Selection, Learning, and Nomination: Essays on Supermodular Games, Design, and Political Theory

Thesis by

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In Partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California

2008

(Defended May 15th, 2008)

For my parents, Denise and René Mathevet

Acknowledgements

Many people have helped me along the way. Christelle has been a source of happiness, and without her uncompromising support and faith in me, I would have been miserable. My parents, to whom this dissertation is dedicated, have shown continual and unconditional support despite the distance. They are a constant source of inspiration. I also wish to thank my advisors, Federico Echenique and Matthew Jackson, for their help and encouragement. Federico has spent numerous hours advising me, meeting with me nearly every week since my second year as a graduate student. He has endeavored to extract the most out of me, discarding unpromising thoughts and results, reading my drafts countless times and demanding (countless!) rewrites. Most importantly, he found the right words in those difficult times of a PhD graduate student. Matt has also been a wonderful advisor, available, demanding, and encouraging. I have benefited greatly from his comments, which are the kind that shapes the core of a research project. Matt has funded me for several years, which allowed me to concentrate on research. I also thank him for his invitation to spend a term at Stanford to further my research. I am indebted to Preston McAfee, whose personality, charisma, and sharpness have made meetings with him among the most fruitful and enjoyable. I have learned a great deal from John Ledyard, whose love for mechanism design is contagious. I also thank him for funding my last year at Caltech, which made a difference during the stressful times on the job market. Kim Border patiently met with me any time I had a technical question and excited me with his passion for theory. To this day, I have yet to find a question to which he has no answer. I am grateful to my fellow graduate students, in particular Laura and David for their help and support. I thank my family whose thoughts are with me; and my friends from California and France for their presence and generosity, and in particular, Martina, Rohini, Bertrand, Jérémy, Manuel, and Philippe.

Finally, I cannot forget where I come from and those who made my admission to Caltech possible, hence this dissertation. First, I wish to thank Philippe Solal, my undergraduate advisor, for raising me as a theoretical economist. He pushed me as few have and instigated the idea of studying abroad. Michel Bellet and many in the Economics department at the University of Saint-Etienne strongly supported my application process to American universities. I could not have been admitted to Caltech without them, and without Alan Kirman who endorsed my application. Of course, I thank the HSS division at Caltech and the members of the recruiting committee that year (such as Federico!) for giving a chance to an unheard of student, from an unheard of university, from an unheard of city. While unusual, I am ending this long acknowledgement section with a quote by Jean Guitton (1961), which I see as an acknowledgement:

> "Saint-Etienne has nothing to seduce: no rivers, no sites, no past, no monuments. She is only sumptuous for those who love labor and pain, and the familiarity of simple people who work happily, as if they were under the sun, whereas coal mines even cover the face of the land. But one enjoys oneself more in privation. And the miner who goes back up drinks the light."

Une mère en sa vallée, p. 47–48, translated by Laurent Mathevet.

Abstract

Games with strategic complementarities (GSC) possess nice properties in terms of learning and structure of the equilibria. Two major concerns in the theory of GSC and mechanism design are addressed. Firstly, complementarities often result in multiple equilibria, which requires a theory of equilibrium selection for GSC to have predictive power. Chapter 2 deals with *global games*, a selection paradigm for GSC. I provide a new proof of equilibrium uniqueness in a wide class of global games. I show that the joint best-response in these games is a contraction. The uniqueness result then follows as a corollary of the contraction principle. Furthermore, the contraction-mapping approach provides an intuition for why uniqueness arises: Complementarities generate multiple equilibria, but the global-games structure dampens complementarities until one equilibrium survives. Secondly, there is a concern in mechanism design about the assumption of equilibrium play. Chapter 3 examines the problem of designing mechanisms that induce supermodular games, thereby guiding agents to play desired equilibrium strategies via learning. In quasilinear environments, I prove that if a scf can be implemented by a mechanism that generates bounded substitutes — as opposed to strategic complementarities — then this mechanism can be converted into a supermodular mechanism that implements the scf. If the scf also satisfies some efficiency criterion, then it admits a supermodular mechanism that balances budget. Then I provide general sufficient conditions for a scf to be implementable with a supermodular mechanism whose equilibria are contained in the smallest interval among all supermodular mechanisms. I also give conditions for the equilibrium to be unique. Finally, a supermodular revelation principle is provided for general preferences. The final chapter is an independent chapter on political economics. It provides three different processes by which two political parties nominate candidates for a general election: Nominations by party leaders, by a vote of party members, and by a spending competition. It is shown that more extreme outcomes can emerge from spending competition and that non-median outcomes can result via any process. Under endogenous party membership, median outcomes ensue when nominations are decided by a vote but not with spending competition.

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Chapter 1

Introduction

The concept of complementarity, whereby two objects are complements if having more of one increases the marginal value of the other, is an old preoccupation of literary and mathematical economists. It motivated Hicks and Allen [49] and [50] to perform their classical reconsideration of ordinal demand theory in 1934 and Samuelson [95] asserted in 1974 that "the time [was] ripe for a fresh, modern look at the concept of complementarity." Since then, a large mathematical toolbox to dealing with this notion has been developed, in particular using lattice theory (Topkis [?]). The concept of complementarity has reached game theory with the class of games with strategic complementarities (GSC) (Bulow et al. [13]), a.k.a supermodular games. Similar in spirit, players' strategies in a game are complements if the marginal utility of each player increases as other players increase their strategies. Interestingly, many economic situations naturally present strategic complementarities. In team production models, it becomes more desirable for a worker in a firm to increase her effort when other workers put more effort into their job. In oligopoly theory, some models with differentiated products are such that, when a firm's competitors raise their prices, the marginal profitability of the firm's own price increase rises. In technology adoption models, when more users adopt a communication system, it increases the marginal return to others of doing the same. In geopolitical models, if a country increases its stock of armament, it may increase the value for additional arms of its neighbors.

Beyond their intuitive interpretation, these games have been shown to possess technically nice properties. First, the monotonicity inherent in GSC guarantees that a Nash equilibrium always exists. On top of this, the set of equilibria has a lattice structure; in particular, there exist a largest and a smallest equilibrium. Second, GSC have remarkable comparative statics and learning properties. If players are given the opportunity to play the game over and over again and if they learn adaptively, then they will end up playing profiles within the equilibrium set. Adaptive learning means that players regard past play as the best predictor of their opponents' future play and they best-respond to their forecast. Convergence extends to sophisticated learners, who react optimally to what their opponents may next best-respond (Milgrom and Roberts [77]). The different learning behaviors covered by adaptive and sophisticated learning are so wide that GSC have very robust learning properties. For example, players always learn to play an equilibrium of a GSC if it is unique.

Unfortunately, complementarities often result in multiple equilibria, which undermines the predictive power of GSC. In team production models, it is a Nash equilibrium for everyone to shirk, but so it is for everyone to work hard. Indeed, a worker's motivation to put effort into her job strongly depends on her co-workers' performance. Similarly, in technology adoption, it is an equilibrium for everyone to adopt technology A, but so it is for everyone to adopt technology B. In arms race models, war can be the outcome if the countries build up their arm levels, but so can peace if no one buys weapons.

In parallel to the literature on GSC, a field of economics, called mechanism design, has been flourishing since the 1970s. Mechanism design involves creating institutions (or games or mechanisms) so that a group of agents reach some desired objective in equilibrium. In a public goods context, a government may want to design a tax system so that the level of a public good is efficient. The mechanism here is the tax system and the objective is to reach an efficient public good level. In allocation problems, a seller may want to set up an auction so that the expected price at which she sells her good is maximized. In a team production model, a manager may want to craft the contracts of her employees so that the revenue generated by the team is maximized. Although this literature has been successful at identifying appropriate incentives, it is becoming increasingly aware of its assumption of equilibrium play. *Everything* indeed relies on the assumption that agents play their equilibrium strategies: Efficient public good levels, maximal revenue and expected price are attained in equilibrium. This important gap in the literature is emphasized in Jackson [56]: "Issues such as how well various mechanisms perform when players are not at equilibrium but learning or adjusting are quite important [...] and yet have not even been touched by implementation theory. [This topic] has not been looked at from the perspective of designing mechanisms to have nice learning or dynamic properties."

The question becomes: Do players play a Nash (or Bayesian Nash) equilibrium or not?

This question is a game-theoretic concern that has received considerable attention. Two justifications for equilibrium play are usually suggested. The first one is epistemic and provides conditions on the players' abilities, such as rationality and knowledge, for an equilibrium to arise directly. Equilibrium play here is a one-shot affair. The second one is evolutive and dynamic in nature. It tries to justify Nash equilibria as limit points of "simple" dynamic processes. As one can imagine, not all games are such that their equilibria are learnable under a wide range of dynamics; but GSC are.

Chapter 2 studies the theory of equilibrium selection engineered by Carlsson and van Damme ([19]) and Frankel, Morris and Pauzner ([36]). Their method, known as *global games*, offers a way to choose a single equilibrium in GSC, thereby restoring some predictive power to GSC. Although global games are widely used in economics,[1] the existing proofs of uniqueness are intricate and have left us with a limited understanding of the uniqueness result. The main contribution of the chapter is to identify a general class of global games where contraction principles apply, which provides a more instructive and intuitively appealing proof of uniqueness.

The contraction result formalizes the intuition of the global games community (see Vives [107]) that the noise structure lessens complementarities until a unique equilibrium survives. GSC have a coordination-game "flavor" that leads to multiple equilibria, and this relationship can be traced to how strong complementarities are. Since global games yield a unique equilibrium in GSC (in the limit), the complementarities must somehow be lessened. The chapter thereby provides the first result that formalizes how the global-games structure dampens complementarities in a general framework: The slope of the best-reply measures the strength of complementarities, hence the complementarities cannot be too strong because the best-reply is a contraction.

Chapter 3 develops the theory of supermodular Bayesian implementation. As previously mentioned, mechanism design has identified convincing incentives, yet it has neglected to take into account the likelihood of equilibrium play. GSC have strong learning properties and, for this reason, a designer may want to look at those supermodular mechanisms producing the right incentives. To be more precise, think of a mechanism as describing the rules of a game: It assigns feasible strategies to the agents and specifies how these strategies

[1]Global games are used, for example, in models of debt pricing (Morris and Shin [82]), currency crises (Morris and Shin [83]), bank runs (Goldstein and Pauzner [39]), and merger waves (Toxvaerd [102]).

map into enforceable outcomes. Since players have preferences over the different outcomes, a mechanism induces a game in the traditional sense. Supermodular mechanisms are those that induce supermodular games. By virtue of the GSC, boundedly rational agents find their way to equilibrium. This theory thus contributes to fill the gap in the literature raised by Jackson. For example, a principal may actually attain revenue maximization by offering the agents a contract that they will face repeatedly for a sufficiently long time. A government may reach an optimal public goods level by repeatedly applying a supermodular tax system.

The chapter studies the class of objective functions, called *social choice functions* (scf), that the designer can implement using a supermodular mechanism. The key result the chapter uncovers is that all scf that can be implemented by a mechanism with bounded strategic substitutes — as opposed to strategic complements — can be implemented by a supermodular mechanism. What drives this result is the ability to add complementarities into a mechanism, hence "undoing" substitutes effects, without affecting the incentives. But only compensable amount of substitutes can be compensated for, which warrants the condition of bounded substitutes. Fortunately, this condition is general and satisfied in many environments of interest. At this point, adding complementarities in a mechanism may seem to require external resources. The chapter demonstrates that it is not the case. The designer can create supermodular mechanisms whose complementarities are financed by the agents themselves. In the language of mechanism design, budget balancing is possible under supermodular Bayesian implementation.

Chapter 3 and 2 are woven together through the multiple equilibrium problem. The main issue facing Chapter 3 is the existence of potentially many equilibria. Supermodular games are those games a mechanism should induce, however, these games are often plagued with multiple equilibria. The major difficulty this causes is that only one equilibrium is known to deliver the desired outcome, while learning is predicted to end up in between the extremal equilibria. Therefore, it is not clear which profile the agents will learn to play. In the context of the above examples, the agents may well end up playing a profile that does not produce an efficient public goods level or does not maximize the manager's revenue. So there is a strong motive for building supermodular mechanisms with tight equilibrium sets. The chapter addresses this issue by developing two notions of supermodular implementation: Optimal and unique supermodular implementation. Optimality refers to the design of a supermodular mechanism whose equilibrium set is the smallest among the class of supermodular mechanisms. Uniqueness refers to the design of supermodular mechanisms with a unique equilibrium, thereby avoiding the problems entailed by multiplicity. The chapter then studies the class of scf that the designer can implement using supermodular mechanisms that meet these additional constraints.

Chapter 4 is motivated by an independent concern from political economy. Given a set of alternatives, social choice theory studies voting rules for how individual preferences over these alternatives are aggregated to form a collective preference. While considering the set of alternatives as given is reasonable in some situations, there are contexts where it is not appropriate. In particular, it is hard to justify in political contexts such as elections, where the alternatives are endogenously determined. For example, the candidates in the United States presidential election are the result of primaries, a nomination process by which a party chooses its candidate. While the modeling of elections is extensive, there are no systematic studies of how the specifics of the nomination process affect election outcomes. The chapter develops and analyzes three simple models of prominent nomination processes, all within the same basic election setting. Two political parties simultaneously nominate candidates for an election out of their respective memberships. If elected, a candidate chooses her most preferred policy and the vote over the two nominees is by majority rule. It is shown that the differences in nomination process can have a large impact on the election outcome. The first nomination process is a dictatorship. A party leader, who is a member of the party and a potential candidate, unilaterally chooses the party's nominee. The second is majority voting. Party members vote over who should be the party's nominee. The last nomination process is campaign spending. The right to be the nominee can be understood as being auctioned off within the party and the member who spends or is willing to spend the most money wins the nomination. As the nomination process varies, the main characterizations of the election outcomes are as follows: In the nomination by party leaders, the winner can come from either party, but lies between the overall median and the leader of the party that contains the median. The outcome can range anywhere between these points. Then it is established that nominations by party vote are equivalent to situations where nominations are made by party leaders, but where the party leaders are the medians of the parties. This provides an intuitive relationship between nominations by a party vote and nominations by party leaders. This then implies that the election outcome when nominations are by a vote by party members always lie between the overall median and the median voter of the party which contains the overall median voter. In contrast, the outcome under spending competition is not constrained to any particular interval. Depending on the intensity of voters' preferences, the outcome can be almost anywhere. Elections by spending competition differ more dramatically from the other nomination processes, have more complicated equilibrium existence issues, and depend on the preferences of various party members in complex and subtle ways. In particular, nominations by spending competition can lead to extremist nominees from either or both parties, and can lead to extreme policy outcomes. Finally, party membership is endogenized, which leads to a convergence to the median in the case of nomination by votes, while if nominations are by spending competition, extremist outcomes can still ensue. [5]

Chapter 2

Global Games

2.1 Introduction

As mentioned in Chapter 1,[1] complementarities often result in many equilibria, which requires a theory of equilibrium selection for GSC to have predictive power. Pioneering work by Carlsson and van Damme ([19], CvD hereafter) and Frankel, Morris and Pauzner ([36], FMP hereafter) has provided such a theory of equilibrium selection with the theory of *Global Games*. In global games, a unique profile survives iterative elimination of dominated strategies. Although global games are widely used in economics, the proofs of uniqueness by CvD and FMP are intricate and have left us with a limited understanding of the uniqueness result. The main contribution of this chapter is to identify a general class of global games where contraction principles apply, which provides a more instructive and intuitively appealing proof of uniqueness.

A function $f : \mathcal{C} \to \mathcal{C}$ is a *contraction mapping* if there is a constant $\alpha \in [0, 1)$ such that $d(f(c), f(c')) \leq \alpha d(c, c')$ for all $c, c' \in \mathcal{C}$. According to Banach's fixed point theorem, a contraction mapping has a unique fixed point if $(\mathcal{C}, d)$ is complete. This theorem can be used to show existence of a unique equilibrium in the normal form of some games: Let $(\mathcal{C}, d)$ be the set of strategy profiles and br be the joint best-reply mapping, then show that $(\mathcal{C}, d)$ is complete and br is a contraction.

This chapter takes a different approach. If an equilibrium is already known to exist, then uniqueness follows from the weaker condition on the best-reply mapping br that

[1] This chapter is based on a paper of mine entitled "A Contraction Principle for Finite Global Games." For their comments, I am grateful to Chris Chambers, Sylvain Chassang, Jon Eguia, Chryssi Giannitsarou, Andrea Mattozzi, Stephen Morris, and Flavio Toxvaerd, and the seminar participants of the workshop on global games (Stony Brook, 2007), and the University of Saint-Etienne.

$d(br(c), br(c')) < d(c, c')$ for all distinct $c, c' \in \mathcal{C}$. Such a map is called a *weak contraction.* Since there is no confusion, "contraction" will be used to mean "weak contraction" for ease of reading. The entire argument is as follows:

1. Global games are GSC, which implies the existence of pure-strategy equilibria, as in Milgrom-Roberts [76] and Vives [106].

2. Furthermore, the information structure is such that, as in Van Zandt and Vives [105] (VZV hereafter), (a) best-responses to monotone (in-type) strategies are monotone and (b) the extremal equilibria are in monotone strategies.

3. I prove that the best-reply mapping, restricted to monotone strategies, is a contraction. Therefore, there can be only one equilibrium in monotone strategies. Since the extremal equilibria are in monotone strategies, there can be no other equilibria.

Establishing that the best-reply is a contraction requires certain restrictions on the beliefs. Global games are a class of games with incomplete information in which players receive a noisy signal about a random payoff parameter. Conditional on their signal, players formulate beliefs about their opponents' signals. The third step requires these beliefs to satisfy a translation criterion discussed later. Without these restrictions on the beliefs, I use a limiting argument. Denote by ν the noise parameter in the signal; the precision of each player's signal increases as this parameter vanishes. Parameterize the global game by its noise level and let br_ν be the joint best-response in this game. Here I show that br_ν is a contraction on a set that becomes arbitrarily large as ν gets smaller. That is, the best-reply is a contraction in the limit, because the set on which it contracts approaches the whole set as ν goes to zero. The argument implies that the equilibrium must be unique in the limit, as the noise disappears.

My results require some assumptions in addition to FMP's, but these assumptions are either automatically satisfied or unnecessary in 2×2 games. The results apply to finite global games where the players' utility depends on their opponents' actions through an aggregate. Also, I assume that there always exists a signal making a player indifferent between any two actions. Last, the strategic complementarities move monotonically with the state of nature: Larger states lead to weaker (or always stronger) complementarities. These assumptions define the class of contractive global games. For 2×2 games, the contraction result is, with some qualifications, as general as the previous results of uniqueness in the literature.[2] Those games are of interest because it is the class of games analyzed in the seminal work by CvD. Even though these assumptions come into play in general finite games, the structure is still general enough to allow for applications to well-known models such as currency crises, Diamond's search, and bank runs.[3] In the currency crises model, the first two assumptions are trivially satisfied, and the last one is a natural property of the exchange rate. In the (finite) Diamond's search model, all assumptions are satisfied in the traditional setting with convex cost functions. The bank run model of Morris and Shin [84] also satisfies these assumptions.

Contractive global games possess desirable properties. As mentioned before, equilibrium uniqueness is transparent, because contraction implies dominance-solvability. It thus improves our understanding of CvD's and FMP's uniqueness result. Moreover, contraction is simpler to establish than the results in previous literature. This is argued in Section 2.2. The intuition for the contraction in the 2×2 case is straightforward and carries over to the general case. Finally, FMP's result relies on a two-step argument where uniqueness in the actual Bayesian game is shown from a simplified version of the same game. Uniqueness follows by continuity to this easier environment where uniqueness is more easily verified. This methodology is not fully informative about the underlying mechanisms of the general case. While I also consider both versions of the game, I prove uniqueness separately. One of the advantages of treating the general game directly is that, unlike FMP, I show there is not always need for a vanishing noise to get uniqueness.

I now discuss some of the related literature. While using a contraction argument has been suggested in previous literature (*inter alia* in Levin [66], in unpublished notes by Morris, and implicitly in FMP [36]) this chapter provides the first work on the subject in a more general setting. Mason and Valentinyi [69] used a contraction mapping approach to establish uniqueness of equilibrium in a class of incomplete information games. Their argument requires sufficiently large perturbations from the complete information case and that players' signals be sufficiently independent, or uninformative about others'. Their theory mainly imposes structure on beliefs. On the other hand, this chapter studies global games, and so the uniqueness typically arises from very small perturbations from the complete information

[2]The first two assumptions are trivially satisfied in 2×2 games and the last one turns out to be unnecessary.
[3]See Section 2.5.

game. In global games, players' signals become fully informative and correlated as the noise level goes to zero. My uniqueness result instead imposes structure on payoffs. Finally, in a recent paper, Oury [90] extended the uniqueness result of FMP to multidimensional global games. It is an important step forward in global games, but her argument is a generalization of FMP's proof technique which is different from my contraction approach.

2.2 A Motivating Example

Consider this adaptation of an example from CvD. Two players are deciding whether to invest. Each player receives a net profit that depends not only on her action and her opponent's action, but also on her type, denoted by $s \in \mathbb{R}$. The payoff matrix is the following:

	I	NI
I	s_i, s_j	s_i - 1, 0
NI	0, $s_j - 1$	0, 0

Notice that NI is strictly dominant for types strictly below 0, and I is strictly dominant for types strictly above 1. While there is a unique equilibrium for any $(s_i, s_j) \in \mathbb{R}\backslash[0,1]^2$, there are two strict Nash equilibria when $(s_i, s_j) \in (0,1)^2$. This is the multiple-equilibrium problem described earlier. Which one should be played?

It would be useful to have a criterion to select one of these equilibria for each $(s_i, s_j) \in (0,1)^2$. CvD and FMP provide a selection method by introducing incomplete information and I show that it results in a contractive best-response. Uniqueness in the incomplete information version of the game leads to a selection argument in the complete information game.

The incomplete information arises when types are thought of as functions of common and private random parameters. Player i's type is $s_i = \theta + \nu\epsilon_i$, where $\nu > 0$, θ is the common parameter (or state) and ϵ_i is the private parameter. Common parameter θ is drawn from a continuous distribution ϕ whose support is the real line. Private parameter ϵ_i is distributed according to c.d.f. $F_i[-\frac{1}{2}, \frac{1}{2}]$, independently from other players' private parameter and θ. From the information structure $(\phi, \{F_i\})$, each player i (through Bayes' rule) constructs a distribution function $\Omega_i(s_j|s_i, \nu)$, or simply $\Omega_i(s_j|s_i)$, representing her beliefs about j's type upon receiving s_i. The cdf Ω_i is *translation non-decreasing* if for all s_j and s_i, $\Omega_i(s_j+\Delta|\, s_i+\Delta) \leq \Omega_i(s_j|\, s_i)$ for all $\Delta \geq 0$. It means that i believes that j's type is more likely to be below s_j when her type is s_i than below $s_j+\Delta$ when her type is $s_i+\Delta$.

In the incomplete information game, it is enough to focus on monotone (in-type) strategies, because there exist extremal equilibria in monotone strategies and best-responses to monotone strategies are monotone (see Section 2.3.3). These properties follow from monotonicity of the beliefs and the complementarities (a) among actions and (b) between a player's type and her action. A strategy for j is fully defined by a cutoff $c_j \in [0,1]$:

$$a_j(s_j) \quad = \quad \begin{cases} NI & \text{if} \quad s_j < c_j \\ I & \text{if} \quad s_j \geq c_j. \end{cases}$$

The cutoff between actions NI and I that corresponds to i's best-response to c_j will be denoted by $br_i(c_j)$. I will prove that for any $i \in N$, br_i is a contraction. That is, $|br_i(c_j') - br_i(c_j)| < |c_j' - c_j|$ for all distinct c_j' and c_j in $[0,1]$. Cutoff strategy c_i is the best-response to cutoff strategy c_j, if and only if, i receives a higher expected payoff for playing I than NI for types above c_i and smaller for types below. This translates formally: $c_i \equiv br_i(c_j)$ iff $(c_i-1)\Omega_i(c_j|\, c_i) + c_i(1-\Omega_i(c_j|\, c_i)) = 0$. Note that there is a unique cutoff that can satisfy this equality, hence best-responses are *almost everywhere* functions of the types, not correspondences. Equivalently,

$$c_i \equiv br_i(c_j) \Leftrightarrow \Omega_i(c_j|\, c_i) = c_i. \tag{2.1}$$

From (2.1), it is straightforward to show contraction using translation non-decreasing beliefs. Take any $\Delta > 0$ and $c_j \in [0,1]$ and say $br_i(c_j)$ is some cutoff strategy c_i. By (2.1), we have $\Omega_i(c_j|\, c_i) = c_i$. Consider an increase from c_j to $c_j+\Delta$. By means of contradiction, suppose $br_i(c_j+\Delta) - br_i(c_j) = \Delta^* \geq (c_j+\Delta) - c_j = \Delta$. Since $br_i(c_j) \equiv c_i$, we have $br_i(c_j+\Delta) \equiv c_i+\Delta^*$; that is, cutoff strategy $c_i+\Delta^*$ is a best-response to cutoff strategy $c_j+\Delta$. This implies $c_i+\Delta^* = \Omega_i(c_j+\Delta|\, c_i+\Delta^*)$ by (2.1). Cdfs are increasing so $\Omega_i(c_j+\Delta|\, c_i+\Delta^*) \leq \Omega_i(c_j+\Delta^*|\, c_i+\Delta^*)$ and translation non-decreasing beliefs imply

$\Omega_i(c_j + \Delta^* | \, c_i + \Delta^*) \leq \Omega_i(c_j | \, c_i)$. Therefore $c_i + \Delta^* = \Omega_i(c_j + \Delta | \, c_i + \Delta^*) \leq \Omega_i(c_j | \, c_i) = c_i$, a contradiction because $\Delta^* > 0$. As a result, $br_i(c_j + \Delta) - br_i(c_j) < \Delta$. Moreover, br_i is monotone increasing and so $br_i(c_j) - br_i(c_j + \Delta) < \Delta$. Since c_j, Δ and Δ^* were arbitrary, we conclude $|br_i(c'_j) - br_i(c_j)| < |c'_j - c_j|$ for all distinct c_j and c'_j.

Notice that contraction of the joint best-response $br = \times_{i \in N} br_i$ follows immediately from the above. Take any $c = (c_i, c_j)$, $c' = (c'_i, c'_j)$ in $[0,1]^2$ and let $d(c, c') = \max_{k \in N} |c'_k - c_k|$. Then $d(br(c), br(c')) < d(c, c')$. To complete the proof of uniqueness, recall that global games are GSC and so an equilibrium exists.[4] Contraction implies uniqueness.[5]

Equilibrium selection in the complete-information version of the game is obtained via uniqueness in its incomplete-information version. For all $(s_i, s_j) \in (0,1)^2$, the Bayesian equilibrium strategies prescribe a unique action profile.

The translation property drives the result. How restrictive is it? Many distributions are translation non-decreasing such as those derived from a uniform prior. Examples of such distributions are beliefs Ω_i that are centered around s_i and normal, or double exponential, or from other location-scale families. Beyond flat priors, the beliefs can hardly be translation non-decreasing at every ν when types are linear functions. However, Lemma 2 of Section 2.7.2 shows that, for a general prior, the global-games information structure naturally tends to be non-decreasing in translation for small noise, which is why uniqueness is then reached in the limit only. In Sections 2.4.1.1 and 2.4.2, I interpret the role of this property in the contraction and I formalize the intuition that strategic complementarities are lessened.

2.3 Game and Assumptions

A (finite) global game is a collection $(\Gamma(\nu))_{\nu > 0}$, where each $\Gamma(\nu)$ is a tuple $(N, (A_i, \succeq_i)_N, \phi, (\tau_i(., \nu), f_i)_N, g, (\pi_i)_N)$ with the following meaning. The set and the number of players are denoted by $N < \infty$. Player i's action set is a finite and linearly ordered set $A_i = \{a_{i,1}, \ldots, a_{i,M_i}\}$, where the actions belong a vector space and are indexed in increasing order. A state $\theta \in \mathbb{R}$ is drawn from the real line according to a common prior. This prior is said to be uniform when each realization from the real line is equally likely (see Hartigan [47]

[4]Contraction alone does not ensure the existence of a fixed point without further information on its domain. However, this existence problem is made vacuous by the fact that global games are GSC, and an equilibrium always exists in GSC.

[5]Suppose the joint best-response br has at least two fixed points, c and c'. Then by the contraction property: $d(c, c') = d(br(c), br(c')) < d(c, c')$, a contradiction.

for a discussion of improper priors); otherwise it is assumed to have a continuous density ϕ with convex support on $\mathbb{R}$.

Each player i observes a signal $s_i = \tau_i(.)$ about θ. The signaling technology τ_i maps $\mathbb{R} \times \mathbb{R}_+ \times [-e_i, e_i]$ — a typical element of this set is $(\theta, \nu, \epsilon_i)$ — into $\mathbb{R}$ where $e_i > 0$ and τ_i is continuous and strictly increasing in all its arguments. Let $e^* = \max\{e_i : i \in N\}$. Parameter ν determines the degree of noise in the signal and is sufficiently small for $\underline{\theta} - \nu e_i$ and $2\overline{\theta} - \underline{\theta} + \nu e_i$ to lie in the support of ϕ for all $i \in N$. The actual noise in the signaling technology is ϵ_i. The signal reveals the state of nature if there is no noise or if $\nu = 0$, that is, $\tau_i(\theta, 0, \epsilon_i) = \tau_i(\theta, \nu, 0) = \theta$. For each (ν, ϵ_i), $\theta \mapsto \tau_i(\theta, \nu, \epsilon_i)$ is a homeomorphism between $\mathbb{R}$ and $\mathbb{R}$ and for each (θ, ν), $\epsilon_i \mapsto \tau_i(\theta, \nu, \epsilon_i)$ is a continuous map whose inverse function exists and is continuously differentiable. There is a case of particular interest, $\tau_i(\theta, \nu, \epsilon_i) = \theta + \nu\epsilon_i$ for all $i \in N$. I refer to this case as linear signaling technologies.

The amount of noise ϵ_i is distributed according to cdf F_i whose density f_i has support $[-e_i, e_i]$. These noises are assumed to be conditionally independent of one another and of the state of nature: Each ϵ_i is independent of θ and of ϵ_j for all $j \neq i$.

Players then choose simultaneously an action in their action space and payoffs accrue according to π.

2.3.1 The Payoff Functions

Players only care about an *increasing* and *non-constant* aggregate g of their opponents' actions.[6] This assumption is restrictive, but still allows for a wide range of applications.

Let $(\mathcal{G}_i, \succeq)$ be a totally ordered set and $g_i : \prod_{j \neq i} A_j \to \mathcal{G}_i$ be a continuously increasing surjection that carries action profiles into $\mathcal{G}_i$. When there is no confusion, the subscript i is dropped, so g_i and $\mathcal{G}_i$ become g and $\mathcal{G}$. For example, a player could care about the sum of her opponents' actions. In this case, $g(a_{-i}) = \sum_{j \neq i} a_j$ for all $a_{-i} \in A_{-i}$.[7] Or she could care about the proportion of her opponents playing less than some action c. Then, for all $a_{-i} \in A_{-i}$, $g(a_{-i}) = (\sum_{j \neq i} 1_{a_j \leq c})/(N-1)$ where $c \in \cap_{j \neq i} A_j$.

Player i's payoff is $\pi_i(a_i, g(a_{-i}), \theta, s_i)$ when she receives signal (is of type) s_i, the state of nature is θ and the action profile is a. Since θ is a common component and

[6] The monotonicity of g is crucial to convey the strategic complementarities and the monotonicity of distribution $g|s_i$ (in s_i w.r.t $\succeq_{st}$) to the incomplete information game. And g non-constant is the most interesting case, for otherwise best-replies are constant and uniqueness trivial.

[7] The two-player case with many actions can be modeled (trivially) with the aggregative sum.

s_i a private component, this global-games framework allows for *common* and *private values*. But I assume no mixture of the two: Either players' payoffs depend only on the state of nature, $\pi_i(a_i, g(a_{-i}), \theta, s_i) \equiv \pi_i(a_i, g(a_{-i}), \theta)$, or they depend only on their type, $\pi_i(a_i, g(a_{-i}), \theta, s_i) \equiv \pi_i(a_i, g(a_{-i}), s_i)$. To avoid redundancy, I will only give the assumptions in the common values setup.

Let A_{-i} be endowed with the product order.[8] Let $d\pi_i(a_i, a'_i, g(a_{-i}), \theta) = \pi_i(a_i, g(a_{-i}), \theta) - \pi_i(a'_i, g(a_{-i}), \theta)$, that is, $d\pi_i$ is the difference in player i's utility of playing a_i over a'_i when facing a_{-i} at state θ. The assumptions on the payoff functions are the following.

Assumption **1** [Dominance Regions]
For extreme values of the payoff parameter θ, the extreme actions are strictly dominant: There exist $\underline{\theta}$ and $\overline{\theta}$ in $\mathbb{R}$ with $\underline{\theta} < \overline{\theta}$ such that, for all i and for all a_{-i}, $d\pi_i(a_{i,M_i}, a_i, g(a_{-i}), \theta) > 0$ if $a_i \neq a_{i,M_i}$ and $\theta > \overline{\theta}$, and $d\pi_i(a_{i,1}, a_i, g(a_{-i}), \theta) > 0$ if $a_i \neq a_{i,1}$ and $\theta < \underline{\theta}$.

The support of the prior includes an open interval containing $[\underline{\theta}, 2\overline{\theta} - \underline{\theta}]$.

Assumption **2** [Strategic Complementarities]

The payoff functions have increasing differences in (a_i, a_{-i}): For all a''_i and a'_i in A_i with $a''_i \succeq_i a'_i$, a''_{-i} and a'_{-i} in A_{-i} such that $a''_{-i} \succeq a'_{-i}$, $d\pi_i(a''_i, a'_i, g(a''_{-i}), \theta) \geq d\pi_i(a''_i, a'_i, g(a'_{-i}), \theta)$ for all $\theta \in \mathbb{R}$.

Assumption **3** [State Monotonicity]

The payoff functions have strictly increasing differences in (a_i, θ): For all a''_i and a'_i in A_i with $a''_i \succ_i a'_i$, θ'' and θ' in $[\underline{\theta}, 2\overline{\theta} - \underline{\theta}]$ such that $\theta'' > \theta'$, $d\pi_i(a''_i, a'_i, g(a_{-i}), \theta'') > d\pi_i(a''_i, a'_i, g(a_{-i}), \theta')$ for all $a_{-i} \in A_{-i}$.

Assumption **4** [Monotone State Monotonicity]

The payoff functions exhibit decreasing (increasing) state monotonicity (DSM and ISM, respectively): For all a''_i and a'_i in A_i with $a''_i \succ_i a'_i$, g'' and g' in $\mathcal{G}$ such that $g'' \succ g'$, $d\pi_i(a''_i, a'_i, g'', \theta) - d\pi_i(a''_i, a'_i, g', \theta)$ is weakly decreasing (increasing) in θ on $[\underline{\theta}, 2\overline{\theta} - \underline{\theta}]$.

Assumption **5** [Existence of Cutoffs]

For all i, for any a''_i, a'_i in A_i and for all a_{-i} in A_{-i}, there exists $\tilde{\theta} \in [\underline{\theta}, \overline{\theta}]$ such that $d\pi_i(a''_i, a'_i, g(a_{-i}), \tilde{\theta}) = 0$.

[8] Let $(X_k, \geq_k)$ be a partially ordered set for each k in a set K. Letting $X = \prod_{k \in K} X_k$, the product order $\geq$ on X is the relation where $x'' \geq x'$ if $x''_k \geq_k x'_k$ for each $k \in K$.

Assumption **6** [Payoff Continuity]

Each $\pi_i(a_i, g(a_{-i}), \theta)$ is continuous in θ.

Next I discuss this set of assumptions. (A1)–(A3) and (A6) are common assumptions in the global-games literature. FMP make the same assumptions. CvD require (A1) and (A6). But CvD's result is limited to 2×2 games so (A2) is trivially satisfied when there are multiple equilibria, which is the case of interest in selection theory. Further, Morris and Shin [83] develop a model of currency attacks that satisfies (A1), (A2), (A6), and the weak version of (A3).

(A4) and (A5) are new assumptions. (A4) is a condition on the curvature of the differences. By assumption, the marginal payoff from playing a larger action is a strictly increasing function of the state. Under *DSM*, this function increases less as g increases, and under *ISM*, it increases more as g increases. Equivalently, *DSM* (*ISM*, respectively) asserts that the strategic complementarities decrease (increase) with the state/signal.

(A4) plays the following role. The complementarities imply that a player has an incentive to increase (the cutoff of) her strategy when her opponents do the same. As the player is increasing it, this incentive decreases under *DSM* so that she will tend not to overreact. Conversely, *ISM* gives an upper bound on players' incentive to increase. But suppose that when players overreact, they do not believe they are facing higher opposing strategies (which is the case on average under translation non-decreasing beliefs). Then the greater incentive to increase does not lead to an overreaction.

For three-times continuously differentiable functions, (A4) comes down to checking the sign of a third-derivative.

(A5) is a technical condition that will ensure the existence of all cutoff signals. It says that for any pair of actions and any opposing profile, there is a state/signal at which the player is indifferent between them. (A5) rules out, for one, actions that are dominated for all θ. Requiring cutoff $\tilde{\theta}$ in (A5) to belong to $[\underline{\theta}, \overline{\theta}]$ is not necessary because the existence of such $\tilde{\theta}$ in $\mathbb{R}$ would be sufficient for my purpose.[9]

[9]Suppose we only know that all $\tilde{\theta} \equiv \tilde{\theta}(a_i'', a_i', a_{-i})$ are in $\mathbb{R}$. Since there are only finitely many, they all are in a compact interval $[l, u]$. Then let $\underline{\theta}_2 = \min\{\underline{\theta}, l\}$ and $\overline{\theta}_2 = \max\{\overline{\theta}, u\}$. Therefore, extreme actions are dominant outside $[\underline{\theta}_2, \overline{\theta}_2]$ and the new dominance regions can be readjusted to $(-\infty, \underline{\theta}_2) \cup (\overline{\theta}_2, \infty)$. Now all the cutoffs lie in the non-dominance region as desired. However, by enlarging the non-dominance region, (A3) and (A4) become stronger.

Note (A5) is redundant in two-action games since it is implied by the dominance regions and (A6). Further, payoff functions which are concave in actions satisfy (A5).

I need this hypothesis to derive the existence and the properties of another kind of cutoffs (*real cutoffs*) whose definition relies on the above.

Finally, I wish to emphasize that the contraction result is, with some qualifications, as general as the previous results of uniqueness for 2×2 games, because (A4) and (A5) are not needed in this class of games. These assumptions become effective in general finite games, but the result captures interesting applications (see Section 2.5.). I have not been able to extend the contraction result to general finite games without these additional assumptions.

2.3.2 The Beliefs

There are two categories of beliefs each player formulates upon receiving her signal: Those about the state of nature θ, and those about the signal of her opponents $(s_j)_{j \neq i}$. I abuse notation and represent player i's beliefs about the signal of her opponents by a distribution function $\Omega_i(s_{-i}|\, s_i, \nu)$. Let $\succeq_{st}$ stand for the first-order stochastic dominance ordering.

I impose the following assumption on beliefs.

Assumption **7** [First-Order stochastic Dominance]

Let $\Psi_i(\theta|s_i, \nu)$ be the conditional distribution of θ given s_i. For all i and for each $\nu > 0$, if $s_i'' > s_i'$, then $\Psi_i(.|s_i'', \nu) \succeq_{st} \Psi_i(.|s_i', \nu)$.

This condition says that a player whose signal increases puts more weight on higher states. It is the only assumption I need on beliefs. While it is not a condition directly on the primitives of the model, it is satisfied by many information structures. For instance, it is satisfied if the prior is uniform and the signaling technologies linear.

Stochastic dominance is not typically assumed in the global games literature (CvD [19] and FMP [36]) as it holds in the limit as the noise becomes small and away from the limit under the normality assumption which is common in applications. But it is useful to obtain results away from the limit for general distributions. In particular, it is used in Section 2.3.3 to establish that the best-reply to an increasing strategy is itself increasing.

Unless otherwise specified, assumptions *(A1)–(A7) are in effect throughout this chapter.*

2.3.3 Increasing Strategies

In this section, I argue that it is enough to study Nash equilibria in increasing strategies because $\Gamma(\nu)$ is a GSC and its extremal equilibria are increasing in the signal.

For definitions of lattice, supermodularity and GSC, see, e.g., Milgrom-Roberts [76] or Topkis [?]. The game $\Gamma(\nu)$ is the Bayesian version of a finite GSC, hence it is itself a GSC. Strategic complementarities yield a greatest and a least equilibrium. VZV [105] proves that for Bayesian GSC these extremal equilibria are monotone in signals when signals are affiliated.[10] In addition, the best-reply to any increasing strategy (in the signal) must be an increasing strategy. These two properties define monotone Bayesian GSC. Here I consider a family of such games, as shown by Proposition 1.[11] As a result, if the contraction is established on the set of profiles in increasing strategies, then uniqueness will follow.

The mapping $a_i : \mathbb{R} \to A_i$ from signals to actions denotes a strategy for player i.

PROPOSITION **1** *Assume (A2), (A3), and (A7). Let* $br_{i,\nu}(a_{-i}(.)) : \mathbb{R} \to A_i$ *be i's best-response to* $a_{-i}(.)$ *in* $\Gamma(\nu)$*. If* $a_j(s_j)$ *is increasing in* s_j *for every* $j \neq i$*, then* $br_{i,\nu}(a_{-i}(.))$ *is increasing in* s_i*. Besides, for all* $i \in N$*, the greatest (least)-equilibrium strategy is increasing in* s_i *for all* $\nu > 0$*.*

Remark. Assuming (A2) and (A3), the claim of Proposition 1 holds if the prior is uniform, because (A7) is trivially satisfied. Notice also that (A3) implies that the best-replies are a.e. functions of the signal, not correspondences.

By Proposition 1, the set of increasing strategies is closed under the best-response operation. Any strategy in this set can be represented as a finite sequence of cutoff points. I call those cutoff points *real cutoffs* and a formal definition will be given in Section 2.7.1. From now on, let player i's strategy be $(c^r_{i,k_i})_{k_i=1}^{M_i-1}$ where $c^r_{i,k_i} \in [\underline{\theta}, \overline{\theta}]$ is the threshold below which i plays a_{i,k_i} and above which she plays a_{i,k_i+1}. For notational purposes, I drop the superscript f and denote these cutoff points by c_{i,k_i} or simply c_{k_i}.

The concept of real cutoffs is fundamentally different from the *fictitious cutoffs*. According to the next definition, the fictitious cutoff between two actions is the only signal at

[10]See [105], Theorem 1, p.11.
[11]This proposition applies to the common and private values games.

which a player is indifferent between them.

Definition 1 *For player i, the fictitious cutoff point between $a_{i,m}, a_{i,n} \in A_i$ where $n > m$ is denoted $c^f_{n,m}$ and it is defined as the signal that makes i indifferent between playing $a_{i,n}$ and $a_{i,m}$: $Ed\pi_i(a_{i,n}, a_{i,m}, g(a_{-i}(.)), c^f_{n,m}) = 0$.*

Since there are finitely many actions, (A3) and (A5) imply that, for any pair of actions, there exists a signal below (above) which the expected utility is strictly greater when playing the smaller (larger) action. By continuity of the expected utility, there exists a signal making the player indifferent. All fictitious cutoffs are well-defined. The expected payoffs are continuous in the signal, because the beliefs (Λ_i below) and the complete information payoffs are continuous in the signal (A6).

There are $\ell_i \equiv \binom{M_i}{2}$ fictitious cutoff points for player i, but many of these pairwise comparisons are invisible when observing the resulting (increasing) best-response. This explains the difference with the real cutoffs that only select those fictitious cutoffs that matter to represent the best-reply.

2.4 The Main Result: Contraction of the Best-Reply

In this section, I prove that under my assumptions, the joint best-reply function in global games is a contraction according to the following definition.

Definition 2 *Let (X, d) be a metric space. If $\xi : X \to X$ satisfies the condition $d(\xi(x), \xi(y)) < d(x, y)$ for all $x, y \in X$ with $x \neq y$, then ξ is called a (weak) contraction (or shrinking map).*

Given the two families of cutoffs, I define two metric spaces. Endow the space $[\underline{\theta}, \overline{\theta}]^{\ell_i}$ of i's fictitious cutoff vectors with metric $d(c'^f_i, c^f_i) = \max_{\{n,m \in \{1,\ldots,M_i\}: n>m\}} |c'^f_{n,m} - c^f_{n,m}|$. Now I construct the space $\mathcal{C}$ of profiles in monotone strategies, hence characterized by their real cutoffs. Notice that contraction of the best-response on $\mathcal{C}$ implies uniqueness. Let $\varphi : [\underline{\theta}, \overline{\theta}] \twoheadrightarrow [\underline{\theta}, \overline{\theta}]$ be defined by $\varphi(c_1) = [c_1, \overline{\theta}]$ and denote its graph by $\mathrm{Gr}\varphi$. Then, let $\varphi^k : \mathrm{Gr}\varphi^{k-1} \twoheadrightarrow [\underline{\theta}, \overline{\theta}]$ be defined by $\varphi^k(c_1, \ldots, c_k) = [c_k, \overline{\theta}]$. Let $(\mathcal{C}, d)$ be the metric space where $\mathcal{C} = \prod_{i \in N} \mathrm{Gr}\varphi^{M_i - 2}$ and $d(c', c) = \max_{i \in N} \max_{k_i} |c'_{k_i} - c_{k_i}|$ for all $(c, c') \in \mathcal{C}^2$. Denote by $\mathcal{C}_{-i}$ the set of profiles of i's opponents that are in increasing strategies, and for any $(c'_{-i}, c_{-i}) \in \mathcal{C}^2_{-i}$, define $d(c'_{-i}, c_{-i}) = \max_{j \in N \setminus \{i\}} \max_{k_j} |c'_{k_j} - c_{k_j}|$.

2.4.1 Private Values Case

As players focus on aggregates that summarize their opponents' play, they only formulate beliefs about g. I slightly abuse notation and refer sometimes to g as an element of $\mathcal{G}$. Let $\Lambda_i(g|c_{-i}, s_i, \nu)$ be i's beliefs that the aggregate is *strictly less* than g when she receives signal s_i given her opponents play according to $c_{-i} \in \mathcal{C}_{-i}$. It results from a theorem by Shaked and Shanthikumar [99] that Λ_i is increasing in s_i with respect to $\succeq_{st}$.[12] Indeed, since for any two players i and j, ϵ_j is independent of s_i, and τ_j is strictly increasing in θ, (A7) implies that, for any $s''_i > s'_i$, distributions $s_j|s''_i$ and $s_j|s'_i$ are such that $s_j|s''_i \succeq_{st} s_j|s'_i$. Thus $\Omega_i(.|s''_i, \nu)$ stochastically dominates $\Omega_i(.|s'_i, \nu)$. Since strategies $a_{-i}(.) \in \mathcal{C}_{-i}$ are increasing in signal and g is monotone, $g \circ a_{-i}(.)$ is increasing in s_{-i}, which implies $\Lambda_i(.|c_{-i}, s''_i, \nu) \succeq_{st} \Lambda_i(.|c_{-i}, s'_i, \nu)$.

2.4.1.1 Main Results

I show that the joint best-response function is a contraction under the assumption on beliefs in Definition 3. This assumption holds under CvD's and FMP's structures with a uniform prior, but it may be quite restrictive. I then establish how, in the absence of assumptions on beliefs, the contraction result is obtained in the limit.

Definition **3** *For player $i \in N$, beliefs $\Lambda_i(g|c_{-i}, s_i, \nu)$ are said to be translation non-decreasing if for all $g \in \mathcal{G}$, $c_{-i} \in \mathcal{C}_{-i}$ and $\Delta \in [0, \overline{\theta} - \underline{\theta}]$, then $\Lambda_i(g|\, c_{-i} + \mathbf{\Delta 1}, s_i + \Delta, \nu) \leq \Lambda_i(g|\, c_{-i}, s_i, \nu)$.*

Beliefs are translation non-decreasing if, whenever player i's signal increases as does every dimension of the opposing cutoff vector, then she believes larger aggregates are more likely. It plays an important role in global games as it prevents players from "overreacting" to complementarities. This property guarantees that a player who increases her strategy as do her opponents believes lower aggregates are more likely: $\Lambda_i(g|\, c_{-i}, c_i, \nu) \leq \Lambda_i(g|\, c_{-i} - \mathbf{\Delta 1}, c_i - \Delta, \nu)$. The complementarities between her action and the aggregate moderates her increase. On the contrary, if she believed larger aggregates were more likely, then the complementarities could make her increase optimal.

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[12] Shaked and Shanthikumar [99] Theorem 4.B.10, p.120: For any two n-dimensional random vectors X and Y, if Y stochastically dominates X and $h : \mathbb{R}^n \to \mathbb{R}$ is any increasing function, then $h(Y) \succeq_{st} h(X)$.

I present two contraction results. Proposition 2 says that every single fictitious cutoff is a contraction of the opposing profile. But many fictitious cutoffs are vacuous in the description of a strategy. So, following from a selection argument, Theorem 1 shows that the joint best-reply is a contraction. I defer the discussion of this issue to Section 2.7.1.

Recall that $\mathcal{C}_{-i}$ is the set of opposing profiles to player i that are in increasing strategies, and so every profile in this set is represented by a vector of real cutoffs. On the other hand, the superscript r marks the fictitious cutoffs.

PROPOSITION **2** *Let $c_i^f(\nu)$ and $c_i'^f(\nu)$ be, respectively, the vectors of fictitious cutoffs to c_{-i} and c'_{-i} in $\mathcal{C}_{-i}$. If for any $i \in N$, beliefs $\Lambda_i(g|c_{-i}, s_i, \nu)$ are translation non-decreasing, then for all $c'_{-i}, c_{-i} \in \mathcal{C}_{-i}$ and all $i \in N$, $d(c_i'^f(\nu), c_i^f(\nu)) < d(c'_{-i}, c_{-i})$.*

THEOREM **1** *Let $br_\nu : \mathcal{C} \to \mathcal{C}$ be the joint best-response function. If for any $i \in N$, beliefs $\Lambda_i(g|\, c_{-i}, s_i, \nu)$ are translation non-decreasing, then br_ν is a contraction.*

Theorem 1 implies equilibrium uniqueness in global games with uniform prior and (A1)–(A7), and in FMP's 2×2 environments. This is Proposition 3, which I describe next. First, $\Gamma(\nu)$ is a GSC, so an equilibrium exists. By Corollary 2 of Section 2.7.2, Theorem 1 then implies that there can be only one equilibrium under a uniform prior and linear signaling technologies.

Second, in 2×2 games, (A4) is dispensable and (A5) is trivially satisfied. The proposition states that the joint best-response in 2×2 global games is contractive, and so there is a unique equilibrium in these games (possibly in the limit) because the joint best-reply is a contraction. This result is established under the traditional assumptions, which is important given that the seminal work in global games by CvD dealt with 2×2 games.[13] Note that it is a private values environment, but it is sufficient for uniqueness in the limit (see FMP [36]).

DEFINITION **4** *A Bayesian-Nash equilibrium $(a_i^*(.))$ is essentially unique if $a_i^*(.)$ is single-valued for a.e. $s_i \in \mathbb{R}$, for all $i \in N$.*

PROPOSITION **3**

[13] Proposition 3 also uses translation non-decreasing beliefs, but the beliefs may not satisfy this condition. Nevertheless, there is an analogous result of this proposition without the translation hypothesis, along the lines of Theorem 2 and Proposition 4: Under the traditional assumptions, the best-reply in 2×2 global games is a contraction in the limit.

1. *If the prior is uniform and the signaling technologies are linear,* $\Gamma(\nu)$ *has an essentially unique Nash equilibrium for every* $\nu > 0$.

2. *Let* $\Gamma(\nu)$ *be a* 2×2 *global game. Under (A1)–(A3) and (A6), if for any* $i \in \{1, 2\}$, *beliefs* $\Omega_i(c_j|s_i, \nu)$ *are translation non-decreasing, then* br_ν *is a contraction and so an essentially unique equilibrium exists.*

The translation condition on beliefs requires that shifting the full set of opponents' cutoffs to the right by Δ does not change the beliefs on average, if the signal is raised by Δ. If we are agnostic about where interim beliefs come from and use them as primitives, then this condition is intuitive and satisfied by many distributions. All those distributions $\Omega_i(s_{-i}|\, s_i, \nu)$ that are centered around $s_i\mathbf{1}$ and normal or from other location-scale families, generate translation non-decreasing beliefs.[14] The approach of uniqueness from interim beliefs can be fruitful, as shown by Morris and Shin [85] and [86]. There are also nice examples of interim beliefs that satisfy asymptotic non-decreasingness but do not come from a signaling model (See Izmalkov and Yildiz [54]).

Extracting translation non-decreasing beliefs from general signaling technologies and priors is more difficult. Beliefs can hardly be translation non-decreasing with linear signals beyond flat priors and there are natural information structures that do not generate translation non-decreasing beliefs. Consider the example of Section 2.2 and take $\theta \sim N(\mu, \sigma^2)$ and $\varepsilon_i \sim N(0, \nu^2)$. Then, $s_j|s_i$ is normally distributed with mean $(\nu^2\mu + \sigma^2 s_i)/(\nu^2 + \sigma^2)$ and variance $(2\nu^2\sigma^2 + \nu^4)/(\nu^2 + \sigma^2)$. Those beliefs are not translation non-decreasing.[15] Further, not only does the best-response fail to be a contraction, but there exist multiple symmetric equilibria for values of μ, ν, and σ. Therefore translation non-decreasing beliefs are not only fundamental for contraction, but they also play a role in uniqueness in global games.

The literature on global games has concentrated on linear signaling technologies and vanishing noise because, as I show in Lemma 2 of Section 2.7.2, it makes beliefs approximately translation non-decreasing. Existing results in global games can thus be thought of as proving that the belief structure becomes asymptotically translation non-decreasing. That is, reducing the study to linear signals and vanishing noise is essentially a way of

[14] Bounded support is needed to fit the framework but not for translation non-decreasing distributions.

[15] Notice distribution Ψ_i is normal with the same mean as $s_j|s_i$ and thus it is not centered around s_i.

circumventing this difficulty, and I use it in the remainder of this section. Here I obtain the limiting result that the best-reply tends to shrink on the whole set of increasing strategies as the noise disappears. This is Theorem 2.

DEFINITION **5** *Let* $\mathcal{C}_\nu = \{(c, c') \in \mathcal{C}^2\colon\ c \neq c'$ *and* $d(br_{\nu'}(c), br_{\nu'}(c')) < d(c, c')$ *for all positive* $\nu' < \nu\}$ *be the contraction domain of the best-response at* ν.

Notice that if $\nu' < \nu$ then $\mathcal{C}_\nu \subset \mathcal{C}_{\nu'}$.

THEOREM **2** *The contraction domain approaches the whole space, that is,* $\cup_{\nu>0}\, \mathcal{C}_\nu = \mathcal{C}^2$.

Equilibrium uniqueness in the limit is a corollary of this result. The greatest and the smallest equilibrium must converge towards one another as ν vanishes.

PROPOSITION **4** *Let* $\overline{e}_\nu$ *and* $\underline{e}_\nu$ *be respectively the largest and least equilibrium in* $\Gamma(\nu)$. *Then,* $d(\overline{e}_\nu, \underline{e}_\nu) \to 0$ *as* $\nu \to 0$ *and so there is an essentially unique equilibrium in the limit.*

2.4.2 Common Values Case

I show that, under translation non-decreasing beliefs, the joint best-response function is a contraction for all ν below some threshold. Hence there is not always need for a vanishing noise to get uniqueness, even under common values. Then, I relax the assumption on beliefs and prove that the contraction domain approaches the whole space. As a corollary, the equilibrium is unique in the limit. The intuition behind uniqueness is that the global-games structure lessens complementarities as ν vanishes, so I provide additional formal insight into this argument.

When beliefs are not assumed to be translation non-decreasing, the signaling technologies are linear. Moreover, the analysis is done under Assumption 7'.

ASSUMPTION 7' [STRICT FIRST-ORDER STOCHASTIC DOMINANCE]

For all i and $\nu > 0$, if $s_i'' > s_i'$, then $\Psi_i(.|s_i'', \nu) \succeq_{st} \Psi_i(.|s_i', \nu)$ and $\Psi_i(\theta|s_i'', \nu) < \Psi_i(\theta|s_i', \nu)$ for all θ in a set of positive measure.

This assumption is not more restrictive than (A7) if for each (s_i, ν), $\Psi_i(.|s_i, \nu)$ has full support. Then (A7) implies (A7') because, for (s_i, ν), $\Psi_i : [s_i - \nu e_i, s_i + \nu e_i] \to [0, 1]$ has a strictly increasing support in s_i. For example, Ψ_i always has full support when signals are linear and the prior is uniform, since f_i has full support by assumption.

Player i's beliefs about (s_{-i}, θ) are given by the joint conditional distribution function $\Omega_i((a_j < s_j < b_j)_{j\neq i}, \theta|\, s_i, \nu)$ for $a < b$ in $\overline{\mathbb{R}}^{n-1}$, which is defined as

$$\int_{-\infty}^{\theta} \prod_{j\in J} \left(F_j\left(\frac{b_j - t}{\nu}\right) - F_j\left(\frac{a_j - t}{\nu}\right)\right) \Psi_i'(t|s_i, \nu)dt. \tag{2.2}$$

The joint distribution function $\Lambda_i(g, \theta|\, c_{-i}, s_i, \nu)$[16] represents i's beliefs about (g, θ) given s_i. This is the probability that the aggregate is strictly less than g and the state less than θ, and it is a finite sum of probabilities $\Omega_i((a_j < s_j < b_j)_{j\neq i}, \theta|\, s_i, \nu)$.

For an arbitrary $\eta > 0$, I strengthen assumptions (A3) and (A4) by replacing the existing $[\underline{\theta}, 2\overline{\theta} - \underline{\theta}]$ with $[\underline{\theta} - 2\eta, 2\overline{\theta} - \underline{\theta} + 6\eta]$, and I assume $\nu < \eta/e^*$. This joint condition ensures that all the fictitious cutoffs are well-defined and is without loss of generality for limiting results.

Now I define the analog of translation non-decreasing beliefs for the common values.

Definition **6** *For player $i \in N$, beliefs $\Lambda_i(g, \theta|\, c_{-i}, s_i, \nu)$ are said to be translation non-decreasing if for all $c_{-i} \in \mathcal{C}_{-i}$ and $\Delta \in [0, \overline{\theta} - \underline{\theta}]$, then $\Lambda_i(g, \theta|\, c_{-i} + \Delta, s_i + \Delta, \nu) \leq \Lambda_i(g, \theta|\, c_{-i}, s_i, \nu)$ for all $g \in \mathcal{G}$, $\theta \in \mathbb{R}$.*

Definition 6 says that whenever player i's signal increases more than does every dimension of the opposing cutoff vector, then her beliefs about the aggregate and the state increase with respect to $\succeq_{st}$. It implies that, when player i increases her strategy more than her opponent, she thinks that greater aggregative values and higher states are less likely. Proposition 5 and Lemma 3[17] show that the information structure of global games implies this form of translation non-decreasing beliefs when the prior is uniform or in the limit (as ν goes to zero) with a general prior.

Proposition **5** *In the common values case with uniform prior, beliefs $\Lambda_i(g, \theta|c_{-i}, s_i, \nu)$ are translation non-decreasing.*[18]

The next theorem applies when beliefs are non-decreasing in translation. In this context, there is ν small enough so that the joint best-response function is a contraction. Unlike

[16] Note $\Lambda_i(g, \theta|\, c_{-i}, s_i, \nu)$ is a finite sum of probabilities of the form $\Omega_i((a_j < s_j < b_j)_{j\neq i}, \theta|\, s_i, \nu)$ which are continuous in s_i (this can be established in a similar way as under private values.). Then Λ_i is also continuous in s_i. The continuity of Λ_i in (θ, s_i) is actually uniform for $s_i \in [\underline{\theta} - \eta, \overline{\theta} + \eta]$ for all i, as θ is in $[s_i - \nu e^*, s_i + \nu e^*]$.

[17] see Section 2.7.2

[18] Given Λ_i is a sum of probabilities of the form (2.2), the claim is fairly clear. The proof is in Mathevet [70].

Theorem 4, a unique equilibrium is actually reached for each $\nu < \eta/e^*$; hence there is no need for a vanishing noise to get uniqueness.

THEOREM **3** *Let $br_\nu : \mathcal{C} \to \mathcal{C}$ be the joint best-response function. If for any $i \in N$, beliefs $\Lambda_i(g, \theta | c_{-i}, s_i, \nu)$ are translation non-decreasing, then for every $\nu < \eta/e^*$, br_ν is a contraction and an essentially unique equilibrium exists.*

The fundamental result in common values is the next theorem which leads to a unique equilibrium in the limit. From Theorem 4, the proof of Corollary 1 is similar to that of Proposition 4, hence it is omitted.

THEOREM **4** *The contraction domain approaches the whole space, that is, $\cup_{\nu>0} \mathcal{C}_\nu = \mathcal{C}^2$.*

COROLLARY **1** *(FMP [36]) Let $\overline{e}_\nu$ and $\underline{e}_\nu$ be respectively the largest and least equilibrium in $\Gamma(\nu)$. Then, $d(\overline{e}_\nu, \underline{e}_\nu) \to 0$ as $\nu \to 0$ and so there is an essentially unique equilibrium in the limit.*

If the global game is one where the payoff functions and the density functions ϕ and ψ_i' are bounded and continuously differentiable in θ (and in s_i when applicable), then I can prove the following related result that sheds some light on uniqueness.[19] There exists a game $\hat{\Gamma}(\nu) = (N, (\hat{u}_i(., \nu) : [\underline{\theta}, \overline{\theta}]^{\ell_i} \times \mathcal{C}_{-i} \mapsto \mathbb{R})_N)$ which is best-response equivalent to global game $\Gamma(\nu)$ and where the complementarities in $\hat{\Gamma}(\nu)$, $\partial^2 \hat{u}_i(., \nu)/\partial c_i^f \partial c_{k_j}$, become strictly smaller than concavity, $-\partial^2 \hat{u}_i(., \nu)/\partial (c_i^f)^2$, as ν vanishes.[20] In differentiable games, the ratio between complementarities and concavity give the slope of the best-response. So, this result formalizes the idea that the global-games structure dampens complementarities to the point where a unique equilibrium exists, that is, to the point where they become weak relatively to concavity.

Remark. FMP proved that as the signal noise shrinks, the sequence of largest equilibria and the sequence of smallest equilibria in common values converge to the unique outcome of $\Gamma(\nu)$ under private values and uniform prior.[21] In this last case, the sequence of unique

[19] The proof is available upon request.

[20] The existence of two types of cutoffs makes the definition of $\hat{\Gamma}(\nu)$ somewhat unusual. But what underlies the contraction in the limit is that, for any player, the complementarities between her fictitious cutoffs and her opponents' real cutoffs become relatively weak.

[21] This is Lemma 5 in FMP [36].

equilibrium converges to a limit, hence all three sequences have the same limit which is the well-known uniqueness prediction of global games.

2.5 Examples

2.5.1 Currency Crises

This is a version of the model by Morris and Shin [83]. Here, a finite number n of speculators decide whether to attack a fixed-exchange regime by selling short one unit of the currency. The current value of the currency is r^*. The economy is characterized by a state of fundamentals θ which is distributed according to ϕ on a convex subset of $\mathbb{R}$. The currency will float to the shadow rate ζ if there is no intervention from the monetary authority. The cost of attacking the currency $c(\theta)$ is strictly increasing in θ, reflecting the fact that stronger fundamentals make an attack costlier. Writing 1 for the action "not attack" and 0 for the action "attack," let $\Delta = (\sum_{j\neq i} 1_{a_j<1})/(n-1)$ for each $i \in N$. The monetary authority defends the currency if this intervention is not too costly. Therefore, the cost of defending the currency is assumed to be increasing in the proportion of speculators who attack, Δ, and decreasing in the state of the fundamentals. There is a minimal proportion of speculators who must attack, $a(\theta)$, for a devaluation to occur.[22] The payoffs are given by:

$$\begin{aligned} u(1,\Delta,\theta) &= 0 \\ u(0,\Delta,\theta) &= \begin{cases} -c(\theta), & \text{if} \quad \Delta < a(\theta), \\ r^* - \zeta(\theta,\Delta) - c(\theta), & \text{if} \quad \Delta \geq a(\theta), \end{cases} \end{aligned}$$

where $a(\theta)$ and $\zeta \equiv \zeta(\theta,\Delta)$ are increasing in θ. Let $\zeta(\theta,\Delta) = r^*$ if $\Delta = a(\theta)$, and then suppose that u is continuous in θ for any Δ.[23] Let $M > 1$ and $r^* - c(\theta) > \zeta(\theta,\Delta)$ for $\Delta > a(\theta) + \frac{1}{M}$.[24] Finally, the shadow exchange rate is increasing with Δ for each θ,[25] and for any $\Delta'' > \Delta'$, $\zeta(\theta,\Delta') - \zeta(\theta,\Delta'')$ is decreasing in θ which accounts for the larger resistance of the exchange rate to changes in Δ when fundamentals are stronger.[26]

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[22] Here, I assume implicitly that both $a(\theta)$ and n make it unnecessary to know whether or not i herself attacks. In other words, given Δ, a player cannot alone spark or prevent the devaluation. For example, assume $a(\mathbb{R}) \subset (-\infty,0) \cup (1/n,(n-1)/n) \cup (1,\infty)$.

[23] Note that this condition allows for some discontinuities of a when ζ and c are continuous.

[24] Suppose M is big enough so the model is interesting for a large range of θ.

[25] This means that $\zeta(\theta,\Delta') - \zeta(\theta,\Delta'') \geq 0$, whenever $\Delta' \succ_{st} \Delta''$ or equivalently $\Delta'' > \Delta'$.

[26] Morris and Shin [83] originally assumed $\zeta \equiv \zeta(\theta)$ is increasing in θ which satisfies all my requirements for ζ except $\zeta(\theta,\Delta) = r^*$ if $\Delta = a(\theta)$; but their setting does not allow continuity of u in θ for all Δ.

This is a model where a devaluation does not always benefit the speculators. There exists an interval after $a(\theta)$ where the devaluation is not substantial enough to be profitable. Even if attacking results in a devaluation, it is only optimal after Δ has passed some threshold $a(\theta) + \frac{1}{M}$.

This game satisfies the standard assumptions of global games, (A1) through (A3) and (A6). Also notice the traditional tripartite division of fundamentals. If $\theta < a^{-1}(-\frac{1}{M})$, each player has a dominant strategy to attack and if $\theta > a^{-1}(1)$, each of them has a dominant strategy not to attack. In addition, there is an interval in $(a^{-1}(-\frac{1}{M}), a^{-1}(1))$ where the two symmetric profiles are both equilibria. There are only two actions so that (A5) is automatically satisfied, and DSM also holds.

2.5.2 Diamond's Search Model

This is a version of the Diamond-type search model of Milgrom and Roberts [76]. There are a finite number of players $\{1, \ldots, n\}$ who exert effort $a_i \in A_i \equiv \{0, \ldots, \overline{a}\}$ searching for trading partners. The probability of any trader to find a partner is proportional to her own effort and the sum of the efforts of the others denoted Σ_{-i}. The economy is characterized by a state of the fundamentals θ which is distributed according to ϕ on a convex subset of $\mathbb{R}$, and each player receives a private signal about θ with the properties described earlier. The positive cost to individual i for exerting a level of search a_i is $c_i(a_i, \theta)$ and so her payoff is defined by:

$$\pi_i(a_i, \Sigma_{-i}, \theta) = \alpha a_i \Sigma_{-i} - c_i(a_i, \theta) \tag{2.3}$$

where $\alpha > 0$. Since $\partial^2 \pi_i / \partial a_i \partial \Sigma = \alpha$ for $i \neq j$, this is a supermodular game in (a_i, Σ) and so (A2) is satisfied. For simplicity, cost function c_i is assumed to be continuous in θ which verifies (A6), and it is C^2 on $Co(A_i)$ for each θ. Moreover, c_i has strictly decreasing differences in (a_i, θ), and in particular, there exist $\underline{\theta}$ and $\overline{\theta}$ with $\underline{\theta} < \overline{\theta}$ such that for all $a_i \in A_i$, $c_i'(a_i, \theta) > \alpha(n-1)(\overline{a} + \varepsilon)$ for some $\varepsilon > 0$ and all $\theta < \underline{\theta}$, and $c_i'(a_i, \theta) < 0$ for all $\theta > \overline{\theta}$. For example, costs may be decreasing when learning is taken into account. The state of the world θ could summarize knowledge in the economy and as θ increases the marginal cost strictly decreases, which accounts for (A3). In particular, there exists a state $\overline{\theta}$ from which learning (by doing) is so high that each unit produced decreases the cost. As a result, (A1) holds and since c_i is traditionally convex in a_i, (A5) is satisfied. Finally, (A4) is also satisfied.

The theory of Section 2.4 then applies. Consider the symmetric version of the game. Notice that there exists $\tilde{\theta} > \underline{\theta}$ such that $c(.,\theta)$ is strictly increasing in a_i for all $\theta \in (\underline{\theta}, \tilde{\theta})$ because A_i is finite, $c'(a_i,\theta) > \alpha(n-1)(\overline{a}+\varepsilon)$ and c' is continuous. Therefore, $a_i = 0$ for all i is an equilibrium. Since c is smooth and convex, if A_i is rich enough, then there is also a symmetric equilibrium where effort levels across players are the same. This equilibrium is derived from the optimality condition in the case where $A_i = [0, \overline{a}]$: $\alpha(n-1)a^* = c'(a^*, \theta)$. Consequently, there is an interval in $(\underline{\theta}, \tilde{\theta})$ for which there are multiple equilibria.

2.6 Beliefs and Payoff Uncertainty

In this section I compare global games under private values with two other models of payoff uncertainty, namely Harsanyi's games with randomly disturbed payoffs and global games with common values.

2.6.1 Private Values vs. Harsanyi's Disturbed Payoffs

There is a relationship between Harsanyi's games with randomly disturbed payoffs and global games, yet their results differ largely in their conclusions (CvD [19]). Translation-increasing beliefs provide an explanation for this divergence.

Harsanyi [46] shows that any equilibrium of a strategic-form game is the limit of a sequence of pure strategy equilibria of slightly perturbed games. He perturbs the payoffs π_i^* of a strategic-form game by adding a random variable s_i^a denoting the player's type at action profile a: $\pi_i(a_i, a_{-i}, s_i^a) = \pi_i^*(a_i, a_{-i}) + s_i^a$. For all $a \in A$, $s_i^a \equiv \nu\epsilon_i^a$ where ϵ_i^a is a random variable with compact range such that different players' types are statistically independent and ν is a scale parameter. Thence Harsanyi's setup with $s_i^a \equiv \nu\epsilon_i$ or 0 for any $a \in A$ is a special case of global games with private values and degenerate distribution for the underlying state of nature. Taking linear signals and a degenerate prior at 0 indeed gives $s_i = \nu\epsilon_i$, so the resulting global game with private values and payoffs π_i amounts to Harsanyi's setup. Despite this relationship, the two models provide dramatically different conclusions. Harsanyi considers the sequence of disturbed games as $\nu \to 0$ and shows that all equilibria can be approximated in the limit by the beliefs associated with equilibria of the disturbed games. So, multiple equilibria can subsist in the limit which contrasts with the global games of Section 2.3. An explanation can be found in players' beliefs. Degenerate priors never generate translation-increasing beliefs, a central property for contraction and uniqueness in global games. Rather, beliefs are translation decreasing and do not tend to be translation increasing as $\nu \to 0$. In Harsanyi's setup, the point mass removes any prior uncertainty and, together with type independence, implies that players' signals are uninformative about their opponents' signals. Therefore, the rationale behind translation increasingness, that signals get so precise as the noise vanishes that each player knows her opponents' signals must vary in the same amount as hers, no longer holds.[27] So, there are belief incentives for non-contractive best-responses which are at the source of multiplicity of equilibria.

2.6.2 Private Values vs. Common Values

The private and common values versions of a global game converge towards one another as the noise vanishes and their unique equilibrium in the limit is the same. As ν goes to zero, players indeed discard any state θ that is too far from their signal. Thus the private values case, where the signals enter payoffs directly, becomes close to common values. In terms of beliefs, translation increasingness is equally restrictive in both setups under stochastic dominance. Linear signaling technologies and vanishing noise also remain a way of obtaining increasing beliefs in common values environments. Under common values, (A7') implies that a player expects higher states for higher signals, so that beliefs become strictly increasing in translation when the prior is uniform (or in the limit). Since payoffs do not depend directly on types, this strictness is important in order to exploit the assumptions on payoffs.

2.7 Appendix

2.7.1 On the Real Cutoff Points

The real cutoffs are the threshold signals that separate an action from its successor, and they are sufficient to represent any increasing (simple) function. Since $M_i - 1$ cutoffs suffice to represent i's strategy and there are many more available fictitious cutoffs, the dispensable

[27] The argument that beliefs in Harsanyi's model are translation decreasing also applies to the quantal response model [73]. In the quantal response model, players' signals are also uncorrelated and there are possibly many equilibria as the noise vanishes. Only one of those equilibria traces back to the centroid (infinite noise), which is how selection operates.

ones must be excluded. This is done in Definition 7. For each player, the subset of fictitious cutoffs which is selected depends on the opposing profile, and thus it is not obvious that the contraction survives *across* those selected fictitious cutoffs. This explains the difference between Proposition 2 and Theorem 1. I begin with an example.

EXAMPLE **1** Suppose $\Gamma(\nu)$ is a global game with $N = \{1, 2\}$ and $A_1 = A_2 = \{0, 1, 2\}$. Further, suppose $\underline{\theta} = 0$ and $\overline{\theta} = 1$. Here, there will be three fictitious cutoffs: $c^f_{1,0}$, $c^f_{2,0}$, and $c^f_{2,1}$. But we only need two of them to represent a player's best-response. For example, suppose strategy $(0.2, 0.8)$ is a best-response to $c_j \in \mathcal{C}_j$. It consists in playing 0 for signals below 0.2, 2 for signals above 0.8, and 1 in between. In this case, the first real cutoff, c_0, that separates 0 and 1 is $0.2 = c^f_{1,0}$. The second real cutoff, c_1, that separates 1 and 2 is $0.8 = c^f_{2,1}$. Now consider the following best-response $(0.4, 0.4)$ to $c'_j \in \mathcal{C}_j$. In this case, the player never plays 1 except possibly on a set of measure zero (when receiving exactly signal 0.4). Then the first real cutoff, c'_0, that separates 0 and 1 is $0.4 = c'^f_{2,0}$, but the second real cutoff, c'_1, is also $c'^f_{2,0}$ since 1 is not played.

Therefore, the real cutoff points take on values of fictitious cutoffs, but they might switch fictitious cutoffs from one opposing profile to the other, which makes it harder to identify any contraction. If each real cutoff was attached to the same fictitious cutoff all the way through, then contraction would follow immediately from Proposition 2.

The derivation of the real cutoffs from a set of fictitious cutoffs works as follows.

DEFINITION **7** *I define the real cutoffs inductively. For any $i \in N$, given $c_{-i} \in \mathcal{C}_{-i}$, the greatest real cutoff point, denoted c_{M_i-1}, is the fictitious cutoff $c^f_{M_i,m}$ such that: $Ed\pi_i(a_{i,M_i}, a_i, g(c_{-i}), s_i) > 0$ for all $a_i \neq a_{i,M_i}$, and $s_i > c^f_{M_i,m}$, and $Ed\pi_i(a_{i,m}, a_i, g(c_{-i}), s_i) > 0$ for all $a_i \neq a_{i,m}$, and $s_i \in (c^f_{M_i,m} - \varepsilon, c^f_{M_i,m})$, for some $\varepsilon > 0$. Given $c_{k_i} = c^f_{n,m}$ with $n > m$, then the real cutoff that precedes, denoted c_{k_i-1}, is also equal to $c^f_{n,m}$ if $k_i > m$ or, if $k_i = m$, it is the fictitious cutoff $c^f_{k_i,\gamma}$ with $k_i > \gamma$ such that: $Ed\pi_i(a_{k_i}, a_i, g(c_{-i}), s_i) > 0$ for all $a_i \neq a_{k_i}$, and $s_i \in (c^f_{k_i,\gamma}, c^f_{k_i,\gamma} + \varepsilon)$, and $Ed\pi_i(a_{i,\gamma}, a_i, g(c_{-i}), s_i) > 0$ for all $a_i \neq a_{i,\gamma}$, and $s_i \in (c^f_{k_i,\gamma} - \varepsilon, c^f_{k_i,\gamma})$ for some $\varepsilon > 0$.*

2.7.2 Proofs

Without further notice, most of the proofs are given assuming *DSM*. The proofs under *ISM* can be found in Mathevet [70] or are available to the reader upon request.

The next remark is an implication of translation non-decreasing beliefs (Definition 3) which is used in the proofs.

REMARK. For all $\Delta^* \in [0, \overline{\theta} - \underline{\theta}]$ and vectors Δ such that $\Delta \leq \Delta^* \mathbf{1}$, we have $\Lambda_i(g|\, c_{-i} + \Delta, s_i + \Delta^*, \nu) \leq \Lambda_i(g|\, c_{-i} + \Delta^* \mathbf{1}, s_i + \Delta^*, \nu)$, because g is an increasing aggregate and player i believes greater values of the aggregate are more likely when her opponents play greater strategies (meaning lower their cutoffs). If beliefs are translation non-decreasing, then $\Lambda_i(g|\, c_{-i} + \Delta, s_i + \Delta^*, \nu) \leq \Lambda_i(g|\, c_{-i}, s_i, \nu)$.

Proof of Proposition 2: Let $\overline{g} = g((a_{j,M_j}))$ and $\sigma : \mathcal{G}_i \to \mathcal{G}_i$ be the successor function defined for all $g \in \mathcal{G}_i$ by $\sigma(g) = \min\{g' \in \mathcal{G}_i : g' \succ g\}$. Let $\underline{g} = g((a_{j,1}))$. Take any $c'_{-i}, c_{-i} \in \mathcal{C}_{-i}$ with $c'_{-i} \neq c_{-i}$. Player i's expected utility of playing a_i when the other players play according to c_{-i} is:

$$\begin{aligned} E\pi_i(a_i, c_{-i}, s_i) &= \Lambda_i(\sigma(\underline{g})|\, c_{-i}, s_i, \nu)\pi_i(a_i, \underline{g}, s_i) + \sum_{\{g:\, \overline{g} > g > \underline{g}\}} (\Lambda_i(\sigma(g)|\, c_{-i}, s_i, \nu) - \Lambda_i(g|\, c_{-i}, s_i, \nu)) \\ &\quad \pi_i(a_i, g, s_i) + (1 - \Lambda_i(\overline{g}|\, c_{-i}, s_i, \nu))\pi_i(a_i, \overline{g}, s_i) \end{aligned}$$

which can be rewritten,

$$E\pi_i(a_i, c_{-i}, s_i) = \pi_i(a_i, \overline{g}, s_i) + \sum_{\{g:\, \overline{g} > g \geq \underline{g}\}} \Lambda_i(\sigma(g)|\, c_{-i}, s_i, \nu)(\pi_i(a_i, g, s_i) - \pi_i(a_i, \sigma(g), s_i)). \tag{2.4}$$

Now, pick $n, m \in \{1, \ldots, M_i\}$ such that $n > m$. For any g, g' in $\mathcal{G}_i$ and signal s_i, let

$$\delta(g, g', s_i) = d\pi_i(a_{i,m}, a_{i,n}, g, s_i) - d\pi_i(a_{i,m}, a_{i,n}, g', s_i).$$

By definition, $c^f_{n,m}$ is the signal s_i that verifies $Ed\pi_i(a_{i,m}, a_{i,n}, c_{-i}, s_i) = 0$, which by (2.4) is equivalent to the signal s_i such that

$$\sum_{\{g:\, \overline{g} > g \geq \underline{g}\}} \Lambda_i(\sigma(g)|\, c_{-i}, s_i, \nu)\delta(g, \sigma(g), s_i) = d\pi_i(a_{i,n}, a_{i,m}, \overline{g}, s_i). \tag{2.5}$$

Denote by $l(c_{-i}, s_i)$ and $r(s_i)$, respectively, the LHS and RHS of (2.5). Let $\Delta_{j,k_j} = |c'_{j,k_j} - c_{j,k_j}|$, $\Delta = (\Delta_{j,k_j})$ and $\Delta^* \geq \max_{j \neq i} \max_{k_j} \Delta_{j,k_j}$. Suppose, by way of contradiction, that $c'^f_{n,m} = c^f_{n,m} + \Delta^*$. By translation non-decreasing beliefs, $\Lambda_i(\sigma(g)|\, c_{-i} + \Delta, c'^f_{n,m}, \nu) \leq \Lambda_i(\sigma(g)|\, c_{-i}, c^f_{n,m}, \nu)$ for all $g \in \mathcal{G}_i$, and thus $\Lambda_i(\sigma(g)|\, c'_{-i}, c'^f_{n,m}, \nu) \leq \Lambda_i(\sigma(g)|\, c_{-i}, c^f_{n,m}, \nu)$

for all g.[28] By (A4), $\delta(g,\sigma(g),c'^{f}_{n,m}) \leq \delta(g,\sigma(g),c^{f}_{n,m})$ for all g, and supermodularity of π_i in (a_i, g) implies $\delta(.)$ is always positive. Hence, $l(c'_{-i}, c'^{f}_{n,m}) \leq l(c_{-i}, c^{f}_{n,m})$. By (A3), $r(c'^{f}_{n,m}) > r(c^{f}_{n,m})$. Since $l(c_{-i}, c^{f}_{n,m}) = r(c^{f}_{n,m})$, it implies $l(c'_{-i}, c'^{f}_{n,m}) < r(c'^{f}_{n,m})$. By definition, $(c'_{-i}, c'^{f}_{n,m})$ must satisfy (2.5), but it does not, which is a contradiction. Consequently, $c'^{f}_{n,m} - c^{f}_{n,m} < \Delta^*$. Similarly but starting from $(c'_{-i}, c'^{f}_{n,m})$, we obtain $|c'^{f}_{n,m} - c^{f}_{n,m}| < \Delta^*$. Since n, m were chosen arbitrarily in a finite set and Δ^* was any real greater or equal to $d(c'_{-i}, c_{-i})$, then $d(c'^{f}_{i}(\nu), c^{f}_{i}(\nu)) < d(c'_{-i}, c_{-i})$. Q.E.D

Proof of Theorem 1: I want to show that $br_{i,\nu} : \mathcal{C}_{-i} \to \mathcal{C}_i$ for all i and $br_\nu : \mathcal{C} \to \mathcal{C}$ are (weak) contractions for each ν. Pick any $\nu > 0$ that is implicit all throughout. By Proposition 2, $d(c'^{f}_{i}, c^{f}_{i}) < d(c'_{-i}, c_{-i})$ for all distinct $c'_{-i}, c_{-i} \in \mathcal{C}_{-i}$ and all $i \in N$. Let $br_i(c_{-i}) = (c_{k_i})_{k_i}$ and $br_i(c'_{-i}) = (c'_{k_i})_{k_i}$. I show that $|c'^{f}_{n,m} - c^{f}_{n,m}| < d(c'_{-i}, c_{-i})$ for all n, m, i implies $|c'_{k_i} - c_{k_i}| < d(c'_{-i}, c_{-i})$ for all k_i and $i \in N$.

I prove this result by induction for an arbitrary pair $(c'_{-i}, c_{-i}) \in \mathcal{C}^2_{-i}$ of distinct elements. First, I show it is true for the greatest real cutoff point. Recall that (A1) implies that a_{M_i} must be played in any best-response. Suppose that $c_{M_i-1} = c^{f}_{M_i,s}$ and $c'_{M_i-1} = c'^{f}_{M_i,t}$ for some $s, t \in \{1, \ldots, M_i - 1\}$. Then $c'_{M_i-1} - c_{M_i-1} = c'^{f}_{M_i,t} - c^{f}_{M_i,s} = c'^{f}_{M_i,t} - c^{f}_{M_i,t} + c^{f}_{M_i,t} - c^{f}_{M_i,s}$. By Proposition 2, $c'^{f}_{M_i,t} - c^{f}_{M_i,t} < d(c'_{-i}, c_{-i})$. By Definition 7, $c^{f}_{M_i,t} - c^{f}_{M_i,s} \leq 0$ because $c_{M_i-1} = c^{f}_{M_i,s}$ implies that a_{M_i} is played right after a_s in the best-reply, and thus the fictitious cutoff between a_{M_i} and a_t must lie before $c^{f}_{M_i,s}$. As a result, $c'_{M_i-1} - c_{M_i-1} < d(c'_{-i}, c_{-i})$. Clearly, the same argument applies to show $c_{M_i-1} - c'_{M_i-1} < d(c'_{-i}, c_{-i})$ and so $|c'_{M_i-1} - c_{M_i-1}| < d(c'_{-i}, c_{-i})$.

Now suppose $|c'_{k_i} - c_{k_i}| < d(c'_{-i}, c_{-i})$ for k_i. I prove that $|c'_{k_i-1} - c_{k_i-1}| < d(c'_{-i}, c_{-i})$.

Case 1: Action a_{k_i} is played both under $br_i(c_{-i})$ and $br_i(c'_{-i})$. Notice this is similar to the case of the greatest real cutoff and so the proof is analogous.

Case 2: Action a_{k_i} is played neither under $br_i(c_{-i})$ nor $br_i(c'_{-i})$. Then, $c_{k_i-1} = c_{k_i}$ and $c'_{k_i-1} = c'_{k_i}$, and so $|c'_{k_i-1} - c_{k_i-1}| < d(c'_{-i}, c_{-i})$ by induction hypothesis.

Case 3a: Action a_{k_i} is not played under $br_i(c_{-i})$ but it is under $br_i(c'_{-i})$. Then, $c_{k_i-1} = c_{k_i}$ and so $c'_{k_i-1} - c_{k_i-1} = c'_{k_i-1} - c_{k_i} = c'_{k_i-1} - c'_{k_i} + c'_{k_i} - c_{k_i}$. Notice $c'_{k_i} - c_{k_i} < d(c'_{-i}, c_{-i})$ by induction hypothesis and $c'_{k_i-1} - c'_{k_i} \leq 0$ since the best-response is increasing in the signal.

[28]Note $\Lambda_i(\sigma(g)|\, c'_{-i}, c'^{f}_{n,m}, \nu) \leq \Lambda_i(\sigma(g)|\, c_{-i} + \Delta, c'^{f}_{n,m}, \nu)$ because g is an increasing aggregate and, when player i's opponents lower all their cutoffs ($c'_{-i} \leq c_{-i} + \Delta$), player i believes higher actions are more likely and so are greater values of the aggregate.

Therefore, $c'_{k_i-1} - c_{k_i-1} < d(c'_{-i}, c_{-i})$.

Case 3b: By a similar argument, if action a_{k_i} is not played under $br_i(c'_{-i})$ but it is under $br_i(c_{-i})$, $c_{k_i-1} - c'_{k_i-1} < d(c'_{-i}, c_{-i})$.

Case 4a: Action a_{k_i} is played under $br_i(c'_{-i})$ but it is not played under $br_i(c_{-i})$. Then, $c'_{k_i-1} = c'^f_{k_i,t}$ and $c_{k_i-1} = c_{k_i} = c^f_{s,w}$ for some $s, t, w \in \{1, \ldots, M_i\}$ where $s > k_i$ and $w < k_i$ (since a_{k_i} is not played). Then $c_{k_i-1} - c'_{k_i-1} = c^f_{s,w} - c'^f_{k_i,t}$. Note that $c^f_{s,w} - c^f_{k_i,w} \leq 0$ as it is optimal to play a_w until $c^f_{s,w}$ and so, since $w < k_i$, it must be that a_{k_i} can preferred to a_w by player i only for signals higher than $c^f_{s,w}$. By a similar argument, $c'^f_{k_i,w} \leq c'^f_{k_i,t}$. As a result, $c_{k_i-1} - c'_{k_i-1} = c^f_{s,w} - c'^f_{k_i,t} \leq c^f_{k_i,w} - c'^f_{k_i,w}$. By Proposition 2, $c^f_{k_i,w} - c'^f_{k_i,w} < d(c'_{-i}, c_{-i})$, hence $c_{k_i-1} - c'_{k_i-1} < d(c'_{-i}, c_{-i})$.

Case 4b: Similarly, if action a_{k_i} is played under $br_i(c_{-i})$ but it is not under $br_i(c'_{-i})$, then $c'_{k_i-1} - c_{k_i-1} < d(c'_{-i}, c_{-i})$.

Put Cases 3a and 4a (3b and 4b) together to obtain: $|c'_{k_i-1} - c_{k_i-1}| < d(c'_{-i}, c_{-i})$, which completes the induction part of the proof. Therefore, $|c'_{k_i} - c_{k_i}| < d(c'_{-i}, c_{-i})$ for all k_i, and so best-response $br_i : \mathcal{C}_{-i} \to \mathcal{C}_i$ shrinks for all $i \in N$. For all distinct $c', c \in \mathcal{C}$ and for all $i \in N$ such that $c'_{-i} \neq c_{-i}$, $\max_{k_i} |c'_{k_i} - c_{k_i}| < d(c'_{-i}, c_{-i}) \leq d(c', c)$ which implies that $\max_{i \in N} \max_{k_i} |c'_{k_i} - c_{k_i}| < d(c', c)$. Equivalently, $d(br(c'), br(c)) < d(c', c)$ for all $c', c \in \mathcal{C}$ with $c' \neq c$. Q.E.D

The next Lemma draws its main idea and technicalities from Lemma 4 in FMP [36], so its proof is omitted.[29] Lemma 1 says that beliefs Ω_i tend to be translation invariant in the limit and this is achieved uniformly in ν for all parameters. Denote

$$\rho_{\Delta^*}(\nu) = \max_{s_i \in [\underline{\theta}, \overline{\theta}]} \frac{\max\limits_{e \in [-e_i, e_i]} \phi(s_i + \Delta^* - \nu e)}{\min\limits_{e \in [-e_i, e_i]} \phi(s_i + \Delta^* - \nu e)},$$

where $\Delta^* \in [0, \overline{\theta} - \underline{\theta}]$. Define $\varepsilon_{\Delta^*}(\nu) = \max\{1 - 1/\rho_{\Delta^*}(\nu), \rho_{\Delta^*}(\nu) - 1\}$ and note that, for all Δ^*, $\varepsilon_{\Delta^*}(\nu)$ is decreasing in ν because $\rho_{\Delta^*}(\nu)$ is decreasing.

Lemma **1** *Let* $J = N \backslash \{i\}$, *and let* $K = [\underline{\theta} - \nu e^*, 2\overline{\theta} - \underline{\theta} + \nu e^*]$ *be contained in the interior*

[29] The proof can be found in Mathevet [70].

of the support of ϕ. Let $i \in N$ and $\Delta^* \in [0, \overline{\theta} - \underline{\theta}]$. Denote

$$\kappa_{\Delta^*}(c_{-i}, s_i, \nu) = \int_{-\infty}^{\infty} \frac{1}{\nu} \prod_{j \in J} F_j \left(\frac{c_j + \Delta^* - \theta}{\nu} \right) f_i \left(\frac{s_i + \Delta^* - \theta}{\nu} \right) d\theta.$$

Then, $\varepsilon_{\Delta^*}(\nu) \downarrow 0$ and $|\Omega_i(c_{-i} + \Delta^* \mathbf{1} | s_i + \Delta^*, \nu) - \kappa_{\Delta^*}(c_{-i}, s_i, \nu)| < \varepsilon_{\Delta^*}(\nu)$ for all $s_i \in [\underline{\theta}, \overline{\theta}]$, $c_{-i} \in \mathcal{C}_{-i}$.

The next Lemma shows that beliefs about the aggregate tend to be translation non-decreasing in the limit and this is achieved uniformly in ν for all parameters.

LEMMA **2** *Let $K = [\underline{\theta} - \nu e^*, 2\overline{\theta} - \underline{\theta} + \nu e^*]$ be contained in the interior of the support of ϕ. For all $i \in N$, $\Delta^* \in [0, \overline{\theta} - \underline{\theta}]$, $g \in \mathcal{G}_i$, there exists a positive number $M(g)$ such that*

$$\Lambda_i(g | \, c_{-i}, s_i, \nu) \geq \Lambda_i(g | \, c_{-i} + \Delta, s_i + \Delta^*, \nu) - M(g)(\varepsilon_0(\nu) + \varepsilon_{\Delta^*}(\nu))$$

for all $s_i \in [\underline{\theta}, \overline{\theta}]$, $\Delta \in [0, \overline{\theta} - \underline{\theta}]^{(\sum_{j \neq i} M_j - N + 1)}$ such that $\Delta \leq \Delta^* \mathbf{1}$, and all $c_{-i} \in \mathcal{C}_{-i}$.

Proof: I prove the result by induction. I will argue that the claim is true for the greatest value of the aggregate in $\mathcal{G}_i$. But first, suppose the claim is true for all aggregate values greater or equal to $g' \in \mathcal{G}_i$. Let $\mathtt{p}(g') = \max\{g \in \mathcal{G}_i : g \prec g'\}$ be the predecessor function and show $\Lambda_i(\mathtt{p}(g') | \, c_{-i}, s_i, \nu)$ is translation non-decreasing in the limit. There are possibly several (but finitely many) ways to obtain $\mathtt{p}(g')$. Let c^l_{-i} be the l-th combination of cutoffs of i's opponents that leads to $\mathtt{p}(g')$, and recall $\Lambda_i(\mathtt{p}(g') | \, c_{-i}, s_i, \nu) = 1 - (\text{Prob}(g \geq g' | \, c_{-i}, s_i, \nu) + \text{Prob}(g = \mathtt{p}(g') | \, c_{-i}, s_i, \nu)$. Note:

$$\text{Prob}(g = \mathtt{p}(g') | \, c_{-i}, s_i, \nu) = \sum_l \Omega_i((c^l_{k_j} < s_j < c^l_{k_j+1})_{j \neq i} | \, s_i, \nu)$$

for some k_j, and since there are only finitely many combinations of cutoffs that result in $\mathtt{p}(g')$, Lemma 1 implies that the distance between $\text{Prob}(g = \mathtt{p}(g') | \, c_{-i}, s_i, \nu)$ and

$$\sum_l \int_{-\infty}^{\infty} \frac{1}{\nu} \prod_{j \neq i} \left(F_j \left(\frac{c^l_{k_j+1} - \theta}{\nu} \right) - F_j \left(\frac{c^l_{k_j} - \theta}{\nu} \right) \right) f_i \left(\frac{s_i - \theta}{\nu} \right) d\theta \tag{2.6}$$

is bounded by $\sum_l \varepsilon_0(\nu)$ for all $s_i \in [\underline{\theta}, \overline{\theta}]$ and $c_{-i} \in \mathcal{C}_{-i}$. Letting $m(\mathtt{p}(g'))$ be the number of ways of specifying $\mathtt{p}(g')$, $\sum_l \varepsilon_0(\nu) = m(\mathtt{p}(g'))\varepsilon_0(\nu)$. A simple change of variables ($\theta' =$

$\theta - \Delta^*$) would establish that (2.6) is equal to

$$\sum_l \int_{-\infty}^{\infty} \frac{1}{\nu} \prod_{j \neq i} \left(F_j \left(\frac{c^l_{k_j+1} + \Delta^* - \theta}{\nu} \right) - F_j \left(\frac{c^l_{k_j} + \Delta^* - \theta}{\nu} \right) \right) f_i \left(\frac{s_i + \Delta^* - \theta}{\nu} \right) d\theta. \qquad (2.7)$$

Since $\text{Prob}(g \geq g' | \, c_{-i}, s_i, \nu) = 1 - \Lambda_i(g' | \, c_{-i}, s_i, \nu)$, the above argument and the induction hypothesis imply that the distance between $\text{Prob}(g \geq g' | \, c_{-i}, s_i, \nu) + \text{Prob}(g = \mathtt{p}(g') | \, c_{-i}, s_i, \nu)$ and

$$\text{Prob}(g \geq g' | \, c_{-i} + \Delta^*, s_i + \Delta^*, \nu) + \text{Prob}(g = \mathtt{p}(g') | \, c_{-i} + \Delta^*, s_i + \Delta^*, \nu) \qquad (2.8)$$

is less than $(M(g') + m(\mathtt{p}(g')))(\varepsilon_0(\nu) + \varepsilon_{\Delta^*}(\nu))$ and this, for all s_i, and c_{-i}. Let $M(\mathtt{p}(g')) = M(g') + m(\mathtt{p}(g'))$. But (2.8) is less than

$$\text{Prob}(g \geq g' | \, c_{-i} + \Delta, s_i + \Delta^*, \nu) + \text{Prob}(g = \mathtt{p}(g') | \, c_{-i} + \Delta, s_i + \Delta^*, \nu)$$

for all s_i, c_{-i} and $\Delta \in [0, \overline{\theta} - \underline{\theta}]^{(\sum_{j \neq i} M_j - N + 1)}$ such that $\Delta \leq \Delta^* \mathbf{1}$, which follows by definition of $\text{Prob}(g \geq \mathtt{p}(g') | .)$. Indeed, g is increasing in the players' actions and, when player i's opponents lower all their cutoffs, player i believes higher actions are more likely and so are greater values of the aggregate. Consequently,

$$\text{Prob}(g \geq g' | \, c_{-i}, s_i, \nu) + \text{Prob}(g = \mathtt{p}(g') | \, c_{-i}, s_i, \nu) \leq \text{Prob}(g \geq g' | \, c_{-i} + \Delta, s_i + \Delta^*, \nu) +$$

$$\text{Prob}(g = \mathtt{p}(g') | \, c_{-i} + \Delta, s_i + \Delta^*, \nu) + M(\mathtt{p}(g'))(\varepsilon_0(\nu) + \varepsilon_{\Delta^*}(\nu)),$$

for all s_i, c_{-i} and Δ which proves the claim for $\mathtt{p}(g')$.

Second, the claim needs to hold for $\overline{g}$ in order to start the induction argument. The same reasoning as the above applies in this case, because $\overline{g}$ has no successor. Finally, since there are only a finite number of such aggregate values, proceeding inductively completes the proof. Q.E.D

Corollary **2** *In the private values case with uniform prior, beliefs $\Lambda_i(g | \, c_{-i}, s_i, \nu)$ are translation non-decreasing for every $\nu > 0$.*

Proof: It unfolds along the same lines as the proof of Lemma 2, given $\Psi'_i(\theta | \, s_i, \nu) = (1/\nu) f_i((s_i - \theta)/\nu)$. Q.E.D

Proof of Proposition 3: (2) Since (A5) is automatically satisfied in 2×2 games, I only need to show Theorem 1 holds in this context whether or not (A4) is satisfied. Take any $c_j, c'_j \in \mathcal{C}_j$. By definition, for $i \in \{1, 2\}$, $c^f_{2,1}$ is the signal s_i verifying

$$\Omega(c_j|s_i, \nu)\delta(a_{j,1}, a_{j,2}, s_i) = d\pi_i(a_{i,2}, a_{i,1}, a_{j,2}, s_i). \tag{2.9}$$

Denote by $l(c_j, s_i)$ and $r(s_i)$, respectively, the LHS and RHS of (2.9) and thus $l(c_j, c^f_{2,1}) = r(c^f_{2,1})$. Let $\Delta^* \geq |c'_j - c_j|$. Suppose, by way of contradiction, that $c'^f_{2,1} = c^f_{2,1} + \Delta^*$. If $\delta(a_{j,1}, a_{j,2}, c'^f_{2,1}) \leq \delta(a_{j,1}, a_{j,2}, c^f_{2,1})$, then the proof goes similarly to that of Theorem 2 as it is covered by *DSM*. If $\delta(a_{j,1}, a_{j,2}, c'^f_{2,1}) - \delta(a_{j,1}, a_{j,2}, c^f_{2,1}) = \delta^+ > 0$, then translation non-decreasing beliefs imply $l(c'_j, c'^f_{2,1}) - l(c_j, c^f_{2,1}) \leq \delta^+$. But $r(s_i) = \delta(a_{j,1}, a_{j,2}, s_i) + d\pi_i(a_{i,2}, a_{i,1}, a_{j,1}, s_i)$ so that $r(c'^f_{2,1}) - r(c^f_{2,1}) > \delta^+$ by (A3). Therefore, $r(c'^f_{2,1}) > l(c'_j, c'^f_{2,1})$, a contradiction because $(c'_j, c'^f_{2,1})$ should satisfy (2.9) by definition. Since i, c_j, c'_j and the direction (from c_j to c'_j) were taken arbitrarily, contraction is proved because in 2-action games fictitious and real cutoffs coincide. A unique fixed point exists which completes the proof. Q.E.D

Proof of Theorem 2: I prove $\cup_{\nu>0}\mathcal{C}_\nu = \mathcal{C}^2$. Trivially $(c, c) \in \cup_{\nu>0}\mathcal{C}_\nu$ for all $c \in \mathcal{C}$. So, take any $(c, c') \in \mathcal{C}^2$ with $c_{-i} \neq c'_{-i}$. Let $l(c_{-i}, s_i, \nu)$ and $r(s_i)$ be, respectively, the LHS and RHS of (2.5). Let $(c_{-i}, c^f_{n,m}(\nu))$ satisfy (2.5) for all ν, that is, $l(c_{-i}, c^f_{n,m}(\nu), \nu) = r(c^f_{n,m}(\nu))$. Let $\Delta_{j,k_j} = |c'_{j,k_j} - c_{j,k_j}|$, $\Delta = (\Delta_{j,k_j})$, and $\Delta^* = \max_{j\neq i}\max_{k_j}\Delta_{j,k_j}$. By means of contradiction, suppose $c'^f_{n,m}(\nu) = c^f_{n,m}(\nu) + \Delta^*$. Since $(c'_{-i}, c'^f_{n,m}(\nu))$ satisfies (2.5) for all ν, Lemma 2 implies that there exists $\mu(\nu) \downarrow 0$ such that:[30] $\sum(\Lambda_i(\sigma(g)|\, c_{-i}, c^f_{n,m}(\nu), \nu) + \mu(\nu))\delta(g, \sigma(g), c'^f_{n,m}(\nu)) \geq r(c'^f_{n,m}(\nu))$. Because $M_j < \infty$ and π_i is bounded as a continuous function on a compact set, this inequality can be rewritten as

$$\sum \Lambda_i(\sigma(g)|\, c_{-i}, c^f_{n,m}(\nu), \nu)\delta(g, \sigma(g), c'^f_{n,m}(\nu)) + \vartheta_i(\nu) \geq r(c'^f_{n,m}(\nu)) \tag{2.10}$$

where $\vartheta_i(\nu) \to 0$ as $\nu \to 0$. Since $r(s_i+\Delta^*) - r(s_i)$ is strictly positive by (A3) and continuous in s_i, then it has a strictly positive minimum over $[\underline{\theta}, \overline{\theta}]$, call it m^*. Let $m^{**} = m^*/2$. Since

[30] For example, let $\mu(\nu) = \max_g M(g)(\varepsilon_0(\nu) + \varepsilon_{\Delta^*}(\nu))$. By Lemma 2, we know $\Lambda_i(g|\, c_{-i}, s_i, \nu) \geq \Lambda_i(g|\, c_{-i} + \Delta, s_i + \Delta^*, \nu) - \mu(\nu) \geq \Lambda_i(g|\, c'_{-i}, s_i + \Delta^*, \nu) - \mu(\nu)$ and since $\delta(.)$ is positive, then the inequality follows.

$l(c_{-i}, c^f_{n,m}(\nu), \nu) = r(c^f_{n,m}(\nu))$, using (A4) we have

$$\sum \Lambda_i(\sigma(g)|c_{-i}, c^f_{n,m}(\nu), \nu)\delta(g, \sigma(g), c'^f_{n,m}(\nu)) + m^{**} \leq r(c^f_{n,m}(\nu)) + m^{**} < r(c'^f_{n,m}(\nu)). \quad (2.11)$$

Since $\vartheta_i(\nu) \to 0$, (2.10) and (2.11) lead to a contradiction as soon as $\vartheta_i(\nu)$ is less than m^{**}. It means that there exists ν small enough such that $l(c'_{-i}, c'^f_{n,m}(\nu), \nu) < r(c'^f_{n,m}(\nu))$, hence $(c'_{-i}, c'^f_{n,m}(\nu))$ does not verify (2.5) for all ν, a contradiction. Given this last inequality, it is also a contradiction for $\Delta^* > \max_{j\neq i} \max_{k_j} \Delta_{j,k_j}$ by (A2), (A3), (A4), and (A7). Therefore, the ν from which the contradiction is reached only depends on i, n, m and on the direction: from $(c_{-i}, c^f_{n,m})$ to $(c'_{-i}, c'^f_{n,m})$, or the other way. Since there are only finitely many players, actions, and directions, then there exists $\underline{\nu} > 0$ such that for all $i \in N$, $d(c'^f_i(\nu), c^f_i(\nu)) < d(c'_{-i}, c_i)$ for all $\nu < \underline{\nu}$. From the proof of Theorem 1, the contraction property of the fictitious cutoffs extend to the real cutoffs. That is, $d(br_\nu(c), br_\nu(c')) < d(c, c')$ for all $\nu < \underline{\nu}$. Therefore, $(c, c') \in \cup_{\nu>0} \mathcal{C}_\nu$. Q.E.D

Proof of Proposition 4: The proof has two steps. First, I will show that for any $(c, c') \in \mathcal{C}^2$, there exist $\underline{\nu}$ and a neighborhood $\mathcal{U}$ of (c, c') in the product topology of $\mathcal{C}^2$ such that $\mathcal{U} \subset \mathcal{C}_{\underline{\nu}}$. Second, I will show this implies uniqueness in the limit. To simplify notation, I do not indicate the dependance on a_m and a_n, but recall they belong to a finite set.

Let us specify function μ_{Δ^*} from the proof of Theorem 2 and then construct ϑ_i. Let $\mu_{\Delta^*}(\nu) = \max_g M(g)(\varepsilon_0(\nu) + \varepsilon_{\Delta^*}(\nu))$ where $\Delta^* = \max_{j\neq i} \max_{k_j} |c'_{k_j} - c_{k_j}|$. By the Maximum Theorem, $\rho_{\Delta^*}(\nu)$ is continuous in Δ^*[31] and then, so are $\varepsilon_{\Delta^*}(\nu) = \max\{1 - 1/\rho_{\Delta^*}(\nu), \rho_{\Delta^*}(\nu) - 1\}$, and $\mu_{\Delta^*}(\nu)$ for each ν. Moreover, notice that the minimum of $r(s_i + \Delta^*) - r(s_i)$ over $s_i \in [\underline{\theta}, \overline{\theta}]$ only depends on i and Δ^* and thus its half, denoted $m^{**}_i(\Delta^*)$, is continuous in Δ^* by the Maximum Theorem. Now let $b_i = \max_{s_i\in[\underline{\theta},\overline{\theta}]} \sum_g \delta(g, \sigma(g), s_i)$, and define $\vartheta_i(\nu, \Delta^*) = b_i \mu_{\Delta^*}(\nu)$. Since $\mu_{\Delta^*}(\nu) \downarrow 0$, then for any $i \in N$, there exists $\underline{\nu}_i > 0$ such that $\vartheta_i(\underline{\nu}_i, \Delta^*) < m^{**}_i(\Delta^*)$. As Δ^* is continuous in (c, c'), for any $i \in N$ there exists a neighborhood of (c_{-i}, c'_{-i}) such that $\vartheta_i(\underline{\nu}_i, \Delta^*) < m^{**}_i(\Delta^*)$, which implies $\vartheta_i(\nu, \Delta^*) < m^{**}_i(\Delta^*)$ for all $\nu < \underline{\nu}_i$ because $\mu_{\Delta^*}(\nu) \downarrow 0$.[32] Therefore, there is a neighborhood $\mathcal{U}$ of (c, c') and $\underline{\nu} > 0$ such that $\mathcal{U} \subset \mathcal{C}_{\underline{\nu}}$.

[31] The definition of ρ_{Δ^*} precedes the proof of Lemma 1.
[32] Recall that $\varepsilon_{\Delta^*}(\nu)$ is decreasing in ν for all Δ^*.

Now suppose by way of contradiction that there is $\iota > 0$ such that for all $\underline{\nu} > 0$, there exists $\nu < \underline{\nu}$ for which $d(\underline{e}_\nu, \overline{e}_\nu) > \iota$. Since $\{(\underline{e}_\nu, \overline{e}_\nu)\} \subset \mathcal{C}^2$ and $\mathcal{C}^2$ is compact, this sequence has at least one cluster point and all its cluster points are in $\mathcal{C}^2$. For such ι to exist, there must be a cluster point of the sequence, denoted $(\underline{e}, \overline{e})$, such that $\underline{e} \neq \overline{e}$. By Theorem 2, there exists $\underline{\nu}$ such that a neighborhood $\mathcal{U}$ of $(\underline{e}, \overline{e})$ is a subset of $\mathcal{C}_{\underline{\nu}}$. Now, take two disjoint neighborhoods $\underline{\mathcal{U}}$ and $\overline{\mathcal{U}}$ in $\mathcal{C}$, respectively of $\underline{e}$ and $\overline{e}$, such that $\underline{\mathcal{U}} \times \overline{\mathcal{U}} \subset \mathcal{U}$. Notice such neighborhoods exist because $\mathcal{U}$ is an open set in the product topology. Since $(\underline{e}, \overline{e})$ is a cluster point, there exists $\nu' < \underline{\nu}$ such that $(\underline{e}_{\nu'}, \overline{e}_{\nu'}) \in \underline{\mathcal{U}} \times \overline{\mathcal{U}} \subset \mathcal{U} \subset \mathcal{C}_{\underline{\nu}}$ which is a contradiction. Q.E.D

Lemma **3** *Let $K = [\underline{\theta} - 2\eta - \nu e^*, 2\overline{\theta} - \underline{\theta} + 6\eta + \nu e^*]$ be contained in the interior of the support of ϕ. Letting $M(g)$ be some positive number, then for all $i \in N$, $\Delta^* \in [0, \overline{\theta} - \underline{\theta} + 2\eta]$, $g \in \mathcal{G}$,*

$$\Lambda_i(g, \theta |\, c_{-i}, s_i, \nu) \geq \Lambda_i(g, \theta |\, c_{-i} + \Delta, s_i + \Delta^*, \nu) - M(g)(\varepsilon_0(\nu) + \varepsilon_{\Delta^*}(\nu))$$

for all $s_i \in [\underline{\theta} - \eta, \overline{\theta} + \eta]$, $\Delta \in [0, \overline{\theta} - \underline{\theta} + 2\eta]^{(\sum_{j\neq i} M_j - N + 1)}$ such that $\Delta \leq \Delta^* \mathbf{1}$, $\theta \in \mathbb{R}$ and all $c_{-i} \in \mathcal{C}_{-i}$.

The next lemma is a technical result that will be useful in proving Theorem 4.[33]

Lemma **4** *Suppose state monotonicity is decreasing. Then there exists a positive function $\delta^*(g^*, g, \theta)$ from $\mathcal{G}^2 \times \mathbb{R}$ to $\mathbb{R}_+$ that is decreasing in (g^*, θ) and such that for all g, s_i, c_{-i} and ν:*

$$\int_{\mathcal{G}\times\mathbb{R}} \delta^*(g^*, g, \theta) d\Lambda_i(g^*, \theta | c_{-i}, s_i, \nu) = \int_{\theta\in\mathbb{R}} \delta(g, \sigma(g), \theta) \Lambda_i(\sigma(g), d\theta | c_{-i}, s_i, \nu).$$

Proof: Define

$$\delta^*(g^*, g, \theta) = \begin{cases} \delta(g, \sigma(g), \theta) & \text{if } g^* < \sigma(g) \\ 0 & \text{otherwise,} \end{cases}$$

and notice δ^* is decreasing in (g^*, θ) by construction, *DSM* and (A2). It is also positive by (A2). The proof then follows by Fubini's theorem. Q.E.D

Proof of Theorem 3: Recall that for $\nu < \eta / \max\{e_i : i \in N\}$, all real cutoffs exist. Suppose first that state monotonicity is decreasing. By strict first-order stochastic dominance of Ψ_i and Lemma 4, Equation (2.12) has a unique solution.[34] From then, the proof

[33] I thank Federico Echenique for suggesting this lemma.

[34] See the proof of Proposition 2.

is similar to the proof of Theorem 2. All the real cutoffs shrink as functions of the opposing profile which translates to the fictitious cutoffs by Theorem 1. Uniqueness then follows from the contraction of $br(\nu)$.

Secondly, suppose state monotonicity is increasing. Take any $c_{-i}, c'_{-i} \in \mathcal{C}_{-i}$. Pick any $\beta, \alpha \in \{1, \ldots, m_i\}$ such that $\beta > \alpha$. By definition, $c^r_{\beta,\alpha}$ is the signal s_i such that:

$$\sum_{\{\Sigma:\overline{\Sigma}>\Sigma\geq\underline{\Sigma}\}} \int_{\theta\in\mathbb{R}} \delta(\Sigma, \rho(\Sigma), \theta)\Lambda_i(\rho(\Sigma), d\theta|c_{-i}, s_i, \nu)) = \int_{\theta\in\mathbb{R}} (d\pi_i(a_\beta, a_\alpha, \overline{\Sigma}, \theta))d\Psi_i(\theta|s_i,\nu).$$

Denote by $l(c_j, s_i)$ and $r(s_i) = \int r(\theta)d\Psi_i(\theta|s_i, \nu)$ the left- and right-hand sides of that equation. Now, I argue that there is a unique solution to this equation by Lemma 4 and (strict) first-order stochastic dominance, hence the best-responses are *almost everywhere* functions. Consider the version of Lemma 4 for *ISM*. An increase (decrease) from s_i to $s_i + (-)\varepsilon$ raises (lowers) the left-hand side by less than $\bar{l}(s_i + \varepsilon) - \bar{l}(s_i)$ (or $\bar{l}(s_i) - \bar{l}(s_i - \varepsilon)$) where

$$\bar{l}(s_i) = \sum_{\{\Sigma:\overline{\Sigma}>\Sigma\geq\underline{\Sigma}\}} \int_{\theta\in\mathbb{R}} \delta(\Sigma, \rho(\Sigma), \theta)d\Psi_i(\theta|s_i, \nu).$$

To see why, notice that by Lemma 4 under (ISM), a variation of s_i has two contradictory effects on δ^*: It is now increasing in θ and decreasing in Σ^*.[35] Since $r(\theta) = \sum_{\{\Sigma:\overline{\Sigma}>\Sigma\geq\underline{\Sigma}\}} \delta(\Sigma, \rho(\Sigma), \theta) + d\pi_i(a_{i,\beta}, a_{i,\alpha}, \underline{\Sigma}, \theta)$, then by (A3), $r(s_i + \varepsilon) - r(s_i) > \bar{l}(s_i + \varepsilon, \nu) - \bar{l}(s_i, \nu)$ and so only one s_i can be solution.

Finally, I complete the proof by showing the contraction. Let $x_{j,k_j} = |c'_{j,k_j} - c_{j,k_j}|$, $x = (x_{j,k_j})_{j\neq i}$ and $x^* \geq \max_j \max_{k_j} x_{j,k_j}$. Suppose by way of contradiction that $c'^r_{\beta,\alpha} = c^r_{\beta,\alpha} + x^*$. Then by translation invariance and Lemma 7, $l(c_j + x, c'^r_{\beta,\alpha}) - l(c_j, c^r_{\beta,\alpha}) \leq \bar{l}(c'^r_{\beta,\alpha}) - \bar{l}(c^r_{\beta,\alpha})$ for the same reason as the above. But we know the right-hand side increases strictly more than $\bar{l}$ which leads to a contradiction. Therefore, all the real cutoffs shrink as functions of the opposing profile which translates to the fictitious cutoffs by Theorem 1. Uniqueness then follows from the contraction of $br(\nu)$. Q.E.D

Proof of Theorem 4: This proof is an adaptation of the proof of Theorem 2, so I only sketch the main steps. First, take any $(c, c') \in \mathcal{C}^2$ with $c_{-i} \neq c'_{-i}$, and pick $n, m \in$

[35] A complete proof is available upon request.

$\{1, \ldots, M_i\}$ such that $n > m$. After rewriting the payoffs, $c^f_{n,m}(\nu)$ is uniquely defined[36] as the signal s_i satisfying:

$$\sum \int_{\theta \in \mathbb{R}} \delta(g, \sigma(g), \theta)\Lambda_i(\sigma(g), d\theta | c_{-i}, s_i, \nu)) = \int_{\theta \in \mathbb{R}} (d\pi_i(a_n, a_m, \overline{g}, \theta))d\Psi_i(\theta | s_i, \nu). \quad (2.12)$$

Define Δ_{j,k_j}, Δ, Δ^* as in the proof of Theorem 2. Let $l(c_{-i}, s_i, \nu)$ and $r(s_i, \nu)$ be, respectively, the LHS and RHS of (2.12). Suppose, by way of contradiction, $c'^f_{n,m}(\nu) = c^f_{n,m}(\nu) + \Delta^*$. Second, (A3), (A6), (A7'), and the continuity of $\Psi_i(\theta | s_i, \nu)$ in $s_i \in [\underline{\theta} - \eta, \overline{\theta} + \eta]$ imply that, for each ν, $r(s_i + \Delta^*, \nu) - r(s_i, \nu)$ is strictly bounded above zero for all s_i. So, using the fact that $r(s_i, \nu)$ converges uniformly (in s_i) to the RHS of (2.5), I show $r(c'^f_{n,m}(\nu), \nu) - r(c^f_{n,m}(\nu), \nu)$ converges to a strictly positive number as $\nu \to 0$. Third, Lemma 3 implies that as $\nu \to 0$ there is a vanishing upper bound on how much $\Lambda_i(g, \theta | \, c_{-i}, c^f_{n,m}(\nu), \nu)$ can stochastically dominate $\Lambda_i(g, \theta | \, c'_{-i}, c'^f_{n.m}(\nu), \nu)$. Thus Lemma 4 allows to conclude that $l(c_{-i}, c'^f_{n,m}(\nu), \nu) - l(c_{-i}, c^f_{n,m}(\nu), \nu)$ tends to be at most zero.[37] Given (2.12), there exists $\underline{\nu}$ below which $l(c_{-i}, c'^f_{n,m}(\nu), \nu) < r(c'^f_{n,m}(\nu), \nu)$, a contradiction. Q.E.D

The proof of Corollary 1 follows similarly to Proposition 4. An analogous argument applies because both $\varepsilon_{\Delta^*}(\nu)$ and the RHS of (2.12) are continuous (respectively, in Δ^* and s_i) for each ν, and ε_{Δ^*} converges monotonically to zero in ν.

[36]There is only one s_i verifying this equation for each (c_{-i}, ν) and m, n by Lemma 4 and (A7').

[37]Lemma 4 allows to apply the stochastic dominance properties of $d\Lambda_i(g, \theta|.)$ to the LHS of (2.12), even though it is a function of a different measure $\Lambda_i(\sigma, d\theta|.)$.

Chapter 3

Supermodular Implementation

3.1 Introduction

The question of how an equilibrium outcome arises in a mechanism is largely open in implementation theory and mechanism design.[1] This literature has produced numerous mechanisms that implement many social choice functions, but theoretical and experimental works reveal that many mechanisms suffer from learning and stability issues.[2] Often mechanisms do not enable boundedly rational agents to *achieve* an equilibrium outcome by learning, if used repeatedly over time. Likewise, slight perturbations in beliefs or behaviors often result in a departure from an equilibrium outcome, posing stability problems. This is particularly troublesome, because the idea behind mechanism design is usually practical in nature: Incentive design explicitly aims to construct mechanisms that *achieve* some desirable outcome in equilibrium. In reality, a static mechanism must sometimes be used repeatedly to reach an outcome. For example, the traffic authorities may set up a toll-system which in the long-run will minimize congestion and allocate users with higher benefits from driving to

[1]This chapter is based on a paper of mine entitled "Supermodular Bayesian Implementation: Learning and Incentive Design." Special thanks are due to Rabah Amir for his insightful comments as a discussant at the IESE conference on Complementarities and Information, and to Morgan Kousser, Thomas Palfrey, and Eran Shmaya for helpful advice and conversations. I also wish to thank Chris Chambers, Paul J. Healy, Bong Chan Koh, Serkan Kucuksenel, Paul Milgrom, Leeat Yariv, and seminar/conference participants at Caltech, UBC, UC Irvine, CMU, UT Austin, Oxford, INSEAD, Pompeu Fabra, Autonoma, IESE, London Business School, the SWET conference, the Game Theory Workshop at Stanford GSB, the IESE conference on Complementarities and Information, the fifth conference on Logic, Game Theory and Social Choice, and the international conference on Game Theory at Stony Brook. Andrea Mattozzi is gratefully acknowledged for financial support. Finally, part of the paper was written while I visited Stanford Economics Department and I am grateful for their hospitality.

[2]Muench and Walker [87] , Cabrales [14] and Cabrales and Ponti [15] show that learning and stability may be serious issues in, respectively, the Groves-Ledyard [42], Abreu-Matsushima [2], and Sjöström [100] mechanisms. On the experimental side, Healy [48] and Chen and Tang [24] provide evidence that convergence of learning dynamics may fail in various mechanisms, such as proportional tax or the paired-difference mechanism.

better roads (Sandholm [96] and [97]). A manager may design the agents' contracts to approach revenue maximization over time. A procurement department may allocate different jobs sequentially to contractors by running an auction several times. A group of scientists may create a control system for planetary exploration vehicles, so that the different units function more efficiently as the mission progresses.[3]

In this chapter, I develop the theory of supermodular Bayesian implementation to improve learning and stability in mechanism design. Think of a mechanism as describing the rules of a game: It assigns feasible strategies (or messages) to the agents and specifies how these strategies map into enforceable outcomes. Since players have preferences over the different outcomes, a mechanism induces a game in the traditional sense. If this induced game is supermodular, then the mechanism is said to be supermodular. Then I define a scf to be supermodular implementable if there is a supermodular mechanism whose equilibrium strategies yield that scf as an outcome. Assuming strategies are numbers, a supermodular game is a game with strategic complementarities, i.e a game in which the marginal utility of an agent increases as other players increase their strategies. The complementarities imply that an agent wants to play a larger strategy when the others do the same. For instance, it becomes more desirable for a worker in a firm to increase her effort when others put more effort into their job.

Supermodular implementation has interesting dynamic properties. Best-replies are always increasing in supermodular games; this feature helps boundedly rational agents find their way to equilibrium, for most learning dynamics inherit some monotonicity that guides them "near" the equilibria. This theory thus contributes to fill the important gap in the literature emphasized in Jackson [56]: "Issues such as how well various mechanisms perform when players are not at equilibrium but learning or adjusting are quite important [. . .] and yet have not even been touched by implementation theory. [This topic] has not been looked at from the perspective of designing mechanisms to have nice learning or dynamic properties." For example, a principal may actually attain revenue maximization by offering the agents a contract that they will face repeatedly for a sufficiently long time. A government may reach an optimal public goods level by repeatedly applying a supermodular tax system.

Supermodular mechanisms are appealing because they receive the theoretical properties of supermodular games. Milgrom and Roberts [76] show that supermodular games have

[3] See issues related to cognitive intelligence (Parkes [92] and Tumer and Wolpert [103]).

a largest and a smallest equilibrium and adaptive learners end up playing profiles in between, regardless of their starting point. Vives [106] reports a related result for learning à la Cournot. Adaptive learners regard past play as the best predictor of their opponents' future play and best-respond to their forecast. Cournot dynamics, fictitious play, and Bayesian learning are examples of adaptive learning. This convergence result extends to sophisticated learners, who react optimally to what their opponents may next best-respond (Milgrom and Roberts [77]). If a supermodular game has a unique equilibrium, then convergence to the equilibrium is ensured. Adaptive and sophisticated learning encompass such a wide range of backward- and forward-looking behaviors that supermodular mechanisms have very robust learning properties. Supermodular games are also attractive in an implementation framework because their mixed strategy equilibria are locally unstable under monotone adaptive dynamics like Cournot dynamics and fictitious play (Echenique and Edlin [34]). Ruling out mixed strategy equilibria is common in implementation theory and often arbitrary; but it is sensible in supermodular implementation. To the contrary, many pure-strategy equilibria are stable. In a parameterized supermodular game, all those equilibria that are increasing in the parameter are stable, such as the extremal equilibria (Echenique [32]).

Supermodular games and mechanisms are supported by strong experimental evidence. Healy [48] tests five public goods mechanisms in a repeated game setting and observes convergence only in those mechanisms that induce a supermodular game. Experiments on the Groves-Ledyard mechanism have shown that convergence is far better when the punishment parameter is high than when it is low (Chen and Plott [23] and Chen and Tang [24]). The Groves-Ledyard mechanism turns out to be supermodular when the punishment parameter is high. Finally, Chen and Gazzale [22] present experiments on a game where a parameter determines the degree of complementarity. In this game, they observe that convergence is significantly better when the parameter lies in the range where the game is supermodular.

The methodology used to derive properties of a mechanism may be promising for mechanism design theory. One striking feature of the traditional design approach is how much it relies on solution concepts to reach certain objectives. For example, if the designer wants the mechanism to be robust to misspecifications of the prior, then she will likely choose implementation in dominant strategies or ex-post equilibrium. Conversely, if the designer targets full efficiency in some quasilinear environment, then she will prefer implementation in Bayesian equilibrium. Economists have attempted to solve nearly all design problems by introducing a solution concept into the implementation framework: Subgame-perfect equilibrium, undominated Nash equilibrium, coalition-proof equilibrium, etc. However, there are interesting properties for mechanisms that are attached to families of games rather than solution concepts.[4] So why focus on the solution concept? This chapter proposes an alternative approach by using a weak solution concept — Bayesian equilibrium — and by instead focusing on a class of games with nice theoretical and experimental properties.

The centerpiece of my analysis is Theorem 5. In quasilinear environments with real type spaces, I prove that if a scf can be implemented by a direct mechanism that generates bounded strategic substitutes — as opposed to strategic complementarities — then this mechanism can be turned into a direct supermodular mechanism that implements the scf. The condition of bounded substitutes is always satisfied on finite type spaces and in twice-continuously differentiable environments with compact type spaces. So, the result is fairly general. The transformation technique is constructive and simple, yet powerful. I explain it in the next section in the context of a public goods example. The transfers can be appended a piece that turns the agents' announcements into complements, and that vanishes in expectation when the opponents play truthfully; thus truthtelling remains an equilibrium after the transformation. That piece is a coordination device that rewards the agent for conforming to the direction and amplitude of her opponents' report and that punishes her for not doing so.

In quasilinear environments, the mechanism designer is often interested in that there be no transfers into or out of the system. This is known as the budget balance condition and it plays an important role in (full) efficiency. Achieving budget balancing is difficult under dominant strategy implementation (Green and Laffont [40]) but possible under Bayesian implementation (Arrow [7] and d'Aspremont and Gérard-Varet [28]). Theorem 6 shows that budget balancing is also possible under supermodular Bayesian implementation. If a scf contains an (allocation) efficient decision rule and admits a mechanism producing bounded substitutes, then it is supermodular implementable with balanced transfers. Interestingly, there are cases where dominant strategy implementation cannot balance the budget, whereas it is possible to balance the budget and induce a supermodular game with

[4]Sandholm [96] and [97] successfully use implementation in potential games to obtain evolutionary properties of the mechanism.

a unique equilibrium.

Complementarities help guide agents towards the equilibrium, but they are source of new equilibria with possibly bad outcomes on which agents may coordinate. Supermodular implementation relies on weak implementation, i.e only the truthful equilibrium is known to deliver the desired outcome. Yet the mechanisms here generate a largest and a smallest equilibrium. There is a multiple equilibrium problem and I deal with it by developing optimal and unique supermodular implementation. Optimal supermodular implementation involves designing a supermodular mechanism that generates the weakest complementarities among all supermodular mechanisms. I prove that the interval between the largest and the smallest equilibrium decreases with the complementarities, hence optimal implementation produces the tightest interval around the truthful equilibrium (Proposition 7). Since this interval is "small," learning leads to a profile close to truthtelling and to the desired outcome. The intuition is that agents should be rewarded or punished to adopt monotone behaviors but no more than necessary, otherwise they tend to overreact. The main result (Theorem 7) is that all twice-differentiable scf whose decision rule depends on types through an aggregate are optimally supermodular implementable. Unique supermodular implementation describes that situation where the truthful equilibrium is the unique equilibrium of the induced supermodular game. All dynamics converge to the equilibrium. Theorem 8 gives conditions for unique supermodular implementation. As a by-product, it implies coalition-proof Nash implementation by Milgrom and Roberts [79].

The theory applies to traditional models of public goods or principal multi-agent models. In a public goods example with quadratic preferences, suppose that a designer uses the expected externality mechanism to implement some decision rule (Section 3.2). In the induced game, many learning dynamics fail to converge to the truthful equilibrium. Nevertheless, the mechanism can be modified to induce a supermodular game where the truthful equilibrium is unique and all dynamics converge to it. In a team-production example, a principal contracts with a set of agents and monitors their contribution to maximize net profits (Section 3.6.1). The scf is optimally implementable and truthtelling is the unique equilibrium of the induced supermodular game. But there are also challenging applications for the theory such as binary-choice models of auctions and public goods. A possible way around this problem is approximate implementation, where the objective becomes to supermodularly implement scf that are arbitrarily *close* to a "target scf." Most bounded scf admit nearby scf that are supermodular implementable (Section 3.5.4). The results apply, for instance, to auctions, public goods, and bargaining (Myerson and Satterthwaite [88]).

Supermodular implementation is widely applicable in quasilinear environments even though the chapter limits attention to direct mechanisms. For general preferences, however, direct mechanisms may be restrictive. The Revelation Principle says that direct mechanisms cause no loss of generality under traditional weak implementation. It is particularly relevant to examine the revelation principle for supermodular implementation, because the space of mechanisms to consider is very large. The Supermodular Revelation Principle (Theorem 9) says that if there exists a mechanism that supermodularly implements a scf such that the range of the equilibrium strategies in the desired equilibrium is a lattice, then there is a direct mechanism that supermodularly implements that scf truthfully. I give an example of a supermodular implementable scf where this range is not a lattice and that cannot be supermodularly implemented by any direct mechanism. This suggests that the condition of the theorem is somewhat minimally sufficient. Although this revelation principle is not as general as the traditional one, it measures the restriction imposed by supermodular direct mechanisms and gives conditions for their use.

A number of papers are related to learning and stability in the context of implementation or mechanism design. Chen [21] deserves mention because it is one of the first papers explicitly aimed at learning and stability in mechanism design. In a complete information environment with quasilinear utilities, she constructs a mechanism that Nash implements Lindahl allocations and induces a supermodular game. Here I build the framework of supermodular Bayesian implementation and I generalize her result in incomplete information. Abreu and Matsushima [1] establishes that for any scf f and positive ϵ, there is an ϵ-close scf f_ϵ that admits a mechanism where iterative deletion of strictly dominated strategies leads to a unique profile whose outcome is f_ϵ. Even though their result is general and strong,[5] it can be questioned on the basis of learning and stability. Following Cabrales [14], when the mechanism implements f_ϵ, it actually implements it in iteratively strictly ϵ-undominated strategies. In other words, elimination of weakly dominated strategies is the solution concept that underlies the exact-implementation problem for f (Abreu and Matsushima [2]); virtual implementation is a way of turning it into elimination of strictly dominated strate-

[5]The solution concept is strong enough to predict convergence of many learning dynamics to the unique equilibrium outcome (See e.g [77]). Note that there are games where some adaptive dynamics from [76] do not converge to a uniquely rationalizable profile.

gies for f_ϵ.[6] Another criticism is that it does not seem to extend to infinite sets of types, which is related to important theoretical questions (Duggan [30]). Their mechanism also employs a message space whose dimension increases to infinity as ϵ vanishes. In contrast, this chapter studies exact implementation with direct mechanisms on finite or infinite type sets. Cabrales [14] and Cabrales and Serrano [16] demonstrate that there are learning dynamics that converge to desired equilibrium outcomes in a general framework of (Bayesian) Nash implementation. But those dynamics require players to strictly randomize over all improvements on past play.[7] This rules out many natural learning dynamics considered here. Finally, there are general impossibility results on the stability of equilibrium outcomes in Nash implementation (Jordan [58] and Kim [61]).

The chapter is organized as follows. Section 3.2 presents the leading public goods example. Section 3.3 gives the basic definitions of lattice theory and Section 3.4 lays out the framework of supermodular implementation. Section 3.5 contains the main results. Section 3.6 provides several applications of the theory to traditional models and introduces approximate supermodular implementation. Section 3.7 presents the supermodular revelation principle. Finally, Section 3.8.2 gives an interpretation of learning in Bayesian games and Section 3.9 concludes.

3.2 Motivation and Intuition

This section provides an economic example of a designer who uses the expected externality mechanism (Arrow [7] and d'Aspremont and Gérard-Varet [28]) to implement a scf. The environment is simple: Two agents with smooth utilities and compact real type spaces. Yet the mechanism induces a game where learning and stability fail under many dynamics.

Then I describe a new approach where the existing mechanism is modified in order to induce a supermodular game. In the example, the benefit is immediate: All learning dynamics converge to the truthful equilibrium, and the equilibrium is stable.

Consider a principal who needs to decide the level of a public good, such as the size of a bridge. Let $X = [0, 2]$ denote the possible values of the public good. There are two

[6] Elimination of strictly dominated strategies implies robust learning properties, but not for weakly dominated strategies because it has the perverse consequence of excluding limit points of some learning dynamics.

[7] This feature is crucial, for example, to allow play to exit an integer game after players have fallen into it.

agents, 1 and 2, whose type spaces are $\Theta_1 = \Theta_2 \subset [0,1]$. Types are independently uniformly distributed. The agents' preferences are quasilinear, $u_i(x,\theta_i) = V_i(x,\theta_i) + t_i$, where $x \in X$, $\theta_i \in \Theta_i$, and $t_i \in \mathbb{R}$ is the transfer from the principal to agent i. The valuation functions are $V_1(x,\theta_1) = \theta_1 x - x^2$ and $V_2(x,\theta_2) = \theta_2 x + x^2/2$.

The principal wishes to make an allocation-efficient decision, i.e. she aims to maximize the sum of the valuation functions by choosing $x^*(\theta) = \theta_1 + \theta_2$. To this end, she wants the agent to reveal their true type, so she opts for the expected externality mechanism.[8] The transfers are set as follows:

$$t_1(\hat\theta_1, \hat\theta_2) = \frac{1}{2} + \frac{\hat\theta_2}{2} + \hat\theta_2^2 + \hat\theta_1 + \frac{\hat\theta_1^2}{2}, \qquad t_2(\hat\theta_1, \hat\theta_2) = -t_1(\hat\theta_1, \hat\theta_2).$$

Consider the straightforward application of learning to the (ex-ante) Bayesian game induced by this mechanism (see Section 3.8.2).[9] I will study convergence and stability of learning dynamics. Time proceeds in discrete periods $t \in \{0, 1, \ldots\}$ and agents are assumed to learn as time passes according to some rule. The strategies at time 0 are given exogenously. The agents observe the history of play from 0 to $t-1$ and then publicly play a strategy at t. More precisely, from the strategies played in the past, each agent updates her beliefs about her opponent's future strategy using some specified rule; then, given those updated beliefs, she plays the strategy which maximizes her current expected payoffs in the mechanism. In this context, a strategy is a *deception*, which is a contingent plan that specifies a type to be announced for each of an agent's possible types, and that she commits to follow after learning her type. Formally, a deception for i at period t is a function $\hat\theta_i^t : \Theta_i \to \Theta_i$.[10]

The questions are: Will the profile played at t converge to the truthful equilibrium as $t \to \infty$? If players were in the truthful equilibrium, will they return to this equilibrium after an exogenous perturbation? The first question asks whether the agents ever learn to play truthfully. The second one asks whether truthtelling is a stable equilibrium.

The players' payoffs determine the answers. For $i = 1, 2$, define the set of deceptions

[8] This mechanism allows truthful implementation of allocation-efficient decision rules (see [7], [28], or Section 23.D in Mas-Colell et al. [68]) i.e. truthtelling is a Bayesian equilibrium of the mechanism.

[9] See Chapter 1 of Fudenberg and Levine [37] for a justification and discussion of myopic learning.

[10] Announcing a deception in the Bayesian game might seem more realistic when type sets are finite (the example has similar conclusions in the finite case), but here it will come down to choosing an intercept between -1 and 1.

Σ_i as the set of measurable functions from Θ_i into Θ_i, and let $\mathcal{P}(\Sigma_i)$ be the set of (Borel) probability measures over Σ_i. Let $\mu_i^t \in P(\Sigma_j)$ be player i's beliefs about player j's deceptions at time t. A learning model is defined by a rule that takes the history of play as input and that generates beliefs μ_i^t as output.

Letting $cplt(i) = 2(-1)^i/i$, player i's expected utility in the mechanism is

$$E[u_i|\mu_i^t] = -\frac{\hat{\theta}_i^2}{2} + \big(\theta_i + cplt(i)E\big[\,E_{\theta_j}[\hat{\theta}_j^t(\theta_j)]\,|\,\mu_i^t\,\big] - (-1)^i/i\,\big)\hat{\theta}_i \tag{3.1}$$

up to a constant, where $E[.|\,\mu_i^t]$ is i's expectation over Σ_j (j's deceptions) given her beliefs μ_i^t.

In (3.1), *cplt* determines how players' strategies depend on one another. Since $cplt(1) < 0$ and $cplt(2) > 0$, if player 1 believes player 2's strategy has increased on average, then 1 decreases her strategy and vice-versa; whereas 2 tries to match any average-variation in 1's strategy. Players essentially chase one another, and so this game has a flavor of "matching-pennies" that will be the source of instability and learning deficiency.

Learning often fails to occur in this example. There are many learning dynamics for which, not only do the agents not converge to truth-revealing but the play cycles forever. Consider first weighted fictitious play (see e.g. Ho [52]) in the case where, for simplicity, types are in $\{0, .5, 1\}$. So Σ_i is finite. Deceptions are initially assigned arbitrary weights and the beliefs are given by the frequencies of the different deceptions in the total weight. Given $0 < \pi < 1$, beliefs are updated each period by multiplying all weights by $1-\pi$ and by adding one to the weight of the opponent's deception played at the last period. If players use an identical rule π, the profile converges to the truthful equilibrium unless π is too high ($\pi > .8$), in which case cycling occurs. But there is no reason a priori for both players to use the same learning rule. For asymmetric rules, learning becomes more uncertain. The player with the highest π often outweighs the other one in a non-linear fashion and prevents learning.[11]

Consider now the model with continuous types in $[0, 1]$. A dynamics is said to be Cournot if each player believes that her opponents will play at t what they played at $t-1$. In the example, Cournot dynamics cycles and this conclusion holds wherever the dynamics starts (except truthtelling) (see Figure 3.1). Besides, if the agents were to play the truthful

[11] If 1 learned according to a fictitious play rule with π_1 while 2 used π_2, then the sequence would enter a cycle for many values of $\pi_1 \geq .9, \pi_2 \geq .55$.

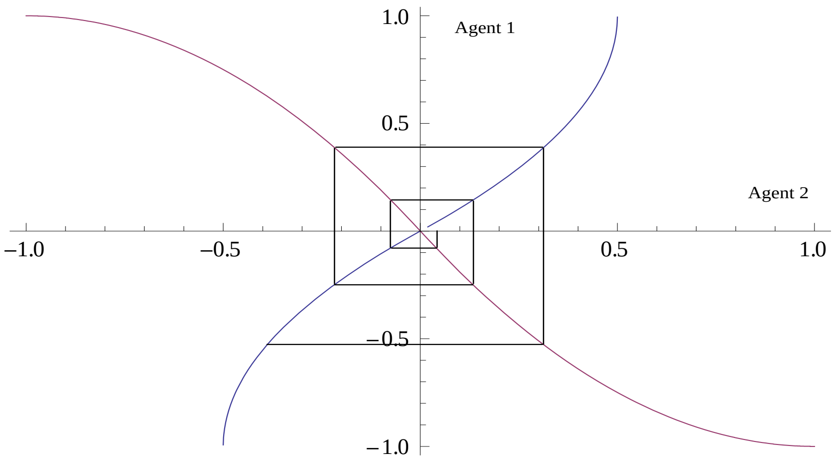

Figure 3.1: Cycling under Cournot Dynamics

equilibrium, the slightest belief perturbation would destabilize it under Cournot adjustment.

Given (3.1) is quadratic, the best-reply of an agent is always an affine function whose intercept varies in response to the opponent's strategy (the slope is always one). Figure 3.1 depicts the Cournot adjustment between two agents whose strategies are affine functions with slope one. The curves give the optimal choice of an intercept in response to the other agent's intercept. The downward-sloping curve is that of agent 1. The upward-sloping curve is that of agent 2 and should be understood as a function mapping 1's intercept (y-axis) to 2's intercept (x-axis). Clearly, Cournot dynamics cycles and this conclusion holds wherever the dynamics starts, except at zero, the truthtelling equilibrium. Cournot dynamics is prone to cycling, because the past only matters through the last period. But cycling prevails for many families of dynamics with a larger memory size, where for example players remember the last T periods and believe that a probability distribution over their opponents' past strategies best describe their future behavior.[12]

Learning also fails for other forms of learning dynamics than adaptive dynamics, such

[12]Consider dynamics where players remember the last T periods. They assign a probability π to the deception played at $t-1$ and $(1-\pi)\delta^k/C$ to that played at $t-k$ where C is normalized so that the probabilities add up to one. Simulations reveal that learning fails under many values of the parameters. Let $(\hat{\theta}_1^0(.), \hat{\theta}_2^0(.))$ be the pair of zero-functions. For $T \in \{2,3\}, \delta = .9$ and $\pi \geq .5$, the process enters a cycle even though the last few periods are weighted almost equally. This suggests that increasing the memory size may improve learning. For $T = 4, \delta = .8$, and $\pi \leq .65$, the profile converges to the truthful equilibrium, but it cycles for $\pi \geq .7$. But a larger memory does not necessarily improve learning, as cycling reappears when $T = \{5,6\}, \delta = .8$ for values of π below .65.

as the sophisticated learning dynamics à la Milgrom-Roberts [77].

Although strategic complementarities are not necessary for convergence, their absence clearly causes the learning failures in the example.

The theory I develop suggests to transform an existing mechanism into one which induces a supermodular game. The main insight is to use transfers to create complementarities between agents' announcements. The general transformation technique is simple and efficient. After transforming the mechanism, all adaptive and sophisticated dynamics converge to the truthful equilibrium, and the equilibrium is stable.

Consider the above two-agent environment and recall that truthtelling is a Bayesian equilibrium in the expected externality mechanism. Now player 1 could be subsidized if she accepts to change the value of the public good as 2 wishes, and taxed otherwise. From 1's point of view, 2 prefers large values of the public good when 2 reports large types on average, i.e. $E_{\theta_2}[\hat{\theta}_2(.)] \geq E_{\theta_2}[\theta_2]$. If 2 prefers small values, then the inequality is reversed. The new tax system could subsidize 1 if 1 reports large types when 2 does so, and tax 1 if 1 still reports large types when 2 does not. Possible transfers $t_1^{SM}(.)$ accomplishing this task are constructed by appending $\rho_1\hat{\theta}_1(\hat{\theta}_2 - E_{\theta_2}[\theta_2])$ to the current transfers, where ρ_1 is an arbitrary parameter capturing the punishment or reward intensity:

$$t_1^{SM}(\hat{\theta}) = E_{\theta_2}[t_1(\hat{\theta}_1, \theta_2)] + \rho_1\hat{\theta}_1(\hat{\theta}_2 - E_{\theta_2}[\theta_2]).$$

Agent 2's transfers are modified similarly with parameter ρ_2. The intuition is that there should be ρ_1 large enough such that, regardless of 1's original incentives, the reward (punishment) for (not) following 2 now is so high that 1 becomes willing to follow 2 along any learning dynamics. But by doing so, we actually created a supermodular mechanism. Note $\partial^2 t_1^{SM}(\hat{\theta})/\partial\hat{\theta}_1\partial\hat{\theta}_2 = \rho_1$. Thus, if $\partial^2 V_1(x_1(\hat{\theta}), \theta_1)/\partial\hat{\theta}_1\partial\hat{\theta}_2$ is bounded below, a condition called *bounded substitutes*,[13] then there is ρ_1 large enough such that

$$\frac{\partial^2 V_1(x_1(\hat{\theta}_1, \hat{\theta}_2), \theta_1)}{\partial\hat{\theta}_1\partial\hat{\theta}_2} + \frac{\partial^2 t_1^{SM}(\hat{\theta}_1, \hat{\theta}_2)}{\partial\hat{\theta}_1\partial\hat{\theta}_2} \geq 0, \text{ for all } \hat{\theta}, \theta_1. \tag{3.2}$$

A similar equation holds for agent 2, which implies that the Bayesian game induced by the

[13]This condition is satisfied in the present public goods example.

mechanism is supermodular.[14] Further, t_1^{SM} and t_1 have the same expectation when the opponents play truthfully: $E_{\theta_2}[t_1^{SM}(.,\theta_2)] = E_{\theta_2}[t_1(.,\theta_2)]$. Thus if 1's best-reply under t_1 was to tell the truth when 2 played truthfully, then it must be the case under t_1^{SM}. So truthtelling is an equilibrium after modifying the transfers.

In addition to its intuitive appeal, this technique can be powerful. Theorem 9 of Section 3.5.3 implies that there are values ρ_1 and ρ_2 for which truthtelling is the unique equilibrium of the supermodular mechanism in this example. All adaptive dynamics now converge to the truthful equilibrium, and the equilibrium is stable.

3.3 Lattice-Theoretic Definitions and Supermodular Games

The basic definitions of lattice theory in this section are discussed in Milgrom-Roberts [76] and Topkis [?].

A set M with a transitive, reflexive, antisymmetric binary relation $\succeq$ is a *lattice* if for any $x, y \in M$, $x \vee y \equiv \sup_M\{x, y\}$ and $x \wedge y \equiv \inf_M\{x, y\}$ exist. It is *complete* if for every non-empty subset A of M, $\inf_M A$ and $\sup_M A$ exist. A nonempty subset A of M is a *sublattice* if for all $x, y \in A$, $x \vee y, x \wedge y \in A$. A *closed interval* $[x, y]$ in M is the set of $m \in M$ such that $y \succeq m \succeq x$. The *order-interval* topology on a lattice is the topology whose subbasis for the closed sets is the set of closed intervals. All lattices are endowed with their order-interval topology. In Euclidean spaces the order-interval topology coincides with the usual topology.

Let T be a partially ordered set; $g : M \to \mathbb{R}$ is *supermodular* if, for all $m, m' \in M$, $g(m)+g(m') \leq g(m\wedge m')+g(m\vee m')$; $g : M\times T \to \mathbb{R}$ has *increasing (decreasing) differences* in (m, t) if, whenever $m \succeq m'$ and $t \succeq t'$, $g(m, t) - g(m', t) \geq (\leq) g(m, t') - g(m', t')$; $g : M \times T \to \mathbb{R}$ satisfies the *single-crossing property* in (m, t) if, whenever $m \succeq m'$ and $t \succeq t'$, $g(m'', t') \geq g(m', t')$ implies $g(m'', t'') \geq g(m', t'')$ and $g(m'', t') > g(m', t')$ implies $g(m'', t'') > g(m', t'')$. If g has decreasing differences in (m, t), then variables m and t are said to be substitutes. If g has increasing differences or satisfies the single-crossing property in (m, t), then m and t are said to be complements.

A game is described by a tuple $(N, \{(M_i, \succeq_i)\}, u)$, where N is a finite set of players; each

[14] If the complete information payoffs define a supermodular game for each $\theta \in \Theta$, then the (ex-ante) Bayesian game is supermodular. Loosely speaking, supermodular games are characterized by utility functions whose cross-partial derivatives are positive.

$i \in N$ has a strategy space M_i with an order $\succeq_i$ and a payoff function $u_i : \prod_{i \in N} M_i \to \mathbb{R}$; and $u = (u_i)$.

Definition **8** *A game $\mathcal{G} = (N, \{(M_i, \succeq_i)\}, u)$ is supermodular if for all $i \in N$,*

1. *$(M_i, \succeq_i)$ is a complete lattice;*
2. *u_i is bounded, supermodular in m_i for each m_{-i}, and has increasing differences in (m_i, m_{-i});*
3. *u_i is upper-semicontinuous in m_i for each m_{-i}, and continuous in m_{-i} for each m_i.*

3.4 Supermodular Implementation: The Framework

Let $N = \{1, \ldots n\}$ denote a collection of agents. A planner faces a measurable set Y of alternatives with generic element $y \in Y$. For each agent $i \in N$, let Θ_i be the measurable space of i's possible types. Let $\Theta_{-i} = \prod_{j \neq i} \Theta_j$. Agents have a common prior ϕ on Θ known to the planner. The planner's desired outcomes are represented by a measurable social choice function $f : \Theta \to Y$.

A mechanism is a tuple $\Gamma = (\{(M_i, \succeq_i)\}, g)$ where each agent i's message space M_i is endowed with an order $\succeq_i$ and is a measurable space; $g : M \to Y$ is a measurable outcome function. A strategy for agent i is a measurable function $m_i : \Theta_i \to M_i$. Denote by $\Sigma_i(M_i)$ the set of equivalence classes of measurable functions from $(\Theta_i, \mathcal{F}_i)$ to M_i. This set is endowed with the pointwise order, also denoted $\succeq_i$. A direct mechanism is one for which each $M_i = \Theta_i$ and $g = f$. In this case, $\Sigma_i(\Theta_i)$ is called the set of i's deceptions and its elements are denoted $\hat{\theta}_i(.)$. Direct mechanisms vary by the order on type spaces.

Each agent i's preferences over alternatives are given by a measurable utility function $u_i : Y \times \Theta_i \to \mathbb{R}$. These utility functions are uniformly bounded by some $\overline{u}$. For $m_{-i} \in \prod_{j \neq i} M_j$, agent i's preferences over messages in M_i are given by her ex-post payoffs $u_i(g(m_i, m_{-i}), \theta_i)$. Agent i's ex-ante payoffs are defined as $u_i^g(m_i(.), m_{-i}(.)) = E_\theta[u_i(g(m_i(\theta_i), m_{-i}(\theta_{-i}), \theta_i)]$ for any profile $m(.)$, where $E_\theta[.]$ is the expectation with respect to ϕ.

There are three stages at which it is relevant to formulate the game induced by mechanism Γ: Ex-ante, interim, and ex-post (complete information). This chapter mostly adopts an ex-ante perspective, as the objective is that the ex-ante induced game $\mathcal{G} = (N, \{(\Sigma_i(M_i), \succeq_i)\}, u^g)$ be supermodular (see Section 3.8.2). However, if message sets are compact sublattices of some Euclidean space, then a sufficient condition for $\mathcal{G}$ to be supermodular is that the complete information game induced by Γ be supermodular for every profile of true types. This explains the next definitions. If a scf is Bayesian implementable with a mechanism that always induces an ex-post supermodular game, then it is supermodular implementable.

DEFINITION **9** *The mechanism Γ supermodularly implements the scf $f(.)$ if there exists a Bayesian equilibrium $m^*(.)$ such that $g(m^*(\theta)) = f(\theta)$ for all $\theta \in \Theta$, and if the induced game $\mathcal{G}(\theta) = (N, \{(M_i, \succeq_i)\}, u(g(.), \theta))$ is supermodular for all $\theta \in \Theta$. The scf f is said to be supermodular implementable.*[15]

DEFINITION **10** *A scf is truthfully supermodular implementable if there exists a direct mechanism that supermodularly implements the scf $f(.)$ such that $\hat{\theta}(\theta) = \theta$ for all $\theta \in \Theta$ is a Bayesian equilibrium.*

Since I am mostly concerned with direct Bayesian mechanisms, I often omit the qualifications of "truthful(ly)" and "Bayesian."

3.5 Supermodular Implementation on Quasilinear Domains

This section deals with supermodular implementation when agents have quasilinear utility functions. The objective is to give general conditions under which a scf is supermodular implementable and the mechanism satisfies some further requirements. There are four main results. The first provides general conditions for supermodular implementability. The second answers the question of supermodular implementation and budget balancing. The third gives sufficient conditions for a scf to be supermodular implementable with a game whose interval between extremal equilibria is the smallest possible. The fourth offers sufficient conditions for supermodular implementability in unique equilibrium.

[15]Definitions 9 and 10 are also simplifying definitions. It is sufficient but not necessary that $\mathcal{G}(\theta)$ be supermodular for each θ in order for the ex-ante Bayesian game to be supermodular. For example, if the prior is mostly concentrated on some subset Θ' of Θ, it may not be necessary to make the ex-post payoffs supermodular for types in $\Theta \backslash \Theta'$. Of course, the possibility of neglecting $\Theta \backslash \Theta'$ depends on how unlikely that set is compared to how submodular the utility function may be for types in that set.

3.5.1 Environment and Definitions

An alternative y is a vector $(x, t_1, \ldots, t_n)$ where x is an element of a compact set $X \subset \mathbb{R}^m$ and $t_i \in \mathbb{R}$ for all i. Each agent i has a compact type space $\Theta_i \subset \mathbb{R}$ (finite or infinite). Endow Θ_i with the usual order.[16]

Let X_i be a compact subset of $\mathbb{R}^{m_i}$ such that $\prod_{i \in N} X_i = X$. In this environment, a scf $f = (x, t)$ is composed of a decision rule $x : \Theta \mapsto (x_i(\theta))$ where $x_i : \Theta \to X_i$ and transfer functions $t_i : \Theta \to \mathbb{R}$.

For all i, preferences are quasilinear with utility function $u_i(x, \theta_i) = V_i(x_i, \theta_i) + t_i$. The function $V_i : X_i \times \Theta_i \to \mathbb{R}$ is called i's valuation and the vector of those valuations is denoted V.

Agents' types are assumed to be independently distributed. For all i, the distribution of i's types admits a bounded density with full support.

A scf is *supermodular implementable* if truthtelling is a Bayesian equilibrium of the supermodular game induced by the direct mechanism.

The next definitions describe conditions on the composition of the valuation functions and the decision rule.

The valuation functions and the decision rule form a *continuous family* if V is bounded and for all i and θ_i, $V_i(x_i(\hat{\theta}), \theta_i)$ is continuous in $\hat{\theta}_{-i}$ for fixed $\hat{\theta}_i$, and $V_i(x_i(\hat{\theta}), \theta_i)$ is upper-semicontinuous in $\hat{\theta}_i$ for fixed $\hat{\theta}_{-i}$.

The valuation functions and the decision rule are (twice) continuously differentiable if for all i, there exist open sets $O_i \supset \Theta_i$ and $U_i \supset X_i$, such that $V_i : U_i \times O_i \to \mathbb{R}$ and $x_i : \prod_{i \in N} O_i \to U_i$ are (twice) continuously differentiable.

For any θ_i'' and θ_i', let $\Delta V_i(\theta_i'', \theta_i', \theta_{-i}, \theta_i) = V_i(x_i(\theta_i'', \theta_{-i}), \theta_i) - V_i(x_i(\theta_i', \theta_{-i}), \theta_i)$. The same notation is used for the utility functions.

Say that the valuations and the decision rule have *bounded substitutes* if, for each i and θ_i, there is a real number $T_i(\theta_i)$ such that $\Delta V_i(\theta_i'', \theta_i', \theta_{-i}'', \theta_i) - \Delta V_i(\theta_i'', \theta_i', \theta_{-i}', \theta_i) \geq T_i(\theta_i)(\theta_i'' - \theta_i') \sum_{j \neq i}(\theta_j'' - \theta_j')$ for all $\theta_i'' \geq \theta_i'$ and $\theta_{-i}'' \geq \theta_{-i}'$. Substitutes are *uniformly bounded* if the lower bound $T_i(.)$ can be chosen to be constant across types.

In twice-differentiable environments, this condition is equivalent to the existence of a

[16]Notice that $\Sigma_i(\Theta_i)$ is a complete lattice with the pointwise order (see Lemma 1 in Van Zandt [104]).

uniform lower bound on the cross-partial derivatives. Bounded substitutes mean that, if agents' announcements are strategic substitutes in the game with no transfers,[17] then at least there is a bound on the negative magnitude of these cross-partial derivatives. This assumption is always satisfied when type sets are finite and whenever the decision rule and the valuations are twice-continuously differentiable on compact type sets.

The valuations and the decision rule have *strong differences* if for each i and θ_i, there is $\gamma_i(\theta_i)$ such that for all $\hat{\theta}_i'' \geq \hat{\theta}_i'$ and $\theta_i'' \geq \theta_i'$, $\Delta V_i(\hat{\theta}_i'', \hat{\theta}_i', \theta_{-i}, \theta_i'') - \Delta V_i(\hat{\theta}_i'', \hat{\theta}_i', \theta_{-i}, \theta_i') \geq \gamma_i(\theta_i)(\hat{\theta}_i'' - \hat{\theta}_i')(\theta_i'' - \theta_i')$.

The utility functions and the scf generate *bounded complements* if for each i and θ_i, there is a real number $K_i(\theta_i)$ such that $\Delta u_i(\theta_i'', \theta_i', \theta_{-i}'', \theta_i) - \Delta u_i(\theta_i'', \theta_i', \theta_{-i}', \theta_i) \leq K_i(\theta_i)(\theta_i'' - \theta_i') \sum_{j \neq i}(\theta_j'' - \theta_j')$ for all $\theta_i'' \geq \theta_i'$ and $\theta_{-i}'' \geq \theta_{-i}'$. Complements are *uniformly bounded* if there is a uniform upper bound K_i. These definitions apply similarly to valuation functions.

Note that the conditions of bounded substitutes and complements, and strong differences are simple bounds on derivatives, generalized to hold in non-differentiable environments.

3.5.2 General Result and Implementation with Budget Balance

This subsection contains two main results. According to the first theorem, if the scf and the utility functions are relatively well-behaved, in the sense of continuous families and bounded substitutes, then a decision rule is implementable with transfers if and only if it is supermodular implementable with transfers. The second theorem provides sufficient conditions to satisfy budget balancing.

Theorem **5** *Let decision rule and the valuation functions form a continuous family with uniformly bounded substitutes. There exist transfers t such that $f = (x, t)$ is implementable and $E_{\theta_{-i}}[t_i(., \theta_{-i})]$ is upper-semicontinuous, if and only if, there are transfers t^{SM} such that (x, t^{SM}) is supermodular implementable and $E_{\theta_{-i}}[t_i^{SM}(., \theta_{-i})]$ is upper-semicontinuous. Moreover, transfers t_i and t_i^{SM} have the same expected value.*

Proof: Sufficiency is immediate. So suppose that $f = (x, t)$ is Bayesian implementable and transfers t are such that $E_{\theta_{-i}}[t_i(., \theta_{-i})]$ is usc for all i. Then,

$$E_{\theta_{-i}}[V_i(x_i(\theta_i, \theta_{-i}), \theta_i)] + E_{\theta_{-i}}[t_i(\theta_i, \theta_{-i})] \geq E_{\theta_{-i}}[V_i(x_i(\hat{\theta}_i, \theta_{-i}), \theta_i)] + E_{\theta_{-i}}[t_i(\hat{\theta}_i, \theta_{-i})] \quad (3.3)$$

[17] That is $\partial^2 V_i(x_i(\hat{\theta}), \theta_i)/\partial \hat{\theta}_i \partial \hat{\theta}_j < 0$ (Section 3.3).

for all $\hat{\theta}_i$. For $\rho_i \in \mathbb{R}$, let

$$\delta_i(\hat{\theta}_i, \hat{\theta}_{-i}) = \sum_{j \neq i} \rho_i \hat{\theta}_i \hat{\theta}_j, \tag{3.4}$$

and define

$$t_i^{SM}(\hat{\theta}_i, \hat{\theta}_{-i}) = \delta_i(\hat{\theta}_i, \hat{\theta}_{-i}) + E_{\theta_{-i}}[t_i(\hat{\theta}_i, \theta_{-i})] - E_{\theta_{-i}}[\delta_i(\hat{\theta}_i, \theta_{-i})]. \tag{3.5}$$

Note that transfers t_i and t_i^{SM} have the same expected value: $E_{\theta_{-i}}[t_i^{SM}(., \theta_{-i})] = E_{\theta_{-i}}[t_i(., \theta_{-i})]$. Thus (x, t^{SM}) is Bayesian implementable by (3.3). Moreover, $\delta_i : \Theta \rightarrow \mathbb{R}$ is continuous and bounded. So it follows from the bounded convergence theorem that $E_\theta[\delta_i(\hat{\theta}_i(\theta_i), \hat{\theta}_{-i}(\theta_{-i})) - E_{\theta_{-i}}[\delta_i(\hat{\theta}_i(\theta_i), \theta_{-i})]]$ is continuous in $\hat{\theta}(.)$. Since transfers t are such that $E_{\theta_{-i}}[t_i(., \theta_{-i})]$ is usc, Fatou's lemma implies that $E_\theta[t_i^{SM}(\hat{\theta}_i(\theta_i), \hat{\theta}_{-i}(\theta_{-i}))]$ is usc in $\hat{\theta}_i(.)$ for each $\hat{\theta}_{-i}(.)$. Therefore, payoffs u_i^f satisfy the continuity requirements for supermodular games. Next I show that it is possible to choose ρ_i so that u_i^f has increasing differences in $(\hat{\theta}_i(.), \hat{\theta}_{-i}(.))$. Since substitutes are uniformly bounded, there exists T_i such that, for all $\theta_i'' \geq \theta_i'$ and $\theta_{-i}'' \geq \theta_{-i}'$, $\Delta V_i(\theta_i'', \theta_i', \theta_{-i}'', \theta_i) - \Delta V_i(\theta_i'', \theta_i', \theta_{-i}', \theta_i) \geq T_i(\theta_i'' - \theta_i') \sum(\theta_j'' - \theta_j')$ for all $\theta_i \in \Theta_i$. Set $\rho_i > -T_i$. Choose any $\theta_i'' \geq_i \theta_i'$ and $\theta_{-i}'' \geq_{-i} \theta_{-i}'$. The function $u_i(x_i(\hat{\theta}_i, \hat{\theta}_{-i}), \theta_i)$ has increasing differences in $(\hat{\theta}_i, \hat{\theta}_{-i})$ for each θ_i, if the following expression is positive for all θ_i,

$$\Delta V_i(\theta_i'', \theta_i', \theta_{-i}'', \theta_i) - \Delta V_i(\theta_i'', \theta_i', \theta_{-i}', \theta_i) + \sum_{j \neq i} \rho_i \big(\theta_i'' \theta_j'' + \theta_i' \theta_j' - \theta_i'' \theta_j' - \theta_i' \theta_j''\big). \tag{3.6}$$

Given $\rho_i > -T_i$, (3.6) is greater than

$$\Delta V_i(\theta_i'', \theta_i', \theta_{-i}'', \theta_i) - \Delta V_i(\theta_i'', \theta_i', \theta_{-i}', \theta_i) - T_i \sum_{j \neq i} (\theta_i'' - \theta_i')(\theta_j'' - \theta_j'). \tag{3.7}$$

Bounded substitutes immediately imply that (3.7) is positive for all θ_i, hence so is (3.6). By Lemma 5, the utility function u_i^f has increasing differences in $(\hat{\theta}_i(.), \hat{\theta}_{-i}(.))$. Finally, since Θ_i is a chain, Lemma 5 implies u_i^f is supermodular in $\hat{\theta}_i(.)$. Q.E.D

Theorem 5 shows that the class of implementable scf that can be supermodularly implemented in Bayesian equilibrium is large, as there are only mild boundedness and continuity conditions on the utility functions and the scf. The transfers are at the heart of the result: It is always possible to add complementarities into the transfers without affecting the incentives that appear in the expected value.

Remark. Since players receive the same expected utility in equilibrium from (x,t) and (x,t^{SM}), if (x,t) satisfies some ex-ante or interim participation constraints, then so does (x,t^{SM}).

Recall that, if type spaces are finite or if the valuations and the decision rule are twice-continuously differentiable on compact type sets, then the assumptions of uniformly bounded substitutes and continuity are satisfied. This leads to the following important corollaries which cover many cases of interest.

Corollary **3** *Let type spaces be finite. For any valuation functions, if the scf* $f=(x,t)$ *is implementable, then there exist transfers* t^{SM} *such that* (x,t^{SM}) *is supermodular implementable.*

Corollary **4** *Let* $f=(x,t)$ *be an implementable scf such that* $E_{\theta_{-i}}[t_i(.,\theta_{-i})]$ *is upper-semicontinuous. If the decision rule and the valuations are twice-continuously differentiable, then there exist transfers* t^{SM} *such that* (x,t^{SM}) *is supermodular implementable.*

The previous results state conditions that apply to Bayesian implementable scf. In some instances it may not be obvious whether the decision rule admits implementing transfers whose expected value is usc. Standard implementation results in differentiable environments demonstrate that the expected value of the transfers in an implementable scf takes an explicit form.[18] This leads to the next proposition.

Proposition **6** *Let type spaces be intervals. Let the decision rule and the valuations form a continuous family with uniformly bounded substitutes. If* $E_{\theta_{-i}}[V_i(x_i(\hat{\theta}_i,\,\theta_{-i}),\theta_i)]$ *is continuous in* $(\hat{\theta}_i,\theta_i)$ *and* $\partial E_{\theta_{-i}}[V_i(x_i(\hat{\theta}_i,\theta_{-i}),\theta_i)]/\partial\theta_i$ *is increasing in* $\hat{\theta}_i$*, then there are transfers* t^{SM} *such that* (x,t^{SM}) *is supermodular implementable.*

To identify supermodular implementable decision rules, Proposition 6 suggests to choose those rules that lead each agent i's expected marginal valuation to be nondecreasing. By Theorem 5 and Proposition 12 in Appendix B, any such rule is supermodular implementable with transfers t^{SM}, combining (3.5) and (3.13).

[18]See, e.g., Proposition 23.D.2 in Mas Colell et al. [68] for linear utility functions.

The rest of this section investigates supermodular implementation under the budget balance condition. In some design problems, the planner should not realize a net gain from the mechanism. While the planner cannot sustain deficits, full efficiency requires there be no waste of numéraire. A scf is *fully efficient* if it maximizes the sum of the utility functions (not only the valuation functions) subject to the feasibility constraint $\sum t_i \leq 0$. So the transfers must add up to zero for each vector of true types. However, complementarities between agents' announcements may be irreconcilable with budget balancing, as shown in the next example.

Example **2** Consider the public goods example of Section 3.2. In this example, if there exist transfers $\{t_i^{SM}(.)\}_{i=1,2}$ such that the resulting scf (x, t^{SM}) is supermodular implementable, then inequality (3.2) must hold for both agents. That is, the cross-partial derivatives of $t_1(\hat{\theta})$ must be greater than 2 and the cross-partial derivatives of $t_2(\hat{\theta})$ must be greater than -1; hence their sum will be strictly greater than 0. The budget balance condition requires $\sum_{i=1,2} t_i(\hat{\theta}) = 0$, so the sum of the cross-partial derivatives of the transfers must be null. As a result, budget balancing must be violated in this example if there is supermodular implementation.

This example points to the difficulty of balancing budget in some situations with two players. The next theorem provides sufficient conditions for a scf to be supermodular implementable using balanced transfers. Say that a decision rule x is *allocation-efficient*, if $x(\theta) \in \text{argmax}_{x \in X} \sum_{i \in N} V_i(x_i, \theta_i)$ for all $\theta \in \Theta$. Basically, if substitutes are bounded, any allocation-efficient decision rule can be paired with a transfer scheme to give a fully efficient supermodular-implementable scf.

Theorem **6** *Let $n \geq 3$. Let valuation functions and the decision rule form a continuous family with uniformly bounded substitutes. If the decision rule is allocation-efficient, then there are balanced transfers t^{BB} such that (x, t^{BB}) is supermodular implementable.*

The proof appears in Appendix B and it is constructive. Transfers t^{BB} correspond to a transformation of the transfers in the expected externality mechanism, and they rely on two observations. First, any player's transfer in the expected externality mechanism displays no complementarities or substitutes, because transfers are separable in announcements. Second, there is a transformation of these transfers similar to that in Theorem 5 that enables to add complementarities while preserving incentives and budget balancing. The key observation is that the transfers from Theorem 5 add complementarity between agents' announcements in a pairwise fashion. As soon as there is a third agent, it is possible to subtract from each individual's transfer those complementarities that come from the other agents' transfers and that do not concern that individual, thus balancing the whole system.

Theorem 6 can be modified to apply to situations where, for every realization of types, enough taxes need to be raised to pay the cost of x. This constraint takes the form $\sum_{i \in N} t_i(\theta) \geq C(x(\theta))$ for all θ, where C is the cost function mapping X into $\mathbb{R}^+$.[19]

3.5.3 Optimal and Unique Supermodular Implementation

This subsection deals with the multiple equilibrium problem in supermodular implementation. Even if a mechanism has an equilibrium outcome with some desirable property, it may have other equilibrium outcomes that are undesirable. The concept of supermodular implementation relies on weak implementation: For direct mechanisms, only the truthful equilibrium is known to have the desired outcome. It follows from [76] that adaptive dynamics lead to play between the greatest and the least equilibrium, so players may learn to play an untruthful equilibrium associated with a bad outcome. Therefore, it is important to minimize the size of the interval between the extremal equilibria, called the interval prediction, and to take the number of equilibria into consideration. If the interval prediction is small, then learning leads to a profile close to truthtelling and to the desired outcome. For these reasons, supermodular implementation is particularly powerful when truth-revealing is the unique equilibrium.

3.5.3.1 Optimal Implementation

I begin with an example that explains the foundations of this section.

Example **3** Consider the public goods example of Section 3.2. Suppose that transfers are defined as $t_i(\hat{\theta}) = \rho_i \hat{\theta}_i \hat{\theta}_j + E_{\theta_j}[t_i(\hat{\theta}_i, \theta_j)] - \rho_i \hat{\theta}_i E_{\theta_j}[\theta_j]$ for $i = 1, 2$ and $j \neq i$, where t_i is given by the expected externality mechanism. If $\rho_1 = 2\frac{1}{2}$ and $\rho_2 = -1/2$, the game induced by

[19] An additional sufficient condition to apply the theorem is that $C(.)$ and $x(.)$ produce bounded substitutes. See, e.g., Lemma 2 in Ledyard and Palfrey [64] for transfers satisfying this budget balance condition. Note that these transfers are separable in types except (possibly) for $C(x(\theta))$, so there are no complementarities or substitutes beyond those contained in $C(x(\theta))$.

the mechanism is supermodular and truthtelling is the unique Bayesian equilibrium (see Example 4). For $\rho_1 = 3\frac{1}{5}$ and $\rho_2 = 1/2$, the supermodular game induced by the mechanism has now a smallest and a largest equilibrium. In the smallest equilibrium, agent 1 announces 0 for any type below $c_1 \approx 0.47$ and $\theta_1 - c_1$ for types above, and agent 2 announces 0 for any type below $c_2 \approx 0.55$ and $\theta_2 - c_2$ for types above. In the largest equilibrium, agent 1 announces $\theta_1 + c_1$ for any type below $1 - c_1$ and 1 for types above, and agent 2 announces $\theta_2 + c_2$ for any type below $1 - c_2$ and 1 for types above. Moreover, increasing ρ_1 to 4 and ρ_2 to 1 produces extremal equilibria with $c_1 = c_2 = 1$ and $c_1 = c_2 = 0$; the smallest equilibrium is the smallest profile of the entire space where each agent always announces her smallest type, and the largest equilibrium is the largest profile of the entire space where each agent always announces her largest type. Increasing ρ_1 and ρ_2 has had three negative consequences: *i*) By increasing these parameters above, respectively, 5/2 and -1/2, we have generated two new equilibria. By increasing them more, *ii*) we have enlarged the size of the interval prediction to be the whole space, so the Milgrom-Roberts theorem is of little help now; *iii*) the truthful equilibrium has become locally unstable.

Before presenting the formal definitions and the results, I discuss some new concepts. Think of the degree of complementarity between the variables of a function as given by its cross-partial derivatives. Large cross-partials mean that the degree of complementarity is high, and vice-versa. In Example 3, the transfers produce more complementarities as ρ_i increases. *Optimal supermodular implementation* involves designing a mechanism whose induced supermodular game has the weakest complementarities among supermodular mechanisms. The rationale behind optimal supermodular implementation is clear from Example 3. First, it is the best compromise between learning, stability, and multiplicity of equilibria. Adding complementarities improves learning and stability, but too much complementarity may yield untruthful equilibria. Second, optimal supermodular implementation provides the tightest interval prediction around the truthful equilibrium (Proposition 7). This is hinted at by Example 3, because the extremal equilibria move apart as the degree of complementarity increases.

Next I define those concepts formally and I prove the claim that relates the size of interval prediction to the degree of complementarity. As mentioned above, the cross-partial derivatives offer a way of measuring complementarities in twice-differentiable environments.

It is natural to say that a transfer function $\tilde{t}$ generates larger complementarities than t, denoted $\tilde{t} \succeq_{\text{ID}} t$, if $\partial^2 \tilde{t}_i(\hat{\theta})/\partial\hat{\theta}_i\partial\hat{\theta}_j \geq \partial^2 t_i(\hat{\theta})/\partial\hat{\theta}_i\partial\hat{\theta}_j$ for all $\hat{\theta}$, j, and i. The next definition formalizes this idea and extends it to non-differentiable transfers.

Definition **11** *Define the ordering relation $\succeq_{ID}$ on the space of transfer functions such that $\tilde{t} \succeq_{ID} t$ if, for all $i \in N$ and for all $\theta''_i > \theta'_i$ and $\theta''_{-i} >_{-i} \theta'_{-i}$, $\tilde{t}_i(\theta''_i, \theta''_{-i}) - \tilde{t}_i(\theta''_i, \theta'_{-i}) - \tilde{t}_i(\theta'_i, \theta''_{-i}) + \tilde{t}_i(\theta'_i, \theta'_{-i}) \geq t_i(\theta''_i, \theta''_{-i}) - t_i(\theta''_i, \theta'_{-i}) - t_i(\theta'_i, \theta''_{-i}) + t_i(\theta'_i, \theta'_{-i})$.*

For twice-differentiable transfers, this definition is equivalent to the condition that the cross-partial derivatives of each $\tilde{t}_i$ are larger than those of t_i.

While $\succeq_{\text{ID}}$ is transitive and reflexive on the space of transfer functions, it is not anti-symmetric. Consider the set of $\succeq_{\text{ID}}$-equivalence classes of transfers, denoted $\mathcal{T}$.[20]

The next proposition shows that if a transfer function generates more complementarities than another transfer function, then it induces a game whose interval prediction is larger than the interval prediction of the game induced by the other transfer. This result is also interesting for the theory of supermodular games, as it relates the degree of complementarity to the size of the interval prediction.[21]

For any $t \in \mathcal{T}$ and supermodular implementable $f = (x, t)$, let $\overline{\theta}^t(.)$ and $\underline{\theta}^t(.)$ denote the extremal equilibria of the induced game.

Proposition **7** *Let the decision rule and the valuation functions be such that $E_{\theta_{-i}}[V_i (x_i(\hat{\theta}_i, \theta_{-i}), \theta_i)]$ is continuous in $(\hat{\theta}_i, \theta_i)$. For any supermodular implementable scf (x, t'') and (x, t') with $t'', t' \in \mathcal{T}$, if $t'' \succeq_{ID} t'$, then $[\underline{\theta}^{t'}(.), \overline{\theta}^{t'}(.)] \subset [\underline{\theta}^{t''}(.), \overline{\theta}^{t''}(.)]$.*

This proposition provides the foundation for the next definition. If a scf is supermodular implementable and its transfers generate the weakest complementarities, then it is optimally supermodular implementable. This gives the tightest interval prediction around the truthful equilibrium.

Definition **12** *A scf $f = (x, t^*)$ is optimally supermodular implementable if it is supermodular implementable and $t \succeq_{ID} t^*$ for all transfers $t \in \mathcal{T}$ such that (x, t) is supermodular implementable.*

The next result determines which decision rules are optimally supermodular implementable. The result uses the following property of decision rules. A decision rule x :

[20] Any quasi-order is transformed into a partially ordered set using equivalence classes.

[21] See Milgrom and Roberts [78] (pp.189–190) for a related result.

$\Theta \mapsto (x_i(\theta))$ is *dimensionally reducible* if, for each $i \in N$, there are twice-continuously differentiable functions $h_i : \mathbb{R}^2 \to X_i$ and $r_i : \Theta_{-i} \to \mathbb{R}$ such that $r_i(.)$ is increasing and $x_i(\theta) = h_i(\theta_i, r_i(\theta_{-i}))$ for all $\theta \in \Theta$. The condition is trivially true when there are two individuals. If there are more than two, a player's decision rule can depend on her own type directly, but it must depend on her opponents' types indirectly through a real-valued aggregate. Taking types in $[0, 1]$, it excludes, for example, x for which $x_1(\theta) = \theta_1^{\theta_2\theta_3} + \theta_1 + \theta_2 + \theta_3$.[22]

THEOREM **7** *Let the valuation functions be twice-continuously differentiable and $f = (x, t)$ be a scf whose decision rule is dimensionally reducible. If f is implementable, then there are transfers t^* such that (x, t^*) is optimally supermodular implementable.*

The theorem gives a powerful conclusion: In twice-continuously differentiable environments, all implementable scf whose decision rule satisfies the dimensionality condition admit a supermodular mechanism with the tightest equilibrium set.

3.5.3.2 Unique Implementation and Full Efficiency

After providing conditions for the smallest interval prediction, it is natural to study situations where truthtelling is the unique equilibrium of the induced supermodular game. All learning dynamics then converge to the equilibrium. This is the concept of *unique supermodular implementation.* As a by-product, it implies coalition-proof Nash implementation by Milgrom and Roberts [79].

This section also supports what appears to be a conflict between full efficiency and learning. Example 2 already delivered the message: Sometimes the designer must sacrifice either learning or efficiency; either she modifies the expected externality mechanism and secures learning at the price of a balanced budget (full efficiency), or she loses the strong learning properties by balancing budget via the expected externality mechanism.

DEFINITION **13** *A scf $f = (x, t)$ is uniquely supermodular implementable if it is supermodular implementable and the truthful equilibrium is the unique Bayesian equilibrium.*

The main result (Theorem 8) gives sufficient conditions for a scf to be uniquely supermodular implementable. If the conditions fail, the theorem provides a simple way to compute bounds on the equilibrium set.

[22]To see why, $h_1(\theta_1, r_1) = \theta_1^{z(r_1)} + \theta_1 + r_1$ for some $z : \mathbb{R} \to \mathbb{R}$ and $r_1(\theta_{-1}) = \theta_2 + \theta_3$. But there is no z such that $z(\theta_2 + \theta_3) = \theta_2\theta_3$ for all θ_{-1}, because $z(0 + 1) \neq z(.5 + .5)$.

The theorem imposes a condition on the *matrix of complementarities*. Assuming bounded complements and strong differences (see Section 3.5.1), the matrix of complementarities is the $n \times n$ matrix whose i-th row contains $(n-1)$ times the element $E_{\theta_i}[K_i(\theta_i)]$ and has $-E_{\theta_{-i}}[\gamma_i(\theta_{-i})]$ as its i-th element.

THEOREM **8** *Let $f = (x, t)$ be a supermodular implementable scf. Let the valuations and the decision rule be continuously differentiable. Assume bounded complements and strong differences.*

1. *If the matrix of complementarities is negative-definite, then f is uniquely supermodular implementable.*
2. *There exist two systems of $2n$ equations whose extremal solutions bound the equilibrium set.*

The matrix of complementarities indicates how sensitive players are as a whole to their own type versus their opponents' announcements. On the one hand, when the complementarities between own announcement and type are strong (large $E[\gamma]$), players tend to announce high types regardless of their opponents' strategies. This favors uniqueness. On the other hand, when the complementarities between players' announcements are strong (large $E[K]$), it is source of multiplicity. The dominating effect is captured by the definiteness of the matrix of complementarities. If it is negative definite, the first effect is stronger, hence there is uniqueness. For example, if the sum of the entries on each row is negative, then the matrix is negative-definite.

The theorem also provides a way to bound the equilibrium set, which is useful when the uniqueness condition fails. Finding these bounds simply involves solving a system of equations. This system is formed exclusively from the primitives of the model. The ability to compute bounds on the equilibrium set is attractive for welfare analysis, as it becomes possible to measure the loss in efficiency caused by learning.

The focus of the next proposition is on optimal transfers and unique implementation. Optimal transfers produce the smallest interval prediction, so a natural question to ask is when they actually lead to unique implementation. Unlike Theorem 8, Proposition 8 has the advantage not to involve conditions on transfers. So one has information on the size of the equilibrium set beforehand.

Proposition **8** *Let the valuation functions be twice-continuously differentiable, and let $f = (x, t)$ be a scf with a dimensionally reducible decision rule. Assume strong differences. Letting*

$$K_i(\theta_i) = \max_{j \neq i} \max_{\hat{\theta} \in \Theta} \left(\frac{\partial^2 V_i(x_i(\hat{\theta}), \theta_i)}{\partial \hat{\theta}_i \partial \hat{\theta}_j} - \min_{\theta_i \in \Theta_i} \frac{\partial^2 V_i(x_i(\hat{\theta}), \theta_i)}{\partial \hat{\theta}_i \partial \hat{\theta}_j} \right), \tag{3.8}$$

if the resulting matrix of complementarities is negative-definite, then (x, t^*) is uniquely supermodular implementable.

Once the bound on complementarities (3.8) is computed for the optimal transfers, then Theorem 8 can be used to establish uniqueness or to get an idea of the size of the equilibrium set under optimal transfers (see Example 7).

The following examples show that there are cases where it is straightforward to apply the previous results. In Example 5, the original mechanism induces a non-supermodular game with multiple equilibria where the truthful equilibrium is unstable. Nonetheless this mechanism can be turned into a supermodular mechanism with a unique equilibrium. Interestingly, this also illustrates how weak implementation can be turned into strong implementation.

Example **4** Consider the public goods example of Section 3.2. Recall that agents' valuation functions are $V_1(x, \theta_1) = \theta_1 x - x^2$ and $V_2(x, \theta_2) = \theta_2 x + x^2/2$. The decision rule is $x(\theta) = \theta_1 + \theta_2$. Since $\partial x_i(\theta)/\partial \theta_i = 1$ and $\partial^2 V_i(x, \theta_i)/\partial x \partial \theta_i = 1$ for $i = 1, 2$, it implies $\gamma_i(\theta_{-i}) = 1$, $i = 1, 2$. Moreover, $\partial^2 V_i(x(\hat{\theta}), \theta_i)/\partial \hat{\theta}_1 \hat{\theta}_2 = -2$ if $i = 1$ and 1 if $i = 2$. By Proposition 8, $K_i(\theta_i) = 0$ for $i = 1, 2$, so the resulting matrix of complementarities is clearly negative-definite and (x, t^*) is uniquely supermodular implementable.

Example **5** Reconsider the public goods example of Section 3.2. Instead of starting with the expected externality transfers t, suppose that the designer used $\tilde{t}_i(\hat{\theta}) = t_i(\hat{\theta}) - 3\hat{\theta}_i \hat{\theta}_j + \frac{3}{2}\hat{\theta}_i$ for $i = 1, 2$. The game induced by this mechanism is not supermodular with respect to the natural order on $\mathbb{R}$ and it has many equilibria. There are two equilibria where one agent always announces 0 while the other always announces 1. Moreover, truthtelling is an unstable equilibrium; any perturbation results in a departure from it. We already know this example falls into Theorem 8.

The rest of this subsection deals with the multiple equilibrium problem under the budget balance condition. The proposition gives sufficient conditions in order for the transfers identified in Theorem 6 to yield truthtelling as a unique equilibrium.

PROPOSITION **9** *Let $n \geq 3$. Let the valuation functions and the allocation-efficient decision rule be continuously differentiable. Assume substitutes and complements are uniformly bounded by, respectively, T_i and τ_i. Under strong differences, if $(\tau_i - T_i) < E_{\theta_i}[\gamma_i(\theta_i)]$, then there are transfers t^{BB} such that (x, t^{BB}) is uniquely supermodular implementable.*

As the proposition describes, unique supermodular implementation is compatible with balancing budget. So there are situations where learning is very robust and can be auto-financed by the agents themselves. The next example is an application in a public good context.

EXAMPLE **6** Consider the same setting as the public goods example of Section 3.2 with an additional player, player 3, whose type is independently distributed from the other player's types in $\Theta_3 = [0, 1]$. Player 3's valuation function is $V_3(x, \theta_3) = \theta_3 x$. Let $X = [0, 3]$ and $x(\theta) = \sum_i \theta_i$. x is allocation-efficient and $\gamma_i(\theta_i) = 1$ for all i. The valuations and the decision rule produce complements and substitutes which admit the same bounds. Proposition 9 says that there exist $\{\rho_i\}$ such that (x, t^{BB}) is uniquely supermodular implementable with a balanced budget.

Although unique supermodular implementation is compatible with balancing budget, there is sometimes a conflict between the size of the equilibrium set and the budget constraint. This is illustrated in the next example where a designer must choose between meeting her budget requirement and making sure learning ends up close to the efficient public good level.

EXAMPLE **7** Consider the public goods setting of Section 3.2 with a third player whose type is uniformly and independently distributed in $[0, 1]$. Her valuation function is $V_3(x, \theta_3) = \theta_3 x - x^3/10$. The decision rule $x(\theta) = \frac{5}{3}(\sqrt{1 + 6/5(\sum_i \theta_i)} - 1)$ is allocation-efficient and dimensionally reducible. The designer has the choice between the budget balanced transfers t^{BB} and the optimal transfers t^*. On the one hand, if she prefers to have full efficiency, then she chooses $\rho_1 \geq 8$, $\rho_2 \geq 5$, and $\rho_3 \geq 6$ in order for transfers t^{BB} to induce a supermodular game. But the interval prediction of the resulting game is always the entire space. On the other hand, if she prefers to have strong learning properties, then she uses the optimal transfers t^*. These transfers give $K_i(\theta_i) = 3/5(1-\theta_i)$ and $\gamma_i(\theta_{-i}) = 1/\sqrt{1+6/5(1+\sum\theta_j)}$, hence the matrix of complementarities is

$$\mathcal{C} = \begin{pmatrix} -.55 & .3 & .3 \\ .3 & -.55 & .3 \\ .3 & .3 & -.55 \end{pmatrix}.$$

Even though $\mathcal{C}$ is not negative-definite, we can find bounds on the equilibrium set by solving two simple systems of $2n$ quadratic equations (Theorem 8). All the equilibria are contained in between the profile where for all i,

$$\underline{s}_i = \begin{cases} 1.1\theta_i - 0.1 & \text{if } 0 \le \theta_i \le \frac{1.1}{0.1} \\ \overline{\theta}_i & \text{otherwise} \end{cases}$$

and the truthtelling equilibrium. Welfare analysis reveals that learning can at most result in a .03 loss in total utility under the optimal transfers.

Before turning to applications, some remarks are in order.

Remarks.

1. To obtain uniqueness in a Bayesian game, either one imposes conditions on the utility functions or on the information structure. Without making assumptions on the beliefs, any result in the first class, such as Theorem 8, will involve a tradeoff between different types of complementarities.

2. Optimal supermodular implementation imposes the weakest "admissible" amount of complementarity, which might imply a low speed of convergence of learning dynamics towards truthtelling. This is not necessarily true. Sometimes, optimal transfers deliver the fastest convergence (Example in Section 3.2) and sometimes they do not; although convergence is possible in one period in Example 7, it takes longer under the optimal transfers.

3. Neither unique nor optimal supermodular implementation implies the other. The truthful equilibrium may be unique, although the transfers are not optimal, and the

transfers could be optimal but the truthful equilibrium not unique.

3.5.4 Approximate Supermodular Implementation

In this section, I generalize some results within the context of approximate (or virtual) implementation.[23] In well-behaved environments, the results are general and compelling. They apply to a variety of contexts such as principal multi-agent (Section 3.6) and public goods models. However there are interesting situations, where discontinuities arise naturally, that fall outside the scope of the current results. A way around this problem is approximate implementation where the objective becomes to supermodularly implement (well-behaved) scf that are arbitrarily close to a "target scf." I now describe the failure of bounded substitutes in the basic auction setting and I present results that accommodate these situations.

Consider the following auction model. There is one unit of an indivisible good to be allocated among two buyers $\{1, 2\}$ whose types lie in $[\underline{\theta}, \overline{\theta}]$. An outcome is represented by the vector (x_1, x_2) where $x_i = 1$ if i gets the good and 0 otherwise. Buyer i's utility function is $u_i(x_i, \theta_i) = \theta_i x_i + t_i$. The allocation-efficient decision rule x^* attributes the good to the buyer with the highest type:

$$x_1^*(\theta) = \begin{cases} 1 & \text{if } \theta_1 \geq \theta_2 \\ 0 & \text{otherwise} \end{cases} \quad \text{and } x_2^*(\theta) = 1 - x_1^*(\theta). \tag{3.9}$$

Note that for any types such that $\theta_2'' > \theta_1'' > \theta_2' > \theta_1'$, we have $x_1(\theta_1'', \theta_2'') - x_1(\theta_1', \theta_2'') - x_1(\theta_1'', \theta_2') + x_1(\theta_1', \theta_2') = -1$. Hence the assumption of bounded substitutes would require the existence of T such that $-\theta_1 \geq T(\theta_1'' - \theta_1')(\theta_2'' - \theta_2')$ for all $\theta_1 \in \Theta_1$. This is clearly impossible as we can maintain the above order of types while $\theta_1' \uparrow \theta_2'$ and $\theta_1'' \downarrow \theta_2'$. Substitutes are unbounded and none of the results apply.

Clearly, the problem is caused by the lack of smoothness of the decision rule. So the idea is to approximate the scf by smooth implementable functions which are known to satisfy the desired conditions.

Definition **14** *A decision rule x is approximately (optimally) supermodular implementable, if there exists a sequence of (optimally) supermodular implementable scf $\{(x_n, t_n)\}$ such that,*

[23] See Abreu and Matsushima [1] and Duggan [30].

for $1 \leq p < \infty$, $||x_{n,i} - x_i||_p = 0$ *for all* i *where* $||\cdot||_p$ *is the* L_p*-norm.*

The next results provide conditions so that a scf can be approached by a sequence of supermodular implementable scf. The intuition is that the space of smooth functions is dense in L_p-spaces, hence a smooth approximation exists. Now smooth scf satisfy the bounded substitutes assumption, so there only remains to establish that incentive-compatibility can be preserved along the sequence. Further, if the decision rule also satisfies the dimensionality condition from Section 3.5.3.1, then it is approachable by scf whose supermodular game form gives the tightest interval prediction.

Proposition **10** *Let the valuation functions be twice-continuously differentiable such that* $\partial V_i(x_i, \theta_i)/\partial \theta_i$ *is increasing in* x_i *for all* i*. If the decision rule is such that* $x_i \in L_p$ *is increasing in* $\hat{\theta}_i$ *for all* i*, then it is approximately supermodular implementable.*

Proposition **11** *Let the valuation functions be twice-continuously differentiable such that* $\partial V_i(x_i, \theta_i)/\partial \theta_i$ *is increasing in* x_i *for all* i*. If decision rule is such that, for all* i*, there exist* $h_i : \mathbb{R}^2 \rightarrow X_i$ *and* $r_i : \Theta_{-i} \rightarrow \mathbb{R}$ *such that*

1. h_i *is bounded and increasing in its first variable,*
2. r_i *is continuous and strictly increasing,*[24]
3. $x_i(\theta) = h_i(\theta_i, r_i(\theta_{-i}))$,

then it is approximately optimally supermodular implementable.

These results apply to many discontinuous models of interest such as public goods, auctions (Section 3.6) and bilateral trading (Myerson and Satterthwaite [88]. Besides, they suggest that there may be a dilemma between close implementability and stability or learning. This supports Cabrales [14] where a similar trade-off is formalized for Abreu and Matsushima [1] and [2].

[24]Function r_i is strictly increasing if $r_i(\theta''_{-i}) > r_i(\theta'_i)$ whenever $\theta''_{-i} \gg \theta'_{-i}$.

3.6 Applications

3.6.1 Principal Multi-Agent Problem

This subsection applies the theory to the traditional principal-multiagent problem with hidden information. A principal contracts with n agents. Agent i's type lies in $[\underline{\theta}_i, \overline{\theta}_i]$. Types are independently distributed according to a common prior. Each agent i exerts some observable effort $x_i \in X_i$ and bears a cost $c_i(x_i, \theta_i)$ when her type is θ_i. From all the efforts $x = (x_1, \ldots, x_n)$ and types, the principal receives utility $w(x, \theta)$. The principal faces the problem of designing a profit-maximizing contract subject to incentive and reservation-utility constraints. A contract is a function that maps types into effort and transfer levels for each agent. The principal's problem can be stated as

$$(\hat{x}, \hat{t}) \in \underset{f=(x,t)}{\operatorname{argmax}} \, E_\theta \left[w(x(\theta), \theta) - \sum_{i=1}^{n} t_i(\theta) \right] \tag{3.10}$$

subject to

$$E_{\theta_{-i}}[t_i(\theta_i, \theta_{-i}) - c_i(x_i(\theta_i, \theta_{-i}), \theta_i)] \geq E_{\theta_{-i}}[t_i(\theta_i', \theta_{-i}) - c_i(x_i(\theta_i', \theta_{-i}), \theta_i)] \tag{3.11}$$

for all θ_i and θ_i', and

$$E_{\theta_{-i}}[t_i(\theta_i, \theta_{-i}) - c_i(x_i(\theta_i, \theta_{-i}), \theta_i)] \geq \overline{u}_i \tag{3.12}$$

for all θ_i. Condition (3.11) requires truthtelling to be an equilibrium. Condition (3.12) is an interim participation constraint, as agents may opt out of the mechanism if it does not meet their reservation utility.

If the underlying functions w, c_i, and the prior are smooth and guarantee the existence of a solution whose decision rule is dimensionally reducible, then the contract is optimally supermodular implementable in virtue of Theorem 7. In other words, if the principal is in a position to engage in a smooth revenue-maximizing and incentive-compatible contract which allows voluntary participation, then she can turn that contract into a supermodular contract which retains properties (3.10), (3.11), and (3.12), and minimizes the size of the equilibrium set.

3.6.2 Auctions, Bilateral Trading, and Allocation Schemes

Auctions are notorious for lacking complementarities. In a first- or second-price auction, suppose an agent goes from losing to winning the object after increasing her bid. Now assume her opponents increase their bids to a level where that increase does not suffice to get the good. Then the agent's marginal utility, whether she pays her bid or the second-highest one, decreases as her opponents increase their bids. This configuration displays a failure of complementarities. If the agent's increase was large enough to secure the object in both situations, then her marginal utility would (weakly) increase. This points to the existence of local substitute effects in auctions.

Consider the auction setting of Section 3.5.4. Two buyers, 1 and 2, have utility function $v(\theta_i)x_i + t_i$, $i = 1, 2$, where v is strictly increasing. The allocation-efficient decision rule x^* is unchanged and, according to Proposition 11,[25] it is approachable by a sequence of optimally supermodular implementable decision rules. An example of such a sequence is

$$x_{1,\epsilon}(\hat{\theta}) = c \int_0^{\frac{\sum_i (a_i \hat{\theta}_i + b_i)}{\epsilon}} \frac{1}{1+t^2} dt + k \text{ and } x_{2,\epsilon} = 1 - x_{1,\epsilon},$$

where constants c, k and a_i, b_i, $i = 1, 2$ are chosen appropriately. From there, the optimal transfers t^*_ϵ and the matrix of complementarities $\mathcal{C}_\epsilon$ can be formed. Since substitutes are unbounded in the exact case, it is not surprising that their bound in the approximated case decreases infinitely as ϵ vanishes. To compensate for this, the transfers add more complementarities as ϵ vanishes, which leads complementarities to explode on those parts of the space where reports were already complements. As a consequence, the interval prediction of the game induced by $\{x_\epsilon, t^*_\epsilon\}$ must be the entire space in the limit. What is interesting, however, is to find the smallest amount of inefficiency $\underline{\epsilon}$ for which $\mathcal{C}_{\underline{\epsilon}}$ is still negative-definite and uniqueness is preserved. Interestingly, this illustrates nicely the forces at work in Theorem 8. Letting $v(\theta) = \theta^\omega$ with $\omega > 0$, it always takes a larger ω to obtain a smaller $\underline{\epsilon}$. That is, the more sensitive an agent is to her own type, the higher is the degree of complementarities that can be compatible with uniqueness; hence the lower ϵ can be. Unfortunately, ω has to be fairly large to accommodate for accurate approximations. In the standard case with $\omega = 1$, $\underline{\epsilon} \approx .45$. When $\omega = 6$, $\underline{\epsilon} \approx .15$. Although uniqueness may fail for some

[25] In the case with $n \geq 3$, let $h_i(\theta_i, r_i) = 1$ if $\theta_i > r_i$, and 0 otherwise for all i. Note h_i is bounded and increasing in θ_i. Now choose $r_i(\theta_{-i}) = \max\{\theta_j : j \neq i\}$ which is continuous and strictly increasing.

ϵ, it can make sense to use this if the size of the equilibrium set is not too wide (Theorem 8).

The results for approximate implementation also apply to public goods situations where a society of agents decide whether or not to undertake a public project. The bargaining mechanism of Myerson and Satterthwaite ([88], pp.274) also satisfies the assumptions of Proposition 11 and as such, the decision rule is approachable by a sequence of optimally supermodular implementable decision rules. The expected gains from trade along the sequence converge to the maximal expected gains.

3.7 A Revelation Principle

Supermodular implementation is widely applicable in quasilinear environments even though the chapter limits attention to direct mechanisms. For general preferences, however, direct mechanisms may be restrictive. The Revelation Principle says that direct mechanisms cause no loss of generality under traditional weak implementation. But how restrictive are direct mechanisms in supermodular Bayesian implementation for general preferences?

It is particularly relevant to analyze this question, because the challenge in any supermodular design problem is to specify an ordered message space and an outcome function so that agents adopt monotone best-responding behaviors. The set of all possible message spaces and orders on those spaces is so large that it might seem intractably complex. A Supermodular Revelation Principle gives conditions so that, if a scf is supermodular implementable, then there exists a direct-revelation mechanism that supermodularly implements this scf truthfully. It is a technical insight which reduces the space of mechanisms to consider to the space of direct-revelation mechanisms. The question is complex because it is combinatorial in essence; it pertains to the existence of orders on type spaces that make the (induced) direct-revelation game supermodular.

Example 10 in Appendix A shows that, unfortunately, there exist supermodular implementable scf that are not truthfully supermodular implementable. Consequently, the revelation principle fails to hold in general for supermodular implementation. Nevertheless, it exists in a weaker form, as captured by the next theorem. Although it is not as general as the traditional revelation principle, it measures the restriction imposed by direct mechanisms and gives conditions that may warrant their use.

As mentioned in Section 3.4, there are issues related to non-Euclidean message spaces that justify the next and more general definition of supermodular implementability.

Definition **15** *The mechanism Γ supermodularly implements the scf $f(.)$ if there exists a Bayesian equilibrium $m^*(.)$ such that $g(m^*(\theta)) = f(\theta)$ for all $\theta \in \Theta$, and if the (ex-ante) induced game $\mathcal{G}$ is supermodular.*

Theorem **9** *(*The Supermodular Revelation Principle for Finite Types[26]*) Let type space Θ_i be a finite set for $i \in N$. If there exists a mechanism $(\{(M_i, \succeq_i)\}, g)$ that supermodularly implements the scf f such that there is a Bayesian equilibrium $m^*(.)$ for which $g \circ m^* = f$ and $m_i^*(\Theta_i)$ is a lattice, then f is supermodular implementable.*

Corollary **5** *Let type spaces be finite sets. If there exists a mechanism $(\{(M_i, \succeq_i)\}, g)$ with totally ordered message spaces that supermodularly implements the scf f such that there is a Bayesian equilibrium $m^*(.)$ for which $g \circ m^* = f$, then f is supermodular implementable.*

According to the supermodular revelation principle, limiting attention to direct mechanisms amounts to restricting one's scope to mechanisms where the equilibrium strategies are lattice-ranged. When the range of the equilibrium strategies is a lattice, it is possible to construct an order on each player's type space that makes it order-isomorphic to the range of her equilibrium strategy. By order-isomorphism, type spaces become lattices under this order and it also preserves supermodularity from the indirect mechanism to its direct version. Therefore, the transmission channel is the range of the equilibrium strategies. Besides, the theorem states conditions that are verifiable a posteriori. It may be useful to know when a complex mechanism can be replaced with a simpler direct mechanism.

Corollary 5 says that if the designer is only interested in mechanisms where the message spaces are totally ordered, then she can look at direct mechanisms without loss of generality.

The theorem only gives sufficient conditions for revelation; but in those cases where a supermodular direct mechanism exists while the lattice condition is violated, the existence of an order has little or nothing to do with a revelation principle. In the spirit of Echenique [33], there may be conditions on the scf and the utility functions such that an order exists for which the game is supermodular. Since this existence would not follow from implementability, it is not a revelation approach.

[26]In Mathevet [**?**], I generalize the definition of supermodular implementability to incorporate orders that are not pointwise orders. This allows proving a supermodular revelation principle for continuous types.

3.8 Discussion

3.8.1 Dominant Strategy vs. Supermodular Implementation

Learning is one of the main arguments in favor of supermodular implementation. So, even though dominant strategy mechanisms do not necessarily induce games with strong learning properties,[27] it is important to highlight the advantage of supermodular implementation over these mechanisms.

Strategy proofness, requiring truthtelling to be a dominant strategy, is not always possible even in smooth environments. Mookherjee and Reichelstein [81] have shown that a sufficient and "nearly" necessary condition for strategy-proofness is that the valuation functions and the decision rule satisfy some single-crossing property (Proposition 2 and Definition 5 in [81]). They make an assumption, the one-dimensional condensation property, which makes it easier to satisfy that necessary condition.

The next example is an application of Theorem 8 in a principal-multiagent context inspired from McAfee and McMillan [72]. The problem violates the one-dimensional condensation property and the necessary condition for strategy-proofness. So the decision rule is not dominant strategy implementable, yet it is uniquely supermodular implementable.

Example **8** Two agents, 1 and 2, whose types are independently and uniformly distributed in $[0, 3]$, exert some effort to produce an observable contribution x_i. The amount of effort e_i necessary for x_i is $e_1(x, \theta_1) = (3 - \theta_1)(x_1 - x_2) + x_1 + \frac{9}{2}$ and $e_2(x, \theta_2) = (3 - \theta_2)(x_2 + x_1)$. Agent 2 has positive externalities on her counterpart, whereas 1 has negative externalities. Before transfers, the principal has utility $w(x, \theta) = u(x, \theta) - c_p(x, \theta)$ where c_p represents the production costs. The principal's objective is to solve (3.10) subject to (3.11) and the ex-ante minimum wage $E_\theta[t_i(\theta)] \geq 0$ on the economy. For simplicity, let u and the production costs be such that the optimal decision rule is $x^*(\theta) = (\theta_2\theta_1 - 3/2\theta_1, \theta_2 - \theta_1)$.[28] Agent i's valuation is $V_i(x, \theta_i) = -e_i(x, \theta_i)$. Proposition 12 of Appendix B applies, so there exist transfers that implement x^*. Constructing optimal transfers from (3.21) and (3.22) gives $t_1^*(\hat{\theta}) = -\hat{\theta}_1^2/2 - 3\hat{\theta}_1 + 2\hat{\theta}_2\hat{\theta}_1 - \frac{13}{2}$ and $t_2^*(\hat{\theta}) = -5\hat{\theta}_2^2/4 + 3\hat{\theta}_2 + 3\hat{\theta}_2\hat{\theta}_1 - \frac{13}{2}$. Given $K_i(\theta_i) = \theta_i$

[27] Learning dynamics may still converge to "unwanted" equilibria, in dominant strategies or not (e.g. Nash), whose outcomes differ from the scf or they may converge to non-equilibrium profiles or simply cycle. Saijo et al. [94] report situations where this concept has serious drawbacks.

[28] For example, let $u(x, \theta) = \theta_2(\theta_1 x_1 + x_2)$ and $c_p(x, \theta) = (x_1^2 + x_2^2)/2 + \theta_1(3/2x_1 + x_2)$.

and $\gamma_1(\theta_2) = \theta_2 - \frac{1}{2}$, $\gamma_2(\theta_1) = \theta_1 + 1$, we can check that the matrix of complementarities

$$\mathcal{C} = \begin{pmatrix} -1 & 3/2 \\ 3/2 & -5/2 \end{pmatrix}$$

is negative-definite and so truthtelling is the unique equilibrium.

It is well-known that dominant-strategy implementation is often incompatible with balancing budget (Green and Laffont [40] and Laffont and Maskin [62]). The next example depicts a situation where unique supermodular implementation allows balancing budget in a case where dominant strategies cannot. In other words, by weakening the solution concept and putting structure on the game form, it is possible to balance the budget and maintain the likelihood of equilibrium play.

EXAMPLE **9** In the public goods example of Section 3.2, let $\Theta_1 = \Theta_2 = [2, 3]$. Add a third player, player 3, whose type is independently distributed from the other player's types in $\Theta_3 = [2, 3]$. Player 3's valuation function is $V_3(x, \theta_3) = \theta_3 x - \ln x$. Letting $X = [5, 10]$, the allocation-efficient decision rule is $x(\theta) = \frac{1}{2}(\sum_i \theta_i + \sqrt{(\sum_i \theta_i)^2 - 4})$. By Theorem 3.1 in [62], the decision rule is dominant strategy implementable only if transfers are of the Groves form. However these transfers cannot balance budget, because they violate the necessary condition from Laffont and Maskin (Theorem 4.1 in [62]). Nevertheless, since $\tau_i - T_i < .03$ and $\gamma_i > 1$ for all i, Proposition 9 implies that x is uniquely supermodular implementable with a balanced budget.

Dominant strategies require conditions on cross-partial derivatives whereas basic smoothness and dimensionality conditions allow for optimal supermodular implementation. As shown in Example 8, there are even scf that are not strategy-proof and yet can be uniquely implemented with a supermodular mechanism.

One reason for moving towards Bayesian implementation was to balance budget in situations dominant strategies cannot. Balancing transfers under supermodular implementation is nearly as general as Bayesian implementation allows. As shown in Example 9, it is even possible to balance budget using a supermodular mechanism with a unique equilibrium in a case where dominant strategies cannot balance budget. Ideally one would like all strategy-proof scf to be uniquely supermodular implementable, but such a result is not known. Most likely, neither dominant strategy nor supermodular implementation implies the other.

3.8.2 Learning in Bayesian Games

The learning literature has a straightforward application to games of incomplete information which is the approach taken in this chapter. In the context of Bayesian implementation, the learning results of supermodular games find a natural interpretation in the ex-ante Bayesian game. Loosely, learning at the ex-ante stage may be interpreted as pre-playing the mechanism. At this stage, agents do not know their own type and they can be viewed as practicing the induced game repeatedly. Each agent submits a deception at each round until the designer collects the agreed-upon profile of deceptions, and types are revealed. Until then, no outcome is actually implemented. Learning at the ex-ante stage may also mean that agents are actually playing the mechanism repeatedly with independently and identically distributed types across periods. As a round begins, the agents do not know their own type yet, hence they submit a deception. By the end of the round, they learn their type and behave according to their deception. An outcome is then implemented at the end of each round. Here the designer is only interested in implementing the desired outcome in the long-run.

Although the learning results only apply directly to the ex-ante Bayesian game, they can be interpreted in the interim formulation. The interim Bayesian game inherits the complementarities, because most results work by showing that the ex-post game is supermodular for every type profile. However, the problem at this stage comes from the interpretation of learning and the technical difficulties related to the Milgrom-Roberts learning theorem. To illustrate the first difficulty, suppose that there are two agents. At the interim stage, each agent knows her own type and so she makes a single announcement at each period that the mechanism is repeated. But to compute her expected utility, an agent uses the prior distribution and the opponent's deception telling her what is played for each type. Since the opponent no longer announces a deception, an agent is unable to compute her expected utility. One way of interpreting learning then is to consider that there is a continuum of agents, and that prior belief ϕ actually represents the distribution of types in this population. An agent now faces a continuum of announcements (one for each opponent) as the mechanism is repeated, hence she can compute her expected utility. The interpretation of the process, however, becomes evolutionary in nature. We are now interested in that the observed proportions of types converge to the true proportions in the population. On the technical side, Van Zandt [104] shows that there are issues in applying some results of supermodular games to interim Bayesian games. But his results can be used to show that the Milgrom-Roberts learning theorem applies to the interim Bayesian game.[29]

3.9 Conclusion

This chapter introduces a theory of implementation where the mechanisms implement scf in supermodular game forms. Supermodular implementation differs from the previous literature by its explicit purpose and methodology. The chapter does not put an end to the question of learning and stability in incentive design and implementation, but it explicitly attacks it and provides answers to this important, yet neglected, question. Given that mechanisms are designed to achieve some equilibrium outcome, it is rather important to design mechanisms that enable boundedly rational agents to learn to play some equilibrium outcome. The methodology consists in inducing supermodular games rather than starting explicitly with a solution concept. Of course, supermodularity implies properties of iterative dominance, but it has stronger theoretical and experimental implications (see, e.g., Camerer [18]). The mechanisms derive their properties from the game that they induce and not directly from the solution concept.

Beyond the results, this chapter brings out basic questions about learning and the design problem. We may wonder whether there is a price to pay for learning or stability in terms of efficiency. The trade-off appears quite clearly in this framework; sometimes the designer must sacrifice learning for full efficiency or vice-versa. In the public goods example, the designer can modify the expected externality mechanism and secure learning at the price of a balanced budget, or she can use the expected externality mechanism to balance budget but she loses the strong learning properties. This may be related to the specifics of supermodular implementation, but it is an interesting issue. We may also wonder whether there is a price to pay for learning or stability in terms of closeness of the decision rule implemented. This has obvious implications in terms of efficiency. Cabrales [14] also suggests a dilemma between learning and close implementability for the Abreu-Matsushima mechanism, and it is verified in the supermodular implementation framework. Although this dilemma may

[29]The results in [76] can only be directly applied to the ex-ante version $\mathcal{G}$. However, Lemma 2, Proposition 3, Lemma 5 and Proposition 5 in Van Zandt [104] are particularly useful to apply the learning theorem to the interim formulation. This requires, however, that the utility functions (with transfers) be continuous in the announcement profile.

be related to the specifics of these frameworks, it is a question with potentially important consequences.

This chapter raises issues that have not been discussed. The multiple equilibrium problem in supermodular implementation suggests an alternative solution, namely strong implementation. Strong implementation requires all equilibria of the mechanisms to yield desired outcomes. Instead of relying on weak implementation, supermodular implementation could be based on strong implementation which would justify indirect mechanisms. Even under strong implementation, learning dynamics may cycle within the interval prediction and players may learn to play a non-equilibrium profile. Although strong supermodular implementation cannot substitute for unique supermodular implementation, it is an avenue to explore.

Like many Bayesian mechanisms, the present mechanisms are parametric in the sense that they rely on agents' prior beliefs. Thus the designer uses information other than that received from the agents (Hurwicz [6]). It may be interesting to design nonparametric supermodular mechanisms. This is yet another justification for indirect mechanisms, as nonparametric direct Bayesian mechanisms impose dominant-strategy incentive-compatibility (Ledyard [63]).

Finally, it is important to pursue testing supermodular games. Since supermodular Bayesian implementation provides a general framework, it is a good candidate for experimental tests. From a practical viewpoint, discretizing type spaces may simplify the players' task of announcing deceptions at each round. But there are also simple environments with continuous types where announcing a deception is equivalent to choosing a real number, such as the leading public goods and the team-production examples.[30].

3.10 Appendix

3.10.1 A Counterexample to the Revelation Principle

This example shows that the revelation principle fails to hold in general for supermodular Bayesian implementation.

[30]In the public goods example of Section 3.2, announcing an optimal deception comes down to choosing an intercept in a compact set (see (3.1)). In Example 8, optimal deceptions are characterized by a positive slope.

Example **10** Consider two agents, 1 and 2, with type spaces $\Theta_1 = \{\theta_1^1, \theta_1^2\}$ and $\Theta_2 = \{\theta_2^1, \theta_2^2, \theta_2^3\}$. Prior beliefs assign equal probabilities to all $\theta \in \Theta$. Let $X = \{x_1, \ldots, x_{12}\}$ be the outcome space. Agent 1's preferences are given by utility function $u_1(x_n, \theta_1)$ such that:

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}
$u_1(x_n, \theta_1^1)$	-10	0	16	-13	-2	33	-21	-2	18	-19	0	36
$u_1(x_n, \theta_1^2)$	-10	0	16	-21	-2	18	-13	-2	33	-19	0	36.

Let u_2 be a constant function. Let the scf f be defined as follows

$f(.,.)$	θ_2^1	θ_2^2	θ_2^3
θ_1^1	x_4	x_5	x_6
θ_1^2	x_7	x_8	x_9.

Consider the following indirect mechanism $\Gamma = ((M_1, \succeq_1), (M_2, \succeq_2), g)$. Agent 1's message space is $M_1 = \{\underline{m}_1, m_1^1, m_1^2, \overline{m}_1\}$; $\succeq_1$ is such that m_1^1 and m_1^2 are unordered, $\overline{m}_1$ is the greatest element and $\underline{m}_1$ is the smallest element. Agent 2's message space is $M_2 = \{\underline{m}_2, m_2^1, \overline{m}_2\}$; $\succeq_2$ is such that $\overline{m}_2 \succeq_2 m_2^1 \succeq_2 \underline{m}_2$. The outcome function g is given by

$g(.,.)$	$\underline{m}_2$	m_2^1	$\overline{m}_2$
$\underline{m}_1$	x_1	x_2	x_3
m_1^1	$f(\theta_1^1, \theta_2^1)$	$f(\theta_1^1, \theta_2^2)$	$f(\theta_1^1, \theta_2^3)$
m_1^2	$f(\theta_1^2, \theta_2^1)$	$f(\theta_1^2, \theta_2^2)$	$f(\theta_1^2, \theta_2^3)$
$\overline{m}_1$	x_{10}	x_{11}	x_{12}.

I show that mechanism Γ supermodularly implements f in Bayesian equilibrium. Given u_2 is constant, any strategy $m_2 : \Theta_2 \to M_2$ is a best-response to any strategy of 1. So, consider strategy $m_2^*(.)$ such that $m_2^*(\theta_2^1) = \underline{m}_2$, $m_2^*(\theta_2^2) = m_2^1$ and $m_2^*(\theta_2^3) = \overline{m}_2$. Since for all m_1 we have

$$\begin{aligned} \textstyle\sum_{m_2} u_1(g(m_1^1, m_2), \theta_1^1) &> \textstyle\sum_{m_2} u_1(g(m^1, m_2), \theta_1^1) \\ \textstyle\sum_{m_2} u_1(g(m_1^2, m_2), \theta_1^2) &> \textstyle\sum_{m_2} u_1(g(m_1, m_2), \theta_1^2) \end{aligned}$$

1's best-response $m_1^*(.)$ to $m_2^*(.)$ is such that $m_1^*(\theta_1^1) = m_1^1$ and $m_1^*(\theta_1^2) = m_1^2$. So $(m_1^*(.), m_2^*(.))$ is a Bayesian equilibrium and $g \circ m^* = f$. Moreover, for each θ_1, $u_1(g(m_1, m_2), \theta_1)$ is supermodular in m_1 and has increasing differences in (m_1, m_2). This implies that u_1^g is supermodular in $m_1(.)$ and has increasing differences in $(m_1(.), m_2(.))$, because $\Sigma_1(\Theta_1)$ is endowed with the pointwise order. Therefore, Γ supermodularly implements f in Bayesian equilibrium, because 2's utility is constant.

Does this imply that there exists a mechanism $(\{(\Theta_i, \geq_i)\}, f)$ which truthfully implements f in supermodular game form? By means of contradiction, suppose there is such a mechanism. Then $(\Theta_1, \geq_1)$ must be totally ordered, for otherwise $\Sigma_1(\Theta_1)$ cannot be a lattice. Assume $\theta_1^2 >_1 \theta_1^1$. Let $\theta_i^k(.) = \theta_i^k$ regardless of i's true type. Let $\theta_1^T(.)$ be the truthful strategy for 1 and let $\theta_1^L(.)$ be constant lying. Note $\theta_1^1(.) <_1 \theta_1^T(.), \theta_1^L(.)$. Moreover, θ_2^1 and θ_2^2 must be ordered, because $\Sigma_2(\Theta_2)$ is a lattice. Thus $\theta_2^1(.)$ and $\theta_2^2(.))$ are ordered.

Since the direct mechanism must induce a supermodular game, $u_1^f(\hat{\theta}_1(.), \hat{\theta}_2(.))$ must satisfy the single-crossing property in $(\hat{\theta}_1(.), \hat{\theta}_2(.))$.[31] Given

$$\begin{aligned} -2 = u_1^f(\theta_1^T(.), \theta_2^2(.)) &\geq u_1^f(\theta_1^1(.), \theta_2^2(.)) = -2 \\ -13 = u_1^f(\theta_1^T(.), \theta_2^1(.)) &> u_1^f(\theta_1^1(.), \theta_2^1(.)) = -17 \end{aligned}$$

u_1^f satisfies the single-crossing property in $(\hat{\theta}_1(.), \hat{\theta}_2(.))$ only if $\theta_2^1 >_2 \theta_2^2$. But

$$-2 = u_1^f(\theta_1^L(.), \theta_2^2(.)) \geq u_1^f(\theta_1^1(.), \theta_2^2(.)) = -2$$

does not imply $-21 = u_1^f(\theta_1^L(.), \theta_2^1(.)) \geq u_1^f(\theta_1^1(.), \theta_2^1(.)) = -17$. The single-crossing property is violated. Now assume $\theta_1^1 >_1 \theta_1^2$. Note $\theta_1^1(.) >_1 \theta_1^T(.), \theta_1^L(.)$. Given

$$\begin{aligned} -2 = u_1^f(\theta_1^1(.), \theta_2^2(.)) &\geq u_1^f(\theta_1^L(.), \theta_2^2(.)) = -2 \\ -17 = u_1^f(\theta_1^1(.), \theta_2^1(.)) &> u_1^f(\theta_1^L(.), \theta_2^1(.)) = -21 \end{aligned}$$

u_1^f satisfies the single-crossing property in $(\hat{\theta}_1(.), \hat{\theta}_2(.))$ only if $\theta_2^1 >_2 \theta_2^2$. But

$$-2 = u_1^f(\theta_1^1(.), \theta_2^2(.)) \geq u_1^f(\theta_1^T(.), \theta_2^2(.)) = -2$$

does not imply $-17 = u_1^f(\theta_1^1(.), \theta_2^1(.)) \geq u_1^f(\theta_1^T(.), \theta_2^1(.)) = -13$. The single-crossing property is violated. The scf f is not truthfully supermodular implementable, although it is supermodular implementable.

This example suggests that the conditions of Theorem 9 are somewhat minimally sufficient. Agent 1's equilibrium strategy is indeed not lattice-ranged and the scf is not truthfully supermodular implementable. Whereas this example might indicate that the pointwise-order structure causes revelation to fail, this is not the case. Allowing more general order

[31] The single-crossing property, defined in Section 3.3, is implied by increasing differences.

structures does not weaken the conditions for a revelation principle (see Mathevet [?]).

3.10.2 Proofs

The following lemma shows that if the complete information payoffs are supermodular and have increasing differences, then the ex-ante payoffs are supermodular and have increasing differences.

LEMMA **5** *Assume $(M_i, \geq_i)$ is a lattice. Suppose that for each θ_i, $u_i(g(m_i, m_{-i}), \theta_i)$ is supermodular in m_i for each m_{-i} and has increasing differences in (m_i, m_{-i}). Then u_i^g is supermodular in $m_i(.)$ for each $m_{-i}(.)$ and has increasing differences in $(m_i(.), m_{-i}(.))$.*

The next proposition is a standard result whose proof is omitted (see Proposition 23.D.2 in Mas-Colell et al. [68]).

PROPOSITION **12** *Consider valuation functions and a decision rule such that $E_{\theta_{-i}}[V_i(x_i(\hat{\theta}_i, \theta_{-i}), \theta_i)]$ is continuous in $(\hat{\theta}_i, \theta_i)$.*

1. *If the scf $f = (x, t)$ is implementable, then for all $\hat{\theta}_i$*

$$E_{\theta_{-i}}[t_i(\hat{\theta}_i, \theta_{-i})] = -E_{\theta_{-i}}[V_i(x_i(\hat{\theta}_i, \theta_{-i}), \hat{\theta}_i)] + \int_{\underline{\theta}_i}^{\hat{\theta}_i} \frac{\partial E_{\theta_{-i}}[V_i(x_i(s, \theta_{-i}), s)]}{\partial \theta_i} ds + \epsilon(\underline{\theta}_i) \tag{3.13}$$

2. *Let the decision rule be such that $\partial E_{\theta_{-i}}[V_i(x_i(\hat{\theta}_i, \theta_{-i}), \theta_i)]/\partial \theta_i$ is increasing in $\hat{\theta}_i$ for each θ_i and i. If transfers t satisfy (3.13), then $f = (x, t)$ is implementable.*

PROPOSITION **13** *If the valuation functions and the decision are twice-continuously differentiable, then $V_i \circ x_i(.)$ has bounded substitutes, is κ_i-Lipschitz in $(\hat{\theta}_i, \hat{\theta}_{-i})$ and is ω_i-Lipschitz in $\hat{\theta}_i$ for all $i \in N$.*

Proof: Since V_i and $x_i(.)$ are twice continuously differentiable, $V_i \circ x_i$ is C^2 in $\prod_{k \in N} O_k \times O_i$. As a result, $\partial V_i(x_i(\hat{\theta}), \theta_i)/\partial \hat{\theta}_i$ is continuous in $(\hat{\theta}_i, \hat{\theta}_{-i}, \theta_i)$ and so $\omega_i \equiv \max_{(\hat{\theta}, \theta_i) \in \Theta \times \Theta_i} \partial V_i(x_i(\hat{\theta}), \theta_i)/\partial \hat{\theta}_i$ exists. Then $V_i \circ x_i(.)$ is ω_i-Lipschitz. Now let

$$\mathcal{D}_{-i}^{w} V_i(x_i(\hat{\theta}), \theta_i) = \lim_{t \to 0} \frac{V_i(x_i(\hat{\theta}_i, \hat{\theta}_{-i} + tw), \theta_i) - V_i(x_i(\hat{\theta}), \theta_i)}{t}. \tag{3.14}$$

The limit in (3.14) is the directional derivative of $V_i \circ x_i(.)$ in $\hat{\theta}_{-i}$ at $(\hat{\theta}, \theta_i)$, in the direction of vector w. Since

$$\mathcal{D}^w_{-i} V_i(x_i(\hat{\theta}), \theta_i) = \sum_{j \neq i} \frac{\partial V_i(x_i(\hat{\theta}), \theta_i)}{\partial \hat{\theta}_j} w_j, \tag{3.15}$$

the directional derivatives of $V_i \circ x_i(.)$ are all well-defined. By (3.15),

$$\mathcal{D}^w_{-i} \frac{\partial V_i(x_i(\hat{\theta}), \theta_i)}{\partial \hat{\theta}_i} = \sum_{j \neq i} \frac{\partial^2 V_i(x_i(\hat{\theta}), \theta_i)}{\partial \hat{\theta}_i \partial \hat{\theta}_j} w_j \text{ for all } (\hat{\theta}, \theta_i), \tag{3.16}$$

and it is well-defined because V_i and $x_i(.)$ are C^2. Let

$$\kappa_i = \max_{j \neq i} \max_{(\hat{\theta}, \theta_i) \in \Theta \times \Theta_i} \frac{\partial^2 V_i(x_i(\hat{\theta}), \theta_i)}{\partial \hat{\theta}_i \partial \hat{\theta}_j} \text{ and } T_i = \min_{j \neq i} \min_{(\hat{\theta}, \theta_i) \in \Theta \times \Theta_i} \frac{\partial^2 V_i(x_i(\hat{\theta}), \theta_i)}{\partial \hat{\theta}_i \partial \hat{\theta}_j} \tag{3.17}$$

which are well-defined because V_i and $x_i(.)$ are C^2 and Θ_k is compact for all $k \in N$. Pick any $\hat{\theta}''_i > \hat{\theta}'_i$, $\hat{\theta}''_{-i} > \hat{\theta}'_{-i}$, and let $w = \hat{\theta}''_{-i} - \hat{\theta}'_{-i}$. By (3.16),

$$\kappa_i \, \mathbf{1}.w \geq \mathcal{D}^w_{-i} \frac{\partial V_i(x_i(\hat{\theta}), \theta_i)}{\partial \hat{\theta}_i} \geq T_i \, \mathbf{1}.w \text{ for all } (\hat{\theta}, \theta_i),$$

hence

$$\kappa_i \, \mathbf{1}.w(\hat{\theta}''_i - \hat{\theta}'_i) = \int_{\hat{\theta}'_i}^{\hat{\theta}''_i} \int_0^1 \kappa_i \, \mathbf{1}.w \, dt \, d\hat{\theta}_i \geq \int_{\hat{\theta}'_i}^{\hat{\theta}''_i} \int_0^1 \mathcal{D}^w_{-i} \frac{\partial V_i(x_i(\hat{\theta}_i, \hat{\theta}'_{-i} + tw), \theta_i)}{\partial \hat{\theta}_i} \, dt \, d\hat{\theta}_i$$

$$= \Delta V_i(\hat{\theta}''_i, \hat{\theta}'_{-i}, \hat{\theta}''_{-i}, \theta_i) - \Delta V_i(\hat{\theta}''_i, \hat{\theta}'_{-i}, \hat{\theta}'_{-i}, \theta_i) \geq T_i \, \mathbf{1}.w(\hat{\theta}''_i - \hat{\theta}'_i), \tag{3.18}$$

for all θ_i. Letting $g(t) = \partial V_i(x_i(\hat{\theta}_i, \hat{\theta}'_{-i} + tw), \theta_i)/\partial \hat{\theta}_i$, the second equality in (3.18) follows from

$$g'(t) = \sum_{j \neq i} \frac{\partial^2 V_i(x_i(\hat{\theta}_i, \hat{\theta}'_{-i} + tw), \theta_i)}{\partial \hat{\theta}_i \partial \hat{\theta}_j} w_j \equiv \mathcal{D}^w_{-i} \frac{\partial V_i(x_i(\hat{\theta}_i, \hat{\theta}'_{-i} + tw), \theta_i)}{\partial \hat{\theta}_i}$$

and

$$\int_0^1 \mathcal{D}^w_{-i} \frac{\partial V_i(x_i(\hat{\theta}_i, \hat{\theta}'_{-i} + tw), \theta_i)}{\partial \hat{\theta}_i} \, dt = g(1) - g(0) =$$

$$= \frac{\partial V_i(x_i(\hat{\theta}_i, \hat{\theta}''_{-i}), \theta_i)}{\partial \hat{\theta}_i} - \frac{\partial V_i(x_i(\hat{\theta}_i, \hat{\theta}'_{-i}), \theta_i)}{\partial \hat{\theta}'_i}.$$

By (3.18), $V_i \circ x_i(.)$ is κ_i-Lipschitz in $(\hat{\theta}_i, \hat{\theta}_{-i})$ and has bounded substitutes for all $i \in N$.Q.E.D

Proof of Proposition 7: Let (x, t'') and (x, t') be any supermodular implementable scf such that $t'', t' \in \mathcal{T}$ and $t'' \succeq_{\text{ID}} t'$. For any supermodular implementable scf, the induced game has a smallest and a greatest equilibrium along with a truthful equilibrium in between. Let $\theta_i^T(.)$ denote player i's truthful strategy, that is, $\theta_i^T(\theta_i) = \theta_i$ for all θ_i. Let $\mathcal{G}_\ell$ be the game $\mathcal{G}$ where the strategy spaces are restricted from $\Sigma_i(\Theta_i)$ to $[\inf \Sigma_i(\Theta_i), \theta_i^T(.)]$, and let $\mathcal{G}_u$ be the game $\mathcal{G}$ where the strategy spaces are restricted from $\Sigma_i(\Theta_i)$ to $[\theta_i^T(.), \sup \Sigma_i(\Theta_i)]$. Since closed intervals are sublattices and $\mathcal{G}$ is supermodular, those modified games $\mathcal{G}_\ell$ and $\mathcal{G}_u$ are supermodular games. Moreover, $\mathcal{G}_\ell$ must have the same least equilibrium as game $\mathcal{G}$ and the truthful equilibrium is its largest equilibrium. Likewise, $\mathcal{G}_u$ has the same greatest equilibrium as game $\mathcal{G}$ and the truthful equilibrium is its smallest equilibrium. Let $u_i^f(\hat{\theta}(.), t) = E_\theta[V_i(x_i(\hat{\theta}(\theta)), \theta_i)] + E_\theta[t_i(\hat{\theta}(\theta))]$. I show that (i) In $\mathcal{G}_\ell$, $u_i^f(\hat{\theta}_i(.), \hat{\theta}_{-i}(.), t)$ has decreasing differences in $(\hat{\theta}_i(.), t)$ for each $\hat{\theta}_{-i}(.)$ and (ii) In $\mathcal{G}_u$, $u_i^f(\hat{\theta}_i(.), \hat{\theta}_{-i}(.), t)$ has increasing differences in $(\hat{\theta}_i(.), t)$ for each $\hat{\theta}_{-i}(.)$. In those modified games, this shows how the untruthful extremal equilibrium varies in response to changes in transfers with respect to $\succeq_{\text{ID}}$. Before proving (i) and (ii), note that Proposition 12 implies that all transfers t_i such that (x, t) is implementable have the same expected value $E_{\theta_{-i}}[t_i(\hat{\theta}_i, \theta_{-i})]$ up to a constant. Taking any implementable scf $(x, \tilde{t})$, those transfers can thus be written $t_i(\hat{\theta}_i, \hat{\theta}_{-i}) = \delta_i(\hat{\theta}_i, \hat{\theta}_{-i}) - E_{\theta_{-i}}[\delta_i(\hat{\theta}_i, \theta_{-i})] + E_{\theta_{-i}}[\tilde{t}_i(\hat{\theta}_i, \theta_{-i})]$ for some function $\delta_i : \Theta \rightarrow \mathbb{R}$. First consider $\mathcal{G}_\ell$ and let δ'' and δ' be the δ functions corresponding to t'' and t'. Choose any $\theta_i''(.) > \theta_i'(.)$ and notice that for any deception $\hat{\theta}_{-i}(.)$, $\hat{\theta}_j(\theta_j) \leq \theta_j$ for all θ_j and $j \neq i$. Moreover, note $t'' \succeq_{\text{ID}} t'$ implies $\delta'' \succeq_{\text{ID}} \delta'$. Hence for all $i \in N$,

$$\begin{aligned} &E_\theta[\delta_i''(\theta_i''(\theta_i), \theta_{-i}) - \delta_i''(\theta_i''(\theta_i), \hat{\theta}_{-i}(\theta_{-i}))] - E_\theta[\delta_i''(\theta_i'(\theta_i), \theta_{-i}) - \delta_i''(\theta_i'(\theta_i), \hat{\theta}_{-i}(\theta_{-i}))] - \\ &- E_\theta[\delta_i'(\theta_i''(\theta_i), \theta_{-i}) - \delta_i'(\theta_i''(\theta_i), \hat{\theta}_{-i}(\theta_{-i}))] + E_\theta[\delta_i'(\theta_i'(\theta_i), \theta_{-i}) - \delta_i'(\theta_i'(\theta_i), \hat{\theta}_{-i}(\theta_{-i}))] \geq 0. \end{aligned} \tag{3.19}$$

But (3.19) is equivalent to

$$u_i^f(\theta_i''(.), \hat{\theta}_{-i}(.), t'') + u_i^f(\theta_i'(.), \hat{\theta}_{-i}(.), t') - u_i^f(\theta_i''(.), \hat{\theta}_{-i}(.), t') - u_i^f(\theta_i'(.), \hat{\theta}_{-i}(.), t'') \leq 0 \tag{3.20}$$

for each $\hat{\theta}_{-i}(.)$, which implies that $u_i^f(\hat{\theta}_i(.), \hat{\theta}_{-i}(.), t)$ has decreasing differences in $(\hat{\theta}_i(.), t)$ for each $\hat{\theta}_{-i}(.)$. It follows from Theorem 6 in Milgrom-Roberts [76] that the smallest equilibrium in $\mathcal{G}_\ell$ is decreasing in t. The same argument applies to $\mathcal{G}_u$. There, all deceptions $\hat{\theta}_{-i}(.)$ are such that $\hat{\theta}_j(\theta_j) \geq \theta_j$ for all θ_j and $j \neq i$. As a result, the sign in (3.19) is reversed, which implies $u_i^f(\hat{\theta}_i(.), \hat{\theta}_{-i}(.), t)$ has increasing differences in $(\hat{\theta}_i(.), t)$ for each $\hat{\theta}_{-i}(.)$. The greatest equilibrium in $\mathcal{G}_u$ is thus increasing in t. Q.E.D

Proof of Theorem 7: Suppose $f = (x, t)$ is implementable and x is dimensionally reducible. Letting

$$\delta_i(\hat{\theta}_i, \hat{\theta}_{-i}) = -\int_{\underline{\theta}_i}^{\hat{\theta}_i} \int_{r_i(\underline{\theta}_{-i})}^{r_i(\hat{\theta}_{-i})} \min_{\theta_i \in \Theta_i} \frac{\partial^2 V_i(h_i(s_i, r_i), \theta_i)}{\partial r_i \partial s_i} \, dr_i \, ds_i \tag{3.21}$$

for all $\hat{\theta} \in \Theta$, I show that

$$t_i^*(\hat{\theta}_i, \hat{\theta}_{-i}) = \delta_i(\hat{\theta}_i, \hat{\theta}_{-i}) - E_{\theta_{-i}}[\delta_i(\hat{\theta}_i, \theta_{-i})] + E_{\theta_{-i}}[t_i(\hat{\theta}_i, \theta_{-i})] \tag{3.22}$$

is well-defined and that (x, t^*) is optimally supermodular implementable. By Proposition 6, $E_{\theta_{-i}}[t_i(\hat{\theta}_i, \theta_{-i})]$ is well-defined and given by (3.13). Since V_i and h_i are C^2 on an open set containing compact set Θ_i, $\min_{\theta_i \in \Theta_i} \partial^2 V_i(h_i(s_i, r_i), \theta_i)/\partial r_i \partial s_i$ exists, it is continuous in (r_i, s_i) by the maximum theorem and it is bounded. Hence $\delta_i : \Theta \to \mathbb{R}$ is continuous, which implies that it is Borel-measurable. Since δ_i is also bounded, $E_{\theta_{-i}}[\delta_i(., \theta_{-i})]$ is well-defined and so is $t_i^* : \Theta \to \mathbb{R}$. The next step is to verify the continuity requirements. As a continuous function on a compact set, δ_i is uniformly continuous in $\hat{\theta}$, and so $E_\theta[t^*(\hat{\theta}(\theta))]$ is continuous in $\hat{\theta}_{-i}(.)$. Since V is C^2, (3.13) is usc in $\hat{\theta}_i$ and so is $E_{\theta_{-i}}[t_i(\hat{\theta}_i, \theta_{-i})]$ by Proposition 6, which implies $E_\theta[t_i^*(\hat{\theta}(\theta))]$ is usc in $\hat{\theta}_i(.)$. Put together, u_i^f satisfies the continuity requirements. Finally I prove that (x, t^*) is optimally supermodular implementable. Note $E_{\theta_{-i}}[t_i^*(\hat{\theta}_i, \theta_{-i})] = E_{\theta_{-i}}[t_i(\hat{\theta}_i, \theta_{-i})]$ and thus (x, t^*) is implementable. By construction, t_i^* is twice-differentiable[32] and

$$\frac{\partial^2 t_i^*(\hat\theta_i,\hat\theta_{-i})}{\partial\hat\theta_i\partial\hat\theta_j} = \frac{\partial^2 \delta_i(\hat\theta_i,\hat\theta_{-i})}{\partial\hat\theta_i\partial\hat\theta_j} = \frac{\partial}{\partial\hat\theta_j}\int_{r_i(\underline{\theta}_{-i})}^{r_i(\hat\theta_{-i})} -\min_{\theta_i\in\Theta_i}\frac{\partial^2 V_i(h_i(\hat\theta_i,r_i),\theta_i)}{\partial r_i\partial s_i}\,dr_i$$

$$= -\left(\min_{\theta_i\in\Theta_i}\frac{\partial^2 V_i(h_i(\hat\theta_i,r_i(\hat\theta_{-i})),\theta_i)}{\partial r_i\partial s_i}\right)\frac{\partial r_i(\hat\theta_{-i})}{\partial\hat\theta_j}. \quad (3.23)$$

Because

$$-\min_{\theta_i\in\Theta_i}\frac{\partial^2 V_i(x_i(\hat\theta_i,\hat\theta_{-i}),\theta_i)}{\partial\hat\theta_i\partial\hat\theta_j} = -\min_{\theta_i\in\Theta_i}\left(\frac{\partial^2 V_i(h_i(\hat\theta_i,r_i(\hat\theta_{-i})),\theta_i)}{\partial r_i\partial s_i}\frac{\partial r_i(\hat\theta_{-i})}{\partial\hat\theta_j}\right) \quad (3.24)$$

and $r_i(.)$ is an increasing function, (3.23) and (3.24) are equal. Therefore, $\partial^2[V_i(x_i(\hat\theta),\theta_i) + t_i^*(\hat\theta)]/\partial\hat\theta_i\partial\hat\theta_j$ is equal to

$$\left(\frac{\partial^2 V_i(h_i(\hat\theta_i,r_i(\hat\theta_{-i})),\theta_i)}{\partial r_i\partial s_i} - \min_{\theta_i\in\Theta_i}\frac{\partial^2 V_i(h_i(\hat\theta_i,r_i(\hat\theta_{-i})),\theta_i)}{\partial r_i\partial s_i}\right)\frac{\partial r_i(\hat\theta_{-i})}{\partial\hat\theta_j} \geq 0 \quad (3.25)$$

for all $\hat\theta$, θ_i and j, i, and so (x, t^*) is supermodular implementable. Moreover, for all transfers $t \in \mathcal{T}$ such that (x, t) is implementable, it must be that

$$\frac{\partial^2 t_i(\hat\theta)}{\partial\hat\theta_i\partial\hat\theta_j} \geq -\min_{\theta_i\in\Theta_i}\frac{\partial^2 V_i(x_i(\hat\theta_i,\hat\theta_{-i}),\theta_i)}{\partial\hat\theta_i\partial\hat\theta_j} = \frac{\partial^2 t_i^*(\hat\theta)}{\partial\hat\theta_i\partial\hat\theta_j}$$

for all $\hat\theta$ and j, i. This implies that (x, t^*) is optimally supermodular implementable.

Q.E.D

Proof of Theorem 8: 1. By way of contradiction, suppose that the truthful equilibrium is not the unique Bayesian equilibrium. Since the scf is supermodular implementable, there exist a greatest and a smallest equilibrium in the game induced by the mechanism. So, one of these extremal equilibria must be strictly greater/smaller than the truthful one. Suppose that the greatest equilibrium, denoted $(\overline{\theta}_i(.))_{i\in N}$, is strictly greater than the truthful equilibrium. That is, for all i, $\overline{\theta}_i(\theta_i) \geq \theta_i$ for a.e θ_i, and there exists $N^* \neq \emptyset$ such that, for all $i \in N^*$, $\overline{\theta}_i(\theta_i) > \theta_i$ for all θ_i in some subset of types with positive measure.

[32]See previous footnote.

I evaluate the first-order condition of agent i's maximization program at the greatest equilibrium; then, I bound it from above by an expression which cannot be positive for all players (hence the contradiction). Consider player i's interim utility at type θ_i against $\overline{\theta}_{-i}(.)$:

$$E_{\theta_{-i}}[V_i(x_i(\hat{\theta}_i, \overline{\theta}_{-i}(\theta_{-i})), \theta_i)] + E_{\theta_{-i}}[t_i(\hat{\theta}_i, \overline{\theta}_{-i}(\theta_{-i}))]. \tag{3.26}$$

Since V_i, x_i, and t_i (by Proposition 12) are continuously differentiable, we can show that for any deception $\hat{\theta}_{-i}(.)$ the first-derivative of (3.26) with respect to $\hat{\theta}_i$ is

$$E_{\theta_{-i}}\left[\frac{\partial V_i(x_i(\hat{\theta}_i, \overline{\theta}_{-i}(\theta_{-i})), \theta_i)}{\partial \hat{\theta}_i}\right] + E_{\theta_{-i}}\left[\frac{\partial t_i(\hat{\theta}_i, \overline{\theta}_{-i}(\theta_{-i}))}{\partial \hat{\theta}_i}\right]. \tag{3.27}$$

By assumption, the utility functions and the decision rule produce bounded complements, so we have

$$E_{\theta_{-i}}\left[\frac{\partial V_i(x_i(\hat{\theta}_i, \overline{\theta}_{-i}(\theta_{-i})), \theta_i)}{\partial \hat{\theta}_i} + \frac{\partial t_i(\hat{\theta}_i, \overline{\theta}_{-i}(\theta_{-i}))}{\partial \hat{\theta}_i} - \frac{\partial V_i(x_i(\hat{\theta}_i, \theta_{-i}), \theta_i)}{\partial \hat{\theta}_i} - \frac{\partial t_i(\hat{\theta}_i, \theta_{-i})}{\partial \hat{\theta}_i}\right] \tag{3.28}$$

$$\leq \int_{\Theta_{-i}} K_i(\theta_i) \sum_{j \neq i} (\overline{\theta}_j(\theta_j) - \theta_j) \phi_{-i}(\theta_{-i}) d\theta_{-i} = K_i(\theta_i) \sum_{j \neq i} E_{\theta_j}[\overline{\theta}_j(\theta_j) - \theta_j]. \tag{3.29}$$

By (3.28) and (3.29),

$$(3.27) \leq K_i(\theta_i) \sum_{j \neq i} E_{\theta_j}[\overline{\theta}_j(\theta_j) - \theta_j] + E_{\theta_{-i}}\left[\frac{\partial V_i(x_i(\hat{\theta}_i, \theta_{-i}), \theta_i)}{\partial \hat{\theta}_i}\right] + E_{\theta_{-i}}\left[\frac{\partial t_i(\hat{\theta}_i, \theta_{-i})}{\partial \hat{\theta}_i}\right]. \tag{3.30}$$

By part (i) of Proposition 12,

$$E_{\theta_{-i}}\left[\frac{\partial t_i(\hat{\theta}_i, \theta_{-i})}{\partial \hat{\theta}_i}\right] = -E_{\theta_{-i}}\left[\left.\frac{\partial V_i(x_i(\theta'_i, \theta_{-i}), \hat{\theta}_i)}{\partial \theta'_i}\right|_{\theta'_i = \hat{\theta}_i}\right].$$

Therefore, (3.30) implies

$$\begin{aligned} (3.27) \quad &\leq \quad K_i(\theta_i) \sum_{j \neq i} E_{\theta_j}[\overline{\theta}_j(\theta_j) - \theta_j] + E_{\theta_{-i}}\left[\frac{\partial V_i(x_i(\hat{\theta}_i, \theta_{-i}), \theta_i)}{\partial \hat{\theta}_i} - \frac{\partial V_i(x_i(\hat{\theta}_i, \theta_{-i}), \hat{\theta}_i)}{\partial \theta'_i}\right] \\ &\leq \quad K_i(\theta_i) \sum_{j \neq i} E_{\theta_j}[\overline{\theta}_j(\theta_j) - \theta_j] + E_{\theta_{-i}}[\gamma_i(\theta_{-i})](\theta_i - \hat{\theta}_i), \end{aligned} \tag{3.31}$$

where the last inequality follows from strong differences.

Since it is optimal for each player i to play $\overline{\theta}_i(\theta_i)$ for a.e type θ_i, then the RHS of (3.31) evaluated at $\hat{\theta}_i = \overline{\theta}_i(\theta_i)$ must be positive for a.e θ_i and all i. To see why, let $\Theta_i^* \subset \Theta_i$ be the set of θ_i for which the RHS of (3.31) is strictly negative when playing $\overline{\theta}_i(\theta_i)$. Note Θ_i^* is measurable by definition, because the RHS of (3.31) is a measurable function in θ_i when plugging $\overline{\theta}_i(\theta_i)$. If there were a player i for whom Θ_i^* had strictly positive measure, then playing $\overline{\theta}_i(\theta_i)$ would lead (3.27) to be strictly negative for all $\theta_i \in \Theta_i^*$. But for types in Θ_i^*, player can announce types in $[\underline{\theta}_i, \overline{\theta}_i(\theta_i)]$ and so she would strictly prefer playing $\theta_i^*(\theta_i) = \overline{\theta}_i(\theta_i) - \varepsilon \mathbf{1}_{\Theta_i^*}$ for some small ε.[33]

Since the RHS of (3.31) is positive for a.e θ_i when playing $\overline{\theta}_i(\theta_i)$, then it must be true in expectation for all i,

$$0 \leq E_{\theta_i}[K_i(\theta_i)] \sum_{j \neq i} E_{\theta_j}[\overline{\theta}_j(\theta_j) - \theta_j] + E_{\theta_{-i}}[\gamma_i(\theta_{-i})] E_{\theta_i}[\theta_i - \overline{\theta}_i(\theta_i)] \tag{3.32}$$

Letting

$$\mathcal{C} = \begin{pmatrix} -E_{\theta_{-1}}[\gamma_1(\theta_{-1})] & E_{\theta_1}[K_1(\theta_1)] & \cdots & E_{\theta_1}[K_1(\theta_1)] \\ E_{\theta_2}[K_2(\theta_2)] & -E_{\theta_{-2}}[\gamma_2(\theta_{-2})] & \cdots & E_{\theta_2}[K_2(\theta_2)] \\ \vdots & \vdots & \ddots & \vdots \\ E_{\theta_n}[K_n(\theta_n)] & E_{\theta_n}[K_n(\theta_n)] & \cdots & -E_{\theta_{-n}}[\gamma_n(\theta_{-n})] \end{pmatrix},$$

(3.32) implies the existence of a positive solution w^* to the system $\mathcal{C}.w \geq 0$. But then it must be that $w^{*T}\mathcal{C}.w^* \geq 0$, a contradiction because $\mathcal{C}$ is the matrix of complementarities and it is negative definite. The same argument applies to show that there is no equilibrium that is smaller than the truthful equilibrium.

2. Find real numbers a_i and b_i such that $K_i(\theta_i) \leq a_i\theta_i + b_i$ for all θ_i. Define

$$U_i(\hat{\theta}_i, \hat{\theta}_{-i}(.), \theta_i) = (a_i\theta_i + b_i)\hat{\theta}_i \sum_{j \neq i} E_{\theta_j}[\hat{\theta}_j(\theta_j) - \theta_j] + E_{\theta_{-i}}[\gamma_i(\theta_{-i})]\hat{\theta}_i \left(\theta_i - \frac{\hat{\theta}_i}{2}\right). \tag{3.33}$$

It follows from (3.31) that $\partial U_i / \partial \hat{\theta}_i$ is larger than $\partial u_i / \partial \hat{\theta}_i$ when agents announce types above their true types. A similar argument to the proof of Proposition 7 then implies that the game $(N, \{\Sigma_i(\Theta_i), U_i\})$ has a greatest equilibrium[34] which is larger than the greatest

[33] Note $\theta_i^*(.) \in \Sigma_i(\Theta_i)$ because $\overline{\theta}_i(.) \in \Sigma_i(\Theta_i)$, so $\theta_i^*(.)$ is a possible choice of deception.

[34] From (3.33), this game is obviously supermodular.

equilibrium of $\mathcal{G}$. Letting

$$A_i = 1 + \frac{a_i(\sum_{j\neq i} E_{\theta_j}[\hat{\theta}_j(\theta_j)] - E_{\theta_j}[\theta_j])}{E_{\theta_{-i}}[\gamma_i(\theta_{-i})]},\ B_i = \frac{b_i(\sum_{j\neq i} E_{\theta_j}[\hat{\theta}_j(\theta_j)] - E_{\theta_j}[\theta_j])}{E_{\theta_{-i}}[\gamma_i(\theta_{-i})]} \tag{3.34}$$

and given $K_i(\theta_i) \geq 0$, we obtain i's best-response to $\hat{\theta}_{-i}(.)$ from (3.33),

$$br_i[\hat{\theta}_{-i}(.)] = \begin{cases} A_i\theta_i + B_i & \text{if } \underline{\theta}_i \leq \theta_i \leq \frac{\overline{\theta}_i - B_i}{A_i} \\ \overline{\theta}_i & \text{otherwise.} \end{cases} \tag{3.35}$$

Computing the expected value of j's best-response from (3.35) gives

$$E_{\theta_j}[br_j[\hat{\theta}_{-j}(.)]] = A_j \int_{\underline{\theta}_j}^{\frac{\overline{\theta}_j - B_j}{A_j}} \theta_j\phi_j(\theta_j)d\theta_j + \Phi_j\left(\frac{\overline{\theta}_j - B_j}{A_j}\right)(B_j - \overline{\theta}_j) + \overline{\theta}_j. \tag{3.36}$$

Plugging (3.36) back into (3.34) results in a system of $2n$ equations: For $i = 1, \ldots, n$,

$$\begin{cases} A_i = 1 + \dfrac{a_i\left(\sum_{j\neq i} A_j \int_{\underline{\theta}_j}^{\frac{\overline{\theta}_j - B_j}{A_j}} \theta_j\phi_j(\theta_j)d\theta_j + \Phi_j\left(\frac{\overline{\theta}_j - B_j}{A_j}\right)(B_j - \overline{\theta}_j) + \overline{\theta}_j - E_{\theta_j}[\theta_j]\right)}{E_{\theta_{-i}}[\gamma_i(\theta_{-i})]} \\ B_i = \dfrac{b_i\left(\sum_{j\neq i} A_j \int_{\underline{\theta}_j}^{\frac{\overline{\theta}_j - B_j}{A_j}} \theta_j\phi_j(\theta_j)d\theta_j + \Phi_j\left(\frac{\overline{\theta}_j - B_j}{A_j}\right)(B_j - \overline{\theta}_j) + \overline{\theta}_j - E_{\theta_j}[\theta_j]\right)}{E_{\theta_{-i}}[\gamma_i(\theta_{-i})]}. \end{cases}$$

This system has at least one solution where $A_i = 1$ and $B_i = 0$ for $i = 1, \ldots, n$ and it corresponds to the truthtelling equilibrium. There also exists a solution which corresponds to the greatest equilibrium of $(N, \{\Sigma_i(\Theta_i), U_i\})$. This solution defines strategies that bound the equilibrium set of $\mathcal{G}$ from above. Similarly, by looking at strategies of the form

$$\begin{cases} \underline{\theta}_i & \text{if } \underline{\theta}_i < \theta_i < \underline{\theta}_i - \frac{B_i}{A_i} \\ A_i\theta_i + B_i & \text{otherwise} \end{cases} \tag{3.37}$$

where $B_i \leq 0$ instead of (3.35), we can construct a system of $2n$ equations whose smallest solution provides a lower bound for the equilibrium set of $\mathcal{G}$: For $i = 1, \ldots, n$,

$$\begin{cases} A_i &= 1 + \dfrac{a_i \left(\displaystyle\sum_{j \neq i} A_j \int_{\underline{\theta}_j - \frac{B_j}{A_j}}^{\overline{\theta}_j} \theta_j \phi_j(\theta_j) d\theta_j + B_j + \Phi_j\!\left(\underline{\theta}_j - \frac{B_j}{A_j}\right)(\underline{\theta}_j - B_j) - E_{\theta_j}[\theta_j] \right)}{E_{\theta_{-i}}[\gamma_i(\theta_{-i})]} \\ B_i &= \dfrac{b_j \left(\displaystyle\sum_{j \neq i} A_j \int_{\underline{\theta}_j - \frac{B_j}{A_j}}^{\overline{\theta}_j} \theta_j \phi_j(\theta_j) d\theta_j + B_j + \Phi_j\!\left(\underline{\theta}_j - \frac{B_j}{A_j}\right)(\underline{\theta}_j - B_j) - E_{\theta_j}[\theta_j] \right)}{E_{\theta_{-i}}[\gamma_i(\theta_{-i})]}. \end{cases}$$

Q.E.D

Proof of Proposition 8: Since the valuations and the decision rule produce γ-increasing differences, $\partial E_{\theta_{-i}}[V_i(x_i(\hat{\theta}_i, \theta_{-i}),\ \theta_i)]/\partial\theta_i$ is strictly increasing in $\hat{\theta}_i$. Let transfers be the optimal transfers defined by (3.21) and (3.22), where t_i is given by (3.13). By assumption, $E_{\theta_{-i}}[V_i(x_i(\hat{\theta}_i, \theta_{-i}), \theta_i)]$ is continuous in $(\hat{\theta}_i, \theta_i)$, so Proposition 12 and Theorem 7 imply (x, t^*) is supermodular implementable. It follows from (3.25) that $u_i \circ f$ has bounded complements, because V and x are C^2. The bound κ_i on complements is computed as follows,

$$\kappa_i = \max_{j \neq i} \max_{(\hat{\theta}, \theta_i) \in \Theta \times \Theta_i} \left(\frac{\partial^2 V_i(x_i(\hat{\theta}), \theta_i)}{\partial \hat{\theta}_i \partial \hat{\theta}_j} - \min_{\theta_i \in \Theta_i} \frac{\partial^2 V_i(x_i(\hat{\theta}), \theta_i)}{\partial \hat{\theta}_i \partial \hat{\theta}_j} \right).$$

Since x is dimensionally reducible and V is C^2, the first derivative of t_i^* in $\hat{\theta}_i$ is uniformly bounded above. Hence transfers are β_i-Lipschitz in $\hat{\theta}_i$. Applying Theorem 8 completes the proof. Q.E.D

Proof of Theorem 6: Let

$$H_i(\hat{\theta}_{-i}) = -\left(\frac{1}{n-1}\right) \sum_{j \neq i} E_{\tilde{\theta}_{-j}} \left[\sum_{k \neq j} V_k(x_k(\hat{\theta}_j, \tilde{\theta}_{-j}), \tilde{\theta}_k) \right],$$

and for $\rho_i \in \mathbb{R}$, let

$$\delta_i(\hat{\theta}_i, \hat{\theta}_{-i}) = \sum_{j \neq i} \rho_i \hat{\theta}_i \hat{\theta}_j.$$

Define

$$t_i^{BB}(\hat{\theta}_i, \hat{\theta}_{-i}) = \delta_i(\hat{\theta}_i, \hat{\theta}_{-i}) - E_{\theta_{-i}}[\delta_i(\hat{\theta}_i, \theta_{-i})] + E_{\tilde{\theta}_{-i}}\left[\sum_{j\neq i} V_j(x_j(\hat{\theta}_i, \tilde{\theta}_{-i}), \tilde{\theta}_j)\right] + H_i(\hat{\theta}_{-i}) -$$
$$- \frac{1}{n-2}\sum_{j\neq i}\sum_{k\neq i,j} \rho_j \hat{\theta}_j \hat{\theta}_k + \frac{1}{n-2}\sum_{j\neq i}\sum_{k\neq i,j} \rho_j \hat{\theta}_j E(\theta_k). \quad (3.38)$$

First, (x, t^{BB}) is implementable because $x(.)$ is allocation-efficient and

$$E_{\theta_{-i}}[t_i^{BB}(\hat{\theta}_i, \theta_{-i})] = E_{\tilde{\theta}_{-i}}\left[\sum_{j\neq i} V_j(x_j(\hat{\theta}_i, \tilde{\theta}_{-i}), \tilde{\theta}_j)\right] + E_{\theta_{-i}}[H_i(\theta_{-i})],$$

which is the expectation of the transfers in the expected externality mechanism. Second, note that for all θ,

$$\sum_{i\in N}\left(\delta_i(\theta_i, \theta_{-i}) - \frac{1}{n-2}\sum_{j\neq i}\sum_{k\neq i,j} \rho_j \theta_j \theta_k\right) = \sum_{i\in N}\delta_i(\theta_i, \theta_{-i}) - \frac{1}{n-2}\sum_{i\in N}\sum_{j\neq i}(n-2)\rho_i\theta_i\theta_j = 0$$

and

$$\sum_{i\in N}\left(\frac{1}{n-2}\sum_{j\neq i}\sum_{k\neq i,j} \rho_j \theta_j E(\theta_k) - E_{\theta_{-i}}[\delta_i(\theta_i, \theta_{-i})]\right) =$$
$$= \frac{1}{n-2}\sum_{i\in N}\sum_{j\neq i}(n-2)\rho_i\theta_i E(\theta_j) - \sum_{i\in N} E_{\theta_{-i}}[\delta_i(\theta_i, \theta_{-i})] = 0,$$

hence

$$\sum_{i\in N} t_i^{BB}(\theta) = \sum_{i\in N} E_{\tilde{\theta}_{-i}}\left[\sum_{j\neq i} V_j(x_j(\theta_i, \tilde{\theta}_{-i}), \tilde{\theta}_j)\right] + \sum_{i\in N} H_i(\theta_{-i}) = 0,$$

because transfers are balanced in the expected externality mechanism. Furthermore, t_i^{BB} is clearly continuous in $\hat{\theta}_{-i}$ for each $\hat{\theta}_i$ and usc in $\hat{\theta}_i$ for each $\hat{\theta}_{-i}$. From standard arguments, $E_\theta[t_i^{SM}(\hat{\theta}_i(\theta_i), \hat{\theta}_{-i}(\theta_{-i}))]$ is continuous in $\hat{\theta}_{-i}(.)$ and usc in $\hat{\theta}_i(.)$. Next I show that it is possible to take ρ_i so that the complete information payoffs have increasing differences in $(\hat{\theta}_i, \hat{\theta}_{-i})$. Since substitutes are uniformly bounded, there exists T_i such that, for all $\theta_i'' \geq \theta_i'$ and $\theta_{-i}'' \geq \theta_{-i}'$, $\Delta V_i(\theta_i'', \theta_i', \theta_{-i}'', \theta_i) - \Delta V_i(\theta_i'', \theta_i', \theta_{-i}', \theta_i) \geq T_i(\theta_i'' - \theta_i')\sum(\theta_j'' - \theta_j')$ for all $\theta_i \in \Theta_i$.

Set $\rho_i > -T_i$. Choose any $\theta''_{-i} \geq_{-i} \theta'_{-i}$ and $\theta''_i > \theta'_i$. From (3.38), note

$$t_i^{BB}(\theta''_i, \theta''_{-i}) - t_i^{BB}(\theta''_i, \theta'_{-i}) - t_i^{BB}(\theta'_i, \theta''_{-i}) + t_i^{BB}(\theta'_i, \theta'_{-i}) = \\ = \delta_i(\theta''_i, \theta''_{-i}) - \delta_i(\theta''_i, \theta'_{-i}) - \delta_i(\theta'_i, \theta''_{-i}) + \delta_i(\theta'_i, \theta'_{-i}). \quad (3.39)$$

If the following expression is positive, then $u_i(x_i(\hat{\theta}_i, \hat{\theta}_{-i}), \theta_i)$ has increasing differences in $(\hat{\theta}_i, \hat{\theta}_{-i})$ for all θ_i,

$$\Delta V_i(\theta''_i, \theta'_i, \theta''_{-i}, \theta_i) - \Delta V_i(\theta''_i, \theta'_i, \theta'_{-i}, \theta_i) + \sum_{j \neq i} \rho_i\big(\theta''_i\theta''_j + \theta'_i\theta'_j - \theta''_i\theta'_j - \theta'_i\theta''_j\big). \quad (3.40)$$

The proof then follows similarly to that of Theorem 5. Q.E.D

Proof of Proposition 9: Since $\tau_i - T_i < \gamma_i/(n-1)$, then $\rho_i = -T_i$ implies $\rho_i + \tau_i < \gamma_i/(n-1)$.

By Theorem 6, (x, t^{BB}) is supermodular implementable whenever $\rho_i \geq -T_i$.

Because $V_i \circ x_i(.)$ has complements bounded by τ_i, the definition of t_i^{BB} implies that $u_i \circ f$ has complements bounded by $\rho_i + \tau_i$. Theorem 8 completes the proof. Q.E.D

Proof of Proposition 10: Let $O \supset \Theta$ be some open set and define the extension of $x(.)$ from Θ to O. For any $\theta \in O$, let $\iota_1(\theta) = \{j \in N : \theta_j \in [\underline{\theta}_j, \overline{\theta}_j]\}$, $\iota_2(\theta) = \{j \in N : \theta_j < \underline{\theta}_j\}$, and $\iota_3(\theta) = \{j \in N : \theta_j > \overline{\theta}_j\}$. The extension of $x(.)$ from Θ to O, denoted x^e, is such that for all $\theta \in O$, $x^e_{(i,k)}(\theta) = x_{(i,k)}((\theta_j)_{\iota_1(\theta)}, (\underline{\theta}_j)_{\iota_2(\theta)}, (\overline{\theta}_j)_{\iota_3(\theta)})$ for all k and $i \in N$. Note that $x^e_{(i,k)} \in L_p(O)$ and it is increasing in $\hat{\theta}_i$ because $x_{(i,k)}$ is increasing in $\hat{\theta}_i$. By Theorem 12.10 in [4], the space of C^2-functions on O is norm dense in $L_p(O)$, hence there exists a sequence $\{x_n\}$ of C^2-functions from O into $\mathbb{R}$ such that $\lim_{n\to\infty}(\int_O |x_{n,(i,k)} - x^e_{(i,k)}|^p)^{1/p} = 0$ for all k and i. This implies $\lim_{n\to\infty}(\int_\Theta |x_{n,(i,k)} - x_{(i,k)}|^p)^{1/p} = 0$ for all k and all i. Moreover, we can take $\{x_n\}$ such that $x_{n,(i,k)}$ is increasing in θ_i on O_i for all k and i.[35] By definition, V and x are C^2 if there exist open sets $U_i \supset X_i$, $i = 1, \ldots, n$, such that $V : U_i \times O_i \to \mathbb{R}$ and $x : \prod_{i\in N} O_i \to U_i$ are C^2. Therefore, since each Θ_i is compact and V and x_n are C^2, then they form a continuous family, $\partial E_{\theta_{-i}}[V_i(x_{n,(i,k)}(\hat{\theta}), \theta_i)]/\partial\theta_i = E_{\theta_{-i}}[\partial V_i(x_{n,(i,k)}(\hat{\theta}), \theta_i)/\partial\theta_i]$ is increasing in $\hat{\theta}_i$ on Θ_i and substitutes are bounded. Proposition 6 and Theorem 5 imply

[35] Since $x_{i,k}$ is increasing in θ_i, it is always possible to take the members of the approximating sequence to be increasing (see Mas-Colell [67]).

that, for all n, there exist t_n^{SM} such that $f = (x_n, t_n^{SM})$ is supermodular implementable.Q.E.D

Proof of Proposition 11: The proof begins with an approximation of the functions $h_{(i,k)} : \mathbb{R}^2 \to \mathbb{R}$ and $r_i : \mathbb{R}^n \to \mathbb{R}$ by C^2-functions, and it studies the convergence of the resulting composite function. Let μ^n denote the Lebesgue measure on $\mathbb{R}^n$. Because type sets are compact and h_i is bounded, Theorem 12.10 in [4] guarantees the existence of a sequence of C^2-functions that converges to $h_{(i,k)}$ in $L_1(\mu^2)$-norm. Since h_i is bounded, we can take that sequence so that each element is (uniformly) bounded. From this sequence, Theorem 12.6 in [4] implies that we can extract a subsequence $\{h^m_{(i,k)}\}$ of C^2-functions that converges pointwise to $h_{(i,k)}$ for μ^2-almost all (θ_i, r_i). Now consider function $r_i(.)$. By the Stone-Weierstrass theorem, for all $i \in N$ there exists a sequence of C^2-increasing functions $\{r_i^q\}$ that uniformly converges to r_i.[36] The triangle inequality gives

$$\int_\Theta |h^m_{(i,k)}(\theta_i, r_i^q(\theta_{-i})) - h_{(i,k)}(\theta_i, r_i(\theta_{-i}))| d\mu^n \leq \int_\Theta |h^m_{(i,k)}(\theta_i, r_i^q(\theta_{-i})) - h^m_{(i,k)}(\theta_i, r_i(\theta_{-i}))| d\mu^n + \int_\Theta |h^m_{(i,k)}(\theta_i, r_i(\theta_{-i})) - h_{(i,k)}(\theta_i, r_i(\theta_{-i}))| d\mu^n. \quad (3.41)$$

The next step is to demonstrate that the second integral in the RHS of (3.41) converges to zero, as a result of the μ^2-a.e convergence of $h^m_{(i,k)}$.[37] Note that

$$\int_\Theta |h^m_{(i,k)}(\theta_i, r_i(\theta_{-i})) - h_{(i,k)}(\theta_i, r_i(\theta_{-i}))| d\mu^n = \int_{\Theta_i \times r_i(\Theta_{-i})} |h^m_{(i,k)}(\theta_i, t) - h_{(i,k)}(\theta_i, t)| d\mu \times \mu_{r_i} \quad (3.42)$$

where $\mu_{r_i} = \mu^{n-1} \circ r_i^{-1}$. One way to proceed is to apply the Radon-Nikodym theorem. To this end, I show that μ_{r_i} is absolutely continuous with respect to μ. By way of contradiction, suppose that $\mu(A) = 0$ for some set A and that there are countable unions of intervals, $\cup_k I_j^k \subset \mathbb{R}$, such that $r_i(\theta_{-i}) \in A$ for all $\theta_{-i} \in \prod_j (\cup_k I_j^k)$. Since $r_i(.)$ is continuous and strictly increasing, $r_i(\prod_j(\cup_k I_j^k))$ must contain some interval I, in which case $I \subset A$ and $\mu(A) > 0$. This is a contradiction. Therefore, for any A such that $\mu(A) = 0$, there is no $\{\cup_k I_j^k\}$ such that $r_i^{-1}(A) \subset \prod_j(\cup_k I_j^k)$, which implies $\mu_{r_i}(A) = 0$. As a result, μ_{r_i} is absolutely continuous with respect to μ. Clearly, both μ_{r_i} and μ are (totally) finite on $r_i(\Theta_{-i})$. By the

[36] Since r_i is increasing, recall that we can take the members of the approximating sequence to be increasing.

[37] This is indeed not immediate. Suppose $\lim_{m\to\infty} h^m_{(i,k)} = h_{(i,k)}$ except for $\{(\theta_i, r_i^*) : \theta_i \in I\}$ where I is some interval. If $r_i(.)$ is constant and equal to r_i^*, then $\lim_{m\to\infty} h^m_{(i,k)} = h_{(i,k)}$ μ^2-a.e, but $\int_\Theta |h^m_{(i,k)}(\theta_i, r_i(\theta_{-i})) - h_{(i,k)}(\theta_i, r_i(\theta_{-i}))| d\mu^n$ does not converge to 0.

Radon-Nikodym theorem, there exists f on $r_i(\Theta_{-i})$ such that $\mu_{r_i}(A) = \int_A f d\mu$ for every measurable set $A \subset r_i(\Theta_{-i})$. From (3.42), it gives

$$\int_\Theta |h^m_{(i,k)}(\theta_i, r_i(\theta_{-i})) - h_{(i,k)}(\theta_i, r_i(\theta_{-i}))| d\mu^n = \int_{\Theta_i \times r_i(\Theta_{-i})} |h^m_{(i,k)}(\theta_i, t) - h_{(i,k)}(\theta_i, t)| f(t) d\mu^2. \tag{3.43}$$

Since $|h^m_{(i,k)}(\theta_i, t) - h_{(i,k)}(\theta_i, t)|f(t)$ is integrable and dominated a.e by $Hf(t)$ for $H > 0$ sufficiently large, the limit of the RHS of (3.43) as $m \to \infty$ is given by the (integral of the) limit of the integrand, and this limit is 0. This result allows to construct the following subsequence from $\{h^m_i(\theta_i, r^q_i(\theta_{-i}))\}$:

1. For each m, take $\alpha(m)$ such that $\int_\Theta |h^{\alpha(m)}_{(i,k)}(\theta_i, r_i(\theta_{-i})) - h_{(i,k)}(\theta_i, r_i(\theta_{-i}))|\, d\mu^n < 1/2m$.

2. Since $h^{\alpha(m)}$ is C^2, $h^{\alpha(m)}_{(i,k)}(\theta_i, r^q_i(\theta_{-i}))$ converges uniformly to $h^{\alpha(m)}_i(\theta_i, r_i(\theta_{-i}))$ as $q \to \infty$; thus choose $\beta(m)$ such that $\int_\Theta |h^{\alpha(m)}_i(\theta_i, r^{\beta(m)}_i(\theta_{-i})) - h_i(\theta_i, r_i(\theta_{-i}))|\, d\mu^n < 1/2m$.

Along the subsequence so constructed, the LHS of (3.41) is less than $1/m$ for all m and thus it converges to $h_i(., r_i(.))$ in L_1-norm. In other words, there is a sequence of dimensionally-reducible decision rules $\{x^m_i\}$ that converges to x_i in L_1-space. Implementability of each x^m follows from the fact that $\partial V_i(x_i, \theta_i)/\partial\theta_i$ is increasing in x_i and $x^m_i(.)$ is increasing in $\hat{\theta}_i$ for each m. Hence $x^m(.)$ is implementable. Theorem 7 completes the proof. Q.E.D

Lemma **6** *Let $(X, \geq)$ be a complete lattice. For $Y \supset X$, let $\phi : X \twoheadrightarrow Y$ be a correspondence whose range is Y and such that for all $x \in X$, $x \in \phi(x)$ and $\phi(x') \cap \phi(x) = \emptyset$ for all $x' \neq x$. Then, there exists an extension $\geq^*$ of $\geq$ such that:*

(i) $(Y, \geq^*)$ is a complete lattice,

(ii) For all distinct $x, x' \in X$, and all $y \in \phi(x)$, $y' \in \phi(x')$, $y \geq^* y'$ iff $x \geq x'$,

(iii) For all $x \in X$, $\phi(x)$ is a complete chain.

Proof: Define $\geq^*$ on X as $\geq$. Then, for all distinct $x, x' \in X$, and all $y \in \phi(x)$, $y' \in \phi(x')$, let $\geq^*$ be such that $y \geq^* y'$ iff $x \geq x'$. So (ii) is satisfied. Finally, complete the definition of $\geq^*$ by using the well ordering principle of set theory. This result implies that, for all $x \in X$, there exists $\succeq$ on $\phi(x)$ such that $(\phi(x), \succeq)$ is a chain, and such that any $B \subset \phi(x)$ has a least upper bound and a greatest lower bound in $\phi(x)$.[38] Define $\geq^*$ to be equal to

[38]Take $\omega \in \phi(x)$. By the well ordering principle, there is an order that well orders $\phi(x) \backslash \{\omega\}$. Extend this order to all of $\phi(x)$ by setting ω as the greatest element. Let $\succeq$ be the extension. Since $(\phi(x), \succeq)$ is also well

$\succeq$ on $\phi(x)$ for each $x \in X$. Therefore, for all $x \in X$, $\phi(x)$ is a complete chain and (iii) is satisfied. I show next that $(Y, \geq^*)$ is a complete lattice with the order $\geq^*$ just defined on all of Y.

First, I prove that it is a partially ordered set. For all $x \in X$, $x \in \phi(x)$ and thus $x \geq^* x$ because $(\phi(x), \geq^*)$ is a chain. This proves reflexivity. Now take $y_1, y_2, y_3 \in Y$ such that $y_1 \geq^* y_2$ and $y_2 \geq^* y_3$. If $y_1 \in \phi(x_1)$, $y_2 \in \phi(x_2)$, and $y_3 \in \phi(x_3)$ where x_1, x_2, x_3 are distinct, then $y_1 \geq^* y_2$ implies $x_1 > x_2$ and $y_2 \geq^* y_3$ implies $x_2 > x_3$. By transitivity of $\geq$, we have $x_1 > x_3$, which implies $y_1 \geq^* y_3$. Suppose that $y_1, y_2 \in \phi(x_1)$ and $y_3 \in \phi(x_3)$ for distinct $x_1, x_3 \in X$. Since $y_2 \geq^* y_3$, we have $x_1 > x_3$ which implies $y_1 \geq^* y_3$. If $y_1, y_2, y_3 \in \phi(x_1)$, then $y_1 \geq^* y_3$ because $(\phi(x_1), \geq^*)$ is a chain, which shows transitivity. Now, if $y_1 \geq^* y_2$ and $y_2 \geq^* y_1$ for some $y_1 \in \phi(x_1)$ and $y_2 \in \phi(x_2)$, then $x_1 = x_2$. Therefore, $y_1, y_2 \in \phi(x_1)$ and so $y_1 = y_2$ because $(\phi(x_1), \geq^*)$ is a chain. This establishes antisymmetry.

Secondly, I prove that $\sup_Y S$ and $\inf_Y S$ exist, so $(Y, \geq^*)$ is a complete lattice. Let $\mathcal{X} \subset X$ be the set of x's whose image intersects S: $x \in \mathcal{X}$ iff $S \cap \phi(x) \neq \emptyset$. If $|\mathcal{X}| = 1$, then $S \subset \phi(x)$ where x is the unique element of $\mathcal{X}$. By definition of $\geq^*$, S has an infimum and a supremum in $\phi(x) \subset Y$. Now assume $|\mathcal{X}| \geq 2$ and let $S(x) = S \cap \phi(x)$ for all $x \in \mathcal{X}$. Note $\{S(x)\}_{x \in \mathcal{X}}$ forms a partition of S. Define $\overline{s}(x) = \sup_Y S(x)$ and $\underline{s}(x) = \inf_Y S(x)$, which exist and belong to $\phi(x)$ by definition of $\geq^*$. Note that if $\sup_Y S$ and $\inf_Y S$ exist, then $\sup_Y S = \sup_Y(\cup_{\mathcal{X}} \overline{s}(x))$ and $\inf_Y S = \inf_Y(\cup_{\mathcal{X}} \underline{s}(x))$ by associativity. Since $(X, \geq)$ is a complete lattice, $\sup_X \mathcal{X}$ exists; call it $\overline{x}$. If $\overline{x} \in \mathcal{X}$, then $\overline{s}(\overline{x}) = \sup_Y(\cup_{\mathcal{X}} \overline{s}(x))$ and so $\sup_Y S$ exists. So suppose $\overline{x} \notin \mathcal{X}$. Define $s^* = \inf_Y \phi(\overline{x})$ and note $s^* \in \phi(\overline{x})$. I show $s^* = \sup_Y(\cup_{\mathcal{X}} \overline{s}(x))$. Since $\overline{x} \notin \mathcal{X}$, $\overline{x} > x$ for all $x \in \mathcal{X}$. This implies $s^* \geq^* \overline{s}(x)$ for all $x \in \mathcal{X}$. Hence s^* is an upper bound for $\cup_{\mathcal{X}} \overline{s}(x)$. Take any upper bound $\overline{y} \neq s^*$ for $\cup_{\mathcal{X}} \overline{s}(x)$. Then $\overline{y} \notin \cup_{\mathcal{X}} \overline{s}(x)$, for if there were $x' \in \mathcal{X}$ such that $\overline{y} = \overline{s}(x')$ then $x' \geq x$ for all $x \in \mathcal{X}$ would imply that $\overline{x} \equiv \sup_X \mathcal{X} = x'$ is in $\mathcal{X}$, a contradiction. Therefore, $\overline{y} \in \phi(\tilde{x})$ for some $\tilde{x} \in X \backslash \mathcal{X}$ and since $\overline{y} \geq^* \overline{s}(x)$ for all $x \in \mathcal{X}$, $\tilde{x} > x$ for all $x \in \mathcal{X}$. Hence $\tilde{x} \geq \overline{x}$. If $\tilde{x} \neq \overline{x}$, then $\overline{y} >^* s^*$, and if $\tilde{x} = \overline{x}$, then $\overline{y} \in \phi(\overline{x})$ implies $\overline{y} \geq^* s^*$. As a result, $s^* = \sup_Y(\cup_{\mathcal{X}} \overline{s}(x))$. Finally, $\inf_Y S$ exists by a similar argument. Since $(X, \geq)$ is a complete lattice, $\inf_X \mathcal{X}$ exists; call it $\underline{x}$. If $\underline{x} \in \mathcal{X}$, then $\inf_Y(\cup_{\mathcal{X}} \underline{s}(x)) = \underline{s}(\underline{x})$. Otherwise $\inf_Y(\cup_{\mathcal{X}} \underline{s}(x)) = \sup_Y \phi(\underline{x})$.

Q.E.D

ordered, $\inf_{\phi(x)}(S)$ exists for any $S \subset \phi(x)$. Since the set of upper bounds of S contains ω, it has a least element because $\phi(x)$ is well ordered. Hence $\sup_{\phi(x)}(S)$ exists.

Proof of Theorem 9: By the traditional revelation principle, (Θ, f) truthfully implements f in Bayesian equilibrium with any order on Θ_i. It remains to prove that there is an order $\geq_i^*$ on Θ_i such that the game induced by $(\{(\Theta, \geq_i^*)\}, f)$ is supermodular. I prove first that, for any $i \in N$, the order $\succeq_i$ on M_i induces an order $\geq_i^*$ on Θ_i such that $(\Theta_i, \geq_i^*)$ is a (complete) lattice. So, $\Sigma_i(\Theta_i)$ is a (complete) lattice with the pointwise order. Second, I establish that under $\geq_i^*$, $u_i^f(\hat{\theta}_i(.), \hat{\theta}_{-i}(.))$ is supermodular in $\hat{\theta}_i(.)$ and has increasing differences in $(\hat{\theta}_i(.), \hat{\theta}_{-i}(.))$.

Denote $M_i^* = m_i^*(\Theta_i)$ for all $i \in N$. Define correspondence $[\,] : M_i^* \twoheadrightarrow \Theta_i$ where $[m_i] = \{\theta_i \in \Theta_i : m_i^*(\theta_i) = m_i\}$ is the equivalence class of $m_i \in M_i^*$. Let $\theta^s : M_i^* \to \Theta_i$ be a selection from $[\,]$. As a mapping from M_i^* to $\theta^s(M_i^*)$, θ^s is a bijection because $m_i \neq m_i'$ necessarily implies $[m_i] \cap [m_i'] = \emptyset$. Since θ^s is a bijection, we can define $\geq_i$ on a subset of Θ_i such that $\theta^s(m_i'') \geq_i \theta^s(m_i')$ if and only if $m_i'' \succeq_i m_i'$. Because θ^s is an order-isomorphism from $(M_i^*, \succeq_i)$ to $(\theta^s(M_i^*), \geq_i)$, it preserves all existing joins and meets. This implies that $(\theta^s(M_i^*), \geq_i)$ is a (complete) lattice because $(M_i^*, \succeq_i)$ is a (complete) lattice. Define the extension $\geq_i^*$ (or simply $\geq^*$) of $\geq_i$ to all of Θ_i, as follows:

1. For any distinct $m_i, m_i' \in M_i^*$ and for all $\theta_i \in [m_i]$, $\theta_i' \in [m_i']$, $\theta_i \geq^* \theta_i'$ if and only if $\theta^s(m_i) \geq_i \theta^s(m_i')$.

2. For all $m_i \in M_i^*$, $([m_i], \geq^*)$ is a complete chain.

By Lemma 6, $(\Theta_i, \geq^*)$ is a (complete) lattice. Thus, $\Sigma_i(\Theta_i)$ is a (complete) lattice with the pointwise order. Endow those lattices with their order-interval topology and the Borel σ-algebra so that all functions are trivially continuous and measurable.

The next step of the proof will use the fact that $m_i^*(.)$ preserves meets and joins, which I prove now. Take any $T \subset \Theta_i$. Since $(M_i^*, \succeq_i)$ and $(\Theta_i, \geq^*)$ are complete lattices, $\sup_{M_i^*}(m_i^*(T))$ and $\sup_{\Theta_i} T$ exist. Denote $\overline{m}_T = \sup_{M_i^*}(m_i^*(T))$. Since $\sup_{\Theta_i} T$ is an upper bound for T, $\geq^*$ implies $m_i^*(\sup_{\Theta_i} T)$ is an upper bound for $m_i^*(T)$ in M_i^*. Thus, $m_i^*(\sup_{\Theta_i} T) \succeq_i \overline{m}_T$. But $\overline{m}_T$ is an upper bound for $m_i^*(T)$, hence $\sup_{[\overline{m}_T]}([\overline{m}_T])$ is an upper bound for T. So, $\sup_{[\overline{m}_T]}([\overline{m}_T]) \geq^* \sup_{\Theta_i} T$, and therefore, $\overline{m}_T \succeq_i m_i^*(\sup_{\Theta_i} T)$. A similar argument applies to show $\inf_{M_i^*}(m_i^*(T)) = m_i^*(\inf_{\Theta_i} T)$.

Now I show that $u_i^f(\hat{\theta}_i(.), \hat{\theta}_{-i}(.))$ is supermodular in $\hat{\theta}_i(.)$ and has increasing differences in $(\hat{\theta}_i(.), \hat{\theta}_{-i}(.))$. Take any $i \in N$ and for all $j \neq i$, endow Θ_j with $\geq_j^*$ and $\Sigma_j(\Theta_j)$ with the corresponding pointwise order. Endow $\prod \Sigma_j(\Theta_j)$ with the product order. The first step is to show that $u_i^f(\hat{\theta}_i(.), \hat{\theta}_{-i}(.))$ is supermodular in $\hat{\theta}_i(.)$. For any $\theta_i''(.)$ and $\theta_i'(.)$, we know $m_i^*(\theta_i'(.)) \vee m_i^*(\theta_i''(.)) = m_i^*(\theta_i'(.) \vee \theta_i''(.))$ and similarly for $\wedge$. Since the mechanism $(\{(M_i, \succeq_i)\}, g)$ supermodularly implements f, $u_i^g(m_i(.), m_{-i}(.))$ is supermodular in $m_i(.)$ for each $m_{-i}(.)$. For any $\hat{\theta}_{-i}(.)$,

$$u_i^g(m_i^*(\theta_i'(.) \vee \theta_i''(.)), m_{-i}^*(\hat{\theta}_{-i}(.))) + u_i^g(m_i^*(\theta_i'(.) \wedge \theta_i''(.)), m_{-i}^*(\hat{\theta}_{-i}(.)))$$
$$\geq u_i^g(m_i^*(\theta_i'(.)), m_{-i}^*(\hat{\theta}_{-i}(.))) + u_i^g(m_i^*(\theta_i''(.)), m_{-i}^*(\hat{\theta}_{-i}(.))),$$

which implies that for any $\hat{\theta}_{-i}(.)$,

$$u_i^f(\theta_i'(.) \vee \theta_i''(.), \hat{\theta}_{-i}(.)) + u_i^f(\theta_i'(.) \wedge \theta_i''(.), \hat{\theta}_{-i}(.)) \geq u_i^f(\theta_i'(.), \hat{\theta}_{-i}(.)) + u_i^f(\theta_i''(.), \hat{\theta}_{-i}(.)).$$

The second step is to show that $u_i^f(\hat{\theta}_i(.), \hat{\theta}_{-i}(.))$ has increasing differences in $(\hat{\theta}_i(.), \hat{\theta}_{-i}(.))$. For any $\theta_i''(.) \geq_i^* \theta_i'(.)$ and $\theta''_{-i}(.) \geq_{-i}^* \theta'_{-i}(.)$, we know $m_i^*(\theta_i''(.)) \succeq_i m_i^*(\theta_i'(.))$ and $m_{-i}^*(\theta''_{-i}(.)) \succeq_{-i} m_{-i}^*(\theta'_{-i}(.))$. Since the mechanism $(\{(M_i, \succeq_i)\}, g)$ supermodularly implements f, $u_i^g(m_i(.), m_{-i}(.))$ has increasing differences in $(m_i(.), m_{-i}(.))$. For any θ_i,

$$u_i^g(m_i^*(\theta_i''(.)), m_{-i}^*(\theta''_{-i}(.))) - u_i^g(m_i^*(\theta_i'(.)), m_{-i}^*(\theta''_{-i}(.))) \geq$$
$$\geq u_i^g(m_i^*(\theta_i''(.)), m_{-i}^*(\theta'_{-i}(.))) - u_i^g(m_i^*(\theta_i'(.)), m_{-i}^*(\theta'_{-i}(.))),$$

which implies that for any θ_i,

$$u_i^f(\theta_i''(.), \theta''_{-i}(.)) - u_i^f(\theta_i'(.), \theta''_{-i}(.)) \geq u_i^f(\theta_i''(.), \theta'_{-i}(.)) - u_i^f(\theta_i''(.), \theta'_{-i}(.)),$$

and it completes the proof. Q.E.D

Chapter 4

Nomination Processes and Policy Outcomes

4.1 Foreword

This chapter reproduces the work of “Nomination Processes and Policy Outcomes,” a paper written jointly by Matthew O. Jackson, Kyle Mattes, and Laurent Mathevet. The part of the work conducted by the present author cuts across all aspects of the paper and may be summarized as follows. The paper began as a class project, so the idea of studying election models is due to Jackson. The present author helped identify what body of theoretical models to consider. In particular, he proposed the primitive version of the spending competition model in an example. The authors split, more or less equally, the ideas on how to analyze the models, their mathematical treatment, and the proofs of the results.

4.2 Introduction

Nominations are a critical part of many elections. While the modeling of elections is extensive, there are no systematic studies of how the specifics of the nomination process affect election outcomes. This chapter develops and analyzes three simple models of prominent nomination processes, all within the same basic election setting. It is shown that the differences in nomination process can have a large impact on the election outcome.

The setting consists of two competing political parties simultaneously nominating candidates (out of their respective memberships) for an election. If elected, a candidate chooses her most preferred policy from a one-dimensional set of potential policies. All voters (who compose the parties and are thus also the potential candidates) have single-peaked preferences. The vote over the two nominees is by majority rule. The three different nomination processes are as follows:

1. A party leader, who is a member of the party (and thus one of the potential candidates), unilaterally chooses the party's nominee.

2. Party members vote over who should be the party's nominee.

3. Party members compete for the nomination by spending. The party member who spends (or is willing to spend) the most money wins the nomination.

These models of processes should prove useful beyond this chapter, especially when models of nomination processes become part of more general election models.

In each case an equilibrium is defined to be a pair of nominees, one for each party, such that the following is true.

- **Nomination by party leaders:** neither party leader would want to change her nominee, given the nominee put forth by the other party and anticipating the eventual election against the other party's nominee.

- **Nomination by a vote of party members:** there is no other party member who would defeat the party's nominee in a majority vote of the party's members, anticipating the eventual election against the other party's nominee.

- **Nomination by spending competition:** no other party member would be willing to spend more than the party's nominee in order to secure the party nomination, anticipating the eventual election against the other party's nominee.

As the nomination process varies, the main characterizations of the election outcomes are as follows. First, nomination by party leaders is analyzed. In this case, the winner can come from either party, but lies between the overall median and the leader of the party that contains the median. The outcome can range anywhere between these points. Then it is shown that nominations by party vote are equivalent to situations where nominations are made by party leaders, but where the party leaders are the medians of the parties. This provides an intuitive relationship between nominations by a party vote and nominations by party leaders. This then implies that the election outcome when nominations are by a vote by party members always lie between the overall median and the median voter of the party which contains the overall median voter. In contrast, the outcome under spending competition is not constrained to any particular interval. Depending on the intensity of voters' preferences, the outcome can be almost anywhere. Elections by spending competition differ more dramatically from the other nomination processes, have more complicated equilibrium existence issues, and depend on the preferences of various party members in complex and subtle ways. In particular, nominations by spending competition can lead to extremist nominees from either or both parties, and can lead to extreme policy outcomes. Finally, party membership is endogenized, which leads to a convergence to the median in the case of nomination by votes, while if nominations are by spending competition, extremist outcomes can still ensue.

Although this political framework is a simple one-dimensional left-right spectrum with single-peaked voter preferences, the median voter's preferences do not always determine the outcome.[1] This analysis offers a different explanation from other models exhibiting non-median outcomes, as it is shown that incorporating parties and nominations into the electoral model can create non-median outcomes.[2] Alternative models with non-median outcomes include analyses that expand the dimensions of the outcomes (e.g., Hinich [51]), include valence (Groseclose [41], Aragones and Palfrey [5]), have more than two candidates (e.g., Hotelling [53], Palfrey [91]), have citizen candidates who run at a cost (Osborne and Slivinsk [89], Besley and Coate [12]), are based on probabilistic voting (e.g., Coughlin [25]), or focus on candidate signalling and character (Callander [**?**], Kartik and McAfee [59]). In the model it is candidates' willingness to spend to influence the nomination and ultimately the election that drives the outcome away from the median.

Nomination processes are the focus of empirical work by Gerber and Morton [38], who show that differences in the laws governing electoral primaries can have an effect on the outcome. They examine the consequences of different primary laws across states in the U.S. and show that closed primaries can lead to more extreme nominations, while semi-closed

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[1] In fact, observed political outcomes often deviate significantly from the median. For example, Stone and Rapoport [101] show that the candidates competing for and winning U.S. Presidential nominations cover a wide range of political ideologies. (See also Arterton [8], Aldrich [3], and Gurian [43] for more discussion of the nomination process.)

[2] Independent work by Serra [98] also shows a model with primaries and non-median outcomes. However, that model is very different from any model of this chapter, having two Downsian candidates in each party, with uncertainty over voter preferences, or incumbency or dogmatic preferences generating nonmedian outcomes.

primaries (allowing voters to declare a party on election day and for independents to vote in a primary) lead to even more moderate nominees than completely open primaries (where strategic voting across parties can occur). The present model is one where party members are the only ones who vote, and so it is a closed system. However, the differences between nomination by party leadership and nomination by party members' vote can be seen as reflecting different degrees of closure. Moreover, once party membership is endogenized, we move closest to a semi-closed system. In that case, the outcome converges to the overall median, which is consistent with their finding that semi-closed systems are the most moderate. The analysis of nomination by spending competition is harder to connect to their classification.

The rest of the literature that has examined primaries and nomination processes, has focused on other aspects, such as relating the nomination process to party structure (e.g., Ranney [93], Jewell [57], Epstein [35]), or modeling information dispersion and acquisition through primaries (e.g., Callander [?], Meirowitz [74], Bartels [11]). Thus, this chapter presents a first systematic modeling of how nomination procedures relate to electoral outcomes.

4.3 The General Model

The model is related to a citizen-candidate framework,[3] but one where the citizens cannot simply decide to run but must be nominated through their parties. There are n voters, and voter i's preferences are represented by a utility function $u_i : [0, 1] \to I\!R$. Voters have single-peaked preferences over the interval $[0, 1]$, and the peak of voter i is denoted x_i.

Without loss of generality, order voters by their labels, so that $x_1 \leq x_2 \leq \cdots \leq x_n$. To keep things simple, assume that n is odd. Also, assume that no voter is indifferent between any distinct candidates i and j.

Voters are divided into two parties, P_1 and P_2, that partition $\{0, 1, \ldots, n\}$. In the first part of the chapter, the two parties are fixed; later party formation is studied. Notation P_ℓ and $P_{-\ell}$ indicates a generic party ℓ and its competitor.

In general, party structures are arbitrary, so that it could be that the parties are not simply left- and right parties, but overlap. For instance, it could be that one party has some

[3]See Osborne and Slivinski [89] and Besley and Coate [12].

left and right-minded voters, and the other party has some centrists. This means that it is possible for some voters to vote for the other party in the final outcome. There is *no overlap* in parties if for each $\ell \in \{1, 2\}$ and any i and $j \in P_\ell$, there does not exist any $k \in P_{-\ell}$ such that $x_i \leq x_k \leq x_j$.

Let $M = (n + 1)/2$ be the overall median voter out of $P_1 \cup P_2$, and let M_ℓ denote a median of party ℓ.[4] Let $W[i, j]$ denote the majority winner among any two candidates i and j. Given that a candidate is identified with her ideal point, let $u_i(j)$ denote $u_i(x_j)$, or the utility that i gets if j wins the overall election. Finally, let

$$d_i(j, k) = u_i(j) - u_i(k). \tag{4.1}$$

This is the difference in utility between what i gets if j is the overall winner vs. what i gets if k is the overall winner.

The political process is as follows:

(1) Each party (simultaneously) nominates one of its members to serve as its candidate.

(2) Voters vote for one of the two candidates, and a candidate is elected by majority rule with ties broken by a fair coin toss.

(3) The policy outcome is the elected candidate's most preferred policy.

Nomination processes are carefully modeled in (1) through equilibrium definitions, where everyone anticipates the election and outcome in (2) and (3). Given just two parties, it is a (weakly) dominant strategy for each voter to vote for her preferred candidate in (2). (3) is motivated by a standard argument that candidates cannot credibly commit to follow any policy other than their most preferred policies.[5]

4.4 Nominations with a Fixed Party Structure

Here, the distribution of voters across the two parties is fixed. As discussed above, three different processes are given for the ways that parties nominate a candidate.

[4]One of the parties will have two medians, and we are explicit in cases where that matters.

[5]It is sufficient to have voters have well-defined expectations regarding what policy each candidate would implement before the nomination process takes place.

- A party leader (one of the party members) unilaterally chooses the candidate,
- party members vote over who should be their candidate, and
- party members compete for the nomination by spending, with the nominated candidate being the party member who spent the most.

Each of these requires a corresponding definition of equilibrium.

4.4.1 Equilibrium Definitions for the Three Nomination Procedures

The definitions of equilibrium for each of the nomination procedures are as follows.

Equilibrium with Nominations by Party Leaders

An *equilibrium* in the case of nominations by party leaders is a pair of nominations, denoted $Nom(P_1) \in P_1$ and $Nom(P_2) \in P_2$, such that for each party ℓ, $W[Nom(P_\ell), Nom(P_{-\ell})]$ is preferred by the leader of party ℓ to $W[x, Nom(P_{-\ell})]$, for any $x \in P_\ell$.

This definition requires that neither party leader can benefit by changing her nomination.

Equilibrium with Nominations by a Vote of Party Members

An *equilibrium* in the case of nominations by a vote of party members is a pair of nominations $Nom(P_1) \in P_1$ and $Nom(P_2) \in P_2$ such that there does not exist any $x \in P_\ell$ such that $W[x, Nom(P_{-\ell})]$ is preferred by a strict majority of voters in P_ℓ to $W[Nom(P_\ell), Nom(P_{-\ell})]$.[6]

This definition requires that a party's nominee not be beaten in a head-to-head vote with some other potential nominee, given the other party's nomination. Thus, the nominee of a party must be a sort of internal Condorcet winner, given that voters anticipate the eventual election and overall outcome. This yields some intuitive interactions between the parties' nominees, as candidates who appeal to the party in the abstract might still be defeated for the nomination if they lack a chance of winning the subsequent election. Even though most of the interesting interaction under nomination by voting is between candidates that are viable given anticipations of what the other party will do, parties' nominees can still drift away from the party and overall median voters.

[6]Note that this definition is related to Duggan [31]'s definition of "group stable" equilibrium, which he defines for abstract games played between groups of players.

Equilibrium with Nominations by Spending Competition

An *equilibrium* in the case of spending competition by party members is a pair of nominations $i = Nom(P_1) \in P_1$ and $k = Nom(P_2) \in P_2$ such that

$$u_i(W[i,k]) - u_i(W[j,k]) \geq u_j(W[j,k]) - u_j(W[i,k]) \tag{4.2}$$

for all $j \in P_1$ and

$$u_k(W[k,i]) - u_k(W[h,i]) \geq u_h(W[h,i]) - u_h(W[k,i]) \tag{4.3}$$

for all $h \in P_2$.

This definition captures competition by candidates through spending. It requires that a party's nominee would not be beaten by some other nominee from the same party in a head-to-head spending competition, given the other party's nomination. That is, the party's nominee would be willing to outspend any challenger in order to keep the nomination. Here, for instance, $u_i(W[i,k]) - u_i(W[j,k])$ represents the maximum that i is willing to spend in order to win the nomination instead of having j win it, given that k is the nominee of party 2. The definition is somewhat subtle since how much a candidate would be willing to spend can depend on whom they are bidding against. A candidate might be willing to spend more to defeat a candidate who differs more drastically from their own stance, than a candidate who is closer in stance.

This definition captures the essential aspect of competition by spending, namely how much different candidates would be willing to pay in order to gain a nomination, without getting caught up in a detailed model of the process itself. One could explicitly model this via an auction process. One natural process would be an "all-pay" auction, where each candidate spends as they wish and the winner is the candidate that spends the most. An equilibrium of an alternating move version of that auction where candidates are aware of each other's willingness to pay corresponds to the equilibrium is defined here. That is, a candidate that is willing to spend more than each other candidate would win the auction by spending a minimal amount as no other candidate would want to spend given that they anticipate eventually being outspent. The setting here is slightly more complicated, as a candidate's willingness to spend depends on whom they are bidding against, but the equilibrium is an extension of that where there are private values. More details are provided in the appendix.

The important difference between nomination by spending competition and the other nomination processes is that intensity of preferences matter under spending competition, while it is only ordinal and not cardinal preferences that matter in the party leadership and voting nomination settings. This is what allows for a wide variety of outcomes under this setting, depending on how much different candidates are willing to spend to win office. Also, there are some other effects that arise, as candidates might seek the nomination even though they would lose the subsequent election in cases where they wish to prevent another nominee from obtaining office.

4.4.2 Nomination by Party Leaders

Equilibrium under each of the nomination procedures are now characterized, starting with the case of a nomination by party leaders.

EXAMPLE **11** *Multiple Equilibria Under Party Leaders, No Overlap*

There are seven voters, $N = \{1, \dots, 7\}$, and two parties that partition N as follows: $P_1 = \{1, 2\}$ and $P_2 = \{3, 4, 5, 6, 7\}$. The voters' ideal points are ordered by their labels.

First, note that in this example, the winner will come from P_2 regardless of who the leaders are. This follows since if 3 is nominated, 3 will win against any nominee from P_1, and all members of P_2 prefer 3 to either nominee of P_1.

In this example, there are multiple equilibria, but all equilibria have the same outcome: the winner is the member of P_2 who is most preferred by the leader of P_2 out of those who beat 2. The winner must always lie in the interval between 4 (the median) and the leader of P_2. For example, if the leader of P_2 is 3, then 3 is the outcome. Note the multiplicity of equilibria; P_1 is willing to nominate either 1 or 2, as it is irrelevant. Either nomination leads to the same outcome. If the leader of P_2 is 4, then 4 is the equilibrium outcome. If the leader is 5, then the outcome is either 4 if 2 beats 5, but is 5 if 5 beats 2. If the leader is 6, then the outcome is in $\{4, 5, 6\}$, and is the highest indexed member of this set that beats 2.

Some features of this example generalize. There may be a multiplicity of equilibria, but they always lie in a well-defined interval between the overall median and the party leader of the party containing the overall median.

PROPOSITION **14** *There always exists an equilibrium under a nomination by party leaders. The winning candidate in any equilibrium lies in the interval between (and including) the overall median voter and the leader of the party which contains the overall median voter.*

The proof appears in the appendix.

The fact that the winner always comes from the interval between the overall median, M, and the leader k of the party that contains M is relatively straightforward. If the winner came from the other side of the median from k, then k could improve by nominating M. If the winner came from the other side of k, then k could improve by nominating him or herself. The more specific details of the equilibrium structure are complicated and there is no simple formula. A simple characterization can be derived in the case of no overlap.

PROPOSITION **15** *If there is no overlap in parties, then there is a unique equilibrium winner. The winning candidate comes from the party that contains the overall median, and the outcome is that party's leader's most preferred member from the set of those who beat all members of the other party.*

The proof is straightforward, following the logic of Example 11, and is left to the reader. The idea is that each party leader prefers its bordering member to any candidate of the other party. The larger party (the one with the median), then necessarily wins, as its leader has a nomination available that will beat all candidates of the other party, and he or she prefers to any nomination of the other party. The rest of the proposition then follows easily.

It is important to emphasize that, even in the case where the parties have no overlap and split so that one party includes all voters up to the median and the other has all voters from the median onward, the outcome might not be the median. As a simple example, consider a society with three voters, and party 1 is voter 1, and party 2 is voters 2 and 3 with 3 being the leader. If voter 2 prefers 3 to 1, then the outcome will be that voters 1 and 3 are nominated and 3 wins. So the median is not the outcome, even in this most central case.

While the case with no overlap produces a unique winner, things are more complicated when there is overlap in parties. In that case there can exist multiple equilibrium outcomes, and depending on the configuration of parties, the winning nominee can come from either party. To get some feeling for this, consider the following example.

EXAMPLE **12** *Multiple Equilibria Under Party Leaders*

There are seven voters, $N = \{1, \ldots, 7\}$, and two parties that partition N as follows: $P_1 = \{2, 3, 6\}$ and $P_2 = \{1, 4, 5, 7\}$. The voters' ideal points are ordered by their labels. The party leaders are 6 and 7. Let preferences be such that $W[i, 5] = i$ unless $i = 6$ or $i = 7$.

There is an equilibrium where the nominees are 6 and 7. There is also an equilibrium where the nominees are 3 and 4. This is an equilibrium even though both leaders would prefer the other equilibrium.[7] Note that these two equilibria have different parties winning. Note also that the set of equilibria is not connected in the sense that there is no equilibrium where 5 is the winner. The only equilibrium outcomes are 4 or 6.

Refine the set of equilibria using strong equilibrium. Then it leads to selecting equilibria where the winner lies between the peaks of the party leaders. The details of this refinement are provided in the appendix.

4.4.3 Nomination by a Vote of Party Members

Nomination processes by a vote of party members is the subject of this subsection. As shown below, nominations by a vote of party members are equivalent to having nominations by party leaders where the party leaders are the medians of the parties.

EXAMPLE **13** *Nomination by Voting*

Reconsider Example 11 where are seven voters, $N = \{1, \ldots, 7\}$, and two parties, $P_1 = \{1, 2\}$ and $P_2 = \{3, 4, 5, 6, 7\}$. The voters' ideal points are ordered by their labels.

In the case where 5 beats 2 in an election, then the unique equilibrium outcome and nominee from P_2 is 5, while there are two equilibria in that P_1 can nominate either 1 or 2. To verify this, it is enough to check that 5 would be the nominee of party 2 regardless of party 1's nomination. Voters 5,6, and 7 prefer to have 5 nominated than either 3 or 4 (either of whom would win in the subsequent election against either candidate from party 1), and so it is clear that 5 would defeat 3 and 4 for the nomination, regardless of party

[7] Note that this is an equilibrium in undominated strategies given that 1 beats 6 (as 1 beats 5).

1's nomination. So consider, a nominee of 6 or 7. If that nominee would win against the nominee of party 1, then 3, 4 and 5 would all rather have 5 nominated. If that nominee would lose against the nominee of party 1, then 5, 6, and 7 would all prefer to have 5 nominated. This leaves 5 as the equilibrium nomination from party 2 in all equilibria.

If 2 beats 5, then one can verify that all equilibria have P_2 nominate 4, who wins the subsequent election.

It is now shown that at least one equilibrium always exists and the equilibrium structure under voting is related to the nominations by party leaders.

PROPOSITION **16** *There always exists an equilibrium under a vote by party members. The set of equilibria coincides with that where the median voter in a party is a "party leader."*[8] *The winning candidate lies between the overall median and the median*[9] *of the party containing the overall median.*

The proof appears in the appendix.

The intuition for a party acting as if the median were a party leader is much more subtle than it would seem. For example, note that it is *not* always true that given a comparison between two arbitrary candidates, if the median prefers one to the other then so does a majority. It is possible, when comparing candidates from opposite sides of the median, that the median's preferences are in the minority.[10] Nonetheless, the claim is true. To understand this, consider the nomination of one party taking the nomination of the other party as given.[11] The set of possible nominees who could defeat the nominee of the other party is either (i) an interval including the median of the party, or (ii) an interval lying entirely to one side of the party median (which then must be on the side of the other party's nominee). In case (i) where the set of viable nominees includes the party median, then the party median would be preferred to the nominee from the party by a majority of the voters of the party, as the comparison would always boil down to a comparison of the party median and some other outcome. In that case, the party median is the only

[8]Given that one party will have two medians, this refers to a union of the sets of equilibria where each one of the two medians is party leader.

[9]This is the furthest median voter of the party, if there are an even number of voters.

[10]For example, voters other than the median may prefer candidates to their right over candidates to their left, while the median's preferences run in the other direction.

[11]Consider the case where the first party has a single median and see the appendix for the case with two medians.

possible nominee in response to the other party's nominee. If instead case (ii) applies and the interval is entirely on one side of the median (the same side of the party median as the other party's nominee), then any two viable nominees from that interval both lie on the same side of the party median and so a majority of the party will have preferences that agree with the party median's preferences.

Although there could be a discontinuity here when the median voter changes from one party to another, as this can move the winning candidate from one side of the median to the other, in many cases the change will not be very substantial. For instance, if each party has a fairly dense set of potential nominees near the median voter, then the eventual winner must be very close to the median. The discontinuity here comes from the finite set of potential nominees and the fact that a candidate is not allowed to do anything other than institute her most preferred policy.

While the nomination by party voting allows for non-median outcomes overall, the chosen candidate still comes from a well-defined interval between the overall median and the median of the party containing the overall median. The equilibrium looks very different when nominations by party spending are considered.

4.4.4 Nomination by Spending Competition

The analysis of nomination by spending competition begins with some examples. First, an equilibrium is shown where there is an extreme outcome in terms of each party's nominee and the overall winner.

Example **14** *Nomination by Spending Competition*

Again, reconsider Example 11 where there are seven voters, $N = \{1, \ldots, 7\}$, and two parties, $P_1 = \{1, 2\}$ and $P_2 = \{3, 4, 5, 6, 7\}$. The voters' ideal points are ordered by their labels.

Note first, that there are preference configurations where the nominee of P_2 is 3, even though all other members of party 2 would prefer to nominate 4, and even though that nominee does not lie between the overall median and the median of P_2 (in contrast to the case of nomination by voting). For example, if $d_3(3, i) > d_i(i, 3)$ for all $i > 3$, then 3 wins the nomination of P_2 and the overall election.

It is also possible to have extremists from both parties nominated. For instance, suppose that all members of P_2 prefer any member of P_2 to any member of P_1. In this case, the nominee of party 2 will win the election and so it is as if there were just one party and spending competition among its members. If $d_7(7,i) > d_i(i,7)$ for each $i \in \{3,4,5,6\}$, then the unique equilibrium outcome would be that 7 wins the nomination and then the overall election. As the nominee from P_1 is irrelevant, extreme nominees from both parties could occur.

This example shows the contrast between nomination by spending competition and nomination by voting. Under spending competition the outcome could be any member of P_2, while in the voting case it would have to be either 4 or 5.

While the possible outcomes under nominations by spending competition are more varied than under nominations by voting, it is still possible to say something about the outcome, at least in the case where there is no overlap in the parties which is a very natural case to consider.

Proposition **17** *If there is no overlap in parties, then any equilibrium winner under nomination by spending competition is from the party containing the median, and is a candidate who defeats all candidates from the other party.*

The proof again appears in the appendix, but is easy to explain. In this case, all members of the party containing the median prefer the candidate k closest to the other party to any nominee of the other party. This means that any candidate willing to outspend k must also be able to win the election.

Proposition 17 does not mention the issue of existence. This is because of another contrast between nomination under spending competition and the other nomination procedures. Under spending competition an equilibrium need not always exist, as shown in the next example. In fact, the example shows nonexistence even in the no overlap case.

Example **15** *Non-Existence of Equilibrium Under Party Spending*

There are five voters $N = \{1, \ldots, 5\}$ and two parties, $P_1 = \{1,2\}$ and $P_2 = \{3,4,5\}$. Consider the utility functions in Figure 1 for voters $3,4$ and 5. Every member of P_2 prefers any member of P_2 to any member of P_1. So, it is clear that the nominee of P_1 is irrelevant. Let $d_4(4,3) > d_3(3,4)$. Then 3 cannot be the nominee as 3 would be outspent by 4. Also, let $d_5(5,4) > d_4(4,5)$. Then 4 cannot be the nominee as 4 would be outspent by 5. This leaves only 5 as the potential nominee. However, if $d_3(3,5) > d_5(5,3)$, then 5 cannot be the nominee either. Thus, there are situations where there is no equilibrium.

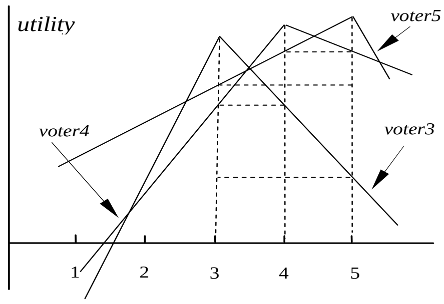

Figure 4.1: Preferences Generating an Intra-Party Cycle

The nonexistence of equilibrium in the case of spending competition follows from the fact that intensities of preferences matter and might not be ordered across party members in any nice way.

4.4.4.1 Sufficient Conditions for Existence Under Party Spending with No Overlap in Parties

An equilibrium may not exist under nominations by spending competition, even in a five-voter[12] world with single-peaked preferences and no overlap in parties. Sufficient conditions on preferences are provided for an equilibrium to exist.

In the case of no overlap, an intuitive condition is sufficient to rule out the cycle exhibited in the above example and to restore existence. Let $i < j$ denote that x_i is to the left of x_j.

Let us say that preferences satisfy the *extremist* condition if $d_i(i,k) \geq d_j(j,k)$ whenever $i \leq j \leq k$ or $i \geq j \geq k$. This condition says that if one voter is willing to spend a given amount to move the outcome in a given direction (say to the left), then voters further to the left would be willing to spend at least as much for the same change. Under this condition, there is a consistent ordering to the intensity of voters preferences and this is enough to avoid the cycles from the example above and guarantee existence.

The extremist condition is clearly very strong, and one would expect to find many

[12]One could even simplify the example further having only one party, and reduce it to a three voter world.

settings where it fails. However, as seen from Example 15, something on the order of this condition is really needed to establish equilibrium existence. There are cases where the extremist condition is satisfied. For instance, if preferences are Euclidean (so that utility is just the opposite of the distance between the outcome and the peak, as is often assumed in the literature), then the condition is clearly satisfied.

PROPOSITION **18** *If there is no overlap in parties and the extremist condition is satisfied, then there exists an equilibrium under nomination by spending competition.*

The proof of the proposition is constructive and appears in the appendix. The idea is that under the extremist condition, the relevant candidates are only extreme ones. We have to be a bit careful, as the relevant ones in some cases need to be defined relative to those who win against nominees of the other party.

4.4.4.2 Sufficient Conditions for Existence Under Party Spending: The General Case

When there is an overlap in parties, cycles turn out to be surprisingly robust to preference restrictions. Even the nice ordering of preferences under the extremist condition fails to be sufficient to guarantee existence. In fact, equilibria may fail to exist even under stronger preference restrictions. Two preference restrictions are examined: First, a "strong extremist" property (that is a strengthening of the extremist condition), and second, an ordered preference intensities condition. The failures of these two conditions to guarantee existence helps illustrate another condition, which is called the "directional party" condition, which ensures existence.

Preferences satisfy the *strong extremist* condition if for all players i, j, k such that $i \leq j \leq k$ and all alternatives h, t with $i \leq h \leq t \leq k$,

1. $d_i(h, t) > d_k(t, h)$ implies $d_i(h', t') > d_j(t', h')$ for all $i \leq h' \leq t' \leq j$ and,

2. $d_k(t, h) > d_i(h, t)$ implies $d_k(t', h') > d_j(h', t')$ for all $j \leq h' \leq t' \leq k$.

The strong extremist condition says that if one voter i has more intense preferences than another voter k regarding pairs of candidates in between those two (h and t), then voter i

has more intense preferences than some other voter j who lies in the same direction as k, over pairs of alternatives between i and j. This, again, is a strong condition that imposes some consistency on preferences to rule out cycles. Similar to the extremist condition, while it is strong and only satisfied in special cases, it is satisfied by Euclidean preferences that are directly proportional to distance between an alternative and a voter's peak. Even with this strengthening of the extremist condition, there are situations where no equilibrium exists, provided there is overlap between the parties.

EXAMPLE **16** *Non-Existence of Equilibrium Under the Strong Extremist Condition*

There are seven voters with ideal points at locations: $x_1 = 0, x_2 = 1, x_3 = 3, x_4 = 6, x_5 = 7, x_6 = 9, x_7 = 10$. Voters' preferences are distance based, so they prefer candidates who are closer to their ideal points to those farther away. Two parties partition N as follows: $P_1 = \{1, 3\}$ and $P_2 = \{2, 4, 5, 6, 7\}$.

Suppose that the strong extremist condition is satisfied in terms of preference intensities and the following are true:[13]

$d_7(7, 2) > d_2(2, 7)$

$d_1(2, 3) > d_3(3, 2)$

$d_2(3, 6) > d_6(6, 3)$.

Let us show that there is no equilibrium. Start by showing that there is no equilibrium with 1 as the nominee of P_1. Every candidate in P_2 beats 1. Thus, by the strong extremist condition, the only candidates for nomination from P_2 are 2 and 7. The nominee for P_2 must then be 7, since $d_7(7, 2) > d_2(2, 7)$. However, if 7 is nominated by P_2, then both 1 and 3 in P_1 would rather have 3 be nominated over 1. Thus, it is impossible to have an equilibrium with 1 as the nominee of P_1. So, let us consider 3 as the nominee of P_1. 2 cannot be the nominee of P_2, as then $d_1(2, 3) > d_3(3, 2)$ implies that 1 would outbid 3 for the nomination of P_1. So, the nominee of P_2 must come from $\{4, 5, 6, 7\}$. It cannot be 6, since 2 would outbid 6 given that $d_2(3, 6) > d_6(6, 3)$. By the strong extremist condition, this also means that it cannot be 5 or 4 for the same reason. So, we are left with 7. However if 7 is nominated, then 3 wins. 6 would then wish to outbid 7 (and 7 would be happy to be outbid). Thus, there is no equilibrium.

[13] These three relationships are consistent with the strong extremist condition.

Suppose now that the intensity of candidate preferences can be ordered. Preferences satisfy the *ordered preference intensity* condition if every distinct pair of voters i and j can be ordered in terms of preference intensity such that either $|d_i(h,k)| > |d_j(h,k)|$ (for all $h \neq k$)[14] or $|d_j(h,k)| > |d_i(h,k)|$ (for all $h \neq k$). Notice that having more intense preferences is a transitive relationship. Even this strong a condition is not enough to guarantee existence.

Example **17** *Non-Existence of Equilibrium when Preference Intensities are Ordered*

There are seven voters with ideal points $x_1 = 1, x_2 = 2, x_3 = 4, x_4 = 7, x_5 = 8, x_6 = 9, x_7 = 11$, and who prefer outcomes closest to their own peaks. Two parties partition N as follows: $P_1 = \{1,4,5,6,7\}$ and $P_2 = \{2,3\}$. Preference intensities are ordered so that $2 > 3 > 7 > 1 > 6 > 5 > 4$, where '$i > j$' means '$i$ has more intense preferences than j'.

Check that there is no equilibrium. No equilibrium can support the nomination of voter 2 in P_2 without the nomination of 7 in P_1 because 7 could win the final election and has the most intense preferences in P_1. But the pair $(7,2)$ is not an equilibrium either since voter 2 would be outspent by voter 3, as 3 is the best outcome that 2 can rationally expect given the next round. Following the same logic, (7, 3) is not an equilibrium because 7 would be outspent by 4, 5 or 6. Furthermore, in each of (4, 3), (5, 3), (6, 3), voter 1 would outspend these other potential nominees from P_1 as she has the most intense preferences in P_1 after 7. Finally, voter 2 would not let voter 3 win the nomination under (1, 3), so that cannot be an equilibrium.

These last two examples suffer similar cycling issues: We first begin to move in one direction, but then someone on the opposite side breaks the directional trend by stealing the nomination, and starts a cycle. The following condition is sufficient to prevent cycling, thus implying equilibrium existence.

Preferences satisfy the *directional-party condition* if, for each party ℓ, either

1. $d_i(h,t) \geq d_j(t,h)$ for all $i \in P_\ell$ and $j \in P_\ell$ and $h,t \in N$ such that $i \leq h < t \leq j$, or

2. $d_i(h,t) \leq d_j(t,h)$ for all $i \in P_\ell$ and $j \in P_\ell$ and $h,t \in N$ such that $i \leq h < t \leq j$.

[14] It would be more natural to require this only when h and k lie to one side of i and to one side of j, but even under this very strong condition equilibria fail to exist.

The directional party condition says that there is a consistent direction with respect to which a party's preferences can be ordered. Either it is always voters more to the left that care at least as much as voters to the right, or vice versa. Again, this condition is very strong, but satisfied when preferences are Euclidean (the opposite of the distance between an alternative and the voter's peak).

PROPOSITION **19** *If preferences satisfy the directional-party condition, then an equilibrium under nomination by spending competition exists.*

The proof is in the appendix, and uses an algorithm that identifies an equilibrium under the directional party condition.

The results have shown that nominations by party leaders and party vote have an interesting relationship, in that nominations by party vote look as if the party median was a party leader. These then lead to outcomes lying between the overall median and the leader (or median) of the party containing the overall median. In both cases equilibria exist. In contrast, the case of spending competition brings in preference intensity which leads to a wider variety of possible outcomes, as well as existence problems.

Party membership will now be endogenized. This is important in order to understand how anticipated outcomes will affect incentives for voters to switch parties and try to affect the overall outcome.

4.5 Endogenous Parties

Interestingly, it turns out that with nominations by voting, endogenizing parties leads to median outcomes, while under nomination by spending competition, it is still possible to get extreme outcomes in both nominations and the overall winner.[15]

Equilibrium with Endogenous Parties

Consider a partition of the population into two parties, (P_1, P_2), with the possibility that one of these is empty. Say that (P_1', P_2') is adjacent to (P_1, P_2) if there exists i such that $(P_1', P_2') = (P_1 \setminus \{i\}, P_2 \cup \{i\})$ or $(P_1', P_2') = (P_1 \cup \{i\}, P_2 \setminus \{i\})$. Thus, adjacent pairs of parties are those where the only difference is that one voter has switched parties.

[15] In this section we do not consider endogenous parties with party leaders, as it is not so clear how to properly define equilibrium in that case (e.g., who are the leaders if a leader switches parties?). Moreover, we already see an interesting contrast between the voting and spending competition cases, which is the more central focus of this chapter.

An equilibrium with endogenous parties is a pair of parties (P_1, P_2), with the possibility that one is empty, that partition the set of voters, and a pair of nominations that form an equilibrium $(Nom(P_1), Nom(P_2))$,[16] as well as a specification of an equilibrium $(Nom(P_1^*), Nom(P_2^*))$ for every adjacent partition into two parties (P_1^*, P_2^*), such that:

$$u_i(W[Nom(P_\ell), Nom(P_{-\ell})]) \geq u_i(W[Nom(P_\ell \backslash \{i\}), Nom(P_{-\ell} \cup \{i\})]), \qquad (4.4)$$

for each P_ℓ and $i \in P_\ell$. A party structure together with specifications of (equilibrium) nominations for that party structure and all adjacent ones is in equilibrium if no member of one party wishes to switch to the other party, anticipating the equilibrium that would ensue.[17]

4.5.1 Endogenous Parties and Nomination by Voting

First, nominations by party voting is revisited. Consider the following example.

EXAMPLE **18** *Every Equilibrium Outcome is the Median with Endogenous Parties, but not with Exogenous Parties*

There are seven voters, $N = \{1, \ldots, 7\}$, and two parties that partition N as follows: $P_1 = \{1, 2, 3, 7\}$ and $P_2 = \{4, 5, 6\}$. Let 6 beat 3 in an election. One equilibrium *when these are exogenous parties* is (3, 5), with candidate 5 winning. This is not, however, part of an equilibrium with endogenous parties. Candidate 4, the median, can join P_1. With the new lineup of $P_1' = \{1, 2, 3, 4, 7\}$ and $P_2' = \{5, 6\}$, (4, 5) is an equilibrium (with either exogenous or endogenous parties). Let us check that $P_1' = \{1, 2, 3, 4, 7\}$ and $P_2' = \{5, 6\}$, (4, 5) is part of an equilibrium with endogenous parties. Clearly, candidate 4 would not wish to switch, as 4 wins the election. Candidates 1, 2, 3, and 7 would have no effect on the outcome by switching to P_2 as it is still an equilibrium to have 4 nominated by P_1 against 5 from P_2; and candidates 5 and 6 would have no effect on the outcome by switching to P_1 as it is then still an equilibrium to have 4 nominated against the remaining candidate in P_2.

[16] In the case where one of the parties is empty, then its nomination is ignored, and the other party's nominee wins the election by default.

[17] One might consider other sorts of equilibrium definitions, where coalitions of voters can separate and form new parties, etc. That is certainly of interest, but beyond the scope of this analysis, given the complications introduced by handling three or more parties.

This feature that the median is the winner is not just an artifact of this example, but is true of all equilibria under nominations by voting when parties are endogenous.

PROPOSITION **20** *When nominations are by votes, then in every equilibrium with endogenous parties* $W[nom(P_1), nom(P_2)] = M$*. Moreover, such an equilibrium exists.*

The proof is in the appendix. The intuition is roughly as follows. Suppose the outcome were not the median. Then we know from Proposition 16 that it lies between the median and the median of the party containing the median. It must then be that the other party (not having the median) has a majority which would prefer the median over the current outcome. Then by switching, the median would be nominated and win. This last part takes some proof, as one has to worry about what possible other equilibria could arise if the median switched parties, and one has to show that the only possibility is to have the median nominated.

While the outcome is necessarily the median once parties are endogenized under nominations by voting, the parties can still have a variety of configurations. For instance, it could be that the equilibrium is to have the median alone in one party, or instead at the other extreme to have all voters in the same party. What is tied down is that unless one of the nominees is the median, then the party structure will turn out to be unstable. This emphasizes that the *equilibrium party structure cannot be separated from what the equilibrium nominees are*. It could be that parties are stable with one pair of nominees, but not with another.

4.5.2 Endogenous Parties and Nomination by Spending Competition

Parties under spending competition are now endogenized. Here, it turns out that non-median outcomes are possible, as shown next.

EXAMPLE **19** *Existence of Extreme Equilibrium Outcomes with Endogenous Parties*

There are five voters $N = \{1, \ldots, 5\}$, and two parties that partition N as follows: $P_1 = \{1, 3\}$ and $P_2 = \{2, 4, 5\}$. Voters' ideal points are ordered by their labels. Moreover, assume that $d_1(2,3) > d_3(3,2)$, and $d_2(i,j) > d_h(k,t)$ for all $h \in \{3,4,5\}$, and all i, j, k, t such that $2 \geq i > j$ and $2 \geq k > t$.

For P_1 and P_2 above, $(1,2)$ is a pair of nominations that form an equilibrium where the general winner is voter 2. Let us check that there is some specification of equilibria for each possible switching of some voter, so that no voter would desire to switch parties. If voter 1 switches party then P_1 only consists of voter 3, the median. In this case, regardless of the nominee from P_2, the final winner is voter 3, and voter 1 is made worse off. If instead voter 3 switched parties, then voter 1 would become the only possible nomination in P_1. In P_2, voter 2 outbids any member, so she is nominated as part of any equilibrium. Voter 3 is not strictly better off since voter 2 is still the general winner. It is clear that voter 2 will not gain by switching parties, regardless of the equilibrium specification. So, we are left only to consider what happens if voter 4 (or 5) switches parties. Here, $(1,2)$ is still an equilibrium because then 4 (5) does not want to outspend 1 as they would still lose to 2 (and 3 still does not want to outspend 1 given that $d_1(2,3) > d_3(3,2)$); and voter 2 continues to outbid the members of her party.

Example 19 shows that, in contrast to nominations by voting, nomination by spending can provide non-median outcomes that are robust to party switching. Just as with fixed parties, there are issues with equilibrium existence, but the directional party condition is again sufficient to guarantee existence.

Proposition **21** *Suppose that nominations are by spending competition. If preferences satisfy the directional party condition and are in the same direction for each party, and* $N \geq 5$*, then an equilibrium with endogenous parties exists.*[18]

The proof of the proposition involves an explicit construction of the two parties and nominations, putting the two most extreme voters (in terms of the directional preference) in different parties. For instance, if the lowest indexed voters are those who have stronger preferences under the directional preference condition, then the constructed equilibrium parties have 1 and 3 together in one party and 2 and 4 together in the other, with any allocation of the remaining voters between the parties. 1 and 2 are nominated and 2 wins the election. None of the remaining voters can switch the outcome by changing parties. 2

[18] In the case where $N = 3$, there need not always exist an equilibrium. For instance, suppose that 1 cares most, then 2, then 3, where 2 is the median. Suppose also that 1 beats 3 in an election. If 1 and 2 are in the same party, then the nomination of that party must be 1 (regardless of whether 3 is present). That is not stable as then 2 would rather switch parties and win the nomination and then the election. It is also not stable to have 1 and 2 in separate parties, as then 1 would like to join the party that 2 is in, to win that nomination and the overall election.

clearly has no gain from changing, and if 1 changes parties, then 3 wins the nomination and the election, which cannot be improving for 1.

Example 19 and the proof of Proposition 21 show us that even with endogenous parties, it is possible to have extreme outcomes under nomination by spending competition. This contrasts with nominations by party votes, where Proposition 20 shows a median outcome. This makes the point that how nominations are conducted can have a big impact on election outcomes, and that if spending plays a substantial role in the nomination process, then outcomes can differ dramatically from a pure voting setting.

4.6 Concluding Remarks

The nomination process is important in determining the outcome of elections, even in a simple single-peaked world. When parties of fixed configurations vote over their nominees, the outcomes that emerge from the election are as if the party medians were party leaders, so the outcomes lie between one of those median peaks and the overall median, but can differ from the overall median. The divergence from the median depends on the specific configurations of parties and voters' ideal points. If parties are endogenous, then the outcome must be the overall median voter. Depending on preferences, a wider range of outcomes are possible under nominations by spending competition, even when parties are endogenous. There it is a very different process that determines the outcome, and intensity of preference determines the outcome. This chapter provides insight into the diversity of outcomes that can occur even in settings where the election is well ordered on one dimension and there are only two parties. This suggests that it is important to model nomination processes in order to understand electoral outcomes, even in the starkest settings.

There is much room for further research, and important ways in which the analysis should be extended. We close by mentioning a few of the most obvious directions for further study.

First, the chapter has modeled extreme versions of nomination processes, where either there are party leaders, there is a vote among party members, or there is simply a spending competition among party members. Reality is, of course, more complex, and involves combinations of these three elements. Party leadership has some discretion in identifying potential nominees, the electorate has substantial input, and spending by potential nominees can also clearly have an effect. Identifying how these different influences interact is of interest.[19]

Second, the current analysis has been confined to elections of single representatives or officials from two-party settings. While this has wide application (even beyond the U.S.), it is also important to understand nomination processes in multi-party systems, as well as things like selections of party lists and platform design and their influence on electoral competition.

Along with multi-party analyses, it would also be important to allow for independent voters who are not affiliated with any party. The results for fixed party structures are easily modified to accommodate the existence of independent voters. For the cases of party leaders and spending competitions, there is no change in the statement of the results as the independent voters would simply be incorporated through the determination of the eventual voting outcome (the W function). In the case of nominations by a party vote, there would be a small modification to the statement of the results. Consider a case where there are independent voters who cannot run as their own candidate or participate in the nomination process of either party. These voters only enter the political process by voting for one of the two candidates in the overall election, and are thus subsumed in the function $W[\cdot]$. The analysis only requires changes in the case where the general median is an independent voter. There, it is important that a unique Condorcet winner exist among all potential candidates from either party.[20] That party member then plays a central role in locating the final winner; the existence part of the results remains unchanged. For example, the second part of Proposition 14 then reads: The winning candidate in any equilibrium lies in the interval between (and including) the Condorcet winner among all party members and the leader of the party which contains the Condorcet winner. While this indicates how independent voters can be addressed when they are fixed, allowing voters to choose whether or not to join a party would be another interesting avenue to follow.

Third, general forms of stability with endogenous parties, where one allows either more than two parties or more than one voter to change at a time, face substantial existence

[19] Related to this, a referee suggested examining situations where party leaders are chosen endogenously, potentially by a vote. Here this maps indirectly into choices of candidates, and thus looks like voting over the nominee; but with richer institutional detail this could be an interesting variation to consider.

[20] Take the closest party member to the left of the overall median and the closest one to the right (even if they are from the same party). If one beats the other then that agent is the overall Condorcet winner among potential candidates.

hurdles. Nonetheless this needs to be investigated, as in situations where two parties are nominating extreme candidates, there are strong incentives for centrist voters to split off and form their own party. This again points to an interest in the modeling of multiple-party systems, even for the understanding of two-party systems.[21] Although modeling party formation has generally been a difficult task and there is a paucity of workable models, it is such a important aspect of electoral competition that it begs for further analysis.

4.7 Appendix

4.7.1 Justifying the Equilibrium in a Spending Competition via an All-Pay Auction

Consider the following all-pay auction.[22,23]

Time proceeds in discrete periods $t \in \{1, 2, \ldots\}$. Candidates alternate in their moves in the order of their indices. Spending amounts start at 0 and fall on a grid with increments of $\varepsilon > 0$. A candidate who is called upon to move can choose to raise her spending to any higher feasible amount or to leave it unchanged. If a candidate does not match the current highest spending amount on her turn, then he or she is out of the auction.[24] The auction ends at the first time where there is a single candidate remaining, or where candidates have each exhausted their budgets. The candidates each have a budget of B. In the case where candidates each hit their budget, the winner is the last candidate.

Consider a case where $u_i(W[h,k]) - u_i(W[j,k])$ is not a positive multiple of ε for any i, j, h, and k, $u_i(j)$ is not a multiple of ε for any i, and no "ties" occur (so that each $u_i(j)$ and $u_i(j) - u_i(k)$ takes a distinct value across all i, j and any $k \neq j$).[25]

[21] There are economies of scale and other aspects of parties (branding, reputation, etc.) that may make it hard to form new parties (or even to switch parties), and so the endogenous party equilibrium analysis may still be a good starting point. But understanding party formation more generally is clearly important.

[22] While this has some specific features to make the analysis relatively easy, some other variations in terms of tie-breaking and rules for dropping out of the auction can be handled with additional arguments.

[23] The timing of this auction provides for different conclusions from simultaneous or sealed bid auction models of electoral contests (e.g., Meirowitz [75]).

[24] On the first round, the current highest spending amount is considered to be $\varepsilon > 0$, so that a candidate is dropped from the auction if he or she does not initiate some minimal spending. The last mover has a slight advantage in that if the other players have not spent anything, then that candidate wins without spending the ε.

[25] Note that generically, ties only occur in a situation where the outcomes are the same, so for instance if $W[i,k] = W[j,k] = k$, and in that case the eventual election outcome is not dependent on which of i or j wins.

The following claim is shown when $|P_1| = 3$. The extension to more candidates holds by an extension with an inductive argument, although a proof is not included.

CLAIM **1** *Let* $|P_1| = 3$ *and suppose that there is a candidate* $i = Nom(P_1) \in P_1$ *and a candidate* $k = Nom(P_2) \in P_2$ *such that*

$$u_i(W[i,k]) - u_i(W[j,k]) > u_j(W[j,k]) - u_j(W[i,k]) \tag{4.5}$$

for all $j \in P_1$*, and that also*

$$u_j(W[h,k]) - u_j(W[i,k]) \leq u_h(W[h,k]) - u_h(W[i,k]) \tag{4.6}$$

for any $h \in P_1$ *and* $j \in P_1$ *both distinct from* i*. Then there is a small enough* ε *and a large enough* B *such that there is a unique subgame perfect equilibrium outcome (anticipating* $Nom(P_2) = k$*) where* i *wins the above described all-pay auction and spends at most* ε*.*

Before moving to prove the claim, let us discuss the importance of (4.6). It is illustrated in the following example.

EXAMPLE **20** *Critical Spending by a Non-Winning Candidate.*

In this example, $i = 3$ satisfies (4.5). Yet, in every equilibrium candidate 2 wins, and candidate 1 campaigns even though 1 loses. Candidate 1 campaigns in order to drag candidate 2 into the race.

Let $W[1,k] = k,\ W[2,k] = 2,\ W[3,k] = 3$. Let ε be in units and consider the following preferences:

- Candidate 1's preferences are $u_1(k) = -1.4$, $u_1(2) = 6.5$, $u_1(3) = 0.3$.
- Candidate 2's preferences are $u_2(k) = 0.2$, $u_2(2) = 8.5$, $u_2(3) = 7.6$.
- Candidate 3's preferences are $u_3(k) = 0.4$, $u_3(2) = 1.5$, $u_3(3) = 4.5$.

Let us provide the insight to why even though $i = 3$ satisfies (4.5), in every equilibrium candidate 2 wins. Note that if candidate 1 spends 5 at the first opportunity, then candidate 3 will surely end up dropping out of the auction. Also, candidate 2 will respond to outbid candidate 1, as candidate 2 is willing to spend 8.3 to change the winner from 1 to 2. Thus, if candidate 1 spends 5 at the first opportunity, candidate 2 will respond, candidate 3 will then drop out and candidate 2 will win. This means that it costs at most 5 for candidate 1 to ensure that candidate 2 wins. The equilibrium then cannot be that candidate 1 will win.

Effectively, even though there is a candidate who is willing to outspend every other candidate in a head-to-head race (4.5), that is not enough to guarantee that the candidate wins. If there is some other candidate who strongly prefers to see yet another candidate win, there are situations where the head-to-head winner does not prevail. That is ruled out by condition (4.6).

To prove Claim 1, we make use of the following observation: By Proposition 2 in Dekel, Jackson, and Wolinsky [29], for large enough B and small enough ε there is a unique equilibrium outcome of this game if there are only two candidates, i and j, in the spending competition.[26] That outcome is the candidate for whom

$$u_i(W[i,k]) - u_i(W[j,k]) > u_j(W[j,k]) - u_j(W[i,k])$$

if there is such a candidate.[27] This also holds in any subgame where we start with large enough remaining budgets, and a current standing bid such that if it is i's turn to move i does not have to bid more than $u_i(W[i,k]) - u_i(W[j,k])$ in order to stay in the auction.

Proof of Claim 1: Let us label the candidates in P_1 as 1,2,3. We need to verify the claim for each choice of i, because the game is not fully symmetric due to the starting order of moves.

First, note that the game is finite and given the distinct payoffs, there are only equal payoffs between actions at a node if the outcome is the same across actions. This implies that there is a unique equilibrium outcome.[28]

Let $v = max_{j \neq i, j \in P_1} u_j(W[j,k]) - u_j(W[i,k])$.

Let us show that there is a large enough budget so that in any subgame where i moves

[26] As emphasized by Dekel, Jackson, and Wolinsky [29], this result is a variation on a result originally shown by Leininger [65], who examines ϵ-equilibria in a slightly different auction.

[27] Otherwise $W[i,k] = W[j,k] = k$ and then the first mover drops out. The tie-breaking rule here is slightly different than that in Dekel, Jackson, and Wolinsky [29]; but in a way that actually makes things slightly easier to see.

[28] There is a possibility that $W[h,k] = W[\ell,k]$ and there is indifference over which of h or ℓ wins, so here we mean eventual outcome of the election against k. Note, however, that by (4.5), such indifference cannot involve i.

last, and there is at least that budget remaining, then i wins without any additional bidding. Consider a subgame where i moves last and the current starting bid is the smallest increment larger than v larger than the current bid of any $j \leq i$. (Note that in this case the total budget must be at least the current starting bid.) Then it is clear that i will win, as all other bidders will drop out at their turns by the definition of v and condition (4.6). Let us now proceed by induction. Suppose that i wins at no additional cost in any subgame where i moves last and the current starting bid is at least $k\varepsilon$ larger than the current bid of any $j \leq i$ for some $k > 0$, and the remaining budget is at least some B_k. We show the same is true for $(k-1)\varepsilon$ when the remaining budget is at least $B_k + v + \varepsilon$. If the other bidders bid, they must expect that there is an outcome other than i winning, as otherwise by dropping out, the first bidder would be sure that i would win in the continuation by the observation above and would save whatever payment. Likewise, the second bidder would then drop out. So, if the other bidders bid, they must be expecting some $j \neq i$ to win. Supposing that the equilibrium continuation is to have some other bidder bid with an expectation of j winning, it follows that no bidder raises their bid by more than $u_j(W[j,k]) - u_j(W[i,k])$ (recalling (4.6)). But if i next bids the minimal increment larger than $u_j(W[j,k]) - u_j(W[i,k])$, it follows from the induction step that i will win, and from (4.5) that i gains more than $u_j(W[j,k]) - u_j(W[i,k])$. This contradicts j winning in the continuation, and the claim is established.

Next, we show that if $i = 1$ and there is a large enough budget, it follows that i will win with a bid of ε. Note that if i bids ε, then by the claim above, i will be the last mover and will win in the continuation with no additional payment. If i does not bid, then some other candidate will win. By assumption, i strictly prefers to win (and by more than ε).

Finally, consider the case where $i = 2$. We show that i will win with a bid of ε. If 1 were to bid, it must be that 1 expects some $j \neq i$ to win, as otherwise 1 could drop out and have $i = 2$ win in the continuation (by the observation above). It must then also be that 1 bids no more than $u_1(W[j,k]) - u_1(W[i,k]) \leq u_j(W[j,k]) - u_j(W[i,k]) < u_i(W[i,k]) - u_i(W[j,k])$. By matching 1's bid, by the claim above $i = 2$ wins in the continuation, which is strictly improving for 2 compared to any outcome where j wins; which is a contradiction. Thus, 1 drops out. Then it is clear that $i = 2$ bids ε and 3 drops out in the continuation. Q.E.D

4.7.2 Proofs of the Propositions

Proof of Proposition 14: Let D_ℓ and $D_{-\ell}$ respectively be the leaders of parties ℓ and $-\ell$. Denote by $(Nom(P_\ell), Nom(P_{-\ell}))$ the pairs of nominations. Without loss of generality, assume $M \in P_\ell$.

Suppose $D_\ell \geq M$. First, we show that the winning candidate in equilibrium lies in $[M, D_\ell]$. By way of contradiction, suppose the winner, call it W^*, is to the left of (less than) M. If D_ℓ nominates M, then $W[W^*, M] = M$ and so D_ℓ is strictly better off by single-peakedness. Outcome W^* could not be supported in equilibrium, a contradiction. If $W^* > D_\ell$, then from a similar argument, D_ℓ is better off nominating herself because $W[W^*, D_\ell] = D_\ell$, a contradiction.

Secondly, we prove existence. If $D_\ell = M$, then it is always an equilibrium for D_ℓ to nominate herself and for $D_{-\ell}$ to choose arbitrarily a nominee in $P_{-\ell}$. If $D_\ell > M$, then take $\hat{x}$ which is defined as the closest point to D_ℓ in $P_\ell \cap [M, D_\ell]$ such that $W[y, \hat{x}] = \hat{x}$ for all $y \in P_{-\ell}$. If $\hat{x} = D_\ell$, then (D_ℓ, y) with any $y \in P_{-\ell}$ is an equilibrium. If $\hat{x} \neq D_\ell$, then for all $x \in P_\ell \cap (\hat{x}, D_\ell]$, there exists $y \in P_{-\ell}$ such that $W[x, y] = y$ (for if this were not true, x would be closer to D_ℓ which violates the definition of $\hat{x}$). Define $x^* \equiv \min(P_\ell \cap (\hat{x}, D_\ell])$. Let $y^* \in P_{-\ell}$ be the closest point to $D_{-\ell}$ in $P_{-\ell}$ such that $W[x^*, y^*] = y^*$. Note that $W[x, y^*] = y^*$ for all $x \in (\hat{x}, D_\ell]$. Now, if $y^* \in (\hat{x}, D_\ell]$, then (x^*, y^*) is an equilibrium because the candidates in P_ℓ that could defeat y^* would make D_ℓ strictly worse off, and so x^* is a best-response for D_ℓ. By definition, y^* is the best nomination for $D_{-\ell}$ when $Nom(P_\ell) = x^*$. But, if $y^* < \hat{x}$, then $(\hat{x}, y^*)$ is an equilibrium because $D_{-\ell}$ is indifferent between all the alternatives in $P_{-\ell}$ while $\hat{x}$ is D_ℓ's best choice when facing y^*.

Now suppose $D_\ell < M$ and let X be the set of voters' peaks. Consider the dual $(X', >')$ of $(X, >)$ where i's peak in X' is greater than j's if and only if it is smaller than j's in X. The above argument completes the proof as $D_\ell >' M$ in X'. Q.E.D

Proof of Proposition 16: First, we prove that a pair of nominations is an equilibrium under a vote by party members if and only if this pair is an equilibrium with nomination by medians as party leaders. Then we show existence and conclude.

Let us first show that if a pair of nominations is an equilibrium with medians as party leaders, then it is an equilibrium under nomination by voting. So, let (one of) the medians of each party be a party leader: $D_\ell = M_\ell$ and $D_{-\ell} = M_{-\ell}$. Suppose $(Nom(P_\ell), Nom(P_{-\ell})) = (i, j)$ is an equilibrium with medians as party leaders. This means that $W[i,j] \succeq_{M_\ell} W[x,j]$ for all $x \in P_\ell$. If $W[M_\ell, j] = M_\ell$, then it must be that $i = M_\ell$. In that case, for any x, since M_ℓ is a median of the party and preferences are single peaked, there is not a strict majority of the party that prefers $W[x,j]$ to M_ℓ, and so it remains an equilibrium nomination for ℓ under voting. So consider the case where $W[M_\ell, j] = j$. There, it must be that either j lies between the overall median and M_ℓ, or on the other side of the median from M_ℓ. This means that for any x (including i), $W[x,j]$ lies to the same side of M_ℓ as j. In that case, a (weak) majority has the same preferences as M_ℓ over the pair $W[i,j]$ and $W[x,j]$. Thus, if $W[i,j] \succeq_{M_\ell} W[x,j]$, then this is true for at least a weak majority of member of party P_ℓ and so no other nominee would defeat i as a nominee. Since ℓ was arbitrary, any (i,j) which is an equilibrium with medians as party leaders is an equilibrium under a nomination by voting.

To see the converse, consider an equilibrium (i,j) under nomination by voting. By means of contraction, suppose that this is not an equilibrium for any choice of medians as party leaders. That is, suppose there exists a party ℓ such that i is not the choice of the party median(s) in response to j. As argued above, the only possible outcomes as a function of the nominations of party ℓ either include at least one of the party medians, or all lie on the same side of the party median(s) as j. Consider the former case first, that is, $W[M_\ell, j] = M_\ell$ where M_ℓ is a median of P_ℓ. In that case, the median closest to $W[i,j]$ would also defeat j.[29] Therefore, i must be the median closest to $W[i,j]$; otherwise a strict majority of P_ℓ would prefer that median to $W[i,j]$, contradicting that (i,j) is an equilibrium. But if i is a median and $W[i,j] = i$, then she must be the choice of a party median in response to j, a contradiction. Second, consider the latter case where neither median would win against j. There, all of the party members to the opposite side of the party median(s) to j have the same preferences as the party median(s) over all the possible outcomes since all possible outcomes are to one side of the party median(s). In that case, it must be that if i is not defeated by a strict majority, then there is no other nomination that the median (or either median if there is more than one) would prefer to i. Agent i is thus the choice of a median of P_ℓ, a contradiction.

[29]We know both M_ℓ and $W[i,j]$ defeat j (at least weakly). Since the set of winners against any candidate is a connected set, the median closest to $W[i,j]$ also beats j.

By Proposition 14, we know that there exists an equilibrium under nominations by any pair of leaders, and so there exists one where the medians are party leaders. Therefore, by the first part of the proof, an equilibrium exists under vote by party members. The third part of our claim follows immediately from Proposition 14. Q.E.D

Proof of Proposition 17: Without loss of generality, let P_2 be the party containing the median. Suppose to the contrary of the claim, that the winner j was from P_1. Let k be the member of P_2 closest to P_1. Since there is no-overlap and $M \in P_2$, k would defeat any member of P_1. So, that the winner j is in P_1 implies $Nom(P_2) \equiv i \neq k$. That is, some i losing to j outspends k. But $d_k(k,j) > d_i(j,k)$, as it must be that $d_i(j,k) < 0$ and $d_k(k,j) > 0$. This is a contradiction, because no such i would outbid k.

Next suppose that the winner $Nom(P_2)$ could be beaten by some member of P_1. A similar argument as the one just given reaches a contradiction. For $Nom(P_2)$ to win, a member of P_1 losing to $Nom(P_2)$ would have to outspend a member who could defeat $Nom(P_2)$, a contradiction since there is no overlap. Q.E.D

Proof of Proposition 18: Without loss of generality, let P_1 contain the median and lie to the left, and order voters by their labels. Let k be the minimal labeled voter in P_2. Let S_1 be the subset of voters in P_1 who would beat k in the election (and this set is non-empty given that the median is in this set). Let $k = Nom(P_2)$. Note that all voters in $P_1 \setminus S_1$ prefer any nominee from S_1 to k and so will not wish to outbid any nominee in S_1; and changing the nominee from P_2 (given that $Nom(P_1) \in S_1$) will not change the outcome. Thus, to complete the specification of an equilibrium, it is enough to find a nominee from S_1 that would not be outbid by any other nominee from S_1. Consider the two extreme candidates from S_1, and label them i and j. If $d_i(i,j) \geq d_j(j,i)$, then set $Nom(P_1) = i$ and otherwise set $Nom(P_1) = j$. Q.E.D

Proof of Proposition 19: With directional parties, there are two cases: either (I) preference intensities for both parties (weakly) increase in the same direction, or (II) preference intensities for the parties increase in opposite directions.

We show that for both cases an equilibrium can be found.

Case I. Without loss of generality, assume that preference intensity in both parties (weakly) increases as the candidates move leftward. Now, choose $1 \equiv \min P_\ell$ and $2 \equiv \min P_{-\ell}$, the leftmost candidates from each party. If $1 \leq M$ and $2 \leq M$, then it is straightforward to check that $(Nom(P_\ell) = 1, Nom(P_{-\ell}) = 2)$ is an equilibrium. However if (say) $2 > M$, then pick the leftmost candidate from Party ℓ who can defeat candidate 2 in a pairwise election. In this case, $M \in P_\ell$ and so such a candidate exists. Call this candidate 3. It is straightforward to check that $(Nom(P_\ell) = 3, Nom(P_{-\ell}) = 2)$ is an equilibrium.

Case II. Let C_ℓ be the direction set of party ℓ, which contains all candidates on the side of the median corresponding to the direction of that party's increasing preferences. Wlog, assume that party ℓ's preference intensities increase for candidates to the right and party $-\ell$'s preferences are increasing to the left. Formally, $C_\ell = \{i \in P_\ell : M \leq i\}$ and $C_{-\ell} = \{i \in P_{-\ell} : i < M\}$. Furthermore, assume wlog that $M \in P_\ell$.

Case IIa: $C_{-\ell} = \emptyset$. Let $2 = \min C_{-\ell}$ be the candidate from $C_{-\ell}$ that is closest to the median. If $C_\ell \backslash [M, 2] \neq \emptyset$, then choose the candidate closest to 2 in that set and call her 1. Then $(Nom(P_\ell) = 1, Nom(P_{-\ell}) = 2)$ is an equilibrium. Otherwise, if $C_\ell \backslash [M, 2] = \emptyset$, then choose the candidate from C_ℓ that is closest to 2, call her 1, and notice $(Nom(P_\ell) = 1, Nom(P_{-\ell}) = 2)$ is an equilibrium.

Case IIb: $C_{-\ell} \neq \emptyset$. Let $2 = \max C_{-\ell}$ be the candidate from $C_{-\ell}$ that is closest to the median. Denote by 1 the candidate from C_ℓ that is furthest from the median *and* can defeat candidate 2.[30] Now, if $C_{-\ell} \cap [M, 1] = \emptyset$, then $(Nom(P_\ell) = 1, Nom(P_{-\ell}) = 2)$ is an equilibrium; otherwise, let $3 = \max C_{-\ell} \cap [M, 1]$ and note $(Nom(P_\ell) = 1, Nom(P_{-\ell}) = 3)$ is an equilibrium. Q.E.D

Proof of Proposition 20: We prove that for every equilibrium partition into parties (P_1, P_2) it must be that $W[Nom(P_1), Nom(P_2)] = M$.

Suppose to the contrary, that $W^* = W[Nom(P_1), Nom(P_2)] \neq M$ in some equilibrium. Without loss of generality, suppose that $M \in P_1$, where $M_1 \leq M \leq M_2$. The possible alignments for W^* can be divided into two distinct cases.

(1) $W^* < M$. Since $M \in P_1$, we know from our characterization of equilibrium that $M_1 \leq W^* < M$. Now let $P_1' = P_1/\{M\}$ and $P_2' = P_2 \cup \{M\}$. If $(Nom(P_1'), Nom(P_2')) = (i, M)$ for any i, then M would win and so there would be a profitable deviation for M contradicting equilibrium. Thus, it must be that $(Nom(P_1'), Nom(P_2')) = (i, j)$, where

[30] Such a candidate can always be found since it is possible to choose the median.

$j \neq M$. In particular, $j \in (M, M_2]$ without loss of generality, which we explain next. We know that $W[i,j] \in (M, M_2]$, by our characterization of equilibrium. So, if $j = W[i,j]$, then $j \in (M, M_2]$. Now suppose $i = W[i,j]$. Note that the median of P_1' is to the left of M_1, as removing members to the right of M_1 (such as M) moves the median to the left. So, for (i,j) to be a pair of nominations in equilibrium, $j > M$ because otherwise a majority in P_1' would always be strictly better off nominating M_1 over i. In equilibrium, j must be such that any nomination which a majority in P_1' prefers to $i = W[i,j]$ also loses to j. Therefore, if (i,j) is an equilibrium where $j > M_2$, then there exists $j^* \in (i, M_2]$ such that (i, j^*) is also equilibrium. As a result, $j \in (M, M_2]$ without loss of generality. Now we show that $(Nom(P_1'), Nom(P_2')) = (i,j)$ with $j \in (M, M_2]$ cannot be an equilibrium. First, $Nom(P_1)$ beats j, or else j would have been the nominee of P_2 since $j \in (M, M_2]$ is preferred to W^* by a majority in P_2. Second, $Nom(P_1) \leq M$. To see why, notice that if $Nom(P_1) = W^*$ then $Nom(P_1) \in [M_1, M]$, and if $Nom(P_2) = W^*$ then $Nom(P_1) > M$ would imply that a majority in P_2 is strictly better off nominating M_2 rather than $Nom(P_2)$, a contradiction. Now we have two cases:

Case I. $Nom(P_1) \geq M_1$: Then, $M_1 \leq Nom(P_1) \leq M$. Recall that the median of P_1' lies to the left of M_1. Therefore, a majority in P_1' prefers $Nom(P_1)$ to $W[i,j] \in (M, M_2]$, and since $Nom(P_1) = W[Nom(P_1), j]$, (i,j) cannot be an equilibrium.

Case II. $Nom(P_1) < M_1$: Because $W^* \in [M_1, M]$, $W^* = Nom(P_2)$. For $(Nom(P_1), Nom(P_2))$ to be a pair of nominations in equilibrium, $Nom(P_1)$ must be such that any nomination which a majority in P_2 prefers to $Nom(P_2)$ loses to $Nom(P_1)$. Thus, if $(Nom(P_1), Nom(P_2))$ is an equilibrium where $Nom(P_1) < M_1$, then there exists $Nom^*(P_1) \in [M_1, M]$ such that $(Nom^*(P_1), Nom(P_2))$ is also equilibrium. We have seen in case I that it is not possible either.

(2) $W^* > M$. This cannot be since the outcome must lie between the party median of the party containing M and M, and so must lie between M_1 and M.

To complete the proof, we argue that there exists a partition into parties with the median as the outcome, and corresponding equilibria (for all possible adjacent party structures). To see this, choose parties with no overlap such that the median is the most extreme voter in one of the parties. Let h be the voter immediately to the right of the median and t be the voter immediately to the left of the median. If h defeats t, then have the median be in the party that contains t (and nominations be M and h), and otherwise have the median be in the party that contains h (and nominations be M and t). Regardless of the deviation by any voter, let the median be nominated. Q.E.D

Proof of Proposition 21: Without loss of generality, suppose that preference intensity increases leftwards (left directional parties). Since $N \geq 5$, there exists a partition of N into (P_1^*, P_2^*) such that $\min P_1^* < \min P_2^* < M$ and no $i \in N$ is such that $\min P_1^* < i < \min P_2^*$. Let $c_1 = \min P_1^*$ (i.e., the leftmost voter in P_1^*) and $c_2 = \min P_2^*$. By the algorithm in the proof of Proposition 19, (c_1, c_2) is an equilibrium of the nomination process and so $((P_1^*, P_2^*), (c_1, c_2))$ may be an equilibrium with endogenous parties. We prove next that it actually is an equilibrium. First, take any voter $x > c_2$. If x switches party, then the algorithm predicts that (c_1, c_2) is still an equilibrium. Therefore, x cannot be strictly better off in all the equilibria of the game with partition $(P_x \backslash \{x\}, P_{-x} \cup \{x\})$. Secondly, if c_1 changes party, then $Nom(P_1^* \backslash \{c_1\}) > c_2$ because c_1 and c_2 are the leftmost candidates in each party. Since $c_1 < c_2$, by single-peakedness this cannot benefit c_1 as it could only push the final winner to the right. Finally, $W[c_1, c_2] = c_2$ and thus there is no equilibrium that could make c_2 strictly better off after switching. Q.E.D

4.7.3 Nomination by Party Leaders and Strong Equilibria

A *strong equilibrium* in the case of nominations by a vote of party leaders is a pair of nominations $Nom(P_1) \in P_1$ and $Nom(P_2) \in P_2$ such that:

(1) The pair is an equilibrium in the case of nominations by a voter of party leaders.

(2) There does not exist any pair of nominees (i, j) where $i \in P_1$ and $j \in P_2$ such that $W[i, j]$ is preferred to $W[Nom(P_1), Nom(P_2)]$ by the leader of P_1 and the leader of P_2.

The idea is that the party leaders cannot get a better outcome by agreeing to change strategies.

Returning to Example 12, there are seven voters, $N = \{1, \ldots, 7\}$, and two parties that partition N as follows: $P_1 = \{2, 3, 6\}$ and $P_2 = \{1, 4, 5, 7\}$. The voters' ideal points are ordered by their labels. The party leaders are 6 and 7. Let preferences be such that $W[i, 5] = i$ unless $i = 6$ or $i = 7$.

The equilibria are $(6, 7)$ and $(3, 4)$. However,$(3, 4)$ is not a strong equilibrium because both party leaders prefer $W[6, 7] = 6$ to $W[3, 4] = 4$.

Proposition **22** *If the pairs of nominees (i,j) and (i', j') are both strong equilibria in the case of nominations by a vote of party leaders, then W[i,j] = W[i', j'].*

Proof: The possible locations of party leaders can be divided into two cases.

(1) Party leaders are on the same side of the median. Let D_ℓ and $D_{-\ell}$, respectively, be the leaders of parties ℓ and $-\ell$. Without loss of generality, assume that $M \in P_\ell$, and $D_{-\ell} < D_\ell \leq M$. We know that $W[i, D_\ell] = D_\ell$ is an equilibrium outcome whenever $i < D_\ell$, and we will show that D_ℓ is the only strong equilibrium outcome. Suppose that W^* is a strong equilibrium outcome different from D_ℓ. Then $W^* \in [D_\ell, M]$, since whenever $W^* < D_\ell$, D_ℓ can improve the outcome by nominating himself, and whenever $W^* > M$, D_ℓ can improve the outcome by nominating M. So $W^* \in [D_\ell, M]$. But, then both $D_{-\ell}$ and D_ℓ would prefer that $i < D_\ell$ and D_ℓ are their respective parties' nominees. Thus, the outcome $W^* \neq D_\ell$ is not supportable as a strong equilibrium, which is a contradiction.

(2) Party leaders are on opposite sides of the median. Without loss of generality, assume that $M \in P_\ell$, and $D_{-\ell} < M < D_\ell$. We will show that whenever $D_{-\ell} < M < D_\ell$, there is always exactly one equilibrium outcome, and hence only one strong equilibrium outcome. Recall, from the proof of Proposition 1, that $\hat{x}$ is defined as the closest candidate to D_ℓ in $P_\ell \cap [M, D_\ell]$ such that $W[\hat{x}, y] = \hat{x}$ for all $y \in P_{-\ell}$. First of all, we know that for any equilibrium outcome W^*, $W^* \in [\hat{x}, D_\ell]$; otherwise D_ℓ could strictly improve the outcome. Trivially, if $\hat{x} = D_\ell$, then the only possible equilibrium outcome is $W[D_\ell, nom(P_{-\ell})] = D_\ell$. Now, let $\hat{x} \neq D_\ell$, and (as in the proof of Proposition 1), define $x^* \equiv \min(P_\ell \cap (\hat{x}, D_\ell])$ and $y^* \in P_{-\ell}$ as the closest point to $D_{-\ell}$ in $P_{-\ell}$ such that $W[x^*, y^*] = y^*$. Whenever $y^* \in [D_{-\ell}, M]$, D_ℓ's best-response is to nominate $\hat{x}$ which, by definition, defeats all of $P_{-\ell}$. So, in this case, the only equilibrium outcome is $W^* = \hat{x}$. Suppose instead that $y^* \in [\hat{x}, D_\ell]$. Then, $W[x^*, y^*] = y^*$ is the only possible equilibrium outcome. Q.E.D

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