6\sin(2.0t + 5\pi/18)$. Both lift forces and added mass forces are significant components of the thrust.

Despite these omissions, this model does a fairly good job of describing the qualitative behavior of the system. To compare the experiment to the simulation model, the computer simulation received as input the actual joint angle trajectories for each gait as reported by the optical shaft encoders. Based on these inputs, the model equations of motion were integrated by a Runge-Kutta method, and the simulated fish's displacement is superimposed on the analogous graph of experimental data. I carried out these comparisons for a large number of forward-swimming gaits with inputs sinusoidal in ϕ_1 and ϕ_2. Examples are shown in Figures 4.10 and 4.12. See also Figures 4.17-4.18. For most of these large-amplitude gaits, lift forces acting on the tailfin are the predominant source of thrust, except for the very beginning of motion when added mass forces on the tailfin are significant. See Figure 4.11. Occasionally both lift forces and added mass forces will be significant in the steady state, as in Figure 4.12.

These large-amplitude gaits would have generated large angular oscillations, if I had allowed it: for these gaits, the fish was only allowed to swim forward longitudinally. The lateral and yaw degrees of freedom were removed, so stability of the gait with respect to those degrees of freedom was not an issue. The practical utility of this propulsion in one dimension is that it is the limiting case of a biomimetic swimmer with a large moment of inertia, e.g., a boat with a flapping propulsor small relative to the boat. These gaits could also be used by a set of paired propulsors, or by a biomimetic swimmer with some other means of stabilization. Our fish robot as currently configured, however, does not have a large enough moment of inertia to use these gaits effectively in full planar motion.

Instead, to propel itself in the plane while keeping a stable heading, the robot fish must use small-amplitude high-frequency strokes. Presumably this will also hold true for future small free-swimming robots. A successful gait for forward propulsion of a small robot that is free to rotate or deviate from its heading is illustrated in Figure 4.13. The inputs shown are $\phi_1 = 0.4\sin(8t)$, $\phi_2 = 0.4\sin(8t + \pi/3)$. Note that I have centered the heading θ around $\theta = \pi$ so that positive motion in the x-direction is forward motion. The achievable speeds are notably lower than for the large-amplitude

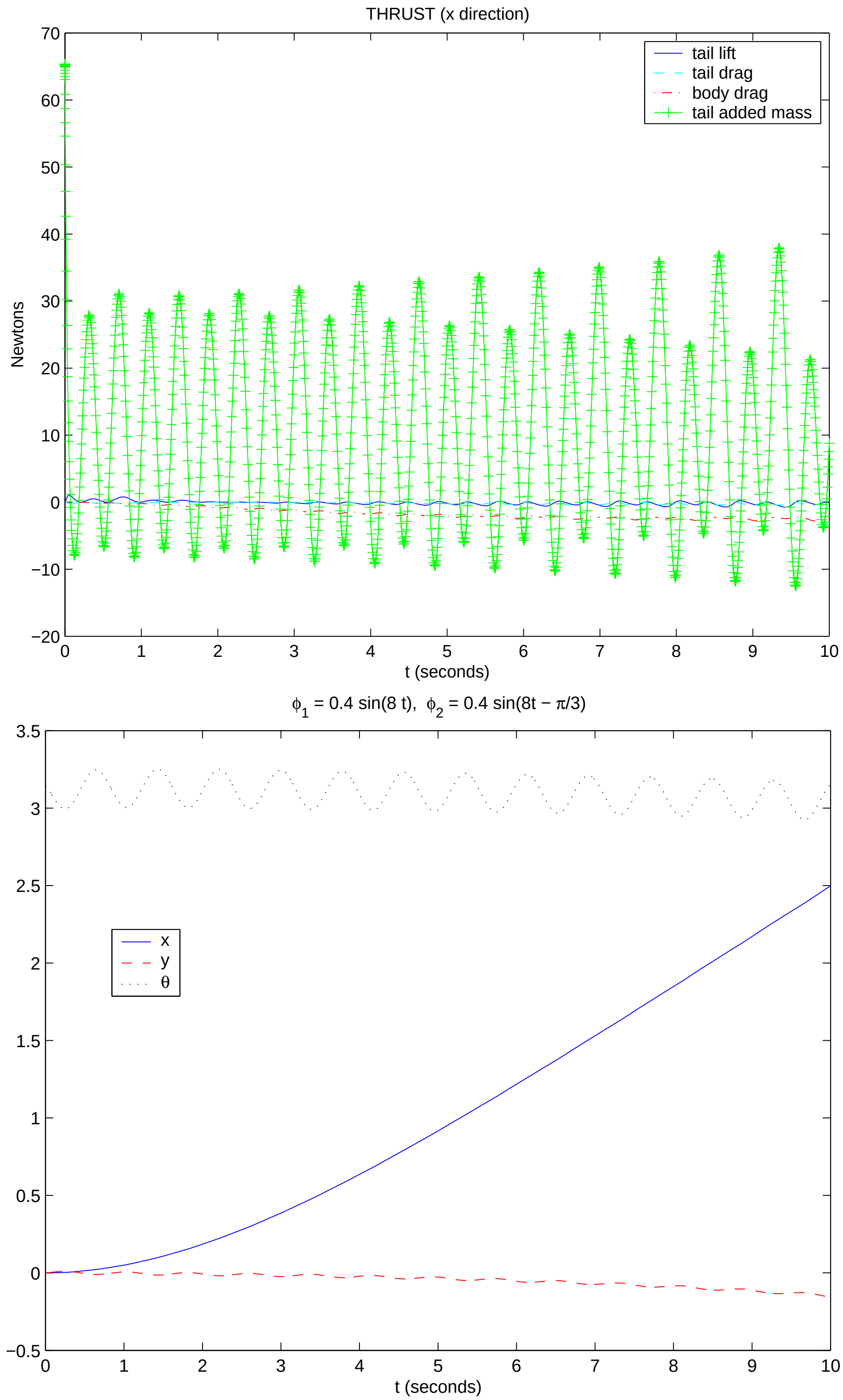


Figure 4.13: A gait which results in stable forward propulsion even when the fish is free to rotate or deviate. Note that added mass forces dominate lift and drag forces. Since θ is near π, the forward direction is along the positive x-axis.

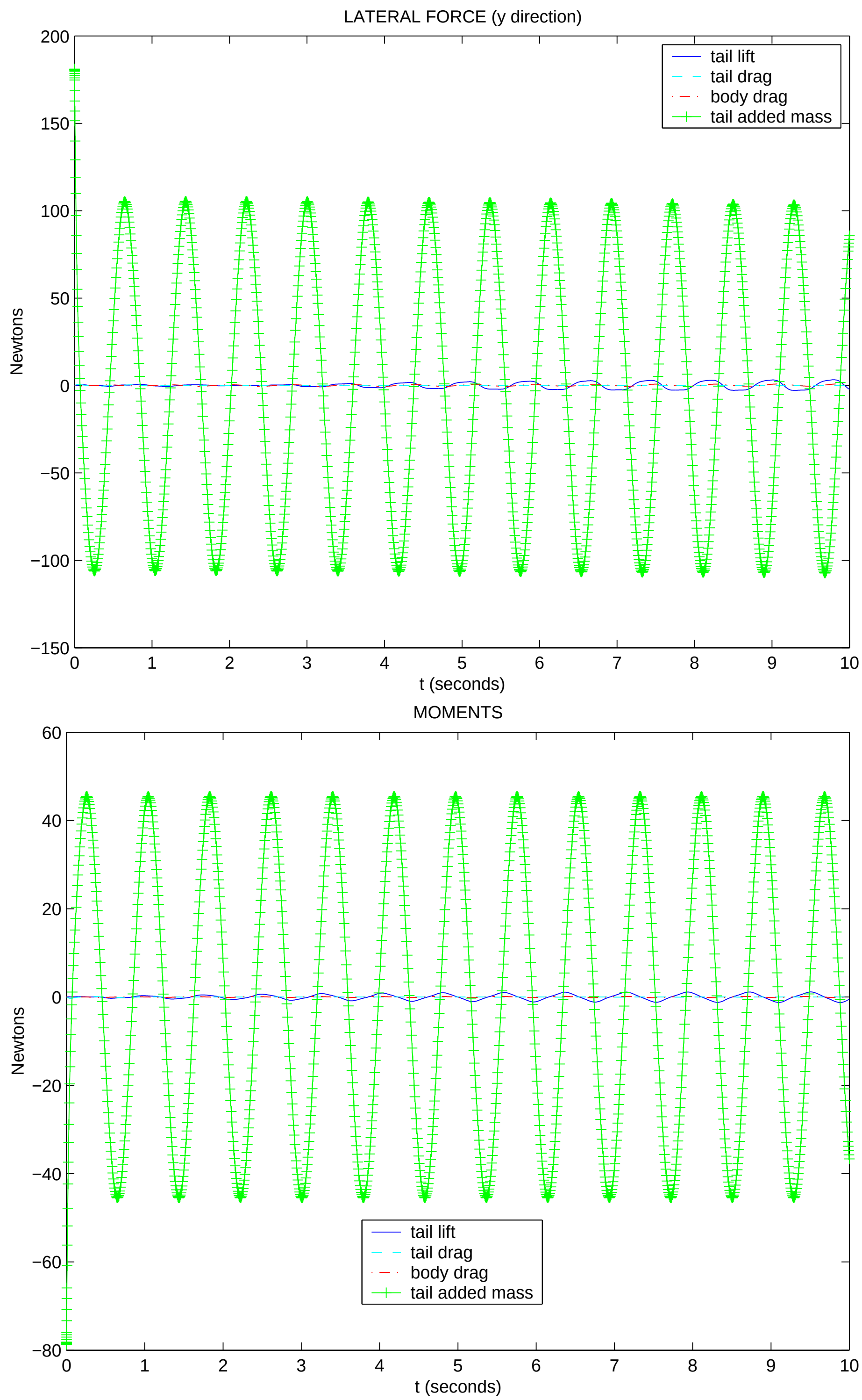


Figure 4.14: Lateral forces and moment generated by the gait in Figure 4.13.

gaits constrained to one dimension. Interestingly, for this free-swimming gait, added mass forces, not lift forces, are the predominant source of thrust.

Also, added mass forces play a vital role in turning maneuvers. Morgansen [MDM+01] suggested the turning gait illustrated in Figure 4.15 with $\phi_1 = 0.4\sin(3.5t)$ and $\phi_2 = 0.4\sin(7t)$.

The reasonably good validation of the model by experiment suggests that this simple set of ODEs can be used as a design tool in biomimetic swimming without necessarily having to resort to computationally intensive fluid models. Perhaps most importantly, Morgansen, Duindam, Vela, and Burdick [MDM+01, MVB02] were able to use this platform and model to perform feedback control experiments: to our knowledge the first instance of a robot fish maintaining a trajectory by feedback control.

4.4 Miscellaneous Experimental Observations

I experimented with a large number of possible joint trajectories, especially sinusoidal inputs of varying amplitude, frequency, and phase, to discover the "best" gaits, which I usually defined as being those with the highest acceleration and forward speed, although I was also interested to note especially smooth or jerky gaits, and later gaits with turning behavior. Figures 4.17–4.18 show steady-state velocities for some of the gaits tested.

Triantafyllou et al. [TTG93] emphasized the importance of the dimensionless Strouhal number, defined as $\mathrm{St} = \frac{\mathrm{fA}}{\mathrm{V}}$, where f is the frequency of the tailfin oscillation, A is the double amplitude of the tail-to-tip excursion, and V is the average forward swimming velocity. Triantafyllou et al. advanced arguments that the Strouhal number of an oscillating foil system should be in the range 0.25–0.35 for optimum thrust. They also cited biological observations suggesting that a wide range of fish and cetaceans actually do operate in this range. In Figure 4.20, I plot the Strouhal number for the same set of sinusoidal gaits represented in Figure 4.19. It will be seen that the Strouhal number for our system is precisely in the optimum range, particularly for

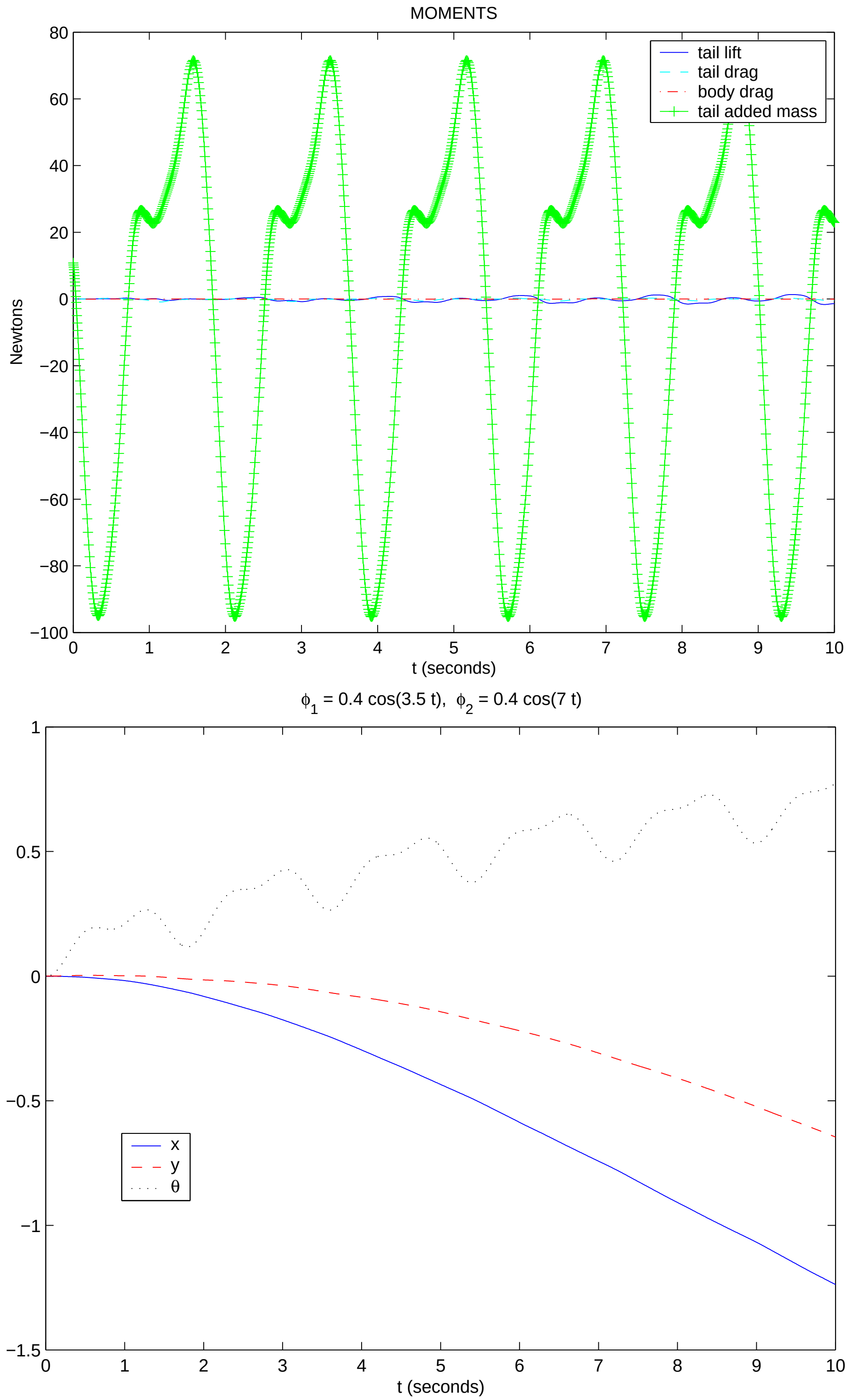


Figure 4.15: A turning gait. Note that added mass effects dominate the moment. Since θ starts near zero, the forward direction is initially aligned with the negative x-axis.

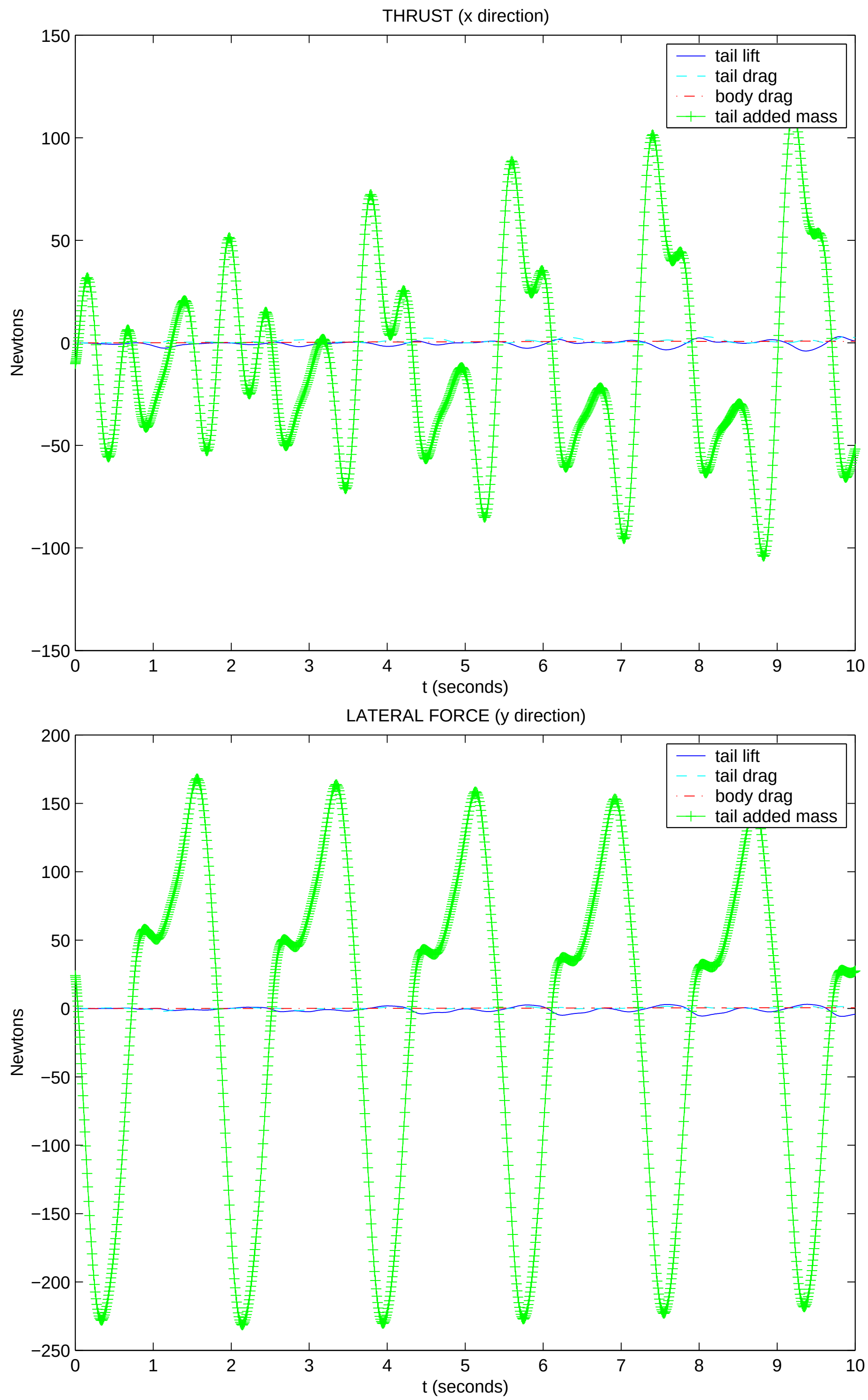


Figure 4.16: Forces experienced in the turn of Figure 4.15. The x and y forces are in a world frame, not fixed in the body of the fish.

the fastest gaits. (Note that the horizontal axes in Figures 4.20 and 4.19 are reversed, to increase visibility of the surfaces.) This tends to further support Triantafyllou's observations and also suggests that our system is operating in a fluid mechanical regime fairly representative of biological fish swimming.

After identifying successful gaits for forward swimming, we began to experiment with unconstrained maneuvers such as turns. Some early turning gaits were found by trial and error (see Figure 4.21), and later some were derived more formally from the model [MDM+01] (see Figure 4.15.) However to date, turning maneuvers are relatively awkward and disappointing relative to the dramatic turning performance of biological fish [Wei72], although the robot's ability to turn in place may be competitive relative to other artificial vehicles of similar power and complexity. Biological fish curve their whole spines during turns, including significant motion of the head, and it might be that a three-link model with rigid head and body is insufficient to properly capture this aspect of fish swimming.

Changes in the stroke's range of motion to cause turns should be synchronized to the periodic motion of the tail, so probably the tail should only be re-oriented to a new range of motion in between propulsive tail beats. This suggests a planning method. If a variety of strokes are each known, through simulation or experiment, to produce a given impulse and angular impulse per beat, then the fish could choose individual beats as necessary to add a quantized amount of momentum or angular momentum. Maneuvers could then be built up of series of such individual strokes. We could draw an analogy between these quantized strokes and the "impulse bits" of thrusters used to orient spacecraft.

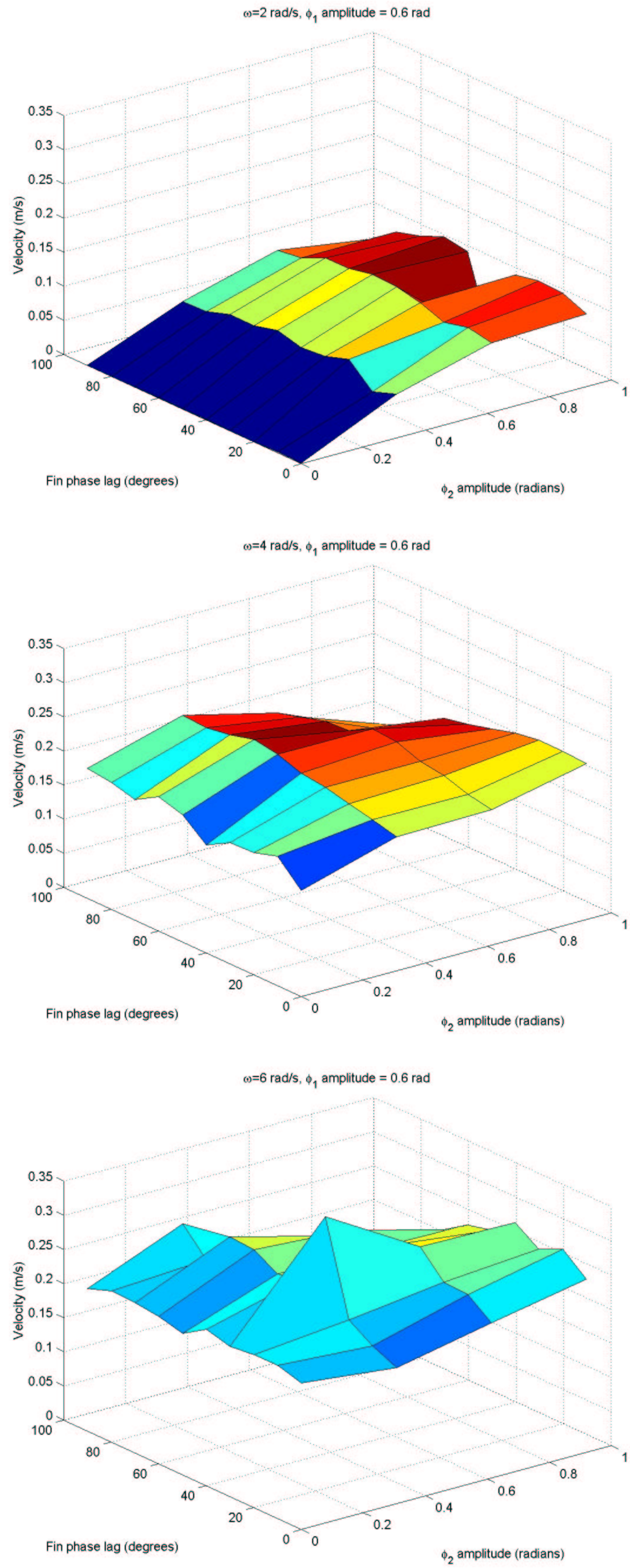


Figure 4.17: Steady-state velocities for single-frequency sinusoidal gaits. In each subplot, frequency and peduncle amplitude are held constant while steady-state velocity

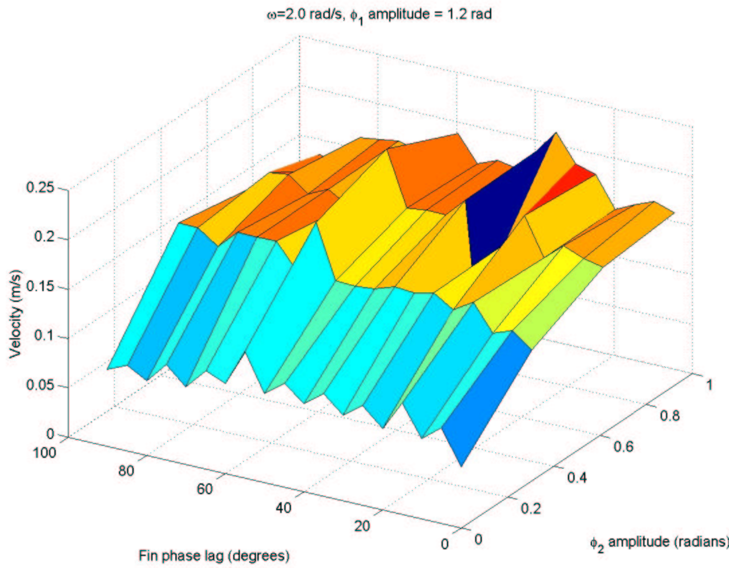


Figure 4.18: Steady-state velocities of another set of gaits with higher amplitude of peduncle oscillation.

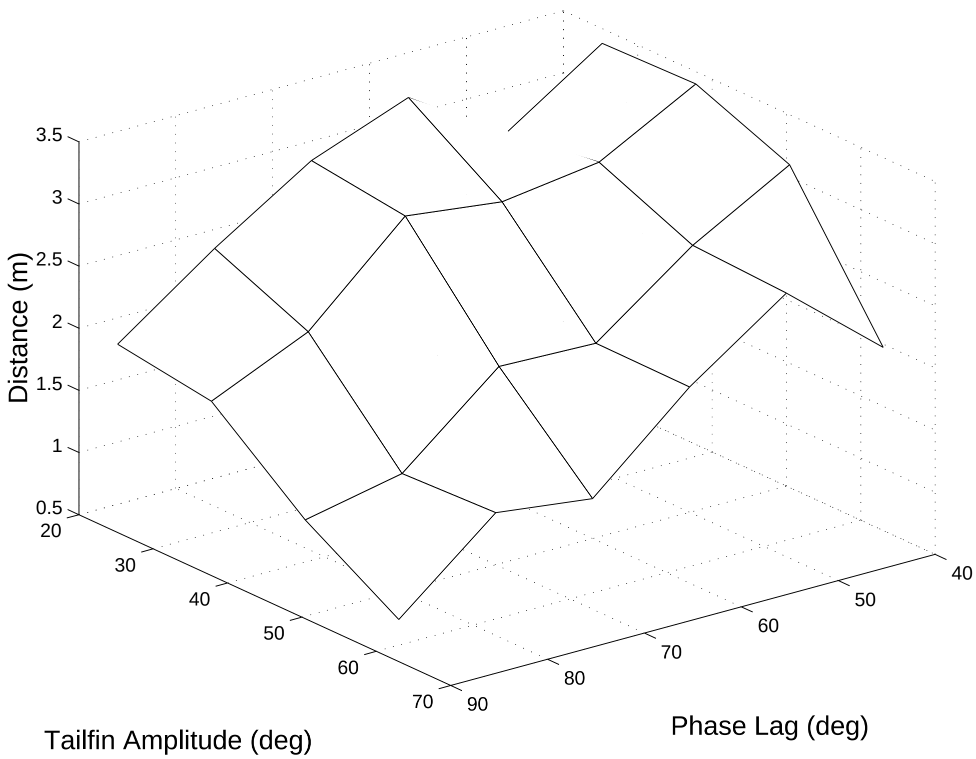


Figure 4.19: Distance traveled by the fish in eight seconds, for gaits of the form $\theta_1 = 1.3\sin(3.5t)$ and $\theta_2 = A\sin(3.5t + \psi)$. The horizontal axes represent the parameters A and ψ.

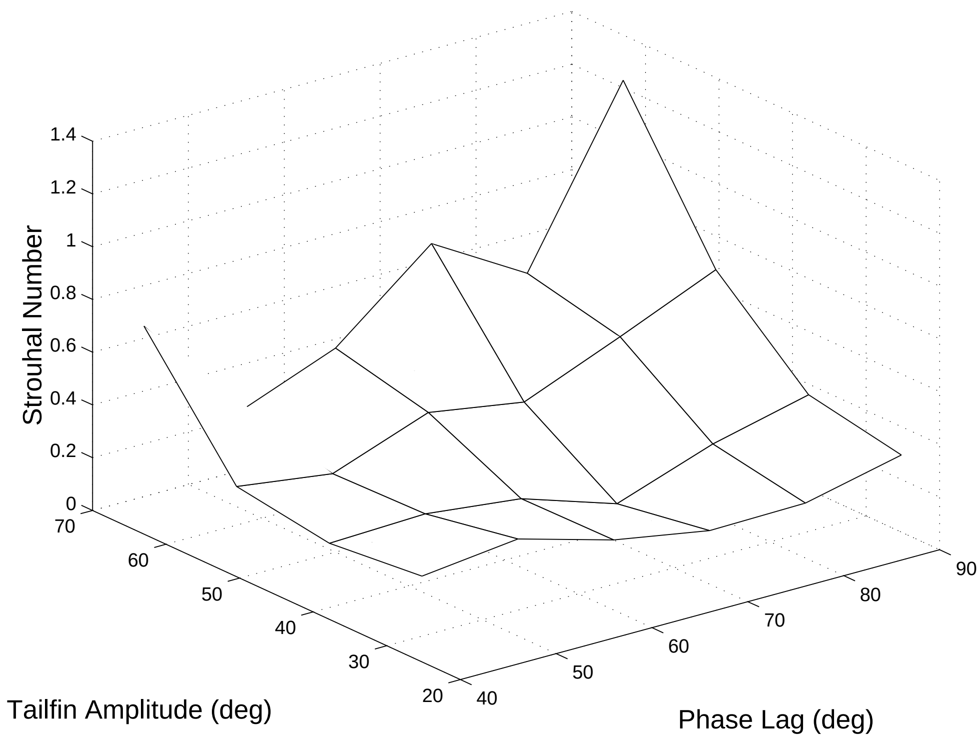


Figure 4.20: Strouhal numbers, for gaits of the form $\theta_1 = 1.3\sin(3.5t)$ and $\theta_2 = A\sin(3.5t + \psi)$. The horizontal axes represent the parameters A and ψ.

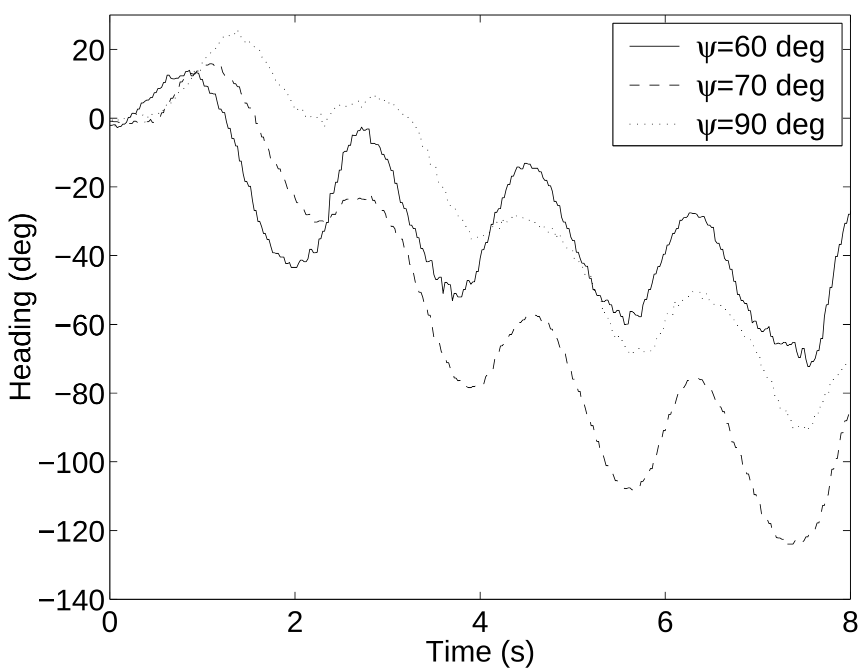


Figure 4.21: Orientation change over time, for gaits of the form $\theta_1 = -0.6 + 1.3\sin(3.5t)$ and $\theta_2 = 1.1\sin(3.5t + \psi)$.

Chapter 5 Optimal Control

As was illustrated in Figures 4.13 and Figures 4.15, our experience with the robot carangiform fish indicates that for orientational stability, small biomimetic swimmers may have to operate in a low-speed, high-frequency regime where added mass forces dominate lift and drag forces. Turns and low-speed maneuvers, the operations where fishlike vehicles are most likely to outperform conventional watercraft, also seem to fall into this regime.

With this in mind, in this chapter I will look at the equations of motion for the three-link fish if only added mass effects are retained and lift and drag forces are neglected, and briefly consider how trajectories under these equations of motion could be designed using optimal control. If lift and drag forces are assumed negligible relative to added mass forces, the carangiform fish's equation of motion becomes:

$$I^{\text{total}} \begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\theta} \end{pmatrix} = \begin{pmatrix} f_{am} \\ \tau_{am} + (x_m - x, y_m - y) \times f_{am} \end{pmatrix}, \tag{5.1}$$

where f_{am}, τ_{am}, x_m, y_m, and I^{total} are as defined in Equations (4.18)-(4.24).

Taking the state of the system to be specified by

$$q = (x, y, \theta, \dot{x}, \dot{y}, \dot{\theta}, \phi_1, \phi_2, \dot{\phi}_1, \dot{\phi}_2) \tag{5.2}$$

and regarding the joint accelerations as inputs:

$$u_1 = \ddot{\phi}_1 \tag{5.3}$$

$$u_2 = \ddot{\phi}_2 \tag{5.4}$$

I find that the state equation of the system is

$$\frac{d}{dt}(x, y, \theta, \dot{x}, \dot{y}, \dot{\theta}, \phi_1, \phi_2, \dot{\phi}_1, \dot{\phi}_2) = f^T = (\dot{x}, \dot{y}, \dot{\theta}, f_4, f_5, f_6, \dot{\phi}_1, \dot{\phi}_2, u_1, u_2)^T, \tag{5.5}$$

where:

$$\begin{aligned}
f_6 = \ddot{\theta} = \{ & {l_f}^2 \pi \rho_f \Big(-512\, {l_p}^2 m^2 \ddot{\phi}_1 - 128\, l^2 {l_p}^2 m \pi \rho_f \ddot{\phi}_1 - 288\, {l_f}^2 m^2 \ddot{\phi}_2 - 72\, l^2 {l_f}^2 m \pi \rho_f \ddot{\phi}_2 \\
& -8\, {l_f}^4 m \pi \rho_f \ddot{\phi}_2 - l^2 {l_f}^4 \pi^2 {\rho_f}^2 \ddot{\phi}_2 - 128\, l_b l_p m \left(4 m + l^2 \pi \rho_f\right) \ddot{\phi}_1 \cos(\phi_1) \\
& -128\, l_b l_p m \left(4 m + l^2 \pi \rho_f\right) \ddot{\phi}_1 \cos(\phi_1 - 2\phi_2) - 512\, l_f l_p m^2 \ddot{\phi}_1 \cos(\phi_1 - \phi_2) \\
& -128\, l^2 l_f l_p m \pi \rho_f \ddot{\phi}_1 \cos(\phi_1 - \phi_2) - 512\, l_f l_p m^2 \ddot{\phi}_2 \cos(\phi_1 - \phi_2) \\
& -128\, l^2 l_f l_p m \pi \rho_f \ddot{\phi}_2 \cos(\phi_1 - \phi_2) - 512\, {l_p}^2 m^2 \ddot{\phi}_1 \cos(2\,(\phi_1 - \phi_2)) \\
& -128\, l^2 {l_p}^2 m \pi \rho_f \ddot{\phi}_1 \cos(2\,(\phi_1 - \phi_2)) - 512\, l_b l_f m^2 \ddot{\phi}_2 \cos(\phi_2) \\
& -128\, l^2 l_b l_f m \pi \rho_f \ddot{\phi}_2 \cos(\phi_2) + l^2 {l_f}^4 \pi^2 {\rho_f}^2 \ddot{\phi}_2 \cos(2\phi_2) \\
& +512\, l_b l_p m^2 \dot{\phi}_1^2 \sin(\phi_1) + 128\, l^2 l_b l_p m \dot{\phi}_1^2 \pi \rho_f \sin(\phi_1) \\
& +1024\, l_b l_p m^2 \dot{\phi}_1 \dot{\theta} \sin(\phi_1) + 256\, l^2 l_b l_p m \dot{\phi}_1 \pi \rho_f \dot{\theta} \sin(\phi_1) \\
& +512\, l_b l_p m^2 \dot{\phi}_1^2 \sin(\phi_1 - 2\phi_2) + 128\, l^2 l_b l_p m \dot{\phi}_1^2 \pi \rho_f \sin(\phi_1 - 2\phi_2) \\
& +1024\, l_b l_p m^2 \dot{\phi}_1 \dot{\theta} \sin(\phi_1 - 2\phi_2) + 256\, l^2 l_b l_p m \dot{\phi}_1 \pi \rho_f \dot{\theta} \sin(\phi_1 - 2\phi_2) \\
& +1024\, l_b l_p m^2 \dot{\theta}^2 \sin(\phi_1 - 2\phi_2) + 256\, l^2 l_b l_p m \pi \rho_f \dot{\theta}^2 \sin(\phi_1 - 2\phi_2) \\
& +512\, l_f l_p m^2 \dot{\phi}_1^2 \sin(\phi_1 - \phi_2) + 128\, l^2 l_f l_p m \dot{\phi}_1^2 \pi \rho_f \sin(\phi_1 - \phi_2) \\
& +1024\, l_f l_p m^2 \dot{\phi}_1 \dot{\theta} \sin(\phi_1 - \phi_2) + 256\, l^2 l_f l_p m \dot{\phi}_1 \pi \rho_f \dot{\theta} \sin(\phi_1 - \phi_2) \\
& +512\, l_f l_p m^2 \dot{\theta}^2 \sin(\phi_1 - \phi_2) + 128\, l^2 l_f l_p m \pi \rho_f \dot{\theta}^2 \sin(\phi_1 - \phi_2) \\
& +512\, {l_p}^2 m^2 \dot{\phi}_1^2 \sin(2\,(\phi_1 - \phi_2)) + 128\, l^2 {l_p}^2 m \dot{\phi}_1^2 \pi \rho_f \sin(2\,(\phi_1 - \phi_2)) \\
& +1024\, {l_p}^2 m^2 \dot{\phi}_1 \dot{\theta} \sin(2\,(\phi_1 - \phi_2)) + 256\, l^2 {l_p}^2 m \dot{\phi}_1 \pi \rho_f \dot{\theta} \sin(2\,(\phi_1 - \phi_2)) \\
& +512\, {l_p}^2 m^2 \dot{\theta}^2 \sin(2\,(\phi_1 - \phi_2)) + 128\, l^2 {l_p}^2 m \pi \rho_f \dot{\theta}^2 \sin(2\,(\phi_1 - \phi_2)) \\
& -512\, l_b l_f m^2 \dot{\theta}^2 \sin(\phi_2) - 128\, l^2 l_b l_f m \pi \rho_f \dot{\theta}^2 \sin(\phi_2) \\
& -512\, {l_b}^2 m^2 \dot{\theta}^2 \sin(2\phi_2) - 128\, l^2 {l_b}^2 m \pi \rho_f \dot{\theta}^2 \sin(2\phi_2) \Big) \} / \{\Upsilon\}
\end{aligned} \tag{5.6}$$

and:

$$
\begin{aligned}
f_4 = \ddot{x} = \{ & {l_f}^2 \, \pi \, \rho_f \, \Big(128 \, I_\theta \, l_f \, \ddot{\phi}_2 + l^4 \, l_f \, \pi \, \rho_f \, \ddot{\phi}_2 - {l_f}^5 \, \pi \, \rho_f \, \ddot{\phi}_2 + {l_f}^5 \, \pi^2 \, {\rho_f}^2 \, \ddot{\phi}_2 \\
& +2 \, l_p \, \Big(128 \, I_\theta \, \ddot{\phi}_1 + \pi \, \rho_f \, \Big(l^4 \, \ddot{\phi}_1 + {l_f}^4 \, \Big(\pi \, \rho_f \, \ddot{\phi}_1 - \ddot{\phi}_2 \Big) \Big) \Big) \cos(\phi_1 - \phi_2) - 2 \, l_b \, {l_f}^4 \, \pi \, \rho_f \, \ddot{\phi}_2 \, \cos(\phi_2) \\
& -256 \, I_\theta \, l_p \, \dot{\phi}_1^2 \, \sin(\phi_1 - \phi_2) - 2 \, l^4 \, l_p \, \dot{\phi}_1^2 \, \pi \, \rho_f \, \sin(\phi_1 - \phi_2) - 2 \, {l_f}^4 \, l_p \, \dot{\phi}_1^2 \, \pi^2 \, {\rho_f}^2 \, \sin(\phi_1 - \phi_2) \\
& -512 \, I_\theta \, l_p \, \dot{\phi}_1 \, \dot{\theta} \, \sin(\phi_1 - \phi_2) - 4 \, l^4 \, l_p \, \dot{\phi}_1 \, \pi \, \rho_f \, \dot{\theta} \, \sin(\phi_1 - \phi_2) - 4 \, {l_f}^4 \, l_p \, \dot{\phi}_1 \, \pi^2 \, {\rho_f}^2 \, \dot{\theta} \, \sin(\phi_1 - \phi_2) \\
& -256 \, I_\theta \, l_p \, \dot{\theta}^2 \, \sin(\phi_1 - \phi_2) - 2 \, l^4 \, l_p \, \pi \, \rho_f \, \dot{\theta}^2 \, \sin(\phi_1 - \phi_2) - 2 \, {l_f}^4 \, l_p \, \pi^2 \, {\rho_f}^2 \, \dot{\theta}^2 \, \sin(\phi_1 - \phi_2) \\
& +256 \, I_\theta \, l_b \, \dot{\theta}^2 \, \sin(\phi_2) + 2 \, l^4 \, l_b \, \pi \, \rho_f \, \dot{\theta}^2 \, \sin(\phi_2) + 2 \, l_b \, {l_f}^4 \, \pi^2 \, {\rho_f}^2 \, \dot{\theta}^2 \, \sin(\phi_2) \Big) \\
& \times \, \Big(2 \, \big(4 \, m + l^2 \, \pi \, \rho_f \big) \, \cos(\theta) \, \sin(\phi_2) + 8 \, m \, \cos(\phi_2) \, \sin(\theta) \Big) \} / (\{ 2 \, \Upsilon \}) \qquad (5.7)
\end{aligned}
$$

and:

$$
\begin{aligned}
f_5 = \ddot{y} = \{ & - \Big({l_f}^2 \, \pi \, \rho_f \, \Big(128 \, I_\theta \, l_f \, \ddot{\phi}_2 + l^4 \, l_f \, \pi \, \rho_f \, \ddot{\phi}_2 - {l_f}^5 \, \pi \, \rho_f \, \ddot{\phi}_2 + {l_f}^5 \, \pi^2 \, {\rho_f}^2 \, \ddot{\phi}_2 \\
& +2 \, l_p \, \Big(128 \, I_\theta \, \ddot{\phi}_1 + \pi \, \rho_f \, \Big(l^4 \, \ddot{\phi}_1 + {l_f}^4 \, \Big(\pi \, \rho_f \, \ddot{\phi}_1 - \ddot{\phi}_2 \Big) \Big) \Big) \cos(\phi_1 - \phi_2) \\
& -2 \, l_b \, {l_f}^4 \, \pi \, \rho_f \, \ddot{\phi}_2 \, \cos(\phi_2) - 256 \, I_\theta \, l_p \, \dot{\phi}_1^2 \, \sin(\phi_1 - \phi_2) - 2 \, l^4 \, l_p \, \dot{\phi}_1^2 \, \pi \, \rho_f \, \sin(\phi_1 - \phi_2) \\
& -2 \, {l_f}^4 \, l_p \, \dot{\phi}_1^2 \, \pi^2 \, {\rho_f}^2 \, \sin(\phi_1 - \phi_2) - 512 \, I_\theta \, l_p \, \dot{\phi}_1 \, \dot{\theta} \, \sin(\phi_1 - \phi_2) - 4 \, l^4 \, l_p \, \dot{\phi}_1 \, \pi \, \rho_f \, \dot{\theta} \, \sin(\phi_1 - \phi_2) \\
& -4 \, {l_f}^4 \, l_p \, \dot{\phi}_1 \, \pi^2 \, {\rho_f}^2 \, \dot{\theta} \, \sin(\phi_1 - \phi_2) - 256 \, I_\theta \, l_p \, \dot{\theta}^2 \, \sin(\phi_1 - \phi_2) - 2 \, l^4 \, l_p \, \pi \, \rho_f \, \dot{\theta}^2 \, \sin(\phi_1 - \phi_2) \\
& -2 \, {l_f}^4 \, l_p \, \pi^2 \, {\rho_f}^2 \, \dot{\theta}^2 \, \sin(\phi_1 - \phi_2) + 256 \, I_\theta \, l_b \, \dot{\theta}^2 \, \sin(\phi_2) \\
& +2 \, l^4 \, l_b \, \pi \, \rho_f \, \dot{\theta}^2 \, \sin(\phi_2) + 2 \, l_b \, {l_f}^4 \, \pi^2 \, {\rho_f}^2 \, \dot{\theta}^2 \, \sin(\phi_2) \Big) \\
& \times \, \Big(8 \, m \, \cos(\phi_2) \, \cos(\theta) - 2 \, \big(4 \, m + l^2 \, \pi \, \rho_f \big) \, \sin(\phi_2) \, \sin(\theta) \Big) \Big) \} / \{ 2 \, \Upsilon \} \qquad (5.8)
\end{aligned}
$$

and:

$$
\begin{aligned}
\Upsilon =\ & 4096\, I_\theta\, m^2 + 1024\, I_\theta\, l^2\, m\, \pi\, \rho_f + 1024\, I_\theta\, {l_f}^2\, m\, \pi\, \rho_f + 32\, l^4\, m^2\, \pi\, \rho_f \\
& + 512\, {l_b}^2\, {l_f}^2\, m^2\, \pi\, \rho_f + 256\, {l_f}^4\, m^2\, \pi\, \rho_f + 512\, {l_f}^2\, {l_p}^2\, m^2\, \pi\, \rho_f + 128\, I_\theta\, l^2\, {l_f}^2\, \pi^2\, {\rho_f}^2 \\
& + 8\, l^6\, m\, \pi^2\, {\rho_f}^2 + 8\, l^4\, {l_f}^2\, m\, \pi^2\, {\rho_f}^2 + 128\, l^2\, {l_b}^2\, {l_f}^2\, m\, \pi^2\, {\rho_f}^2 + 64\, l^2\, {l_f}^4\, m\, \pi^2\, {\rho_f}^2 \\
& + 128\, l^2\, {l_f}^2\, {l_p}^2\, m\, \pi^2\, {\rho_f}^2 + 32\, {l_f}^4\, m^2\, \pi^2\, {\rho_f}^2 + l^6\, {l_f}^2\, \pi^3\, {\rho_f}^3 + 8\, l^2\, {l_f}^4\, m\, \pi^3\, {\rho_f}^3 \\
& + 8\, {l_f}^6\, m\, \pi^3\, {\rho_f}^3 + l^2\, {l_f}^6\, \pi^4\, {\rho_f}^4 + 256\, l_b\, {l_f}^2\, l_p\, m\, \pi\, \rho_f\, \left(4\, m + l^2\, \pi\, \rho_f\right)\, \cos(\phi_1) \\
& + 256\, l_b\, {l_f}^2\, l_p\, m\, \pi\, \rho_f\, \left(4\, m + l^2\, \pi\, \rho_f\right)\, \cos(\phi_1 - 2\, \phi_2) + 1024\, {l_f}^3\, l_p\, m^2\, \pi\, \rho_f\, \cos(\phi_1 - \phi_2) \\
& + 256\, l^2\, {l_f}^3\, l_p\, m\, \pi^2\, {\rho_f}^2\, \cos(\phi_1 - \phi_2) + 512\, {l_f}^2\, {l_p}^2\, m^2\, \pi\, \rho_f\, \cos(2\, (\phi_1 - \phi_2)) \\
& + 128\, l^2\, {l_f}^2\, {l_p}^2\, m\, \pi^2\, {\rho_f}^2\, \cos(2\, (\phi_1 - \phi_2)) + 1024\, l_b\, {l_f}^3\, m^2\, \pi\, \rho_f\, \cos(\phi_2) \\
& + 256\, l^2\, l_b\, {l_f}^3\, m\, \pi^2\, {\rho_f}^2\, \cos(\phi_2) + 512\, {l_b}^2\, {l_f}^2\, m^2\, \pi\, \rho_f\, \cos(2\, \phi_2) - 128\, I_\theta\, l^2\, {l_f}^2\, \pi^2\, {\rho_f}^2\, \cos(2\, \phi_2) \\
& + 128\, l^2\, {l_b}^2\, {l_f}^2\, m\, \pi^2\, {\rho_f}^2\, \cos(2\, \phi_2) - l^6\, {l_f}^2\, \pi^3\, {\rho_f}^3\, \cos(2\, \phi_2) - l^2\, {l_f}^6\, \pi^4\, {\rho_f}^4\, \cos(2\, \phi_2)
\end{aligned}
\tag{5.9}
$$

There are, of course, many ways in which "optimality" of a trajectory between two points could be defined or quantified by the engineer. For the sake of the current example, I will assume that the trajectories being sought are minimum-time trajectories, subject to the constraint that the joint accelerations are bounded:

$$\|\ddot{\phi}_1\| \leq 2\pi \tag{5.10}$$

$$\|\ddot{\phi}_2\| \leq 2\pi \tag{5.11}$$

In the language of optimal control, then, I am seeking to minimize the performance index:

$$J = \int_0^T 1\, dt \tag{5.12}$$

and the appropriate Hamiltonian for the problem is [LS95]:

$$H = 1 + \lambda^T f \tag{5.13}$$

The costate equations are given by

$$\dot{\lambda} = -\frac{\partial H}{\partial q} \tag{5.14}$$

and these could readily be written in closed (if slightly cumbersome) form. I also have Pontryagin's minimum principle:

$$H(q^*, u^*, \lambda^*, t) \leq H(q^*, u, \lambda^*) \tag{5.15}$$

which says that the optimal joint accelerations $u_i = \ddot{\phi}_i$ minimize the Hamiltonian, not just relative to alternative functions for the joint acceleration and the alternative positions and velocities resulting from the state equation, but also relative to other values for the acceleration while keeping the optimal position and velocity functions fixed, consistent with the acceleration bounds but not consistent with the state equation. This implies that the joint accelerations must be at their maximum or minimum allowed limits at all times, unless the value of the joint accelerations has no influence on the Hamiltonian H at all.

$$\begin{aligned}
\ddot{\phi}_1 &= -2\pi\,\mathrm{sign}(\frac{\partial H}{\partial \ddot{\phi}_1}) \\
&= -2\pi\,\mathrm{sign}(\lambda_9 + \lambda_4 \frac{\partial \ddot{x}}{\partial \ddot{\phi}_1} + \lambda_5 \frac{\partial \ddot{y}}{\partial \ddot{\phi}_1} + \lambda_6 \frac{\partial \ddot{\theta}}{\partial \ddot{\phi}_1}) \\
&= -2\pi\,\mathrm{sign}\Big(\lambda_9 + \frac{\lambda_4}{\Upsilon}\{{l_f}^2\, l_p\, \pi\, \rho_f\, \left(128\, I_\theta + \pi\, \rho_f\, \left(l^4 + {l_f}^4\, \pi\, \rho_f\right)\right)\, \cos(\phi_1 - \phi_2) \\
&\quad \times \left(2\, \left(4\, m + l^2\, \pi\, \rho_f\right)\, \cos(\theta)\, \sin(\phi_2) + 8\, m\, \cos(\phi_2)\, \sin(\theta)\right)\} \\
&\quad + \frac{\lambda_5}{\Upsilon}\{-\left({l_f}^2\, l_p\, \pi\, \rho_f\, \left(128\, I_\theta + \pi\, \rho_f\, \left(l^4 + {l_f}^4\, \pi\, \rho_f\right)\right)\, \cos(\phi_1 - \phi_2)\right. \\
&\quad \left.\left(8\, m\, \cos(\phi_2)\, \cos(\theta) - 2\, \left(4\, m + l^2\, \pi\, \rho_f\right)\, \sin(\phi_2)\, \sin(\theta)\right)\right)\} \\
&\quad + \frac{\lambda_6}{\Upsilon}\{{l_f}^2\, \pi\, \rho_f\, \left(-512\, {l_p}^2\, m^2 - 128\, l^2\, {l_p}^2\, m\, \pi\, \rho_f - 128\, l_b\, l_p\, m\, \left(4\, m + l^2\, \pi\, \rho_f\right)\, \cos(\phi_1)\right. \\
&\quad - 128\, l_b\, l_p\, m\, \left(4\, m + l^2\, \pi\, \rho_f\right)\, \cos(\phi_1 - 2\, \phi_2) - 512\, l_f\, l_p\, m^2\, \cos(\phi_1 - \phi_2) \\
&\quad - 128\, l^2\, l_f\, l_p\, m\, \pi\, \rho_f\, \cos(\phi_1 - \phi_2) - 512\, {l_p}^2\, m^2\, \cos(2\, (\phi_1 - \phi_2)) \\
&\quad \left. - 128\, l^2\, {l_p}^2\, m\, \pi\, \rho_f\, \cos(2\, (\phi_1 - \phi_2))\right)\}\Big)
\end{aligned} \tag{5.16}$$

$$
\begin{aligned}
\ddot{\phi}_2 &= -2\pi\,\mathrm{sign}(\frac{\partial H}{\partial \ddot{\phi}_2}) \\
&= -2\pi\,\mathrm{sign}(\lambda_{10} + \lambda_4 \frac{\partial \ddot{x}}{\partial \ddot{\phi}_2} + \lambda_5 \frac{\partial \ddot{y}}{\partial \ddot{\phi}_2} + \lambda_6 \frac{\partial \ddot{\theta}}{\partial \ddot{\phi}_2}) \\
&= -2\pi\,\mathrm{sign}\,(\lambda_{10} \\
&\quad + \frac{\lambda_4}{2\Upsilon}\{{l_f}^2\,\pi\,\rho_f\,\left(128\,I_\theta\,l_f + l^4\,l_f\,\pi\,\rho_f - {l_f}^5\,\pi\,\rho_f + {l_f}^5\,\pi^2\,{\rho_f}^2 - 2\,{l_f}^4\,l_p\,\pi\,\rho_f\,\cos(\phi_1-\phi_2)\right. \\
&\quad \left. -2\,l_b\,{l_f}^4\,\pi\,\rho_f\,\cos(\phi_2)\right)\,\left(2\,\left(4\,m + l^2\,\pi\,\rho_f\right)\,\cos(\theta)\,\sin(\phi_2) + 8\,m\,\cos(\phi_2)\,\sin(\theta)\right)\} \\
&\quad + \frac{\lambda_5}{2\Upsilon}\{-\left({l_f}^2\,\pi\,\rho_f\,\left(128\,I_\theta\,l_f + l^4\,l_f\,\pi\,\rho_f - {l_f}^5\,\pi\,\rho_f + {l_f}^5\,\pi^2\,{\rho_f}^2 - 2\,{l_f}^4\,l_p\,\pi\,\rho_f\,\cos(\phi_1-\phi_2)\right.\right. \\
&\quad \left.\left. -2\,l_b\,{l_f}^4\,\pi\,\rho_f\,\cos(\phi_2)\right)\,\left(8\,m\,\cos(\phi_2)\,\cos(\theta) - 2\,\left(4\,m + l^2\,\pi\,\rho_f\right)\,\sin(\phi_2)\,\sin(\theta)\right)\right)\} \\
&\quad + \frac{\lambda_6}{\Upsilon}\{{l_f}^2\,\pi\,\rho_f\,\left(-288\,{l_f}^2\,m^2 - 72\,l^2\,{l_f}^2\,m\,\pi\,\rho_f - 8\,{l_f}^4\,m\,\pi\,\rho_f - l^2\,{l_f}^4\,\pi^2\,{\rho_f}^2\right. \\
&\quad -512\,l_f\,l_p\,m^2\,\cos(\phi_1-\phi_2) - 128\,l^2\,l_f\,l_p\,m\,\pi\,\rho_f\,\cos(\phi_1-\phi_2) - 512\,l_b\,l_f\,m^2\,\cos(\phi_2) \\
&\quad \left. -128\,l^2\,l_b\,l_f\,m\,\pi\,\rho_f\,\cos(\phi_2) + l^2\,{l_f}^4\,\pi^2\,{\rho_f}^2\,\cos(2\,\phi_2)\right)\}) \qquad (5.17)
\end{aligned}
$$

Thus I can explore the space of configurations reachable in a given time by the fish by systematically choosing trial values of λ at $t = 0$, and shooting forward using the state and costate equations and Equations (5.16)–(5.17) to arrive at some final state at $t = T$. After shooting forward, we will have determined the minimum-time trajectory to whatever point I arrive at. In this way I could build up a catalog of reachable points. Of particular interest will be any pair of trajectories, one of which reaches final values of $\phi_1,\ \phi_2, \dot{\phi}_1, \dot{\phi}_2, \dot{\theta}$, which are the same or approximately the same as the starting values for the other trajectory. These trajectories could be concatenated together to form a larger trajectory, since the equations of motion are invariant with respect to transformations in x, y, θ, and the values of $\dot{x}$ and $\dot{y}$ are not significant either unless they become large enough to invalidate the assumption of added-mass dominance.

Figure 5.1 shows one minimum-time maneuver found by numerical optimization of bang-bang trajectories, which could plausibly be concatenated with itself.

Although I could have used a different model of the system, a different definition

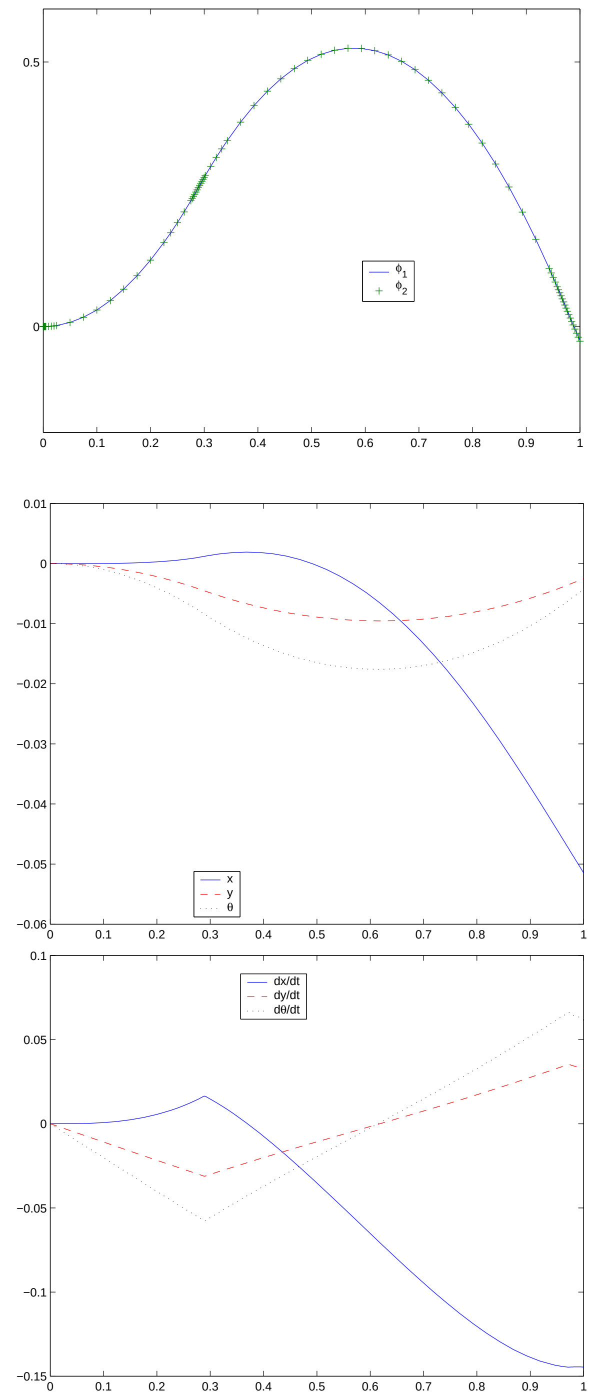


Figure 5.1: A bang-bang maneuver, starting from rest, that produces both acceleration and displacement in the x-direction, while producing less acceleration and almost no displacement in the y- and $\theta-$ directions. The joint accelerations $(\ddot{\phi}_1, \ddot{\phi}_2)$ are $(2\pi, 2\pi)$ from $t = 0$ to $t = 0.29$, then $(-2\pi, -2\pi)$ to $t = 0.97$ The values of $\ddot{\phi}_1$, $\ddot{\phi}_2$ oscillate rapidly just before the end of the trajectory.

of optimality, or a different scheme for searching through the space of trajectories, the broader point is that in the lab, relatively ample computational power can be used to optimize the feasible trajectories of a swimmer over short time horizons. A suitable set of such feasible trajectories can then be used as motion primitives and pieced together to form larger trajectories. This will be the subject of the next chapter.

Chapter 6 Trajectory Planning Using Density Functions

6.1 Motivation

Many mobile robots operate under nonholonomic or other kinodynamic constraints which complicate the task of planning their motions. In this chapter I present an algorithm for planning the gross motion of many types of mobile robots in the presence of such constraints.

There is a large literature on the subject of nonholonomic and kinodynamic motion planning. Many approaches to this problem are based on a decomposition of the planning problem into two parts. In the first step, the kinematic and dynamic constraints are ignored, and the gross path of the system that avoids obstacles is determined. This "holonomic" path can be chosen manually, or by a conventional rigid body holonomic motion planner. Next, the actual control inputs that respect the constraints are determined so that the system approximately tracks the holonomic path. Based on differential geometric concepts, there are many local motion planners that compute the controls inputs which cause the vehicle to approximately track the specified trajectory (e.g., [LS93, MS93b, Gur92, MS91]). Implicitly, this approach assumes that the vehicle is small-time locally controllable (STLC), so that it is capable of locally approximating any trajectory. When this criterion is not satisfied, the success of these methods is uncertain.

The successful probabilistic roadmap motion planning paradigm [KSLO96] has also been adapted to nonholonomic motion planning (e.g., [SO95]). Unfortunately, a local nonholonomic motion planning problem must be solved each time a candidate node is considered for addition to the roadmap. Lavalle and Kuffner [LK00] have also developed a probabilistic kinodynamic planning approach that is based on an incremental simulation of the system's dynamics. A forward simulation of the system's dynamics is used to expand a search tree. This approach circumvents the small time local controllability assumption of many other techniques, and has produced excellent simulation results. However, its probabilistic convergence may require an exponential number of computations.

Here I introduce a different type of algorithm for approximately solving this class of motion planning problems. My deterministic approach is based on constructing an approximation to the vehicle's reachable set of states. I use techniques of fast Fourier transforms on Lie groups to efficiently compute these sets. This approach is motivated by the Ebert-Uphoff algorithm for solving the inverse kinematics of discretely actuated manipulators [CK01]. A solution path can then be constructed from knowledge of the reachable set. Like [LK00], this technique does not require small-time local controllability, and I show that the algorithm has an attractive computational complexity. Because I use a finite set approximation to the continuous mechanics, my technique can also be used to plan the motions of "discrete" nonholonomic systems [CMPB00].

Section 6.2 provides an intuitive overview of the approach, focusing on the vehicle's reachable set of states. Sections 6.3 and 6.5 describe the density of reachable states concept, and algorithms for its computation. Section 6.6 then describes the planning method, and the algorithm is illustrated by an example. Another variation of the algorithm with tighter time and space bounds is summarized in Section 6.6.5. Finally, a third variation of the algorithm to be used in the presence of static obstacles is presented in Section 6.6.6.

6.2 Summary of the Approach

Most nonholonomically constrained wheeled vehicles move by the influence of periodic or quasi-periodic controls. Hopping, walking, and swimming robots move by periodic oscillations of their driving actuators. A common theme among these systems is that their gross trajectories can be naturally broken into individual hops, steps, or

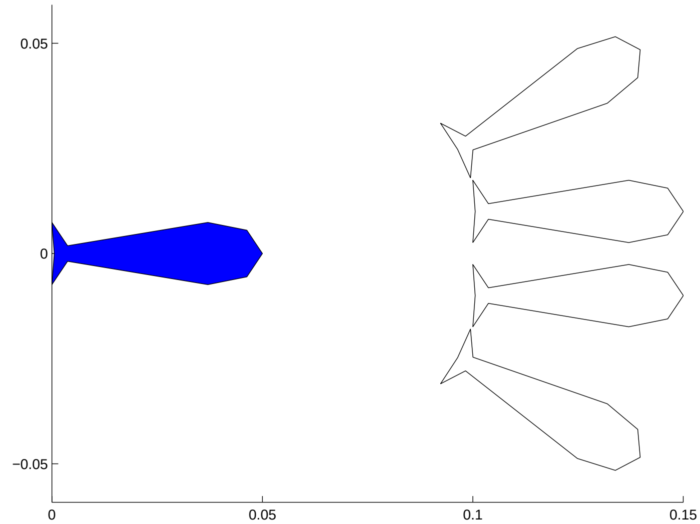


Figure 6.1: This fish robot can reach any of four positions in one tail stroke.

undulations. The trajectories of such robots can therefore often be considered as a sequence of quasi-periodic steps, each one of which causes some change in the robot's position while returning the robot's actuators to approximately the same state at the end of the step as at the beginning. Furthermore, different combinations of the periodic inputs can be interpreted as different vehicle "gaits" that move the vehicle in different directions [OB98]. Individual steps or gaits can be designed using optimal control techniques.

Based on these observations, my approach discretizes the motion group $SE(D)$ (D=2 or 3) in which the robot operates, and plans a trajectory to any given volume element in $SE(D)$ by considering the number of reachable endpoints in that volume element and in intermediate volume elements along the trajectory. My algorithm is largely inspired by the Ebert-Uphoff algorithm for solving the inverse kinematics of discrete multistage manipulators [CK01]. I adapt this algorithm to motion planning purposes, and improve upon its computational complexity. I also draw on ideas from Dijkstra's algorithm, as applied to path planning in a discretized configuration space by Barraquand and Latombe [BL89]. Like those algorithms, my method requires a memory-intensive mapping of the workspace, but generates trajectories to chosen goal points very quickly once the map is constructed.

I consider a mobile robot which moves by discrete steps, transforming its location

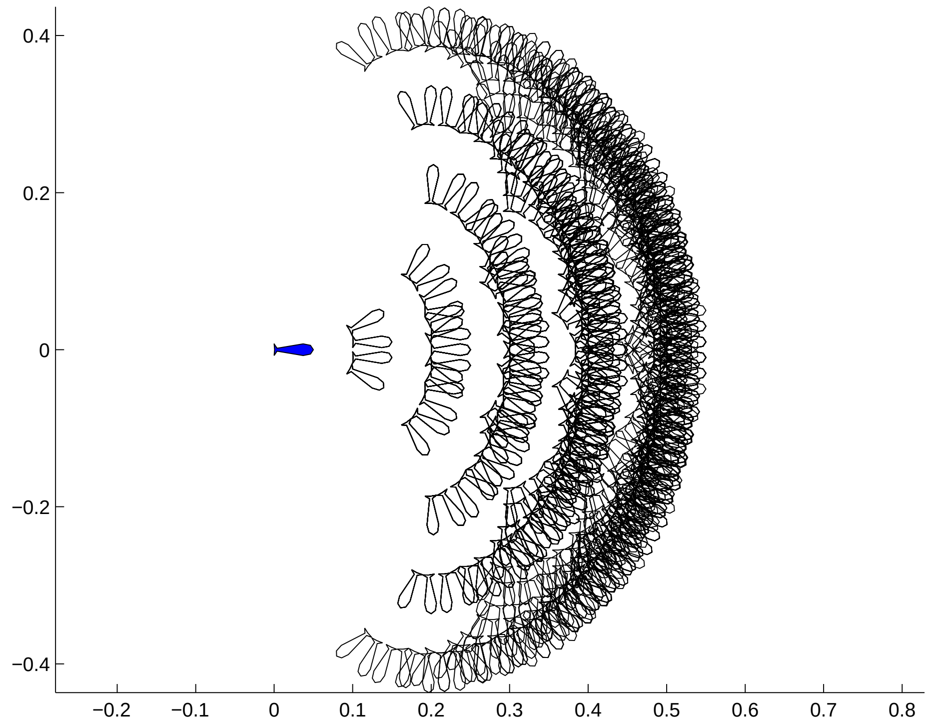


Figure 6.2: Positions reachable in five or fewer tail strokes.

in $SE(D)$ by any one of K motions in the finite set $A \subset SE(D)$. This discreteness of movement may arise because the robot's actuators are actually discrete, or because it is convenient to consider a restricted set of "optimal" motions at each step (e.g., a carlike robot whose optimal motions are either straight ahead, left arcs, or right arcs [RS90]), or solely for the sake of approximating the reachable set of states. In the case of complex kinodynamic constraints, the finite set of motions can be crafted from a forward simulation of the system for each of a finite set of inputs. This procedure avoids the need for small-time local controllability. For example, Figure 6.1 depicts a hypothetical robot fish which can stroke its tail to make one of four motions: turn left, move forward and sideslip left, move forward and sideslip right, or turn right. These motions are consistent with its complex hydrodynamics.

For a sequence of P such steps, the robot can follow K^P potential trajectories to K^P endpoints in $SE(D)$, with some of the endpoints possibly being duplicates. Figure 6.2 shows how the reachable configurations of the robot fish grow quickly. Figure 6.3 shows the same set of configurations as points in (x, y, θ) configuration space.

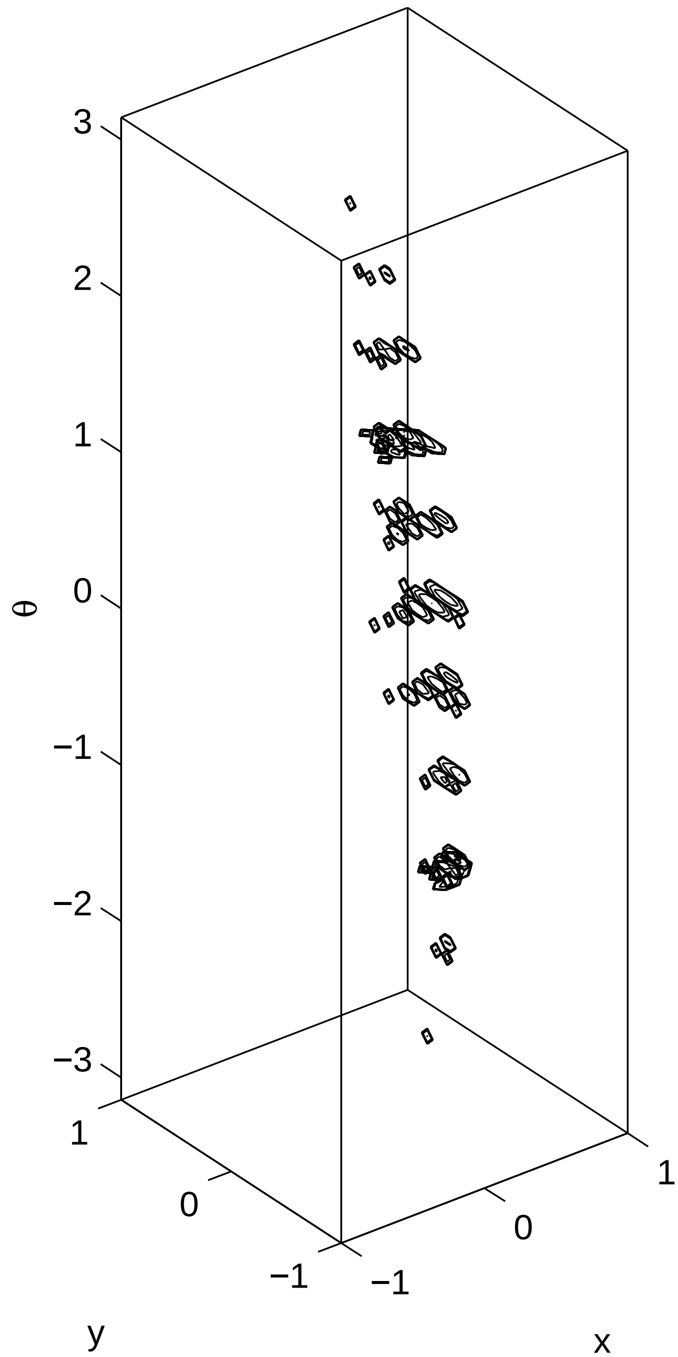


Figure 6.3: States in Figure 6.2 viewed as a density function in configuration space.

6.3 Density Functions

I will be concerned with functions expressing the density of a cloud of points in a continuous group. Like any nonuniform density, this density function is not well-defined unless I specify the volume elements over which the density is measured.

More precisely, let F be a Lie group, and $G \subseteq F$ be a relevant subset of the group. I begin with the concept of a *disjoint discretization* V_0 of G. This is a collection of disjoint, finite, connected subsets (or "volume elements") of G which covers G, together with a function which associates points in G with volume elements. I will call the set of volume elements $V_0(G)$. The elements in $V_0(G)$ are chosen *a priori* and might, for example, form a regular grid covering G. The associated function $V_0(x) : G \to V_0(G)$ maps any element $x \in G$ to the unique volume element $V_0(x) \in V_0(G)$ which contains x.

For reasons to be made clear below, I also need the more general concept of a *non-disjoint discretization* V of G. This again involves an arbitrarily chosen set $V(G)$ of finite, connected subsets of G. The elements of $V(G)$ cover G but are not necessarily disjoint. The function $V(x) : G \to V(G)$ is chosen to associate one volume element $V(x)$ with each element $x \in G$. The function $V(x)$ must be chosen so that $x \in V(x) \, \forall x \in G$. However, since x may lie in more than one element of $V(G)$, this requirement is not enough to wholly determine the function $V(x)$ (as it was in the disjoint case). There is, therefore, an element of freedom in choosing the function $V(x)$.

A non-disjoint discretization V can be thought of as "built up" from a disjoint discretization V_0, if for any $x \in G$, $V(x)$ consists of the connected union of $V_0(x)$ with a number of neighboring elements of $V_0(G)$. Again, V may be "designed" in one way or another by the choice of different sets of neighboring elements to form the union.

Once I have a set of volume elements, I can proceed to define a density function. Let $n(H)$ for any subset $H \subset G$ be the number of discrete points that lie in H. And let $\|H\| \in \mathbb{R}$ be a measure of the volume of H. The density function $\rho(x) : G \to \mathbb{R}$ is defined as

$$\rho(x) = \frac{n(V(x))}{\|V(x)\|}. \tag{6.1}$$

How large should the volume elements that define the density function be? The infinitesimal limit of the smallest possible volume element is not necessarily the most useful scale to choose. It is not useful to take the volume elements small relative to the spacing of the discrete points and then say that the density is zero almost everywhere.

I wish to find a reachable configuration close to a specified group element (i.e. the goal). Consequently, the volume elements should be sufficiently large so that each one contains one or more reachable points. Figure 6.4 shows the density function of Figure 6.3, but considered with a coarser discretization so that more volume elements are perceived to have nonzero density. But if the volume elements are both large and disjoint, the discretization of G is coarse, and the resulting paths will be correspondingly imprecise. I can partly avoid this problem by using non-disjoint volume elements.

To use a geographic analogy, a census taker might define the population density "at" each location, x, as the number of people within a one kilometer radius of x, and then measure and record the population density at locations spaced ten meters apart. In this case the range of V would consist of overlapping one-kilometer-radius circles with centers ten meters apart.

The choice of V determines two different kinds of spatial resolution. The first resolution involves the number of volume elements in a particular volume of G. The second resolution is the size of the volume elements. If the volume elements are disjoint, these two resolutions are essentially the same. The main virtue of non-disjoint volume elements is that the first resolution can remain fine while the second one is made coarse.[1]

[1] Another advantage of using non-disjoint volume elements is that their shape can be chosen freely. For example, one could make a good case for using volume elements whose projection on $\mathbb{R}^D$ was spherical. A spherical volume element would treat translational distance isotropically, unlike a rectangular volume element which contains more points in some directions (the corners) than in others. But $\mathbb{R}^D$ cannot be covered by disjoint spherical volume elements of the same size.

I start with a discretization V_0 where the volume elements are disjoint, and then construct coarser discretizations as necessary. For example a coarser discretization V_1 on G can be defined as follows. (See Figure 6.5.) Let $V_0(G)$ be the range of V. Then for any $x \in G$,

$$V_1(x) = \bigcup \{H \in V_0(G) | \overline{H} \cap \overline{V_0(x)} \neq \emptyset\}.$$

Note that if H borders $V_0(x)$, then the intersection of $\overline{H}$ with $\overline{V_0(x)}$ is nonempty. We can iterate this coarsening procedure. For $k > 1$,

$$V_k(x) = \bigcup \{H \in V_{k-1}(G) | \overline{H} \cap \overline{V_{k-1}(x)} \neq \emptyset\}. \tag{6.2}$$

If the volume elements in the range of V_0 have uniform volume, then the volume of the elements in the range of V_k will increase exponentially with k.

6.4 Order Notation

In measuring the computational cost of various algorithms, I will use the following order notation. If $g(n)$ is a function from $\mathcal{N}$ to $\Re$, then $\mathcal{O}(g(n))$ means the set of all functions $f(n)$ from $\mathcal{N}$ to $\Re$ such that for some constants $c > 0$ and $n_0 \geq 0$,

$$f(n) \leq c\, g(n) \text{ for all } n \geq n_0. \tag{6.3}$$

So if a function is in $\mathcal{O}(g(n))$, then that function is bounded from above by a fixed multiple of $g(n)$ for sufficiently large values of n. Conversely, to speak of bounds from below I use $\Omega(g(n))$, which is the set of all functions $f(n)$ from $\mathcal{N}$ to $\Re$ such that for some constants $c > 0$ and $n_0 \geq 0$,

$$f(n) \geq c\, g(n) \text{ for all } n \geq n_0. \tag{6.4}$$

The set $\Theta(g(n))$ is the intersection of $\mathcal{O}(g(n))$ and $\Omega(g(n))$ [LD91].

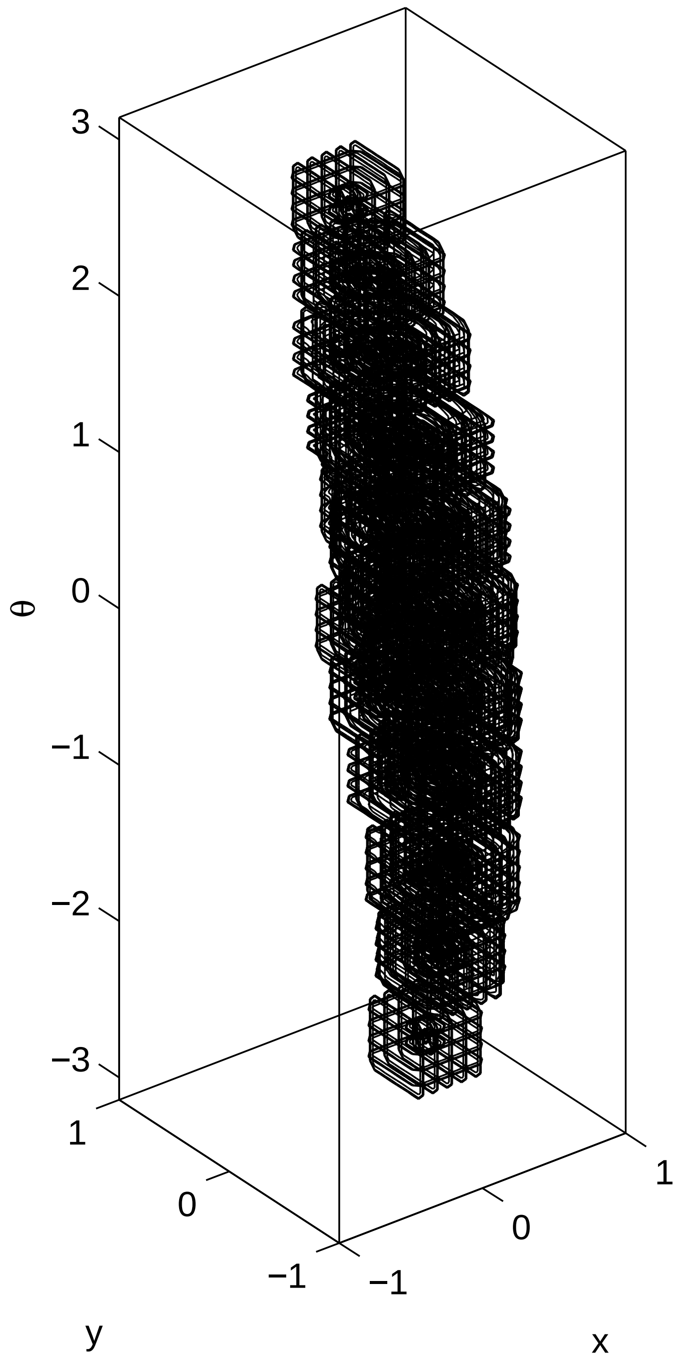
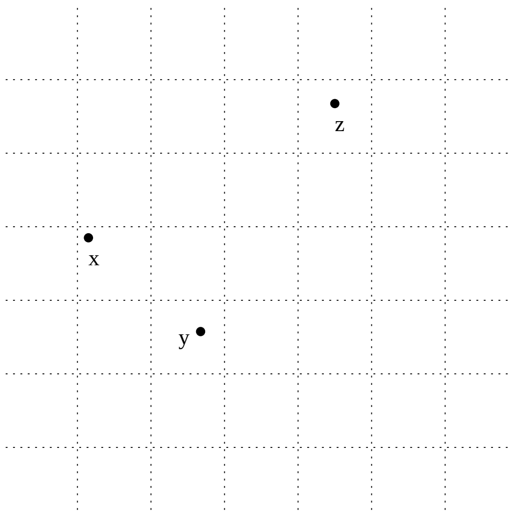
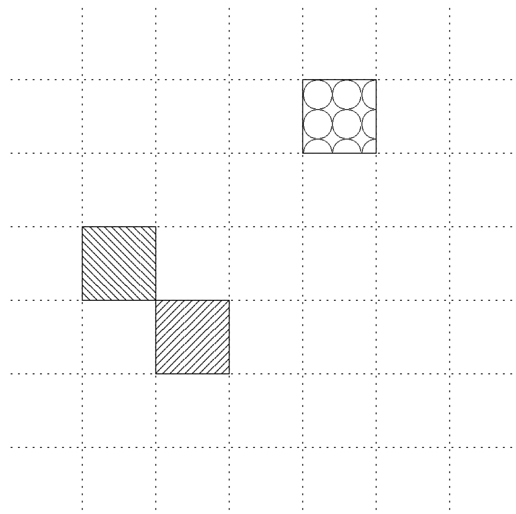


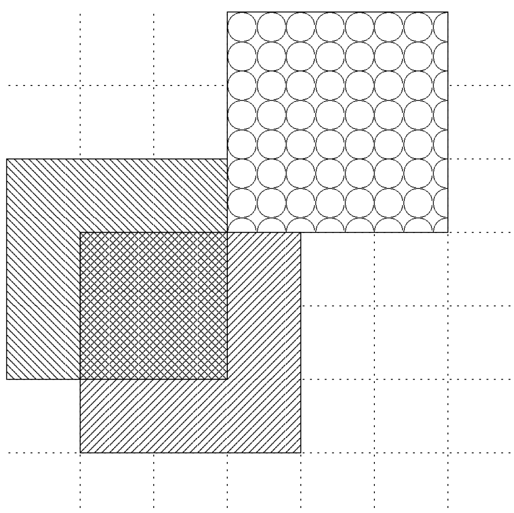
Figure 6.4: Density from Figure 6.3 using a coarser discretization.



(a) Points $x, y, z \in G$.



(b) Sets $V_0(x), V_0(y), V_0(z) \subset G$.



(c) Sets $V_1(x), V_1(y), V_1(z) \subset G$.

Figure 6.5: Discretization V_0 and coarser discretization V_1.

6.5 Computing the Density of Reachable States

Recall the premise that the robot transforms its location in $SE(D)$ by one of K possible motions. There are K^P sequences of P steps which will take the robot to as many as K^P endpoints in $SE(D)$. I write ρ^P to designate the function such that $\rho^P(g)$ for a configuration $g \in SE(D)$ is the number of endpoints in the volume element $V(g)$ centered on g divided by the volume of $V(g)$. (See Figure 6.3.) In other words, $\rho^P(g)$ is proportional to the number of different ways in which the robot can reach the volume element $V(g)$ in P steps.

For small P, I can compute ρ^P by simply enumerating the K^P points that can be reached. Obviously this method becomes unfeasible as P grows large. For larger P, the density function ρ^P can be found by taking the convolution of two known density functions representing smaller numbers of steps.

$$\rho^P(H) = \left(\rho^Q * \rho^R\right)(H) \tag{6.5}$$

$$= \int_{SE(D)} \rho^Q(\mathcal{H})\, \rho^R\left(\mathcal{H}^{-1}H\right) d(\mathcal{H}), \tag{6.6}$$

where $H, \mathcal{H} \in SE(D)$ and $Q + R = P$. Since $\rho^k * \rho^k = \rho^{2k}$, I can produce ρ^P with a number of convolutions in $\mathcal{O}(\log P)$ operations.

Since ρ^1 has at most K nonzero elements, straightforward convolution of ρ^1 with any other function can be performed in $\mathcal{O}(NK)$ time. If K is not large[2], this may actually be the most sensible way in which to proceed. However even if K is not large, this method has the drawback that ρ^P can only be computed by serially convolving ρ^1 with $\rho^1, \rho^2, \ldots, \rho^{(P-1)}$, requiring $\mathcal{O}(P)$ convolutions.

Performing a convolution by numerical integration of Equation (6.6) should require $\mathcal{O}(N^2)$ operations, where N is the total number of volume elements in the discretization of $SE(D)$. For approximation error to be kept low, N must be quite

[2] If K is large, in particular if K is an appreciable fraction of N, then this method may be computationally prohibitive. The more sophisticated methods of convolution described later will definitely be superior if K is in $\Theta(N^\gamma)$ where $\gamma > \frac{1}{3}$.

N	Total number of samples on $SE(2)$	$\mathcal{O}(S^3)$
N_r	Number of samples on $SO(2)$	$\mathcal{O}(S)$
N_R	Number of samples on $I\!R^2$	$\mathcal{O}(S^2)$
N_p	Number of samples on p interval	$\mathcal{O}(S)$
N_u	Number of samples on $[0, 2\pi)$	$\mathcal{O}(S)$
N_F	Total number of harmonics	$\mathcal{O}(S^2)$

Table 6.1: Resolution measurements in $SE(2)$.

large, even for $D = 2$ and especially for $D = 3$, so naive convolution becomes prohibitively expensive. Fortunately, a faster method of convolution is available.

6.5.1 The Fourier Transform

It is common knowledge that in $I\!R^n$, the Fourier transform of two convolved functions is simply the product of their individual Fourier transforms.

$$\mathcal{F}(f_1 * f_2) = \mathcal{F}(f_1)\mathcal{F}(f_2) \tag{6.7}$$

The fast Fourier transform (FFT) algorithm provides an efficient way to numerically approximate a function's Fourier transform. Since the FFT, the multiplication of the resulting Fourier transforms, and the inverse FFT can each be performed in subquadratic time, this is fast way to perform convolution in $I\!R^n$. To generalize these concepts to $SE(D)$, we apply concepts from the field of noncommutative harmonic analysis. The remainder of this section summarizes material from [CK01].

I can express the matrix elements of the Fourier transform of a function $f(\mathbf{r}, \theta)$ on $SE(2)$ by [CK01]

$$\hat{f}_{mn}(p) = \int_{\mathbf{r} \in I\!R^2} \int_{\theta=0}^{2\pi} \int_{\psi=0}^{2\pi} f(\mathbf{r}, \theta) \times e^{in\psi} e^{i(\mathbf{p}\cdot\mathbf{r})} e^{-im(\psi-\theta)} \, d^2r \, d\theta \, d\psi. \tag{6.8}$$

In practice I use a band-limited approximation of the Fourier transform and only compute elements with $|m|, |n| < S$ for some $S \in \mathcal{Z}+$. The group $SE(2)$ is discretized with the number of samples described in Table 6.1. The Fourier transform can be carried out numerically in the following way. First, assuming that $f(\mathbf{r}, \theta)$ is sampled on a Cartesian grid of $\mathbf{r} \in I\!R^2$ values, the integration over $I\!R^2$ is

$$f_1(\mathbf{p}, \theta) = \int_{I\!R^2} f(\mathbf{r}, \theta) e^{i(\mathbf{p}\cdot\mathbf{r})} \, d^2r \tag{6.9}$$

using the usual FFT in $\mathcal{O}\left(N_R N_r \log(N_r)\right)$ time. Then I interpolate from the Cartesian grid to sample $f_1(\mathbf{p}, \theta)$ on a polar-coordinate grid of $\mathbf{p}$ values. This set of $f_1(p, \psi, \theta)$ values can be found in $\mathcal{O}\left(N_R N_r\right) = \mathcal{O}\left(N\right)$ time assuming that the interpolation of each individual grid point can be done in constant time. (See Table 6.1 for definitions of N, N_R, et cetera.) Then the usual one-dimensional FFT can be used to integrate, first along the θ dimension in $\mathcal{O}\left(N_r N_R \log(N_R)\right)$ time

$$f_2^{(m)}(p, \psi) = \int_{SO(2)} f_1(p, \psi, \theta) e^{im\theta} \, d\theta \tag{6.10}$$

and along the ψ dimension in $\mathcal{O}\left(N_R N_p N_u \log(N_u)\right)$ time

$$\hat{f}_{mn}(p) = \int_0^{2\pi} \left[f_2^{(m)}(p, \psi) e^{-im\psi} \right] e^{in\psi} \, d\psi. \tag{6.11}$$

The entire process takes $\mathcal{O}\left(N \log(N)\right)$ time. The recovery of a function from its band-limited Fourier transform also takes $\mathcal{O}\left(N \log(N)\right)$ time.

$$f(\mathbf{r}, \theta) = \frac{1}{2\pi} \int_0^{\infty} p \, dp \sum_{m,n=-S}^{S} \left[\hat{f}_{mn}(p) \times \int_0^{2\pi} e^{i(m-n)\psi} e^{-i\mathbf{r}\cdot\mathbf{p}} e^{-im\theta} \, d\psi \right] \tag{6.12}$$

The Fourier transform on $SE(2)$ has the crucial convolution property [KC99]

$$\mathcal{F}(f * g) = \mathcal{F}(g)\mathcal{F}(f), \tag{6.13}$$

$$\mathcal{F}(f * g)_{mn}(p) = \sum_{k=-S}^{S} \hat{g}_{mk}(p) \hat{f}_{kn}(p). \tag{6.14}$$

Thus, the convolution of functions f and g on $SE(2)$ can be performed by finding the Fourier transforms in $\mathcal{O}\left(N \log(N)\right)$ time, multiplying the Fourier matrices, and performing the inverse Fourier transform in $\mathcal{O}\left(N \log(N)\right)$ time. The most computationally expensive step is matrix multiplication, which takes $\mathcal{O}(NN_R) = \mathcal{O}\left(N^{4/3}\right)$ time if the "obvious" method of matrix multiplication is used. This bound can be slightly improved by using more advanced matrix multiplication algorithms.[3] In any case, for large N this method is clearly a drastic improvement over $\mathcal{O}(N^2)$ straightforward numerical convolution.

6.6 Trajectory Planning

6.6.1 Dijkstra's Algorithm in Trajectory Planning

I begin by reviewing a well-known mobile robot planning method discussed in [Lat90], based on Dijsktra's algorithm. By comparing this algorithm to the density-based methods to be introduced in this chapter, I will show the advantages (and disadvantages) of density-based planning relative to existing methods. Later, in Section 6.6.6, we will show how ideas from this algorithm can be combined with density-based planning to obtain some of the advantages of both.

The Dijkstra-based idea is to divide the configuration space into N cells in a disjoint discretization V_0, then construct a directed graph whose edges are feasible paths between configurations, and whose nodes are configurations with no more than one configuration per cell. Once constructed, the graph serves as a "map": a feasible path from the start cell to any cell reached by the graph can be recovered from the graph in $\mathcal{O}(P)$ time, where P is the number of steps in the path. The graph is a tree and only contains one path to one configuration in any given cell. This one path is intended to be the "best" path according to some cost function (possibly, but not necessarily, distance or number of steps) used to guide construction of the path.

The graph is constructed according to the following well-known procedure [Lat90]. Maintain a list OPEN of configurations which have been reached but whose successors have not yet been considered. Also maintain a list CLOSED of configurations whose successors have been considered, and mark each cell in V_0 which contains a CLOSED

[3] The matrix multiplication step will require $\mathcal{O}\left(N^{(\gamma+1)/3}\right)$ computations, where $\gamma = 3$ for "standard" matrix multiplication, but $\gamma = \log_2 7 \approx 2.81$ using Strassen's algorithm [CK01].

configuration. Initially OPEN will only contain the start configuration and CLOSED will be empty. Given a cell in V_0, it can be determined in constant time whether it contains a CLOSED configuration (and if so what the configuration is.) The OPEN list is maintained as a balanced tree ordered by cost and can be accessed in at most logarithmic time.[4]

Select the node in OPEN with the lowest cost to reach, transfer it to CLOSED, and compute its K successors. For each successor, if it lies in a cell which already contains a CLOSED configuration, discard it. Otherwise, add it to OPEN. Keep a record of the immediate ancestor of every node which is not discarded. The algorithm terminates when every cell that the planner desired to map a path to is marked CLOSED, or when OPEN is empty and no more cells can be reached. Because the algorithm explores the successors of at most one configuration per cell, the time to construct the graph is in $\mathcal{O}(KN\log(N))$, or $\mathcal{O}(KN)$ if OPEN can be accessed in constant time.

The complete path to a CLOSED cell can be recovered from the graph by finding the reached configuration in that cell and then recursively tracing the ancestry of that configuration backwards.

Limitations of the Latombe-Barraquand-Dijkstra Algorithm

A drawback of the Latombe-Barraquand-Dijkstra (LBD) algorithm is that, since it prunes what would otherwise be an exponentially growing tree of possible paths by recording at most one configuration per cell, there is a danger that it will prune away a desirable solution.

Suppose that the system is capable of reaching configurations g_1 and g_2, where $g_2 = g_1(I + \sum_{i=1}^{D(D+1)} \epsilon_i X_i)$ for some small scalars ϵ_i and X_i are elements of $se(D)$.

[4] If the cost function used to select nodes for exploration is sufficiently simple, then OPEN can be accessed in constant time. Three examples of sufficiently simple cost functions are (a) number of steps along the path; (b) number of steps along the path which require any change of steering from the previous step; (c) number of steps along the path which require a *specific* change of steering from the previous step, e.g., a change from forward to reverse for a robot car. The relevant facts about these cost functions are that they take integer values and the value changes by at most one between a configuration and it successors. In each of these cases, OPEN should be maintained not as a tree but as a collection of linked lists, each linked list containing OPEN nodes of equal cost.

Since g_1 and g_2 are in the same cell $V(g_1) = V(g_2)$, and g_2 requires an equal or greater cost to reach, the LBD algorithm will not record g_2 or explore its successors. This means that if the precise desired goal is g_2, the LBD algorithm will only return the approximate solution g_1. But more importantly, after a further transformation h, corresponding to one or more additional steps, the successor of g_2 would have been:

$$g_2\,h = g_1(I + \sum_{i=1}^{D(D+1)} \epsilon_i X_i)\,h = g_1\,h + \sum_{i=1}^{D(D+1)} \epsilon_i X_i h \tag{6.15}$$

Even if ϵ is small, for h sufficiently large the error between successors of g_1 and g_2 will grow large compared to the size of the cells $V_0(g_1 h)$ and $V_0(g_2 h)$ and therefore the successors will lie in different cells. It is possible, therefore, that the successors of g_1 reach $V_0(g_2 h)$ only at higher cost than $g_2 h$, or that they never reach $V_0(g_2 h)$ at all. The LBD algorithm, which aims to find the lowest-cost path to each cell, may thus lose the desired solution, particularly if the system is not STLC.

This problem is aggravated as h grows large, suggesting that the optimality of solutions found by the LBD algorithm becomes more suspect as the total path length grows larger. The problem also grows worse as the size of the cells of V_0 increases, so it may not be possible to use the LBD algorithm on a very coarse scale: past a certain point of coarseness I may get, not just a more approximate solution, but no solution at all.

I will now suggest another planning algorithm, using density functions. Part of this algorithm's appeal is that (a) the density function represents all solutions and not just the ones nominally with lowest cost; and (b) a density function constructed by convolution can represent long paths formed by the concatenation of two large motions rather than by the incremental addition of many smaller motions; and for these two reasons the density-based algorithm is less likely to "lose" solutions due to round-off error. The density-based algorithm should also function at a variety of scales, both fine and coarse. Informally speaking, it can be applied not only to the parking problem for a car-like robot, but also to the cross-country trip problem.

6.6.2 The Ebert-Uphoff Algorithm

Let V_0^j denote the discretization of $SE(D)$ chosen for a j step density function[5]. Suppose that the density functions ρ^j and their associated discretizations V_0^j for $1 \le j \le P$ are known. To find a trajectory which ends as closely as possible to the goal $g_{\text{des}} \in SE(D)$ in P steps, one sequentially chooses the j^{th} motion step to maximize the reachable state density function of the remaining $P - j$ steps around the goal. That is, the first step g_1 is chosen from the set of K allowable motions in $SE(D)$ to maximize $\rho^{(P-1)}(g_1^{-1} g_{\text{des}})$. With g_1 fixed, g_2 is chosen to maximize $\rho^{(P-2)}(g_2^{-1} g_1^{-1} g_{\text{des}})$. One goes on to choose the jth step g_j to maximize $\rho^{(P-k)}(g_j^{-1} \cdots g_2^{-1} g_1^{-1} g_{\text{des}})$. After $P - 1$ steps, the final step g_P is chosen in order to minimize a distance measure

$$e = d(g_{\text{des}}, g_1 g_2 \cdots g_P) \tag{6.16}$$

This scheme for finding a trajectory that approximately reaches the goal in P steps is identical to the Ebert-Uphoff algorithm for solving the inverse kinematics of a discretely actuated P-link manipulator [CK01]. It returns an approximate solution in $\mathcal{O}(KP)$ time.

6.6.3 Implicit Storage of Paths in Graph

Because this algorithm starts with a goal reachable in P steps, makes a transformation to a point reachable in $P - 1$ steps, then $P - 2$ steps and so on, it is vaguely reminiscent of the BLD algorithm [BL89]. I can increase the resemblance by explicitly constructing a tree connecting reached volume elements.

For every configuration g_0 where $\rho^j(g_0)$ is nonzero, let $v = \{g | \; V^j(g) = V^j(g_0)\}$. To construct the graph, search for the $g \in v$ and $h \in A$ which maximize $\rho^{j-1}(h^{-1} g)$, then store a pointer between $V(g)$ and $V(h^{-1} g)$. This step requires $\mathcal{O}\left(2^D KNP\right)$ time for P density functions. Alternatively, instead of searching over both g and h, one can specify a "representative" value of $g \in v$; in this case, the graph construction

[5]Generally, the discretizations V_0 and V_0^j are the same. However, I allow for the option that one may choose to resample $SE(D)$ for purposes of refining the density function ρ^j.

takes $\mathcal{O}(KNP)$ time. In either case, by following edges of the constructed graph instead of explicitly checking K density values at each step of the trajectory, paths can be found in $\mathcal{O}(P)$ rather than $\mathcal{O}(KP)$.

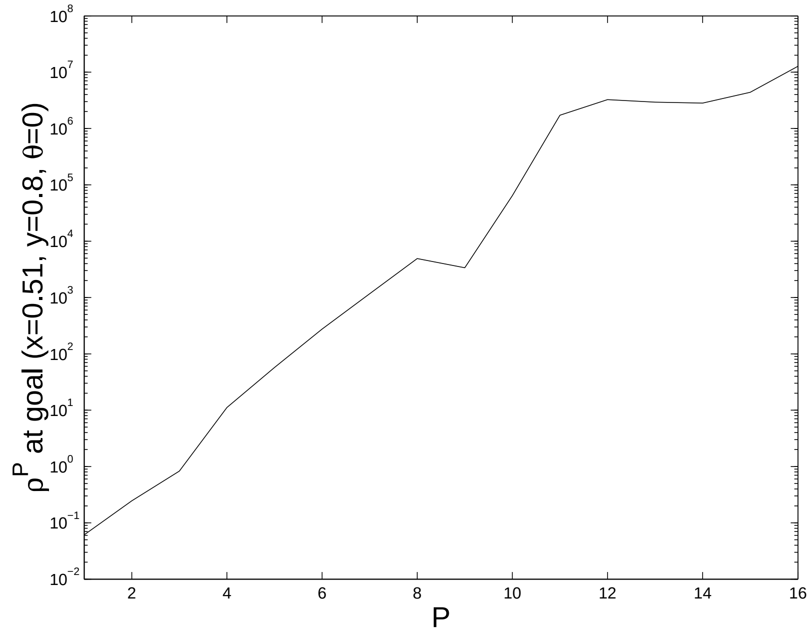


Figure 6.6: The density ρ^P in the vicinity of a particular goal, as a function of P. The path length P can be chosen so that ρ^P is sufficiently high. Often, ρ^P increases with increasing P, so a longer path may yield a better approximation to the goal.

When such an explicit graph is used, the method is still distinct from the traditional application of Dijkstra's algorithm in at least two ways. Dijkstra's algorithm chooses steps backwards from the goal to the trajectory, whereas this method chooses steps in a forward fashion. This means *inter alia* that while both methods require $\mathcal{O}(P)$ time to construct the whole path, if only the next step in the path is required then this method can find it in $\mathcal{O}(1)$ time. Also, instead of associating a single pointer with each volume element and implicitly storing only the shortest path to each volume element, up to P pointers and implicitly up to P successful trajectories are stored for every volume element. With this construction, one can either choose the shortest path which approximately reaches the goal, or possibly choose a longer path which reaches the goal more exactly.

Example. Figures 6.6–6.9 show the output from an implementation of this algorithm as applied to the fish robot of Figure 6.1 trying to reach the goal $(x, y, \theta) =$

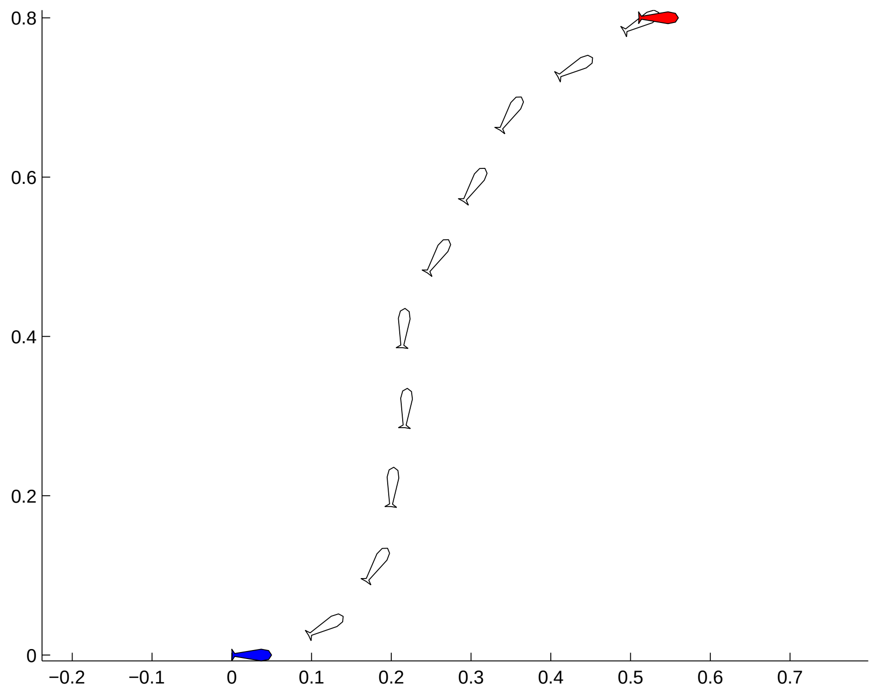


Figure 6.7: The algorithm finds a 10-step path to the goal. While the final heading is not quite right, this is roughly the shortest path with possibly acceptable error.

$(0.51, 0.8, 0)$. Figures 6.7–6.9 illustrate the trade-off between path length and final goal accuracy.

6.6.4 Multiscale Densities

This section introduces a multiscale extension of the basic algorithm. Recall that ρ_k^j denotes the density function ρ^j defined at the coarser scale V_k^j. To motivate the multiscale approach, note that each density function ρ^j is associated to a discretization V^j. The different discretizations $V^1, \cdots, V^P$ need not be identical. The algorithm, as described so far, neglects the possibility that at the j^{th} step g_j, the density function $\rho^{(P-j)}(g_j^{-1} \cdots g_2^{-1} g_1^{-1} g_{\text{des}})$ is zero for all K possible choices of g_j. In this case the goal g_{des} lies in a volume element which is not considered reachable.

It might then be said that the resolution of V^j was chosen too fine. Since I wish to reach the closest configuration to g_{des}, I should have chosen the volume elements big enough so that the one which contains g_{des} also contains some reachable point. Unfortunately, in advance of knowing g_{des}, it is not necessarily possible to know how

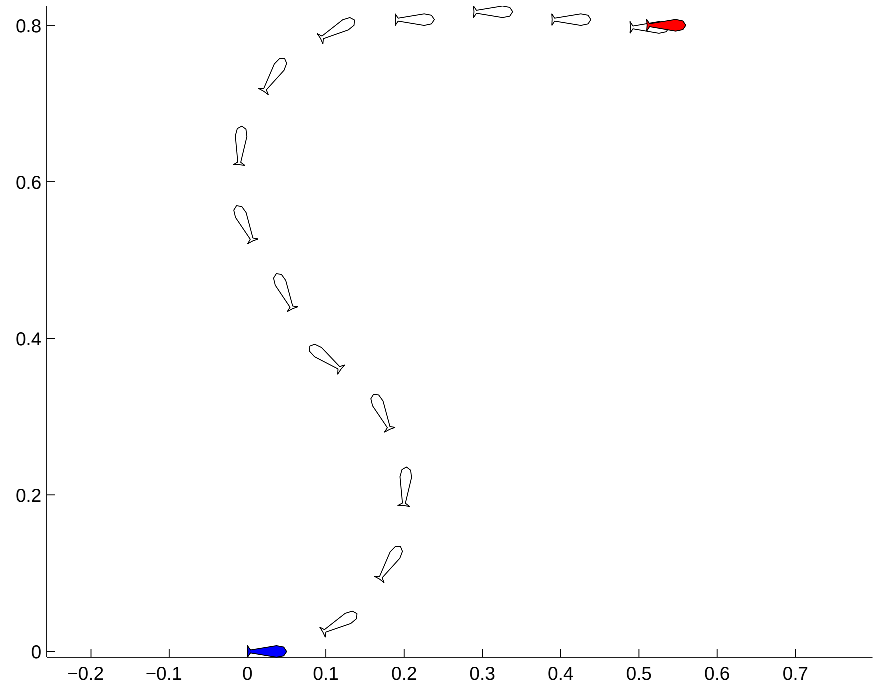


Figure 6.8: Higher accuracy can be obtained by varying P to obtain a more dense ρ^P. This 14-step path ends in a better approximation to the goal.

fine a resolution is too fine.

I will now adapt the algorithm to use density functions at a coarser resolution when necessary. Recall that according to the coarsening scheme outlined earlier, for $k \geq 1$,

$$V_k^j(x) = \bigcup \left\{ H \in V_{k-1}^j(G) | \overline{H} \cap \overline{V_{k-1}^j(x)} \neq \emptyset \right\}. \tag{6.17}$$

I want to find the smallest k such that $\rho_k^{(P-j)}(g_j^{-1} \cdots g_2^{-1} g_1^{-1} g_{\text{des}})$ is nonzero for some valid choice of g_j. Then I choose g_j to maximize this value.

In the worst case, it might be necessary to consider $\mathcal{O}(\log(N))$ scales.[6] All $\mathcal{O}(\log(N))$ scales can be constructed from the finest scale in $\mathcal{O}(N \log(N))$ time, which is dominated by the time required to construct the density function at the finest scale in the first place. To store the density function at all $\mathcal{O}(\log(N))$ scales would require $\mathcal{O}(N \log(N))$ space. But not all of this information need be retained.

[6] I have defined a series of scales whose individual volume elements increase geometrically in size. Only $\mathcal{O}(\log(N))$ scales can be defined before reaching the coarsest scale, where the density function is uniform across the whole environment. I could instead have coarsened the scale more gradually, say linearly, and defined the density function at $\mathcal{O}(N)$ different scales. But the complexity of this multiscale algorithm would rise to $\mathcal{O}(KNP)$ in time, and the time required to construct the different density functions in the first place would be $\Omega(N^2)$.

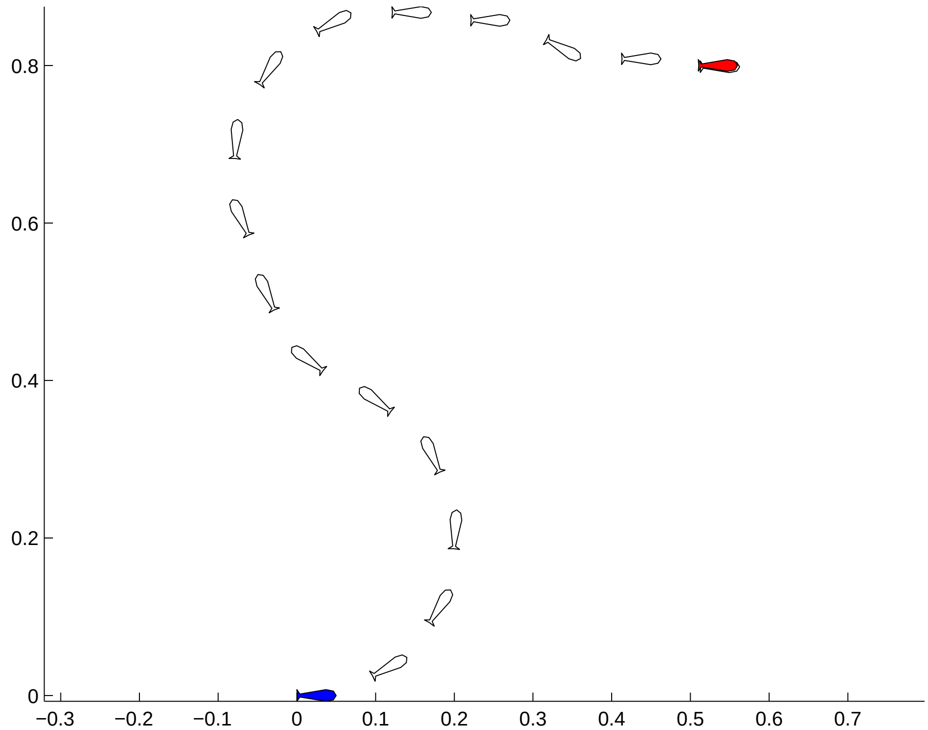


Figure 6.9: A 16-step path reaches the goal configuration almost perfectly.

The construction of the density functions at coarser scales only requires knowledge of the next finer scale. I only need to record for each volume element the finest scale at which the density function assumes nonzero value there, and that nonzero value. Consequently, the multiscale algorithm requires $\mathcal{O}(N)$ space.

Assume that the multiscale density functions are precomputed, and that the finest scale at which each volume element has a nonzero density is known. Then the algorithm can be employed as before, with the proviso that arguments are evaluated at the finest scale which displays nonzero density. Since this can be done in constant time for each argument, the worst-case performance of the modified algorithm is still $\mathcal{O}(KP)$, or $\mathcal{O}(P)$ if a graph is used to store trajectories.

Error Bounds

The worst-case error can be bounded at the outset by the size of the volume element $V_k^P(g_{\text{des}})$ for the smallest value of k such that $\rho_k^P(g_{\text{des}})$ is nonzero. After j steps have already been taken, the remaining expected error is bounded by the size of $V_k^{(P-j)}(g_j^{-1}\cdots g_2^{-1}g_1^{-1}g_{\text{des}})$ for the smallest k such that $\rho_k^{(P-j)}(g_j^{-1}\cdots g_2^{-1}g_1^{-1}g_{\text{des}})$ is nonzero. And of course, when the algorithm is complete the actual error is $e = d(g_{\text{des}}, g_1 g_2 \cdots g_P)$.

Limitations of Ebert-Uphoff Methods

At this point, I should acknowledge some advantages which the Dijkstra-based BLD algorithm retains over the density-based Ebert-Uphoff method.

- For single-processor implementations, the time requirements of the BLD algorithm are somewhat lower.
- The BLD algorithm has more flexibility to assign different costs to different allowed motions. The density-based method treats all steps equally and cannot easily account for path cost if the cost function cannot be related to "number of steps."
- The Dijkstra-based methods can take account of world-fixed constraints like static obstacles and space-dependent currents, provided these are known at the precomputation/mapping stage.

In Section 6.6.5, I will present a different density-based algorithm with different computational bounds, offering the potential for faster performance over long paths if the system's density functions are reasonably well-behaved.

Adjusting density functions to account for obstacles and other conditions located at particular points in space is problematic. Both the concept of constructing density functions from convolution, and the Ebert-Uphoff planning algorithm, rely on the idea that the density of allowed transformations is invariant under transformations of the origin in $SE(D)$. This clearly does not hold for the density of solutions in the presence of obstacles which destroy the symmetry of the configuration space.

Nevertheless, the unaltered density functions for the "open-space" problem, without obstacles, can be used to aid a BLD-like attack on the problem with obstacles. In Section 6.6.6, I describe how to use density functions to navigate around obstacles in this way.

Method	Time to Construct Map	Time to Plan Path
Barraquand and Latombe	$\mathcal{O}(NK)$	$\mathcal{O}(P)$
Ebert-Uphoff	$\mathcal{O}\left(N^{(\gamma+1)/3}P\right)$ or $\mathcal{O}(NKP)$	$\mathcal{O}(KP)$
Multiscale Ebert-Uphoff	$\mathcal{O}\left(N^{(\gamma+1)/3}P\right)$ or $\mathcal{O}(NKP + NP\log N)$	$\mathcal{O}(KP)$
Multiscale E-U plus graph	$\mathcal{O}\left(N^{(\gamma+1)/3}P + NKP\right)$	$\mathcal{O}(P)$
Log method	$\mathcal{O}\left(N^{(\gamma+1)/3}\log P\right)$	$\mathcal{O}(\tau P)$
Log method plus graph	$\mathcal{O}\left(N^{(\gamma+1)/3}\log P + N\tau\log P\right)$	$\mathcal{O}(P)$
Log method with $\mathcal{O}(P)$ processors	$\mathcal{O}\left(N^{(\gamma+1)/3}(\log P)/P\right)$	$\mathcal{O}(\tau\log P)$
Log method, graph, $\mathcal{O}(P)$ processors	$\mathcal{O}\left(N^{(\gamma+1)/3}(\log P)/P + N\tau(\log P)/P\right)$	$\mathcal{O}(\log P)$

Method	Space
Barraquand and Latombe	$\mathcal{O}(N)$
Ebert-Uphoff	$\mathcal{O}(NP)$
Multiscale Ebert-Uphoff	$\mathcal{O}(NP)$
Multiscale E-U plus graph	$\mathcal{O}(NP)$
D&C method	$\mathcal{O}(N\log P)$
D&C method plus graph	$\mathcal{O}(N\log P)$
D&C method with $\mathcal{O}(P)$ processors	$\mathcal{O}(N\log P)$
D&C method, graph, $\mathcal{O}(P)$ processors	$\mathcal{O}(N\log P)$

Table 6.2: Time and space bounds, where N is the number of volume elements, P is the number of trajectory steps, K is the number of possible motions at each step, and the constants τ and γ depend upon the chosen methods of function maximization and matrix multiplication. Although the Barraquand/Latombe algorithm has the lowest time and space requirements, it does not have all the same utility as the algorithms presented here; notably it does not allow task-by-task trade-offs between path length and accuracy.

6.6.5 A Divide-and-Conquer Algorithm

I now present a variation of the algorithm with logarithmic time complexity. This version is based on the observation that if the densities $\rho^j, \rho^{j+1}, \cdots, \rho^{2j}$ are known, then simple addition yields the function $\rho^{[j,2j-1]}$, which is the density of endpoints reachable by trajectories of at least j but no more than $2j-1$ steps. Convolving $\rho^{[j,2j-1]}$ with ρ^j yields the function $\rho^{[2j,3j-1]}$, and convolution of $\rho^{[j,2j-1]}$ with ρ^{2j} results in $\rho^{[3j,4j-1]}$. The functions $\rho^{[2j,3j-1]}$ and $\rho^{[3j,4j-1]}$ can be added to find $\rho^{[2j,4j-1]}$. Finally convolution of ρ^{2j} with itself produces ρ^{4j}. Thus, three convolutions and one addition extends our knowledge of ρ^j, ρ^{2j}, and $\rho^{[j,2j-1]}$, to ρ^{4j} and $\rho^{[2j,4j-1]}$. A further addition obtains $\rho^{[2j,4j]}$. Iteration of this procedure produces the sequence of functions $\rho^{[1,2]}, \rho^{[2,4]}, \rho^{[4,8]}, \rho^{[8,16]}, \cdots, \rho^{[P/2,P]}$ in time and space logarithmic in P.

Armed with these functions, a trajectory to $g_{\text{des}} \in SE(D)$ is planned as follows.

Choose a trajectory length P large enough so that the density $\rho^{[P/2,P]}(g_{\text{des}})$ is sufficiently high; the higher it is, the more closely the goal is likely to be reached. Using a function maximization algorithm, search for an element $h \in SE(D)$ which maximizes the product $\rho^{[P/4,P/2]}(h)\rho^{[P/4,P/2]}(h^{-1}g_{\text{des}})$. Having found such an element h, search for a path of no more than $P/2$ steps which approximates h, and another path of no more than $P/2$ steps which approximates $h^{-1}g_{\text{des}}$. In this manner, by recursively finding the midpoint of each unknown path segment, I eventually find P points along the path, which describe an entire trajectory which approximately reaches g_{des}.

This approach exploits the exponential properties of convolution better than the "linear" Ebert-Uphoff algorithm, and therefore only requires the computation and storage of $\mathcal{O}(\log P)$ density functions. The main drawback is the use of the function-maximization search. Depending on the details of the system, it may be difficult to predict the time τ required by this step. In the theoretical worst case, if it is necessary to check every possible volume element to find the maximum product of densities, then τ could be $\mathcal{O}(N)$. But if the density functions of the system are fairly well-behaved, then in practice τ will be manageably small. For example, if direction set maximization methods are usable, then τ should only be quadratic in the dimension of the configuration space.

Using a single processor, the path-finding procedure takes $\mathcal{O}(\tau P)$ time to find a trajectory. However, the algorithm lends itself to parallelization. If $\mathcal{O}(P)$ processors are available, then the trajectory can be found in $\mathcal{O}(\tau \log(P))$ time.

6.6.6 Obstacle Navigation with Density Functions

Suppose that the density functions ρ^P for the system in the absence of obstacles are already computed for a wide range of P. Then the goal g_{des} is provided but a certain number of the cells in the discretization V_0 are now occupied by obstacles and may not be entered. Paths to g_{des} which would succeed in the absence of obstacles may now be blocked. Here I will describe an algorithm—a sort of fusion of the BLD algorithm and the linear Ebert-Uphoff algorithm—which uses the density functions to inform a search for a path through the obstacles.

Begin by finding the lowest value of P such that $\rho^P(g_{\text{des}})$ is nonzero (or acceptably high.) That is, there is known to be an acceptable path of length P to the goal in the absence of obstacles, but there is no path of length $P-1$ in either the absence or presence of obstacles.

Maintain three lists:

- A list OPEN of configurations which have been reached, and which lie on a path of length P to the goal in the absence of obstacles, but whose successors have not *all* yet been considered;

- A list DEFERRED of configurations which have been reached, and whose successors have not been considered, but which does not lie on a path of length P to the goal, even in the absence of obstacles. (These configurations are "deferred" because they may still lie on a path of length greater than P which reaches the goal.)

- A list CLOSED of configurations whose successors have all been considered.

For every configuration $c = g_1 g_2 \cdots g_j$ which is in OPEN, DEFERRED, or CLOSED, I record the integer $j(c)$, which is the number of steps j by which c was reached, and also the configuration cg_j^{-1} which was the immediate ancestor of c. Every cell in V_0 that contains a configuration in either OPEN, DEFERRED, or CLOSED, has a pointer to that configuration. Thus, given any cell in V_0, it can be determined in constant time if it has been reached, and if so the path to the cell can be recovered in $\mathcal{O}(j)$ time. Both the OPEN and DEFERRED lists are maintained as a collection of linked lists ordered by j. Initially OPEN contains only the start configuration (with $j = 0$) and DEFERRED and CLOSED are empty.

Then proceed as follows:

1. Take the configuration c in OPEN with maximum $j(c)$.

2. Find the allowable transformation $g \in A$ such that cg does not intersect an OPEN or DEFERRED or CLOSED cell, which maximizes[7] $\rho^{(P-(j(cg))}(g^{-1}c^{-1}g_{\text{des}})$.

3. If there is no such g (i.e., all K successors of c lie in obstacles or cells that have already been reached) then remove c from OPEN and place it in CLOSED.

4. If $\rho^{(P-j-1)}(g^{-1}c^{-1}g_{\text{des}})$ is zero, then there is no path of length P from the origin to the goal which passes through c. Remove c from OPEN and place it in DEFERRED.

5. Otherwise, $\rho^{(P-j-1)}(g^{-1}c^{-1}g_{\text{des}})$ is greater than zero. Add cg to OPEN. Leave c in OPEN. Since $j(cg) = j(c) + 1$, the configuration cg is now at the top of the OPEN list and will be considered next.

6. Go back to step 1 and repeat until the goal is reached or until OPEN is empty. If the goal is reached, the algorithm terminates (success.)

7. If OPEN is empty, then there is no path of length P from origin to goal in the presence of the obstacles. If DEFERRED is also empty, then the algorithm terminates (failure.) If DEFERRED is not empty, there may be a longer path from the origin to the goal. Increment P by one. Move all the configurations in DEFERRED back into OPEN. Return to step 1.

With two caveats, this algorithm will find the minimum length path to the goal if one exists. **Proof:** Whenever a given cell is reached for the first time, and a configuration c in the cell is added to OPEN, the path of length j to c is known to be part of a path of length P that would reach the goal in the absence of obstacles. Furthermore, all paths of length less than P that would reach the goal in the absence of obstacles have been explored until they either collided with an obstacle or began retracing a previously explored path of length less than P (which in turn was explored

[7] Finding the maximizing successor once takes only $\mathcal{O}(K)$ time, but if the node c is revisited K times and must produce K unexplored successors in order of decreasing solution density, this could take $\mathcal{O}(K^2)$ time. For better asymptotic performance, the successors should all be sorted by solution density when c is first considered, in $\mathcal{O}(K \log K)$ time. Of course, K may be small enough that this is a niggling distinction.

until it collided with an obstacle). Therefore, there cannot be a feasible path of length $k < j$ to the cell containing c, because if so the cell containing c would have been previously entered while the algorithm was searching for paths of length $P - (j - k)$. So in particular, when the goal cell is reached the path to the goal cell will be of minimum length. Since every reachable cell will eventually be reached, the minimum length path to the goal will be found if a path exists.

The two caveats or holes in the proof concern the possibility of numerical error. First, the proof assumes that the information in the density functions is reliable, and in particular, that the density function will be nonzero when there is a solution and zero when there is no solution. Obviously, to the extent that there is approximation error or any other kind of error in the density functions, the path search may be misdirected accordingly. Second, and perhaps more importantly, the algorithm assumes that multiple configurations in the same cell do not have to be retained, because further motion starting from those similar configurations will reach substantially similar cells. But as we have seen, over long paths even tiny differences in starting orientation can lead to large differences in the final destination. This algorithm thus inherits the same vulnerability to "round-off" error as the original BLD algorithm. In the absence of small time local controllability, there is no obvious remedy except to keep the cell resolution as high as computationally feasible.

Time Bounds

In the worst case, this algorithm does not perform any better than the BLD algorithm, because there is no guarantee that searching where solutions disregarding obstacles are known to be dense will yield a solution which accounts for obstacles. In the *best case*, however, when the obstacles turn out not to obstruct the standard Ebert-Uphoff path from the origin to g_{des}, this algorithm will find the path in $\mathcal{O}(KP)$, while the conservative breadth-first search of the BLD algorithm may still take a long time (in $\mathcal{O}(NK)$) to complete even in this optimistic case.

The information provided by the density functions ρ provides this algorithm with two advantages relative to the uninformed BLD algorithm.

- The density-based algorithm can deduce in $\mathcal{O}(P)$ time that the length of the path to the goal must be at least P, considering only motion constraints and not the effect of obstacles. This algorithm can therefore immediately begin a depth-first search for a path of length P. The BLD algorithm, with no "knowledge" of the constraints placed on path length by the allowed motion set, will perform a breadth-first search to avoid missing a (nonexistent) path of length less than P.

- If exploration from a given node cannot possibly result in a successful path of length P (because of motion constraints), then the density-based algorithm will not explore from that node until after it has checked all nodes that possibly could result in a path of length P. The uninformed BLD algorithm will explore equally from all nodes equally distant from the origin.

So I expect that the density-based algorithm will typically be faster than the uninformed BLD algorithm, because it avoids checking fruitless paths that the BLD algorithm will check before finding a solution. On the other hand, the density-based algorithm will *never* perform significantly worse than the BLD, because the density-based algorithm never explores any path longer than the path actually required to reach the goal, but the BLD in general will search *every* path of length less than or equal to the solution path before finding the solution. In the worst case, where there is no solution, the BLD algorithm will take $\mathcal{O}(NK)$ time and the density based algorithm will take $\mathcal{O}(NK \log K)$ time. The additional factor of $\log(K)$ arises from the need to sort paths by solution density—obviously, it is of no real significance unless K is unusually large.

Chapter 7 Conclusions and Future Work

I have presented general closed-form expressions for the force and moment on a deformable Joukowski foil, and the equations of motion for a few examples of deformable swimming bodies. I have built and modelled a three-link planar robot fish—the robot fish itself is the first such platform to be well suited for feedback control experiments, while the model is useful insofar as it assists feedback control and path planning, and gives insight into what physical processes are dominant during any particular fish maneuver. I have outlined how sets of reachable points, and the trajectories necessary to reach them in a minimum time, could be computed for the three-link fish. And I have shown how the density of these sets of reachable points can be convolved to obtain the density of endpoints of longer trajectories, and how these functions can be used as tools in plotting motions for the fish, or for many other mobile robots, with or without the presence of obstacles.

There are still many directions in which this work should proceed. Numerical optimization of desirable trajectories for the three-link robot, and for the self-propelled deformable Joukowski foil, is still unfinished. It would probably be possible to extend the treatment given to the deformable Joukowski foil and apply it to more general kinds of deformable foil shapes. More ambitiously, it would be interesting to evaluate the potential around multiple foil shapes, so that a fish, for example, could be modelled as a collection of hydrodynamically interacting fins and streamlined body parts. (This contrasts with my current treatment of the the three-link robot in which the three links are assumed to by hydrodynamically separate.)

In the realm of path-planning using density functions, the ideas suggested here remain untested on real-world problems. The divide-and-conquer algorithm for charting a path with a logarithmic number of density functions, in particular, requires that the system's density functions, and their products, be reasonably well-behaved and amenable to standard searching techniques. It would be interesting to see whether this holds true of the density functions for a real system of interest (as opposed to an elementary model constructed to illustrate the algorithm).

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Appendix A Integrals for Unsteady Flow Forces

These integrals arise in determining the force on a deformable Joukowski foil due to the time-varying potential.

$$\begin{aligned}\int_\Sigma \frac{\partial}{\partial t}(w+\overline{w})\,dz = \dot{U}\left(\int_\Sigma (w_1+\overline{w_1})\,dz\right) + \dot{V}\left(\int_\Sigma (w_2+\overline{w_2})\,dz\right) + \dot{\Omega}\left(\int_\Sigma (w_3+\overline{w_3})\,dz\right) \\ + \dot{\gamma}_c\left(\int_\Sigma (w_4+\overline{w_4}\,dz\right) + \sum_k \dot{\gamma}_k\left(2\int_\Sigma w_5^k\,dz\right) \\ + \ddot{\zeta}_x\left(\int_\Sigma (w_1^s+\overline{w_1^s}\,dz\right) + \ddot{\zeta}_y\left(\int_\Sigma (w_2^s+\overline{w_2^s}\,dz\right) + \ddot{a}\left(\int_\Sigma (w_3^s+\overline{w_3^s}\,dz\right) \\ + U\left(\int_\Sigma (\frac{\partial w_1}{\partial t}+\frac{\partial \overline{w_1}}{\partial t}\right) + V\left(\int_\Sigma (\frac{\partial w_2}{\partial t}+\frac{\partial \overline{w_2}}{\partial t}\right) + \Omega\left(\int_\Sigma (\frac{\partial w_3}{\partial t}+\frac{\partial \overline{w_3}}{\partial t}\right) \\ + \gamma_c\left(2\int_\Sigma \frac{\partial w_4}{\partial t}\right) + \sum_k \gamma_k\left(2\int_\Sigma \frac{\partial w_5^k}{\partial t}\,dz\right) \\ + \dot{\zeta}_x\left(\int_\Sigma (\frac{\partial w_1^s}{\partial t}+\frac{\partial \overline{w_1^s}}{\partial t}\right) + \dot{\zeta}_y\left(\int_\Sigma (\frac{\partial w_2^s}{\partial t}+\frac{\partial \overline{w_2^s}}{\partial t}\right) + \dot{a}\left(\int_\Sigma (\frac{\partial w_3^s}{\partial t}+\frac{\partial \overline{w_3^s}}{\partial t}\right)\end{aligned} \tag{A.1}$$

$$\begin{aligned}\int_\Sigma w_1\,dz &= \int_C \left(-\frac{r_c^2}{\zeta}+\frac{a^2}{\zeta+\zeta_c}\right)\left(1-\frac{a^2}{(\zeta+\zeta_c)^2}\right)d\zeta \\ &= 2\pi i(a^2-r_c^2)\end{aligned} \tag{A.2}$$

$$\begin{aligned}
\int_\Sigma \overline{w_1}\,dz &= \overline{\int_\Sigma w_1\,\overline{dz}} \\
&= \overline{\int_C \left(-\frac{r_c^2}{\zeta} + \frac{a^2}{\zeta+\zeta_c}\right)\left[-\frac{r_c^2}{\zeta^2} + \frac{a^2 r_c^2}{(r_c^2+\overline{\zeta_c}\zeta)^2}\right] d\zeta} \\
&= \overline{2\pi i(-a^2 + \frac{a^4 r_c^2}{(r_c^2-\delta^2)^2})} \\
&= 2\pi i(a^2 - \frac{a^4 r_c^2}{(r_c^2-\delta^2)^2})
\end{aligned} \tag{A.3}$$

$$\begin{aligned}
\int_\Sigma w_2\,dz &= \int_C \left(-i\frac{r_c^2}{\zeta} - i\frac{a^2}{\zeta+\zeta_c}\right)\left(1 - \frac{a^2}{(\zeta+\zeta_c)^2}\right) d\zeta \\
&= 2\pi(a^2 + r_c^2)
\end{aligned} \tag{A.4}$$

$$\begin{aligned}
\int_\Sigma \overline{w_2}\,dz &= \overline{\int_\Sigma w_2\,\overline{dz}} \\
&= \overline{\left(-i\frac{r_c^2}{\zeta} - i\frac{a^2}{\zeta+\zeta_c}\right)\left[-\frac{r_c^2}{\zeta^2} + \frac{a^2 r_c^2}{(r_c^2+\overline{\zeta_c}\zeta)^2}\right] d\zeta} \\
&= \overline{2\pi(a^2 + \frac{a^4 r_c^2}{(r_c^2-\delta^2)^2})} \\
&= 2\pi(a^2 + \frac{a^4 r_c^2}{(r_c^2-\delta^2)^2})
\end{aligned} \tag{A.5}$$

$$\begin{aligned}
\int_\Sigma w_3\,dz &= \frac{-i}{2}\int_\Sigma \left(\frac{2\zeta_c r_c^2}{\zeta} + \frac{2a^2 r_c^2}{\zeta(\zeta+\zeta_c)} + \frac{2a^2\overline{\zeta_c}}{(\zeta+\zeta_c)} + \frac{a^4(\zeta-\zeta_c)}{(\zeta+\zeta_c)(r_c^2-\delta^2)}\right)\left(1 - \frac{a^2}{(\zeta+\zeta_c)^2}\right) d\zeta \\
&= 2\pi\left(r_c^2\zeta_c + a^2\overline{\zeta_c} - \frac{a^4\zeta_c}{(r_c^2-\delta^2)}\right)
\end{aligned} \tag{A.6}$$

$$
\begin{aligned}
\int_\Sigma \overline{w_3}\, dz &= \overline{\int_\Sigma w_3 \,\overline{dz}} \\
&= \overline{\frac{-i}{2}\int_\Sigma \left(\frac{2\zeta_c r_c^2}{\zeta} + \frac{2a^2 r_c^2}{\zeta(\zeta+\zeta_c)} + \frac{2a^2\overline{\zeta_c}}{(\zeta+\zeta_c)} + \frac{a^4(\zeta-\zeta_c)}{(\zeta+\zeta_c)(r_c^2-\delta^2)}\right)\left[-\frac{r_c^2}{\zeta^2} + \frac{a^2 r_c^2}{(r_c^2+\overline{\zeta_c}\zeta)^2}\right] d\zeta} \\
&= \overline{2\pi\left[\zeta_c a^2 + \frac{a^4}{\zeta_c} - \frac{a^4 r_c^4}{\zeta_c(r_c^2-\delta^2)^2} + \frac{a^4 r_c^2\overline{\zeta_c}}{(r_c^2-\delta^2)^2} - \frac{a^6\zeta_c r_c^2}{(r_c^2-\delta^2)^3}\right]} \\
&= 2\pi\overline{\left[\zeta_c a^2 + -\frac{a^4\overline{\zeta_c}}{(r_c^2-\delta^2)} - \frac{a^6\zeta_c r_c^2}{(r_c^2-\delta^2)^3}\right]} \\
&= 2\pi\left[\overline{\zeta_c} a^2 - \frac{a^4\zeta_c}{(r_c^2-\delta^2)} - \frac{a^6\overline{\zeta_c} r_c^2}{(r_c^2-\delta^2)^3}\right] \qquad \text{(A.7)}
\end{aligned}
$$

In order to perform the integrals involving logarithmic terms in w_4 and w_1^s, w_2^s, w_3^s, we do integration by parts, in each case choosing the branch cut associated with the logarithmic potential to pass through the point $z_{\text{cut}} = 2a$, i.e., the trailing edge of the foil.

$$
\begin{aligned}
\int_\Sigma \log(\frac{\zeta}{r_c})\, dz &= [\log(\frac{\zeta+\zeta_c}{r_c})z]_-^+ - \int_C z\frac{d\zeta}{\zeta} \\
&= [2\pi i z_{\text{cut}}] - \int_C \left(\zeta + \zeta_c + \frac{a^2}{(\zeta+\zeta_c)}\right)\frac{d\zeta}{\zeta} \\
&= [2\pi i z_{\text{cut}}] - 2\pi i\zeta_c \qquad \text{(A.8)}
\end{aligned}
$$

$$
\begin{aligned}
\int_\Sigma \log(\frac{\zeta+\zeta_c}{r_c})\, dz &= [\log(\frac{\zeta+\zeta_c}{r_c})z]_-^+ - \int_C z\frac{d\zeta}{(\zeta+\zeta_c)} \\
&= [2\pi i z_{\text{cut}}] - \int_C \left(\zeta + \zeta_c + \frac{a^2}{(\zeta+\zeta_c)}\right)\frac{d\zeta}{(\zeta+\zeta_c)} \\
&= [2\pi i z_{\text{cut}}] \qquad \text{(A.9)}
\end{aligned}
$$

$$\begin{aligned}\int_\Sigma \log(\frac{\zeta+\zeta_c}{\zeta})\,dz &= -\int_C z\frac{(-\zeta_c)}{\zeta(\zeta+\zeta_c)}\,d\zeta \\ &= \int_C (\zeta+\zeta_c+\frac{a^2}{\zeta+\zeta_c})\frac{\zeta_c}{\zeta(\zeta+\zeta_c)}\,d\zeta \\ &= [2\pi i\zeta_c]\end{aligned} \tag{A.10}$$

$$\begin{aligned}\int_\Sigma w_4\,dz &= -2\pi(z_{\text{cut}}-\zeta_c) \\ &= -2\pi(2a-\zeta_c)\end{aligned} \tag{A.11}$$

$$\begin{aligned}\int_\Sigma w_1^s\,dz &= \int_\Sigma \left[-\frac{r_c^2}{\zeta}+\frac{a^2r_c^2}{\zeta\zeta_c^2}+\frac{a^2}{(\zeta+\zeta_c)}-\frac{a^4\zeta_c^2}{(\zeta+\zeta_c)(r_c^2-\delta^2)^2}\right. \\ &\quad \left.-\left(\frac{a^2r_c^2}{\zeta_c^3}-\frac{a^4r_c^2}{\zeta_c(r_c^2-\delta^2)^2}\right)\frac{\zeta}{(\zeta+\zeta_c)}\right]\left(1-\frac{a^2}{(\zeta+\zeta_c)^2}\right)\,d\zeta \\ &\quad -\frac{2a^4r_c^2\zeta_c}{(r_c^2-\delta^2)^3}\int_\Sigma \log((\zeta+\zeta_c)/r_c)\,dz+\frac{2a^4r_c^2(i\zeta_y)}{(r_c^2-\delta^2)^3}\log(\frac{\zeta}{r_c}) \\ &\quad -2a^2r_c^2\left(\frac{1}{\zeta_c^3}+\frac{\overline{\zeta_c}a^2}{(r_c^2-\delta^2)^3}\right)\int_\Sigma \log(\frac{\zeta+\zeta_c}{\zeta})\,dz \\ &= 2\pi i\left[-r_c^2+a^2+\frac{2a^2r_c^2}{\zeta_c^2}-\frac{a^4(\zeta_c^2+r_c^2)}{(r_c^2-\delta^2)^2}-\frac{2a^4r_c^2\zeta_c}{(r_c^2-\delta^2)^3}z_{\text{cut}}\right. \\ &\quad \left.+\frac{2a^4r_c^2(i\zeta_y)}{(r_c^2-\delta^2)^3}(z_{\text{cut}}-\zeta_c)-2a^2r_c^2\left(\frac{1}{\zeta_c^2}+\frac{\delta^2a^2}{(r_c^2-\delta^2)^3}\right)\right] \\ &= 2\pi i\left[-r_c^2+a^2+\frac{2a^2r_c^2}{\zeta_c^2}-\frac{a^4(\zeta_c^2+r_c^2)}{(r_c^2-\delta^2)^2}-\frac{4a^5r_c^2\zeta_c}{(r_c^2-\delta^2)^3}\right. \\ &\quad \left.+\frac{2a^4r_c^2(i\zeta_y)}{(r_c^2-\delta^2)^3}(2a-\zeta_c)-2a^2r_c^2\left(\frac{1}{\zeta_c^2}+\frac{\delta^2a^2}{(r_c^2-\delta^2)^3}\right)\right]\end{aligned} \tag{A.12}$$

$$\begin{aligned}
\int_\Sigma w_1^s \, \overline{dz} &= \int_\Sigma \left[-\frac{r_c^2}{\zeta} + \frac{a^2 r_c^2}{\zeta \zeta_c^2} + \frac{a^2}{(\zeta+\zeta_c)} - \frac{a^4 \zeta_c^2}{(\zeta+\zeta_c)(r_c^2-\delta^2)^2} \right. \\
&\quad \left. -\left(\frac{a^2 r_c^2}{\zeta_c^3} - \frac{a^4 r_c^2}{\zeta_c (r_c^2-\delta^2)^2} \right) \frac{\zeta}{(\zeta+\zeta_c)} \right] \left[-\frac{r_c^2}{\zeta^2} + \frac{a^2 r_c^2}{(r_c^2+\overline{\zeta_c}\zeta)^2} \right] d\zeta \\
&\quad -\frac{2a^4 r_c^2 \zeta_c}{(r_c^2-\delta^2)^3} \int_\Sigma \log((\zeta+\zeta_c)/r_c) \, \overline{dz} + \frac{2a^4 r_c^2 (i\zeta_y)}{(r_c^2-\delta^2)^3} \int_\Sigma \log(\frac{\zeta}{r_c}) \, \overline{dz} \\
&\quad -2a^2 r_c^2 \left(\frac{1}{\zeta_c^3} + \frac{\overline{\zeta_c} a^2}{(r_c^2-\delta^2)^3} \right) \int_\Sigma \log(\frac{\zeta+\zeta_c}{\zeta}) \, \overline{dz} \\
&= 2\pi i \left[-a^2 + \frac{a^4}{\zeta_c^2} + \frac{a^4 r_c^2}{(r_c^2-\delta^2)^2} - \frac{a^6 r_c^2 \zeta_c^2}{(r_c^2-\delta^2)^4} + \frac{a^4 r_c^4}{\zeta_c^2 (r_c^2-\delta^2)^2} - \frac{a^6 r_c^4}{(r_c^2-\delta^2)^4} \right. \\
&\quad -\frac{2a^4 r_c^2 \zeta_c}{(r_c^2-\delta^2)^3} \left(\overline{z}_{\text{cut}} - \overline{\zeta_c} + \frac{a^2 \zeta_c}{(r_c^2-\delta^2)} \right) + \frac{2a^4 r_c^2 (i\zeta_y)}{(r_c^2-\delta^2)^3} \left(\overline{z}_{\text{cut}} - \overline{\zeta_c} \right) \\
&\quad \left. -2a^4 r_c^2 \left(\frac{1}{\zeta_c^3} + \frac{\overline{\zeta_c} a^2}{(r_c^2-\delta^2)^3} \right) \left(\frac{\zeta_c}{(r_c^2-\delta^2)} \right) \right] \qquad \text{(A.13)}
\end{aligned}$$

$$\begin{aligned}
\int_\Sigma \overline{w_1^s} \, dz &= 2\pi i \left[a^2 - \frac{a^4}{\overline{\zeta_c}^2} - \frac{a^4 r_c^2}{(r_c^2-\delta^2)^2} + \frac{a^6 r_c^2 \overline{\zeta_c}^2}{(r_c^2-\delta^2)^4} - \frac{a^4 r_c^4}{\overline{\zeta_c}^2 (r_c^2-\delta^2)^2} + \frac{a^6 r_c^4}{(r_c^2-\delta^2)^4} \right. \\
&\quad +\frac{2a^4 r_c^2 \overline{\zeta_c}}{(r_c^2-\delta^2)^3} \left(2a - \zeta_c + \frac{a^2 \overline{\zeta_c}}{(r_c^2-\delta^2)} \right) + \frac{2a^4 r_c^2 (i\zeta_y)}{(r_c^2-\delta^2)^3} \left(2a - \zeta_c \right) \\
&\quad \left. +2a^4 r_c^2 \left(\frac{1}{\overline{\zeta_c}^3} + \frac{\zeta_c a^2}{(r_c^2-\delta^2)^3} \right) \left(\frac{\overline{\zeta_c}}{(r_c^2-\delta^2)} \right) \right] \qquad \text{(A.14)}
\end{aligned}$$

$$\begin{aligned}
\int_\Sigma w_2^s\, dz &= \int_\Sigma (-i)\left[\frac{r_c^2}{\zeta} - \frac{a^2 r_c^2}{\zeta\zeta_c^2} + \frac{a^2}{(\zeta+\zeta_c)} - \frac{a^4\zeta_c^2}{(\zeta+\zeta_c)(r_c^2-\delta^2)^2}\right.\\
&\quad \left. + \left(\frac{a^2 r_c^2}{\zeta_c^3} - \frac{a^4 r_c^2}{\zeta_c(r_c^2-\delta^2)^2}\right)\frac{\zeta}{(\zeta+\zeta_c)}\right]\left(1 - \frac{a^2}{(\zeta+\zeta_c)^2}\right)\, d\zeta\\
&\quad + i\frac{2a^4 r_c^2\zeta_c}{(r_c^2-\delta^2)^3}\int_\Sigma \log(\frac{\zeta+\zeta_c}{r_c})\, dz - i\frac{2a^4 r_c^2\zeta_x}{(r_c^2-\delta^2)^3}\int_\Sigma \log(\frac{\zeta}{r_c})\, dz\\
&\quad - i2a^2 r_c^2\left(\frac{1}{\zeta_c^3} + \frac{\overline{\zeta_c}a^2}{(r_c^2-\delta^2)^3}\right)\int_\Sigma \log(\frac{\zeta+\zeta_c}{\zeta})\, dz\\
&= 2\pi\left[r_c^2 + a^2 - \frac{2a^2 r_c^2}{\zeta_c^2} - \frac{a^4(\zeta_c^2 - r_c^2)}{(r_c^2-\delta^2)^2} - \frac{2a^4 r_c^2\zeta_c}{(r_c^2-\delta^2)^3} z_{\text{cut}}\right.\\
&\quad \left. + \frac{2a^4 r_c^2\zeta_x}{(r_c^2-\delta^2)^3}(z_{\text{cut}} - \zeta_c) + 2a^2 r_c^2\left(\frac{1}{\zeta_c^2} + \frac{\delta^2 a^2}{(r_c^2-\delta^2)^3}\right)\right]\\
&= 2\pi\left[r_c^2 + a^2 - \frac{2a^2 r_c^2}{\zeta_c^2} - \frac{a^4(\zeta_c^2 - r_c^2)}{(r_c^2-\delta^2)^2} - \frac{4a^5 r_c^2\zeta_c}{(r_c^2-\delta^2)^3}\right.\\
&\quad \left. + \frac{2a^4 r_c^2\zeta_x}{(r_c^2-\delta^2)^3}(2a - \zeta_c) + 2a^2 r_c^2\left(\frac{1}{\zeta_c^2} + \frac{\delta^2 a^2}{(r_c^2-\delta^2)^3}\right)\right] \qquad \text{(A.15)}
\end{aligned}$$

$$\begin{aligned}
\int_\Sigma w_2^s\, \overline{dz} &= \int_C (-i)\left[\frac{r_c^2}{\zeta} - \frac{a^2 r_c^2}{\zeta\zeta_c^2} + \frac{a^2}{(\zeta+\zeta_c)} - \frac{a^4\zeta_c^2}{(\zeta+\zeta_c)(r_c^2-\delta^2)^2}\right.\\
&\quad \left. + \left(\frac{a^2 r_c^2}{\zeta_c^3} - \frac{a^4 r_c^2}{\zeta_c(r_c^2-\delta^2)^2}\right)\frac{\zeta}{(\zeta+\zeta_c)}\right]\left[-\frac{r_c^2}{\zeta^2} + \frac{a^2 r_c^2}{(r_c^2 + \overline{\zeta_c}\zeta)^2}\right]\, d\zeta\\
&\quad + i\frac{2a^4 r_c^2\zeta_c}{(r_c^2-\delta^2)^3}\int_\Sigma \log(\frac{\zeta+\zeta_c}{r_c})\, \overline{dz} - i\frac{2a^4 r_c^2\zeta_x}{(r_c^2-\delta^2)^3}\int_\Sigma \log(\frac{\zeta}{r_c})\, \overline{dz}\\
&\quad - i2a^2 r_c^2\left(\frac{1}{\zeta_c^3} + \frac{\overline{\zeta_c}a^2}{(r_c^2-\delta^2)^3}\right)\int_\Sigma \log(\frac{\zeta+\zeta_c}{\zeta})\, \overline{dz}\\
&= 2\pi\left[a^2 - \frac{a^4}{\zeta_c^2} + \frac{a^4 r_c^2}{(r_c^2-\delta^2)^2} - \frac{a^6 r_c^2\zeta_c^2}{(r_c^2-\delta^2)^4} - \frac{a^4 r_c^4}{\zeta_c^2(r_c^2-\delta^2)^2} + \frac{a^6 r_c^4}{(r_c^2-\delta^2)^4}\right.\\
&\quad - \frac{2a^4 r_c^2\zeta_c}{(r_c^2-\delta^2)^3}\left(\overline{z}_{\text{cut}} - \overline{\zeta_c} + \frac{a^2\zeta_c}{(r_c^2-\delta^2)}\right) + \frac{2a^4 r_c^2\zeta_x}{(r_c^2-\delta^2)^3}\left(\overline{z}_{\text{cut}} - \overline{\zeta_c}\right)\\
&\quad \left. + 2a^4 r_c^2\left(\frac{1}{\zeta_c^3} + \frac{\overline{\zeta_c}a^2}{(r_c^2-\delta^2)^3}\right)\left(\frac{\zeta_c}{(r_c^2-\delta^2)}\right)\right] \qquad \text{(A.16)}
\end{aligned}$$

$$\int_\Sigma \overline{w_2^s}\,dz = 2\pi\left[a^2 - \frac{a^4}{\overline{\zeta_c}^2} + \frac{a^4 r_c^2}{(r_c^2-\delta^2)^2} - \frac{a^6 r_c^2 \overline{\zeta_c}^2}{(r_c^2-\delta^2)^4} - \frac{a^4 r_c^4}{\overline{\zeta_c}^2 (r_c^2-\delta^2)^2} + \frac{a^6 r_c^4}{(r_c^2-\delta^2)^4} \right.$$
$$-\frac{2a^4 r_c^2 \overline{\zeta_c}}{(r_c^2-\delta^2)^3}\left(2a - \zeta_c + \frac{a^2\overline{\zeta_c}}{(r_c^2-\delta^2)}\right) + \frac{2a^4 r_c^2 \zeta_x}{(r_c^2-\delta^2)^3}(2a-\zeta_c)$$
$$\left. +2a^4 r_c^2 \left(\frac{1}{\overline{\zeta_c}^3} + \frac{\zeta_c a^2}{(r_c^2-\delta^2)^3}\right)\left(\frac{\overline{\zeta_c}}{(r_c^2-\delta^2)}\right)\right] \tag{A.17}$$

$$\begin{aligned}
\int_\Sigma w_3^s\,dz &= \int_C 2a\left(-\frac{r_c^2}{\zeta\zeta_c} - \frac{a^2\zeta_c}{(\zeta+\zeta_c)(r_c^2-\delta^2)}\right)\left(1 - \frac{a^2}{(\zeta+\zeta_c)^2}\right)d\zeta \\
&\quad -2\frac{a^3 r_c^2}{(r_c^2-\delta^2)^2}\int_\Sigma \log((\zeta+\zeta_c)/r_c)\,dz \\
&\quad +2a\frac{r_c^2((r_c^2-\delta^2)^2 - a^2\zeta_c^2)}{\zeta_c^2(r_c^2-\delta^2)^2}\int_\Sigma \log\left(\frac{(\zeta+\zeta_c)}{\zeta}\right)dz \\
&= 4\pi a i\left(-\frac{r_c^2}{\zeta_c} - \frac{a^2\zeta_c}{(r_c^2-\delta^2)} - \frac{a^2 r_c^2}{(r_c^2-\delta^2)^2} z_{\text{cut}} + \frac{r_c^2((r_c^2-\delta^2)^2 - a^2\zeta_c^2)}{\zeta_c(r_c^2-\delta^2)^2}\right) \\
&= 4\pi a i\left(-\frac{r_c^2}{\zeta_c} - \frac{a^2\zeta_c}{(r_c^2-\delta^2)} - \frac{2a^3 r_c^2}{(r_c^2-\delta^2)^2} + \frac{r_c^2((r_c^2-\delta^2)^2 - a^2\zeta_c^2)}{\zeta_c(r_c^2-\delta^2)^2}\right)
\end{aligned} \tag{A.18}$$

$$\begin{aligned}
\int_\Sigma w_3^s\,\overline{dz} &= \int_C 2a\left(-\frac{r_c^2}{\zeta\zeta_c} - \frac{a^2\zeta_c}{(\zeta+\zeta_c)(r_c^2-\delta^2)}\right)\left[-\frac{r_c^2}{\zeta^2} + \frac{a^2 r_c^2}{(r_c^2+\overline{\zeta_c}\zeta)^2}\right]d\zeta \\
&\quad -2\frac{a^3 r_c^2}{(r_c^2-\delta^2)^2}\int_\Sigma \log((\zeta+\zeta_c)/r_c)\,\overline{dz} \\
&\quad +2a\frac{r_c^2((r_c^2-\delta^2)^2 - a^2\zeta_c^2)}{\zeta_c^2(r_c^2-\delta^2)^2}\int_\Sigma \log\left(\frac{(\zeta+\zeta_c)}{\zeta}\right)\overline{dz} \\
&= 4\pi a i\left(-\frac{a^2}{\zeta_c} - \frac{a^4 r_c^2\zeta_c}{(r_c^2-\delta^2)^3}\right) \\
&\quad -4\pi i\frac{a^3 r_c^2}{(r_c^2-\delta^2)^2}\left(\overline{z}_{\text{cut}} - \overline{\zeta_c} + \frac{a^2\zeta_c}{(r_c^2-\delta^2)}\right) \\
&\quad +4\pi i a^3\frac{r_c^2((r_c^2-\delta^2)^2 - a^2\zeta_c^2)}{\zeta_c^2(r_c^2-\delta^2)^2}\left(\frac{\zeta_c}{(r_c^2-\delta^2)}\right)
\end{aligned} \tag{A.19}$$

$$\int_\Sigma \overline{w_3^s}\, dz = 4\pi a i \left(\frac{a^2}{\overline{\zeta_c}} + \frac{a^4 r_c^2 \overline{\zeta_c}}{(r_c^2 - \delta^2)^3} \right) + 4\pi i \frac{a^3 r_c^2}{(r_c^2-\delta^2)^2} \left(2a - \zeta_c + \frac{a^2 \overline{\zeta_c}}{(r_c^2 - \delta^2)} \right) - 4\pi i a^3 \frac{r_c^2((r_c^2-\delta^2)^2 - a^2 \overline{\zeta_c}^2)}{\overline{\zeta_c}^2 (r_c^2-\delta^2)^2} \left(\frac{\overline{\zeta_c}}{(r_c^2-\delta^2)} \right) \tag{A.20}$$

Now we evaluate the integrals involving time derivatives of the Kirchoff potentials.

$$\begin{aligned} \int_\Sigma \frac{\partial w_1}{\partial t}\, dz &= \int_\Sigma \left(\frac{-2\, r_c\, \dot{r_c}}{\zeta} + \frac{2\, a\, \dot{a}}{\zeta + \zeta_c} - \frac{a^2\, \dot{\zeta}_x}{(\zeta+\zeta_c)^2} - \frac{a^2\, i\, \dot{\zeta}_y}{(\zeta+\zeta_c)^2} \right) dz \\ &= 4\pi i\, (\, a\, \dot{a} - \, r_c\, \dot{r_c}) \\ &= 2\pi i \frac{d}{dt} \left(a^2 - r_c^2 \right) \end{aligned} \tag{A.21}$$

$$\begin{aligned} \int_\Sigma \frac{\partial \overline{w_1}}{\partial t}\, dz &= \int_\Sigma \frac{2\, a\, \dot{a}\, {r_c}^2\, \zeta}{\left({r_c}^2 + \zeta\, \overline{\zeta_c}\right)^2} - \frac{2\, {r_c}^3\, \dot{r_c}\, \zeta}{\left({r_c}^2 + \zeta\, \overline{\zeta_c}\right)^2} + \frac{2\, a\, \dot{a}\, \zeta^2\, \overline{\zeta_c}}{\left({r_c}^2 + \zeta\, \overline{\zeta_c}\right)^2} - \frac{4\, r_c\, \dot{r_c}\, \zeta^2\, \overline{\zeta_c}}{\left({r_c}^2 + \zeta\, \overline{\zeta_c}\right)^2} - \frac{2\, \dot{r_c}\, \zeta^3\, \overline{\zeta_c}^2}{r_c \left({r_c}^2 + \zeta\, \overline{\zeta_c}\right)^2} \\ &\quad - \frac{a^2\, \zeta^2\, \dot{\zeta}_x}{\left({r_c}^2 + \zeta\, \overline{\zeta_c}\right)^2} + \frac{a^2\, i\, \zeta^2\, \dot{\zeta}_y}{\left({r_c}^2 + \zeta\, \overline{\zeta_c}\right)^2}\, dz \\ &= \frac{4\, a^2\, i\, \pi\, \left(\dot{r_c}\, ({r_c}^2 - \delta^2)^3 + a\, \dot{a}\, \left(-{r_c}^5 + {r_c}^3\, \delta^2\right) + a^2\, {r_c}^3\, \zeta_c \left(-\dot{\zeta}_x + i\, \dot{\zeta}_y \right) \right)}{r_c\, ({r_c}^2 - \delta^2)^3} \end{aligned} \tag{A.22}$$

$$\begin{aligned} \int_\Sigma \frac{\partial w_2}{\partial t}\, dz &= 4\pi\, (\, a\, \dot{a} + \, r_c\, \dot{r_c}) \\ &= 2\pi \frac{d}{dt} \left(a^2 + {r_c}^2 \right) \end{aligned} \tag{A.23}$$

$$\int_\Sigma \frac{\partial \overline{w_2}}{\partial t}\, dz = \frac{4a^2\pi \left(a\dot{a}{r_c}^3 \left({r_c}^2 - \delta^2\right) + \dot{r_c}({r_c}^2 - \delta^2)^3 + a^2 {r_c}^3 \zeta_c \left(\dot{\zeta}_x - i\dot{\zeta}_y \right) \right)}{r_c({r_c}^2 - \delta^2)^3} \tag{A.24}$$

$$\int_\Sigma \frac{\partial w_3}{\partial t}\,dz \;=\; 2\pi\left[2r_c\dot{r_c}\zeta_c + 2a\dot{a}\overline{\zeta_c} + \frac{2a^4 r_c\dot{r_c}\zeta_c}{\left({r_c}^2-\delta^2\right)^2} - \frac{4a^3\dot{a}\zeta_c}{{r_c}^2-\delta^2} + {r_c}^2\dot{\zeta}_x \right.$$
$$\left. -\frac{a^4\delta^2\dot{\zeta}_x}{\left({r_c}^2-\delta^2\right)^2} - \frac{a^4\dot{\zeta}_x}{{r_c}^2-\delta^2} + i{r_c}^2\dot{\zeta}_y - \frac{a^4 i\delta^2\dot{\zeta}_y}{\left({r_c}^2-\delta^2\right)^2} - \frac{a^4 i\dot{\zeta}_y}{{r_c}^2-\delta^2}\right] \quad \text{(A.25)}$$

$$\int_\Sigma \frac{\partial \overline{w_3}}{\partial t}\,dz \;=\; \frac{2\,a^2\,\pi}{r_c\left({r_c}^2-\delta^2\right)^4}\left(-4\,a^3\,\dot{a}\,{r_c}^3\,\overline{\zeta_c}\,\left({r_c}^2-\delta^2\right) - 2\,a\,\dot{a}\,r_c\,\zeta_c\left({r_c}^2-\delta^2\right)^3\right.$$
$$+\left({r_c}^2-\delta^2\right)^4\left(2\,\dot{r_c}\,\overline{\zeta_c} + r_c\left(\dot{\zeta}_x - i\,\dot{\zeta}_y\right)\right)$$
$$-a^2\,\zeta_c\left({r_c}^2-\delta^2\right)^2\left(4\,{r_c}^2\,\dot{r_c} - 2\,\dot{r_c}\,\delta^2 + r_c\,\zeta_c\left(\dot{\zeta}_x - i\,\dot{\zeta}_y\right)\right)$$
$$\left.+a^4\,{r_c}^3\left(2\,r_c\,\dot{r_c}\,\overline{\zeta_c} - 2\,\delta^2\left(\dot{\zeta}_x - i\,\dot{\zeta}_y\right) + {r_c}^2\left(-\dot{\zeta}_x + i\,\dot{\zeta}_y\right)\right)\right) \quad \text{(A.26)}$$

$$\int_\Sigma \frac{\partial w_4}{\partial t}\,dz = \int_\Sigma i\left(\frac{r_c}{\zeta}\right)\left(\frac{-\zeta}{r_c^2}\right)\dot{r_c}\,dz = 0$$

$$\int_\Sigma \frac{\partial w_5^k}{\partial t}\,dz \;=\; i\int_\Sigma\left[-\frac{\zeta_k}{r_c^2}\left(\frac{-2r_c\dot{r_c}}{\zeta_k} + \frac{r_c^2\dot{\zeta}_k}{\zeta_k^2}\right) - \frac{1}{\zeta-\zeta_k}\dot{\zeta}_k\right.$$
$$\left. -\frac{1}{\zeta - r_c^2/\overline{\zeta_k}}\left(\frac{-2r_c\dot{r_c}}{\overline{\zeta_k}} + \frac{r_c^2\dot{\overline{\zeta_k}}}{\overline{\zeta_k}^2}\right)\right]dz$$
$$=\; 2\,\pi\left(\frac{r_c\left(-2\,\dot{r_c}\,\overline{\zeta_k} + r_c\,\dot{\overline{\zeta_k}}\right)}{\overline{\zeta_k}^2} + \frac{a^2\,\dot{\zeta}_k}{\left(\zeta_c+\zeta_k\right)^2}\right) \quad \text{(A.27)}$$

$$\int_\Sigma \frac{\partial w_1^s}{\partial t}\,dz \;=\; \dot{\zeta}_c\int_\Sigma \frac{\partial w_1^s}{\partial \zeta_c}\,dz + \dot{a}\int_\Sigma \frac{\partial w_1^s}{\partial a}\,dz + \dot{r_c}\int_\Sigma \frac{\partial w_1^s}{\partial r_c}\,dz \quad \text{(A.28)}$$

$$\int_\Sigma \frac{\partial w_1^s}{\partial \zeta_c}\,dz = \Bigg[4a^4\pi\left(\frac{-2\,a\,i\,{r_c}^4}{({r_c}^2-\delta^2)^4}-\frac{2\,i\,{r_c}^4\,\zeta_c}{({r_c}^2-\delta^2)^4}-\frac{2\,i\,{r_c}^4\,\overline{\zeta_c}}{({r_c}^2-\delta^2)^4}-\frac{4\,a\,i\,{r_c}^2\,\delta^2}{({r_c}^2-\delta^2)^4}\right.$$
$$\left.+\frac{2\,i\,{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}}{({r_c}^2-\delta^2)^4}-\frac{i\,{r_c}^2\,\zeta_c\,\overline{\zeta_c}^2}{({r_c}^2-\delta^2)^4}-\frac{6\,a\,{r_c}^2\,\overline{\zeta_c}\,\zeta_y}{({r_c}^2-\delta^2)^4}+\frac{3\,{r_c}^2\,\delta^2\,\zeta_y}{({r_c}^2-\delta^2)^4}\right)\Bigg] \tag{A.29}$$

$$\int_\Sigma \frac{\partial w_1^s}{\partial a}\,dz = \Bigg[4\pi\left(a\,i-\frac{8\,a^4\,i\,{r_c}^2\,\zeta_c}{({r_c}^2-\delta^2)^3}-\frac{4\,a^3\,i\,{r_c}^2\,\delta^2}{({r_c}^2-\delta^2)^3}-\frac{2\,a^3\,i\,{r_c}^2}{({r_c}^2-\delta^2)^2}\right.$$
$$\left.-\frac{2\,a^3\,i\,{\zeta_c}^2}{({r_c}^2-\delta^2)^2}-\frac{8\,a^4\,{r_c}^2\,\zeta_y}{({r_c}^2-\delta^2)^3}+\frac{4\,a^3\,{r_c}^2\,\zeta_c\,\zeta_y}{({r_c}^2-\delta^2)^3}\right)\Bigg] \tag{A.30}$$

$$\int_\Sigma \frac{\partial w_1^s}{\partial r_c}\,dz = \Bigg[4\pi\left(-(i\,r_c)+\frac{2\,a^2\,i\,r_c}{{\zeta_c}^2}-\frac{2\,a^2\,i\,{r_c}^9}{{\zeta_c}^2\,({r_c}^2-\delta^2)^4}+\frac{8\,a^5\,i\,{r_c}^3\,\zeta_c}{({r_c}^2-\delta^2)^4}\right.$$
$$+\frac{8\,a^2\,i\,{r_c}^7\,\overline{\zeta_c}}{\zeta_c\,({r_c}^2-\delta^2)^4}+\frac{4\,a^4\,i\,{r_c}^3\,\delta^2}{({r_c}^2-\delta^2)^4}+\frac{4\,a^5\,i\,r_c\,{\zeta_c}^2\,\overline{\zeta_c}}{({r_c}^2-\delta^2)^4}$$
$$-\frac{12\,a^2\,i\,{r_c}^5\,\overline{\zeta_c}^2}{({r_c}^2-\delta^2)^4}+\frac{2\,a^4\,i\,r_c\,{\zeta_c}^2\,\overline{\zeta_c}^2}{({r_c}^2-\delta^2)^4}+\frac{8\,a^2\,i\,{r_c}^3\,\zeta_c\,\overline{\zeta_c}^3}{({r_c}^2-\delta^2)^4}$$
$$-\frac{2\,a^2\,i\,r_c\,{\zeta_c}^2\,\overline{\zeta_c}^4}{({r_c}^2-\delta^2)^4}+\frac{a^4\,i\,{r_c}^3}{({r_c}^2-\delta^2)^3}+\frac{2\,a^4\,i\,r_c\,{\zeta_c}^2}{({r_c}^2-\delta^2)^3}$$
$$+\frac{a^4\,i\,r_c\,\delta^2}{({r_c}^2-\delta^2)^3}+\frac{8\,a^5\,{r_c}^3\,\zeta_y}{({r_c}^2-\delta^2)^4}-\frac{4\,a^4\,{r_c}^3\,\zeta_c\,\zeta_y}{({r_c}^2-\delta^2)^4}$$
$$\left.+\frac{4\,a^5\,r_c\,\delta^2\,\zeta_y}{({r_c}^2-\delta^2)^4}-\frac{2\,a^4\,r_c\,{\zeta_c}^2\,\overline{\zeta_c}\,\zeta_y}{({r_c}^2-\delta^2)^4}\right)\Bigg] \tag{A.31}$$

$$\int_\Sigma \overline{\frac{\partial w_1^s}{\partial \zeta_c}}\,dz = \frac{4a^4\pi}{(r_c^2-\delta^2)^5}\left[-2\,a\,{r_c}^6 + 3\,a^2\,i\,{r_c}^4\,\zeta_c + {r_c}^6\,\zeta_c - i\,{r_c}^6\,\zeta_c \right.$$
$$-i\,{r_c}^4\,{\zeta_c}^3 - a^2\,{r_c}^4\,\overline{\zeta_c} + 2\,a^2\,i\,{r_c}^4\,\overline{\zeta_c} - 2\,a\,{r_c}^4\,\delta^2 + 3\,a^2\,i\,{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}$$
$$+{r_c}^4\,{\zeta_c}^2\,\overline{\zeta_c} + 2\,i\,{r_c}^4\,{\zeta_c}^2\,\overline{\zeta_c}$$
$$+2\,i\,{r_c}^2\,{\zeta_c}^4\,\overline{\zeta_c} - 2\,a^2\,{r_c}^2\,\zeta_c\,\overline{\zeta_c}^2 + a^2\,i\,{r_c}^2\,\zeta_c\,\overline{\zeta_c}^2 + 4\,a\,{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}^2$$
$$-2\,{r_c}^2\,{\zeta_c}^3\,\overline{\zeta_c}^2 - i\,{r_c}^2\,{\zeta_c}^3\,\overline{\zeta_c}^2 - i\,{\zeta_c}^5\,\overline{\zeta_c}^2 - 6\,a\,{r_c}^4\,\zeta_c\,\zeta_y$$
$$\left.+3\,{r_c}^4\,{\zeta_c}^2\,\zeta_y + 6\,a\,{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}\,\zeta_y - 3\,{r_c}^2\,{\zeta_c}^3\,\overline{\zeta_c}\,\zeta_y\right] \tag{A.32}$$

$$\int_\Sigma \overline{\frac{\partial w_1^s}{\partial a}}\,dz = \frac{-4a^3\pi}{(r_c^2-\delta^2)^4}\left[-2\,a^2\,{r_c}^2\,\left(-2\,\overline{\zeta_c}^2 + i\,\left({r_c}^2 + 2\,\delta^2 + \overline{\zeta_c}^2\right)\right) + 8\,a\,{r_c}^2\,\left({r_c}^2-\delta^2\right)\,\left(\overline{\zeta_c}+\zeta_y\right)\right.$$
$$\left.+\left({r_c}^2-\delta^2\right)\,\left(i\,\left({r_c}^2+{\zeta_c}^2\right)\,\left({r_c}^2-\delta^2\right) - 4\,{r_c}^2\,\zeta_c\,\left(\overline{\zeta_c}+\zeta_y\right)\right)\right] \tag{A.33}$$

$$\int_\Sigma \overline{\frac{\partial w_1^s}{\partial r_c}}\,dz = \frac{-4a^2\pi}{r_c(r_c^2-\delta^2)^5}\left[a^4\,i\,{r_c}^6 - i\,{r_c}^{10} + a^2\,i\,{r_c}^6\,{\zeta_c}^2 - 8\,a^3\,{r_c}^6\,\overline{\zeta_c} + 5\,a^4\,i\,{r_c}^4\,\delta^2\right.$$
$$+4\,a^2\,{r_c}^6\,\delta^2 + 5\,i\,{r_c}^8\,\delta^2 - 3\,a^2\,i\,{r_c}^4\,{\zeta_c}^3\,\overline{\zeta_c} - 4\,a^4\,{r_c}^4\,\overline{\zeta_c}^2 + 2\,a^4\,i\,{r_c}^4\,\overline{\zeta_c}^2$$
$$+4\,a^3\,{r_c}^4\,\zeta_c\,\overline{\zeta_c}^2 + 2\,a^4\,i\,{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}^2 - 2\,a^2\,{r_c}^4\,{\zeta_c}^2\,\overline{\zeta_c}^2 - 10\,i\,{r_c}^6\,{\zeta_c}^2\,\overline{\zeta_c}^2 + 3\,a^2\,i\,{r_c}^2\,{\zeta_c}^4\,\overline{\zeta_c}^2$$
$$-2\,a^4\,{r_c}^2\,\zeta_c\,\overline{\zeta_c}^3 + 4\,a^3\,{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}^3 - 2\,a^2\,{r_c}^2\,{\zeta_c}^3\,\overline{\zeta_c}^3$$
$$+10\,i\,{r_c}^4\,{\zeta_c}^3\,\overline{\zeta_c}^3 - a^2\,i\,{\zeta_c}^5\,\overline{\zeta_c}^3 - 5\,i\,{r_c}^2\,{\zeta_c}^4\,\overline{\zeta_c}^4 + i\,{\zeta_c}^5\,\overline{\zeta_c}^5 - 8\,a^3\,{r_c}^6\,\zeta_y + 4\,a^2\,{r_c}^6\,\zeta_c\,\zeta_y$$
$$\left.+4\,a^3\,{r_c}^4\,\delta^2\,\zeta_y - 2\,a^2\,{r_c}^4\,{\zeta_c}^2\,\overline{\zeta_c}\,\zeta_y + 4\,a^3\,{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}^2\,\zeta_y - 2\,a^2\,{r_c}^2\,{\zeta_c}^3\,\overline{\zeta_c}^2\,\zeta_y\right] \tag{A.34}$$

$$\int_\Sigma \frac{\partial w_2^s}{\partial \zeta_c}\,dz = \frac{4a^4\pi r_c^2}{(r_c^2-\delta^2)^4}\left[-2\,{r_c}^2\,\left(\zeta_c-\overline{\zeta_c}\right) + \delta^2\,\left(2\,\zeta_c+\overline{\zeta_c}-3\,i\,\zeta_y\right) - 2\,a\,\left({r_c}^2+2\,\delta^2-3\,i\,\overline{\zeta_c}\,\zeta_y\right)\right] \tag{A.35}$$

$$\int_\Sigma \frac{\partial w_2^s}{\partial a}\,dz \;=\; 4\,a\,\pi - \frac{32\,a^4\,\pi\,{r_c}^2\,\zeta_c}{\left({r_c}^2-\delta^2\right)^3} + \frac{16\,a^3\,\pi\,{r_c}^2\,\delta^2}{\left({r_c}^2-\delta^2\right)^3}$$
$$+\frac{8\,a^3\,\pi\,{r_c}^2}{\left({r_c}^2-\delta^2\right)^2} - \frac{8\,a^3\,\pi\,{\zeta_c}^2}{\left({r_c}^2-\delta^2\right)^2} + \frac{16\,a^3\,i\,\pi\,{r_c}^2\,\left(2\,a-\zeta_c\right)\,\zeta_y}{\left({r_c}^2-\delta^2\right)^3} \quad \text{(A.36)}$$

$$\int_\Sigma \frac{\partial w_2^s}{\partial r_c}\,dz \;=\; 4\pi r_c\left[1 + \frac{12\,a^5\,{r_c}^2\,\zeta_c}{\left({r_c}^2-\delta^2\right)^4} - \frac{6\,a^4\,{r_c}^2\,\delta^2}{\left({r_c}^2-\delta^2\right)^4} - \frac{2\,a^4\,{r_c}^2}{\left({r_c}^2-\delta^2\right)^3} - \frac{4\,a^5\,\zeta_c}{\left({r_c}^2-\delta^2\right)^3} + \frac{2\,a^4\,{\zeta_c}^2}{\left({r_c}^2-\delta^2\right)^3}\right.$$
$$+\frac{2\,a^4\,\delta^2}{\left({r_c}^2-\delta^2\right)^3} + \frac{a^4}{\left({r_c}^2-\delta^2\right)^2} - \frac{12\,a^5\,i\,{r_c}^2\,\zeta_y}{\left({r_c}^2-\delta^2\right)^4} + \frac{6\,a^4\,i\,{r_c}^2\,\zeta_c\,\zeta_y}{\left({r_c}^2-\delta^2\right)^4}$$
$$\left.+\frac{4\,a^5\,i\,\zeta_y}{\left({r_c}^2-\delta^2\right)^3} - \frac{2\,a^4\,i\,\zeta_c\,\zeta_y}{\left({r_c}^2-\delta^2\right)^3}\right] \quad \text{(A.37)}$$

$$\int_\Sigma \overline{\frac{\partial w_2^s}{\partial \zeta_c}}\,dz \;=\; \frac{-4a^4\pi}{(r_c^2-\delta^2)^5}\left[a^2\,{r_c}^2\,\left(\delta^2\,\left(-3\,\zeta_c+\overline{\zeta_c}+2\,i\,\overline{\zeta_c}\right)+{r_c}^2\,\left(-3\,\zeta_c+(2+i)\,\overline{\zeta_c}\right)\right)\right.$$
$$+2\,a\,i\,{r_c}^2\,\left({r_c}^2-\delta^2\right)\,\left({r_c}^2+2\,\delta^2+3\,\zeta_c\,\zeta_y\right)$$
$$\left.-\zeta_c\,\left({r_c}^2-\delta^2\right)\,\left((1+i)\,{r_c}^4+{\zeta_c}^3\,\overline{\zeta_c}-{r_c}^2\,\zeta_c\,\left(\zeta_c+\overline{\zeta_c}-2\,i\,\overline{\zeta_c}-3\,i\,\zeta_y\right)\right)\right] \quad \text{(A.38)}$$

$$\int_\Sigma \overline{\frac{\partial w_2^s}{\partial a}}\,dz \;=\; \frac{4a^3\pi}{(r_c^2-\delta^2)^4}\left[2\,a^2\,{r_c}^2\,\left({r_c}^2-\overline{\zeta_c}\,\left(-2\,\zeta_c+\overline{\zeta_c}+2\,i\,\overline{\zeta_c}\right)\right)\right.$$
$$-8\,a\,i\,{r_c}^2\,\left({r_c}^2-\delta^2\right)\,\left(\overline{\zeta_c}+\zeta_y\right)$$
$$\left.+\left({r_c}^2-\delta^2\right)\,\left({r_c}^4+{\zeta_c}^3\,\overline{\zeta_c}-{r_c}^2\,\zeta_c\,\left(\zeta_c+\overline{\zeta_c}-4\,i\,\overline{\zeta_c}-4\,i\,\zeta_y\right)\right)\right] \quad \text{(A.39)}$$

$$
\begin{aligned}
\int_{\Sigma} \overline{\frac{\partial w_2^s}{\partial r_c}}\, dz \quad = \quad & 4a^2\pi \left[\frac{1}{r_c} - \frac{a^2}{r_c\,\overline{\zeta_c}^2} - \frac{a^4\, {r_c}^5}{\left({r_c}^2 - \delta^2\right)^5} + \frac{12\, a^2\, {r_c}^5\, {\zeta_c}^2}{\left({r_c}^2 - \delta^2\right)^5} \right. \\
& + \frac{2\, a^2\, {r_c}^9}{\overline{\zeta_c}^2 \left({r_c}^2 - \delta^2\right)^5} - \frac{8\, a^2\, {r_c}^7\, \zeta_c}{\overline{\zeta_c} \left({r_c}^2 - \delta^2\right)^5} - \frac{5\, a^4\, {r_c}^3\, \delta^2}{\left({r_c}^2 - \delta^2\right)^5} \\
& - \frac{8\, a^2\, {r_c}^3\, {\zeta_c}^3\, \overline{\zeta_c}}{\left({r_c}^2 - \delta^2\right)^5} + \frac{4\, a^4\, i\, {r_c}^3\, \overline{\zeta_c}^2}{\left({r_c}^2 - \delta^2\right)^5} - \frac{2\, a^4\, r_c\, {\zeta_c}^2\, \overline{\zeta_c}^2}{\left({r_c}^2 - \delta^2\right)^5} \\
& + \frac{2\, a^2\, r_c\, {\zeta_c}^4\, \overline{\zeta_c}^2}{\left({r_c}^2 - \delta^2\right)^5} + \frac{2\, a^4\, i\, r_c\, \zeta_c\, \overline{\zeta_c}^3}{\left({r_c}^2 - \delta^2\right)^5} + \frac{8\, a^3\, i\, {r_c}^3\, \overline{\zeta_c}}{\left({r_c}^2 - \delta^2\right)^4} \\
& - \frac{4\, a^2\, i\, {r_c}^3\, \delta^2}{\left({r_c}^2 - \delta^2\right)^4} + \frac{4\, a^3\, i\, r_c\, \zeta_c\, \overline{\zeta_c}^2}{\left({r_c}^2 - \delta^2\right)^4} \\
& - \frac{2\, a^2\, i\, r_c\, {\zeta_c}^2\, \overline{\zeta_c}^2}{\left({r_c}^2 - \delta^2\right)^4} - \frac{a^2\, {r_c}^3}{\overline{\zeta_c}^2 \left({r_c}^2 - \delta^2\right)^2} \\
& - \frac{2\, a^4\, {r_c}^3\, \overline{\zeta_c}^2}{\left({r_c}^2 - \delta^2\right)^2 \left(-{r_c}^2 + \delta^2\right)^3} + \frac{8\, a^3\, i\, {r_c}^3\, \zeta_y}{\left({r_c}^2 - \delta^2\right)^4} \\
& - \frac{4\, a^2\, i\, {r_c}^3\, \zeta_c\, \zeta_y}{\left({r_c}^2 - \delta^2\right)^4} + \frac{4\, a^3\, i\, r_c\, \delta^2\, \zeta_y}{\left({r_c}^2 - \delta^2\right)^4} \\
& \left. - \frac{2\, a^2\, i\, r_c\, {\zeta_c}^2\, \overline{\zeta_c}\, \zeta_y}{\left({r_c}^2 - \delta^2\right)^4} \right]
\end{aligned} \tag{A.40}
$$

$$
\int_{\Sigma} \frac{\partial w_3^s}{\partial \zeta_c}\, dz = \frac{-4\, a^3\, i\, \pi\, {r_c}^2\, \left(3\, {r_c}^2 + 4\, a\, \overline{\zeta_c} - \zeta_c\, \overline{\zeta_c}\right)}{\left({r_c}^2 - \zeta_c\, \overline{\zeta_c}\right)^3} \tag{A.41}
$$

$$
\int_{\Sigma} \frac{\partial w_3^s}{\partial a}\, dz = \frac{-12\, a^2\, i\, \pi\, \left(2\, a\, {r_c}^2 + \zeta_c\, \left(2\, {r_c}^2 - \zeta_c\, \overline{\zeta_c}\right)\right)}{\left({r_c}^2 - \zeta_c\, \overline{\zeta_c}\right)^2} \tag{A.42}
$$

$$
\int_{\Sigma} \frac{\partial w_3^s}{\partial r_c}\, dz = \frac{16\, a^3\, i\, \pi\, r_c\, \left({r_c}^2\, \zeta_c + a\, \left({r_c}^2 + \zeta_c\, \overline{\zeta_c}\right)\right)}{\left({r_c}^2 - \zeta_c\, \overline{\zeta_c}\right)^3} \tag{A.43}
$$

$$
\begin{aligned}
\int_{\Sigma} \overline{\frac{\partial w_3^s}{\partial \zeta_c}}\, dz \quad = \quad & \frac{4a^3\pi}{(r_c^2 - \delta^2)^4} \left[-4\, a\, {r_c}^2\, \overline{\zeta_c}\, \left({r_c}^2 - \delta^2\right) + \overline{\zeta_c}^2\, \left({r_c}^2 - \delta^2\right)\, \left((2+i)\, {r_c}^2 - i\, \delta^2\right) \right. \\
& \left. - a^2\, {r_c}^2\, \left(3\, i\, {r_c}^2 + 2\, \delta^2 + 4\, i\, \delta^2\right) \right]
\end{aligned} \tag{A.44}
$$

$$\int_{\Sigma} \overline{\frac{\partial w_3^s}{\partial a}}\, dz = \frac{4a^2\pi}{(r_c^2-\delta^2)^3}\left[-6\,a\,{r_c}^4 + 3\,{r_c}^4\,\zeta_c - i\,{r_c}^4\,\zeta_c - 3\,a^2\,{r_c}^2\,\overline{\zeta_c} + 6\,a^2\,i\,{r_c}^2\,\overline{\zeta_c}\right.$$
$$\left. + 6\,a\,{r_c}^2\,\delta^2 - 3\,{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c} + 2\,i\,{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c} - i\,{\zeta_c}^3\,\overline{\zeta_c}^2\right] \quad \text{(A.45)}$$

$$\int_{\Sigma} \overline{\frac{\partial w_3^s}{\partial r_c}}\, dz = \frac{-8a^3\pi}{r_c(r_c^2-\delta^2)^4}\left[a^2\,{r_c}^2\,\overline{\zeta_c}\,\left((-1+2\,i)\,{r_c}^2 + (-1+i)\,\delta^2\right)\right.$$
$$+\zeta_c\,\left({r_c}^2-\delta^2\right)\left((1+i)\,{r_c}^4 + (1-2\,i)\,{r_c}^2\,\delta^2 + i\,{\zeta_c}^2\,\overline{\zeta_c}^2\right)$$
$$\left. -2\,a\,\left({r_c}^6 - {r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}^2\right)\right] \quad \text{(A.46)}$$

Appendix B Integrals for Bernoulli Effect Forces

These integrals arise in the force on a deformable Joukowski foil due to the Bernoulli effect.

$$\int_{C_+} \frac{dw_1}{d\zeta}\overline{\frac{dw_1}{d\zeta}} \left(\frac{(\zeta+\zeta_c)^2}{(\zeta+\zeta_c)^2-a^2}\right)\left(\frac{-r_c^2}{\zeta^2}\right) d\zeta = \frac{-4\,a^2\,i\,\pi\,\overline{\zeta_c}\,({r_c}^2-\delta^2)\left(({r_c}^2-\delta^2)^3+a^2\left(-{r_c}^4+\zeta_c\,\overline{\zeta_c}^3\right)\right)}{{r_c}^2\left({r_c}^2+(a-\zeta_c)\,\overline{\zeta_c}\right)^2\left({r_c}^2-(a+\zeta_c)\,\overline{\zeta_c}\right)^2} \tag{B.1}$$

$$\int_{C_+} \frac{dw_1}{d\zeta}\overline{\frac{dw_2}{d\zeta}} \left(\frac{(\zeta+\zeta_c)^2}{(\zeta+\zeta_c)^2-a^2}\right)\left(\frac{-r_c^2}{\zeta^2}\right) d\zeta = \frac{4\,a^2\,\pi\,\overline{\zeta_c}\,({r_c}^2-\delta^2)\left(({r_c}^2-\delta^2)^3+a^2\left(-{r_c}^4+\zeta_c\,\overline{\zeta_c}^3\right)\right)}{{r_c}^2\left({r_c}^2+(a-\zeta_c)\,\overline{\zeta_c}\right)^2\left({r_c}^2-(a+\zeta_c)\,\overline{\zeta_c}\right)^2} \tag{B.2}$$

$$\int_{C_+} \frac{dw_2}{d\zeta}\overline{\frac{dw_1}{d\zeta}} \left(\frac{(\zeta+\zeta_c)^2}{(\zeta+\zeta_c)^2-a^2}\right)\left(\frac{-r_c^2}{\zeta^2}\right) d\zeta = \frac{4\,a^2\,\pi\,\overline{\zeta_c}\,({r_c}^2-\delta^2)\left(({r_c}^2-\delta^2)^3+a^2\left({r_c}^4+\zeta_c\,\overline{\zeta_c}^3\right)\right)}{{r_c}^2\left({r_c}^2+(a-\zeta_c)\,\overline{\zeta_c}\right)^2\left({r_c}^2-(a+\zeta_c)\,\overline{\zeta_c}\right)^2} \tag{B.3}$$

$$\int_{C_+} \frac{dw_2}{d\zeta}\overline{\frac{dw_2}{d\zeta}} \left(\frac{(\zeta+\zeta_c)^2}{(\zeta+\zeta_c)^2-a^2}\right)\left(\frac{-r_c^2}{\zeta^2}\right) d\zeta = \frac{4\,a^2\,i\,\pi\,\overline{\zeta_c}\,({r_c}^2-\delta^2)\,\left(({r_c}^2-\delta^2)^3+a^2\,\left({r_c}^4+\zeta_c\,\overline{\zeta_c}^3\right)\right)}{{r_c}^2\left({r_c}^2+(a-\zeta_c)\,\overline{\zeta_c}\right)^2\left({r_c}^2-(a+\zeta_c)\,\overline{\zeta_c}\right)^2} \tag{B.4}$$

$$\int_{C_+} \frac{dw_1}{d\zeta}\overline{\frac{dw_3}{d\zeta}} \left(\frac{(\zeta+\zeta_c)^2}{(\zeta+\zeta_c)^2-a^2}\right)\left(\frac{-r_c^2}{\zeta^2}\right) d\zeta = \frac{4\,a^2\,\pi\,\left({r_c}^4-2\,{r_c}^2\,\delta^2+\left(a^2+{\zeta_c}^2\right)\overline{\zeta_c}^2\right)\left(-({r_c}^2-\delta^2)^3+a^2\,\left({r_c}^4-\zeta_c\,\overline{\zeta_c}^3\right)\right)}{{r_c}^2\left({r_c}^2+(a-\zeta_c)\,\overline{\zeta_c}\right)^2\left({r_c}^2-(a+\zeta_c)\,\overline{\zeta_c}\right)^2} \tag{B.5}$$

$$\int_{C_+} \frac{dw_3}{d\zeta}\overline{\frac{dw_1}{d\zeta}} \left(\frac{(\zeta+\zeta_c)^2}{(\zeta+\zeta_c)^2-a^2}\right)\left(\frac{-r_c^2}{\zeta^2}\right) d\zeta = \frac{-4\,a^2\,\pi\,\overline{\zeta_c}\,\left(a^2\,{r_c}^2\,\overline{\zeta_c}\,(2\,{r_c}^2-3\,\delta^2)\,({r_c}^2-\delta^2)-\zeta_c\,({r_c}^2-\delta^2)^4+a^4\,\left({r_c}^4\,\zeta_c-{r_c}^2\,\overline{\zeta_c}^3+\zeta_c\,\overline{\zeta_c}^4\right)\right)}{{r_c}^2\left({r_c}^2+(a-\zeta_c)\,\overline{\zeta_c}\right)^2\left({r_c}^2-(a+\zeta_c)\,\overline{\zeta_c}\right)^2} \tag{B.6}$$

$$\int_{C_+} \frac{dw_2}{d\zeta}\overline{\frac{dw_3}{d\zeta}} \left(\frac{(\zeta+\zeta_c)^2}{(\zeta+\zeta_c)^2-a^2}\right)\left(\frac{-r_c^2}{\zeta^2}\right) d\zeta = \frac{-4\,a^2\,i\,\pi\,\left({r_c}^4-2\,{r_c}^2\,\delta^2+\left(a^2+{\zeta_c}^2\right)\overline{\zeta_c}^2\right)\left(({r_c}^2-\delta^2)^3+a^2\,\left({r_c}^4+\zeta_c\,\overline{\zeta_c}^3\right)\right)}{{r_c}^2\left({r_c}^2+(a-\zeta_c)\,\overline{\zeta_c}\right)^2\left({r_c}^2-(a+\zeta_c)\,\overline{\zeta_c}\right)^2} \tag{B.7}$$

$$\int_{C_+} \frac{dw_3}{d\zeta}\overline{\frac{dw_2}{d\zeta}}\left(\frac{(\zeta+\zeta_c)^2}{(\zeta+\zeta_c)^2-a^2}\right)\left(\frac{-r_c^2}{\zeta^2}\right)d\zeta = \frac{-4\,a^2\,i\,\pi\,\overline{\zeta_c}\left(a^2\,{r_c}^2\,\overline{\zeta_c}\,\left(2\,{r_c}^2-3\,\delta^2\right)\,\left({r_c}^2-\delta^2\right)-\zeta_c\left({r_c}^2-\delta^2\right)^4+a^4\,\left({r_c}^4\,\zeta_c-{r_c}^2\,\overline{\zeta_c}^3+\zeta_c\,\overline{\zeta_c}^4\right)\right)}{{r_c}^2\left({r_c}^2+(a-\zeta_c)\,\overline{\zeta_c}\right)^2\left({r_c}^2-(a+\zeta_c)\,\overline{\zeta_c}\right)^2} \tag{B.8}$$

$$\int_{C_+} \frac{dw_3}{d\zeta}\overline{\frac{dw_3}{d\zeta}}\left(\frac{(\zeta+\zeta_c)^2}{(\zeta+\zeta_c)^2-a^2}\right)\left(\frac{-r_c^2}{\zeta^2}\right)d\zeta = \frac{4\,a^2\,i\,\pi\,\left({r_c}^4-2\,{r_c}^2\,\delta^2+\left(a^2+{\zeta_c}^2\right)\,\overline{\zeta_c}^2\right)\times\left(a^2\,{r_c}^2\,\overline{\zeta_c}\,\left(2\,{r_c}^2-3\,\delta^2\right)\,\left({r_c}^2-\delta^2\right)-\zeta_c\left({r_c}^2-\delta^2\right)^4+a^4\,\left({r_c}^4\,\zeta_c-{r_c}^2\,\overline{\zeta_c}^3+\zeta_c\,\overline{\zeta_c}^4\right)\right)}{{r_c}^2\left({r_c}^2+(a-\zeta_c)\,\overline{\zeta_c}\right)^2\left({r_c}^2-\delta^2\right)\left({r_c}^2-(a+\zeta_c)\,\overline{\zeta_c}\right)^2} \tag{B.9}$$

$$\int_{C_+} \frac{dw_1}{d\zeta}\overline{\frac{dw_4}{d\zeta}}\left(\frac{(\zeta+\zeta_c)^2}{(\zeta+\zeta_c)^2-a^2}\right)\left(\frac{-r_c^2}{\zeta^2}\right)d\zeta = 0 \tag{B.10}$$

$$\int_{C_+} \frac{dw_4}{d\zeta}\overline{\frac{dw_1}{d\zeta}}\left(\frac{(\zeta+\zeta_c)^2}{(\zeta+\zeta_c)^2-a^2}\right)\left(\frac{-r_c^2}{\zeta^2}\right)d\zeta = \frac{-\pi}{r_c^2}\left(\frac{2\,{r_c}^2\,{\zeta_c}^2-2\,a^2\,\left({r_c}^2+{\zeta_c}^2\right)}{(a-\zeta_c)\,(a+\zeta_c)} + \frac{a^3\,{r_c}^4}{(a-\zeta_c)\,\left({r_c}^2+(a-\zeta_c)\,\overline{\zeta_c}\right)^2}+\frac{a^3\,{r_c}^4}{(a+\zeta_c)\,\left({r_c}^2-(a+\zeta_c)\,\overline{\zeta_c}\right)^2}\right) \tag{B.11}$$

$$\int_{C_+} \frac{dw_2}{d\zeta}\overline{\frac{dw_4}{d\zeta}}\left(\frac{(\zeta+\zeta_c)^2}{(\zeta+\zeta_c)^2-a^2}\right)\left(\frac{-r_c^2}{\zeta^2}\right)d\zeta = 0 \tag{B.12}$$

$$\int_{C_+} \frac{dw_4}{d\zeta}\overline{\frac{dw_2}{d\zeta}}\left(\frac{(\zeta+\zeta_c)^2}{(\zeta+\zeta_c)^2-a^2}\right)\left(\frac{-r_c^2}{\zeta^2}\right)d\zeta = \frac{-2\,i\,\pi\,\left({r_c}^2\,({r_c}^2-\delta^2)^4+a^4\,\left({r_c}^4\,\overline{\zeta_c}^2+(r_c-\zeta_c)\,(r_c+\zeta_c)\,\overline{\zeta_c}^4\right)\right)}{{r_c}^2\,\left({r_c}^2+(a-\zeta_c)\,\overline{\zeta_c}\right)^2\left({r_c}^2-(a+\zeta_c)\,\overline{\zeta_c}\right)^2} - 2i\pi\frac{\left(a^2\,({r_c}^2-\delta^2)^2\,\left({r_c}^4+{\zeta_c}^2\,\overline{\zeta_c}^2-2\,{r_c}^2\,\overline{\zeta_c}\,\left(\zeta_c+\overline{\zeta_c}\right)\right)\right)}{{r_c}^2\,\left({r_c}^2+(a-\zeta_c)\,\overline{\zeta_c}\right)^2\left({r_c}^2-(a+\zeta_c)\,\overline{\zeta_c}\right)^2} \quad \text{(B.13)}$$

$$\int_{C_+} \frac{dw_3}{d\zeta}\overline{\frac{dw_4}{d\zeta}}\left(\frac{(\zeta+\zeta_c)^2}{(\zeta+\zeta_c)^2-a^2}\right)\left(\frac{-r_c^2}{\zeta^2}\right)d\zeta = 0 \quad \text{(B.14)}$$

$$\int_{C_+} \frac{dw_4}{d\zeta}\overline{\frac{dw_3}{d\zeta}}\left(\frac{(\zeta+\zeta_c)^2}{(\zeta+\zeta_c)^2-a^2}\right)\left(\frac{-r_c^2}{\zeta^2}\right)d\zeta = -2\,i\,\pi\,\left(-\left(\frac{a^2\,\zeta_c\,\left(a^2+{\zeta_c}^2\right)}{{r_c}^2\,\left(a^2-{\zeta_c}^2\right)}\right) + \overline{\zeta_c} + \frac{a^4\,{r_c}^2}{(a-\zeta_c)\,\left({r_c}^2+(a-\zeta_c)\,\overline{\zeta_c}\right)^2} - \frac{a^4\,{r_c}^2}{(a+\zeta_c)\,\left({r_c}^2-(a+\zeta_c)\,\overline{\zeta_c}\right)^2}\right) \quad \text{(B.15)}$$

$$\int_{C_+} \frac{dw_4}{d\zeta}\overline{\frac{dw_4}{d\zeta}}\left(\frac{(\zeta+\zeta_c)^2}{(\zeta+\zeta_c)^2-a^2}\right)\left(\frac{-r_c^2}{\zeta^2}\right)d\zeta = 0 \quad \text{(B.16)}$$

$$\int_{C_+} \frac{dw_1}{d\zeta}\overline{\frac{dw_5^k}{d\zeta}}\left(\frac{(\zeta+\zeta_c)^2}{(\zeta+\zeta_c)^2-a^2}\right)\left(\frac{-r_c^2}{\zeta^2}\right)d\zeta = \frac{-2\,\pi\,\left(-\left(a^2\,\overline{\zeta_k}^2\right)+{r_c}^2\,\left(\zeta_c+\overline{\zeta_k}\right)^2\right)}{\left(a-\zeta_c-\overline{\zeta_k}\right)\,\overline{\zeta_k}^2\,\left(a+\zeta_c+\overline{\zeta_k}\right)} \quad \text{(B.17)}$$

$$\int_{C_+} \frac{dw_5^k}{d\zeta}\overline{\frac{dw_1}{d\zeta}}\left(\frac{(\zeta+\zeta_c)^2}{(\zeta+\zeta_c)^2-a^2}\right)\left(\frac{-r_c^2}{\zeta^2}\right)\,d\zeta =$$

$$2\pi \left({r_c}^2 \left({r_c}^2-\delta^2\right)^4 (\zeta_c+\zeta_k)^2 \left(\overline{\zeta_c}+\overline{\zeta_k}\right)^2 - a^6\,\overline{\zeta_c}^4 \left({r_c}^2+\zeta_c\,\overline{\zeta_k}\right)^2 \right.$$
$$-a^2 \left({r_c}^2-\delta^2\right)^2 (\zeta_c+\zeta_k)^2 \left({r_c}^4\,\overline{\zeta_k}^2 + {\zeta_c}^2\,\overline{\zeta_c}^2\,\overline{\zeta_k}^2 + 2\,{r_c}^2\,\overline{\zeta_c}\left(\overline{\zeta_c}^3 + 2\,\overline{\zeta_c}^2\,\overline{\zeta_k} - \zeta_c\,\overline{\zeta_k}^2 + \overline{\zeta_c}\,\overline{\zeta_k}^2\right)\right)$$
$$+a^4\,\overline{\zeta_c}\left({\zeta_c}^2\,\overline{\zeta_c}^3\left(2\,{\zeta_c}^2+2\,\zeta_c\,\zeta_k+{\zeta_k}^2\right)\overline{\zeta_k}^2 - {r_c}^8\left(\overline{\zeta_c}+2\,\overline{\zeta_k}\right)\right.$$
$$+2\,{r_c}^6\left(\zeta_k\left(\overline{\zeta_c}+\overline{\zeta_k}\right)^2 + 2\,\zeta_c\,\overline{\zeta_c}\left(\overline{\zeta_c}+2\,\overline{\zeta_k}\right)\right)$$
$$-{r_c}^4\,\overline{\zeta_c}\left({\zeta_k}^2\,\overline{\zeta_k}\left(2\,\overline{\zeta_c}+\overline{\zeta_k}\right) + {\zeta_c}^2\left(3\,\overline{\zeta_c}^2 + 8\,\overline{\zeta_c}\,\overline{\zeta_k} - 2\,\overline{\zeta_k}^2\right) + 2\,\zeta_c\,\zeta_k\left(\overline{\zeta_c}^2 + 4\,\overline{\zeta_c}\,\overline{\zeta_k} + 2\,\overline{\zeta_k}^2\right)\right)$$
$$+{r_c}^2\,\overline{\zeta_c}^2\left(2\,{\zeta_c}^3\left(\overline{\zeta_c}-2\,\overline{\zeta_k}\right)\overline{\zeta_k} + \overline{\zeta_c}\,{\zeta_k}^2\left(\overline{\zeta_c}+\overline{\zeta_k}\right)^2 + 2\,\zeta_c\,\overline{\zeta_c}\,\zeta_k\left(\overline{\zeta_c}^2 + 2\,\overline{\zeta_c}\,\overline{\zeta_k} + \overline{\zeta_k}\left(\zeta_k+\overline{\zeta_k}\right)\right)\right.$$
$$\left.\left.\left.+{\zeta_c}^2\,\overline{\zeta_c}\left(\overline{\zeta_c}^2 + 2\,\overline{\zeta_c}\,\overline{\zeta_k} + \overline{\zeta_k}\left(4\,\zeta_k+\overline{\zeta_k}\right)\right)\right)\right)\right)$$
$$/\left({r_c}^2\left({r_c}^2+(a-\zeta_c)\,\overline{\zeta_c}\right)^2\left({r_c}^2-(a+\zeta_c)\,\overline{\zeta_c}\right)^2 (a-\zeta_c-\zeta_k)\,(a+\zeta_c+\zeta_k)\left(\overline{\zeta_c}+\overline{\zeta_k}\right)^2\right) \tag{B.18}$$

$$\int_{C_+} \frac{dw_2}{d\zeta}\overline{\frac{dw_5^k}{d\zeta}}\left(\frac{(\zeta+\zeta_c)^2}{(\zeta+\zeta_c)^2-a^2}\right)\left(\frac{-r_c^2}{\zeta^2}\right)\,d\zeta \quad = \quad \frac{-2\,i\,\pi\left(a^2\,\overline{\zeta_k}^2 + {r_c}^2\left(\zeta_c+\overline{\zeta_k}\right)^2\right)}{\left(a-\zeta_c-\overline{\zeta_k}\right)\overline{\zeta_k}^2\left(a+\zeta_c+\overline{\zeta_k}\right)} \tag{B.19}$$

$$\int_{C_+} \frac{dw_5^k}{d\zeta} \overline{\frac{dw_2}{d\zeta}} \left(\frac{(\zeta+\zeta_c)^2}{(\zeta+\zeta_c)^2 - a^2} \right) \left(\frac{-r_c^2}{\zeta^2} \right) d\zeta =$$

$$\left(\frac{-2\,i\,\pi}{{r_c}^2 \left({r_c}^2 + (a-\zeta_c)\,\overline{\zeta_c}\right)^2 \left({r_c}^2 - (a+\zeta_c)\,\overline{\zeta_c}\right)^2 (a-\zeta_c-\zeta_k)\,(a+\zeta_c+\zeta_k)\,\left(\overline{\zeta_c}+\overline{\zeta_k}\right)^2} \right)$$

$$\times \Big\{ {r_c}^2 \left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)^4 (\zeta_c+\zeta_k)^2 \left(\overline{\zeta_c}+\overline{\zeta_k}\right)^2 + a^6\,\overline{\zeta_c}^4 \left({r_c}^2 + \zeta_c\,\overline{\zeta_k}\right)^2$$

$$+a^2 \left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)^2 (\zeta_c+\zeta_k)^2 \left({r_c}^4\,\overline{\zeta_k}^2 + {\zeta_c}^2\,\overline{\zeta_c}^2\,\overline{\zeta_k}^2 - 2\,{r_c}^2 \left(\overline{\zeta_c}^4 + 2\,\overline{\zeta_c}^3\,\overline{\zeta_k} + \overline{\zeta_c}\,\left(\zeta_c+\overline{\zeta_c}\right)\,\overline{\zeta_k}^2\right)\right)$$

$$+a^4\,\overline{\zeta_c} \left(-\left({\zeta_c}^2\,\overline{\zeta_c}^3 \left(2\,{\zeta_c}^2 + 2\,\zeta_c\,\zeta_k + {\zeta_k}^2\right)\overline{\zeta_k}^2\right) + {r_c}^8 \left(\overline{\zeta_c} + 2\,\overline{\zeta_k}\right)\right.$$

$$+{r_c}^4\,\overline{\zeta_c} \left(\zeta_c\,\overline{\zeta_c}^2 \left(3\,\zeta_c + 2\,\zeta_k\right) + 2\,\overline{\zeta_c}\,(2\,\zeta_c+\zeta_k)^2\,\overline{\zeta_k} + \left(-2\,{\zeta_c}^2 + 4\,\zeta_c\,\zeta_k + {\zeta_k}^2\right)\overline{\zeta_k}^2\right)$$

$$+{r_c}^2\,\overline{\zeta_c}^2 \left(\overline{\zeta_c}^3\,(\zeta_c+\zeta_k)^2 + 2\,\overline{\zeta_c}\left(-\zeta_c+\overline{\zeta_c}\right)(\zeta_c+\zeta_k)^2\,\overline{\zeta_k} + \left({\zeta_c}^2 \left(4\,\zeta_c+\overline{\zeta_c}\right) + 2\,\zeta_c\,\overline{\zeta_c}\,\zeta_k + \overline{\zeta_c}\,{\zeta_k}^2\right)\overline{\zeta_k}^2\right)$$

$$\left.-2\,{r_c}^6 \left(\zeta_k \left(\overline{\zeta_c}+\overline{\zeta_k}\right)^2 + 2\,\zeta_c\,\overline{\zeta_c}\,\left(\overline{\zeta_c} + 2\,\overline{\zeta_k}\right)\right)\right)\Big\} \quad \text{(B.20)}$$

$$\int_{C_+} \frac{dw_3}{d\zeta} \overline{\frac{dw_5^k}{d\zeta}} \left(\frac{(\zeta+\zeta_c)^2}{(\zeta+\zeta_c)^2 - a^2} \right) \left(\frac{-r_c^2}{\zeta^2} \right) d\zeta =$$

$$\frac{-2\,i\,\pi \left(a^4\,\zeta_c\,\overline{\zeta_k}^2 - {r_c}^2\,\zeta_c \left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)\left(\zeta_c+\overline{\zeta_k}\right)^2 - a^2 \left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)\left(\overline{\zeta_c}\,\overline{\zeta_k}^2 + {r_c}^2 \left(\zeta_c + 2\,\overline{\zeta_k}\right)\right)\right)}{\left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)\overline{\zeta_k}^2 \left(-a+\zeta_c+\overline{\zeta_k}\right)\left(a+\zeta_c+\overline{\zeta_k}\right)} \quad \text{(B.21)}$$

$$\int_{C_+} \frac{dw_5^k}{d\zeta}\overline{\frac{dw_3}{d\zeta}}\left(\frac{(\zeta+\zeta_c)^2}{(\zeta+\zeta_c)^2-a^2}\right)\left(\frac{-r_c^2}{\zeta^2}\right)d\zeta =$$

$$\frac{2\,i\,\pi}{{r_c}^2\left({r_c}^2+(a-\zeta_c)\,\overline{\zeta_c}\right)^2\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)\left({r_c}^2-(a+\zeta_c)\,\overline{\zeta_c}\right)^2(-a+\zeta_c+\zeta_k)\,(a+\zeta_c+\zeta_k)\,\left(\overline{\zeta_c}+\overline{\zeta_k}\right)^2}$$

$$\times\Big\{{r_c}^2\,\overline{\zeta_c}\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^5(\zeta_c+\zeta_k)^2\left(\overline{\zeta_c}+\overline{\zeta_k}\right)^2-a^8\,\overline{\zeta_c}^5\left({r_c}^2+\zeta_c\,\overline{\zeta_k}\right)^2$$

$$-a^6\,\overline{\zeta_c}^2\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)\Big(\zeta_c\,\overline{\zeta_c}^2(\zeta_c+\zeta_k)^2\,\overline{\zeta_k}^2+2\,{r_c}^6\left(\overline{\zeta_c}+\overline{\zeta_k}\right)$$

$$-{r_c}^2\,\overline{\zeta_c}\left(\overline{\zeta_c}^2(\zeta_c+\zeta_k)^2+2\,{\zeta_c}^2\,\overline{\zeta_c}\,\overline{\zeta_k}+2\,{\zeta_c}^2\,\overline{\zeta_k}^2\right)-2\,{r_c}^4\left(\overline{\zeta_c}+\overline{\zeta_k}\right)\left(2\,\zeta_c\,\overline{\zeta_c}+\zeta_k\left(\overline{\zeta_c}+\overline{\zeta_k}\right)\right)\Big)$$

$$+a^2\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^3(\zeta_c+\zeta_k)^2\Big({\zeta_c}^3\,\overline{\zeta_c}^2\,\overline{\zeta_k}^2+{r_c}^6\left(\overline{\zeta_c}+2\,\overline{\zeta_k}\right)+{r_c}^4\,\zeta_c\left(-2\,\overline{\zeta_c}^2-4\,\overline{\zeta_c}\,\overline{\zeta_k}+\overline{\zeta_k}^2\right)$$

$$+{r_c}^2\,\overline{\zeta_c}\left(-2\,\overline{\zeta_c}^2\left(\overline{\zeta_c}+\overline{\zeta_k}\right)^2+{\zeta_c}^2\left(\overline{\zeta_c}^2+2\,\overline{\zeta_c}\,\overline{\zeta_k}-2\,\overline{\zeta_k}^2\right)\right)\Big)$$

$$-a^4\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)\Big({\zeta_c}^5\,\overline{\zeta_c}^4\,\overline{\zeta_k}^2+{r_c}^{10}\left(\overline{\zeta_c}+2\,\overline{\zeta_k}\right)$$

$$-{r_c}^2\,\overline{\zeta_c}^3\Big(\overline{\zeta_c}^2\left(-2\,{\zeta_c}^2+\overline{\zeta_c}^2\right)(\zeta_c+\zeta_k)^2+2\,\overline{\zeta_c}\left(-3\,{\zeta_c}^2+\overline{\zeta_c}^2\right)(\zeta_c+\zeta_k)^2\,\overline{\zeta_k}$$

$$+\left({\zeta_c}^4-8\,{\zeta_c}^3\,\zeta_k+2\,\zeta_c\,\overline{\zeta_c}^2\,\zeta_k+\overline{\zeta_c}^2\,{\zeta_k}^2+{\zeta_c}^2\left(\overline{\zeta_c}^2-4\,{\zeta_k}^2\right)\right)\overline{\zeta_k}^2\Big)$$

$$-{r_c}^8\left(2\,\zeta_k\left(\overline{\zeta_c}+\overline{\zeta_k}\right)^2+5\,\zeta_c\,\overline{\zeta_c}\left(\overline{\zeta_c}+2\,\overline{\zeta_k}\right)\right)-{r_c}^4\,\zeta_c\,\overline{\zeta_c}^2\Big(4\,{\zeta_k}^2\left(\overline{\zeta_c}+\overline{\zeta_k}\right)\left(\overline{\zeta_c}+2\,\overline{\zeta_k}\right)$$

$$+2\,\zeta_c\,\zeta_k\left(\overline{\zeta_c}+\overline{\zeta_k}\right)\left(5\,\overline{\zeta_c}+9\,\overline{\zeta_k}\right)+{\zeta_c}^2\left(7\,\overline{\zeta_c}^2+18\,\overline{\zeta_c}\,\overline{\zeta_k}+\overline{\zeta_k}^2\right)\Big)$$

$$+{r_c}^6\,\overline{\zeta_c}\Big(2\,{\zeta_k}^2\left(\overline{\zeta_c}+\overline{\zeta_k}\right)\left(\overline{\zeta_c}+2\,\overline{\zeta_k}\right)+4\,\zeta_c\,\zeta_k\left(\overline{\zeta_c}+\overline{\zeta_k}\right)\left(2\,\overline{\zeta_c}+3\,\overline{\zeta_k}\right)$$

$$+{\zeta_c}^2\left(9\,\overline{\zeta_c}^2+20\,\overline{\zeta_c}\,\overline{\zeta_k}+\overline{\zeta_k}^2\right)\Big)\Big)\Big\} \quad \text{(B.22)}$$

$$\int_{C_+} \frac{dw_4}{d\zeta}\overline{\frac{dw_5^k}{d\zeta}}\left(\frac{(\zeta+\zeta_c)^2}{(\zeta+\zeta_c)^2-a^2}\right)\left(\frac{-r_c^2}{\zeta^2}\right)d\zeta = \frac{2\,i\,\pi\left(\zeta_c+\overline{\zeta_k}\right)^2}{\overline{\zeta_k}\left(-a+\zeta_c+\overline{\zeta_k}\right)\left(a+\zeta_c+\overline{\zeta_k}\right)} \quad \text{(B.23)}$$

$$\int_{C_+} \frac{dw_5^k}{d\zeta}\overline{\frac{dw_4}{d\zeta}}\left(\frac{(\zeta+\zeta_c)^2}{(\zeta+\zeta_c)^2-a^2}\right)\left(\frac{-r_c^2}{\zeta^2}\right)d\zeta = \frac{2\,i\,\pi\,(\zeta_c+\zeta_k)^2}{\zeta_k\,(-a+\zeta_c+\zeta_k)\,(a+\zeta_c+\zeta_k)} \quad \text{(B.24)}$$

$$\int_{C_+} \frac{dw_5^k}{d\zeta} \overline{\frac{dw_5^j}{d\zeta}} \left(\frac{(\zeta+\zeta_c)^2}{(\zeta+\zeta_c)^2 - a^2} \right) \left(\frac{-r_c^2}{\zeta^2} \right) d\zeta =$$

$$\frac{2\,i\,\pi}{\left(a-\zeta_c-\overline{\zeta_j}\right)\,\left(a+\zeta_c+\overline{\zeta_j}\right)\,(a-\zeta_c-\zeta_k)\,(a+\zeta_c+\zeta_k)\,(-{r_c}^2+\zeta_j\,\zeta_k)\,\left(-{r_c}^2+\overline{\zeta_j}\,\overline{\zeta_k}\right)}$$

$$\times\ \Big\{\left(\zeta_c+\overline{\zeta_j}\right)^2(\zeta_c+\zeta_k)^2\ \left(-\left(\zeta_j\ \left(\overline{\zeta_j}+\zeta_k\right)\ \overline{\zeta_k}\right)+{r_c}^2\ \left(\zeta_j+\overline{\zeta_k}\right)\right)$$

$$+a^2\ \left(-\left({r_c}^4\ \left(2\,\zeta_c+\overline{\zeta_j}+\zeta_k\right)\right)+\zeta_c\,\zeta_j\ \left(2\,\overline{\zeta_j}\,\zeta_k+\zeta_c\ \left(\overline{\zeta_j}+\zeta_k\right)\right)\ \overline{\zeta_k}-{r_c}^2\ \left({\zeta_c}^2-\overline{\zeta_j}\,\zeta_k\right)\ \left(\zeta_j+\overline{\zeta_k}\right)\right)\Big\} \tag{B.25}$$

$$\int_{C_+} \frac{dw_5^k}{d\zeta} \overline{\frac{dw_5^k}{d\zeta}} \left(\frac{(\zeta+\zeta_c)^2}{(\zeta+\zeta_c)^2 - a^2} \right) \left(\frac{-r_c^2}{\zeta^2} \right) d\zeta = 2\,i\,\pi\left(r_c^2-\zeta_k\,\overline{\zeta_k}\right)\times$$

$$\frac{\left(-\left((\zeta_c+\zeta_k)^2\left(\zeta_c+\overline{\zeta_k}\right)^2\ \left(\zeta_k+\overline{\zeta_k}\right)\right)+a^2\ \left({r_c}^2\ \left(2\,\zeta_c+\zeta_k+\overline{\zeta_k}\right)+\zeta_c\ \left(2\,\zeta_k\,\overline{\zeta_k}+\zeta_c\ \left(\zeta_k+\overline{\zeta_k}\right)\right)\right)\right)}{(a-\zeta_c-\zeta_k)\ (a+\zeta_c+\zeta_k)\ \left(-{r_c}^2+{\zeta_k}^2\right)\ \left(a-\zeta_c-\overline{\zeta_k}\right)\ \left(a+\zeta_c+\overline{\zeta_k}\right)\ \left({r_c}^2-\overline{\zeta_k}^2\right)} \tag{B.26}$$

$$\int_{C_+} \frac{dw_1}{d\zeta} \overline{\frac{dw_1^s}{d\zeta}} \left(\frac{(\zeta+\zeta_c)^2}{(\zeta+\zeta_c)^2 - a^2} \right) \left(\frac{-r_c^2}{\zeta^2} \right) d\zeta =$$

$$\frac{4\,a^2\,i\,\pi\ \left(a^2\,{r_c}^4\ \left(\zeta_c+\overline{\zeta_c}\right)+\overline{\zeta_c}\left(-{r_c}^2+\delta^2\right)^3\right)}{{r_c}^2\left({r_c}^2-\delta^2\right)^3} \tag{B.27}$$

$$\int_{C_+} \frac{dw_1^s}{d\zeta}\overline{\frac{dw_1}{d\zeta}}\left(\frac{(\zeta+\zeta_c)^2}{(\zeta+\zeta_c)^2-a^2}\right)\left(\frac{-r_c^2}{\zeta^2}\right)d\zeta = \frac{4\,a^2\,i\,\pi}{{r_c}^2\left({r_c}^2+(a-\zeta_c)\,\overline{\zeta_c}\right)^2\left({r_c}^2-\delta^2\right)^3\left({r_c}^2-(a+\zeta_c)\,\overline{\zeta_c}\right)^2}$$
$$\times\left(\overline{\zeta_c}\left(-{r_c}^2+\delta^2\right)^7+a^2\left({r_c}^2-\delta^2\right)^4\left(2\,{r_c}^2\,\overline{\zeta_c}^3-2\,\zeta_c\,\overline{\zeta_c}^4+{r_c}^4\left(\zeta_c+\overline{\zeta_c}-i\,\zeta_y\right)\right)\right.$$
$$-a^6\,{r_c}^2\,\overline{\zeta_c}^2\left({r_c}^4\left(\zeta_c-i\,\zeta_y\right)+{r_c}^2\,\overline{\zeta_c}^2\left(-\zeta_c+i\,\zeta_y\right)+{\zeta_c}^2\,\overline{\zeta_c}^2\left(\overline{\zeta_c}+i\,\zeta_y\right)\right)$$
$$-a^4\left({r_c}^2-\delta^2\right)^2\left(-\left(\zeta_c\,\overline{\zeta_c}^6\right)+{r_c}^6\left(\zeta_c+\overline{\zeta_c}-i\,\zeta_y\right)+{r_c}^4\,\overline{\zeta_c}\left(\zeta_c\left(\zeta_c+4\,\overline{\zeta_c}\right)+2\,i\left(\zeta_c-\overline{\zeta_c}\right)\zeta_y\right)\right.$$
$$\left.\left.+{r_c}^2\,\overline{\zeta_c}^2\left(\overline{\zeta_c}^3-{\zeta_c}^2\left(\overline{\zeta_c}+i\,\zeta_y\right)\right)\right)\right) \quad \text{(B.28)}$$

$$\int_{C_+} \frac{dw_2}{d\zeta}\overline{\frac{dw_1^s}{d\zeta}}\left(\frac{(\zeta+\zeta_c)^2}{(\zeta+\zeta_c)^2-a^2}\right)\left(\frac{-r_c^2}{\zeta^2}\right)d\zeta = \frac{4\,a^2\,\pi\left(a^2\,{r_c}^4\left(\zeta_c+\overline{\zeta_c}\right)+\overline{\zeta_c}\left({r_c}^2-\delta^2\right)^3\right)}{{r_c}^2\left({r_c}^2-\delta^2\right)^3} \quad \text{(B.29)}$$

$$\int_{C_+} \frac{dw_1^s}{d\zeta}\overline{\frac{dw_2}{d\zeta}}\left(\frac{(\zeta+\zeta_c)^2}{(\zeta+\zeta_c)^2-a^2}\right)\left(\frac{-r_c^2}{\zeta^2}\right)d\zeta = \frac{-4\,a^2\,\pi}{{r_c}^2\left({r_c}^2+(a-\zeta_c)\,\overline{\zeta_c}\right)^2\left({r_c}^2-\delta^2\right)^3\left({r_c}^2-(a+\zeta_c)\,\overline{\zeta_c}\right)^2}\times\left(\overline{\zeta_c}\left(-{r_c}^2+\delta^2\right)^7\right.$$
$$+a^6\,{r_c}^2\,\overline{\zeta_c}^2\left(-\left(\zeta_c\left({r_c}^4+{r_c}^2\,\overline{\zeta_c}^2+\zeta_c\,\overline{\zeta_c}^3\right)\right)+i\left({r_c}^4+(r_c-\zeta_c)\,(r_c+\zeta_c)\,\overline{\zeta_c}^2\right)\zeta_y\right)$$
$$+a^2\left({r_c}^2-\delta^2\right)^4\left(2\,{r_c}^2\,\overline{\zeta_c}^3-2\,\zeta_c\,\overline{\zeta_c}^4+{r_c}^4\left(-\zeta_c+\overline{\zeta_c}+i\,\zeta_y\right)\right)$$
$$+a^4\left({r_c}^2-\delta^2\right)^2\left(\zeta_c\,\overline{\zeta_c}^6-{r_c}^6\left(\zeta_c+\overline{\zeta_c}-i\,\zeta_y\right)+{r_c}^4\,\overline{\zeta_c}\left(-{\zeta_c}^2-2\,i\left(\zeta_c+\overline{\zeta_c}\right)\zeta_y\right)\right.$$
$$\left.\left.+{r_c}^2\,\overline{\zeta_c}^2\left(-\overline{\zeta_c}^3+{\zeta_c}^2\left(\overline{\zeta_c}+i\,\zeta_y\right)\right)\right)\right) \quad \text{(B.30)}$$

$$\int_{C_+} \frac{dw_3}{d\zeta}\overline{\frac{dw_1^s}{d\zeta}} \left(\frac{(\zeta+\zeta_c)^2}{(\zeta+\zeta_c)^2-a^2}\right)\left(\frac{-r_c^2}{\zeta^2}\right) d\zeta =$$

$$\frac{-4\,a^2\,\pi\,\left(a^4\,{r_c}^4\,\zeta_c\,\left(\zeta_c+\overline{\zeta_c}\right)-\zeta_c\,\overline{\zeta_c}\,({r_c}^2-\delta^2)^4+a^2\,({r_c}^2-\delta^2)^2\,\left({r_c}^4+2\,{r_c}^2\,\overline{\zeta_c}^2-\zeta_c\,\overline{\zeta_c}^3\right)\right)}{{r_c}^2\,({r_c}^2-\delta^2)^4} \tag{B.31}$$

$$\int_{C_+} \frac{dw_1^s}{d\zeta}\overline{\frac{dw_3}{d\zeta}} \left(\frac{(\zeta+\zeta_c)^2}{(\zeta+\zeta_c)^2-a^2}\right)\left(\frac{-r_c^2}{\zeta^2}\right) d\zeta =$$

$$\frac{-4\,a^2\,\pi}{{r_c}^2\left({r_c}^2+(a-\zeta_c)\,\overline{\zeta_c}\right)^2({r_c}^2-\delta^2)^4\left({r_c}^2-(a+\zeta_c)\,\overline{\zeta_c}\right)^2}$$

$$\begin{aligned}
\times\,\Big(&\left({r_c}^2-\delta^2\right)^9-a^2\left({r_c}^2-\delta^2\right)^5\left({r_c}^6-2\,{r_c}^2\,\zeta_c\,\overline{\zeta_c}^3+{\zeta_c}^2\,\overline{\zeta_c}^4+{r_c}^4\,\overline{\zeta_c}\,\left(\overline{\zeta_c}-i\,\zeta_y\right)\right)\\
&+a^8\,{r_c}^2\,\zeta_c\,\overline{\zeta_c}^4\,\left({r_c}^2\,\left(\zeta_c-i\,\zeta_y\right)+\zeta_c\,\overline{\zeta_c}\,\left(\overline{\zeta_c}+i\,\zeta_y\right)\right)\\
&+a^4\left({r_c}^2-\delta^2\right)^3\Big({r_c}^8-{\zeta_c}^2\,\overline{\zeta_c}^6-{r_c}^4\,\overline{\zeta_c}\,\Big(2\,{\zeta_c}^3+3\,{\zeta_c}^2\,\overline{\zeta_c}-3\,\zeta_c\,\overline{\zeta_c}^2+\overline{\zeta_c}^3+2\,i\,\left({\zeta_c}^2+\overline{\zeta_c}^2\right)\zeta_y\Big)\\
&+{r_c}^6\left(2\,{\zeta_c}^2-\overline{\zeta_c}^2+\zeta_c\,\left(\overline{\zeta_c}+i\,\zeta_y\right)\right)+{r_c}^2\,\zeta_c\,\overline{\zeta_c}^2\left(2\,\overline{\zeta_c}^3+{\zeta_c}^2\,\left(\overline{\zeta_c}+i\,\zeta_y\right)\right)\Big)\\
&+a^6\,\overline{\zeta_c}\,\left({r_c}^2-\delta^2\right)\Big(-2\,{r_c}^2\,\zeta_c\,\overline{\zeta_c}^6+{\zeta_c}^2\,\overline{\zeta_c}^7+{r_c}^8\,\left(4\,\zeta_c+\overline{\zeta_c}-4\,i\,\zeta_y\right)+{r_c}^6\,\zeta_c\,\overline{\zeta_c}\,\left(-5\,\zeta_c+\overline{\zeta_c}+8\,i\,\zeta_y\right)\\
&+{r_c}^4\,\overline{\zeta_c}^2\left({\zeta_c}^3-\zeta_c\,\overline{\zeta_c}^2+\overline{\zeta_c}^2\,\left(\overline{\zeta_c}+i\,\zeta_y\right)-2\,{\zeta_c}^2\,\left(\overline{\zeta_c}+2\,i\,\zeta_y\right)\right)\Big)\Big)
\end{aligned} \tag{B.32}$$

$$\int_{C_+} \frac{dw_4}{d\zeta}\overline{\frac{dw_1^s}{d\zeta}} \left(\frac{(\zeta+\zeta_c)^2}{(\zeta+\zeta_c)^2-a^2}\right)\left(\frac{-r_c^2}{\zeta^2}\right) d\zeta = \frac{-2\,\pi\,\left(a-r_c\right)\,\left(a+r_c\right)}{{r_c}^2} \tag{B.33}$$

$$\int_{C_+} \frac{dw_1^s}{d\zeta}\overline{\frac{dw_4}{d\zeta}} \left(\frac{(\zeta+\zeta_c)^2}{(\zeta+\zeta_c)^2-a^2}\right)\left(\frac{-r_c^2}{\zeta^2}\right) d\zeta = 0 \tag{B.34}$$

$$\int_{C_+} \frac{dw_5^k}{d\zeta} \overline{\frac{dw_1^s}{d\zeta}} \left(\frac{(\zeta + \zeta_c)^2}{(\zeta+\zeta_c)^2 - a^2} \right) \left(\frac{-r_c^2}{\zeta^2} \right) d\zeta =$$

$$\Big\{ {r_c}^2 \left({r_c}^2 - \zeta_c \overline{\zeta_c}\right)^3 \zeta_k \left(\zeta_c + \zeta_k\right)^2 \left(\overline{\zeta_c} + \overline{\zeta_k}\right)^2 - a^2 \left({r_c}^2 - \zeta_c \overline{\zeta_c}\right)^3 \zeta_k \left(\zeta_c + \zeta_k\right)^2 \left({r_c}^2 + \overline{\zeta_k}^2\right)$$

$$-a^4 \left(-\left({r_c}^8 \zeta_k\right) + {\zeta_c}^3 \overline{\zeta_c}^3 \zeta_k \overline{\zeta_k}^2 + {r_c}^6 \overline{\zeta_c} \zeta_k \left(3 \zeta_c + \overline{\zeta_c} + 2 \overline{\zeta_k}\right) \right.$$

$$- {r_c}^2 {\zeta_c}^2 \overline{\zeta_c} \zeta_k \overline{\zeta_k} \left(2 \zeta_c \overline{\zeta_c} + \zeta_c \overline{\zeta_k} + 3 \overline{\zeta_c} \overline{\zeta_k}\right)$$

$$+ {r_c}^4 \left(\zeta_c \left(\overline{\zeta_c}^2 \left(\overline{\zeta_c} - 2 \zeta_k\right) \left(2 \zeta_c + \zeta_k\right) + 2 \overline{\zeta_c} \left(2 \zeta_c \overline{\zeta_c} - \zeta_c \zeta_k + \overline{\zeta_c} \zeta_k - 2 {\zeta_k}^2\right) \overline{\zeta_k} \right.\right.$$

$$\left.\left.\left. + \left(2 \overline{\zeta_c} - \zeta_k\right) \left(\zeta_c + 2 \zeta_k\right) \overline{\zeta_k}^2 \right) + 2 i \left(\zeta_c + \zeta_k\right)^2 \left(\overline{\zeta_c} + \overline{\zeta_k}\right)^2 \zeta_y \right) \right) \Big\}$$

$$\times \frac{-2\pi}{{r_c}^2 \left({r_c}^2 - \zeta_c \overline{\zeta_c}\right)^3 \zeta_k \left(-a + \zeta_c + \zeta_k\right) \left(a + \zeta_c + \zeta_k\right) \left(\overline{\zeta_c} + \overline{\zeta_k}\right)^2} \quad \text{(B.35)}$$

$$\int_{C_+} \frac{dw_1^s}{d\zeta} \overline{\frac{dw_5^k}{d\zeta}} \left(\frac{(\zeta + \zeta_c)^2}{(\zeta+\zeta_c)^2 - a^2} \right) \left(\frac{-r_c^2}{\zeta^2} \right) d\zeta =$$

$$\frac{2\pi}{\left({r_c}^2 - \zeta_c \overline{\zeta_c}\right)^3 \overline{\zeta_k}^2 \left(-a + \zeta_c + \overline{\zeta_k}\right) \left(a + \zeta_c + \overline{\zeta_k}\right)}$$

$$\times \Big\{ {r_c}^2 \left({r_c}^2 - \zeta_c \overline{\zeta_c}\right)^3 \left(\zeta_c + \overline{\zeta_k}\right)^2 - a^2 \left({r_c}^2 - \zeta_c \overline{\zeta_c}\right)^3 \left({r_c}^2 + \overline{\zeta_k}^2\right)$$

$$+ a^4 \overline{\zeta_k} \left({r_c}^4 \overline{\zeta_k} - {\zeta_c}^3 \overline{\zeta_c} \overline{\zeta_k} + {r_c}^2 \left(\zeta_c \left(2 \zeta_c \overline{\zeta_c} - \zeta_c \overline{\zeta_k} + \overline{\zeta_c} \overline{\zeta_k} - 2 \overline{\zeta_k}^2 \right) + 2 i \left(\zeta_c + \overline{\zeta_k}\right)^2 \zeta_y \right) \right) \Big\} \quad \text{(B.36)}$$

$$\int_{C_+} \frac{dw_1}{d\zeta} \overline{\frac{dw_2^s}{d\zeta}} \left(\frac{(\zeta + \zeta_c)^2}{(\zeta+\zeta_c)^2 - a^2} \right) \left(\frac{-r_c^2}{\zeta^2} \right) d\zeta =$$

$$\frac{4 a^2 \pi \left(a^2 {r_c}^4 \left(\zeta_c - \overline{\zeta_c}\right) + \overline{\zeta_c} \left({r_c}^2 - \delta^2\right)^3 \right)}{{r_c}^2 \left({r_c}^2 - \delta^2\right)^3} \quad \text{(B.37)}$$

$$\int_{C_+} \frac{dw_2^s}{d\zeta}\overline{\frac{dw_1}{d\zeta}}\left(\frac{(\zeta+\zeta_c)^2}{(\zeta+\zeta_c)^2-a^2}\right)\left(\frac{-r_c^2}{\zeta^2}\right)d\zeta =$$
$$\frac{4\,a^2\,\pi}{{r_c}^2\left({r_c}^2+(a-\zeta_c)\,\overline{\zeta_c}\right)^2\left({r_c}^2-\delta^2\right)^3\left({r_c}^2-(a+\zeta_c)\,\overline{\zeta_c}\right)^2}\times\Big\{\overline{\zeta_c}\left({r_c}^2-\delta^2\right)^7$$
$$+a^2\left({r_c}^2-\delta^2\right)^4\left(-2\,{r_c}^2\,\overline{\zeta_c}^3+2\,\zeta_c\,\overline{\zeta_c}^4+{r_c}^4\,\left(\zeta_c+\overline{\zeta_c}-i\,\zeta_y\right)\right)$$
$$+a^6\,{r_c}^2\,\overline{\zeta_c}^2\left({r_c}^2\,\overline{\zeta_c}^2\,\left(\zeta_c-i\,\zeta_y\right)+{\zeta_c}^2\,\overline{\zeta_c}^2\,\left(\overline{\zeta_c}-i\,\zeta_y\right)+{r_c}^4\,\left(-\zeta_c+i\,\zeta_y\right)\right)$$
$$+a^4\left({r_c}^2-\delta^2\right)^2\left(-\left(\zeta_c\,\overline{\zeta_c}^6\right)+{r_c}^6\,\left(-\zeta_c+\overline{\zeta_c}+i\,\zeta_y\right)\right.$$
$$\left.-{r_c}^4\,\overline{\zeta_c}\,\left({\zeta_c}^2+2\,i\,\left(\zeta_c-\overline{\zeta_c}\right)\,\zeta_y\right)+{r_c}^2\,\overline{\zeta_c}^2\,\left(\overline{\zeta_c}^3+{\zeta_c}^2\,\left(-\overline{\zeta_c}+i\,\zeta_y\right)\right)\right)\Big\} \quad \text{(B.38)}$$

$$\int_{C_+} \frac{dw_2}{d\zeta}\overline{\frac{dw_2^s}{d\zeta}}\left(\frac{(\zeta+\zeta_c)^2}{(\zeta+\zeta_c)^2-a^2}\right)\left(\frac{-r_c^2}{\zeta^2}\right)d\zeta =$$
$$\frac{-4\,a^2\,i\,\pi\,\left(a^2\,{r_c}^4\,\left(\zeta_c-\overline{\zeta_c}\right)+\overline{\zeta_c}\left(-{r_c}^2+\delta^2\right)^3\right)}{{r_c}^2\left({r_c}^2-\delta^2\right)^3} \quad \text{(B.39)}$$

$$\int_{C_+} \frac{dw_2^s}{d\zeta}\overline{\frac{dw_2}{d\zeta}}\left(\frac{(\zeta+\zeta_c)^2}{(\zeta+\zeta_c)^2-a^2}\right)\left(\frac{-r_c^2}{\zeta^2}\right)d\zeta =$$
$$\frac{4\,a^2\,i\,\pi}{{r_c}^2\left({r_c}^2+(a-\zeta_c)\,\overline{\zeta_c}\right)^2\left({r_c}^2-\delta^2\right)^3\left({r_c}^2-(a+\zeta_c)\,\overline{\zeta_c}\right)^2}\times\Big(\overline{\zeta_c}\left({r_c}^2-\delta^2\right)^7$$
$$+a^6\,{r_c}^2\,\overline{\zeta_c}^2\left({\zeta_c}^2\,\overline{\zeta_c}^2\,\left(\overline{\zeta_c}-i\,\zeta_y\right)+{r_c}^4\,\left(-\zeta_c+i\,\zeta_y\right)+{r_c}^2\,\overline{\zeta_c}^2\,\left(-\zeta_c+i\,\zeta_y\right)\right)$$
$$+a^2\left({r_c}^2-\delta^2\right)^4\left(-2\,{r_c}^2\,\overline{\zeta_c}^3+2\,\zeta_c\,\overline{\zeta_c}^4+{r_c}^4\,\left(-\zeta_c+\overline{\zeta_c}+i\,\zeta_y\right)\right)$$
$$+a^4\left({r_c}^2-\delta^2\right)^2\left(-\left(\zeta_c\,\overline{\zeta_c}^6\right)+{r_c}^6\,\left(-\zeta_c+\overline{\zeta_c}+i\,\zeta_y\right)+{r_c}^4\,\overline{\zeta_c}\,\left(-\left(\zeta_c\,\left(\zeta_c-4\,\overline{\zeta_c}\right)\right)\right.\right.$$
$$\left.\left.-2\,i\,\left(\zeta_c+\overline{\zeta_c}\right)\,\zeta_y\right)+{r_c}^2\,\overline{\zeta_c}^2\,\left(\overline{\zeta_c}^3+{\zeta_c}^2\,\left(-\overline{\zeta_c}+i\,\zeta_y\right)\right)\right)\Big) \quad \text{(B.40)}$$

$$\int_{C_+} \frac{dw_3}{d\zeta}\overline{\frac{dw_2^s}{d\zeta}} \left(\frac{(\zeta+\zeta_c)^2}{(\zeta+\zeta_c)^2-a^2}\right)\left(\frac{-r_c^2}{\zeta^2}\right) d\zeta =$$
$$\frac{4\,a^2\,i\,\pi\,\left(a^4\,{r_c}^4\,\zeta_c\,\left(\zeta_c-\overline{\zeta_c}\right)+\zeta_c\,\overline{\zeta_c}\,({r_c}^2-\delta^2)^4+a^2\,({r_c}^2-\delta^2)^2\,\left({r_c}^4-2\,{r_c}^2\,\overline{\zeta_c}^2+\zeta_c\,\overline{\zeta_c}^3\right)\right)}{{r_c}^2\,({r_c}^2-\delta^2)^4} \tag{B.41}$$

$$\int_{C_+} \frac{dw_2^s}{d\zeta}\overline{\frac{dw_3}{d\zeta}} \left(\frac{(\zeta+\zeta_c)^2}{(\zeta+\zeta_c)^2-a^2}\right)\left(\frac{-r_c^2}{\zeta^2}\right) d\zeta =$$
$$\begin{aligned} &\frac{-4\,a^2\,i\,\pi}{{r_c}^2\,\left({r_c}^2+(a-\zeta_c)\,\overline{\zeta_c}\right)^2\,({r_c}^2-\delta^2)^4\,\left({r_c}^2-(a+\zeta_c)\,\overline{\zeta_c}\right)^2}\times\Big\{\left({r_c}^2-\delta^2\right)^9 \\ &+a^8\,{r_c}^2\,\zeta_c\,\overline{\zeta_c}^4\,\left(\zeta_c\,\overline{\zeta_c}\,\left(\overline{\zeta_c}-i\,\zeta_y\right)+{r_c}^2\,\left(-\zeta_c+i\,\zeta_y\right)\right) \\ &+a^2\,\left({r_c}^2-\delta^2\right)^5\,\left({r_c}^6+2\,{r_c}^2\,\zeta_c\,\overline{\zeta_c}^3-{\zeta_c}^2\,\overline{\zeta_c}^4-{r_c}^4\,\overline{\zeta_c}\,\left(\overline{\zeta_c}+i\,\zeta_y\right)\right) \\ &+a^4\,\left({r_c}^2-\delta^2\right)^3\,\Big({r_c}^8-{\zeta_c}^2\,\overline{\zeta_c}^6-{r_c}^4\,\overline{\zeta_c}\,\Big(-2\,{\zeta_c}^3+3\,{\zeta_c}^2\,\overline{\zeta_c}+3\,\zeta_c\,\overline{\zeta_c}^2+\overline{\zeta_c}^3-2\,i\,\left({\zeta_c}^2+\overline{\zeta_c}^2\right)\,\zeta_y \\ &+{r_c}^6\,\left(-2\,{\zeta_c}^2+\overline{\zeta_c}^2+\zeta_c\,\left(\overline{\zeta_c}-i\,\zeta_y\right)\right)+{r_c}^2\,\zeta_c\,\overline{\zeta_c}^2\,\left(2\,\overline{\zeta_c}^3+{\zeta_c}^2\,\left(\overline{\zeta_c}-i\,\zeta_y\right)\right)\Big)\Big) \\ &-a^6\,\overline{\zeta_c}\,\left({r_c}^2-\delta^2\right)\,\Big(2\,{r_c}^2\,\zeta_c\,\overline{\zeta_c}^6-{\zeta_c}^2\,\overline{\zeta_c}^7-{r_c}^6\,\zeta_c\,\overline{\zeta_c}\,\left(5\,\zeta_c+\overline{\zeta_c}-8\,i\,\zeta_y\right) \\ &+{r_c}^8\,\left(4\,\zeta_c-\overline{\zeta_c}-4\,i\,\zeta_y\right)+{r_c}^4\,\overline{\zeta_c}^2\,\left({\zeta_c}^3-\zeta_c\,\overline{\zeta_c}^2+2\,{\zeta_c}^2\,\left(\overline{\zeta_c}-2\,i\,\zeta_y\right)+\overline{\zeta_c}^2\,\left(-\overline{\zeta_c}+i\,\zeta_y\right)\right)\Big)\Big\} \end{aligned} \tag{B.42}$$

$$\int_{C_+} \frac{dw_4}{d\zeta}\overline{\frac{dw_2^s}{d\zeta}} \left(\frac{(\zeta+\zeta_c)^2}{(\zeta+\zeta_c)^2-a^2}\right)\left(\frac{-r_c^2}{\zeta^2}\right) d\zeta = \frac{-2\,i\,\pi\,\left(a^2+{r_c}^2\right)}{{r_c}^2} \tag{B.43}$$

$$\int_{C_+} \frac{dw_2^s}{d\zeta}\overline{\frac{dw_4}{d\zeta}} \left(\frac{(\zeta+\zeta_c)^2}{(\zeta+\zeta_c)^2-a^2}\right)\left(\frac{-r_c^2}{\zeta^2}\right) d\zeta = 0 \tag{B.44}$$

$$\int_{C_+} \frac{dw_5^k}{d\zeta}\overline{\frac{dw_2^s}{d\zeta}}\left(\frac{(\zeta+\zeta_c)^2}{(\zeta+\zeta_c)^2-a^2}\right)\left(\frac{-r_c^2}{\zeta^2}\right)d\zeta =$$

$$\frac{2\,i\,\pi}{{r_c}^2\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^3\zeta_k\,\left(-a+\zeta_c+\zeta_k\right)\,\left(a+\zeta_c+\zeta_k\right)\,\left(\overline{\zeta_c}+\overline{\zeta_k}\right)^2}$$

$$\times\Big\{{r_c}^2\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^3\zeta_k\left(\zeta_c+\zeta_k\right)^2\left(\overline{\zeta_c}+\overline{\zeta_k}\right)^2+a^2\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^3\zeta_k\left(\zeta_c+\zeta_k\right)^2\,\left(-{r_c}^2+\overline{\zeta_k}^2\right)$$

$$+a^4\,\Big({r_c}^8\,\zeta_k+{\zeta_c}^3\,\overline{\zeta_c}^3\,\zeta_k\,\overline{\zeta_k}^2+{r_c}^6\,\overline{\zeta_c}\,\zeta_k\,\left(-3\,\zeta_c+\overline{\zeta_c}+2\,\overline{\zeta_k}\right)+{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}\,\zeta_k\,\overline{\zeta_k}\,\left(2\,\zeta_c\,\overline{\zeta_c}+\zeta_c\,\overline{\zeta_k}-3\,\overline{\zeta_c}\,\overline{\zeta_k}\right)$$

$$+{r_c}^4\,\Big(\zeta_c\,\Big(\overline{\zeta_c}^2\,\left(2\,\zeta_c+\zeta_k\right)\,\left(\overline{\zeta_c}+2\,\zeta_k\right)+2\,\overline{\zeta_c}\,\left(2\,\zeta_c\,\overline{\zeta_c}+\left(\zeta_c+\overline{\zeta_c}\right)\,\zeta_k+2\,{\zeta_k}^2\right)\,\overline{\zeta_k}$$

$$+\left(2\,\overline{\zeta_c}+\zeta_k\right)\,\left(\zeta_c+2\,\zeta_k\right)\,\overline{\zeta_k}^2\Big)+2\,i\left(\zeta_c+\zeta_k\right)^2\left(\overline{\zeta_c}+\overline{\zeta_k}\right)^2\zeta_y\Big)\Big)\Big\} \quad \text{(B.45)}$$

$$\int_{C_+} \frac{dw_2^s}{d\zeta}\overline{\frac{dw_5^k}{d\zeta}}\left(\frac{(\zeta+\zeta_c)^2}{(\zeta+\zeta_c)^2-a^2}\right)\left(\frac{-r_c^2}{\zeta^2}\right)d\zeta =$$

$$\frac{2\,i\,\pi}{\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^3\overline{\zeta_k}^2\,\left(-a+\zeta_c+\overline{\zeta_k}\right)\,\left(a+\zeta_c+\overline{\zeta_k}\right)}$$

$$\times\,\Big\{{r_c}^2\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^3\left(\zeta_c+\overline{\zeta_k}\right)^2+a^2\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^3\,\left(-{r_c}^2+\overline{\zeta_k}^2\right)$$

$$+a^4\,\overline{\zeta_k}\,\Big({r_c}^4\,\overline{\zeta_k}+{\zeta_c}^3\,\overline{\zeta_c}\,\overline{\zeta_k}+{r_c}^2\,\Big(\zeta_c\,\Big(2\,\zeta_c\,\overline{\zeta_c}+\left(\zeta_c+\overline{\zeta_c}\right)\,\overline{\zeta_k}+2\,\overline{\zeta_k}^2\Big)-2\,i\left(\zeta_c+\overline{\zeta_k}\right)^2\zeta_y\Big)\Big)\Big\} \quad \text{(B.46)}$$

$$\int_{C_+} \frac{dw_1}{d\zeta}\overline{\frac{dw_3^s}{d\zeta}}\left(\frac{(\zeta+\zeta_c)^2}{(\zeta+\zeta_c)^2-a^2}\right)\left(\frac{-r_c^2}{\zeta^2}\right)d\zeta =$$

$$\frac{4\,a\,i\,\pi}{{r_c}^2\left({r_c}^2+\left(a-\zeta_c\right)\,\overline{\zeta_c}\right)^2\left({r_c}^2-\delta^2\right)^2\left({r_c}^2-\left(a+\zeta_c\right)\,\overline{\zeta_c}\right)^2}\times\,\Big\{2\,a^6\,{r_c}^4\,\overline{\zeta_c}^4$$

$$-{r_c}^2\left({r_c}^2-\delta^2\right)^6+a^2\left({r_c}^2-\delta^2\right)^4\,\left({r_c}^4+5\,{r_c}^2\,\overline{\zeta_c}^2-2\,\zeta_c\,\overline{\zeta_c}^3\right)$$

$$+a^4\,\overline{\zeta_c}^2\left({r_c}^2-\delta^2\right)^2\,\left(-5\,{r_c}^4-2\,{r_c}^2\,\overline{\zeta_c}^2+2\,\zeta_c\,\overline{\zeta_c}^3\right)\Big\} \quad \text{(B.47)}$$

$$\int_{C_+} \frac{dw_3^s}{d\zeta}\overline{\frac{dw_1}{d\zeta}}\left(\frac{(\zeta+\zeta_c)^2}{(\zeta+\zeta_c)^2-a^2}\right)\left(\frac{-r_c^2}{\zeta^2}\right)d\zeta =$$
$$\frac{-4\,a^3\,i\,\pi}{{r_c}^2\left({r_c}^2+(a-\zeta_c)\,\overline{\zeta_c}\right)^2({r_c}^2-\delta^2)^2\left({r_c}^2-(a+\zeta_c)\,\overline{\zeta_c}\right)^2}\times\Big\{a^4\,{r_c}^2\,\overline{\zeta_c}^2\,\left({r_c}^4-{r_c}^2\,\overline{\zeta_c}^2+{\zeta_c}^2\,\overline{\zeta_c}^2\right)$$
$$-\left({r_c}^2-\delta^2\right)^4\,\left({r_c}^4+3\,{r_c}^2\,\overline{\zeta_c}^2-2\,\zeta_c\,\overline{\zeta_c}^3\right)$$
$$+a^2\left({r_c}^2-\delta^2\right)^2\,\left({r_c}^6-2\,\zeta_c\,\overline{\zeta_c}^5+2\,{r_c}^4\,\overline{\zeta_c}\,\left(2\,\zeta_c+\overline{\zeta_c}\right)+{r_c}^2\,\left(-\left({\zeta_c}^2\,\overline{\zeta_c}^2\right)+\overline{\zeta_c}^4\right)\right)\Big\} \tag{B.48}$$

$$\int_{C_+} \frac{dw_2}{d\zeta}\overline{\frac{dw_3^s}{d\zeta}}\left(\frac{(\zeta+\zeta_c)^2}{(\zeta+\zeta_c)^2-a^2}\right)\left(\frac{-r_c^2}{\zeta^2}\right)d\zeta =$$
$$\frac{4\,a\,\pi}{{r_c}^2\left({r_c}^2+(a-\zeta_c)\,\overline{\zeta_c}\right)^2({r_c}^2-\delta^2)^2\left({r_c}^2-(a+\zeta_c)\,\overline{\zeta_c}\right)^2}\times\,\Big\{2\,a^6\,{r_c}^4\,\overline{\zeta_c}^4$$
$$+{r_c}^2\left({r_c}^2-\delta^2\right)^6+a^2\left({r_c}^2-\delta^2\right)^4\,\left({r_c}^4-5\,{r_c}^2\,\overline{\zeta_c}^2+2\,\zeta_c\,\overline{\zeta_c}^3\right)$$
$$-a^4\,\overline{\zeta_c}^2\left({r_c}^2-\delta^2\right)^2\,\left(5\,{r_c}^4-2\,{r_c}^2\,\overline{\zeta_c}^2+2\,\zeta_c\,\overline{\zeta_c}^3\right)\Big\} \tag{B.49}$$

$$\int_{C_+} \frac{dw_3^s}{d\zeta}\overline{\frac{dw_2}{d\zeta}}\left(\frac{(\zeta+\zeta_c)^2}{(\zeta+\zeta_c)^2-a^2}\right)\left(\frac{-r_c^2}{\zeta^2}\right)d\zeta =$$
$$\frac{4\,a^3\,\pi}{{r_c}^2\left({r_c}^2+(a-\zeta_c)\,\overline{\zeta_c}\right)^2({r_c}^2-\delta^2)^2\left({r_c}^2-(a+\zeta_c)\,\overline{\zeta_c}\right)^2}\times\Big(a^4\,{r_c}^2\,\overline{\zeta_c}^2\,\left({r_c}^4+\left({r_c}^2+{\zeta_c}^2\right)\,\overline{\zeta_c}^2\right)$$
$$+\left({r_c}^2-\delta^2\right)^4\,\left({r_c}^4-3\,{r_c}^2\,\overline{\zeta_c}^2+2\,\zeta_c\,\overline{\zeta_c}^3\right)$$
$$+a^2\left({r_c}^2-\delta^2\right)^2\,\left({r_c}^6+2\,{r_c}^4\,\left(2\,\zeta_c-\overline{\zeta_c}\right)\,\overline{\zeta_c}-2\,\zeta_c\,\overline{\zeta_c}^5+{r_c}^2\,\left(-\left({\zeta_c}^2\,\overline{\zeta_c}^2\right)+\overline{\zeta_c}^4\right)\right)\Big) \tag{B.50}$$

$$\int_{C_+} \frac{dw_3}{d\zeta}\overline{\frac{dw_3^s}{d\zeta}} \left(\frac{(\zeta+\zeta_c)^2}{(\zeta+\zeta_c)^2-a^2}\right)\left(\frac{-r_c^2}{\zeta^2}\right) d\zeta =$$
$$\frac{-4\,a\,\pi}{{r_c}^2\left({r_c}^2+(a-\zeta_c)\;\overline{\zeta_c}\right)^2\left({r_c}^2-\delta^2\right)^3\left({r_c}^2-(a+\zeta_c)\;\overline{\zeta_c}\right)^2}\times\Big\{2\,a^8\,{r_c}^4\,\zeta_c\,\overline{\zeta_c}^4-$$
$${r_c}^2\,\zeta_c\left({r_c}^2-\delta^2\right)^7+a^2\,\overline{\zeta_c}\left({r_c}^2-\delta^2\right)^5\,\left({r_c}^4+4\,{r_c}^2\,\delta^2-2\,{\zeta_c}^2\,\overline{\zeta_c}^2\right)$$
$$+a^4\,{r_c}^2\left({r_c}^2-\delta^2\right)^4\,\left({r_c}^2\,\zeta_c-7\,\overline{\zeta_c}^3\right)-a^6\,\overline{\zeta_c}^2\left({r_c}^2-\delta^2\right)^2\,\left(5\,{r_c}^4\,\zeta_c-4\,{r_c}^2\,\overline{\zeta_c}^3+2\,\zeta_c\,\overline{\zeta_c}^4\right)\Big\}$$
(B.51)

$$\int_{C_+} \frac{dw_3^s}{d\zeta}\overline{\frac{dw_3}{d\zeta}} \left(\frac{(\zeta+\zeta_c)^2}{(\zeta+\zeta_c)^2-a^2}\right)\left(\frac{-r_c^2}{\zeta^2}\right) d\zeta =$$
$$\frac{-4\,a^3\,\pi}{{r_c}^2\left({r_c}^2+(a-\zeta_c)\;\overline{\zeta_c}\right)^2\left({r_c}^2-\delta^2\right)^3\left({r_c}^2-(a+\zeta_c)\;\overline{\zeta_c}\right)^2}\times\Big\{a^6\,{r_c}^2\,\zeta_c\,\overline{\zeta_c}^4\,\left({r_c}^2+\delta^2\right)$$
$$+\overline{\zeta_c}\left(-{r_c}^2+\delta^2\right)^5\,\left(4\,{r_c}^4-5\,{r_c}^2\,\delta^2+2\,{\zeta_c}^2\,\overline{\zeta_c}^2\right)$$
$$+a^2\,{r_c}^2\,\zeta_c\left({r_c}^2-\delta^2\right)^3\,\left(5\,{r_c}^4-6\,{r_c}^2\,\delta^2+{\zeta_c}^2\,\overline{\zeta_c}^2+2\,\overline{\zeta_c}^4\right)$$
$$+a^4\,\overline{\zeta_c}\,\left({r_c}^2-\delta^2\right)\,\left(4\,{r_c}^8-2\,{r_c}^6\,\delta^2-2\,{r_c}^4\,{\zeta_c}^2\,\overline{\zeta_c}^2-3\,{r_c}^2\,\zeta_c\,\overline{\zeta_c}^5+2\,{\zeta_c}^2\,\overline{\zeta_c}^6\right)\Big\} \quad \text{(B.52)}$$

$$\int_{C_+} \frac{dw_4}{d\zeta}\overline{\frac{dw_3^s}{d\zeta}} \left(\frac{(\zeta+\zeta_c)^2}{(\zeta+\zeta_c)^2-a^2}\right)\left(\frac{-r_c^2}{\zeta^2}\right) d\zeta =$$
$$\frac{4\,a^3\,\pi\,\overline{\zeta_c}\,\left(2\,{r_c}^6-5\,{r_c}^4\,\delta^2+4\,{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}^2+(a-\zeta_c)\;\zeta_c\;(a+\zeta_c)\;\overline{\zeta_c}^3\right)}{{r_c}^2\left({r_c}^2+(a-\zeta_c)\;\overline{\zeta_c}\right)^2\left({r_c}^2-(a+\zeta_c)\;\overline{\zeta_c}\right)^2} \quad \text{(B.53)}$$

$$\int_{C_+} \frac{dw_3^s}{d\zeta}\overline{\frac{dw_4}{d\zeta}} \left(\frac{(\zeta+\zeta_c)^2}{(\zeta+\zeta_c)^2-a^2}\right)\left(\frac{-r_c^2}{\zeta^2}\right) d\zeta = 0 \quad \text{(B.54)}$$

$$\int_{C_+} \frac{dw_5^k}{d\zeta}\overline{\frac{dw_3^s}{d\zeta}} \left(\frac{(\zeta+\zeta_c)^2}{(\zeta+\zeta_c)^2-a^2}\right)\left(\frac{-r_c^2}{\zeta^2}\right) d\zeta =$$

$$\frac{-4\,a\,\pi}{{r_c}^2\left({r_c}^2+(a-\zeta_c)\,\overline{\zeta_c}\right)^2\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^2\left({r_c}^2-(a+\zeta_c)\,\overline{\zeta_c}\right)^2\zeta_k\,(-a+\zeta_c+\zeta_k)\,(a+\zeta_c+\zeta_k)\,\left(\overline{\zeta_c}+\overline{\zeta_k}\right)^2}$$
$$\times\Big\{{r_c}^2\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^6\zeta_k\,(\zeta_c+\zeta_k)^2\,\left(\overline{\zeta_c}+\overline{\zeta_k}\right)-a^2\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^4\,\Big(\zeta_c\,\overline{\zeta_c}^2\,\zeta_k\,(\zeta_c+\zeta_k)^2\,\overline{\zeta_k}^2$$
$$+{r_c}^6\,\zeta_k\,\left(\overline{\zeta_c}+\overline{\zeta_k}\right)+{r_c}^2\,\overline{\zeta_c}\,\zeta_k\,\Big(2\,\overline{\zeta_c}^2\,(\zeta_c+\zeta_k)^2+\overline{\zeta_c}\,\zeta_k\,(2\,\zeta_c+\zeta_k)\,\overline{\zeta_k}-\left(3\,{\zeta_c}^2+4\,\zeta_c\,\zeta_k+2\,{\zeta_k}^2\right)\overline{\zeta_k}^2\Big)$$
$$+{r_c}^4\,\zeta_c\,\left(\overline{\zeta_c}+\overline{\zeta_k}\right)\,\left(2\,\zeta_k\,\overline{\zeta_k}+\zeta_c\,\left(\overline{\zeta_c}+\overline{\zeta_k}\right)\right)\Big)-a^6\,\overline{\zeta_c}^4\,\Big(-3\,{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}\,\zeta_k\,\overline{\zeta_k}^2+{\zeta_c}^3\,\overline{\zeta_c}^2\,\zeta_k\,\overline{\zeta_k}^2$$
$$+{r_c}^6\,\zeta_k\,\left(\overline{\zeta_c}+2\,\overline{\zeta_k}\right)+{r_c}^4\,\Big({\zeta_c}^2\left(\overline{\zeta_c}+\overline{\zeta_k}\right)^2-{\zeta_k}^2\left(\overline{\zeta_c}+\overline{\zeta_k}\right)^2-\zeta_c\,\zeta_k\,\Big(\overline{\zeta_c}^2+2\,\overline{\zeta_c}\,\overline{\zeta_k}-2\,\overline{\zeta_k}^2\Big)\Big)\Big)$$
$$+a^4\,\overline{\zeta_c}^2\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^2\,\Big(\zeta_c\,\overline{\zeta_c}^2\,\zeta_k\,\left(2\,{\zeta_c}^2+2\,\zeta_c\,\zeta_k+{\zeta_k}^2\right)\overline{\zeta_k}^2+{r_c}^6\,\zeta_k\,\left(4\,\overline{\zeta_c}+5\,\overline{\zeta_k}\right)$$
$$+{r_c}^2\,\overline{\zeta_c}\,\zeta_k\,\left(\overline{\zeta_c}+2\,\overline{\zeta_k}\right)\,\left(\overline{\zeta_c}\,(\zeta_c+\zeta_k)^2-3\,{\zeta_c}^2\,\overline{\zeta_k}\right)$$
$$+{r_c}^4\,\Big(2\,{\zeta_c}^2\left(\overline{\zeta_c}+\overline{\zeta_k}\right)^2-3\,{\zeta_k}^2\left(\overline{\zeta_c}+\overline{\zeta_k}\right)^2+\zeta_c\,\zeta_k\,\Big(-5\,\overline{\zeta_c}^2-4\,\overline{\zeta_c}\,\overline{\zeta_k}+4\,\overline{\zeta_k}^2\Big)\Big)\Big)\Big\}$$
$$\text{(B.55)}$$

$$\int_{C_+} \frac{dw_3^s}{d\zeta}\overline{\frac{dw_5^k}{d\zeta}} \left(\frac{(\zeta+\zeta_c)^2}{(\zeta+\zeta_c)^2-a^2}\right)\left(\frac{-r_c^2}{\zeta^2}\right) d\zeta =$$
$$\frac{4\,a\,\pi\,\Big(\zeta_c\left({r_c}^3-r_c\,\zeta_c\,\overline{\zeta_c}\right)^2+{r_c}^2\,\Big(a^2\,{\zeta_c}^2+\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^2\Big)\,\overline{\zeta_k}+a^2\,\zeta_c\,\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)\,\overline{\zeta_k}^2-a^2\,{r_c}^2\,\overline{\zeta_k}^3\Big)}{\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^2\overline{\zeta_k}^2\,\left(-a+\zeta_c+\overline{\zeta_k}\right)\,\left(a+\zeta_c+\overline{\zeta_k}\right)}$$
$$\text{(B.56)}$$

$$\int_{C_+} \frac{dw_1^s}{d\zeta}\overline{\frac{dw_1^s}{d\zeta}} \left(\frac{(\zeta+\zeta_c)^2}{(\zeta+\zeta_c)^2-a^2}\right)\left(\frac{-r_c^2}{\zeta^2}\right) d\zeta =$$
$$\frac{-4\,a^2\,i\,\pi}{{r_c}^2\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^5}\times\Big(\overline{\zeta_c}\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^5+a^2\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^2\,\Big(-2\,{r_c}^2\,\overline{\zeta_c}^3+\zeta_c\,\overline{\zeta_c}^4$$
$$+{r_c}^4\,\left(-2\,\left(\zeta_c+\overline{\zeta_c}\right)+i\,\zeta_y\right)\Big)+a^4\,{r_c}^2\,\Big({r_c}^4\,\left(2\,\zeta_c+\overline{\zeta_c}-i\,\zeta_y\right)-{\zeta_c}^2\,\overline{\zeta_c}^2\,\left(\overline{\zeta_c}+i\,\zeta_y\right)$$
$$+2\,{r_c}^2\,\zeta_c\,\Big({\zeta_c}^2+\zeta_c\,\overline{\zeta_c}+\overline{\zeta_c}^2+i\,\overline{\zeta_c}\,\zeta_y\Big)\Big)\Big)\quad\text{(B.57)}$$

$$\int_{C_+} \frac{dw_1^s}{d\zeta}\overline{\frac{dw_2^s}{d\zeta}} \left(\frac{(\zeta+\zeta_c)^2}{(\zeta+\zeta_c)^2-a^2}\right)\left(\frac{-r_c^2}{\zeta^2}\right) d\zeta = \frac{-4\,a^2\,\pi}{{r_c}^2\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^5}$$
$$\times\left(\overline{\zeta_c}\left(-{r_c}^2+\zeta_c\,\overline{\zeta_c}\right)^5+a^2\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^2\left(2\,{r_c}^2\,\overline{\zeta_c}^3-\zeta_c\,\overline{\zeta_c}^4+{r_c}^4\,\left(-2\,\zeta_c+i\,\zeta_y\right)\right)\right.$$
$$\left.+a^4\,{r_c}^2\,\left({r_c}^4\,\left(-\overline{\zeta_c}+i\,\zeta_y\right)+{\zeta_c}^2\,\overline{\zeta_c}^2\,\left(\overline{\zeta_c}+i\,\zeta_y\right)+2\,{r_c}^2\,\zeta_c\,\left({\zeta_c}^2-\overline{\zeta_c}\,\left(\overline{\zeta_c}+i\,\zeta_y\right)\right)\right)\right) \tag{B.58}$$

$$\int_{C_+} \frac{dw_2^s}{d\zeta}\overline{\frac{dw_1^s}{d\zeta}} \left(\frac{(\zeta+\zeta_c)^2}{(\zeta+\zeta_c)^2-a^2}\right)\left(\frac{-r_c^2}{\zeta^2}\right) d\zeta = \frac{4\,a^2\,\pi}{{r_c}^2\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^5}$$
$$\times\left(\overline{\zeta_c}\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^5+a^2\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^2\left(-2\,{r_c}^2\,\overline{\zeta_c}^3+\zeta_c\,\overline{\zeta_c}^4+{r_c}^4\,\left(2\,\zeta_c-i\,\zeta_y\right)\right)\right.$$
$$\left.+a^4\,{r_c}^2\,\left({\zeta_c}^2\,\overline{\zeta_c}^2\,\left(-\overline{\zeta_c}+i\,\zeta_y\right)+{r_c}^4\,\left(\overline{\zeta_c}+i\,\zeta_y\right)-2\,{r_c}^2\,\zeta_c\,\left({\zeta_c}^2-\overline{\zeta_c}^2+i\,\overline{\zeta_c}\,\zeta_y\right)\right)\right) \tag{B.59}$$

$$\int_{C_+} \frac{dw_2^s}{d\zeta}\overline{\frac{dw_2^s}{d\zeta}} \left(\frac{(\zeta+\zeta_c)^2}{(\zeta+\zeta_c)^2-a^2}\right)\left(\frac{-r_c^2}{\zeta^2}\right) d\zeta = \frac{4\,a^2\,i\,\pi}{{r_c}^2\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^5}$$
$$\times\left(\overline{\zeta_c}\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^5+a^2\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^2\left(-2\,{r_c}^2\,\overline{\zeta_c}^3+\zeta_c\,\overline{\zeta_c}^4+{r_c}^4\,\left(-2\,\zeta_c+2\,\overline{\zeta_c}+i\,\zeta_y\right)\right)\right.$$
$$\left.+a^4\,{r_c}^2\,\left({\zeta_c}^2\,\overline{\zeta_c}^2\,\left(-\overline{\zeta_c}+i\,\zeta_y\right)+{r_c}^4\,\left(-2\,\zeta_c+\overline{\zeta_c}+i\,\zeta_y\right)+2\,{r_c}^2\,\zeta_c\,\left({\zeta_c}^2-\zeta_c\,\overline{\zeta_c}+\overline{\zeta_c}^2-i\,\overline{\zeta_c}\,\zeta_y\right)\right)\right) \tag{B.60}$$

$$\int_{C_+} \frac{dw_1^s}{d\zeta}\overline{\frac{dw_3^s}{d\zeta}} \left(\frac{(\zeta+\zeta_c)^2}{(\zeta+\zeta_c)^2-a^2}\right)\left(\frac{-r_c^2}{\zeta^2}\right) d\zeta = \frac{-4\,a\,i\,\pi}{{r_c}^2\left({r_c}^2+(a-\zeta_c)\,\overline{\zeta_c}\right)^2\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^4\left({r_c}^2-(a+\zeta_c)\,\overline{\zeta_c}\right)^2}$$
$$\times\left({r_c}^2\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^8-a^2\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^6\left({r_c}^4+6\,{r_c}^2\,\overline{\zeta_c}^2-2\,\zeta_c\,\overline{\zeta_c}^3\right)\right.$$
$$+a^4\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^4\left({r_c}^6+9\,{r_c}^2\,\overline{\zeta_c}^4-4\,\zeta_c\,\overline{\zeta_c}^5+{r_c}^4\left(3\,{\zeta_c}^2+2\,\zeta_c\,\overline{\zeta_c}+5\,\overline{\zeta_c}^2\right)\right)$$
$$+2\,a^8\,{r_c}^2\,\overline{\zeta_c}^4\left({r_c}^4-{\zeta_c}^2\,\overline{\zeta_c}\left(\overline{\zeta_c}+i\,\zeta_y\right)+{r_c}^2\,\zeta_c\left(2\left(\zeta_c+\overline{\zeta_c}\right)+i\,\zeta_y\right)\right)$$
$$+a^6\,\overline{\zeta_c}\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^2\left(2\,\zeta_c\,\overline{\zeta_c}^6+{r_c}^6\left(-4\,\zeta_c-5\,\overline{\zeta_c}+4\,i\,\zeta_y\right)-{r_c}^4\,\overline{\zeta_c}\left(9\,{\zeta_c}^2+10\,\zeta_c\,\overline{\zeta_c}+2\,\overline{\zeta_c}^2+6\,i\,\zeta_c\,\zeta_y\right)\right.$$
$$\left.\left.+2\,{r_c}^2\,\overline{\zeta_c}^2\left(-2\,\overline{\zeta_c}^3+{\zeta_c}^2\left(\overline{\zeta_c}+i\,\zeta_y\right)\right)\right)\right) \quad \text{(B.61)}$$

$$\int_{C_+} \frac{dw_3^s}{d\zeta}\overline{\frac{dw_1^s}{d\zeta}} \left(\frac{(\zeta+\zeta_c)^2}{(\zeta+\zeta_c)^2-a^2}\right)\left(\frac{-r_c^2}{\zeta^2}\right) d\zeta = \frac{-4\,a^3\,i\,\pi}{{r_c}^2\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^4}$$
$$\times\left(-\left(\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^2\left(2\,{r_c}^4+3\,{r_c}^2\,\overline{\zeta_c}^2-2\,\zeta_c\,\overline{\zeta_c}^3\right)\right)+a^2\left({r_c}^6-{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}^2+4\,{r_c}^4\,\zeta_c\left(\zeta_c+\overline{\zeta_c}\right)\right)\right) \quad \text{(B.62)}$$

$$\int_{C_+} \frac{dw_2^s}{d\zeta}\overline{\frac{dw_3^s}{d\zeta}} \left(\frac{(\zeta+\zeta_c)^2}{(\zeta+\zeta_c)^2-a^2}\right)\left(\frac{-r_c^2}{\zeta^2}\right) d\zeta = \frac{4\,a\,\pi}{{r_c}^2\left({r_c}^2+(a-\zeta_c)\,\overline{\zeta_c}\right)^2\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^4\left({r_c}^2-(a+\zeta_c)\,\overline{\zeta_c}\right)^2}$$
$$\times\left({r_c}^2\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^8+a^2\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^6\left({r_c}^4-6\,{r_c}^2\,\overline{\zeta_c}^2+2\,\zeta_c\,\overline{\zeta_c}^3\right)\right.$$
$$+a^4\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^4\left({r_c}^6+9\,{r_c}^2\,\overline{\zeta_c}^4-4\,\zeta_c\,\overline{\zeta_c}^5+{r_c}^4\left(-3\,{\zeta_c}^2+2\,\zeta_c\,\overline{\zeta_c}-5\,\overline{\zeta_c}^2\right)\right)$$
$$+2\,a^8\,{r_c}^2\,\overline{\zeta_c}^4\left({r_c}^4-{r_c}^2\,\zeta_c\left(2\,\zeta_c-2\,\overline{\zeta_c}+i\,\zeta_y\right)+{\zeta_c}^2\,\overline{\zeta_c}\left(-\overline{\zeta_c}+i\,\zeta_y\right)\right)$$
$$+a^6\,\overline{\zeta_c}\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^2\left(2\,\zeta_c\,\overline{\zeta_c}^6+{r_c}^6\left(4\,\zeta_c-5\,\overline{\zeta_c}-4\,i\,\zeta_y\right)+{r_c}^4\,\overline{\zeta_c}\left(9\,{\zeta_c}^2-10\,\zeta_c\,\overline{\zeta_c}+2\,\overline{\zeta_c}^2+6\,i\,\zeta_c\,\zeta_y\right)\right.$$
$$\left.\left.+2\,{r_c}^2\,\overline{\zeta_c}^2\left(-2\,\overline{\zeta_c}^3+{\zeta_c}^2\left(\overline{\zeta_c}-i\,\zeta_y\right)\right)\right)\right) \quad \text{(B.63)}$$

$$\int_{C_+} \frac{dw_3^s}{d\zeta}\overline{\frac{dw_2^s}{d\zeta}} \left(\frac{(\zeta+\zeta_c)^2}{(\zeta+\zeta_c)^2-a^2}\right)\left(\frac{-r_c^2}{\zeta^2}\right) d\zeta = \frac{4\,a^3\,\pi}{{r_c}^2\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^4}$$
$$\times\left(a^2\,\left({r_c}^6-4\,{r_c}^4\,\zeta_c\,\left(\zeta_c-\overline{\zeta_c}\right)-{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}^2\right)+\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^2\,\left(2\,{r_c}^4-3\,{r_c}^2\,\overline{\zeta_c}^2+2\,\zeta_c\,\overline{\zeta_c}^3\right)\right) \tag{B.64}$$

$$\int_{C_+} \frac{dw_3^s}{d\zeta}\overline{\frac{dw_3^s}{d\zeta}} \left(\frac{(\zeta+\zeta_c)^2}{(\zeta+\zeta_c)^2-a^2}\right)\left(\frac{-r_c^2}{\zeta^2}\right) d\zeta =$$
$$\frac{8\,a^2\,i\,\pi}{{r_c}^2\left({r_c}^2+(a-\zeta_c)\,\overline{\zeta_c}\right)^2\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^3\left({r_c}^2-(a+\zeta_c)\,\overline{\zeta_c}\right)^2}$$
$$\times\,\left({r_c}^2\,\overline{\zeta_c}\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^6+a^6\,{r_c}^2\,\zeta_c\,\overline{\zeta_c}^4\,\left(-5\,{r_c}^2+\zeta_c\,\overline{\zeta_c}\right)\right.$$
$$-a^2\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^4\,\left(3\,{r_c}^4\,\zeta_c+6\,{r_c}^2\,\overline{\zeta_c}^3-2\,\zeta_c\,\overline{\zeta_c}^4\right)$$
$$\left.+a^4\,\overline{\zeta_c}\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^2\,\left(2\,{r_c}^6+12\,{r_c}^4\,\zeta_c\,\overline{\zeta_c}-2\,\zeta_c\,\overline{\zeta_c}^5+{r_c}^2\,\left(-\left({\zeta_c}^2\,\overline{\zeta_c}^2\right)+3\,\overline{\zeta_c}^4\right)\right)\right) \tag{B.65}$$

Appendix C Coefficients in Equation (3.156)

$$\mathcal{B}_1 = \frac{\pi}{(r_c^2-\delta^2)^4}\Big[2\,a^2\left({r_c}^2-\delta^2\right)^4 - {r_c}^2\left({r_c}^2-\delta^2\right)^4 + a^6\,{r_c}^2\,\left({r_c}^2+\overline{\zeta_c}\,\left(2\,\zeta_c+\overline{\zeta_c}+2\,i\,\overline{\zeta_c}\right)\right)$$
$$-4\,a^5\,{r_c}^2\,\left({r_c}^2-\delta^2\right)\left(\zeta_c - i\,\overline{\zeta_c} - i\,\zeta_y - i\,\zeta_y\right)$$
$$-2\,a^4\,\left({r_c}^2-\delta^2\right)\left({r_c}^4 - {\zeta_c}^3\,\overline{\zeta_c} + {r_c}^2\,\zeta_c\,\left(\zeta_c + i\,\left(\overline{\zeta_c}+2\,\zeta_y\right)\right)\right)\Big]$$

$$\mathcal{B}_2 = \frac{\pi}{(r_c^2-\delta^2)^4}\Big[4\,a^5\,{r_c}^2\,\left(i\,\zeta_c-\overline{\zeta_c}\right)\,\left({r_c}^2-\delta^2\right) - 2\,a^2\,i\left({r_c}^2-\delta^2\right)^4 - i\,{r_c}^2\left({r_c}^2-\delta^2\right)^4$$
$$-2\,a^4\,\left({r_c}^2-\delta^2\right)\left(-\left({r_c}^2\,\delta^2\right) + i\,\left({r_c}^4 - {r_c}^2\,{\zeta_c}^2 + {\zeta_c}^3\,\overline{\zeta_c}\right)\right)$$
$$-a^6\,{r_c}^2\,\left(2\,\overline{\zeta_c}^2 + i\,\left({r_c}^2 + \left(2\,\zeta_c - \overline{\zeta_c}\right)\,\overline{\zeta_c}\right)\right)\Big]$$

$$\mathcal{B}_3 = \frac{2a^3\pi}{(r_c^2-\delta^2)^3}\left[a^2\,(2+i)\,{r_c}^2\,\overline{\zeta_c} + 2\,a\,(-1+i)\,{r_c}^2\,\left({r_c}^2-\delta^2\right)\right.$$
$$\left.-\zeta_c\,\left((3+i)\,{r_c}^2 - 2\,\delta^2\right)\left({r_c}^2-\delta^2\right)\right]$$

$$\mathcal{C}_1 = \frac{2\,a^2\,\pi\,\zeta_c\,\left(r_c{}^2-\delta^2\right)\,\left(\left(r_c{}^2-\delta^2\right)^3+a^2\,\left(-r_c{}^4+\zeta_c{}^3\,\overline{\zeta_c}\right)\right)}{r_c{}^2\left(r_c{}^2+\zeta_c\,\left(a-\overline{\zeta_c}\right)\right)^2\left(r_c{}^2-\zeta_c\,\left(a+\overline{\zeta_c}\right)\right)^2}$$

$$\mathcal{C}_2 = \frac{-4\,a^2\,i\,\pi\,\zeta_c\,\left(r_c{}^2-\delta^2\right)\,\left(r_c{}^6-3\,r_c{}^4\,\delta^2+3\,r_c{}^2\,\zeta_c{}^2\,\overline{\zeta_c}^2+\zeta_c{}^3\,\overline{\zeta_c}\,\left(a^2-\overline{\zeta_c}^2\right)\right)}{r_c{}^2\left(r_c{}^2+\zeta_c\,\left(a-\overline{\zeta_c}\right)\right)^2\left(r_c{}^2-\zeta_c\,\left(a+\overline{\zeta_c}\right)\right)^2}$$

$$\mathcal{C}_3 = \frac{-2\,a^2\,\pi\,\zeta_c\,\left(r_c{}^2-\delta^2\right)\,\left(\left(r_c{}^2-\delta^2\right)^3+a^2\,\left(r_c{}^4+\zeta_c{}^3\,\overline{\zeta_c}\right)\right)}{r_c{}^2\left(r_c{}^2+\zeta_c\,\left(a-\overline{\zeta_c}\right)\right)^2\left(r_c{}^2-\zeta_c\,\left(a+\overline{\zeta_c}\right)\right)^2}$$

$$\begin{aligned}\mathcal{C}_4 = {} & -2\,a^2\,i\,\pi\,\Big(-\Big(\left(r_c{}^2-2\,\delta^2\right)\,\left(r_c{}^2-\delta^2\right)^4\Big)+a^4\,\zeta_c\,\Big(r_c{}^2\,\zeta_c{}^3+r_c{}^4\,\left(\zeta_c-\overline{\zeta_c}\right)\\ & -2\,\zeta_c{}^4\,\overline{\zeta_c}\Big)+a^2\,\Big(r_c{}^8-4\,r_c{}^2\,\zeta_c{}^4\,\overline{\zeta_c}^2+r_c{}^4\,\zeta_c{}^2\,\overline{\zeta_c}\,\left(7\,\zeta_c+\overline{\zeta_c}\right)-r_c{}^6\,\zeta_c\,\left(3\,\zeta_c+2\,\overline{\zeta_c}\right)\Big)\Big)\\ & /\Big(r_c{}^2\left(r_c{}^2+\zeta_c\,\left(a-\overline{\zeta_c}\right)\right)^2\left(r_c{}^2-\zeta_c\,\left(a+\overline{\zeta_c}\right)\right)^2\Big)\end{aligned}$$

$$\begin{aligned}\mathcal{C}_5 = {} & 2\,a^2\,\pi\,\Big(\left(r_c{}^2-2\,\delta^2\right)\,\left(r_c{}^2-\delta^2\right)^4\\ & +a^2\,\Big(r_c{}^8+r_c{}^6\,\zeta_c\,\left(3\,\zeta_c-2\,\overline{\zeta_c}\right)+4\,r_c{}^2\,\zeta_c{}^4\,\overline{\zeta_c}^2+r_c{}^4\,\zeta_c{}^2\,\overline{\zeta_c}\,\left(-7\,\zeta_c+\overline{\zeta_c}\right)\Big)\\ & +a^4\,\zeta_c\,\left(-\left(r_c{}^2\,\zeta_c{}^3\right)+2\,\zeta_c{}^4\,\overline{\zeta_c}+r_c{}^4\,\left(\zeta_c+\overline{\zeta_c}\right)\right)\Big)\\ & /\Big(r_c{}^2\left(r_c{}^2+\zeta_c\,\left(a-\overline{\zeta_c}\right)\right)^2\left(r_c{}^2-\zeta_c\,\left(a+\overline{\zeta_c}\right)\right)^2\Big)\end{aligned}$$

$$\begin{aligned}\mathcal{C}_6 = {} & -2\,a^2\,\pi\,\Big(-\Big(\overline{\zeta_c}\left(r_c{}^2-\delta^2\right)^6\Big)+a^6\,\left(-\left(r_c{}^2\,\zeta_c{}^5\right)+r_c{}^4\,\zeta_c{}^2\,\overline{\zeta_c}+\zeta_c{}^6\,\overline{\zeta_c}\right)\\ & +a^2\,\zeta_c\left(r_c{}^2-\delta^2\right)^3\,\left(2\,r_c{}^4-4\,r_c{}^2\,\delta^2+\zeta_c{}^2\,\overline{\zeta_c}^2\right)\\ & +a^4\,\Big(r_c{}^8\,\overline{\zeta_c}+\zeta_c{}^6\,\overline{\zeta_c}^3+r_c{}^6\,\left(\zeta_c{}^3-2\,\zeta_c\,\overline{\zeta_c}^2\right)+r_c{}^4\,\left(-2\,\zeta_c{}^4\,\overline{\zeta_c}+\zeta_c{}^2\,\overline{\zeta_c}^3\right)\Big)\Big)\\ & /\Big(r_c{}^2\left(r_c{}^2+\zeta_c\,\left(a-\overline{\zeta_c}\right)\right)^2\,\left(r_c{}^2-\delta^2\right)\,\left(r_c{}^2-\zeta_c\,\left(a+\overline{\zeta_c}\right)\right)^2\Big)\end{aligned}$$

$$\mathcal{D}_1 = 2\,a^2\,\pi\,\Big(2\,\zeta_c\left({r_c}^2-\delta^2\right)^7 - a^2\left({r_c}^2-\delta^2\right)^4\,\left(4\,{r_c}^2\,{\zeta_c}^3 - 4\,{\zeta_c}^4\,\overline{\zeta_c} + {r_c}^4\,\left(3\,\zeta_c + 2\,\overline{\zeta_c} - i\,\zeta_y\right)\right)$$
$$+a^6\,{r_c}^2\,{\zeta_c}^2\,\Big({r_c}^4\,\left(\overline{\zeta_c} - i\,\zeta_y\right) + {\zeta_c}^2\,\overline{\zeta_c}^2\,\left(\zeta_c - i\,\zeta_y\right) + {r_c}^2\,{\zeta_c}^2\,\left(-2\,\zeta_c - 2\,\overline{\zeta_c} - i\,\zeta_y\right)\Big)$$
$$+a^4\left({r_c}^2-\delta^2\right)^2\,\Big(-2\,{\zeta_c}^6\,\overline{\zeta_c} + {r_c}^6\,\left(\zeta_c + \overline{\zeta_c} - i\,\zeta_y\right)$$
$$+{r_c}^4\,\zeta_c\,\Big(4\,{\zeta_c}^2 + 6\,\delta^2 + \overline{\zeta_c}^2 + 2\,i\,\zeta_c\,\zeta_y - 2\,i\,\overline{\zeta_c}\,\zeta_y\Big) + {r_c}^2\,{\zeta_c}^2\,\Big(2\,{\zeta_c}^3 - \zeta_c\,\overline{\zeta_c}^2 + i\,\overline{\zeta_c}^2\,\zeta_y\Big)\Big)\Big)$$
$$/\Big({r_c}^2\left({r_c}^2 + \zeta_c\,\left(a - \overline{\zeta_c}\right)\right)^2\left({r_c}^2-\delta^2\right)^3\left({r_c}^2 - \zeta_c\,\left(a + \overline{\zeta_c}\right)\right)^2\Big)$$

$$\mathcal{D}_2 = -2\,a^2\,\pi\,\Big(2\,i\,\zeta_c\left({r_c}^2-\delta^2\right)^7 + a^2\left({r_c}^2-\delta^2\right)^4\,\left(-i\,{r_c}^4\,\zeta_c + 2\,i\,\left(-2\,{r_c}^2\,{\zeta_c}^3 + {r_c}^4\,\overline{\zeta_c} + 2\,{\zeta_c}^4\,\overline{\zeta_c}\right)\right.$$
$$\left.-{r_c}^4\,\zeta_y\right) + a^6\,{r_c}^2\,{\zeta_c}^2\,\Big(-i\,{r_c}^2\,{\zeta_c}^3 + i\,\Big(-\left({r_c}^2\,{\zeta_c}^2\,\left(\zeta_c - 2\,\overline{\zeta_c}\right)\right) - {r_c}^4\,\overline{\zeta_c} + {\zeta_c}^3\,\overline{\zeta_c}^2\Big)$$
$$+\Big({r_c}^4 - {r_c}^2\,{\zeta_c}^2 - {\zeta_c}^2\,\overline{\zeta_c}^2\Big)\,\zeta_y\Big) + a^4\left({r_c}^2-\delta^2\right)^2\,\Big(2\,i\,{r_c}^4\,{\zeta_c}^3 + i\,\Big({r_c}^6\,\left(\zeta_c - \overline{\zeta_c}\right)$$
$$-2\,{\zeta_c}^6\,\overline{\zeta_c} + {r_c}^4\,\zeta_c\,\Big(2\,{\zeta_c}^2 - 2\,\delta^2 - \overline{\zeta_c}^2\Big) + {r_c}^2\,\Big(2\,{\zeta_c}^5 - {\zeta_c}^3\,\overline{\zeta_c}^2\Big)\Big)$$
$$+{r_c}^2\,\Big({r_c}^4 + 2\,{r_c}^2\,\zeta_c\,\left(\zeta_c - \overline{\zeta_c}\right) + {\zeta_c}^2\,\overline{\zeta_c}^2\Big)\,\zeta_y\Big)\Big)$$
$$/\Big({r_c}^2\left({r_c}^2 + \zeta_c\,\left(a - \overline{\zeta_c}\right)\right)^2\left({r_c}^2-\delta^2\right)^3\left({r_c}^2 - \zeta_c\,\left(a + \overline{\zeta_c}\right)\right)^2\Big)$$

$$\mathcal{D}_3 = 2\,a\,\pi\,\Big(2\,{r_c}^2\left({r_c}^2-\delta^2\right)^6 - a^2\left({r_c}^2-\delta^2\right)^4\,\left(3\,{r_c}^4 + 10\,{r_c}^2\,{\zeta_c}^2 - 4\,{\zeta_c}^3\,\overline{\zeta_c}\right)$$
$$+a^6\,{r_c}^2\,{\zeta_c}^2\,\Big({r_c}^4 - 4\,{r_c}^2\,{\zeta_c}^2 + {\zeta_c}^2\,\overline{\zeta_c}^2\Big)$$
$$+a^4\left({r_c}^2-\delta^2\right)^2\,\Big({r_c}^6 - 4\,{\zeta_c}^5\,\overline{\zeta_c} + {r_c}^4\,\zeta_c\,\left(9\,\zeta_c + 4\,\overline{\zeta_c}\right) + {r_c}^2\,\Big(4\,{\zeta_c}^4 - {\zeta_c}^2\,\overline{\zeta_c}^2\Big)\Big)\Big)$$
$$/\Big({r_c}^2\left({r_c}^2 + \zeta_c\,\left(a - \overline{\zeta_c}\right)\right)^2\left({r_c}^2-\delta^2\right)^2\left({r_c}^2 - \zeta_c\,\left(a + \overline{\zeta_c}\right)\right)^2\Big)$$

$$\mathcal{D}_4 = \frac{2\,a^2\,\pi}{r_c} - 2\,\pi\,r_c$$

$$\mathcal{D}_5 = -2\,a^2\,\pi\,\Big(2\,i\,\zeta_c\left({r_c}^2-\delta^2\right)^7 + a^2\left({r_c}^2-\delta^2\right)^4\,\left(i\,\left(-4\,{r_c}^2\,{\zeta_c}^3 + 4\,{\zeta_c}^4\,\overline{\zeta_c} + {r_c}^4\,\left(\zeta_c + 2\,\overline{\zeta_c}\right)\right)\right.$$
$$\left.-{r_c}^4\,\zeta_y\right) + a^6\,{r_c}^2\,{\zeta_c}^2\,\Big(i\,\Big({r_c}^4\,\overline{\zeta_c} + {\zeta_c}^3\,\overline{\zeta_c}^2 + 2\,{r_c}^2\,{\zeta_c}^2\,\left(\zeta_c + \overline{\zeta_c}\right)\Big) - \Big({r_c}^4 + {r_c}^2\,{\zeta_c}^2 - {\zeta_c}^2\,\overline{\zeta_c}^2\Big)\,\zeta_y\Big)$$
$$+a^4\left({r_c}^2-\delta^2\right)^2\,\Big(i\,\Big(-2\,{\zeta_c}^6\,\overline{\zeta_c} + {r_c}^6\,\left(\zeta_c + \overline{\zeta_c}\right) + {r_c}^4\,\zeta_c\,\Big(-4\,{\zeta_c}^2 - 2\,\delta^2 + \overline{\zeta_c}^2\Big)$$
$$+{r_c}^2\,\Big(2\,{\zeta_c}^5 - {\zeta_c}^3\,\overline{\zeta_c}^2\Big)\Big) - {r_c}^2\,\Big({r_c}^4 + {\zeta_c}^2\,\overline{\zeta_c}^2 - 2\,{r_c}^2\,\zeta_c\,\left(\zeta_c + \overline{\zeta_c}\right)\Big)\,\zeta_y\Big)\Big)$$
$$/\Big({r_c}^2\left({r_c}^2 + \zeta_c\,\left(a - \overline{\zeta_c}\right)\right)^2\left({r_c}^2-\delta^2\right)^3\left({r_c}^2 - \zeta_c\,\left(a + \overline{\zeta_c}\right)\right)^2\Big)$$

$$
\begin{aligned}
\mathcal{D}_6 &= -2\,a^2\,\pi\,\Big(2\,\zeta_c\left({r_c}^2-\delta^2\right)^7+a^2\left({r_c}^2-\delta^2\right)^4\,\left(-4\,{r_c}^2\,{\zeta_c}^3+4\,{\zeta_c}^4\,\overline{\zeta_c}\right.\\
&\quad\left.+{r_c}^4\,\left(3\,\zeta_c-2\,\overline{\zeta_c}-i\,\zeta_y\right)\right)+a^6\,{r_c}^2\,{\zeta_c}^2\,\Big({\zeta_c}^2\,\overline{\zeta_c}^2\,\left(\zeta_c-i\,\zeta_y\right)\\
&\quad+{r_c}^2\,{\zeta_c}^2\,\left(2\,\zeta_c-2\,\overline{\zeta_c}-i\,\zeta_y\right)+{r_c}^4\,\left(-\overline{\zeta_c}-i\,\zeta_y\right)\Big)+a^4\left({r_c}^2-\delta^2\right)^2\,\Big(-2\,{\zeta_c}^6\,\overline{\zeta_c}\\
&\quad+{r_c}^6\,\left(\zeta_c-\overline{\zeta_c}-i\,\zeta_y\right)-{r_c}^4\,\zeta_c\,\left(4\,{\zeta_c}^2-6\,\delta^2+\overline{\zeta_c}^2-2\,i\,\zeta_c\,\zeta_y-2\,i\,\overline{\zeta_c}\,\zeta_y\right)\\
&\quad+{r_c}^2\,{\zeta_c}^2\,\left(2\,{\zeta_c}^3-\zeta_c\,\overline{\zeta_c}^2-i\,\overline{\zeta_c}^2\,\zeta_y\right)\Big)\Big)\\
&\quad/\left({r_c}^2\left({r_c}^2+\zeta_c\,\left(a-\overline{\zeta_c}\right)\right)^2\left({r_c}^2-\delta^2\right)^3\left({r_c}^2-\zeta_c\,\left(a+\overline{\zeta_c}\right)\right)^2\right)\\
\mathcal{D}_7 &= -2\,a\,i\,\pi\,\Big(2\,{r_c}^2\left({r_c}^2-\delta^2\right)^6+a^2\left({r_c}^2-\delta^2\right)^4\,\left(3\,{r_c}^4-10\,{r_c}^2\,{\zeta_c}^2+4\,{\zeta_c}^3\,\overline{\zeta_c}\right)\\
&\quad+a^6\,{r_c}^2\,{\zeta_c}^2\,\left({r_c}^4+4\,{r_c}^2\,{\zeta_c}^2+{\zeta_c}^2\,\overline{\zeta_c}^2\right)\\
&\quad+a^4\left({r_c}^2-\delta^2\right)^2\,\left({r_c}^6-4\,{\zeta_c}^5\,\overline{\zeta_c}+{r_c}^4\,\zeta_c\,\left(-9\,\zeta_c+4\,\overline{\zeta_c}\right)+{r_c}^2\,\left(4\,{\zeta_c}^4-{\zeta_c}^2\,\overline{\zeta_c}^2\right)\right)\Big)\\
&\quad/\left({r_c}^2\left({r_c}^2+\zeta_c\,\left(a-\overline{\zeta_c}\right)\right)^2\left({r_c}^2-\delta^2\right)^2\left({r_c}^2-\zeta_c\,\left(a+\overline{\zeta_c}\right)\right)^2\right)\\
\mathcal{D}_8 &= \frac{-2\,i\,\pi\,\left(a^2+{r_c}^2\right)}{r_c}\\
\mathcal{D}_9 &= \Big[-\left(i\,\pi\,{r_c}^4\left({r_c}^2-\delta^2\right)^8\right)+a^2\,i\left({r_c}^2-\delta^2\right)^6\,\left(2\,\pi\,{r_c}^4\,{\zeta_c}^2+\pi\,\left({r_c}^2-4\,\delta^2\right)\,\left({r_c}^2-\delta^2\right)^2\right)\\
&\quad+a^4\left({r_c}^2-\delta^2\right)^4\,\Big(i\,\Big(\pi\,{r_c}^4\,\left({r_c}^4-{\zeta_c}^4-2\,{r_c}^2\,\delta^2+{\zeta_c}^2\,\overline{\zeta_c}^2\right)+\pi\,\zeta_c\,\Big({r_c}^6\,\left(5\,\zeta_c-2\,\overline{\zeta_c}\right)\\
&\quad-3\,{r_c}^2\,{\zeta_c}^3\,\overline{\zeta_c}^2+4\,{\zeta_c}^4\,\overline{\zeta_c}^3+2\,{r_c}^4\,\delta^2\,\left(-3\,\zeta_c+\overline{\zeta_c}\right)\Big)\Big)+2\,\pi\,{r_c}^4\,\zeta_c\,\left({r_c}^2-\delta^2\right)\,\zeta_y\Big)\\
&\quad+a^{10}\,\pi\,{r_c}^2\,{\zeta_c}^4\,\left(i\,\left({r_c}^4+2\,{\zeta_c}^2\,\overline{\zeta_c}^2+4\,{r_c}^2\,\overline{\zeta_c}\,\left(\zeta_c+\overline{\zeta_c}\right)\right)+2\,\overline{\zeta_c}\,\left(-{r_c}^2+\delta^2\right)\,\zeta_y\right)\\
&\quad+a^8\,\zeta_c\,\left({r_c}^2-\delta^2\right)\,\Big(i\,\Big(\pi\,{r_c}^4\,{\zeta_c}^3\,\left({r_c}^2-\delta^2\right)+\pi\,\Big(8\,{r_c}^8\,\overline{\zeta_c}-11\,{r_c}^2\,{\zeta_c}^6\,\overline{\zeta_c}+4\,{\zeta_c}^7\,\overline{\zeta_c}^2\\
&\quad+2\,{r_c}^6\,\zeta_c\,\left({\zeta_c}^2-2\,\delta^2-7\,\overline{\zeta_c}^2\right)+{r_c}^4\,{\zeta_c}^2\,\left(7\,{\zeta_c}^3-4\,{\zeta_c}^2\,\overline{\zeta_c}+4\,\zeta_c\,\overline{\zeta_c}^2+6\,\overline{\zeta_c}^3\right)\Big)\Big)\\
&\quad+2\,\pi\,{r_c}^4\,\left(-4\,{r_c}^4+{\zeta_c}^4+8\,{r_c}^2\,\delta^2-4\,{\zeta_c}^2\,\overline{\zeta_c}^2\right)\,\zeta_y\Big)\\
&\quad+a^6\left({r_c}^2-\delta^2\right)^3\,\Big(i\,\Big(-2\,\pi\,{r_c}^4\,{\zeta_c}^2\,\left({r_c}^2-\delta^2\right)+\pi\,\Big(3\,{r_c}^8-4\,{\zeta_c}^6\,\overline{\zeta_c}^2\\
&\quad+{r_c}^6\,\left(-6\,{\zeta_c}^2+5\,\delta^2+6\,\overline{\zeta_c}^2\right)-{r_c}^4\,\zeta_c\,\left(13\,{\zeta_c}^3-10\,{\zeta_c}^2\,\overline{\zeta_c}+10\,\zeta_c\,\overline{\zeta_c}^2+6\,\overline{\zeta_c}^3\right)\\
&\quad+{r_c}^2\,\left(17\,{\zeta_c}^5\,\overline{\zeta_c}+2\,{\zeta_c}^3\,\overline{\zeta_c}^3\right)\Big)\Big)+2\,\pi\,{r_c}^2\,\left({r_c}^4\,\overline{\zeta_c}+{\zeta_c}^2\,\overline{\zeta_c}^3-2\,{r_c}^2\,\zeta_c\,\left({\zeta_c}^2+\overline{\zeta_c}^2\right)\right)\,\zeta_y\Big)\Big]\\
&\quad/\left({r_c}^2\left({r_c}^2+\zeta_c\,\left(a-\overline{\zeta_c}\right)\right)^2\left({r_c}^2-\delta^2\right)^4\left({r_c}^2-\zeta_c\,\left(a+\overline{\zeta_c}\right)\right)^2\right)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{D}_{10} &= \Big[\pi\, {r_c}^4 \left({r_c}^2-\delta^2\right)^8 + a^2 \left({r_c}^2-\delta^2\right)^6 \left(-2\,\pi\, {r_c}^4\, {\zeta_c}^2 + \pi\, \left({r_c}^2-4\,\delta^2\right) \left({r_c}^2-\delta^2\right)^2\right) \\
&\quad + a^{10}\, \pi\, {r_c}^2\, {\zeta_c}^4 \left({r_c}^4 + 2\,\zeta_c\, {\overline{\zeta_c}}^2\, \left(\zeta_c - i\,\zeta_y\right) + 2\, {r_c}^2\, \overline{\zeta_c}\, \left(2\,\zeta_c - 2\,\overline{\zeta_c} - i\,\zeta_y\right)\right) \\
&\quad - a^4 \left({r_c}^2-\delta^2\right)^4 \left(\pi\, {r_c}^4 \left({r_c}^4 - {\zeta_c}^4 - 2\,{r_c}^2\,\delta^2 + {\zeta_c}^2\, {\overline{\zeta_c}}^2\right)\right. \\
&\quad \left. - \pi\,\zeta_c\, \left(-{r_c}^2+\delta^2\right) \left({r_c}^2\, {\zeta_c}^2\, \overline{\zeta_c} + 4\,{\zeta_c}^3\, {\overline{\zeta_c}}^2 + {r_c}^4\, \left(-5\,\zeta_c - 2\,\overline{\zeta_c} - 2\,i\,\zeta_y\right)\right)\right) \\
&\quad + a^6 \left({r_c}^2-\delta^2\right)^3 \left(2\,\pi\, {r_c}^4\, {\zeta_c}^2\, \left({r_c}^2-\delta^2\right) + \pi\, \left(3\,{r_c}^8 - 4\,{\zeta_c}^6\, {\overline{\zeta_c}}^2\right.\right. \\
&\quad + {r_c}^2\, {\zeta_c}^2\, \overline{\zeta_c}\, \left(17\,{\zeta_c}^3 + 2\,\zeta_c\, {\overline{\zeta_c}}^2 + 2\,i\,{\overline{\zeta_c}}^2\,\zeta_y\right) + {r_c}^6\, \left(6\,{\zeta_c}^2 + 5\,\delta^2 - 2\,\overline{\zeta_c}\, \left(3\,\overline{\zeta_c} - i\,\zeta_y\right)\right) \\
&\quad \left.\left. + {r_c}^4\, \zeta_c\, \left(-13\,{\zeta_c}^3 - 10\,\zeta_c\, {\overline{\zeta_c}}^2 + 2\,{\overline{\zeta_c}}^2\, \left(3\,\overline{\zeta_c} - 2\,i\,\zeta_y\right) + {\zeta_c}^2\, \left(-10\,\overline{\zeta_c} - 4\,i\,\zeta_y\right)\right)\right)\right) \\
&\quad - a^8\,\zeta_c\, \left({r_c}^2-\delta^2\right) \left(\pi\, {r_c}^4\, {\zeta_c}^3\, \left({r_c}^2-\delta^2\right) + \pi\, \left(11\,{r_c}^2\, {\zeta_c}^6\, \overline{\zeta_c} - 4\,{\zeta_c}^7\, {\overline{\zeta_c}}^2\right.\right. \\
&\quad + 8\,{r_c}^8\, \left(\overline{\zeta_c} - i\,\zeta_y\right) - {r_c}^4\, {\zeta_c}^2\, \left(7\,{\zeta_c}^3 + 4\,\zeta_c\, {\overline{\zeta_c}}^2 - 6\,{\overline{\zeta_c}}^3 - 8\,i\,{\overline{\zeta_c}}^2\,\zeta_y + {\zeta_c}^2\, \left(4\,\overline{\zeta_c} + 2\,i\,\zeta_y\right)\right) \\
&\quad \left.\left. + 2\,{r_c}^6\,\zeta_c\, \left({\zeta_c}^2 + 2\,\delta^2 + \overline{\zeta_c}\, \left(-7\,\overline{\zeta_c} - 8\,i\,\zeta_y\right)\right)\right)\right)\Big] \\
&\quad / \left({r_c}^2 \left({r_c}^2 + \zeta_c\, \left(a - \overline{\zeta_c}\right)\right)^2 \left({r_c}^2-\delta^2\right)^4 \left({r_c}^2 - \zeta_c\, \left(a + \overline{\zeta_c}\right)\right)^2\right) \\
\mathcal{D}_{11} &= 2\,a\,i\, \left(-\left(\left(\pi+\pi\right)\, {r_c}^2\, \overline{\zeta_c} \left({r_c}^2-\delta^2\right)^7\right) + a^8\,\pi\, {r_c}^2\, {\zeta_c}^4\, \overline{\zeta_c}\, \left(5\,{r_c}^2 + \delta^2\right)\right. \\
&\quad + 2\,a^2\,\zeta_c \left({r_c}^2-\delta^2\right)^5 \left(\pi\, {r_c}^4 - \pi\, \left({r_c}^4 - 4\,{r_c}^2\,\delta^2 + 2\,{\zeta_c}^2\, {\overline{\zeta_c}}^2\right)\right) \\
&\quad + a^4\, {r_c}^2 \left({r_c}^2-\delta^2\right)^3 \left(\pi\,{\zeta_c}^3\, \left(-4\,{r_c}^2 + 3\,\delta^2\right) + \pi\, \left(8\,{r_c}^4\,\overline{\zeta_c} - 9\,{r_c}^2\,\zeta_c\, \left({\zeta_c}^2 + {\overline{\zeta_c}}^2\right)\right.\right. \\
&\quad \left.\left. + {\zeta_c}^2\,\overline{\zeta_c}\, \left(11\,{\zeta_c}^2 + {\overline{\zeta_c}}^2\right)\right)\right) + a^6\,\zeta_c\, \left({r_c}^2-\delta^2\right)\, \left(2\,\pi\,{r_c}^2\,{\zeta_c}^4\, \left({r_c}^2-\delta^2\right)\right. \\
&\quad \left.\left. + \pi\, \left(4\,{r_c}^8 - 11\,{r_c}^6\,\delta^2 - 10\,{r_c}^2\,{\zeta_c}^5\,\overline{\zeta_c} + 4\,{\zeta_c}^6\,{\overline{\zeta_c}}^2 + {r_c}^4\, \left(5\,{\zeta_c}^4 + 7\,{\zeta_c}^2\,{\overline{\zeta_c}}^2\right)\right)\right)\right) \\
&\quad / \left({r_c}^2 \left({r_c}^2 + \zeta_c\, \left(a - \overline{\zeta_c}\right)\right)^2 \left({r_c}^2-\delta^2\right)^3 \left({r_c}^2 - \zeta_c\, \left(a + \overline{\zeta_c}\right)\right)^2\right) \\
\mathcal{D}_{12} &= \frac{-2\,i}{r_c \left({r_c}^2-\delta^2\right)^4}\, \left(a^6\,\pi\,{r_c}^4\,\overline{\zeta_c} + \pi\,{r_c}^2\,\zeta_c \left({r_c}^2-\delta^2\right)^4 + a^2\,\pi\,\overline{\zeta_c} \left({r_c}^2-\delta^2\right)^4\right. \\
&\quad \left. + a^4\,\zeta_c \left({r_c}^2-\delta^2\right)^2\, \left(\pi\,{r_c}^2 - 2\,\pi\,{r_c}^2 + \pi\,\delta^2\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{E}_1 &= 2\,a^2\,\pi\,\Big(\zeta_c\left({r_c}^2-\delta^2\right)^5+2\,a^3\,{r_c}^4\,\left({r_c}^2-\delta^2\right)\,\left((-1+i)\,{r_c}^2+3\,i\,\overline{\zeta_c}\,\zeta_y\right.\\
&\quad\left.+\zeta_c\,\left(2\,(-1+i)\,\overline{\zeta_c}+3\,i\,\zeta_y\right)\right)-a^2\,\left({r_c}^2-\delta^2\right)\,\Big(-4\,{r_c}^2\,{\zeta_c}^4\,\overline{\zeta_c}+{\zeta_c}^5\,\overline{\zeta_c}^2\\
&\quad+{r_c}^6\,\left((5+i)\,\zeta_c+4\,\overline{\zeta_c}-i\,\zeta_y\right)+{r_c}^4\,\zeta_c\,\left(3\,{\zeta_c}^2+\zeta_c\,\left(-5\,\overline{\zeta_c}+2\,i\,\overline{\zeta_c}+3\,i\,\zeta_y\right)\right.\\
&\quad\left.\left.-\overline{\zeta_c}\,\left(\overline{\zeta_c}-2\,i\,\zeta_y\right)\right)\right)+a^4\,{r_c}^2\,\Big({r_c}^4\,\left(4\,\zeta_c+(4+i)\,\overline{\zeta_c}+i\,\zeta_y\right)-{\zeta_c}^2\,\overline{\zeta_c}^2\,\left(\zeta_c-i\,\zeta_y\right)\\
&\quad+{r_c}^2\,\overline{\zeta_c}\,\Big(5\,{\zeta_c}^2+2\,\overline{\zeta_c}^2+\zeta_c\,\left((3+2\,i)\,\overline{\zeta_c}-2\,i\,\zeta_y\right)\Big)\Big)\Big)\\
&\quad/\Big({r_c}^2\left({r_c}^2-\delta^2\right)^5\Big)\\
\mathcal{E}_2 &= (-2-2\,i)\,a^2\,\pi\,\Big(\overline{\zeta_c}\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^5+a^2\,\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)\,\Big(-\Big({\zeta_c}^2\,\overline{\zeta_c}^5\Big)\\
&\quad+{r_c}^2\,\Big((1-i)\,{\zeta_c}^4\,\overline{\zeta_c}+3\,\zeta_c\,\overline{\zeta_c}^4\Big)+{r_c}^6\,\left(-2\,i\,\zeta_c+(3-i)\,\overline{\zeta_c}-\zeta_y\,\right)+{r_c}^4\,\Big((-1+i)\,{\zeta_c}^3\\
&\quad+2\,i\,{\zeta_c}^2\,\overline{\zeta_c}-2\,\overline{\zeta_c}^3+\zeta_c\,\overline{\zeta_c}\,\left(-2\,i\,\overline{\zeta_c}+\zeta_y\right)\Big)\Big)+a^4\,{r_c}^2\,\Big({r_c}^4\,\left((2-4\,i)\,\zeta_c+\overline{\zeta_c}-\zeta_y\right)\\
&\quad-{\zeta_c}^2\,\overline{\zeta_c}^2\,\left(\overline{\zeta_c}+\zeta_y\right)+2\,{r_c}^2\,\zeta_c\,\left(i\,{\zeta_c}^2+(1-2\,i)\,\zeta_c\,\overline{\zeta_c}+\overline{\zeta_c}\,\left(\overline{\zeta_c}+\zeta_y\right)\,\right)\Big)\Big)\\
&\quad/\Big({r_c}^2\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^5\Big)\\
\mathcal{E}_3 &= -2\,a^2\,\pi\,\Big(\zeta_c\left({r_c}^2-\delta^2\right)^5-2\,a^3\,{r_c}^4\,\left({r_c}^2-\delta^2\right)\,\left((-1+i)\,{r_c}^2+3\,i\,\overline{\zeta_c}\,\zeta_y\right.\\
&\quad\left.+\zeta_c\,\left(2\,(-1+i)\,\overline{\zeta_c}+3\,i\,\zeta_y\right)\right)+a^4\,{r_c}^2\,\Big({\zeta_c}^2\,\overline{\zeta_c}^2\,\left(-\zeta_c-i\,\zeta_y\right)\\
&\quad+{r_c}^4\,\left(4\,\zeta_c+(-4-i)\,\overline{\zeta_c}-i\,\zeta_y\right)+{r_c}^2\,\overline{\zeta_c}\,\Big(5\,{\zeta_c}^2+2\,\overline{\zeta_c}^2+\zeta_c\,\left((-3-2\,i)\,\overline{\zeta_c}+2\,i\,\zeta_y\right)\Big)\Big)\\
&\quad+a^2\,\left({r_c}^2-\delta^2\right)\,\Big(4\,{r_c}^2\,{\zeta_c}^4\,\overline{\zeta_c}-{\zeta_c}^5\,\overline{\zeta_c}^2+{r_c}^6\,\left((5+i)\,\zeta_c-4\,\overline{\zeta_c}-i\,\zeta_y\right)\\
&\quad+{r_c}^4\,\zeta_c\,\left(-3\,{\zeta_c}^2+\zeta_c\,\left(-5\,\overline{\zeta_c}+2\,i\,\overline{\zeta_c}+3\,i\,\zeta_y\right)+\overline{\zeta_c}\,\left(\overline{\zeta_c}+3\,i\,\zeta_y-i\,\zeta_y\right)\right)\Big)\Big)\\
&\quad/\Big({r_c}^2\left({r_c}^2-\delta^2\right)^5\Big)
\end{aligned}
$$

$$\begin{aligned}
\mathcal{E}_4 \;=\; & -2\,i\,a^3\,\pi\,\Big(a^2\left(r_c{}^2-\zeta_c\,\overline{\zeta_c}\right)^3\,\Big(3\,r_c{}^8-8\,\zeta_c{}^2\,\overline{\zeta_c}^6+r_c{}^6\,\left(-7\,\zeta_c{}^2-(1-2\,i)\,\zeta_c\,\overline{\zeta_c}\right.\\
&\left.+(5+4\,i)\,\overline{\zeta_c}^2\right)+r_c{}^2\,\left(-9\,\zeta_c{}^3\,\overline{\zeta_c}^3+22\,\zeta_c\,\overline{\zeta_c}^5\right)+r_c{}^4\,\overline{\zeta_c}\,\left(7\,\zeta_c{}^3\right.\\
&\left.+(7+2\,i)\,\zeta_c{}^2\,\overline{\zeta_c}-14\,\overline{\zeta_c}^3+\zeta_c\,\overline{\zeta_c}\,\left((7+4\,i)\,\overline{\zeta_c}+16\,i\,\zeta_y\right)\right)\Big)\\
&+a^6\,r_c{}^2\,\overline{\zeta_c}^4\,\Big(2\,r_c{}^4+r_c{}^2\,\left(-8\,\zeta_c{}^2+(2+4\,i)\,\overline{\zeta_c}^2+2\,i\,\zeta_c\,\left(\overline{\zeta_c}-\zeta_y\right)\right)\\
&+\zeta_c{}^2\,\overline{\zeta_c}\,\left(3\,\overline{\zeta_c}+2\,i\,\zeta_y\right)\Big)+4\,i\,a^5\,r_c{}^4\,\overline{\zeta_c}^4\,\left(r_c{}^2-\zeta_c\,\overline{\zeta_c}\right)\,\left((1+2\,i)\,\zeta_c\right.\\
&\left.+(2+i)\,\overline{\zeta_c}+4\,\zeta_y\right)-8\,i\,a^3\,r_c{}^4\,\overline{\zeta_c}^2\left(r_c{}^2-\zeta_c\,\overline{\zeta_c}\right)^3\,\left((1+2\,i)\,\zeta_c\right.\\
&\left.+(2+i)\,\overline{\zeta_c}+4\,\zeta_y\right)+4\,i\,a\,r_c{}^4\left(r_c{}^2-\zeta_c\,\overline{\zeta_c}\right)^5\,\left((1+2\,i)\,\zeta_c+(2+i)\,\overline{\zeta_c}\right.\\
&\left.+4\,\zeta_y\right)-\left(r_c{}^2-\zeta_c\,\overline{\zeta_c}\right)^5\,\Big(3\,r_c{}^6-4\,\zeta_c{}^2\,\overline{\zeta_c}^4+r_c{}^2\,\left(-4\,\zeta_c{}^3\,\overline{\zeta_c}+11\,\zeta_c\,\overline{\zeta_c}^3\right)\\
&+r_c{}^4\,\left((4+2\,i)\,\zeta_c{}^2+(3+4\,i)\,\zeta_c\,\overline{\zeta_c}-7\,\overline{\zeta_c}^2+8\,i\,\zeta_c\,\zeta_y\right)\Big)\\
&+a^4\,\overline{\zeta_c}\,\left(r_c{}^2-\zeta_c\,\overline{\zeta_c}\right)\,\Big(4\,\zeta_c{}^2\,\overline{\zeta_c}^7+r_c{}^2\,\zeta_c\,\overline{\zeta_c}^3\,\left(-11\,\overline{\zeta_c}^3+\zeta_c{}^2\,\left(8\,\overline{\zeta_c}+2\,i\,\zeta_y\right)\right)\\
&+r_c{}^8\,\left(4\,\zeta_c-3\,\overline{\zeta_c}-4\,i\,\zeta_y\right)+r_c{}^6\,\overline{\zeta_c}\,\left(13\,\zeta_c{}^2+(5-4\,i)\,\zeta_c\,\overline{\zeta_c}-(6+8\,i)\,\overline{\zeta_c}^2+10\,i\,\zeta_c\,\zeta_y\right)\\
&+r_c{}^4\,\overline{\zeta_c}^2\,\left(-17\,\zeta_c{}^3-(10-2\,i)\,\zeta_c{}^2\,\overline{\zeta_c}+4\,i\,\zeta_c\,\overline{\zeta_c}^2+7\,\overline{\zeta_c}^3-8\,i\,\zeta_c\,\left(\zeta_c+\overline{\zeta_c}\right)\,\zeta_y\right)\Big)\Big)\\
&/\left(r_c{}^2\left(r_c{}^2+(a-\zeta_c)\,\overline{\zeta_c}\right)^2\left(r_c{}^2-\zeta_c\,\overline{\zeta_c}\right)^4\left(r_c{}^2-(a+\zeta_c)\,\overline{\zeta_c}\right)^2\right)\\
\mathcal{E}_5 \;=\; & -2\,a\,\pi\,\Big(4\,i\,a^7\,r_c{}^4\,\left((1+2\,i)\,\zeta_c-(2+i)\,\overline{\zeta_c}\right)\,\overline{\zeta_c}^4\,\left(r_c{}^2-\zeta_c\,\overline{\zeta_c}\right)\\
&-8\,i\,a^5\,r_c{}^4\,\left((1+2\,i)\,\zeta_c-(2+i)\,\overline{\zeta_c}\right)\,\overline{\zeta_c}^2\left(r_c{}^2-\zeta_c\,\overline{\zeta_c}\right)^3+4\,i\,a^3\,r_c{}^4\,\left((1+2\,i)\,\zeta_c\right.\\
&\left.-(2+i)\,\overline{\zeta_c}\right)\,\left(r_c{}^2-\zeta_c\,\overline{\zeta_c}\right)^5+2\,r_c{}^2\left(r_c{}^2-\zeta_c\,\overline{\zeta_c}\right)^8+a^2\left(r_c{}^2-\zeta_c\,\overline{\zeta_c}\right)^5\,\Big(9\,r_c{}^6-4\,\zeta_c{}^2\,\overline{\zeta_c}^4\\
&-r_c{}^4\,\left((4+2\,i)\,\zeta_c{}^2+(3-4\,i)\,\zeta_c\,\overline{\zeta_c}+11\,\overline{\zeta_c}^2\right)+r_c{}^2\,\left(4\,\zeta_c{}^3\,\overline{\zeta_c}+15\,\zeta_c\,\overline{\zeta_c}^3\right)\Big)\\
&+a^4\left(r_c{}^2-\zeta_c\,\overline{\zeta_c}\right)^3\,\Big(7\,r_c{}^8+8\,\zeta_c{}^2\,\overline{\zeta_c}^6+r_c{}^6\,\left(-7\,\zeta_c{}^2+(7+2\,i)\,\zeta_c\,\overline{\zeta_c}-(23+4\,i)\,\overline{\zeta_c}^2\right)\\
&+r_c{}^4\,\overline{\zeta_c}\,\left(7\,\zeta_c{}^3-(7-2\,i)\,\zeta_c{}^2\,\overline{\zeta_c}+(11-4\,i)\,\zeta_c\,\overline{\zeta_c}^2+16\,\overline{\zeta_c}^3\right)-r_c{}^2\,\left(7\,\zeta_c{}^3\,\overline{\zeta_c}^3+24\,\zeta_c\,\overline{\zeta_c}^5\right)\Big)\\
&+a^8\,r_c{}^2\,\overline{\zeta_c}^4\,\Big(8\,r_c{}^4+\zeta_c{}^2\,\overline{\zeta_c}\,\left(-3\,\overline{\zeta_c}+2\,i\,\zeta_y\right)-2\,r_c{}^2\,\Big(4\,\zeta_c{}^2-(8+i)\,\zeta_c\,\overline{\zeta_c}+(1+2\,i)\,\overline{\zeta_c}^2\\
&+i\,\zeta_c\,\zeta_y\Big)\Big)+a^6\,\overline{\zeta_c}\,\left(r_c{}^2-\zeta_c\,\overline{\zeta_c}\right)\,\Big(-4\,\zeta_c{}^2\,\overline{\zeta_c}^7-r_c{}^4\,\overline{\zeta_c}^2\,\Big(17\,\zeta_c{}^3+(8+4\,i)\,\zeta_c\,\overline{\zeta_c}^2+7\,\overline{\zeta_c}^3\\
&+\zeta_c{}^2\,\left((-34-2\,i)\,\overline{\zeta_c}+8\,i\,\zeta_y\right)\Big)+r_c{}^8\,\left(4\,\zeta_c-17\,\overline{\zeta_c}-4\,i\,\zeta_y\right)+r_c{}^6\,\overline{\zeta_c}\,\Big(13\,\zeta_c{}^2-(17+4\,i)\,\zeta_c\,\overline{\zeta_c}\\
&+(14+8\,i)\,\overline{\zeta_c}^2+10\,i\,\zeta_c\,\zeta_y\Big)+r_c{}^2\,\left(11\,\zeta_c\,\overline{\zeta_c}^6+2\,i\,\zeta_c{}^3\,\overline{\zeta_c}^3\,\zeta_y\right)\Big)\Big)\\
&/\left(r_c{}^2\left(r_c{}^2+(a-\zeta_c)\,\overline{\zeta_c}\right)^2\left(r_c{}^2-\zeta_c\,\overline{\zeta_c}\right)^4\left(r_c{}^2-(a+\zeta_c)\,\overline{\zeta_c}\right)^2\right)
\end{aligned}$$

$$
\begin{aligned}
\mathcal{E}_6 &= 2\,a^2\,\pi\,\Big(a^6\,{r_c}^2\,{\zeta_c}^4\,\overline{\zeta_c}\,\left((16+3\,i)\,{r_c}^2-2\,\delta^2\right)+6\,a^5\,(-1+i)\,{r_c}^4\,{\zeta_c}^4\,\left({r_c}^2-\delta^2\right)\\
&\quad -12\,a^3\,(-1+i)\,{r_c}^4\,{\zeta_c}^2\,\left({r_c}^2-\delta^2\right)^3+6\,a\,(-1+i)\,{r_c}^4\,\left({r_c}^2-\delta^2\right)^5\\
&\quad -3\,{r_c}^2\,\zeta_c\,\left((3+i)\,{r_c}^2-2\,\delta^2\right)\,\left({r_c}^2-\delta^2\right)^5+a^2\,\left({r_c}^2-\delta^2\right)^3\,\Big(3\,(4+i)\,{r_c}^6\,\overline{\zeta_c}\\
&\quad -24\,{r_c}^2\,{\zeta_c}^4\,\overline{\zeta_c}+4\,{\zeta_c}^5\,\overline{\zeta_c}^2+{r_c}^4\,\zeta_c\,\left((26+6\,i)\,{\zeta_c}^2-3\,(4+i)\,\overline{\zeta_c}^2\right)\Big)\\
&\quad -a^4\,\zeta_c\,\left({r_c}^2-\delta^2\right)\,\Big(4\,{r_c}^8+2\,(16+3\,i)\,{r_c}^6\,\delta^2+4\,{\zeta_c}^6\,\overline{\zeta_c}^2+2\,{r_c}^2\,{\zeta_c}^3\,\overline{\zeta_c}\,\left(-7\,{\zeta_c}^2+\overline{\zeta_c}^2\right)\\
&\quad +{r_c}^4\,{\zeta_c}^2\,\left((13+3\,i)\,{\zeta_c}^2-2\,(19+3\,i)\,\overline{\zeta_c}^2\right)\Big)\Big)\\
&\quad /\left({r_c}^2\,\left({r_c}^2+\zeta_c\,\left(a-\overline{\zeta_c}\right)\right)^2\,\left({r_c}^2-\delta^2\right)^3\,\left({r_c}^2-\zeta_c\,\left(a+\overline{\zeta_c}\right)\right)^2\right)\\
\mathcal{E}_7 &= 2\,i\,\pi\,\Big(-\left(a^2\,\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^5\right)+{r_c}^2\,\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^5\\
&\quad +a^6\,{r_c}^2\,\left({r_c}^4+2\,\zeta_c\,\left(\zeta_c+i\,\overline{\zeta_c}\right)\,\overline{\zeta_c}^2+{r_c}^2\,\left(5\,\zeta_c\,\overline{\zeta_c}+(2+4\,i)\,\overline{\zeta_c}^2\right)\right)\\
&\quad -a^4\,\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)\,\Big({r_c}^6-{\zeta_c}^4\,\overline{\zeta_c}^2+{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}\,\left((1+2\,i)\,\overline{\zeta_c}+4\,i\,\zeta_y\right)\\
&\quad +{r_c}^4\,\zeta_c\,\left(\zeta_c+(4+4\,i)\,\overline{\zeta_c}+8\,i\,\zeta_y\right)\Big)\\
&\quad -4\,a^5\,{r_c}^2\,\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)\,\left(2\,{r_c}^2+\zeta_c\,\overline{\zeta_c}\right)\,\left(\zeta_c-i\,\left(\overline{\zeta_c}+2\,\zeta_y\right)\right)\Big)\\
&\quad /\left(r_c\,\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^5\right)\\
\mathcal{E}_8 &= 2\,\pi\,\Big(-\left(a^2\,\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^5\right)-{r_c}^2\,\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^5\\
&\quad -4\,a^5\,{r_c}^2\,\left(\zeta_c+i\,\overline{\zeta_c}\right)\,\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)\,\left(2\,{r_c}^2+\zeta_c\,\overline{\zeta_c}\right)\\
&\quad +a^4\,\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)\,\left({r_c}^6+(1+2\,i)\,{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}^2+{\zeta_c}^4\,\overline{\zeta_c}^2+{r_c}^4\,\zeta_c\,\left(-\zeta_c+(4+4\,i)\,\overline{\zeta_c}\right)\right)\\
&\quad +a^6\,{r_c}^2\,\left({r_c}^4+2\,\zeta_c\,\left(\zeta_c-i\,\overline{\zeta_c}\right)\,\overline{\zeta_c}^2+{r_c}^2\,\left(5\,\zeta_c\,\overline{\zeta_c}-(2+4\,i)\,\overline{\zeta_c}^2\right)\right)\Big)\\
&\quad /\left(r_c\,\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^5\right)\\
\mathcal{E}_9 &= 4\,a^3\,\pi\,\Big((-1-i)\,{r_c}^6\,\left(2\,a+i\,\zeta_c\right)+i\,{r_c}^4\,\left((2+i)\,a^2-{\zeta_c}^2\right)\,\overline{\zeta_c}\\
&\quad +(1+i)\,{r_c}^2\,\zeta_c\,\left(i\,a^2+2\,a\,\zeta_c+(1+2\,i)\,{\zeta_c}^2\right)\,\overline{\zeta_c}^2-i\,{\zeta_c}^4\,\overline{\zeta_c}^3\Big)\\
&\quad /\left(r_c\,\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^4\right)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{F}_1 &= \frac{\pi}{2r_c^2}\left[\frac{2\,{r_c}^2\,{\zeta_c}^2 - 2\,a^2\,\left({r_c}^2 + {\zeta_c}^2\right)}{(a-\zeta_c)\,(a+\zeta_c)}\right. \\
&\quad \left. + \frac{a^3\,{r_c}^4}{(a-\zeta_c)\,\left({r_c}^2 + (a-\zeta_c)\,\overline{\zeta_c}\right)^2} + \frac{a^3\,{r_c}^4}{(a+\zeta_c)\,\left({r_c}^2 - (a+\zeta_c)\,\overline{\zeta_c}\right)^2}\right] \\
\mathcal{F}_2 &= \frac{i\pi}{2}\left[2 + \frac{2\,a^2\,{\zeta_c}^2}{{r_c}^2\,\left(-a^2 + {\zeta_c}^2\right)}\right. \\
&\quad \left. + a^3\,{r_c}^2\,\left(\frac{1}{(a-\zeta_c)\,\left({r_c}^2 + (a-\zeta_c)\,\overline{\zeta_c}\right)^2} + \frac{1}{(a+\zeta_c)\,\left({r_c}^2 - (a+\zeta_c)\,\overline{\zeta_c}\right)^2}\right)\right] \\
\mathcal{F}_3 &= i\,\pi\,\left(-\left(\frac{a^2\,\zeta_c\,\left(a^2 + {\zeta_c}^2\right)}{{r_c}^2\,\left(a^2 - {\zeta_c}^2\right)}\right) + \overline{\zeta_c}\right. \\
&\quad \left. + \frac{a^4\,{r_c}^2}{(a-\zeta_c)\,\left({r_c}^2 + (a-\zeta_c)\,\overline{\zeta_c}\right)^2} - \frac{a^4\,{r_c}^2}{(a+\zeta_c)\,\left({r_c}^2 - (a+\zeta_c)\,\overline{\zeta_c}\right)^2}\right) \\
\mathcal{F}_4 &= \pi\,\left(-1 + \frac{a^2}{{r_c}^2}\right) \\
\mathcal{F}_5 &= i\,\pi\,\left(1 + \frac{a^2}{{r_c}^2}\right) \\
\mathcal{F}_6 &= \frac{-2\,a^3\,\pi\,\overline{\zeta_c}\,\left(2\,{r_c}^6 - 5\,{r_c}^4\,\zeta_c\,\overline{\zeta_c} + 4\,{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}^2 + (a-\zeta_c)\,\zeta_c\,(a+\zeta_c)\,\overline{\zeta_c}^3\right)}{{r_c}^2\left({r_c}^2 + (a-\zeta_c)\,\overline{\zeta_c}\right)^2\left({r_c}^2 - (a+\zeta_c)\,\overline{\zeta_c}\right)^2}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{G}_1^k = \pi\, \Big(& a^2\,(a-\zeta_c)\,\zeta_c{}^2\,(a+\zeta_c)\,\overline{\zeta_c}^4\,\zeta_k{}^2\,\left(a^2-(\zeta_c+\zeta_k)^2\right)\overline{\zeta_k}^2 + r_c{}^{12}\,(\zeta_c+\zeta_k)^2\left(\overline{\zeta_c}+\overline{\zeta_k}\right)^2 \\
& -r_c{}^{10}\,\left(4\,\zeta_c{}^3\,\overline{\zeta_c}+8\,\zeta_c{}^2\,\overline{\zeta_c}\,\zeta_k+\left(a^2+\zeta_c\,\left(\zeta_c+4\,\overline{\zeta_c}\right)\right)\zeta_k{}^2+2\,\zeta_c\,\zeta_k{}^3+\zeta_k{}^4\right)\left(\overline{\zeta_c}+\overline{\zeta_k}\right)^2 \\
& +r_c{}^8\,\Big(2\,\zeta_c\,\overline{\zeta_c}\,(\zeta_c+\zeta_k)^2\,\left(3\,\zeta_c\,\overline{\zeta_c}+2\,\zeta_k{}^2\right)\left(\overline{\zeta_c}+\overline{\zeta_k}\right)^2+a^4\,\overline{\zeta_c}\,\zeta_k{}^2\,\left(\overline{\zeta_c}+2\,\overline{\zeta_k}\right) \\
& +a^2\,\Big(-2\,\overline{\zeta_c}^3\,\left(\zeta_c{}^2\,\overline{\zeta_c}+2\,\zeta_c\,\left(\overline{\zeta_c}-\zeta_k\right)\zeta_k+\overline{\zeta_c}\,\zeta_k{}^2\right)-4\,\overline{\zeta_c}^2\,\left(\zeta_c{}^2\,\overline{\zeta_c}+2\,\zeta_c\,\left(\overline{\zeta_c}-\zeta_k\right)\zeta_k+\overline{\zeta_c}\,\zeta_k{}^2\right)\overline{\zeta_k} \\
& -\left(2\,\zeta_c{}^2\,\overline{\zeta_c}^2+4\,\zeta_c\,\overline{\zeta_c}^2\,\zeta_k-\left(\zeta_c{}^2+4\,\zeta_c\,\overline{\zeta_c}-2\,\overline{\zeta_c}^2\right)\zeta_k{}^2-2\,\zeta_c\,\zeta_k{}^3-\zeta_k{}^4\right)\overline{\zeta_k}^2\Big)\Big) \\
& +r_c{}^4\,\overline{\zeta_c}^2\,\Big(a^6\,\overline{\zeta_c}^2\,\zeta_k{}^2+\zeta_c{}^3\,\overline{\zeta_c}\,(\zeta_c+\zeta_k)^2\,\left(\zeta_c\,\overline{\zeta_c}+4\,\zeta_k{}^2\right)\left(\overline{\zeta_c}+\overline{\zeta_k}\right)^2 \\
& +a^4\,\Big(\overline{\zeta_c}^2\,\left(2\,\zeta_c\left(\overline{\zeta_c}-\zeta_k\right)^2\zeta_k+\overline{\zeta_c}^2\,\zeta_k{}^2+\zeta_c{}^2\,\left(\overline{\zeta_c}^2+3\,\zeta_k{}^2\right)\right) \\
& +2\,\overline{\zeta_c}\,\left(\zeta_c{}^2\,\overline{\zeta_c}^2+2\,\zeta_c\,\overline{\zeta_c}^2\,\zeta_k+\left(-2\,\zeta_c+\overline{\zeta_c}\right)^2\zeta_k{}^2+4\,\zeta_c\,\zeta_k{}^3+\zeta_k{}^4\right)\overline{\zeta_k} \\
& +\left(\zeta_c{}^2\,\left(\overline{\zeta_c}^2-2\,\zeta_k{}^2\right)+\zeta_k{}^2\,\left(\overline{\zeta_c}^2+\zeta_k{}^2\right)+2\,\zeta_c\,\zeta_k\,\left(\overline{\zeta_c}^2-2\,\overline{\zeta_c}\,\zeta_k+2\,\zeta_k{}^2\right)\right)\overline{\zeta_k}^2\Big) \\
& -2\,a^2\,\zeta_c\,\Big(2\,\overline{\zeta_c}\,\zeta_k{}^4\left(\overline{\zeta_c}+\overline{\zeta_k}\right)^2+\zeta_c{}^3\,\left(\overline{\zeta_c}^4+2\,\overline{\zeta_c}^3\,\overline{\zeta_k}+\left(\overline{\zeta_c}^2-3\,\zeta_k{}^2\right)\overline{\zeta_k}^2\right) \\
& +2\,\zeta_c{}^2\,\zeta_k\,\left(\overline{\zeta_c}^4+2\,\overline{\zeta_c}^3\,\overline{\zeta_k}+\left(\overline{\zeta_c}^2-3\,\zeta_k{}^2\right)\overline{\zeta_k}^2\right)+\zeta_c\,\zeta_k{}^2\,\Big(\overline{\zeta_c}^3\,\left(\overline{\zeta_c}+4\,\zeta_k\right)+2\,\overline{\zeta_c}^2\,\left(\overline{\zeta_c}+4\,\zeta_k\right)\overline{\zeta_k} \\
& +\left(\overline{\zeta_c}^2+4\,\overline{\zeta_c}\,\zeta_k-3\,\zeta_k{}^2\right)\overline{\zeta_k}^2\Big)\Big)\Big)-2\,r_c{}^6\,\overline{\zeta_c}\,\Big(\zeta_c{}^2\,\overline{\zeta_c}\,(\zeta_c+\zeta_k)^2\,\left(2\,\zeta_c\,\overline{\zeta_c}+3\,\zeta_k{}^2\right)\left(\overline{\zeta_c}+\overline{\zeta_k}\right)^2 \\
& +a^4\,\zeta_k{}^2\,\left(-\left(\left(\overline{\zeta_c}-\zeta_k\right)\left(\overline{\zeta_c}+\overline{\zeta_k}\right)^2\right)+2\,\zeta_c\,\overline{\zeta_c}\,\left(\overline{\zeta_c}+2\,\overline{\zeta_k}\right)\right)-a^2\,\Big(\overline{\zeta_c}\,\zeta_k{}^4\left(\overline{\zeta_c}+\overline{\zeta_k}\right)^2 \\
& +2\,\zeta_c{}^3\,\left(\overline{\zeta_c}^2+\overline{\zeta_c}\,\overline{\zeta_k}-\zeta_k\,\overline{\zeta_k}\right)\left(\overline{\zeta_c}^2+\left(\overline{\zeta_c}+\zeta_k\right)\overline{\zeta_k}\right)-2\,\zeta_c{}^2\,\zeta_k\,\Big(-2\,\overline{\zeta_c}^4+\overline{\zeta_c}^3\,\left(\zeta_k-4\,\overline{\zeta_k}\right) \\
& +2\,\overline{\zeta_c}^2\,\left(\zeta_k-\overline{\zeta_k}\right)\overline{\zeta_k}+\overline{\zeta_c}\,\zeta_k\,\overline{\zeta_k}^2+2\,\zeta_k{}^2\,\overline{\zeta_k}^2\Big) \\
& +2\,\zeta_c\,\zeta_k{}^2\,\left(\overline{\zeta_c}^3\,\left(\overline{\zeta_c}+\zeta_k\right)+2\,\overline{\zeta_c}^2\,\left(\overline{\zeta_c}+\zeta_k\right)\overline{\zeta_k}+\left(\overline{\zeta_c}^2+\overline{\zeta_c}\,\zeta_k-\zeta_k{}^2\right)\overline{\zeta_k}^2\right)\Big)\Big) \\
& -r_c{}^2\,\overline{\zeta_c}^3\,\zeta_k{}^2\,\Big(\zeta_c{}^4\,\overline{\zeta_c}\,(\zeta_c+\zeta_k)^2\left(\overline{\zeta_c}+\overline{\zeta_k}\right)^2+a^6\,\overline{\zeta_c}\,\left(-2\,\zeta_c\,\overline{\zeta_k}+\left(\overline{\zeta_c}+\overline{\zeta_k}\right)^2\right) \\
& +a^2\,\zeta_c{}^2\,\Big(4\,\zeta_c{}^3\,\overline{\zeta_k}^2-2\,\overline{\zeta_c}\,\zeta_k{}^2\left(\overline{\zeta_c}+\overline{\zeta_k}\right)^2-\zeta_c{}^2\,\left(\overline{\zeta_c}^3+2\,\overline{\zeta_c}^2\,\overline{\zeta_k}+\left(\overline{\zeta_c}-8\,\zeta_k\right)\overline{\zeta_k}^2\right) \\
& -4\,\zeta_c\,\zeta_k\,\left(\overline{\zeta_c}^3+2\,\overline{\zeta_c}^2\,\overline{\zeta_k}+\left(\overline{\zeta_c}-\zeta_k\right)\overline{\zeta_k}^2\right)\Big)+a^4\,\Big(2\,\zeta_c{}^3\,\left(\overline{\zeta_c}-2\,\overline{\zeta_k}\right)\overline{\zeta_k}+\overline{\zeta_c}\,\zeta_k{}^2\left(\overline{\zeta_c}+\overline{\zeta_k}\right)^2 \\
& -\zeta_c{}^2\,\overline{\zeta_c}\,\left(\overline{\zeta_c}^2+2\,\overline{\zeta_c}\,\overline{\zeta_k}+\overline{\zeta_k}\,\left(-4\,\zeta_k+\overline{\zeta_k}\right)\right)+2\,\zeta_c\,\overline{\zeta_c}\,\zeta_k\,\left(\overline{\zeta_c}^2+2\,\overline{\zeta_c}\,\overline{\zeta_k}+\overline{\zeta_k}\,\left(\zeta_k+\overline{\zeta_k}\right)\right)\Big)\Big)\Big) \\
& /\left(r_c{}^2\left(r_c{}^2+(a-\zeta_c)\,\overline{\zeta_c}\right)^2\left(r_c{}^2-(a+\zeta_c)\,\overline{\zeta_c}\right)^2(a-\zeta_c-\zeta_k)\,\zeta_k{}^2\,(a+\zeta_c+\zeta_k)\,\left(\overline{\zeta_c}+\overline{\zeta_k}\right)^2\right)
\end{aligned}
$$

$$\mathcal{G}_2^k = i\,\pi\,\Big(a^2\,(a-\zeta_c)\,{\zeta_c}^2\,(a+\zeta_c)\,\overline{\zeta_c}^4\,{\zeta_k}^2\,\big(a^2-(\zeta_c+\zeta_k)^2\big)\,\overline{\zeta_k}^2+{r_c}^{12}\,(\zeta_c+\zeta_k)^2\,\big(\overline{\zeta_c}+\overline{\zeta_k}\big)^2$$

$$-{r_c}^{10}\,\big(4\,{\zeta_c}^3\,\overline{\zeta_c}+8\,{\zeta_c}^2\,\overline{\zeta_c}\,\zeta_k-\big(a^2+\zeta_c\,(\zeta_c-4\,\overline{\zeta_c})\big)\,{\zeta_k}^2-2\,\zeta_c\,{\zeta_k}^3-{\zeta_k}^4\big)\,\big(\overline{\zeta_c}+\overline{\zeta_k}\big)^2$$

$$+{r_c}^8\,\Big(2\,\zeta_c\,\overline{\zeta_c}\,(\zeta_c+\zeta_k)^2\,\big(3\,\zeta_c\,\overline{\zeta_c}-2\,{\zeta_k}^2\big)\,\big(\overline{\zeta_c}+\overline{\zeta_k}\big)^2+a^4\,\overline{\zeta_c}\,{\zeta_k}^2\,\big(\overline{\zeta_c}+2\,\overline{\zeta_k}\big)$$

$$-a^2\,\Big(2\,\overline{\zeta_c}^3\,\big({\zeta_c}^2\,\overline{\zeta_c}+\overline{\zeta_c}\,{\zeta_k}^2+2\,\zeta_c\,\zeta_k\,\big(\overline{\zeta_c}+\zeta_k\big)\big)+4\,\overline{\zeta_c}^2\,\big({\zeta_c}^2\,\overline{\zeta_c}+\overline{\zeta_c}\,{\zeta_k}^2+2\,\zeta_c\,\zeta_k\,\big(\overline{\zeta_c}+\zeta_k\big)\big)\,\overline{\zeta_k}$$

$$+\Big(2\,{\zeta_c}^2\,\overline{\zeta_c}^2+4\,\zeta_c\,\overline{\zeta_c}^2\,\zeta_k-\Big({\zeta_c}^2-4\,\zeta_c\,\overline{\zeta_c}-2\,\overline{\zeta_c}^2\Big)\,{\zeta_k}^2-2\,\zeta_c\,{\zeta_k}^3-{\zeta_k}^4\Big)\,\overline{\zeta_k}^2\Big)\Big)$$

$$+{r_c}^4\,\overline{\zeta_c}^2\,\Big(a^6\,\overline{\zeta_c}^2\,{\zeta_k}^2+{\zeta_c}^3\,\overline{\zeta_c}\,(\zeta_c+\zeta_k)^2\,\big(\zeta_c\,\overline{\zeta_c}-4\,{\zeta_k}^2\big)\,\big(\overline{\zeta_c}+\overline{\zeta_k}\big)^2$$

$$+a^4\,\Big(\overline{\zeta_c}^2\,\Big(\overline{\zeta_c}^2\,{\zeta_k}^2+2\,\zeta_c\,\zeta_k\,\big(\overline{\zeta_c}+\zeta_k\big)^2+{\zeta_c}^2\,\Big(\overline{\zeta_c}^2+3\,{\zeta_k}^2\Big)\Big)$$

$$+2\,\overline{\zeta_c}\,\Big({\zeta_c}^2\,\overline{\zeta_c}^2+2\,\zeta_c\,\overline{\zeta_c}^2\,\zeta_k+\big(2\,\zeta_c+\overline{\zeta_c}\big)^2\,{\zeta_k}^2+4\,\zeta_c\,{\zeta_k}^3+{\zeta_k}^4\Big)\,\overline{\zeta_k}$$

$$+\Big({\zeta_c}^2\,\Big(\overline{\zeta_c}^2-2\,{\zeta_k}^2\Big)+{\zeta_k}^2\,\Big(\overline{\zeta_c}^2+{\zeta_k}^2\Big)+2\,\zeta_c\,\zeta_k\,\Big(\overline{\zeta_c}^2+2\,\overline{\zeta_c}\,\zeta_k+2\,{\zeta_k}^2\Big)\Big)\,\overline{\zeta_k}^2\Big)$$

$$-2\,a^2\,\zeta_c\,\Big(-2\,\overline{\zeta_c}\,{\zeta_k}^4\,\big(\overline{\zeta_c}+\overline{\zeta_k}\big)^2+{\zeta_c}^3\,\Big(\overline{\zeta_c}^4+2\,\overline{\zeta_c}^3\,\overline{\zeta_k}+\Big(\overline{\zeta_c}^2-3\,{\zeta_k}^2\Big)\,\overline{\zeta_k}^2\Big)$$

$$+2\,{\zeta_c}^2\,\zeta_k\,\Big(\overline{\zeta_c}^4+2\,\overline{\zeta_c}^3\,\overline{\zeta_k}+\Big(\overline{\zeta_c}^2-3\,{\zeta_k}^2\Big)\,\overline{\zeta_k}^2\Big)+\zeta_c\,{\zeta_k}^2\,\Big(\overline{\zeta_c}^3\,\big(\overline{\zeta_c}-4\,\zeta_k\big)+2\,\overline{\zeta_c}^2\,\big(\overline{\zeta_c}-4\,\zeta_k\big)\,\overline{\zeta_k}$$

$$+\Big(\overline{\zeta_c}^2-4\,\overline{\zeta_c}\,\zeta_k-3\,{\zeta_k}^2\Big)\,\overline{\zeta_k}^2\Big)\,\Big)\Big)+{r_c}^2\,\overline{\zeta_c}^3\,{\zeta_k}^2\,\Big({\zeta_c}^4\,\overline{\zeta_c}\,(\zeta_c+\zeta_k)^2\,\big(\overline{\zeta_c}+\overline{\zeta_k}\big)^2$$

$$+a^6\,\overline{\zeta_c}\,\Big(2\,\zeta_c\,\overline{\zeta_k}+\big(\overline{\zeta_c}+\overline{\zeta_k}\big)^2\Big)-a^2\,{\zeta_c}^2\,\Big(4\,{\zeta_c}^3\,\overline{\zeta_k}^2+2\,\overline{\zeta_c}\,{\zeta_k}^2\,\big(\overline{\zeta_c}+\overline{\zeta_k}\big)^2$$

$$+4\,\zeta_c\,\zeta_k\,\Big(\overline{\zeta_c}^3+2\,\overline{\zeta_c}^2\,\overline{\zeta_k}+\big(\overline{\zeta_c}+\zeta_k\big)\,\overline{\zeta_k}^2\Big)+{\zeta_c}^2\,\Big(\overline{\zeta_c}^3+2\,\overline{\zeta_c}^2\,\overline{\zeta_k}+\big(\overline{\zeta_c}+8\,\zeta_k\big)\,\overline{\zeta_k}^2\Big)\Big)$$

$$+a^4\,\Big(-2\,{\zeta_c}^3\,\big(\overline{\zeta_c}-2\,\overline{\zeta_k}\big)\,\overline{\zeta_k}+\overline{\zeta_c}\,{\zeta_k}^2\,\big(\overline{\zeta_c}+\overline{\zeta_k}\big)^2+2\,\zeta_c\,\overline{\zeta_c}\,\zeta_k\,\Big(\overline{\zeta_c}^2+2\,\overline{\zeta_c}\,\overline{\zeta_k}+\overline{\zeta_k}\,\big(-\zeta_k+\overline{\zeta_k}\big)\,\Big)$$

$$-{\zeta_c}^2\,\overline{\zeta_c}\,\Big(\overline{\zeta_c}^2+2\,\overline{\zeta_c}\,\overline{\zeta_k}+\overline{\zeta_k}\,\big(4\,\zeta_k+\overline{\zeta_k}\big)\Big)\Big)\Big)\Big)-2\,{r_c}^6\,\overline{\zeta_c}\,\Big({\zeta_c}^2\,\overline{\zeta_c}\,(\zeta_c+\zeta_k)^2\,\big(2\,\zeta_c\,\overline{\zeta_c}-3\,{\zeta_k}^2\big)\,\big(\overline{\zeta_c}+\overline{\zeta_k}\big)^2$$

$$+a^4\,{\zeta_k}^2\,\Big(\big(\overline{\zeta_c}+\zeta_k\big)\,\big(\overline{\zeta_c}+\overline{\zeta_k}\big)^2+2\,\zeta_c\,\overline{\zeta_c}\,\big(\overline{\zeta_c}+2\,\overline{\zeta_k}\big)\Big)+a^2\,\Big(\overline{\zeta_c}\,{\zeta_k}^4\,\big(\overline{\zeta_c}+\overline{\zeta_k}\big)^2$$

$$-2\,{\zeta_c}^3\,\Big(\overline{\zeta_c}^2+\overline{\zeta_c}\,\overline{\zeta_k}-\zeta_k\,\overline{\zeta_k}\Big)\,\Big(\overline{\zeta_c}^2+\big(\overline{\zeta_c}+\zeta_k\big)\,\overline{\zeta_k}\Big)+2\,\zeta_c\,{\zeta_k}^2\,\Big(-\overline{\zeta_c}^4+\overline{\zeta_c}^3\,\big(\zeta_k-2\,\overline{\zeta_k}\big)$$

$$+\overline{\zeta_c}^2\,\big(2\,\zeta_k-\overline{\zeta_k}\big)\,\overline{\zeta_k}+\overline{\zeta_c}\,\zeta_k\,\overline{\zeta_k}^2+{\zeta_k}^2\,\overline{\zeta_k}^2\Big)$$

$$-2\,{\zeta_c}^2\,\zeta_k\,\Big(2\,\overline{\zeta_c}^4+\overline{\zeta_c}\,\zeta_k\,\overline{\zeta_k}^2-2\,{\zeta_k}^2\,\overline{\zeta_k}^2+2\,\overline{\zeta_c}^2\,\overline{\zeta_k}\,\big(\zeta_k+\overline{\zeta_k}\big)+\overline{\zeta_c}^3\,\big(\zeta_k+4\,\overline{\zeta_k}\big)\Big)\Big)\Big)\Big)\Big)$$

$$/\Big({r_c}^2\,\big({r_c}^2+(a-\zeta_c)\,\overline{\zeta_c}\big)^2\,\big({r_c}^2-(a+\zeta_c)\,\overline{\zeta_c}\big)^2\,(a-\zeta_c-\zeta_k)\,{\zeta_k}^2\,(a+\zeta_c+\zeta_k)\,\big(\overline{\zeta_c}+\overline{\zeta_k}\big)^2\Big)$$

$$\mathcal{G}_3^k = i\,\pi\,\Big(-\Big({r_c}^2\,({r_c}^2-\zeta_c\,\overline{\zeta_c})^5\,(\zeta_c+\zeta_k)^2\,({r_c}^2\,\zeta_c+\overline{\zeta_c}\,{\zeta_k}^2)\,(\overline{\zeta_c}+\overline{\zeta_k})^2\Big)$$

$$+a^8\,\overline{\zeta_c}^4\,{\zeta_k}^2\,\Big({r_c}^4\,\overline{\zeta_c}+{\zeta_c}^2\,\overline{\zeta_c}\,\overline{\zeta_k}^2+{r_c}^2\,\zeta_c\,\big(\overline{\zeta_c}^2+4\,\overline{\zeta_c}\,\overline{\zeta_k}+\overline{\zeta_k}^2\big)\Big)$$

$$+a^6\,\overline{\zeta_c}^2\,({r_c}^2-\zeta_c\,\overline{\zeta_c})\,\Big(\zeta_c\,\overline{\zeta_c}^2\,{\zeta_k}^2\,(\zeta_c+\zeta_k)^2\,\overline{\zeta_k}^2+2\,{r_c}^6\,{\zeta_k}^2\,(\overline{\zeta_c}+\overline{\zeta_k})$$

$$-{r_c}^4\,(\overline{\zeta_c}+\overline{\zeta_k})\,\Big(2\,\zeta_k\,\big(\overline{\zeta_c}^2+{\zeta_k}^2\big)\,(\overline{\zeta_c}+\overline{\zeta_k})+\zeta_c\,\big(\overline{\zeta_c}^3+6\,\overline{\zeta_c}\,{\zeta_k}^2+\overline{\zeta_c}^2\,\overline{\zeta_k}+2\,{\zeta_k}^2\,\overline{\zeta_k}\big)\Big)$$

$$-{r_c}^2\,\overline{\zeta_c}^2\,{\zeta_k}^2\,\Big(2\,\zeta_c\,\overline{\zeta_c}\,\zeta_k-{\zeta_c}^2\,(\overline{\zeta_c}+2\,\overline{\zeta_k})+\overline{\zeta_c}\,\big({\zeta_k}^2+(\overline{\zeta_c}+\overline{\zeta_k})^2\big)\Big)\Big)$$

$$+a^4\,({r_c}^2-\zeta_c\,\overline{\zeta_c})\,\Big({\zeta_c}^5\,\overline{\zeta_c}^4\,{\zeta_k}^2\,\overline{\zeta_k}^2+{r_c}^{10}\,{\zeta_k}^2\,(\overline{\zeta_c}+2\,\overline{\zeta_k})+{r_c}^2\,\overline{\zeta_c}^3\,{\zeta_k}^2\,\Big(-2\,\zeta_c\,\overline{\zeta_c}^2\,\zeta_k\,(\overline{\zeta_c}+\overline{\zeta_k})^2$$

$$-\overline{\zeta_c}^2\,{\zeta_k}^2\,(\overline{\zeta_c}+\overline{\zeta_k})^2+4\,{\zeta_c}^3\,\zeta_k\,(\overline{\zeta_c}+\overline{\zeta_k})\,(\overline{\zeta_c}+2\,\overline{\zeta_k})+{\zeta_c}^2\,(\overline{\zeta_c}+\overline{\zeta_k})\,\big(\overline{\zeta_c}^3+2\,\overline{\zeta_c}\,{\zeta_k}^2+\overline{\zeta_c}^2\,\overline{\zeta_k}+4\,{\zeta_k}^2\,\overline{\zeta_k}\big)$$

$$+{\zeta_c}^4\,\big(\overline{\zeta_c}^2+4\,\overline{\zeta_c}\,\overline{\zeta_k}-2\,\overline{\zeta_k}^2\big)\Big)+{r_c}^8\,\Big(-2\,\zeta_k\,\big(-2\,\overline{\zeta_c}^2+{\zeta_k}^2\big)\,(\overline{\zeta_c}+\overline{\zeta_k})^2+\zeta_c\,\Big(2\,\overline{\zeta_c}^2\,\big(\overline{\zeta_c}^2-2\,{\zeta_k}^2\big)$$

$$+4\,\overline{\zeta_c}\,\big(\overline{\zeta_c}^2-2\,{\zeta_k}^2\big)\,\overline{\zeta_k}+\big(2\,\overline{\zeta_c}^2+{\zeta_k}^2\big)\,\overline{\zeta_k}^2\Big)\Big)+{r_c}^4\,\zeta_c\,\overline{\zeta_c}^2\,\Big(2\,\zeta_c\,\zeta_k\,(\overline{\zeta_c}+\overline{\zeta_k})\,\Big(\overline{\zeta_c}^3-5\,\overline{\zeta_c}\,{\zeta_k}^2+\overline{\zeta_c}^2\,\overline{\zeta_k}$$

$$-9\,{\zeta_k}^2\,\overline{\zeta_k}\Big)-{\zeta_k}^2\,(\overline{\zeta_c}+\overline{\zeta_k})\,\big(5\,\overline{\zeta_c}^3+4\,\overline{\zeta_c}\,{\zeta_k}^2+5\,\overline{\zeta_c}^2\,\overline{\zeta_k}+8\,{\zeta_k}^2\,\overline{\zeta_k}\big)$$

$$+{\zeta_c}^2\,\Big(\overline{\zeta_c}^4-4\,\overline{\zeta_c}^2\,{\zeta_k}^2+2\,\overline{\zeta_c}\,\big(\overline{\zeta_c}^2-6\,{\zeta_k}^2\big)\,\overline{\zeta_k}+\big(\overline{\zeta_c}^2+2\,{\zeta_k}^2\big)\,\overline{\zeta_k}^2\Big)\Big)$$

$$+2\,{r_c}^6\,\overline{\zeta_c}\,\Big({\zeta_k}^2\,(\overline{\zeta_c}+\overline{\zeta_k})\,\big(\overline{\zeta_c}^3+\overline{\zeta_c}\,{\zeta_k}^2+\overline{\zeta_c}^2\,\overline{\zeta_k}+2\,{\zeta_k}^2\,\overline{\zeta_k}\big)$$

$$+2\,\zeta_c\,\zeta_k\,(\overline{\zeta_c}+\overline{\zeta_k})\,\big(-2\,\overline{\zeta_c}^3+2\,\overline{\zeta_c}\,{\zeta_k}^2-2\,\overline{\zeta_c}^2\,\overline{\zeta_k}+3\,{\zeta_k}^2\,\overline{\zeta_k}\big)$$

$$-{\zeta_c}^2\,\Big(2\,\overline{\zeta_c}^4+4\,\overline{\zeta_c}^3\,\overline{\zeta_k}-7\,\overline{\zeta_c}\,{\zeta_k}^2\,\overline{\zeta_k}+{\zeta_k}^2\,\overline{\zeta_k}^2+\overline{\zeta_c}^2\,\big(-3\,{\zeta_k}^2+2\,\overline{\zeta_k}^2\big)\Big)\Big)\Big)\Big)$$

$$-a^2\,({r_c}^2-\zeta_c\,\overline{\zeta_c})^3\,\Big({\zeta_c}^3\,\overline{\zeta_c}^2\,{\zeta_k}^2\,(\zeta_c+\zeta_k)^2\,\overline{\zeta_k}^2+{r_c}^8\,(\zeta_c+2\,\zeta_k)\,(\overline{\zeta_c}+\overline{\zeta_k})^2$$

$$-{r_c}^2\,\overline{\zeta_c}\,{\zeta_k}^2\,\Big(4\,\zeta_c\,\overline{\zeta_c}^2\,\zeta_k\,(\overline{\zeta_c}+\overline{\zeta_k})^2+2\,\overline{\zeta_c}^2\,{\zeta_k}^2\,(\overline{\zeta_c}+\overline{\zeta_k})^2-{\zeta_c}^4\,\big(\overline{\zeta_c}^2+2\,\overline{\zeta_c}\,\overline{\zeta_k}-2\,\overline{\zeta_k}^2\big)$$

$$-2\,{\zeta_c}^3\,\zeta_k\,\big(\overline{\zeta_c}^2+2\,\overline{\zeta_c}\,\overline{\zeta_k}-2\,\overline{\zeta_k}^2\big)+{\zeta_c}^2\,\Big(\overline{\zeta_c}^4+2\,\overline{\zeta_c}^3\,\overline{\zeta_k}-2\,\overline{\zeta_c}\,{\zeta_k}^2\,\overline{\zeta_k}+2\,{\zeta_k}^2\,\overline{\zeta_k}^2+\overline{\zeta_c}^2\,\big(-{\zeta_k}^2+\overline{\zeta_k}^2\big)\Big)\Big)$$

$$+{r_c}^6\,\Big({\zeta_c}^2\,\Big(-2\,\overline{\zeta_c}^3-4\,\overline{\zeta_c}^2\,\overline{\zeta_k}+2\,{\zeta_k}^2\,\overline{\zeta_k}+\overline{\zeta_c}\,\big({\zeta_k}^2-2\,\overline{\zeta_k}^2\big)\Big)+2\,\zeta_c\,\zeta_k\,\Big(-2\,\overline{\zeta_c}^3-4\,\overline{\zeta_c}^2\,\overline{\zeta_k}+2\,{\zeta_k}^2\,\overline{\zeta_k}$$

$$+\overline{\zeta_c}\,\big({\zeta_k}^2-2\,\overline{\zeta_k}^2\big)\Big)+{\zeta_k}^2\,\Big(\overline{\zeta_c}^3+2\,\overline{\zeta_c}^2\,\overline{\zeta_k}+2\,{\zeta_k}^2\,\overline{\zeta_k}+\overline{\zeta_c}\,\big({\zeta_k}^2+\overline{\zeta_k}^2\big)\Big)\Big)$$

$$-{r_c}^4\,\zeta_c\,\Big({\zeta_c}^2\,\Big(\overline{\zeta_c}^4+2\,\overline{\zeta_c}^3\,\overline{\zeta_k}+4\,\overline{\zeta_c}\,{\zeta_k}^2\,\overline{\zeta_k}-{\zeta_k}^2\,\overline{\zeta_k}^2+\overline{\zeta_c}^2\,\big(2\,{\zeta_k}^2+\overline{\zeta_k}^2\big)\Big)$$

$$+2\,\zeta_c\,\zeta_k\,\Big(\overline{\zeta_c}^4+2\,\overline{\zeta_c}^3\,\overline{\zeta_k}+4\,\overline{\zeta_c}\,{\zeta_k}^2\,\overline{\zeta_k}-{\zeta_k}^2\,\overline{\zeta_k}^2+\overline{\zeta_c}^2\,\big(2\,{\zeta_k}^2+\overline{\zeta_k}^2\big)\Big)$$

$$+{\zeta_k}^2\,\Big(4\,\overline{\zeta_c}^4+8\,\overline{\zeta_c}^3\,\overline{\zeta_k}+4\,\overline{\zeta_c}\,{\zeta_k}^2\,\overline{\zeta_k}-{\zeta_k}^2\,\overline{\zeta_k}^2+2\,\overline{\zeta_c}^2\,\big({\zeta_k}^2+2\,\overline{\zeta_k}^2\big)\Big)\Big)\Big)\Big)$$

$$/\Big({r_c}^2\,\big({r_c}^2+(a-\zeta_c)\,\overline{\zeta_c}\big)^2\,({r_c}^2-\zeta_c\,\overline{\zeta_c})\,\big({r_c}^2-(a+\zeta_c)\,\overline{\zeta_c}\big)^2\,{\zeta_k}^2\,\big((\zeta_c+\zeta_k)^2-a^2\big)\,(\overline{\zeta_c}+\overline{\zeta_k})^2\Big)$$

$$\mathcal{G}_4^k = \pi\,\Big\{-\Big({r_c}^2\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^3\,(r_c-\zeta_k)\,(r_c+\zeta_k)\,(\zeta_c+\zeta_k)^2\left(\overline{\zeta_c}+\overline{\zeta_k}\right)^2\Big)$$
$$+a^2\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^3\,\Big(-\Big({\zeta_k}^2\,(\zeta_c+\zeta_k)^2\,\overline{\zeta_k}^2\Big)+{r_c}^4\left(\overline{\zeta_c}+\overline{\zeta_k}\right)^2$$
$$-{r_c}^2\,{\zeta_k}^2\,\left(\zeta_c-\overline{\zeta_c}+\zeta_k-\overline{\zeta_k}\right)\,\left(\zeta_c+\overline{\zeta_c}+\zeta_k+\overline{\zeta_k}\right)\Big)+a^4\,\zeta_k\,\Big({r_c}^8\,\zeta_k-{\zeta_c}^3\,\overline{\zeta_c}^3\,\zeta_k\,\overline{\zeta_k}^2$$
$$-{r_c}^6\,\zeta_k\,\Big(3\,\zeta_c\,\overline{\zeta_c}+2\,\overline{\zeta_c}^2+4\,\overline{\zeta_c}\,\overline{\zeta_k}+\overline{\zeta_k}^2\Big)+{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}\,\zeta_k\,\Big(3\,\overline{\zeta_c}\,\overline{\zeta_k}^2+\zeta_c\,\Big(\overline{\zeta_c}^2+4\,\overline{\zeta_c}\,\overline{\zeta_k}+2\,\overline{\zeta_k}^2\Big)\Big)$$
$$+{r_c}^4\,\Big(\zeta_c\,\Big(\zeta_c\,\Big(-4\,\overline{\zeta_c}^3+\overline{\zeta_c}^2\,\left(5\,\zeta_k-8\,\overline{\zeta_k}\right)+4\,\overline{\zeta_c}\,\left(\zeta_k-\overline{\zeta_k}\right)\,\overline{\zeta_k}+2\,\zeta_k\,\overline{\zeta_k}^2\,\Big)$$
$$+\zeta_k\,\Big(-2\,\overline{\zeta_c}^3+4\,\overline{\zeta_c}^2\,\left(\zeta_k-\overline{\zeta_k}\right)+\overline{\zeta_c}\,\left(8\,\zeta_k-5\,\overline{\zeta_k}\right)\,\overline{\zeta_k}+4\,\zeta_k\,\overline{\zeta_k}^2\,\Big)\Big)-4\,i\,(\zeta_c+\zeta_k)^2\left(\overline{\zeta_c}+\overline{\zeta_k}\right)^2\zeta_y\Big)\Big)\Big\}$$
$$/\Big({r_c}^2\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^3{\zeta_k}^2\,(-a+\zeta_c+\zeta_k)\,(a+\zeta_c+\zeta_k)\,\left(\overline{\zeta_c}+\overline{\zeta_k}\right)^2\Big)$$

$$\mathcal{G}_5^k = -\Big\{i\,\pi\,\Big({r_c}^2\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^3(\zeta_c+\zeta_k)^2\,\left({r_c}^2+{\zeta_k}^2\right)\,\left(\overline{\zeta_c}+\overline{\zeta_k}\right)^2$$
$$+a^2\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^3\,\Big({\zeta_k}^2\,(\zeta_c+\zeta_k)^2\,\overline{\zeta_k}^2-{r_c}^4\left(\overline{\zeta_c}+\overline{\zeta_k}\right)^2$$
$$-{r_c}^2\,{\zeta_k}^2\,\left(\zeta_c-\overline{\zeta_c}+\zeta_k-\overline{\zeta_k}\right)\,\left(\zeta_c+\overline{\zeta_c}+\zeta_k+\overline{\zeta_k}\right)\Big)+a^4\,\zeta_k\,\Big({r_c}^8\,\zeta_k+{\zeta_c}^3\,\overline{\zeta_c}^3\,\zeta_k\,\overline{\zeta_k}^2$$
$$+{r_c}^6\,\zeta_k\,\Big(-3\,\zeta_c\,\overline{\zeta_c}+2\,\overline{\zeta_c}^2+4\,\overline{\zeta_c}\,\overline{\zeta_k}+\overline{\zeta_k}^2\Big)+{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}\,\zeta_k\,\Big(\zeta_c\,\overline{\zeta_c}^2+4\,\zeta_c\,\overline{\zeta_c}\,\overline{\zeta_k}+\left(2\,\zeta_c-3\,\overline{\zeta_c}\right)\,\overline{\zeta_k}^2\Big)$$
$$+{r_c}^4\,\zeta_c\,\Big(\overline{\zeta_c}^2\,\left(4\,\zeta_c\,\overline{\zeta_c}+5\,\zeta_c\,\zeta_k+2\,\overline{\zeta_c}\,\zeta_k+4\,{\zeta_k}^2\right)+4\,\overline{\zeta_c}\,\left(2\,\zeta_c\,\overline{\zeta_c}+\left(\zeta_c+\overline{\zeta_c}\right)\,\zeta_k+2\,{\zeta_k}^2\right)\,\overline{\zeta_k}$$
$$+\left(4\,\zeta_c\,\overline{\zeta_c}+2\,\zeta_c\,\zeta_k+5\,\overline{\zeta_c}\,\zeta_k+4\,{\zeta_k}^2\right)\,\overline{\zeta_k}^2\Big)\Big)\Big)\Big\}$$
$$/\Big({r_c}^2\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^3{\zeta_k}^2\,(-a+\zeta_c+\zeta_k)\,(a+\zeta_c+\zeta_k)\,\left(\overline{\zeta_c}+\overline{\zeta_k}\right)^2\Big)$$

$$\mathcal{G}_6^k = 2\,a\,\pi\,\Big\{\Big[{r_c}^4\,\zeta_c\left({r_c}^2+(a-\zeta_c)\,\overline{\zeta_c}\right)^2\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^2\left({r_c}^2-(a+\zeta_c)\,\overline{\zeta_c}\right)^2$$
$$+{r_c}^4\left({r_c}^2+(a-\zeta_c)\,\overline{\zeta_c}\right)^2\left({r_c}^2-(a+\zeta_c)\,\overline{\zeta_c}\right)^2\,\Big(2\,a^2\,{\zeta_c}^2+\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^2\Big)\,\zeta_k$$
$$+a^2\,\zeta_c\left({r_c}^2+(a-\zeta_c)\,\overline{\zeta_c}\,\right)^2\,\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)\,\left(3\,{r_c}^2-\zeta_c\,\overline{\zeta_c}\right)\,\left({r_c}^2-(a+\zeta_c)\,\overline{\zeta_c}\right)^2{\zeta_k}^2$$
$$-a^2\,\Big({r_c}^{12}-{r_c}^8\,\left(5\,a^2+12\,{\zeta_c}^2\right)\,\overline{\zeta_c}^2+2\,{r_c}^6\,\zeta_c\,\left(5\,a^2+14\,{\zeta_c}^2\right)\,\overline{\zeta_c}^3$$
$$+{r_c}^4\,\left(2\,a^2-9\,{\zeta_c}^2\right)\,\left(a^2+3\,{\zeta_c}^2\right)\,\overline{\zeta_c}^4-4\,{r_c}^2\,{\zeta_c}^3\,\left(a^2-3\,{\zeta_c}^2\right)\,\overline{\zeta_c}^5+2\,\left(a-\zeta_c\right)\,{\zeta_c}^4\,\left(a+\zeta_c\right)\,\overline{\zeta_c}^6\Big)\,{\zeta_k}^3$$
$$-a^2\,\overline{\zeta_c}\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^2\,\Big(2\,{r_c}^6-5\,{r_c}^4\,\zeta_c\,\overline{\zeta_c}+4\,{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}^2+(a-\zeta_c)\,\zeta_c\,\left(a+\zeta_c\right)\,\overline{\zeta_c}^3\,\Big)\,{\zeta_k}^4\Big]$$
$$/\left[(a-\zeta_c-\zeta_k)\,{\zeta_k}^2\,\left(a+\zeta_c+\zeta_k\right)\right]+\frac{a^2\,\overline{\zeta_c}^3\,\left({r_c}^2+(a-\zeta_c)\,\overline{\zeta_c}\right)\,\left(-{r_c}^2+\zeta_c\,\overline{\zeta_c}\right)^3\,\left(-{r_c}^2+(a+\zeta_c)\,\overline{\zeta_c}\right)}{\left(\overline{\zeta_c}+\overline{\zeta_k}\right)^2}$$
$$+\frac{\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^3}{\overline{\zeta_c}+\overline{\zeta_k}}\,\Big({r_c}^8-3\,{r_c}^6\,\zeta_c\,\overline{\zeta_c}+{r_c}^4\,\left(-5\,a^2+3\,{\zeta_c}^2\right)\,\overline{\zeta_c}^2-{r_c}^2\,\zeta_c\,\left(-7\,a^2+{\zeta_c}^2\right)\,\overline{\zeta_c}^3$$
$$+2\,a^2\,\left(a-\zeta_c\right)\,\left(a+\zeta_c\right)\,\overline{\zeta_c}^4\Big)\Big\}\,/\,\Big({r_c}^2\left({r_c}^2+(a-\zeta_c)\,\overline{\zeta_c}\right)^2\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^2\left({r_c}^2-(a+\zeta_c)\,\overline{\zeta_c}\right)^2\Big)$$

$$\mathcal{G}_7^k = \frac{4\,\pi\,r_c}{\overline{\zeta_k}}$$
$$\mathcal{G}_8^k = \frac{-2\,a^2\,\pi}{\left(\zeta_c+\zeta_k\right)^2}$$
$$\mathcal{G}_9^k = \frac{-2\,\pi\,{r_c}^2}{\overline{\zeta_k}^2}$$
$$\mathcal{G}_{10}^k = \frac{-2\,i\,\pi\left(\zeta_c+\zeta_k\right)^2}{\zeta_k\,\left(-a+\zeta_c+\zeta_k\right)\,\left(a+\zeta_c+\zeta_k\right)}$$
$$\mathcal{G}_{11}^k = i\,\pi\,\Big\{\left(\zeta_c+\overline{\zeta_j}\right)^2\left(\zeta_c+\zeta_k\right)^2\,\left(\zeta_j\,\left(\overline{\zeta_j}+\zeta_k\right)\,\overline{\zeta_k}-{r_c}^2\,\left(\zeta_j+\overline{\zeta_k}\right)\right)$$
$$+a^2\,\left({r_c}^4\,\left(2\,\zeta_c+\overline{\zeta_j}+\zeta_k\right)-\zeta_c\,\zeta_j\,\left(2\,\overline{\zeta_j}\,\zeta_k+\zeta_c\,\left(\overline{\zeta_j}+\zeta_k\right)\right)\,\overline{\zeta_k}+{r_c}^2\,\left({\zeta_c}^2-\overline{\zeta_j}\,\zeta_k\right)\,\left(\zeta_j+\overline{\zeta_k}\right)\right)\Big\}$$
$$/\left(\left(a-\zeta_c-\overline{\zeta_j}\right)\,\left(a+\zeta_c+\overline{\zeta_j}\right)\,\left(a-\zeta_c-\zeta_k\right)\,\left(a+\zeta_c+\zeta_k\right)\,\left(-{r_c}^2+\zeta_j\,\zeta_k\right)\,\left(-{r_c}^2+\overline{\zeta_j}\,\overline{\zeta_k}\right)\right)$$
$$\mathcal{G}_{12}^k = \frac{2\,i\,\pi\,\left(\zeta_c+\zeta_k\right)\,\left(\left(\zeta_c+\zeta_k\right)^3\overline{\zeta_k}-a^2\,\left({r_c}^2+\zeta_c\,\overline{\zeta_k}\right)\right)}{\left(-a+\zeta_c+\zeta_k\right)^2\left(a+\zeta_c+\zeta_k\right)^2\,\left(-{r_c}^2+\zeta_k\,\overline{\zeta_k}\right)}$$

$$
\begin{aligned}
\mathcal{H}_1 &= 2\pi(2a - \zeta_c) \\
\mathcal{H}_2^k &= \frac{2\,\pi\,\left(r_c{}^2 + \overline{\zeta_k}\,\left(-2\,a - \zeta_k + \left(\zeta + \zeta_c + \frac{a^2}{(\zeta+\zeta_c)^2}\right)_k\right)\right)}{\overline{\zeta_k}}
\end{aligned}
$$

Appendix D Coefficients in Equation (3.160)

$$\mathcal{J}_1 = \mathsf{Re}\left\{\frac{-i\pi}{(r_c^2-\delta^2)^3}\left[a^6\,{r_c}^2\,\left(\zeta_c+2\,\overline{\zeta_c}\right)+2\,a^2\,\zeta_c\left({r_c}^2-\delta^2\right)^3-{r_c}^2\,\zeta_c\left({r_c}^2-\delta^2\right)^3\right.\right. \quad \text{(D.1)}$$

$$\left.\left.+a^4\,\left(-2\,{\zeta_c}^2\,\overline{\zeta_c}^3-{r_c}^4\,\left(\zeta_c+3\,\overline{\zeta_c}\right)+{r_c}^2\,\delta^2\,\left(\zeta_c+5\,\overline{\zeta_c}\right)\right)\right]\right\}$$

$$= \pi\zeta_y\left[(2\,a^2-r_c^2)+2\frac{a^4}{(r_c^2-\delta^2)}-\frac{a^6\,{r_c}^2}{(r_c^2-\delta^2)^3}\right] \quad \text{(D.2)}$$

$$\mathcal{J}_2 = \mathsf{Re}\left\{\frac{\pi}{(r_c^2-\delta^2)^3}\left[a^6\,{r_c}^2\,\left(\zeta_c-2\,\overline{\zeta_c}\right)+{r_c}^2\,\zeta_c\left({r_c}^2-\delta^2\right)^3-2\,a^2\,\zeta_c\left(-{r_c}^2+\delta^2\right)^3\right.\right. \quad \text{(D.3)}$$

$$\left.\left.-a^4\,\left(-\left({r_c}^4\,\left(\zeta_c-3\,\overline{\zeta_c}\right)\right)+{r_c}^2\,\zeta_c\,\left(\zeta_c-5\,\overline{\zeta_c}\right)\,\overline{\zeta_c}+2\,{\zeta_c}^2\,\overline{\zeta_c}^3\right)\right]\right\}$$

$$= \pi\zeta_x\left(2\,a^2+{r_c}^2-\frac{2\,a^4}{(r_c^2-\delta^2)}-\frac{a^6\,{r_c}^2}{(r_c^2-\delta^2)^3}\right) \quad \text{(D.4)}$$

$$\mathcal{J}_3 = \pi\mathsf{Re}\left\{2a^2\zeta_c^2+r_c^2\delta^2+\frac{2a^4(r_c^2-2\delta^2)}{(r_c^2-\delta^2)}+\frac{2a^6\overline{\zeta_c}^2}{(r_c^2-\delta^2)^2}+\frac{a^8r_c^2\delta^2}{(rc^2-\delta^2)^4}\right\} \quad \text{(D.5)}$$

$$\mathcal{K}_1 = \mathsf{Re}\left\{\mathcal{K}_{1a}\log(\frac{r_c^2+a\zeta_c-\delta^2}{r_c^2})+\mathcal{K}_{1b}\log(\frac{r_c^2-\delta^2}{r_c^2})\right.$$

$$\left.+\mathcal{K}_{1c}\log(\frac{a}{\overline{\zeta_c}})+\frac{\mathcal{K}_{1d}}{(r_c^2+a\overline{\zeta_c}-\delta^2)(r_c^2-\delta^2)^5}-\frac{\mathcal{K}_{1e}}{(r_c^2-\delta^2)^5}\right\} \quad \text{(D.6)}$$

$$\mathcal{K}_{1a} = \frac{-2\,i\,a^6\,\pi\,{r_c}^4\,\left({r_c}^4\,\zeta_c-a^2\,{\zeta_c}^3-{r_c}^4\,\overline{\zeta_c}+a^2\,{\zeta_c}^2\,\overline{\zeta_c}-2\,{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}+2\,{r_c}^2\,\zeta_c\,\overline{\zeta_c}^2+{\zeta_c}^3\,\overline{\zeta_c}^2-{\zeta_c}^2\,\overline{\zeta_c}^3\right)}{{\zeta_c}^2\,\left({r_c}^2-\delta^2\right)^5}$$

$$+\frac{4\,a^6\,\pi\,{r_c}^4\,\left(-{r_c}^4+a^2\,{\zeta_c}^2+2\,{r_c}^2\,\delta^2-{\zeta_c}^2\,\overline{\zeta_c}^2\right)\,\zeta_y}{{\zeta_c}^2\,\left({r_c}^2-\delta^2\right)^5} \quad \text{(D.7)}$$

$$\mathcal{K}_{1b} = \frac{2\,i\,\left(-\left(a^4\,\pi\,{r_c}^4\right)+a^6\,\pi\,{\zeta_c}^2\right)}{{\zeta_c}^2\,{\overline{\zeta_c}}^3} + \frac{2\,i\,\left(-\left(a^4\,\pi\,{r_c}^6\right)+a^6\,\pi\,{\zeta_c}^4\right)}{{r_c}^2\,{\zeta_c}^3\,{\overline{\zeta_c}}^2} - \frac{2\,i\,a^6\,\pi}{{r_c}^2\,\overline{\zeta_c}}$$
$$-\frac{4\,i\,\left(a^8\,\pi\,{r_c}^6+a^8\,\pi\,{r_c}^4\,{\zeta_c}^2\right)}{\zeta_c\left(-{r_c}^2+\delta^2\right)^5} - \frac{4\,i\,a^8\,\pi\,{r_c}^4}{\zeta_c\left(-{r_c}^2+\delta^2\right)^4} + \frac{2\,i\,\left(a^6\,\pi\,{r_c}^6+a^6\,\pi\,{r_c}^4\,{\zeta_c}^2+a^6\,\pi\,{r_c}^2\,{\zeta_c}^4+a^6\,\pi\,{\zeta_c}^6\right)}{{\zeta_c}^3\left(-{r_c}^2+\delta^2\right)^3}$$
$$+\frac{2\,i\,\left(2\,a^6\,\pi\,{r_c}^6-a^6\,\pi\,{r_c}^2\,{\zeta_c}^4-a^6\,\pi\,{\zeta_c}^6\right)}{{r_c}^2\,{\zeta_c}^3\left(-{r_c}^2+\delta^2\right)^2} + \frac{2\,i\,a^6\,\pi\,\zeta_c}{{r_c}^2\,\left(-{r_c}^2+\delta^2\right)} \quad \text{(D.8)}$$

$$\mathcal{K}_{1c} = \frac{-2\,i\,a^6\,\pi\,{r_c}^4\,\left({r_c}^2+a\,\overline{\zeta_c}-\delta^2\right)\left(-\left({r_c}^2\,\zeta_c\right)+{r_c}^2\,\overline{\zeta_c}+a\,\delta^2+{\zeta_c}^2\,\overline{\zeta_c}-a\,{\overline{\zeta_c}}^2-\zeta_c\,{\overline{\zeta_c}}^2\right)}{{\overline{\zeta_c}}^2\left({r_c}^2-\delta^2\right)^5}$$
$$-\frac{4\,a^6\,\pi\,{r_c}^4\,\left({r_c}^2+a\,\overline{\zeta_c}-\delta^2\right)\left(-{r_c}^2+a\,\overline{\zeta_c}+\delta^2\right)\zeta_y}{{\overline{\zeta_c}}^2\left({r_c}^2-\delta^2\right)^5} \quad \text{(D.9)}$$

$$\mathcal{K}_{1d} = -i\,\pi\,\Big(a\,{r_c}^2\,{\zeta_c}^2\,{\overline{\zeta_c}}^4\left({r_c}^2-\delta^2\right)^5+{r_c}^2\,{\zeta_c}^2\,{\overline{\zeta_c}}^3\left({r_c}^2-\delta^2\right)^6+a^3\,\delta^2\left({r_c}^2-\delta^2\right)^5\left({r_c}^4-\zeta_c\,{\overline{\zeta_c}}^3\right)$$
$$+a^2\,\zeta_c\left({r_c}^2-\delta^2\right)^6\left({r_c}^4-\zeta_c\,{\overline{\zeta_c}}^3\right)+a^9\,{r_c}^2\,\zeta_c\,{\overline{\zeta_c}}^3\left({r_c}^4+\zeta_c\,{\overline{\zeta_c}}^3\right)$$
$$-a^6\left({r_c}^2-\delta^2\right)^3\Big({r_c}^6\,\zeta_c+\zeta_c\,{\overline{\zeta_c}}^6-{r_c}^2\,{\overline{\zeta_c}}^2\left({\zeta_c}^3-{\zeta_c}^2\,\overline{\zeta_c}+\zeta_c\,{\overline{\zeta_c}}^2+2\,{\overline{\zeta_c}}^3\right)+{r_c}^4\,\delta^2\,\left(\zeta_c+2\,i\,\left(i+\beta-\pi\right)\zeta_x\right)\Big)$$
$$+a^8\,{r_c}^2\,\zeta_c\,{\overline{\zeta_c}}^2\left({r_c}^2-\delta^2\right)\left({r_c}^4+\zeta_c\,{\overline{\zeta_c}}^3+2\,{r_c}^2\,\overline{\zeta_c}\,\zeta_x\right)$$
$$+a^4\,{r_c}^2\,\overline{\zeta_c}\left({r_c}^2-\delta^2\right)^3\Big(2\,{r_c}^6-6\,{r_c}^4\,\delta^2+{\zeta_c}^2\,{\overline{\zeta_c}}^3\left(-2i\zeta_y\right)+{r_c}^2\,\zeta_c\,{\overline{\zeta_c}}^2\left(7\,\zeta_c+\overline{\zeta_c}+2\,i\,\beta\,\overline{\zeta_c}-2\,i\,\pi\,\overline{\zeta_c}-2\,\beta\,\zeta_y+2\,\pi\,\zeta_y\right)\Big)$$
$$-a^7\,\overline{\zeta_c}\left({r_c}^2-\delta^2\right)^2\Big({r_c}^6\,\zeta_c+\zeta_c\,{\overline{\zeta_c}}^6-{r_c}^2\,{\overline{\zeta_c}}^2\left({\zeta_c}^3-{\zeta_c}^2\,\overline{\zeta_c}+\zeta_c\,{\overline{\zeta_c}}^2+2\,{\overline{\zeta_c}}^3\right)$$
$$+{r_c}^4\,\overline{\zeta_c}\,\left({\zeta_c}^2-2\,\overline{\zeta_c}\,\zeta_x+2\,\left(-\beta+\pi\right)\zeta_c\,\left(-i\zeta_x\right)\right)\Big)$$
$$+a^5\,{r_c}^2\,{\overline{\zeta_c}}^2\left({r_c}^2-\delta^2\right)^2\Big(2\,{r_c}^6-6\,{r_c}^4\,\delta^2+{\zeta_c}^2\,{\overline{\zeta_c}}^3\left(-2i\zeta_y\right)$$
$$+{r_c}^2\,\delta^2\,\left(\zeta_c\,\left(9\,\overline{\zeta_c}+2\,i\,\zeta_y\right)+\overline{\zeta_c}\,\left(\overline{\zeta_c}+2\,i\,\beta\,\overline{\zeta_c}-2\,i\,\pi\,\overline{\zeta_c}-2\,\beta\,\zeta_y+2\,\pi\,\zeta_y\right)\right)\Big)\Big)\Big/\left(\zeta_c\,{\overline{\zeta_c}}^3\right) \quad \text{(D.10)}$$

$$
\begin{aligned}
\mathcal{K}_{1e} = i\,a^2\,\pi\, \Big(& \left({r_c}^4-{\zeta_c}^4\right) \overline{\zeta_c}^2 \left({r_c}^2-\delta^2\right)^5 + a^6\,{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}^2 \left({r_c}^4+2\,{\zeta_c}^2\,\overline{\zeta_c}\left(3\,\zeta_c+\overline{\zeta_c}-i\,\zeta_y\right)\right.\\
&\left.+{r_c}^2\,\zeta_c\left(-2\,\zeta_c+\overline{\zeta_c}+2\,i\,\zeta_y\right)\right) - 2\,a^5\,{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}^2\left({r_c}^2-2\,\delta^2\right)\left({r_c}^2-\delta^2\right)\left(\zeta_c-i\,\zeta_y\right)\\
&+2\,a^3\,{\zeta_c}^3\,\overline{\zeta_c}\left({r_c}^2-2\,\delta^2\right)\left({r_c}^3-r_c\,\delta^2\right)^2\left(\zeta_c-i\,\zeta_y\right)\\
&-a^4\left({r_c}^2-\delta^2\right)^2\Big({r_c}^6\,\overline{\zeta_c}^2+{\zeta_c}^3\,\overline{\zeta_c}^5+{r_c}^4\,\zeta_c\left(\overline{\zeta_c}^3+2\,\left(1-i\,\beta+i\,\pi\right){\zeta_c}^2\left(\zeta_c-i\,\zeta_y\right)\right)\\
&-{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}\left(4\,{\zeta_c}^3+2\,{\zeta_c}^2\,\overline{\zeta_c}+\overline{\zeta_c}^3-2\,i\left({\zeta_c}^2+\overline{\zeta_c}^2\right)\zeta_y\right)\Big)\\
&+2\,a^2\,\delta^2\left({r_c}^3-r_c\,\delta^2\right)^2\Big({r_c}^6-3\,{r_c}^4\,\delta^2+{\zeta_c}^3\,\overline{\zeta_c}^2\left(\zeta_c-\overline{\zeta_c}-i\,\zeta_y\right)\\
&+{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}\left(3\,\overline{\zeta_c}+\left(-\beta+\pi\right)\left(i\,\zeta_c+\zeta_y\right)\right)\Big)\Big) \Big/ \left({\zeta_c}^3\,\overline{\zeta_c}^2\right)
\end{aligned}
\tag{D.11}
$$

$$
\begin{aligned}
\mathcal{K}_2 = \mathsf{Re}\Big\{ & \mathcal{K}_{2a}\log(\frac{r_c^2+a\zeta_c-\delta^2}{r_c^2}) + \mathcal{K}_{2b}\log(\frac{r_c^2-\delta^2}{r_c^2})\\
&+\mathcal{K}_{2c}\log(\frac{a}{\overline{\zeta_c}}) + \frac{\mathcal{K}_{2d}}{(r_c^2+a\overline{\zeta_c}-\delta^2)(r_c^2-\delta^2)^5} - \frac{\mathcal{K}_{2e}}{(r_c^2-\delta^2)^5}\Big\}
\end{aligned}
\tag{D.12}
$$

$$
\mathcal{K}_{2a} = \frac{2\,a^6\,\pi\,{r_c}^4\left(-\left({r_c}^4\,\zeta_c\right)+a^2\,{\zeta_c}^3-{r_c}^4\,\overline{\zeta_c}+a^2\,{\zeta_c}^2\,\overline{\zeta_c}+2\,{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}+2\,{r_c}^2\,\zeta_c\,\overline{\zeta_c}^2-{\zeta_c}^3\,\overline{\zeta_c}^2-{\zeta_c}^2\,\overline{\zeta_c}^3\right)}{{\zeta_c}^2\left({r_c}^2-\delta^2\right)^5}
\tag{D.13}
$$

$$
\begin{aligned}
\mathcal{K}_{2b} = & \frac{2\left(-\left(a^4\,\pi\,{r_c}^4\right)+a^6\,\pi\,{\zeta_c}^2\right)}{{\zeta_c}^2\,\overline{\zeta_c}^3} + \frac{2\left(a^4\,\pi\,{r_c}^6+a^6\,\pi\,{\zeta_c}^4\right)}{{r_c}^2\,{\zeta_c}^3\,\overline{\zeta_c}^2} + \frac{2\,a^6\,\pi}{{r_c}^2\,\overline{\zeta_c}} + \frac{4\left(a^8\,\pi\,{r_c}^6-a^8\,\pi\,{r_c}^4\,{\zeta_c}^2\right)}{\zeta_c\left(-{r_c}^2+\delta^2\right)^5}\\
&+\frac{4\,a^8\,\pi\,{r_c}^4}{\zeta_c\left(-{r_c}^2+\delta^2\right)^4} - \frac{2\left(a^6\,\pi\,{r_c}^6-a^6\,\pi\,{r_c}^4\,{\zeta_c}^2+a^6\,\pi\,{r_c}^2\,{\zeta_c}^4-a^6\,\pi\,{\zeta_c}^6\right)}{{\zeta_c}^3\left(-{r_c}^2+\delta^2\right)^3}\\
&-\frac{2\left(2\,a^6\,\pi\,{r_c}^6-a^6\,\pi\,{r_c}^2\,{\zeta_c}^4+a^6\,\pi\,{\zeta_c}^6\right)}{{r_c}^2\,{\zeta_c}^3\left(-{r_c}^2+\delta^2\right)^2} - \frac{2\,a^6\,\pi\,\zeta_c}{{r_c}^2\left(-{r_c}^2+\delta^2\right)}
\end{aligned}
\tag{D.14}
$$

$$
\mathcal{K}_{2c} = \frac{-2\,a^6\,\pi\,{r_c}^4\left({r_c}^2+a\,\overline{\zeta_c}-\delta^2\right)\left(-\left({r_c}^2\,\zeta_c\right)-{r_c}^2\,\overline{\zeta_c}+a\,\delta^2+{\zeta_c}^2\,\overline{\zeta_c}+a\,\overline{\zeta_c}^2+\zeta_c\,\overline{\zeta_c}^2\right)}{\overline{\zeta_c}^2\left({r_c}^2-\delta^2\right)^5}
\tag{D.15}
$$

$$\mathcal{K}_{2d} = \pi\,\Big(-\Big(a\,{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}^4\left({r_c}^2-\delta^2\right)^5\Big) - {r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}^3\left({r_c}^2-\delta^2\right)^6 + a^9\,{r_c}^2\,\zeta_c\,\overline{\zeta_c}^3\,\Big(-{r_c}^4+\zeta_c\,\overline{\zeta_c}^3\Big)$$
$$-a^2\,\zeta_c\left({r_c}^2-\delta^2\right)^6\,\Big({r_c}^4+\zeta_c\,\overline{\zeta_c}^3\Big) + a^3\,\delta^2\left(-{r_c}^2+\delta^2\right)^5\,\Big({r_c}^4+\zeta_c\,\overline{\zeta_c}^3\Big)$$
$$-a^8\,{r_c}^2\,\zeta_c\,\overline{\zeta_c}^2\,\left({r_c}^2-\delta^2\right)\,\Big({r_c}^4-\zeta_c\,\overline{\zeta_c}^3-2\,{r_c}^2\,\overline{\zeta_c}\,\zeta_x\Big) + a^4\,{r_c}^2\,\overline{\zeta_c}\left({r_c}^2-\delta^2\right)^3\,\Big(-2\,{r_c}^6+6\,{r_c}^4\,\delta^2+{\zeta_c}^2\,\overline{\zeta_c}^3\,\left(\zeta_c+\overline{\zeta_c}\right)$$
$$+{r_c}^2\,\zeta_c\,\overline{\zeta_c}^2\,\left(-7\,\zeta_c+\overline{\zeta_c}+2\,i\,\beta\,\overline{\zeta_c}-2\,i\,\pi\,\overline{\zeta_c}-2\,\beta\,\zeta_y+2\,\pi\,\zeta_y\,\right)\Big)$$
$$+a^5\,{r_c}^2\,\overline{\zeta_c}^2\left({r_c}^2-\delta^2\right)^2\,\Big(-2\,{r_c}^6+6\,{r_c}^4\,\delta^2+{\zeta_c}^2\,\overline{\zeta_c}^3\,\left(\zeta_c+\overline{\zeta_c}\right)$$
$$+{r_c}^2\,\delta^2\,\left(\overline{\zeta_c}\,\left(-5\,\zeta_c+\overline{\zeta_c}+2\,i\,\beta\,\overline{\zeta_c}-2\,i\,\pi\,\overline{\zeta_c}\right)+2\,\left(i\,\zeta_c-\beta\,\overline{\zeta_c}+\pi\,\overline{\zeta_c}\right)\,\zeta_y\right)\Big)$$
$$+a^6\left({r_c}^2-\delta^2\right)^3\,\Big({r_c}^6\,\zeta_c-\zeta_c\,\overline{\zeta_c}^6-{r_c}^2\,\overline{\zeta_c}^2\,\Big({\zeta_c}^3+{\zeta_c}^2\,\overline{\zeta_c}+\zeta_c\,\overline{\zeta_c}^2-2\,\overline{\zeta_c}^3\Big)+{r_c}^4\,\delta^2\,\left(\zeta_c+2\,\left(i+\beta-\pi\right)\,\left(-i\zeta_x\right)\right)\Big)$$
$$+a^7\,\overline{\zeta_c}\left({r_c}^2-\delta^2\right)^2\,\Big({r_c}^6\,\zeta_c-\zeta_c\,\overline{\zeta_c}^6-{r_c}^2\,\overline{\zeta_c}^2\,\Big({\zeta_c}^3+{\zeta_c}^2\,\overline{\zeta_c}+\zeta_c\,\overline{\zeta_c}^2-2\,\overline{\zeta_c}^3\Big)$$
$$+{r_c}^4\,\overline{\zeta_c}\,\left({\zeta_c}^2+2\,\overline{\zeta_c}\,\zeta_x+2\,\left(\beta-\pi\right)\,\zeta_c\,\left(-i\zeta_x\right)\right)\Big)\Big)\,/\,\Big(\zeta_c\,\overline{\zeta_c}^3\Big) \qquad \text{(D.16)}$$

$$\mathcal{K}_{2e} = -\Big(a^2\,\pi\,\Big(\left({r_c}^4-{\zeta_c}^4\right)\,\overline{\zeta_c}^2\left({r_c}^2-\delta^2\right)^5 + a^6\,{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}^2\,\left({r_c}^4+2\,{\zeta_c}^2\,\overline{\zeta_c}\,\left(-3\,\zeta_c+\overline{\zeta_c}+i\,\zeta_y\right)\right.$$
$$\left.+{r_c}^2\,\zeta_c\,\left(2\,\zeta_c+\overline{\zeta_c}-2\,i\,\zeta_y\right)\right) + 2\,a^5\,{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}^2\,\left({r_c}^2-2\,\delta^2\right)\,\left({r_c}^2-\delta^2\right)\,\left(\zeta_c-i\,\zeta_y\right)$$
$$-2\,a^3\,{\zeta_c}^3\,\overline{\zeta_c}\,\left({r_c}^2-2\,\delta^2\right)\,\left({r_c}^3-r_c\,\delta^2\right)^2\,\left(\zeta_c-i\,\zeta_y\right)$$
$$+2\,a^2\,\delta^2\left({r_c}^3-r_c\,\delta^2\right)^2\,\Big({r_c}^6-3\,{r_c}^4\,\delta^2-{\zeta_c}^3\,\overline{\zeta_c}^2\,\left(\zeta_c+\overline{\zeta_c}-i\,\zeta_y\right)+{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}\,\left(3\,\overline{\zeta_c}+\left(\beta-\pi\right)\,\left(i\,\zeta_c+\zeta_y\right)\right)\Big)$$
$$-a^4\left({r_c}^2-\delta^2\right)^2\,\Big({r_c}^6\,\overline{\zeta_c}^2+{\zeta_c}^3\,\overline{\zeta_c}^5+{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}\,\Big(4\,{\zeta_c}^3-2\,{\zeta_c}^2\,\zeta_x-\overline{\zeta_c}^2\,\left(\overline{\zeta_c}+2\,i\,\zeta_y\right)\Big)$$
$$+{r_c}^4\,\zeta_c\,\Big(\overline{\zeta_c}^3+2\,\left(i+\beta-\pi\right)\,{\zeta_c}^2\,\left(i\,\zeta_c+\zeta_y\right)\Big)\Big)\Big)\,\Big)\,/\,\Big({\zeta_c}^3\,\overline{\zeta_c}^2\Big) \qquad \text{(D.17)}$$

$$\mathcal{K}_3 = \mathsf{Re}\left\{\mathcal{K}_{3a}\log\Big(\frac{r_c^2-\delta^2}{r_c^2}\Big) + \frac{\mathcal{K}_{3b}}{(r_c^2+a\overline{\zeta_c}-\delta^2)(r_c^2-\delta^2)^4} + \frac{\mathcal{K}_{3c}}{(r_c^2-\delta^2)^4}\right\} \qquad \text{(D.18)}$$

$$\mathcal{K}_{3a} = \frac{-2\,i\,\left(-2\,a^3\,\pi\,{r_c}^4+3\,a^5\,\pi\,{\zeta_c}^2\right)}{{\zeta_c}^2\,\overline{\zeta_c}^2} - \frac{12\,i\,a^5\,\pi\,\zeta_c}{{r_c}^2\,\overline{\zeta_c}} + \frac{8\,i\,a^7\,\pi\,{r_c}^4}{\left(-{r_c}^2+\delta^2\right)^4} - \frac{6\,i\,\left(a^5\,\pi\,{r_c}^4+a^5\,\pi\,{\zeta_c}^4\right)}{{\zeta_c}^2\left(-{r_c}^2+\delta^2\right)^2} + \frac{12\,i\,a^5\,\pi\,{\zeta_c}^2}{{r_c}^2\,\left(-{r_c}^2+\delta^2\right)} \qquad \text{(D.19)}$$

$$\mathcal{K}_{3b} = -2\,i\,a^3\,\pi\,\left(a^5\,{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}^4 - \overline{\zeta_c}\left({r_c}^2-\delta^2\right)^3\,\left({r_c}^6 + i\,\left(2\,i-\beta+\pi\right)\,{r_c}^4\,\delta^2 - 2\,{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}^2 + {\zeta_c}^3\,\overline{\zeta_c}^3\right)\right.$$
$$+a^3\,\overline{\zeta_c}\left({r_c}^2-\delta^2\right)^2\,\left(i\,\left(-\beta+\pi\right)\,{r_c}^4\,\zeta_c + {r_c}^2\,\left({r_c}^2+{\zeta_c}^2\right)\,\overline{\zeta_c} + 2\,{r_c}^2\,\overline{\zeta_c}^3 - \zeta_c\,\overline{\zeta_c}^4\right)$$
$$+a^4\,\left({r_c}^6\,\zeta_c\,\overline{\zeta_c}^2 - {r_c}^2\,{\zeta_c}^3\,\overline{\zeta_c}^4\right) - a\,\overline{\zeta_c}^2\left({r_c}^2-\delta^2\right)^2\,\left({r_c}^6 - 2\,{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}^2 + {\zeta_c}^3\,\overline{\zeta_c}^3\right.$$
$$\left.+{r_c}^4\,\zeta_c\,\left(-\zeta_c + i\,\left(2\,i-\beta+\pi\right)\,\overline{\zeta_c}\right)\right)$$
$$\left.+a^2\left({r_c}^2-\delta^2\right)^3\,\left(\left(1-i\,\beta+i\,\pi\right)\,{r_c}^4\,\zeta_c - \zeta_c\,\overline{\zeta_c}^4 + {r_c}^2\,\overline{\zeta_c}\,\left({\zeta_c}^2 + 2\,\overline{\zeta_c}^2\right)\right)\right)/\left(\zeta_c\,\overline{\zeta_c}^2\right) \quad \text{(D.20)}$$

$$\mathcal{K}_{3c} = 2\,i\,a^3\,\pi\,\left(a^4\,{r_c}^2\,\zeta_c\,\overline{\zeta_c}^2\,\left({r_c}^2 - 5\,\delta^2\right) + a^3\,{r_c}^2\,\overline{\zeta_c}^2\,\left({r_c}^2 - 2\,\delta^2\right)\,\left({r_c}^2-\delta^2\right)\right.$$
$$-a\,\delta^2\,\left({r_c}^2 - 2\,\delta^2\right)\,\left({r_c}^3 - r_c\,\delta^2\right)^2 + \overline{\zeta_c}\left({r_c}^2-\delta^2\right)^2\,\left({r_c}^6 + i\,\left(i+\beta-\pi\right)\,{r_c}^4\,\delta^2 - 2\,{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}^2 + {\zeta_c}^3\,\overline{\zeta_c}^3\right)$$
$$\left.+a^2\left({r_c}^2-\delta^2\right)^2\,\left(\left(1-i\,\beta+i\,\pi\right)\,{r_c}^4\,\zeta_c + \zeta_c\,\overline{\zeta_c}^4 - {r_c}^2\,\left(3\,{\zeta_c}^2\,\overline{\zeta_c} + 2\,\overline{\zeta_c}^3\right)\right)\right)/\left(\zeta_c\,\overline{\zeta_c}^2\right) \quad \text{(D.21)}$$

$$\mathcal{L}_1 = \mathsf{Re}\left[i\,\pi\,\left(-\left(a^8\,{r_c}^4\,\overline{\zeta_c}^4\right) + {r_c}^4\left({r_c}^2-\delta^2\right)^6 + 2\,a^6\,\overline{\zeta_c}^2\left({r_c}^2-\delta^2\right)^2\,\left(2\,{r_c}^4 - \zeta_c\,\overline{\zeta_c}^3\right)\right.\right.$$
$$-2\,a^2\left({r_c}^2-\delta^2\right)^4\,\left({r_c}^6 + 3\,{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}^2 - {\zeta_c}^3\,\overline{\zeta_c}^3 + {r_c}^4\,\overline{\zeta_c}\,\left(-3\,\zeta_c + \overline{\zeta_c}\right)\right)$$
$$\left.+a^4\left({r_c}^3 - r_c\,\delta^2\right)^2\,\left({r_c}^6 - 2\,{\zeta_c}^2\,\overline{\zeta_c}^4 - 2\,{r_c}^4\,\overline{\zeta_c}\,\left(\zeta_c + \overline{\zeta_c}\right) + {r_c}^2\,\overline{\zeta_c}^2\,\left({\zeta_c}^2 + 4\,\delta^2 + \overline{\zeta_c}^2\right)\right)\right)$$
$$\left./\left({r_c}^2\left({r_c}^2 + \left(a-\zeta_c\right)\,\overline{\zeta_c}\right)^2\left({r_c}^2-\delta^2\right)^2\left({r_c}^2 - \left(a+\zeta_c\right)\,\overline{\zeta_c}\right)^2\right)\right] \quad \text{(D.22)}$$

$$\mathcal{L}_2 = \mathsf{Re}\left[\frac{-4\,a^2\,\pi\,\overline{\zeta_c}\,\left({r_c}^2-\delta^2\right)\,\left({r_c}^6\,\zeta_c - 3\,{r_c}^4\,\left(a^2+{\zeta_c}^2\right)\,\overline{\zeta_c} + 3\,{r_c}^2\,\zeta_c\,\left(a^2+{\zeta_c}^2\right)\,\overline{\zeta_c}^2 + \left(a^4-{\zeta_c}^4\right)\,\overline{\zeta_c}^3\right)}{{r_c}^2\left({r_c}^2 + \left(a-\zeta_c\right)\,\overline{\zeta_c}\right)^2\left({r_c}^2 - \left(a+\zeta_c\right)\,\overline{\zeta_c}\right)^2}\right] \quad \text{(D.23)}$$

$$\mathcal{L}_3 = \mathsf{Re}\left[i\,\pi\,\left(-\left(a^8\,{r_c}^4\,\overline{\zeta_c}^4\right) + {r_c}^4\left({r_c}^2-\delta^2\right)^6 + 2\,a^6\,\overline{\zeta_c}^2\left({r_c}^2-\delta^2\right)^2\,\left(2\,{r_c}^4 + \zeta_c\,\overline{\zeta_c}^3\right)\right.\right.$$
$$+2\,a^2\left({r_c}^2-\delta^2\right)^4\,\left({r_c}^6 + 3\,{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}^2 - {\zeta_c}^3\,\overline{\zeta_c}^3 - {r_c}^4\,\overline{\zeta_c}\,\left(3\,\zeta_c + \overline{\zeta_c}\right)\right)$$
$$\left.+a^4\left({r_c}^3 - r_c\,\delta^2\right)^2\,\left({r_c}^6 + 2\,{\zeta_c}^2\,\overline{\zeta_c}^4 + 2\,{r_c}^4\,\overline{\zeta_c}\,\left(-2i\zeta_y\right) + {r_c}^2\,\overline{\zeta_c}^2\,\left({\zeta_c}^2 - 4\,\delta^2 + \overline{\zeta_c}^2\right)\right)\right)$$
$$\left./\left({r_c}^2\left({r_c}^2 + \left(a-\zeta_c\right)\,\overline{\zeta_c}\right)^2\left({r_c}^2-\delta^2\right)^2\left({r_c}^2 - \left(a+\zeta_c\right)\,\overline{\zeta_c}\right)^2\right)\right] \quad \text{(D.24)}$$

$$\mathcal{L}_4 = \mathsf{Re}\Big[\pi\,\Big(a^{10}\,{r_c}^4\,\left(\zeta_c-\overline{\zeta_c}\right)\,\overline{\zeta_c}^4+{r_c}^4\,\left(\zeta_c-\overline{\zeta_c}\right)\,\left(-{r_c}^2+\delta^2\right)^7-2\,a^8\,\overline{\zeta_c}^2\,\left({r_c}^2-\delta^2\right)^2\,\left({r_c}^4\,\left(2\,\zeta_c-\overline{\zeta_c}\right)\right.$$
$$+2\,\zeta_c\,\overline{\zeta_c}^4-{r_c}^2\,\overline{\zeta_c}^2\,\left(\zeta_c+2\,\overline{\zeta_c}\right)\Big)+2\,a^2\left({r_c}^2-\delta^2\right)^5\,\Big({r_c}^6\,\left(2\,\zeta_c-\overline{\zeta_c}\right)+{r_c}^2\,{\zeta_c}^2\,\left(6\,\zeta_c-\overline{\zeta_c}\right)\,\overline{\zeta_c}^2-2\,{\zeta_c}^4\,\overline{\zeta_c}^3$$
$$-{r_c}^4\,\overline{\zeta_c}\,\left(6\,{\zeta_c}^2-3\,\delta^2+\overline{\zeta_c}^2\right)\Big)-a^6\left({r_c}^2-\delta^2\right)^3\,\Big({r_c}^6\,\left(\zeta_c-7\,\overline{\zeta_c}\right)-4\,{\zeta_c}^2\,\overline{\zeta_c}^5+2\,{r_c}^2\,\overline{\zeta_c}^3\,\left(-2\,{\zeta_c}^2-2\,\delta^2+\overline{\zeta_c}^2\right)$$
$$+{r_c}^4\,\overline{\zeta_c}\,\left(-{\zeta_c}^2+11\,\delta^2+6\,\overline{\zeta_c}^2\right)\Big)+a^4\left({r_c}^2-\delta^2\right)^3\,\Big(2\,{r_c}^8\,\left(\zeta_c-5\,\overline{\zeta_c}\right)-4\,{\zeta_c}^4\,\overline{\zeta_c}^5$$
$$-2\,{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}^3\,\left({\zeta_c}^2-12\,\delta^2-2\,\overline{\zeta_c}^2\right)+{r_c}^4\,\overline{\zeta_c}^2\,\left(6\,{\zeta_c}^3-46\,{\zeta_c}^2\,\overline{\zeta_c}-9\,\zeta_c\,\overline{\zeta_c}^2+\overline{\zeta_c}^3\right)$$
$$+{r_c}^6\,\left(-6\,{\zeta_c}^2\,\overline{\zeta_c}+36\,\zeta_c\,\overline{\zeta_c}^2+4\,\overline{\zeta_c}^3\right)\Big)\Big)$$
$$/\Big({r_c}^2\left({r_c}^2+\left(a-\zeta_c\right)\,\overline{\zeta_c}\right)^2\left({r_c}^2-\delta^2\right)^3\left({r_c}^2-\left(a+\zeta_c\right)\,\overline{\zeta_c}\right)^2\Big)\Big] \quad \text{(D.25)}$$

$$\mathcal{L}_5 = \mathsf{Re}\Big[i\,\pi\,\Big(a^{10}\,{r_c}^4\,\overline{\zeta_c}^4\,\left(\zeta_c+\overline{\zeta_c}\right)+{r_c}^4\,\left(\zeta_c+\overline{\zeta_c}\right)\,\left({r_c}^2-\delta^2\right)^7-2\,a^8\,\overline{\zeta_c}^2\left({r_c}^2-\delta^2\right)^2\,\Big({r_c}^2\,\left(\zeta_c-2\,\overline{\zeta_c}\right)\,\overline{\zeta_c}^2$$
$$+2\,\zeta_c\,\overline{\zeta_c}^4+{r_c}^4\,\left(2\,\zeta_c+\overline{\zeta_c}\right)\Big)-a^6\left({r_c}^2-\delta^2\right)^3\,\Big(-4\,{\zeta_c}^2\,\overline{\zeta_c}^5+{r_c}^6\,\left(\zeta_c+7\,\overline{\zeta_c}\right)-{r_c}^4\,\overline{\zeta_c}\,\left({\zeta_c}^2+11\,\delta^2-6\,\overline{\zeta_c}^2\right)$$
$$-2\,{r_c}^2\,\overline{\zeta_c}^3\,\left(-2\,{\zeta_c}^2+2\,\delta^2+\overline{\zeta_c}^2\right)\Big)+2\,a^2\left({r_c}^2-\delta^2\right)^5\,\Big(-2\,{\zeta_c}^4\,\overline{\zeta_c}^3+{r_c}^6\,\left(2\,\zeta_c+\overline{\zeta_c}\right)+{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}^2\,\left(6\,\zeta_c+\overline{\zeta_c}\right)$$
$$-{r_c}^4\,\overline{\zeta_c}\,\left(6\,{\zeta_c}^2+3\,\delta^2+\overline{\zeta_c}^2\right)\Big)-a^4\left({r_c}^2-\delta^2\right)^3\,\Big(4\,{\zeta_c}^4\,\overline{\zeta_c}^5+2\,{r_c}^8\,\left(\zeta_c+5\,\overline{\zeta_c}\right)$$
$$-2\,{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}^3\,\left({\zeta_c}^2+12\,\delta^2-2\,\overline{\zeta_c}^2\right)+2\,{r_c}^6\,\overline{\zeta_c}\,\left(-3\,{\zeta_c}^2-18\,\delta^2+2\,\overline{\zeta_c}^2\right)$$
$$-{r_c}^4\,\overline{\zeta_c}^2\,\left(-6\,{\zeta_c}^3-46\,{\zeta_c}^2\,\overline{\zeta_c}+9\,\zeta_c\,\overline{\zeta_c}^2+\overline{\zeta_c}^3\right)\Big)\Big)$$
$$/\Big({r_c}^2\left({r_c}^2+\left(a-\zeta_c\right)\,\overline{\zeta_c}\right)^2\left({r_c}^2-\delta^2\right)^3\left({r_c}^2-\left(a+\zeta_c\right)\,\overline{\zeta_c}\right)^2\Big)\Big] \quad \text{(D.26)}$$

$$\mathcal{L}_6 = \mathsf{Re}\Big[i\,\pi\,\Big(-\Big(a^{12}\,{r_c}^4\,\zeta_c\,\overline{\zeta_c}^5\Big)+{r_c}^4\,\delta^2\left({r_c}^2-\delta^2\right)^8+a^{10}\,\overline{\zeta_c}^3\left({r_c}^2-\delta^2\right)^2\,\Big(2\,{r_c}^4\,\zeta_c+2\,\zeta_c\,\overline{\zeta_c}^4$$
$$+{r_c}^2\,\overline{\zeta_c}\,\left({\zeta_c}^2-3\,\overline{\zeta_c}^2\right)\Big)+a^2\left({r_c}^2-\delta^2\right)^6\,\Big(-2\,{\zeta_c}^5\,\overline{\zeta_c}^3-4\,{r_c}^4\,\delta^2\,\left(2\,{\zeta_c}^2+\overline{\zeta_c}^2\right)+{r_c}^6\,\left(3\,{\zeta_c}^2+\overline{\zeta_c}^2\right)$$
$$+{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}^2\,\left(7\,{\zeta_c}^2+\overline{\zeta_c}^2\right)\Big)+a^8\,\overline{\zeta_c}\left({r_c}^2-\delta^2\right)^3\,\Big(7\,{r_c}^6\,\zeta_c+2\,{r_c}^2\,{\zeta_c}^3\,\overline{\zeta_c}^2-4\,{\zeta_c}^2\,\overline{\zeta_c}^5+{r_c}^4\,\left(-9\,{\zeta_c}^2\,\overline{\zeta_c}+2\,\overline{\zeta_c}^3\right)\Big)$$
$$+a^4\left({r_c}^2-\delta^2\right)^4\,\left(2\,{r_c}^2-\delta^2\right)\,\Big(2\,{r_c}^8-12\,{r_c}^6\,\delta^2+4\,{\zeta_c}^4\,\overline{\zeta_c}^4+2\,{r_c}^2\,\zeta_c\,\overline{\zeta_c}^3\,\left(-8\,{\zeta_c}^2+\overline{\zeta_c}^2\right)$$
$$+{r_c}^4\,\left(22\,{\zeta_c}^2\,\overline{\zeta_c}^2-\overline{\zeta_c}^4\right)\Big)+a^6\,{r_c}^2\left({r_c}^2-\delta^2\right)^4\,\Big({r_c}^4\,\left({\zeta_c}^2+5\,\overline{\zeta_c}^2\right)-2\,{r_c}^2\,\delta^2\,\left({\zeta_c}^2+6\,\overline{\zeta_c}^2\right)$$
$$+\overline{\zeta_c}^2\,\left({\zeta_c}^4+8\,{\zeta_c}^2\,\overline{\zeta_c}^2+\overline{\zeta_c}^4\right)\Big)\Big)/\Big({r_c}^2\left({r_c}^2+\left(a-\zeta_c\right)\,\overline{\zeta_c}\right)^2\left({r_c}^2-\delta^2\right)^4\left({r_c}^2-\left(a+\zeta_c\right)\,\overline{\zeta_c}\right)^2\Big)\Big]$$
$$\text{(D.27)}$$

$$\mathcal{M}_1 = \mathsf{Re}\Big[2\,i\,\pi\,\Big({r_c}^4\left({r_c}^2-\delta^2\right)^8 - 2\,a^2\left({r_c}^2-\delta^2\right)^6\,\Big({r_c}^6+3\,{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}^2-{\zeta_c}^3\,\overline{\zeta_c}^3+{r_c}^4\,\overline{\zeta_c}\,\left(-3\,\zeta_c+\overline{\zeta_c}\right)\Big)$$
$$+a^6\left({r_c}^2-\delta^2\right)^3\,\Big(\overline{\zeta_c}^2\,\Big(4\,{r_c}^6+2\,{\zeta_c}^2\,\overline{\zeta_c}^4+{r_c}^2\,\delta^2\,\Big({\zeta_c}^2-6\,\overline{\zeta_c}^2\Big)-{r_c}^4\,\Big({\zeta_c}^2+6\,\delta^2-4\,\overline{\zeta_c}^2\Big)\Big)$$
$$-i\,{r_c}^2\,\zeta_c\,\Big({r_c}^4+{\zeta_c}^2\,\overline{\zeta_c}^2+2\,{r_c}^2\,\overline{\zeta_c}\,\left(-2i\zeta_y\right)\Big)\,\zeta_y\Big)+a^{10}\,{r_c}^2\,\overline{\zeta_c}^4\,\Big({r_c}^4-{\zeta_c}^2\,\overline{\zeta_c}\,\zeta_x$$
$$+{r_c}^2\,\Big({\zeta_c}^2+2\,\delta^2+3\,\overline{\zeta_c}^2+i\,\zeta_c\,\zeta_y\Big)\Big)+a^4\left({r_c}^2-\delta^2\right)^4\,\Big({r_c}^8-2\,{\zeta_c}^3\,\overline{\zeta_c}^5+{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}^2\,\Big({\zeta_c}^2+5\,\overline{\zeta_c}^2\Big)$$
$$+{r_c}^6\,\Big({\zeta_c}^2-\delta^2+\overline{\zeta_c}^2+i\,\zeta_c\,\zeta_y\Big)+{r_c}^4\,\Big(-2\,{\zeta_c}^3\,\overline{\zeta_c}-4\,\zeta_c\,\overline{\zeta_c}^3+\overline{\zeta_c}^4-i\,{\zeta_c}^2\,\overline{\zeta_c}\,\zeta_y\Big)\Big)$$
$$-a^8\,\overline{\zeta_c}\,\left({r_c}^2-\delta^2\right)\,\Big(2\,{\zeta_c}^2\,\overline{\zeta_c}^7+{r_c}^2\,\zeta_c\,\overline{\zeta_c}^4\,\Big({\zeta_c}^2-5\,\overline{\zeta_c}^2\Big)+{r_c}^8\,\left(4\,\zeta_c+3\,\overline{\zeta_c}-4\,i\,\zeta_y\right)$$
$$+{r_c}^6\,\overline{\zeta_c}\,\Big(-{\zeta_c}^2+3\,\delta^2+6\,\overline{\zeta_c}^2+8\,i\,\zeta_c\,\zeta_y\Big)-{r_c}^4\,\overline{\zeta_c}^2\,\Big(3\,{\zeta_c}^3+7\,{\zeta_c}^2\,\overline{\zeta_c}$$
$$+7\,\zeta_c\,\overline{\zeta_c}^2-3\,\overline{\zeta_c}^3+i\,\zeta_c\,\left(4\,\zeta_c+\overline{\zeta_c}\right)\,\zeta_y\Big)\Big)\Big)$$
$$/\Big({r_c}^2\left({r_c}^2+\left(a-\zeta_c\right)\,\overline{\zeta_c}\right)^2\left({r_c}^2-\delta^2\right)^4\Big({r_c}^2-\left(a+\zeta_c\right)\,\overline{\zeta_c}\Big)^2\Big)\Big] \tag{D.28}$$

$$\mathcal{M}_2 = \mathsf{Re}\Big[2\,a^2\,\pi\,\Big(-2\,\delta^2\left({r_c}^2-\delta^2\right)^8+a^8\,{r_c}^2\,\overline{\zeta_c}^4\,\Big({\zeta_c}^2\,\overline{\zeta_c}\,\left(\overline{\zeta_c}-i\,\zeta_y\right)+{r_c}^2\,\Big({\zeta_c}^2-3\,\overline{\zeta_c}^2+i\,\zeta_c\,\zeta_y\Big)\Big)$$
$$+a^2\left({r_c}^2-\delta^2\right)^5\,\Big(-2\,{r_c}^6-2\,{\zeta_c}^2\,\overline{\zeta_c}^4-{r_c}^2\,\delta^2\,\Big({\zeta_c}^2+\overline{\zeta_c}^2\Big)+{r_c}^4\,\Big({\zeta_c}^2+\delta^2+3\,\overline{\zeta_c}^2+i\,\zeta_c\,\zeta_y\Big)\Big)$$
$$+a^4\left({r_c}^2-\delta^2\right)^3\,\Big({r_c}^8-2\,{\zeta_c}^2\,\overline{\zeta_c}^6+{r_c}^2\,\zeta_c\,\overline{\zeta_c}^2\,\Big(8\,\overline{\zeta_c}^3+{\zeta_c}^2\,\left(3\,\overline{\zeta_c}-i\,\zeta_y\right)\Big)+{r_c}^6\,\left(\overline{\zeta_c}\,\left(\zeta_c+4\,\overline{\zeta_c}\right)-i\,\zeta_c\,\zeta_y\right)$$
$$+{r_c}^4\,\overline{\zeta_c}\,\Big(-\Big(\overline{\zeta_c}\,\Big(5\,{\zeta_c}^2+2\,\delta^2+6\,\overline{\zeta_c}^2\Big)\Big)+2\,i\,\zeta_c\,\left(\zeta_c-\overline{\zeta_c}\right)\,\zeta_y\Big)\Big)-a^6\,\overline{\zeta_c}\,\left({r_c}^2-\delta^2\right)\,\Big(-2\,{\zeta_c}^2\,\overline{\zeta_c}^7$$
$$+{r_c}^2\,\zeta_c\,\overline{\zeta_c}^4\,\Big({\zeta_c}^2+5\,\overline{\zeta_c}^2\Big)+{r_c}^8\,\left(4\,\zeta_c-\overline{\zeta_c}-4\,i\,\zeta_y\right)-{r_c}^6\,\overline{\zeta_c}\,\Big({\zeta_c}^2+\delta^2+3\,\overline{\zeta_c}^2-8\,i\,\zeta_c\,\zeta_y\Big)$$
$$+{r_c}^4\,\overline{\zeta_c}^2\,\Big(-3\,{\zeta_c}^3+{\zeta_c}^2\,\overline{\zeta_c}+4\,\zeta_c\,\overline{\zeta_c}^2-3\,\overline{\zeta_c}^3-i\,\zeta_c\,\left(4\,\zeta_c+\overline{\zeta_c}\right)\,\zeta_y\Big)\Big)\Big)$$
$$/\Big({r_c}^2\left({r_c}^2+\left(a-\zeta_c\right)\,\overline{\zeta_c}\right)^2\left({r_c}^2-\delta^2\right)^4\Big({r_c}^2-\left(a+\zeta_c\right)\,\overline{\zeta_c}\Big)^2\Big)\Big] \tag{D.29}$$

$$\mathcal{M}_3 = \mathsf{Re}\Big[2\,i\,a\,\pi\,\Big(2\,{r_c}^2\,\zeta_c\left({r_c}^2-\delta^2\right)^7+a^2\left({r_c}^2-\delta^2\right)^5\,\Big(4\,{\zeta_c}^2\,\overline{\zeta_c}^3+{r_c}^4\,\left(3\,\zeta_c+2\,\overline{\zeta_c}\right)-2\,{r_c}^2\,\delta^2\,\left(\zeta_c+5\,\overline{\zeta_c}\right)\Big)$$
$$+a^8\,{r_c}^2\,\overline{\zeta_c}^4\,\left(-\left({\zeta_c}^2\,\overline{\zeta_c}\right)+{r_c}^2\,\left(3\,\zeta_c+5\,\overline{\zeta_c}\right)\right)-a^6\,\overline{\zeta_c}\,\left({r_c}^2-\delta^2\right)\,\Big(4\,{r_c}^8+2\,{r_c}^2\,\zeta_c\,\left(\zeta_c-5\,\overline{\zeta_c}\right)\,\overline{\zeta_c}^4+4\,{\zeta_c}^2\,\overline{\zeta_c}^6$$
$$+2\,{r_c}^6\,\overline{\zeta_c}\,\left(3\,\zeta_c+5\,\overline{\zeta_c}\right)+{r_c}^4\,\overline{\zeta_c}^2\,\Big(-10\,{\zeta_c}^2-13\,\delta^2+6\,\overline{\zeta_c}^2\,\Big)\Big)-a^4\,{r_c}^2\left({r_c}^2-\delta^2\right)^3\,\Big({r_c}^4\,\left(\zeta_c-\overline{\zeta_c}\right)$$
$$+{r_c}^2\,\overline{\zeta_c}\,\Big(-2\,{\zeta_c}^2+7\,\delta^2-12\,\overline{\zeta_c}^2\Big)+\zeta_c\,\overline{\zeta_c}^2\,\Big({\zeta_c}^2-4\,\delta^2+10\,\overline{\zeta_c}^2\Big)\,\Big)\Big)$$
$$/\Big({r_c}^2\left({r_c}^2+\left(a-\zeta_c\right)\,\overline{\zeta_c}\right)^2\left({r_c}^2-\delta^2\right)^3\Big({r_c}^2-\left(a+\zeta_c\right)\,\overline{\zeta_c}\Big)^2\Big)\Big] \tag{D.30}$$

$$\mathcal{M}_4 = \mathsf{Re}\left[\frac{-2\,i\,\pi\,\left(-\left(a^2\,\zeta_c\,\left({r_c}^2-\delta^2\right)^2\right)+\zeta_c\,\left({r_c}^3-r_c\,\delta^2\right)^2+a^4\,\left(-\left(\zeta_c\,\overline{\zeta_c}^2\right)+{r_c}^2\,\left(\zeta_c+2\,\overline{\zeta_c}\right)\right)\right)}{r_c\,\left({r_c}^2-\delta^2\right)^2}\right] \tag{D.31}$$

$$\begin{aligned}\mathcal{M}_5 = \mathsf{Re}\Big[&-2\,a^2\,\pi\,\Big(2\,\delta^2\,\left({r_c}^2-\delta^2\right)^8-a^6\,\overline{\zeta_c}\,\left({r_c}^2-\delta^2\right)\,\Big(2\,{\zeta_c}^2\,\overline{\zeta_c}^7-{r_c}^2\,\zeta_c\,\overline{\zeta_c}^4\,\left({\zeta_c}^2+5\,\overline{\zeta_c}^2\right)\\&+{r_c}^4\,\overline{\zeta_c}^2\,\Big(-3\,{\zeta_c}^3+3\,\overline{\zeta_c}^3+\delta^2\,\left(4\,\overline{\zeta_c}+i\,\zeta_y\right)-{\zeta_c}^2\,\left(\overline{\zeta_c}+4\,i\,\zeta_y\right)\Big)+{r_c}^6\,\overline{\zeta_c}\,\Big(-{\zeta_c}^2-3\,\overline{\zeta_c}^2+\zeta_c\,\left(\overline{\zeta_c}+8\,i\,\zeta_y\right)\Big)\\&+{r_c}^8\,\left(4\,\zeta_c+\overline{\zeta_c}-4\,i\,\zeta_y\right)\Big)+a^8\,{r_c}^2\,\overline{\zeta_c}^4\,\Big(-\left({\zeta_c}^2\,\overline{\zeta_c}\,\zeta_x\right)+{r_c}^2\,\Big({\zeta_c}^2-3\,\overline{\zeta_c}^2+i\,\zeta_c\,\zeta_y\Big)\Big)\\&+a^2\,\left({r_c}^2-\delta^2\right)^5\,\Big(-2\,{r_c}^6+2\,{\zeta_c}^2\,\overline{\zeta_c}^4+{r_c}^2\,\delta^2\,\left({\zeta_c}^2+\overline{\zeta_c}^2\right)-{r_c}^4\,\Big({\zeta_c}^2-\delta^2+3\,\overline{\zeta_c}^2+i\,\zeta_c\,\zeta_y\Big)\Big)\\&+a^4\,\left({r_c}^2-\delta^2\right)^3\,\Big(-{r_c}^8+2\,{\zeta_c}^2\,\overline{\zeta_c}^6-{r_c}^2\,\zeta_c\,\overline{\zeta_c}^2\,\Big(8\,\overline{\zeta_c}^3+{\zeta_c}^2\,\left(3\,\overline{\zeta_c}+i\,\zeta_y\right)\Big)-{r_c}^6\,\left(\left(\zeta_c-4\,\overline{\zeta_c}\right)\,\overline{\zeta_c}+i\,\zeta_c\,\zeta_y\right)\\&+{r_c}^4\,\overline{\zeta_c}\,\Big(\overline{\zeta_c}\,\Big(5\,{\zeta_c}^2-2\,\delta^2+6\,\overline{\zeta_c}^2\Big)+2\,i\,\zeta_c\,\left(\zeta_c+\overline{\zeta_c}\right)\,\zeta_y\Big)\Big)\Big)\\&/\Big({r_c}^2\,\left({r_c}^2+\left(a-\zeta_c\right)\,\overline{\zeta_c}\right)^2\,\left({r_c}^2-\delta^2\right)^4\,\left({r_c}^2-\left(a+\zeta_c\right)\,\overline{\zeta_c}\right)^2\Big)\Big]\end{aligned} \tag{D.32}$$

$$\begin{aligned}\mathcal{M}_6 = \mathsf{Re}\Big[&-2\,i\,\pi\,\Big(-\left({r_c}^4\,\left({r_c}^2-\delta^2\right)^8\right)-2\,a^2\,\left({r_c}^2-\delta^2\right)^6\,\Big({r_c}^6+3\,{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}^2-{\zeta_c}^3\,\overline{\zeta_c}^3-{r_c}^4\,\overline{\zeta_c}\,\left(3\,\zeta_c+\overline{\zeta_c}\right)\Big)\\&+a^6\,\left({r_c}^2-\delta^2\right)^3\,\Big(\overline{\zeta_c}^2\,\Big(-4\,{r_c}^6+2\,{\zeta_c}^2\,\overline{\zeta_c}^4+{r_c}^2\,\delta^2\,\left({\zeta_c}^2-6\,\overline{\zeta_c}^2\right)+{r_c}^4\,\Big(-{\zeta_c}^2+6\,\delta^2+4\,\overline{\zeta_c}^2\Big)\Big)\\&+i\,{r_c}^2\,\zeta_c\,\Big({r_c}^4+{\zeta_c}^2\,\overline{\zeta_c}^2-2\,{r_c}^2\,\overline{\zeta_c}\,\left(\zeta_c+\overline{\zeta_c}\right)\Big)\,\zeta_y\Big)+a^{10}\,{r_c}^2\,\overline{\zeta_c}^4\,\Big({r_c}^4-{\zeta_c}^2\,\overline{\zeta_c}\,\left(\overline{\zeta_c}-i\,\zeta_y\right)\\&-{r_c}^2\,\Big({\zeta_c}^2-2\,\delta^2+3\,\overline{\zeta_c}^2+i\,\zeta_c\,\zeta_y\Big)\Big)+a^8\,\overline{\zeta_c}\,\left({r_c}^2-\delta^2\right)\,\Big(-2\,{\zeta_c}^2\,\overline{\zeta_c}^7+{r_c}^2\,\Big(-\left({\zeta_c}^3\,\overline{\zeta_c}^4\right)+5\,\zeta_c\,\overline{\zeta_c}^6\,\Big)\\&+{r_c}^4\,\overline{\zeta_c}^2\,\Big(-3\,{\zeta_c}^3-3\,\overline{\zeta_c}^3+\delta^2\,\left(-7\,\overline{\zeta_c}+i\,\zeta_y\right)+{\zeta_c}^2\,\left(7\,\overline{\zeta_c}-4\,i\,\zeta_y\right)\Big)+{r_c}^8\,\left(4\,\zeta_c-3\,\overline{\zeta_c}-4\,i\,\zeta_y\right)\\&+{r_c}^6\,\overline{\zeta_c}\,\Big(-{\zeta_c}^2-3\,\delta^2+6\,\overline{\zeta_c}^2+8\,i\,\zeta_c\,\zeta_y\Big)\Big)+a^4\,\left({r_c}^2-\delta^2\right)^4\,\Big(-{r_c}^8-2\,{\zeta_c}^3\,\overline{\zeta_c}^5+{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}^2\,\left({\zeta_c}^2+5\,\overline{\zeta_c}^2\right)\\&+{r_c}^6\,\Big({\zeta_c}^2+\delta^2+\overline{\zeta_c}^2+i\,\zeta_c\,\zeta_y\Big)-{r_c}^4\,\overline{\zeta_c}\,\Big(2\,{\zeta_c}^3+4\,\zeta_c\,\overline{\zeta_c}^2+\overline{\zeta_c}^3+i\,{\zeta_c}^2\,\zeta_y\Big)\Big)\Big)\\&/\Big({r_c}^2\,\left({r_c}^2+\left(a-\zeta_c\right)\,\overline{\zeta_c}\right)^2\,\left({r_c}^2-\delta^2\right)^4\,\left({r_c}^2-\left(a+\zeta_c\right)\,\overline{\zeta_c}\right)^2\Big)\Big]\end{aligned} \tag{D.33}$$

$$\mathcal{M}_7 = \mathsf{Re}\Big[-2\,a\,\pi\,\Big(2\,{r_c}^2\,\zeta_c\left({r_c}^2-\delta^2\right)^7 + a^8\,{r_c}^2\,\overline{\zeta_c}^4\,\left({r_c}^2\,\left(3\,\zeta_c - 5\,\overline{\zeta_c}\right) - {\zeta_c}^2\,\overline{\zeta_c}\right)$$
$$-a^2\left({r_c}^2-\delta^2\right)^5\,\Big({r_c}^4\,\left(3\,\zeta_c - 2\,\overline{\zeta_c}\right) - 2\,{r_c}^2\,\zeta_c\,\left(\zeta_c - 5\,\overline{\zeta_c}\right)\,\overline{\zeta_c} - 4\,{\zeta_c}^2\,\overline{\zeta_c}^3\Big)$$
$$-a^6\,\overline{\zeta_c}\,\left({r_c}^2-\delta^2\right)\,\Big(4\,{r_c}^8 + 2\,{r_c}^6\,\left(3\,\zeta_c - 5\,\overline{\zeta_c}\right)\,\overline{\zeta_c} + 4\,{\zeta_c}^2\,\overline{\zeta_c}^6 - 2\,{r_c}^2\,\zeta_c\,\overline{\zeta_c}^4\,\left(\zeta_c + 5\,\overline{\zeta_c}\right)$$
$$+{r_c}^4\,\overline{\zeta_c}^2\,\Big(-10\,{\zeta_c}^2 + 13\,\delta^2 + 6\,\overline{\zeta_c}^2\,\Big)\Big)$$
$$-a^4\,{r_c}^2\left({r_c}^2-\delta^2\right)^3\,\Big({r_c}^4\,\left(\zeta_c + \overline{\zeta_c}\right) + \zeta_c\,\overline{\zeta_c}^2\,\Big({\zeta_c}^2 + 4\,\delta^2 + 10\,\overline{\zeta_c}^2\Big) - {r_c}^2\,\overline{\zeta_c}\,\Big(2\,{\zeta_c}^2 + 7\,\delta^2 + 12\,\overline{\zeta_c}^2\Big)\,\Big)\Big)$$
$$/\Big({r_c}^2\left({r_c}^2 + \left(a-\zeta_c\right)\,\overline{\zeta_c}\right)^2\left({r_c}^2-\delta^2\right)^3\left({r_c}^2 - \left(a+\zeta_c\right)\,\overline{\zeta_c}\right)^2\Big)\Big] \quad \text{(D.34)}$$

$$\mathcal{M}_8 \;=\; \mathsf{Re}\left[\frac{-2\,\pi\,\Big(a^2\,\zeta_c\left({r_c}^2-\delta^2\right)^2 + \zeta_c\left({r_c}^3 - r_c\,\delta^2\right)^2 + a^4\,\Big({r_c}^2\,\left(\zeta_c - 2\,\overline{\zeta_c}\right) + \zeta_c\,\overline{\zeta_c}^2\Big)\Big)}{r_c\left({r_c}^2-\delta^2\right)^2}\right] \quad \text{(D.35)}$$

$$\mathcal{M}_9 = \mathsf{Re}\Big[\pi\,\Big(-2\,{r_c}^4\,\left(2\,\zeta_c - \overline{\zeta_c}\right)\,\left({r_c}^2-\delta^2\right)^9 + 2\,a^2\left({r_c}^2-\delta^2\right)^7\,\Big({r_c}^6\,\left(3\,\zeta_c - 2\,\overline{\zeta_c}\right) + {r_c}^2\,{\zeta_c}^2\,\left(11\,\zeta_c - 2\,\overline{\zeta_c}\right)\,\overline{\zeta_c}^2$$
$$-4\,{\zeta_c}^4\,\overline{\zeta_c}^3 - 2\,{r_c}^4\,\overline{\zeta_c}\,\Big(5\,{\zeta_c}^2 - 4\,\delta^2 + \overline{\zeta_c}^2\Big)\Big) + a^{12}\,{r_c}^2\,\overline{\zeta_c}^4\,\Big({r_c}^4\,\left(\zeta_c - 3\,\overline{\zeta_c}\right) - 4\,{\zeta_c}^2\,\overline{\zeta_c}^2\,\zeta_x$$
$$-{r_c}^2\,\delta^2\,\left(\zeta_c + 7\,\overline{\zeta_c} - 4\,i\,\zeta_y\right)\,\Big) + 2\,a^{10}\,\overline{\zeta_c}^2\left({r_c}^2-\delta^2\right)^2\,\Big(4\,\zeta_c\,\overline{\zeta_c}^6 + {r_c}^4\,\overline{\zeta_c}\,\Big({\zeta_c}^2 - 6\,\overline{\zeta_c}^2 + 4\,\zeta_c\,\left(\overline{\zeta_c} - i\,\zeta_y\right)\Big)$$
$$-{r_c}^2\,\overline{\zeta_c}^2\,\Big(2\,{\zeta_c}^3 + 9\,\overline{\zeta_c}^3 + 2\,{\zeta_c}^2\,\zeta_x\Big) + {r_c}^6\,\left(-9\,\zeta_c + 3\,\overline{\zeta_c} + 8\,i\,\zeta_y\right)\Big)$$
$$+a^4\,{r_c}^2\left({r_c}^2-\delta^2\right)^5\,\Big(3\,{r_c}^6\,\left(\zeta_c - 3\,\overline{\zeta_c}\right) + 4\,{\zeta_c}^2\,\overline{\zeta_c}^4\,\left(5\,\zeta_c + 2\,\overline{\zeta_c}\right) + {r_c}^2\,\overline{\zeta_c}^2\,\Big(3\,{\zeta_c}^3 - 20\,\zeta_c\,\overline{\zeta_c}^2 + 2\,\overline{\zeta_c}^3$$
$$-{\zeta_c}^2\,\left(53\,\overline{\zeta_c} + 4\,i\,\zeta_y\right)\Big) + 2\,{r_c}^4\,\overline{\zeta_c}\,\Big(-3\,{\zeta_c}^2 + 21\,\delta^2 + 4\,\overline{\zeta_c}^2 + 2\,i\,\zeta_c\,\zeta_y\Big)\Big)$$
$$+a^8\left({r_c}^3 - r_c\,\delta^2\right)^2\,\Big(4\,{\zeta_c}^2\,\overline{\zeta_c}^4\,\Big(2\,{\zeta_c}^3 + 5\,\overline{\zeta_c}^3 + {\zeta_c}^2\,\zeta_x\Big)$$
$$-{r_c}^2\,\zeta_c\,\overline{\zeta_c}^3\,\Big(33\,{\zeta_c}^3 - 21\,\zeta_c\,\overline{\zeta_c}^2 + \overline{\zeta_c}^2\,\left(43\,\overline{\zeta_c} - 4\,i\,\zeta_y\right) + {\zeta_c}^2\,\left(35\,\overline{\zeta_c} + 36\,i\,\zeta_y\right)\Big)$$
$$+{r_c}^4\,\overline{\zeta_c}^2\,\Big(27\,{\zeta_c}^3 - 45\,\zeta_c\,\overline{\zeta_c}^2 + 27\,\overline{\zeta_c}^3 + {\zeta_c}^2\,\left(39\,\overline{\zeta_c} + 76\,i\,\zeta_y\right)\Big)$$
$$+{r_c}^8\,\left(-15\,\zeta_c - 19\,\overline{\zeta_c} + 16\,i\,\zeta_y\right) + {r_c}^6\,\overline{\zeta_c}\,\Big(13\,{\zeta_c}^2 + 11\,\delta^2 + 24\,\overline{\zeta_c}^2 - 60\,i\,\zeta_c\,\zeta_y\Big)\Big)$$
$$+2\,a^6\left({r_c}^2-\delta^2\right)^4\,\Big(-8\,{\zeta_c}^3\,\overline{\zeta_c}^6 + {r_c}^8\,\left(-3\,\zeta_c + 2\,\overline{\zeta_c}\right)$$
$$+2\,{r_c}^2\,\zeta_c\,\overline{\zeta_c}^2\,\Big(-{\zeta_c}^4 + 12\,\zeta_c\,\overline{\zeta_c}^3 + \overline{\zeta_c}^4 + {\zeta_c}^3\,\zeta_x\Big) + {r_c}^6\,\zeta_c\,\Big(-2\,{\zeta_c}^2 + 8\,\delta^2 - 7\,\overline{\zeta_c}^2 + 2\,i\,\zeta_c\,\zeta_y\Big)$$
$$+{r_c}^4\,\overline{\zeta_c}\,\Big(4\,{\zeta_c}^4 - 7\,{\zeta_c}^3\,\overline{\zeta_c} + 5\,{\zeta_c}^2\,\overline{\zeta_c}^2 - 20\,\zeta_c\,\overline{\zeta_c}^3 - 2\,\overline{\zeta_c}^4 - 4\,i\,\zeta_c\,\Big({\zeta_c}^2 + \overline{\zeta_c}^2\Big)\,\zeta_y\Big)\Big)\,\Big)$$
$$/\Big(2\,{r_c}^2\left({r_c}^2 + \left(a-\zeta_c\right)\,\overline{\zeta_c}\right)^2\left({r_c}^2-\delta^2\right)^5\left({r_c}^2 - \left(a+\zeta_c\right)\,\overline{\zeta_c}\right)^2\Big)\Big] \quad \text{(D.36)}$$

$$
\begin{aligned}
\mathcal{M}_{10} = \mathsf{Re}\Big[& -i\,\pi\,\Big(-2\,r_c{}^4\,(2\,\zeta_c + \overline{\zeta_c})\,(r_c{}^2 - \delta^2)^9 - 2\,a^2\,(r_c{}^2 - \delta^2)^7\,\Big(-4\,\zeta_c{}^4\,\overline{\zeta_c}^3 + r_c{}^6\,(3\,\zeta_c + 2\,\overline{\zeta_c}) \\
& + r_c{}^2\,\zeta_c{}^2\,\overline{\zeta_c}^2\,(11\,\zeta_c + 2\,\overline{\zeta_c}) - 2\,r_c{}^4\,\overline{\zeta_c}\,\Big(5\,\zeta_c{}^2 + 4\,\delta^2 + \overline{\zeta_c}^2\Big)\,\Big) + a^{12}\,r_c{}^2\,\overline{\zeta_c}^4\,\Big(r_c{}^4\,(\zeta_c + 3\,\overline{\zeta_c}) + 4\,\zeta_c{}^2\,\overline{\zeta_c}^2\,(\overline{\zeta_c} - i\,\zeta_y) \\
& + r_c{}^2\,\delta^2\,(-\zeta_c + 7\,\overline{\zeta_c} + 4\,i\,\zeta_y)\,\Big) + 2\,a^{10}\,\overline{\zeta_c}^2\,(r_c{}^2 - \delta^2)^2\,\Big(-4\,\zeta_c\,\overline{\zeta_c}^6 + r_c{}^2\,\overline{\zeta_c}^2\,\Big(-2\,\zeta_c{}^3 + 9\,\overline{\zeta_c}^3 + 2\,\zeta_c{}^2\,(\overline{\zeta_c} - i\,\zeta_y)\Big) \\
& + r_c{}^4\,\overline{\zeta_c}\,\Big(\zeta_c{}^2 - 6\,\overline{\zeta_c}^2 - 4\,\zeta_c\,\zeta_x\Big) + r_c{}^6\,(-9\,\zeta_c - 3\,\overline{\zeta_c} + 8\,i\,\zeta_y)\Big) \\
& + a^4\,r_c{}^2\,(r_c{}^2 - \delta^2)^5\,\Big(4\,\zeta_c{}^2\,\overline{\zeta_c}^4\,(-5\,\zeta_c + 2\,\overline{\zeta_c}) + 3\,r_c{}^6\,(\zeta_c + 3\,\overline{\zeta_c}) + r_c{}^2\,\overline{\zeta_c}^2\,\Big(3\,\zeta_c{}^3 - 20\,\zeta_c\,\overline{\zeta_c}^2 - 2\,\overline{\zeta_c}^3 \\
& + \zeta_c{}^2\,(53\,\overline{\zeta_c} - 4\,i\,\zeta_y)) + 2\,r_c{}^4\,\overline{\zeta_c}\,\Big(-3\,\zeta_c{}^2 - 21\,\delta^2 + 4\,\overline{\zeta_c}^2 \\
& + 2\,i\,\zeta_c\,\zeta_y)) + a^8\,(r_c{}^3 - r_c\,\delta^2)^2\,\Big(4\,\zeta_c{}^2\,\overline{\zeta_c}^4\,\Big(2\,\zeta_c{}^3 - 5\,\overline{\zeta_c}^3 - \zeta_c{}^2\,(\overline{\zeta_c} - i\,\zeta_y)\Big) \\
& + r_c{}^2\,\zeta_c\,\overline{\zeta_c}^3\,\Big(-33\,\zeta_c{}^3 + 21\,\zeta_c\,\overline{\zeta_c}^2 + \overline{\zeta_c}^2\,(43\,\overline{\zeta_c} + 4\,i\,\zeta_y) + \zeta_c{}^2\,(35\,\overline{\zeta_c} - 36\,i\,\zeta_y)\Big) \\
& + r_c{}^4\,\overline{\zeta_c}^2\,\Big(27\,\zeta_c{}^3 - 45\,\zeta_c\,\overline{\zeta_c}^2 - 27\,\overline{\zeta_c}^3 + \zeta_c{}^2\,(-39\,\overline{\zeta_c} + 76\,i\,\zeta_y)\Big) + r_c{}^8\,(-15\,\zeta_c + 19\,\overline{\zeta_c} + 16\,i\,\zeta_y) \\
& + r_c{}^6\,\overline{\zeta_c}\,\Big(13\,\zeta_c{}^2 - 11\,\delta^2 + 24\,\overline{\zeta_c}^2 - 60\,i\,\zeta_c\,\zeta_y\Big)\Big) \\
& + 2\,a^6\,(r_c{}^2 - \delta^2)^4\,\Big(8\,\zeta_c{}^3\,\overline{\zeta_c}^6 + r_c{}^8\,(3\,\zeta_c + 2\,\overline{\zeta_c}) - 2\,r_c{}^2\,\zeta_c\,\overline{\zeta_c}^2\,\Big(\zeta_c{}^4 + 12\,\zeta_c\,\overline{\zeta_c}^3 - \overline{\zeta_c}^4 + \zeta_c{}^3\,(\overline{\zeta_c} - i\,\zeta_y)\Big) \\
& - r_c{}^6\,\zeta_c\,\Big(2\,\zeta_c{}^2 + 8\,\delta^2 + 7\,\overline{\zeta_c}^2 - 2\,i\,\zeta_c\,\zeta_y\Big) + r_c{}^4\,\overline{\zeta_c}\,(4\,\zeta_c{}^4 + 7\,\zeta_c{}^3\,\overline{\zeta_c} \\
& + 5\,\zeta_c{}^2\,\overline{\zeta_c}^2 + 20\,\zeta_c\,\overline{\zeta_c}^3 - 2\,\overline{\zeta_c}^4 - 4\,i\,\zeta_c\,\Big(\zeta_c{}^2 + \overline{\zeta_c}^2\Big)\,\zeta_y\Big)\Big)\,\Big) \\
& / \Big(2\,r_c{}^2\,(r_c{}^2 + (a - \zeta_c)\,\overline{\zeta_c})^2\,(r_c{}^2 - \delta^2)^5\,(r_c{}^2 - (a + \zeta_c)\,\overline{\zeta_c})^2\Big)\Big]
\end{aligned}
\tag{D.37}
$$

$$
\begin{aligned}
\mathcal{M}_{11} = \mathsf{Re}\Big[& -2\,a\,\pi\,\Big(2\,r_c{}^2\,\zeta_c{}^2\,(r_c{}^2 - \delta^2)^8 + a^{10}\,r_c{}^2\,\zeta_c\,\overline{\zeta_c}^5\,(3\,r_c{}^2 + \delta^2) \\
& + a^2\,(r_c{}^2 - \delta^2)^6\,(2\,r_c{}^2 - \delta^2)\,\Big(r_c{}^4 - 2\,r_c{}^2\,\delta^2 - 4\,\zeta_c{}^2\,\overline{\zeta_c}^2\Big) + a^6\,(r_c{}^2 - \delta^2)^2\,(4\,r_c{}^{10} + 7\,r_c{}^8\,\delta^2 \\
& - 31\,r_c{}^6\,\zeta_c{}^2\,\overline{\zeta_c}^2 + 25\,r_c{}^4\,\zeta_c{}^3\,\overline{\zeta_c}^3 - 5\,r_c{}^2\,(2\,r_c{}^4 + \zeta_c{}^4)\,\overline{\zeta_c}^4 + 11\,r_c{}^4\,\zeta_c\,\overline{\zeta_c}^5 + 2\,r_c{}^2\,\zeta_c{}^2\,\overline{\zeta_c}^6 - 4\,\zeta_c{}^3\,\overline{\zeta_c}^7\Big) \\
& + a^4\,(r_c{}^2 - \delta^2)^4\,\Big(4\,\zeta_c{}^3\,\overline{\zeta_c}^5 + r_c{}^2\,\zeta_c{}^2\,\overline{\zeta_c}^2\,\Big(\zeta_c{}^2 - 32\,\overline{\zeta_c}^2\Big) + r_c{}^6\,\Big(\zeta_c{}^2 - 14\,\overline{\zeta_c}^2\Big) + r_c{}^4\,\Big(-2\,\zeta_c{}^3\,\overline{\zeta_c} + 46\,\zeta_c\,\overline{\zeta_c}^3\Big)\Big) \\
& + a^8\,\overline{\zeta_c}^2\,(r_c{}^2 - \delta^2)^2\,\Big(4\,r_c{}^6 - 6\,r_c{}^4\,\delta^2 - 4\,\zeta_c\,\overline{\zeta_c}^5 + r_c{}^2\,\Big(3\,\zeta_c{}^2\,\overline{\zeta_c}^2 + 10\,\overline{\zeta_c}^4\Big)\Big)\Big) \\
& / \Big(r_c{}^2\,(r_c{}^2 + (a - \zeta_c)\,\overline{\zeta_c})^2\,(r_c{}^2 - \delta^2)^4\,(r_c{}^2 - (a + \zeta_c)\,\overline{\zeta_c})^2\Big)\Big]
\end{aligned}
\tag{D.38}
$$

$$\mathcal{M}_{12} = \mathsf{Re}\Big[2\,\pi\,\Big(a^8\,{r_c}^4\,\delta^2 - {r_c}^2\,\delta^2\left({r_c}^2-\delta^2\right)^5 + a^2\,{\zeta_c}^2\left(-{r_c}^2+\delta^2\right)^5 \\ -a^4\left({r_c}^2-\delta^2\right)^3\left(2\,{r_c}^4 - 3\,{r_c}^2\,\delta^2 + 2\,{\zeta_c}^2\,\overline{\zeta_c}^2\right) + a^6\left({r_c}^2-\delta^2\right)^2\left(\zeta_c\,\overline{\zeta_c}^3 + {r_c}^2\left({\zeta_c}^2 - 2\,\overline{\zeta_c}^2\right)\right)\Big) \\ \Big/\left(r_c\left({r_c}^2-\delta^2\right)^5\right)\Big] \quad \text{(D.39)}$$

$$\mathcal{N}_1 = \mathsf{Re}\Big\{\mathcal{N}_{1a}\log(\frac{r_c^2 + a\zeta_c - \delta^2}{r_c^2}) + \mathcal{N}_{1b}\log(\frac{r_c^2-\delta^2}{r_c^2}) \\ +\mathcal{N}_{1c}\log(\frac{a}{\overline{\zeta_c}}) + \frac{\mathcal{N}_{1d}}{(r_c^2 + a\overline{\zeta_c} - \delta^2)(r_c^2-\delta^2)^6} + \frac{\mathcal{N}_{1e}}{(r_c^2-\delta^2)^6}\Big\} \quad \text{(D.40)}$$

$$\mathcal{N}_{1a} = \frac{6\,a^6\,\pi\,{r_c}^4\left(-\left({r_c}^4\,\zeta_c\right) + a^2\,{\zeta_c}^3 - {r_c}^4\,\overline{\zeta_c} + a^2\,{\zeta_c}^2\,\overline{\zeta_c} + 2\,{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c} + 2\,{r_c}^2\,\zeta_c\,\overline{\zeta_c}^2 - {\zeta_c}^3\,\overline{\zeta_c}^2 - {\zeta_c}^2\,\overline{\zeta_c}^3\right)\zeta_y}{{\zeta_c}^2\left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)^6} \quad \text{(D.41)}$$

$$\mathcal{N}_{1b} = \frac{-6\,i\left(-\left(a^4\,\pi\,{r_c}^4\right) + a^6\,\pi\,{\zeta_c}^2\right)}{{\zeta_c}^2\,\overline{\zeta_c}^4} - \frac{12\,i\,a^6\,\pi\,\zeta_c}{{r_c}^2\,\overline{\zeta_c}^3} - \frac{2\,i\left(-3\,a^4\,\pi\,{r_c}^8 + a^6\,\pi\,{r_c}^2\,{\zeta_c}^4 + 9\,a^6\,\pi\,{\zeta_c}^6\right)}{{r_c}^4\,{\zeta_c}^4\,\overline{\zeta_c}^2} \\ - \frac{12\,i\left(a^6\,\pi\,{r_c}^2\,\zeta_c + 2\,a^6\,\pi\,{\zeta_c}^3\right)}{{r_c}^6\,\overline{\zeta_c}} + \frac{6\,i\left(a^8\,\pi\,{r_c}^8 + 2\,a^8\,\pi\,{r_c}^6\,{\zeta_c}^2 + a^8\,\pi\,{r_c}^4\,{\zeta_c}^4\right)}{{\zeta_c}^2\left(-{r_c}^2 + \zeta_c\,\overline{\zeta_c}\right)^6} \\ + \frac{4\,i\left(3\,a^8\,\pi\,{r_c}^6 + 2\,a^8\,\pi\,{r_c}^4\,{\zeta_c}^2\right)}{{\zeta_c}^2\left(-{r_c}^2 + \zeta_c\,\overline{\zeta_c}\right)^5} + \frac{6\,i\left(a^8\,\pi\,{r_c}^4 - a^6\,\pi\,{r_c}^6 - 2\,a^6\,\pi\,{r_c}^4\,{\zeta_c}^2 - a^6\,\pi\,{r_c}^2\,{\zeta_c}^4\right)}{{\zeta_c}^2\left(-{r_c}^2 + \zeta_c\,\overline{\zeta_c}\right)^4} - \frac{4\,i\left(a^6\,\pi\,{r_c}^4 - 2\,a^6\,\pi\,{\zeta_c}^4\right)}{{\zeta_c}^2\left(-{r_c}^2 + \zeta_c\,\overline{\zeta_c}\right)^3} \\ - \frac{2\,i\,a^6\left(3\,\pi\,{r_c}^8 + 5\,\pi\,{r_c}^2\,{\zeta_c}^6 + 3\,\pi\,{\zeta_c}^8\right)}{{r_c}^4\,{\zeta_c}^4\left(-{r_c}^2 + \zeta_c\,\overline{\zeta_c}\right)^2} + \frac{12\,i\,a^6\left(\pi\,{r_c}^2\,{\zeta_c}^2 + 2\,\pi\,{\zeta_c}^4\right)}{{r_c}^6\left(-{r_c}^2 + \zeta_c\,\overline{\zeta_c}\right)} \quad \text{(D.42)}$$

$$\mathcal{N}_{1c} = \frac{-6\,a^6\,\pi\,{r_c}^4\left(-\left({r_c}^4\,\zeta_c\right) - {r_c}^4\,\overline{\zeta_c} + 2\,{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c} + a^2\,\zeta_c\,\overline{\zeta_c}^2 + 2\,{r_c}^2\,\zeta_c\,\overline{\zeta_c}^2 - {\zeta_c}^3\,\overline{\zeta_c}^2 + a^2\,\overline{\zeta_c}^3 - {\zeta_c}^2\,\overline{\zeta_c}^3\right)\zeta_y}{\overline{\zeta_c}^2\left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)^6} \quad \text{(D.43)}$$

$$
\begin{aligned}
\mathcal{N}_{1d} = i\, a^2\, \pi\, {r_c}^2\, \Big(& 3\, a\, {r_c}^2\, \zeta_c\, \overline{\zeta_c} \left({r_c}^2 - \zeta_c\, \overline{\zeta_c}\right)^6 + 3\, {r_c}^2\, \zeta_c \left({r_c}^2 - \zeta_c\, \overline{\zeta_c}\right)^7 \\
& - a^7\, \zeta_c\, \overline{\zeta_c}^3\, \left(-{r_c}^6 + 4\, {r_c}^4\, \zeta_c\, \overline{\zeta_c} + {\zeta_c}^2\, \overline{\zeta_c}^4 + 2\, {r_c}^2\, \zeta_c\, \overline{\zeta_c}^2\, \left(\zeta_c + 2\, \overline{\zeta_c}\right)\right) \\
& + a^6\, \zeta_c\, \overline{\zeta_c}^2\, \left({r_c}^2 - \zeta_c\, \overline{\zeta_c}\right) \Big({r_c}^6 - {\zeta_c}^2\, \overline{\zeta_c}^4 - 2\, {r_c}^4\, \overline{\zeta_c}\, \left(2\, \zeta_c + \overline{\zeta_c}\right) - 2\, {r_c}^2\, \zeta_c\, \overline{\zeta_c}^2\, \left(\zeta_c + 4\, \overline{\zeta_c} + 3\, i\, \zeta_y\right)\Big) \\
& + a^2\, \overline{\zeta_c} \left({r_c}^2 - \zeta_c\, \overline{\zeta_c}\right)^3 \Big(6\, {r_c}^8 - 22\, {r_c}^6\, \zeta_c\, \overline{\zeta_c} - {\zeta_c}^3\, \overline{\zeta_c}^5 + {r_c}^4\, \zeta_c\, \overline{\zeta_c}^2\, \left(30\, \zeta_c + (-1 - 2\, i\, \beta + 2\, i\, \pi)\, \overline{\zeta_c}\right) \\
& - 2\, {r_c}^2\, {\zeta_c}^2\, \overline{\zeta_c}^3\, \left(10\, \zeta_c + (2 + 2\, i\, \beta - 2\, i\, \pi)\, \overline{\zeta_c} + 3\, (-\beta + \pi)\, \zeta_y\right)\Big) \\
& - a^5\, \overline{\zeta_c} \left({r_c}^2 - \zeta_c\, \overline{\zeta_c}\right)^2 \Big(3\, {r_c}^6\, \zeta_c + 2\, {r_c}^4\, \overline{\zeta_c}^2\, \left(i\, (-\beta + \pi)\, \zeta_c + \overline{\zeta_c}\right) + \zeta_c\, \left(2\, \zeta_c - \overline{\zeta_c}\right)\, \overline{\zeta_c}^3\, \left({\zeta_c}^2 + \overline{\zeta_c}^2\right) \\
& + {r_c}^2\, \overline{\zeta_c}^2\, \Big(-6\, {\zeta_c}^3 + 2\, \overline{\zeta_c}^3 + 3\, \zeta_c\, \overline{\zeta_c}\, \left(\overline{\zeta_c} + 2\, i\, \zeta_y\right) + 2\, (\beta - \pi)\, {\zeta_c}^2\, \left(-2\, i\, \overline{\zeta_c} + 3\, \zeta_y\right)\Big)\Big) \\
& - a^4 \left({r_c}^2 - \zeta_c\, \overline{\zeta_c}\right)^3 \Big(3\, {r_c}^6\, \zeta_c + 2\, (1 - i\, \beta + i\, \pi)\, {r_c}^4\, \zeta_c\, \overline{\zeta_c}^2 + \zeta_c\, \left(2\, \zeta_c - \overline{\zeta_c}\right)\, \overline{\zeta_c}^3\, \left({\zeta_c}^2 + \overline{\zeta_c}^2\right) \\
& + {r_c}^2\, \overline{\zeta_c}^2\, \Big(-6\, {\zeta_c}^3 - \zeta_c\, \overline{\zeta_c}^2 + 2\, \overline{\zeta_c}^3 + 2\, (i + \beta - \pi)\, {\zeta_c}^2\, \left(-2\, i\, \overline{\zeta_c} + 3\, \zeta_y\right)\Big)\Big) \\
& + a^3\, \overline{\zeta_c}^2 \left({r_c}^2 - \zeta_c\, \overline{\zeta_c}\right)^2 \Big(6\, {r_c}^8 - 22\, {r_c}^6\, \zeta_c\, \overline{\zeta_c} - {\zeta_c}^3\, \overline{\zeta_c}^5 + {r_c}^4\, \zeta_c\, \overline{\zeta_c}^2\, \left(28\, \zeta_c + (-1 - 2\, i\, \beta + 2\, i\, \pi)\, \overline{\zeta_c}\right) \\
& + 2\, {r_c}^2\, {\zeta_c}^2\, \overline{\zeta_c}^2\, \left(-3\, \zeta_c\, \left(4\, \overline{\zeta_c} + i\, \zeta_y\right) + \overline{\zeta_c}\, \left(2\, i\, (i - \beta + \pi)\, \overline{\zeta_c} + 3\, (\beta - \pi)\, \zeta_y\right)\right)\Big)\Big) \Big/ \left(\zeta_c\, \overline{\zeta_c}^4\right)
\end{aligned}
\tag{D.44}
$$

$$
\begin{aligned}
\mathcal{N}_{1e} = i\, \pi\, \Big(& {r_c}^4\, {\zeta_c}^4\, \overline{\zeta_c}^2 \left({r_c}^2 - \zeta_c\, \overline{\zeta_c}\right)^6 + a^2\, \overline{\zeta_c}^2 \left({r_c}^2 - \zeta_c\, \overline{\zeta_c}\right)^6\, \left(3\, {r_c}^6 - 2\, {r_c}^2\, {\zeta_c}^4 + 2\, {\zeta_c}^5\, \overline{\zeta_c}\right) \\
& + 2\, a^7\, {r_c}^4\, {\zeta_c}^3\, \overline{\zeta_c}^2\, \left({r_c}^2 - 2\, \zeta_c\, \overline{\zeta_c}\right)\, \left({r_c}^2 - \zeta_c\, \overline{\zeta_c}\right)\, \left({r_c}^2 + 2\, \zeta_c\, \overline{\zeta_c} - 3\, i\, \overline{\zeta_c}\, \zeta_y\right) \\
& - 2\, a^5\, {r_c}^4\, {\zeta_c}^4\, \overline{\zeta_c}\, \left({r_c}^2 - 2\, \zeta_c\, \overline{\zeta_c}\right)\, \left({r_c}^2 - \zeta_c\, \overline{\zeta_c}\right)^2\, \left({r_c}^2 + 2\, \zeta_c\, \overline{\zeta_c} - 3\, i\, \overline{\zeta_c}\, \zeta_y\right) \\
& - a^6 \left({r_c}^2 - \zeta_c\, \overline{\zeta_c}\right)^2 \Big(2\, i\, (i + \beta - \pi)\, {r_c}^8\, {\zeta_c}^4 + 3\, {r_c}^{10}\, \overline{\zeta_c}^2 + 2\, {\zeta_c}^5\, \overline{\zeta_c}^7 \\
& + {r_c}^2\, {\zeta_c}^4\, \overline{\zeta_c}^2\, \left({\zeta_c}^4 - 7\, \overline{\zeta_c}^4 - 2\, {\zeta_c}^2\, \overline{\zeta_c}\, \left(\overline{\zeta_c} + i\, \zeta_y\right)\right) \\
& + {r_c}^4\, {\zeta_c}^3\, \overline{\zeta_c}^2\, \left(11\, {\zeta_c}^3 - 5\, \zeta_c\, \overline{\zeta_c}^2 + 2\, \overline{\zeta_c}^2\, \left(\overline{\zeta_c} - 3\, i\, \zeta_y\right) + {\zeta_c}^2\, \left(6\, \overline{\zeta_c} - 4\, i\, \zeta_y\right)\right) \\
& + {r_c}^6\, {\zeta_c}^2\, \overline{\zeta_c}\, \left(4\, i\, \beta\, {\zeta_c}^3 - 4\, i\, \pi\, {\zeta_c}^3 + {\zeta_c}^2\, \overline{\zeta_c} + 2\, \zeta_c\, \overline{\zeta_c}^2 - 6\, \overline{\zeta_c}^3 + 6\, (i + \beta - \pi)\, {\zeta_c}^2\, \zeta_y\right)\Big) \\
& + a^8\, {r_c}^2\, {\zeta_c}^2\, \overline{\zeta_c}^2\, \Big({r_c}^8 + {r_c}^6\, \zeta_c\, \left(\zeta_c - 4\, \overline{\zeta_c}\right) + 2\, {\zeta_c}^4\, \overline{\zeta_c}^3\, \left(\overline{\zeta_c} + i\, \zeta_y\right) \\
& - 2\, {r_c}^2\, {\zeta_c}^3\, \overline{\zeta_c}\, \left({\zeta_c}^2 + 8\, \zeta_c\, \overline{\zeta_c} + 6\, \overline{\zeta_c}^2 + i\, \zeta_c\, \zeta_y\right) - {r_c}^4\, {\zeta_c}^2\, \Big(9\, {\zeta_c}^2 + 11\, \overline{\zeta_c}^2 + 2\, \zeta_c\, \left(8\, \overline{\zeta_c} - 5\, i\, \zeta_y\right) \\
& + 2\, i\, \overline{\zeta_c}\, \zeta_y - 4\, {\zeta_y}^2\Big)\Big) + a^4\, \zeta_c\, \overline{\zeta_c} \left({r_c}^3 - r_c\, \zeta_c\, \overline{\zeta_c}\right)^2\, \Big(6\, {r_c}^{10} - 22\, {r_c}^8\, \zeta_c\, \overline{\zeta_c} + {r_c}^6\, {\zeta_c}^2\, \overline{\zeta_c}\, \left((3 + 2\, i\, \beta - 2\, i\, \pi)\, \zeta_c + 30\, \overline{\zeta_c}\right) \\
& + 2\, {\zeta_c}^5\, \overline{\zeta_c}^3\, \left({\zeta_c}^2 + \overline{\zeta_c}^2\right) - {r_c}^2\, {\zeta_c}^4\, \overline{\zeta_c}^2\, \left(4\, {\zeta_c}^2 + 3\, \zeta_c\, \overline{\zeta_c} + 2\, \overline{\zeta_c}^2 + 2\, i\, \left(\zeta_c - 3\, \overline{\zeta_c}\right)\, \zeta_y\right) \\
& + 2\, {r_c}^4\, {\zeta_c}^3\, \overline{\zeta_c}\, \left({\zeta_c}^2 + i\, \zeta_c\, \left((3\, i + 2\, \beta - 2\, \pi)\, \overline{\zeta_c} + \zeta_y\right) + \overline{\zeta_c}\, \left(-7\, \overline{\zeta_c} + 3\, \beta\, \zeta_y - 3\, \pi\, \zeta_y\right)\right)\Big)\Big) / {r_c}^2\, {\zeta_c}^4\, \overline{\zeta_c}^2
\end{aligned}
\tag{D.45}
$$

$$\mathcal{N}_2 = \mathsf{Re}\left\{\mathcal{N}_{2b}\log(\frac{r_c^2-\delta^2}{r_c^2}) + \mathcal{N}_{2c}\log(\frac{a}{\overline{\zeta_c}}) + \frac{\mathcal{N}_{2d}}{(r_c^2+a\overline{\zeta_c}-\delta^2)(r_c^2-\delta^2)^6} + \frac{\mathcal{N}_{2e}}{(r_c^2-\delta^2)^6}\right\} \tag{D.46}$$

$$\begin{aligned}\mathcal{N}_{2b} = & \frac{-12\,\left(-\left(a^4\,\pi\,{r_c}^4\right)+a^6\,\pi\,{\zeta_c}^2\right)}{{\zeta_c}^2\,\overline{\zeta_c}^4} - \frac{24\,a^6\,\pi\,\zeta_c}{{r_c}^2\,\overline{\zeta_c}^3} - \frac{12\,\left(a^4\,\pi\,{r_c}^8+3\,a^6\,\pi\,{\zeta_c}^6\right)}{{r_c}^4\,{\zeta_c}^4\,\overline{\zeta_c}^2} \\ & - \frac{48\,a^6\,\pi\,{\zeta_c}^3}{{r_c}^6\,\overline{\zeta_c}} - \frac{12\,\left(a^8\,\pi\,{r_c}^8-a^8\,\pi\,{r_c}^4\,{\zeta_c}^4\right)}{{\zeta_c}^2\,\left(-{r_c}^2+\zeta_c\,\overline{\zeta_c}\right)^6} - \frac{24\,a^8\,\pi\,{r_c}^6}{{\zeta_c}^2\,\left(-{r_c}^2+\zeta_c\,\overline{\zeta_c}\right)^5} \\ & - \frac{12\,a^8\,\pi\,{r_c}^4}{{\zeta_c}^2\,\left(-{r_c}^2+\zeta_c\,\overline{\zeta_c}\right)^4} + \frac{12\,a^6\,\left(\pi\,{r_c}^8-\pi\,{\zeta_c}^8\right)}{{r_c}^4\,{\zeta_c}^4\,\left(-{r_c}^2+\zeta_c\,\overline{\zeta_c}\right)^2} + \frac{48\,a^6\,\pi\,{\zeta_c}^4}{{r_c}^6\,\left(-{r_c}^2+\zeta_c\,\overline{\zeta_c}\right)}\end{aligned} \tag{D.47}$$

$$\begin{aligned}\mathcal{N}_{2d} = & -2\,a^2\,\pi\,{r_c}^6\,\left(-\left(a^4\,{r_c}^4\right)+4\,a^7\,\zeta_c-80\,a^3\,{r_c}^4\,\zeta_c-2\,a^6\,{\zeta_c}^2+182\,a^2\,{r_c}^4\,{\zeta_c}^2+14\,a^5\,{\zeta_c}^3\right. \\ & \left.+60\,a\,{r_c}^4\,{\zeta_c}^3-24\,a^4\,{\zeta_c}^4-105\,{r_c}^4\,{\zeta_c}^4\right) - \frac{6\,\left(a^6\,\pi\,{r_c}^{14}-a^2\,\pi\,{r_c}^{18}\right)}{\overline{\zeta_c}^4} \\ & - \frac{6\,\left(-2\,a^4\,\pi\,{r_c}^{16}+a^7\,\pi\,{r_c}^{12}\,\zeta_c-a^3\,\pi\,{r_c}^{16}\,\zeta_c-3\,a^6\,\pi\,{r_c}^{12}\,{\zeta_c}^2+7\,a^2\,\pi\,{r_c}^{16}\,{\zeta_c}^2\right)}{\zeta_c\,\overline{\zeta_c}^3} \\ & + \frac{2\,\left(6\,a^5\,\pi\,{r_c}^{14}+a^8\,\pi\,{r_c}^{10}\,\zeta_c-40\,a^4\,\pi\,{r_c}^{14}\,\zeta_c+6\,a^7\,\pi\,{r_c}^{10}\,{\zeta_c}^2-18\,a^3\,\pi\,{r_c}^{14}\,{\zeta_c}^2-3\,a^6\,\pi\,{r_c}^{10}\,{\zeta_c}^3+63\,a^2\,\pi\,{r_c}^{14}\,{\zeta_c}^3\right)}{\zeta_c\,\overline{\zeta_c}^2} \\ & + \frac{2\,\left(a^9\,\pi\,{r_c}^8-34\,a^5\,\pi\,{r_c}^{12}-5\,a^8\,\pi\,{r_c}^8\,\zeta_c+114\,a^4\,\pi\,{r_c}^{12}\,\zeta_c+3\,a^7\,\pi\,{r_c}^8\,{\zeta_c}^2\right)}{\overline{\zeta_c}} \\ & + \frac{2\left(45\,a^3\,\pi\,{r_c}^{12}\,{\zeta_c}^2-17\,a^6\,\pi\,{r_c}^8\,{\zeta_c}^3-105\,a^2\,\pi\,{r_c}^{12}\,{\zeta_c}^3\right)}{\overline{\zeta_c}} \\ & - 2\,a^2\,\pi\,{r_c}^4\,\left(-\left(a^5\,{r_c}^4\right)+5\,a^4\,{r_c}^4\,\zeta_c+2\,a^7\,{\zeta_c}^2+102\,a^3\,{r_c}^4\,{\zeta_c}^2-2\,a^6\,{\zeta_c}^3\right. \\ & \left.-172\,a^2\,{r_c}^4\,{\zeta_c}^3-10\,a^5\,{\zeta_c}^4-45\,a\,{r_c}^4\,{\zeta_c}^4+12\,a^4\,{\zeta_c}^5+63\,{r_c}^4\,{\zeta_c}^5\right)\,\overline{\zeta_c} \\ & -2\,a^2\,\pi\,{r_c}^2\,\zeta_c\,\left(4\,a^5\,{r_c}^4-9\,a^4\,{r_c}^4\,\zeta_c-70\,a^3\,{r_c}^4\,{\zeta_c}^2+90\,a^2\,{r_c}^4\,{\zeta_c}^3+2\,a^5\,{\zeta_c}^4+18\,a\,{r_c}^4\,{\zeta_c}^4-2\,a^4\,{\zeta_c}^5-21\,{r_c}^4\,{\zeta_c}^5\right)\,\overline{\zeta_c}^2 \\ & +2\,a^2\,\pi\,{r_c}^4\,{\zeta_c}^2\,\left(5\,a^5-7\,a^4\,\zeta_c-20\,a^3\,{\zeta_c}^2+20\,a^2\,{\zeta_c}^3+3\,a\,{\zeta_c}^4-3\,{\zeta_c}^5\right)\,\overline{\zeta_c}^3-4\,a^6\,\pi\,{r_c}^2\,\left(a-\zeta_c\right)\,{\zeta_c}^3\,\overline{\zeta_c}^4\end{aligned} \tag{D.48}$$

$$\mathcal{N}_{2e} = (1+i)\,\pi\,\Big(r_c{}^4\,\zeta_c{}^4\,\overline{\zeta_c}\left(r_c{}^2-\zeta_c\,\overline{\zeta_c}\right)^6 + (1+i)\,a^2\,\overline{\zeta_c}\left(r_c{}^2-\zeta_c\,\overline{\zeta_c}\right)^6\,\left(3\,i\,r_c{}^6 + r_c{}^2\,\zeta_c{}^4 - (1-i)\,\zeta_c{}^5\,\overline{\zeta_c}\right)$$
$$-(1-i)\,a^4\,r_c{}^2\,\zeta_c\left(r_c{}^2-\zeta_c\,\overline{\zeta_c}\right)^3\,\Big(6\,r_c{}^8 - 16\,r_c{}^6\,\zeta_c\,\overline{\zeta_c} + (1-i)\,\zeta_c{}^6\,\overline{\zeta_c}^2 - 2\,\zeta_c{}^4\,\overline{\zeta_c}^4 + r_c{}^4\,\zeta_c{}^2\,\overline{\zeta_c}\,\left(-i\,\zeta_c + 14\,\overline{\zeta_c}\right)$$
$$+r_c{}^2\,\zeta_c{}^4\,\overline{\zeta_c}\,\left((-1+i)\,\zeta_c - \overline{\zeta_c} - (1+i)\,\zeta_y\right)\Big)$$
$$+a^6\,\overline{\zeta_c}\left(r_c{}^2-\zeta_c\,\overline{\zeta_c}\right)^2\,\Big((3-3\,i)\,r_c{}^{10} + 2\,\zeta_c{}^5\,\overline{\zeta_c}^5 + (2-2\,i)\,r_c{}^6\,\left(\zeta_c{}^4 - 3\,\zeta_c{}^2\,\overline{\zeta_c}^2\right)$$
$$-(1+i)\,r_c{}^4\,\zeta_c{}^3\,\left(\zeta_c{}^3 - 3\,\zeta_c\,\overline{\zeta_c}^2 + 2\,i\,\overline{\zeta_c}^3 + \zeta_c{}^2\,\left(8\,i\,\overline{\zeta_c} - (1-i)\,\zeta_y\right)\right)$$
$$+i\,r_c{}^2\,\left(\zeta_c{}^8 + (2+7\,i)\,\zeta_c{}^4\,\overline{\zeta_c}^4 + 2\,i\,\zeta_c{}^6\,\overline{\zeta_c}\,\left(\overline{\zeta_c}+\zeta_y\right)\right)\Big) + a^8\,r_c{}^2\,\zeta_c{}^2\,\overline{\zeta_c}\,\Big((-1+i)\,r_c{}^8 - r_c{}^6\,\zeta_c\,\left(\zeta_c - (4-4\,i)\,\overline{\zeta_c}\right)$$
$$-2\,\zeta_c{}^4\,\overline{\zeta_c}^3\,\left(\overline{\zeta_c}+\zeta_y\right) + r_c{}^4\,\zeta_c{}^2\,\Big(9\,i\,\zeta_c{}^2 - (4+4\,i)\,\zeta_c\,\overline{\zeta_c} + (11-8\,i)\,\overline{\zeta_c}^2$$
$$+2\,\zeta_c\,\zeta_y - 4\,\overline{\zeta_c}\,\zeta_y + 4\,i\,\zeta_y{}^2\Big) + r_c{}^2\,\zeta_c{}^3\,\overline{\zeta_c}\,\Big(2\,i\,\zeta_c{}^2 + 6\,\overline{\zeta_c}\,\left((2-i)\,\overline{\zeta_c} + \zeta_y\right)$$
$$-\zeta_c\,\left((2+3\,i)\,\overline{\zeta_c} + 2\,\zeta_y\,\right)\Big)\Big)\Big)\,/\,\left(r_c{}^2\,\zeta_c{}^4\,\overline{\zeta_c}\right) \quad \text{(D.49)}$$

$$\mathcal{N}_3 = \mathsf{Re}\Big\{\mathcal{N}_{3a}\log(\frac{r_c^2 + a\zeta_c - \delta^2}{r_c^2}) + \mathcal{N}_{3b}\log(\frac{r_c^2-\delta^2}{r_c^2})$$
$$+\mathcal{N}_{3c}\log(\frac{a}{\overline{\zeta_c}}) + \frac{\mathcal{N}_{3d}}{(r_c^2 + a\overline{\zeta_c} - \delta^2)(r_c^2-\delta^2)^6} + \frac{\mathcal{N}_{3e}}{(r_c^2-\delta^2)^6}\Big\} \quad \text{(D.50)}$$

$$\mathcal{N}_{3a} = \frac{6\,a^6\,\pi\,r_c{}^4\,\left(-\left(r_c{}^4\,\zeta_c\right) + a^2\,\zeta_c{}^3 - r_c{}^4\,\overline{\zeta_c} + a^2\,\zeta_c{}^2\,\overline{\zeta_c} + 2\,r_c{}^2\,\zeta_c{}^2\,\overline{\zeta_c} + 2\,r_c{}^2\,\zeta_c\,\overline{\zeta_c}^2 - \zeta_c{}^3\,\overline{\zeta_c}^2 - \zeta_c{}^2\,\overline{\zeta_c}^3\right)\,\zeta_y}{\zeta_c{}^2\,\left(r_c{}^2-\zeta_c\,\overline{\zeta_c}\right)^6} \quad \text{(D.51)}$$

$$\mathcal{N}_{3b} = \frac{6\,i\,\left(-\left(a^4\,\pi\,r_c{}^4\right) + a^6\,\pi\,\zeta_c{}^2\right)}{\zeta_c{}^2\,\overline{\zeta_c}^4} + \frac{12\,i\,a^6\,\pi\,\zeta_c}{r_c{}^2\,\overline{\zeta_c}^3} - \frac{2\,i\,\left(3\,a^4\,\pi\,r_c{}^8 + a^6\,\pi\,r_c{}^2\,\zeta_c{}^4 - 9\,a^6\,\pi\,\zeta_c{}^6\right)}{r_c{}^4\,\zeta_c{}^4\,\overline{\zeta_c}^2}$$
$$-\frac{12\,i\,\left(a^6\,\pi\,r_c{}^2\,\zeta_c - 2\,a^6\,\pi\,\zeta_c{}^3\right)}{r_c{}^6\,\overline{\zeta_c}} - \frac{6\,i\,\left(a^8\,\pi\,r_c{}^8 - 2\,a^8\,\pi\,r_c{}^6\,\zeta_c{}^2 + a^8\,\pi\,r_c{}^4\,\zeta_c{}^4\right)}{\zeta_c{}^2\,\left(-r_c{}^2+\zeta_c\,\overline{\zeta_c}\right)^6} - \frac{4\,i\,\left(3\,a^8\,\pi\,r_c{}^6 - 2\,a^8\,\pi\,r_c{}^4\,\zeta_c{}^2\right)}{\zeta_c{}^2\,\left(-r_c{}^2+\zeta_c\,\overline{\zeta_c}\right)^5}$$
$$-\frac{6\,i\,\left(a^8\,\pi\,r_c{}^4 + a^6\,\pi\,r_c{}^6 - 2\,a^6\,\pi\,r_c{}^4\,\zeta_c{}^2 + a^6\,\pi\,r_c{}^2\,\zeta_c{}^4\right)}{\zeta_c{}^2\,\left(-r_c{}^2+\zeta_c\,\overline{\zeta_c}\right)^4} - \frac{4\,i\,\left(a^6\,\pi\,r_c{}^4 - 2\,a^6\,\pi\,\zeta_c{}^4\right)}{\zeta_c{}^2\,\left(-r_c{}^2+\zeta_c\,\overline{\zeta_c}\right)^3}$$
$$+\frac{2\,i\,\left(3\,a^6\,\pi\,r_c{}^8 - 5\,a^6\,\pi\,r_c{}^2\,\zeta_c{}^6 + 3\,a^6\,\pi\,\zeta_c{}^8\right)}{r_c{}^4\,\zeta_c{}^4\,\left(-r_c{}^2+\zeta_c\,\overline{\zeta_c}\right)^2} + \frac{12\,i\,\left(a^6\,\pi\,r_c{}^2\,\zeta_c{}^2 - 2\,a^6\,\pi\,\zeta_c{}^4\right)}{r_c{}^6\,\left(-r_c{}^2+\zeta_c\,\overline{\zeta_c}\right)} \quad \text{(D.52)}$$

$$\mathcal{N}_{3c} = \frac{-6\,a^6\,\pi\,{r_c}^4\,\left(-\left({r_c}^4\,\zeta_c\right) - {r_c}^4\,\overline{\zeta_c} + 2\,{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c} + a^2\,\zeta_c\,\overline{\zeta_c}^2 + 2\,{r_c}^2\,\zeta_c\,\overline{\zeta_c}^2 - {\zeta_c}^3\,\overline{\zeta_c}^2 + a^2\,\overline{\zeta_c}^3 - {\zeta_c}^2\,\overline{\zeta_c}^3\right)\,\zeta_y}{\overline{\zeta_c}^2\,\left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)^6} \tag{D.53}$$

$$\begin{aligned}
\mathcal{N}_{3d} = {} & -i\,a^2\,\pi\,{r_c}^2\,\Big(3\,a\,{r_c}^2\,\zeta_c\,\overline{\zeta_c}\left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)^6 + 3\,{r_c}^2\,\zeta_c\left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)^7 \\
& + a^7\,\zeta_c\,\overline{\zeta_c}^3\,\Big({r_c}^6 - 4\,{r_c}^4\,\zeta_c\,\overline{\zeta_c} - 2\,{r_c}^2\,\zeta_c\,\left(\zeta_c - 2\,\overline{\zeta_c}\right)\,\overline{\zeta_c}^2 + {\zeta_c}^2\,\overline{\zeta_c}^4\Big) \\
& + a^6\,\zeta_c\,\overline{\zeta_c}^2\,\left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)\,\Big({r_c}^6 + {\zeta_c}^2\,\overline{\zeta_c}^4 + 2\,{r_c}^4\,\overline{\zeta_c}\,\left(-2\,\zeta_c + \overline{\zeta_c}\right) - 2\,{r_c}^2\,\zeta_c\,\overline{\zeta_c}^2\,\left(\zeta_c - 4\,\overline{\zeta_c} - 3\,i\,\zeta_y\right)\Big) \\
& + a^2\,\overline{\zeta_c}\left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)^3\,\Big(6\,{r_c}^8 - 22\,{r_c}^6\,\zeta_c\,\overline{\zeta_c} + {\zeta_c}^3\,\overline{\zeta_c}^5 + {r_c}^4\,\zeta_c\,\overline{\zeta_c}^2\,\left(30\,\zeta_c + \overline{\zeta_c} + 2\,i\,\beta\,\overline{\zeta_c} - 2\,i\,\pi\,\overline{\zeta_c}\right) \\
& + 2\,{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}^3\,\left(-10\,\zeta_c + \left(2 + 2\,i\,\beta - 2\,i\,\pi\,\right)\,\overline{\zeta_c} + 3\,\left(-\beta + \pi\right)\,\zeta_y\right)\Big) \\
& - a^5\,\overline{\zeta_c}\left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)^2\,\Big(3\,{r_c}^6\,\zeta_c + 2\,i\,{r_c}^4\,\left(\beta\,\zeta_c - \pi\,\zeta_c + i\,\overline{\zeta_c}\right)\,\overline{\zeta_c}^2 \\
& + \zeta_c\,\overline{\zeta_c}^3\,\Big(2\,{\zeta_c}^3 + {\zeta_c}^2\,\overline{\zeta_c} + 2\,\zeta_c\,\overline{\zeta_c}^2 + \overline{\zeta_c}^3\Big) - {r_c}^2\,\overline{\zeta_c}^2\,\Big(6\,{\zeta_c}^3 + 2\,\overline{\zeta_c}^3 + \zeta_c\,\overline{\zeta_c}\,\left(5\,\overline{\zeta_c} + 6\,i\,\zeta_y\right) \\
& + 2\,\left(\beta - \pi\right)\,{\zeta_c}^2\,\left(-2\,i\,\overline{\zeta_c} + 3\,\zeta_y\right)\Big)\Big) \\
& - a^4\,\left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)^3\,\Big(3\,{r_c}^6\,\zeta_c + 2\,i\,\left(i + \beta - \pi\right)\,{r_c}^4\,\zeta_c\,\overline{\zeta_c}^2 \\
& + \zeta_c\,\overline{\zeta_c}^3\,\Big(2\,{\zeta_c}^3 + {\zeta_c}^2\,\overline{\zeta_c} + 2\,\zeta_c\,\overline{\zeta_c}^2 + \overline{\zeta_c}^3\Big) - {r_c}^2\,\overline{\zeta_c}^2\,\Big(6\,{\zeta_c}^3 + \zeta_c\,\overline{\zeta_c}^2 + 2\,\overline{\zeta_c}^3 + 2\,\left(i + \beta - \pi\right)\,{\zeta_c}^2\,\left(-2\,i\,\overline{\zeta_c} + 3\,\zeta_y\right)\Big)\Big) \\
& + a^3\,\overline{\zeta_c}^2\,\left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)^2\,\Big(6\,{r_c}^8 - 22\,{r_c}^6\,\zeta_c\,\overline{\zeta_c} + {\zeta_c}^3\,\overline{\zeta_c}^5 + {r_c}^4\,\zeta_c\,\overline{\zeta_c}^2\,\left(32\,\zeta_c + \overline{\zeta_c} + 2\,i\,\beta\,\overline{\zeta_c} - 2\,i\,\pi\,\overline{\zeta_c}\right) \\
& + 2\,{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}^2\,\left(\zeta_c\,\left(-8\,\overline{\zeta_c} + 3\,i\,\zeta_y\right) + \overline{\zeta_c}\,\left(\left(2 + 2\,i\,\beta - 2\,i\,\pi\right)\,\overline{\zeta_c} + 3\,\left(-\beta + \pi\right)\,\zeta_y\right)\right)\,\Big)\Big)\Big/\Big(\zeta_c\,\overline{\zeta_c}^4\Big)
\end{aligned} \tag{D.54}$$

$$\mathcal{N}_{3e} = -i\,\pi\,\left(-\left(r_c{}^4\,\zeta_c{}^4\,\overline{\zeta_c}^2\,\left(r_c{}^2-\zeta_c\,\overline{\zeta_c}\right)^6\right)+a^2\,\overline{\zeta_c}^2\,\left(r_c{}^2-\zeta_c\,\overline{\zeta_c}\right)^6\,\left(3\,r_c{}^6-2\,r_c{}^2\,\zeta_c{}^4+2\,\zeta_c{}^5\,\overline{\zeta_c}\right)\right.$$
$$+2\,a^5\,r_c{}^4\,\zeta_c{}^4\,\overline{\zeta_c}\,\left(r_c{}^2-2\,\zeta_c\,\overline{\zeta_c}\right)\,\left(r_c{}^2-\zeta_c\,\overline{\zeta_c}\right)^2\,\left(r_c{}^2+2\,\zeta_c\,\overline{\zeta_c}-3\,i\,\overline{\zeta_c}\,\zeta_y\right)$$
$$-2\,a^7\,r_c{}^4\,\zeta_c{}^3\,\overline{\zeta_c}^2\,\left(r_c{}^4-3\,r_c{}^2\,\zeta_c\,\overline{\zeta_c}+2\,\zeta_c{}^2\,\overline{\zeta_c}^2\right)\,\left(r_c{}^2+2\,\zeta_c\,\overline{\zeta_c}-3\,i\,\overline{\zeta_c}\,\zeta_y\right)$$
$$-a^6\,\left(r_c{}^2-\zeta_c\,\overline{\zeta_c}\right)^2\,\left(2\,\left(1-i\,\beta+i\,\pi\right)\,r_c{}^8\,\zeta_c{}^4+3\,r_c{}^{10}\,\overline{\zeta_c}^2+2\,\zeta_c{}^5\,\overline{\zeta_c}^7\right.$$
$$+r_c{}^2\,\zeta_c{}^4\,\overline{\zeta_c}^2\,\left(\zeta_c{}^4-7\,\overline{\zeta_c}^4-2\,\zeta_c{}^2\,\overline{\zeta_c}\,\left(\overline{\zeta_c}-i\,\zeta_y\right)\right)$$
$$+r_c{}^4\,\zeta_c{}^3\,\overline{\zeta_c}^2\,\left(-11\,\zeta_c{}^3+5\,\zeta_c\,\overline{\zeta_c}^2+2\,\overline{\zeta_c}^2\,\left(\overline{\zeta_c}+3\,i\,\zeta_y\right)+\zeta_c{}^2\,\left(6\,\overline{\zeta_c}+4\,i\,\zeta_y\right)\right)$$
$$\left.+r_c{}^6\,\zeta_c{}^2\,\overline{\zeta_c}\,\left(-4\,i\,\beta\,\zeta_c{}^3+4\,i\,\pi\,\zeta_c{}^3+\zeta_c{}^2\,\overline{\zeta_c}-2\,\zeta_c\,\overline{\zeta_c}^2-6\,\overline{\zeta_c}^3-6\,i\,\zeta_c{}^2\,\zeta_y-6\,\beta\,\zeta_c{}^2\,\zeta_y+6\,\pi\,\zeta_c{}^2\,\zeta_y\right)\right)$$
$$+a^8\,r_c{}^2\,\zeta_c{}^2\,\overline{\zeta_c}^2\,\left(r_c{}^8-r_c{}^6\,\zeta_c\,\left(\zeta_c+4\,\overline{\zeta_c}\right)+2\,\zeta_c{}^4\,\overline{\zeta_c}^3\,\left(\overline{\zeta_c}-i\,\zeta_y\right)\right.$$
$$\left.-2\,r_c{}^2\,\zeta_c{}^3\,\overline{\zeta_c}\,\left(\zeta_c{}^2-8\,\zeta_c\,\overline{\zeta_c}+6\,\overline{\zeta_c}^2+i\,\zeta_c\,\zeta_y\right)+r_c{}^4\,\zeta_c{}^2\,\left(-9\,\zeta_c{}^2+16\,\zeta_c\,\overline{\zeta_c}-11\,\overline{\zeta_c}^2+2\,i\,\zeta_c\,\zeta_y+2\,i\,\overline{\zeta_c}\,\zeta_y-4\,\zeta_y{}^2\right)\right)$$
$$+a^4\,\zeta_c\,\overline{\zeta_c}\,\left(r_c{}^3-r_c\,\zeta_c\,\overline{\zeta_c}\right)^2\,\left(6\,r_c{}^{10}-22\,r_c{}^8\,\zeta_c\,\overline{\zeta_c}+r_c{}^6\,\zeta_c{}^2\,\overline{\zeta_c}\,\left(\left(-3-2\,i\,\beta+2\,i\,\pi\right)\,\zeta_c+30\,\overline{\zeta_c}\right)\right.$$
$$+2\,\zeta_c{}^5\,\overline{\zeta_c}^3\,\left(\zeta_c{}^2+\overline{\zeta_c}^2\right)-r_c{}^2\,\zeta_c{}^4\,\overline{\zeta_c}^2\,\left(4\,\zeta_c{}^2+\zeta_c\,\left(-3\,\overline{\zeta_c}+2\,i\,\zeta_y\right)+2\,\overline{\zeta_c}\,\left(\overline{\zeta_c}+3\,i\,\zeta_y\right)\right)$$
$$+2\,r_c{}^4\,\zeta_c{}^3\,\overline{\zeta_c}\,\left(\zeta_c{}^2+i\,\zeta_c\,\left(\left(-3\,i-2\,\beta+2\,\pi\right)\,\overline{\zeta_c}+\zeta_y\right)\right.$$
$$\left.\left.\left.+\overline{\zeta_c}\,\left(-7\,\overline{\zeta_c}+3\,\left(-\beta+\pi\right)\,\zeta_y\right)\right)\right)\right)\Big/\left(r_c{}^2\,\zeta_c{}^4\,\overline{\zeta_c}^2\right) \qquad \text{(D.55)}$$

$$\mathcal{N}_4 = \mathsf{Re}\left\{\mathcal{N}_{4a}\log\left(\frac{r_c^2+a\zeta_c-\delta^2}{r_c^2}\right)+\mathcal{N}_{4b}\log\left(\frac{r_c^2-\delta^2}{r_c^2}\right)\right.$$
$$+\mathcal{N}_{4c}\log\left(\frac{a}{\overline{\zeta_c}}\right)+\frac{\mathcal{N}_{4d}}{(r_c^2+a\overline{\zeta_c}-\delta^2)^2(r_c^2-a\overline{\zeta_c}-\delta^2)^2(r_c^2-\delta^2)^5}$$
$$\left.+\frac{\mathcal{N}_{4e}}{(r_c^2+a\overline{\zeta_c}-\delta^2)(r_c^2-\delta^2)^5}+\frac{\mathcal{N}_{4f}}{(r_c^2-\delta^2)^5}\right\} \qquad \text{(D.56)}$$

$$\mathcal{N}_{4a} = \frac{4\,i\,a^5\,\pi\,r_c{}^4\,\left(r_c{}^4-2\,r_c{}^2\,\zeta_c\,\overline{\zeta_c}+\zeta_c{}^2\,\left(-a^2+\overline{\zeta_c}^2\right)\right)\,\left(\zeta_c-\overline{\zeta_c}-4\,i\,\zeta_y\right)}{\zeta_c{}^2\,\left(-r_c{}^2+\zeta_c\,\overline{\zeta_c}\right)^5} \qquad \text{(D.57)}$$

$$\mathcal{N}_{4b} = \frac{8\,i\,\left(-\left(a^3\,\pi\,{r_c}^4\right)+a^5\,\pi\,{\zeta_c}^2\right)}{{\zeta_c}^2\,{\overline{\zeta_c}}^3} + \frac{8\,i\,\left(-\left(a^3\,\pi\,{r_c}^6\right)+a^5\,\pi\,{\zeta_c}^4\right)}{{r_c}^2\,{\zeta_c}^3\,{\overline{\zeta_c}}^2}$$
$$- \frac{16\,i\,a^5\,\pi}{{r_c}^2\,\overline{\zeta_c}} - \frac{24\,i\,\left(a^7\,\pi\,{r_c}^6+a^7\,\pi\,{r_c}^4\,{\zeta_c}^2\right)}{\zeta_c\left(-{r_c}^2+\zeta_c\,\overline{\zeta_c}\right)^5}$$
$$- \frac{24\,i\,a^7\,\pi\,{r_c}^4}{\zeta_c\left(-{r_c}^2+\zeta_c\,\overline{\zeta_c}\right)^4} + \frac{8\,i\,\left(a^5\,\pi\,{r_c}^6+2\,a^5\,\pi\,{r_c}^4\,{\zeta_c}^2+2\,a^5\,\pi\,{r_c}^2\,{\zeta_c}^4+a^5\,\pi\,{\zeta_c}^6\right)}{{\zeta_c}^3\left(-{r_c}^2+\zeta_c\,\overline{\zeta_c}\right)^3}$$
$$+ \frac{8\,i\,\left(2\,a^5\,\pi\,{r_c}^6-2\,a^5\,\pi\,{r_c}^2\,{\zeta_c}^4-a^5\,\pi\,{\zeta_c}^6\right)}{{r_c}^2\,{\zeta_c}^3\left(-{r_c}^2+\zeta_c\,\overline{\zeta_c}\right)^2} + \frac{16\,i\,a^5\,\pi\,\zeta_c}{{r_c}^2\,\left(-{r_c}^2+\zeta_c\,\overline{\zeta_c}\right)} \quad \text{(D.58)}$$

$$\mathcal{N}_{4c} = \frac{4\,i\,a^5\,\pi\,{r_c}^4\,\left(-{r_c}^4+2\,{r_c}^2\,\zeta_c\,\overline{\zeta_c}+\left(a^2-{\zeta_c}^2\right)\,{\overline{\zeta_c}}^2\right)\,\left(-\zeta_c+\overline{\zeta_c}+4\,i\,\zeta_y\right)}{{\overline{\zeta_c}}^2\,\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^5} \quad \text{(D.59)}$$

$$\mathcal{N}_{4d} = -\frac{2\,i\,a\,\pi}{r_c^2}\,\left(-\left({r_c}^2\,\zeta_c\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^9\right)\right.$$
$$-a^2\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^7\,\left(2\,{\zeta_c}^2\,{\overline{\zeta_c}}^3+{r_c}^4\,\left(\zeta_c+2\,\overline{\zeta_c}\right)-{r_c}^2\,\zeta_c\,\overline{\zeta_c}\,\left(\zeta_c+6\,\overline{\zeta_c}\right)\right)$$
$$+a^4\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^5\,\left(2\,{\zeta_c}^2\,{\overline{\zeta_c}}^5+{r_c}^6\,\left(\zeta_c+\overline{\zeta_c}\right)\right.$$
$$\left.+{r_c}^4\,\left({\zeta_c}^3-{\zeta_c}^2\,\overline{\zeta_c}+\zeta_c\,{\overline{\zeta_c}}^2-{\overline{\zeta_c}}^3\right)-{r_c}^2\,\zeta_c\,\overline{\zeta_c}\,\left({\zeta_c}^3+2\,\zeta_c\,{\overline{\zeta_c}}^2+2\,{\overline{\zeta_c}}^3\right)\right)$$
$$+a^{10}\,{r_c}^2\,{\overline{\zeta_c}}^4\,\left(-2\,{\zeta_c}^2\,{\overline{\zeta_c}}^2\,\left(\overline{\zeta_c}+i\,\zeta_y\right)+3\,{r_c}^4\,\left(2\,\zeta_c+\overline{\zeta_c}-2\,i\,\zeta_y\right)+3\,{r_c}^2\,\zeta_c\,\overline{\zeta_c}\,\left(\zeta_c+2\,\overline{\zeta_c}+2\,i\,\zeta_y\right)\right)$$
$$+a^6\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^4\,\left(-2\,\zeta_c\,{\overline{\zeta_c}}^6+{r_c}^2\,{\overline{\zeta_c}}^2\,\left(-2\,{\zeta_c}^3-\zeta_c\,{\overline{\zeta_c}}^2+8\,{\overline{\zeta_c}}^3+2\,{\zeta_c}^2\,\left(\overline{\zeta_c}+i\,\zeta_y\right)\right)\right.$$
$$\left.+{r_c}^6\,\left(2\,\zeta_c-\overline{\zeta_c}-2\,i\,\zeta_y\right)-{r_c}^4\,\overline{\zeta_c}\,\left({\zeta_c}^2+4\,\zeta_c\,\overline{\zeta_c}-6\,{\overline{\zeta_c}}^2+2\,i\,\zeta_c\,\zeta_y\right)\right)$$
$$-a^8\,{\overline{\zeta_c}}^2\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^2\,\left(-2\,\zeta_c\,{\overline{\zeta_c}}^6+{r_c}^2\,\left(-\left({\zeta_c}^3\,{\overline{\zeta_c}}^2\right)+5\,{\overline{\zeta_c}}^5\right)\right.$$
$$\left.\left.+2\,{r_c}^6\,\left(8\,\zeta_c+3\,\overline{\zeta_c}-8\,i\,\zeta_y\right)+{r_c}^4\,\overline{\zeta_c}\,\left(6\,{\zeta_c}^2+11\,\zeta_c\,\overline{\zeta_c}+3\,{\overline{\zeta_c}}^2+12\,i\,\zeta_c\,\zeta_y\right)\right)\right) \quad \text{(D.60)}$$

$$
\begin{aligned}
\mathcal{N}_{4e} = -2\,i\,a\,\pi\, \Big(& a\,\zeta_c\,\overline{\zeta_c}\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^5 \left({r_c}^4-\zeta_c\,\overline{\zeta_c}^3\right)+\zeta_c\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^6\left({r_c}^4-\zeta_c\,\overline{\zeta_c}^3\right) \\
& +a^7\,{r_c}^2\,\zeta_c\,\overline{\zeta_c}^3\left(2\,{r_c}^4+3\,{r_c}^2\,\zeta_c\,\overline{\zeta_c}+\zeta_c\,\overline{\zeta_c}^2\,\left(\zeta_c+2\,\overline{\zeta_c}\right)\right) \\
& -a^4\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^3\left({r_c}^6\,\zeta_c+2\,\zeta_c\,\overline{\zeta_c}^6+{r_c}^2\,\overline{\zeta_c}^2\left(-2\,{\zeta_c}^3+{\zeta_c}^2\,\overline{\zeta_c}-2\,\zeta_c\,\overline{\zeta_c}^2-4\,\overline{\zeta_c}^3\right)\right. \\
& +{r_c}^4\,\zeta_c\,\overline{\zeta_c}\,\left(\left(1+2\,i\,\beta-2\,i\,\pi\right)\,\zeta_c+4\,i\,\left(i+\beta-\pi\right)\,\left(\overline{\zeta_c}+i\,\zeta_y\right)\right)) \\
& +a^6\,{r_c}^2\,\zeta_c\,\overline{\zeta_c}^2\,\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)\,\left(2\,{r_c}^4+\zeta_c\,\overline{\zeta_c}^2\,\left(\zeta_c+2\,\overline{\zeta_c}\right)\right. \\
& +{r_c}^2\,\overline{\zeta_c}\,\left(5\,\zeta_c+4\,\overline{\zeta_c}+4\,i\,\zeta_y\right))+a^2\,\overline{\zeta_c}\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^3\,\left(4\,{r_c}^8-12\,{r_c}^6\,\zeta_c\,\overline{\zeta_c}-{\zeta_c}^3\,\overline{\zeta_c}^5+{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}^3\,\left(\zeta_c+4\,\overline{\zeta_c}\right)\right. \\
& \left.+{r_c}^4\,\zeta_c\,\overline{\zeta_c}^2\,\left(\left(15+2\,i\,\beta-2\,i\,\pi\right)\,\zeta_c+\overline{\zeta_c}+4\,i\,\beta\,\overline{\zeta_c}-4\,i\,\pi\,\overline{\zeta_c}-4\,\beta\,\zeta_y+4\,\pi\,\zeta_y\right)\right) \\
& -a^5\,\overline{\zeta_c}\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^2\,\left({r_c}^6\,\zeta_c+2\,\zeta_c\,\overline{\zeta_c}^6+{r_c}^2\,\overline{\zeta_c}^2\,\left(-2\,{\zeta_c}^3+{\zeta_c}^2\,\overline{\zeta_c}-2\,\zeta_c\,\overline{\zeta_c}^2-4\,\overline{\zeta_c}^3\right)\right. \\
& -{r_c}^4\,\overline{\zeta_c}\,\left(\left(-3-2\,i\,\beta+2\,i\,\pi\right)\,{\zeta_c}^2+4\,\overline{\zeta_c}\,\left(\overline{\zeta_c}+i\,\zeta_y\right)+2\,\zeta_c\,\left(\overline{\zeta_c}-2\,i\,\beta\,\overline{\zeta_c}+2\,i\,\pi\,\overline{\zeta_c}+2\,\beta\,\zeta_y-2\,\pi\,\zeta_y\right)\right)) \\
& +a^3\,\overline{\zeta_c}^2\,\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^2\,\left(4\,{r_c}^8-12\,{r_c}^6\,\zeta_c\,\overline{\zeta_c}-{\zeta_c}^3\,\overline{\zeta_c}^5+{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}^3\,\left(\zeta_c+4\,\overline{\zeta_c}\right)\right. \\
& +{r_c}^4\,\zeta_c\,\overline{\zeta_c}\,\left(2\,{\zeta_c}^2+\zeta_c\,\left(\left(19+2\,i\,\beta-2\,i\,\pi\right)\,\overline{\zeta_c}+4\,i\,\zeta_y\right)\right. \\
& \left.+\overline{\zeta_c}\,\left(\overline{\zeta_c}+4\,i\,\beta\,\overline{\zeta_c}-4\,i\,\pi\,\overline{\zeta_c}-4\,\beta\,\zeta_y+4\,\pi\,\zeta_y\right)\right)\Big)\Big)\Big)\,/\left(\zeta_c\,\overline{\zeta_c}^3\right)
\end{aligned}
\tag{D.61}
$$

$$
\begin{aligned}
\mathcal{N}_{4f} = -2\,i\,a\,\pi\, \Big(& {r_c}^6\,\overline{\zeta_c}^2\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^5-a^4\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^2\left({r_c}^8\,\overline{\zeta_c}^2-2\,{\zeta_c}^4\,\overline{\zeta_c}^6+{r_c}^2\,{\zeta_c}^3\,\overline{\zeta_c}^2\,\left(-{\zeta_c}^3+{\zeta_c}^2\,\overline{\zeta_c}+6\,\overline{\zeta_c}^3\right)\right. \\
& +{r_c}^6\,\left(\left(4-4\,i\,\beta+4\,i\,\pi\right)\,{\zeta_c}^4+3\,\zeta_c\,\overline{\zeta_c}^3+2\,\left(1-i\,\beta+i\,\pi\right)\,{\zeta_c}^3\,\left(\overline{\zeta_c}-2\,i\,\zeta_y\right)\right) \\
& \left.+{r_c}^4\,{\zeta_c}^2\,\overline{\zeta_c}\,\left(-8\,{\zeta_c}^3-4\,\overline{\zeta_c}^2\,\left(\overline{\zeta_c}-i\,\zeta_y\right)+{\zeta_c}^2\,\left(-13\,\overline{\zeta_c}+4\,i\,\zeta_y\right)\right)\right) \\
& +a^6\,{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}^2\,\left(2\,{r_c}^6-{\zeta_c}^3\,\overline{\zeta_c}^3+2\,{r_c}^4\,\zeta_c\,\left(\zeta_c+4\,\overline{\zeta_c}+i\,\zeta_y\right)+{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}\,\left(15\,\zeta_c+19\,\overline{\zeta_c}-4\,i\,\zeta_y\right)\right) \\
& +2\,a^3\,{r_c}^4\,{\zeta_c}^3\,\overline{\zeta_c}\,\left({r_c}^2-2\,\zeta_c\,\overline{\zeta_c}\right)\,\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^2\,\left(2\,\zeta_c+\overline{\zeta_c}-2\,i\,\zeta_y\right) \\
& -2\,a^5\,{r_c}^4\,{\zeta_c}^2\,\overline{\zeta_c}^2\,\left({r_c}^4-3\,{r_c}^2\,\zeta_c\,\overline{\zeta_c}+2\,{\zeta_c}^2\,\overline{\zeta_c}^2\right)\,\left(2\,\zeta_c+\overline{\zeta_c}-2\,i\,\zeta_y\right) \\
& +a^2\,\zeta_c\,\overline{\zeta_c}\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^2\,\left(4\,{r_c}^{10}-12\,{r_c}^8\,\zeta_c\,\overline{\zeta_c}+2\,{\zeta_c}^5\,\overline{\zeta_c}^5-{r_c}^2\,{\zeta_c}^4\,\overline{\zeta_c}^3\,\left(\zeta_c+8\,\overline{\zeta_c}\right)\right. \\
& +{r_c}^4\,{\zeta_c}^3\,\overline{\zeta_c}^2\,\left(7\,\zeta_c+10\,\overline{\zeta_c}-4\,i\,\zeta_y\right)+2\,{r_c}^6\,{\zeta_c}^2\,\overline{\zeta_c}\,\left(\left(-1-2\,i\,\beta+2\,i\,\pi\right)\,\zeta_c+\left(3-i\,\beta+i\,\pi\right)\,\overline{\zeta_c}\right. \\
& +2\,\left(-\beta+\pi\right)\,\zeta_y\Big)\Big)\Big)\,/{r_c}^2\,{\zeta_c}^3\,\overline{\zeta_c}^2
\end{aligned}
\tag{D.62}
$$

$$\mathcal{N}_5 = \mathsf{Re}\Big\{\mathcal{N}_{5a}\log(\frac{r_c^2 + a\zeta_c - \delta^2}{r_c^2}) + \mathcal{N}_{5b}\log(\frac{r_c^2 - \delta^2}{r_c^2})$$
$$+\mathcal{N}_{5c}\log(\frac{a}{\overline{\zeta_c}}) + \frac{\mathcal{N}_{5d}}{(r_c^2 + a\overline{\zeta_c} - \delta^2)^2(r_c^2 - a\overline{\zeta_c} - \delta^2)^2(r_c^2 - \delta^2)^5}$$
$$+\frac{\mathcal{N}_{5e}}{(r_c^2 + a\overline{\zeta_c} - \delta^2)(r_c^2 - \delta^2)^5} + \frac{\mathcal{N}_{5f}}{(r_c^2 - \delta^2)^5}\Big\} \quad \text{(D.63)}$$

$$\mathcal{N}_{5a} = \frac{4\,a^5\,\pi\,{r_c}^4\,\left(\zeta_c + \overline{\zeta_c}\right)\,\left({r_c}^4 - 2\,{r_c}^2\,\zeta_c\,\overline{\zeta_c} + {\zeta_c}^2\,\left(-a^2 + \overline{\zeta_c}^2\right)\right)}{{\zeta_c}^2\,\left(-{r_c}^2 + \zeta_c\,\overline{\zeta_c}\right)^5} \quad \text{(D.64)}$$

$$\mathcal{N}_{5b} = \frac{8\,\left(-\left(a^3\,\pi\,{r_c}^4\right) + a^5\,\pi\,{\zeta_c}^2\right)}{{\zeta_c}^2\,\overline{\zeta_c}^3} + \frac{8\,\left(a^3\,\pi\,{r_c}^6 + a^5\,\pi\,{\zeta_c}^4\right)}{{r_c}^2\,{\zeta_c}^3\,\overline{\zeta_c}^2} + \frac{16\,a^5\,\pi}{{r_c}^2\,\overline{\zeta_c}}$$
$$+\frac{24\,\left(a^7\,\pi\,{r_c}^6 - a^7\,\pi\,{r_c}^4\,{\zeta_c}^2\right)}{\zeta_c\,\left(-{r_c}^2 + \zeta_c\,\overline{\zeta_c}\right)^5} + \frac{24\,a^7\,\pi\,{r_c}^4}{\zeta_c\,\left(-{r_c}^2 + \zeta_c\,\overline{\zeta_c}\right)^4} - \frac{8\,\left(a^5\,\pi\,{r_c}^6 - 2\,a^5\,\pi\,{r_c}^4\,{\zeta_c}^2 + 2\,a^5\,\pi\,{r_c}^2\,{\zeta_c}^4 - a^5\,\pi\,{\zeta_c}^6\right)}{{\zeta_c}^3\,\left(-{r_c}^2 + \zeta_c\,\overline{\zeta_c}\right)^3}$$
$$-\frac{8\,\left(2\,a^5\,\pi\,{r_c}^6 - 2\,a^5\,\pi\,{r_c}^2\,{\zeta_c}^4 + a^5\,\pi\,{\zeta_c}^6\right)}{{r_c}^2\,{\zeta_c}^3\,\left(-{r_c}^2 + \zeta_c\,\overline{\zeta_c}\right)^2} - \frac{16\,a^5\,\pi\,\zeta_c}{{r_c}^2\,\left(-{r_c}^2 + \zeta_c\,\overline{\zeta_c}\right)} \quad \text{(D.65)}$$

$$\mathcal{N}_{5c} = \frac{-4\,a^5\,\pi\,{r_c}^4\,\left(\zeta_c + \overline{\zeta_c}\right)\,\left({r_c}^4 - 2\,{r_c}^2\,\zeta_c\,\overline{\zeta_c} + \left(-a^2 + {\zeta_c}^2\right)\,\overline{\zeta_c}^2\right)}{\overline{\zeta_c}^2\,\left(-{r_c}^2 + \zeta_c\,\overline{\zeta_c}\right)^5} \quad \text{(D.66)}$$

$$\mathcal{N}_{5d} = -2\,a\,\pi\,\Big({r_c}^2\,\zeta_c\,\left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)^9 - a^2\,\left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)^7\,\Big({r_c}^4\,\left(\zeta_c - 2\,\overline{\zeta_c}\right) - {r_c}^2\,\zeta_c\,\left(\zeta_c - 6\,\overline{\zeta_c}\right)\,\overline{\zeta_c} - 2\,{\zeta_c}^2\,\overline{\zeta_c}^3\Big)$$
$$-a^4\,\left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)^5\,\Big({r_c}^6\,\left(\zeta_c - \overline{\zeta_c}\right) + 2\,{\zeta_c}^2\,\overline{\zeta_c}^5 + {r_c}^2\,\zeta_c\,\overline{\zeta_c}\,\left({\zeta_c}^3 + 2\,\zeta_c\,\overline{\zeta_c}^2 - 2\,\overline{\zeta_c}^3\right)$$
$$-{r_c}^4\,\left({\zeta_c}^3 + {\zeta_c}^2\,\overline{\zeta_c} + \zeta_c\,\overline{\zeta_c}^2 + \overline{\zeta_c}^3\right)\Big) + a^{10}\,{r_c}^2\,\overline{\zeta_c}^4\,\Big({r_c}^4\,\left(6\,\zeta_c - 3\,\left(\overline{\zeta_c} + 2\,i\,\zeta_y\right)\right)$$
$$+2\,{\zeta_c}^2\,\overline{\zeta_c}^2\,\left(\overline{\zeta_c} - i\,\zeta_y\right) + 3\,{r_c}^2\,\zeta_c\,\overline{\zeta_c}\,\left(\zeta_c - 2\,\overline{\zeta_c} + 2\,i\,\zeta_y\right)\Big)$$
$$+a^6\,\left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)^4\,\Big(2\,\zeta_c\,\overline{\zeta_c}^6 - {r_c}^2\,\overline{\zeta_c}^2\,\left(2\,{\zeta_c}^3 + \zeta_c\,\overline{\zeta_c}^2 + 8\,\overline{\zeta_c}^3 + 2\,{\zeta_c}^2\,\left(\overline{\zeta_c} - i\,\zeta_y\right)\right)$$
$$+{r_c}^6\,\left(2\,\zeta_c + \overline{\zeta_c} - 2\,i\,\zeta_y\right) + {r_c}^4\,\overline{\zeta_c}\,\left(-{\zeta_c}^2 + 4\,\zeta_c\,\overline{\zeta_c} + 6\,\overline{\zeta_c}^2 - 2\,i\,\zeta_c\,\zeta_y\right)\Big)$$
$$-a^8\,\overline{\zeta_c}^2\,\left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)^2\,\Big(2\,\zeta_c\,\overline{\zeta_c}^6 - {r_c}^2\,\overline{\zeta_c}^2\,\left({\zeta_c}^3 + 5\,\overline{\zeta_c}^3\right)$$
$$+2\,{r_c}^6\,\left(8\,\zeta_c - 3\,\overline{\zeta_c} - 8\,i\,\zeta_y\right) + {r_c}^4\,\overline{\zeta_c}\,\left(6\,{\zeta_c}^2 - 11\,\zeta_c\,\overline{\zeta_c} + 3\,\overline{\zeta_c}^2 + 12\,i\,\zeta_c\,\zeta_y\right)\Big)\Big)\Big)/{r_c}^2 \quad \text{(D.67)}$$

$$\mathcal{N}_{5e} = -2\,a\,\pi\,\Big(a^7\,{r_c}^2\,\zeta_c\,\overline{\zeta_c}^3\,\left(2\,{r_c}^4+3\,{r_c}^2\,\zeta_c\,\overline{\zeta_c}+\zeta_c\,\left(\zeta_c-2\,\overline{\zeta_c}\right)\,\overline{\zeta_c}^2\right)$$

$$+a\,\zeta_c\,\overline{\zeta_c}\,\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^5\,\left({r_c}^4+\zeta_c\,\overline{\zeta_c}^3\right)+\zeta_c\,\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^6\,\left({r_c}^4+\zeta_c\,\overline{\zeta_c}^3\right)$$

$$+a^6\,{r_c}^2\,\zeta_c\,\overline{\zeta_c}^2\,\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)\,\left(2\,{r_c}^4+\zeta_c\,\left(\zeta_c-2\,\overline{\zeta_c}\right)\,\overline{\zeta_c}^2+{r_c}^2\,\overline{\zeta_c}\,\left(5\,\zeta_c-4\,\left(\overline{\zeta_c}+i\,\zeta_y\right)\right)\right)$$

$$+a^2\,\overline{\zeta_c}\,\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^3\,\Big(4\,{r_c}^8-12\,{r_c}^6\,\zeta_c\,\overline{\zeta_c}+{r_c}^2\,{\zeta_c}^2\,\left(\zeta_c-4\,\overline{\zeta_c}\right)\,\overline{\zeta_c}^3+{\zeta_c}^3\,\overline{\zeta_c}^5$$

$$+{r_c}^4\,\zeta_c\,\overline{\zeta_c}^2\,\left(\left(15+2\,i\,\beta-2\,i\,\pi\right)\,\zeta_c+\left(-1-4\,i\,\beta+4\,i\,\pi\right)\,\overline{\zeta_c}+4\,\left(\beta-\pi\right)\,\zeta_y\right)\Big)$$

$$-a^4\,\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^3\,\Big({r_c}^6\,\zeta_c-2\,\zeta_c\,\overline{\zeta_c}^6-{r_c}^2\,\overline{\zeta_c}^2\,\left(2\,{\zeta_c}^3+{\zeta_c}^2\,\overline{\zeta_c}+2\,\zeta_c\,\overline{\zeta_c}^2-4\,\overline{\zeta_c}^3\right)$$

$$+{r_c}^4\,\zeta_c\,\overline{\zeta_c}\,\left(\left(1+2\,i\,\beta-2\,i\,\pi\right)\,\zeta_c+4\,\left(i+\beta-\pi\right)\,\left(-i\,\overline{\zeta_c}+\zeta_y\right)\right)\Big)$$

$$-a^5\,\overline{\zeta_c}\,\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^2\,\Big({r_c}^6\,\zeta_c-2\,\zeta_c\,\overline{\zeta_c}^6-{r_c}^2\,\overline{\zeta_c}^2\,\left(2\,{\zeta_c}^3+{\zeta_c}^2\,\overline{\zeta_c}+2\,\zeta_c\,\overline{\zeta_c}^2-4\,\overline{\zeta_c}^3\right)$$

$$+{r_c}^4\,\overline{\zeta_c}\,\left(\left(3+2\,i\,\beta-2\,i\,\pi\right)\,{\zeta_c}^2+4\,\overline{\zeta_c}\,\left(\overline{\zeta_c}+i\,\zeta_y\right)+\zeta_c\,\left(\left(-2-4\,i\,\beta+4\,i\,\pi\right)\,\overline{\zeta_c}+4\,\left(\beta-\pi\right)\,\zeta_y\right)\right)\Big)$$

$$+a^3\,\overline{\zeta_c}^2\,\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^2\,\Big(4\,{r_c}^8-12\,{r_c}^6\,\zeta_c\,\overline{\zeta_c}+{r_c}^2\,{\zeta_c}^2\,\left(\zeta_c-4\,\overline{\zeta_c}\right)\,\overline{\zeta_c}^3+{\zeta_c}^3\,\overline{\zeta_c}^5$$

$$+{r_c}^4\,\zeta_c\,\overline{\zeta_c}\,\Big(2\,{\zeta_c}^2+\zeta_c\,\left(\left(11+2\,i\,\beta-2\,i\,\pi\right)\,\overline{\zeta_c}-4\,i\,\zeta_y\right)$$

$$+\overline{\zeta_c}\,\left(\left(-1-4\,i\,\beta+4\,i\,\pi\right)\,\overline{\zeta_c}+4\,\left(\beta-\pi\right)\,\zeta_y\right)\Big)\Big)\Big)/\zeta_c\,\overline{\zeta_c}^3 \quad \text{(D.68)}$$

$$\mathcal{N}_{5f} = 2\,a\,\pi\,\Big({r_c}^6\,\overline{\zeta_c}^2\,\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^5+a^6\,{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}^2\,\Big(2\,{r_c}^6-{\zeta_c}^3\,\overline{\zeta_c}^3$$

$$-2\,{r_c}^4\,\zeta_c\,\left(\zeta_c-4\,\overline{\zeta_c}+3\,i\,\zeta_y\right)+{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}\,\left(-15\,\zeta_c+19\,\overline{\zeta_c}+4\,i\,\zeta_y\right)\Big)$$

$$-2\,a^3\,{r_c}^4\,{\zeta_c}^3\,\overline{\zeta_c}\,\left({r_c}^2-2\,\zeta_c\,\overline{\zeta_c}\right)\,\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^2\,\left(2\,\zeta_c-\overline{\zeta_c}-2\,i\,\zeta_y\right)$$

$$+2\,a^5\,{r_c}^4\,{\zeta_c}^2\,\overline{\zeta_c}^2\,\left({r_c}^4-3\,{r_c}^2\,\zeta_c\,\overline{\zeta_c}+2\,{\zeta_c}^2\,\overline{\zeta_c}^2\right)\,\left(2\,\zeta_c-\overline{\zeta_c}-2\,i\,\zeta_y\right)$$

$$+a^2\,\zeta_c\,\overline{\zeta_c}\,\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^2\,\Big(4\,{r_c}^{10}-12\,{r_c}^8\,\zeta_c\,\overline{\zeta_c}+{r_c}^2\,{\zeta_c}^4\,\left(\zeta_c-8\,\overline{\zeta_c}\right)\,\overline{\zeta_c}^3+2\,{\zeta_c}^5\,\overline{\zeta_c}^5$$

$$+{r_c}^4\,{\zeta_c}^3\,\overline{\zeta_c}^2\,\left(-7\,\zeta_c+10\,\overline{\zeta_c}+4\,i\,\zeta_y\right)+2\,{r_c}^6\,{\zeta_c}^2\,\overline{\zeta_c}\,\left(\zeta_c+2\,i\,\beta\,\zeta_c-2\,i\,\pi\,\zeta_c+3\,\overline{\zeta_c}-i\,\beta\,\overline{\zeta_c}+i\,\pi\,\overline{\zeta_c}+2\,\beta\,\zeta_y-2\,\pi\,\zeta_y\right)\Big)$$

$$-a^4\,\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^2\,\Big({r_c}^8\,\overline{\zeta_c}^2-2\,{\zeta_c}^4\,\overline{\zeta_c}^6+{r_c}^2\,{\zeta_c}^3\,\overline{\zeta_c}^2\,\left({\zeta_c}^3+{\zeta_c}^2\,\overline{\zeta_c}+6\,\overline{\zeta_c}^3\right)$$

$$+{r_c}^4\,{\zeta_c}^2\,\overline{\zeta_c}\,\left(8\,{\zeta_c}^3-4\,\overline{\zeta_c}^2\,\left(\overline{\zeta_c}+i\,\zeta_y\right)-{\zeta_c}^2\,\left(13\,\overline{\zeta_c}+4\,i\,\zeta_y\right)\right)$$

$$+{r_c}^6\,\left(4\,i\,\left(i+\beta-\pi\right)\,{\zeta_c}^4+3\,\zeta_c\,\overline{\zeta_c}^3+2\,\left(i+\beta-\pi\right)\,{\zeta_c}^3\,\left(-i\,\overline{\zeta_c}+2\,\zeta_y\right)\right)\Big)\Big)/\left({r_c}^2\,{\zeta_c}^3\,\overline{\zeta_c}^2\right) \quad \text{(D.69)}$$

$$\mathcal{N}_6 = \mathsf{Re}\left\{\mathcal{N}_{6b}\log(\frac{r_c^2-\delta^2}{r_c^2}) + \frac{\mathcal{N}_{6d}}{(r_c^2+a\overline{\zeta_c}-\delta^2)^2(r_c^2-a\overline{\zeta_c}-\delta^2)^2(r_c^2-\delta^2)^4} + \frac{\mathcal{N}_{6e}}{(r_c^2+a\overline{\zeta_c}-\delta^2)(r_c^2-\delta^2)^4} + \frac{\mathcal{N}_{6f}}{(r_c^2-\delta^2)^4}\right\} \quad \text{(D.70)}$$

$$\mathcal{N}_{6b} = \frac{-2\,i\,\left(-2\,a^2\,\pi\,{r_c}^4 + 7\,a^4\,\pi\,{\zeta_c}^2\right)}{{\zeta_c}^2\,\overline{\zeta_c}^2} - \frac{28\,i\,a^4\,\pi\,\zeta_c}{{r_c}^2\,\overline{\zeta_c}} + \frac{24\,i\,a^6\,\pi\,{r_c}^4}{\left(-{r_c}^2+\zeta_c\,\overline{\zeta_c}\right)^4} - \frac{14\,i\,\left(a^4\,\pi\,{r_c}^4 + a^4\,\pi\,{\zeta_c}^4\right)}{{\zeta_c}^2\left(-{r_c}^2+\zeta_c\,\overline{\zeta_c}\right)^2} + \frac{28\,i\,a^4\,\pi\,{\zeta_c}^2}{{r_c}^2\,\left(-{r_c}^2+\zeta_c\,\overline{\zeta_c}\right)} \quad \text{(D.71)}$$

$$\begin{aligned}\mathcal{N}_{6d} = -4\,i\,a^2\,\pi\,\Big(&-\left({r_c}^2\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^8\right) + a^8\,{r_c}^2\,\overline{\zeta_c}^4\left(3\,{r_c}^4 + 6\,{r_c}^2\,\zeta_c\,\overline{\zeta_c} - {\zeta_c}^2\,\overline{\zeta_c}^2\right)\\ &+a^4\,{r_c}^2\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^4\left({r_c}^4 - 2\,{r_c}^2\,\zeta_c\,\overline{\zeta_c} - {\zeta_c}^2\,\overline{\zeta_c}^2 + 5\,\overline{\zeta_c}^4\right)\\ &+a^6\,\overline{\zeta_c}^2\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^2\left(-8\,{r_c}^6 - 12\,{r_c}^4\,\zeta_c\,\overline{\zeta_c} + 2\,\zeta_c\,\overline{\zeta_c}^5 + {r_c}^2\,\overline{\zeta_c}^2\left({\zeta_c}^2 - 4\,\overline{\zeta_c}^2\right)\right)\\ &+a^2\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^6\left(-2\,\zeta_c\,\overline{\zeta_c}^3 + {r_c}^2\left({\zeta_c}^2 + 4\,\overline{\zeta_c}^2\right)\right)\Big)/{r_c}^2\end{aligned} \quad \text{(D.72)}$$

$$\begin{aligned}\mathcal{N}_{6e} = -2\,i\,a^2\,\pi\,\Big(&3\,a^5\,{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}^4 + 3\,a^4\,{r_c}^2\,\zeta_c\,\overline{\zeta_c}^2\left({r_c}^4 - {\zeta_c}^2\,\overline{\zeta_c}^2\right)\\ &-\overline{\zeta_c}\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^3\left({r_c}^6 + \left(-2-3\,i\,\beta+3\,i\,\pi\right)\,{r_c}^4\,\zeta_c\,\overline{\zeta_c} - 8\,{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}^2 + 3\,{\zeta_c}^3\,\overline{\zeta_c}^3\right)\\ &+a^3\,\overline{\zeta_c}\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^2\left(-3\,i\,\beta\,{r_c}^4\,\zeta_c + 3\,i\,\pi\,{r_c}^4\,\zeta_c + 3\,{r_c}^4\,\overline{\zeta_c} + {r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c} + 6\,{r_c}^2\,\overline{\zeta_c}^3 - 3\,\zeta_c\,\overline{\zeta_c}^4\right)\\ &-a\,\overline{\zeta_c}^2\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^2\left({r_c}^6 - 8\,{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}^2 + 3\,{\zeta_c}^3\,\overline{\zeta_c}^3 - {r_c}^4\,\zeta_c\left(3\,\zeta_c + \left(2+3\,i\,\beta-3\,i\,\pi\right)\overline{\zeta_c}\right)\right)\\ &+a^2\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^3\left(3\,\left(1-i\,\beta+i\,\pi\right)\,{r_c}^4\,\zeta_c - 3\,\zeta_c\,\overline{\zeta_c}^4 + {r_c}^2\,\overline{\zeta_c}\left({\zeta_c}^2 + 6\,\overline{\zeta_c}^2\right)\right)\Big)/\zeta_c\,\overline{\zeta_c}^2\end{aligned} \quad \text{(D.73)}$$

$$\begin{aligned}\mathcal{N}_{6f} = 2\,i\,a^2\,\pi\,\Big(&3\,a^4\,{r_c}^2\,\zeta_c\,\overline{\zeta_c}^2\left({r_c}^2 - 5\,\zeta_c\,\overline{\zeta_c}\right) - 3\,a\,\zeta_c\,\overline{\zeta_c}\left({r_c}^2 - 2\,\zeta_c\,\overline{\zeta_c}\right)\left({r_c}^3 - r_c\,\zeta_c\,\overline{\zeta_c}\right)^2\\ &+3\,a^3\,{r_c}^2\,\overline{\zeta_c}^2\left({r_c}^4 - 3\,{r_c}^2\,\zeta_c\,\overline{\zeta_c} + 2\,{\zeta_c}^2\,\overline{\zeta_c}^2\right)\\ &+\overline{\zeta_c}\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^2\left({r_c}^6 + i\,\left(i+3\,\beta-3\,\pi\right)\,{r_c}^4\,\zeta_c\,\overline{\zeta_c} - 4\,{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}^2 + {\zeta_c}^3\,\overline{\zeta_c}^3\right)\\ &+a^2\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^2\left(3\,\left(1-i\,\beta+i\,\pi\right)\,{r_c}^4\,\zeta_c + \zeta_c\,\overline{\zeta_c}^4 - {r_c}^2\left(9\,{\zeta_c}^2\,\overline{\zeta_c} + 4\,\overline{\zeta_c}^3\right)\right)\Big)/\zeta_c\,\overline{\zeta_c}^2\end{aligned} \quad \text{(D.74)}$$

$$\mathcal{N}_7 = \mathsf{Re}\left\{\mathcal{N}_{7a}\log(\frac{r_c^2+a\zeta_c-\delta^2}{r_c^2}) + \mathcal{N}_{7b}\log(\frac{r_c^2-\delta^2}{r_c^2}) + \mathcal{N}_{7c}\log(\frac{a}{\overline{\zeta_c}}) + \frac{\mathcal{N}_{7d}}{(r_c^2-\delta^2)^5}\right\} \tag{D.75}$$

$$\mathcal{N}_{7a} = \frac{8\,a^6\,\pi\,{r_c}^3\,\left(2\,{r_c}^2+\zeta_c\,\overline{\zeta_c}\right)\,\left({r_c}^4-2\,{r_c}^2\,\zeta_c\,\overline{\zeta_c}+{\zeta_c}^2\,\left(-a^2+\overline{\zeta_c}^2\right)\right)\,\left(i\,\zeta_c+\zeta_y\right)}{{\zeta_c}^2\,\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^6} \tag{D.76}$$

$$\begin{aligned}\mathcal{N}_{7b} = {} & \frac{8\,i\,\left(-\left(a^4\,\pi\,{r_c}^6\right)+2\,a^6\,\pi\,{\zeta_c}^4\right)}{{r_c}^3\,{\zeta_c}^3\,\overline{\zeta_c}^2} + \frac{8\,i\,\left(2\,a^6\,\pi\,{r_c}^2+9\,a^6\,\pi\,{\zeta_c}^2\right)}{{r_c}^5\,\overline{\zeta_c}} - \frac{24\,i\,\left(a^8\,\pi\,{r_c}^7+a^8\,\pi\,{r_c}^5\,{\zeta_c}^2\right)}{\zeta_c\,\left(-{r_c}^2+\zeta_c\,\overline{\zeta_c}\right)^6} \\ & - \frac{8\,i\,\left(4\,a^8\,\pi\,{r_c}^5+a^8\,\pi\,{r_c}^3\,{\zeta_c}^2\right)}{\zeta_c\,\left(-{r_c}^2+\zeta_c\,\overline{\zeta_c}\right)^5} - \frac{8\,i\,\left(a^8\,\pi\,{r_c}^3-3\,a^6\,\pi\,{r_c}^3\,{\zeta_c}^2-3\,a^6\,\pi\,r_c\,{\zeta_c}^4\right)}{\zeta_c\,\left(-{r_c}^2+\zeta_c\,\overline{\zeta_c}\right)^4} - \frac{8\,i\,\left(2\,a^6\,\pi\,{r_c}^2\,\zeta_c+5\,a^6\,\pi\,{\zeta_c}^3\right)}{r_c\,\left(-{r_c}^2+\zeta_c\,\overline{\zeta_c}\right)^3} \\ & + \frac{8\,i\,\left(a^6\,\pi\,{r_c}^6+2\,a^6\,\pi\,{r_c}^2\,{\zeta_c}^4+7\,a^6\,\pi\,{\zeta_c}^6\right)}{{r_c}^3\,{\zeta_c}^3\,\left(-{r_c}^2+\zeta_c\,\overline{\zeta_c}\right)^2} - \frac{8\,i\,\left(2\,a^6\,\pi\,{r_c}^2\,\zeta_c+9\,a^6\,\pi\,{\zeta_c}^3\right)}{{r_c}^5\,\left(-{r_c}^2+\zeta_c\,\overline{\zeta_c}\right)}\end{aligned} \tag{D.77}$$

$$\mathcal{N}_{7c} = \frac{8\,a^6\,\pi\,{r_c}^3\,\left(2\,{r_c}^2+\zeta_c\,\overline{\zeta_c}\right)\,\left(-{r_c}^4+2\,{r_c}^2\,\zeta_c\,\overline{\zeta_c}+\left(a^2-{\zeta_c}^2\right)\,\overline{\zeta_c}^2\right)\,\left(i\,\zeta_c+\zeta_y\right)}{\overline{\zeta_c}^2\,\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^6} \tag{D.78}$$

$$\begin{aligned}\mathcal{N}_{7d} = {} & 4\,i\,a^2\,\pi\,\Big(-\left(\left({r_c}^4-{\zeta_c}^4\right)\,\overline{\zeta_c}^2\,\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^6\right)+a^6\,{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}^2\,\Big({r_c}^6+2\,{\zeta_c}^3\,\overline{\zeta_c}^2\,\left(2\,\zeta_c+\overline{\zeta_c}-i\,\zeta_y\right) \\ & +{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}\,\left(11\,\zeta_c+5\,\overline{\zeta_c}-3\,i\,\zeta_y\right)+{r_c}^4\,\zeta_c\,\left(-5\,\zeta_c+2\,\overline{\zeta_c}+5\,i\,\zeta_y\right)\Big) \\ & +2\,a^3\,{\zeta_c}^3\,\overline{\zeta_c}\,\left({r_c}^3-r_c\,\zeta_c\,\overline{\zeta_c}\right)^2\,\left(2\,{r_c}^4-3\,{r_c}^2\,\zeta_c\,\overline{\zeta_c}-2\,{\zeta_c}^2\,\overline{\zeta_c}^2\right)\,\left(\zeta_c-i\,\zeta_y\right) \\ & -2\,a^5\,{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}^2\,\left(2\,{r_c}^6-5\,{r_c}^4\,\zeta_c\,\overline{\zeta_c}+{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}^2+2\,{\zeta_c}^3\,\overline{\zeta_c}^3\right)\,\left(\zeta_c-i\,\zeta_y\right) \\ & -a^2\,\zeta_c\,\overline{\zeta_c}\,\left({r_c}^3-r_c\,\zeta_c\,\overline{\zeta_c}\right)^2\,\Big(2\,{r_c}^8-6\,{r_c}^6\,\zeta_c\,\overline{\zeta_c}+2\,{\zeta_c}^4\,\overline{\zeta_c}^3\,\left(-\zeta_c+\overline{\zeta_c}+i\,\zeta_y\right) \\ & +{r_c}^4\,{\zeta_c}^2\,\overline{\zeta_c}\,\left(i\,\left(i+4\,\beta-4\,\pi\right)\,\zeta_c+10\,\overline{\zeta_c}+\left(i+4\,\beta-4\,\pi\right)\,\zeta_y\right) \\ & +{r_c}^2\,{\zeta_c}^3\,\overline{\zeta_c}^2\,\left(i\,\left(3\,i+2\,\beta-2\,\pi\right)\,\zeta_c-8\,\overline{\zeta_c}+\left(3\,i+2\,\beta-2\,\pi\right)\,\zeta_y\right)\Big) \\ & +a^4\,\left({r_c}^2-\zeta_c\,\overline{\zeta_c}\right)^2\,\Big({r_c}^8\,\overline{\zeta_c}^2-{\zeta_c}^4\,\overline{\zeta_c}^6+2\,{r_c}^2\,{\zeta_c}^3\,\overline{\zeta_c}^2\,\left(2\,{\zeta_c}^3+\zeta_c\,\overline{\zeta_c}^2+{\zeta_c}^2\,\left(\overline{\zeta_c}-i\,\zeta_y\right)+\overline{\zeta_c}^2\,\left(\overline{\zeta_c}-i\,\zeta_y\right)\right) \\ & +4\,\left(i+\beta-\pi\right)\,{r_c}^6\,{\zeta_c}^3\,\left(i\,\zeta_c+\zeta_y\right) \\ & +2\,{r_c}^4\,{\zeta_c}^2\,\overline{\zeta_c}\,\left(\left(3+i\,\beta-i\,\pi\right)\,{\zeta_c}^3+2\,\zeta_c\,\overline{\zeta_c}^2-\overline{\zeta_c}^2\,\left(\overline{\zeta_c}+2\,i\,\zeta_y\right)+{\zeta_c}^2\,\left(2\,\overline{\zeta_c}+\left(-i+\beta-\pi\right)\,\zeta_y\right)\right)\Big)\Big)\Big/\left(r_c\,{\zeta_c}^3\,\overline{\zeta_c}^2\right)\end{aligned} \tag{D.79}$$

$$\mathcal{N}_8 \quad = \quad \mathsf{Re}\left\{\mathcal{N}_{8a}\log(\frac{r_c^2+a\zeta_c-\delta^2}{r_c^2})+\mathcal{N}_{8b}\log(\frac{r_c^2-\delta^2}{r_c^2})+\mathcal{N}_{8c}\log(\frac{a}{\overline{\zeta_c}})+\frac{\mathcal{N}_{8d}}{(r_c^2-\delta^2)^6}\right\} \tag{D.80}$$

$$\mathcal{N}_{8a} = \frac{8\,a^6\,\pi\,r_c{}^3\,\left(2\,r_c{}^2+\zeta_c\,\overline{\zeta_c}\right)\,\left(r_c{}^4-2\,r_c{}^2\,\zeta_c\,\overline{\zeta_c}+\zeta_c{}^2\,\left(-a^2+\overline{\zeta_c}^2\right)\right)\,\left(\zeta_c-i\,\zeta_y\right)}{\zeta_c{}^2\,\left(r_c{}^2-\zeta_c\,\overline{\zeta_c}\right)^6} \tag{D.81}$$

$$\begin{aligned}\mathcal{N}_{8b} = {} & \frac{8\,\left(a^4\,\pi\,r_c{}^6+2\,a^6\,\pi\,\zeta_c{}^4\right)}{r_c{}^3\,\zeta_c{}^3\,\overline{\zeta_c}^2}-\frac{8\,\left(2\,a^6\,\pi\,r_c{}^2-9\,a^6\,\pi\,\zeta_c{}^2\right)}{r_c{}^5\,\overline{\zeta_c}}+\frac{24\,\left(a^8\,\pi\,r_c{}^7-a^8\,\pi\,r_c{}^5\,\zeta_c{}^2\right)}{\zeta_c\,\left(-r_c{}^2+\zeta_c\,\overline{\zeta_c}\right)^6} \\ & +\frac{8\,\left(4\,a^8\,\pi\,r_c{}^5-a^8\,\pi\,r_c{}^3\,\zeta_c{}^2\right)}{\zeta_c\,\left(-r_c{}^2+\zeta_c\,\overline{\zeta_c}\right)^5}+\frac{8\,\left(a^8\,\pi\,r_c{}^3-3\,a^6\,\pi\,r_c{}^3\,\zeta_c{}^2+3\,a^6\,\pi\,r_c\,\zeta_c{}^4\right)}{\zeta_c\,\left(-r_c{}^2+\zeta_c\,\overline{\zeta_c}\right)^4}+\frac{8\,\left(2\,a^6\,\pi\,r_c{}^2\,\zeta_c-5\,a^6\,\pi\,\zeta_c{}^3\right)}{r_c\,\left(-r_c{}^2+\zeta_c\,\overline{\zeta_c}\right)^3} \\ & -\frac{8\,\left(a^6\,\pi\,r_c{}^6+2\,a^6\,\pi\,r_c{}^2\,\zeta_c{}^4-7\,a^6\,\pi\,\zeta_c{}^6\right)}{r_c{}^3\,\zeta_c{}^3\,\left(-r_c{}^2+\zeta_c\,\overline{\zeta_c}\right)^2}+\frac{8\,\left(2\,a^6\,\pi\,r_c{}^2\,\zeta_c-9\,a^6\,\pi\,\zeta_c{}^3\right)}{r_c{}^5\,\left(-r_c{}^2+\zeta_c\,\overline{\zeta_c}\right)}\end{aligned} \tag{D.82}$$

$$\mathcal{N}_{8c} = \frac{-8\,a^6\,\pi\,r_c{}^3\,\left(2\,r_c{}^2+\zeta_c\,\overline{\zeta_c}\right)\,\left(r_c{}^4-2\,r_c{}^2\,\zeta_c\,\overline{\zeta_c}+\left(-a^2+\zeta_c{}^2\right)\,\overline{\zeta_c}^2\right)\,\left(\zeta_c-i\,\zeta_y\right)}{\overline{\zeta_c}^2\,\left(r_c{}^2-\zeta_c\,\overline{\zeta_c}\right)^6} \tag{D.83}$$

$$\begin{aligned}\mathcal{N}_{8d} = {} & -4\,a^2\,\pi\,\Big(-\left(\left(r_c{}^4-\zeta_c{}^4\right)\,\overline{\zeta_c}^2\,\left(r_c{}^2-\zeta_c\,\overline{\zeta_c}\right)^6\right)+a^6\,r_c{}^2\,\zeta_c{}^2\,\overline{\zeta_c}^2\,\Big(r_c{}^6+2\,\zeta_c{}^3\,\overline{\zeta_c}^2\,\left(-2\,\zeta_c+\overline{\zeta_c}+i\,\zeta_y\right) \\ & +r_c{}^2\,\zeta_c{}^2\,\overline{\zeta_c}\,\left(-11\,\zeta_c+5\,\overline{\zeta_c}+3\,i\,\zeta_y\right)+r_c{}^4\,\zeta_c\,\left(5\,\zeta_c+2\,\overline{\zeta_c}-5\,i\,\zeta_y\right)\Big) \\ & +2\,a^3\,\zeta_c{}^3\,\overline{\zeta_c}\,\left(r_c{}^3-r_c\,\zeta_c\,\overline{\zeta_c}\right)^2\,\left(-2\,r_c{}^4+3\,r_c{}^2\,\zeta_c\,\overline{\zeta_c}+2\,\zeta_c{}^2\,\overline{\zeta_c}^2\right)\,\left(\zeta_c-i\,\zeta_y\right) \\ & +2\,a^5\,r_c{}^2\,\zeta_c{}^2\,\overline{\zeta_c}^2\,\left(2\,r_c{}^6-5\,r_c{}^4\,\zeta_c\,\overline{\zeta_c}+r_c{}^2\,\zeta_c{}^2\,\overline{\zeta_c}^2+2\,\zeta_c{}^3\,\overline{\zeta_c}^3\right)\,\left(\zeta_c-i\,\zeta_y\right) \\ & -a^2\,\zeta_c\,\overline{\zeta_c}\,\left(r_c{}^3-r_c\,\zeta_c\,\overline{\zeta_c}\right)^2\,\Big(2\,r_c{}^8-6\,r_c{}^6\,\zeta_c\,\overline{\zeta_c}+2\,\zeta_c{}^4\,\overline{\zeta_c}^3\,\left(\zeta_c+\overline{\zeta_c}-i\,\zeta_y\right) \\ & +r_c{}^2\,\zeta_c{}^3\,\overline{\zeta_c}^2\,\left(\left(3-2\,i\,\beta+2\,i\,\pi\right)\,\zeta_c-8\,\overline{\zeta_c}+\left(-3\,i-2\,\beta+2\,\pi\right)\,\zeta_y\right) \\ & +r_c{}^4\,\zeta_c{}^2\,\overline{\zeta_c}\,\left(\left(1-4\,i\,\beta+4\,i\,\pi\right)\,\zeta_c+10\,\overline{\zeta_c}+\left(-i-4\,\beta+4\,\pi\right)\,\zeta_y\right)\Big) \\ & +a^4\,\left(r_c{}^2-\zeta_c\,\overline{\zeta_c}\right)^2\,\Big(r_c{}^8\,\overline{\zeta_c}^2-\zeta_c{}^4\,\overline{\zeta_c}^6+2\,r_c{}^2\,\zeta_c{}^3\,\overline{\zeta_c}^2\,\Big(-2\,\zeta_c{}^3-\zeta_c\,\overline{\zeta_c}^2+\zeta_c{}^2\,\left(\overline{\zeta_c}+i\,\zeta_y\right) \\ & +\overline{\zeta_c}^2\,\left(\overline{\zeta_c}+i\,\zeta_y\right)\Big)+4\,\left(1-i\,\beta+i\,\pi\right)\,r_c{}^6\,\zeta_c{}^3\,\left(\zeta_c-i\,\zeta_y\right) \\ & +2\,r_c{}^4\,\zeta_c{}^2\,\overline{\zeta_c}\,\Big(i\,\left(3\,i-\beta+\pi\right)\,\zeta_c{}^3-2\,\zeta_c\,\overline{\zeta_c}^2-\overline{\zeta_c}^2\,\left(\overline{\zeta_c}-2\,i\,\zeta_y\right) \\ & +\zeta_c{}^2\,\left(2\,\overline{\zeta_c}+\left(i-\beta+\pi\right)\,\zeta_y\right)\Big)\Big)\Big)\Big)\,/\,\left(r_c\,\zeta_c{}^3\,\overline{\zeta_c}^2\right)\end{aligned} \tag{D.84}$$

$$\mathcal{N}_9 = \mathsf{Re}\left\{\mathcal{N}_{9a}\log(\frac{r_c^2 + a\zeta_c - \delta^2}{r_c^2}) + \mathcal{N}_{9b}\log(\frac{r_c^2 - \delta^2}{r_c^2}) + \mathcal{N}_{9c}\log(\frac{a}{\overline{\zeta_c}}) + \frac{\mathcal{N}_{9d}}{(r_c^2 - \delta^2)^6}\right\} \quad (D.85)$$

$$\mathcal{N}_{9a} = \frac{16\,i\,a^7\,\pi\,{r_c}^5}{\left(-{r_c}^2 + \zeta_c\,\overline{\zeta_c}\right)^5} + \frac{8\,i\,a^7\,\pi\,{r_c}^3}{\left(-{r_c}^2 + \zeta_c\,\overline{\zeta_c}\right)^4} - \frac{16\,i\,a^5\,\pi\,{r_c}^5}{{\zeta_c}^2\left(-{r_c}^2 + \zeta_c\,\overline{\zeta_c}\right)^3} - \frac{8\,i\,a^5\,\pi\,{r_c}^3}{{\zeta_c}^2\left(-{r_c}^2 + \zeta_c\,\overline{\zeta_c}\right)^2} \quad (D.86)$$

$$\begin{aligned}\mathcal{N}_{9b} = \frac{8\,i\,\left(a^3\,\pi\,{r_c}^4 + 2\,a^5\,\pi\,{\zeta_c}^2\right)}{r_c\,{\zeta_c}^2\,\overline{\zeta_c}^2} + \frac{64\,i\,a^5\,\pi\,\zeta_c}{{r_c}^3\,\overline{\zeta_c}} + \frac{32\,i\,a^7\,\pi\,{r_c}^5}{\left(-{r_c}^2 + \zeta_c\,\overline{\zeta_c}\right)^5} \\ + \frac{16\,i\,a^7\,\pi\,{r_c}^3}{\left(-{r_c}^2 + \zeta_c\,\overline{\zeta_c}\right)^4} - \frac{32\,i\,a^5\,\pi\,r_c\,{\zeta_c}^2}{\left(-{r_c}^2 + \zeta_c\,\overline{\zeta_c}\right)^3} \\ - \frac{8\,i\,\left(a^5\,\pi\,{r_c}^4 - 6\,a^5\,\pi\,{\zeta_c}^4\right)}{r_c\,{\zeta_c}^2\left(-{r_c}^2 + \zeta_c\,\overline{\zeta_c}\right)^2} - \frac{64\,i\,a^5\,\pi\,{\zeta_c}^2}{{r_c}^3\,\left(-{r_c}^2 + \zeta_c\,\overline{\zeta_c}\right)}\end{aligned} \quad (D.87)$$

$$\mathcal{N}_{9c} = \frac{8\,i\,a^5\,\pi\,{r_c}^3\,\left({r_c}^2 + \zeta_c\,\overline{\zeta_c}\right)\left({r_c}^4 - 2\,{r_c}^2\,\zeta_c\,\overline{\zeta_c} + \left(-a^2 + {\zeta_c}^2\right)\,\overline{\zeta_c}^2\right)}{\overline{\zeta_c}^2\left(-{r_c}^2 + \zeta_c\,\overline{\zeta_c}\right)^5} \quad (D.88)$$

$$\begin{aligned}\mathcal{N}_{9d} = -4\,i\,a^3\,\pi\,\Big(3\,a^4\,{r_c}^2\,\zeta_c\,\overline{\zeta_c}^2\left({r_c}^4 - 3\,{r_c}^2\,\zeta_c\,\overline{\zeta_c} - 2\,{\zeta_c}^2\,\overline{\zeta_c}^2\right) \\ +2\,a\,\zeta_c\,\overline{\zeta_c}\left({r_c}^3 - r_c\,\zeta_c\,\overline{\zeta_c}\right)^2\left(-{r_c}^4 + {r_c}^2\,\zeta_c\,\overline{\zeta_c} + 2\,{\zeta_c}^2\,\overline{\zeta_c}^2\right) \\ +2\,a^3\,{r_c}^2\,\overline{\zeta_c}^2\left({r_c}^6 - 2\,{r_c}^4\,\zeta_c\,\overline{\zeta_c} - {r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}^2 + 2\,{\zeta_c}^3\,\overline{\zeta_c}^3\right) - \overline{\zeta_c}\left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)^2\Big(2\,{r_c}^8 + (-3 - 2\,i\,\beta + 2\,i\,\pi)\,{r_c}^6\,\zeta_c\,\overline{\zeta_c} \\ + (1 - 2\,i\,\beta + 2\,i\,\pi)\,{r_c}^4\,{\zeta_c}^2\,\overline{\zeta_c}^2 + 6\,{r_c}^2\,{\zeta_c}^3\,\overline{\zeta_c}^3 - 2\,{\zeta_c}^4\,\overline{\zeta_c}^4\Big) \\ +2\,a^2\,\zeta_c\left({r_c}^2 - \zeta_c\,\overline{\zeta_c}\right)^2\left((1 - i\,\beta + i\,\pi)\,{r_c}^6 + i\,(2\,i - \beta + \pi)\,{r_c}^4\,\zeta_c\,\overline{\zeta_c} + \zeta_c\,\overline{\zeta_c}^5 - 3\,{r_c}^2\,\overline{\zeta_c}^2\left({\zeta_c}^2 + \overline{\zeta_c}^2\right)\right)\Big) \\ /\left(r_c\,\zeta_c\,\overline{\zeta_c}^2\right)\end{aligned} \quad (D.89)$$

$$\mathcal{P}_1 = \mathsf{Re}\Bigg[-(\pi\,\zeta_c) + \frac{a^2\,\pi\,\zeta_c}{{r_c}^2} - \frac{4\,a^4\,\pi\,{r_c}^4\,\overline{\zeta_c}}{\left({r_c}^2 + (a-\zeta_c)\,\overline{\zeta_c}\right)^2\left({r_c}^2 - (a+\zeta_c)\,\overline{\zeta_c}\right)^2}$$
$$+\frac{10\,a^4\,\pi\,{r_c}^2\,\zeta_c\,\overline{\zeta_c}^2}{\left({r_c}^2 + (a-\zeta_c)\,\overline{\zeta_c}\right)^2\left({r_c}^2 - (a+\zeta_c)\,\overline{\zeta_c}\right)^2} - \frac{8\,a^4\,\pi\,{\zeta_c}^2\,\overline{\zeta_c}^3}{\left({r_c}^2 + (a-\zeta_c)\,\overline{\zeta_c}\right)^2\left({r_c}^2 - (a+\zeta_c)\,\overline{\zeta_c}\right)^2}$$
$$-\frac{2\,a^6\,\pi\,\zeta_c\,\overline{\zeta_c}^4}{{r_c}^2\left({r_c}^2 + (a-\zeta_c)\,\overline{\zeta_c}\right)^2\left({r_c}^2 - (a+\zeta_c)\,\overline{\zeta_c}\right)^2} + \frac{2\,a^4\,\pi\,{\zeta_c}^3\,\overline{\zeta_c}^4}{{r_c}^2\left({r_c}^2 + (a-\zeta_c)\,\overline{\zeta_c}\right)^2\left({r_c}^2 - (a+\zeta_c)\,\overline{\zeta_c}\right)^2}\Bigg] \tag{D.90}$$

$$\mathcal{P}_2 = \mathsf{Re}\Bigg[i\,\pi\,\zeta_c + \frac{i\,a^2\,\pi\,\zeta_c}{{r_c}^2} - \frac{4\,i\,a^4\,\pi\,{r_c}^4\,\overline{\zeta_c}}{\left({r_c}^2 + (a-\zeta_c)\,\overline{\zeta_c}\right)^2\left({r_c}^2 - (a+\zeta_c)\,\overline{\zeta_c}\right)^2}$$
$$+\frac{10\,i\,a^4\,\pi\,{r_c}^2\,\zeta_c\,\overline{\zeta_c}^2}{\left({r_c}^2 + (a-\zeta_c)\,\overline{\zeta_c}\right)^2\left({r_c}^2 - (a+\zeta_c)\,\overline{\zeta_c}\right)^2} - \frac{8\,i\,a^4\,\pi\,{\zeta_c}^2\,\overline{\zeta_c}^3}{\left({r_c}^2 + (a-\zeta_c)\,\overline{\zeta_c}\right)^2\left({r_c}^2 - (a+\zeta_c)\,\overline{\zeta_c}\right)^2}$$
$$-\frac{2\,i\,a^6\,\pi\,\zeta_c\,\overline{\zeta_c}^4}{{r_c}^2\left({r_c}^2 + (a-\zeta_c)\,\overline{\zeta_c}\right)^2\left({r_c}^2 - (a+\zeta_c)\,\overline{\zeta_c}\right)^2} + \frac{2\,i\,a^4\,\pi\,{\zeta_c}^3\,\overline{\zeta_c}^4}{{r_c}^2\left({r_c}^2 + (a-\zeta_c)\,\overline{\zeta_c}\right)^2\left({r_c}^2 - (a+\zeta_c)\,\overline{\zeta_c}\right)^2}\Bigg] \tag{D.91}$$

$$\mathcal{P}_3 = \mathsf{Re}\Bigg[\frac{-i\,a^6\,\pi}{{r_c}^2\,\left(a^2 - {\zeta_c}^2\right)} - \frac{2\,i\,a^4\,\pi\,{\zeta_c}^2}{{r_c}^2\,\left(a^2 - {\zeta_c}^2\right)} - \frac{i\,a^2\,\pi\,{\zeta_c}^4}{{r_c}^2\,\left(a^2 - {\zeta_c}^2\right)} + i\,\pi\,\delta^2$$
$$+\frac{2\,i\,a^5\,\pi\,{r_c}^2}{(a-\zeta_c)\,\left({r_c}^2 + (a-\zeta_c)\,\overline{\zeta_c}\right)^2} - \frac{i\,a^4\,\pi}{-{r_c}^2 + \delta^2} + \frac{2\,i\,a^5\,\pi\,{r_c}^2}{(a+\zeta_c)\,\left({r_c}^2 - (a+\zeta_c)\,\overline{\zeta_c}\right)^2}\Bigg] \tag{D.92}$$

$$\mathcal{P}_4 = \mathsf{Re}\Big[\pi\,\Big(a^2\,\zeta_c\left({r_c}^2 - \delta^2\right)^3 - {r_c}^2\,\zeta_c\left({r_c}^2 - \delta^2\right)^3$$
$$+a^4\,\left(-\left({\zeta_c}^2\,\overline{\zeta_c}^3\right) + {r_c}^2\,\delta^2\,\left(\zeta_c + 3\,\overline{\zeta_c}\right) + {r_c}^4\,\left(-5\,\zeta_c - 2\,\overline{\zeta_c} + 4\,i\,\zeta_y\right)\right)\Big)\Big/\left({r_c}^2\left({r_c}^2 - \delta^2\right)^3\right)\Big] \tag{D.93}$$

$$\mathcal{P}_5 = \mathsf{Re}\Big[i\,\pi\,\Big(a^2\,\zeta_c\left({r_c}^2 - \delta^2\right)^3 + {r_c}^2\,\zeta_c\left({r_c}^2 - \delta^2\right)^3$$
$$+a^4\,\left({r_c}^4\,\left(5\,\zeta_c - 2\,\overline{\zeta_c}\right) - {r_c}^2\,\zeta_c\,\left(\zeta_c - 3\,\overline{\zeta_c}\right)\,\overline{\zeta_c} - {\zeta_c}^2\,\overline{\zeta_c}^3\right)\Big)\Big/\left({r_c}^2\left({r_c}^2 - \delta^2\right)^3\right)\Big] \tag{D.94}$$

$$\mathcal{P}_6 = \mathsf{Re}\left[\frac{-2\,a^5\,\pi}{{r_c}^2\,\left(a^2-{\zeta_c}^2\right)} - \frac{2\,a^3\,\pi\,{\zeta_c}^2}{{r_c}^2\,\left(a^2-{\zeta_c}^2\right)} + \frac{2\,a^4\,\pi\,{r_c}^2}{(a-\zeta_c)\,\left({r_c}^2+(a-\zeta_c)\,\overline{\zeta_c}\right)^2} \right.$$
$$\left. -\frac{4\,a^3\,\pi\,{r_c}^2}{\left({r_c}^2-\delta^2\right)^2} - \frac{2\,a^3\,\pi}{{r_c}^2-\delta^2} + \frac{2\,a^4\,\pi\,{r_c}^2}{(a+\zeta_c)\,\left({r_c}^2-(a+\zeta_c)\,\overline{\zeta_c}\right)^2}\right] \quad \text{(D.95)}$$

$$\mathcal{P}_7 = \mathsf{Re}\left[2\,\pi\,r_c - \frac{2\,a^4\,\pi\,r_c}{\left({r_c}^2-\delta^2\right)^2}\right] = 2\,\pi\,r_c - \frac{2\,a^4\,\pi\,r_c}{\left({r_c}^2-\delta^2\right)^2} \quad \text{(D.96)}$$

$$\mathcal{Q}_1 = \mathsf{Re}\left\{\frac{\mathcal{Q}_{1a}}{(r_c^2+a\overline{\zeta_c}-\delta^2)^2(r_c^2-a\overline{\zeta_c}-\delta^2)(a-\zeta_c-\zeta_k)(a+\zeta_c+\zeta_k)(\overline{\zeta_c}+\overline{\zeta_k})^2} \right.$$
$$\left. +\frac{\mathcal{Q}_{1b}}{(a-\zeta_c-\zeta_k)(a+\zeta_c+\zeta_k)(\zeta_c+\zeta_k)}\right\} \quad \text{(D.97)}$$

$$
\begin{aligned}
\mathcal{Q}_{1a} = \pi \Big(& a^8 \overline{\zeta_c}^4 \overline{\zeta_k}^2 \left(r_c{}^2 + \zeta_c \overline{\zeta_k}\right) - r_c{}^2 \left(r_c{}^2 - \zeta_c \overline{\zeta_c}\right)^4 (\zeta_c + \zeta_k)^2 \left(\overline{\zeta_c} + \overline{\zeta_k}\right)^2 \left(r_c{}^2 + \zeta_c \overline{\zeta_k}\right) - a^4 \Big(r_c{}^{10} \overline{\zeta_k}^2 + \zeta_c{}^5 \overline{\zeta_c}^4 \overline{\zeta_k}^3 \\
& + r_c{}^8 \left(2 \overline{\zeta_c}^4 + 4 \overline{\zeta_c}^3 \overline{\zeta_k} - 2 \overline{\zeta_c}^2 \left(\zeta_c + \zeta_k - \overline{\zeta_k}\right) \overline{\zeta_k} - 4 \overline{\zeta_c} \left(2 \zeta_c + \zeta_k\right) \overline{\zeta_k}^2 - \left(\zeta_c + 2 \zeta_k\right) \overline{\zeta_k}^3 \right) \\
& + r_c{}^2 \zeta_c \overline{\zeta_c}^3 \overline{\zeta_k} \Big(5 \zeta_c{}^3 \overline{\zeta_c} \overline{\zeta_k} + \overline{\zeta_c} \zeta_k{}^2 \left(\overline{\zeta_c} + \overline{\zeta_k}\right)^2 - \zeta_c{}^2 \left(\overline{\zeta_c} + \overline{\zeta_k}\right) \left(\overline{\zeta_c}^2 + \overline{\zeta_c} \overline{\zeta_k} - 8 \zeta_k \overline{\zeta_k}\right) \\
& - 2 \zeta_c \zeta_k \left(\overline{\zeta_c} + \overline{\zeta_k}\right) \left(\overline{\zeta_c}^2 + \overline{\zeta_c} \overline{\zeta_k} - 2 \zeta_k \overline{\zeta_k}\right) \Big) + r_c{}^4 \overline{\zeta_c}^2 \Big(\overline{\zeta_c}^2 \zeta_k{}^2 \left(\overline{\zeta_c} + \overline{\zeta_k}\right)^2 - 2 \zeta_c{}^3 \overline{\zeta_k} \left(\overline{\zeta_c}^2 + 8 \overline{\zeta_c} \overline{\zeta_k} + 2 \overline{\zeta_k}^2\right) \\
& + \zeta_c{}^2 \left(\overline{\zeta_c} + \overline{\zeta_k}\right) \left(3 \overline{\zeta_c}^3 + 7 \overline{\zeta_c}^2 \overline{\zeta_k} - 2 \overline{\zeta_c} \left(\zeta_k - 2 \overline{\zeta_k}\right) \overline{\zeta_k} - 18 \zeta_k \overline{\zeta_k}^2\right) \\
& + 2 \zeta_c \zeta_k \left(\overline{\zeta_c} + \overline{\zeta_k}\right) \left(\overline{\zeta_c}^3 + 5 \overline{\zeta_c}^2 \overline{\zeta_k} + 4 \overline{\zeta_c} \overline{\zeta_k}^2 - 4 \zeta_k \overline{\zeta_k}^2\right) \Big) \\
& + 2 r_c{}^6 \overline{\zeta_c} \Big(-2 \zeta_k \overline{\zeta_k} \left(\overline{\zeta_c} + \overline{\zeta_k}\right) \left(\overline{\zeta_c}^2 + \overline{\zeta_c} \overline{\zeta_k} - \zeta_k \overline{\zeta_k}\right) \\
& + \zeta_c{}^2 \overline{\zeta_k} \left(2 \overline{\zeta_c}^2 + 9 \overline{\zeta_c} \overline{\zeta_k} + 2 \overline{\zeta_k}^2\right) - \zeta_c \left(\overline{\zeta_c} + \overline{\zeta_k}\right) \left(2 \overline{\zeta_c}^3 + 3 \overline{\zeta_c}^2 \overline{\zeta_k} - 6 \zeta_k \overline{\zeta_k}^2 + \overline{\zeta_c} \overline{\zeta_k} \left(-2 \zeta_k + \overline{\zeta_k}\right)\right) \Big) \Big) \\
& - a^6 \overline{\zeta_c}^2 \Big(\zeta_c \overline{\zeta_c}^2 (\zeta_c + \zeta_k)^2 \overline{\zeta_k}^3 + 4 r_c{}^6 \overline{\zeta_k} \left(\overline{\zeta_c} + \overline{\zeta_k}\right) \\
& - r_c{}^4 \left(\overline{\zeta_c} + \overline{\zeta_k}\right) \left(\overline{\zeta_c}^3 + \overline{\zeta_c}^2 \overline{\zeta_k} + 2 \overline{\zeta_c} \left(3 \zeta_c + \zeta_k\right) \overline{\zeta_k} + 2 \left(-\zeta_c + \zeta_k\right) \overline{\zeta_k}^2\right) + r_c{}^2 \overline{\zeta_c} \overline{\zeta_k} \Big(-\left(\zeta_c{}^2 \overline{\zeta_k} \left(3 \overline{\zeta_c} + 4 \overline{\zeta_k}\right)\right) \\
& + \zeta_c \overline{\zeta_c} \left(\overline{\zeta_c}^2 + 2 \overline{\zeta_c} \overline{\zeta_k} + \overline{\zeta_k} \left(2 \zeta_k + \overline{\zeta_k}\right)\right) + \overline{\zeta_c} \zeta_k \left(2 \overline{\zeta_c}^2 + 4 \overline{\zeta_c} \overline{\zeta_k} + \overline{\zeta_k} \left(\zeta_k + 2 \overline{\zeta_k}\right)\right) \Big) \Big) \\
& + a^2 \left(r_c{}^2 - \zeta_c \overline{\zeta_c}\right)^2 \Big(\zeta_c{}^3 \overline{\zeta_c}^2 (\zeta_c + \zeta_k)^2 \overline{\zeta_k}^3 + r_c{}^8 \left(\overline{\zeta_c} + \overline{\zeta_k}\right)^2 + r_c{}^6 \Big(\zeta_c{}^2 \overline{\zeta_k}^2 + \zeta_k \overline{\zeta_k} \left(-2 \overline{\zeta_c}^2 - 4 \overline{\zeta_c} \overline{\zeta_k} + \left(\zeta_k - 2 \overline{\zeta_k}\right) \overline{\zeta_k}\right) \\
& - \zeta_c \left(2 \overline{\zeta_c}^3 + 5 \overline{\zeta_c}^2 \overline{\zeta_k} + 4 \overline{\zeta_c} \overline{\zeta_k}^2 + \overline{\zeta_k}^2 \left(-2 \zeta_k + \overline{\zeta_k}\right)\right) \Big) + r_c{}^4 \Big(\zeta_c{}^3 \overline{\zeta_k}^2 \left(-2 \overline{\zeta_c} + \overline{\zeta_k}\right) \\
& + 2 \overline{\zeta_c}^2 \zeta_k{}^2 \left(\overline{\zeta_c} + \overline{\zeta_k}\right)^2 + \zeta_c \zeta_k \left(4 \overline{\zeta_c}^4 + 12 \overline{\zeta_c}^3 \overline{\zeta_k} + 12 \overline{\zeta_c}^2 \overline{\zeta_k}^2 - 2 \overline{\zeta_c} \left(\zeta_k - 2 \overline{\zeta_k}\right) \overline{\zeta_k}^2 + \zeta_k \overline{\zeta_k}^3\right) \\
& + \zeta_c{}^2 \left(3 \overline{\zeta_c}^4 + 8 \overline{\zeta_c}^3 \overline{\zeta_k} + 7 \overline{\zeta_c}^2 \overline{\zeta_k}^2 + 2 \zeta_k \overline{\zeta_k}^3 + 2 \overline{\zeta_c} \overline{\zeta_k}^2 \left(-2 \zeta_k + \overline{\zeta_k}\right)\right) \Big) \\
& + r_c{}^2 \zeta_c \overline{\zeta_c} \overline{\zeta_k} \Big(\zeta_c{}^3 \left(\overline{\zeta_c} - 2 \overline{\zeta_k}\right) \overline{\zeta_k} + 2 \overline{\zeta_c} \zeta_k{}^2 \left(\overline{\zeta_c} + \overline{\zeta_k}\right)^2 + \zeta_c{}^2 \left(\overline{\zeta_c}^3 + 2 \overline{\zeta_c}^2 \overline{\zeta_k} - 4 \zeta_k \overline{\zeta_k}^2 + \overline{\zeta_c} \overline{\zeta_k} \left(2 \zeta_k + \overline{\zeta_k}\right)\right) \\
& + \zeta_c \zeta_k \left(2 \overline{\zeta_c}^3 + 4 \overline{\zeta_c}^2 \overline{\zeta_k} - 2 \zeta_k \overline{\zeta_k}^2 + \overline{\zeta_c} \overline{\zeta_k} \left(\zeta_k + 2 \overline{\zeta_k}\right)\right) \Big) \Big) \Big) / \left(r_c{}^2 \overline{\zeta_k}\right)
\end{aligned} \tag{D.98}
$$

$$
\mathcal{Q}_{1b} = -\left(\frac{\pi \left(a^2 + (\zeta_c + \zeta_k)^2\right) \left(a^2 \zeta_k{}^2 - r_c{}^2 (\zeta_c + \zeta_k)^2\right)}{\zeta_k{}^2} \right) \tag{D.99}
$$

$$
\begin{aligned}
\mathcal{Q}_2 = \mathsf{Re} \Big\{ & \frac{\mathcal{Q}_{2a}}{(r_c^2 + a\overline{\zeta_c} - \delta^2)^2 (r_c^2 - a\overline{\zeta_c} - \delta^2)(a - \zeta_c - \zeta_k)(a + \zeta_c + \zeta_k)(\overline{\zeta_c} + \overline{\zeta_k})^2} \\
& + \frac{\mathcal{Q}_{2b}}{(a - \zeta_c - \zeta_k)(a + \zeta_c + \zeta_k)(\zeta_c + \zeta_k)} \Big\}
\end{aligned} \tag{D.100}
$$

$$
\begin{aligned}
\mathcal{Q}_{2a} = i\,\pi\, \Big(& a^8\,\overline{\zeta_c}^4\,\overline{\zeta_k}^2\,\left(r_c{}^2+\zeta_c\,\overline{\zeta_k}\right)+r_c{}^2\left(r_c{}^2-\zeta_c\,\overline{\zeta_c}\right)^4(\zeta_c+\zeta_k)^2\left(\overline{\zeta_c}+\overline{\zeta_k}\right)^2\,\left(r_c{}^2+\zeta_c\,\overline{\zeta_k}\right) \\
& +a^6\,\overline{\zeta_c}^2\,\Big(-\left(\zeta_c\,\overline{\zeta_c}^2\,(\zeta_c+\zeta_k)^2\,\overline{\zeta_k}^3\right)-4\,r_c{}^6\,\overline{\zeta_k}\,\left(\overline{\zeta_c}+\overline{\zeta_k}\right)-r_c{}^4\,\left(\overline{\zeta_c}+\overline{\zeta_k}\right)\,\Big(\overline{\zeta_c}^3+\overline{\zeta_c}^2\,\overline{\zeta_k} \\
& -2\,\overline{\zeta_c}\,\left(3\,\zeta_c+\zeta_k\right)\,\overline{\zeta_k}+2\,\left(\zeta_c-\zeta_k\right)\,\overline{\zeta_k}^2\Big)+r_c{}^2\,\overline{\zeta_c}\,\overline{\zeta_k}\,\Big(\zeta_c{}^2\,\overline{\zeta_k}\,\left(3\,\overline{\zeta_c}+4\,\overline{\zeta_k}\right) \\
& +\zeta_c\,\overline{\zeta_c}\,\left(\overline{\zeta_c}^2+2\,\overline{\zeta_c}\,\overline{\zeta_k}+\overline{\zeta_k}\,\left(-2\,\zeta_k+\overline{\zeta_k}\right)\right)+\overline{\zeta_c}\,\zeta_k\,\left(2\,\overline{\zeta_c}^2+4\,\overline{\zeta_c}\,\overline{\zeta_k}+\overline{\zeta_k}\,\left(-\zeta_k+2\,\overline{\zeta_k}\right)\right)\Big)\Big)\ \Big) \\
& +a^4\,\Big(-\left(r_c{}^{10}\,\overline{\zeta_k}^2\right)-\zeta_c{}^5\,\overline{\zeta_c}^4\,\overline{\zeta_k}^3+r_c{}^8\,\Big(2\,\overline{\zeta_c}^4+4\,\overline{\zeta_c}^3\,\overline{\zeta_k}+4\,\overline{\zeta_c}\,\left(2\,\zeta_c+\zeta_k\right)\,\overline{\zeta_k}^2+\left(\zeta_c+2\,\zeta_k\right)\,\overline{\zeta_k}^3 \\
& +2\,\overline{\zeta_c}^2\,\overline{\zeta_k}\,\left(\zeta_c+\zeta_k+\overline{\zeta_k}\right)\ \Big) \\
& -r_c{}^2\,\zeta_c\,\overline{\zeta_c}^3\,\overline{\zeta_k}\,\Big(5\,\zeta_c{}^3\,\overline{\zeta_c}\,\overline{\zeta_k}-\overline{\zeta_c}\,\zeta_k{}^2\left(\overline{\zeta_c}+\overline{\zeta_k}\right)^2+2\,\zeta_c\,\zeta_k\,\left(\overline{\zeta_c}+\overline{\zeta_k}\right)\,\left(\overline{\zeta_c}^2+\overline{\zeta_c}\,\overline{\zeta_k}+2\,\zeta_k\,\overline{\zeta_k}\right) \\
& +\zeta_c{}^2\,\left(\overline{\zeta_c}+\overline{\zeta_k}\right)\,\left(\overline{\zeta_c}^2+\overline{\zeta_c}\,\overline{\zeta_k}+8\,\zeta_k\,\overline{\zeta_k}\right)\ \Big) \\
& -2\,r_c{}^6\,\overline{\zeta_c}\,\Big(2\,\zeta_k\,\overline{\zeta_k}\,\left(\overline{\zeta_c}+\overline{\zeta_k}\right)\,\left(\overline{\zeta_c}^2+\overline{\zeta_c}\,\overline{\zeta_k}+\zeta_k\,\overline{\zeta_k}\right)+\zeta_c{}^2\,\overline{\zeta_k}\,\left(2\,\overline{\zeta_c}^2+9\,\overline{\zeta_c}\,\overline{\zeta_k}+2\,\overline{\zeta_k}^2\right) \\
& +\zeta_c\,\left(\overline{\zeta_c}+\overline{\zeta_k}\right)\,\left(2\,\overline{\zeta_c}^3+3\,\overline{\zeta_c}^2\,\overline{\zeta_k}+6\,\zeta_k\,\overline{\zeta_k}^2+\overline{\zeta_c}\,\overline{\zeta_k}\,\left(2\,\zeta_k+\overline{\zeta_k}\right)\right)\ \Big) \\
& +r_c{}^4\,\overline{\zeta_c}^2\,\Big(\overline{\zeta_c}^2\,\zeta_k{}^2\left(\overline{\zeta_c}+\overline{\zeta_k}\right)^2+2\,\zeta_c{}^3\,\overline{\zeta_k}\,\left(\overline{\zeta_c}^2+8\,\overline{\zeta_c}\,\overline{\zeta_k}+2\,\overline{\zeta_k}^2\right) \\
& +2\,\zeta_c\,\zeta_k\,\left(\overline{\zeta_c}+\overline{\zeta_k}\right)\,\left(\overline{\zeta_c}^3+5\,\overline{\zeta_c}^2\,\overline{\zeta_k}+4\,\overline{\zeta_c}\,\overline{\zeta_k}^2+4\,\zeta_k\,\overline{\zeta_k}^2\right) \\
& +\zeta_c{}^2\,\left(\overline{\zeta_c}+\overline{\zeta_k}\right)\,\left(3\,\overline{\zeta_c}^3+7\,\overline{\zeta_c}^2\,\overline{\zeta_k}+18\,\zeta_k\,\overline{\zeta_k}^2+2\,\overline{\zeta_c}\,\overline{\zeta_k}\,\left(\zeta_k+2\,\overline{\zeta_k}\right)\right)\ \Big)\Big) \\
& -a^2\left(r_c{}^2-\zeta_c\,\overline{\zeta_c}\right)^2\,\Big(-\left(\zeta_c{}^3\,\overline{\zeta_c}^2\,(\zeta_c+\zeta_k)^2\,\overline{\zeta_k}^3\right)+r_c{}^8\left(\overline{\zeta_c}+\overline{\zeta_k}\right)^2 \\
& +r_c{}^2\,\zeta_c\,\overline{\zeta_c}\,\overline{\zeta_k}\,\Big(2\,\overline{\zeta_c}\,\zeta_k{}^2\left(\overline{\zeta_c}+\overline{\zeta_k}\right)^2+\zeta_c{}^3\,\overline{\zeta_k}\,\left(-\overline{\zeta_c}+2\,\overline{\zeta_k}\right)+\zeta_c{}^2\,\left(\overline{\zeta_c}^3+2\,\overline{\zeta_c}^2\,\overline{\zeta_k}+4\,\zeta_k\,\overline{\zeta_k}^2+\overline{\zeta_c}\,\overline{\zeta_k}\,\left(-2\,\zeta_k+\overline{\zeta_k}\right)\right) \\
& +\zeta_c\,\zeta_k\,\left(2\,\overline{\zeta_c}^3+4\,\overline{\zeta_c}^2\,\overline{\zeta_k}+2\,\zeta_k\,\overline{\zeta_k}^2+\overline{\zeta_c}\,\overline{\zeta_k}\,\left(-\zeta_k+2\,\overline{\zeta_k}\right)\right)\ \Big) \\
& -r_c{}^6\,\Big(\zeta_c{}^2\,\overline{\zeta_k}^2+\zeta_c\,\left(2\,\overline{\zeta_c}^3+5\,\overline{\zeta_c}^2\,\overline{\zeta_k}+4\,\overline{\zeta_c}\,\overline{\zeta_k}^2+\overline{\zeta_k}^2\,\left(2\,\zeta_k+\overline{\zeta_k}\right)\right) \\
& +\zeta_k\,\overline{\zeta_k}\,\left(2\,\overline{\zeta_c}^2+4\,\overline{\zeta_c}\,\overline{\zeta_k}+\overline{\zeta_k}\,\left(\zeta_k+2\,\overline{\zeta_k}\right)\right)\Big)+r_c{}^4\,\Big(\zeta_c{}^3\,\left(2\,\overline{\zeta_c}-\overline{\zeta_k}\right)\,\overline{\zeta_k}^2+2\,\overline{\zeta_c}^2\,\zeta_k{}^2\left(\overline{\zeta_c}+\overline{\zeta_k}\right)^2 \\
& +\zeta_c{}^2\,\left(3\,\overline{\zeta_c}^4+8\,\overline{\zeta_c}^3\,\overline{\zeta_k}+7\,\overline{\zeta_c}^2\,\overline{\zeta_k}^2-2\,\zeta_k\,\overline{\zeta_k}^3+2\,\overline{\zeta_c}\,\overline{\zeta_k}^2\,\left(2\,\zeta_k+\overline{\zeta_k}\right)\right) \\
& +\zeta_c\,\zeta_k\,\left(4\,\overline{\zeta_c}^4+12\,\overline{\zeta_c}^3\,\overline{\zeta_k}+12\,\overline{\zeta_c}^2\,\overline{\zeta_k}^2-\zeta_k\,\overline{\zeta_k}^3+2\,\overline{\zeta_c}\,\overline{\zeta_k}^2\,\left(\zeta_k+2\,\overline{\zeta_k}\right)\right)\ \Big)\Big)\Big)\,/\,\left(r_c{}^2\,\overline{\zeta_k}\right)
\end{aligned}
\tag{D.101}
$$

$$
\mathcal{Q}_{2b}=\frac{i\,\pi\,\left(a^2+(\zeta_c+\zeta_k)^2\right)\,\left(a^2\,\zeta_k{}^2+r_c{}^2\,(\zeta_c+\zeta_k)^2\right)}{\zeta_k{}^2}
\tag{D.102}
$$

$$
\begin{aligned}
\mathcal{Q}_3 = \mathsf{Re}\Bigg\{ & \frac{\mathcal{Q}_{3a}}{(r_c^2+a\overline{\zeta_c}-\delta^2)^2(r_c^2-a\overline{\zeta_c}-\delta^2)(-a+\zeta_c+\zeta_k)(a+\zeta_c+\zeta_k)(\overline{\zeta_c}+\overline{\zeta_k})^2} \\
& +\frac{\mathcal{Q}_{3b}}{(-a+\zeta_c+\zeta_k)(a+\zeta_c+\zeta_k)(\zeta_c+\zeta_k)}\Bigg\}
\end{aligned}
\tag{D.103}
$$

$$
\begin{gathered}
\mathcal{Q}_{3a} = -i\,\pi\,\Big(-\Big(a^{10}\,\overline{\zeta_c}^{5}\,\overline{\zeta_k}^{2}\,(r_c^{2}+\zeta_c\,\overline{\zeta_k})\Big)+r_c^{2}\,\overline{\zeta_c}\,(r_c^{2}-\zeta_c\,\overline{\zeta_c})^{5}\,(\zeta_c+\zeta_k)^{2}\,(\overline{\zeta_c}+\overline{\zeta_k})^{2}\,(r_c^{2}+\zeta_c\,\overline{\zeta_k}) \\
+a^{8}\,\overline{\zeta_c}^{3}\,\Big(\zeta_c\,\overline{\zeta_c}^{2}\,\zeta_k\,(2\,\zeta_c+\zeta_k)\,\overline{\zeta_k}^{3}+r_c^{6}\,\Big(-\overline{\zeta_c}^{2}+2\,\overline{\zeta_c}\,\overline{\zeta_k}+2\,\overline{\zeta_k}^{2}\Big)+r_c^{4}\,\zeta_c\,\Big(\overline{\zeta_c}^{3}-3\,\overline{\zeta_c}^{2}\,\overline{\zeta_k}+\overline{\zeta_c}\,\overline{\zeta_k}^{2}+2\,\overline{\zeta_k}^{3}\Big) \\
+r_c^{2}\,\overline{\zeta_c}\,\overline{\zeta_k}\,\Big(\overline{\zeta_c}\,\zeta_k^{2}\,\overline{\zeta_k}-2\,\zeta_c\,\zeta_k\,\Big(\overline{\zeta_c}^{2}+\overline{\zeta_c}\,\overline{\zeta_k}+\overline{\zeta_k}^{2}\Big)-\zeta_c^{2}\,\Big(\overline{\zeta_c}^{2}+6\,\overline{\zeta_c}\,\overline{\zeta_k}+3\,\overline{\zeta_k}^{2}\Big)\Big)\Big) \\
+a^{2}\,(r_c^{2}-\zeta_c\,\overline{\zeta_c})^{3}\,\Big(\zeta_c^{4}\,\overline{\zeta_c}^{2}\,(\zeta_c+\zeta_k)^{2}\,\overline{\zeta_k}^{3}+r_c^{8}\,\Big(\overline{\zeta_c}\,(\zeta_c-\overline{\zeta_c}+\zeta_k)\,(\zeta_c+\overline{\zeta_c}+\zeta_k) \\
+2\,(\zeta_c-\overline{\zeta_c}+\zeta_k)\,(\zeta_c+\overline{\zeta_c}+\zeta_k)\,\overline{\zeta_k}-\overline{\zeta_c}\,\overline{\zeta_k}^{2}\Big)+r_c^{6}\,\Big(-2\,\zeta_c\,\overline{\zeta_c}^{2}\,(\zeta_c-\overline{\zeta_c}+\zeta_k)\,(\zeta_c+\overline{\zeta_c}+\zeta_k) \\
+\overline{\zeta_c}\,\Big(-3\,\zeta_c^{3}+5\,\zeta_c\,\overline{\zeta_c}^{2}-6\,\zeta_c^{2}\,\zeta_k+2\,\overline{\zeta_c}^{2}\,\zeta_k-3\,\zeta_c\,\zeta_k^{2}\Big)\,\overline{\zeta_k}+(\zeta_c+\zeta_k)\,\Big(4\,\overline{\zeta_c}^{2}+3\,\zeta_c\,(\zeta_c+\zeta_k)\Big)\,\overline{\zeta_k}^{2} \\
+\overline{\zeta_c}\,(\zeta_c+2\,\zeta_k)\,\overline{\zeta_k}^{3}\Big)+r_c^{4}\,\Big(-2\,\overline{\zeta_c}^{3}\,\zeta_k^{2}\,(\overline{\zeta_c}+\overline{\zeta_k})^{2} \\
-4\,\zeta_c\,\overline{\zeta_c}^{2}\,\zeta_k\,(\overline{\zeta_c}+\overline{\zeta_k})^{3}+\zeta_c^{4}\,\Big(\overline{\zeta_c}^{3}-6\,\overline{\zeta_c}\,\overline{\zeta_k}^{2}+\overline{\zeta_k}^{3}\Big)+2\,\zeta_c^{3}\,\zeta_k\,\Big(\overline{\zeta_c}^{3}-6\,\overline{\zeta_c}\,\overline{\zeta_k}^{2}+\overline{\zeta_k}^{3}\Big) \\
+\zeta_c^{2}\,\Big(-3\,\overline{\zeta_c}^{5}-8\,\overline{\zeta_c}^{4}\,\overline{\zeta_k}-6\,\overline{\zeta_c}\,\zeta_k^{2}\,\overline{\zeta_k}^{2}-2\,\overline{\zeta_c}^{2}\,\overline{\zeta_k}^{3}+\zeta_k^{2}\,\overline{\zeta_k}^{3}+\overline{\zeta_c}^{3}\,\Big(\zeta_k^{2}-7\,\overline{\zeta_k}^{2}\Big)\Big)\Big) \\
+r_c^{2}\,\zeta_c\,\overline{\zeta_c}\,\overline{\zeta_k}\,\Big(-2\,\zeta_c\,\overline{\zeta_c}^{2}\,\zeta_k\,(\overline{\zeta_c}+\overline{\zeta_k})^{2}-2\,\overline{\zeta_c}^{2}\,\zeta_k^{2}\,(\overline{\zeta_c}+\overline{\zeta_k})^{2}+\zeta_c^{4}\,\Big(\overline{\zeta_c}^{2}+3\,\overline{\zeta_c}\,\overline{\zeta_k}-2\,\overline{\zeta_k}^{2}\Big) \\
+2\,\zeta_c^{3}\,\zeta_k\,\Big(\overline{\zeta_c}^{2}+3\,\overline{\zeta_c}\,\overline{\zeta_k}-2\,\overline{\zeta_k}^{2}\Big)-\zeta_c^{2}\,\Big(\overline{\zeta_c}^{4}+2\,\overline{\zeta_c}^{3}\,\overline{\zeta_k}-3\,\overline{\zeta_c}\,\zeta_k^{2}\,\overline{\zeta_k}+2\,\zeta_k^{2}\,\overline{\zeta_k}^{2}+\overline{\zeta_c}^{2}\,\Big(-\zeta_k^{2}+\overline{\zeta_k}^{2}\Big)\Big)\Big)\Big) \\
+a^{6}\,\overline{\zeta_c}\,(r_c^{2}-\zeta_c\,\overline{\zeta_c})\,\Big(-\Big(\zeta_c^{2}\,\overline{\zeta_c}^{3}\,(2\,\zeta_c^{2}+2\,\zeta_c\,\zeta_k+\zeta_k^{2})\,\overline{\zeta_k}^{3}\Big) \\
+r_c^{8}\,\Big(2\,\overline{\zeta_c}^{2}+8\,\overline{\zeta_c}\,\overline{\zeta_k}+7\,\overline{\zeta_k}^{2}\Big)+r_c^{4}\,\overline{\zeta_c}\,\Big(\overline{\zeta_c}^{3}\,\Big(3\,\zeta_c^{2}-\overline{\zeta_c}^{2}+2\,\zeta_c\,\zeta_k+\zeta_k^{2}\Big) \\
+2\,\overline{\zeta_c}^{2}\,\Big(9\,\zeta_c^{2}-\overline{\zeta_c}^{2}+8\,\zeta_c\,\zeta_k+\zeta_k^{2}\Big)\,\overline{\zeta_k}+\overline{\zeta_c}\,\Big(23\,\zeta_c^{2}-\overline{\zeta_c}^{2}+36\,\zeta_c\,\zeta_k+6\,\zeta_k^{2}\Big)\,\overline{\zeta_k}^{2}+(\zeta_c^{2}+20\,\zeta_c\,\zeta_k+4\,\zeta_k^{2})\,\overline{\zeta_k}^{3}\Big) \\
-r_c^{6}\,\Big(8\,\zeta_k\,\overline{\zeta_k}\,(\overline{\zeta_c}+\overline{\zeta_k})^{2}+\zeta_c\,\Big(4\,\overline{\zeta_c}^{3}+22\,\overline{\zeta_c}^{2}\,\overline{\zeta_k}+23\,\overline{\zeta_c}\,\overline{\zeta_k}^{2}+\overline{\zeta_k}^{3}\Big)\Big) \\
+r_c^{2}\,\overline{\zeta_c}^{2}\,\overline{\zeta_k}\,\Big(2\,\overline{\zeta_c}^{2}\,\zeta_k\,(\overline{\zeta_c}+\overline{\zeta_k})^{2}-\zeta_c^{3}\,\Big(\overline{\zeta_c}^{2}+4\,\overline{\zeta_c}\,\overline{\zeta_k}-3\,\overline{\zeta_k}^{2}\Big) \\
-2\,\zeta_c^{2}\,\zeta_k\,\Big(\overline{\zeta_c}^{2}+7\,\overline{\zeta_c}\,\overline{\zeta_k}+4\,\overline{\zeta_k}^{2}\Big)+\zeta_c\,\Big(\overline{\zeta_c}^{4}+2\,\overline{\zeta_c}^{3}\,\overline{\zeta_k}-3\,\overline{\zeta_c}\,\zeta_k^{2}\,\overline{\zeta_k}-2\,\zeta_k^{2}\,\overline{\zeta_k}^{2}+\overline{\zeta_c}^{2}\,\Big(\zeta_k^{2}+\overline{\zeta_k}^{2}\Big)\Big)\Big)\Big) \\
+a^{4}\,(r_c^{2}-\zeta_c\,\overline{\zeta_c})\,\Big(\zeta_c^{4}\,\overline{\zeta_c}^{4}\,\zeta_k\,(2\,\zeta_c+\zeta_k)\,\overline{\zeta_k}^{3}-r_c^{12}\,(\overline{\zeta_c}+2\,\overline{\zeta_k})+r_c^{10}\,\zeta_c\,\Big(4\,\overline{\zeta_c}^{2}+7\,\overline{\zeta_c}\,\overline{\zeta_k}-3\,\overline{\zeta_k}^{2}\Big) \\
+r_c^{8}\,\Big(-2\,\overline{\zeta_c}^{3}\,\Big(4\,\zeta_c^{2}-\overline{\zeta_c}^{2}+2\,\zeta_c\,\zeta_k+\zeta_k^{2}\Big)-2\,\overline{\zeta_c}^{2}\,\Big(7\,\zeta_c^{2}-2\,\overline{\zeta_c}^{2} \\
+5\,\zeta_c\,\zeta_k+2\,\zeta_k^{2}\Big)\,\overline{\zeta_k}+\overline{\zeta_c}\,\Big(9\,\zeta_c^{2}+2\,\overline{\zeta_c}^{2}-2\,\zeta_c\,\zeta_k+\zeta_k^{2}\Big)\,\overline{\zeta_k}^{2}+(\zeta_c^{2}+6\,\zeta_c\,\zeta_k+4\,\zeta_k^{2})\,\overline{\zeta_k}^{3}\Big) \\
+r_c^{6}\,\overline{\zeta_c}\,\Big(4\,\zeta_c\,\overline{\zeta_c}^{3}\,\Big(2\,\zeta_c^{2}-\overline{\zeta_c}^{2}+2\,\zeta_c\,\zeta_k+\zeta_k^{2}\Big)+2\,\overline{\zeta_c}^{2}\,\Big(7\,\zeta_c^{3}-5\,\zeta_c\,\overline{\zeta_c}^{2}+9\,\zeta_c^{2}\,\zeta_k-2\,\overline{\zeta_c}^{2}\,\zeta_k+3\,\zeta_c\,\zeta_k^{2}\Big)\,\overline{\zeta_k} \\
-\overline{\zeta_c}\,\Big(15\,\zeta_c^{3}+8\,\zeta_c\,\overline{\zeta_c}^{2}+6\,\zeta_c^{2}\,\zeta_k+8\,\overline{\zeta_c}^{2}\,\zeta_k+9\,\zeta_c\,\zeta_k^{2}\Big)\,\overline{\zeta_k}^{2}-\Big(5\,\zeta_c^{3}+2\,\zeta_c\,\overline{\zeta_c}^{2}+24\,\zeta_c^{2}\,\zeta_k+4\,\overline{\zeta_c}^{2}\,\zeta_k+15\,\zeta_c\,\zeta_k^{2}\Big)\,\overline{\zeta_k}^{3}\Big) \\
+r_c^{2}\,\zeta_c\,\overline{\zeta_c}^{3}\,\overline{\zeta_k}\,\Big(-2\,\zeta_c\,\overline{\zeta_c}^{2}\,\zeta_k\,(\overline{\zeta_c}+\overline{\zeta_k})^{2}+\overline{\zeta_c}^{2}\,\zeta_k^{2}\,(\overline{\zeta_c}+\overline{\zeta_k})^{2}-\zeta_c^{4}\,\Big(\overline{\zeta_c}^{2}+6\,\overline{\zeta_c}\,\overline{\zeta_k}+3\,\overline{\zeta_k}^{2}\Big) \\
-2\,\zeta_c^{3}\,\zeta_k\,\Big(\overline{\zeta_c}^{2}+5\,\overline{\zeta_c}\,\overline{\zeta_k}+8\,\overline{\zeta_k}^{2}\Big)-\zeta_c^{2}\,\Big(\overline{\zeta_c}^{4}+2\,\overline{\zeta_c}^{3}\,\overline{\zeta_k}+7\,\overline{\zeta_c}\,\zeta_k^{2}\,\overline{\zeta_k}+9\,\zeta_k^{2}\,\overline{\zeta_k}^{2}+\overline{\zeta_c}^{2}\,\Big(2\,\zeta_k^{2}+\overline{\zeta_k}^{2}\Big)\Big)\Big) \\
+r_c^{4}\,\overline{\zeta_c}^{2}\,\Big(\overline{\zeta_c}^{3}\,\zeta_k^{2}\,(\overline{\zeta_c}+\overline{\zeta_k})^{2}+2\,\zeta_c\,\overline{\zeta_c}^{2}\,\zeta_k\,(\overline{\zeta_c}+\overline{\zeta_k})^{2}\,(\overline{\zeta_c}+4\,\overline{\zeta_k})+\zeta_c^{4}\,\Big(-3\,\overline{\zeta_c}^{3}-4\,\overline{\zeta_c}^{2}\,\overline{\zeta_k}+15\,\overline{\zeta_c}\,\overline{\zeta_k}^{2}+7\,\overline{\zeta_k}^{3}\Big) \\
+2\,\zeta_c^{3}\,\zeta_k\,\Big(-2\,\overline{\zeta_c}^{3}-3\,\overline{\zeta_c}^{2}\,\overline{\zeta_k}+9\,\overline{\zeta_c}\,\overline{\zeta_k}^{2}+16\,\overline{\zeta_k}^{3}\Big) \\
+\zeta_c^{2}\,\Big(3\,\overline{\zeta_c}^{5}+10\,\overline{\zeta_c}^{4}\,\overline{\zeta_k}+15\,\overline{\zeta_c}\,\zeta_k^{2}\,\overline{\zeta_k}^{2}+4\,\overline{\zeta_c}^{2}\,\overline{\zeta_k}^{3}+19\,\zeta_k^{2}\,\overline{\zeta_k}^{3}+\overline{\zeta_c}^{3}\,\Big(-2\,\zeta_k^{2}+11\,\overline{\zeta_k}^{2}\Big)\Big)\Big)\Big)\Big) \\
/\,\Big(r_c^{2}\,(r_c^{2}-\zeta_c\,\overline{\zeta_c})\,\overline{\zeta_k}\Big) \qquad \text{(D.104)}
\end{gathered}
$$

$$\mathcal{Q}_{3b} = \frac{-i\,\pi\,\left(a^2+{\zeta_c}^2\right)\,\left(a^4\,\zeta_c - 6\,{r_c}^4\,\zeta_c - a^2\,{r_c}^2\,\overline{\zeta_c} + 6\,{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c} + a^2\,\zeta_c\,{\overline{\zeta_c}}^2\right)}{-{r_c}^2+\zeta_c\,\overline{\zeta_c}}$$
$$- \frac{i\,\left(a^4\,\pi\,{r_c}^2\,\zeta_c + 2\,a^2\,\pi\,{r_c}^2\,{\zeta_c}^3 + \pi\,{r_c}^2\,{\zeta_c}^5\right)}{{\zeta_k}^2} - \frac{2\,i\,\left(a^4\,\pi\,{r_c}^2 + 3\,a^2\,\pi\,{r_c}^2\,{\zeta_c}^2 + 2\,\pi\,{r_c}^2\,{\zeta_c}^4\right)}{\zeta_k}$$
$$- \frac{2\,i\,\pi\,\left(-\left(a^2\,{r_c}^4\right) + a^4\,{\zeta_c}^2 - 2\,{r_c}^4\,{\zeta_c}^2 + 2\,{r_c}^2\,{\zeta_c}^3\,\overline{\zeta_c} + a^2\,{\zeta_c}^2\,{\overline{\zeta_c}}^2\right)\,\zeta_k}{-{r_c}^2+\zeta_c\,\overline{\zeta_c}}$$
$$- \frac{i\,\pi\,\left(a^4\,\zeta_c - {r_c}^4\,\zeta_c - a^2\,{r_c}^2\,\overline{\zeta_c} + {r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c} + a^2\,\zeta_c\,{\overline{\zeta_c}}^2\right)\,{\zeta_k}^2}{-{r_c}^2+\zeta_c\,\overline{\zeta_c}} \quad \text{(D.105)}$$

$$\mathcal{Q}_4 = \mathsf{Re}\Bigg(\pi\,\Bigg(-\Bigg(\left({r_c}^2-\delta^2\right)\,\left(a-\zeta_c-\zeta_k\right)\,\left(a+\zeta_c+\zeta_k\right)\,\Bigg(\left({r_c}^3 - r_c\,\delta^2\right)^2\,\left(\zeta_c+\zeta_k\right)\,\left(\overline{\zeta_c}+\overline{\zeta_k}\right)^2\,\Big(\zeta_c\,{\zeta_k}^2\,\overline{\zeta_k}$$
$$+{r_c}^2\,\left({\zeta_k}^2 + \left(\zeta_c+\zeta_k\right)\,\overline{\zeta_k}\right)\Big) + a^2\,\left({r_c}^2-\delta^2\right)^2\,\Bigg(-\left(\zeta_c\,{\zeta_k}^2\,\left(\zeta_c+\zeta_k\right)\,{\overline{\zeta_k}}^3\right) + {r_c}^2\,{\zeta_k}^2\,\overline{\zeta_k}\,\Big({\overline{\zeta_c}}^2 - \zeta_c\,\left(\zeta_c+\zeta_k\right)$$
$$-\left(\zeta_c - 2\,\overline{\zeta_c} + \zeta_k\right)\,\overline{\zeta_k} + {\overline{\zeta_k}}^2\Big) + {r_c}^4\,\left(-\left({\zeta_k}^2\,\left(\zeta_c+\zeta_k\right)\right) + \overline{\zeta_k}\,\left(\overline{\zeta_c}+\overline{\zeta_k}\right)^2\right)\Bigg) + a^4\,{\zeta_k}^2\,\overline{\zeta_k}\,\Bigg(-\left(\zeta_c\,{\overline{\zeta_c}}^2\,\left(\zeta_c+\zeta_k\right)\,{\overline{\zeta_k}}^2\right)$$
$$-{r_c}^4\,\left({\overline{\zeta_c}}^2 + 2\,\overline{\zeta_c}\,\overline{\zeta_k} + \overline{\zeta_k}\,\left(\zeta_c+\zeta_k+\overline{\zeta_k}\right)\right) + {r_c}^2\,\Big(-\left({\zeta_c}^2\,{\overline{\zeta_c}}^2\right) + \overline{\zeta_c}\,\zeta_k\,\overline{\zeta_k}\,\left(\overline{\zeta_c} + 2\,\overline{\zeta_k}\right) + \zeta_c\,\overline{\zeta_k}\,\Big({\overline{\zeta_c}}^2 + \zeta_k\,\overline{\zeta_k}$$
$$+2\,\overline{\zeta_c}\,\left(\zeta_k+\overline{\zeta_k}\right)\Big)\Big)\Bigg)\Bigg)\Bigg) + 4\,i\,a^4\,{r_c}^4\,\zeta_k\,\left(\zeta_c+\zeta_k\right)^2\,\left(a^2+\left(\zeta_c+\zeta_k\right)^2\right)\,\overline{\zeta_k}\,\left(\overline{\zeta_c}+\overline{\zeta_k}\right)^2\,\zeta_y\Bigg)$$
$$/\left({r_c}^2\,\left({r_c}^2-\delta^2\right)^3\,{\zeta_k}^2\,\left(\zeta_c+\zeta_k\right)\,\left(-a+\zeta_c+\zeta_k\right)\,\left(a+\zeta_c+\zeta_k\right)\,\overline{\zeta_k}\,\left(\overline{\zeta_c}+\overline{\zeta_k}\right)^2\right)\Bigg) \quad \text{(D.106)}$$

$$\mathcal{Q}_5 = \mathsf{Re}\left\{\frac{\mathcal{Q}_{5a}}{(r_c^2-\delta^2)^3(-a+\zeta_c+\zeta_k)(a+\zeta_c+\zeta_k)(\overline{\zeta_c}+\overline{\zeta_k})^2} + \frac{\mathcal{Q}_{5b}}{(r_c^2-\delta^2)^3(-a+\zeta_c+\zeta_k)(a+\zeta_c+\zeta_k)(\zeta_c+\zeta_k)}\right\} \quad \text{(D.107)}$$

$$
\begin{aligned}
\mathcal{Q}_{5a} = -i\,\pi\,\Big(& r_c{}^2\left(r_c{}^2-\zeta_c\,\overline{\zeta_c}\right)^3\zeta_k\left(\zeta_c+\zeta_k\right)^2\left(\overline{\zeta_c}+\overline{\zeta_k}\right)^2\left(r_c{}^2+\zeta_c\,\overline{\zeta_k}\right)\\
&+a^6\,\overline{\zeta_k}\,\Big(r_c{}^6\,\zeta_k\,\overline{\zeta_k}+\zeta_c{}^2\,\overline{\zeta_c}^3\,\zeta_k\,\overline{\zeta_k}^2+r_c{}^4\,\Big(2\,\overline{\zeta_c}^2\left(i\,\zeta_y\,\zeta_k+\zeta_c\left(\overline{\zeta_c}+i\,\zeta_y+\zeta_k\right)\right)\\
&+\overline{\zeta_c}\left(4\,\zeta_c\left(\overline{\zeta_c}+i\,\zeta_y\right)+\left(\zeta_c+\overline{\zeta_c}+4\,i\,\zeta_y\right)\zeta_k\right)\overline{\zeta_k}\\
&+\left(2\left(\overline{\zeta_c}+i\,\zeta_y\right)\zeta_k+\zeta_c\left(2\,\overline{\zeta_c}+2\,i\,\zeta_y+\zeta_k\right)\right)\overline{\zeta_k}^2\Big)\\
&+r_c{}^2\,\zeta_c\,\overline{\zeta_c}\,\zeta_k\,\overline{\zeta_k}\left(\zeta_c\left(2\,\overline{\zeta_c}+\overline{\zeta_k}\right)-\overline{\zeta_c}\left(\overline{\zeta_c}+3\,\overline{\zeta_k}\right)\right)\Big)\\
&+a^4\,\Big(r_c{}^{10}\,\zeta_k-\zeta_c{}^2\,\overline{\zeta_c}^3\,\zeta_k{}^2\left(2\,\zeta_c+\zeta_k\right)\overline{\zeta_k}^3-r_c{}^8\,\zeta_k\,\Big(3\,\zeta_c\,\overline{\zeta_c}-\zeta_c\,\overline{\zeta_k}+\overline{\zeta_k}^2\Big)\\
&+r_c{}^6\,\zeta_k\,\Big(3\,\zeta_c{}^2\,\overline{\zeta_c}^2+\overline{\zeta_c}\left(-3\,\zeta_c{}^2+2\,\zeta_c\,\overline{\zeta_c}+2\,\overline{\zeta_c}\,\zeta_k\right)\overline{\zeta_k}\\
&-\left(\zeta_c{}^2-7\,\zeta_c\,\overline{\zeta_c}+2\,\zeta_c\,\zeta_k-4\,\overline{\zeta_c}\,\zeta_k+\zeta_k{}^2\right)\overline{\zeta_k}^2+\left(\zeta_c+2\,\zeta_k\right)\overline{\zeta_k}^3\Big)+r_c{}^4\,\Big(-\Big(\zeta_c{}^3\,\overline{\zeta_c}^3\,\zeta_k\Big)\\
&+\overline{\zeta_c}^2\left(2\,i\,\zeta_y\,\zeta_k{}^3+2\,\zeta_c\,\zeta_k{}^2\left(3\,i\,\zeta_y+\zeta_k\right)+2\,\zeta_c{}^2\,\zeta_k\left(\overline{\zeta_c}+3\,i\,\zeta_y+\zeta_k\right)+\zeta_c{}^3\left(2\,\overline{\zeta_c}+2\,i\,\zeta_y+3\,\zeta_k\right)\right)\overline{\zeta_k}\\
&+\overline{\zeta_c}\left(4\,\zeta_c{}^3\left(\overline{\zeta_c}+i\,\zeta_y\right)+3\,\zeta_c{}^2\left(\zeta_c+4\,i\,\zeta_y\right)\zeta_k+2\,\zeta_c\left(5\,\zeta_c-\overline{\zeta_c}+6\,i\,\zeta_y\right)\zeta_k{}^2\right.\\
&\left.+\left(7\,\zeta_c-\overline{\zeta_c}+4\,i\,\zeta_y\right)\zeta_k{}^3\right)\overline{\zeta_k}^2+\left(-2\left(\overline{\zeta_c}-i\,\zeta_y\right)\zeta_k{}^3+\zeta_c{}^3\left(2\,\overline{\zeta_c}+2\,i\,\zeta_y+\zeta_k\right)+\zeta_c\,\zeta_k{}^2\left(-4\,\overline{\zeta_c}+6\,i\,\zeta_y+3\,\zeta_k\right)\right.\\
&\left.+\zeta_c{}^2\,\zeta_k\left(3\,\overline{\zeta_c}+6\,i\,\zeta_y+4\,\zeta_k\right)\right)\overline{\zeta_k}^3\Big)+r_c{}^2\,\zeta_c\,\overline{\zeta_c}\,\zeta_k\,\overline{\zeta_k}\,\Big(\zeta_c{}^3\left(\overline{\zeta_c}+\overline{\zeta_k}\right)^2+2\,\zeta_c{}^2\,\overline{\zeta_c}^2\left(\zeta_k+\overline{\zeta_k}\right)\\
&+\overline{\zeta_c}\,\zeta_k{}^2\,\overline{\zeta_k}\left(\overline{\zeta_c}+3\,\overline{\zeta_k}\right)-\zeta_c\,\zeta_k\,\overline{\zeta_k}\,\Big(-2\,\overline{\zeta_c}^2+2\,\overline{\zeta_c}\,\zeta_k-6\,\overline{\zeta_c}\,\overline{\zeta_k}+\zeta_k\,\overline{\zeta_k}\Big)\Big)\Big)\\
&+a^2\left(r_c{}^2-\zeta_c\,\overline{\zeta_c}\right)^3\zeta_k\,\Big(\zeta_c\left(\zeta_c+\zeta_k\right)^2\overline{\zeta_k}^3-r_c{}^4\,\Big(\left(\zeta_c+\zeta_k\right)^2+\left(\overline{\zeta_c}+\overline{\zeta_k}\right)^2\Big)\\
&+r_c{}^2\,\overline{\zeta_k}\,\Big(-\zeta_c{}^3+\zeta_c{}^2\left(-2\,\zeta_k+\overline{\zeta_k}\right)+\zeta_c\,\Big(\overline{\zeta_c}^2-\zeta_k{}^2+2\left(\overline{\zeta_c}+\zeta_k\right)\overline{\zeta_k}+\overline{\zeta_k}^2\Big)\\
&+\zeta_k\,\Big(2\,\overline{\zeta_c}^2+4\,\overline{\zeta_c}\,\overline{\zeta_k}+\overline{\zeta_k}\left(\zeta_k+2\,\overline{\zeta_k}\right)\Big)\Big)\Big)\Big)\Big/\left(r_c{}^2\,\zeta_k\,\overline{\zeta_k}\right)
\end{aligned}
\tag{D.108}
$$

$$
\begin{aligned}
\mathcal{Q}_{5b} = -i\,\pi\,\Big(a^2+\left(\zeta_c+\zeta_k\right)^2\Big)\,\Big(& r_c{}^2\left(r_c{}^2-\zeta_c\,\overline{\zeta_c}\right)^3\left(\zeta_c+\zeta_k\right)^2\\
&+a^2\left(r_c{}^2-\zeta_c\,\overline{\zeta_c}\right)^3\left(-r_c{}^2+\zeta_k{}^2\right)+a^4\,\zeta_k\left(r_c{}^4\,\zeta_k+\zeta_c{}^3\,\overline{\zeta_c}\,\zeta_k+r_c{}^2\left(2\,\zeta_c{}^2\left(\overline{\zeta_c}-i\,\zeta_y\right)\right.\right.\\
&\left.\left.+\zeta_c\left(\zeta_c+\overline{\zeta_c}-4\,i\,\zeta_y\right)\zeta_k+2\left(\zeta_c-i\,\zeta_y\right)\zeta_k{}^2\right)\right)\Big)/\zeta_k{}^2
\end{aligned}
\tag{D.109}
$$

$$
\begin{aligned}
\mathcal{Q}_6 = \mathsf{Re}\Bigg\{&\frac{\mathcal{Q}_{6a}}{(r_c^2-\delta^2)^2(r_c^2+a\overline{\zeta_c}-\delta^2)^2(r_c^2-a\overline{\zeta_c}-\delta^2)(-a+\zeta_c+\zeta_k)(a+\zeta_c+\zeta_k)(\overline{\zeta_c}+\overline{\zeta_k})^2}\\
&+\frac{\mathcal{Q}_{6b}}{(r_c^2-\delta^2)^2(-a+\zeta_c+\zeta_k)(a+\zeta_c+\zeta_k)(\zeta_c+\zeta_k)}\Bigg\}
\end{aligned}
\tag{D.110}
$$

$$
\begin{aligned}
\mathcal{Q}_{6a} = 2\,a\,\pi\,\Big(& r_c^{\,2}\left(r_c^{\,2} - \zeta_c\,\overline{\zeta_c}\right)^6 \zeta_k\left(\zeta_c+\zeta_k\right)^2\left(\overline{\zeta_c}+\overline{\zeta_k}\right)\left(r_c^{\,2}+\zeta_c\,\overline{\zeta_k}\right) \\
& -a^8\,\overline{\zeta_c}^4\,\overline{\zeta_k}\,\Big(\zeta_c^{\,2}\,\overline{\zeta_c}^2\,\zeta_k\,\overline{\zeta_k}^2 - r_c^{\,2}\,\zeta_c\,\overline{\zeta_c}\,\zeta_k\,\overline{\zeta_k}\left(\overline{\zeta_c}+3\,\overline{\zeta_k}\right) \\
& +r_c^{\,4}\left(\zeta_c\left(\overline{\zeta_c}+\overline{\zeta_k}\right)^2 - \zeta_k\left(\overline{\zeta_c}^2+\overline{\zeta_c}\,\overline{\zeta_k}-\overline{\zeta_k}^2\right)\right)\Big) \\
& +a^6\,\overline{\zeta_c}^2\,\Big(\zeta_c^{\,2}\,\overline{\zeta_c}^4\,\zeta_k\left(\zeta_c+\zeta_k\right)^2\overline{\zeta_k}^3 - r_c^{\,8}\left(\overline{\zeta_c}+\overline{\zeta_k}\right)\Big(\overline{\zeta_c}^2\,\zeta_k - 2\,\overline{\zeta_c}\left(\zeta_c-2\,\zeta_k\right)\overline{\zeta_k} \\
& -2\left(\zeta_c+\zeta_k\right)\overline{\zeta_k}^2\Big) + r_c^{\,6}\,\zeta_c\,\overline{\zeta_c}\left(\overline{\zeta_c}+\overline{\zeta_k}\right)\left(2\,\overline{\zeta_c}^2\,\zeta_k+\overline{\zeta_c}\left(-4\,\zeta_c+7\,\zeta_k\right)\overline{\zeta_k}-2\left(2\,\zeta_c+5\,\zeta_k\right)\overline{\zeta_k}^2\right) \\
& +r_c^{\,2}\,\zeta_c\,\overline{\zeta_c}^3\,\zeta_k\,\overline{\zeta_k}\,\Big(2\,\zeta_c\,\zeta_k\left(\overline{\zeta_c}-\overline{\zeta_k}\right)\left(\overline{\zeta_c}+2\,\overline{\zeta_k}\right)-\zeta_k^{\,2}\,\overline{\zeta_k}\left(\overline{\zeta_c}+3\,\overline{\zeta_k}\right) \\
& +\zeta_c^{\,2}\left(\overline{\zeta_c}^2-4\,\overline{\zeta_c}\,\overline{\zeta_k}-7\,\overline{\zeta_k}^2\right)\Big) + r_c^{\,4}\,\overline{\zeta_c}^2\,\Big(\zeta_c^{\,3}\,\overline{\zeta_k}\left(\overline{\zeta_c}+\overline{\zeta_k}\right)^2+\zeta_c\,\zeta_k^{\,2}\,\overline{\zeta_k}\left(-\overline{\zeta_c}^2+3\,\overline{\zeta_k}^2\right) \\
& +\zeta_k^{\,3}\,\overline{\zeta_k}\left(\overline{\zeta_c}^2+3\,\overline{\zeta_c}\,\overline{\zeta_k}+3\,\overline{\zeta_k}^2\right)-\zeta_c^{\,2}\,\zeta_k\left(\overline{\zeta_c}^3+6\,\overline{\zeta_c}^2\,\overline{\zeta_k}-7\,\overline{\zeta_c}\,\overline{\zeta_k}^2-13\,\overline{\zeta_k}^3\right)\Big)\Big) \\
& -a^2\left(r_c^{\,2}-\zeta_c\,\overline{\zeta_c}\right)^4\Big(\zeta_c^{\,2}\,\overline{\zeta_c}^2\,\zeta_k\left(\zeta_c+\zeta_k\right)^2\overline{\zeta_k}^3+r_c^{\,8}\,\zeta_k\left(\overline{\zeta_c}+\overline{\zeta_k}\right)-r_c^{\,6}\,\zeta_c\,\zeta_k\left(2\,\overline{\zeta_c}-\overline{\zeta_k}\right)\left(\overline{\zeta_c}+\overline{\zeta_k}\right) \\
& +r_c^{\,2}\,\zeta_c\,\overline{\zeta_c}\,\zeta_k\,\overline{\zeta_k}\left(\overline{\zeta_c}^2\left(\zeta_c^{\,2}+2\,\zeta_c\,\zeta_k+2\,\zeta_k^{\,2}\right)+\overline{\zeta_c}\,\zeta_k\left(2\,\zeta_c+3\,\zeta_k\right)\overline{\zeta_k}-\left(\zeta_c+\zeta_k\right)\left(3\,\zeta_c+\zeta_k\right)\overline{\zeta_k}^2\right) \\
& +r_c^{\,4}\,\Big(\zeta_c^{\,3}\,\overline{\zeta_k}\left(\overline{\zeta_c}+\overline{\zeta_k}\right)^2+\zeta_k^{\,3}\left(2\,\overline{\zeta_c}^3+\overline{\zeta_c}^2\,\overline{\zeta_k}-3\,\overline{\zeta_c}\,\overline{\zeta_k}^2-\overline{\zeta_k}^3\right) \\
& +\zeta_c\,\zeta_k^{\,2}\left(4\,\overline{\zeta_c}^3+5\,\overline{\zeta_c}^2\,\overline{\zeta_k}+\overline{\zeta_k}^3\right)+\zeta_c^{\,2}\,\zeta_k\left(3\,\overline{\zeta_c}^3+4\,\overline{\zeta_c}^2\,\overline{\zeta_k}+3\,\overline{\zeta_c}\,\overline{\zeta_k}^2+3\,\overline{\zeta_k}^3\right)\Big)\Big) \\
& +a^4\left(r_c^{\,2}-\zeta_c\,\overline{\zeta_c}\right)^2\Big(\zeta_c^{\,4}\,\overline{\zeta_c}^4\,\zeta_k\,\overline{\zeta_k}^3+r_c^{\,8}\left(2\,\overline{\zeta_c}^3\,\zeta_k+\overline{\zeta_c}^2\left(-\zeta_c+\zeta_k\right)\overline{\zeta_k}-\overline{\zeta_c}\left(2\,\zeta_c+3\,\zeta_k\right)\overline{\zeta_k}^2-\left(\zeta_c+\zeta_k\right)\overline{\zeta_k}^3\right) \\
& +r_c^{\,2}\,\zeta_c\,\overline{\zeta_c}^3\,\zeta_k\,\overline{\zeta_k}\left(-\left(\zeta_c^{\,2}\left(\overline{\zeta_c}-\overline{\zeta_k}\right)^2\right)-2\,\zeta_c\,\zeta_k\left(\overline{\zeta_c}-4\,\overline{\zeta_k}\right)\left(\overline{\zeta_c}+\overline{\zeta_k}\right)+\zeta_k^{\,2}\left(\overline{\zeta_c}+\overline{\zeta_k}\right)\left(\overline{\zeta_c}+6\,\overline{\zeta_k}\right)\right) \\
& +r_c^{\,4}\,\overline{\zeta_c}^2\,\Big(\zeta_c^{\,3}\,\overline{\zeta_k}\left(\overline{\zeta_c}+\overline{\zeta_k}\right)^2+\zeta_k^{\,3}\left(\overline{\zeta_c}-4\,\overline{\zeta_k}\right)\left(\overline{\zeta_c}+\overline{\zeta_k}\right)\left(\overline{\zeta_c}+2\,\overline{\zeta_k}\right) \\
& +\zeta_c^{\,2}\,\zeta_k\left(\overline{\zeta_c}-2\,\overline{\zeta_k}\right)\left(3\,\overline{\zeta_c}^2+6\,\overline{\zeta_c}\,\overline{\zeta_k}+\overline{\zeta_k}^2\right)+2\,\zeta_c\,\zeta_k^{\,2}\left(\overline{\zeta_c}^3-8\,\overline{\zeta_c}\,\overline{\zeta_k}^2-7\,\overline{\zeta_k}^3\right)\Big) \\
& +r_c^{\,6}\,\overline{\zeta_c}\left(2\,\zeta_c^{\,2}\,\overline{\zeta_k}\left(\overline{\zeta_c}+\overline{\zeta_k}\right)^2+4\,\zeta_k^{\,2}\,\overline{\zeta_k}\left(\overline{\zeta_c}+\overline{\zeta_k}\right)^2+\zeta_c\,\zeta_k\left(-4\,\overline{\zeta_c}^3+4\,\overline{\zeta_c}^2\,\overline{\zeta_k}+17\,\overline{\zeta_c}\,\overline{\zeta_k}^2+5\,\overline{\zeta_k}^3\right)\right)\Big)\Big)/r_c^{\,2}\,\zeta_k\,\overline{\zeta_k}
\end{aligned}
\tag{D.111}
$$

$$
\begin{aligned}
\mathcal{Q}_{6b} = & \frac{2\,a\,\pi\,\left(-\left(\zeta_c\left(r_c^{\,3}-r_c\,\zeta_c\,\overline{\zeta_c}\right)^2\right)-r_c^{\,2}\left(a^2\,\zeta_c^{\,2}+\left(r_c^{\,2}-\zeta_c\,\overline{\zeta_c}\right)^2\right)\zeta_k\right)}{\zeta_k^{\,2}} \\
& + \frac{2a\pi\left(a^2\,\zeta_c\left(-r_c^{\,2}+\zeta_c\,\overline{\zeta_c}\right)\zeta_k^{\,2}+a^2\,r_c^{\,2}\,\zeta_k^{\,3}\right)\left(a^2+\left(\zeta_c+\zeta_k\right)^2\right)}{\zeta_k^{\,2}}
\end{aligned}
\tag{D.112}
$$

$$\mathcal{Q}_7 = \mathsf{Re}\left(\frac{-2\,\pi\,r_c\,\left(-a^4+{r_c}^4-2\,{r_c}^2\,\delta^2+{\zeta_c}^2\,\overline{\zeta_c}^2\right)}{\left({r_c}^2-\delta^2\right)^2}+\frac{4\,\left(a^2\,\pi\,{r_c}^4-2\,a^2\,\pi\,{r_c}^2\,\delta^2+a^4\,\pi\,\overline{\zeta_c}^2+a^2\,\pi\,{\zeta_c}^2\,\overline{\zeta_c}^2\right)}{r_c\,\left({r_c}^2-\delta^2\right)\,\left(\overline{\zeta_c}+\overline{\zeta_k}\right)^2}\right.$$
$$\left.-\frac{4\,\left(-\left(a^2\,\pi\,{r_c}^4\,\zeta_c\right)+2\,a^4\,\pi\,{r_c}^2\,\overline{\zeta_c}+2\,a^2\,\pi\,{r_c}^2\,{\zeta_c}^2\,\overline{\zeta_c}-a^4\,\pi\,\zeta_c\,\overline{\zeta_c}^2-a^2\,\pi\,{\zeta_c}^3\,\overline{\zeta_c}^2\right)}{r_c\,\left({r_c}^2-\delta^2\right)^2\,\left(\overline{\zeta_c}+\overline{\zeta_k}\right)}\right) \quad \text{(D.113)}$$

$$\mathcal{Q}_8 = \mathsf{Re}\left(\frac{-2\,\pi\,{r_c}^2\,\left(a^2\,\left({r_c}^2-\delta^2\right)^2+a^4\,\zeta_c\,\zeta_k+\zeta_c\,\left({r_c}^2-\delta^2\right)^2\,\left(\zeta_c+\zeta_k\right)\right)}{\left({r_c}^2-\delta^2\right)^2\,{\zeta_k}^2\,\left(\zeta_c+\zeta_k\right)}\right) \quad \text{(D.114)}$$

$$\mathcal{Q}_9 = \mathsf{Re}\left(\frac{-2\,a^2\,\pi\,\left({r_c}^6+\zeta_c\,\left(a^2+{\zeta_c}^2\right)\,\overline{\zeta_c}^2\,\overline{\zeta_k}+{r_c}^4\,\zeta_c\,\left(-2\,\overline{\zeta_c}+\overline{\zeta_k}\right)+{r_c}^2\,\overline{\zeta_c}\,\left({\zeta_c}^2\,\left(\overline{\zeta_c}-2\,\overline{\zeta_k}\right)-a^2\,\left(\overline{\zeta_c}+2\,\overline{\zeta_k}\right)\right)\right)}{\left({r_c}^2-\delta^2\right)^2\,\overline{\zeta_k}\,\left(\overline{\zeta_c}+\overline{\zeta_k}\right)^2}\right) \quad \text{(D.115)}$$

$$\mathcal{Q}_{10} = 0 \quad \text{(D.116)}$$

$$\mathcal{Q}_{11} = \mathsf{Re}\left(i\,\pi\,\left(a^4\,\left(-{r_c}^4+\zeta_j\,\left(\zeta_c\,\overline{\zeta_j}+\left(\zeta_c+\overline{\zeta_j}\right)\,\zeta_k\,\right)\,\overline{\zeta_k}-{r_c}^2\,\zeta_c\,\left(\zeta_j+\overline{\zeta_k}\right)\right)\right.\right.$$
$$+\left(\zeta_c+\overline{\zeta_j}\right)^2\left(\zeta_c+\zeta_k\right)^2\,\left({r_c}^4-\zeta_j\,\left(\zeta_c\,\overline{\zeta_j}+\left(\zeta_c+\overline{\zeta_j}\right)\,\zeta_k\right)\,\overline{\zeta_k}+{r_c}^2\,\zeta_c\,\left(\zeta_j+\overline{\zeta_k}\right)\right)$$
$$+a^2\,\left(2\,\zeta_c+\overline{\zeta_j}+\zeta_k\right)\,\left(-\left({r_c}^4\,\left(2\,\zeta_c+\overline{\zeta_j}+\zeta_k\right)\right)-\zeta_j\,\left(\overline{\zeta_j}\,\zeta_k\,\left(\overline{\zeta_j}+\zeta_k\right)+\zeta_c\,\left(\overline{\zeta_j}^2+{\zeta_k}^2\right)\right)\,\overline{\zeta_k}\right.$$
$$\left.\left.\left.+{r_c}^2\,\left(2\,\overline{\zeta_j}\,\zeta_k+\zeta_c\,\left(\overline{\zeta_j}+\zeta_k\right)\,\right)\,\left(\zeta_j+\overline{\zeta_k}\right)\right)\right)$$
$$/\left(\left(a-\zeta_c-\overline{\zeta_j}\right)\,\left(a+\zeta_c+\overline{\zeta_j}\right)\,\left(a-\zeta_c-\zeta_k\right)\,\left(a+\zeta_c+\zeta_k\right)\,\left(-{r_c}^2+\zeta_j\,\zeta_k\right)\,\left(-{r_c}^2+\overline{\zeta_j}\,\overline{\zeta_k}\right)\right)\right) \quad \text{(D.117)}$$

$$\mathcal{Q}_{12} = \mathsf{Re}\left(\frac{i\,\pi\,\left(4\,a^2\,\left(\zeta_c+\zeta_k\right)^2\,\left({r_c}^2-\zeta_k\,\overline{\zeta_k}\right)+a^4\,\left({r_c}^2+\left(2\,\zeta_c+\zeta_k\right)\,\overline{\zeta_k}\right)-\left(\zeta_c+\zeta_k\right)^4\,\left({r_c}^2+\left(2\,\zeta_c+\zeta_k\right)\,\overline{\zeta_k}\right)\right)}{\left(-a+\zeta_c+\zeta_k\right)^2\left(a+\zeta_c+\zeta_k\right)^2\,\left(-{r_c}^2+\zeta_k\,\overline{\zeta_k}\right)}\right) \quad \text{(D.118)}$$

$$\mathcal{R}_1 = \mathsf{Re}\left\{ \left(\frac{-2\,a^2\,\pi\,{r_c}^2}{{\zeta_c}^2} + \frac{2\,a^4\,\pi\,{r_c}^2}{\left({r_c}^2-\delta^2\right)^2} \right) \log(\frac{{r_c}^2 + a\,\zeta_c - \delta^2}{{r_c}^2}) + \left(\frac{2\,a^2\,\pi\,{r_c}^2}{\overline{\zeta_c}^2} - \frac{2\,a^4\,\pi\,{r_c}^2}{\left({r_c}^2-\delta^2\right)^2} \right) \log(\frac{a}{\overline{\zeta_c}}) \right.$$
$$+\frac{2\,\pi\,\left(2\,a^3 + i\,\beta\,{r_c}^2\,\zeta_c - i\,\pi\,{r_c}^2\,\zeta_c + 2\,a\,{\zeta_c}^2\right)}{\zeta_c} + \frac{2\,i\,\left(-i\,a^2\,\pi\,{r_c}^2 - a^2\,\beta\,\pi\,{r_c}^2 + a^2\,\pi^2\,{r_c}^2\right)}{\overline{\zeta_c}^2} - \frac{2\,\left(a\,\pi\,{r_c}^2 + a^2\,\pi\,\zeta_c\right)}{\overline{\zeta_c}}$$
$$\left. -\frac{2\,\pi\,\left(a^2 + {\zeta_c}^2\right)\,\overline{\zeta_c}}{\zeta_c} - \frac{2\,\left(-\left(a^3\,\pi\,{r_c}^2\right) + a^4\,\pi\,\zeta_c\right)}{\zeta_c\,\left(-{r_c}^2 + \delta^2\right)} \right\} \quad \text{(D.119)}$$

$$\mathcal{R}_2 = \mathsf{Re}\Big(a^6\,\zeta_c\,\overline{\zeta_c}^2\,\zeta_k\,\left(\overline{\zeta_c} - \overline{\zeta_k}\right) + {\zeta_c}^2\,\overline{\zeta_c}\left({r_c}^2 - \delta^2\right)^2\,\left(\delta^2\,\zeta_k + {r_c}^2\,\left(2\,\zeta_c + \zeta_k\right)\,\right)\,\left(\overline{\zeta_c} + \overline{\zeta_k}\right)$$
$$-a\,{\zeta_c}^2\,\left(-{r_c}^2 + \delta^2\right)\,\left({r_c}^2\,\delta^2\,\left(4\,\overline{\zeta_c} - 3\,\zeta_k\right) + 2\,\zeta_c\,\overline{\zeta_c}^2\,\left(\zeta_c + \overline{\zeta_c}\right)\,\zeta_k + {r_c}^4\,\left(-2\,\overline{\zeta_c} + \zeta_k\right)\right)\,\left(\overline{\zeta_c} + \overline{\zeta_k}\right)$$
$$+2\,a^3\,\left({r_c}^2 - \delta^2\right)\,\left(\delta^2\,\zeta_k\,\left(\overline{\zeta_c}\left(\zeta_c + \overline{\zeta_c}\right)^2 + \left(-{\zeta_c}^2 + 2\,\delta^2 + \overline{\zeta_c}^2\right)\,\overline{\zeta_k}\right)\right.$$
$$\left.+{r_c}^2\,\zeta_c\,\left(2\,\overline{\zeta_c}^3 - \overline{\zeta_c}^2\,\left(\zeta_k - 2\,\overline{\zeta_k}\right) + \zeta_c\,\zeta_k\,\overline{\zeta_k} - \overline{\zeta_c}\,\zeta_k\,\left(2\,\zeta_c + \overline{\zeta_k}\right)\,\right) - {r_c}^4\,\left(-\left(\zeta_c\,\zeta_k\right) + \overline{\zeta_c}\,\left(\overline{\zeta_c} + \overline{\zeta_k}\right)\right)\right)$$
$$+a^5\,\overline{\zeta_c}^2\,\zeta_k\,\left({r_c}^2\,\left(2\,\zeta_c + \overline{\zeta_c} + \overline{\zeta_k}\right) - 2\,\zeta_c\,\left(\zeta_c\,\left(\overline{\zeta_c} - \overline{\zeta_k}\right) + \overline{\zeta_c}\,\left(\overline{\zeta_c} + \overline{\zeta_k}\right)\right)\right)$$
$$+a^2\,\left({r_c}^2 - \delta^2\right)\,\left(-\left({\zeta_c}^2\,\overline{\zeta_c}\,\zeta_k\,\left({\zeta_c}^2\,\left(\overline{\zeta_c} - \overline{\zeta_k}\right) + 4\,\delta^2\,\left(\overline{\zeta_c} + \overline{\zeta_k}\right) + 2\,\overline{\zeta_c}^2\,\left(\overline{\zeta_c} + \overline{\zeta_k}\right)\right)\right)\right.$$
$$+{r_c}^4\,\left(-2\,{\zeta_c}^2\,\zeta_k + \zeta_c\,\left(2\,\overline{\zeta_c} - \zeta_k\right)\,\left(\overline{\zeta_c} + \overline{\zeta_k}\right) + \overline{\zeta_c}\,\zeta_k\,\left(\overline{\zeta_c} + \overline{\zeta_k}\right)\right)$$
$$\left.+{r_c}^2\,{\zeta_c}^2\,\left(-4\,\overline{\zeta_c}^3 + 4\,\overline{\zeta_c}^2\,\left(\zeta_k - \overline{\zeta_k}\right) - \zeta_c\,\zeta_k\,\overline{\zeta_k} + \overline{\zeta_c}\,\zeta_k\,\left(3\,\zeta_c + 4\,\overline{\zeta_k}\right)\right)\right)$$
$$+a^4\,\left({r_c}^4\,\left(-2\,\overline{\zeta_c}^3 + \overline{\zeta_c}^2\,\left(\zeta_k - 2\,\overline{\zeta_k}\right) - \zeta_c\,\zeta_k\,\overline{\zeta_k} + \overline{\zeta_c}\,\zeta_k\,\left(\zeta_c + \overline{\zeta_k}\right)\,\right)\right.$$
$$+\zeta_c\,\overline{\zeta_c}^2\,\zeta_k\,\left(2\,{\zeta_c}^2\,\left(\overline{\zeta_c} - \overline{\zeta_k}\right) + 4\,\delta^2\,\left(\overline{\zeta_c} + \overline{\zeta_k}\right) + \overline{\zeta_c}^2\,\left(\overline{\zeta_c} + \overline{\zeta_k}\right)\right)$$
$$\left.-{r_c}^2\,\overline{\zeta_c}\,\left(-2\,\delta^2\,\left(\overline{\zeta_c} - 2\,\zeta_k\right)\,\left(\overline{\zeta_c} + \overline{\zeta_k}\right) + \overline{\zeta_c}^2\,\zeta_k\,\left(\overline{\zeta_c} + \overline{\zeta_k}\right) + {\zeta_c}^2\,\left(4\,\overline{\zeta_c}\,\zeta_k - 2\,\zeta_k\,\overline{\zeta_k}\right)\right)\right)$$
$$/\left(2\,\left(a - \zeta_c\right)\,\delta^2\,\left({r_c}^2 + \left(a - \zeta_c\right)\,\overline{\zeta_c}\right)\,\left(-{r_c}^2 + \delta^2\right)\,\zeta_k\,\left(\overline{\zeta_c} + \overline{\zeta_k}\right)\right)\Big) \quad \text{(D.120)}$$