Aspects of Topological String Theory

Thesis by
Paul L.H. Cook

In Partial Fulfillment of the Requirements
for the Degree of
Doctor of Philosophy

California Institute of Technology
Pasadena, California

2008

(Defended May 14, 2008)

Acknowledgements

I am deeply grateful to my advisor, Hirosi Ooguri, for his generous support, guidance, collaboration and many patient explanations.

I have benefited greatly from enjoyable and informative interaction with many members of the Caltech string theory group, past and present: professors Sergei Gukov, Anton Kapustin and John Schwarz; postdoctoral fellows Andrew Frey, Matthew Johnson, Joseph Marsano, Andrei Mikhailov, Yutaka Ookouchi, Kirill Saraikin, Natalia Saulina and Sakura Schafer-Nameki; and graduate students Lotty Ackerman, Miguel Bandres, Tudor Dimofte, Yi Li, Donal O'Connell, Takuya Okuda, Chang Soon Park, Kevin Setter and Jaewon Song. My special thanks to collaborator and office-mate Jie Yang, and to Ketan Vyas.

My thanks to all those with whom I have had the privilege of interacting at conferences, workshops and seminars. My particular gratitude to Robert de Mello Koch for inspiration and endless generosity over many years.

For support and encouragement, my thanks to all my friends in both Pasadena and Johannesburg, amongst them my house-mates Michael Adams and Greg Ver Steeg, Louise Coney, Paul Franklyn, Rhiannon Gwyn, Valerie Karplus, Michelle Robbins and especially Ellen Juhlin.

Last but certainly not least, for their unending support and inspiration, my thanks to my parents and brother, who have made this thesis possible.

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Abstract

Two aspects of the topological string and its applications are considered in this thesis. Firstly, non-perturbative contributions to the OSV conjecture relating four-dimensional extremal black holes and the closed topological string partition function are studied. A new technique is formulated for encapsulating these contributions for the case of a Calabi-Yau manifold constructed by fibering two line bundle over a torus, with the unexpected property that the resulting non-perturbative completion of the topological string partition function is such that the black hole partition function is equal to a product of a chiral and an anti-chiral function. This new approach is considered both in the context of the requirement of background independence for the topological string, and for more general Calabi-Yau manifolds. Secondly, this thesis provides a microscopic derivation of the open topological string holomorphic anomaly equations proposed by Walcher in arXiv:0705.4098 under the assumption that open string moduli do not contribute. In doing so, however, new anomalies are found for compact Calabi-Yau manifolds when the disk one-point functions (string to boundary amplitudes) are non-zero. These new anomalies introduce coupling to wrong moduli (complex structure moduli in A-model and Kähler moduli in B-model), and spoil the recursive structure of the holomorphic anomaly equations. For vanishing disk one-point functions, the open string holomorphic anomaly equations can be integrated to solve for amplitudes recursively, using a Feynman diagram approach, for which a proof is presented.

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Chapter 1

Introduction

String theory has wide-ranging applications in both physics and mathematics. As a proposal for a theory of quantum gravity, string theory offers the possibility of realising a "theory of everything," unifying general relativity and quantum field theory. There are good reasons to expect that a theory of quantum gravity should require a substantial change in how we think of the nature of space and time. In particular, the holographic principle, as embodied in, for example, the AdS/CFT correspondence [1], suggests that spacetime can carry far less information than would be naively expected — the information content scales by a dimension less than that of the volume of spacetime.

Black holes present an ideal test environment for studying quantum gravity and the holographic principle, as the Bekenstein-Hawking entropy of a (classical) black hole scales with black hole surface area rather than volume. There now exist many string theory constructions which realise spacetime black holes from sufficiently dense collections of strings or branes, and which reproduce at leading order the Bekenstein-Hawking entropy. A complete microscopic description of general black holes is, however, still lacking. In its absence, a natural question arises: are there simpler theories that are still "stringy," and which can be used to describe subclasses of general black holes? Just such a theory, topological string theory, will form the backdrop of this thesis.

Topological string theory describes a sub-sector of full string theory. It is still a conformal field theory describing mappings of a string propagating through time (giving a two-dimensional *worldsheet*) into a *target space*, which could be spacetime plus some compactification manifold. The key ingredient, however, is the application of a *topological twist*, which removes dependence on the (worldsheet) metric from calculations, even before one sums over such metrics during coupling to gravity. As we review below, this yields a much simplified theory which admits exact solution in some cases. Furthermore, being topological, it has no dynamics and is thus sensitive only to details of the topology of the target space — precisely the details of relevance in black hole partition functions.

Indeed, in a major step towards a complete *microscopic* description of black holes, the OSV (Ooguri-Strominger-Vafa [2]) conjecture posits a correspondence between the partition function of supersymmetric (extremal) black holes, and a product of the partition function of topological string theory on the same space and its complex conjugate; schematically,

$$Z_{\mathrm{BH}} = \psi\overline{\psi}. \tag{1.1}$$

We will review the precise form of the conjecture in Section 2.2.

Chapter 2 will use a specific realisation of an extremal black hole, due to Vafa [3], which provides successful explicit tests of the OSV conjecture at large black hole charge N. Non-perturbative (small N) effects, however, strongly suggest [4] that the appropriate gravitational object to which the OSV conjecture refers is not a single black hole, but rather the sum of multi-centre gravitational solutions, with the charges of the centres (classical singularities) summing to the total charge. In the near horizon limit, these centres are "baby universes," so the OSV conjecture can be viewed as a statement in a "third quantised" framework, involving sums over states with different numbers of universes.

An alternate and novel approach to handling non-perturbative corrections to the OSV conjecture for this system will be presented in Section 2.5. The key difference is that while the existing approach mixes the holomorphic topological string partition function and its conjugate at each order in the non-perturbative corrections, the new approach maintains the factorised form of the right-hand side of the OSV relation (1.1). Consequently, this approach can be thought of as providing a non-perturbative completion of the topological string partition function itself, which can be expressed in terms of the perturbative topological string partition function using a "chiral" recursion relation. Furthermore, since the topological string partition function can be viewed as a wavefunction for the universe [5], this new approach manifestly maintains quantum coherence under tunnelling to multiple baby-universe states. We also briefly describe attempts to generalise the result beyond the setup of [6], and discuss the complications introduced by our use of non-compact Calabi-Yau target spaces as the background.

A deeper puzzle results from the wavefunction interpretation of the topological string partition function ψ[7]. The function ψ depends on a choice of background, or values for the parameters of the target space of the topological string. One would expect these choices not to affect physical observables, and indeed the variation of ψ and $\overline{\psi}$ cancel as the choice of background is modified. The chiral recursion relation complicates the interpretation of background independence, as discussed in Section 2.6, as it is not manifest that the proposed non-perturbative completion of the topological string partition function has the expected transformation properties under change of background.

Taking a step back, the wavefunction interpretation of the topological string depends on the holomorphic anomaly equations of BCOV [8], which form the backdrop to the second half of this thesis. The holomorphic anomaly equations capture the anomalous dependence of the topological string partition function on anti-holomorphic Calabi-Yau (target space) moduli. In addition to a relation to the wavefunction interpretation of the topological string, the anomaly equations allow efficient calculation of the partition function, in terms of a genus-by-genus recursion relation, up to a holomorphic function (the *holomorphic ambiguity*) at each genus. Fixing these requires additional data. One source thereof makes use of the wavefunction interpretation to change *polarisation*, or choice of background, for the topological string. A particularly useful choice gives a partition function that is holomorphic, but suffers from a modular anomaly (since, as we will see, the complex structure that defines holomorphicity is related to a choice of three-cycles on the manifold, and hence to monodromies around points in moduli space where three-cycles shrink). The interplay between holomorphicity and modularity has been used to fix the holomorphic ambiguity to very high genus using conditions from special points in moduli space [9, 10, 11]. These computations are also of interest mathematically, as they allow the extraction of topological invariants, such as Gromov-Witten invariants, that count the number of maps of various kinds from

Riemann surfaces into Calabi-Yau manifolds.

Recently Walcher [12, 13] proposed extended holomorphic anomaly equations for the *open* topological string (that is, in the presence of D-branes), under the assumptions that open string moduli are absent and that the disk one-point functions (i.e., closed strings ending on branes) vanish. A detailed microscopic derivation of the extended holomorphic anomaly equations will be presented in Chapter 3, confirming the conjectured result. This is followed by an analysis of the decoupling, or lack thereof, of moduli from the "wrong" model — that is, moduli which should be irrelevant for the topological twisting under consideration. For both the anti-holomorphic and wrong model moduli we demonstrate, as reported in [14], that additional anomalies are present *unless* the disk one-point functions vanish.

Armed with the extended holomorphic anomaly equations, Chapter 4 provides a proof of a recursive solution for open topological string amplitudes genus-by-genus in terms of Feynman diagrams, as reported in [15]. The proof makes use of the analogous approach for solving for closed string topological string amplitudes recursively.

The remainder of this chapter will provide a brief review and definition of the topological string and associated mathematical objects. Chapter 2 is concerned with the chiral recursion relation in the context of the OSV conjecture relating black holes and the topological string. Chapter 3 covers the extended holomorphic anomaly equations for open topological string theory, and the new anomalies that are present for non-vanishing disk one-point functions. Chapter 4 presents and proves a method for solving the open string holomorphic anomaly equations recursively. Appendix A provides some calculations used in Section 2.7, for extending the results of Chapter 2 to more general target manifolds.

1.1 Calabi-Yau manifolds

String theory, and especially topological string theory, is often considered on a background that is a *Calabi-Yau manifold*, a manifold that preserves an unusually large amount of symmetry. Usually Calabi-Yau three-folds (that is, having six real dimensions) are considered, and we will soon restrict our attention to this case. In physical string theory, three-folds allow dimensional reduction from ten dimensions to the four dimensions of spacetime, while preserving some unbroken supersymmetry. Furthermore, topological string theory has the richest mathematical structure on a Calabi-Yau three-fold.

In essence, a Calabi-Yau manifold is a Kähler manifold with vanishing first Chern class. We unpack aspects of this definition below; more details can be found in [16] or any of the canonical texts. We start with a complex n-fold, which is an orientable $2n$-dimensional manifold with a *complex structure*, allowing the consistent definition of holomorphic coordinates z^i and anti-holomorphic coordinates $\bar{z}^{\bar{i}}$, $i = 1, \cdots, n$. Manifolds may have many complex structures, or indeed none. A metric compatible with the complex structure satisfies

$$g_{ij} = g_{\bar{i}\bar{j}} = 0. \tag{1.2}$$

Such a metric is termed *Hermitian*, and defines the Kähler form

$$\omega = \frac{i}{2} g_{i\bar{j}} dz^i \wedge d\bar{z}^{\bar{j}}. \tag{1.3}$$

The metric is termed *Kähler* if $d\omega = 0$, in which case one can locally find a function K, called the *Kähler potential*, satisfying

$$\omega \equiv 2i\partial\bar{\partial}K, \tag{1.4}$$

where $\partial_i \equiv \frac{\partial}{\partial z^i}$. ω is a global $(1,1)$ form, and defines a cohomology class in $H^{1,1}(M)$, termed the Kähler class. A complex manifold with Kähler metric is a Kähler manifold. It has the important property that the Levi-Civita connection vanishes for mixed indices, so that holomorphic vectors remain holomorphic under parallel transport, thereby restricting the holonomy to a $U(n)$ subgroup of $SO(2n, \mathbb{R})$.

The Calabi-Yau condition for a manifold can be expressed in a number of equivalent forms. Amongst them are:

- The first Chern class vanishes, $c_1(M) = 0$.
- There exists a unique (up to rescaling), nowhere vanishing global holomorphic $(n, 0)$ form, Ω. The volume of the manifold is then $\int_M \Omega \wedge \overline{\Omega}$.

- There exists a unique metric such that the Ricci tensor vanishes — and hence the manifold is a solution to the vacuum Einstein equations, admits covariant-constant spinors, and so preserves some spacetime supersymmetry after compactification.

- The manifold has $SU(n)$ holonomy.

Consider now Calabi-Yau three-folds in particular. The above definitions imply that for a given topology, choosing a Kähler class and complex structure uniquely determine a Calabi-Yau manifold. The space of such choices is termed the *moduli space* $\mathcal{M}$ of the Calabi-Yau, with Kähler and complex structure moduli as coordinates on the moduli space. The term "Calabi-Yau manifold" is frequently used to refer to the entire class of manifolds of given topology but arbitrary moduli.

The moduli space is intimately related to the cohomology classes of the manifold. Define the *Hodge numbers* $h^{p,q} = \dim H^{p,q}(M)$ as the dimensions of the cohomology classes of the manifold. The following properties can be shown:

$$h^{p,q} = h^{q,p}, \qquad h^{p,q} = h^{n-p,n-q}, \tag{1.5}$$

$$h^{1,0} = h^{2,0} = 0. \tag{1.6}$$

The existence of the unique holomorphic top form Ω implies $h^{3,0} = 1$, and of course $h^{0,0} = h^{3,3} = 1$, so the only unfixed Hodge numbers are $h^{1,1}$ and $h^{2,1}$. The Kähler class (1.3) can be deformed by the addition of arbitrary elements of $H^{1,1}(M)$, and so locally the Kähler moduli space is isomorphic to $H^{1,1}(M)$, with dimension $h^{1,1}$. The volumes of $2p$-cycles C_{2p} of the manifold are calculated as

$$\int_{C_{2p}} \omega \wedge \cdots \wedge \omega,$$

where there are p factors of ω. That all volumes be positive constrains the moduli to the *Kähler cone*, with boundaries corresponding to singular degenerations of the Calabi-Yau.

Complex structure deformations are best studied through their effects on the unique holomorphic top-form $\Omega = f(z)dz^1 \wedge \cdots \wedge dz^n$. Deformations mix holomorphic and anti-holomorphic coordinates,

$$z^i \to a^i{}_j z^j + b^i{}_{\bar{j}} \bar{z}^{\bar{j}}.$$

Infinitesimally, then, complex structure deformations transform Ω to a $(3,0)+(2,1)$-form. It can be shown that the tangent space of infinitesimal complex structure deformations is indeed isomorphic to $H^{2,1}(M)$, with dimension $h^{2,1}$. In Chapter 2 we will need coordinates on this moduli space, which can be defined as follows: From the above discussion and Poincaré duality,

$$\dim H^3(M) = h^{3,0} + h^{2,1} + h^{1,2} + h^{0,3} = 2h^{2,1} + 2 = \dim H_3(M).$$

We can choose a canonical basis for the group of three-cycles $H_3(M)$ as a set of $2h^{2,1}+2$ three-cycles A^I, B_J, $(I, J = 0, ..., h^{2,1})$. In six dimensions three-cycles generically intersect at points, so by choosing signs according to orientation the basis has intersection numbers

$$\begin{aligned} &A^I \cap A^J = 0, \qquad\qquad B_I \cap B_J = 0, \\ &A^I \cap B_J, = \delta^I_J. \end{aligned} \tag{1.7}$$

From these define coordinates,

$$X^I = \int_{A^I} \Omega, \qquad\qquad F_I = \int_{B_I} \Omega, \tag{1.8}$$

and a dual basis α_I, β^J for $H^3(M)$,

$$\int_{A^I} \alpha_J = \delta^I_J, \qquad\qquad \int_{B_I} \alpha_J = 0, \tag{1.9}$$

and likewise for β^I. Now a theorem by de Rham shows that Ω is completely determined by its integrals over this basis of three-cycles, so

$$\Omega = X^I \alpha_I + F_I \beta^I, \tag{1.10}$$

with Einstein summation assumed. There are more variables X^I and F_I than complex structure moduli. Indeed, it can be shown that the F_I are dependent variables,

$$F_I = \partial_I \mathcal{F}_0, \tag{1.11}$$

where $\mathcal{F}_0 = \frac{1}{2}X^J F_J$ is termed the *prepotential*. Lastly, recall that Ω is defined only up to rescaling by a complex number, which defines a complex line bundle $\mathcal{L}$ on the moduli space of complex structures. The X^I are therefore homogeneous coordinates on the projective space of complex structures. $\mathcal{F}_0$ is homogeneous of degree two in the X^I, and so is a section of $\mathcal{L}^2$.

There is a natural metric on the moduli space of complex structures, the *Weil-Petersson* metric,

$$G_{i\bar{j}} = \frac{\int \chi_i \wedge \overline{\chi}_{\bar{j}}}{\int \Omega \wedge \overline{\Omega}}, \tag{1.12}$$

where χ_i and $\overline{\chi}_{\bar{j}}$ are $(2,1)$ and $(1,2)$ forms, respectively. This metric is itself Kähler (that is, both the Calabi-Yau and its moduli space are Kähler manifolds), with Kähler potential,

$$K = -\log i \int \Omega \wedge \overline{\Omega} = -\log i \left(X^I \overline{\partial_I \mathcal{F}_0} - \overline{X}^I \partial_I \mathcal{F}_0 \right), \tag{1.13}$$

where bars are complex conjugation. It follows that

$$e^{-K} = \int \Omega \wedge \overline{\Omega} \tag{1.14}$$

has the natural structure of an inner product or metric on the line bundle $\mathcal{L}$ identified in the previous paragraph.

1.2 Topological string theory

String theory can be treated by considering the quantum field theory living on the *worldsheet* of the string, the two-dimensional Riemann surface that the string traverses in spacetime. Since the choice of coordinates on the worldsheet is arbitrary, the theory is a two-dimensional conformal field theory, coupled to two-dimensional gravity. Conformal field theories are in addition topological if their correlators are independent of the worldsheet metric, *before* the path integral is performed. Coupling a topological field theory to gravity produces a topological string theory. These turn out to be much simpler than physical string theories, and can in some cases be exactly solved. This section will very briefly review the construction of a topological string theory, from a topological field theory of the cohomological or "Witten" type. More detail can be found in the reviews [16, 17, 18, 19].

The theory starts as a non-linear sigma model, a quantum field theory of maps of the string worldsheet Σ (locally $\mathbb{C}$), into a *target space* manifold M. The mapping is

$$X : \Sigma \to M.$$

$X = X^i$ can be treated as a bosonic field on the worldsheet, taking values in the target space, with i running over the dimension of the target space. Supersymmetry can be included by adding "fermionic" directions to the worldsheet, such that the worldsheet fields are now superfields. Taylor expanding these with respect to the fermionic directions gives a finite set of bosonic and fermionic fields, as the anti-commuting nature of fermionic variables truncates the Taylor expansion. Bosonic fields at order great than zero in the Taylor expansion are auxiliary and can be integrated out, so supersymmetry can be handled by just introducing fermionic fields ψ^i on the worldsheet. We will be interested in specifically $\mathcal{N} = (2, 2)$ supersymmetric nonlinear sigma models, which have four supercharges on the worldsheet. Moving to a light-cone gauge on the worldsheet distinguishes left- and right-moving operators and fields, so there are two supercharges in each sector, denoted $G_\pm$ and $\overline{G}_\pm$, respectively. These obey the commutation relations

$$\begin{aligned} &\{G^\pm, \overline{G}^\pm\} = 0, && [G^\pm, H_L] = 0, \\ &\{G^+, G^-\} = 2T, && \{\overline{G}^+, \overline{G}^-\} = 2\overline{T}, \end{aligned} \tag{1.15}$$

where T and $\overline{T}$ are the left- and right-moving energy-momentum charges, related to the Hamiltonian and momentum by $H = T + \overline{T}$ and $P = T - \overline{T}$, respectively. Supercurrents $G^\pm_z$ and $\overline{G}^\pm_{\bar{z}}$ corresponding to the supercharges can be defined,

$$G^+ = \oint dz\, G^+_z(z), \qquad \overline{G}^+ = \oint dz\, \overline{G}^+_{\bar{z}}(z),$$

and likewise for G^-, $\overline{G}^-$. The action is

$$S = \int d^2z \left(\frac{g_{i\bar{j}}}{2}(\overline{\partial}_{\bar{z}} X^i \partial_z X^{\bar{j}} - \partial_z X^i \overline{\partial}_{\bar{z}} X^{\bar{j}}) + \frac{i}{2} g_{i\bar{j}} \psi^i_+ D_{\bar{z}} \psi^{\bar{j}}_+ + \frac{i}{2} g_{i\bar{j}} \psi^i_- D_z \psi^{\bar{j}}_- + \frac{1}{4} R_{i\bar{j}k\bar{l}} \psi^i_+ \psi^{\bar{j}}_+ \psi^k_- \psi^{\bar{l}}_- \right), \quad (1.16)$$

where $g_{i\bar{j}}$ is the metric on the target space, $i, \bar{j} = 1, 2, 3$ are holomorphic and anti-holomorphic indices on the target space, $\psi^i_\pm$ and $\psi^{\bar{i}}_\pm$ are left- and right-moving fermionic fields, respectively, D_z is the (pulled-back) covariant derivative and $R_{i\bar{j}k\bar{l}}$ is the Riemann tensor.

Requiring $\mathcal{N} = (2, 2)$ supersymmetry and worldsheet superconformal symmetry restricts the manifold M to be not only Kähler, but Calabi-Yau; and as discussed above, the three-fold is the most interesting case. For physical superstring theory, in particular type IIA and IIB theories, the total target space is ten dimensional: a Calabi-Yau compactification manifold fibred over a non-compact $(3, 1)$-dimensional spacetime. Four dimensional effective field theories follow from taking the former to be small. The physics depends on the geometry of the Calabi-Yau, thus we will be interested in the dependence of the topological string theory on the Calabi-Yau moduli.

To construct a topological field theory, we want a global supercharge on the worldsheet, but this requires covariantly constant spinors, and therefore a flat worldsheet. This restriction can be circumvented by *topologically twisting* the theory. By suitably modifying the Lorentz group, this produces fermionic fields that transform as scalars, as required. The Lorentz group of the worldsheet, in Euclidean signature, is $SO(2) = U(1)_E$. The action (1.16) has two $U(1)_R$ R-symmetries (where the terminology reflects that these symmetries commute with the supersymmetry), referred to as the *axial* and *vector* R-symmetries. The charge assignments of the supercharges are shown in Table 1.2. In the connection, replacing $U(1)_E$ with the diagonal subgroup $U(1)'_E$ of $U(1)_E \times U(1)_R$ changes the transformation properties of the supercharges such that, depending on the choice of $U(1)_R$, two of the supercharges become (anti-commuting) scalars, and the other two become vectors. Twisting with $U(1)_V$ produces the so-called A-twist, and $U(1)_A$ the B-twist. Each case has a scalar combination

	Before twisting				A-twist		B-twist	
	$U(1)_V$	$U(1)_A$	$U(1)_E$	$\mathcal{L}$	$U(1)'_E$	$\mathcal{L}'$	$U(1)'_E$	$\mathcal{L}'$
G^+	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\sqrt{K}$	0	$\mathbb{C}$	0	$\mathbb{C}$
G^-	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\sqrt{K}$	-1	K	-1	K
$\overline{G}^+$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\sqrt{\overline{K}}$	0	$\mathbb{C}$	1	$\overline{K}$
$\overline{G}^-$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\sqrt{\overline{K}}$	1	$\overline{K}$	0	$\mathbb{C}$

Table 1.1: *Charges of the supercharges before and after twisting. $U(1)_V$ and $U(1)_A$ are the axial and vector R-symmetries and $U(1)_E$ is the Lorentz group. $\mathcal{L}$ is the bundle of which the supercharges are sections, where K is the canonical bundle and $\mathbb{C}$ is the trivial bundle.*

of left- and right-moving supercharges:

$$\text{A-twist:} \qquad Q_A = G^+ + \overline{G}^+,$$
$$\text{B-twist:} \qquad Q_B = G^+ + \overline{G}^-. \tag{1.17}$$

In the following we choose $Q = Q_A$, but by a simple change of notation the statements hold equally for the B-twist. The supercurrents G^+_z and $\overline{G}^+_{\bar{z}}$ are now holomorphic and anti-holomorphic one-forms, respectively, and supercurrents G^-_{zz} and $\overline{G}^-_{\bar{z}\bar{z}}$ are two-forms (or rather tensors with two cotangent holomorphic and anti-holomorphic indices, respectively).

The theory is now a cohomological topological quantum field theory, with operator Q as the cohomology charge. This statement has four requirements. Firstly, there must exist a fermionic symmetry operator satisfying

$$Q^2 = 0. \tag{1.18}$$

From (1.15), Q is such an operator. This construction is similar to the Faddeev-Popov method of gauge fixing, for example in the bosonic string, in which case Q is termed a BRST operator. Secondly, physical operators $\mathcal{O}_i$ are defined to be closed under the action of Q,

$$\{Q, \mathcal{O}_i\} = 0. \tag{1.19}$$

Thirdly, vacua of the theory should not spontaneously break the Q symmetry, $Q|0\rangle = 0$. This implies

$$\mathcal{O}_i \sim \mathcal{O}_i + \{Q, \Lambda\}, \tag{1.20}$$

where Λ is any operator, and here and in the following the anti-commutator is used to represent both commutator and anti-commutator, as appropriate. This relation follows from noting that in an expectation value, (1.19) means that Q can be anti-commuted past any other operators in the expectation value to annihilate the vacuum. The physical operators are thus Q-cohomology classes.

The fourth requirement for a cohomological theory is that the energy-momentum tensor is Q-exact,

$$T_{\alpha\beta} \equiv \frac{\delta S}{\delta h^{\alpha\beta}} = \{Q, V_{\alpha\beta}\}, \tag{1.21}$$

for some operator $V_{\alpha\beta}$. It is this condition that makes the theory topological: as long as the operators $\mathcal{O}_i$ are independent of the metric, the only source of metric dependence in the path integral is the action. Now, however, (1.21) implies that such dependence is Q-exact, and the Q can be anti-commuted using (1.19) to annihilate the vacuum. Metric invariance has the useful implication that we can freely deform the worldsheet, or equivalently move operator insertions around the worldsheet, without affecting correlators. This will be particularly useful in Chapter 3 when worldsheet deformations will be used to dramatically simplify calculations. A second key benefit of topological theories follows from restoring $\hbar$ dependence in the action. Consider an unnormalised expectation value,

$$\langle \mathcal{O} \rangle = \int \mathcal{D}\phi \; \mathcal{O} \; \exp\left(\frac{i}{\hbar} S(\phi)\right), \tag{1.22}$$

where ϕ represents the set of fields of the theory. An easy way to satisfy (1.21), which turns out to be possible for our theories, is to write $S = \{Q, V\}$ for some operator V. Then the $\hbar$ derivative of (1.22) is a correlator including $\{Q, V\}$, which vanishes as above. Thus semi-classical calculations are exact.

The topological field theory constructed above can be upgraded to a topological string theory by coupling to gravity — that is, by making the worldsheet metric a dynamical field, and integrating over it, to give the (vacuum) amplitude at worldsheet genus g, F_g. The moduli space of a genus g Riemann surface has complex dimension $m_g = 3(g-1)$, which can be thought of as specifying the locations of the endpoints and the period matrix of each handle, with the first handle not contributing moduli but merely fixing the remaining symmetry of the sphere. Integrating over this moduli space requires a measure that is invariant under coordinate transformations of both the worldsheet and moduli space. In two dimensions, conformal transformations correspond to holomorphic transformations, so the worldsheet moduli space corresponds to *changes* of the complex structure on the worldsheet. These can be parametrised by holomorphic one-forms with anti-holomorphic vector indices, termed *Beltrami differentials*, defined by

$$dz \mapsto dz + \epsilon \mu_{\bar{z}}^{z}(z) d\bar{z}. \tag{1.23}$$

The indices of $\mu_{\bar{z}}^{z}$ are not suitable for integration. The resolution can be motivated by, for instance, analogy with the very similar structures in bosonic string theory [20], and it is to contract with the supercurrent G^-_{zz}. Thus the integration over the worldsheet moduli space is

$$\int_{\mathcal{M}_g} \prod_{a=1}^{3g-3} \left(dm^a d\bar{m}^a \int_\Sigma G^-_{zz} (\mu_a)^z_{\bar{z}} \int_\Sigma \overline{G}^-_{\bar{z}\bar{z}} (\bar{\mu}_a)^{\bar{z}}_z \right). \tag{1.24}$$

The factors of G^- and $\overline{G}^-$ introduce axial and vector $U(1)_R$ charge into the measure. This turns out to be desirable, as this charge exactly absorbs the fermion zero modes corresponding to zero eigenvalues of the twisted covariant derivative appearing in the action in terms of the form $\psi^i_+ D_{\bar{z}} \psi^{\bar{j}}_+$, that would otherwise result in vanishing fermionic integrals in the path integral (recalling that $\int d\psi = 0$, while $\int d\psi\, \psi = 1$). In order to track charge, it is convenient to define generators for left- and right-moving $U(1)_R$ symmetries,

$$F_L = F_A + F_V, \qquad F_R = F_A - F_V, \tag{1.25}$$

where F_A and F_V are the generators of the axial and vector $U(1)$ symmetries, respectively.

Consulting Table 1.2 gives the charge assignments,

$$G^+ : \quad (+1,0), \qquad \overline{G}^+ : \quad (0,+1), \qquad G^- : \quad (-1,0), \qquad \overline{G}^- : \quad (0,-1). \tag{1.26}$$

Any additional insertions in the amplitudes must maintain the overall charge $(3g-3, 3g-3)$ of the amplitude. Note that $g = 0$ and $g = 1$ are special cases, where additional insertions (three for $g = 0$ and one for $g = 1$) are required in order to fix the remaining rotational symmetry of the worldsheet — the vacuum amplitude F_g vanishes.

As in physical string theory, the genus g amplitude comes with $2g-2$ powers of the string coupling λ. Including worldsheets of all genus produces the *topological string partition function*,

$$Z = \exp\left(\sum_{g=0}^{\infty} \lambda^{2g-2} F_g\right). \tag{1.27}$$

We are interested in the dependence of the amplitudes F_g on the moduli of the Calabi-Yau target space. Furthermore, we would like to understand the physical operators that may be inserted into the vacuum amplitudes. It turns out that studying the latter question provides insight into the former, as described in the next section.

1.3 Chiral rings

The invariance of the theory under worldsheet metric deformation allows a very explicit realisation of the operator-state correspondence. Inserting a physical operator ϕ_I on a hemisphere gives a definite ground state on the boundary by stretching out the hemisphere to be infinitely long. This ground state can be identified with the operator inserted,

$$|I\rangle = \phi_I |0\rangle. \tag{1.28}$$

The appropriate physical operators are, as discussed above, those satisfying

$$\{G^+, \phi_I\} = 0, \qquad \{\overline{G}^+, \phi_I\} = 0. \tag{1.29}$$

The range of the index I will become clear below. These operators form a ring, termed the (c, c) chiral ring, with natural multiplication either as operators or as states by "gluing" two long hemispheres together and taking the path integral over the resulting "cigar." This defines the topological metric,

$$\eta_{IJ} = \langle J|I\rangle. \tag{1.30}$$

The conjugate operators also form a ring, called the (a, a) or anti-chiral ring, satisfying

$$\{G^-, \overline{\phi}_{\bar{I}}\} = 0, \qquad \{\overline{G}^-, \overline{\phi}_{\bar{I}}\} = 0.$$

Since the chiral rings correspond to the same set of vacua, there must be a change of basis transformation relating the two, which defines the Hermitian metric,

$$g_{I\bar{J}} = \langle \bar{J}|I\rangle. \tag{1.31}$$

The ring is such that

$$\phi_I \phi_J = C_{IJ}^K \phi_K + \{Q, \Lambda\}, \tag{1.32}$$

where the C_{IJ}^K are the structure constants. From the operator-state correspondence it follows that $C_{IJK} = \langle \phi_I \phi_J \phi_K \rangle$ is the three-point function or Yukawa coupling, and it can be shown to be holomorphic. By choosing the other twisting (1.17), one can also form the *twisted chiral* (c, a) and *twisted anti-chiral* (a, c) rings, swapping the right-moving commutation properties above.

To construct the chiral primary operators explicitly, it is convenient to rename the worldsheet fermions to better indicate the bundles to which they belong, after twisting:

$$\begin{aligned} \psi_+^i \equiv \psi^i &\quad \in \quad X^*(T^{(1,0)}M) \\ \psi_+^{\bar{i}} \equiv \overline{\psi}^{\bar{i}} &\quad \in \quad X^*(T^{(0,1)}M) \\ \psi_-^i \equiv \eta^i &\quad \in \quad \Omega^{1,0} \otimes X^*(T^{(1,0)}M) \\ \psi_-^{\bar{i}} \equiv \overline{\eta}^{\bar{i}} &\quad \in \quad \Omega^{0,1} \otimes X^*(T^{(0,1)}M), \end{aligned} \tag{1.33}$$

where "∈" means "is a section of," and X^* is the pullback of the map from worldsheet to target space. To satisfy (1.29), ϕ_I may not include factors of η or $\overline{\eta}$. Thus a general (local) (c, c) chiral ring operator is

$$\phi_I = \omega_{i_1 \cdots i_p, \bar{j}_1 \cdots \bar{j}_q} \psi^{i_1} \cdots \psi^{i_p} \overline{\psi}^{\bar{j}_1} \cdots \overline{\psi}^{\bar{j}_q}, \tag{1.34}$$

where anti-symmetry of the target space indices imply that

$$0 \leq p, q \leq 3,$$

where (p, q) corresponds to the (left,right) $U(1)_R$ charge of the operator. Non-local chiral operators can be constructed using the *descent equations*. For our purposes, we can construct a two-form operator using the one-form supercharges,

$$\phi_I^{(2)} = \{G^-, [\overline{G}^-, \phi_I]\}. \tag{1.35}$$

The chiral operators with charge $(1, 1)$ are particularly important, and we denote them ϕ_i, and $\phi_i^{(2)}$ for the two-form descendant. These descendants have vanishing overall $U(1)_R$ charge, and so can be inserted into the correlator. Indeed, these operators are termed *marginal*, as they generate marginal deformations of the conformal field theory. Explicitly, we can add to the action the term

$$\delta S = t^i \int_\Sigma \phi_i^{(2)}. \tag{1.36}$$

To determine the physical effects of these deformations, note that the anti-commuting fermions give the chiral primaries (1.34) the structure of (p, q) forms, with Q identified as the de Rham cohomology operator. Their relation with forms on the target space M depends on the choice of twist, giving so-called A- and B-model topological string theory, as follows:

- **A-model:** Charge (p, q) chiral primaries are identified with $H^{p,q}(M)$. Marginal operators correspond to $H^{1,1}(M)$ cohomology elements, and thus generate deformations of the Kähler form. A-model is therefore sensitive to Kähler moduli ("volume" moduli) of the Calabi-Yau, and is independent of complex structure moduli.

- **B-model:** Charge (p,q) chiral primaries are identified with $H^p_{\bar\partial}(M, \wedge^q TM)$, that is $(0,p)$ forms with values in the antisymmetrised product of q tangent spaces. On a Calabi-Yau there is a unique holomorphic top-form Ω (three form in our case), which can be contracted with the vector indices to map to the cohomology class $H^{3-q,p}(M)$. Marginal operators thus correspond to $H^{2,1}(M)$ cohomology elements, and so generate deformations of complex structure. B-model is therefore sensitive to complex structure moduli ("shape" moduli) of the Calabi-Yau, and independent of Kähler moduli.

The above statements of course need proof, for which we refer to the references, particularly [8, 16]. In particular, the statements that A-model is independent of complex structure moduli and B-model of Kähler moduli (henceforth termed "wrong" moduli) follows naively from demonstrating that deformations of the form (1.36), but with operators from the chiral rings (c,a) and (a,c), are BRST trivial (that is, Q-exact), and so are zero in the correlator. A more careful analysis confirms this intuition, but reveals further structure in the case of (a,a) chiral ring insertions, corresponding to the addition to the action of the term

$$\delta S' = \bar{t}^{\bar{i}} \int_\Sigma \overline{\phi}^{(2)}_{\bar{i}}. \tag{1.37}$$

This insertion is BRST trivial, but [8] showed that non-zero contributions arise at the boundaries of moduli space. Thus A-model depends anomalously on anti-holomorphic Kähler moduli and B-model on anti-holomorphic complex structure moduli, through the so-called *holomorphic anomaly equations*,

$$\frac{\partial}{\partial \bar{t}^{\bar{i}}} F_g = \frac{1}{2} \overline{C}^{jk}_{\bar{i}} \left[\sum_{r=1}^{g-1} D_j F_r D_k F_{g-r} + D_j D_k F_{g-1} \right], \tag{1.38}$$

where D_j is the natural covariant derivative on the vacuum bundle, or the space of theories under variation of the Calabi-Yau moduli, and $\overline{C}^{jk}_{\bar{i}}$ is the anti-topological Yukawa coupling, with indices raised using the topological metric. We discuss the derivation of this result, or rather its extension to the open string case, in Chapter 3; and briefly discuss the interpretation of the anomaly in Section 2.6.

The above discussion has shown a close relation between marginal chiral primary fields and forms (equivalently moduli) on the Calabi-Yau target space. Indeed, the chiral primary topological metric restricted to marginal operators, when appropriately normalised, is exactly the Weil-Petersson metric on the Calabi-Yau moduli space (1.12),

$$\frac{g_{i\bar{j}}}{\langle\bar{0}|0\rangle} = e^{K} g_{i\bar{j}} = G_{i\bar{j}}. \tag{1.39}$$

Chapter 2

Non-perturbative topological strings and black holes

One of the more remarkable applications of the topological string is the conjectured relationship, the *OSV conjecture* [2], between the statistical partition function of four-dimensional BPS black holes constructed by compactifying type II superstrings on Calabi-Yau three-folds, and the topological string partition function on the same Calabi-Yau three-fold. The conjecture takes the schematic form

$$Z_{\mathrm{BH}} = |\psi_{\mathrm{top}}|^2, \tag{2.1}$$

where Z_{BH} is the partition function of the black hole, calculated using the grand canonical ensemble for electric charges and the microcanonical ensemble for magnetic charges, and ψ_{top} is the topological string amplitude.

Explicit tests of this correspondence have not yet been performed for compact Calabi-Yau manifolds. However, adapting the conjecture to the case of non-compact Calabi-Yau manifolds allows explicit calculation of both the gravity and topological string sides. This was first done [6] for a toric Calabi-Yau three-fold, constructed as the sum of two complex line bundles over a torus T^2. As we review below, the black hole partition function reduces to that of the two-dimensional Yang-Mills theory on T^2. At large N, this partition function can be decomposed into chiral and anti-chiral components, which can be identified as topological string amplitudes.

An obvious follow-up question is how to address non-perturbative corrections to the OSV conjecture, that is, small N effects. In the above system, the black hole partition function is exactly computable, but the topological string amplitudes at finite N over-count the states. However, these topological string amplitudes are only defined perturbatively, and so we can use this mismatch in counting to investigate the non-perturbative implications of the OSV conjecture. This was done in [4] for the T^2-based target space, and extended in [21] to somewhat more general toric Calabi-Yau manifolds. In their approach, discussed in Section 2.4, the overcounted states were systematically removed by subtracting terms with additional factors of $|\psi_{\rm top}|^2$. The *full* black hole partition function $Z_{\rm BH}$ is then equivalent to a sum over multi-centred black hole solutions, each interpretable as a "baby universe," with charges summing to the overall $Z_{\rm BH}$ charge.

In Section 2.5, we propose an alternate scheme for including the non-perturbative corrections in the OSV conjecture. This approach also yields a sum over terms with arbitrary numbers of factors, with charges summing to the overall black hole charge. However, in contrast to the previous scheme, the topological string side of the OSV conjecture is still in the form of a product of chiral and anti-chiral functions. Specifically, we define a new quantity Ψ, which satisfies the following conditions:

1. $\Psi = \psi_{\rm top}$ in the large N limit;
2. The appropriate form of the OSV conjecture using Ψ (schematically $Z_{\rm BH} = |\Psi|^2$) is *exact* non-perturbatively; and
3. Ψ is expressible as an infinite sum of positive powers of $\psi_{\rm top}$, using a recursion relation.

As a result, we will call Ψ the *non-perturbative completion of the topological string partition function* for the system we consider. This approach also raises interesting questions related to background independence of the topological string, and quantum coherence of the baby universes, which we discuss in Section 2.6.

One can attempt to extend this approach to more general classes of Calabi-Yau — in particular, replacing the base T^2 with an arbitrary genus G surface. For these systems, we find a chiral function satisfying the first two of the above conditions. For the third condition, we find qualitative behaviour that is suggestively similar to other approaches to the problem [21]. In particular, the non-compact Calabi-Yau forces the introduction of representations coupling the various partition functions, due to “ghost” branes manifesting the presence of non-normalisable Kähler moduli corresponding to boundary conditions at asymptotic infinity in the Calabi-Yau. We find, in agreement with [21], two classes of such representations: P-type representations coupling the Ψ and $\overline{\Psi}$ factors; and S-type representations coupling the factors in the expansion of Ψ in terms of (perturbative) topological string amplitudes ψ_{top}. Less helpfully, the quantitative matching of classical prefactors in the recursion relation fails. These prefactors are in general somewhat ambiguous in non-compact manifolds, so further progress may require concrete models involving compact Calabi-Yau manifolds.

This chapter is organised as follows. Section 2.1 reviews the construction of four-dimensional black holes in type IIB string theory, allowing the OSV conjecture to be precisely stated in Section 2.2. Section 2.3 describes an explicit test system for the OSV conjecture. Sections 2.4 and 2.5 describe the two approaches to handling non-perturbative corrections, the former breaking chiral factorisation and the latter maintaining it. A challenge to the interpretation of the latter results is outlined in Section 2.6, having to do with the wavefunction interpretation of the topological string and background independence. Finally, Section 2.7 and Appendix A describe progress towards extending our treatment to somewhat more general Calabi-Yau manifolds.

2.1 Black holes from type IIB string theory

Superstring theory can produce “black” objects (black holes as well as black strings, rings, and so forth) in many different ways — with excellent agreement with classical predictions like the Bekenstein-Hawking entropy. In this section we review the construction of four-dimensional supersymmetric black holes in the context of type IIB superstring theory, by wrapping D-branes on cycles in a small Calabi-Yau. Type IIB is chosen for convenience here, and in Section 2.3.2 we will see a type IIA construction.

Type IIB superstring theory has odd D-branes. Since we want a black hole localised in Lorentzian space and extended in time, the branes must wrap odd cycles in the Calabi-

Yau — and more specifically, three-cycles, as one- and five-cycles are homologically trivial in Calabi-Yau manifolds. Using the basis of three-cycles defined in Section 1.1, wrap a D3-brane on the three-cycle,

$$\mathcal{C} = q_I A^I - p^J B_J,$$

where q_I and p^J are the *wrapping numbers* which count wrapping multiplicity for each cycle. The D3-brane is at a point in non-compact space, with the D-brane configuration preserving $\mathcal{N} = 2$ supersymmetry in spacetime, producing a four-dimensional black hole. In fact, it is a Reissner-Nordström (charged, non-rotating) black hole: D3-branes couple to four-form gauge fields, which become one-form gauge fields in spacetime after compactification (three indices are along the internal directions of the D-brane worldvolume), so the black hole is charged, with charges q_I and p^J, under the gauge group $U(1)^{h^{2,1}+1}$. The q and p charges are dual through Hodge duality of the forms dual to the cycles A^I and B_J, so we term the q_I *electric* and the p^J *magnetic* charges. The preservation of supersymmetry implies that the black hole is *extremal*, i.e., it has the least mass among black holes of the same charge, or equivalently the inner and outer horizons of the Reissner-Nordström solution coincide. This means the black hole emits no Hawking radiation, as such radiation would lower the mass but not the charge. From the supersymmetry perspective, the black hole is BPS, and the lack of Hawking radiation corresponds to the fact that supersymmetric solutions are necessarily of lowest possible energy. The microscopic entropy of the black hole is the Boltzmann entropy: the logarithm of the number of BPS states of the given brane configuration. It is one of the major successes of string theory that the leading order of this entropy exactly matches the Bekenstein-Hawking entropy for macroscopic black holes.

The mass of the black hole is the energy required to "stretch" the D-brane over the volume of this cycle, motivating the result that the BPS mass is

$$M_{BPS}^2 = e^K |\mathcal{Q}|^2, \qquad \mathcal{Q} = \int_{\mathcal{C}} \Omega = q_I X^I - p^I F_I. \tag{2.2}$$

The Bekenstein-Hawking entropy is

$$S_{BH} = \frac{\pi}{4} \int_{CY} \Omega \wedge \overline{\Omega}. \tag{2.3}$$

We know, however, that Ω depends on the values of the complex structure moduli, so these need to be fixed — and their values should only depend on the parameters (mass and charge) of the four-dimensional black hole. Luckily, the moduli are indeed fixed by the charges of the black hole, through the *attractor equations* [22, 23]: minimising the mass (2.2) gives that at the event horizon,

$$\mathrm{Re}(\lambda^{-1}X^I) = p^I, \qquad \mathrm{Re}(\lambda^{-1}F_I) = q_I. \tag{2.4}$$

These $2h^{2,1}+2$ real equations fix all complex structure moduli, and hence Ω. Physically, the charges of the black hole fix the geometry (or more precisely, the complex structure) of the Calabi-Yau at the location of the black hole, independent of the values of the moduli at spatial infinity.

So far we have not considered Kähler moduli. Upon compactifying type IIB on the Calabi-Yau, the moduli become the lowest components of supersymmetry multiplets. The dictionary is

$$\begin{aligned} \text{Vector multiplets} &\leftrightarrow h^{2,1} \text{ complex structure moduli}, \\ \text{Graviphoton multiplet} &\leftrightarrow \text{rescaling of } \Omega, \\ \text{Hypermultiplets} &\leftrightarrow h^{1,1} \text{ Kähler moduli}. \end{aligned}$$

The hypermultiplets decouple in the effective action, as we expect from noting that the black hole does not depend on the volume of the two cycles. The decoupling of the Kähler moduli is suggestive of a link with the B-model topological string, as will be realised below.

String theory predicts corrections to the classical Einstein solution. These are encoded in higher-order terms in the effective action, that include the graviphoton multiplet (the highest component of which is the Riemann tensor). The relevant terms are F-terms of the form

$$\int dx^4 \int d^4\theta F_g(X^I)(\mathcal{W}^2)^g), \tag{2.5}$$

where $\mathcal{W}$ is the Weyl multiplet, and $F_g(X^I)$ turns out to be exactly the genus g B-model topological string amplitude, written in terms of the vector multiplets X^I.

2.2 The OSV conjecture

The suggestive connections between four-dimensional BPS black holes in type IIB superstring theory, and the B-model topological string, were made explicit in [2]. The first step is to introduce the imaginary part of X^I as

$$\lambda^{-1}X^I = p^I + \frac{i}{\pi}\phi^I, \tag{2.6}$$

where ϕ^I will be interpreted as a chemical potential for the electric charges q_I. The black hole partition function can now be written,

$$Z_{\rm BH}(\phi^I, p^I) = \sum_{q_I} \Omega(p^I, q_I) e^{-\phi^I q_I}, \tag{2.7}$$

where $\Omega(p^I, q_I)$ is the number of states (or to be precise, the Witten index) for a BPS black hole of the given charges.[1] $Z_{\rm BH}$ is thus a mixed ensemble partition function — microcanonical for the fixed magnetic charges, and grand canonical for the electric charges. The separation of electric and magnetic charges arises naturally on the topological string side from considerations of background independence, to which we return below.

For the topological string, the full partition function can be written as a genus expansion,

$$\psi_{\rm top}(\lambda, t^i) = \exp \sum_{g\geq 0} \lambda^{2g-2} F_g(t^i), \tag{2.8}$$

where $F_g(t^i)$ is genus g amplitude or "free energy," at the values of the moduli t^i, where $t^i = X^i/X^0$, $i = 1, ..., h^{2,1}(X)$. The statement of the OSV conjecture is

$$Z_{\rm BH}(\phi^I, p^I) = \left|\psi_{\rm top}(\lambda, t^i)\right|^2, \tag{2.9}$$

at the attractor point, which sets

$$t^i = \frac{p^i + i\phi^i/\pi}{p^0 + i\phi^0/\pi}, \qquad \lambda = \frac{4\pi}{p^0 + i\phi^0/\pi}. \tag{2.10}$$

[1]Note that $\Omega(p^I, q_I)$ is not related to the holomorphic top form Ω on the Calabi-Yau!

It is worth emphasising the distinction between the two sides of (2.9): the black hole partition function is related to the counting of microstates of a four-dimensional extremal black hole, while the topological string partition function is a path integral of a sigma model with Calabi-Yau target space. The parameters of the two sides are matched by fixing the moduli of the Calabi-Yau at the values provided by the attractor equations from the black hole charges.

The derivation of the OSV conjecture in [2] is perturbative in higher-order corrections to the Riemann tensor, or equivalently in large charges (i.e., large black holes). Certainly the black hole partition function should have a non-perturbative definition, and so the non-perturbative implications of (2.9) are of interest. A major difficulty is that most existing tools for calculating the topological string partition function use genus expansions of the form (2.8). In the rest of this chapter, we will consider a setup where it is possible to see some hints of what the non-perturbative implications of (2.9) are. The OSV conjecture has been tested in other systems [24, 25], and several general proofs have been presented [26, 27, 28].

2.3 Realising the OSV conjecture

In this section we describe a specific compactification due to [6] in which both the black hole and topological string partition functions can be explicitly calculated in terms of free fermions moving on a circle. The process is illustrated in Figure 2.1.

2.3.1 Topological strings

Consider the A-model topological string, with target space the non-compact toric Calabi-Yau,

$$X = \mathcal{O}(m) \oplus \mathcal{O}(-m) \to T^2. \tag{2.11}$$

Here $\mathcal{O}(m)$ is a degree m complex line bundle over the base T^2, that is, a holomorphic section of this bundle has a divisor of degree m on T^2 which denotes the zeros of the corresponding holomorphic section. $\mathcal{O}(-m)$ is the inverse bundle, such that each meromorphic section of the bundle has m poles.

The topological string partition function ψ_{top} on this space is a function of the string

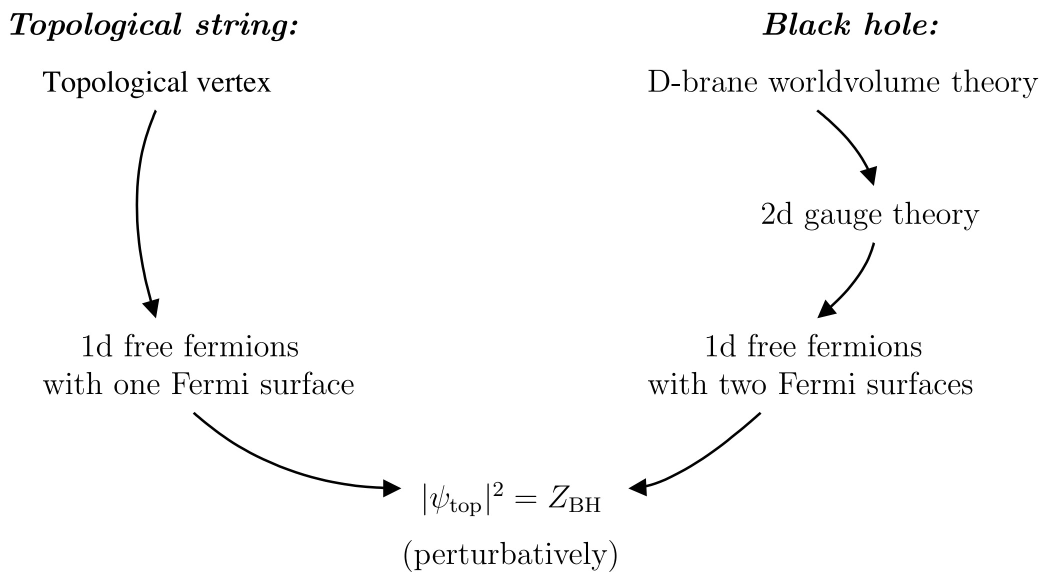


Figure 2.1: *Realising the OSV conjecture explicitly*

coupling λ, as well as the cohomology class $t \in H^{1,1}(T^2)$ of the complexified Kähler form k on T^2. Dependence on the non-compact two-cycle drops out as their Kähler moduli are infinite. The exact expression can be found using the topological vertex [29]. The details are beyond the scope of our discussion, but the essence is that topological vertex provides rules for extracting the topological string partition function from a toric diagram, which in turn is a line diagram with trivalent vertices, encoding the degeneration loci of a fibration. For the geometry (2.8) the toric diagram is a simple line connected to itself on a periodically identified plane, the T^2. Toric diagrams have an additional ambiguity associated with each edge, referred to as a choice of "framing," which in this case is identified with the degree m of the bundles in (2.8). The result is that the perturbative topological string partition function is

$$\begin{aligned} \psi_{\text{top}} &= \psi_0 \sum_{\mathcal{R} \text{ of } U(\infty)} q^{m\kappa_{\mathcal{R}}/2} e^{-t|\mathcal{R}|}, \\ \kappa_{\mathcal{R}} &= 2 \sum_{\square \text{ of } \mathcal{R}} (i(\square) - j(\square)), \end{aligned} \tag{2.12}$$

where $q = e^{-\lambda}$; $\mathcal{R}$ is a Young diagram with arbitrary number of rows, all of positive or zero length, that is, a representation of $(S)U(\infty)$; $i(\square)$ and $j(\square)$ are the row and column, respectively, of the box in the Young diagram; and $|\mathcal{R}|$ is the total number of boxes in the Young diagram. ψ_0 contains the classical contributions (constant maps in A-model) at genus zero and one, as opposed to the higher genus worldsheet instanton contributions captured in (2.12). The classical contributions are somewhat ambiguous for a non-compact target space, but in this case the appropriate choice for the topological string–black hole correspondence is [6]

$$\psi_0 = \exp\left(\frac{F_0(t)}{\lambda^2} + F_1(t)\right) = \exp\left(-\frac{t^3}{6m^2\lambda^2} + \frac{t}{24}\right). \tag{2.13}$$

2.3.2 Black holes and two-dimensional Yang-Mills theory

In Section 2.1 we considered type IIB string theory; here we need type IIA string theory so as to make contact with the A-model results of the previous section. Type IIA has D_p-branes, with p even, which can be wrapped on even cycles of the Calabi-Yau to form a four-dimensional BPS black hole. Considering the Calabi-Yau X (2.11), wrap N D4-branes on $\mathcal{O}(m) \to T^2$. Then consider bound states which have in addition N_2 D2-branes wrapping the T^2, and N_0 D0-branes scattered on the D4-branes. The D6-brane charge is set to zero, as are the charges of the two- and four-cycles not already mentioned. The results of Section 2.1 go through, once we identify the wrapping numbers of zero- and two-cycles as electric and those of four- and six-cycles as magnetic.

As noted above, the black hole partition function should be calculated in the mixed ensemble: fix the magnetic charge N, and sum over the electric charges N_2 and N_0, describing a gas of D2- and D0-branes. To proceed, we consider the gauge theory on the D4-brane worldvolume C_4, as considered in [6]; see also a generalisation in [3]. It is a topologically twisted $\mathcal{N} = 4$ $U(N)$ Yang-Mills theory, as considered in [30]. The D2- and D0-brane gas can be modelled by turning on observables,

$$S_{4d} = \frac{1}{2\lambda}\int_{C_4} \mathrm{tr} F \wedge F + \frac{\theta}{\lambda}\int_{C_4} \mathrm{tr} F \wedge K, \tag{2.14}$$

where K is the unit volume form on T^2. These observables correspond to turning on chemical potentials for the D0- and D2-branes, respectively, of

$$\phi^0 = \frac{4\pi^2}{\lambda}, \quad \phi^1 = \frac{2\pi\theta}{\lambda}. \tag{2.15}$$

This theory can be reduced to a two-dimensional theory on the T^2. Consider the holonomy of the gauge theory around the circle at infinity in the fibre, over a point z in the base T^2,

$$\Phi(z) = \int_{S^1_{z,|w|=\infty}} A, \tag{2.16}$$

where w is the coordinate in the fibre. With some assumptions on the reasonableness of the gauge configuration, it follows that

$$\int_{\text{fiber}} F_{w\bar{w}}(z,w)\, dw\, d\bar{w} = \Phi(z),$$

and hence that the action (2.14) reduces to

$$\int_{T^2} \left(\frac{1}{\lambda} \mathrm{Tr} F\Phi + \frac{\theta}{\lambda} \mathrm{Tr}\Phi \right). \tag{2.17}$$

Recall, however, that the fibre has m zeroes. At these points we have additional massless states, which should manifest as topological point-like observables on the reduced theory. As argued in [6], this adds to the action (2.17) the term

$$\int_{T^2} \frac{m}{2\lambda} \mathrm{Tr}\Phi^2.$$

Integrating out Φ and the fermions from this topologically twisted theory gives two-dimensional bosonic $U(N)$ Yang-Mills on a torus, with action [31]

$$S_{\text{2d}} = -\int_{T^2} \frac{1}{g^2_{\text{YM}}} \left(\frac{1}{2} \mathrm{Tr} F^2 + \theta \mathrm{Tr} F \right), \tag{2.18}$$

where the coupling constant is identified as

$$g^2_{\text{YM}} = m\lambda. \tag{2.19}$$

This theory has been exactly solved [32, 33, 34, 35, 36]. The partition function is

$$Z_{\mathrm{BH}} = \alpha(\lambda,\theta) \sum_{R \text{ of } U(N)} \exp\left(-\frac{1}{2} g_{\mathrm{YM}}^2 C_2(R) + i\theta C_1(R)\right), \tag{2.20}$$

where $C_1(R)$ and $C_2(R)$ are the first and second Casimirs of representation R, which can be related to the quantities in (2.12),

$$\begin{aligned} C_1(R) &= |R|, \\ C_2(R) &= \kappa_R + N|R|, \quad \kappa_R = \sum_{i=1}^{N} R_i(R_i - 2i + 1). \end{aligned} \tag{2.21}$$

The normalisation $\alpha(\lambda,\theta)$ of (2.20) has ambiguities, coming in part from the choice of regularisation. The appropriate choice for our purposes is [3],

$$\alpha(\lambda,\theta) = \exp\left(-\frac{1}{24} m\lambda(N^3 - N) + \frac{N\theta^2}{2m\lambda}\right). \tag{2.22}$$

Making the identification

$$t = \frac{1}{2} m\lambda N - i\theta, \tag{2.23}$$

gives the partition function of the charge N black hole in this setup,

$$Z_{\mathrm{BH}} = q^{m(N^3-N)/24} e^{N\theta^2/2m\lambda} \sum_{R \text{ of } U(N)} q^{m\kappa_R/2} e^{-t|R|}. \tag{2.24}$$

Note that (2.24) is nearly (2.12). There is one key difference besides the prefactor: the representations $\mathcal{R}$ of $U(\infty)$ in (2.12) are unlimited in their number of rows, while representations R of $U(N)$ in (2.24) have at most N rows, but they can have negative length.

2.3.3 One-dimensional free fermions

Two-dimensional bosonic $U(N)$ Yang-Mills on a torus has an illuminating reformulation in terms of N non-relativistic free fermions moving on a circle [37, 38]. Representations R of $U(N)$ are in one-to-one correspondence with fermion configurations, as follows: Denote the momenta of the fermions by $p_i \in \frac{1}{2} + \mathbb{Z}$, $i = 1, \cdots, N$, with each momentum state containing at most one fermion. Let the empty Young diagram correspond to the ground state configuration, with fermion states from $p = -\frac{N}{2} + \frac{1}{2}$ to $\frac{N}{2} - \frac{1}{2}$ filled, so that for a general young diagram R with row lengths R_i (which may be negative),

$$p_i = \frac{1}{2} - i + R_i. \tag{2.25}$$

For simplicity we henceforth assume that N is even — the generalisation is straightforward. The Casimirs in (2.21) have a simple interpretation as the total momentum and energy of a configuration,

$$\begin{aligned} C_1(R) &= P = \sum_{i=1}^{N} p_i, \\ C_2(R) &= E - E_0 = \sum_{i=1}^{N} \frac{1}{2} p_i^2 - E_0, \\ E_0 &= \frac{1}{24}(N^3 - N). \end{aligned} \tag{2.26}$$

where E_0 is the energy of the ground state configuration.

The topological string partition function $\psi(t)$, equation (2.12), can likewise be interpreted as a system of non-relativistic fermions on a circle — but infinitely many, as we started with $U(\infty)$ Yang-Mills. As will become clear below, the appropriate "ground" state corresponding to trivial representation $\mathcal{R}$ is all fermion states with momenta $p \leq \frac{N}{2} - \frac{1}{2}$ filled. This state is stable only perturbatively, as the addition of momentum greater than N allows fermions from the infinite sea of negative momentum (but positive energy) to lower their energy by filling states of positive momentum. The classical contribution (2.13) contains the zero-point energy, regulated by treating the negative momentum states as zero energy when filled. For the conjugate partition function $\overline{\psi}(\bar{t})$, the sign of the θ-dependent term is reversed, and so the fermion momenta change sign.

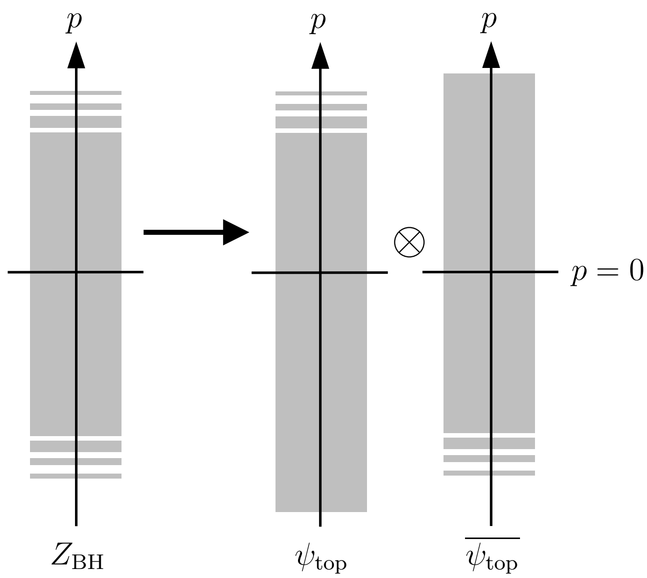


Figure 2.2: *The (perturbative) OSV relation in terms of free fermions, understood as a decoupling of Fermi surfaces*

2.3.4 Perturbative OSV

The OSV relation follows from the free fermion description. As shown in Figure 2.2, small deviations from the ground state of the black hole can be thought of as fermion excitations at the two Fermi surfaces, one each for positive and negative momentum states. The topological string partition functions ψ and $\overline{\psi}$, on the other hand, have a single Fermi surface each, at positive and negative momenta, respectively. Thus the OSV relation is simply the decoupling of the two Fermi surfaces. In this section we make this statement rigorous, by specifying how to split a representation R describing a black hole state into two representations $\mathcal{R}^+$ and $\mathcal{R}^-$ describing holomorphic and anti-holomorphic topological string states, respectively.

The first complication is that Young diagrams R for $U(N)$ may have negative length rows, so there exists a "shift" operation for the black hole, where all the rows of the Young diagram are lengthened or shortened by a unit, or equivalently the centre of mass of the fermion distribution is shifted in momentum space. To fix this degree of freedom, define integer $-\frac{N}{2} \leq l \leq \frac{N}{2}$ as in Figure 2.3 to be the edge of the largest square that can be inserted between the boundary of the Young diagram R and the point $(\frac{N}{2}, 0)$ (that is, row $\frac{N}{2}$, column zero). Explicitly,

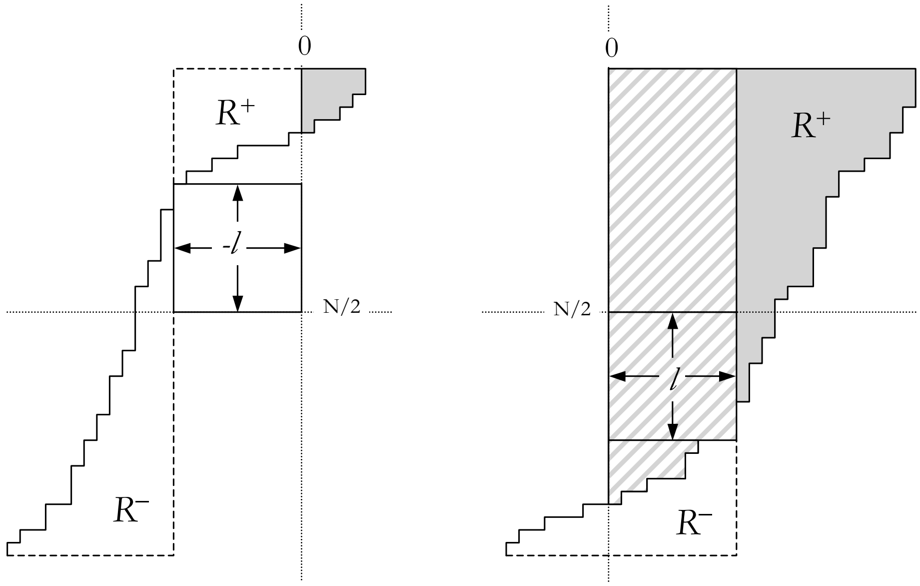


Figure 2.3: *Decomposition of a $U(N)$ representation R corresponding to a black hole microstate into an $SU(N/2+l)$ representation R^+ and an $SU(N/2-l)$ representation R^-. The longest vertical line (dotted) is the zero of row length. The left diagram is a case with negative l, and the right has positive l. The diagonal striped region is in R but not in either R^+ or R^-.*

- If $R_{N/2} \geq 0$, let $l \geq 0$ be the largest number such that $R_{N/2+l} \geq l$;
- If $R_{N/2} < 0$, let $l < 0$ be the smallest number such that $R_{N/2-l} \leq l$.

Now define representation R^+ of $SU(\frac{N}{2}+l)$ as the rest of the first $\frac{N}{2}+l$ rows of R, starting at column l; and R^- of $SU(\frac{N}{2}-l)$ as the conjugate of the remaining $\frac{N}{2}-l$ rows, starting at column l. That is,

$$R_i^+ = R_i - l, \qquad i = 1, \cdots, \frac{N}{2}+l,$$
$$R_i^- = l - R_{N+1-i}, \qquad i = 1, \cdots, \frac{N}{2}-l.$$

The reverse map takes $\frac{N}{2} \leq l \leq \frac{N}{2}$, R^+ of $SU(\frac{N}{2}+l)$ and R^- of $SU(N-l)$, to R of $U(N)$, as follows: Fill the first l columns of the Young diagram of R. Then add the diagram of R^+ to complete the first l rows, and subtract R^- from the remaining rows, as shown in Figure 2.3, to complete R. This completes the definition of a one-to-one map between R and (l, R^+, R^-).

Using this decomposition, the quantities in the black hole partition function (2.24) can be rewritten as follows:

$$\begin{aligned}
&\sum_R \to \sum_{l=N/2}^{N/2} \sum_{R^+,R^-}, \\
&|R| = Nl + |R^+| -| R^-|, \\
&\kappa_R = \kappa_{R^+} + \kappa_{R^-} + 2|R^+|l + 2|R^-|(N-l) + Nl^2 - N^2 l,
\end{aligned} \tag{2.27}$$

where the last result follows most easily from an alternate expression for κ_R, $\kappa_R = \sum_i R_i(R_i - 2i+1)$. Thus (2.24) can be written

$$Z_{\rm BH} = q^{m(N^3-N)/24} e^{N\theta^2/2m\lambda} \sum_{l,R^+,R^-} q^{\frac{m}{2}\left[\kappa_{R^+}+\kappa_{R^-}+(N+2l)|R^+|+(N-2l)|R^-|+Nl^2\right]} e^{i\theta\left(Nl+|R^+|-|R^-|\right)}. \tag{2.28}$$

To treat the factors free of representation dependence, use the identities

$$\begin{aligned}
q^{m(N^3-N)/24} e^{N\theta^2/2m\lambda} &= \exp\left(-\frac{(t^3+\bar{t}^3)}{6m^2\lambda^2} + \frac{(t+\bar{t})}{24}\right), \\
q^{\frac{m}{2}Nl^2} e^{i\theta Nl} &= \exp\left(-\frac{(t^2-\bar{t}^2)l}{2m\lambda} - \frac{(t+\bar{t})l^2}{2}\right).
\end{aligned} \tag{2.29}$$

The second line above may be absorbed into the first, at the cost of substituting $t \to t + m\lambda l$ and $\bar{t} \to \bar{t} - m\lambda l$ in the first line. Indeed, this choice of variables can be used throughout (2.28), to yield

$$Z_{\rm BH} = \sum_{l=-N/2}^{N/2} \Psi_{N/2+l}(t+m\lambda l)\overline{\Psi}_{N/2-l}(\bar{t}-m\lambda l), \tag{2.30}$$

$$\Psi_k(t) = e^{-t^3/6m^2\lambda^2 + t/24} \sum_{R \text{ of } SU(k)} q^{m\kappa_R/2} e^{-t|R|}, \tag{2.31}$$

where we have defined the chiral function $\Psi_k(t)$. Note that the subscript k, denoting the maximum number of rows of the representations in the sum, is included for clarity only, since it is exactly $k = \text{Re}(t)/m\lambda$.

This definition is identical to the (perturbative) topological string partition function ψ, equation (2.12), except that the sum over representations R includes only those with at most $k = \frac{N}{2} + l$ rows. This difference is non-perturbative in N: terms with l of order $\frac{N}{2}$ or $-\frac{N}{2}$ are exponentially suppressed by the t^3 term in the prefactor exponential, so any perturbatively significant k is of order N. Then, however, a box in row $(k+1)$ in a Young diagram requires all boxes in the first column up to row k to be present, so $|\mathcal{R}| > k$ is order N, which gives exponential suppression from the $e^{-t|\mathcal{R}|}$ factor. Thus up to non-perturbative corrections,

$$Z_{\mathrm{BH}} = \sum_{l=-N/2}^{N/2} \psi(t + m\lambda l)\overline{\psi}(\bar{t} - m\lambda l), \tag{2.32}$$

which is a statement of the OSV conjecture, up to sum over l.

The sum over l merits further discussion. The literature differs in the limits of the l summation: [3, 6] sum over $l \in \mathbb{Z}$, while the restricted range of (2.32) appears in [4]. The difference is due to there being different ways to decompose the black hole representation R of (2.24). The technique used above yields representations R^+ of $SU(N/2 + l)$ and R^- of $SU(N/2-l)$, with finite summation range. One can also define l as the length of row $R_{N/2}$, R^+ as the rest of the first $\frac{N}{2}$ rows, starting at column l, and R^- as the conjugate of the second $\frac{N}{2}$ rows starting at column l. This gives representations R^+ and R^- of $SU(N/2)$, and infinite summation range for l. Both techniques yield decompositions that give the perturbative topological string partition function (2.12), and so the choice is arbitrary for perturbative results. For our purposes, however, the classical prefactors in the chiral recursion relation derived below, equation (2.46), match only for our chosen decomposition. The summation over l in (2.32) was interpreted in [6] in terms of RR-fluxes through the base T^2, which would of course appear in a physical black hole constructed by compactification on the manifold (2.11).

The perturbative OSV relation (2.32) follows from decoupling of the two Fermi surfaces of the black hole. Allowing non-perturbative corrections, or equivalently "deep" excitations (a finite fraction of N fermions below the Fermi surfaces), spoils the decoupling. For example, as shown in Figure 2.4, an excitation deep within the black hole Fermi tower can be interpreted as an excitation of either Fermi surface, and therefore be associated with either of the topological string Fermi towers. The right-hand side of (2.32) thus overcounts this state.

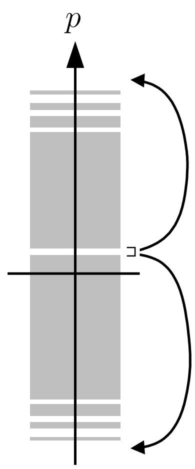


Figure 2.4: *Overcounting in the perturbative OSV relation (2.32): excitations deep in the black hole Fermi sea can be associated with either Fermi surface.*

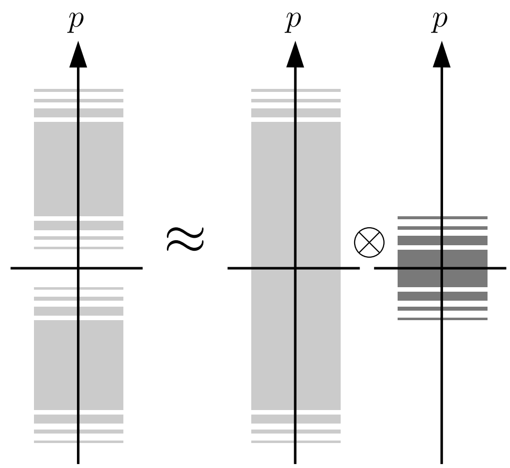


Figure 2.5: *Removing overcounted states: a black hole with deep excitations is the product of two black holes without deep excitations, the latter (in dark grey) composed of holes rather than fermions.*

Furthermore, excitations more than N fermions down in the topological string Fermi sea have no corresponding states on the black hole side. The rest of the this chapter will develop techniques to handle these non-perturbative corrections.

2.4 Non-chiral baby universes

In this section we review an approach to handling the overcounting, due to [4], that replaces the left-hand side of (2.32) with a sum over all collections of black holes with the same overall charge as the single black hole considered previously. A different approach to subtracting the overcounted states is presented in Section 2.5, where we will interpret (2.32) as providing the correct way to define the non-perturbative completion of the topological string partition function.

2.4.1 A non-chiral recursion relation

The approach of [4] is grounded in the intuition shown in Figure 2.5. Overcounted states are exactly those with holes deep within the black hole Fermi sea. These excitations can be viewed as a sea of holes within the sea of fermions, and so treated as a black hole Fermi sea of holes superimposed on a black hole Fermi sea without deep excitations. This motivates the relation

$$Z_N = \sum_{l=-N/2}^{N/2} \psi_{N/2+l}\overline{\psi}_{N/2-l} - \sum_{n=1}^{\infty} Z_{N+n} Z_{-n}, \tag{2.33}$$

where the subscripts on the black hole partition functions Z indicate the number of fermions in the Fermi sea, and Z_{-n} is a partition function of n holes. The first term on the right is the perturbative result, and the second term subtracts overcounted states, that is, those with one or more deep excitations. A hole is the *absence* of a fermion with the given energy and momentum, so from the above definitions it follows that

$$Z_{-n}(\theta, \lambda) = Z_n(-\theta, -\lambda). \tag{2.34}$$

Z_{-n} is clearly unstable, so by a process of analytic continuation, (2.33) can be written as [4]

$$Z_N = \sum_{l=-N/2}^{N/2} \psi_{N/2+l}\overline{\psi}_{N/2-l} - \sum_{n=1}^{\infty} Z_{N-n} Z_n, \tag{2.35}$$

where this expression can be trusted when all subscripts are large, that is $N \gg 1$, $|l| \ll N$ and $n \gg 1$.

Equation (2.35) is a recursion relation, which can be expanded to give

$$Z_N = \sum_{n=1}^{\infty} (-1)^{n-1} C_{n-1} \sum_{N_1^+ + \ldots + N_n^+ + N_1^- + \ldots + N_n^- = N} \psi_{N_1^+} \ldots \psi_{N_n^+} \overline{\psi}_{N_1^-} \ldots \overline{\psi}_{N_n^-} \tag{2.36}$$

where $C_n = \frac{(2n)!}{n!(n+1)!}$ is a *Catalan number*, capturing the combinatorics of the expansion [4].

2.4.2 Gravitational interpretation

We return now to the counting of BPS states of the black hole. ψ_N can be Laplace transformed with respect to the chemical potentials for the electric charges,

$$\psi_{N,N_0,N_2}(\theta,\lambda) = \exp\left(\frac{2\pi^2 N_0}{\lambda} + \frac{N_2 \pi\theta}{\lambda}\right)\psi(t^i,\lambda), \tag{2.37}$$

with t^i fixed by the attractor equations (2.4), and N_2 and N_0 the electric charges for this system. Then from (2.7) and (2.9) it follows [4] that the number of microstates of the extremal black hole is

$$\Omega(N,N_2,N_0) = \int d\left(\frac{1}{\lambda}\right) d\left(\frac{\theta}{\lambda}\right) |\psi_{N,N_2,N_0}|^2 . \tag{2.38}$$

Each pair $\psi_{N_i^+}\overline{\psi}_{N_i^-}$ thus corresponds to a black hole of magnetic charge $N_i^+ + N_i^-$, and so (2.36) is a statement that the full non-perturbative black hole partition function is a sum over multiple black hole configurations with charges summing to the total black hole charge N. This section will develop this intuition from the gravity side.

Starting with a single-centred solution, a static spherically symmetric black hole has metric [4]

$$ds^2 = -\frac{\pi}{S(r)}dt^2 + \frac{S(r)}{\pi}\sum_{a=1,2,3}(dx^a)^2 + ds^2_{\mathrm{CY}}, \tag{2.39}$$

where $r = |x|$, and ds^2_{CY} is the metric of the Calabi-Yau, which depends on r through the attractor mechanism. The asymptotic (near horizon) behaviour of $S(x)$ is

$$S(x) \sim \frac{S^{(0)}_{BH}(P,Q)}{|x|^2}, \qquad x \to 0, \tag{2.40}$$

where $S^{(0)}_{BH}(P,Q)$ is the semi-classical entropy of the solution, with charge vectors P and Q. From (2.3) and the attractor equations (2.4), $S^{(0)}_{BH}$ is fixed in terms of the charges.

The key point is that the geometry (2.39) is not the only solution with the given charges which preserves the requisite supersymmetry. Multi-centre solutions partition the charges between multiple black holes locally, but have the same behaviour at spacial infinity as the single-centre solution. More precisely, the charges are partitioned,

$$P^I = \sum_{i=1}^{n} p_i^I, \qquad\qquad Q_I = \sum_{i=1}^{n} q_{iI}, \tag{2.41}$$

where I labels the charges (or equivalently the homologically distinct cycles in the Calabi-Yau), and i labels the n distinct centres. Let us specialise to the case of a single electric charge. Define a scalar function,

$$S(x) = \pi \left(c + \sum_{i=1}^{n} \frac{q_i}{|x - x_i|} \right)^2, \tag{2.42}$$

where all the charges q_i are assumed positive. Inserting this into the metric (2.39) gives a solution which near a given "centre," at x_i, behaves as

$$S(x) \sim \frac{\pi q_i^2}{|x - x_i|^2}, \qquad\qquad x \to x_i,$$

but towards spacial infinity goes as

$$S(x) \sim \pi \left(c + \frac{Q}{|x|} \right)^2, \qquad\qquad |x| \to \infty.$$

Thus this solution looks like a single black hole of charge q_i near x_i, but like a black hole of total charge Q at spacial infinity. Indeed, by Wick rotating (2.39) and interpreting $S(x)$ as the Euclidean time, one can interpret this [39] in terms of tunnelling from a single-centred solution at $S \to 0$ to a multi-centred solution at $S \to \infty$. The Euclidean action is proportional to the difference in entropy of the configurations, and thus to the square of the charges, leading to exponential suppression at large charge.

The solution can be generalised to the case with multiple charges [40, 41], with two complications: firstly, the locations x_i of the centres are no longer arbitrary, as it costs energy for the scalar fields corresponding to the Calabi-Yau moduli to interpolate between their values at each horizon. Secondly, the magnetic charges need to have the same sign, which is conveniently satisfied by the gauge theory result (2.36). Taking these considerations into account, stable supersymmetric solutions still exist, with qualitatively similar behaviour to the example above. The implication of the non-perturbative result (2.36) is thus: the square of the topological string partition function calculates the partition function of not just a single-centred black hole, but rather the partition function of all gravitational solutions that preserve the appropriate supersymmetry and have the given total charges at spacial infinity.

Taking the low energy near horizon limit, in the style of the AdS/CFT correspondence, splits the geometry (2.39) into multiple near horizon regions, or *baby universes*. An ensemble of all possible numbers of universes is thus dual to a *single* gauge theory. The near horizon limit is particularly relevant in the light of the interpretation [5] of ψ_{top} (or rather its transform $\psi_{P,Q}$) as the Hartle-Hawking wavefunction of the universe in the mini-superspace sector of string theory, which describes BPS (supersymmetry-protected) quantities. These wavefunctions are the ground states of the theory, after long time evolution, which implies the near-horizon limit. We return to the wavefunction interpretation in Section 2.6.

Summing over universe number seems to imply a loss of quantum coherence for the Hartle-Hawking state for a given universe, due to the coupling of the wavefunctions of different universes through the overall charge conservation. This conclusion is perhaps unnecessarily strong: measuring the charges (or equivalently coupling constants) of one universe determines the wavefunction of that universe as a pure state, in agreement with arguments [42] about quantum coherence in situations with baby universe creation.

2.5 Chiral completion of the topological string

We turn now to a novel approach to treating the overcounting identified in Section 2.3.4. This approach has the merit that the results are exact for all values of the parameters — no large charge assumptions are required. The result will provide a non-perturbative statement of the OSV relation for the system described in Section 2.3, but with the topological string side still in the form of a product of a chiral and anti-chiral function. This motivates the interpretation that these functions, denoted Ψ and $\overline{\Psi}$, are the non-perturbative completion of the topological string partition function. Furthermore, these functions will be related to the perturbative topological string partition function ψ by a recursion relation similar to the

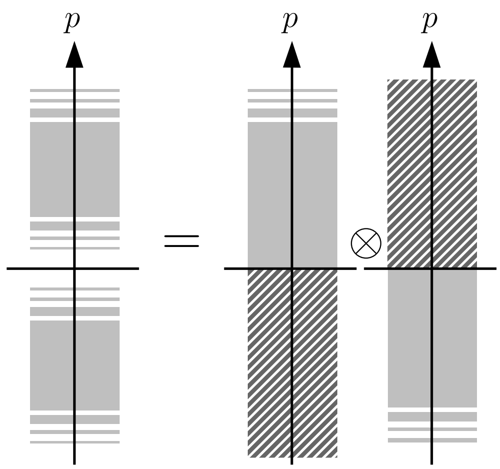


Figure 2.6: *The black hole partition function (on left) equals two modified topological string partitions functions* Ψ *and* $\overline{\Psi}$ *(on right). The hashed regions may not contain holes.*

chiral (or anti-chiral) part of (2.36).

Recall that the non-perturbative corrections to (2.32) are excitations deep within the black hole Fermi sea, as such excitations can be assigned to either of the topological string partition functions. Rather than systematically subtract the overcounted states as in the previous section, however, the topological string partition function can be modified to allow excitations only to a given depth, so that each black hole excitation is assigned unambiguously to either the chiral or anti-chiral partition function, as illustrated in Figure 2.6. The derivation in Section 2.3.4 used exactly this approach: equation (2.30) is an exact result, and so $\Psi(t)$ defined by (2.31) is the candidate non-perturbative completion of the topological string partition function $\psi(t)$.

To support this interpretation, we need a relation between $\psi(t)$ and $\Psi(t)$. The two differ by the presence in ψ of "deep" excitations, which are forbidden in Ψ. These, however, can be described using an inverted tower of holes, as shown in Figure 2.7. To make this concrete, consider an arbitrary $U(\infty)$ representation $\mathcal{R}$, as shown in Figure 2.8. There is a one-to-one correspondence between such representations and pairs of representations (R^1, R^2) of $SU(k+r)$ and $SU(r)$, respectively, as follows: Define r to be the largest number such that a rectangle of width r and height $k+r$ (the hashed region in Figure 2.8) fits within the Young diagram. Remove the rectangle. The remainder of the first $(k+r)$ rows is R^1; while the

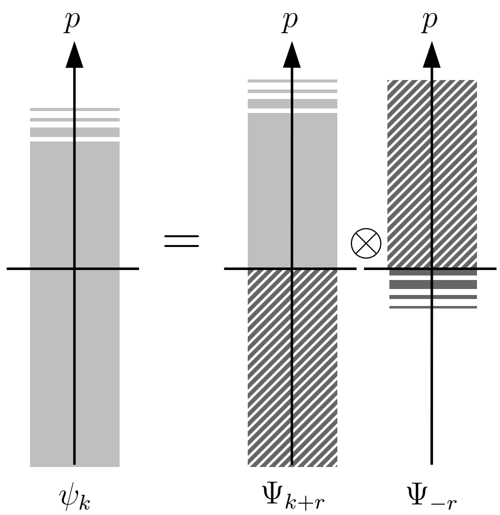


Figure 2.7: *Decomposing the perturbative topological string partition function ψ into non-perturbative completions thereof, Ψ, in terms of free fermions. The hashed region is inaccessible to excitations, and the darker (non-hashed) region represents holes (negative energy fermions).*

rest of the diagram is the *transpose* of R^2. Conversely, such a pair of representations, along with the number r, uniquely defines representation $\mathcal{R}$. With this definition, let $|R^1| = n$, $|R^2| = m$, and so $|\mathcal{R}| = n + m + (k+r)r$. Furthermore,

$$\begin{aligned} \kappa_{\mathcal{R}} &= (\kappa_{R^1} + 2rn) - (\kappa_{R^2} + 2(k+r)m) + \kappa_{(k+r)\times r} \\ &= \kappa_{R^1} - \kappa_{R^2} + 2(rn - km - rm) - k^2 r - kr^2, \end{aligned} \tag{2.43}$$

where the minus sign before κ_{R^2} follows from using the transpose of the representation.

Using the above decomposition, and neglecting the prefactor ψ_0, we can write

$$\begin{aligned} \psi_k(t) &= \sum_r \sum_{R^1,R^2} q^{\frac{1}{2}m\left(\kappa_{R^1} - \kappa_{R^2} + 2(rn-km-rm) - k^2 r - kr^2\right)} e^{-\left(\frac{1}{2}m\lambda k - i\theta\right)(n+m+(k+r)r)} \\ &= \sum_r e^{-\left(\frac{1}{2}mk\lambda - i\theta\right)r(k+r)} \Psi_{k+r}(t + mr\lambda)\Psi_{-r}(t - m(k+r)\lambda), \end{aligned}$$

where the non-perturbative partition function for holes is defined as

$$\Psi_{k<0}(t) = e^{-t^3/6m^2\lambda^2 + t/24} \sum_{R \text{ of } SU(|k|)} q^{-m\kappa_R/2} e^{-t|R|}, \tag{2.44}$$

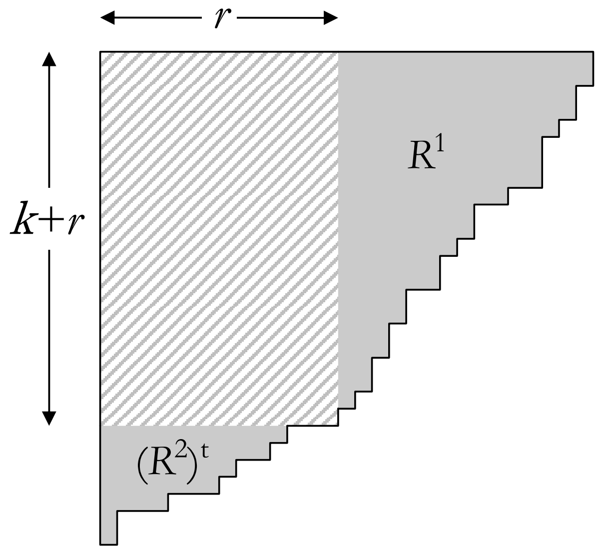


Figure 2.8: *Decomposing a $U(\infty)$ representation into an $SU(k+r)$ representation R^1 and a (transposed) $SU(r)$ representation R^2.*

where the sign preceding κ_R follows from (2.12) by treating the representation R as transposed, which interchanges fermions and holes. Note that the momentum of the fermions is also sign-flipped, and so Ψ_{-r} above is indeed chiral, as opposed to anti-chiral. The prefactors from classical maps satisfy

$$\psi_0\left(t=\frac{1}{2}mk\lambda - i\theta\right) = \psi_0\left(t+mr\lambda\right)\,\psi_0\left(t-m(k+r)\lambda\right)\, e^{\left(\frac{1}{2}mk\lambda-i\theta\right)r(k+r)}e^{i\theta^3/6m^2\lambda^2}e^{i\theta/24}.$$

Including the prefactor gives

$$\psi_k(t) = \Theta\sum_{r=0}^{\infty}\Psi_{k+r}(t+mr\lambda)\Psi_{-r}(t-m(k+r)\lambda), \qquad (2.45)$$
$$\Theta = e^{i\theta^3/6m^2\lambda^2}e^{i\theta/24},$$

where Θ is pure imaginary and so cancels between topological and anti-topological partition functions. It owes its existence to the difference between ψ_0 and the naive zero-point energy of the topological string partition function fermion tower.

Equation (2.45) gives a recursion relation for $\Psi(t)$ in terms of $\psi(t)$. The factor $\Psi_{-r}(t-m(k+r)\lambda)$ above, for $r=0$, is just $\Psi_0(t-mk\lambda)=\Theta$, as the representation sum ranges over representations with zero rows, i.e., only the trivial representation. We can thus write

$$\Psi_k(t) = \psi_k(t) - \Theta \sum_{r=1}^{\infty} \Psi_{k+r}(t + mr\lambda)\Psi_{-r}(t - m(k+r)\lambda). \tag{2.46}$$

As an example, expanding each factor of Ψ once gives

$$\begin{aligned}\Psi_k(t) = \psi_k(t) - \Theta \sum_{r=1}^{\infty} \Bigg[&\psi_{k+r}\psi_{-r} - \Theta\, \psi_{-r} \left(\sum_{u=1}^{\infty} \Psi_{k+r+u}\Psi_{-u} \right) \\ &- \Theta\, \psi_{k+r} \left(\sum_{v=1}^{\infty} \Psi_{-r-v}\Psi_v \right) + \Theta^2 \left(\sum_{u=1}^{\infty} \Psi_{k+r+u}\Psi_{-u} \right) \left(\sum_{v=1}^{\infty} \Psi_{-r-v}\Psi_v \right) \Bigg].\end{aligned}$$

The corresponding anti-chiral recursion relation follows immediately by replacing $\psi \to \overline{\psi}$, $\Psi \to \overline{\Psi}$, $\Theta \to \overline{\Theta}$.

The recursion relation (2.46) is *chiral* (contains only factors of ψ, not $\overline{\psi}$), in contrast to the factorisation in Section 2.4. Substituting (2.46) into (2.30) gives

$$Z_N = \sum_{n,l=1}^{\infty} (-1)^{n+l} C_{n-1} C_{l-1} \sum_{N_1^+ + ... + N_n^+ + N_1^- + ... + N_l^- = N} \psi_{N_1^+} ... \psi_{N_n^+} \overline{\psi}_{N_1^-} ... \overline{\psi}_{N_l^-}, \tag{2.47}$$

where the charges in the sum have arbitrary sign, except that at least one of the N_i^+ is positive, and if $n > 1$ then at least one is negative; and likewise for the N_i^- charges. Note that the appearance of negative charges is natural [21]: they can be interpreted as D4-branes in child universes wrapping the opposite choice of four-cycle in the manifold X. This four-cycle has negative intersection number with the base T^2, giving negative effective D4-brane charge.

2.6 Background independence

The proposal (2.47) raises some puzzles related to the question of background independence, to which we now turn. The A- and B-model topological strings (and indeed type IIA and IIB physical string theory) are related through *mirror symmetry*. As discussed in detail in [16], the A-model topological string on a Calabi-Yau manifold X is dual to the B-model topological string on a different Calabi-Yau manifold $\widetilde{X}$. Since A-model depends on Kähler moduli and B-model on complex structure moduli, this implies that $h^{1,1}(X) = h^{2,1}(\widetilde{X})$ and $h^{2,1}(X) = h^{1,1}(\widetilde{X})$. Recall from Section 1.1, however, that Kähler structure deformations are exactly elements of $H^{1,1}(X)$, while the correspondence between complex structure deformations and $H^{2,1}(X)$ arises from considering infinitesimal deformations of the holomorphic top form Ω. Thus the complex structure moduli space (but not, apparently, the Kähler moduli space) is defined relative to a choice of *background*, or reference complex structure.

This apparent contradiction was resolved in [7]. The Lagrangian of the untwisted $\mathcal{N} = 2$ supersymmetric theory can be written

$$L = L_0 - t^i \int_\Sigma \phi_i^{(2)} - \bar{t}^{\bar{i}} \int_\Sigma \overline{\phi}_{\bar{i}}^{(2)}, \tag{2.48}$$

where all terms other than those involving the chiral primaries are in L_o. After twisting, as discussed in Section 1.3, the last term becomes BRST-trivial. Deformations of the theory with marginal operators modify only the holomorphic moduli, $t^i \to t^i + u^i$. Naively, then, amplitudes depend on $t^i + u^i$, and the original choice of t^i is irrelevant. The holomorphic anomaly equations (1.38), however, show that the last term in (2.48) is not completely irrelevant, and the theory still depends on $\bar{t}^{\bar{i}}$ — from which one can recover t^i. Thus both A- and B-model depend on the initial choice of a background point. Standard calculations in A-model do not show this dependence, but in fact these calculations are implicitly performed with background $\bar{t}^{\bar{i}} \to \infty$ [7, 8].

A quantum theory of gravity should, however, be independent of the background. Luckily, a more sophisticated version of background independence is indeed present [7]. Consider quantum mechanics formulated on the phase space of conjugate position and momentum variables. The wavefunction is expressed in terms of half of these variables, usually either positions or momenta, giving the phase space a natural symplectic structure. Changing the symplectic structure does not affect the *physical* wavefunction, but its functional form undergoes a Bogoliubov transformation. The topological string also requires the choice of a symplectic structure, the complex structure on the Calabi-Yau, through the choice of decomposition $H^3(X) = H^{3,0}(X) \otimes H^{2,1}(X) \otimes H^{1,2}(X) \otimes H^{0,3}(X)$. The holomorphic anomaly equations (1.38) for the amplitudes can be rewritten as equations for the partition function (1.27),

$$\left(\frac{\partial}{\partial \bar{t}^{\bar{i}}} - \frac{\lambda^2}{2}\overline{C}_{\bar{i}}^{jk} D_j D_k\right) Z = 0, \tag{2.49}$$

with genus one contributions suppressed for simplicity. Equation (2.49), however, is exactly the Bogoliubov transformation for a wavefunction Z with phase space $H^3(X)$ [7]! Thus the *physical* wavefunction (which we have denoted ψ) is independent of the background.

Returning to the OSV conjecture, the modifications of ψ and $\overline{\psi}$ under change of background cancel, such that Z_{BH} is left invariant. The result (2.47) now presents a puzzle: since powers of ψ and $\overline{\psi}$ are not required to match, is this result still background independent? The calculations in this chapter were performed for a particular choice of background, so one possibility is that this choice renders trivial some additional contribution to (2.47), which order-by-order transforms in such a way as to restore background independence.

A related issue is the interpretation of (2.47) in terms of physical black holes, along the lines of the discussion in Section 2.4.2. Recall that the near-horizon limit of a four-dimensional black hole is the space $AdS_2 \times S^2$. AdS_2 is unique amongst anti-de-Sitter spaces for having two independent boundaries, and so it is tempting to interpret the topological string partition function as the dual gauge theory living on one of the boundaries. The result (2.47) then suggests universe-creating instantons which split just one of the AdS_2 boundaries, leaving baby universes which "share" the other boundary.

To consider these questions further, it would be useful to have results for somewhat more general Calabi-Yau target spaces, to which we turn in the next section. However, the local (non-compact) nature of the Calabi-Yau manifolds under discussion, while necessary for the application of topological vertex techniques, makes the derivation of further results difficult and ultimately inconclusive.

In any event, the chiral factorisation has interesting implications for the question of quantum coherence of the wavefunction of the universe, as discussed at the end of Section 2.4.2. The structure $Z_{\mathrm{BH}} = \Psi\overline{\Psi}$ includes all contributions regardless of the number of the universes, so tunnelling to a multi-centre (multi-universe) configuration does not destroy quantum coherence. The universe (and each baby universe) remains in a pure state, without the need to measure the local charges.

2.7 Extension to other genus target spaces

A natural extension of the work above is to consider more general Calabi-Yau target spaces. In particular, (2.11) can be generalised to

$$X_G = \mathcal{O}(m + 2g - 2) \oplus \mathcal{O}(-m) \to \Sigma_G, \tag{2.50}$$

where Σ_G is a genus G surface (with G distinct from the genus g of the worldsheet), and the degrees of the line bundles are chosen to give overall vanishing first Chern class. The OSV conjecture in this setup was considered in [3], and a non-chiral baby universes interpretation in the style of Section 2.4 was proposed in [21]. We seek a chiral, non-perturbative realisation of the OSV conjecture.

Unfortunately, while there is such a realisation, with chiral partition functions Ψ that are perturbatively the topological string partition function ψ_{top}, the chiral recursion relation between ψ_{top} and Ψ suffers from a mismatch in classical prefactors. Below we will demonstrate the former statement, and attempt to motivate the latter. The chief complication arising is that, in addition to the sum over l that appears in (2.32), the chiral and anti-chiral topological string partition functions are coupled by a sum over representations, referred to as (P-type) "ghost brane" representations. Physically, these can be interpreted as boundary conditions at infinity of the non-compact Calabi-Yau. In the chiral recursion relation we will need to account for two sets of representations which couple the decomposed partition functions: one set (S-type) come from decomposing the chiral topological string partition function itself; the other set (P-type) come from decomposing the P-type ghost brane representations we found while decomposing the black hole. These will be interpreted physically below.

2.7.1 Realising the OSV conjecture

The topological string partition function on the space (2.50) can be derived by starting with simple annulus, pants and caps diagrams, ending on stacks of D-branes. More complicated worldsheets can then be constructed by gluing the boundaries together. The result is [3]

$$\begin{aligned} Z_{\text{top}}(q,t) &= Z_0(q,t) \sum_{\mathcal{R} \text{ of } U(\infty)} \left(\frac{1}{d_q(\mathcal{R})}\right)^{2G-2} q^{(m+G-1)\kappa_{\mathcal{R}}/2} e^{-t|\mathcal{R}|}, \\ Z_0(q,t) &= M(q)^{1-G} \exp\left(-\frac{t^3}{6m(m+2G-2)\lambda^2} + \frac{(m+2G-2)t}{24m}\right), \end{aligned} \tag{2.51}$$

where $M(q) = \prod_{n=1}^{\infty}(1-q^n)^{-n}$ is the McMahon function, and Z_0 captures the contributions from constant maps. Compared to the result for $G = 1$, equation (2.12), the major new ingredient above is the *quantum dimension* $d_q(\mathcal{R})$ of the symmetric group representation corresponding to the Young diagram $\mathcal{R}$ (with arbitrary column lengths),

$$d_q(\mathcal{R}) = \prod_{\square \in \mathcal{R}} \frac{1}{[h(\square)]_q}, \tag{2.52}$$

where $h(\square)$ is the *hook length* of the corresponding box in the Young diagram, that is the number of boxes directly below the box, but in the same column, plus those directly to the right of the box, but in the same row, plus one. The q-analogue $[x]_q$ is defined as

$$[x]_q = q^{x/2} - q^{-x/2}. \tag{2.53}$$

This partition function can still be interpreted as that of a sea of fermions on a circle, however the fermions are now interacting due to the presence of the quantum dimension.

On the black hole side, the additional subtlety is that the holonomy Φ of the gauge theory around points on the base Σ_G, as defined by equation (2.16), is periodic, so it is only $e^{i\Phi}$ that is a good variable.

Taking this into account, the black hole partition function is [3]

$$Z_{\text{BH}} = \alpha(\lambda,\theta) \sum_{R \text{ of } U(N)} S_{0R}^{2-2G} q^{mC_2(R)/2} e^{i\theta C_1(R)}, \tag{2.54}$$

with

$$\alpha(\lambda,\theta) = q^{\frac{(m+2G-2)^2}{2m}\left(\frac{N^3}{12}-\frac{N}{12}\right)} q^{(2G-2)\left(\frac{N^3}{12}-\frac{N}{24}\right)} e^{\frac{N\theta^2}{2m\lambda}},$$

which reduces to the results of Section 2.3.2 for $G = 1$. S_{0R} is a quantity best known in Chern-Simons theory, related to entries of the S-matrix of the $U(N)_k$ WZW model (for non-integer level k). It is related to the quantum dimension for finite N,

$$\frac{S_{0R}}{S_{00}} = \dim_q(R) = \prod_{1\le i<j\le N} \frac{[R_i - R_j + j - i]_q}{[j-i]_q}, \tag{2.55}$$

where R_i is the length of the i^{th} row of the Young diagram, and S_{00} is the denominator on the right. It is worth emphasising the difference between $d_q(R)$ and $\dim_q(R)$: the Young diagram R is treated as having infinitely many rows (most of them empty) for the former, but only N rows for the latter. Taking N to infinity makes them equal.

To find the appropriate value for the Kähler modulus t, we need to consider the wrapping more carefully. Consider the following divisors:

$$\mathcal{D} = \mathcal{O}(m + 2G - 2) \to \Sigma_G, \qquad \mathcal{D}' = \mathcal{O}(-m) \to \Sigma_G. \tag{2.56}$$

The N D4-branes are wrapped on $\mathcal{D}$, giving an effective magnetic charge proportional to the intersection number of $\mathcal{D}$ with the two-cycle wrapped by the (electric) D2-branes, Σ_G. The intersection number is

$$\#(\mathcal{D} \cap \Sigma_G) = m + 2G - 2,$$

so that the Kähler modulus should be fixed to be

$$t = \frac{1}{2}(m + 2G - 2)N\lambda - i\theta. \tag{2.57}$$

The black hole partition function (2.54) can be decomposed as in Section 2.3.4 and Figure 2.6. The factors of S_{0R} complicate the mathematics significantly; the derivation can be found in Appendix A. The result, equations (A.7) and (A.8), is

$$Z_{\mathrm{BH}}(\Sigma_g) = \sum_{l=-N/2}^{N/2} \sum_{P_1,\ldots,P_{|2-2G|}} \Psi^{N/2+l}_{P_1,\ldots,P_{|2-2G|}}(t+ml\lambda)\overline{\Psi}^{N/2-l}_{P_1,\ldots,P_{|2-2G|}}(\bar{t}-ml\lambda), \tag{2.58}$$

with both holomorphic and anti-holomorphic chiral partition functions given by

$$\Psi^k_{P_1,\ldots,P_{|2G-2|}}(t) = \widehat{Z}_0(q,t)\exp\left(-\frac{t(|P_1|+\cdots+|P_{|2G-2|}|)}{m+2G-2}\right) \times \sum_{R \text{ of } SU(k)} \left(\frac{1}{d_q(R)}\right)^{2G-2} q^{(m+G-1)k_R/2} e^{-t|R|} \prod_{n=1}^{|2G-2|} s_{P_n}(q^{R_i+\frac{1}{2}-i}), \tag{2.59}$$

except for $G = 0$, where the anti-holomorphic chiral partition function is

$$\overline{\Psi}^k_{P_1,P_2}(\bar{t}) = (-1)^{|P_1|+|P_2|}\Psi^k_{P_1^t,P_2^t}(t).$$

The prefactor is $\widehat{Z}_0(q,t) = Z_0(q,t)\eta^{t(2G-2)/(m+2G-2)\lambda}$.

The most noteworthy aspect of (2.58) is the sum over representations P_n coupling the two chiral partition functions. These have a physical interpretation: the additional factors in (2.59) compared to (2.51) are exactly those required to describe *open* topological strings ending on $|2G-2|$ stacks of D-branes in the fibre above the base Σ_G [3]! As described in the next chapter, these branes must wrap Lagrangian three-cycles, so they meet the D4-branes in a circle in the fibre, and wrap the other fibre, $\mathcal{O}(-m)$. A similar phenomenon was interpreted in [21], as follows. The topological string is sensitive to the choice of boundary conditions of the non-compact Calabi-Yau. In particular, as discussed in [43], there are infinitely many (non-normalisable) Kähler moduli not supported by compact two-cycles, which we should integrate over. They can be viewed as the eigenvalues of representations of $U(\infty)$ corresponding to open strings ending on branes — that is, the variation of the geometry captured by the Kähler moduli is given by the backreaction of stacks of branes. These moduli are present on the topological string side, but not on the black hole side, since there we consider only the D4-brane worldvolume gauge theory. We label the additional branes as P-type "ghost" branes. There are $|2-2G|$ such branes due to there being that many invariant points on the base of the divisor $\mathcal{D}$.

The definition (2.59) is, after taking into account the ghost branes, perturbatively equivalent to the perturbative topological string partition function, as the only difference is the restriction on the height of the Young diagram R. The height restriction, however, means that the OSV relation (2.58) does not suffer from overcounting, and Ψ is thus a candidate non-perturbative completion of the topological string partition function.

2.7.2 The chiral recursion relation

To interpret Ψ as the non-perturbative completion of the topological string partition function, it would be useful to have a chiral recursion relation like (2.46), which in turn used the result (2.45). Since we want the lowest-order expansion of ψ_{top} to be Ψ with $|2G-2|$ ghost representations, we should start with a perturbative topological string partition function with $|2G-2|$ ghost representations, namely (2.59) without restriction on R, and then decompose both the representation R and the ghost representations. That is, we wish to show

$$\psi^{k}_{P_1,\ldots,P_{|2G-2|}}(t) \sim \sum_{r=0}^{\infty} \Psi^{k+r}_{P_1^a,\ldots,P^a_{|2G-2|}}(t')\Psi^{-r}_{P_1^b,\ldots,P^b_{|2G-2|}}(t''), \tag{2.60}$$

with t', t'' to be found. The P-type representations on the left and right should be related, and there may be a prefactor depending on $i\theta$.

The quantum dimension $d_q(\mathcal{R})$ and the Schur function $s_{P_n}(q^{R_i+\frac{1}{2}-i})$ can indeed be decomposed as desired, as shown in Appendix A, Section A.2. The prescription for decomposing P_i to P_i^a and P_i^b is that we should sum over all representations P_i^a and P_i^b giving non-vanishing Littlewood-Richardson coefficients $N^{P_i}_{P_i^a P_i^b}$. Note that this implies $|P_i| = |P_i^a| + |P_i^b|$. Turning to $d_q(\mathcal{R})$, decomposing the representation $\mathcal{R}$ introduces a sum over a second type of ghost branes, expressing additional correlations between Ψ^{k+r} and Ψ^{-r}. The chiral recursion relation is thus schematically of the form

$$\psi^k_{P_1,\ldots,P_{|2G-2|}}(t) \sim \sum_{r=0}^{\infty} \sum_{S_1,\ldots,S_{|2G-2|}} \sum_{\substack{P^a_1,\ldots,P^a_{|2G-2|} \\ P^b_1,\ldots,P^b_{|2G-2|}}} \Psi^{k+r}_{P^a_1,\ldots,P^a_{|2G-2|},S_1,\ldots,S_{|2G-2|}}(t') \times \Psi^{-r}_{P^b_1,\ldots,P^b_{|2G-2|},S_1,\ldots,S_{|2G-2|}}(t''). \quad (2.61)$$

The S-type ghost branes can be physically interpreted [21] as the insertion of non-compact D2-branes wrapping the fibre of the divisor $\mathcal{D}$ wrapped by the D4-branes. The original black hole setup had trivial boundary conditions at infinity for the non-compact D4-branes, and D2-branes wrapping only the compact base. When creating baby universes, however, the non-compact D2-brane charge may be non-zero in each universe, as long as the sum of the charges vanishes. The S-type representations express the presence of these non-compact D2-branes, with the coupling between universes due to the overall charge cancellation constraint. In closed string language, the eigenvalues of the S-type representations are the values of infinitely many non-normalisable Kähler moduli for the boundary conditions at infinity of the D4-branes.

We need now to find the values of the Kähler modulus t' and t'' on the right. The only dependence on $|P_i|$ in (2.59) appears in the exponential prefactor, so recalling that $|P_i| = |P^a_i| + |P^b_i|$, it follows that $t = t' + t''$. Taking into account all the additional prefactors, however, one can show that with this constraint the classical factors $\widehat{Z}_0(q,t)$ on left and right of (2.61) do not match — and the mismatch does not cancel between the holomorphic and anti-holomorphic partition functions.

One can attempt other approaches to realising a chiral recursion relation, or even modified definitions of Ψ that are perturbatively equivalent to ψ_{top}, but there remain mismatches in the classical prefactors. The one-dimensional fermion system provides an intuitive understanding of the mismatch, as follows: The OSV relation relates the topological string and black hole partition functions, equations (2.51) and (2.54), respectively. The factors corresponding to the energy of the free fermions are, respectively, $q^{(m+G-1)\kappa_{\mathcal{R}}/2}$ and $q^{mC_2(R)/2}$, with $C_2(R) = \kappa_R + N|R|$. The fermion energies are multiplied by different factors, $(m+G-1)$ versus m, implying that the process of splitting the black hole rescales the energies of the fermions.

The effective Kähler modulus on the right-hand side of (2.58) is

$$t + ml\lambda = \frac{1}{2}(m + 2G - 2)N\lambda + ml\lambda, \tag{2.62}$$

thus the classical prefactor Z_0 given by (2.51) no longer calculates a simple zero-point energy as it did in the $G = 1$ case, as the fermions do not have a uniform energy scaling.

That the classical prefactors no longer map to simple zero-point energies is the heart of the difficulty in finding a chiral recursion relation. As illustrated in Figure 2.7, for the $G = 1$ case the chiral recursion relation follows from a simple mapping of fermions and holes, including the zero-point energies, between ψ_{top} and Ψ^2. For example, in equation (2.46), either replacing m with $(m + 2G - 2)$, or leaving it unchanged, will introduce mismatches elsewhere, since the effective Kähler modulus (2.62) does not have a single scaling factor.

2.7.3 Outlook

For the more general Calabi-Yau (2.50), we have found a realisation of the OSV conjecture, (2.58), that is correct non-perturbatively, and for which the chiral partition function Ψ is perturbatively the topological string partition function. It is thus tempting to label Ψ the non-perturbative completion of the topological string partition function. However, there does not seem to be a natural chiral recursion relation for expressing Ψ in terms of the perturbative topological string partition function. Our results therefore do not shed much light on the questions of background independence or of tunnelling to baby universes that were raised earlier.

There are, however, suggestive partial successes in realising the chiral partition function — the representations decompose appropriately, as shown in Section A.2. More to the point, the core difficulty is the presence of ghost branes corresponding to non-normalisable Kähler moduli of the non-compact Calabi-Yau. It is thus possible that a proper understanding of the results of the $G = 1$ case will be achieved only with an explicit realisation of the OSV conjecture, at the non-perturbative level, on a compact Calabi-Yau manifold.

Chapter 3

Anomalies in open topological string theory

The BCOV [8] holomorphic anomaly equations (1.38) for the closed topological string have already appeared in the previous chapter as crucial ingredients in the wavefunction interpretation of the topological string partition function, and hence in background independence of the partition function and its use as the Hartle-Hawking wavefunction of the universe in the mini-superspace sector. In the next chapter, the holomorphic anomaly equations will be used to calculate higher-genus amplitudes by direct integration.

This chapter takes a step back, and considers the anomalies themselves. Section 1.2 discussed how naive BRST arguments indicate that A-model topological string amplitudes are independent of anti-holomorphic Kähler moduli, as well as all complex structure moduli; and B-model amplitudes are independent of anti-holomorphic complex structure moduli, and all Kähler moduli. The BCOV holomorphic anomaly equations capture the anomalous dependence of A- and B-model amplitudes on their anti-holomorphic but still "right" moduli (that is, Kähler for A-model, and complex structure for B-model), but confirmed independence from "wrong" moduli — thus showing decoupling of two models. Walcher [13] recently proposed an extension of the BCOV holomorphic anomaly equations to the *open* topological string case (that is, in the presence of D-branes), under the additional assumptions that open string moduli do not contribute to factorisations in open string channels, and that disk one-point functions (closed string states terminating on a boundary) vanish.

The novel material presented in this chapter, and reported in [14], is a careful derivation on the worldsheet of Walcher's proposed holomorphic anomaly equations (3.23) for the open string, under the assumption that open string moduli do not contribute and that disk one-point functions vanish. Relaxing the second assumption, however, leads us to find a new set of anomalies: non-vanishing disk one-point functions on compact Calabi-Yau manifolds generate new terms (3.24) and (3.29) in the holomorphic anomaly equations and spoil their recursive structure, and moreover can lead to string amplitudes developing dependence on the wrong moduli.

Some salient facts about open topological string theory are presented in Section 3.1. In Section 3.2, the holomorphic anomaly equations for the open string are derived, and the existence of new anomalies when disk one-point functions are non-vanishing is demonstrated. In Section 3.3, the dependence of amplitudes on wrong moduli is investigated, leading to further new anomalies. Finally, the relevance and consistency of the new anomalies with existing results, particularly matrix models and large N duality, is discussed in Section 3.4.

3.1 The open topological string

From the worldsheet perspective, open topological string theory adds boundaries or holes to the theory discussed in Section 1.2, such that worldsheet topologies are classified by both genus g and hole number h, with vacuum amplitudes $F_{g,h}$. The boundaries correspond to attachment points to D-branes in the target space. The Dirichlet boundary conditions identify the left- and right-moving sectors of the string theory, and hence the supercharges. In order to preserve the supersymmetry $Q = Q^+ + \overline{Q}^+$, the appropriate boundary condition is $Q|B\rangle = 0$, where $|B\rangle$ is the boundary. In terms of the supercurrents, this is

$$(G^+_z dz + \overline{G}^+_{\bar{z}} d\bar{z})|_{\partial\Sigma} = 0, \qquad \text{and} \qquad (G^-_{zz}\chi^z dz + \overline{G}^-_{\bar{z}\bar{z}}\bar{\chi}^{\bar{z}} d\bar{z})|_{\partial\Sigma} = 0, \tag{3.1}$$

where χ is a holomorphic vector along the boundary direction.

A careful analysis of these boundary conditions [44] can be used to constrain the allowed D-brane configurations consistent with supersymmetry. The result is that A-model requires D-branes wrapping special Lagrangian sub-manifolds, of three (real) dimensions, while B-model requires D-branes wrapping even-dimensional holomorphic cycles. This result is somewhat surprising: 3-cycles are dual to 3-forms, which are related to complex structure deformations, as discussed in Section 1.2. A-model (closed) string theory should, however, be independent of complex structure moduli. Conversely, 2- and 4-cycles are dual to $(1,1)$ forms, which are related to Kähler structure deformation, of which the B-model should be independent. Thus we may be concerned that boundaries could spoil the decoupling of moduli in the two models — and indeed this chapter will identify anomalies that match this expectation, in compact manifolds with non-vanishing D-brane topological charge (that is, non-cancelling winding numbers around non-trivial cycles).

Since boundaries identify the left- and right-moving sectors, the $U(1)_R$ charge constraints identified in Section 1.3 reduce to a single constraint. The form of this constraint can be determined using a doubling construction: take two copies of the Riemann surface $\Sigma_{g,h}$ and glue the matching boundaries of the copies together, to form a closed surface $\Sigma'_{2g+h-1,0}$, with the boundaries of $\Sigma_{g,h}$ on the fixed plane of a $\mathbb{Z}_2$ involution of $\Sigma'_{2g+h-1,0}$. The combined worldsheet must have total charge $p = q + \bar{q} = 12g - 12 + 6h$, so on $\Sigma_{g,h}$ the constraint for the total charge of all insertions is

$$\sum_i p_i = \sum_i (q_i + \bar{q}_i) = (6g - 6 + 3h). \tag{3.2}$$

Recall that the term (1.36) in the action represents marginal deformations. In the presence of boundaries, however, this term may not be supersymmetry invariant, due to (worldsheet) boundary terms. This is termed the *Warner problem.* To resolve this problem, one can add a boundary term to the action, such that the total contribution corresponding to t^i variation is

$$\delta S = t^i \left(\int_\Sigma \{G^-, [\overline{G}^-, \phi_i]\} - \int_{\partial\Sigma} \psi_i^{(1)} \right), \tag{3.3}$$

where the first term corresponds to ϕ_i insertion in the bulk, and the second term is a one-form descendant of an open string state. To find ψ_i, consider the supersymmetric variation of the first term,

$$\delta_Q\left(t^i\int_\Sigma \phi_i^{(2)}\right) = t^i\int_\Sigma [G^+ + \overline{G}^+, \phi_i^{(2)}]$$
$$= t^i\int_\Sigma \left(2T[\overline{G}^-, \phi_i] + [G^-, 2\overline{T}\phi_i]\right),$$

where we have used (1.15), and two-form descendant $\phi_i^{(2)}$ defined in (1.35). T and $\overline{T}$ can be replaced in the path integral by worldsheet partial derivatives, so applying Stokes theorem gives

$$\delta_Q\left(t^i\int_\Sigma \phi_i^{(2)}\right) = t^i\int_{\partial\Sigma} [G^- + \overline{G}^-, \phi_i]. \tag{3.4}$$

This term must be cancelled by the supersymmetric variation of the second term in (3.3). That is,

$$\phi_i^{(1)}\Big|_{\partial\Sigma} = [Q_{\text{bndry}},\ \psi_i^{(1)}]\Big|_{\partial\Sigma}, \tag{3.5}$$

where Q_{bndry} is the boundary part of the supercharge.

By the operator-state correspondence, $\psi_i^{(1)}$ corresponds to an open string state, and so the second term in (3.4) corresponds to variation of open string moduli, that is deformations of the D-brane. There are two cases, as argued in [13]: open string moduli with relations like (3.5) are cases where the D-brane deforms with the bulk — bulk and boundary moduli are not independent. On the other hand, open string moduli not fixed by relations like (3.5) can be lifted (made massive) by small bulk deformations. Thus [13] argues that open string moduli are either lifted for generic values of the bulk moduli, or do not appear as independent moduli, and so they drop out of $F_{g,h}$. We will not argue this further, but will take as assumed that open string moduli need not be considered — and so there are no marginal ($p = 1, 2$) open string states.

3.2 Extended holomorphic anomaly equations

3.2.1 Boundaries of moduli space

Our starting point is to generalise the integration over worldsheet moduli, expression (1.24), for the open topological string amplitude at genus g and with h boundaries. The vacuum amplitude is [13]

$$F_{g,h} = \int_{\mathcal{M}_{g,h}} [dm][dl] \left\langle \prod_{a=1}^{3g-3+h} \int \mu_a G^- \int \bar{\mu}_a \overline{G}^- \prod_{b=1}^{h} \int (\lambda_b G^- + \bar{\lambda}_b \overline{G}^-) \right\rangle , \qquad (3.6)$$

where the worldsheet indices have been suppressed. Each handle is associated with the integration of three supercurrents (i.e., G^- or $\overline{G}^-$), folded with Beltrami differentials μ_a. For thin handles, these moduli can be interpreted as the endpoints of the handle on the Riemann surface, plus the period matrix describing the shape of the handle. Each hole has a complex modulus specifying its location on the worldsheet, as well as an additional real modulus l specifying its boundary circumference, corresponding to the integration of the supercharge combination $(G^- + \overline{G}^-)$ that is preserved at the boundary, folded with a real differential λ_b that has support near the boundary.

Variation of the correlation function with respect to the anti-holomorphic moduli $\bar{t}^{\bar{i}}$ corresponds to inserting a BRST trivial operator,

$$\int_\Sigma \overline{\phi}_{\bar{i}}^{(2)} = \int_\Sigma \{G^+, [\overline{G}^+, \bar{\phi}_{\bar{i}}]\}. \qquad (3.7)$$

Following the approach of [45], it will often be convenient to phrase arguments in terms of the supercurrents defined by (1.16), rather than the supercharges. For example, (3.7) can be written

$$\int_\Sigma \oint_{C_w} G_z^+ \oint_{C_w'} \overline{G}_{\bar{z}}^+ \overline{\phi}_{\bar{i}}(w), \qquad (3.8)$$

where C_z and C_z' are contours around the point z. The contours can then be deformed using the standard techniques of complex analysis. In the following, the identification of supercharge commutators and supercurrent contours is assumed.

An immediate obstacle to deforming the contours in (3.7) is that G^+ and $\overline{G}^+$ do not annihilate the boundary. We thus rewrite the insertion as

$$-\frac{1}{2}\int_{\Sigma}\{G^{+}+\overline{G}^{+},[G^{+}-\overline{G}^{+},\overline{\phi}_{\bar{i}}]\}. \tag{3.9}$$

The outer contour can now be deformed around the worldsheet, picking up contributions from the commutation relationships (1.15) when crossing insertions of G^{-} or $\overline{G}^{-}$. Explicitly,

$$\begin{aligned}\bar{\partial}_{\bar{t}^{\bar{i}}}F_{g,h} = -\int\limits_{\mathcal{M}_{g,h}}[dm][dl]\Bigg[\sum_{c=1}^{3g-3+h}\Big\langle\int\overline{\phi}_{\bar{i}}^{(1)}\Big(2\int\mu_c T\int\bar{\mu}_c\overline{G}^{-}+2\int\mu_c G^{-}\int\bar{\mu}_c\bar{T}\Big)\times \\ \times\prod_{a\neq c}\int\mu_a G^{-}\int\bar{\mu}_a\overline{G}^{-}\prod_{b=1}^{h}\int(\lambda_b G^{-}+\bar{\lambda}_b\overline{G}^{-})\Big\rangle \\ +\sum_{c=1}^{h}\Big\langle\int\overline{\phi}_{\bar{i}}^{(1)}\int 2\left(\lambda_c T+\bar{\lambda}_c\bar{T}\right)\times \\ \times\prod_{a=1}^{3g-3+h}\int\mu_a G^{-}\int\bar{\mu}_a\overline{G}^{-}\prod_{b\neq c}\int(\lambda_b G^{-}+\bar{\lambda}_b\overline{G}^{-})\Big\rangle\Bigg],\end{aligned} \tag{3.10}$$

where we have defined

$$\overline{\phi}_{\bar{i}}^{(1)}=\frac{1}{2}[G^{+}-\overline{G}^{+},\bar{\phi}_{\bar{i}}]. \tag{3.11}$$

The Beltrami differentials μ_i parametrise the change in the Kähler metric under infinitesimal change of the coordinates on the moduli space, through the definition (1.23). Recalling that

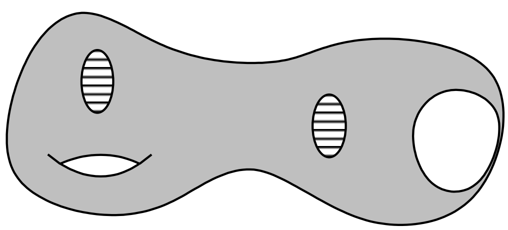
$$T_{\alpha\beta}=\frac{\partial S}{\partial h^{\alpha\beta}},$$

one arrives at the following "chain rule:"

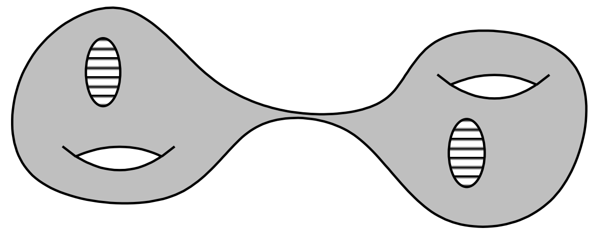
$$\mu_i T=\frac{\partial S}{\partial m^i}. \tag{3.12}$$

Thus the combinations $\mu_i T$ and $\bar{\mu}_i\bar{T}$ can be converted into derivatives with respect to the moduli m, $\bar{m}$ and l. By Cauchy's theorem, this restricts the integral to the boundaries of the moduli space, with boundaries corresponding to degenerations of both the complex and real moduli, that is, both open and closed string degenerations.

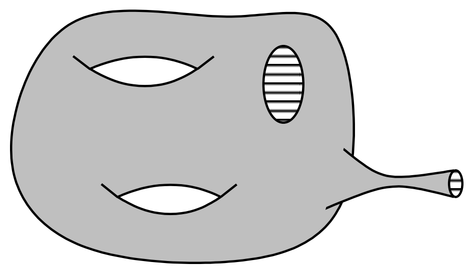
To enumerate all the moduli space boundaries, a useful technique is to consider the



(a) Case A: A handle pinches off



(b) Case B: An equator pinches off

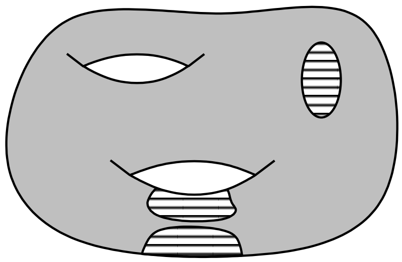


(c) Case C: A boundary shrinks

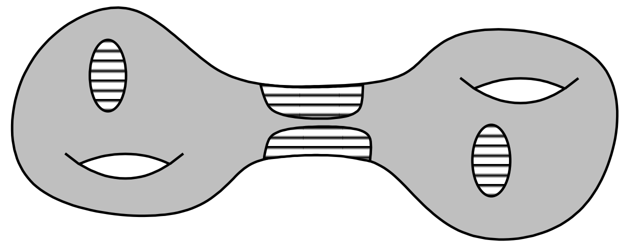
Figure 3.1: *Moduli space boundaries resulting from the degeneration of a closed one-cycle on the Riemann surface* $\Sigma_{g,h}$.

Case	Description
A	A handle shrinks to zero diameter (pinches off), leaving $\Sigma_{g-1,h}$ plus a degenerating thin tube.
B	An equator shrinks to zero diameter (pinches off), splitting the Riemann surface into two non-trivial daughter surfaces $\Sigma_{r,s}$ and $\Sigma_{g-r,h-s}$, joined by a degenerating thin tube. Both daughter surfaces have $2g+h \geq 2$.
C	A cycle around a boundary shrinks, that is, the boundary closes off. Conformally this is a boundary on the end of a degenerating thin tube attached to the remaining surface $\Sigma_{g,h-1}$.

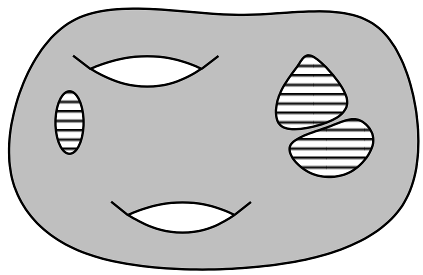
Table 3.1: *Moduli space boundaries resulting from the degeneration of a closed one-cycle on the Riemann surface* $\Sigma_{g,h}$.



(a) Case D: A boundary expanding around a handle



(b) Case E: A surface splitting by a boundary expanding around an equator



(c) Case F: Two boundaries merging

Figure 3.2: *Moduli space boundaries resulting from the degeneration of an open one-path on the Riemann surface* $\Sigma_{g,h}$.

Case	Description
D	A path from a boundary, around a handle, and back to the same boundary, degenerates to leave $\Sigma_{g-1,h+1}$, with the two child boundaries joined by a degenerating thin strip.
E	A path from a boundary, around an equation, to the same boundary, degenerates to leave two surfaces $\Sigma_{r,s}$ and $\Sigma_{g-r,h-s+1}$, with the two daughter surfaces joined by a degenerating thin strip. Both daughter surfaces have $2g+h \geq 2$.
F	A path between two different boundaries degenerates, leaving $\Sigma_{g,h-1}$, with a degenerating thin strip across the newly joined boundary.

Table 3.2: *Moduli space boundaries resulting from the degeneration of an open one-path on the Riemann surface* $\Sigma_{g,h}$.

degeneration, in turn, of all closed one-cycles, and open one-paths with endpoints on (possibly distinct) boundaries, as described in [46]. The cases resulting from degenerations of closed one-cycles are shown in Table 3.1 and Figure 3.1, and those of open one-cycles are shown in Table 3.2 and Figure 3.2. Degenerations which split the surface have the constraint that $2g + h \geq 2$ on each daughter surface. This follows from the fact that genus zero and one amplitudes vanish due to unfixed translational invariance on the worldsheet unless enough additional points are fixed on the surface — three for the sphere and one for the torus. Boundaries each fix a point, and as we will see the degenerations also leave a point fixed.

The key ingredients in all of these cases are tubes which shrink to zero diameter, or strips which shrink to zero width, or boundaries which shrink. In the next three subsections, we consider these cases, before putting it all together to get the extended holomorphic anomaly equations.

3.2.2 Degenerating tubes

Consider the case where, at the boundary of the moduli space, a closed string tube becomes infinitely long and narrow. This is the case that was considered in [8], and our results are the same, though our arguments are arranged slightly differently. The three complex moduli, (τ, v, w) corresponding to the handle can be identified as follows (see Figure 3.3): v and w are the attachments points of the end of the tube to the remainder of the Riemann surface. The Beltrami differentials localise to the attachment points, giving, for example,

$$\int \mu_w G^- \int \bar{\mu}_w \overline{G}^- \to \oint_{C_w} G^- \oint_{C'_w} \overline{G}^- \tag{3.13}$$

where w is the insertion point of one of the ends of the handle and C_w and C'_w are contours around w. The third complex modulus, τ, parametrises the shape of the handle, such that $\tau \to \infty$ at the boundary of moduli space. The twist of the handle remains as a real modulus, represented by the insertion of

$$\int (\mu_\tau G^- - \bar{\mu}_\tau \overline{G}^-). \tag{3.14}$$

The tube's infinite length projects all intermediate states to closed string ground states at

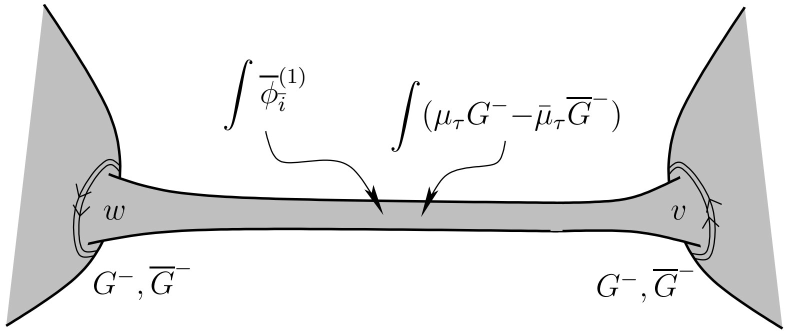


Figure 3.3: *The operators on a degenerating tube, for $\bar{t}^{\bar{i}}$ derivative. The supercurrent combination around $\overline{\phi}_{\bar{i}}$ may now be anti-commuted past the supercurrents folded with Beltrami differentials.*

the end of the handle, and so each of the attachment points of the handle can be replaced with a complete set of ground states, $\sum_{I,\bar{J}} |I\rangle g^{I\bar{J}} \langle \bar{J}|$. The currents (3.13) around the endpoints annihilate $|I\rangle$ unless it is an operator ϕ_I in the (c, c) chiral ring with charge at least $(1, 1)$. Recall from Section 1.3, however, that operators inserted on the worldsheet must be neutral with respect to the $U(1)_R$ charge. The operators (3.13) contribute $q = -2$, so $\phi_I = \phi_i$ must be exactly charge $(1, 1)$, i.e., marginal. $\langle \bar{J}| = \langle \bar{j}|$ is thus a charge $(-1, -1)$ state from the (a, a) chiral ring. Such states will be annihilated by (3.14) on the tube unless the insertion $\overline{\phi}_{\bar{i}}^{(1)}$ is on the tube.

Inserting another complete set of ground states $\sum_{\bar{k}i} |\bar{k}\rangle g^{\bar{k}i'} \langle i'|$ at the other end of the tube, we can write the amplitude near the middle of the tube as

$$\begin{aligned}
&\int d^2z \langle \bar{j}| [G^+ - \overline{G}^+, \overline{\phi}_{\bar{i}}(z)] \int (\mu_\tau G^- - \bar{\mu}_\tau \overline{G}^-) |\bar{k}\rangle \\
&= \int d^2z \langle \bar{j}| \overline{\phi}_{\bar{i}}(z) \int 2(\mu_\tau T - \bar{\mu}_\tau \bar{T}) |\bar{k}\rangle \\
&= \frac{\partial}{\partial \operatorname{Im} \tau} \int d^2z \langle \bar{j}| \overline{\phi}_{\bar{i}}(z) |\bar{k}\rangle,
\end{aligned}$$

where the absence of boundaries has allowed us to deform the contour of $(G^+ - \overline{G}^+)$, picking up commutators as per (1.15), which were transformed into derivatives with respect to the moduli using (3.12). As we are already at the boundary of moduli space, the additional derivative is in the direction normal to the boundary, namely $\operatorname{Im} \tau$. In the limit $\tau \to \infty$, the volume of the domain of the integral of $\overline{\phi}_{\bar{i}}$ is $\operatorname{Im} \tau$, which cancels the derivative. Thus the total contribution from the handle is equivalent to the insertion of

$$\phi_i g^{i\bar{j}} \langle \bar{j} | \overline{\phi}_{\bar{i}} | \bar{k} \rangle g^{\bar{k}i'} \phi_{i'} = \bar{C}_{\bar{i}\bar{j}\bar{k}} e^{2K} G^{\bar{j}i} G^{\bar{k}i'} \phi_i \phi_{i'}, \tag{3.15}$$

with the marginal operators ϕ_i and $\phi_{i'}$ surrounded by integrals (3.13), which can be identified as the descendants $\phi_i^{(2)}$ in equation (1.35).

3.2.3 Degenerating thin strips

A narrow strip, or propagating open string state, is associated with three real moduli (r, s, l), identifiable as the location of the two endpoints r and s of the strip on the boundary, and the length l of the strip. At the boundary of moduli space $l \to \infty$. The Beltrami differentials corresponding to moduli r and s are localised near the respective endpoints of the strip, leaving these points surrounded by the contour integral,

$$\int_{C_r} (G^- + \overline{G}^-). \tag{3.16}$$

The attachment points of the long strip can be replaced with complete sets of open string ground states, $\sum_\alpha |\alpha\rangle\langle\alpha|$ and $\sum_\beta |\beta\rangle\langle\beta|$. However, as discussed in Section 3.1, we assume there are no marginal open string states, leaving only charge $p = 0, 3$ states. The moduli at the endpoints annihilate charge 0 states, and charge 3 states violate the overall charge constraint. Thus this case gives zero contribution, regardless of the location of the insertion $\bar{\phi}_{\bar{i}}^{(1)}$.

3.2.4 Shrinking boundary

The last closed 1-cycle degeneration, case C, covers the case when a hole shrinks, or equivalently becomes separated from the Riemann surface by a long tube. This case arises from the second term in (3.10). That such a degeneration is part of the boundary of moduli space can be seen from the doubling method discussed in Section 3.1: the pinching off of a $\Sigma'_{2g+h-1,0}$ handle which crosses the $\mathbb{Z}_2$ fixed plane is equivalent to a shrinking boundary in $\Sigma_{g,h}$.

A worldsheet boundary is associated with three real moduli insertions, specifying the location of the boundary and its length, so after degeneration there are two supercharges localised to the attachment point of the tube, as per (3.13), with the boundary at the other end of the tube. The absence of additional supercharges corresponding to moduli on the tube itself distinguishes this class from the closed string factorisation class above, and furthermore allows the remaining insertion $\bar{\phi}_{\bar{i}}^{(1)}$, as defined by (3.11), to be anywhere on the worldsheet.

Firstly, $\overline{\phi}_{\bar{i}}^{(1)}$ may nevertheless be on the tube. The degeneration $\tau \to \infty$ projects the intermediate states on both sides of the insertion to ground states, which are annihilated by supercharges. Thus the $(G^+ - \overline{G}^+)$ part of $\overline{\phi}_{\bar{i}}^{(1)}$ annihilates the ground states, so this case is zero.

Secondly, $\overline{\phi}_{\bar{i}}^{(1)}$ may be near the shrinking boundary. As in Section 3.2.2, considerations on the rest of the Riemann surface mean that the tube is replaced with a complete set of closed string marginal ground states, $\sum_{j,\bar{j}} |j\rangle g^{i\bar{j}} \langle \bar{j}|$. The correlator of the degenerating region is then

$$e^K G^{j\bar{j}} \langle \bar{j} | \bar{\phi}_{\bar{i}}^{(1)} | B \rangle, \tag{3.17}$$

where B is the boundary. We choose coordinates on the disk where $\langle \bar{j}| = \bar{\phi}_{\bar{j}}$ is at $r = 0$, and the boundary is at $r = 1$. $\bar{\phi}_{\bar{i}}^{(1)}$ is a worldsheet 1-form, so the amplitude is explicitly

$$e^K G^{j\bar{j}} \int_o^1 dr \langle \bar{\phi}_{\bar{i}}^{(1)}(r) \bar{\phi}_{\bar{j}}(0) \rangle = \Delta_{\bar{i}}^{j}, \tag{3.18}$$

namely the anti-topological disk two-point function.

Lastly, $\overline{\phi}_{\bar{i}}^{(1)}$ may be inserted somewhere else on the Riemann surface, as shown in Figure 3.4. The tube is again replaced with a complete set of ground states $\sum_{I,\bar{J}} |I\rangle g^{I\bar{J}} \langle \bar{J}|$. To avoid annihilation by the supercurrents G^- and $\overline{G}^-$ localised as per (3.13) to the tube attachment point, $|I\rangle$ must be in the (c,c) chiral ring and have $q_I, \bar{q}_I \neq 0$. Furthermore, $\overline{\phi}_{\bar{i}}^{(1)}$ is a state with total $U(1)_R$ charge $p = 1$, and the tube end-point supercurrents contribute charge $(-1,-1)$, so $|I\rangle$ is required to be charge $p = 3$ — specifically, a charge $(1,2)$ or $(2,1)$ state to avoid annihilation by the supercurrents at the end-points. We denote these ω_a, index a running over charge $(1,2)$ and $(2,1)$ chiral primaries. Note that the ω_a are not associated with marginal deformations of the topological string in question; they are associated with deformations of the opposite model. In the A-model, as discussed in Section 1.3, they

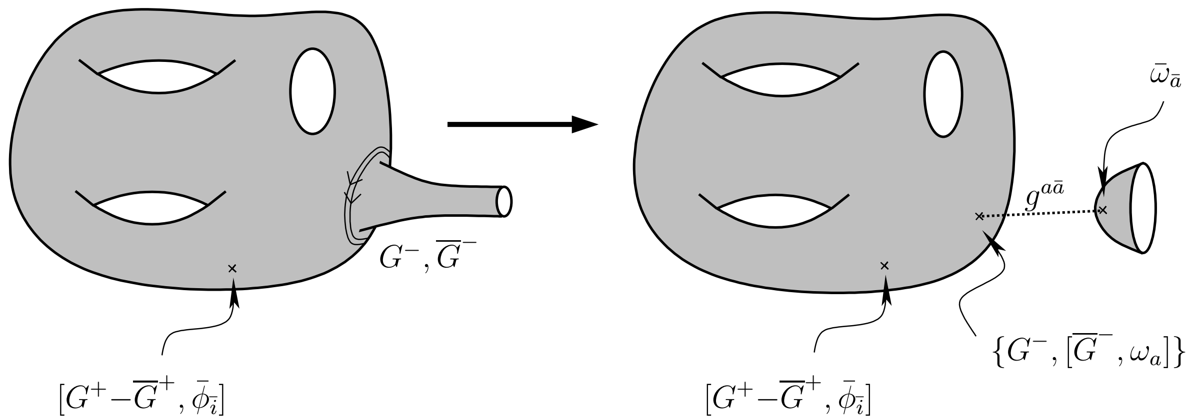


Figure 3.4: *Riemann surface for the shrinking boundary degenerating for $\bar{t}^{\bar{i}}$ derivative, with the insertion $\overline{\phi}^{(1)}_{\bar{i}}$ located away from the shrinking boundary. On the right we have replaced the tube with a sum over states ω_a of charge $(1,2)$ and $(2,1)$, rendering the near-boundary region a disk one-point function.*

correspond to target space 3-forms, and hence to complex structure deformation, and in the B-model they are $(1,1)$ forms, and so correspond to Kähler deformations. Near the shrinking boundary the resulting amplitude is the disk one-point function,

$$g^{a\bar{a}}\overline{C}_{\bar{a}} = g^{a\bar{a}}\langle\bar{\omega}_{\bar{a}}|B\rangle. \tag{3.19}$$

Thus this contribution represents a new anomaly, if the disk one-point functions do not vanish.

3.2.5 Putting it together

We can now return to the degenerations enumerated in Tables 3.1 and 3.2. Cases A and B appear in [8]. Case A is a handle degenerating, leaving the surface $\Sigma_{g-1,h}$ plus two closed string insertions $\phi^{(2)}_i$ corresponding to the remnants of the handle. The handle contribution is given by (3.15). The amplitude of the Riemann surface is thus

$$\bar{C}_{\bar{i}\bar{j}\bar{k}}e^{2K}G^{\bar{j}j}G^{\bar{k}k}\int_{\mathcal{M}_{g-1,h}}[dm'][dl]\left\langle\int\phi^{(2)}_j\int\phi^{(2)}_k\cdots\right\rangle,$$

where the dots are the integrals of G^- and $\overline{G}^-$ folded with Beltrami differentials that were left unaffected by the degeneration, as shown in (3.10). This amplitude should be integrated over the remaining moduli m' and l at the boundary of the moduli space, and the locations of the insertions $\phi_j^{(2)}$ and $\phi_k^{(2)}$, with a factor $\frac{1}{2}$ to take account of the interchange symmetry of the endpoints of the tube. Being marginal, the insertions $\int \phi_j^{(2)}$ and $\int \phi_k^{(2)}$ generate deformations corresponding to infinitesimal change of moduli, so they can be replaced by covariant derivatives D_j and D_k of the amplitude without insertions, $F_{g-1,h}$ [8]. The contribution of degenerations to $\Sigma_{g-1,h}$ is thus

$$\frac{1}{2}\bar{C}_{\bar{i}\bar{j}\bar{k}} e^{2K} G^{\bar{j}j} G^{\bar{k}k} D_j D_k F_{g-1,h}. \tag{3.20}$$

In case B the Riemann surface splits into components $\Sigma_{r,s}$ and $\Sigma_{g-r,h-s}$, where r and s count how many of the handles and boundaries are located in each of the respective "daughter" Riemann surfaces. The two components are joined by a narrow tube of the sort in Section 3.2.2. The resulting contribution is

$$\frac{1}{2}\bar{C}_{\bar{i}\bar{j}\bar{k}} e^{2K} G^{\bar{j}j} G^{\bar{k}k} \sum_{r=0}^{g}\sum_{s=0}^{h} \int_{\mathcal{M}_{r,s}} [dm'][dl'] \left\langle \int \phi_j^{(2)} \cdots \right\rangle \int_{\mathcal{M}_{g-r,h-s}} [dm''][dl''] \left\langle \int \phi_k^{(2)} \cdots \right\rangle$$
$$= \frac{1}{2}\bar{C}_{\bar{i}\bar{j}\bar{k}} e^{2K} G^{\bar{j}j} G^{\bar{k}k} \sum_{r=0}^{g}\sum_{s=0}^{h} D_j F_{r,s} D_k F_{g-r,h-s} \tag{3.21}$$

where the sets of moduli m' and m'', and l' and l'' correspond to the remaining moduli on each of the daughter surfaces, and $D_j F_{g,h} = 0$ for $2g + h < 2$. The insertions $\phi_j^{(2)}$ and $\phi_k^{(2)}$ are integrated over the surfaces on which they are respectively inserted, and have been converted to covariant derivatives. The overall factor $\frac{1}{2}$ comes from the $\mathbb{Z}_2$ symmetry of the sum generated by simultaneously taking $r \to (g-r)$, $s \to (h-s)$ and $j \leftrightarrow k$.

The last non-zero degeneration is case C, discussed in Section 3.2.4. Consider first vanishing disk one-point functions. The insertion of $|j\rangle$ on the remainder of the Riemann surface can be written as a covariant derivative, so the Riemann surface amplitude is,

$$-\Delta_{\bar{i}\bar{j}} e^K G^{\bar{j}j} D_j F_{g,h-1}. \tag{3.22}$$

Putting all the pieces together and using the metrics to raise indices, we arrive at the extended holomorphic anomaly equations, subject to the assumptions of vanishing disk one-point functions and the absence of open string moduli,

$$\bar{\partial}_{\bar{i}} F_{g,h} = \frac{1}{2}\overline{C}_{\bar{i}}^{jk}\left[\sum_{r=0}^{g}\sum_{s=0}^{h} D_j F_{r,s} D_k F_{g-r,h-s} + D_j D_k F_{g-1,h}\right] - \Delta_{\bar{i}}^{j} D_j F_{g,h}. \tag{3.23}$$

If, on the other hand, the disk one-point functions do not vanish, equation (3.19) means that an additional anomalous term is present:

$$g^{\bar{a}b}\overline{C}_{\bar{a}} \int_{\mathcal{M}_{g,h-1}} [dm][dl] \left\langle \{G^-, [\overline{G}^-, \omega_b]\} \; [G^+ - \overline{G}^+, \bar{\phi}_{\bar{i}}] \right\rangle_{\Sigma_{g,h-1}}. \tag{3.24}$$

This term, shown in Figure 3.4, represents an anomalous coupling to wrong moduli through the charge $(2,1)$ or $(1,2)$ chiral primary operator ω_b. Furthermore, since neither insertion is marginal, (3.24) cannot be written as a recursion relation on lower genus amplitudes. We will consider the implications of the anomaly further in Section 3.4. Note finally that the torus and annulus amplitudes are special cases, as the worldsheets contain residual unfixed rotational symmetry, and so the vacuum amplitudes vanish. The amplitudes with insertions are non-zero and have holomorphic anomaly equations, equations (4.4) and (4.5), respectively. Their derivations require the consideration of propagating zero modes, as there are not enough moduli to provide localised Beltrami differentials folded with supercurrents, which normally kill the zero modes. The derivations can be found in [8, 13].

3.3 Decoupling of moduli from the other model

As discussed in Section 1.3, we expect that wrong moduli (that is, Kähler moduli in B-model and complex structure moduli in A-model) should decouple from topological string amplitudes. In this section we show that this is indeed the case for the open topological string, under the assumptions that open string moduli are absent and that the disk one-point functions vanish; but if the latter condition is not satisfied, a new anomaly is present.

Consider the dependence of the amplitude $F_{g,h}$ on wrong moduli y^a — results for the moduli $\bar{y}^{\bar{a}}$ can be obtained by complex conjugation. The derivative with respect to y^a corresponds to inserting the operator

$$\int_\Sigma \{\overline{G}^+, [G^-, \varphi_a]\} + 2\int_{\partial\Sigma} \varphi_a, \tag{3.25}$$

where φ_a is a charge $(1,-1)$ marginal operator from the (c,a) ring. The second term is the boundary term required to resolve the Warner problem, introduced in Section 3.1. By the commutation properties of the supercharges, $2\int_{\partial\Sigma}\varphi_a = \int_\Sigma \{G^+, [G^-, \varphi_a]\}$, and so (3.25) can be rewritten as

$$\int_\Sigma \{G^+ + \overline{G}^+, [G^-, \varphi_a]\}. \tag{3.26}$$

The outer contour can now be deformed past boundaries on the worldsheet, producing terms corresponding to all possible degenerations of the Riemann surface, as for the $\bar{t}^{\bar{i}}$ derivative in the previous section. For each degeneration, there remains the insertion

$$\int_\Sigma [G^-, \varphi_a], \tag{3.27}$$

located somewhere on the worldsheet.

As in the previous section, consider the three building-block degenerations:

1. **Degenerating tube** (Cases A and B): The tube degeneration vanishes if the insertion (3.27) is not on the tube. If it is on the tube, then the amplitude of the tube is

$$-\langle\bar{j}|[G^-, \bar{\varphi}_{\bar{a}}]\int(\mu_\tau G^- - \bar{\mu}_\tau \overline{G}^-)|\bar{k}\rangle, \tag{3.28}$$

   where the second integral is due to the remaining modulus of the tube. Recall that $\langle\bar{j}|$ and $|\bar{k}\rangle$ are charge $(-1,-1)$ operators, and so we are working in the anti-topological twist. This means that G^- as a supercurrent is a worldsheet one-form, so the contour can be deformed. G^- annihilates charge $(-1,-1)$ states and commutes with the remaining modulus integral, so (3.28) vanishes.

2. **Degenerating strip** (Cases D, E and F): As in the previous section, the vanishing of these contributions follows from the assumption that open string moduli do not

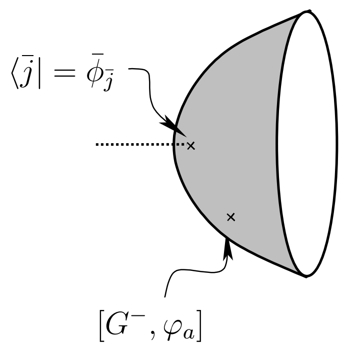


Figure 3.5: *The near-boundary region of the shrinking boundary degeneration for y^a derivative, with insertion (3.27) near the boundary. This amplitude vanishes, as described in the text.*

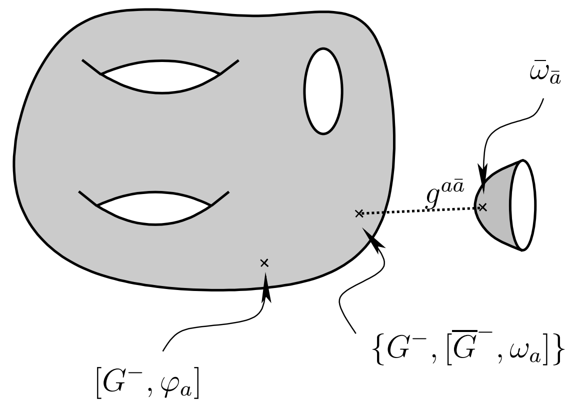


Figure 3.6: *Riemann surface for the shrinking boundary degeneration for y^a derivative, with insertion (3.27) elsewhere on the Riemann surface. This is non-zero unless the disk one-point function vanishes.*

contribute.

3. **Boundary shrinking** (Case C): As previously, the boundary is attached to the Riemann surface with the complete set of marginal ground states $\sum_{j,\bar{j}} |j\rangle g^{j\bar{j}} \langle \bar{j}|$, with insertion (3.27) near the boundary or elsewhere on the Riemann surface. The former case is shown in Figure 3.5. Near the boundary the theory is anti-topologically twisted, making G^- and $\overline{G}^-$ dimension one as supercurrents, and so allowing contour deformation. Using the properties of the chiral rings, (3.27) can be written $\int_\Sigma [G^- + \overline{G}^-, \varphi_a]$, which annihilates both $\langle \bar{j}|$ and the boundary, so this case is zero by contour deformation.

   Secondly, (3.27) may be elsewhere on the Riemann surface, as shown in Figure 3.6. As in the $\bar{t}^{\bar{i}}$ case, the near-boundary amplitude is the disk one-point function,

$$g^{a\bar{a}} \overline{C}_{\bar{a}} = g^{a\bar{a}} \langle \bar{\omega}_{\bar{a}} | B \rangle,$$

   and on the rest of the worldsheet $|a\rangle = \omega_a$ is a charge $(1,2)$ or $(2,1)$ state.

The only potentially non-vanishing contribution is the final case, which is non-zero if the disk one-point functions $\overline{C}_{\bar{a}}$ are non-zero. In this case, then,

$$\frac{\partial}{\partial y^a} F_{g,h} = g^{\bar{a}b} \overline{C}_{\bar{a}} \int_{\mathcal{M}_{g,h-1}} [dm][dl] \left\langle \{G^-, [\overline{G}^-, \omega_b]\} \, [G^-, \varphi_a] \cdots \right\rangle_{\Sigma_{g,h-1}} . \tag{3.29}$$

Note that the G^- and $\overline{G}^-$ contours around ω_b and φ_a cannot be deformed as they are dimension 2 as supercurrents. Equation (3.29) represents an anomalous coupling of the amplitude to wrong moduli. Furthermore, as the insertions are not marginal operators, (3.29) cannot be written in the form of a recursion relation for lower genus amplitudes.

3.4 Implications of the new anomalies

The previous sections have uncovered the existence of new anomalies of the open topological string, terms (3.24) and (3.29), in addition to the usual anti-holomorphic anomaly (3.23). The new anomalies are present when the disk one-point functions are non-zero, and they introduce coupling to wrong moduli, in such a way that the resulting anomaly cannot be expressed as a recursion relation with respect to worldsheet genus. In this section, we point out some connections between these new anomalies and existing results.

If overall D-brane charge vanishes, then contributions to the disk one-point function will cancel, removing the contribution of the new anomalies. Evidence for the presence of the new anomalies is provided by the observation that D-brane charge cancellation appears to be required for the successful counting of the number of BPS states in M-theory using the topological string partition function. In [47], it was conjectured that the partition function of the closed topological string can be interpreted as counting BPS states in M-theory compactified to five dimensions on a Calabi-Yau manifold. This conjecture was extended to cases with D-branes in [48, 49]. In [50] Walcher applied the formulae of [48] to examples of *compact* Calabi-Yau manifolds and found that the integrality of BPS state counting can be assured only when the topological charges of the D-branes were cancelled by introducing orientifold planes [50], such that the disk one-point functions vanish. Our result gives a microscopic explanation of this observation. Furthermore, in [13], Walcher considered compensating stacks of D-branes and anti-D-branes wrapping homologically equivalent cycles, such that the disk one-point functions vanished, and found integral Gromov-Witten invariants without new anomalies.

Large N duality, at its simplest, is the observation that an open string theory with $U(N)$ gauge group, N large, should be dual to a closed string theory. This idea has many realisations, notably the AdS/CFT correspondence. In topological string theory a key example is the geometric transition, where open topological string on a deformed conifold with D3-branes is dual to closed topological string theory on a resolved conifold, that is, with singularity blown up by the presence of the fluxes left from the back-reaction of the D-branes. The absence of the new anomalies appears to be a prerequisite for large N duality. Specifically, duality implies that topological string amplitudes in both theories should obey the same equations, notably the holomorphic anomaly equations, and should not depend on the wrong moduli. In the next chapter, indeed, we will see that the holomorphic anomaly equations (3.23) are similar to that for the closed string, after appropriate shifts of closed string moduli by amounts proportional to the 't Hooft coupling. Conversely, the presence of the new anomalies is correlated with the breakdown of large N duality. For compact Calabi-Yau manifolds, the conifold geometric transition requires homology relations among vanishing cycles [51, 52]. For example, if a single three-cycle of non-trivial homology shrinks and the singularity is blown up, the resulting manifold cannot be Kähler. Thus, the presence of D-branes with non-trivial topological charge implies a topological string theory without closed string dual — and simultaneously the disk one-point functions do not vanish, and so the new anomalies are present.

The derivation of the anomalies may not seem to distinguish between compact and non-compact Calabi-Yau target spaces. In fact, the anomalies need only appear in the compact case, as we demonstrate by example in the next paragraph. Beforehand, note that this agrees with our expectations: D-branes wrapped on cycles in compact Calabi-Yau manifolds and filling spacetime (or perhaps even two directions in spacetime [50]) give an inconsistent setup unless there are sinks for the topological D-brane charges. Simultaneously, these sinks cancel the disk one-point functions, and so the appearance of the new anomalies is correlated with invalid spacetime constructions.

Furthermore, the standard results of the Chern-Simons gauge theory and matrix models as open topological field theories are not affected by the new anomalies. Consider, for example, N D-branes wrapping the S^3 of the space T^*S^3, again giving $\overline{C}_{\bar{a}} \neq 0$. The total space of T^*S^3 is Calabi-Yau and non-compact, with the S^3 radius as the complex structure modulus. It is well known that open topological string theory on this space is the $U(N)$ Chern-Simons theory, which is topological and should be independent of the S^3 radius. To resolve this apparent contradiction, consider embedding T^*S^3 in a compact space containing a second 3-cycle in the same homology class as the base S^3, wrapped by N anti-D-branes. The boundary states of the two stacks combine to give $\overline{C}_{\bar{a}} = 0$, and the new anomalies do not appear. Now take the limit where the second 3-cycle moves infinitely far away from the base S^3 to recover an anomaly-free local Calabi-Yau construction. The point is that in non-compact Calabi-Yau manifolds, the new anomalies can be removed by an appropriate choice of boundary conditions at infinity.

Chapter 4

Solving for amplitudes using Feynman rules

The extended holomorphic anomaly equations of the previous chapter express the anti-holomorphic derivatives of amplitudes at worldsheet genus g and boundary number h in terms of amplitudes with lower genus and/or boundary number, plus insertions corresponding to insertions of marginal closed string states, or equivalently covariant derivatives. Integrating these equations thus allows the recursive solving of amplitudes, up to a holomorphic function (integration constant) at each order. This chapter will develop and prove a technique for doing so, using a set of rules which take the form of Feynman rules. Note that we assume the vanishing of disk one-point functions throughout, so the new anomalies identified in the last chapter are not present — indeed, as they do not admit a recursive structure, the approaches in this chapter *require* their absence.

The Feynman rule approach for solving *closed* string amplitudes was described in [8]. [13] conjectured a Feynman rule approach for the open topological string, that is, in the presence of worldsheet boundaries. The novel material in the following treatment, first reported in [15], is a proof of form of the open string Feynman rules. The key insight is that the open string Feynman rules follow directly from the closed string treatment by shifting the closed string moduli by amounts proportional to the 't Hooft coupling. An interpretation of this result in terms of background independence of the topological string partition function was proposed in [53], to which we will return at the end of this chapter.

In Section 4.1 we formulate the holomorphic anomaly equations for correlation functions, that is, amplitudes containing closed string insertions. Section 4.2 describes the Feynman rules, which are then proved for the closed string in Section 4.3, and for the open string in Section 4.4. Section 4.4 concludes with some discussion on the implications of the simple relationship between the proofs of the open and closed string rules.

4.1 Holomorphic anomaly equations of correlation functions

In order to iterate the holomorphic anomaly equation (3.23), we need a holomorphic anomaly equation for amplitudes of genus g boundary number h worldsheets, with n marked points (corresponding to covariant derivatives or marginal closed string states), denoted $F^{(g,h)}_{i_1\cdots i_n}$. This requires two modifications to the holomorphic anomaly equation: firstly, when a Riemann surface splits after degenerating, the marked points are restricted to one of the daughter surfaces — requiring a sum over possible assignments of marked points to daughter surfaces. The first term of (3.23), which expresses the worldsheet-splitting degenerations, can thus be written

$$\frac{1}{2}\sum_{r=0}^{g}\sum_{s=0}^{h}\overline{C}_{\bar{i}}^{jk}\sum_{p,\sigma}\frac{1}{p!(n-p)!}F^{(r,s)}_{ji_{\sigma(1)}\cdots i_{\sigma(p)}}F^{(g-r,h-s)}_{ki_{\sigma(s+1)}\cdots i_{\sigma(n)}}, \tag{4.1}$$

where σ is a permutation of the n marked points, and the factorials correct for overcounting due to permutations within each daughter surface.

Secondly, a new class of moduli space boundaries exist in addition to those listed in Tables 3.1 and 3.2, corresponding to deforming the contour integral of $(G^+ - \overline{G}^+)$ past a marked point. Recall that the marked points are insertions of the form $\phi_i^{(2)} = \{G^-, [\overline{G}^-, \phi_i]\}$, so by the standard arguments the supercurrent commutators give a total derivative. The boundary of the moduli space is the collision of the remaining operators, $\phi_i^{(1)}$ and $\bar{\phi}_{\bar{i}}^{(1)}$, corresponding to the marked point and the deformation insertion (3.11), respectively. The resulting contribution was found in [8],

$$-(2g-2+h+n-1)\sum_{s=1}^{n}G_{\bar{i}i_s}F^{(g,h)}_{i_1\cdots i_{s-1}i_{s+1}\cdots i_n}. \tag{4.2}$$

Using (4.1) and (4.2), the holomorphic anomaly equation for correlation functions is thus

$$\begin{aligned}\frac{\partial}{\partial \bar{t}^{\bar{i}}} F^{(g,h)}_{i_1\cdots i_n} &= \frac{1}{2}\sum_{r=0}^{g}\sum_{s=0}^{h}\overline{C}^{jk}_{\bar{i}}\sum_{p,\sigma}\frac{1}{p!(n-p)!}F^{(r,s)}_{ji_{\sigma(1)}\cdots i_{\sigma(p)}}F^{(g-r,h-s)}_{ki_{\sigma(s+1)}\cdots i_{\sigma(n)}}+\frac{1}{2}\overline{C}^{jk}_{\bar{i}}F^{(g-1,h)}_{jki_1\cdots i_n}\\ &\quad -\Delta^{j}_{\bar{i}}F^{(g,h-1)}_{ji_1\cdots i_n}-(2g-2+h+n-1)\sum_{p=1}^{n}G_{i_p\bar{i}}F^{(g,h)}_{i_1\cdots i_{p-1},i_{p+1}\cdots i_n}. \end{aligned} \tag{4.3}$$

This equation is valid for $(2g-2+h+n)>0$, except for $F^{(1,0)}_i$ and $F^{(0,2)}_i$, which we write separately below. Note that by definition $C_{ijk}=F^{(0,0)}_{ijk}$.

The holomorphic anomaly equation for the torus with one marked point, $F^{(1,0)}_i$, is [45]

$$\frac{\partial}{\partial \bar{t}^{\bar{i}}} F^{(1,0)}_j = \frac{1}{2}\overline{C}^{kl}_{\bar{i}}C_{jkl}-\left(\frac{\chi}{24}-1\right)G_{\bar{i}j}. \tag{4.4}$$

The first term is the expected contribution from (4.3), due to the handle degenerating. The second part of the second term, $G_{\bar{i}j}=\overline{C}^{k0}_{\bar{i}}C_{jk0}$, is due to the unique state of charge $(0,0)$ propagating in the degenerating handle. Normally, as discussed in Section 3.2.2, the zero charge state is annihilated by the supercurrent insertions G^- and $\overline{G}^-$ integrated around the handle endpoints, but the sphere three-point function has no additional moduli and hence no such supercurrent insertions, so this state is not removed. The final contribution, $-\frac{\chi}{24}G_{\bar{i}j}$, is a contact term resulting from collision of the holomorphic and anti-holomorphic insertions, with χ the Euler characteristic of the Calabi-Yau manifold target space.

The annulus with one marked point was considered in [13], giving

$$\frac{\partial}{\partial \bar{t}^{\bar{i}}} F^{(1,0)}_j = -\Delta_{jk}\Delta^{k}_{\bar{i}}+\frac{N}{2}G_{\bar{i}j}. \tag{4.5}$$

The first term is expected from (4.3), and corresponds to the annulus pinching off with one insertion point near each boundary. The annulus pinching off with both insertion points near the same boundary leaves a disk one-point function for the other boundary, which gives an anomaly of the sort discussed in the previous chapter, which we neglect by assuming vanishing disk one-point functions. The last term in (4.5) is due to factorisation in the open string channel. The boundaries colliding produce a thin strip, but the annulus has no remaining moduli to provide supercurrents integrated around the endpoints of the thin strip, so the arguments of Section 3.2.3 are insufficient to eliminate charge 0 open string states. N is the number of such states.

4.2 The Feynman rules

The holomorphic anomaly equations for the correlation function, (4.3), can in principle be integrated as-is to solve for the correlation function to all genus and hole number, up to the *holomorphic ambiguity*, namely a holomorphic function at each genus and hole number. In practice, however, it would be useful to have an algorithm that automates the integration of the holomorphic anomaly equations. In this section we describe such an algorithm, which gives $F_{g,h}$ by summing Feynman diagrams, with propagators, terminators (sources) and vertices depending on the properties of the target space. This approach was developed for the closed string in [8], with open string extension proposed in [13] and proven in [15].

The starting point for integrating the holomorphic anomaly equation is the Yukawa coupling, $\overline{C}_{\bar{i}\bar{j}\bar{k}} \in \mathrm{Sym}^3 T \otimes \mathcal{L}^{-2}$, where $\mathcal{L}$ is the line bundle corresponding to rescaling of Ω, identified in Section 1.1[1]. The Yukawa coupling is symmetric in its indices, and satisfies a tt^* equation [8],

$$D_{\bar{i}}\overline{C}_{\bar{j}\bar{k}\bar{l}} = D_{\bar{j}}\overline{C}_{\bar{i}\bar{k}\bar{l}}.$$

It can be integrated locally to give

$$C_{\bar{i}\bar{j}\bar{k}} = e^{-2K} D_{\bar{i}} D_{\bar{j}} \partial_{\bar{k}} S, \tag{4.6}$$

where S is a local section of $\mathcal{L}^2$, and derivatives are with respect to the moduli $\bar{t}^{\bar{i}}$. From S we define,

$$\begin{aligned} S_{\bar{i}} &= \bar{\partial}_{\bar{i}} S, & S^j_{\bar{i}} &= \bar{\partial}_{\bar{i}} S^j, \\ S^j &= G^{j\bar{j}} S_{\bar{j}}, & S^{jk} &= G^{j\bar{j}} G^{k\bar{k}} S_{\bar{j}\bar{k}}, \end{aligned} \tag{4.7}$$

[1]Note that $\mathcal{L}$ is the dual bundle to the $\mathcal{L}$ of [8].

which gives, for example,

$$\overline{C}_{\bar{i}}^{jk} = \overline{C}_{\bar{i}\bar{j}\bar{k}} e^{2K} G^{j\bar{j}} G^{k\bar{k}} = \bar{\partial}_{\bar{i}} S^{jk}. \tag{4.8}$$

On the open string side, since $D_{\bar{i}} \Delta_{\bar{j}\bar{k}} \in \mathrm{Sym}^3 T \otimes \mathcal{L}^{-1}$ is symmetric in all three indices, it can be locally integrated to give

$$\Delta_{\bar{i}\bar{j}} = e^{-K} D_{\bar{i}} D_{\bar{j}} \Delta, \tag{4.9}$$

with $\Delta \in \mathcal{L}$. Using this,

$$\begin{aligned} \Delta_{\bar{i}}^{j} &= e^{K} G^{\bar{j}j} \Delta_{\bar{i}\bar{j}} = \bar{\partial}_{\bar{i}} G^{\bar{k}j} \bar{\partial}_{\bar{k}} \Delta = \bar{\partial}_{\bar{i}} \Delta^{j}, \\ \Delta^{j} &= G^{\bar{j}j} \bar{\partial}_{\bar{j}} \Delta. \end{aligned} \tag{4.10}$$

Using (4.8) and (4.10) in the anomaly equations (4.3), and then integrating the right-hand side by parts, yields a right-hand side which is the sum of a total anti-holomorphic derivative and terms with anti-holomorphic derivatives acting on correlation functions with lower $(2g-2+h+n)$. The same process can now be applied on the latter terms, and the final result after repeated iteration is a total anti-holomorphic derivative on the right-hand side. See [8, 13] for further details and examples; as a sample the result for $F^{(1,1)}$ is [13]

$$F^{(1,1)} = \frac{1}{2} S^{jk} \Delta_{jk} - F_j^{(1,0)} \Delta^j + \frac{1}{2} C_{jkl} S^{kl} \Delta^j - \left(\frac{\chi}{24} - 1 \right) \Delta + f(t), \tag{4.11}$$

where $f(t)$ is the holomorphic ambiguity. Thus result can be interpreted as a sum of Feynman diagrams, as shown in Figure 4.1, with S, S^j and S^{ij} as propagators, Δ and Δ^j as terminators (tadpoles), and $F^{(g,h)}_{i_1 \cdots i_n}$ as loop-corrected vertices. The interpretation of the term involving Δ follows from (4.4), and will be made more exact below.

Let us make the statement more exact (with proof in the next section). The dotted lines in Figure 4.1, corresponding to the presence of S, S^j, Δ or Δ^j, can be interpreted as the dilaton propagating [8], as opposed to the solid lines which represent marginal (c, c) fields.

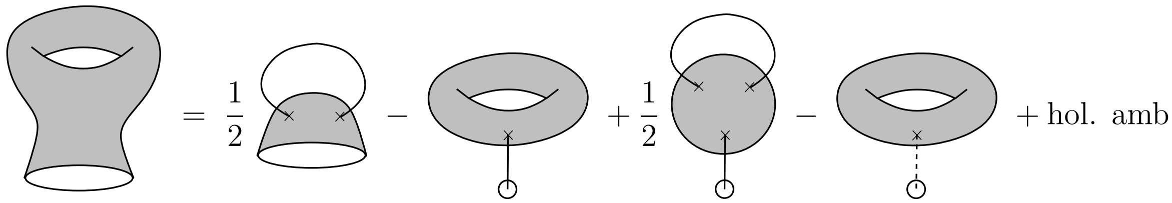


Figure 4.1: *Feynman diagram expansion of $F^{(1,1)}$. Compare to equation (4.11). The dotted line indicates an absent index, or propagating dilaton.*

To formalise this, define new propagators and terminators,

$$K^{ij} = -S^{ij}, \qquad K^{i\varphi} = -S^{i}, \qquad K^{\varphi\varphi} = -2S,$$
$$T^{i} = -\Delta^{i}, \qquad T^{\varphi} = \Delta. \tag{4.12}$$

Vertices are given by

$$\begin{aligned}
\tilde{F}^{(g,h)}_{i_1\cdots i_n} &= F^{(g,h)}_{i_1\cdots i_n}, & \tilde{F}^{(g,h)}_{i_1\cdots i_n,\varphi^m} &= 0 \text{ for } (2g-2+h+n+m) \le 0,\\
\tilde{F}^{(0,2)}_{\varphi} &= \frac{N}{2}, & \tilde{F}^{(1,0)}_{\varphi} &= \frac{\chi}{24} - 1,\\
\tilde{F}^{(0,1)}_{ij} &= \Delta_{ij}, & \tilde{F}^{(0,0)}_{ijk} &= C_{ijk},\\
\tilde{F}^{(g,h)}_{i_1\cdots i_n,\varphi^{m+1}} &= (2g-2+h+n+m)\tilde{F}^{(g,h)}_{i_1\cdots i_n,\varphi^m} \quad \text{otherwise.}
\end{aligned} \tag{4.13}$$

For the amplitude $F^{(g,h)}$, the diagrams that contribute are all those such that g equals the number of loops in the Feynman diagram plus the sum of the genera of all vertices; and h equals the number of terminators plus the sum of the boundaries in all (non-terminator) vertices. Then using the standard Feynman rules, including symmetry factors, produces integrated expressions for topological string amplitudes, as we will prove in the next section. For example, (4.11) can be written

$$F^{(1,1)} = \frac{1}{2}\tilde{F}^{(0,1)}_{jk}K^{jk} + \tilde{F}^{(1,0)}_{j}T^{i} + \frac{1}{2}\tilde{F}^{(0,0)}_{jkl}K^{kl}T^{j} + \tilde{F}^{(1,0)}_{\varphi}T^{\varphi} + f(t),$$

with all other terms zero.

To use these formulae, explicit expressions for the propagators and terminators are required. The former follows from special geometry [8], specifically the relation

$$\bar{\partial}_{\bar{i}}\left(S^{jk}C_{klm}\right) = \bar{\partial}_{\bar{i}}\left(\partial_l K \delta^j_m + \partial_m K \delta^j_l + \Gamma^j_{lm}\right), \tag{4.14}$$

where K is the Kähler form and Γ^j_{lm} the Christoffel connection. Integrating this and inverting C_{klm} gives an expression for S^{jk}, and S^j and S follow from integrating (4.7). The terminators follow from the holomorphic anomaly of the disk amplitude [13],

$$\bar{\partial}_{\bar{i}}\Delta_{jk} = -C_{jkl}\Delta^l_{\bar{i}}, \tag{4.15}$$

which can be integrated as C_{jkl} is pure holomorphic. Inverting C_{jkl} gives an expression for Δ^j in terms of the disk amplitude Δ_{jk}, and integrating (4.10) gives Δ. Thus the propagators can be calculated in terms of the geometry of the Calabi-Yau moduli space and the terminators depend additionally on the boundary conditions, up to a holomorphic ambiguity in both cases.

4.3 Proof of closed string Feynman rules

The statements in the previous section are most easily proved for the closed string case first, as the open string case follows by a shift of variables. Following [8], define the generating function for the holomorphic anomalies of $F^{(g)}_{i_1\cdots i_n}$,

$$\widehat{W}(x,\varphi;t,\bar{t}) = \sum_{g,n} \frac{1}{n!}\lambda^{2g-2}F^{(g)}_{i_1\cdots i_n}x^{i_1}\cdots x^{i_n}\left(\frac{1}{1-\varphi}\right)^{2g-2+n} + \left(\frac{\chi}{24}-1\right)\log\left(\frac{1}{1-\varphi}\right), \tag{4.16}$$

where the sum is over $g,n \geq 0$ such that $(2g-2+n) > 0$ and λ is the topological string coupling constant. $\widehat{W}$ satisfies

$$\frac{\partial}{\partial \bar{t}^{\bar{i}}} e^{\widehat{W}(x,\varphi;t,\bar{t})} = \left(\frac{\lambda^2}{2}\overline{C}^{jk}_{\bar{i}}\frac{\partial^2}{\partial x^j \partial x^k} - G_{\bar{i}j}x^j\frac{\partial}{\partial\varphi}\right)e^{\widehat{W}(x,\varphi;t,\bar{t})}, \tag{4.17}$$

with expansion order-by-order in λ, x and φ yielding (4.3) and (4.4) (with $h = 0$).

There exists another function of x^i and φ which satisfies almost the same equation as

(4.17), namely

$$\widehat{Y}(x,\varphi;t,\bar{t}) = -\frac{1}{2\lambda^2}\left(\Theta_{ij}x^i x^j + 2\Theta_{i\varphi}x^i\varphi + \Theta_{\varphi\varphi}\varphi^2\right) + \frac{1}{2}\log\left(\frac{\det\Theta}{\lambda^2}\right). \tag{4.18}$$

Here Θ (denoted Δ in [8]) is the inverse of the propagator K defined in (4.12),

$$K^{IJ}\Theta_{JK} = \delta^I_K, \tag{4.19}$$

where capital indices are the corresponding lower case indices plus φ. From (4.7) it can be shown that

$$\frac{\partial}{\partial \bar{t}^{\bar{i}}} e^{\widehat{Y}(x,\varphi;t,\bar{t})} = \left(-\frac{\lambda^2}{2}\overline{C}_{\bar{i}}^{jk}\frac{\partial^2}{\partial x^j \partial x^k} - G_{\bar{i}j}x^j\frac{\partial}{\partial\varphi}\right)e^{\widehat{Y}(x,\varphi;t,\bar{t})}. \tag{4.20}$$

Now consider the integral

$$Z = \int dx d\varphi\, \exp(\widehat{Y} + \widehat{W}). \tag{4.21}$$

Treating x and φ as dynamical variables, the expansion of Z can be evaluated as a perturbation expansion in λ, using the the usual Feynman rules. The result is the Feynman diagram solutions for $F^{(g)}$ described in the previous section. For example, $F^{(2)}$ is given (up to holomorphic ambiguity) by the order λ^2 terms in the expansion of Z, the first few of which are

$$F^{(2)} - \frac{1}{2}S^{ij}F^{(1)}_{ij} - \frac{1}{2}F^{(1)}_i S^{ij} F^{(1)}_i - \frac{1}{8}S^{jk}S^{mn}F^{(0)}_{jkmn} + \cdots .$$

$F^{(3)}$ is given by the λ^4 terms, and so forth. Formulating it this way, however, provides a means of proof for all g. Using (4.17) and (4.20),

$$\begin{aligned}\frac{\partial}{\partial \bar{t}^{\bar{i}}}Z &= \int dx d\varphi\left(e^{\widehat{Y}}\frac{\partial}{\partial \bar{t}^{\bar{i}}}e^{\widehat{W}} + e^{\widehat{W}}\frac{\partial}{\partial \bar{t}^{\bar{i}}}e^{\widehat{Y}}\right)\\ &= \int dx d\varphi\left(\frac{\lambda^2}{2}\overline{C}_{\bar{i}}^{jk}\left[\frac{\partial}{\partial x^j}\left(e^{\widehat{Y}}\frac{\partial}{\partial x^k}e^{\widehat{W}}\right) - \frac{\partial}{\partial x^j}\left(e^{\widehat{W}}\frac{\partial}{\partial x^k}e^{\widehat{Y}}\right)\right] - G_{\bar{i}j}\frac{\partial}{\partial\varphi}\left[e^{\widehat{Y}+\widehat{W}}\right]\right).\end{aligned}$$

The right-hand side of this equation is a total derivative with respect to x^i and φ. All the integrals are Gaussian, and integration by parts gives

$$\frac{\partial}{\partial \bar{t}^{\bar{i}}} Z = 0. \tag{4.22}$$

This proves that $\frac{\partial}{\partial \bar{t}^{\bar{i}}} F^{(g)}$ is the anti-holomorphic derivative of all the other terms in the Feynman rule expansion of Z at order λ^{2g-2} — and thus that the Feynman rules give $F^{(g)}$ up to an arbitrary holomorphic function, the holomorphic ambiguity.

4.4 Extension to open strings

Following the approach of the previous section, and as reported in [15], define the generating function for open topological string amplitudes,

$$\begin{aligned} W(x, \varphi; t, \bar{t}) = \sum_{g,h,n} \frac{1}{n!} \lambda^{2g-2} \mu^h F^{(g,h)}_{i_1 \cdots i_n} x^{i_1} \cdots x^{i_n} \left(\frac{1}{1-\varphi} \right)^{2g-2+h+n} \\ + \left(\frac{\chi}{24} - 1 - \frac{N}{2} \lambda^{-2} \mu^2 \right) \log \left(\frac{1}{1-\varphi} \right), \end{aligned} \tag{4.23}$$

where the sum is over $g, h, n \geq 0$ such that $(2g - 2 + h + n) > 0$ and μ is the 't Hooft coupling constant, namely λ times the topological string Chan-Paton factor. The last term contributes to the torus and annulus holomorphic anomalies, (4.4) and (4.5), respectively. The generating function W satisfies an extension of (4.17) by a μ-dependent term, namely,

$$\frac{\partial}{\partial \bar{t}^{\bar{i}}} e^{W(x,\varphi;t,\bar{t})} = \left(\frac{\lambda^2}{2} \overline{C}_{\bar{i}}^{jk} \frac{\partial^2}{\partial x^j \partial x^k} - G_{\bar{i}j} x^j \frac{\partial}{\partial \varphi} - \mu \Delta_{\bar{i}}^j \frac{\partial}{\partial x^j} \right) e^{W(x,\varphi;t,\bar{t})}, \tag{4.24}$$

which reproduces the open topological string holomorphic anomaly equation (4.3) for each genus and boundary number.

The key result is that equation (4.24) can be rewritten in the same form as the closed topological string analogue, equation (4.17), by shifting

$$x^i \to x^i + \mu \Delta^i, \quad \varphi \to \varphi + \mu \Delta. \tag{4.25}$$

After this shift equation (4.24) becomes

$$\frac{\partial}{\partial \bar{t}^{\bar{i}}} e^{W(x+\mu\Delta,\varphi+\mu\Delta;t,\bar{t})} = \left(\frac{\lambda^2}{2} \overline{C}_{\bar{i}}^{jk} \frac{\partial^2}{\partial x^j \partial x^k} - G_{\bar{i}j} x^j \frac{\partial}{\partial \varphi} \right) e^{W(x+\mu\Delta,\varphi+\mu\Delta;t,\bar{t})}, \tag{4.26}$$

which is exactly (4.17). This result follows from a straightforward application of the chain rule: recalling that $\bar{\partial}_{\bar{i}} \Delta^j = \Delta^j_{\bar{i}}$, the shift (4.25) produces two new terms on the left,

$$\left(\mu \Delta^j_{\bar{i}} \frac{\partial}{\partial x^j} + \mu \Delta_{\bar{i}} \frac{\partial}{\partial \varphi} \right) e^W. \tag{4.27}$$

The first is the additional μ-dependent term on the right of (4.24). Using $G_{\bar{i}j}\Delta^j = \Delta_{\bar{i}}$, the second term combines with the second term on the right of (4.26) to give $-G_{\bar{i}j}(x^j + \mu\Delta^j)\frac{\partial}{\partial\varphi}e^W$, which is required for matching powers of $x + \mu\Delta$ in the expansion of the generating function. We have thus reproduced the open topological string holomorphic anomaly equations from the closed topological string holomorphic anomaly equations, simply by a shift of variables.

A proof of the open string Feynman rules follows immediately. Since the shifted W satisfies the closed string differential equation (4.17), the proof of the closed string Feynman rules presented in Section 4.3 applies here too. The shift has indeed an elegant interpretation in terms of the Feynman rules — in field theory language, the shift effectively generates the vacuum expectation values $\langle x^i \rangle = \Delta^i$ and $\langle \varphi \rangle = \Delta$, and so terms containing Δ^i and Δ correspond to diagrams with tadpoles, as we saw in Figure 4.1.

The shift we use above is, strictly speaking, a shift of the variables x and φ, rather than the closed string moduli t and λ. However, the two sets of variables are simply related. Firstly, from equation (4.23) one can see that a shift of φ is equivalent to a re-scaling of λ and x. Secondly, equation (3.18) in [8] is

$$\left[\frac{\partial}{\partial t^i} + \Gamma^k_{ij} x^j \frac{\partial}{\partial x^k} + \frac{\partial K}{\partial t^i} \left(\frac{\chi}{24} - 1 - \lambda \frac{\partial}{\partial \lambda} \right) \right] e^{\widehat{W}} = \left(\frac{\partial}{\partial x^i} - \frac{\partial F_1}{\partial t^i} - \frac{1}{2\lambda^2} C_{ijk} x^j x^k \right) e^{\widehat{W}}. \tag{4.28}$$

This equation encodes the fact that

$$F^{(g)}_{i_1 \cdots i_n} = D_{i_1} \cdots D_{i_n} F^{(g)},$$

as well as the low genus result (4.4). We can now adopt Kähler normal coordinates. As explained in Section 2.6 of [8], we can choose coordinates of the closed string moduli space and a section of the vacuum line bundle so that, at a given point $(t_0, \bar{t}_0)$,

$$\partial_{i_1} \cdots \partial_{i_n} \Gamma^k_{ij} = 0, \qquad \partial_{i_1} \cdots \partial_{i_n} K = 0. \tag{4.29}$$

This removes all but the first term on the left of equation (4.28). On the right of equation (4.28), the second and third terms contribute at low genus only, and can be absorbed by redefining the sum in equation (4.23) to have only the restrictions $g \geq 1$ and $n \geq 0$. With these choices,

$$\frac{\partial}{\partial t^i} \widehat{W} = \frac{\partial}{\partial x^i} \widehat{W}, \tag{4.30}$$

that is, $\widehat{W} = \widehat{W}(t + x; \bar{t})$.

An elegant and computationally more efficient reformulation of the closed string Feynman rules for calculating topological string amplitudes was provided by [54]. Subsequent work by [55, 56] has provided a similar reformulation of the above open string Feynman rules.

Neitzke and Walcher [53] noted that the shift (4.25) for W does not give a generating function which satisfies equation (4.28). This fact does not affect the proof of the open string Feynman rules. However, they provide a slightly different shift, and absorb a factor appearing at genus 1 into the topological string partition function, to give a shifted W which satisfies both (4.17) and (4.28). This is then interpreted as the statement that the open string partition functions are boundary-condition dependent states in the Hilbert space of Witten's picture of background independence of the topological string, as discussed in Section 2.6. Recall that the closed topological string partition function was identified as a single wavefunction on the phase space $H^3(X)$. The open topological string partition function is thus a different wavefunction on this phase space for each choice of worldsheet boundary conditions (encoded in the form of the terminator Δ).

It would be interesting to develop a better understanding of the holomorphic ambiguities of the open string amplitudes, along the lines of the powerful techniques that are now available [10] for the closed string. Furthermore, the open-closed relationship demonstrated by the shift (4.25) is reminiscent of large N duality, where the background is shifted by an amount proportional to the 't Hooft coupling. It would be interesting to explore the implications of this for the Gromov-Witten and Gopakumar-Vafa topological invariants.

Appendix A

Chiral non-perturbative OSV at other genus

This appendix supplements Section 2.7. We derive results of use in realising the OSV conjecture on the non-compact genus G Calabi-Yau manifold X defined by equation (2.50).

Useful in these derivations will be the *Schur functions* $s_R(x)$, which are completely symmetric functions (polynomials) of n (possibly infinite) variables x. The function is determined by the row lengths of the Young diagram R. For details refer to, e.g., [57]; properties we will use include

$$\prod_{i,j}(1 - Qx_iy_j) = \sum_P (-1)^{|P|} Q^{|P|} s_P(x) s_{P^t}(y)$$
$$\prod_{i,j} \frac{1}{(1 - Qx_iy_j)} = \sum_P Q^{|P|} s_P(x) s_P(y)$$
$$d_q(\mathcal{R}) = q^{\kappa_{\mathcal{R}}/4} s_{\mathcal{R}}(q^{j-\frac{1}{2}}) = (-1)^{|\mathcal{R}|} q^{-\kappa_{\mathcal{R}}/4} s_{\mathcal{R}}(q^{\frac{1}{2}-j}), \tag{A.1}$$

where if the indices i, j run to infinity, then the representation P is a $U(\infty)$ representation; else it is a representation of $SU(k)$, where $k = \min(\max(i), \max(j))$. Q is an arbitrary factor, and the superscript in P^t denotes the transposed Young diagram (which has arbitrary column lengths, and so is a $U(\infty)$ representation). In the last equation, $q^{j-\frac{1}{2}}$ denotes the set of variables generated by the range of the index j.

A.1 Decomposition of the black hole partition function

Our first result is the decomposition of the black hole partition function (2.54). We start by decomposing the (finite N) quantum dimension,

$$\dim_q(R) = \prod_{1\le i<j\le N} \frac{[R_i - R_j + j - i]_q}{[j-i]_q}$$
$$= \dim_q^{(\frac{N}{2}+l)}(R^+)\dim_q^{(\frac{N}{2}-l)}(R^-)\left(\prod_{i=1}^{\frac{N}{2}+l}\prod_{j=1}^{\frac{N}{2}-l}\frac{[R_i^+ + R_j^- + N + 1 - j - i]_q}{[N+1-j-i]_q}\right), \quad \text{(A.2)}$$

where the notation $\dim_q^{(k)}(R)$ means that R should be treated as a Young diagram of a $U(k)$ representation, and so $d_q(R) = \dim_q^{(\infty)}(R)$. Recalling that the row lengths of R^+ and R^- are non-negative, the last term in (A.2) can be rewritten

$$\left(\prod_{i,j=1}^{\infty}\frac{[R_i^+ + R_j^- + N + 1 - j - i]_q}{[N+1-j-i]_q}\right)\left(\prod_{i=1}^{\frac{N}{2}+l}\prod_{j=1}^{R_i^+}\frac{1}{[\frac{N}{2}+l+j-i]_q}\right)\left(\prod_{j=1}^{\frac{N}{2}-l}\prod_{i=1}^{R_j^-}\frac{1}{[\frac{N}{2}-l+i-j]_q}\right),$$

where the last two factors required the use of "telescoping factors" — all other values of the indices give cancelling contributions between numerator and denominator. These last two factors are, however, exactly the ratios $d_q(R^+)/\dim_q^{(\frac{N}{2}+l)}(R^+)$ and $d_q(R^-)/\dim_q^{(\frac{N}{2}-l)}(R^-)$. Recalling that $[x]_q = q^{x/2} - q^{-x/2}$, the first factor in the above expression can be rewritten using the first of the identities (A.1) as

$$\prod_{i,j=1}^{\infty} q^{-(R_i^+ + R_j^-)/2}\left(\frac{q^{R_i^+ + R_j^- + N + 1 - j - i} - 1}{q^{N+1-j-i} - 1}\right)$$
$$= q^{-\frac{N}{2}(|R^+|+|R^-|)}\sum_P (-1)^{|P|} q^{N|P|} s_P(q^{R_i^+ + \frac{1}{2} - i}) s_{P^t}(q^{R_j^- + \frac{1}{2} - j}) \prod_{i,j=1}^{\infty}\left(\frac{1}{1-q^{N+1-j-i}}\right). \quad \text{(A.3)}$$

Finally, we have the identity [3]

$$q^{\frac{N^3}{12} - \frac{N}{24}} S_{00} = M(q)\eta^N(q)\prod_{i,j=1}^{\infty}\left(1 - q^{N+1-j-i}\right), \quad \text{(A.4)}$$

where $M(q)$ is the McMahon function and $\eta(q)$ the Dedekind eta function. Putting this together, and recalling $S_{0R} = S_{00}\dim_q(R)$,

$$q^{\frac{N^3}{12}-\frac{N}{24}} S_{0R} = M(q)\,\eta(q)\,d_q(R^+)\,d_q(R^-)\,(-1)^{|R^+|+|R^-|} q^{-\frac{N}{2}(|R^+|+|R^-|)} q^{-\frac{1}{4}(\kappa_{R^+}+\kappa_{R^-})} \times \sum_P (-1)^{|P|} q^{N|P|} s_P(q^{R_i^+ + \frac{1}{2}-i}) s_{P^t}(q^{R_j^- + \frac{1}{2}-j}). \quad \text{(A.5)}$$

This result is useful for $G = 0$. For $G \geq 2$, we can use the second of the identities (A.1) instead of the first in equation (A.3), giving

$$q^{\frac{N^3}{12}-\frac{N}{24}} S_{0R} = M(q)\,\eta(q)\,d_q(R^+)\,d_q(R^-)\,(-1)^{|R^+|+|R^-|} q^{-\frac{N}{2}(|R^+|+|R^-|)} q^{-\frac{1}{4}(\kappa_{R^+}+\kappa_{R^-})} \times \sum_P q^{-N|P|} \left(s_P(q^{R_i^+ + \frac{1}{2}-i}) s_P(q^{R_j^- + \frac{1}{2}-j}) \right)^{-1}. \quad \text{(A.6)}$$

We can now decompose the black hole partition function (2.54) into topological string partition functions. The approach is that of Section 2.3.4, though the additional prefactors for $G \neq 1$ require modifying the result (2.29). It is now

$$q^{\frac{(m+2G-2)^2}{2m}\left(\frac{N^3}{12}-\frac{N}{12}\right)} e^{\frac{N\theta^2}{2m\lambda}} = \exp\left(-\frac{t^3+\bar{t}^3}{6m(m+2G-2)\lambda^2} + \frac{(m+2G-2)(t+\bar{t})}{24m} \right),$$
$$q^{\frac{m}{2}Nl^2} e^{i\theta N l} = \exp\left(-\frac{(t^2-\bar{t}^2)l}{2(m+2G-2)\lambda} + -\frac{m(t+\bar{t})l^2}{2(m+2G-2)} \right),$$

where the second line can be absorbed into the first by taking $t \to t + m\lambda l$ and $\bar{t} \to \bar{t} - m\lambda l$. Putting it all together,

$$Z_{\rm BH}(\Sigma_g) = \sum_{l=-N/2}^{N/2} \sum_{P_1,\ldots,P_{|2-2G|}} \Psi^{N/2+l}_{P_1,\ldots,P_{|2-2G|}}(t+m\lambda l) \overline{\Psi}^{N/2-l}_{P_1,\ldots,P_{|2-2G|}}(\bar{t}-m\lambda l), \quad \text{(A.7)}$$

where the P_i are $U(\infty)$ representations, and the non-perturbative completion of the topological string partition function is

$$\Psi^k_{P_1,\dots,P_{|2G-2|}}(t) = \widehat{Z}_0(q,t)\exp\left(-\frac{t(|P_1|+\cdots+|P_{|2G-2|}|)}{m+2G-2}\right)$$
$$\times \sum_{R \text{ of } SU(k)} \left(\frac{1}{d_q(R)}\right)^{2G-2} q^{(m+G-1)k_R/2} e^{-t|R|} \prod_{n=1}^{|2G-2|} s_{P_n}(q^{R_i+\frac{1}{2}-i}), \quad \text{(A.8)}$$

where $\widehat{Z}_0(q,t) = Z_0(q,t)\eta^{t(2G-2)/(m+2G-2)\lambda}$, $Z_0(q,t)$ defined in equation (2.51). As noted in [3], the eta function contributes only at genus zero and perturbatively, hence we need not worry that it did not appear in the perturbative Z_{top}. The anti-holomorphic partitions function $\overline{\Psi}$ is also given by (A.8), *except* for the $G=0$ case, where the representations are transposed:

$$\overline{\Psi}^k_{P_1,P_2}(\bar{t}) = (-1)^{|P_1|+|P_2|}\Psi^k_{P_1^t,P_2^t}(t). \quad \text{(A.9)}$$

A.2 The chiral recursion relation

We now wish to express (A.8) in terms of the perturbative topological string partition function, (2.51), as discussed in Section 2.7.2 and illustrated in Figure 2.8. Starting with the quantum dimension of a $U(\infty)$ representation $\mathcal{R}$,

$$\begin{aligned} d_q(\mathcal{R}) &= \prod_{\square\in\mathcal{R}} \frac{1}{[h(\square)]_q} \\ &= d_q(R^1)d_q(R^2)\prod_{i=1}^{k+r}\prod_{j=1}^{r} \frac{1}{[R^1_i+r-j+R^2_j+k+r-i+1]_q} \\ &= d_q(R^1)d_q(R^2)(-1)^{r(r+k)}q^{\frac{1}{2}(r|R^1|+(k+r)|R^2|)}q^{\frac{1}{4}r(k^2+3kr+2r^2)} \qquad \text{(A.10)} \\ &\qquad \times \sum_{S \text{ of } U(r)} q^{(2r+k)|S|}\tilde{s}_S(q^{R^1_i+\frac{1}{2}-i})\tilde{s}_S(q^{R^2_i+\frac{1}{2}-i}), \qquad \text{(A.11)} \end{aligned}$$

where in the last line, the tilde on the Schur functions $\tilde{s}_S(q^{R^1_i+\frac{1}{2}-i})$ denotes that there are only finitely many variables for the function, since the indices on the product above range only over finite values.

The second result is for the interaction with the ghost brane coupling the chiral and anti-chiral Fermi surfaces, $s_P(q^{\mathcal{R}_i-i+\frac{1}{2}})$, where i runs over all positive integers. The set of variables $x_i = q^{\mathcal{R}_i-i+\frac{1}{2}}$ can be split up as

$$q^{\mathcal{R}_i-i+\frac{1}{2}} = \begin{cases} q^{R^1_i+r-i+\frac{1}{2}} & i \leq k+r \\ q^{(R^2)^t_{i-k-r}-i+\frac{1}{2}} & i > k+r. \end{cases}$$

We can now use an identity [57] for Schur functions of disjoint sets of variables,

$$\begin{aligned} s_P(q^{\mathcal{R}_i-i+\frac{1}{2}}) &= \sum_{P_a,P_b}{}' N^P_{P_aP_b} s_{P_a}(q^{(R^2)^t_i-i-k-r+\frac{1}{2}}) s_{P_b}(q^{R^1_i+r-i+\frac{1}{2}}) \\ &= \sum_{P_a,P_b} N^P_{P_aP_b} q^{-(k+r)|P_a|+r|P_b|} s_{P_a}(q^{(R^2)^t_i-i+\frac{1}{2}}) s_{P_b}(q^{R^1_i-i+\frac{1}{2}}), \end{aligned}$$

where the sum runs over partitions satisfying $P \supset P_b \supset P_a$, where $P_b \supset P_a$ requires that all the rows of P_b are at least as long as the corresponding rows of P_a, i.e., $P_{b,i} \geq P_{a,i}$ for all i. Furthermore, the Littlewood-Richardson coefficients $N^P_{P_aP_b}$ vanish unless $|P| = |P_a| + |P_b|$.

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