Raman Transitions in Cavity QED

Thesis by

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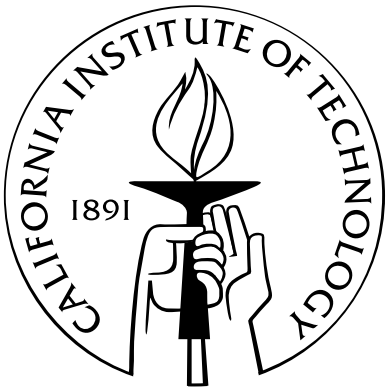

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This thesis is dedicated to all theses that are not dedicated to themselves.

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The cavity QED lab, which was used to perform the experiments presented in this thesis, is the product of several generations of students and postdoctoral fellows. David Vernooy and June Ye built the core of the lab and achieved the crucial first step of trapping atoms inside the cavity. Jason McKeever, Joe Buck, Alex Kuzmich, Christoph Nägerl, and Dan Stamper-Kurn made a series of important advances, transforming the lab into a powerful tool for studying cavity QED. Andreea Boca and Russell Miller worked closely with me on the Raman project, and I owe them a special debt of gratitude. The results presented here would not have been possible without their hard work and dedication.

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Abstract

In order to study quantum effects such as state superposition and entanglement, one would like to construct simple systems for which the damping rates are slow relative to the rate of coherent evolution. One such system is strong-coupling cavity quantum electrodynamics (QED), in which a single atom is coupled to a single mode of a high finesse optical cavity. In recent years, optical trapping techniques have been applied to the cavity QED system, allowing an individual atom to remain coupled to the cavity for long periods of time. For the purpose of future cavity QED experiments, one would like to gain as much control over the trapped atom as possible; in particular, one would like to cool the center of mass motion of the atom, to measure the magnetic field at the location of the atom, and to be able to prepare the atom in a given internal state. In the first part of this thesis, I present a scheme for driving Raman transitions inside the cavity that can be used to achieve these goals. After giving a detailed theoretical treatment of the Raman scheme, I describe how it can be implemented in the lab and discuss some preliminary experimental results.

In the second part of this thesis, I present a number of simple field theory models. These models were developed in an attempt to understand some of the central ideas of theoretical physics by looking at how the ideas work in a highly simplified context. The hope is that by reducing the mathematical complexity of an actual theory, the underlying physical concepts can be more easily understood.

Note on units: this thesis uses natural units ($\hbar = c = 1$).

Contents

Chapter 1

Atomic physics

1.1 Introduction

1.1.1 Cavity QED

Many quantum systems can be described in terms of a simple system with a few degrees of freedom interacting with a complicated environment with many degrees of freedom. If the simple system were isolated it would evolve coherently, but because of the coupling to the environment the coherent evolution is damped at a rate determined by the strength of the coupling. The damping destroys quantum effects such as state superposition and entanglement, so to study these effects one would like to construct systems where damping is slow relative to the rate of coherent evolution.

Two candidates for building such systems are atoms and photons. Atoms can store quantum information for long periods of time in internal states and are easily manipulated with laser beams, while photons are useful for transporting quantum information from one location to another. One system that combines the advantages of both is cavity quantum electrodynamics, in which a single atom is coupled to a resonant mode of an optical cavity. For cavity QED, the internal states of the atom and the Fock states of the quantized cavity mode constitute the degrees of freedom of the system, while the modes of the electromagnetic field to which the atom and cavity couple serve as the environment.

An optical cavity is formed by two mirrors separated by a distance L. The mirrors confine light to a finite region, resulting in a set of modes at integer multiples of the free spectral range $\nu_{FSR} = 1/2L$. Each mode acts like a separate harmonic oscillator, which may be driven by coupling resonant light into the cavity and which is damped at a rate κ determined by the mirror reflectivities. By adjusting the mirror separation L, one of the modes can be tuned into resonance with an atomic transition; for $\nu_{FSR} \gg \gamma$, where γ is the spontaneous decay rate of the atom, the non-resonant modes are so far detuned that they don't interact with the atom and may be neglected. The resonant mode couples

to the atom via the Hamiltonian

$$H_i = -\vec{p} \cdot \vec{E}$$

where $\vec{p}$ is the atomic dipole moment and $\vec{E}$ is the electric field at the position of the atom.

We would like to compare the coherent coupling strength to the damping rates γ and κ. The coherent coupling strength is determined by the magnitude of H_i, which can be estimated as follows. If there are n photons inside the cavity, then the field energy is

$$n\omega = \frac{1}{8\pi} V E^2$$

where ω is the angular frequency of the mode and V is the mode volume. Thus, the electric field is given by

$$E = (8\pi n\omega / V)^{1/2}$$

The spontaneous decay rate of the atom is

$$\gamma = \frac{4}{3}\omega^3 p^2$$

Thus, the atomic dipole moment is given by

$$p = (3\gamma / 4\omega^3)^{1/2}$$

Substituting these results into H_i, we obtain

$$H_i \sim g\sqrt{n}$$

where

$$g = (3/16\pi^2)^{1/2} \, (Q\lambda^3/V)^{1/2} \, \gamma$$

characterizes the strength of the atom/cavity coupling. Here $Q = \omega/\gamma$ is the quality factor of the atom and $\lambda = 2\pi/\omega$. Note that to generate a large coupling we want a cavity with a small mode volume and an atom with a large Q. For our cavity $V/\lambda^3 = 3.6 \times 10^4$ and $\kappa = (2\pi)(8.4 \text{ MHz})$, and for the atomic transition we use $Q = 6.8 \times 10^7$ and $\gamma = (2\pi)(5.2 \text{ MHz})$. Thus, the coherent coupling strength is $g = (2\pi)(32 \text{ MHz})$.

Having estimated the coherent coupling strength, let us now consider the dynamics of the system. For simplicity, we will assume the atom has only two levels: a ground state g and excited state

e. We can define raising and lowering operators for the atom:

$$\begin{aligned} \sigma_+ &= |e\rangle\langle g| \\ \sigma_- &= |g\rangle\langle e| \end{aligned}$$

By substituting for the dipole moment of the atom and for the electric field of the light, one can show that

$$H_i = -\vec{p} \cdot E = g(a\sigma_+ + a^\dagger \sigma_-)$$

where g is the coherent coupling rate we just calculated, and $a^\dagger$ and a are creation and annihilation operators for the cavity mode. If we include terms for the energy of the atom and the field, we find that the total Hamiltonian is

$$H = \omega\sigma_+\sigma_- + \omega a^\dagger a + g(a\sigma_+ + a^\dagger\sigma_-)$$

This is known as the Jaynes-Cummings Hamiltonian [1]. We can form a basis of states for the system by taking tensor products of the atomic states $|g\rangle$, $|e\rangle$ and cavity Fock states $|n\rangle$ to obtain product states $|g, n\rangle, |e, n\rangle$. The ground state is $|g, 0\rangle$, and the two lowest excited eigenstates are

$$|\pm\rangle = \frac{1}{\sqrt{2}}(|e, 0\rangle \pm |g, 1\rangle)$$

with eigenvalues $\omega \pm g$. If the atom is decoupled from the cavity, these states are degenerate, but in the presence of the coupling they are split by $2g$. This splitting of the lowest two excited states is called the vacuum Rabi splitting.

Now suppose we drive the cavity with a laser beam tuned to the atomic resonance. Because the vacuum Rabi splitting shifts the eigenstates of the system out of resonance with the driving field, we expect the presence of an atom to suppress the coupling of light into the cavity. We can calculate this effect as follows. To include the effects of the driving laser, we add a term to the Hamiltonian:

$$H = \omega\sigma_+\sigma_- + \omega a^\dagger a + g(a\sigma_+ + a^\dagger\sigma_-) + \lambda(a + a^\dagger)\cos\omega t$$

where λ gives the driving strength. The Hamiltonian can be simplified by applying some unitary transformations and making the rotating wave approximation:

$$H = g(a\sigma_+ + a^\dagger\sigma_-) + (\lambda/2)(a + a^\dagger)$$

We can obtain an approximate steady state solution by viewing the coupled atom-cavity system as an isolated atom described by an effective Hamiltonian H_a, and an isolated cavity described by an effective Hamiltonian H_c. The Hamiltonian H_a is obtained from H by replacing the operator a with a parameter $\alpha = \langle a \rangle$ characterizing the field amplitude:

$$H_a = g(\alpha\sigma_+ + \alpha^*\sigma_-)$$

Similarly, H_c is obtained from H by replacing the operator σ_- by a parameter $\beta = \langle \sigma_- \rangle$ characterizing the atomic dipole:

$$H_c = g(\beta^* a + \beta a^\dagger) + (\lambda/2)(a + a^\dagger)$$

If we define an effective Rabi frequency $\Omega_E = 2g\alpha$ and an effective driving strength $\lambda_E = 2g\beta + \lambda$, we can express H_a and H_c as

$$\begin{aligned} H_a &= \frac{1}{2}(\Omega_E\,\sigma_+ + \Omega_E^*\,\sigma_-) \\ H_c &= \frac{1}{2}(\lambda_E\,a^\dagger + \lambda_E^*\,a) \end{aligned}$$

For weak driving, the atomic dipole and the field amplitude are given by the ratio of the driving rates (Ω_E and λ_E) to the damping rates (γ and κ):

$$\beta = \langle \sigma_- \rangle = -i\Omega_E/\gamma = -2ig\alpha/\gamma$$

$$\alpha = \langle a \rangle = -i\lambda_E/\kappa = -\alpha/N_A - i\lambda/\kappa$$

where $N_A = \kappa\gamma/4g^2$ is called the critical atom number. The field amplitude is therefore

$$\alpha = -i(\lambda/\kappa)(1 + N_A^{-1})^{-1}$$

and the number of photons inside the cavity is

$$n = |\alpha|^2 = (\lambda/\kappa)^2(1 + N_A^{-1})^{-2} = n_0(1 + N_A^{-1})^{-2}$$

where $n_0 = (\lambda/\kappa)^2$ is the photon number for an empty cavity. Thus, the coupling to the atom reduces the number of photons in the cavity by a factor of $\sim N_A^{-2}$.

In the limit $g \ll \kappa$, we can give a second interpretation to the critical atom number. Suppose we start the system in state $|e, 0\rangle$. The excitation can decay via one of two channels: either the atom spontaneously emits a photon $|e, 0\rangle \to |g, 0\rangle$ (rate γ), or the atom coherently transfers its exci-

tation to the cavity $|e,0\rangle \rightarrow |g,1\rangle$ (Rabi frequency $2g$), and the cavity emits the photon $|g,1\rangle \rightarrow |g,0\rangle$ (rate κ). In the limit $g \ll \kappa$, decay via the second channel proceeds at a rate $\Gamma = (2g)^2/\kappa$. Thus, if we repeatedly start the system with the atom in its excited state, then the ratio of photons emitted by the atom to photons emitted by the cavity is $\gamma/\Gamma = \kappa\gamma/4g^2 = N_A$, the critical atom number.

So far we have considered the effect of the atom on the light in the cavity, but what about the effect of the light on the atom? The atom saturates when the effective Rabi frequency Ω_E equals the spontaneous decay rate γ. Substituting our expression for Ω_E, we see that the field needed to saturate the atom is $\alpha = \gamma/2g$, so the number of photons needed to saturate the atom is

$$N_\gamma = |\alpha|^2 = \gamma^2/4g^2$$

This is called the saturation photon number. Substituting for g, we find

$$N_\gamma \sim V/Q\lambda^3$$

Thus, for a small enough cavity ($V < Q\lambda^3$) a single photon will saturate the atom.

In summary, by using a cavity with high finesse and low mode volume we can enhance the effects of single quanta, so that a single atom strongly effects the intra-cavity field and a single photon strongly effects the atom.

1.1.2 Optical trapping

To study cavity QED in the lab, we need a way of delivering atoms to the optical cavity. One delivery method is to pass an atomic beam through the cavity; another is to drop a cloud of cold atoms on it. For both these methods, an individual atom passes rapidly through the cavity and is only briefly coupled to the cavity mode. A better method is to trap an atom at a well-defined location inside the cavity, so it remains coupled for a long time. To accomplish this, we create an optical trap by driving a cavity mode that has a resonant frequency much lower than that of the atom. This type of trap is known as a far off resonance trap, or FORT [2], [3].

Roughly, the FORT works as follows. The electric field $\vec{E}$ of the FORT light induces a dipole $\vec{p}$ in the atom, which couples back to the field, yielding a potential

$$U(\vec{r}) = -\vec{p} \cdot \vec{E}(\vec{r})$$

For red detuned light (atom driven below resonance), the induced dipole oscillates in phase with the field, so the atom is pulled toward regions of high field intensity. Because of the standing wave structure of the cavity mode, the high intensity regions occur at the center of pancake shaped wells, any one of which is capable of trapping an atom. We can create a nearly conservative trap by using light that is highly detuned; for light with intensity I and detuning Δ, the trap depth is $\sim I/\Delta$ while the scattering rate is $\sim I/\Delta^2$, so by increasing both the detuning and the intensity we can hold the trap depth constant while reducing the scattering rate. Using a FORT, we have succeeded in trapping atoms inside our cavity for $\sim 3\,s$, which is $\sim 10^8$ times longer than the timescale $1/g$ that characterizes the atom/cavity coupling [4].

1.1.3 Some applications of cavity QED

A wide variety of experiments can be performed using cavity QED with trapped atoms. Two applications that we have implemented in the lab are the single atom laser [5], [6] and single photon generation [7].

By using a cavity with high finesse and low mode volume, one can enhance the effects of single quanta to such a degree that a single atom can serve as a lasing medium [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18]. In a conventional laser, a lasing medium consisting of many atoms is coupled to one or more modes of an optical cavity and is driven by an external pumping mechanism. Because of the atom/cavity coupling, light that is present in the cavity can stimulate the atoms to coherently emit into the cavity modes instead of spontaneously emitting into free space, and for strong enough coupling emission into the cavity dominates the emission into free space. As we make the atom/cavity coupling stronger and stronger, fewer and fewer atoms are needed for the laser to operate, until ultimately a single atom will suffice. If the critical atom number is small but the critical photon number is large, then the single atom laser has a sharp threshold, obeys the semiclassical laser equations, and exhibits other laser-like properties. For our cavity, both the critical number and the critical photon number are small; thus, deviations from conventional laser-like behavior are observed, such as photon antibunching and the lack of a well-defined threshold.

A second application of cavity QED is single photon generation. In free space, an atom with two ground states a and b and one excited state e can be used to generate single photons in the following way. First we optically pump the atom into state a by applying a field on the $b-e$ transition. Next, we apply a field on the $a-e$ transition. The field excites the atom to state e, from which it can either decay to a, where it will be re-excited by the field, or to b, where it decouples from the field. Eventually the atom is optically pumped into b, with a single photon emitted at frequency ω_{be}. This scheme for generating single photons is not very useful, because the photon is emitted in a random

direction. However, by introducing a cavity that has a mode resonant with the $b - e$ transition, we can collect the photon and direct it into a single spatial mode. Furthermore, the cavity can be used to implement an adiabatic passage scheme, which allows the temporal profile of the photon to be controlled.

1.1.4 Raman scheme

For the purpose of future cavity QED experiments, one would like to gain as much control over the trapped atom as possible. In particular, one would like to cool the center of mass motion of the atom, to measure the magnetic field at the location of the atom, and to be able to prepare the atom in a given internal state. As the main topic of this thesis, I present a scheme I have developed for driving Raman transitions inside the cavity that can be used to achieve all of these goals. Raman transitions are a standard tool for cooling and manipulating atoms [19], [20], [21], [22], [23], and have been used to cool trapped ions to the motional ground state [24]. Here I show how this powerful technique can be applied to the cavity QED system.

Before introducing the scheme and discussing its applications, I want to briefly review the concept of a Raman transition by using a simple model. Consider a three-level atom that has an excited state e and two degenerate ground states a and b (see Figure 1.1). If we drive both the $a - e$ and $b - e$ transitions with classical fields that have Rabi frequency Ω and detuning Δ, then the Hamiltonian for the system is

$$H = -\Delta|e\rangle\langle e| + \frac{\Omega}{2}(|e\rangle\langle a| + |a\rangle\langle e|) + \frac{\Omega}{2}(|e\rangle\langle b| + |b\rangle\langle e|)$$

We can simplify H by introducing states that are superpositions of a and b:

$$|\pm\rangle = \frac{1}{\sqrt{2}}(|a\rangle \pm |b\rangle)$$

Thus,

$$H = -\Delta|e\rangle\langle e| + \frac{\Omega}{2}(|e\rangle\langle +| + |+\rangle\langle e|)$$

For large detunings ($\Delta \gg \Omega$), the first term dominates, and we can treat the second term as a perturbation. To lowest order, the eigenstates are $|e\rangle$, $|+\rangle$, $|-\rangle$ with eigenvalues $E_e = -\Delta$, $E_+ = \Omega^2/4\Delta$, $E_- = 0$. Thus, we can approximate H by

$$H \simeq -\Delta|e\rangle\langle e| + \frac{\Omega^2}{4\Delta}|+\rangle\langle +|$$

Or, substituting for $|+\rangle$,

$$H \simeq -\Delta|e\rangle\langle e| + U(|a\rangle\langle a| + |b\rangle\langle b|) + \frac{\Omega_R}{2}(|a\rangle\langle b| + |b\rangle\langle a|)$$

where $U \equiv \Omega^2/4\Delta$ and $\Omega_R \equiv \Omega^2/2\Delta$.

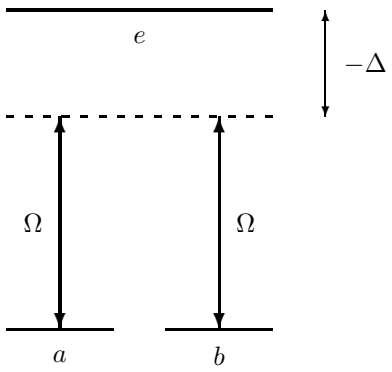

Figure 1.1: Three-level atom.

Note that in the approximate Hamiltonian there is no coupling of the ground states to the excited state; the effects of the couplings that were present in the original Hamiltonian are taken into account in the second and third terms. The second term describes a level shift to the ground states by an amount U, and the third term describes an effective field with Rabi frequency Ω_R, which couples the ground states to one another. It is this third term that is of interest; in general, one can couple two ground states of an atom by driving the atom with a pair of far detuned beams, where the relative detuning of the two beams is equal to the splitting between the two ground states. The coupling generated by such a method is called a Raman coupling, and the resulting transitions between ground states are called Raman transitions.

The Raman scheme presented in this thesis involves using the FORT trapping light itself as one leg of a Raman pair. To form the other leg, we pulse on a second, much weaker beam, which I will call the Raman beam. The relative detuning between the FORT and Raman beams is chosen so as to drive Raman transitions between the two ground state hyperfine manifolds of Cesium. By controlling the power, detuning, and duration of the Raman pulse, one can apply various unitary transformations to the ground state manifold of the atom. This has a number of applications.

One application is Raman spectroscopy. If we repeatedly start the atom in one ground state manifold, apply a Raman pulse with a given detuning, and then check if the atom has been transfered

to the other ground state manifold, we can determine the transfer probability for that detuning. By measuring the transfer probability for many different detunings we can map out a Raman spectrum. This is a useful diagnostic tool, which provides information about the magnetic fields at the location of the atom and about the distribution of ground state populations.

A second application is atomic state preparation. One can create arbitrary superpositions of ground states by optically pumping the atom into a known initial state and then applying an appropriate Raman pulse. The ability to create superposition states is a key ingredient in many entanglement and teleportation schemes.

A third application is cooling the atomic center of mass motion. The strength of the Raman coupling depends on the intensities of the FORT and Raman beams at the position of the atom, and therefore varies as the atom moves around inside the cavity. Thus, the Raman coupling gives a coupling of the internal atomic state to the center of mass motion, which can be exploited to cool the atom.

1.2 Two-level systems

One can often model an atom as a two-level system with one ground state and one excited state. In this section, I discuss some general results that are useful in working with such two-level systems.

1.2.1 Representations of pure states

Here I discuss several ways of representing pure states, which are states of a single, isolated quantum system. The state of a two-level system can be described by a wavefunction

$$|\psi\rangle = c_e|e\rangle + c_g|g\rangle$$

where c_e and c_g are complex numbers. Two complex numbers correspond to four real degrees of freedom, but only two of these are physical. One degree of freedom is removed by requiring that the wavefunction be normalized:

$$\langle\psi|\psi\rangle = |c_e|^2 + |c_g|^2 = 1$$

A second degree of freedom corresponds to the freedom to make phase transformations $|\psi\rangle \rightarrow e^{i\theta}|\psi\rangle$, which give different wavefunctions that describe the same physical state. Thus, a two-level system in a pure state has two physical degrees of freedom.

The state of the system may also be represented by a density matrix

$$\rho = |\psi\rangle\langle\psi| = \sum_{ij} \rho_{ij} |i\rangle\langle j|$$

where

$$\rho_{ij} = c_i \, c_j^*$$

Note that for pure states, the density matrix is idempotent ($\rho^2 = \rho$), hermitian ($\rho^\dagger = \rho$), and has unit trace ($Tr[\rho] = 1$). Conversely, any idempotent, hermitian matrix with unit trace is the density matrix for a pure state. To see this, consider the eigenstates of such a matrix. Since the matrix is hermitian, it has a pair of eigenstates $|\phi_1\rangle$, $|\phi_2\rangle$ with eigenvalues λ_1, λ_2:

$$\rho = \lambda_1 |\phi_1\rangle\langle\phi_1| + \lambda_2 |\phi_2\rangle\langle\phi_2|$$

Since the matrix is idempotent,

$$\rho|\phi_n\rangle = \lambda_n|\phi_n\rangle = \rho^2|\phi_n\rangle = \lambda_n^2|\phi_n\rangle$$

so

$$\lambda_n(1 - \lambda_n)|\phi_n\rangle = 0$$

Thus, $\lambda_n \in \{0, 1\}$. Since the matrix has unit trace, one eigenstate, $|\phi_a\rangle$, must have eigenvalue 1, and the other, $|\phi_b\rangle$, must have eigenvalue 0. Thus,

$$\rho = |\phi_a\rangle\langle\phi_a|$$

which is the density matrix corresponding to the pure state $|\phi_a\rangle$.

Wavefunctions and idempotent density matrices can be used to represent the pure states of quantum systems with any number of levels. But there are additional representations that apply only to two-level systems. Consider the normalization condition for the wavefunction of a two-level system:

$$|c_e|^2 + |c_g|^2 = 1$$

This is the equation for the three sphere S^3, which has two unique properties. First, it is isomorphic to $SU(2)$:

$$(c_e, c_g) \in S^3 \Leftrightarrow \begin{pmatrix} c_e & c_g \\ -c_g^* & c_e^* \end{pmatrix} \in SU(2)$$

Second, S^3 forms a $U(1)$ bundle over S^2 via the Hopf fibration. These properties give two additional representations for pure states. First, a pure state can be represented as a unitary matrix with unit determinant:

$$U = \begin{pmatrix} c_e & c_g \\ -c_g^* & c_e^* \end{pmatrix}$$

Second, a pure state can be represented as a three-vector with unit length. To see this, note that since the density matrix is hermitian, it can be expressed as

$$\rho = \alpha + \vec{\beta} \cdot \vec{\sigma}$$

were α and $\vec{\beta}$ are real. Here

$$\begin{aligned} \sigma_x &= |e\rangle\langle g| + |g\rangle\langle e| \\ \sigma_y &= -i(|e\rangle\langle g| - |g\rangle\langle e|) \\ \sigma_z &= |e\rangle\langle e| - |g\rangle\langle g| \end{aligned}$$

are the Pauli spin matrices. They satisfy

$$\begin{aligned} [\sigma_i, \sigma_j] &= 2i\,\epsilon_{ijk}\,\sigma_k \\ \{\sigma_i, \sigma_j\} &= 2\,\delta_{ij} \end{aligned}$$

Since the density matrix has unit trace $\alpha = 1/2$, and since it is idempotent,

$$\rho^2 = \frac{1}{4} + \beta^2 + \vec{\beta} \cdot \sigma = \rho$$

This implies that $|\vec{\beta}| = 1/2$, so

$$\rho = \frac{1}{2}(1 + \hat{r} \cdot \sigma)$$

where $\hat{r} = 2\vec{\beta}$ is called the Bloch vector. The wavefunction and Bloch vector are related by

$$\hat{r} = 2Re(c_g\, c_e^*)\,\hat{x} + 2Im(c_g\, c_e^*)\,\hat{y} + (|c_e|^2 - |c_g|^2)\,\hat{z}$$

In fiber bundle language, the mapping $|\psi\rangle \to \hat{r}$ constitutes a projection from S^3 (the total space) onto S^2 (the base space). All the wavefunctions that can be obtained from $|\psi\rangle$ by a phase transformation $|\psi\rangle \to e^{i\theta}|\psi\rangle$ are projected onto the same Bloch vector $\hat{r}$, so the Bloch vector can be thought of as a projective representation. Since the density matrix shares this property, it is also a projective representation.

It is often useful to express the quantum state as a function of two coordinates that correspond to its two physical degrees of freedom. Here is what the quantum state would look like for one possible choice of coordinates, in each of the four representations discussed:

wavefunction:

$$|\psi\rangle = e^{i\phi/2}\sin\theta/2\,|g\rangle + e^{-i\phi/2}\cos\theta/2\,|e\rangle$$

unitary matrix:

$$U = \begin{pmatrix} e^{-i\phi/2}\cos\theta/2 & e^{i\phi/2}\sin\theta/2 \\ -e^{-i\phi/2}\sin\theta/2 & e^{-i\phi/2}\cos\theta/2 \end{pmatrix}$$

density matrix:

$$\rho = \frac{1}{2}\begin{pmatrix} 1+\cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & 1-\cos\theta \end{pmatrix}$$

Bloch vector:

$$\hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$$

In summary, a pure quantum state of a two-level system can be represented as a wavefunction (element of S^3) or, isomorphically, as a unitary matrix (element of $SU(2)$):

$$c_e|e\rangle + c_g|g\rangle \longleftrightarrow \begin{pmatrix} c_e & c_g \\ -c_g^* & c_e^* \end{pmatrix}$$

By projecting out the overall phase factor, a pure quantum state can also be represented as a unit length Bloch vector (element of S^2) or, isomorphically, as an idempotent density matrix (idempotent hermitian matrix with unit trace):

$$\hat{r} \longleftrightarrow \frac{1}{2}\begin{pmatrix} 1+r_z & r_x - ir_y \\ r_x + ir_y & 1-r_z \end{pmatrix}$$

1.2.2 Representations of mixed states

In the previous section we discussed four ways of representing the pure states of a single, isolated two-level system. Two of these representations—the density matrix and the Bloch vector—may be generalized to mixed states, which describe a statistical ensemble of quantum systems.

Suppose that a fraction p_n of the systems in an ensemble are described by the sate vector $|\psi_n\rangle$. The expectation value of an arbitrary operator A is a weighted sum over expectation values of each

of the systems in the ensemble:

$$\langle A\rangle = \sum_n p_n \langle\psi_n|A|\psi_n\rangle = \sum_n p_n Tr[\rho_n A] = Tr[\rho A]$$

where

$$\rho_n \equiv |\psi_n\rangle\langle\psi_n|$$

is the density matrix for the pure state $|\psi_n\rangle$, and

$$\rho \equiv \sum_n p_n \rho_n$$

is the density matrix describing the statistical ensemble. Note that

$$Tr[\rho] = \sum_n p_n Tr[\rho_n] = \sum_n p_n = 1$$

Also, ρ is clearly hermitian. Thus, for both pure states and mixed states, the density matrix is hermitian and has unit trace.

How can we tell if a given density matrix describes a pure state or a mixed state? Since the density matrix is hermitian in both cases, it can always be diagonalized by a suitable unitary transformation. In the basis $\{|\phi_n\rangle\}$ in which it is diagonal, it takes the form

$$\rho = \sum_n \lambda_n |\phi_n\rangle\langle\phi_n|$$

Note that

$$\langle\phi_n|\rho|\phi_n\rangle = \lambda_n = \sum_n p_n |\langle\phi_n|\psi_n\rangle|^2$$

Thus, $\lambda_n \geq 0$. Since ρ has unit trace,

$$Tr[\rho] = \sum_n \lambda_n = 1$$

Thus, $\lambda_n \leq 1$. Note that

$$Tr[\rho^2] = \sum_n \lambda_n^2$$

Since $0 \leq \lambda_n \leq 1$, we have that $\lambda_n^2 \leq \lambda_n$. Thus,

$$Tr[\rho^2] \leq 1$$

If $Tr[\rho^2] = 1$, then for all n we must have that $\lambda_n^2 = \lambda_n$, which implies that $\lambda_n \in \{0, 1\}$. Since ρ has unit trace, one of the λ_n's is one and the rest are zero, so the density matrix describes a pure state. If $Tr[\rho^2] < 1$, then there are several nonzero λ_n's, so the density matrix describes a mixed state.

The Bloch vector representation can also be generalized to describe mixed states. Let $\hat{r}_n$ denote the Bloch vector corresponding to the state $|\psi_n\rangle$. Then the n^{th} quantum system in the ensemble is described by the density matrix

$$\rho_n = |\psi_n\rangle\langle\psi_n| = \frac{1}{2}(1 + \hat{r}_n \cdot \vec{\sigma})$$

Thus, the density matrix for the entire ensemble may be expressed as

$$\rho = \sum_n p_n \, \rho_n = \frac{1}{2}(1 + \vec{r} \cdot \vec{\sigma})$$

where

$$\vec{r} = \sum_n p_n \, \hat{r}_n$$

Note that

$$Tr[\rho^2] = \frac{1}{2}(1 + |\vec{r}|^2)$$

Thus, for pure states $|\vec{r}| = 1$, while for mixed states $|\vec{r}| < 1$; that is, pure states live on the surface of the Bloch sphere, while mixed states live on the interior.

The Bloch sphere and density matrix representations are related by

$$\rho = \frac{1}{2}(1 + \vec{r} \cdot \vec{\sigma}) = \frac{1}{2} \begin{pmatrix} 1 + r_z & r_x - ir_y \\ r_x + ir_y & 1 - r_z \end{pmatrix}$$

and

$$\vec{r} = 2Re(\rho_{21}) \, \hat{x} + 2Im(\rho_{21}) \, \hat{y} + (\rho_{11} - \rho_{22}) \, \hat{z}$$

1.2.3 Composite systems

We often want to describe the state of a system A that is not isolated, but which is part of a larger system that is isolated. The larger system can be described by a pure state, but in general the corresponding state of system A is a mixed state. We may therefore view a single state of the larger system as describing an ensemble of states for system A.

To understand how this works, consider two systems A and B, which are coupled to form a compos-

ite system $A \otimes B$. Let $\{|\alpha_n\rangle\}$ be a basis of states for system A, and let $\{|\beta_m\rangle\}$ be a basis of states for system B. Using these states, we can form a basis of states $\{|\alpha_n, \beta_m\rangle = |\alpha_n\rangle \otimes |\beta_m\rangle\}$ for the composite system. An arbitrary state $|\psi\rangle$ of the composite system can be expanded in this basis

$$|\psi\rangle = \sum_{nm} c_{nm} |\alpha_n, \beta_m\rangle$$

We can define a set of states $\{|\gamma_m\rangle\}$ of system A by

$$|\gamma_m\rangle = b_m^{-1} \sum_n c_{nm} |\alpha_n\rangle$$

where

$$b_m = \left(\sum_n |c_{nm}|^2 \right)^{1/2}$$

The states $\{|\gamma_m\rangle\}$ are normalized, but in general they are not orthogonal to one another. If we define states $\{|\gamma_m, \beta_m\rangle = |\gamma_m\rangle \otimes |\beta_m\rangle\}$ for the composite system, we can express $|\psi\rangle$ as

$$|\psi\rangle = \sum_m b_m |\gamma_m, \beta_m\rangle$$

Note that $|\gamma_m\rangle$ may be viewed as the relative state of $|\beta_m\rangle$; that is, whenever system B is in state $|\beta_m\rangle$, system A is in state $|\gamma_m\rangle$. Because the states $\{|\beta_m\rangle\}$ are orthonormal, the states $\{|\gamma_m, \beta_m\rangle\}$ are also orthonormal. Thus, the probability that the composite system is in state $|\gamma_m, \beta_m\rangle$ is

$$p_m = |\langle \gamma_m, \beta_m | \psi \rangle|^2 = |b_m|^2$$

We may therefore view the single state $|\psi\rangle$ of the composite system as an ensemble of states for system A, where state $|\gamma_m\rangle$ occurs in the ensemble with probability p_m. The density matrix for A is therefore

$$\rho_A = \sum_m p_m |\gamma_m\rangle\langle\gamma_m| = Tr_B \, \rho$$

where

$$\rho = |\psi\rangle\langle\psi| = \sum_m p_m |\gamma_m, \beta_m\rangle\langle\gamma_m, \beta_m|$$

is the density matrix for the composite system.

1.2.4 Time evolution

The wavefunction for an isolated system evolves in time according to the Schrödinger equation:

$$i\partial_t |\psi\rangle = H|\psi\rangle$$

where H is the Hamiltonian for the system. For a two-level system, H can always be expressed in the form

$$H = E_0 + \frac{1}{2}\vec{\Omega} \cdot \vec{\sigma}$$

for some choice of parameters E_0 and $\vec{\Omega}$. Changing E_0 does not alter the physical behavior of the system, so for simplicity I will set $E_0 = 0$.

We can use the equation of motion for the wavefunction to derive the corresponding equation of motion of the density matrix. Since $\rho = |\psi\rangle\langle\psi|$, we have that

$$\dot{\rho} = -i[H, \rho]$$

From the equation of motion for the density matrix, we obtain the equation of motion for the Bloch vector. Recall that the density matrix and Bloch vector are related by

$$\rho = \frac{1}{2}(1 + \hat{r} \cdot \vec{\sigma})$$

Substituting this into the equation of motion for the density matrix, we find

$$\dot{\rho} = \frac{1}{2}\dot{\hat{r}} \cdot \vec{\sigma} = -i[H, \rho] = -\frac{i}{4}[\vec{\Omega} \cdot \vec{\sigma}, \hat{r} \cdot \vec{\sigma}] = \frac{1}{2}(\vec{\Omega} \times \hat{r}) \cdot \vec{\sigma}$$

Thus, the equation of motion for the Bloch vector is

$$\dot{\hat{r}} = \vec{\Omega} \times \hat{r}$$

I now want to solve for the eigenstates and eigenvalues of H for the case where $\vec{\Omega}$ is constant in time. I will denote the eigenstates by $|\pm\rangle$, and the corresponding eigenvalues by $E_\pm$:

$$H|\pm\rangle = E_\pm|\pm\rangle$$

From the equation of motion for the Bloch vector, we see that the eigenstates $|\pm\rangle$ correspond to the Bloch vectors $\pm\hat{\Omega}$. Suppose $\hat{\Omega}$ has the following coordinate representation:

$$\hat{\Omega} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$$

Then, using the results of section 1.2.1, we find that the eigenstate $|+\rangle$ corresponding to $+\hat{\Omega}$ is

$$|+\rangle = e^{i\phi/2}\sin\theta/2\,|g\rangle + e^{-i\phi/2}\cos\theta/2\,|e\rangle$$

and the eigenstate $|-\rangle$ corresponding to $-\hat{\Omega}$ is

$$|-\rangle = e^{i\phi/2}\cos\theta/2\,|g\rangle - e^{-i\phi/2}\sin\theta/2\,|e\rangle$$

The eigenvalues corresponding to these states are $E_\pm = \pm|\vec{\Omega}|/2$.

1.3 Two-level atoms

I now want to apply the general results for two-level systems that we derived in section 1.2 to the special case of a two-level atom. After briefly reviewing the transformation to the interaction picture, I write down the Hamiltonian for a two-level atom coupled to a beam of light, and I write down the master equation for a two-level atom that can spontaneously decay.

1.3.1 Interaction picture

In many situations a Schrödinger picture Hamiltonian H_S can be divided into a simple part H_0 and a complicated part H_i:

$$H_S = H_0 + H_i$$

Usually H_0 describes how the system would evolve freely in isolation, while H_i describes the interaction of the system with an external driving force. Thus, H_0 is called the free Hamiltonian, and H_i is called the interaction Hamiltonian.

The total Hamiltonian H_S determines the time evolution of the Schrödinger picture wavefunction $|\psi_S\rangle$:

$$i\partial_t|\psi_S\rangle = H_S|\psi_S\rangle$$

The wavefunction evolves under the combined influence of H_0 and H_i, but because the evolution under H_0 is usually known and uninteresting, we can simplify the problem by transforming to the interaction picture, in which the evolution under H_0 is already taken into account. We therefore define an interaction picture wavefunction $|\psi_I\rangle$, which is related to $|\psi_S\rangle$ by

$$|\psi_I\rangle = U^\dagger|\psi_S\rangle$$

where

$$U = e^{-iH_0t}$$

is a unitary transformation describing time evolution under H_0. The time evolution of $|\psi_I\rangle$ is given by

$$i\partial_t|\psi_I\rangle = i(\partial_t U^\dagger)|\psi_S\rangle + iU^\dagger\,\partial_t|\psi_S\rangle = i(\partial_t U^\dagger)U|\psi_I\rangle + U^\dagger H_S|\psi_S\rangle = H_I|\psi_I\rangle$$

where I have defined an interaction picture Hamiltonian H_I by

$$H_I = U^\dagger H_S U + i(\partial_t U^\dagger)U = U^\dagger H_i U$$

As an example, consider an atom with a set of internal states $\{|n\rangle\}$. We can define transition operators A_{jk}:

$$A_{jk} = |j\rangle\langle k|$$

and projection operators P_r:

$$P_r = |r\rangle\langle r|$$

Typically, the Hamiltonian for the atom will consist of sums of Hamiltonians of the form

$$H_S = H_0 + H_i$$

where

$$H_0 = \omega_r P_r$$

and

$$H_i = f(A_{jk})$$

for some function f. We can transform to the interaction picture via the unitary transformation

$$U = e^{-iH_0 t} = e^{-iP_r\omega_r t}$$

The interaction picture Hamiltonian is

$$H_I = U^\dagger H_i U = U^\dagger\, f(A_{jk})\, U = f(U^\dagger A_{jk} U) = f(A(t))$$

where

$$A(t) = U^\dagger A_{jk} U = e^{iP_r\omega_r t}\, A_{jk}\, e^{-iP_r\omega_r t}$$

Note that

$$\frac{d}{dt}(U^\dagger A_{jk}U) = i\omega_r U^\dagger[P_r, A_{jk}]U = i\omega_r(\delta_{rj} - \delta_{rk})(U^\dagger A_{jk}U)$$

If we integrate this differential equation subject to the initial condition

$$U^\dagger(0)A_{jk}U(0) = A_{jk}$$

we find that

$$U^\dagger A_{jk}U = A_{jk}\, e^{i(\delta_{rj}-\delta_{rk})\omega_r t}$$

Thus, the interaction picture Hamiltonian is

$$H_I = f(A_{jk}\, e^{i(\delta_{rj}-\delta_{rk})\omega_r t})$$

In summary,

$$H_S = \omega_r P_r + f(A_{jk}) \;\leftrightarrow\; H_I = f(A_{jk}\, e^{i(\delta_{rj}-\delta_{rk})\omega_r t})$$

For a two-level atom, this gives

$$H_S = \omega\sigma_+\sigma_- + f(\sigma_-) \;\leftrightarrow\; H_I = f(\sigma_-\, e^{-i\omega t})$$

As another example, consider a harmonic oscillator Hamiltonian of the form

$$H_S = \omega a^\dagger a + f(a, a^\dagger)$$

If we transform to the interaction picture via the unitary transformation

$$U = e^{-ia^\dagger a\omega t}$$

then the interaction picture Hamiltonian is

$$H_I = f(a\, e^{-i\omega t}, a^\dagger\, e^{i\omega t})$$

So

$$H_S = \omega a^\dagger a + f(a, a^\dagger) \;\leftrightarrow\; H_I = f(a\, e^{-i\omega t}, a^\dagger\, e^{i\omega t})$$

1.3.2 Two-level atom without spontaneous decay

I now want to write down the Hamiltonian for a two-level atom that is driven by a beam of light. The free Hamiltonian for the atom is

$$H_0 = \omega_A|e\rangle\langle e| = \omega_A\sigma_+\sigma_-$$

where ω_A is the splitting between the excited state $|e\rangle$ and the ground state $|g\rangle$. The interaction Hamiltonian describing the coupling of the atom to the light is

$$H_i = e\vec{r} \cdot \vec{E}$$

I will assume the light is a plane wave with polarization $\hat{\epsilon}$:

$$\vec{E} = Re\,\hat{\epsilon}\,E_0\,e^{-i(\omega t - \vec{k}\cdot\vec{r})} = \frac{1}{2}(\hat{\epsilon}\,e^{-i(\omega t - \vec{k}\cdot\vec{r})} + \hat{\epsilon}^*\,e^{i(\omega t - \vec{k}\cdot\vec{r})})\,E_0$$

If the atom is much smaller than the wavelength of the light ($\vec{k}\cdot\vec{r} \ll 1$), then the phase of the electric field is approximately constant over the entire atom, and we can make the dipole approximation:

$$\vec{E} \simeq \frac{1}{2}(\hat{\epsilon}\,e^{-i\omega t} + \hat{\epsilon}^*\,e^{i\omega t})\,E_0$$

It is convenient to express the field in terms of the cycle-averaged intensity I, which is given by

$$I = \frac{1}{8\pi}\langle(E^2 + B^2)\rangle = \frac{1}{8\pi}E_0^2$$

Thus,

$$\vec{E} = \frac{1}{2}(8\pi I)^{1/2}\,(\hat{\epsilon}\,e^{-i\omega t} + \hat{\epsilon}^*\,e^{i\omega t})$$

Substituting this into the Hamiltonian, we obtain

$$H_i = \frac{1}{2}(8\pi\alpha I)^{1/2}\,(\hat{\epsilon}\cdot\vec{r}\,e^{-i\omega t} + \hat{\epsilon}^*\cdot\vec{r}\,e^{i\omega t})$$

where $\alpha = e^2$ is the fine structure constant. If we insert a complete set of states $\{|g\rangle, |e\rangle\}$, and note that parity considerations require $\langle e|\vec{r}|e\rangle = \langle g|\vec{r}|g\rangle = 0$, we find

$$H_i = \frac{1}{2}(\Omega\,\sigma_+\,e^{-i\omega t} + \Omega^*\,\sigma_-\,e^{i\omega t}) + \frac{1}{2}(\Omega_c\,\sigma_+\,e^{i\omega t} + \Omega_c^*\,\sigma_-\,e^{-i\omega t})$$

where

$$\Omega = (8\pi\alpha I)^{1/2}\,\langle e|\hat{\epsilon}\cdot\vec{r}|g\rangle$$

is the Rabi frequency, and

$$\Omega_c = (8\pi\alpha I)^{1/2}\,\langle e|\hat{\epsilon}^*\cdot\vec{r}|g\rangle$$

is a counter-rotating Rabi frequency. Note that for linearly polarized light $\hat{\epsilon}$ is real, so $\Omega = \Omega_c$ and the interaction Hamiltonian can be expressed as

$$H_i = (\Omega\,\sigma_+ + \Omega^*\,\sigma_-)\cos\omega t$$

For elliptically polarized light, this form of H_i only holds in the rotating wave approximation.

The total Schrödinger picture Hamiltonian is

$$\begin{aligned} H_S &= H_0 + H_i \\ &= \omega_A \sigma_+ \sigma_- + \frac{1}{2}(\Omega\, \sigma_+\, e^{-i\omega t} + \Omega^*\, \sigma_-\, e^{i\omega t}) + \frac{1}{2}(\Omega_c\, \sigma_+\, e^{i\omega t} + \Omega_c^*\, \sigma_-\, e^{-i\omega t}) \end{aligned}$$

The corresponding interaction picture Hamiltonian is ($U = e^{-i\sigma_+\sigma_-\omega_A t}$)

$$H_I = \frac{1}{2}(\Omega\, \sigma_+\, e^{-i\Delta t} + \Omega^*\, \sigma_-\, e^{i\Delta t}) + \frac{1}{2}(\Omega_c\, \sigma_+\, e^{i(\omega+\omega_A)t} + \Omega_c^*\, \sigma_-\, e^{-i(\omega+\omega_A)t})$$

where $\Delta = \omega - \omega_A$ is the detuning of the light. In the rotating wave approximation, the second term may be dropped:

$$H_I = \frac{1}{2}(\Omega\, \sigma_+\, e^{-i\Delta t} + \Omega^*\, \sigma_-\, e^{i\Delta t})$$

Transforming back to the Schrödinger picture ($U = e^{i\sigma_+\sigma_-\Delta t}$), we obtain

$$H = -\Delta\sigma_+\sigma_- + \frac{1}{2}(\Omega\, \sigma_+ + \Omega^*\, \sigma_-)$$

It is convenient to express this in the form

$$H = -\frac{1}{2}\Delta + \frac{1}{2}(\Omega\sigma_+ + \Omega^*\sigma_- - \Delta\sigma_z) = -\frac{1}{2}\Delta + \frac{1}{2}\,\vec{\Omega}_E \cdot \vec{\sigma}$$

where

$$\vec{\Omega}_E = Re\,\Omega\,\hat{x} - Im\,\Omega\,\hat{y} - \Delta\,\hat{z}$$

The magnitude of $\vec{\Omega}_E$ is

$$\Omega_E = |\vec{\Omega}_E| = (\Omega^2 + \Delta^2)^{1/2}$$

and

$$\hat{\Omega}_E = \vec{\Omega}_E/\Omega_E = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$$

where I have defined the angles $\phi \in [0, 2\pi]$ and $\theta \in [0, \pi]$ by

$$\Omega = |\Omega|\, e^{-i\phi}$$

and

$$\sin\theta \quad = \quad |\Omega|/\Omega_E$$

$$\cos\theta = -\Delta/\Omega_E$$

Using the results of section 1.2.4, we find that the eigenvectors of H are

$$\begin{aligned} |+\rangle &= e^{i\phi/2}\sin\theta/2\,|g\rangle + e^{-i\phi/2}\cos\theta/2\,|e\rangle \\ |-\rangle &= e^{i\phi/2}\cos\theta/2\,|g\rangle - e^{-i\phi/2}\sin\theta/2\,|e\rangle \end{aligned}$$

and the eigenvalues are

$$E_\pm = \frac{1}{2}(-\Delta \pm \Omega_E)$$

We can express the ground and excited states in terms of $|\pm\rangle$:

$$\begin{aligned} |e\rangle &= e^{i\phi/2}\left(\cos\theta/2\,|+\rangle - \sin\theta/2\,|-\rangle\right) \\ |g\rangle &= e^{-i\phi/2}\left(\sin\theta/2\,|+\rangle + \cos\theta/2\,|-\rangle\right) \end{aligned}$$

Suppose we start the atom in the ground state:

$$|\psi(0)\rangle = |g\rangle = e^{-i\phi/2}\left(\sin\theta/2\,|+\rangle + \cos\theta/2\,|-\rangle\right)$$

Then the wavefunction at time t is

$$\begin{aligned} |\psi(t)\rangle &= e^{-i\phi/2}\left(e^{-iE_+t}\,\sin\theta/2\,|+\rangle + e^{-iE_-t}\,\cos\theta/2\,|-\rangle\right) \\ &= c_g|g\rangle + c_e|e\rangle \end{aligned}$$

where

$$\begin{aligned} c_g &= e^{i\Delta t/2}\left(\cos(\Omega_E t/2) - i\frac{\Delta}{\Omega_E}\sin(\Omega_E t/2)\right) \\ c_e &= -i\frac{\Omega}{\Omega_E}\,e^{i\Delta t/2}\,\sin(\Omega_E t/2) \end{aligned}$$

Thus, the probability of being in the excited state at time t is

$$p_e = |c_e|^2 = \frac{|\Omega|^2}{\Omega_E^2}\,\sin^2(\Omega_E t/2)$$

1.3.3 Two-level atom with spontaneous decay

If we include the spontaneous decay of the two-level atom, then its time evolution is given by the master equation

$$\dot{\rho} = -i[H,\rho] + \frac{\gamma}{2}(2\sigma_-\rho\sigma_+ - \sigma_+\sigma_-\rho - \rho\sigma_+\sigma_-)$$

where

$$H = -\Delta\sigma_+\sigma_- + \frac{1}{2}(\Omega\sigma_+ + \Omega^*\sigma_-)$$

and γ is the spontaneous decay rate of the excited state. The equations of motion for the matrix elements of ρ are

$$\begin{aligned}
\dot{\rho}_{gg} &= \frac{i}{2}(\Omega\rho_{ge} - \Omega^*\rho_{eg}) + \gamma\rho_{ee} \\
\dot{\rho}_{ee} &= \frac{i}{2}(\Omega^*\rho_{eg} - \Omega\rho_{ge}) - \gamma\rho_{ee} \\
\dot{\rho}_{ge} &= \frac{i}{2}\Omega^*(\rho_{gg} - \rho_{ee}) - (\frac{\gamma}{2} + i\Delta)\rho_{ge}
\end{aligned}$$

Note that

$$\langle\sigma_z\rangle = Tr[\rho(|e\rangle\langle e| - |g\rangle\langle g|)] = \rho_{ee} - \rho_{gg}$$

$$\langle\sigma_+\rangle = Tr[\rho|e\rangle\langle g|] = \rho_{ge}$$

Thus, we can write equations of motion for the expectation values of σ_- and σ_z:

$$\begin{aligned}
\frac{d}{dt}\langle\sigma_-\rangle &= i\frac{\Omega}{2}\langle\sigma_z\rangle - (\frac{\gamma}{2} - i\Delta)\langle\sigma_-\rangle \\
\frac{d}{dt}\langle\sigma_z\rangle &= i(\Omega^*\langle\sigma_-\rangle - \Omega\langle\sigma_+\rangle) - \gamma(1 + \langle\sigma_z\rangle)
\end{aligned}$$

In steady state, the density matrix is

$$\begin{aligned}
\rho_{gg} &= \frac{|\Omega/2|^2 + (\gamma/2)^2 + \Delta^2}{2|\Omega/2|^2 + (\gamma/2)^2 + \Delta^2} \\
\rho_{ee} &= \frac{|\Omega/2|^2}{2|\Omega/2|^2 + (\gamma/2)^2 + \Delta^2} \\
\rho_{ge} &= \frac{i(\Omega^*/2)(\gamma/2 - i\Delta)}{2|\Omega/2|^2 + (\gamma/2)^2 + \Delta^2}
\end{aligned}$$

1.4 Optical cavities

In this section I review some basic properties of optical cavities and discuss the coupling of an atom to one of the modes of an optical cavity.

1.4.1 Classical cavity

I first want to consider a classical description of an optical cavity. For simplicity, I'll start by modeling the cavity as a pair of flat mirrors that are parallel to one another and are separated by a distance L, then later I'll consider the effects of mirror curvature.

Suppose a plane wave with wavelength λ is incident on one of the mirrors, and that the coefficients of transmission and reflection for both mirrors are t and r. We want to solve for the electric field $E_c(z)$ inside the cavity at a distance z from the input mirror. Since the light that enters the cavity bounces back and forth between the two mirrors, we can express $E_c(z)$ as sum of the fields for each bounce:

$$\begin{aligned} E_c(z) &= tE_i\, e^{ikz} + trE_i\, e^{ik(2L-z)} + tr^2E_i\, e^{ik(2L+z)} + \cdots \\ &= tE_i(e^{ikz} + r\, e^{-ikz})(1 + r^2\, e^{2ikL} + r^4\, e^{4ikL} + \cdots) \\ &= tE_i(e^{ikz} + r\, e^{-ikz})(1 - r^2\, e^{2ikL})^{-1} \end{aligned}$$

where $k = 2\pi/\lambda$. If we assume that the mirrors are highly reflective ($r \simeq -1$), then we can approximate this as

$$E_c(z) = 2itE_i(1 - r^2 e^{2ikL})^{-1}\, \sin kz$$

I will define $R = r^2$ and $T = t^2$. Assuming the mirrors do not absorb any of the light, R and T satisfy the conservation equation $R + T = 1$. The intensity inside the cavity is then

$$\begin{aligned} I_c(z) &= |E_c(z)/E_i|^2\, I_i \\ &= 4TI_i(1 + R^2 - 2R\cos 2kL)^{-1} \sin^2 kz \\ &= 4TI_i(1 + R^2 - 2R + 4R\sin^2 kL)^{-1} \sin^2 kz \\ &= 4TI_i(T^2 + 4R\sin^2 kL)^{-1} \sin^2 kz \end{aligned}$$

Since $R \simeq 1$, we may approximate this as

$$I_c(z) = (4/T)I_i(1 + (4/T^2)\, \sin^2 kL)^{-1} \sin^2 kz$$

Because the mirrors are highly reflective, the $4/T^2$ factor in the denominator is very large, so light is only coupled into the cavity if its wavelength is tuned such that $kL = n\pi$ for some integer n. We can understand this condition as follows. If the mirrors were perfectly reflective, then the cavity would have normal modes at integer multiples of the free spectral range $\nu_{FSR} = 1/2L$. If we now allow the mirrors to be slightly transmissive, we expect light to be coupled into the cavity when it is tuned into resonance with one of these modes. The width of the resonance can be determined as follows. Assume the light is nearly resonant with mode n, so the detuning of the light from the mode is small compared to the free spectral range. The detuning is given by

$$\Delta = 2\pi(1/\lambda - n\nu_{FSR}) = 2\pi/\lambda - n\pi/L$$

In terms of the detuning,

$$kL = 2\pi L/\lambda = \Delta L + n\pi$$

and

$$I_c(z) = (4/T)I_i(1 + (4/T^2)\,\sin^2\Delta L)^{-1}\sin^2 kz$$

If we expand around $\Delta = 0$, we find

$$I_c(z) = (4/T)I_i(1 + (2\Delta/\kappa)^2)^{-1}\sin^2 kz$$

where $\kappa = T/L$ is the full width at half maximum of the resonance. According to this definition, κ gives the energy decay rate for the entire cavity. Note that some authors define κ differently, so that it represents an amplitude decay rate, or so that it gives the decay rate for an individual mirror.

I now want to solve for the transmitted and reflected fields. The electric field at the face of the output mirror is

$$\begin{aligned} E_t &= t^2 E_i\, e^{ikL} + t^2 r^2 E_i\, e^{3ikL} + \cdots \\ &= t^2 E_i (1 - r^2\, e^{2ikL})^{-1}\, e^{ikL} \end{aligned}$$

Thus, making the same approximations as before, we find that the transmitted intensity is

$$I_t = I_i(1 + (4/T^2)\sin^2 kL)^{-1}$$

The reflected field can be obtained from this by using the conservation equation $I_r + I_t = I_i$.

The results obtained thus far were derived by treating the mirrors as flat planes. If we now include the curvature of the mirrors, the results are modified in several ways. Because of the mirror curvature, light is confined in the transverse direction, resulting in a set of transverse modes. If we drive only the lowest order transverse mode (the gaussian mode) then the intra-cavity intensity is

$$I_c(\vec{r}) = (4/T)(1 + (2\Delta/\kappa)^2)^{-1}|\psi(\vec{r})|^2\, I_i$$

where

$$\psi(\vec{r}) = \sin kz\, e^{-(x^2+y^2)/w_0^2}$$

describes the mode shape. The mode radius w_0 is related to the mirror radius R by

$$w_0^2 = \frac{\lambda}{2\pi}(L(2R - L))^{1/2}$$

For $R \gg L$, this may be approximated as

$$w_0^2 = \frac{\lambda}{2\pi}(2RL)^{1/2}$$

For our cavity, $R = 20$ cm and $L = 45\,\mu$m, so $w_0 = 25\,\mu$m at the FORT wavelength of $\lambda = 895$ nm. We can define an effective mode volume by

$$V = \int |\psi(\vec{r})|^2\, d^3r = \frac{\lambda}{8}(2RL)^{1/2}L = \frac{1}{2}AL$$

where A is an effective area for the mode, obtained by integrating over the transverse mode profile:

$$A = \iint e^{-2(x^2+y^2)/w_0^2}\, dx\, dy = \frac{\pi}{2}w_0^2$$

To couple light into the cavity, the input light must be spatially mode matched to a cavity mode. The input intensity I_i is related to the input power P_i by $I_i = P_i/A$, where A is the effective area. Usually P_i is less than the total power in the input beam, since not all the input power is mode matched into the cavity. We can measure P_i in the lab by tuning the input beam to resonance and measuring the output power, and then including a correction factor to account for light that is absorbed in the mirrors. Note that we can express the intensity inside the cavity in terms of P_i:

$$I_c(\vec{r}) = (2/\kappa V)(1 + (2\Delta/\kappa)^2)^{-1}\, |\psi(\vec{r})|^2\, P_i$$

The total energy inside the cavity is therefore

$$E = \int I_c(\vec{r})\, d^3r = (2/\kappa)(1 + (2\Delta/\kappa)^2)^{-1}\, P_i$$

1.4.2 Quantum cavity

In the previous section we showed that the transmission spectrum of the cavity is sharply peaked around integer multiples of the free spectral range. If the mirrors are highly reflective, then we may treat the system as a set of cavity modes at these resonant frequencies, which are weakly coupled to a continuum of output modes via the mirrors. I want to single out one of these modes and quantize it by introducing creation and annihilation operators a and $a^\dagger$. The Hamiltonian for the chosen mode is

$$H = \omega_c a^\dagger a + \lambda(a + a^\dagger)\cos\omega t$$

where ω_c is the frequency of the mode, and ω and λ are the frequency and pumping strength of the driving field. The Hamiltonian can be simplified by making a series of unitary transformations.

First, make a transformation to eliminate the first term ($U = e^{-ia^\dagger a\omega_c t}$):

$$H \rightarrow \lambda(a\, e^{-i\omega_c t} + a^\dagger\, e^{i\omega_c t})\cos\omega t$$

Next, make the rotating wave approximation:

$$H \rightarrow (\lambda/2)(a\, e^{i\Delta t} + a^\dagger\, e^{-i\Delta t})$$

where $\Delta = \omega - \omega_c$ is the detuning of the light from the cavity resonance. Finally, make a transformation to eliminate the time dependence ($U = e^{-ia^\dagger a\Delta t}$):

$$H = -\Delta a^\dagger a + (\lambda/2)(a + a^\dagger)$$

This Hamiltonian describes the coherent evolution of the mode. To include the damping that arises from the weak coupling to the output modes, we write down a master equation for the system:

$$\dot{\rho} = -i[H, \rho] + \frac{\kappa}{2}(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a)$$

where κ is the cavity decay rate discussed in the previous section. The steady state solution to the master equation is

$$\rho = |\alpha\rangle\langle\alpha|$$

where $|\alpha\rangle$ is a coherent state with amplitude

$$\alpha = \frac{\lambda/2}{\Delta + i\kappa/2}$$

Thus, in steady state, the number of photons in the cavity is

$$n = |\alpha|^2 = \frac{(\lambda/\kappa)^2}{1 + (2\Delta/\kappa)^2}$$

The field energy in the cavity is $E = n\omega$. If we compare this to the classical result from the previous section, we can relate the pumping strength λ to the power P_i of the driving field:

$$\lambda^2 = (2\kappa)(P_i/\omega)$$

Note that when the cavity is on resonance, all the input light is transmitted through the cavity. Thus, the rate at which photons are emitted from the cavity is the same as the rate at which they

are delivered by the input beam, which is

$$\Gamma = P_i/\omega = n\kappa/2$$

Because of the driving field, photons are only emitted from the output mirror. In the absence of the driving field, however, photons that are initially present in the cavity are emitted from both mirrors, so the total emission rate is $n\kappa$.

1.4.3 Cavity QED

So far we have been discussing the case of an empty cavity, but I now want to consider the coupling of the cavity to a two-level atom. The Hamiltonian for the atom/cavity system is

$$H = \omega_a \sigma_+ \sigma_- + \omega_c a^\dagger a + g(a^\dagger \sigma_- + a\sigma_+) + \lambda(a + a^\dagger)\cos\omega t$$

The first term is the Hamiltonian for the atom, the second term is the Hamiltonian for the mode, the third term gives the coupling of the atom to the mode, and the fourth term describes a probe beam that is driving the mode. Here ω_a and ω_c are the resonant frequencies of the atom and cavity mode, g is the atom/cavity coupling strength, and ω and λ are the frequency and pumping strength of the probe beam.

The Hamiltonian can be simplified by making a series of unitary transformations. First, make a transformation on a and $a^\dagger$ to eliminate the second term ($U = e^{-ia^\dagger a\omega_c t}$):

$$H \rightarrow \omega_a \sigma_+ \sigma_- + g(a^\dagger \sigma_- e^{i\omega_c t} + a\sigma_+ e^{-i\omega_c t}) + \lambda(a\, e^{-i\omega_c t} + a^\dagger\, e^{i\omega_c t})\cos\omega t$$

Now make the rotating wave approximation in the last term:

$$H \rightarrow \omega_a \sigma_+ \sigma_- + g(a^\dagger \sigma_-\, e^{i\omega_c t} + a\sigma_+\, e^{-i\omega_c t}) + (\lambda/2)(a\, e^{i\Delta_c t} + a^\dagger\, e^{-i\Delta_c t})$$

where $\Delta_c = \omega - \omega_c$. Make a second unitary transformation on a and $a^\dagger$ to eliminate the time dependence in the last term ($U = e^{-ia^\dagger a\Delta_c t}$):

$$H \rightarrow \omega_a \sigma_+ \sigma_- - \Delta_c a^\dagger a + g(a^\dagger \sigma_-\, e^{i\omega t} + a\sigma_+\, e^{-i\omega t}) + (\lambda/2)(a + a^\dagger)$$

Finally, make a unitary transformation on σ_+ and σ_- to eliminate the time dependence in the third term ($U = e^{-i\sigma_+\sigma_-\Delta_a t}$):

$$H \rightarrow -\Delta_a\sigma_+\sigma_- - \Delta_c a^\dagger a + g(a^\dagger\sigma_- + a\sigma_+) + (\lambda/2)(a + a^\dagger)$$

where $\Delta_a = \omega - \omega_a$. This is the form of the Hamiltonian we will use.

The Hamiltonian gives the coherent part of the evolution. To describe the damping of the system, we write down a master equation that includes terms for the decay of the atom at rate γ and for the decay of the cavity at rate κ:

$$\dot{\rho} = -i[H, \rho] + \frac{\gamma}{2}(2\sigma_-\rho\sigma_+ - \sigma_+\sigma_-\rho - \rho\sigma_+\sigma_-) + \frac{\kappa}{2}(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a)$$

One might worry that the presence of the cavity would alter the mode structure of the vacuum and thereby change the spontaneous decay rate of the atom. We can show, however, that this effect is negligible. From the point of view of an atom located at the focus of the cavity mode, the mode subtends a solid angle $\Omega = 4\pi(1 - \cos\theta)$, where $\theta = \lambda/\pi w_0$ is beam divergence angle. The ratio of the solid angle subtended by the mode to the total solid angle is

$$\Omega/4\pi = 1 - \cos\theta \sim \theta^2/2 \sim 6 \times 10^{-5}$$

where I have substituted the values of λ and w_0 for our cavity. Thus, the majority of the solid angle that the atom sees corresponds to vacuum modes, so the spontaneous decay rate is not significantly different from the free space value.

1.4.4 Semiclassical approximation

In many cases, cavity QED can be described in a semiclassical approximation. As an example, I'll use the semiclassical approximation to solve for the steady state photon number when the cavity is driven by a probe with strength λ. The Hamiltonian for the system is

$$H = -\Delta_a\sigma_+\sigma_- - \Delta_c a^\dagger a + g(a^\dagger\sigma_- + a\sigma_+) + (\lambda/2)(a + a^\dagger)$$

In the semiclassical approximation, we treat the coupled atom-cavity system as an isolated atom described by an effective Hamiltonian H_a, and an isolated cavity described by an effective Hamiltonian H_c. The Hamiltonian H_a is obtained from H by replacing the operator a with a parameter

$\alpha = \langle a \rangle$ characterizing the field amplitude:

$$H_a = -\Delta_a \sigma_+ \sigma_- + g(\alpha \sigma_+ + \alpha^* \sigma_-)$$

Similarly, H_c is obtained from H by replacing the operator σ_- by a parameter $\beta = \langle \sigma_- \rangle$ characterizing the atomic dipole:

$$H_c = -\Delta_c a^\dagger a + g(\beta^* a + \beta a^\dagger) + (\lambda/2)(a + a^\dagger)$$

We can express H_a and H_c as

$$\begin{aligned} H_a &= -\Delta_a \sigma_+ \sigma_- + \frac{1}{2}(\Omega_E \, \sigma_+ + \Omega_E^* \, \sigma_-) \\ H_c &= -\Delta_c a^\dagger a + \frac{1}{2}(\lambda_E \, a^\dagger + \lambda_E^* \, a) \end{aligned}$$

where $\Omega_E = 2g\alpha$ is an effective Rabi frequency and $\lambda_E = 2g\beta + \lambda$ is an effective pumping strength. The master equation for the atom is

$$\dot{\rho}_a = -i[H_a, \rho_a] + \frac{\gamma}{2}(2\sigma_- \rho_a \sigma_+ - \sigma_+ \sigma_- \rho_a - \rho_a \sigma_+ \sigma_-)$$

The master equation for the cavity is

$$\dot{\rho}_c = -i[H_c, \rho_c] + \frac{\kappa}{2}(2a \rho_c a^\dagger - a^\dagger a \rho_c - \rho_c a^\dagger a)$$

We now want to find the steady state solution. For simplicity, let us assume that the cavity is resonant with the atom ($\omega_a = \omega_c$), so $\Delta_a = \Delta_c = \Delta$. From the steady state solution to the atomic master equation (see section 1.3.3), we find

$$\beta = Tr[\rho_a \sigma_-] = \frac{-i(\Omega_E/2)(\gamma/2 + i\Delta)}{2|\Omega_E/2|^2 + (\gamma/2)^2 + \Delta^2} = \frac{-ig\alpha(\gamma/2 + i\Delta)}{2g^2|\alpha|^2 + (\gamma/2)^2 + \Delta^2}$$

From the cavity master equation, we find that in steady state

$$i\dot{\alpha} = \langle [a, H_a] \rangle - \frac{i\kappa}{2}\alpha = 0$$

So the steady state value of α is

$$\alpha = Tr[\rho_c a] = (\lambda_E/2)(\Delta_c + i\kappa/2)^{-1} = \frac{g\beta + \lambda/2}{\Delta + i\kappa/2}$$

If we require that α and β be consistent with one another, we obtain the solution for the coupled atom-cavity system. In the weak driving limit, we can approximate β as

$$\beta = \frac{g\alpha}{\Delta + i\gamma/2}$$

Substituting this result into our equation for α, we find

$$\alpha = \frac{(\Delta + i\gamma/2)(\lambda/2)}{(\Delta + i\kappa/2)(\Delta + i\gamma/2) - g^2}$$

So the photon number is

$$n = |\alpha|^2 = \frac{(\Delta^2 + \gamma^2/4)(\lambda/2)^2}{(g^2 - \Delta^2 + \kappa\gamma/4)^2 + \Delta^2(\gamma/2 + \kappa/2)^2}$$

Note that on resonance, the photon number is

$$n = (\lambda/\kappa)^2/(1 + 1/N_A)^2 = n_0/(1 + 1/N_A)^2$$

where n_0 is the photon number for an empty cavity, and $N_A = \kappa\gamma/4g^2$ is the critical atom number. The excited state population for the atom is

$$p_e = |\beta|^2 = n/N_\gamma$$

where $N_\gamma = \gamma^2/4g^2$ is the critical photon number. For our cavity,

$$\begin{aligned} g &= (2\pi)(32\,\text{MHz}) \\ \kappa &= (2\pi)(8.4\,\text{MHz}) \\ \gamma &= (2\pi)(5.2\,\text{MHz}) \end{aligned}$$

so the critical atom and photon numbers are

$$\begin{aligned} N_A &= 0.0102 \\ N_\gamma &= 0.0066 \end{aligned}$$

Figure 1.2 is a graph of photon number n versus probe detuning Δ, using the parameters relevant to our experiment. We recently performed an experiment to measure this curve in the lab [25].

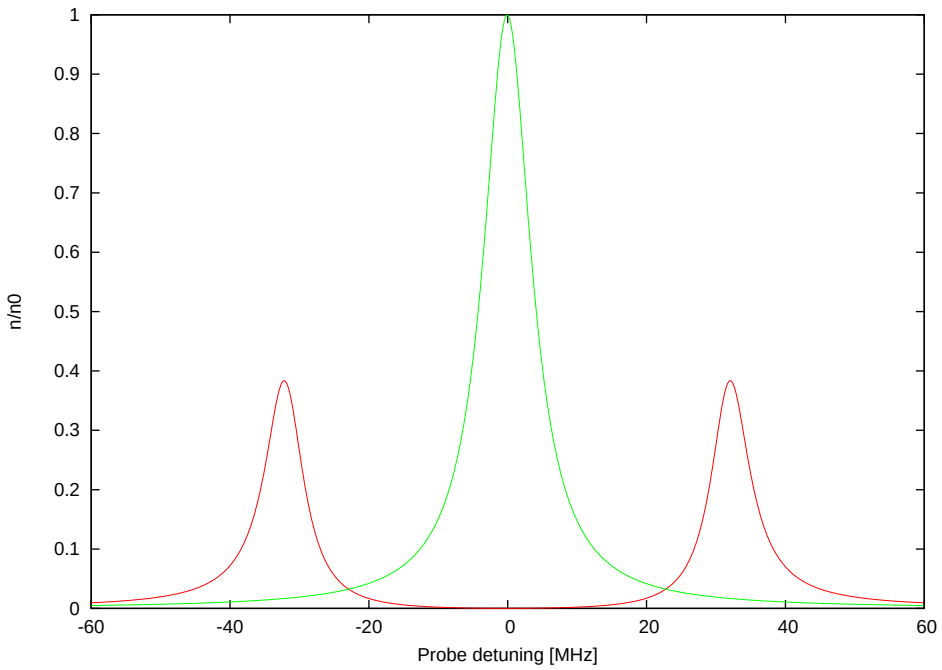

Figure 1.2: Photon number n/n_0 versus probe detuning Δ. The green curve is for an empty cavity; the red curve is for an atom coupled to the cavity with strength g.

1.4.5 Single atom laser

I now want to apply the semiclassical approximation to a simple three-level atom model for a single atom laser. A level diagram of the single atom laser is shown in Figure 1.3. The atom has states 1, 2, and 3, where state 3 decays to state 2 at rate γ and state 2 decays to state 1 at rate Γ. The $2-3$ transition is coupled with strength g to the cavity mode, which is assumed to be resonant with the transition, and the $1-3$ transition is driven on resonance by a classical field with Rabi frequency Ω, where for simplicity we will chose the phase of the field such that Ω is real. Note that the decay $2 \rightarrow 1$ draws population from 2, which in a conventional laser would maintain a population inversion across the $2-3$ lasing transition.

The Hamiltonian for the coupled atom-cavity system is

$$H = g(a\hat{A}_{32} + a^\dagger \hat{A}_{23}) + \frac{\Omega}{2}(\hat{A}_{31} + \hat{A}_{13})$$

where $\hat{A}_{jk} = |j\rangle\langle k|$ are atomic raising and lowering operators. The master equation is

$$\dot{\rho} = -i[H, \rho] + \mathcal{L}_\Gamma \rho + \mathcal{L}_\gamma \rho + \mathcal{L}_\kappa \rho$$

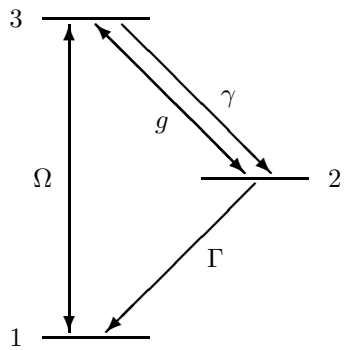

Figure 1.3: Level diagram for the single atom laser.

where

$$\begin{aligned}
\mathcal{L}_\Gamma \rho &= \frac{\Gamma}{2}(2\hat{A}_{12}\rho\hat{A}_{21} - \hat{A}_{22}\rho - \rho\hat{A}_{22}) \\
\mathcal{L}_\gamma \rho &= \frac{\gamma}{2}(2\hat{A}_{23}\rho\hat{A}_{32} - \hat{A}_{33}\rho - \rho\hat{A}_{33}) \\
\mathcal{L}_\kappa \rho &= \frac{\kappa}{2}(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a)
\end{aligned}$$

describe the spontaneous decay on the $1-2$ transition, the spontaneous decay on the $2-3$ transition, and the cavity decay. In the semiclassical approximation, the equations of motion for the expectation values are

$$\begin{aligned}
\dot{A}_{11} &= -i(\Omega/2)(A_{13} - A_{31}) + \Gamma A_{22} \\
\dot{A}_{22} &= -ig(\alpha^* A_{23} - \alpha A_{32}) - \Gamma A_{22} + \gamma A_{33} \\
\dot{A}_{33} &= -ig(\alpha A_{32} - \alpha^* A_{23}) - i(\Omega/2)(A_{31} - A_{13}) - \gamma A_{33} \\
\dot{A}_{12} &= -ig\alpha^* A_{13} + i(\Omega/2)A_{32} - (\Gamma/2)A_{12} \\
\dot{A}_{23} &= -ig\alpha(A_{22} - A_{33}) - i(\Omega/2)A_{21} - (1/2)(\Gamma + \gamma)A_{23} \\
\dot{A}_{31} &= -i(\Omega/2)(A_{33} - A_{11}) + ig\alpha^* A_{21} - (\gamma/2)A_{31} \\
\dot{\alpha} &= -igA_{23} - (\kappa/2)\alpha
\end{aligned}$$

where $A_{jk} = \langle \hat{A}_{jk} \rangle$ and $\alpha = \langle a \rangle$. Note that these equations are invariant under the transformation $(\alpha, A_{21}, A_{23}) \rightarrow (e^{i\phi}\alpha, e^{i\phi}A_{21}, e^{i\phi}A_{23})$, where ϕ is an arbitrary phase. By exploiting this invariance, we can always choose the phase such that α is real. Given that α is real, the equations imply that A_{12} is also real, and that A_{23} and A_{31} are imaginary (of course, A_{11}, A_{22}, and A_{33} are always real, as they represent atomic populations). It is convenient to introduce real valued variables $B_{23} = -iA_{23}$

and $B_{31} = -iA_{31}$. In terms of the new variables, the equations of motion are

$$\begin{aligned}
\dot{A}_{11} &= -\Omega B_{31} + \Gamma A_{22} \\
\dot{A}_{22} &= 2g\alpha B_{23} - \Gamma A_{22} + \gamma A_{33} \\
\dot{A}_{33} &= -2g\alpha B_{23} + \Omega B_{31} - \gamma A_{33} \\
\dot{A}_{12} &= -g\alpha B_{31} + (\Omega/2) B_{23} - (\Gamma/2) A_{12} \\
\dot{B}_{23} &= -g\alpha (A_{22} - A_{33}) - (\Omega/2) A_{12} - (1/2)(\Gamma + \gamma) B_{23} \\
\dot{B}_{31} &= -(\Omega/2)(A_{33} - A_{11}) + g\alpha A_{12} - (\gamma/2) B_{31} \\
\dot{\alpha} &= g B_{23} - (\kappa/2)\alpha
\end{aligned}$$

We now want to look at the steady state solution to these equations. From $\dot{A}_{22} = 0$ and $\dot{\alpha} = 0$, we find that

$$\kappa n + \gamma A_{33} = \Gamma A_{22}$$

We can understand this relation by noting that for every photon spontaneously emitted on the $1-2$ transition, there is either one photon emitted from the cavity or one photon spontaneously emitted on the $2-3$ transition. Thus, the sum of the rates for cavity emission and for spontaneous emission on the $2-3$ transition must equal the rate for spontaneous emission on the $1-2$ transition.

Before continuing our investigation of the steady state solution, it is convenient to introduce some new parameters. Let us define a scaled photon number $m = n/N_\gamma$, a dimensionless pumping strength $p = \Omega^2/\gamma\Gamma$, and a parameter $\eta = \gamma/\Gamma$ that gives the ratio of the spontaneous emission rates for the $2-3$ and $1-2$ transitions. Also, recall that the critical atom and photon numbers are $N_A = \kappa\gamma/4g^2$ and $N_\gamma = \gamma^2/4g^2$.

After a bit of algebra, we find that in steady state the scaled photon number m obeys the equation

$$a\, m^2 + b\, m + c = 0$$

where

$$\begin{aligned}
a &= \eta^2 \\
b &= \eta^2 p + \eta^2 + 2\eta \\
c &= (2+\eta)p^2 + (2\eta + 2/\eta + 3 - N_A^{-1})p + 1 + \eta
\end{aligned}$$

Thus, the laser output as a function of pumping intensity is given by

$$m(p) = \begin{cases} (\sqrt{b^2 - 4ac} - b)/2a & \text{for } c > 0 \\ 0 & \text{for } c < 0 \end{cases}$$

Note that this expression for $m(p)$ is independent of the critical photon number N_γ. We see that the laser threshold occurs at a pumping strength p_t such that $c(p_t) = 0$. Above threshold ($p > p_t$), the populations are

$$\begin{aligned} A_{11} &= 1 - A_{22} - A_{33} \\ A_{22} &= N_A \eta\, m(p) + N_A \eta (p + 1 + 1/\eta) \\ A_{33} &= N_A (p + 1 + 1/\eta) \end{aligned}$$

Below threshold ($p < p_t$),

$$\begin{aligned} A_{11} &= 1 - A_{22} - A_{33} \\ A_{22} &= \eta A_{33} \\ A_{33} &= (2 + \eta + \eta/p)^{-1} \\ A_{12} &= 0 \\ B_{23} &= 0 \\ B_{31} &= (\gamma/\Omega) A_{33} \end{aligned}$$

Now that we have an understanding of the semiclassical theory, let us consider a full quantum treatment of the single atom laser. The degree to which the quantum description agrees with the semiclassical description depends on the value of the critical photon number N_γ. We can see this by considering the behavior of the single atom laser for pumping strengths near the semiclassical threshold p_t. In the limit of large critical photon number, if we are even slightly above the semiclassical threshold then there are many photons in the cavity (recall that $n = N_\gamma\, m$), so the quantum fluctuations are small and the system is well described by the semiclassical theory. As we reduce the critical photon number, there are fewer photons in the cavity and quantum fluctuations begin to smear out the sharp threshold predicted by the semiclassical theory. We can describe this transition quantitatively by using the semiclassical theory to calculate the scaled photon number at twice p_t; when $m(2p_t) \gg 1/N_\gamma$ (which implies that $n(2p_t) \gg 1$), we expect the system to be well approximated by the semiclassical theory. Figure 1.4 shows lines of constant $m(2p_t)$ in the space of parameters (η, N_A) that characterize the system. Note that for large enough values of N_A and η, there is no semiclassical threshold.

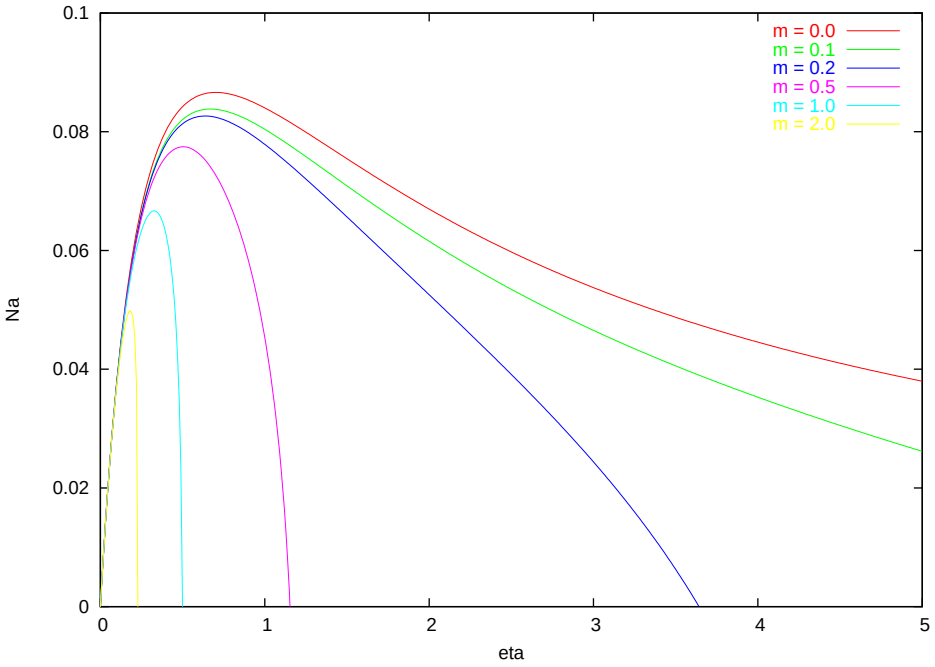

Figure 1.4: Lines of constant $m(2p_t)$. Curves are shown for $m = 0.0, 0.1, 0.2, 0.5, 1.0, 2.0$. Above the $m = 0.0$ curve there is no semiclassical threshold.

Using the quantum optics toolbox [26], it is straightforward to simulate the full quantum system on a computer. For the simulations I present here $N_A = 0.05$ and $\eta = 0.5$, which gives $p_t = 0.128$ and $m(2p_t) = 0.918$. The simulations are performed by truncating the Fock space at 70 photons. Graphs of $m(p)$ for different values of N_γ are shown in Figure 1.5 and Figure 1.6.

1.5 Raman transitions in a three-level system

In this section I present a scheme for driving Raman transitions in a cavity. For simplicity, I'll describe the scheme using a three-level model for the atom; then in a later section (section 1.7) I'll generalize to the case of a many-level atom.

1.5.1 Adiabatic elimination

We will often be interested in the interaction of an atom with light fields that are far detuned from resonance. For far detuned light the excited state populations are small, and we can approximate the full Hamiltonian with an effective Hamiltonian that only involves the ground states. In this section I show how this can be accomplished.

To gain some intuition about the problem I'll start with the case of a two-level atom, which we

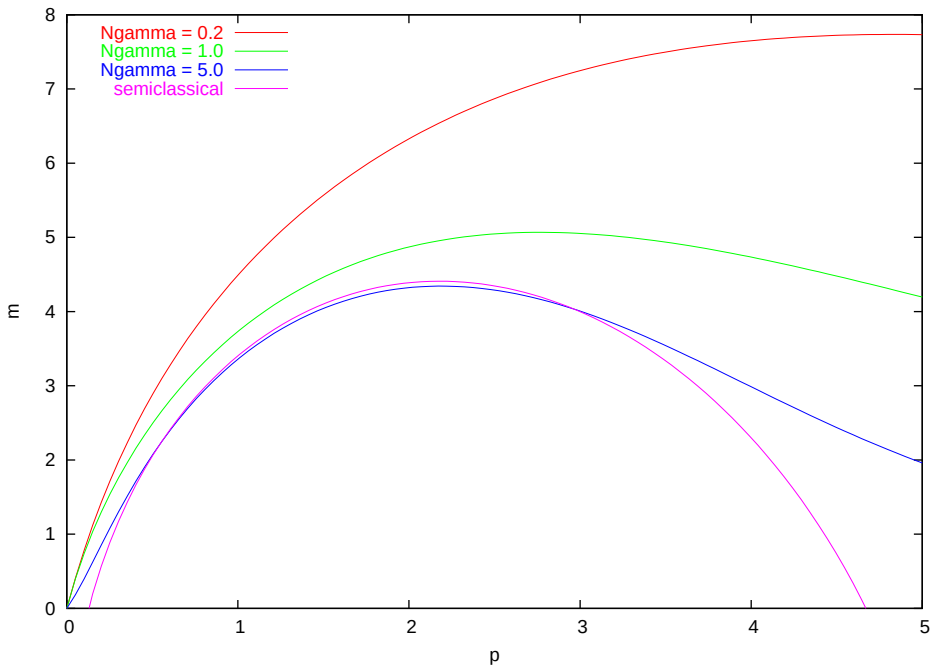

Figure 1.5: Scaled photon number m verses pump strength p for $N_A = 0.05$, $\eta = 0.5$, and $N_\gamma = 0.2, 1.0, 5.0$. The semiclassical result is also shown; the kink in the semiclassical curve at $p = 0.128$ is the laser threshold.

already know how to solve exactly. Suppose the atom is coupled to a light field with Rabi frequency Ω and detuning Δ, where for simplicity I will assume Ω is real and Δ is negative. The Hamiltonian for the atom is

$$\begin{aligned} H &= -\Delta\sigma_+\sigma_- + \frac{\Omega}{2}(\sigma_+ + \sigma_-) \\ &= -\frac{1}{2}\Delta + \frac{1}{2}(\Omega^2 + \Delta^2)^{1/2}(\sin\theta\,\sigma_x + \cos\theta\,\sigma_z) \end{aligned}$$

where θ is determined by

$$\begin{aligned} \cos\theta &= -\Delta\,(\Omega^2 + \Delta^2)^{-1/2} \\ \sin\theta &= \Omega\,(\Omega^2 + \Delta^2)^{-1/2} \end{aligned}$$

The eigenstates and eigenvalues are

$$\begin{aligned} |+\rangle &= \cos\theta/2\,|e\rangle + \sin\theta/2\,|g\rangle \\ |-\rangle &= \cos\theta/2\,|g\rangle - \sin\theta/2\,|e\rangle \end{aligned}$$

and

$$E_\pm = -\frac{\Delta}{2} \pm \frac{1}{2}(\Omega^2 + \Delta^2)^{1/2}$$

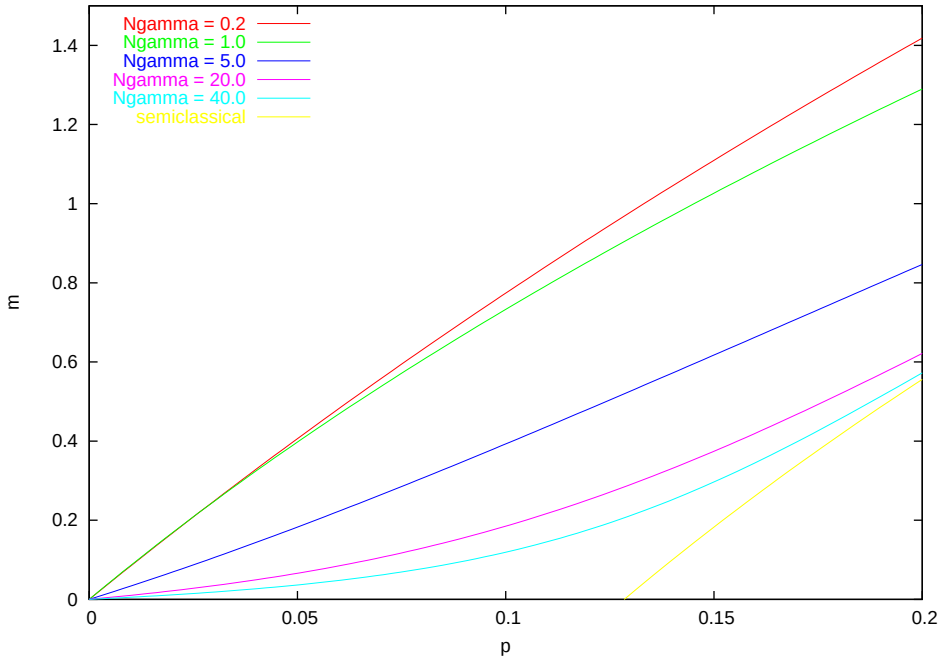

Figure 1.6: Scaled photon number m verses pump strength p (zooming in on the threshold region) for $N_A = 0.05$, $\eta = 0.5$, and $N_\gamma = 0.2, 1.0, 5.0, 20.0, 40.0$. The semiclassical result is also shown. A threshold can be observed for $N_\gamma = 40$ and $N_\gamma = 20$, but for smaller values of N_γ the threshold is smeared out by quantum fluctuations.

For large detunings $|\Delta| \gg \Omega$, the eigenstates and eigenvalues can be approximated as

$$\begin{aligned} |+\rangle &\simeq |e\rangle - \frac{\Omega}{2\Delta}|g\rangle \\ |-\rangle &\simeq |g\rangle + \frac{\Omega}{2\Delta}|e\rangle \end{aligned}$$

and

$$\begin{aligned} E_+ &\simeq -\Delta \\ E_- &\simeq \frac{\Omega^2}{4\Delta} \end{aligned}$$

Thus, one eigenstate consists mostly of the excited state and evolves at a fast rate Δ, while the other eigenstate consists mostly of the ground state and evolves at a slow rate $\Omega^2/4\Delta$.

Now consider a multilevel atom with a set of excited states $\{|e\rangle\}$ and a set of ground states $\{|g\rangle\}$. I'll assume the Hamiltonian for the atom has the form

$$H = -\sum_e \Delta_e |e\rangle\langle e| + \frac{\Omega}{2}(A + A^\dagger)$$

where Δ_e is the detuning of the light from excited state e and A is a generalized lowering operator that connects excited states to ground states.

As in the case of the two-level atom, for large detunings $|\Delta_e| \gg \Omega$ the eigenstates of H can be split into two manifolds $\{|E\rangle\}$ and $\{|G\rangle\}$, where eigenstates in manifold $\{|E\rangle\}$ have eigenvalues $\sim -\Delta_e$ and consist mostly of admixtures of excited states ($\langle g|E\rangle \sim \Omega/\Delta_e$), while eigenstates in manifold $\{|G\rangle\}$ have eigenvalues $\sim \Omega^2/4\Delta_e$ and consist mostly of admixtures of ground states ($\langle e|G\rangle \sim \Omega/\Delta_e$).

Consider a state $|\psi\rangle$ which lies entirely in manifold $\{|G\rangle\}$. It can be expressed as

$$|\psi\rangle = \sum_G |G\rangle = \sum_e c_e|e\rangle + \sum_g c_g|g\rangle$$

The equations of motion for the amplitudes $\{c_e\}$ and $\{c_g\}$ are

$$\begin{aligned} i\dot{c}_e &= -\Delta_e\, c_e + \frac{\Omega}{2}\sum_g \langle e|A^\dagger|g\rangle\, c_g \\ i\dot{c}_g &= \frac{\Omega}{2}\sum_e \langle g|A|e\rangle\, c_e \end{aligned}$$

Because the state $|\psi\rangle$ lies entirely in manifold $\{|G\rangle\}$, it evolves at a rate $\sim \Omega^2/4\Delta_e$, which is much slower than the detunings Δ_e. Thus, $\dot{c}_e \ll \Delta_e c_e$, and we can approximate the first set of equations as

$$-\Delta_e\, c_e + \frac{\Omega}{2}\sum_g \langle e|A^\dagger|g\rangle\, c_g \simeq 0$$

We can therefore solve for the excited state amplitudes in terms of the ground state amplitudes:

$$c_e = \frac{\Omega}{2\Delta_e}\sum_g \langle e|A^\dagger|g\rangle\, c_g$$

Substituting this result into the second set of equations, we find

$$i\dot{c}_g = \sum_e \frac{\Omega^2}{4\Delta_e}\langle g|A|e\rangle\langle e|A^\dagger|g'\rangle\, c_{g'}$$

Thus, the evolution of the ground state amplitudes is given by an effective Hamiltonian

$$H_E = \sum_e \frac{\Omega^2}{4\Delta_e} A|e\rangle\langle e|A^\dagger$$

One can generalize the above result to include Hamiltonians of the form

$$H = H_{gg} - \sum_e \Delta_e |e\rangle\langle e| + \frac{\Omega}{2}(A + A^\dagger)$$

where H_{gg} only couples ground states to ground states, and $H_{gg} \ll |\Delta_e|$. In this case, the effective Hamiltonian is

$$H_E = H_{gg} + \sum_e \frac{\Omega^2}{4\Delta_e} A|e\rangle\langle e|A^\dagger$$

So far I have assumed the evolution is unitary, so the atom can be described by a wavefunction that evolves according to Schrödinger's equation. If we include the spontaneous decay of the excited states, then the atom is described by a density matrix that evolves under the master equation. By employing similar reasoning to that used above, one can adiabatically eliminate the excited states in the master equation, resulting in an effective master equation for the ground states. One finds that the ground states evolve under the effective Hamiltonian given above, but there are also effective decays from one ground state to another. These decays result from off-resonant excitation of the excited states, and the decay rates are $\sim \gamma p_e$, where $p_e \sim \Omega^2/\Delta_e^2$ is the population in the excited state e.

1.5.2 FORT configuration

As an example of the adiabatic elimination technique, I want to calculate the FORT potential for a three-level atom coupled to far detuned light. The atom has ground states a and b with energies $-\delta_A/2$ and $+\delta_A/2$, and an excited state e with energy ω_e. Thus, the Hamiltonian for the atom is

$$H_0 = \omega_e |e\rangle\langle e| + \frac{1}{2}\delta_A \sigma_z$$

where I have defined

$$\sigma_z = |b\rangle\langle b| - |a\rangle\langle a|$$

I will assume that the light drives both the $a - e$ transition and the $b - e$ transition with Rabi frequency Ω. The total Hamiltonian for the system is then

$$H = H_0 + \Omega\,(A + A^\dagger)\,\cos\omega_L t$$

where ω_L is the frequency of the light, and

$$A = (|a\rangle + |b\rangle)\langle e|$$

is a generalized atomic lowering operator (see Figure 1.7).

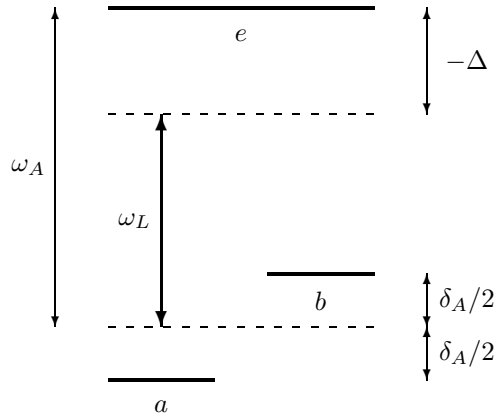

Figure 1.7: Three-level atom, FORT configuration.

By applying some unitary transformations and making the rotating wave approximation, we can cast the Hamiltonian in the form

$$H = -\Delta|e\rangle\langle e| + \frac{1}{2}\delta_A\sigma_z + \frac{\Omega}{2}(A + A^\dagger)$$

where $\Delta = \omega_L - \omega_e$ is the detuning of the light. If we adiabatically eliminate the excited state, we obtain an effective Hamiltonian for the ground states:

$$H_E = \frac{\Omega^2}{4\Delta}AA^\dagger + \frac{1}{2}\delta_A\sigma_z = \frac{1}{2}\delta_A\sigma_z + U + U\sigma_x$$

where $U = \Omega^2/4\Delta$.

We will assume that $\delta_A \gg U$, so to lowest order the eigenstates of H_E are $|a\rangle$ and $|b\rangle$. The second term of H_E gives the state independent FORT potential U, and the third term gives a state dependent correction to the FORT potential of order U^2/δ_A and may be neglected.

1.5.3 Raman scheme

Now I want to generalize the results of the previous section to the case where there are two beams of light driving the atom. I assume the beams have optical frequencies $\omega_\pm = \omega_L \pm \delta_R/2$ and Rabi frequencies $\Omega_\pm$ (see Figure 1.8).

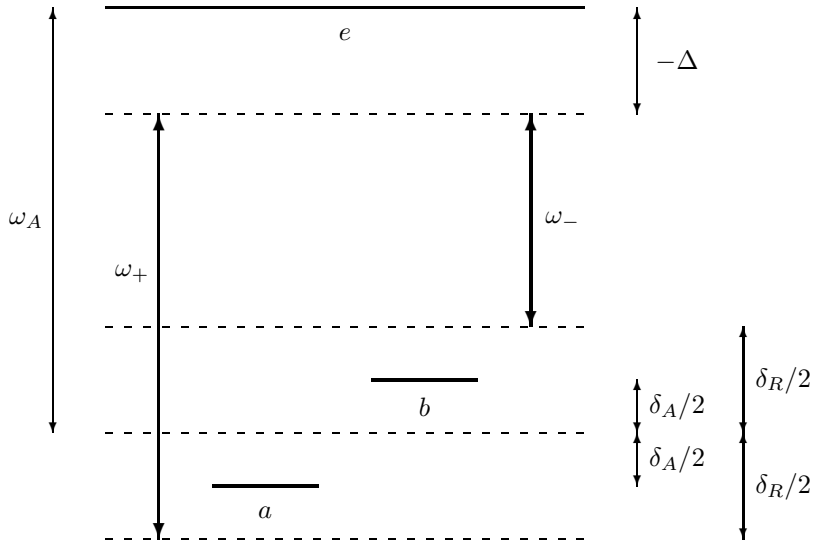

Figure 1.8: Three-level atom, Raman configuration.

The Hamiltonian for the system is

$$H = H_0 + (\Omega_+ \cos\omega_+ t + \Omega_- \cos\omega_- t)(A + A^\dagger)$$

As before, we can simplify the Hamiltonian by applying some unitary transformations and making the rotating wave approximation:

$$H' = -\Delta|e\rangle\langle e| + \frac{1}{2}\delta_A\sigma_z + B + B^\dagger$$

where

$$B = \frac{1}{2}(\Omega_+ e^{i\delta_R t/2} + \Omega_- e^{-i\delta_R t/2})\, A$$

If we adiabatically eliminate the excited state, we obtain the effective Hamiltonian

$$\begin{aligned} H_E &= \frac{1}{\Delta}BB^\dagger + \frac{1}{2}\delta_A\sigma_z \\ &= \frac{1}{4\Delta}(\Omega_+^2 + \Omega_-^2 + 2\Omega_+\Omega_-\cos\delta_R t)(1+\sigma_x) + \frac{1}{2}\delta_A\sigma_z \\ &= \frac{1}{2}\delta_A\sigma_z + U + U\sigma_x + \Omega_E\cos\delta_R t + \Omega_E\,\sigma_x\cos\delta_R t \end{aligned}$$

where

$$U = \frac{1}{4\Delta}(\Omega_+^2 + \Omega_-^2)$$

and

$$\Omega_E = \frac{\Omega_+ \Omega_-}{2\Delta}$$

We will assume that one beam is much stronger than the other, so $U \gg \Omega_E$. Also, let us assume that $\delta_A \gg U$, so to lowest order the eigenstates are $|a\rangle$ and $|b\rangle$. The first three terms in H_E have the same form as in the Hamiltonian from the previous section, which described an atom coupled to a single beam. As before, the second term gives the state independent FORT potential, and the third term gives a state dependent correction to the FORT potential of order U^2/δ_A and may be neglected. The fourth and fifth terms are new; they appear because a second beam is present. The fourth term modulates the state independent FORT potential with frequency δ_R and amplitude Ω_E. Because δ_R is generally much larger than the motional frequency of the atom, this term averages to zero. In addition, because $U \gg \Omega_E$, the amplitude of the modulation is small relative to the static FORT potential. Thus, the fourth term may be neglected, and we are left with

$$H_E = \frac{1}{2}\delta_A \sigma_z + U + \Omega_E \sigma_x \cos \delta_R t$$

This Hamiltonian describes an effective two-level atom with ground state a, excited state b, and resonant frequency δ_A, which is coupled to a classical field with frequency δ_R and Rabi frequency Ω_E. By performing some unitary transformations and making the rotating wave approximation, we can express the effective Hamiltonian as

$$H_E = U - \frac{\delta}{2}\sigma_z + \frac{\Omega_E}{2}\,\sigma_x$$

where $\delta = \delta_R - \delta_A$.

So far we have only considered the internal degrees of freedom of the atom, but I now want to include the center of mass motion as well. If we add a kinetic energy term, and allow U and Ω_E to depend on the position $\vec{r}$ of the atom, then the total Hamiltonian is

$$H = \frac{p^2}{2m} + U(\vec{r}) - \frac{\delta}{2}\sigma_z + \frac{1}{2}\Omega_E(\vec{r})\,\sigma_x$$

To find $U(\vec{r})$ and $\Omega_E(\vec{r})$, we need to know the spatial dependence of the beams. I will assume that the atom is inside a cavity, and that the two beams are obtained by driving one of the cavity modes with a pair of input beams (note that since the beams are at two different frequencies, at least one of the beams must be detuned from the cavity resonance, and its coupling into the cavity will be suppressed). As a first step, I will only consider motion along the cavity axis, so we only include the

axial standing wave structure of the beams:

$$\Omega_{\pm}(z) = \Omega_{\pm}(0)\, \cos kz$$

Thus, the effective Rabi frequency is

$$\Omega_E(z) = \Omega \cos^2 kz$$

where

$$\Omega \equiv \frac{1}{2\Delta}\, \Omega_+(0)\, \Omega_-(0)$$

and the FORT potential is

$$U(z) = -U_0 \cos^2 kz = -U_0 + U_0 \sin^2 kz$$

where

$$U_0 \equiv -\frac{1}{4\Delta}(\Omega_+^2(0) + \Omega_-^2(0))$$

I have chosen the sign convention such that $U_0 > 0$ for red detuning ($\Delta < 0$). Substituting these expressions into the Hamiltonian, we find

$$H = \frac{p_z^2}{2m} + U_0 \sin^2 kz - \frac{\delta}{2}\sigma_z + \frac{\Omega}{2} \cos^2 kz\, \sigma_x$$

where I have dropped the constant term $-U_0$.

1.5.4 Harmonic approximation

We can gain some insight into the model Hamiltonian by considering a harmonic approximation, which should be valid for atomic motion near the bottom of the well. If we expand in z (for a red detuned FORT we are expanding about an antinode), and retain only the lowest order terms, we find

$$H = \frac{p_z^2}{2m} + \frac{1}{2}m\omega^2 z^2 - \frac{\delta}{2}\sigma_z + \frac{\Omega}{2}(1 - (kz)^2)\sigma_x$$

where I have introduced a vibrational frequency ω, defined by

$$\frac{1}{2}m\omega^2 = U_0 k^2$$

Note that the vibrational frequency may be expressed as

$$\omega = 2(U_0 \omega_{rec})^{1/2}$$

where $\omega_{rec} = k^2/2m$ is called the recoil frequency.

The atomic motion can be quantized by introducing creation and annihilation operators $b^\dagger$ and b, which are related to z and p_z by

$$\begin{aligned} z &= (2m\omega)^{-1/2}\,(b + b^\dagger) \\ p_z &= -i(m\omega/2)^{1/2}\,(b - b^\dagger) \end{aligned}$$

Thus,

$$kz = \eta(b + b^\dagger)$$

where η, the Lamb-Dicke parameter, is

$$\eta = k(2m\omega)^{-1/2} = (\omega_{rec}/\omega)^{1/2}$$

Putting all this together, we find that the Hamiltonian is

$$H = \omega b^\dagger b - \frac{\delta}{2}\sigma_z + \frac{\Omega}{2}\,(1 - \eta^2(b + b^\dagger)^2)\sigma_x$$

where I have dropped the constant term $\omega/2$ corresponding to the zero point energy. The first term gives the kinetic energy of the atom, and the second term gives the internal energy. The third term describes a coupling of the internal state of the atom to the center of mass motion. We can calculate the Rabi frequencies for transitions to different motional states by taking matrix elements of this term; for a transition from motional state n to motional state n', the Rabi frequency is

$$\Omega_{n\to n'} = \langle b, n'|\Omega\,(1 - \eta^2(b + b^\dagger)^2)|a, n\rangle$$

The Rabi frequencies for the red sideband, carrier, and blue sideband are

$$\begin{aligned} \Omega_{n\to n-2} &= \eta^2\sqrt{n}\sqrt{n-1}\,\Omega \\ \Omega_{n\to n} &= \Omega - \eta^2(2n+1)\,\Omega \\ \Omega_{n\to n+2} &= \eta^2\sqrt{n+1}\sqrt{n+2}\,\Omega \end{aligned}$$

Thus, the sidebands are suppressed by η^2 relative to the carrier (note that for a blue detuned FORT we would expand around a node, so in this case both the carrier and the sidebands would be of order η^2). Because of symmetry considerations, transitions can only be driven to even sidebands. This a consequence of using the FORT itself as one of the legs in the Raman pair.

1.5.5 Numerical results

I now want to return to general Hamiltonian

$$H = \frac{p_z^2}{2m} + U_0 \sin^2 kz - \frac{\delta}{2}\sigma_z + \frac{\Omega}{2}\cos^2 kz\,\sigma_x$$

and numerically solve for the energy eigenvalues and Rabi frequencies. It is convenient to introduce dimensionless variables $\bar{z}$ and $\bar{p}_z$, which are related to z and p by a canonical transformation:

$$\begin{aligned} \bar{z} &= (m\omega)^{1/2}\, z \\ \bar{p}_z &= (m\omega)^{-1/2}\, p_z \end{aligned}$$

In terms of the dimensionless variables, the Hamiltonian is

$$H = H_{ext} + H_{int}$$

where

$$H_{ext} = \frac{1}{2}\omega(\bar{p}_z^2 + \frac{1}{2\eta^2}\sin^2(\sqrt{2}\eta\bar{z}))$$

describes the center of mass motion, and

$$H_{int} = -\frac{\delta}{2}\sigma_z + \frac{\Omega}{2}\cos^2(\sqrt{2}\eta\bar{z})\,\sigma_x$$

describes the internal state. The eigenstates ψ_n and eigenvalues E_n of H_{ext} are given by

$$E_n\psi_n = H_{ext}\,\psi_n$$

We can define dimensionless eigenvalues $\epsilon_n \equiv E_n/\omega$, and express this as a set of coupled first order differential equations:

$$\begin{aligned} \psi_n' &= \pi_n \\ \pi_n' &= (\frac{1}{2\eta^2}\sin^2(\sqrt{2}\eta\bar{z}) - 2\epsilon_n)\,\psi_n \end{aligned}$$

where the prime denotes a derivative with respect to $\bar{z}$. These equations can be numerically solved to obtain the energy eigenstates and eigenvalues. From the eigenstates, we can calculate the Rabi frequencies for different transitions:

$$\Omega_{r\to n} = \Omega \int \psi_n^*(\bar{z})\,\cos^2(\sqrt{2}\eta\bar{z})\,\psi_r(\bar{z})\,d\bar{z}$$

Note that the only free parameter in the problem is η.

The parameters describing the actual experiment are as follows. At the FORT wavelength of 935 nm, the recoil frequency is $\omega_{rec} = (2\pi)(1.71\,\text{kHz})$. I will assume the FORT depth is $U_0 = (2\pi)(45.0\,\text{MHz})$, which gives a vibrational splitting of $\omega = (2\pi)(0.554\,\text{MHz})$ and a Lamb-Dicke parameter of $\eta = 0.0555$. In terms of the dimensionless variables, the well depth is $U_0/\omega = 1/4\eta^2 = 81.2$ and the potential reaches a maximum at $\bar{z} = \pm\pi/2\sqrt{2}\eta = \pm 20.0$. For this particular value of η, I numerically solve the Schrödinger equation. The resulting spectrum of states is shown in Figure 1.9, and the energy splittings and Rabi frequencies are shown in Figure 1.10. The FORT has ~ 100 bound states, although for the very highest lying states quantum tunneling becomes important and we can no longer consider states to be localized to particular wells.

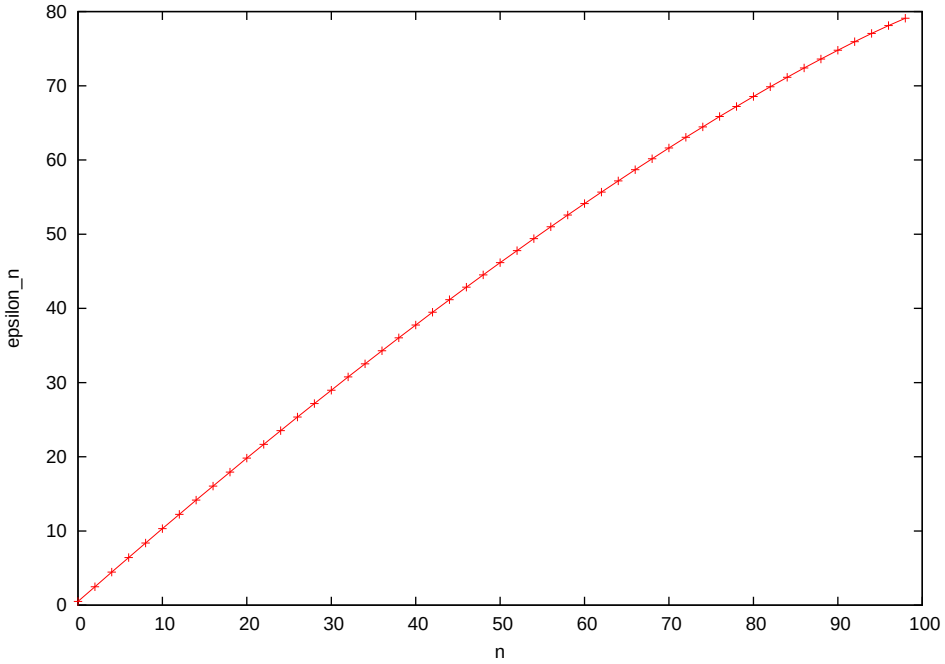

Figure 1.9: Eigenvalues ϵ_n versus n, where I have plotted points for even values of n.

1.5.6 Semiclassical model

To treat the case of full three dimensional motion I've developed a semiclassical approach, in which the internal states of the atom are quantized but the center of mass motion is classical. I will first illustrate the technique for the one dimensional model, so we can compare it against the numerical results obtained in the previous section.

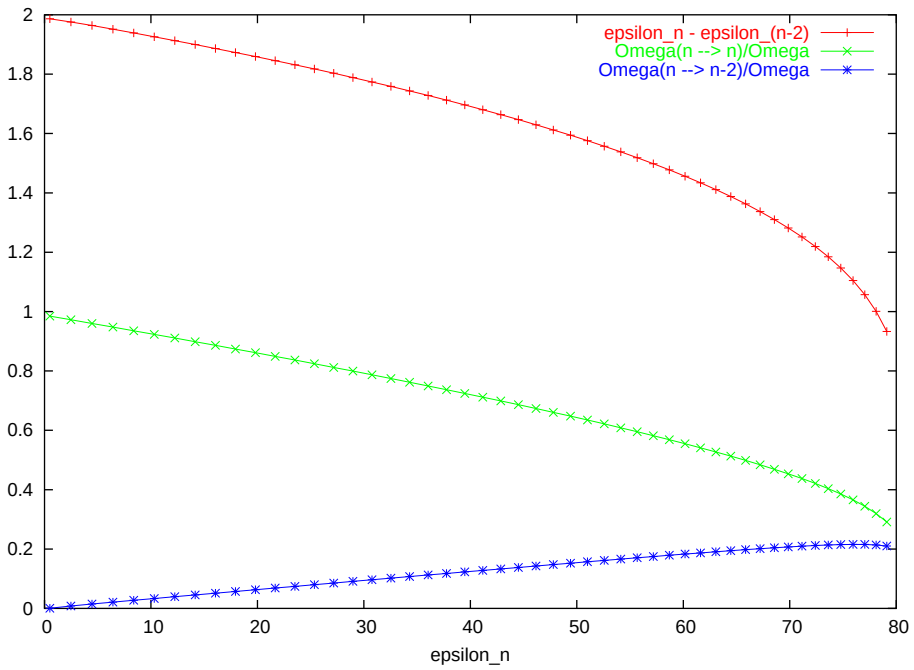

Figure 1.10: The red curve is $\epsilon_n - \epsilon_{n-2}$ versus ϵ_n, the blue curve is $\Omega_{n\to n}/\Omega$ versus ϵ_n, and the green curve is $\Omega_{n\to n-2}/\Omega$ versus ϵ_n. I have plotted points for even values of n.

Recall that the Hamiltonian for the $1D$ model is

$$H = H_{ext} + H_{int}$$

where

$$H_{ext} = \frac{1}{2}\omega(\bar{p}_z^2 + \frac{1}{2\eta^2}\sin^2(\sqrt{2}\eta\bar{z}))$$

and

$$H_{int} = -\frac{\delta}{2}\sigma_z + \frac{\Omega}{2}\cos^2(\sqrt{2}\eta\bar{z})\,\sigma_x$$

If we treat H_{ext} as a classical Hamiltonian, we obtain equations of motion

$$\begin{aligned}\dot{z} &= p_z \\ \dot{p}_z &= -\frac{1}{\sqrt{2}\eta}\sin(\sqrt{2}\eta z)\cos(\sqrt{2}\eta z)\end{aligned}$$

where to simplify the notation I have dropped the bars on z and p_z. If we treat H_{int} as a quantum Hamiltonian, where $z(t)$ is taken as a time-dependent parameter rather than a dynamical variable, we find that the equations of motion for the ground and excited state populations are

$$\dot{c}_b = -\frac{i}{2}(-\delta\, c_b + \Omega_0\cos^2(\sqrt{2}\eta z)\, c_a)$$

$$\dot{c}_a = -\frac{i}{2}(+\delta\, c_a + \Omega_0 \cos^2(\sqrt{2}\eta z)\, c_b)$$

Let us choose initial conditions such that the atom is in its ground state and has kinetic energy E:

$$\begin{aligned} z(0) &= 0 \\ p_z(0) &= \sqrt{2E} \\ c_b(0) &= 0 \\ c_a(0) &= 1 \end{aligned}$$

We can integrate the two sets of equations simultaneously to find the probability $p_b(t)$ of the atom being in the excited state at time t. For a Raman pulse of duration T, I will define an average transfer probability $\bar{p}_b$ by

$$\bar{p}_b = \frac{1}{T}\int_0^T p_b(t)\, dt$$

A graph $\bar{p}_b$ as a function of Raman detuning is shown in Figure 1.11, where curves are shown for both the semiclassical model presented here and for the full quantum solution from the previous section. For this graph, the Rabi frequency is given by $\Omega/\omega = 0.4$ and the duration of the Raman pulse is given by $\omega T = (2\pi)(10.0)$. The curves for the full quantum theory are produced by taking $p_b(t)$ to be the sum of the probabilities for making a transition to the carrier, the second order sideband, and the fourth order sideband:

$$\begin{aligned} p_b(t) = {} & P(\Omega_{n\to n}, \delta, t) + \\ & P(\Omega_{n\to n+2}, \delta - (E_{n+2} - E_n), t) + \\ & P(\Omega_{n\to n+4}, \delta - (E_{n+4} - E_n), t) \end{aligned}$$

where

$$P(\Omega, \Delta, T) = \frac{\Omega^2}{\Omega^2 + \Delta^2}\sin^2((\Omega^2 + \Delta^2)^{1/2}T/2)$$

is the probability that a pulse of Rabi frequency Ω, detuning Δ, and duration T transfers an atom from the ground state to the excited state (see section 1.3.2).

Now let us consider the atomic motion in all three dimensions. The spatial dependence of the beams is given by the mode shape $\psi(\vec{r})$ from section 1.4.1:

$$\Omega_\pm(\vec{r}) = \Omega_\pm\, \psi(\vec{r})$$

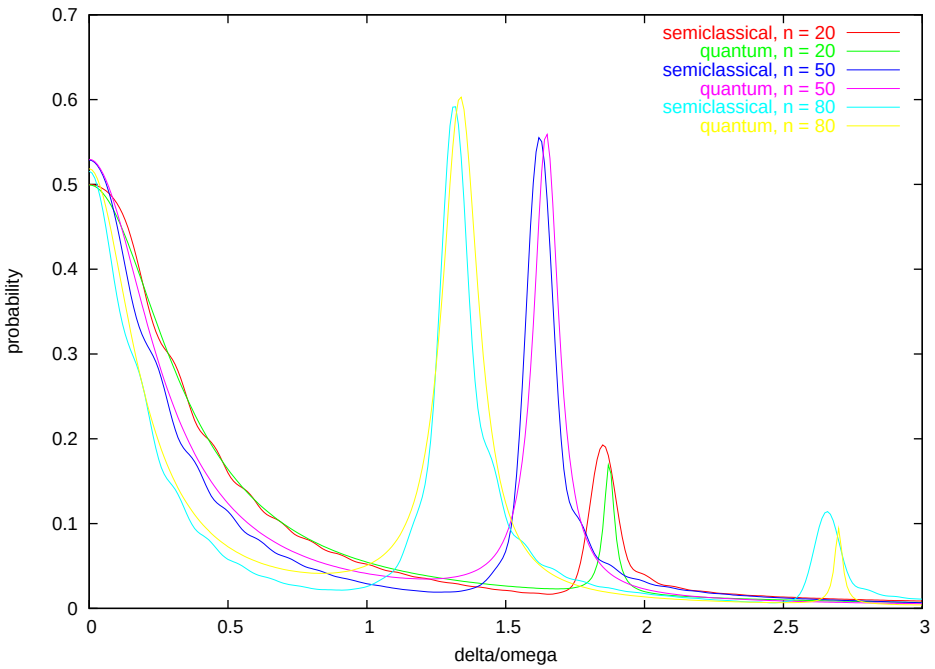

Figure 1.11: Comparison of quantum and semiclassical treatments of the atomic motion. Curves are plotted for atoms with $n = 20, 50, 80$.

Thus, the FORT potential is

$$U(\vec{r}) = -U_0 \, |\psi(\vec{r})|^2 = -U_0 \, \cos^2 kz \, e^{-2(x^2+y^2)/w_0^2}$$

where

$$U_0 \equiv -\frac{\Omega_+^2 + \Omega_-^2}{4\Delta}$$

and the effective Rabi frequency is

$$\Omega_E(\vec{r}) = \Omega \, |\psi(\vec{r})|^2 = \Omega \, \cos^2 kz \, e^{-2(x^2+y^2)/w_0^2}$$

where

$$\Omega \equiv \frac{\Omega_+ \Omega_-}{2\Delta}$$

In these expressions, I have shifted z by $\pi/2k$ relative to the definition of z from section 1.4.1, so that $\vec{r} = 0$ corresponds to the bottom of a FORT well.

The Hamiltonian for the motion is

$$H_{ext}(\vec{r}, \vec{p}) = \frac{p^2}{2m} + U(\vec{r})$$

In the harmonic approximation,

$$\begin{aligned} U(\vec{r}) &\simeq -U_0 + U_0 k^2 z^2 + (2U_0/w_0^2)(x^2+y^2) \\ &\simeq -U_0 + \frac{1}{2} m\omega_a^2 z^2 + \frac{1}{2} m\omega_r^2 (x^2+y^2) \end{aligned}$$

where ω_a and ω_r are the axial and radial vibrational frequencies. Thus, the axial vibrational frequency is given by

$$\frac{1}{2} m\omega_a^2 = U_0 k^2$$

and the radial frequency is given by

$$\frac{1}{2} m\omega_r^2 = 2U_0/w_0^2$$

Thus, the ratio α of radial to axial vibrational frequency is

$$\alpha \equiv \frac{\omega_r}{\omega_a} = \frac{\sqrt{2}}{k w_0} \sim \frac{1}{119}$$

where I have substituted the FORT wavelength $\lambda = 935\,\mathrm{nm}$ and the beam waist size $w_0 = 25.1\,\mu\mathrm{m}$. In order to simulate the motion on a computer, it is convenient to make all the variables dimensionless by scaling them by an appropriate factor:

$$\begin{aligned} t &\rightarrow \omega_a t \\ \vec{r} &\rightarrow k\vec{r} \\ \vec{p} &\rightarrow \frac{k\vec{p}}{m\omega_a} \end{aligned}$$

In terms of the scaled variables, the total energy is

$$E(\vec{r},\vec{p}) = U_0 \left(p^2 - \cos^2 z\, e^{-\alpha^2(x^2+y^2)}\right)$$

and the equations of motion are

$$\begin{aligned} \dot{\vec{r}} &= \vec{p} \\ \dot{p}_x &= -2\alpha^2\, x\, e^{-\alpha^2(x^2+y^2)} \\ \dot{p}_y &= -2\alpha^2\, y\, e^{-\alpha^2(x^2+y^2)} \\ \dot{p}_z &= -\sin z \cos z\, e^{-\alpha^2(x^2+y^2)} \end{aligned}$$

These can be numerically integrated together with the equations of motion for the internal state of the atom to give the transfer probability p_b.

1.6 Many-level atoms

So far we have only considered simple models in which the atom has two or three internal states. I now want to treat the case of a real many-level Cesium atom. After briefly reviewing the Cesium spectrum, I discuss several results that are useful for performing calculations involving many-level atoms, and I apply these results to generalize the calculations performed earlier for two- or three-level atoms to the case of a many-level atom. A good general reference on the Cesium atom is [27].

1.6.1 Cesium spectrum

The portion of the Cesium spectrum that is relevant to the experiments discussed in this thesis is shown in Figure 1.12. The spectrum can be divided into a $6S$ manifold of ground states and a $6P$ manifold of excited states. These manifolds are split into fine structure manifolds by the coupling of the electron spin $\vec{S}$ to the orbital angular momentum $\vec{L}$, which gives total angular momentum $\vec{J} = \vec{L} + \vec{S}$. Since the electron has spin 1/2, there are two excited state fine structure manifolds ($6P_{1/2}$ and $6P_{3/2}$), and a single ground state fine structure manifold ($6S_{1/2}$). The fine structure manifolds are in turn split into hyperfine manifolds by the coupling of $\vec{J}$ to the nuclear spin $\vec{I}$, which gives total angular momentum $\vec{F} = \vec{J} + \vec{I}$. Since the Cesium nucleus has spin 7/2, $6P_{3/2}$ splits into four hyperfine manifolds ($F = 2, 3, 4, 5$), while $6S_{1/2}$ and $6P_{1/2}$ each split into two hyperfine manifolds ($F = 3, 4$).

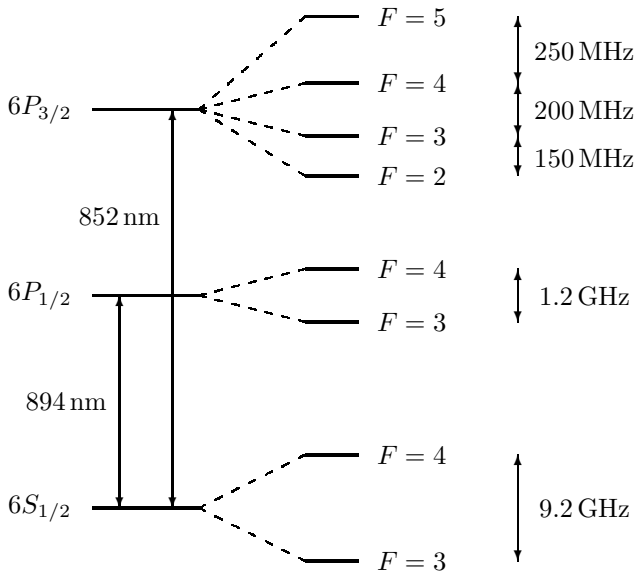

Figure 1.12: Spectrum of the Cesium atom.

The fine structure splitting and hyperfine splitting define important energy scales for the atom.

The fine structure splitting between $6P_{3/2}$ and $6P_{1/2}$ is 16.5 THz, while the hyperfine splitting is much smaller: the $6S_{1/2}, F = 3$ and $6S_{1/2}, F = 4$ manifolds are split by $\Delta_{HF} = (2\pi)(9.2\,\mathrm{GHz})$, the $6P_{1/2}, F = 3$ and $6P_{1/2}, F = 4$ manifolds are split by $\Delta_{HF} = (2\pi)(1.2\,\mathrm{GHz})$, and the hyperfine manifolds within $6P_{3/2}$ are separated by splittings of order hundreds of MHz. States in $6S_{1/2}, F = 4$ can decay to $6S_{1/2}, F = 3$ through a $\vec{\mu} \cdot \vec{B}$ interaction, but because the splitting between the two manifolds is so small this decay is very slow, and to a good approximation both manifolds can be considered stable.

We can form a basis of energy eigenstates for the atom that are also eigenstates of L^2, S^2, I^2, J^2, F^2, and the projection F_z of $\vec{F}$ along an arbitrary axis $\hat{z}$. The states are denoted by their eigenvalues:

$$|6L_J, F, m_F\rangle$$

Sometimes I will shorten this to $|F, m\rangle$ for the ground states and $|F', m'\rangle$ for the excited states, and it will be understood from context to which fine structure manifold the excited state belongs. Due to rotational invariance, the energy of a state is independent of m_F, so each hyperfine manifold F consists of $2F + 1$ degenerate Zeeman states.

1.6.2 Circular polarization vectors

In performing atomic physics calculations with vectors, it is often convenient to choose a basis of circular polarization vectors relative to a coordinate system $\{\hat{x}, \hat{y}, \hat{z}\}$:

$$\hat{e}_{\pm 1} = \mp \frac{1}{\sqrt{2}}(\hat{x} \pm i\hat{y})$$

$$\hat{e}_0 = \hat{z}$$

I will show that the components of a vector with respect to this basis transform in the same way as the spherical harmonics $Y_{1,q}$, and therefore form a spherical tensor of rank one. Note that the circular polarization vectors satisfy the orthonormality relations

$$\hat{e}_q^* \cdot \hat{e}_r = \delta_{qr}$$

and that

$$\hat{e}_q^* = (-1)^q \, \hat{e}_{-q}$$

An arbitrary vector $\vec{A}$ can be expressed as

$$\vec{A} = \sum_q A_q \, \hat{e}_q^*$$

where

$$A_q = \hat{e}_q \cdot \vec{A}$$

If $\vec{A}$ is nonzero, we can define a unit vector $\hat{A} = \vec{A}/|\vec{A}|$, which can be expressed as

$$\hat{A} = \sin\theta \cos\phi \, \hat{x} + \sin\theta \sin\phi \, \hat{y} + \cos\theta \, \hat{z}$$

for some choice of angles θ and ϕ. The spherical harmonics $Y_{1,m}(\hat{A})$ are

$$Y_{1,\pm1}(\hat{A}) = \mp(3/8\pi)^{1/2} \, e^{\pm i\phi} \, \sin\theta = (3/4\pi)^{1/2} \, \hat{A} \cdot \hat{e}_{\pm1}$$

$$Y_{1,0}(\hat{A}) = (3/4\pi)^{1/2} \, \cos\theta = (3/4\pi)^{1/2} \, \hat{A} \cdot \hat{e}_0$$

Thus, the components of $\vec{A}$ are

$$A_q = \hat{e}_q \cdot \vec{A} = (4\pi/3)^{1/2} \, |\vec{A}| \, Y_{1,q}(\hat{A})$$

Therefore, the components A_q transform under rotations in the same way as the spherical harmonics $Y_{1,q}(\hat{A})$.

One must be careful when considering the circular polarization components of the hermitian conjugate of an operator, since the notation $A_q^\dagger$ is ambiguous. I will adopt the convention that

$$A_q^\dagger \equiv \hat{e}_q \cdot \vec{A}^\dagger$$

Note that this is not the same as $(A_q)^\dagger$, which is given by

$$(A_q)^\dagger = (\hat{e}_q \cdot \vec{A})^\dagger = \hat{e}_q^* \cdot \vec{A}^\dagger = (-1)^q \, \hat{e}_{-q} \cdot \vec{A}^\dagger = (-1)^q \, A_{-q}^\dagger$$

1.6.3 Wigner-Eckart theorem and projection theorem

The Wigner-Eckart theorem allows us to express matrix elements of an arbitrary vector operator $\vec{A}$ as a product of a reduced matrix element and a Clebsch-Gordon coefficient (see [28] for a proof of

the theorem):

$$\langle J_2, m_2|A_q|J_1, m_1\rangle = (2J_2+1)^{-1/2}\langle J_2||A||J_1\rangle\langle J_2, m_2|1, q; J_1, m_1\rangle$$

Some authors define their reduced matrix elements such that the $(2J_2+1)^{-1/2}$ factor does not appear, but I prefer this definition, because reduced matrix elements defined in this way have the property that

$$|\langle J_2||A||J_1\rangle| = |\langle J_1||A^\dagger||J_2\rangle|$$

This can be seen from the following considerations. Note that

$$\begin{aligned}
|\langle J_2, m_2|A_q|J_1, m_1\rangle|^2 &= \langle J_1, m_1|\hat{e}_q^* \cdot \vec{A}^\dagger|J_2, m_2\rangle\langle J_2, m_2|\hat{e}_q \cdot \vec{A}|J_1, m_1\rangle \\
&= \langle J_1, m_1|\hat{e}_{-q} \cdot \vec{A}^\dagger|J_2, m_2\rangle\langle J_2, m_2|\hat{e}_{-q}^* \cdot \vec{A}|J_1, m_1\rangle \\
&= |\langle J_1, m_1|A_{-q}^\dagger|J_2, m_2\rangle|^2
\end{aligned}$$

where $A_{-q}^\dagger \equiv \hat{e}_{-q} \cdot \vec{A}^\dagger$. From the Wigner-Eckart theorem,

$$\begin{aligned}
&\sum_q \sum_{m_1} \sum_{m_2} |\langle J_2, m_2|A_q|J_1, m_1\rangle|^2 \\
&\quad = (2J_2+1)^{-1}|\langle J_2||A||J_1\rangle|^2 \sum_q \sum_{m_1} \sum_{m_2} |\langle J_2, m_2|1, q; J_1, m_1\rangle|^2 \\
&\quad = |\langle J_2||A||J_1\rangle|^2
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\sum_q \sum_{m_1} \sum_{m_2} |\langle J_1, m_1|A_{-q}^\dagger|J_2, m_2\rangle|^2 \\
&\quad = (2J_1+1)^{-1}|\langle J_1||A^\dagger||J_2\rangle|^2 \sum_q \sum_{m_1} \sum_{m_2} |\langle J_1, m_1|1, -q; J_2, m_2\rangle|^2 \\
&\quad = |\langle J_1||A^\dagger||J_2\rangle|^2
\end{aligned}$$

Thus, the magnitudes of the two reduced matrix elements are equal.

We can derive a useful corollary to the Wigner-Eckart theorem for the special case $J_1 = J_2 = J$. Note that

$$\begin{aligned}
\langle J, m|\vec{J} \cdot \vec{A}|J, m\rangle &= \sum_q \langle J, m|(\hat{e}_q^* \cdot \vec{J})(\hat{e}_q \cdot \vec{A})|J, m\rangle \\
&= \sum_q \sum_{m'} \langle J, m|\hat{e}_q^* \cdot \vec{J}|J, m'\rangle\langle J, m'|A_q|J, m\rangle
\end{aligned}$$

$$= \quad c_J \langle J||A||J\rangle$$

where

$$c_J \equiv (2J+1)^{-1/2} \sum_q \sum_{m'} \langle J,m|\hat{e}_q^* \cdot \vec{J}|J,m'\rangle \langle J,m'|1,q;J,m\rangle$$

Note that since $\vec{J}\cdot\vec{A}$ is a scalar, c_J does not depend on m. If we substitute $\vec{A} = \vec{J}$ in the above result, we find that

$$\langle J,m|J^2|J,m\rangle = c_J \langle J||J||J\rangle = J(J+1)$$

Thus,

$$c_J = \frac{J(J+1)}{\langle J||J||J\rangle}$$

and

$$\langle J,m|\vec{J}\cdot\vec{A}|J,m\rangle = J(J+1)\frac{\langle J||A||J\rangle}{\langle J||J||J\rangle}$$

From the Wigner-Eckart theorem, we have that

$$\frac{\langle J,m'|A_q|J,m\rangle}{\langle J,m'|J_q|J,m\rangle} = \frac{\langle J||A||J\rangle}{\langle J||J||J\rangle}$$

Thus,

$$\langle J,m'|A_q|J,m\rangle = \frac{1}{J(J+1)} \langle J,m|\vec{J}\cdot\vec{A}|J,m\rangle \langle J,m'|J_q|J,m\rangle$$

This result factors the matrix element of $\vec{A}$ into a matrix element of $\vec{J}$ and a term corresponding to the projection of $\vec{A}$ along $\hat{J}$, and is therefore known as the projection theorem.

1.6.4 Coupling to a magnetic field

The projection theorem can be used to calculate the coupling of an atom to a magnetic field. The coupling Hamiltonian is

$$H_B = -\vec{\mu}\cdot\vec{B}$$

where $\vec{\mu}$, the magnetic moment of the atom, includes contributions from the electron's orbital motion and from its spin:

$$\vec{\mu} = \mu_B(g_L\vec{L} + g_S\vec{S})$$

There is also a contribution from the nuclear spin, but it is smaller by the electron to proton mass ratio and may be neglected. The constant μ_B is the Bohr magneton, which is defined such that $g_L = -1$ for an electron. We can calculate the Bohr magneton as follows. Suppose an electron moves in a circle of radius r at angular frequency ω. The electric current generated by the moving charge is $I = -e\omega/2\pi$, and the angular momentum is $L = m\omega r^2$, so the magnetic moment due to

the electron's orbital motion is

$$\mu = \pi r^2 I = -(e/2m)\, L$$

Thus, the Bohr magneton is given by

$$\mu_B = e/2m = (1/2m)(\alpha\hbar/c)^{1/2} = (2\pi)(1.40\,\mathrm{MHz}\cdot\mathrm{G}^{-1})$$

where I have used the definition of a gauss:

$$\mathrm{G} = \mathrm{g}^{1/2}\cdot\mathrm{cm}^{-1/2}\cdot\mathrm{s}^{-1}$$

The total Hamiltonian for the system is

$$H = H_0 + H_B$$

where H_0 is the Hamiltonian for the atom in the absence of a field. The Hamiltonian H_0 has eigenstates $|F, m\rangle$ that are grouped into hyperfine manifolds F, where the splitting between these manifolds is $\sim 100\,\mathrm{MHz}$ for the excited states and $9.2\,\mathrm{GHz}$ for the ground states. If H_B is small relative to these splittings, then we can approximate its contribution to H using first order perturbation theory:

$$H \simeq H_0 + \sum_F \sum_{m'} \sum_m |F, m'\rangle\langle F, m'|H_B|F, m\rangle\langle F, m|$$

Note that

$$\langle F, m'|H_B|F, m\rangle = -\vec{B}\cdot\langle F, m'|\vec{\mu}|F, m\rangle$$

By the projection theorem,

$$\langle F, m'|\vec{\mu}|F, m\rangle = g_F(F, J, L)\,\mu_B\,\langle F, m'|\vec{F}|F, m\rangle$$

for some constant $g_F(F, J, L)$, which is called the Lande g-factor. Thus, we can express the Hamiltonian as

$$H = H_0 - \mu_B\vec{B}\cdot\vec{F}\sum_F P_F\, g_F$$

where the sum is taken over all hyperfine manifolds F, P_F is a projection operator onto manifold F, and g_F is the Lande g-factor for that manifold.

I now want to evaluate the g-factors. This can be accomplished as follows. By the projection theorem,

$$\langle J, m'|\vec{S}|J, m\rangle = \frac{1}{J(J+1)}\,\langle J, m'|\vec{J}\cdot\vec{S}|J, m\rangle\,\langle J, m'|\vec{J}|J, m\rangle$$

The matrix element of $\vec{J}\cdot\vec{S}$ can be evaluated by using that

$$\vec{J}\cdot\vec{S}=\frac{1}{2}(\vec{J}\cdot\vec{J}+\vec{S}\cdot\vec{S}-|\vec{J}-\vec{S}|^2)=\frac{1}{2}(\vec{J}\cdot\vec{J}+\vec{S}\cdot\vec{S}-\vec{L}\cdot\vec{L})$$

Thus,

$$\langle J,m'|\vec{S}|J,m\rangle=\frac{1}{2}\left(1+\frac{S(S+1)-L(L+1)}{J(J+1)}\right)\,\langle J,m'|\vec{J}|J,m\rangle$$

Similarly,

$$\langle J,m'|\vec{L}|J,m\rangle=\frac{1}{2}\left(1+\frac{L(L+1)-S(S+1)}{J(J+1)}\right)\,\langle J,m'|\vec{J}|J,m\rangle$$

Thus, if we substitute for $\vec{\mu}$, we obtain

$$\begin{aligned}\langle J,m'|\vec{\mu}|J,m\rangle &= \mu_B\left(g_L\,\langle J,m'|\vec{L}|J,m\rangle+g_S\,\langle J,m'|\vec{S}|J,m\rangle\right)\\ &= \mu_B\,g_J(J,L)\,\langle J,m'|\vec{J}|J,m\rangle\end{aligned}$$

where

$$g_J(J,L)=\frac{1}{2}(g_L+g_S)+\frac{1}{2}(g_L-g_S)\left(\frac{L(L+1)-S(S+1)}{J(J+1)}\right)$$

Similarly, using that

$$\langle F,m'|\vec{J}|F,m\rangle=\frac{1}{2}\left(1+\frac{J(J+1)-I(I+1)}{F(F+1)}\right)\,\langle F,m'|\vec{F}|F,m\rangle$$

we find that

$$\langle F,m'|\vec{\mu}|F,m\rangle=\mu_B\,g_F(F,J,L)\,\langle F,m'|\vec{F}|F,m\rangle$$

where

$$g_F(F,J,L)=\frac{1}{2}\left(1+\frac{J(J+1)-I(I+1)}{F(F+1)}\right)\,g_J(J,L)$$

Substituting $g_S=-2$, $g_L=-1$, we obtain the following g-factors:

L_J	F	$g_F(L,J,F)$
$P_{3/2}$	5	$+2/5$
$P_{3/2}$	4	$+4/15$
$P_{3/2}$	3	0
$P_{3/2}$	2	$-2/3$
$P_{1/2}$	4	$+1/12$
$P_{1/2}$	3	$-1/12$
$S_{1/2}$	4	$+1/4$
$S_{1/2}$	3	$-1/4$

1.6.5 Relating reduced matrix elements

Suppose we add angular momentum $\vec{J}$ to angular momentum $\vec{I}$ to get total angular momentum $\vec{F} = \vec{J} + \vec{I}$. I want to show that if $\vec{A}$ is a vector operator that does not couple to $\vec{I}$, then we can relate the reduced matrix elements $\langle F'||A||F\rangle$ and $\langle J'||A||J\rangle$:

$$\langle F'||A||F\rangle = -(-1)^{J+I+F'}\sqrt{2F+1}\sqrt{2F'+1}\begin{Bmatrix} 1 & J & J' \\ I & F' & F \end{Bmatrix}\langle J'||A||J\rangle$$

I will call this the reduced matrix element reduction formula.

The quantity in brackets is a $6J$ symbol, which is defined as follows. Consider a system of three spins. One could describe the system using a product basis:

$$\{|J_1, m_1\rangle|J_2, m_2\rangle|J_3, m_3\rangle\}$$

Alternatively, one could couple spins J_1 and J_2 together to get a spin J_{12}, then couple J_{12} with J_3 to get a total spin J (see Figure 1.13). I will denote these states by

$$\{|((J_1, J_2)J_{12}, J_3)J, m\rangle\}$$

As an example of this notation, consider the Cesium hyperfine states. They are obtained by coupling $\vec{L}$ to $\vec{S}$ to get $\vec{J}$, and then coupling $\vec{J}$ to $\vec{I}$ to get $\vec{F}$. Thus, in this notation they would be expressed as

$$\{|((L, S)J, I)F, m\rangle\}$$

A third possibility is to couple J_2 with J_3 to get J_{23}, and then couple J_1 with J_{23} to get total spin J (see Figure 1.13):

$$\{|(J_1, (J_2, J_3)J_{23})J, m\rangle\}$$

The $6J$ symbols relate these last two descriptions of the system. They are defined by

$$\langle((J_1, J_2)J_{12}, J_3)J, m|(J_1, (J_2, J_3)J_{23})J, m\rangle = (-1)^{J_1+J_2+J_3+J}\sqrt{2J_{12}+1}\sqrt{2J_{23}+1}\begin{Bmatrix} J_1 & J_2 & J_{12} \\ J_3 & J & J_{23} \end{Bmatrix}$$

Note that the overlap does not depend on m. The $6J$ symbol is symmetric under permutation of columns and under exchange of rows for any pair of columns.

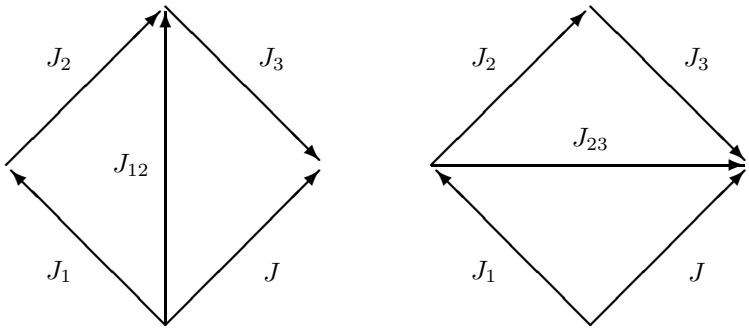

Figure 1.13: Two different methods for coupling J_1, J_2, and J_3 to get total spin J.

Now, let us return to the derivation of the reduced matrix element reduction formula. Note that we may expand matrix elements of the operator $\vec{A}$ as

$$\langle F', m'|A_q|F, m\rangle = \sum_{m'_J}\sum_{m_I}\sum_{m_J}\langle F', m'|J', m'_J; I, m_I\rangle\langle J', m'_J|A_q|J, m_J\rangle\langle J, m_J; I, m_I|F, m\rangle$$

where I have used that $\vec{A}$ does not couple to $\vec{I}$. From the Wigner-Eckart theorem,

$$\langle J', m'_J|A_q|J, m_J\rangle = (2J'+1)^{-1/2}\langle J'||A||J\rangle\langle J', m'_J|1, q; J, m_J\rangle$$

Also,

$$\langle F', m'|A_q|F, m\rangle = (2F'+1)^{-1/2}\langle F'||A||F\rangle\langle F', m'|1, q; F, m\rangle$$

Substituting these results into our expansion of $\langle F', m'|A_q|F, m\rangle$, we obtain the following relation:

$$\frac{(2F'+1)^{-1/2}\langle F'||A||F\rangle}{(2J'+1)^{-1/2}\langle J'||A||J\rangle}\langle F', m'|1, q; F, m\rangle = \sum_{m'_J}\sum_{m_I}\sum_{m_J}\langle F', m'|J', m'_J; I, m_I\rangle\langle J', m'_J|1, q; J, m_J\rangle\langle J, m_J; I, m_I|F, m\rangle$$

The ratio of reduced matrix elements can be expressed in terms of a $6J$ symbol. First,

$$\begin{aligned}
&|(1,(J,I)F)F', m'\rangle \\
&= \sum_q\sum_m\langle 1, q; F, m|F', m'\rangle\,|1, q\rangle\,|(J,I)F, m\rangle \\
&= \sum_m\sum_q\sum_{m_J}\sum_{m_I}\langle 1, q; F, m|F', m'\rangle\,\langle J, m_J; I, m_I|F, m\rangle\,|1, q\rangle\,|J, m_J\rangle\,|I, m_I\rangle
\end{aligned}$$

Next,

$$\begin{aligned}
&\langle((1,J)J',I)F',m'| \\
&\quad= \sum_{m'_J}\sum_{m_I}\langle F',m'|J',m'_J;I,m_I\rangle\,\langle(1,J)J',m'_J|\,\langle I,m_I| \\
&\quad= \sum_{m'_J}\sum_{q}\sum_{m_J}\sum_{m_I}\langle F',m'|J',m'_J;I,m_I\rangle\,\langle J',m'_J|1,q;J,m_J\rangle\,\langle 1,q|\,\langle J,m_J|\,\langle I,m_I|
\end{aligned}$$

Putting all this together, we find that the overlap is

$$\begin{aligned}
&\langle((1,J)J',I)F',m'|(1,(J,I)F)F',m'\rangle \\
&\quad= \frac{(2F'+1)^{-1/2}\,\langle F'||A||F\rangle}{(2J'+1)^{-1/2}\,\langle J'||A||J\rangle}\sum_q\sum_m\langle F',m'|1,q;F,m\rangle\langle 1,q;F,m|F',m'\rangle \\
&\quad= \frac{(2F'+1)^{-1/2}\,\langle F'||A||F\rangle}{(2J'+1)^{-1/2}\,\langle J'||A||J\rangle}
\end{aligned}$$

where I have used completeness on $1\otimes F$. From the definition of the 6J symbols, we also have

$$\begin{aligned}
&\langle((1,J)J',I)F',m'|(1,(J,I)F)F',m'\rangle = \\
&\qquad -(-1)^{J+I+F'}\sqrt{2J'+1}\sqrt{2F+1}\begin{Bmatrix} 1 & J & J' \\ I & F' & F\end{Bmatrix}
\end{aligned}$$

Thus, we obtain the desired result:

$$\langle F'||A||F\rangle = -(-1)^{J+I+F'}\sqrt{2F+1}\sqrt{2F'+1}\begin{Bmatrix} 1 & J & J' \\ I & F' & F\end{Bmatrix}\langle J'||A||J\rangle$$

1.6.6 Dipole matrix elements

In performing atomic physics calculations we often need to evaluate matrix elements of $\vec{r}$ between two atomic states. In this section, I will show how these matrix elements can be evaluated.

I'll start by considering matrix elements of $\vec{r}$ between atomic states of good orbital angular momentum $\vec{L}$. Using the Wigner-Eckart theorem, we can express these matrix elements as

$$\langle 6P,m'|r_q|6S,0\rangle = (1/3)^{1/2}\,\langle 6P||r||6S\rangle\,\langle 1,m'|1,q;0,0\rangle = D\,\delta_{m',q}$$

where

$$D = (1/3)^{1/2}\,\langle 6P||r||6S\rangle$$

is the dipole moment for the $S \rightarrow P$ transition.

Now I want to consider matrix elements of $\vec{r}$ between two fine structure states (that is, states of good $\vec{J}$). From the Wigner-Eckart theorem,

$$\langle J', m'|r_q|J, m\rangle = (2J'+1)^{-1/2} \langle J'||r||J\rangle \langle J', m'|1, q; J, m\rangle$$

We can relate the reduced matrix elements $\langle J'||r||J\rangle$ to the dipole moment D as follows. Since r^2 is a scalar and does not couple to the electronic spin, we have that

$$\langle J', m'|r^2|J, m'\rangle = \langle 6P, 0|r^2|6S, 0\rangle = |\langle 6P, 0|r_0|6S, 0\rangle|^2 = D^2$$

But if we insert a complete set of states and apply the Wigner-Eckart theorem, we can also express this as

$$\begin{aligned}
\langle J', m'|r^2|J, m'\rangle &= \sum_q \sum_m |\langle J', m'|r_q|J, m\rangle|^2 \\
&= (2J'+1) |\langle J'||r||J\rangle|^2 \sum_q \sum_m |\langle J', m'|1, q; J, m\rangle|^2 \\
&= (2J'+1) |\langle J'||r||J\rangle|^2
\end{aligned}$$

where I have used completeness on $1 \otimes J$. Equating the two expressions, we find that

$$D = (2J'+1)^{1/2} \langle J'||r||J\rangle$$

Thus, the dipole matrix elements are

$$\langle J', m'|r_q|J, m\rangle = D \langle J', m'|1, q; J, m\rangle$$

Finally, I'll consider matrix elements of $\vec{r}$ between two hyperfine states (that is, states of good $\vec{F}$). From the Wigner-Eckart theorem,

$$\langle F', m'|r_q|F, m\rangle = (2F'+1)^{-1/2} \langle F'||r||F\rangle \langle F', m'|1, q; F, m\rangle$$

Since $\vec{r}$ does not couple to the nuclear spin $\vec{I}$, we may apply the reduced matrix element reduction formula to express $\langle F'||r||F\rangle$ in terms of $\langle J'||r||J\rangle$:

$$\begin{aligned}
(2F'+1)^{-1/2} \langle F'||r||F\rangle &= (2J'+1)^{-1/2} \langle J'||r||J\rangle \beta_{J'}(F, F') \\
&= D \beta_{J'}(F, F')
\end{aligned}$$

where I have defined

$$\beta_{J'}(F,F') = -(-1)^{J+I+F'}\sqrt{2J'+1}\sqrt{2F+1}\begin{Bmatrix} 1 & J & J' \\ I & F' & F \end{Bmatrix}$$

Thus, the dipole matrix elements are

$$\langle F', m'|r_q|F, m\rangle = D\,\beta_{J'}(F,F')\,\langle F', m'|1, q; F, m\rangle$$

Here are Mathematica routines to calculate the factors $\beta_{J'}(F', F)$ and the dipole matrix elements $\langle F', m'|r_q|F, m\rangle$:

```
beta[Jp_, Fp_, F_] :=
  -(-1)^Fp*Sqrt[2Jp + 1]*Sqrt[2F + 1]*
    SixJSymbol[{1, 1/2, Jp}, {7/2, Fp, F}]

dipole[Jp_, Fp_, mp_, F_, m_] :=
  D*beta[Jp, Fp, F]*
    ClebschGordan[{1, mp - m}, {F, m}, {Fp, mp}]
```

where D should be defined as the value of the dipole moment in whatever system of units is being used. Using the first routine, we find that the factors $\beta_{J'}(F', F)$ are as follows:

For $J' = 3/2$,

F'	2	3	3	4	4	5
F	3	3	4	3	4	4
$\beta_{3/2}$	1	$\sqrt{3/4}$	$\sqrt{1/4}$	$\sqrt{5/12}$	$\sqrt{7/12}$	1

For $J' = 1/2$,

F'	3	3	4	4
F	3	4	3	4
$\beta_{1/2}$	$-\sqrt{1/4}$	$\sqrt{3/4}$	$-\sqrt{7/12}$	$\sqrt{5/12}$

Note that the factors obey the sum rule

$$\sum_F |\beta_{J'}(F', F)|^2 = 1$$

In the next section, I will explain why this is true.

The dipole moment D can be expressed in a number of different ways. I defined D in terms of

the reduced matrix element for the $S \rightarrow P$ transition:

$$D = (1/3)^{1/2} \langle 6P||r||6S\rangle$$

The dipole moment D is also equal to the dipole matrix element for the $|4,4\rangle \leftrightarrow |5',5'\rangle$ cycling transition:

$$D = \langle 6P_{3/2}, 5, 5|r_{+1}|6S_{1/2}, 4, 4\rangle$$

If we express D in terms of the Bohr radius a_0, we find

$$D = 3.166\, a_0$$

Later we shall see that D can be related to a saturation intensity I_{sat}:

$$D = (8\pi\alpha I_{sat})^{-1/2}\, \gamma$$

where

$$\gamma = \gamma_{D2} = (2\pi)(5.22\,\mathrm{MHz})$$

is the decay rate for the $D2$ line and

$$I_{sat} = (1/6\pi)\, \omega_{D2}^3\, \gamma = (\hbar c/6\pi)(2\pi)^3(\gamma/\lambda^3) = 2.19\ \mathrm{mW/cm^2}$$

is the saturation intensity for the $D2$ line.

1.6.7 Atomic raising and lowering operators

We can define an atomic raising operator for the many-level atom by

$$\vec{A}^{\dagger} = D^{-1}\, P(e)\, \vec{r}\, P(g)$$

where

$$P(g) = |6S,0\rangle\langle 6S,0| = \sum_m |J,m\rangle\langle J,m| = \sum_F \sum_m |F,m\rangle\langle F,m|$$

projects onto the manifold of ground states, and

$$P(e) = \sum_{m'} |6P,m'\rangle\langle 6P,m'| = \sum_{J'} \sum_{m'} |J',m'\rangle\langle J',m'| = \sum_{F'} \sum_{m'} |F',m'\rangle\langle F',m'|$$

projects onto the manifold of excited states. If we substitute the different forms of the projection operators into our expression for $\vec{A}^{\dagger}$, and use the results of the previous section to evaluate the

resulting dipole matrix elements of $\vec{r}$, we find that

$$\begin{aligned}
\vec{A}^\dagger &= D^{-1}\sum_q P(e)\, r_q\, P(g)\, \hat{e}_q^* \\
&= \sum_q |6P,q\rangle\langle 6S,0|\, \hat{e}_q^* \\
&= \sum_q \sum_{J'} \sum_{m'} \sum_m \langle J',m'|1,q;J,m\rangle\, |J',m'\rangle\langle J,m|\, \hat{e}_q^* \\
&= \sum_q \sum_{F'} \sum_{m'} \sum_F \sum_m \beta_{J'}(F,F')\, \langle F',m'|1,q;F,m\rangle\, |F',m'\rangle\langle F,m|\, \hat{e}_q^*
\end{aligned}$$

The operator $\vec{A}^\dagger$ is the many-level atom generalization of the σ_+ operator for a two-level atom.

The reduced matrix elements of $\vec{A}^\dagger$ can be obtained from the reduced matrix elements of $\vec{r}$, which were calculated in the previous section:

$$\begin{aligned}
\langle 6P||A^\dagger||6S\rangle &= 3^{1/2} \\
\langle J'||A^\dagger||J\rangle &= (2J'+1)^{1/2} \\
\langle F'||A^\dagger||F\rangle &= (2F'+1)^{1/2}\, \beta_{J'}(F',F)
\end{aligned}$$

By taking the hermitian conjugate of $\vec{A}^\dagger$, we obtain an atomic lowering operator $\vec{A}$. We can express $\vec{r}$ in terms of the raising and lowering operators:

$$\vec{r} = D(\vec{A} + \vec{A}^\dagger)$$

From

$$\vec{A}^\dagger = \sum_q |6P,q\rangle\langle 6S,0|\, \hat{e}_q^*$$

we see that

$$\vec{A}\cdot\vec{A}^\dagger = 3P(g)$$

and

$$\vec{A}^\dagger\cdot\vec{A} = P(e)$$

We can use this last result to prove the sum rule

$$\sum_F |\beta_{J'}(F',F)|^2 = 1$$

The proof goes as follows. Since $\vec{A}^\dagger \cdot \vec{A} = P(e)$, we have

$$\langle F', m'|\vec{A}^\dagger \cdot \vec{A}|F', m'\rangle = 1$$

But if we insert a complete set of states $\{|F, m\rangle\}$, and substitute

$$\vec{A}^\dagger = \sum_q \sum_{F'} \sum_{m'} \sum_F \sum_m \beta_{J'}(F, F') \langle F', m'|1, q; F, m\rangle |F', m'\rangle\langle F, m| \hat{e}_q^*$$

we find

$$\begin{aligned}
\langle F', m'|\vec{A}^\dagger \cdot \vec{A}|F', m'\rangle &= \sum_q \sum_F \sum_m |\langle F', m'|\hat{e}_q \cdot \vec{A}^\dagger|F, m\rangle|^2 \\
&= \sum_F |\beta_{J'}(F, F')|^2 \sum_q \sum_m |\langle F', m'|1, q; F, m\rangle|^2 \\
&= \sum_F |\beta_{J'}(F', F)|^2
\end{aligned}$$

Equating these two expressions, we obtain the sum rule.

The operator $\vec{A}^\dagger$ connects the $6S$ manifold of ground states to the entire $6P$ manifold of excited states. But sometimes we will want an operator that only connects ground states to a particular submanifold of excited states. I will define operators

$$\vec{A}^\dagger(J') = P(J')\, \vec{A}^\dagger$$

which connect ground states to excited states in the fine structure manifold J', and operators

$$\vec{A}^\dagger(J', F') = P(J', F')\, \vec{A}^\dagger$$

which connect ground states to excited states in the hyperfine manifold F' that lies in the fine structure manifold J'. Note that

$$\sum_{F'} \vec{A}^\dagger(J', F') = \vec{A}^\dagger(J')$$

and

$$\sum_{J'} \vec{A}^\dagger(J') = \vec{A}$$

1.6.8 Spontaneous decay rate

I now want to calculate the spontaneous decay rate for Cesium. We will start with a two-level atom, and then generalize to the full Cesium atom. The atom decays via the interaction Hamiltonian

$$H_i = -e\vec{r}\cdot\vec{E}$$

where $\vec{E}$ is the quantized electric field, evaluated at the position of the electron. In the dipole approximation we can just evaluate $\vec{E}$ at the position of the atom, which I take to be at the origin. In the radiation gauge, the electric field is given by

$$\vec{E} = -\partial_t \vec{A}$$

where $\vec{A}$, the vector potential, is obtained by quantizing the modes of a box with periodic boundary conditions:

$$\vec{A} = \sum_{\vec{k}\lambda} (2\pi/\omega_{\vec{k}} V)^{1/2} \left(a_{\vec{k}\lambda}\, \hat{\epsilon}_{\vec{k}\lambda}\, e^{-i(\omega_{\vec{k}} t - \vec{k}\cdot\vec{r})} + a^\dagger_{\vec{k}\lambda}\, \hat{\epsilon}^*_{\vec{k}\lambda}\, e^{i(\omega_{\vec{k}} t - \vec{k}\cdot\vec{r})}\right)$$

Here V is the volume of the box, and $a_{\vec{k}\lambda}$ and $a^\dagger_{\vec{k}\lambda}$ are creation and annihilation operators for mode $\vec{k}\lambda$. The vectors $\hat{\epsilon}_{\vec{k}\lambda}$ are photon polarization vectors; they are orthogonal to each other and to $\hat{k}$, so the set $\{\hat{k}, \hat{\epsilon}_{\vec{k}1}, \hat{\epsilon}_{\vec{k}2}\}$ forms an orthonormal frame. A convenient coordinate representation for these vectors is

$$\begin{aligned}
\hat{k} &= \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z} \\
\hat{\epsilon}_{\vec{k}1} &= \cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z} \\
\hat{\epsilon}_{\vec{k}2} &= -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}
\end{aligned}$$

Using the above expression for the vector potential, we find that the electric field at the position of the atom is

$$\vec{E} = -\partial_t \vec{A} = i \sum_{\vec{k}\lambda} (2\pi\omega_{\vec{k}}/V)^{1/2} \left(a_{\vec{k}\lambda}\, \hat{\epsilon}_{\vec{k}\lambda}\, e^{-i\omega_{\vec{k}} t} - a^\dagger_{\vec{k}\lambda}\, \hat{\epsilon}^*_{\vec{k}\lambda}\, e^{i\omega_{\vec{k}} t}\right)$$

In the Schrödinger picture,

$$\vec{E} = i \sum_{\vec{k}\lambda} (2\pi\omega_{\vec{k}}/V)^{1/2} \left(a_{\vec{k}\lambda}\, \hat{\epsilon}_{\vec{k}\lambda} - a^\dagger_{\vec{k}\lambda}\, \hat{\epsilon}^*_{\vec{k}\lambda}\right)$$

If we substitute this into the interaction Hamiltonian and make the rotating wave approximation, we obtain

$$H_i = -e\,\vec{r}\cdot\vec{E} = i \sum_{\vec{k}\lambda} (g_{\vec{k}\lambda}\, a_{\vec{k}\lambda}\, \sigma_+ - g^*_{\vec{k}\lambda}\, a^\dagger_{\vec{k}\lambda}\, \sigma_-)$$

where

$$g_{\vec{k}\lambda} = (2\pi\alpha\omega_{\vec{k}}/V)^{1/2} \langle e|\vec{r}\cdot\hat{\epsilon}_{\vec{k}\lambda}|g\rangle$$

gives the coupling of the atom to the mode $\vec{k}\lambda$.

We can calculate the spontaneous decay rate from the interaction Hamiltonian by applying Fermi's golden rule:

$$d^3\gamma = 2\pi|\langle f|H_i|i\rangle|^2\,\delta(E_f - E_i)\,d^3n$$

Suppose the atom decays from an excited state e to a ground state g by emitting a photon into mode $\vec{k}\lambda$. The matrix element for this process is

$$\langle g, 1_{\vec{k}\lambda}|H_i|e,0\rangle = ig_{\vec{k}\lambda}$$

The density of final states is given by

$$d^3n = (2\pi)^{-3}\,V\,dk^3 = (2\pi)^{-3}\,V\,\omega^2\,d\omega\,d\Omega$$

Thus, the decay rate per unit solid angle for emitting a photon in direction $\hat{k}$ is

$$\frac{d\gamma}{d\Omega} = (2\pi)^{-1}\alpha\omega^3 \sum_\lambda |\langle e|\vec{r}\cdot\hat{\epsilon}_{\vec{k}\lambda}|g\rangle|^2$$

where $\omega = E_e - E_g$ is the atomic transition frequency. Note that

$$|\langle e|\vec{r}\cdot\hat{\epsilon}_{\vec{k}\lambda}|g\rangle|^2 = \sum_q |\hat{e}_q^*\cdot\hat{\epsilon}_{\vec{k}\lambda}|^2\,|\langle e|r_q|g\rangle|^2$$

Substituting for the circular polarization vectors $\hat{e}_q^*$, we find

$$\begin{aligned}
|\hat{e}_0^*\cdot\hat{\epsilon}_{\vec{k}1}|^2 &= \sin^2\theta \\
|\hat{e}_0^*\cdot\hat{\epsilon}_{\vec{k}2}|^2 &= 0 \\
|\hat{e}_{\pm1}^*\cdot\hat{\epsilon}_{\vec{k}1}|^2 &= \frac{1}{2}\cos^2\theta \\
|\hat{e}_{\pm1}^*\cdot\hat{\epsilon}_{\vec{k}2}|^2 &= \frac{1}{2}
\end{aligned}$$

Thus,

$$\sum_\lambda |\langle e|\vec{r}\cdot\hat{\epsilon}_{\vec{k}\lambda}|g\rangle|^2 = \sin^2\theta\,|\langle e|r_0|g\rangle|^2 + \frac{1}{2}(1+\cos^2\theta)\,(|\langle e|r_{+1}|g\rangle|^2 + |\langle e|r_{-1}|g\rangle|^2)$$

If we substitute this into our expression for the decay rate per unit solid angle and integrate over all possible photon emission directions $\hat{k}$, we obtain the total decay rate:

$$\gamma(e \to g) = \frac{4}{3}\alpha\omega^3 \sum_q |\langle e|r_q|g\rangle|^2 = \frac{4}{3}\alpha\omega^3 \langle g|\vec{r}|e\rangle \cdot \langle e|\vec{r}|g\rangle$$

Now let us consider the full Cesium atom. We can use this result to calculate the decay rate from excited state $|F', m'\rangle$ to ground state $|F, m\rangle$:

$$\gamma(|F', m'\rangle \to |F, m\rangle) = \frac{4}{3}\alpha\omega_{J'}^3 \sum_q |\langle F', m'|r_q|F, m\rangle|^2$$

If we substitute for the dipole matrix element, we obtain

$$\gamma(|F', m'\rangle \to |F, m\rangle) = \gamma_{J'} |\beta_{J'}(F', F)|^2 |\langle F', m'|1, m' - m; F, m\rangle|^2$$

where

$$\gamma_{J'} = \frac{4}{3}\alpha\omega_{J'}^3 D^2$$

From completeness on $1 \otimes F$ we have

$$\sum_m |\langle F', m'|1, m' - m; F, m\rangle|^2 = 1$$

Thus, the total decay rate from $|F', m'\rangle$ to the ground state manifold F is

$$\sum_m \gamma(|F', m'\rangle \to |F, m\rangle) = \gamma_{J'} |\beta_{J'}(F', F)|^2$$

Note that this rate is independent of m'. From the sum rule

$$\sum_F |\beta_{J'}(F', F)|^2 = 1$$

we see that the total decay rate out of $|F', m'\rangle$ is

$$\sum_F \sum_m \gamma(|F', m'\rangle \to |F, m\rangle) = \gamma_{J'}$$

Note that this rate is independent of the excited state hyperfine manifold F'.

We can also use our expression for the quantized atom/field interaction to derive the coupling of a classical beam of light to the atom. Suppose mode $\vec{k}\lambda$ is in a coherent state $\alpha_{\vec{k}\lambda}$, and all other

modes are in their ground states. Then, tracing over the field variables, we obtain

$$Tr_F[H_i] = (1/2)(\Omega\sigma_+ + \Omega^*\sigma_-)$$

where

$$\Omega = 2i\, g_{\vec{k}\lambda}\, \alpha_{\vec{k}\lambda}$$

Thus,

$$|\Omega|^2 = (8\pi\alpha n\omega/V)\,|\langle e|\vec{r}\cdot\hat{\epsilon}_{\vec{k}\lambda}|g\rangle|^2 = (8\pi\alpha I)\,|\langle e|\vec{r}\cdot\hat{\epsilon}_{\vec{k}\lambda}|g\rangle|^2$$

where $n = |\alpha_{\vec{k}\lambda}|^2$ is the number of photons and $I = n\omega/V$ is the intensity of the beam. This agrees with the expression for the Rabi frequency we obtained in section 1.3.2.

1.6.9 Coupling to cavity mode

I now want to consider the coupling of a many-level atom to a cavity mode. The atomic dipole $\vec{p}$ couples to the cavity field $\vec{E}$ at the position of the atom via the Hamiltonian

$$H_i = -\vec{p}\cdot\vec{E}$$

The atomic dipole may be expressed in terms of the generalized atomic raising and lowering operators:

$$\vec{p} = e\vec{r} = eD(\vec{A} + \vec{A}^\dagger)$$

The quantized cavity field is given by

$$\vec{E} = i(2\pi\omega_c/V)^{1/2}\ \psi(\vec{r})\,(a\,\hat{\epsilon}\,e^{-i\omega t} - a^\dagger\,\hat{\epsilon}^*\,e^{i\omega t})$$

where ω_c and $\hat{\epsilon}$ are the frequency and polarization of the cavity mode, $\psi(\vec{r})$ describes the mode shape, and V is the mode volume.

Note that if the birefringent splitting of the cavity is small, as is the case for our cavity, then modes occur in nearly-degenerate pairs. One of the modes in each pair is linearly polarized along one birefringent axis; the other mode is polarized along the orthogonal birefringent axis. To describe the coupling of the atom to the cavity, we need to include coupling Hamiltonians for both of the modes that are resonant with the atom.

If we substitute for the atomic dipole and the quantized cavity field, we find that the coupling

Hamiltonian is

$$H_i = -ig_0\,\psi(\vec{r})\,(a\,\hat{\epsilon}\,e^{-i\omega t} - a^\dagger\,\hat{\epsilon}^*\,e^{i\omega t})\cdot(\vec{A}+\vec{A}^\dagger)$$

where I have defined a coupling constant

$$g_0 = (2\pi\alpha\omega_c/V)^{1/2}\,D$$

If we use our expression for the atomic decay rate

$$\gamma = \frac{4}{3}\alpha\omega_a^3 D^2$$

and approximate $\omega_c \simeq \omega_a$, then we can express the coupling constant as

$$g_0 = (3/16\pi^2)^{1/2}\,(Q\lambda^3/V)^{1/2}\,\gamma$$

where $Q = \omega_a/\gamma$ is the quality factor of the atom, and $\lambda = 2\pi/\omega_a$ is the wavelength.

The operators $\vec{A}$ and $\vec{A}^\dagger$ are Schrödinger picture operators. We can transform to the interaction picture by defining operators

$$\vec{A}^\dagger(F,F') = P(F')\,\vec{A}^\dagger\,P(F)$$

which connect the ground state hyperfine manifold F to the excited state hyperfine manifold F'. The operator $\vec{A}$ can then be expressed as

$$\vec{A}^\dagger = \sum_F \sum_{F'} \vec{A}^\dagger(F,F')$$

In the interaction picture, the operator $\vec{A}^\dagger$ takes the form

$$\vec{A}^\dagger \rightarrow \sum_F \sum_{F'} \vec{A}^\dagger(F,F')\,e^{i\omega_{F,F'}}$$

where $\omega_{F,F'}$ is the $F \rightarrow F'$ transition frequency. If we substitute this into the interaction Hamiltonian H_i and make the rotating wave approximation, we obtain coupling terms for all possible atomic transitions. The terms corresponding to a $F \rightarrow F'$ transition evolve in time at a rate given by the detuning $\Delta_{F,F'} = \omega - \omega_{F,F'}$. For the situations we will be interested in, the cavity is nearly resonant with one of these transitions and is far detuned from all the others. We can neglect the couplings to the far detuned transitions and retain only the coupling to the nearly resonant $F \rightarrow F'$ transition:

$$H_i = -ig_0\,\psi(\vec{r})\,(a\,\hat{\epsilon}\cdot\vec{A}^\dagger(F,F')\,e^{-i\Delta_{F,F'}t} - a^\dagger\,\hat{\epsilon}^*\cdot\vec{A}(F,F')\,e^{i\Delta_{F,F'}t})$$

1.6.10 Coupling to classical fields

We will often want to describe the coupling of the atom to a laser beam. The atomic dipole $\vec{p}$ couples to the electric field $\vec{E}$ of the beam via the coupling Hamiltonian

$$H_i = -\vec{p} \cdot \vec{E}$$

The dipole may be expressed in terms of the generalized atomic raising and lowering operators:

$$\vec{p} = e\vec{r} = eD(\vec{A} + \vec{A}^\dagger)$$

As for the two-level atom we discussed in section 1.3.2, the electric field is given by

$$\vec{E} = \frac{1}{2}(8\pi I)^{1/2} \left(\hat{\epsilon}\, e^{-i\omega t} + \hat{\epsilon}^*\, e^{i\omega t}\right)$$

where I and $\hat{\epsilon}$ are the intensity and polarization of the light at the position of the atom. Thus,

$$H_i = \frac{\Omega_0}{2} \left(\hat{\epsilon}\, e^{-i\omega t} + \hat{\epsilon}^*\, e^{i\omega t}\right) \cdot (\vec{A} + \vec{A}^\dagger)$$

Here

$$\Omega_0 = (8\pi\alpha I)^{1/2}\, D = \gamma (I/I_{sat})^{1/2}$$

where I have defined a saturation intensity I_{sat} relative to the dipole D:

$$\gamma = (8\pi\alpha I_{sat})^{1/2}\, D$$

We will be interested in situations where the light is nearly resonant with a single atomic transition $F \rightarrow F'$ and far detuned from all the others. Thus, if we make the rotating wave approximation, and neglect the couplings to the far detuned transitions, we can approximate the coupling Hamiltonian as

$$H_i = \frac{\Omega_0}{2} \left(\hat{\epsilon} \cdot \vec{A}^\dagger(F, F')\, e^{-i\omega t} + \hat{\epsilon}^* \cdot \vec{A}(F, F')\, e^{i\omega t}\right)$$

In the rotating wave approximation this can also be expressed as

$$H_i = \Omega_0 \left(\hat{\epsilon} \cdot \vec{A}^\dagger(F, F') + \hat{\epsilon}^* \cdot \vec{A}(F, F')\right) \cos\omega t$$

1.7 Far detuned light

Both the FORT and the Raman scheme involve the use of far detuned light. In the context of a three-level model, we have seen how the coupling of far detuned light to the atom can be accounted

for by adiabatically eliminating the excited state. In this section I want to generalize these results the case of a many-level Cesium atom.

1.7.1 Ground state transition amplitudes

When we adiabatically eliminate the excited states for a many-level atom, we obtain ground state transition amplitudes of the form

$$\sum_e \frac{1}{\Delta_e} \langle F_2, m_2 | r_i | e \rangle \langle e | r_j | F_1, m_1 \rangle$$

where the sum is carried out over all excited states e. In general, the excited states can be grouped into a set of manifolds $\{E\}$ such that for each manifold the energy splittings between states within that manifold are small relative to the detuning of the light. For each manifold E in the set, we can neglect the differences in detunings among the states in E and approximate them all by a single detuning Δ_E for the entire manifold. We can then express the sum as

$$\sum_E \frac{1}{\Delta_E} \sum_{e \in E} \langle F_2, m_2 | r_i | e \rangle \langle e | r_j | F_1, m_1 \rangle$$

The case we will be most interested in is one where the detuning of the light is large compared to the excited state hyperfine splitting but small relative to the fine structure splitting, so for this case we would group the excited states into fine structure manifolds $6P_{1/2}$ and $6P_{3/2}$.

To evaluate the ground state transition amplitudes, it is helpful to express $\vec{r}$ in terms of atomic raising and lowering operators that couple ground states to excited states in the manifold E:

$$\vec{r} = D \sum_E (\vec{A}(E) + \vec{A}^\dagger(E))$$

Thus,

$$\sum_{e \in E} \langle F_2, m_2 | r_i | e \rangle \langle e | r_j | F_1, m_1 \rangle = D^2 \, \langle F_2, m_2 | A_i(E) \, A_j^\dagger(E) | F_1, m_1 \rangle$$

The tensor $A_i(E) \, A_j^\dagger(E)$ may be evaluated by decomposing it into its irreducible parts (for a discussion of this type of decomposition, see [28] or [29]):

$$A_i(E) \, A_j^\dagger(E) = T^{(0)}(E) \, \delta_{ij} + T_k^{(1)}(E) \, \epsilon_{kij} + T_{ij}^{(2)}(E)$$

where

$$T^{(0)}(E) = \frac{1}{3} \, \vec{A}(E) \cdot \vec{A}^\dagger(E)$$

transforms as a scalar (spin 0),

$$T_k^{(1)}(E) = \frac{1}{2}\,\epsilon_{ijk}\,A_i(E)\,A_j^\dagger(E)$$

transforms as a vector (spin 1), and

$$T_{ij}^{(2)}(E) = \frac{1}{2}\,(A_i(E)\,A_j^\dagger(E) + A_j(E)\,A_i^\dagger(E)) - \frac{1}{3}\,\vec{A}(E)\cdot\vec{A}^\dagger(E)\,\delta_{ij}$$

transforms as a traceless symmetric tensor (spin 2).

If we choose E to be a hyperfine manifold F' within a fine structure manifold J', then in general all three components will contribute:

$$A_i(J',F')\,A_j^\dagger(J',F') = T^{(0)}(J',F')\,\delta_{ij} + T_k^{(1)}(J',F')\,\epsilon_{kij} + T_{ij}^{(2)}(J',F')$$

If we choose E to be a fine structure manifold J', then we can ignore the contribution of the nuclear spin and consider eigenstates of J^2 and J_z. Since the ground state has $J = 1/2$, and it is not possible to couple spin 1/2 to spin 1/2 via spin 2, the tensor component vanishes:

$$A_i(J')\,A_j^\dagger(J') = T^{(0)}(J')\,\delta_{ij} + T_k^{(1)}(J')\,\epsilon_{kij}$$

Note that

$$\begin{aligned}
\sum_{F'} T_{ij}^{(2)}(J',F') &= 0 \\
\sum_{F'} T_k^{(1)}(J',F') &= T_k^{(1)}(J') \\
\sum_{F'} T^{(0)}(J',F') &= T^{(0)}(J')
\end{aligned}$$

Finally, if we group all the $6P$ excited states into a single manifold, then we can ignore both the nuclear spin and the electron spin and consider eigenstates of L^2 and L_z. Since the ground state has $L = 0$, only the scalar component contributes:

$$A_i\,A_j^\dagger = T^{(0)}\,\delta_{ij}$$

Note that

$$\sum_{J'} \vec{T}^{(1)}(J') = 0$$

$$\sum_{J'} T^{(0)}(J') \quad = \quad T^{(0)}$$

I now want to evaluate the irreducible tensors. The tensor $T^{(0)}$ is given by

$$T^{(0)} = \frac{1}{3}\vec{A}\cdot\vec{A}^{\dagger} = P(g) = 1$$

The tensors $T^{(0)}(J')$ may be evaluated as follows:

$$T^{(0)}(J') = \langle J,m|T^{(0)}(J')|J,m\rangle = \frac{1}{3}\langle J,m|\vec{A}(J')\cdot\vec{A}^{\dagger}(J')|J,m\rangle$$

If we insert a complete set of states, we obtain

$$\begin{aligned} T^{(0)}(J') &= \frac{1}{3}\sum_q\sum_{m'}|\langle J,m|A_q|J',m'\rangle|^2 \\ &= \frac{1}{3}(2J+1)^{-1}\,|\langle J||A||J'\rangle|^2\sum_q\sum_{m'}|\langle J,m|1,q;J',m'\rangle|^2 \\ &= \frac{1}{3}(2J+1)^{-1}\,|\langle J||A||J'\rangle|^2 \end{aligned}$$

The reduced matrix element is given by

$$|\langle J||A||J'\rangle|^2 = |\langle J'||A^{\dagger}||J\rangle|^2 = (2J'+1)^{1/2}$$

So

$$T^{(0)}(J') = (1/3)(2J'+1)(2J+1)^{-1} = (1/6)(2J'+1)$$

Thus,

$$\begin{aligned} T^{(0)}(1/2) &= \frac{1}{3} \\ T^{(0)}(3/2) &= \frac{2}{3} \end{aligned}$$

Note that

$$\sum_{J'} T^{(0)}(J') = T^{(0)}$$

as expected.

It is a little more complicated to evaluate $T^{(1)}(J')$. Since $\vec{T}^{(1)}(J')$ only acts on ground states, which all have $J = 1/2$, we can apply the projection theorem to express it as

$$\vec{T}^{(1)}(J') = c_{J'}\,\vec{J}$$

for some constant $c_{J'}$. Recall that

$$\sum_{J'} \vec{T}^{(1)}(J') = 0$$

Thus, $c_{3/2} = -c_{1/2}$. We can find $c_{1/2}$ by considering a particular matrix element:

$$\langle 6S_{1/2}, 1/2|T_z^{(1)}(1/2)|6S_{1/2}, 1/2\rangle = c_{1/2}\,\langle 6S_{1/2}, 1/2|\vec{J}|6S_{1/2}, 1/2\rangle$$

Note that

$$\begin{aligned} T_z^{(1)}(J') &= \frac{1}{2}(A_x(J')\,A_y^\dagger(J') - A_y(J')\,A_x^\dagger(J')) \\ &= \frac{i}{2}(A_{+1}(J')\,A_{-1}^\dagger(J') - A_{-1}(J')\,A_{+1}^\dagger(J')) \end{aligned}$$

where $A_q^\dagger \equiv \hat{e}_q \cdot \vec{A}^\dagger$. Thus,

$$\begin{aligned} &\langle 6S_{1/2}, 1/2|T_z^{(1)}(1/2)|6S_{1/2}, 1/2\rangle \\ &= \frac{i}{2}\langle 6S_{1/2}, 1/2|A_{+1}(1/2)\,A_{-1}^\dagger(1/2))|6S_{1/2}, 1/2\rangle \\ &= -\frac{i}{2}|\langle 6P_{1/2}, -1/2|A_{-1}^\dagger(1/2)|6S_{1/2}, 1/2\rangle|^2 \\ &= -\frac{i}{2}|\langle 1/2, -1/2|1, -1; 1/2, 1/2\rangle|^2 \\ &= -\frac{i}{3} \end{aligned}$$

Also,

$$\langle 6S_{1/2}, 1/2|J_z|6S_{1/2}, 1/2\rangle = 1/2$$

So

$$c_{1/2} = -\frac{2}{3}i$$

and

$$\vec{T}^{(1)}(3/2) = -\vec{T}^{(1)}(1/2) = +\frac{2}{3}i\vec{J}$$

If we combine our expressions for $T^{(0)}(J')$ and $T^{(1)}(J')$, we find

$$\begin{aligned} A_i(3/2)\,A_j^\dagger(3/2) &= \frac{2}{3}\,\delta_{ij} + \frac{2}{3}i\,\epsilon_{ijk}\,J_k \\ A_i(1/2)\,A_j^\dagger(1/2) &= \frac{1}{3}\,\delta_{ij} - \frac{2}{3}i\,\epsilon_{ijk}\,J_k \end{aligned}$$

Using this result, we can express the ground state transition amplitudes as

$$\sum_{J'} \frac{1}{\Delta_{J'}} A_i(J')\,A_j^\dagger(J') = (1/3)[(2/\Delta_3 + 1/\Delta_1)\,\delta_{ij} + 2i(1/\Delta_3 - 1/\Delta_1)\,\epsilon_{ijk}\,J_k]$$

where Δ_1 and Δ_3 are the detunings of the light from the $D1$ and $D2$ lines:

$$\Delta_k = 2\pi/\lambda_L - 2\pi/\lambda_k$$

It is convenient to introduce some dimensionless detuning parameters

$$C_k = (\lambda_L/\lambda_k - 1)^{-1} = -2\pi/\lambda_L \Delta_k$$

At the FORT wavelength of $\lambda_L = 935\,\mathrm{nm}$, these parameters are

$$\begin{aligned} C_3 &= (\lambda_L/\lambda_3 - 1)^{-1} = 10.3 \\ C_1 &= (\lambda_L/\lambda_1 - 1)^{-1} = 22.1 \end{aligned}$$

In terms of the detuning parameters, the ground state transition amplitudes are

$$\sum_{J'} \frac{1}{\Delta_{J'}} A_i(J')\, A_j^\dagger(J') = -(\lambda_L/6\pi)[(2C_3 + C_1)\,\delta_{ij} + 2i(C_3 - C_1)\,\epsilon_{ijk}\,J_k]$$

1.7.2 Matrix elements of $\vec{J}$

In the previous section, we showed that the ground state transition amplitudes can be expressed in terms of $\vec{J}$. I now want to evaluate the matrix elements of $\vec{J}$ between two hyperfine states:

$$\langle F_2, m_2 | \vec{J} | F_1, m_1 \rangle$$

For $F_1 = F_2 = F$, we can use the projection theorem to express matrix elements of $\vec{J}$ in terms of matrix elements of $\vec{F}$. Recall that g_F is defined by

$$\langle F, m_2 | \vec{\mu} | F, m_1 \rangle = \mu_B\, g_F\, \langle F, m_2 | \vec{F} | F, m_1 \rangle$$

The ground states have no orbital angular momentum, so

$$\vec{\mu} = \mu_B\, g_S\, \vec{S} = \mu_B\, g_S\, \vec{J}$$

Thus, since $g_S = -2$,

$$\langle F, m_2 | \vec{J} | F, m_1 \rangle = -\frac{1}{2} g_F\, \langle F, m_2 | \vec{F} | F, m_1 \rangle$$

In particular

$$\langle F, m | J_z | F, m \rangle = -\frac{1}{2} g_F\, m$$

For the general case of arbitrary F_1 and F_2, we can apply the Wigner-Eckart theorem:

$$\langle F_2, m_2|J_q|F_2, m_1\rangle = (2F_2+1)^{-1/2}\langle F_2||J||F_1\rangle\langle F_2, m_2|1, q; F_1, m_1\rangle$$

Since $\vec{J}$ does not couple to the nuclear spin, we can use the reduced matrix element reduction formula to express $\langle F_2||J||F_1\rangle$ in terms of $\langle J||J||J\rangle$:

$$\langle F_2||J||F_1\rangle = -(-1)^{J+I+F_2}\sqrt{2F_1+1}\sqrt{2F_2+1}\left\{\begin{array}{ccc} 1 & J & J \\ I & F_2 & F_1 \end{array}\right\}\langle J||J||J\rangle$$

The reduced matrix element $\langle J||J||J\rangle$ can be evaluated as follows. Note that

$$\langle J, m|J^2|J, m\rangle = J(J+1)$$

If we insert a complete set of states and apply the Wigner-Eckart theorem, we find

$$\langle J, m|J^2|J, m\rangle = \sum_q\sum_{m'}|\langle J, m|J_q|J, m'\rangle|^2 = (2J+1)^{-1}|\langle J||J||J\rangle|^2$$

Thus,

$$\langle J||J||J\rangle = (J(J+1)(2J+1))^{1/2}$$

Putting this all together, we find

$$\langle F_2, m_2|J_q|F_2, m_1\rangle = \\ -(3/2)^{1/2}(-1)^{F_2}\sqrt{2F_1+1}\left\{\begin{array}{ccc} 1 & 1/2 & 1/2 \\ 7/2 & F_2 & F_1 \end{array}\right\}\langle F_2, m_2|1, q; F_1, m_1\rangle$$

For J_{+1},

$$\begin{aligned} \langle 4, m+1|J_{+1}|4, m\rangle &= +\sqrt{4-m}\sqrt{5+m}/8\sqrt{2} \\ \langle 3, m+1|J_{+1}|3, m\rangle &= -\sqrt{3-m}\sqrt{4+m}/8\sqrt{2} \\ \langle 4, m+1|J_{+1}|3, m\rangle &= -\sqrt{4+m}\sqrt{5+m}/8\sqrt{2} \\ \langle 3, m+1|J_{+1}|4, m\rangle &= +\sqrt{3-m}\sqrt{4-m}/8\sqrt{2} \end{aligned}$$

For J_0,

$$\begin{aligned} \langle 4, m|J_0|4, m\rangle &= -m/8 \\ \langle 3, m|J_0|3, m\rangle &= +m/8 \end{aligned}$$

$$\begin{aligned}\langle 4,m|J_0|3,m\rangle &= -\sqrt{16-m^2}/8 \\ \langle 3,m|J_0|4,m\rangle &= -\sqrt{16-m^2}/8\end{aligned}$$

For J_{-1},

$$\begin{aligned}\langle 4,m-1|J_{-1}|4,m\rangle &= -\sqrt{5-m}\sqrt{4+m}/8\sqrt{2} \\ \langle 3,m-1|J_{-1}|3,m\rangle &= +\sqrt{3+m}\sqrt{4-m}/8\sqrt{2} \\ \langle 4,m-1|J_{-1}|3,m\rangle &= -\sqrt{4-m}\sqrt{5-m}/8\sqrt{2} \\ \langle 3,m-1|J_{-1}|4,m\rangle &= +\sqrt{3+m}\sqrt{4+m}/8\sqrt{2}\end{aligned}$$

The operators $\vec{J}_{\pm 1}$ can be related to the angular momentum raising and lowering operators $J_\pm$ as follows. The raising and lowering operators are defined by

$$J_\pm = J_x \pm iJ_y$$

Thus,

$$J_{\pm 1} = \hat{e}_{\pm 1}\cdot\vec{J} = \mp\frac{1}{\sqrt{2}}J_\pm$$

1.7.3 FORT potential

For the three-level model discussed in section 1.5.2, we found that the effective Hamiltonian for the ground states was

$$H_E = \frac{1}{2}\delta_A\sigma_z + U$$

where

$$U = \frac{\Omega^2}{4\Delta}$$

For the full Cesium atom, the effective Hamiltonian is

$$H_E = \frac{1}{2}\Delta_{HF}(P_4 - P_3) + \hat{U}$$

where

$$\hat{U} = \sum_E \frac{1}{4\Delta_E}\hat{\Omega}(E)\,\hat{\Omega}^\dagger(E)$$

and P_3 and P_4 are projection operators onto the $F=3$ and $F=4$ ground state hyperfine manifolds. The operator $\hat{\Omega}(E)$ is the generalization of the Rabi frequency Ω; it is given by

$$\hat{\Omega}^\dagger(E) = \gamma\,(I/I_{sat})^{1/2}\,\hat{\epsilon}\cdot\vec{A}^\dagger(E)$$

where I and $\hat{\epsilon}$ are the intensity and polarization of the FORT beam. Thus, using the results of section 1.7.1, we find

$$\hat{U} = \frac{\gamma^2}{12}\frac{I}{I_{sat}}\left(\frac{2}{\Delta_3} + \frac{1}{\Delta_1} + 2i\left(\frac{1}{\Delta_3} - \frac{1}{\Delta_1}\right)(\hat{\epsilon}^* \times \hat{\epsilon}) \cdot \vec{J}\right)$$

We will assume that the FORT beam propagates in the $\hat{z}$ direction, so the polarization vector $\hat{\epsilon}$ lies in the $x - y$ plane and can be expressed as

$$\hat{\epsilon} = (\hat{e}_{+1} \cdot \hat{\epsilon})\,\hat{e}^*_{+1} + (\hat{e}_{-1} \cdot \hat{\epsilon})\,\hat{e}^*_{-1}$$

Note that

$$\hat{e}_{+1} \times \hat{e}^*_{+1} = -\hat{e}_{-1} \times \hat{e}^*_{-1} = -i\hat{z}$$

Thus,

$$\hat{\epsilon}^* \times \hat{\epsilon} = -i(|\hat{e}_{+1} \cdot \hat{\epsilon}|^2 - |\hat{e}_{-1} \cdot \hat{\epsilon}|^2)\,\hat{z}$$

So

$$\hat{U} = \frac{\gamma^2}{12}\frac{I}{I_{sat}}\left(\frac{2}{\Delta_3} + \frac{1}{\Delta_1} + 2\left(\frac{1}{\Delta_3} - \frac{1}{\Delta_1}\right)(|\hat{e}_{+1} \cdot \hat{\epsilon}|^2 - |\hat{e}_{-1} \cdot \hat{\epsilon}|^2)\,J_z\right)$$

Note that J_z couples the $F = 3$ and $F = 4$ ground states. However, because $\Delta_{HF} \gg \hat{U}$, this coupling can be neglected, and we can make the replacement

$$J_z \to -\frac{1}{2} g_F F_z$$

where $g_4 = 1/4$ and $g_3 = -1/4$. Thus, the Hamiltonian can be approximated by

$$H_E = \frac{1}{2}\Delta_{HF}(P_4 - P_3) + U(\vec{r}) + g_F\,\mu_B\,\vec{B}_E(\vec{r}) \cdot \vec{F}$$

where

$$U(\vec{r}) = \frac{\gamma^2}{12}\frac{I(\vec{r})}{I_{sat}}\left(\frac{2}{\Delta_3} + \frac{1}{\Delta_1}\right) = -\frac{\gamma^2 \lambda_F}{24\pi}\frac{I(\vec{r})}{I_{sat}}(2C_3 + C_1)$$

is a state independent potential, and

$$\vec{B}_E(\vec{r}) = -\frac{1}{\mu_B}\frac{\gamma^2}{12}\frac{I(\vec{r})}{I_{sat}}\left(\frac{1}{\Delta_3} - \frac{1}{\Delta_1}\right)(|\hat{e}_{+1} \cdot \hat{\epsilon}|^2 - |\hat{e}_{-1} \cdot \hat{\epsilon}|^2)\,\hat{z}$$

is an effective magnetic field. It is convenient to express U in terms of the input power to the cavity. Recall that the ratio of the maximum intensity inside the cavity I_c to the input power P_i is

$$I_c/P_i = 4/\kappa L A$$

The FORT depth U_F is given by

$$U_F = \frac{\gamma^2 \lambda_F}{24\pi} \frac{I_c}{I_{sat}} (2C_3 + C_1)$$

Thus, the ratio of the FORT depth to the input power is

$$U_F/P_i = (2\pi)(12\pi^2)^{-1}(2C_3 + C_1)(\gamma^2 \lambda_F/\kappa L A I_{sat}) = 37.0\,\mathrm{MHz/mW}$$

In deriving the expression for U_F, we assumed that the detunings were the same for the $F = 3$ and $F = 4$ ground states. This is a reasonable approximation, because the hyperfine splitting Δ_{HF} between $F = 3$ and $F = 4$ is much smaller than the detunings. However, because the detuning of the $F = 3$ manifold is slightly larger than the detuning for $F = 4$, the FORT potential is slightly weaker for $F = 3$. Thus, the FORT squeezes the two manifolds together, causing a small reduction δ in the effective hyperfine splitting. We can calculate δ as follows. The FORT depth is

$$U_F = \frac{\gamma^2}{12} \frac{I_c}{I_{sat}} \left(\frac{2}{\Delta_3} + \frac{1}{\Delta_1} \right)$$

So

$$\begin{aligned} \delta &= \frac{\gamma^2}{12} \frac{I_c}{I_{sat}} \left(\frac{2}{\Delta_3} + \frac{1}{\Delta_1} \right) - \frac{\gamma^2}{12} \frac{I_c}{I_{sat}} \left(\frac{2}{\Delta_3 + \Delta_{HF}} + \frac{1}{\Delta_1 + \Delta_{HF}} \right) \\ &\simeq \frac{\gamma^2}{12} \frac{I_c}{I_{sat}} \left(\frac{2}{\Delta_3} \frac{\Delta_{HF}}{\Delta_3} + \frac{1}{\Delta_1} \frac{\Delta_{HF}}{\Delta_1} \right) \\ &\simeq -U \left(\frac{2C_3^2 + C_1^2}{2C_3 + C_1} \right) \frac{\Delta_{HF} \lambda_F}{2\pi c} \end{aligned}$$

Substituting the FORT wavelength of $\lambda_F = 935\,\mathrm{nm}$, we find that the ratio of the shift δ to the FORT depth U_F is

$$\delta/U_F = -470\,\mathrm{Hz/MHz}$$

Although this is a small shift, it is possible to observe it in the lab (see section 1.9.8).

So far I have only considered the Stark shifts of the ground states. The excited states are also shifted by the FORT; however, the situation is more complicated than for the ground states because additional transitions contribute to the shifts. By choosing the FORT wavelength λ_F properly, one can exploit these additional transitions to match the excited state shift to the ground state shift, resulting in a state insensitive FORT. Such a FORT has two advantages over an ordinary, state dependent FORT. First, an atom that makes transitions between ground and excited states will not be heated by a fluctuating FORT potential, as would be the case for an ordinary FORT. Second, and more importantly, the detunings of the atom from any additional light beams that are present

are independent of the position of the atom within the FORT. For Cesium, the magic wavelength that results in a state insensitive FORT is $\lambda_F = 935$ nm, which fortunately lies very close to one of the modes of our cavity. The state insensitive FORT is discussed in depth in [30] and [31].

For the experiments discussed in this thesis we use far detuned light in order to create a state independent trapping potential $U(\vec{r})$, and the effective magnetic field $\vec{B}_E(\vec{r})$ is an unwanted side effect, which we eliminate by using linearly polarized light. But it is worth keeping in mind that far detuned light could also be used for the specific purpose of generating an effective magnetic field. By using three circularly polarized beams propagating along three orthogonal axes one could generate a field of arbitrary magnitude and direction, and if the beams were large enough the field would be nearly uniform over the region of interest. The effective magnetic field would have the advantage that it could be varied or turned on and off much more rapidly than a real field generated by bias coils.

Another possibility would be to create more complicated FORT potentials by driving multiple modes of the cavity. By varying the intensities of the different driving beams, one could even change the shape of the potential in real time. Perhaps this technique could be used to translate atoms along the cavity axis so as to optimize their coupling to the QED probe.

1.7.4 Raman transitions

For the three-level model discussed in section 1.5.3, we found that the Raman coupling was

$$H_i = \Omega_E \, \sigma_x \cos \delta_R t$$

where

$$\Omega_E = \frac{\Omega_R \Omega_F}{2\Delta}$$

For the full Cesium atom, this generalizes to

$$H_i = \hat{\Omega}_E \cos \delta_R t$$

where

$$\hat{\Omega}_E = \sum_E \frac{1}{2\Delta_E} \, \hat{\Omega}_F(E) \, \hat{\Omega}_R^\dagger(E)$$

Here

$$\hat{\Omega}_k^\dagger(E) = \gamma \, (I_k / I_{sat})^{1/2} \, \hat{\epsilon}_k \cdot \vec{A}^\dagger(E)$$

where I_F, I_R and $\hat{\epsilon}_F$, $\hat{\epsilon}_R$ are the intensities and polarizations of the FORT and Raman beams. Thus, using the results of section 1.7.1, we find

$$\begin{aligned}\hat{\Omega}_E &= (\gamma^2/6)\,(I_F I_R/I_{sat}^2)^{1/2}\,((2/\Delta_3 + 1/\Delta_1)\,\hat{\epsilon}_F^* \cdot \hat{\epsilon}_R + \\ &\quad 2i(1/\Delta_3 - 1/\Delta_1)\,(\hat{\epsilon}_F^* \times \hat{\epsilon}_R) \cdot \vec{J})\end{aligned}$$

I will assume that the FORT and Raman beams are linearly polarized and mutually orthogonal, so the first term vanishes and

$$\hat{k} \equiv \hat{\epsilon}_F^* \times \hat{\epsilon}_R$$

is a unit vector that lies along the cavity axis. Thus, $\{\hat{\epsilon}_R, \hat{\epsilon}_F, \hat{k}\}$ forms an orthonormal frame. The quantization axis $\hat{z}$ can be chosen to lie in an arbitrary direction relative to this frame. For a given choice of $\hat{z}$, we can define vectors $\hat{x}$ and $\hat{y}$ such that $\{\hat{x}, \hat{y}, \hat{z}\}$ also forms an orthonormal frame, which is related to $\{\hat{\epsilon}_R, \hat{\epsilon}_F, \hat{k}\}$ by

$$\begin{aligned}\hat{x} &= \cos\theta\cos\phi\,\hat{\epsilon}_F + \cos\theta\sin\phi\,\hat{\epsilon}_R - \sin\theta\,\hat{k} \\ \hat{y} &= -\sin\phi\,\hat{\epsilon}_F + \cos\phi\,\hat{\epsilon}_R \\ \hat{z} &= \sin\theta\cos\phi\,\hat{\epsilon}_F + \sin\theta\sin\phi\,\hat{\epsilon}_R + \cos\theta\,\hat{k}\end{aligned}$$

Note that θ represents the angle between the cavity axis $\hat{k}$ and the quantization axis $\hat{z}$. From these relations, it follows that

$$\hat{k} = \cos\theta\,\hat{z} - \sin\theta\,\hat{x}$$

Let us now calculate the Rabi frequencies for transitions between states that are defined relative to quantization axis $\hat{z}$. The Rabi frequency for $|4, m_2\rangle \leftrightarrow |3, m_1\rangle$ is

$$\Omega(|4, m_2\rangle \leftrightarrow |3, m_1\rangle) = \langle 4, m_2|\hat{\Omega}_E|3, m_1\rangle$$

It is convenient to express $\hat{\Omega}_E$ in terms of the Rabi frequency Ω_0 for the transition $|4, 0\rangle \leftrightarrow |3, 0\rangle$ with quantization axis $\hat{z} = \hat{k}$. This is given by

$$\begin{aligned}\Omega_0 &\equiv \langle 4, 0|\hat{\Omega}_E|3, 0\rangle \\ &= i(\gamma^2/3)\,(I_F I_R/I_{sat}^2)^{1/2}\,(1/\Delta_3 - 1/\Delta_1)\,\langle 4, 0|J_z|3, 0\rangle \\ &= -i(\gamma^2/6)\,(I_F I_R/I_{sat}^2)^{1/2}\,(1/\Delta_3 - 1/\Delta_1)\end{aligned}$$

Thus,

$$\hat{\Omega}_E = -2\Omega_0\,\hat{k} \cdot \vec{J} = -\Omega_0\,(2J_z\,\cos\theta + \sqrt{2}J_{-1}\,\sin\theta - \sqrt{2}J_{+1}\,\sin\theta)$$

The Rabi frequencies for $\Delta m = 0$ transitions are

$$\langle 4, m|\hat{\Omega}_E|3, m\rangle = (1 - m^2/16)^{1/2}\,\Omega_0\,\cos\theta$$

and the Rabi frequencies for $\Delta m = +1$ transitions are

$$\langle 4, m+1|\hat{\Omega}_E|3, m\rangle = -\frac{1}{8}(4+m)^{1/2}(5+m)^{1/2}\,\Omega_0\,\sin\theta$$

I will sometimes speak of the Rabi frequency of the Raman beam; by this I mean the frequency Ω_0 at the bottom of a FORT well. We can relate the value of Ω_0 at the bottom a FORT well to the FORT intensity U_F as follows. Recall that for a linearly polarized FORT,

$$U_F = (\gamma^2/12)(I_F/I_{sat})(2/\Delta_3 + 1/\Delta_1)$$

where I_F is the maximum intensity of the FORT beam inside the cavity. Thus,

$$\Omega_0/U_F = -2i\left(\frac{C_3 - C_1}{2C_3 + C_1}\right)(I_R/I_F)^{1/2} = -i(0.553)(I_R/I_F)^{1/2}$$

where I_R is the maximum intensity of the Raman beam inside the cavity. I will assume the FORT beam is resonant with the cavity, while the Raman beam is detuned by $\Delta_{HF} = (2\pi)(9.2\,\mathrm{GHz})$. Thus, we can relate the ratio of Raman to FORT intracavity intensity to the ratio of Raman to FORT input power by

$$I_R/I_F = (1 + (2\Delta_{HF}/\kappa)^2)^{-1}(P_R/P_F)$$

where P_F and P_R are the input powers of the FORT and Raman beams, and $\kappa = (2\pi)(1.6\,\mathrm{GHz})$ is the full width at half maximum of the cavity resonance at the FORT wavelength. Thus,

$$\begin{aligned}\Omega_0/U_F &= -2i\left(\frac{C_3 - C_1}{2C_3 + C_1}\right)(1 + (2\Delta_{HF}/\kappa)^2)^{-1/2}(P_R/P_F)^{1/2} \\ &= -i(0.048)(P_R/P_F)^{1/2}\end{aligned}$$

1.8 Experimental apparatus

1.8.1 Overview of experiment

At the heart of the experiment is a vacuum system consisting of two chambers: an upper chamber, which is connected to a Cesium source, and a lower chamber, which contains the physics cavity. A typical experimental cycle begins by loading a cloud of cold atoms in a magneto-optical trap (MOT) in the upper chamber. The cloud is then dropped into the lower chamber, where it is recaptured in

a second MOT situated directly above the physics cavity. Finally, this second MOT is dropped on the cavity, and atoms falling into the cavity mode are cooled into a FORT by beams that enter the cavity from the side. Once an atom has been loaded we are ready to perform a cavity QED experiment. This generally involves driving the atom with various light fields (applied either from the side of the cavity or coupled into a cavity mode) and collecting the output light with photon counters. The photon counters generate a stream of clicks, which are recorded by a computer for later analysis.

Many different pieces of equipment are needed to make all this work. Since most of this has been thoroughly documented elsewhere [32],[33],[31], I will just give a brief overview of the main components.

1.8.2 MOTs

The magneto-optical trap (MOT) is a standard tool for cooling and trapping atoms [34], [35], [36]. To form a MOT, we combine trapping light that is red detuned from $4-5'$ transition with a magnetic quadrupole field generated by a pair of anti-Helmholtz coils. The trapping light is delivered by three pairs of counter-propagating beams oriented along three orthogonal axes that intersect at the center of the quadrupole field. Atoms at the center of the trap are driven by light that is red detuned, so they don't scatter very strongly. However, if an atom moves away from the center, it sees a magnetic field that Zeeman shifts it into resonance with the light. If the polarizations of the beams are chosen properly, then atomic selection rules will only permit the atom to absorb light from beams that push it back toward the center of the trap. Thus, the combination of the trapping light and the magnetic field produces a restoring force that confines atoms to the center of the trap. In addition, the red detuned trapping light provides Doppler cooling, which damps the atomic motion and allows the trap to be loaded from background vapor. Using a MOT, a cloud of $\sim 10^7$ atoms can be loaded in $\sim 1\,\mathrm{s}$. The atoms in the MOT have a temperature that is set by the Doppler limit of $\gamma/2$, which for Cesium is $125\,\mu\mathrm{K}$.

The MOT loading rate depends on the pressure of background Cesium gas, which is why we use a double chamber configuration: we can maintain a relatively high Cesium pressure in the upper chamber, so the upstairs MOT loads quickly, and a relatively low pressure in the lower chamber, so the lifetime of atoms trapped in the cavity is not unduly limited by collisions with background gas.

1.8.3 Cavity locking and cavity QED probe

The probe frequency f_p and cavity frequency f_c must be known and stable to a frequency resolution that is small relative to the rates γ, κ, and g that characterize the atom/cavity system. To

achieve this we need to actively servo-lock the cavity length and the probe laser frequency to a reliable frequency reference. A convenient reference is the Cesium atom itself: by performing saturated absorption spectroscopy in a Cesium cell it is possible to resolve narrow lines corresponding to transitions to specific excited state hyperfine manifolds. The width of these lines is set by the spontaneous decay rate $\gamma = (2\pi)(5.2\,\mathrm{MHz})$, so they can be used to stabilize frequencies to within a few hundred kHz.

The locking scheme involves two cavities and two diode lasers; in addition to the physics cavity (PC) and probe laser (PL) that we want to stabilize, there is an ancillary cavity called the transfer cavity (TC) and an ancillary laser called the locking laser (LL). The physics cavity and transfer cavity have modes at integer multiples of their free spectral ranges $\nu_{PC} = 2/L_{PC}$ and $\nu_{TC} = 2/L_{TC}$:

$$\begin{aligned} f_{PC}(n) &= n\,\nu_{PC} \\ f_{TC}(n) &= n\,\nu_{TC} \end{aligned}$$

Since the cavity lengths are $L_{PC} = 42\,\mu\mathrm{m}$ and $L_{TC} \sim 30\,\mathrm{cm}$, the free spectral ranges are $\nu_{PC} = 3.6\,\mathrm{THz}$ and $\nu_{TC} \sim 500\,\mathrm{MHz}$. The probe laser has wavelength 852 nm and drives mode 99 of the physics cavity, while the locking laser has wavelength 836 nm and drives mode 101 of the physics cavity. The basic idea is that the transfer cavity is stabilized by locking it to Cesium, the probe laser and locking laser are stabilized by locking them to the transfer cavity, and the physics cavity is stabilized by locking it to the locking laser. In detail, the locking scheme works as follows (see Figure 1.14).

First, we lock the probe laser to the transfer cavity by monitoring the coupling of light from the probe laser into one of the transfer cavity modes and feeding back to the probe laser. This sets

$$f_{PL} = f_{TC}(n_1)$$

for some mode n_1 of the transfer cavity.

Next, we lock the transfer cavity to a known line in Cesium by monitoring a saturated absorption signal derived from probe light and feeding back to the transfer cavity. This sets

$$f_{TC}(n_1) = f_{PL} = f_{Cs}$$

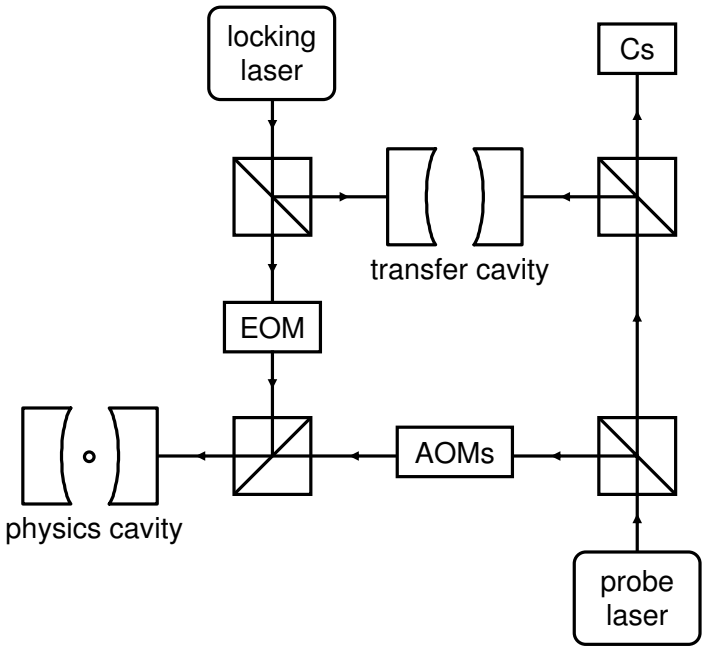

Figure 1.14: Scheme for locking the physics cavity and probe laser.

Thus, the modes of the transfer cavity now occur at fixed frequencies relative to Cesium; that is, for an arbitrary mode n, we have

$$f_{TC}(n) = f_{TC}(n_1) + (n - n_1)\,\nu_{TC} = f_{Cs} + (n - n_1)\,\nu_{TC}$$

where ν_{TC} is held at a constant, stable value by the lock.

The next step is to lock the locking laser to the transfer cavity by monitoring the coupling of locking laser light into one of the transfer cavity modes and then feeding back to the locking laser (the locking laser and probe laser are coupled into opposite sides of the transfer cavity). This sets

$$f_{LL} = f_{TC}(n_2)$$

for some mode n_2 of the transfer cavity.

Finally, we lock the physics cavity to the locking laser. This is accomplished by using an EOM to put sidebands on light from the locking laser, monitoring the coupling of one of the sidebands into the physics cavity, and feeding back to the physics cavity. This sets

$$f_{PC}(101) = f_{LL} + \Delta f_{LL}$$

where Δf_{LL} is the frequency offset of the sideband. The modes of the physics cavity now occur at fixed frequencies relative to Cesium; specifically, for mode 99 at 852 nm, we have

$$f_c \equiv f_{PC}(99) = f_{Cs} + (n_2 - n_1)\,\nu_{TC} - 2\,\nu_{PC} + \Delta f_{LL}$$

where ν_{PC} and ν_{TC} are stabilized by the locks. By adjusting the sideband frequency Δf_{LL}, we can tune the cavity frequency f_c. Note that Δf_{LL} only needs to be tuned over a transfer cavity free spectral range ν_{TC}; if we want to tune f_c further than ν_{TC}, we simply lock the locking laser to a different transfer cavity mode.

The probe beam is obtained by passing light from the probe laser through two double passed AOMs, so the frequency of the probe beam is

$$f_p = f_{PL} + \Delta f_{PL}$$

where Δf_{PL} is the net AOM shift. Thus,

$$f_p = f_{Cs} + \Delta f_{PL}$$

By varying the AOM shift Δf_{PL}, we can tune the probe frequency.

1.8.4 FORT and Raman lasers

The FORT and Raman beams are obtained from two independent diode lasers. Both beams drive cavity mode 90 at 935 nm, but the FORT beam is tuned to the cavity resonance while the Raman beam is detuned by $\Delta_{HF} = (2\pi)(9.2\,\mathrm{GHz})$ (note that the Raman beam is tuned blue of the FORT beam). The cavity mirrors are coated to optimize the reflectivity at 852 nm; at 935 nm the coating is not as effective and the cavity resonance is very broad ($\kappa = (2\pi)(1.6\,\mathrm{GHz})$). Thus, even though the Raman beam is detuned from resonance it is still possible to couple a significant amount of Raman light into the cavity. Also, because the resonance is so broad, the coupling of the FORT beam into the cavity is relatively insensitive to the FORT frequency, and consequently the FORT laser doesn't need to be locked. It is sufficient to monitor the transmission of the FORT light through the cavity and tweak the FORT wavelength by hand over the course of an experimental run.

To drive Raman transitions, the frequency offset between the FORT and Raman beams must be known and stable to a frequency resolution that is small relative to the effective Rabi frequency. This is achieved by phase locking the Raman laser to the FORT laser. The phase lock works by combining light from the two lasers on a fast photo-detector, mixing the resulting beat note down

to DC to obtain an error signal, and then feeding back to the Raman laser.

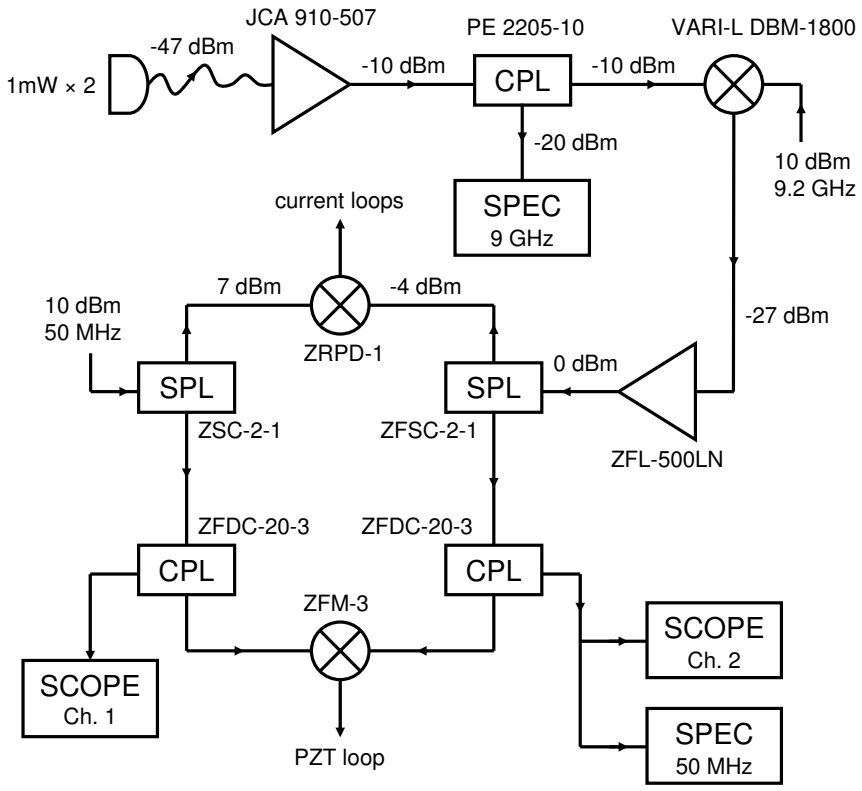

Figure 1.15: Feedback loop for the FORT/Raman phase lock.

A diagram of the feedback loop is shown in Figure 1.15. We combine light from the two lasers using a fiber beam splitter; at the output of the fiber there is ~ 1 mW of light from each laser. The combined light is focused onto a NewFocus detector, which results in a -47 dBm signal at 9 GHz. This signal is amplified and mixed against a 9 GHz synthesizer to get a 50 MHz intermediate frequency, which is more convenient to work with. A 50 MHz synthesizer is used to mix the intermediate frequency signal down to a DC error signal. This error signal is fed back to the laser using three branches: a fast current loop that modulates the laser diode directly through a bias-T, a slow current loop that modulates the laser current through the current controller, and a DC lock that feeds back to the PZT. More details about the FORT/Raman phase lock can be found in [37].

Light from the Raman laser passes through an AOM, which is used for tuning and shuttering the Raman beam, and is then mixed with the FORT beam on a polarizing beam splitting cube before being coupled into the physics cavity.

1.8.5 Photon counters

Output light from the cavity is coupled into a fiber, split into two beams with a fiber beam splitter, and sent to a pair of photon counters. There are many ways light can be lost on the way from the cavity to the photon counters, and the counters themselves have an overall quantum efficiency of ~ 0.5. Including all the loss mechanisms, the detection efficiency (that is, the probability that a photon that starts inside the cavity will be recorded by the computer) is ~ 0.07.

We use two counters because after a counter detects one photon it must wait $\sim 50\,\text{ns}$ before it can detect another. This means that a single photon counter cannot be used to measure time correlation properties of the light over timescales shorter than 50 ns. If two photon counters are used, one can get around this problem by correlating the output of one photon counter with the output of the other.

The photon counters are connected to a computer card that records the arrival times of detected photons. The card has four channels; so since each detector uses a single channel there are two remaining channels that can be used to record timestamps for other events in the experiment. The software that controls the card produces a text file containing a list of all the recorded clicks. Clicks are listed in the form of 32 bit integers, where the two high order bits give the channel number and the remaining bits give the arrival time in units of 2 ns. The card can also be operated in a two-channel mode, which has a time resolution of 1 ns.

1.8.6 Timing

For the experiment to function, a complicated series of precisely timed events must occur: laser beams need to be shuttered on and off, pieces of equipment must be triggered, and so forth. In all, about twenty TTL channels are needed. The pulses for all twenty channels are generated by an AdWin Gold. The AdWin also has four analog output channels, three of which are used to set the currents for the magnetic bias coils.

In order to simplify the specification of timing information, I have written a simple programming language for defining pulses. The system works by running a text file containing the timing information through a pulse compiler program, which results in a set of low-level commands that can be downloaded into the AdWin. The language has a number of convenient features: names can be assigned to channels, events, and durations; pulses can be referenced to other pulses; and subsequences of pulses can be iterated. These features allow the timing information for the entire experiment to be quickly and easily modified.

coils	radius [cm]	separation [cm]	# turns	calibration [G/A]
axial	8.0	8.4	38	2.07
transverse	4.0	20.0	18	0.14

Table 1.1: Bias coil parameters.

coil	calibration [kHz/V]
x	25
y	50
z	360

Table 1.2: Bias coil calibration factors, expressed in units of frequency splitting versus programming voltage

1.8.7 Bias coils

A set of six bias coils is used to control the magnetic field at the location of the atom. The coils are grouped into three pairs that are aligned along three orthogonal axes $\hat{x}$, $\hat{y}$, and $\hat{z}$, where $\hat{z}$ lies along the cavity axis and $\hat{x}$ and $\hat{y}$ are perpendicular to the cavity axis and at a 45 degree angle to the vertical. The total bias field is the sum of the fields produced by the three pairs of coils.

The coil parameters are listed in Table 1.1. I have not been able to locate a record of the number of turns on the coils, but by combining the known coil geometry with information from Raman spectroscopy, we can infer that the axial coils have ~ 38 turns each and the transverse coils have ~ 18 turns each. In the case of the axial coils we can measure the field with a Hall probe, and the results agree well with the Raman scans. Each pair of coils for a given axis is wired in parallel, and the calibration factor in Table 1.1 is the ratio of the field at the position of the cavity to the total current flowing into the pair of coils.

The three coil pairs are driven by three independent current sources, which receive programming voltages from analog output channels on the AdWin Gold. The current supplies for the $\hat{x}$ and $\hat{z}$ axes can supply a maximum of five amps, while the supply for the $\hat{y}$ axis can supply a maximum of ten amps (the ambient field along $\hat{y}$ is larger than along $\hat{x}$ or $\hat{z}$, so more current is needed). The currents are related to their programming voltages by

$$\begin{aligned} I_x &= -(0.5\,A/V)\,V_x \\ I_y &= -(1.0\,A/V)\,V_y \\ I_z &= -(0.5\,A/V)\,V_z \end{aligned}$$

It is convenient to express the calibration factors in terms of frequency splitting versus AdWin output voltage (see Table 1.2).

1.9 Experimental techniques and results

1.9.1 Useful transitions

There are a number of standard techniques for manipulating atoms that involve driving specific atomic transitions with a classical field. Here I review some of the more useful transitions and discuss their applications.

One of the most important transitions is the $4 - 5'$ transition. Because of the dipole selection rule $\Delta F = 0, \pm 1$, an atom that starts in $F = 4$ and is excited by $4 - 5'$ light can only decay back into $F = 4$. Thus, by detecting the light scattered from the atom, we can determine which hyperfine ground state the atom is in: an $F = 4$ atom will scatter $\sim 10^4$ photons before it falls into $F = 3$ via off-resonant excitation of $F' = 4$, while an $F = 3$ atom is decoupled from the light and doesn't scatter any photons.

A second application of the $4 - 5'$ transition is optical pumping. If we drive an $F = 4$ atom with $4 - 5'$ light that is σ_+ polarized, then the atom will be optically pumped into $|4, 4\rangle$. Once there the atom cycles between $|4, 4\rangle$ and $|5', 5'\rangle$; because of the selection rule $\Delta m = 0, \pm 1$, from $|5', 5'\rangle$ the atom can only decay back to $|4, 4\rangle$.

A third application of the $4 - 5'$ transition is Doppler cooling. To Doppler cool the atom, we drive it with counter-propagating laser beams that are red detuned from the $4 - 5'$ transition. For a moving atom, the beam propagating opposite to the atom's direction of motion is Doppler shifted into resonance, causing the atom to scatter light from the beam. Since the atom only absorbs light with momentum opposite to its own, but emits the light in all directions, the net effect is to damp the motion.

Another useful transition is the $4 - 4'$ transition. An atom driven by a $4 - 4'$ beam has a single dark state that depends on the polarization of the beam. This is easy to see in the case of σ_+ polarized light, for which the $|4, 4\rangle$ state is dark. For linearly polarized light the $|4, 0\rangle$ state is dark, where the quantization axis is chosen along the polarization direction of the light. Thus, by properly choosing the polarization of the $4 - 4'$ light, we can optically pump the atom into specific states in the $F = 4$ manifold.

The $4 - 4'$ transition can also be used to cool the atom via polarization gradient cooling. This works by driving the atom with counter-propagating beams of circularly polarized light that are blue detuned from the $4 - 4'$ transition. At every point in space the two beams combine to yield a

linear polarization vector, but the orientation of the polarization vector rotates as one moves along the beam propagation direction. Thus, a stationary atom sees a constant polarization vector and is optically pumped into the corresponding dark state, while a moving atom sees a polarization vector that rotates with angular frequency $\omega = kv$. If the velocity is high enough then the atom won't have enough time to be pumped into the dark state corresponding to its current location, and will therefore couple to the light. Thus, the light exerts a force on the atom that is proportional to the atom's velocity. For blue detuned light, the force is opposite to the velocity and therefore cools the atomic motion.

1.9.2 Loading into the FORT

At the beginning of each experimental cycle, a cloud of atoms is collected in the upper chamber, transfered to the lower chamber, and then dropped on the cavity, resulting in a shower of cold atoms falling through the cavity mode. The falling atoms can be loaded into the FORT using a variety of methods. One possibility is to monitor the transmission of a resonant probe beam through the cavity and to trigger on the FORT when a dip in the transmission indicates that an atom is present. Another possibility is to keep the FORT on at all times and cool the atoms into the trap. One way to provide cooling is to use side beams that are blue detuned from the $4-4'$ and $3-3'$ transitions; experiments indicate that such a scheme gives a maximum loading probability of ~ 0.4 (the loading probability is the probability that for a given drop at least one atom is loaded into the trap). A variation on this scheme is to use side beams that are blue detuned from the $4-4'$ transition together with a strong Raman beam tuned to Raman resonance. This combination seems to be a particularly effective way of loading atoms; loading probabilities of up to ~ 0.9 can easily be achieved. This Raman loading scheme is used for all the experiments discussed here.

We can determine if we succeeded in loading an atom by checking whether a resonant probe beam is transmitted through the cavity. If it is not, we know that at least one atom was loaded. Given that we know at least one atom was loaded, the expected number of loaded atoms can be calculated as follows. I will assume that the probability of loading an atom is independent of how many atoms have already been loaded, so the probability of loading n atoms is given by a Poisson distribution:

$$P_n = \frac{1}{n!} \langle n \rangle^n \, e^{-\langle n \rangle}$$

The loading probability p is then

$$p = \sum_{n=1}^{\infty} P_n = 1 - P_0 = 1 - e^{-\langle n \rangle}$$

Thus, the mean number of atoms is

$$\langle n \rangle = -\log(1-p)$$

Since we know there is at least one atom in the trap, the probability that there are n atoms is P_{n-1}. Thus, the expected number of atoms is

$$N = \sum_{n=1}^{\infty} n\, P_{n-1} = \sum_{n=0}^{\infty} (1+n)\, P_n = 1 + \langle n \rangle = 1 - \log(1-p)$$

If the loading probability is small, then when a transmission dip is observed it is highly likely that there is only one atom in the cavity.

1.9.3 Optical pumping

For many experiments, the atom needs to be prepared in a specific initial state. This is accomplished through optical pumping: we drive the atom with light fields that couple to all the states except the one we want to populate, which is called the dark state. The driving fields cause the atom to randomly scatter from one state to another until it lands in the dark state and decouples from the light. Thus, if we leave the pumping light on long enough, the atom eventually ends up in the desired state.

One state we want to prepare is $|3,0\rangle$, where the quantization axis is chosen to lie along the cavity axis. To prepare this state, we use a resonant $3-3'$ beam that enters the cavity from the side and is linearly polarized along the cavity axis. The dark states for this beam are the $|3,0\rangle$ state and all the $F=4$ states. Thus, we can prepare the atom in $|3,0\rangle$ by combining the $3-3'$ beam with additional beams that repump the atom from $F=4$ to $F=3$. The $4-4'$ cooling beams are used for this purpose, although they are not ideal because they have their own dark state in the $F=4$ manifold. However, because the polarization of the $4-4'$ light depends on the position of the atom, so does the dark state; therefore, a moving atom won't get stuck in the $4-4'$ dark state and will eventually scatter to $F=3$.

Sometimes rather than preparing a specific state we want to randomly choose a state $|3,m\rangle$ in such a way that the m values are uniformly sampled. This can be achieved by alternating pulses of $3-3'$ and $4-4'$ cooling light to pump the atom back and forth between $F=4$ and $F=3$. For each pulse m is randomly changed by $\pm 1, 0$; so if enough pulses are used m is effectively randomized. By using $4-4'$ light for the final pulse, we ensure that the atom ends up in an $F=3$ state.

1.9.4 Measuring the atomic state

Here I present a method for determining which ground state hyperfine manifold the atom is in by measuring the transmission of a resonant probe beam. Both the probe beam and the cavity are tuned to the $4-5'$ transition, so every state in the $F=4$ manifold is coupled to the cavity mode. For atoms in $F=3$ all the probe light is transmitted (since the atoms are decoupled from the cavity mode), while for atoms in $F=4$ the probe transmission is reduced by an amount determined by the strength of the coupling. If the probe is linearly polarized and we choose a quantization axis along the polarization direction, then for an atom in $|4,m\rangle$ the cavity coupling is

$$g = g_0\,\psi(\vec{r})\,\langle 5,m|1,0;4,m\rangle$$

and the transmission is reduced by a factor

$$\eta \sim (\kappa\gamma/4g^2)^2 \sim N_A^2\,(g_0/g)^4$$

For our cavity $N_A^2 \sim 10^{-4}$, so for all Zeeman sublevels, and almost all atom positions, this is a huge reduction. Thus, to a good approximation, the probe is shuttered off any time the atom is in $F=4$.

We can use the shuttering effect to determine which ground state the atom is in by pulsing on a probe and counting the number of photons that are transmitted. For this scheme to work, the atom has to stay in the same ground state for the duration of the pulse. This will clearly be the case if the atom starts in $F=3$; since the atom is decoupled from the cavity, it doesn't interact with the probe light and can't scatter from $F=3$ to $F=4$. It will also be the case if the atom starts in $F=4$, provided the probing interval is short enough, because the scattering rate from $F=4$ to $F=3$ is very low. There are two reasons for the low scattering rate. First, since the $4-5'$ transition is a cycling transition, the atom can only get to $F=3$ through off-resonant excitation of $F'=4$. Second, because of the shuttering effect itself, there is very little light in the cavity to excite the atom.

For this measurement technique we typically use a probe strength corresponding to an empty cavity photon number of $n=0.14$ and a probe duration of $100\,\mu$s. Thus, if the cavity is empty, or if the atom is in $F=3$, the counting rate at the detectors is $en\kappa \sim 500\,$kcps (where $e=0.07$ is the detection efficiency, as discussed in section 1.8.5) and the mean number of clicks recorded during a probe interval is ~ 50. To demonstrate the detection scheme, we performed an experiment where atoms were randomly prepared in $F=3$ or $F=4$ and then interrogated by the probe beam. Figure 1.16 is a record of the detected clicks for a single atom, and Figure 1.17 is a histogram of clicks for the entire data set. We can clearly resolve a peak at ~ 58 clicks due to the $F=4$ atoms, and

another peak at ~ 10 clicks due to the $F = 3$ atoms. The widths of the peaks are set by the shot noise of the light. To determine which state the atom is in, we set a threshold at half the mean number of clicks for an empty cavity. If the number of counts recorded during the probing window is below the threshold, we assume the atom is in $F = 3$; if it is above the threshold, we assume the atom is in $F = 4$.

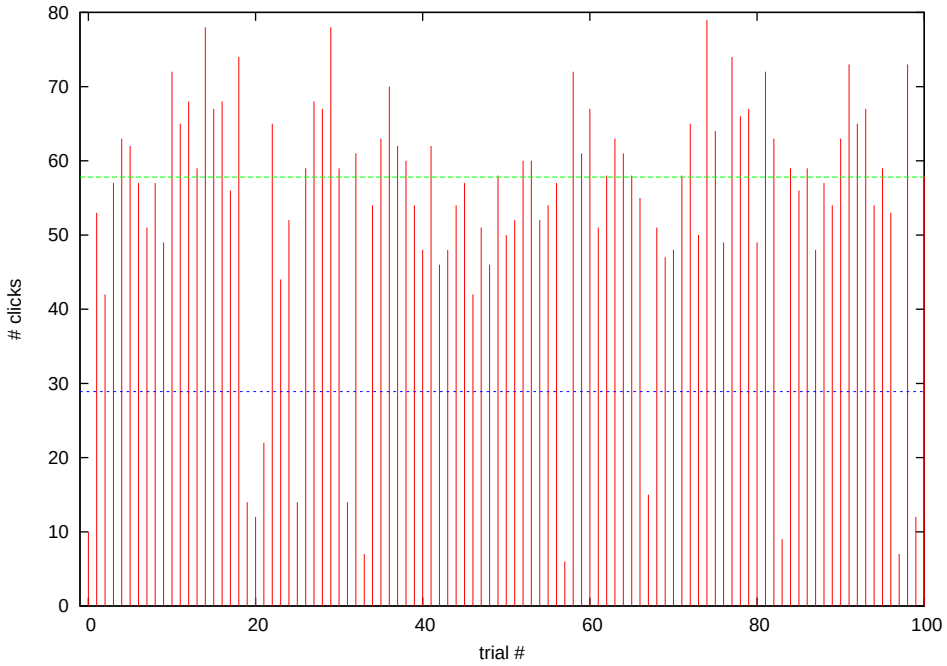

Figure 1.16: State measurements for a single atom. The horizontal line at 57.8 clicks is the mean number of clicks for an empty cavity; the line at $n = 28.9$ clicks is the threshold. For trials where the transmission dips below the threshold, the atom is assumed to be in $F = 4$.

1.9.5 Measuring the Raman transfer probability

We often want to measure the probability that a Raman pulse transfers the atom from one ground state hyperfine manifold to the other. This can be accomplished by repeatedly preparing the atom in $F = 3$, applying the Raman pulse, and then checking if the atom was transfered to $F = 4$ by using the scheme from the previous section. I will call a single transfer attempt a trial. By taking the ratio of the number of successful transfers to the total number of trials, we can determine the Raman transfer probability.

The Raman transfer probability depends on a number of factors, such as the way the initial state is prepared, the magnetic field, and the Rabi frequency, detuning, and duration of the Raman pulse. Thus, for each probability measurement these factors are held constant across all the trials. We can often obtain useful information about the system by repeating the probability measurement for

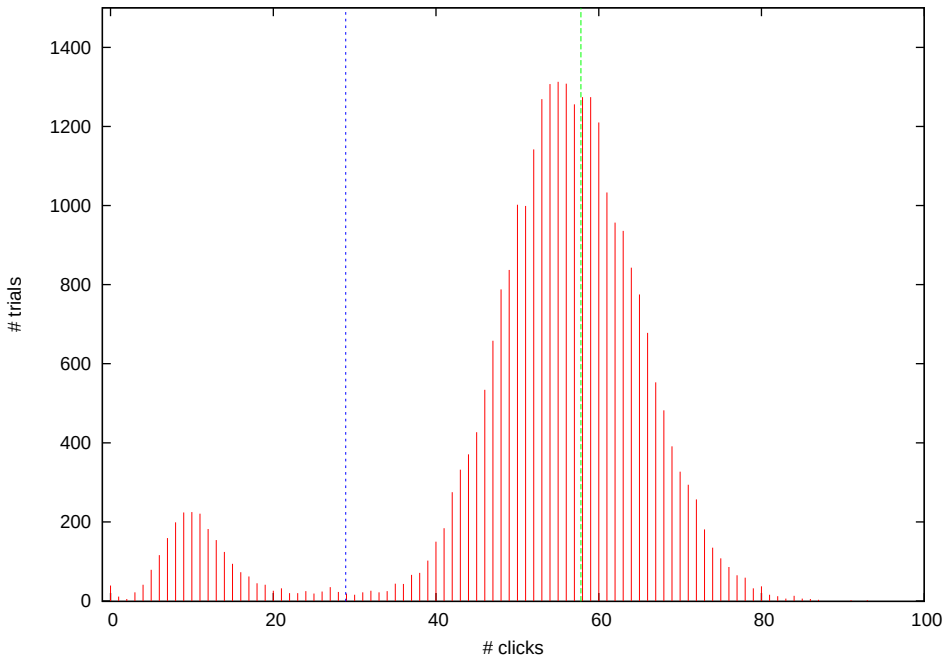

Figure 1.17: Histogram of clicks. The vertical line at 57.8 clicks is the mean number of clicks for an empty cavity; the line at $n = 28.9$ clicks is the threshold used to determine whether an atom is in $F = 3$ or $F = 4$.

different values of a single parameter in order to map out the dependence of the probability on the parameter.

There is one complication I haven't mentioned yet: since the atom can leave the FORT at any time (either by being heated out of the trap or because of collisions with background gas), for each trial we need to know whether or not an atom is present. If the transmission is low during the state detection phase of the trial then we know that an $F = 4$ atom is present, but if transmission is high it could either mean that an $F = 3$ atom is present or that the cavity is empty.

We deal with this problem as follows. In a typical experiment, each atom is subjected to $N = 1000$ trials, after which the atom is released, if it hasn't already left on its own. For each trial k we obtain a measurement result n_k, where n_k is 1 if the transmission was low for that trial (indicating that an atom was present and had made a Raman transition $3 \rightarrow 4$), and n_k is 0 if the transmission was high (indicating either that the atom stayed in $F = 3$, or that the atom had left the cavity). Thus, the total number of transmission dips that have been recorded up to and including trial k is

$$t_k = \sum_{r=1}^{k} n_r$$

If we plot t_k versus k, the result typically looks like Figure 1.18. At first the graph climbs with a slope given by the probability of making a Raman transition, but at some point the graph suddenly flattens out, indicating that the atom has left the cavity. For the trials that occur prior to this kink in the graph we know that an atom was present. Specifically, if r is the largest trial number such that $t_r < t_N - m$ then I assume the atom remained in the cavity for r trials, where m is a small margin to allow for occasional transmission dips after the atom has left (such dips could be caused by shot noise fluctuations in the probe light, for example). Typically we take $m = 5$.

Now that we know when the atom left the cavity, we can easily calculate the Raman transfer probability; it is given by $p = t_r/r$. To get better statistics we typically repeat this measurement protocol for ~ 10 atoms and divide the total number of transitions by the total number of trials to find the probability.

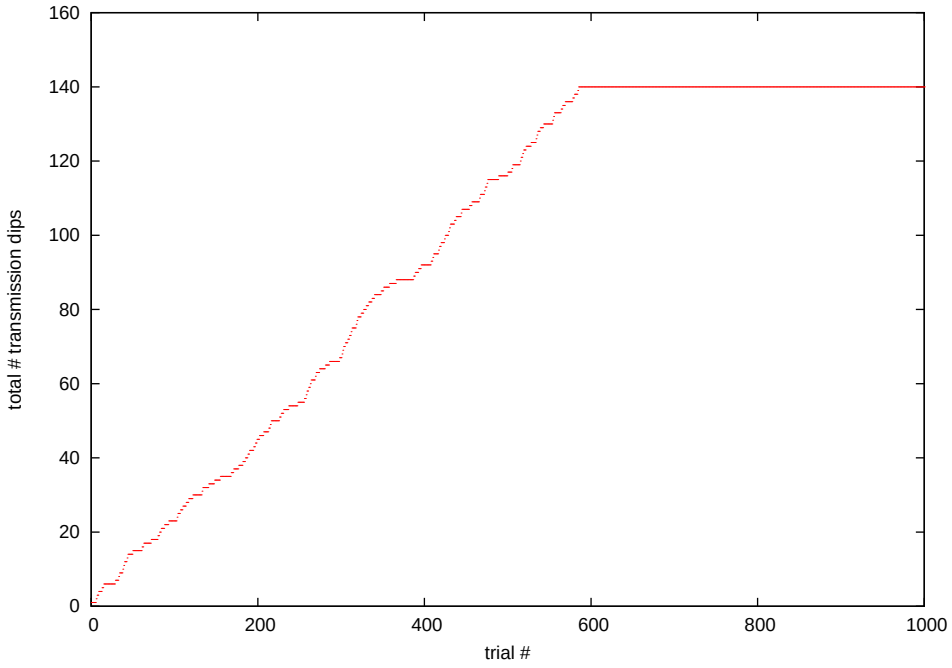

Figure 1.18: Total number of transmission dips versus trial number for a sample atom. From the location of the kink, we infer that the atom left the cavity around trial number 580.

1.9.6 Raman spectroscopy

The technique for measuring the Raman transfer probability can be used to perform Raman spectroscopy. This involves measuring the Raman transfer probability as a function of the detuning of the Raman pulse, while holding the duration and Rabi frequency of the pulse constant

To understand the Raman spectrum, we must first look at the energy spectrum of the ground

state manifold. The Hamiltonian for the ground states is

$$H = \frac{1}{2}\Delta_{HF}(P_4 - P_3) + H_B$$

The first term gives the hyperfine splitting between $F = 3$ and $F = 4$, while the second term describes the coupling of the ground states to the magnetic field $\vec{B}$ at the position of the atom:

$$H_B = g_F \, \mu_B \, \vec{B} \cdot \vec{F}$$

Recall that $g_4 = 1/4$ and $g_3 = -1/4$. Thus, if we choose the quantization axis along the direction of the magnetic field,

$$\begin{aligned} H|4,m\rangle &= (+\frac{1}{2}\Delta_{HF} + m\omega_B)|4,m\rangle \\ H|3,m\rangle &= (-\frac{1}{2}\Delta_{HF} - m\omega_B)|3,m\rangle \end{aligned}$$

where

$$\omega_B = \frac{1}{4}\mu_B|\vec{B}| = (2\pi)(350\,\mathrm{kHz/G})|\vec{B}|$$

is the magnetic splitting between adjacent Zeeman levels. This energy spectrum is shown in Figure 1.19.

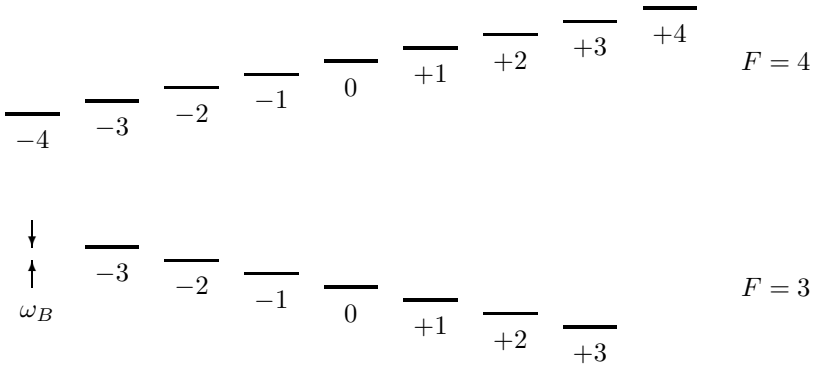

Figure 1.19: Ground state spectrum of Cesium.

Raman transitions between states in the $F = 3$ and $F = 4$ manifolds are subject to the selection rules $\Delta m = 0, \pm 1$ (see Figure 1.20). Note that the $\Delta m = 0$ transitions occur at even multiples of ω_B, and the $\Delta m = \pm 1$ transitions occur at odd multiples of ω_B. Because $g_4 = -g_3$, the $\Delta m = \pm 1$ transitions are twofold degenerate ($|3,m\rangle \leftrightarrow |4,m+1\rangle$ is degenerate with $|3,m+1\rangle \leftrightarrow |4,m\rangle$), except for the edge transitions $|3,3\rangle \leftrightarrow |4,4\rangle$ and $|3,-3\rangle \leftrightarrow |4,-4\rangle$.

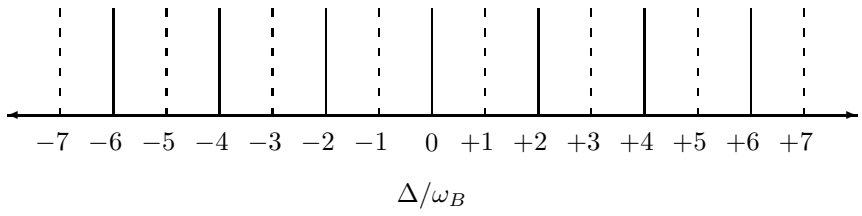

Figure 1.20: Spectrum of allowed transitions. The solid lines are for $\Delta m = 0$ transitions; the dashed lines are for $\Delta m = \pm 1$ transitions.

The Raman spectrum depends on the initial state of the atom and on the strength and duration of the Raman pulse. Suppose we prepare the atom so it is in state $|3, m\rangle$ with probability $p_i(|3, m\rangle)$, and apply a Raman pulse of Rabi frequency Ω, duration T, and detuning Δ. If the Rabi frequency is small relative to the magnetic splitting, then each Zeeman sublevel in $F = 3$ will couple appreciably to at most one sublevel in $F = 4$, and the system can be approximated as a collection of two-level atoms. In this approximation, the probability that the atom is in $F = 4$ after the Raman pulse is

$$p(\Delta) = \sum_m \sum_{m'} P(\Omega_E(|3, m\rangle \leftrightarrow |4, m'\rangle),\, \Delta - (m' - m)\omega_B,\, T)\; p_i(|3, m\rangle)$$

where

$$P(\Omega, \Delta, T) = \frac{\Omega^2}{\Omega^2 + \Delta^2} \sin^2((\Omega^2 + \Delta^2)^{1/2}\, T/2)$$

is the probability that a two-level atom initially in its ground state is transfered to the excited state by a pulse of light with Rabi frequency Ω, detuning Δ, and duration T (see section 1.3.2). If we consider the probability as a function of detuning, we see that it consists of a Lorentzian envelope that is modulated by a $\sin^2$ factor, where the frequency scale of the modulation is $\sim 1/T$. For long pulse durations, this fine scale modulation is averaged out by the atomic motion or other sources of frequency jitter, so we only see the Lorentzian envelope. Thus, we can approximate the Raman spectrum as

$$p(\Delta) = \frac{1}{2} \sum_m \sum_{m'} \left(1 + \frac{(\Delta - (m' - m)\omega_B)^2}{\Omega_E^2(|3, m\rangle \leftrightarrow |4, m'\rangle)}\right)^{-1} p_i(|3, m\rangle)$$

A sample Raman scan is shown in Figure 1.21. Peaks are observed for both $\Delta m = 0$ and $\Delta m = \pm 1$ transitions, but the $\Delta m = 0$ peaks are stronger, indicating that for this scan the magnetic field was more axial than transverse. Figure 1.22 shows a Raman scan taken after the magnetic fields were nulled. Figure 1.23 shows a Raman scan for an axial bias field, and Figure 1.24 shows a Raman scan for a transverse bias field.

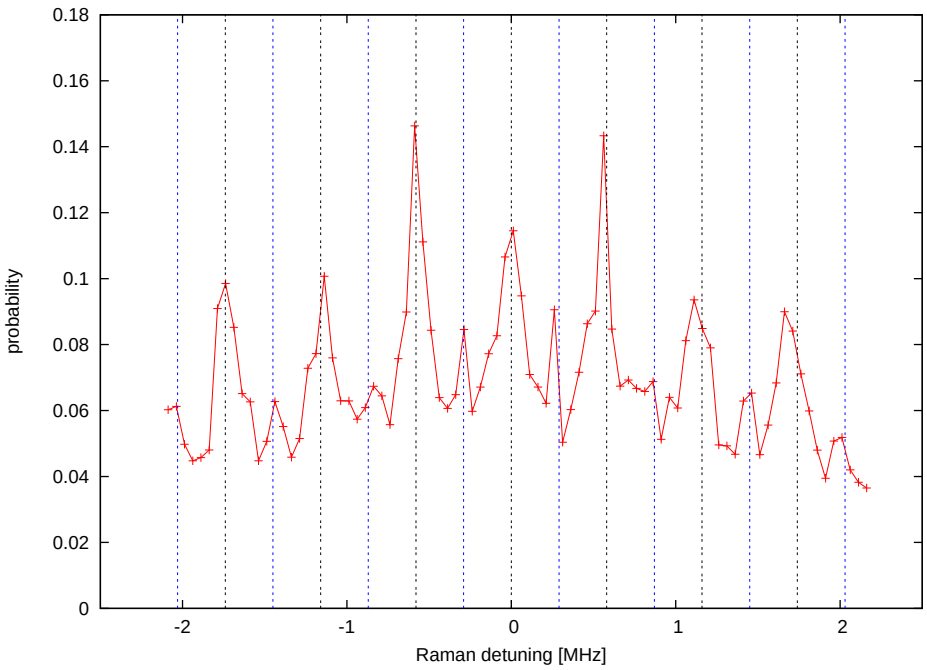

Figure 1.21: Raman scan. The vertical lines are at integer multiples of the 290 kHz magnetic splitting. The Rabi frequency for this scan was $\Omega = (2\pi)(50\,\mathrm{kHz})$.

1.9.7 Measuring and nulling the magnetic field

The total magnetic field $\vec{B}$ at the position of the atom is the sum of the ambient field $\vec{B}_a$ and the field $\vec{B}_b$ produced by the bias coils:

$$\vec{B} = \vec{B}_a + \vec{B}_b$$

The bias field is known under our control, but to control the field $\vec{B}$ that the atom actually sees we need to know the ambient field. Here I present a technique for measuring the ambient field by preparing the atom in a random $F = 3$ state, applying a resonant Raman pulse, and tuning the bias field so as to maximize the Raman transfer probability. The basic idea is that if the bias field exactly cancels the ambient field then the Zeeman splitting vanishes, all the transitions pile up around zero detuning, and the atom is transfered by the Raman pulse regardless of its initial state. If the fields don't cancel then the transitions are split out, so the atom is only transfered if it starts in $|3, 0\rangle$. Thus, if we tune the bias field so as to null the ambient field, we should see an enhancement in the Raman transfer probability.

In section 1.8.7 I described how the magnetic field is broken into components along three orthogonal axes. We null the three components one at a time; for each component, we maximize the Raman transfer probability by varying the current in the pair of bias coils corresponding to that component of the field, while holding the orthogonal components of the field fixed. I'll assume we start with

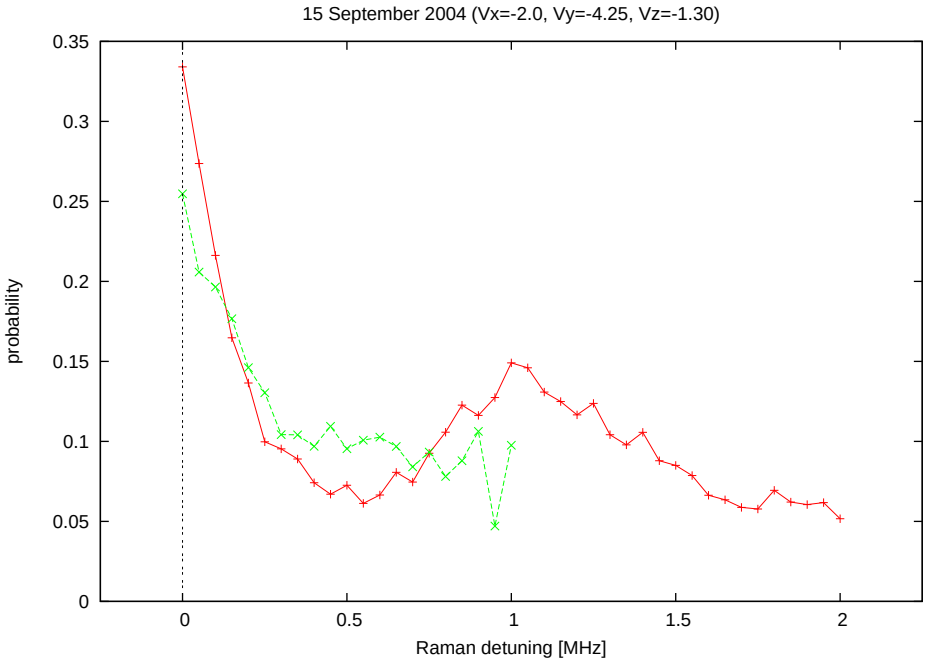

Figure 1.22: Raman scan with no bias field. Two scans are shown; one with Rabi frequency $\Omega = (2\pi)(70\,\mathrm{kHz})$ (red curve), and one with Rabi frequency $\Omega = (2\pi)(35\,\mathrm{kHz})$ (green curve).

the $\hat{z}$ component. The magnitude of the component of $\vec{B}$ parallel to $\hat{z}$ is

$$B_{\|} = \vec{B} \cdot \hat{z}$$

and the magnitude of the component of $\vec{B}$ perpendicular to $\vec{z}$ is

$$B_{\perp} = |\vec{B} - B_{\|}\,\hat{z}|$$

Thus, we can express the magnetic splitting as

$$\omega_B = \frac{1}{4}\mu_B|\vec{B}| = \omega_{B_\perp}(1 + B_{\|}^2/B_{\perp}^2)^{1/2} = \omega_{B_\perp}(1 + (B_{a,\|} + B_{b,\|})^2/B_{\perp}^2)^{1/2}$$

where

$$\omega_{B_\perp} \equiv \frac{1}{4}\mu_B B_{\perp}$$

Suppose we choose a Rabi frequency larger than $\omega_{B_\perp}$ and tune to Raman resonance. For $B_{\|} \gg B_{\perp}$ the magnetic field will split out the Zeeman sublevels, and the Raman beams only addresses the $|3, m\rangle$ state. However, as $B_{\|} \to 0$, the magnetic splitting becomes small enough that several Zeeman sublevels lie within the Lorentzian envelope (recall that the width of the envelope is set by the Rabi frequency), and we see an enhancement in the Raman transfer probability. Thus, if we plot

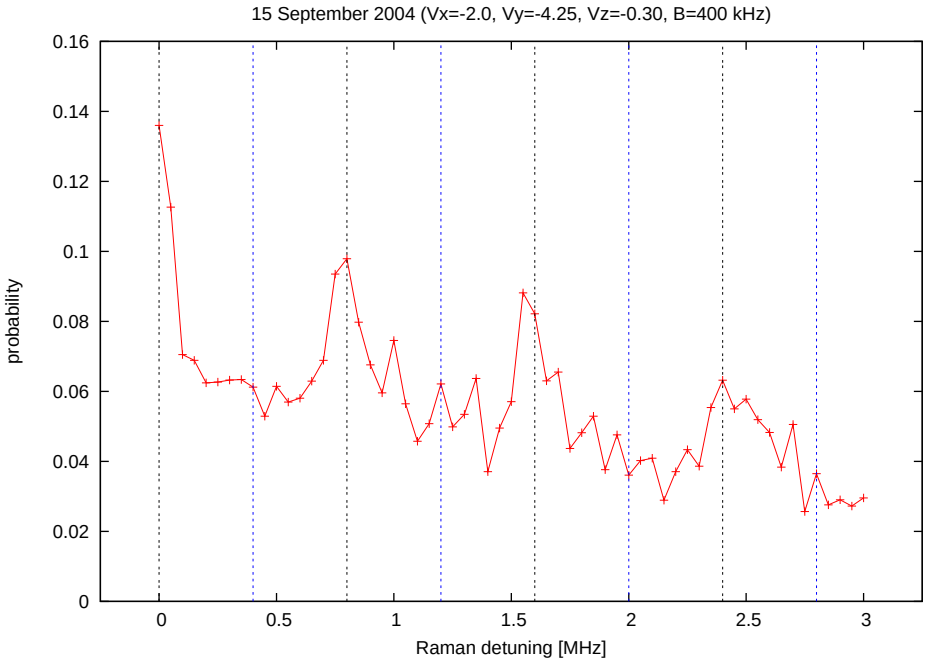

Figure 1.23: Raman scan for an axial bias field. The vertical lines are at integer multiples of the 400 kHz magnetic splitting. The Rabi frequency for this scan was $\Omega = (2\pi)(70\,\mathrm{kHz})$.

the Raman transfer probability as a function of the bias field component along the $\hat{z}$ direction, we should get something that looks like Figure 1.25. The peak of this graph indicates the value of $B_{b,\|}$ that nulls the ambient field along $\hat{z}$. By performing the same sort of scan along the two orthogonal axes (Figure 1.26 and Figure 1.27), we can null the ambient field in all three dimensions. Note that the width of the peak is set by the Rabi frequency, so by reducing the Rabi frequency we can measure the fields to higher resolution.

1.9.8 Rabi flopping

If we apply a large axial bias field to split out the Zeeman sublevels and tune near Raman resonance, then only the $|3,0\rangle \leftrightarrow |4,0\rangle$ transition is driven appreciably, and the system may be approximated as a two-level atom. Recall from section 1.3.2 that if we start a two-level atom in the ground state (in this case the $|3,0\rangle$ state) and apply pulse of coherent light of duration T with Rabi frequency Ω and detuning Δ, then the probability to find the atom in the excited state (in this case $|4,0\rangle$) is

$$P(\Omega,\Delta,T) = \frac{\Omega^2}{\Omega^2+\Delta^2}\sin^2((\Omega^2+\Delta^2)^{1/2}T/2)$$

For our case $\Omega = \Omega(|3,0\rangle \leftrightarrow |4,0\rangle)$, and the detuning Δ is given by

$$\Delta = \omega_R - \omega_F - \Delta_{HF}$$

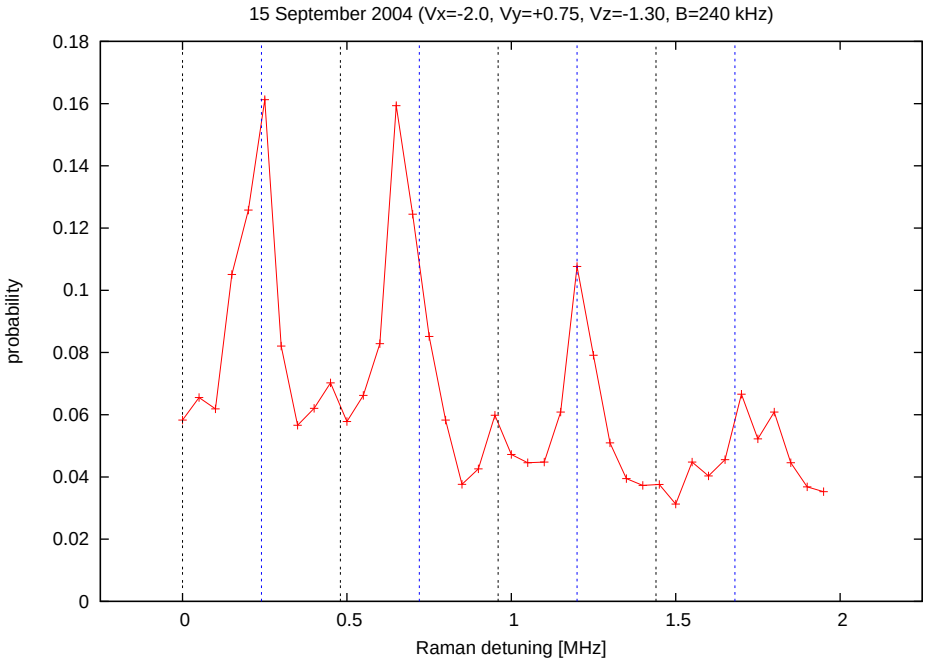

Figure 1.24: Raman scan for a transverse field. The vertical lines are at integer multiples of the 240 kHz magnetic splitting. The Rabi frequency for this scan was $\Omega = (2\pi)(70\,\mathrm{kHz})$.

Figure 1.28 shows a Raman scan over the $|3,0\rangle \leftrightarrow |4,0\rangle$ transition for a Rabi frequency of $\Omega = (2\pi)(50\,\mathrm{kHz})$ and pulse duration of $T = 100\,\mu\mathrm{s}$. For this scan the atoms were prepared by optically pumping them into $|3,0\rangle$ before each Raman pulse. Because the FORT squeezes the hyperfine manifolds together and reduces the effective hyperfine splitting (see section 1.7.3), the peak is shifted away from zero by $\delta \sim -20\,\mathrm{kHz}$. We can see the Lorentzian envelope due to the first term, plus some of the $\sin^2$ modulation from the second term. This fine scale modulation is partially washed out because atoms at different temperatures have different Rabi frequencies, which leads to decoherence of the Rabi oscillations

In Figure 1.29, the Raman detuning was set to zero, and the Raman transfer probability was measured for different pulse durations. The Rabi oscillations can be clearly observed, although they gradually decohere over time. From the observed rate of decoherence and known dependence of Rabi frequency on temperature, one could probably use this curve to infer something about the temperature of the atoms.

1.9.9 Laser noise

So far I have assumed that the FORT and Raman beams are both perfectly monochromatic. In practice this is not the case; because the FORT/Raman phase lock is not perfect, there is broadband noise around the Raman resonance frequency that causes incoherent decays between the $F = 3$ and

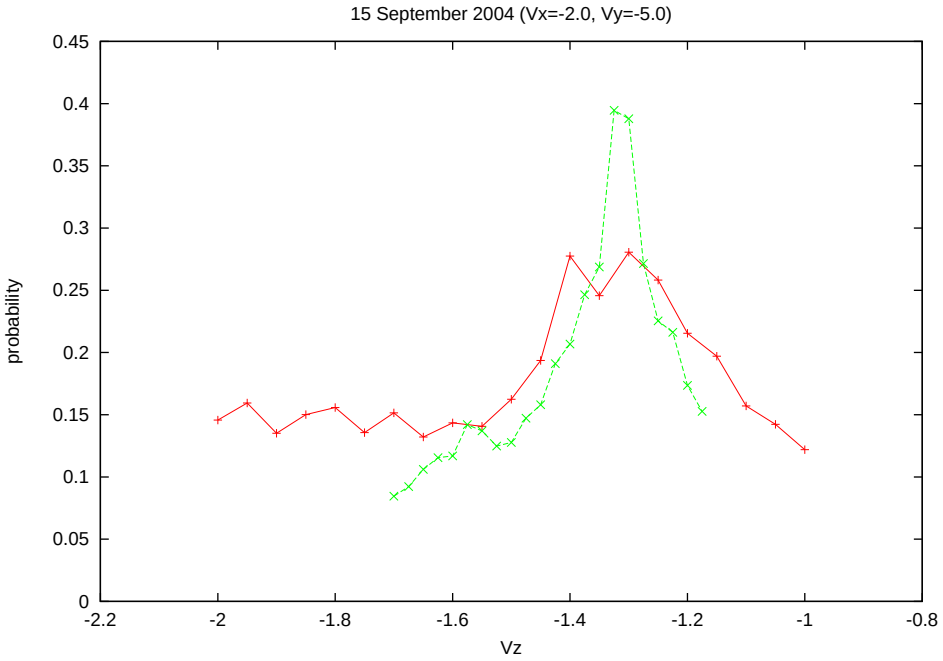

Figure 1.25: Transfer probability versus programming voltage for the $\hat{z}$ coils. Two scans are shown: one with Rabi frequency $\Omega = (2\pi)(70\,\mathrm{kHz})$ (red curve), and one with Rabi frequency $\Omega = (2\pi)(35\,\mathrm{kHz})$ (green curve).

$F = 4$ ground state manifolds. We can calculate the decay rates by using a simple two-level atom model and approximating the broadband noise as a comb of discrete frequencies.

Suppose we start the two-level atom in its ground state and drive it with a comb of classical fields that have frequencies ω_m and Rabi frequencies Ω_m. If we only consider the coupling of the atom to mode m, then the equation of motion for the excited state amplitude is

$$i\dot{c}_e = \frac{\Omega_m}{2}\, e^{-i(\omega_m - \omega_A)t}\, c_g$$

At small times $c_g \sim 1$, so we can approximate this as

$$i\dot{c}_e = \frac{\Omega_m}{2}\, e^{-i(\omega_m - \omega_A)t}$$

If we integrate this, we obtain

$$c_e(t) = \frac{\Omega_m}{2}\, (\omega_m - \omega_A)^{-1} (e^{-i(\omega_m - \omega_A)t} - 1)$$

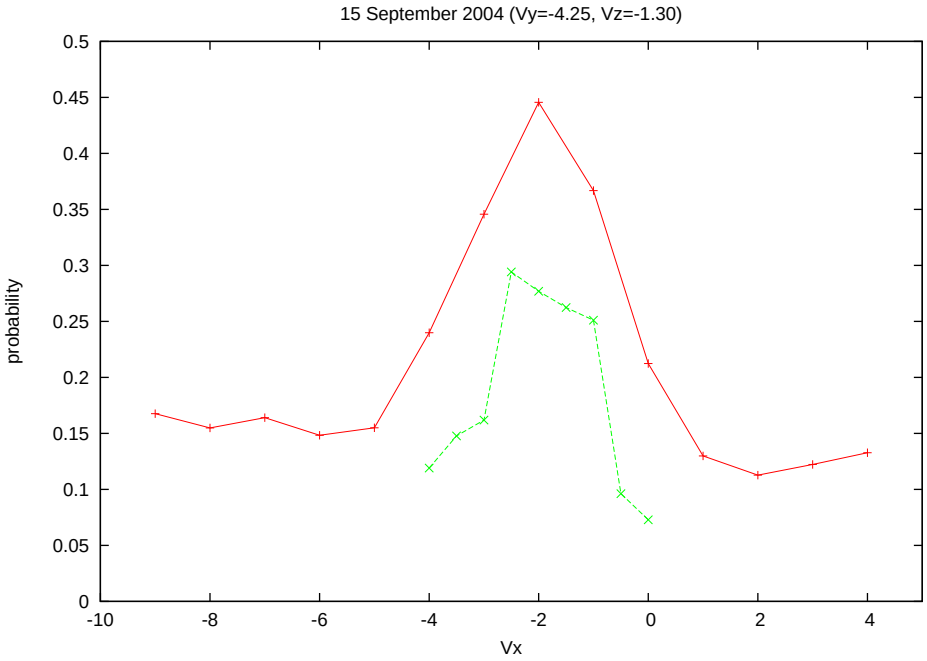

Figure 1.26: Transfer probability versus programming voltage for the $\hat{x}$ coils. Two scans are shown: one with Rabi frequency $\Omega = (2\pi)(70\,\mathrm{kHz})$ (red curve), and one with Rabi frequency $\Omega = (2\pi)(35\,\mathrm{kHz})$ (green curve).

The decay rate $g \to e$ for a single frequency ω_m is therefore

$$\gamma_m = \frac{|c_e(t)|^2}{t} = \frac{\pi}{4} t\, \Omega_m^2\, D((\omega_m - \omega_A)t/2)$$

where

$$D(x) = \frac{\sin^2 x}{\pi x^2}$$

The total decay rate is obtained by summing the decay rates for all the different fields in the comb:

$$\gamma = \sum_m \gamma_m = \frac{\pi}{4} t \sum_m \Omega_m^2\, D((\omega_m - \omega_A)t/2)$$

To calculate the decay rate we need to know the distribution of Rabi frequencies Ω_m. This information can be obtained by forming a beat note between the Raman and FORT beams on a photodetector. If the beams were monochromatic, then the electric fields at the detector would be

$$\begin{aligned} E_R(t) &= E_R^0\, e^{-i\omega_R t} \\ E_F(t) &= E_F^0\, e^{-i\omega_F t} \end{aligned}$$

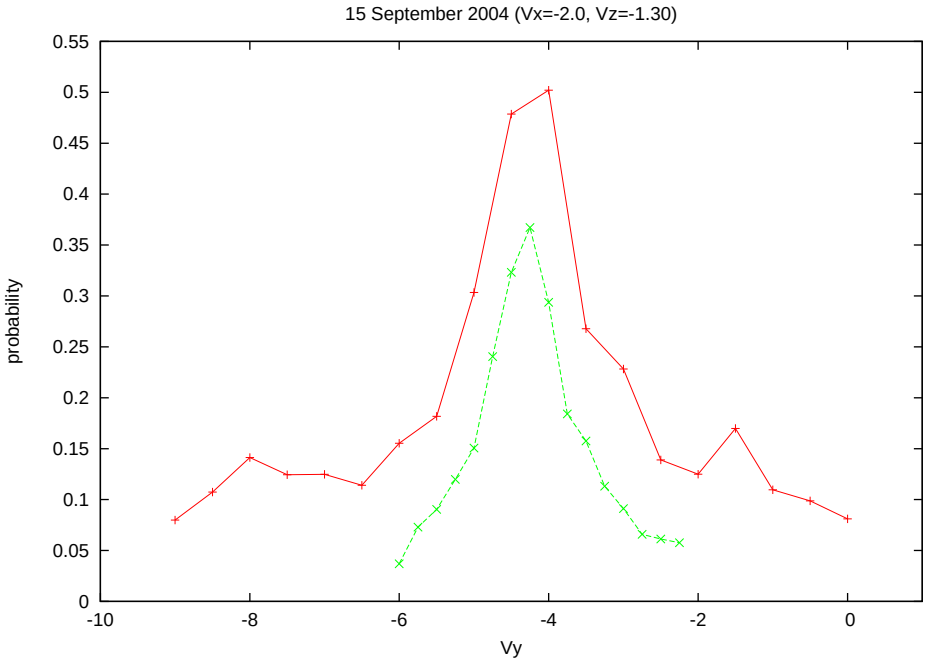

Figure 1.27: Transfer probability versus programming voltage for the $\hat{y}$ coils. Two scans are shown: one with Rabi frequency $\Omega = (2\pi)(70\,\mathrm{kHz})$ (red curve), and one with Rabi frequency $\Omega = (2\pi)(35\,\mathrm{kHz})$ (green curve).

and the resulting photocurrent $I_\gamma(t)$ would be

$$I_\gamma(t) \sim |E_R(t) + E_F(t)|^2 \sim I_R + I_F + 2\cos((\omega_R - \omega_F)t)\,\sqrt{I_R I_F}$$

where I_F and I_R are the cycle-averaged intensities of the FORT and Raman beams. Consider the power spectrum $S(\omega)$ of the photocurrent. In the limit of monochromatic beams, there is a spike at frequency $\omega_R - \omega_F$ that has integrated power proportional to $I_R I_F$. Relative frequency jitter between the two beams produces broadband noise around this spike, resulting in a curve that looks like Figure 1.30. Since the effective Rabi frequency is proportional to $\sqrt{I_R I_F}$, the effective Rabi frequency for comb line m is given by

$$\Omega_m^2 = \alpha\, S(\omega_m)\,\delta\omega$$

where α is a constant that depends on calibration factors and $\delta\omega$ is the spacing between adjacent comb lines. Using this result, we can express the total decay rate as

$$\gamma = \frac{\pi}{4}\alpha t \int S(\omega)\, D((\omega - \omega_A)t/2)\, d\omega$$

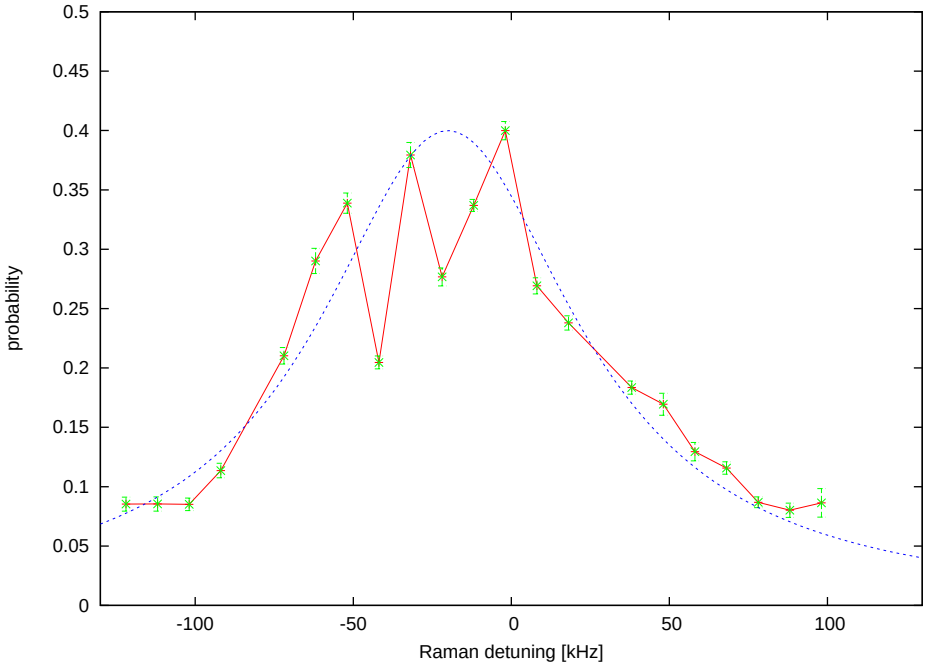

Figure 1.28: Raman scan over the $|3,0\rangle \leftrightarrow |4,0\rangle$ transition. The curve is the Lorentzian envelope one would expect based on the calculated Rabi frequency of $\Omega = (2\pi)(50\,\mathrm{kHz})$ and the calculated FORT-induced shift of $\delta = -(2\pi)(20\,\mathrm{kHz})$, where the height of the curve has been fit to the data.

If the noise spectrum is flat over a bandwidth $\sim 1/t$, then we can approximate D as a delta function:

$$\gamma = \frac{\pi}{2}\alpha\, S(\omega_A) = \frac{\alpha}{4}\, S(\nu_A)$$

We can determine the constant α by expressing it in terms of the Rabi frequency of the coherent oscillations that are driven by the spike at the beat note frequency. If the beams are tuned to Raman resonance, and if the spectral width of the spike is much less than $1/T$, where T is the duration of the Raman pulse, then the spike will drive coherent Rabi oscillations with a Rabi frequency given by

$$\Omega_c^2 = \alpha P_c$$

where P_c is the integrated power in the spike. Thus, the decay rate can be expressed as

$$\gamma(\nu) = \frac{1}{4}\frac{S(\nu)}{P_c}\,\Omega_c^2$$

Spectrum analyzers usually display the power spectrum in terms of the power $P_n(\nu) = B\,S(\nu)$ in some bandwidth B, so we can also write this as

$$\gamma(\nu) = \frac{1}{4}\frac{P_n(\nu)}{P_c}\frac{\Omega_c^2}{B}$$

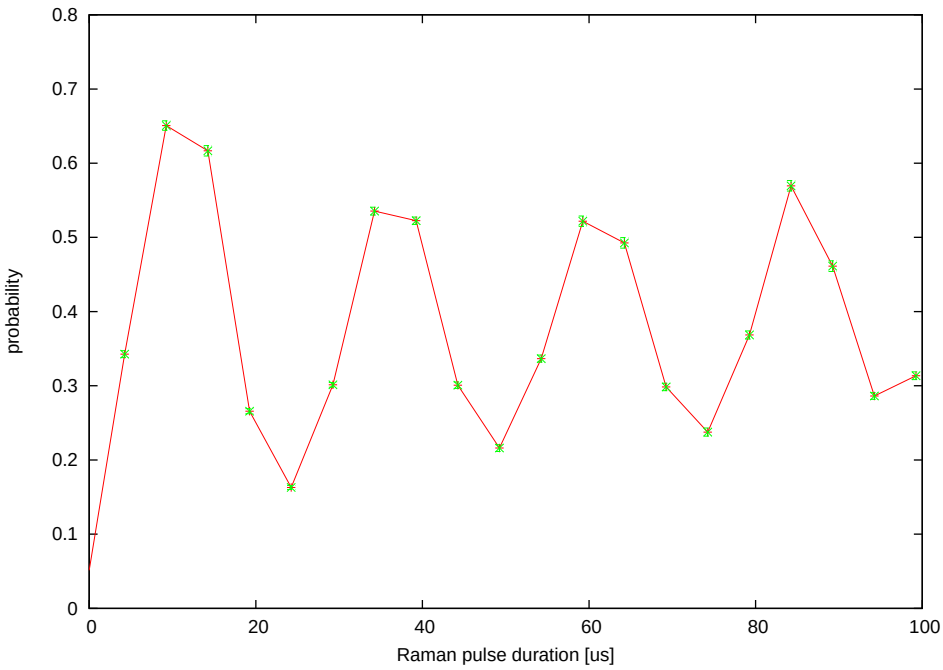

Figure 1.29: Rabi flopping on the $|3,0\rangle \leftrightarrow |4,0\rangle$ transition.

The rate equations for the ground and excited state populations p_g and p_e are

$$\begin{aligned} \dot{p}_e &= \gamma(\nu)\,(p_g - p_e) \\ \dot{p}_g &= \gamma(\nu)\,(p_e - p_g) \end{aligned}$$

Since $p_g + p_e = 1$, we have that

$$\dot{p}_e = \gamma(\nu)\,(1 - 2p_e)$$

If we solve this subject to the initial condition $p_e(0) = 0$, we find

$$p_e(\nu) = \frac{1}{2}(1 - \exp(-2\gamma(\nu)t))$$

Figure 1.31 is a comparison of an actual Raman scan with the spectrum $p_e(\nu)$ we would expect based on the laser noise shown in Figure 1.30. The graphs show reasonable agreement; the discrepancy most likely is due to the fact that the two curves were measured on different days, and the exact shape of the FORT/Raman beat note depends on parameters in the feedback loop that vary on a day to day basis.

Fortunately, the effects of the laser noise are only significant when large Rabi frequencies are used, which means the current setup can be used to perform Raman scans, null the magnetic fields, and observe Rabi flopping, as has been discussed in previous sections. However, it is likely that the

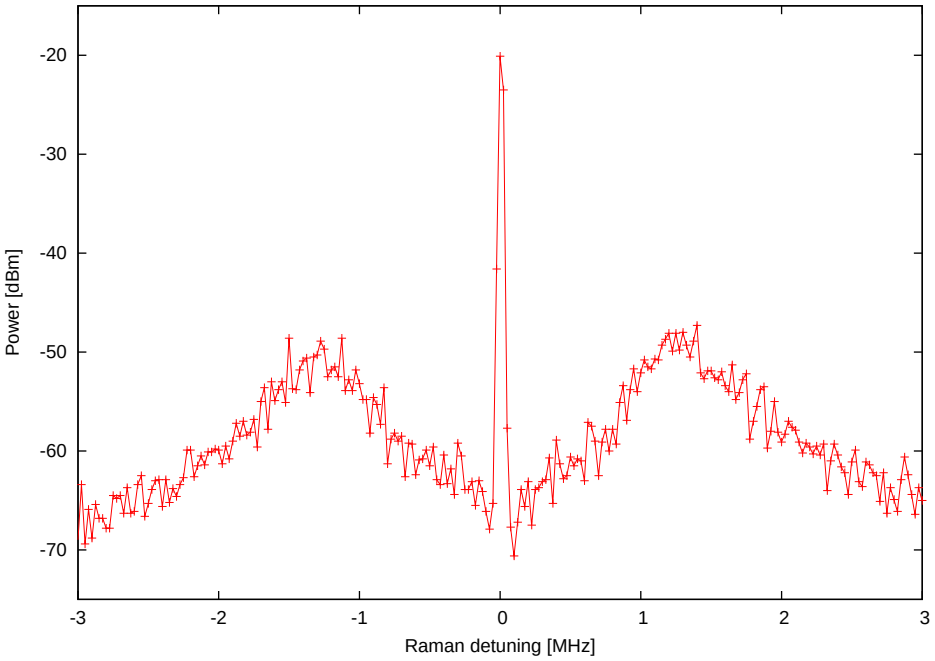

Figure 1.30: Power spectrum of the FORT/Raman beat note ($B = 30\,\mathrm{kHz}$).

laser noise contributes to the large backgrounds observed in the Raman scans (see Figure 1.21, for example). As we will discuss in the following section, the laser noise becomes a serious problem for Raman cooling, because this application requires large Rabi frequencies and also because the bumps in the noise spectrum occur near the frequency of interest—the red sideband at $-1\,\mathrm{MHz}$.

To reduce the laser noise we are planning on changing the Raman configuration; instead of generating the Raman beam by using a separate laser that is phase locked 9.2 GHz away from the FORT laser, we will generate the Raman beam by picking off light from the FORT laser and passing it through a 9.2 GHz EOM and a filtering cavity.

1.9.10 Raman cooling

I now want to show how Raman transitions can be used to cool the atom. To introduce the basic idea, let us first consider Raman cooling using the three-level model we discussed 1.5.3. In the harmonic approximation, the Hamiltonian for the model system is

$$H = H_0 + H_i$$

where

$$H_0 = \omega b^\dagger b + \frac{\delta_A}{2}\sigma_z$$

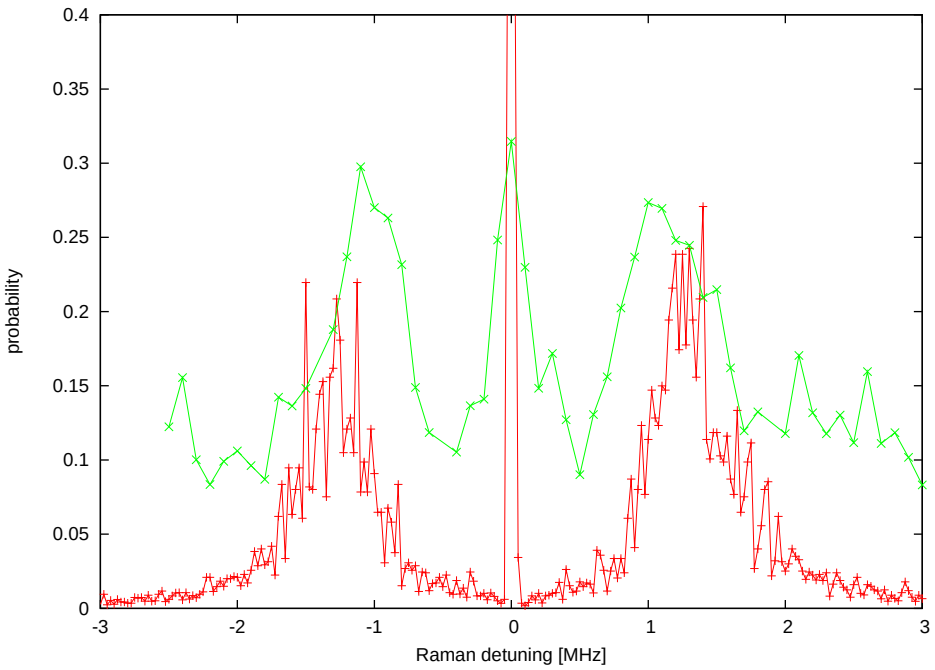

Figure 1.31: Population transfer due to laser noise. The red curve is the observed Raman spectrum for $\Omega = (2\pi)(200\,\mathrm{kHz})$; the green curve is the predicted spectrum based on the power spectrum of the beat note.

is the Hamiltonian for the external and internal degrees of freedom for the atom, and

$$H_i = \Omega\,(1 - \eta^2(b + b^\dagger)^2)\,\sigma_x \cos\delta_R t$$

describes the Raman coupling. The eigenstates of H_0 are $\{|a,n\rangle, |b,n\rangle\}$, and the eigenvalues are $n\omega - \delta_A/2$ for $|a,n\rangle$ and $n\omega + \delta_A/2$ for $|b,n\rangle$. The spectrum of H_0 is shown in Figure 1.32.

Suppose we start the atom in ground state a and vibrational state n. We can lower the vibrational quantum number by driving the atom with a Raman pulse that is tuned to the red sideband ($\delta_R = \delta_A - 2\omega$). The Raman pulse will transfer some of the population from state $|a,n\rangle$ to state $|b,n-2\rangle$. We can then optically pump the atom back into ground state a by applying a classical field on the $b-e$ transition. Note that because n-changing transitions are suppressed by at least $\sim \eta\sqrt{n}$, for small enough n it is unlikely that the atom will change its vibrational state during the optical pumping process. The net effect of the Raman and optical pumping pulses is to move some of the population from state $|a,n\rangle$ to state $|a,n-2\rangle$. By iterating the pulse sequence, we can cool the atom. Although the scheme described here involves alternating Raman pulses with optical pumping pulses, it is also possible to cool the atom by applying the Raman and optical pumping light simultaneously, and it is this method that we actually use in the lab.

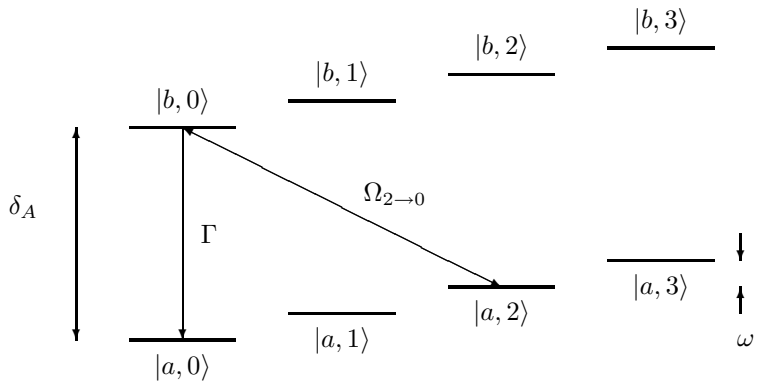

Figure 1.32: Raman cooling. The cooling operates by interleaving Raman pulses tuned to the red sideband with optical pumping pulses on the $b - e$ transition.

Note that the cooling scheme combines a coherent process (the Raman pulse) with a spontaneous process (the spontaneous decay caused by the optical pumping pulse). The spontaneous process is necessary because when we cool the atom we are trying to map all possible initial states to the same final state, which is impossible with unitary evolution alone.

The cooling rate is determined by the Rabi frequency of the Raman pulses. Recall from section 1.5.3 that if the effective Rabi frequency is Ω, then the Rabi frequency for the $|a,n\rangle \leftrightarrow |b,n-2\rangle$ transition is

$$\Omega_{n\to n-2} = \eta^2\sqrt{n}\sqrt{n-1}\,\Omega$$

Thus, we can speed up the cooling by increasing the effective Rabi frequency Ω. However, if we try to go too fast we will off-resonantly excite the transition $|a,n\rangle \leftrightarrow |b,n\rangle$, which has Rabi frequency

$$\Omega_{n\to n} = \Omega - \eta^2(2n+1)\,\Omega$$

Thus, to ensure that we drive the red sideband exclusively, Ω must be smaller than the detuning 2ω of the carrier. This requirement sets the maximum possible cooling rate. Also, note that $\Omega_{n\to n-2}$ is roughly proportional to n. This means that the cooling rate becomes smaller and smaller as the atoms get colder and colder.

So far we have been working in the harmonic approximation, which is only valid near the bottom of the well. When we consider the full $\sin^2 kx$ potential, the situation becomes more complicated, because the vibrational frequency of the atom now depends on the atom's energy. The potential is shallower than harmonic, so hot atoms see a smaller vibrational frequency than cold atoms (see section 1.5.5). To deal with this problem, one can slowly sweep the Raman detuning over the course

of the cooling procedure.

Now let us consider Raman cooling for a real Cesium atom. We will assume that the magnetic field is nulled, so all the Zeeman transitions pile up around zero detuning. If we choose the quantization axis to lie along the cavity axis, then state $|3, m\rangle$ is coupled to state $|4, m\rangle$ by the Raman pulse, and the system can be thought of as a collection of two-level atoms, all with resonant frequency Δ_{HF}. Thus, an arbitrary state $|3, m\rangle$ in the $F = 3$ manifold plays the role of state $|a\rangle$ in the model we just considered, and the corresponding state $|4, m\rangle$ in the $F = 4$ manifold plays the role of state $|b\rangle$.

To implement Raman cooling in the lab, we apply a series of $2,000$ cooling cycles to the atom. Each cooling cycle involves three $100\,\mu$s pulses of light, so the entire series takes 600 ms to complete. The first pulse consists of blue detuned $4-4'$ light, which is applied from the side of the cavity by pairs of circularly polarized beams, together with Raman light, which has detuning $-2\omega_a = -(2\pi)(1\,\text{MHz})$ and Rabi frequency $\Omega_0 = (2\pi)(200\,\text{KHz})$. The $4 - 4'$ light provides polarization gradient cooling in the radial direction, and also acts in tandem with the Raman light to provide Raman cooling in the axial direction (it is the analog of the $b - e$ beam in the three level model). The second pulse consists of Raman light alone, and the third pulse consists of $4 - 5'$ probe light, which is used to detect if the atom is still present in the cavity. The probe light heats the atom, so without the axial cooling provided by the Raman beam the atom would gradually be heated out of the FORT.

Unfortunately, because of the laser noise issues discussed in the previous section, the Raman beam does not drive coherent Rabi flops at the large Rabi frequencies needed for Raman cooling. However, even with the laser noise, the cooling should still be partially effective. For these initial cooling tests we do not worry about sweeping the Raman detuning to account for the anharmonicity of the well; rather, we keep it set at a fixed detuning for the entire cooling process.

To check that the Raman cooling was indeed cooling the atom, we measured the mean lifetime and the probability that the atom survived all $2,000$ cooling cycles for several different values of the Raman detuning. Table 1.3 shows the results we obtained when we tuned the Raman beam to the red sideband, blue sideband, and carrier (-1 MHz, $+1$ MHz, and 0 MHz). The mean lifetime and survival probability are much larger when the Raman beam is tuned to the red sideband than when it is tuned to the carrier or to the blue sideband, which indicates that the Raman cooling is working as intended.

The idea of using the FORT itself as one leg of a Raman pair means that the FORT and Raman beams are perfectly registered, so the effective Rabi frequencies are the same for all the FORT wells.

Raman detuning [MHz]	mean lifetime (# cycles)	survival probability
+1.0	331	0%
0.0	443	2%
-1.0	1,317	36%

Table 1.3: Results of Raman cooling experiment.

This is very convenient for state preparation and for diagnostics such as Raman spectroscopy, but it is a disadvantage when it comes to Raman cooling because, as we discussed earlier, it causes a slowdown in the cooling rate. One could get around this by driving Raman transitions using a pair of Raman beams on a different cavity mode from the FORT. The resulting misregistration of the FORT and Raman beams would allow at least some of the FORT wells to have efficient cooling rates. The ideal cavity mode for the Raman beams would be the mode at exactly half the FORT wavelength (that is, mode number 180 at 468 nm), because at this wavelength there would be a node in the Raman beam at every antinode of the FORT beam. This would result in very efficient cooling for every FORT well.

As discussed in the previous section, we are planning to get rid of the FORT/Raman phase lock and to generate the Raman beam by using an EOM to add 9.2 GHz sidebands to the FORT beam. For this new setup, one could take the laser that currently generates the Raman beam, tune it to a different cavity mode, and put its output light through the same EOM. This way, the FORT and FORT sideband could be used to drive well-insensitive Raman transitions, and the second beam and its sideband could be used for efficient Raman cooling.

1.9.11 Cavity transmission spectrum

As a trapped atom rolls around inside a FORT well its coupling to the cavity changes. The changing cavity coupling will modulate the transmission of a resonant probe beam, so if we look at the power spectrum of the probe transmission we should see a peak at the axial vibrational frequency $\omega_a \sim (2\pi)(500\,\mathrm{kHz})$. We can obtain the power spectrum by taking the click record from a series of Raman measurements and keeping only those trials for which the atom was coupled to the cavity. Each trial is 100 μs long, which is enough time for the atom to undergo ~ 50 axial oscillations. Thus, we can obtain the power spectrum by adding the power spectra for many individual trials. Figure 1.33 shows a power spectrum calculated in this way, using data that was acquired for a Raman scan. We can clearly see a peak at the axial oscillation frequency. The peak is broadened because the trap is anharmonic, so atoms at different temperatures have different vibrational frequencies.

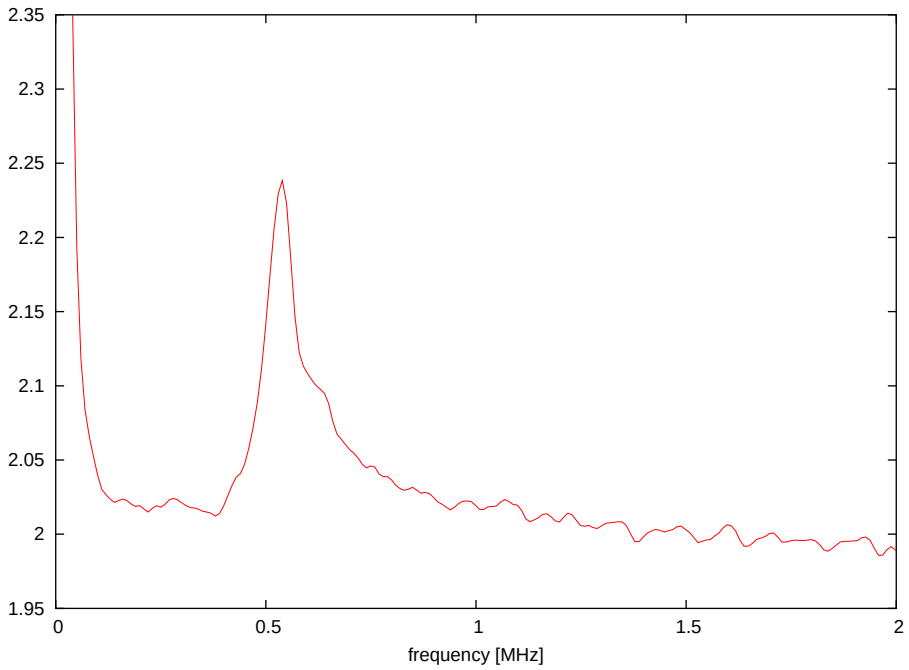

Figure 1.33: Cavity transmission spectrum. The peak at 500 kHz is due to the axial motion of the atom.

1.10 FORT issues

In this section, I discuss several technical issues involving the FORT. I show how the ellipticity of the FORT light can be measured in the lab, and I calculate the rates for atomic scattering due to off-resonant excitation by the FORT light.

1.10.1 FORT ellipticity

If the polarization vector of the linearly polarized FORT input beam is not aligned with a birefringent axis of the cavity, then the cavity will convert the linear polarization to elliptical polarization. As discussed in section 1.7.3, elliptical polarization leads to an unwanted pseudo-magnetic field, so we want to minimize the ellipticity of the FORT by aligning the input polarization as carefully as possible. In this section, I show how this can be accomplished.

Assume the FORT beam propagates in the $\hat{z}$ direction, so the input polarization can be expressed as

$$\hat{\epsilon}_{in} = \cos\theta\,\hat{x} + \sin\theta\,\hat{y}$$

where the axes $\hat{x}$ and $\hat{y}$ are chosen to lie along the birefringent axes of the cavity. The output polarization vector is then given by

$$\hat{\epsilon}_{out} = \cos\theta\,\hat{x} + e^{i\phi}\,\sin\theta\,\hat{y}$$

for some angle ϕ. Note that

$$|\hat{e}_{\pm 1}\cdot\hat{\epsilon}_{out}|^2 = \frac{1}{2}(1 \pm \sin\phi\sin 2\theta)$$

This corresponds to an effective field proportional to

$$|\hat{e}_{+1}\cdot\hat{\epsilon}_{out}|^2 - |\hat{e}_{-1}\cdot\hat{\epsilon}_{out}|^2 = \sin\phi\sin 2\theta = \sqrt{1-e^2}$$

where

$$e = (1-\sin^2\phi\sin^2 2\theta)^{1/2}$$

is the ellipticity of the light.

To measure the ellipticity, we put a polarizing cube in the output beam and measure the transmitted light. Assume the cube is oriented so that it transmits light that is polarized along

$$\hat{e}_\alpha = \cos\alpha\,\hat{x} + \sin\alpha\,\hat{y}$$

Note that

$$\hat{e}_\alpha\cdot\hat{\epsilon}_{out} = \cos\alpha\cos\theta + e^{i\phi}\sin\alpha\sin\theta$$

Thus, the fraction of the power that is transmitted through the cube is

$$\begin{aligned} |\hat{e}_\alpha\cdot\hat{\epsilon}_{out}|^2 &= \cos^2\alpha\,\cos^2\theta + \sin^2\alpha\,\sin^2\theta + \frac{1}{2}\cos\phi\,\sin 2\alpha\,\sin 2\theta \\ &= \frac{1}{2}(1 + \cos 2\alpha\,\cos 2\theta + \sin 2\alpha\,\cos\phi\,\sin 2\theta) \end{aligned}$$

If we define an angle γ by $\cos 2\theta = e\cos\gamma$, then $\cos\phi\sin 2\theta = e\sin\gamma$, and we can express the transmitted power as

$$\begin{aligned} |\hat{e}_\alpha\cdot\hat{\epsilon}_{out}|^2 &= \frac{1}{2}(1 + e\,(\cos\gamma\cos 2\alpha + \sin\gamma\sin 2\alpha)) \\ &= \frac{1}{2}(1 + e\,\cos(2\alpha - \gamma)) \end{aligned}$$

Suppose we rotate the angle α of the polarizing cube and find the maximum and minimum transmitted power P_{max} and P_{min}. Then the ellipticity is given by

$$e = \frac{P_{max} - P_{min}}{P_{max} + P_{min}}$$

We can find the birefringent axes of the cavity by measuring the ellipticity for different values of the input polarization angle θ, as shown in Figure 1.34. The figure plots the measured ellipticity e as a function the angle $\theta_{\lambda/2}$ of a $\lambda/2$ waveplate that is used to rotate the polarization of the input beam. Figure 1.35 shows the results of axial field scans performed before and after the FORT input polarization was aligned with the cavity. Note that when the polarization is misaligned, the spatially inhomogeneous FORT pseudofield broadens out the peak in the field scan.

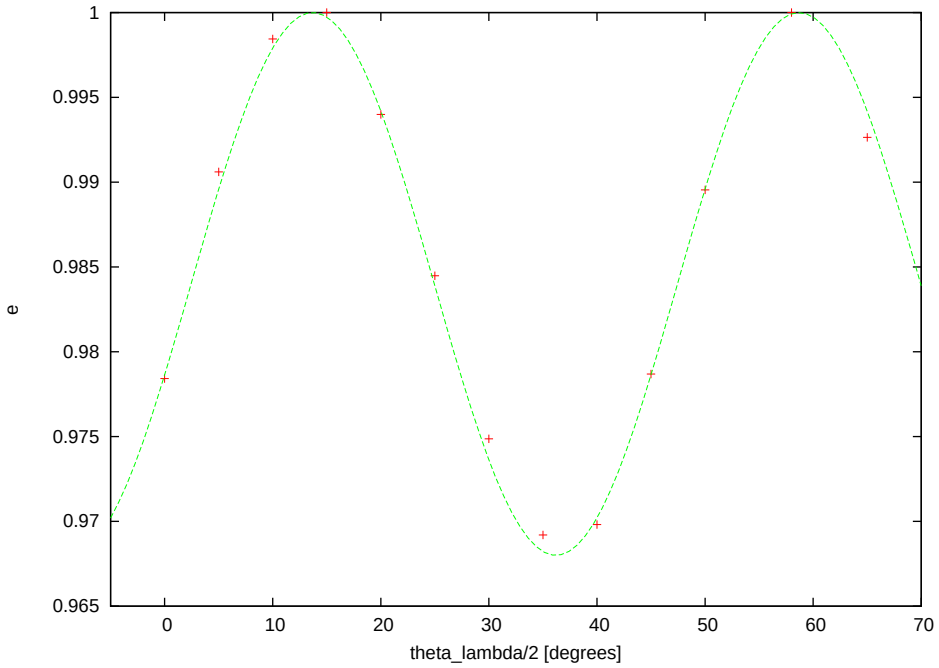

Figure 1.34: Measurement of FORT ellipticity. The curve is a fit to the data, which indicates that a birefringent axis occurs at $\theta_{\lambda/2} = 13.7\,\text{deg}$ and that the maximum ellipticity is $e = 0.032$.

1.10.2 Scattering in a FORT

The FORT is not a perfectly conservative trap; in addition to providing a trapping force for the atom, the FORT light causes scattering among the various atomic ground states. Here I show how to calculate these rates, first for a three-level atom and then for the full Cesium atom.

Consider a three-level atom that has ground states a and b and excited state e, where the $a - e$ transition is driven by a classical field with Rabi frequency Ω and detuning Δ, and the atom can

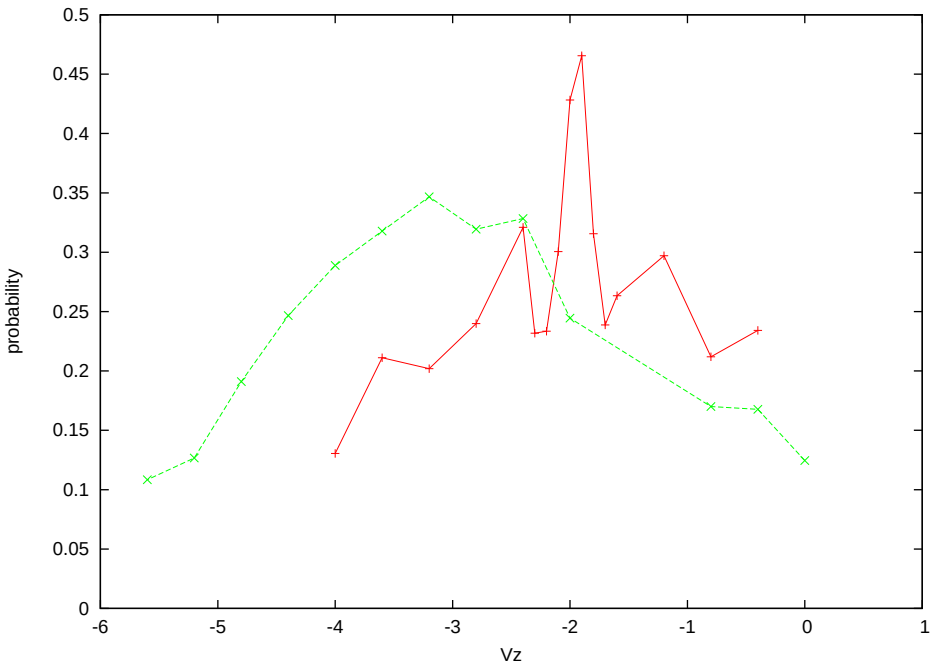

Figure 1.35: Field scans performed before (green curve) and after (red curve) the FORT input polarization was aligned with a birefringent axis of the cavity. A FORT pesudofield broadens the peak.

spontaneously decay from e to b. The Hamiltonian for the system is

$$\begin{aligned} H &= \frac{1}{2}(\Omega|e\rangle\langle a|\, e^{-i\Delta t} + \Omega^*|a\rangle\langle e|\, e^{i\Delta t}) + \\ &\quad \sum_{\vec{k}\lambda} g_{\vec{k}\lambda}(|e\rangle\langle b|\, a_{\vec{k}\lambda}\, e^{-i\delta_{\vec{k}}t} + |b\rangle\langle e|\, a^\dagger_{\vec{k}\lambda}\, e^{i\delta_{\vec{k}}t}) \end{aligned}$$

where $\Delta = \omega_L - \omega_A$ and $\delta_{\vec{k}} = \omega_{\vec{k}} - \omega_A$.

If the atom starts in state a, the classical field will drive it to e, where it can spontaneously decay to b. We want to calculate the resulting $a \rightarrow b$ scattering rate. The wavefunction for the system can be expressed as

$$|\psi(t)\rangle = A(t)|a,0\rangle + E(t)|e,0\rangle + \sum_{\vec{k}\lambda} B_{\vec{k}\lambda}(t)|b,1_{\vec{k}\lambda}\rangle$$

where $A(0) = 1$ and $E(0) = B_{\vec{k}\lambda}(0) = 0$. The equations of motion for $E(t)$ and $B_{\vec{k}\lambda}(t)$ are

$$\begin{aligned} i\dot{E} &= \frac{\Omega}{2}\, e^{-i\Delta t}\, A + \sum_{\vec{k}\lambda} g_{\vec{k}\lambda}\, e^{-i\delta_{\vec{k}}}\, B_{\vec{k}\lambda} \\ i\dot{B}_{\vec{k}\lambda} &= g_{\vec{k}\lambda}\, e^{i\delta_{\vec{k}}t}\, E \end{aligned}$$

For small times, almost all the population is in $|a,0\rangle$, so the equation of motion for E may be approximated as

$$i\dot{E} = \frac{\Omega}{2}\, e^{-i\Delta t}\, A$$

Integrate this subject to the initial condition $E(0) = 0$:

$$E = \frac{\Omega}{2\Delta}(e^{-i\Delta t} - 1)$$

Substitute this into the equations of motion for $\dot{B}_{\vec{k}\lambda}$:

$$i\dot{B}_{\vec{k}\lambda} = \frac{\Omega}{2\Delta}\, g_{\vec{k}\lambda}\, (e^{i(\omega_{\vec{k}}-\omega_L)t} - e^{i\delta_{\vec{k}}t}) \simeq \frac{\Omega}{2\Delta}\, g_{\vec{k}\lambda}\, e^{i(\omega_{\vec{k}}-\omega_L)t}$$

Now integrate this equation, subject to the initial conditions $B_{\vec{k}\lambda}(0) = 0$:

$$B_{\vec{k}\lambda} = -\frac{\Omega}{2\Delta}\, g_{\vec{k}\lambda}\, (\omega_{\vec{k}} - \omega_L)^{-1}\, (e^{i(\omega_{\vec{k}}-\omega_L)t} - 1)$$

Thus,

$$|B_{\vec{k}\lambda}|^2 = \pi t^2 \left|\frac{\Omega}{2\Delta}\, g_{\vec{k}\lambda}\right|^2 D((\omega_{\vec{k}} - \omega_L)t/2)$$

where

$$D(x) = \frac{\sin^2 x}{\pi x^2}$$

Thus, the scattering rate into mode $\vec{k}\lambda$ is

$$\Gamma_{\vec{k}\lambda} = \frac{|B_{\vec{k}\lambda}|^2}{t} = 2\pi \left|\frac{\Omega}{2\Delta}\, g_{\vec{k}\lambda}\right|^2 \delta(\omega_{\vec{k}} - \omega_L)$$

To get the total scattering rate we sum over the scattering rates into all possible modes:

$$\Gamma = \sum_{\vec{k}\lambda} \Gamma_{\vec{k}\lambda} = (2\pi)^{-3}\, V \sum_{\lambda} \int \Gamma_{\vec{k}\lambda}\, d^3k$$

where I have used that the density of modes is given by

$$d^3n = (2\pi)^{-3}\, V\, d^3k$$

The Rabi frequency is

$$\Omega = (8\pi\alpha I)^{1/2}\, \langle e|\hat{\epsilon}\cdot\vec{r}|a\rangle$$

Recall that the mode couplings are (see section 1.6.8):

$$g_{\vec{k}\lambda} = (2\pi\alpha\omega_{\vec{k}}/V)^{1/2}\, \langle e|\vec{r}\cdot\hat{\epsilon}_{\vec{k}\lambda}|b\rangle$$

and that

$$\int |\langle e|\hat{e}_q \cdot \vec{r}|b\rangle|^2 \, d\Omega_{\hat{k}} = \frac{8\pi}{3} \sum_q |\langle e|\hat{e}_q \cdot \vec{r}|b\rangle|^2$$

Thus, the total scattering rate is

$$\Gamma = \frac{8\pi}{3}\alpha^2(\omega_L^3/\Delta) I \sum_q |\langle b|\hat{e}_q^* \cdot \vec{r}|e\rangle\langle e|\hat{\epsilon} \cdot \vec{r}|a\rangle|^2$$

Now consider a full Cesium atom. We can generalize this result to calculate the scattering rate from ground state $|F_1, m_1\rangle$ to ground state $|F_2, m_2\rangle$:

$$\begin{aligned}
&\Gamma(|F_1, m_1\rangle \rightarrow |F_2, m_2\rangle) \\
&= \frac{8\pi}{3}\alpha^2\omega_L^3 I \, D^4 \sum_q \sum_{J'} \left| \frac{1}{\Delta_{J'}} \langle F_2, m_2|(\hat{e}_q^* \cdot \vec{A}(J'))\,(\hat{\epsilon} \cdot \vec{A}^\dagger(J'))|F_1, m_1\rangle \right|^2 \\
&= \frac{\gamma^3}{4}(\omega_L/\omega)^3 (I/I_{sat}) \sum_q \sum_{J'} \left| \frac{1}{\Delta_{J'}} \langle F_2, m_2|(\hat{e}_q^* \cdot \vec{A}(J'))\,(\hat{\epsilon} \cdot \vec{A}^\dagger(J'))|F_1, m_1\rangle \right|^2 \\
&= \frac{\gamma^3}{36\pi^2}(\lambda^3/\lambda_L)(I/I_{sat})(\frac{1}{4}(2C_3 + C_1)^2 \, \delta_{F_1,F_2} \, \delta_{m_1,m_2} + \\
&\quad (C_3 - C_1)^2 \, |\langle F_2, m_2|\hat{\epsilon} \times \vec{J}|F_1, m_1\rangle|^2)
\end{aligned}$$

I will assume the FORT is linearly polarized and that the quantization axis is chosen along the direction of the FORT polarization. It is convenient to express the scattering rates in terms of the FORT potential:

$$U_F = \frac{\gamma^2 \lambda_L}{24\pi} \frac{I}{I_{sat}} (2C_3 + C_1)$$

The scattering rate for $\Delta m = 0$ transitions is

$$\Gamma(|F, m\rangle \rightarrow |F, m\rangle) = \frac{1}{6\pi} \frac{\gamma \lambda^3}{\lambda_L^2} (2C_3 + C_1) \, U_F$$

and the scattering rate for $\Delta m = \pm 1$ transitions is

$$\begin{aligned}
&\Gamma(|F_1, m\rangle \rightarrow |F_2, m \pm 1\rangle) = \\
&\quad \frac{2}{3\pi} \frac{\gamma \lambda^3}{\lambda_L^2} \frac{(C_3 - C_1)^2}{2C_3 + C_1} |\langle F_2, m \pm 1|J_\pm|F_1, m\rangle|^2 \, U_F
\end{aligned}$$

Using these expressions for the scattering rates, we can simulate the depolarization of a trapped atom (these simulations are discussed in [31]).

Chapter 2

Classical field theory model in $1+1$ dimensions

2.1 Introduction

In order to study some basic concepts of classical field theory, I have developed a simple field theory model that describes a scalar field interacting with a non-relativistic particle in $1+1$ dimensions. The model can be viewed as an extreme simplification of classical electrodynamics; specifically, I have simplified classical electrodynamics in three ways: by reducing the number of spatial dimensions from three to one, by replacing the vector field with a scalar field, and by using non-relativistic as opposed to relativistic dynamics.

I chose these specific simplifications for several reasons. Clearly, the fewer dimensions we have to worry about, the simpler the theory will be; thus, we are led to consider theories in one or two spatial dimensions. The reason for choosing one rather than two spatial dimensions is that waves propagate differently in even and odd dimensions: in odd spatial dimensions waves propagate only on the light-cone, while in even spatial dimensions they propagate both on and within the light-cone. Thus, in the interest of preserving as much of the character of $3+1$ dimensional electrodynamics as possible, it is better to work in one spatial dimension. If we simply write down Maxwell's equations in $1+1$ dimensions, we find that the electromagnetic field is not dynamical; it can be solved for in terms of the particle degrees of freedom. Thus, because I was interested in studying field dynamics, I replaced the vector field with a scalar field. However, I found that there are a number of subtleties involving relativistic scalar fields in $1+1$ dimensions, so to avoid these, I made a final simplification by assuming non-relativistic dynamics for the particle motion. The end result is a model that is very simple, but which still exhibits many of the features of $3+1$ dimensional electrodynamics.

2.2 Model field theory

In this section I define the model and then present a detailed analysis of it. I discuss the fields radiated by the particle, the decomposition of the field into *in*/*out* and retarded/advanced fields, scattering, and radiation reaction.

2.2.1 Hamiltonian for the model theory

The model theory describes a particle interacting with a scalar field ϕ and with a static potential V. I will take the Hamiltonian for the system to be

$$H = H_f + H_p + H_i$$

where

$$H_f = \frac{1}{2}\int \left[\pi(t,x)^2 + (\partial_x \phi(t,x))^2\right] dx$$

is the Hamiltonian for a free field,

$$H_p = \frac{p^2}{2m} + V(z)$$

is the Hamiltonian for a free particle, and

$$H_i = 2g \int \phi(t,x)\rho(t,x)\, dx$$

describes the interaction of the particle with the field. Here $\rho(t,x)$ is the charge density; it is given by

$$\rho(t,x) = f(x - z(t))$$

where $f(x)$ is the charge density profile of the particle and $z(t)$ is the position of the particle at time t. I will assume that the charge density profile is normalized such that

$$\int f(x)\, dx = 1$$

Note that

$$\partial_t \rho(t,x) = -v(t)\, f'(x - z(t))$$

and

$$\partial_x \rho(t,x) = f'(x - z(t))$$

so the charge density satisfies the continuity equation

$$\partial_t \rho(t,x) + v(t)\, \partial_x \rho(t,x) = 0$$

From the Hamiltonian, it follows that the equations of motion for the field variables are

$$\begin{aligned} \partial_t \phi(t,x) &= \pi(t,x) \\ \partial_t \pi(t,x) &= \partial_x^2 \phi(t,x) - 2g\rho(t,x) \end{aligned}$$

and the equations of motion for the particle variables are

$$\begin{aligned} \dot{z} &= p/m \\ \dot{p} &= -\frac{dV}{dz} + 2g \int \partial_x \rho(t,x)\, \phi(t,x)\, dx \\ &= -\frac{dV}{dz} - 2g \int \rho(t,x)\, \partial_x \phi(t,x)\, dx \end{aligned}$$

where in the equation of motion for the particle momentum I have integrated by parts and used that the charge density vanishes as $x \to \pm\infty$. From the equations of motion, we find that the field equation is

$$\Box \phi(t,x) = -2g\rho(t,x)$$

and the force on the particle is

$$F = m\ddot{z} = -\frac{dV}{dz} - 2g \int \rho(t,x)\, \partial_x \phi(t,x)\, dx$$

These two equations form a coupled system: the first equation describes how the particle acts as source for the field; the second describes how the field exerts a force on the particle.

We will often want to specialize to the case of a point particle, for which the charge density profile is

$$f(x) = \delta(x)$$

Then the interaction Hamiltonian is

$$H_i = 2g \int \phi(t,x)\, \delta(x - z(t))\, dx = 2g\phi(t, z(t))$$

the equations of motion for the field variables are

$$\begin{aligned} \partial_t \phi(t,x) &= \pi(t,x) \\ \partial_t \pi(t,x) &= \partial_x^2 \phi(t,x) - 2g\, \delta(x - z(t)) \end{aligned}$$

and the equations of motion for the particle variables are

$$\begin{aligned} \dot{z} &= p/m \\ \dot{p} &= -\frac{dV}{dz} - 2g\,\partial_x\phi \end{aligned}$$

where $\partial_x\phi$ is understood to be evaluated at the particle position:

$$\partial_x\phi \equiv \partial_x\phi(t,x)\mid_{x=z(t)}$$

The field equation is

$$\Box\phi(t,x) = -2g\,\delta(x-z(t))$$

and the force on the point particle is

$$F = m\ddot{z} = -\frac{dV}{dz} - 2g\,\partial_x\phi$$

It is interesting to note that a scalar field gives an attractive force between two particles of the same charge. To see this, consider a point charge at rest at z_1. The charge density is

$$\rho(t,x) = \delta(x-z_1)$$

From the field equation, we find that the static field generated by the charge is

$$\phi(t,x) = g|x-z_1|$$

The force that this field exerts on a test charge at z_2 is

$$F = -2g\,\partial_x\phi\mid_{x=z_2} = -2g^2\,\epsilon(z_2-z_1)$$

So the charges attract. Note that for a vector field, like charges repel.

2.2.2 Retarded and advanced fields

As a first step toward understanding the model field theory, I want to calculate the fields radiated by a moving particle. In the full theory, the particle and field form a coupled system, so the particle motion itself depends on the field configuration; however, to start out we will ignore this and assume the particle moves along a prescribed trajectory $z(t)$. Thus, we will view the particle's position not as a dynamical variable, but rather as a time-dependent parameter that enters into the Hamiltonian $H_f + H_i$ for the field. From the equations of motion that follow from this Hamiltonian we obtain

the field equation

$$\Box\phi(t,x) = -2g\rho(t,x)$$

To solve for the field, it is helpful to introduce retarded and advanced Greens functions, which satisfy the equations

$$\Box G_r(t,x) = \Box G_a(t,x) = 2\,\delta(t)\,\delta(x)$$

with boundary conditions $G_r(t,x) = 0$ for $t < 0$ and $G_a(t,x) = 0$ for $t > 0$. The Greens functions are given by

$$\begin{aligned} G_r(t,x) &= \theta(t - |x|) \\ G_a(t,x) &= \theta(-t - |x|) \end{aligned}$$

Using the Greens functions, we can define retarded and advanced fields by

$$\begin{aligned} \phi_r(t,x) &= -g \iint G_r(t-t', x-x')\,\rho(t',x')\,dt'\,dx' + gt \\ \phi_a(t,x) &= -g \iint G_a(t-t', x-x')\,\rho(t',x')\,dt'\,dx' - gt \end{aligned}$$

These fields are solutions to the inhomogeneous wave equation:

$$\Box\phi_r(t,x) = \Box\phi_a(t,x) = -2g\rho(t,x)$$

The gradients of the retarded field are

$$\begin{aligned} \partial_t\phi_r(t,x) &= -g \iint \delta(t-t'-|x-x'|)\,\rho(t',x')\,dt'\,dx' + g \\ &= -g \int \rho(t-|x-y|,y)\,dy + g \end{aligned}$$

$$\begin{aligned} \partial_x\phi_r(t,x) &= g \iint \epsilon(x-x')\,\delta(t-t'-|x-x'|)\,\rho(t',x')\,dt'\,dx' \\ &= g \int \epsilon(x-y)\,\rho(t-|x-y|,y)\,dy \end{aligned}$$

and the gradients of the advanced field are

$$\partial_t\phi_a(t,x) = g \int \rho(t+|x-y|,y)\,dy - g$$

$$\partial_x\phi_a(t,x) = g \int \epsilon(x-y)\,\rho(t+|x-y|,y)\,dy$$

For the case of a point particle, we can explicitly solve for the field gradients. The charge density profile is

$$\rho(t,x) = \delta(x - z(t))$$

So the fields produced by the charge are

$$\begin{aligned}
\phi_r(t,x) &= -g \int \theta(t - t' - |x - z(t')|)\, dt' + gt \\
\phi_a(t,x) &= -g \int \theta(t' - t - |x - z(t')|)\, dt' - gt
\end{aligned}$$

The gradients of the retarded field are

$$\begin{aligned}
\partial_t \phi_r(t,x) &= -g \int \delta(t - t' - |x - z(t')|)\, dt' + g \\
\partial_x \phi_r(t,x) &= g \int \epsilon(x - z(t'))\, \delta(t - t' - |x - z(t')|)\, dt'
\end{aligned}$$

and the gradients of the advanced field are

$$\begin{aligned}
\partial_t \phi_a(t,x) &= g \int \delta(t' - t - |x - z(t')|)\, dt' - g \\
\partial_x \phi_a(t,x) &= g \int \epsilon(x - z(t'))\, \delta(t' - t - |x - z(t')|)\, dt'
\end{aligned}$$

To evaluate these integrals, it is helpful to consider the retarded time $t_r(t,x)$ and the advanced time $t_a(t,x)$ corresponding to the event (t,x). They are defined implicitly by the equations

$$\begin{aligned}
t_r &= t - |x - z(t_r)| \\
t_a &= t + |x - z(t_a)|
\end{aligned}$$

These definitions are unique, provided that the particle always moves more slowly than the speed of light. To see this, suppose that there are two different retarded times t_1 and t_2 corresponding to the event (t,x). Let us choose the labels such that $t_2 > t_1$. Then

$$t_2 - t_1 = |x - z(t_1)| - |x - z(t_2)| \leq |z(t_2) - z(t_1)|$$

where I have used the triangle inequality

$$|a - c| \leq |a - b| + |b - c|$$

Thus, the mean velocity in the time interval $[t_1, t_2]$ is at least the speed of light.

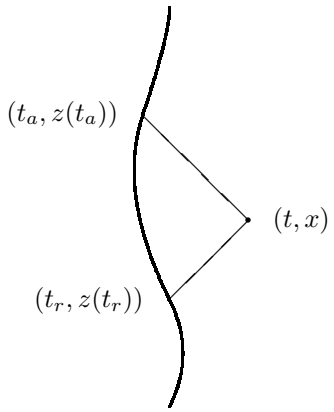

Figure 2.1: The retarded time $t_r(t,x)$ and advanced time $t_a(t,x)$ for the event (t,x).

The retarded time has a simple physical interpretation: if a pulse of radiation is emitted by the particle at the retarded time $t_r(t,x)$, then it will reach point x at time t. Similarly, if a pulse of radiation is emitted from point x at time t, it will reach the particle at the advanced time $t_a(t,x)$ (see Figure 2.1).

The retarded and advanced times can be used to simplify the delta functions in the field gradient integrals:

$$\delta(t - t' - |x - z(t')|) = [1 - v(t_r)\,\epsilon(x - z(t_r))]^{-1}\,\delta(t' - t_r)$$

$$\delta(t' - t - |x - z(t')|) = [1 + v(t_a)\,\epsilon(x - z(t_a))]^{-1}\,\delta(t' - t_a)$$

Thus, performing the t' integration, we find that the gradients of the retarded field are

$$\begin{aligned}
\partial_t \phi_r(t,x) &= -g[1 - v(t_r)\,\epsilon(x - z(t_r))]^{-1}\,v(t_r)\,\epsilon(x - z(t_r)) \\
\partial_x \phi_r(t,x) &= g[1 - v(t_r)\,\epsilon(x - z(t_r))]^{-1}\,\epsilon(x - z(t_r))
\end{aligned}$$

and the gradients of the advanced fields are

$$\begin{aligned}
\partial_t \phi_a(t,x) &= -g[1 + v(t_a)\,\epsilon(x - z(t_a))]^{-1}\,v(t_a)\,\epsilon(x - z(t_a)) \\
\partial_x \phi_a(t,x) &= g[1 + v(t_a)\,\epsilon(x - z(t_a))]^{-1}\,\epsilon(x - z(t_a))
\end{aligned}$$

To simplify the notation, sometimes I will write the gradients of the retarded field as

$$\begin{aligned}
\partial_t \phi_r(t,x) &= -g(1 - v\,\epsilon(x - z))^{-1}\,v\,\epsilon(x - z) \\
\partial_x \phi_r(t,x) &= g(1 - v\,\epsilon(x - z))^{-1}\,\epsilon(x - z)
\end{aligned}$$

where it is understood that z and v are to be evaluated at the retarded time ($v \equiv v(t_r)$, $z \equiv z(t_r)$).

If the particle always moves slower than the speed of light, then

$$\epsilon(x - z(t)) = \epsilon(x - z(t_r))$$

In other words, if the particle is to the left (right) of x at time t, then it was also to the left (right) of x at time $t_r(t, x)$. To see this, note that

$$t - t_r > |z(t) - z(t_r)|$$

since otherwise the average speed of the particle during the time interval $[t_r, t]$ would exceed the speed of light. Thus, from the definition of t_r,

$$|x - z(t_r)| > |z(t) - z(t_r)|$$

So

$$\epsilon(x - z(t_r)) = \epsilon(x - z(t))$$

Thus, we can also express the field gradients as

$$\begin{aligned} \partial_t \phi_r(t, x) &= -g[1 - v(t_r)\,\epsilon(x - z(t))]^{-1}\, v(t_r)\,\epsilon(x - z(t)) \\ \partial_x \phi_r(t, x) &= g[1 - v(t_r)\,\epsilon(x - z(t))]^{-1}\,\epsilon(x - z(t)) \end{aligned}$$

So far, we have assumed that the event (t, x) does not lie on the particle trajectory (so $x \neq z(t)$). But to calculate the radiation reaction force, we will need to know the field gradient at the position of the particle. Thus, I will repeat the derivation of the field gradient for the special case of events that lie on the particle trajectory. For such an event, the field gradient is

$$\begin{aligned} \partial_x \phi_r(t, x)\,|_{x=z(t)} &= g \int \epsilon(z(t) - z(t'))\,\delta(t - t' - |z(t) - z(t')|)\,dt' \\ &= g \int \epsilon(z(t) - z(t - \tau))\,\delta(\tau - |z(t) - z(t - \tau)|)\,d\tau \end{aligned}$$

where $\tau \equiv t - t'$. We can evaluate the integral by expanding in τ; this technique is based on a similar method used in [38] to evaluate the radiation reaction for $3+1$ dimensional electrodynamics. Note that because the particle always moves slower than the speed of light, the delta function is zero for any finite value of τ. The integration region can therefore be restricted to $[-\eta, +\eta]$, where

η is arbitrarily small:

$$\partial_x \phi_r(t,x) \mid_{x=z(t)} = g \int_{-\eta}^{+\eta} \epsilon(z(t) - z(t-\tau))\, \delta(\tau - |z(t) - z(t-\tau)|)\, d\tau$$

For small values of τ, $z(t-\tau)$ can be expanded in a power series:

$$z(t-\tau) = z - v\tau + \cdots$$

where, to simplify the notation, I have defined $z \equiv z(t)$ and $v \equiv v(t)$. If we choose η small enough, then for all values of τ in the integration region the higher order terms in the expansion are negligible. Thus,

$$\partial_x \phi_r(t,x) \mid_{x=z(t)} = g \int_{-\eta}^{+\eta} \epsilon(v\tau)\, \delta(\tau - |v\tau|)\, d\tau$$

For $v > 0$,

$$\begin{aligned} \partial_x \phi_r(t,x) \mid_{x=z(t)} &= g[\int_0^{+\eta} \delta((1-v)\tau)\, d\tau - \int_{-\eta}^{0} \delta((1+v)\tau)\, d\tau] \\ &= g\,(1-v^2)^{-1}\, v \end{aligned}$$

where I have used that

$$\int_0^{+\eta} \delta((1 \pm v)\tau)\, d\tau = (1 \pm v)^{-1} \int_0^{+\eta} \delta(\tau)\, d\tau = \frac{1}{2}(1 \pm v)^{-1}$$

Similarly, for $v < 0$,

$$\begin{aligned} \partial_x \phi_r(t,x) \mid_{x=z(t)} &= -2g^2[\int_0^{+\eta} \delta((1+v)\tau)\, d\tau - \int_{-\eta}^{0} \delta((1-v)\tau)\, d\tau] \\ &= g\,(1-v^2)^{-1}\, v \end{aligned}$$

In either case,

$$\partial_x \phi_r(t,x) \mid_{x=z(t)} = g\,(1-v^2)^{-1}\, v$$

Note that this is the average of the field gradients on either side of the trajectory.

In summary, for points (t,x) that do not lie on the particle trajectory, the field gradient is

$$\partial_x \phi_r(t,x) = g[1 - v(t_r)\, \epsilon(x - z(t))]^{-1}\, \epsilon(x - z(t))$$

where $t_r(t,x)$ is the retarded time for (t,x). For points $(t, x = z(t))$ that do lie on the trajectory, the field gradient is

$$\partial_x \phi_r(t,x) = g\,(1 - v^2(t))^{-1}\, v(t)$$

2.2.3 *in* and *out* fields

Suppose $\phi(t,x)$ is a solution to the inhomogeneous wave equation. We have shown that $\phi_r(t,x)$ and $\phi_a(t,x)$ are also solutions to the inhomogeneous wave equation:

$$\Box\phi(t,x) = \Box\phi_r(t,x) = \Box\phi_a(t,x) = -2g\rho(t,x)$$

Thus, we can define fields $\phi_{in}(t,x)$ and $\phi_{out}(t,x)$ by

$$\begin{aligned} \phi_{in}(t,x) &= \phi(t,x) - \phi_r(t,x) \\ \phi_{out}(t,x) &= \phi(t,x) - \phi_a(t,x) \end{aligned}$$

which satisfy the homogeneous wave equation:

$$\Box\phi_{in}(t,x) = \Box\phi_{out}(t,x) = 0$$

In this section I want to solve for the time evolution of the *in* and *out* fields. I will show that given the initial conditions $\phi_{in}(0,x)$ and $\pi_{in}(0,x)$, the *in* field at time t is

$$\phi_{in}(t,x) = \frac{1}{2}[\phi_{in}(0,x+t) + \phi_{in}(0,x-t) + \int_{x-t}^{x+t} \pi_{in}(0,y)\,dy]$$

A similar result holds for the *out* field. It is straightforward to verify this result by checking that it satisfies the homogeneous wave equation and gives the correct initial conditions at $t = 0$, but it is also instructive to derive it using Greens function methods. The derivation I give here is adapted from an analogous derivation for $3+1$ dimensional electrodynamics that is presented in [39].

The Greens function method for solving the homogeneous wave equation relies on Gauss's theorem, which states that

$$\int_V \partial^\mu C_\mu\, d^2x = \int_{\partial V} C_\mu\, d\sigma^\mu$$

where C_μ is an arbitrary vector field, V is a closed spacetime region, and ∂V is the boundary of V. Suppose we know the value of the fields at time $t = 0$ and want to solve for the fields at a later time $t = t'$. Then we define V to be the rectangular region enclosed by the spacelike lines $(t = 0, x)$ and $(t = t', x)$, and by the timelike lines $(t, -L)$ and $(t, +L)$, where L is chosen to be large enough that

the fields vanish outside of $[-L,+L]$ for all times in the interval $[0,t']$. We choose the vector C_μ to be

$$C_\mu = A\,\partial_\mu B - B\,\partial_\mu A$$

where A and B are given by

$$\begin{aligned} A(t,x) &= \phi_{in}(t,x) \\ B(t,x) &= G_-(t-t',x-x') \end{aligned}$$

and I have defined a Greens function

$$G_-(t,x) = G_r(t,x) - G_a(t,x) = \theta(t-|x|) - \theta(-t-|x|) = \epsilon(t)\,\theta(t^2-x^2)$$

Note that $G_-(t,x)$ satisfies the homogeneous wave equation

$$\Box G_-(t,x) = 0$$

and obeys the boundary conditions

$$\begin{aligned} G_-(t,x)\mid_{t=0} &= 0 \\ \partial_t G_-(t,x)\mid_{t=0} &= 2\delta(x) \end{aligned}$$

Substituting our expression for C_μ into Gauss's theorem, we find that

$$\int_V (A\,\Box B - B\,\Box A)\,d^2x = \int_{\partial V} (A\,\partial_\mu B - B\,\partial_\mu A)\,d\sigma^\mu$$

Because A and B both satisfy the homogeneous wave equation, the right hand side vanishes. Also, because of the way we have defined V, the surface integral vanishes on the two timelike lines. Thus, the surface integrals on the two spacelike lines must be equal:

$$\int [A\,\partial_t B - B\,\partial_t A]_{t=0}\,dx = \int [A\,\partial_t B - B\,\partial_t A]_{t=t'}\,dx$$

Note that

$$\begin{aligned} B\mid_{t=t'} &= 0 \\ \partial_t B\mid_{t=t'} &= 2\delta(x-x') \end{aligned}$$

and

$$\begin{aligned} B\mid_{t=0} &= -\epsilon(t')\,\theta(t'^2-|x-x'|^2) \\ \partial_t B\mid_{t=0} &= \delta(t'+|x-x'|)+\delta(t'-|x-x'|) \end{aligned}$$

We will assume $t'>0$, so

$$\begin{aligned} B\mid_{t=0} &= -\theta(t'^2-|x-x'|^2) \\ \partial_t B\mid_{t=0} &= \delta(t'-|x-x'|) \\ &= \delta(t'-x+x')+\delta(t'-x'+x) \end{aligned}$$

Thus,

$$\begin{aligned} \phi_{in}(t',x') &= \frac{1}{2}\int [A\,\partial_t B - B\,\partial_t A]_{t=0}\,dx \\ &= \frac{1}{2}[\phi_{in}(0,x'+t')+\phi_{in}(0,x'-t')+\int_{x'-t'}^{x'+t'}\pi_{in}(0,x)\,dx] \end{aligned}$$

We can simplify the notation by dropping the primes:

$$\phi_{in}(t,x)=\frac{1}{2}[\phi_{in}(0,x+t)+\phi_{in}(0,x-t)+\int_{x-t}^{x+t}\pi_{in}(0,y)\,dy]$$

2.2.4 Field energy and momentum

Because the model system is invariant under translations of time and space, it obeys conservation laws for energy and momentum. For simplicity, in what follows I will assume the particle is a point charge.

The energy density $u(t,x)$ and momentum density $s(t,x)$ for the field are

$$\begin{aligned} u(t,x) &= \frac{1}{2}(\pi(t,x)^2+(\partial_x\phi(t,x))^2) \\ s(t,x) &= -\pi(t,x)\,\partial_x\phi(t,x) \end{aligned}$$

Note that $s(t,x)$ can also be interpreted as an energy flux, and $u(t,x)$ can be interpreted as a momentum flux. The total energy and momentum of the field can be obtained by integrating the energy density and momentum density over all space:

$$E_f = \int u(t,x)\,dx$$

$$P_f = \int s(t,x)\,dx$$

The total energy of the particle is

$$E_p = \frac{p^2}{2m} + V(z)$$

and the interaction energy due to the coupling of the particle to the field is

$$E_i = 2g\phi(t, z(t))$$

Using the equations of motion, we can calculate the rates of change of these energies. For the particle energy, we find

$$\dot{E}_p = -2g\, v\, \partial_x\phi \mid_{x=z} = vF_f$$

where

$$F_f = -2g\, \partial_x\phi \mid_{x=z}$$

is the force exerted on the particle by the field. For the interaction energy, we find

$$\dot{E}_i = 2g(\pi + v\, \partial_x\phi) \mid_{x=z}$$

Similarly, using the field equation, we can write down a continuity equation for the field energy density:

$$\partial_t u + \partial_x s = -2g\pi\rho = -(\dot{E}_p + \dot{E}_i)\, \delta(x - z(t))$$

If we integrate the continuity equation over all space, and assume there is no outgoing energy flux at $\pm\infty$ (that is, if we assume $s(t,x) \to 0$ as $x \to \pm\infty$), we obtain an energy conservation equation:

$$\frac{d}{dt}(E_f + E_i + E_p) = 0$$

Note that in the energy conservation equation for a vector field the E_i term does not occur.

We can also use the field equations to write down a continuity equation for the field momentum density:

$$\partial_t s + \partial_x u = 2g\rho\, \partial_x\phi = -F_f\, \delta(x - z(t))$$

If we integrate this equation over all space, we find

$$\dot{P}_f + u(t, +\infty) - u(t, -\infty) = -F_f$$

Because the static field of the charge does not fall off with distance, $u(t,x)$ does not vanish as $x \to \pm\infty$. However, we are free to assume that the radiation field does vanish for $x \to \pm\infty$, in which case only the static field is present and $u(t,+\infty) = u(t,-\infty)$. Thus,

$$\dot{P}_f = -F_f$$

and we obtain a momentum conservation equation:

$$\frac{d}{dt}(P_f + p) + \frac{dV}{dz} = 0$$

where I have used that the total force on the particle is

$$F = \dot{p} = -\frac{dV}{dz} + F_f$$

2.2.5 Scattering

We are now ready to consider the problem of scattering in the model theory. The situation I want to consider is the following. In the far past ($t < t_i$, for some time t_i) and the far future ($t > t_f$, for some time t_f), the particle is stationary and is well separated from any radiation fields that are present:

$$z(t) = \begin{cases} z_i & \text{for } t < t_i \\ z_f & \text{for } t > t_f \end{cases}$$

Between t_i and t_f, incoming wave packets (and possibly externally applied driving forces) push the particle around, causing it to emit and absorb radiation. The problem we want to solve is to express the state of the field after t_f in terms of the state of the field before t_i.

For $t < t_i$, the retarded field is just the static field generated by the stationary charge:

$$\phi_r(t,x) = g|x - z_i| \equiv \psi_i(t,x)$$

The total field is

$$\phi(t,x) = \phi_{in}(t,x) + \psi_i(t,x)$$

Thus, for $t < t_i$, the total field cleanly divides into a free radiation field ϕ_{in}, which describes the incoming wave packets, and a static field ψ_i, which is bound to the particle. For $t > t_i$, the *in* field describes what the incoming wave packets would look like if they didn't interact with the particle and continued to evolve freely.

Similarly, for $t > t_f$, we can write down the advanced field:

$$\phi_a(t,x) = g|x - z_f| \equiv \psi_f(t,x)$$

The total field is

$$\phi(t,x) = \phi_{out}(t,x) + \psi_f(t,x)$$

For $t > t_f$, the *out* field describes the outgoing wave packets, while for $t < t_f$, the *out* field describes what the outgoing wave packets would look like if they were freely evolved backward in time, neglecting their interaction with the particle.

We can relate the *in* and *out* fields by noting that for all times

$$\phi(t,x) = \phi_{in}(t,x) + \phi_r(t,x) = \phi_{out}(t,x) + \phi_a(t,x)$$

Thus,

$$\phi_{out}(t,x) = \phi_{in}(t,x) + \phi_-(t,x)$$

where

$$\phi_-(t,x) = \phi_{out}(t,x) - \phi_{in}(t,x) = \phi_r(t,x) - \phi_a(t,x)$$

represents the radiation emitted or absorbed by the particle during the time interval $[t_i, t_f]$.

It will be convenient to restrict our attention to a finite region of spacetime containing the entire scattering process (see Figure 2.2). We will choose a time T big enough that $-T < t_i < t_f < T$. Also, we will choose a length L big enough that for all times $t \in [-T,T]$ the particle and the radiation fields are contained in the spatial region $[-L,L]$; that is, for $t \in [-T,T]$, $z(t) \in [-L,L]$ and the *in* and *out* fields vanish outside of $[-L,L]$. From now on I will only refer to times in the interval $[-T,T]$, so I will assume that these two conditions are met.

I will redefine the field energy E_f, so that only the energy in the region $[-L,L]$ is included:

$$E_f = \int_{-L}^{L} u(t,x)\,dx$$

The total energy of the system is

$$E = E_f + E_p + E_i$$

Because the *in* and *out* fields vanish outside of $[-L,L]$, no field energy can enter or leave this region; therefore, the total energy E is conserved throughout the entire scattering process.

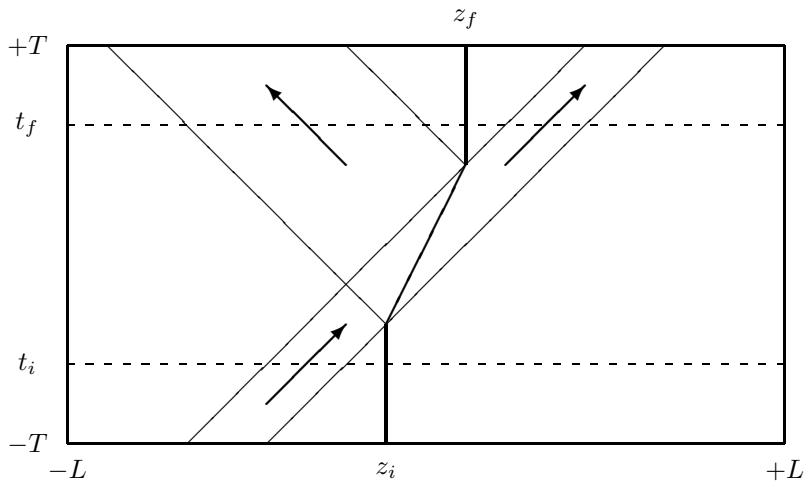

Figure 2.2: Scattering region.

For $t < t_i$, I will define an energy density for the *in* field alone:

$$u_{in}(t,x) = \frac{1}{2}[\pi_{in}^2(t,x) + (\partial_x \phi_{in}(t,x))^2]$$

Note that

$$\phi_{in} = \phi - \psi_i$$

So

$$\begin{aligned}(\partial_x \phi_{in})^2 &= (\partial_x \phi)^2 - 2(\partial_x \phi)(\partial_x \psi_i) + (\partial_x \psi_i)^2 \\ &= (\partial_x \phi)^2 - 2\,\partial_x(\phi\,\partial_x \psi_i) + 2\phi\,\partial_x^2 \psi_i + (\partial_x \psi_i)^2\end{aligned}$$

Substituting for the static field, we obtain

$$(\partial_x \phi_{in})^2 = (\partial_x \phi)^2 - 2g\,\partial_x(\phi\,\epsilon(x - z_i)) + 4g\phi\,\delta(x - z_i) + g^2$$

Also,

$$\pi_{in}^2 = \pi^2$$

Thus, we can relate the *in* field energy density to the total field energy density:

$$u_{in} = u - g\partial_x(\phi\,\epsilon(x - z_i)) + 2g\phi\,\delta(x - z_i) + \frac{1}{2}g^2$$

Since the particle is at rest, $E_p = V(z_i)$, and the total energy of the system is

$$E = E_f + E_i = \int_{-L}^{L} u(t,x)\,dx + V(z_i) + 2g\phi(t,z_i)$$

If we substitute for u, we get

$$E = \int_{-L}^{L} u_{in}(t,x)\,dx - \frac{1}{2}g^2(2L) + g[\phi(t,L) + \phi(t,-L)]$$

Since

$$\phi(t,L) + \phi(t,-L) = \psi_i(t,L) + \psi_i(t,-L) = g|L - z_i| + g|L + z_i| = 2gL$$

we are left with

$$E = \int_{-L}^{L} u_{in}(t,x)\,dx + E_{bound} + V(z_i)$$

where

$$E_{bound} = \frac{1}{2}\int (\partial_x \psi_i)^2\,dx = \frac{1}{2}\int (\partial_x \psi_f)^2\,dx = \frac{1}{2}g^2(2L)$$

is the energy of the static field bound to the stationary charge. Similarly, for $t > t_f$, we can show that

$$E = \int_{-L}^{L} u_{out}(t,x)\,dx + E_{bound} + V(z_f)$$

In summary, for $t < t_i$ the field decomposes into a radiation field ϕ_{in} and a static field ψ_i, while for $t > t_f$ the field decomposes into a radiation field ϕ_{out} and a static field ψ_f. In both cases, the energy of the system also decomposes into a term corresponding to the energy of the radiation field and a term corresponding to the energy of the bound field.

2.2.6 Radiated power

I now want to calculate the power radiated by a particle moving along a specified trajectory $z(t)$. This can be accomplished by evaluating the energy density of the *out* field for $t > t_f$:

$$u_{out}(t,x) = \frac{1}{2}[(\partial_t \phi_{out}(t,x))^2 + (\partial_x \phi_{out}(t,x))^2]$$

For $t > t_f$, the total field may be expressed as

$$\phi(t,x) = \phi_{out}(t,x) + \psi_f(t,x) = \phi_{in}(t,x) + \phi_r(t,x)$$

I will assume that before t_i there is no radiation, so $\phi_{in}(t,x) = 0$. Then

$$\phi_{out}(t,x) = \phi_r(t,x) - \psi_f(t,x)$$

Recall from section 2.2.2 that the gradients of the retarded field are

$$\begin{aligned}\partial_t \phi_r(t,x) &= -g[1 - v\,\epsilon(x - z(t))]^{-1}\, v\,\epsilon(x - z(t)) \\ \partial_x \phi_r(t,x) &= g[1 - v\,\epsilon(x - z(t))]^{-1}\,\epsilon(x - z(t))\end{aligned}$$

where $v \equiv v(t_r)$. For $t > t_f$, $z(t) = z_f$, so

$$\begin{aligned}\partial_t \phi_r(t,x) &= -g[1 - v\,\epsilon(x - z_f)]^{-1}\, v\,\epsilon(x - z_f) \\ \partial_x \phi_r(t,x) &= g[1 - v\,\epsilon(x - z_f)]^{-1}\,\epsilon(x - z_f)\end{aligned}$$

The gradients of the static field are

$$\begin{aligned}\partial_t \psi_f(t,x) &= 0 \\ \partial_x \psi_f(t,x) &= g\,\epsilon(x - z_f)\end{aligned}$$

Thus, the gradients of the *out* field are

$$\begin{aligned}\partial_t \phi_{out}(t,x) &= -g[1 - v\,\epsilon(x - z_f)]^{-1}\, v\,\epsilon(x - z_f) \\ \partial_x \phi_{out}(t,x) &= g[1 - v\,\epsilon(x - z_f)]^{-1}\, v\end{aligned}$$

If we substitute these gradients into the expression for the energy density, we obtain

$$u_{out}(t,x) = g^2 v^2 [1 - v\,\epsilon(x - z_f)]^{-2}$$

Thus, the total energy radiated by the particle during the interval $[t_r, t_r + \Delta t]$ is (see Figure 2.3)

$$\begin{aligned}\Delta E &= u_{out}(t, z(t_r) + (t - t_r))\,(1 - v)\,\Delta t + u_{out}(t, z(t_r) - (t - t_r))\,(1 + v)\,\Delta t \\ &= 2g^2 \frac{v^2}{1 - v^2}\,\Delta t\end{aligned}$$

The radiated power is therefore

$$P = \frac{\Delta E}{\Delta t} = 2g^2 \frac{v^2}{1 - v^2}$$

Note a particle moving at a constant velocity radiates; the theory is not Galilean invariant.

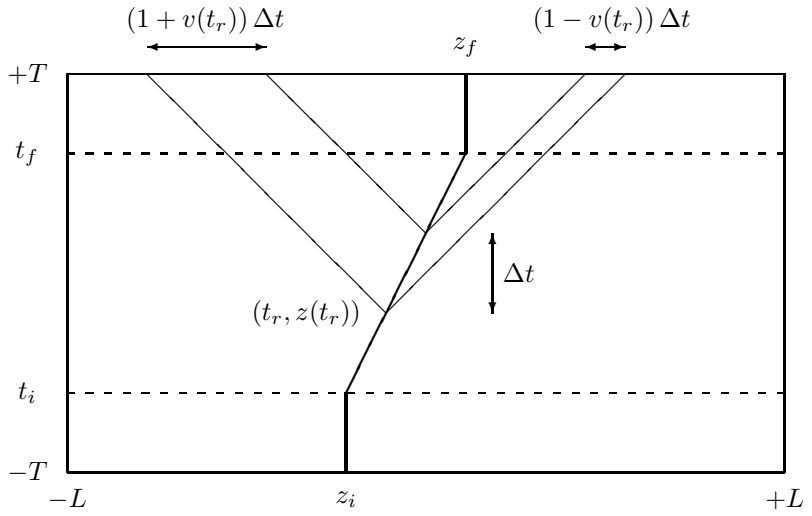

Figure 2.3: Energy radiated by the particle during the time interval $[t_r, t_r + \Delta t]$.

2.2.7 Radiation reaction

A moving particle radiates because the particle does work against a radiation reaction force, which is caused by the coupling of the particle to its own retarded field. Since the total field is

$$\phi(t,x) = \phi_{in}(t,x) + \phi_r(t,x)$$

the total force that the field exerts on the particle is

$$F_f(t) = -2g\,\partial_x\phi(t,x)\mid_{x=z(t)} = F_{in}(t) + F_r(t)$$

where

$$F_{in}(t) = -2g\,\partial_x\phi_{in}(t,x)\mid_{x=z(t)}$$

is the force that the incoming radiation field exerts on the particle, and

$$F_r(t) = -2g\,\partial_x\phi_r(t,x)\mid_{x=z(t)}$$

is the radiation reaction force that the particle exerts on itself. Recall from section 2.2.2 that the gradient of the retarded field evaluated at the position of the particle is

$$\partial_x\phi_r(t,x)\mid_{x=z(t)} = g\,(1 - v^2(t))^{-1}\,v(t)$$

Thus, the radiation reaction force is

$$F_r(t) = -2g^2 \left(1 - v^2(t)\right)^{-1} v(t)$$

For small velocities we approximate this as

$$F_r = -\gamma m \dot{z}$$

where

$$\gamma = 2g^2/m$$

The equation of motion for the particle is then

$$m\ddot{z} + m\gamma\dot{z} = -\frac{dV}{dz} - 2g\,\partial_x \phi_{in}$$

It is sometimes argued that radiation damping in electrodynamics constitutes a time asymmetry of the theory. This is not correct, as can be understood from the model field theory; the time asymmetric damping term only reflects our time asymmetric decomposition of the field into an *in* field and a retarded field. We could equally well decompose the field into an *out* field and an advanced field, in which case the equation of motion would be

$$m\ddot{z} - m\gamma\dot{z} = -\frac{dV}{dz} - 2g\,\partial_x \phi_{out}$$

Note that the sign of the damping term has flipped. The real time asymmetry is not a time asymmetry in the theory; rather, it is a time asymmetry in the asymptotic conditions. In most physical situations, the state of the field is simple in the far past, but complicated in the far future; thus, it is natural to express the equation of motion in terms of the *in* field rather than the *out* field.

2.2.8 Scattering from a harmonically bound particle

I now want to apply the equation of motion derived in the previous section to the problem of scattering from a harmonically bound particle. Consider a wavepacket that starts to the left of the particle and propagates toward it. I will assume the wavepacket is very broad and nearly monochromatic, and may therefore be approximated as a plane wave:

$$\phi_{in}(t, x) = \phi_I\, e^{-i(\omega t - kx)}$$

where $\omega = k$, since the wave propagates to the right. The equation of motion for the particle is

$$\ddot{z} + \gamma\dot{z} + \omega_0^2 z = -(2g/m)\,\partial_x\phi_{in}(t,x)\mid_{x=z} = -i(\gamma\omega/g)\phi_I\, e^{-i(\omega t - kz)}$$

I will assume the displacement of the particle is much less than the wavelength of the radiation ($kz \ll 1$), so we can make the dipole approximation:

$$\ddot{z} + \gamma\dot{z} + \omega_0^2 z = -i(\gamma\omega/g)\phi_I\, e^{-i\omega t}$$

The solution is

$$z(t) = (i/g)\phi_I\, f(\omega)\, e^{-i\omega t}$$

where I have defined

$$f(\omega) = \gamma\omega\,(\omega^2 - \omega_0^2 + i\gamma\omega)^{-1}$$

The velocity of the particle is

$$v(t) = -i\omega\, z(t) = -(\omega/g)\phi_I\, f(\omega)\, e^{-i\omega t}$$

Thus, $v \ll 1$. In this limit, the gradient of the retarded field is

$$\partial_t\phi_r(t,x) = -g\, v(t_r)\,\epsilon(x - z(t_r)) = \omega\phi_I\, f(\omega)\, e^{-i\omega t_r}\,\epsilon(x - z(t_r))$$

Recall that the retarded time $t_r(t,x)$ is given by

$$t_r = t - |x - z(t_r)|$$

Thus, in the dipole approximation

$$e^{-i\omega t_r} \simeq e^{-i\omega(t-|x|)}$$

Also, far away from the particle ($|x| \gg |z|$)

$$\epsilon(x - z(t_r)) = \epsilon(x)$$

Thus, we may simplify our expression for the field gradient:

$$\partial_t\phi_r(t,x) = -\omega\phi_I\, f(\omega)\, e^{-i\omega(t-|x|)}\,\epsilon(x)$$

The total field is given by the sum of the *in* field and the retarded field:

$$\phi = \phi_{in} + \phi_r$$

Thus, the gradient of the total field is

$$\begin{aligned} \partial_t \phi(t,x) &= -i\omega\phi_I\, e^{-i(\omega t - kx)} - \omega\phi_I\, f(\omega)\, e^{-i\omega(t-|x|)}\, \epsilon(x) \\ &= \begin{cases} A_i\, e^{-i(\omega t - kx)} + A_r\, e^{-i(\omega t + kx)} & \text{for } x < 0 \\ A_t\, e^{-i(\omega t - kx)} & \text{for } x > 0 \end{cases} \end{aligned}$$

where

$$\begin{aligned} A_i &= -i\omega\phi_I \\ A_r &= i f(\omega)\, A_i \\ A_t &= (1 - i f(\omega))\, A_i \end{aligned}$$

are the amplitudes of the incoming, reflected, and transmitted waves. The corresponding intensities are

$$\begin{aligned} I_i &= \omega^2 |\phi_I|^2 \\ I_r &= |f(\omega)|^2\, I_i \\ I_t &= (1 - |f(\omega)|^2)\, I_i \end{aligned}$$

where I have used that

$$Imf = -|f|^2$$

Note that the intensities obey the conservation equation $I_i = I_r + I_t$.

2.2.9 Vector field

It is interesting to compare our model field theory for a scalar field with an analogous theory for a vector field. The field equations for the vector field are just Maxwell's equations in $1+1$ dimensions:

$$\begin{aligned} \partial_x E(t,x) &= 2\rho(t,x) \\ \partial_t E(t,x) &= -2J(t,x) \end{aligned}$$

Note that partial derivatives commute:

$$\partial_t \partial_x E(t,x) = \partial_x \partial_t E(t,x)$$

Thus, we obtain a continuity equation for the electric charge:

$$\partial_t \rho(t,x) + \partial_x J(t,x) = 0$$

We can solve for the field explicitly:

$$E(t,x) = E_0 + \int \epsilon(x-y)\,\rho(t,y)\,dy$$

where E_0 is a constant. It is straightforward to check that this satisfies the field equations:

$$\partial_x E(t,x) = 2\int \delta(x-y)\,\rho(t,y)\,dy = 2\rho(t,x)$$

and

$$\begin{aligned}
\partial_t E(t,x) &= -2\int \epsilon(x-y)\,\partial_y J(t,y)\,dy \\
&= -2\int \delta(x-y)\,J(t,y)\,dy \\
&= -2J(t,x)
\end{aligned}$$

where I have used the continuity equation and integrated by parts.

The energy density of the field is

$$u(t,x) = \frac{1}{4}E^2(t,x)$$

and there is no energy flux:

$$s(t,x) = 0$$

Note that

$$\partial_t u(t,x) = \frac{1}{2}E(t,x)\,\partial_t E(t,x) = -J(t,x)E(t,x)$$

and

$$\partial_x u(t,x) = \frac{1}{2}E(t,x)\,\partial_x E(t,x) = \rho(t,x)E(t,x)$$

Thus, we obtain an energy continuity equation

$$\partial_t u(t,x) + \partial_x s(t,x) + J(t,x)E(t,x) = 0$$

and a momentum continuity equation

$$\partial_t s(t,x) + \partial_x u(t,x) - \rho(t,x)E(t,x) = 0$$

For a point particle following a trajectory $z(t)$, the charge density is

$$\rho(t,x) = e\,\delta(x - z(t))$$

and the current is

$$J(t,x) = v(t)\,\rho(t,x)$$

Since we are using nonrelativistic dynamics, the equation of motion for the particle is

$$F(t) = m\ddot{z}(t) = \dot{p}(t) = e\,E(t,z(t))$$

Note that for a point particle,

$$\partial_t u(t,x) = -v(t)\,F(t)\,\delta(x - z(t))$$

and

$$\partial_x u(t,x) = F(t)\,\delta(x - z(t))$$

2.3 Extended particles

So far I have only considered the case of a point particle; I now want to generalize to the case of a spatially extended particle. First I present exact calculations for the self-energy of a stationary extended particle and for the radiation reaction force of an extended particle moving at a uniform velocity, and then I derive approximate results for the self-energy and radiation reaction force of an extended particle in arbitrary motion.

2.3.1 Extended particle at rest

I will begin by calculating the self-energy of a static charge distribution $\rho(x)$. We will assume the distribution is normalized such that

$$\int \rho(x)\,dx = 1$$

and that the charge is confined to a finite region of space, so we can choose a length L such that $\rho(x)$ vanishes outside the interval $[-L, +L]$. The retarded field generated by the charge distribution

is

$$\phi_r(x) = g \int |x - y| \, \rho(y) \, dy + \phi_0$$

where ϕ_0 is an arbitrary constant, which for simplicity I will set to zero. The field energy in the region $[-L, +L]$ is

$$E_f = \frac{1}{2} \int_{-L}^{+L} (\partial_x \phi_r(x))^2 \, dx = \frac{1}{2} \phi_r(x) \, (\partial_x \phi_r(x))|_{x=-L}^{x=+L} - g \int \phi_r(x) \rho(x) \, dx$$

where I have integrated by parts and used the field equation

$$\partial_x^2 \phi_r(x) = 2g\rho(x)$$

The gradient of the retarded field is

$$\partial_x \phi_r(x) = g \int \epsilon(x - y) \, \rho(y) \, dy$$

So

$$\partial_x \phi_r(x) \, |_{x=\pm L} = g \int \epsilon(\pm L - y) \, \rho(y) \, dy = \pm g \int \rho(y) \, dy = \pm g$$

Also,

$$\phi_r(+L) + \phi_r(-L) = g \int (|L - y| + |L + y|) \, \rho(y) \, dy = 2gL \int \rho(y) \, dy = 2gL$$

Thus, the field energy may be expressed as

$$E_f = \frac{1}{2} g^2 (2L) - g \int \phi_r(x) \rho(x) \, dx$$

Note that the first term is just the field energy E_{bound} for a point charge. The interaction energy is

$$E_i = 2g \int \phi_r(x) \rho(x) \, dx$$

So the total energy is

$$\begin{aligned} E_f + E_i &= \frac{1}{2} g^2 (2L) + g \int \phi_r(x) \rho(x) \, dx \\ &= \frac{1}{2} g^2 (2L) + g^2 \iint |x - y| \, \rho(x) \rho(y) \, dx \, dy \end{aligned}$$

Note that the total energy increases if the charge distribution expands; this is because like charges attract.

2.3.2 Extended particle moving at constant velocity

I now want to calculate the radiation reaction force for an extended particle moving at a constant velocity v. The charge density is

$$\rho(t,x) = f(x-vt)$$

where $f(x)$ is the charge density profile of the particle. The gradient of the particle's retarded field is

$$\begin{aligned}
\partial_x\phi_r(t,x) &= g\int \epsilon(x-y)\,\rho(t-|x-y|,y)\,dy \\
&= g\int \epsilon(x-y)\,f(y-v(t-|x-y|))\,dy \\
&= g\int_{-\infty}^{x} f(y-v(t-x+y))\,dy - g\int_{x}^{\infty} f(y-v(t-y+x))\,dy \\
&= g(1-v)^{-1}\int_{-\infty}^{x-vt} f(u)\,du - g(1+v)^{-1}\int_{x-vt}^{\infty} f(u)\,du \\
&= g(1-v^2)^{-1}\int (v+\epsilon(x-vt-u))\,f(u)\,du \\
&= gv(1-v^2)^{-1} + g(1-v^2)^{-1}\int \epsilon(x-vt-u)\,f(u)\,du
\end{aligned}$$

Thus, the retarded force on the particle is

$$F_r(t) = -2g\int \rho(t,x)\,\partial_x\phi_r(t,x)\,dx = -2g^2v(1-v^2)^{-1}$$

This is the same as the retarded force for a point particle.

2.3.3 Extended particle in arbitrary motion

Now let us generalize to the case of arbitrary motion. The calculations presented here are patterned after the discussion of the Abraham-Lorentz self-force that is presented in [40].

Recall that the gradient of the retarded field generated by a charge distribution $\rho(t,x)$ is

$$\partial_x\phi_r(t,x) = g\int \epsilon(x-y)\,\rho(t-|x-y|,y)\,dy$$

Thus, the retarded force is

$$\begin{aligned}
F_r(t) &= -2g\int \rho(t,x)\,\partial_x\phi_r(t,x)\,dx \\
&= -2g^2\iint \rho(t,x)\,\rho(t-|x-y|,y)\,\epsilon(x-y)\,dx\,dy
\end{aligned}$$

We can expand the charge density in a power series in $|x-y|$:

$$\rho(t-|x-y|,y) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} |x-y|^n \, \partial_t^n \rho(t,y)$$

For a particle that moves along a trajectory $z(t)$ and that has a charge density profile $f(x)$, the charge density is

$$\rho(t,y) = f(y-z(t))$$

Thus, if we keep only terms of linear order, then for $n > 0$

$$\partial_t^n \rho(t,y) = -\frac{d^n z(t)}{dt^n} \, \partial_y \rho(t,y)$$

So

$$\rho(t-|x-y|,y) = \rho(t,y) - \partial_y \rho(t,y) \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} |x-y|^n \, \frac{d^n z(t)}{dt^n}$$

Substitute this into the expression for the force:

$$\begin{aligned}
F_r(t) &= 2g^2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{d^n z(t)}{dt^n} \iint \rho(t,x) \, (\partial_y \rho(t,y)) \, |x-y|^n \, \epsilon(x-y) \, dx \, dy \\
&= 2g^2 \sum_{n=1}^{\infty} \frac{(-1)^n}{(n-1)!} \frac{d^n z(t)}{dt^n} \iint \rho(t,x) \, \rho(t,y) \, |x-y|^{n-1} \, dx \, dy \\
&= -2g^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{d^{n+1} z(t)}{dt^{n+1}} \iint \rho(t,x) \, \rho(t,y) \, |x-y|^n \, dx \, dy
\end{aligned}$$

where I have integrated by parts in going from the first to the second line, and used that

$$\partial_y [|x-y|^n \, \epsilon(x-y)] = -n|x-y|^{n-1}$$

For a fixed time t, I will define variables $u = x - z(t)$ and $v = y - z(t)$. Then

$$\begin{aligned}
F_r(t) &= -2g^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{d^{n+1} z(t)}{dt^{n+1}} \iint |u-v|^n \, f(y) f(v) \, du \, dv \\
&= -2g^2 \, \dot{z} - m_S \, \ddot{z} + \cdots
\end{aligned}$$

where

$$m_S \equiv -2g^2 \iint |u-v| \, f(u) f(v) \, du \, dv$$

is the self-energy of the particle.

If the velocities and accelerations are low enough that a linear approximation is justified, and if

the particle is small enough that we can truncate the series expansion of the retarded force at the second order, then the equation of motion for the particle is

$$m\,\ddot{z} = F_{in} + F_r = -2g\partial_x\phi_{in} - 2g^2\,\dot{z} - m_S\,\ddot{z}$$

Thus,

$$m_R\,\ddot{z} + m_R\gamma\dot{z} = -2g\partial_x\phi_{in}$$

where

$$m_R = m + m_S$$

is the renormalized mass, and

$$\gamma = 2g^2/m_R$$

is the damping constant.

2.4 Alternative models for space

In setting up the model field theory, we have assumed that space is both infinitely extended and infinitely divisible, so that points in space can put in one to one correspondence with real numbers. The theory therefore has an infinite number of degrees of freedom, because the value of the field at each point in space constitutes an independent degree of freedom for the system. I now want to consider some alternative models of space. First, I will show how the theory can be adapted to a finite spatial region $[-L, +L]$ on which periodic boundary conditions are imposed. Next, I will write down a discrete version of the theory by introducing a spatial lattice. I consider two different lattice models: one with damped boundary conditions, and one with periodic boundary conditions. The lattice models have a finite number of degrees of freedom, which allows them to be simulated on a computer.

2.4.1 Periodic boundary conditions

The model field theory applies to an infinite spatial region $(-\infty, \infty)$. In this section, I want to show how the theory may be adapted to a finite spatial region $[-L, +L]$, on which periodic boundary conditions are imposed by identifying the points $+L$ and $-L$. For simplicity, I will only consider static charge distributions and fields. The change in spatial topology (from $\mathbb{R}$ to S^1) introduces several new features.

First, because the field gradient is a physical quantity, it must have the same value at $+L$ and

$-L$, as these are just different labels for the same physical point:

$$\partial_x \phi \mid_{x=+L} = \partial_x \phi \mid_{x=-L}$$

From the field equation

$$\partial_x^2 \phi = 2g\rho$$

we see that this implies that the total charge must vanish:

$$\int_{-L}^{+L} \rho \, dx = 0$$

Second, it is not always possible to define a single field ϕ over the entire spatial manifold. Instead, we must introduce an open cover $\{U_\alpha\}$ of the manifold, and define a separate field ϕ_α for each coordinate patch U_α. These fields satisfy the field equations

$$\partial_x^2 \phi_\alpha = 2g\rho$$

In regions where two coordinate patches U_α and U_β overlap, the corresponding fields are related by a constant:

$$\phi_\alpha = \phi_\beta + c_{\alpha\beta}$$

Thus, in the overlap region $U_\alpha \cap U_\beta$ the gradients of the fields ϕ_α and ϕ_β agree:

$$\partial_x \phi_\alpha = \partial_x \phi_\beta$$

As an example, I will consider an open cover consisting of two coordinate patches U_1 and U_2:

$$U_1 = (-L, +L)$$

$$U_2 = [-L, 0) \cup (0, +L]$$

Note that $U_1 \cup U_2 = [-L, +L]$. There are two disjoint regions where U_1 and U_2 overlap, which I will call U_R and U_L:

$$U_L = (-L, 0)$$

$$U_R = (0, +L)$$

Thus, $U_1 \cap U_2 = U_R \cup U_L$.

To find the field $\phi_1(x)$ generated by the charge distribution $\rho(x)$, it is helpful to introduce a Greens

function

$$G(x) = |x| - x^2/2L$$

which satisfies

$$\partial_x^2 G(x) = 2(\delta(x) - 1/2L)$$

Using the Greens function, we can express $\phi_1(x)$ as

$$\phi_1(x) = Ex + c + \int_{-L}^{+L} G(x-y)\,\rho(y)\,dy$$

where E and c are constants. Note that because we need to specify these constants, the charge distribution does not uniquely determine the field or field gradient. If we substitute our expression for the Greens function into this expression for the field, we find that

$$\phi_1(+L) - \phi_1(-L) = 2LE$$

Thus, for $E = 0$ the field is continuous across the boundary.

As an example, consider a configuration of two stationary point particles, one with charge $+g$ at z_+, and one with charge $-g$ at z_-. The field ϕ_1 for coordinate patch U_1 is

$$\begin{aligned}\phi_1(x) &= Ex + c + g|x - z_+| - (g/2L)(x - z_+)^2 - \\ &\quad g|x - z_-| + (g/2L)(x - z_-)^2 \\ &= E'x + c' + g|x - z_+| - g|x - z_-|\end{aligned}$$

where I have defined

$$\begin{aligned}E' &= E + (g/L)(z_+ - z_-) \\ c' &= c - (g/2L)(z_+^2 - z_-^2)\end{aligned}$$

A corresponding field ϕ_2 for the coordinate patch U_2 is given by

$$\phi_2 = \begin{cases} \phi_1(x) + d & \text{for } x < 0 \\ \phi_1(x) + d - 2LE & \text{for } x > 0 \end{cases}$$

where d is an arbitrary constant. The field gradient is

$$\partial_x \phi_1 = \partial_x \phi_2 = E' + g\epsilon(x - z_+) - g\epsilon(x - z_-)$$

2.4.2 Finite lattice with damped boundary conditions

By introducing spatial lattice with a finite number of sites, we can write down a discrete version of the model field theory that has only a finite number of degrees of freedom. The lattice sites are labeled by an integer n that ranges from $-R$ to R, so the total number of sites in the lattice is $N = 2R + 1$. For each site n, we can define field variables $\phi_n(t)$ and $\pi_n(t)$, which evolve under the Hamiltonian

$$H = H_f + H_i + H_p$$

where

$$\begin{aligned} H_f &= \frac{1}{2}\omega_F \sum_n [\pi_n^2 + (\phi_{n+1} - \phi_n)^2] \\ H_i &= 2g \sum_n \phi_n \rho_n \\ H_p &= \frac{p^2}{2m} + V(z) \end{aligned}$$

are the lattice versions of the field, interaction, and particle Hamiltonians. The quantity ρ_n is the charge at lattice site n; it is given by

$$\rho_n(t) = f(n - z(t))$$

where $f(x)$ is the charge density profile of the particle and $z(t)$ is the particle's trajectory. The equations of motion are

$$\begin{aligned} \dot{\phi}_n &= \omega_F \pi_n \\ \dot{\pi}_n &= \omega_F \phi_n'' - 2g\rho_n \\ \dot{z} &= p/m \\ \dot{p} &= -\frac{d}{dz}V(z) - 2g \sum_n \phi_n \frac{d}{dz} f(n - z) \end{aligned}$$

where

$$\phi_n'' \equiv \phi_{n+1} + \phi_{n-1} - 2\phi_n$$

is the lattice equivalent of the second derivative. From the equations of motion, we find that the lattice wave equation is

$$\ddot{\phi}_n - \omega_F^2 \, \phi_n'' = -2g\omega_F \rho_n$$

and the force on the particle is

$$F = m\ddot{z} = -\frac{dV}{dz} - 2g\sum_n \phi_n \frac{d}{dz} f(n-z)$$

Because there are only a finite number of lattice sites, we need boundary conditions for the terminal sites $\pm R$. If the lattice extended beyond site R, then equation of motion for π_R would be

$$\dot{\pi}_R = \omega_F \, \phi_R'' = \omega_F \, (\phi_{R+1} + \phi_{R-1} - 2\phi_R)$$

where for simplicity I have neglected the static field of the charge. If we replace this with an equation of motion that gives the same time evolution for π_R, but which does not contain the field variable ϕ_{R+1}, then waves incident on the terminal lattice site R will be damped. We can obtain such an equation from the following considerations. For the continuum theory, a monochromatic wave has the form

$$\phi(t,x) = \phi_0 \, e^{i(kx-\omega t)}$$

where $\omega = |k|$. Note that

$$(\partial_t \pm \partial_x)\phi(t,x) = 0$$

where the upper sign is for a rightmoving wave ($k > 0$), and the lower sign is for a leftmoving wave ($k < 0$). Thus,

$$\partial_x \phi(t,x) = \mp\pi(t,x)$$

In the lattice theory, this becomes

$$\phi_{n+1} - \phi_n \simeq \mp\pi_n$$

for waves that have a wavelength much longer than the lattice spacing. Thus, for a rightmoving wave incident on site R,

$$\phi_{R+1} \simeq \phi_R - \pi_R$$

We can use this result to substitute for ϕ_{R+1} in the original equation of motion for π_R:

$$\dot{\pi}_R = \omega_F \, (\phi_{R-1} - \phi_R - \pi_R)$$

If we also include the static field of the charge, we get

$$\dot{\pi}_R = \omega_F \, (\phi_{R-1} - \phi_R - \pi_R) + g$$

So, the equations of motions for damped boundary conditions are

$$\begin{aligned}
\dot{\phi}_n &= \omega_F \, \pi_n \\
\dot{\pi}_n &= \begin{cases} \omega_F \left(\phi_{R-1} - \phi_R - \pi_R \right) + g & \text{for } n = R \\ \omega_F \, \phi''_n - 2g\rho_n & \text{for } -R < n < R \\ \omega_F \left(\phi_{-(R-1)} - \phi_{-R} - \pi_{-R} \right) + g & \text{for } n = -R \end{cases}
\end{aligned}$$

The lattice field theory can be simulated on a computer by integrating the equations of motions for some set of initial conditions. For the simulations I present here, the particle follows the trajectory

$$z(t) = \frac{1}{2} a t^2$$

and the charge density of the particle is

$$f(x) = (2\pi\sigma^2)^{-1/2} \, e^{-x^2/2\sigma^2}$$

I integrate the equations of motion for the field variables, subject to the initial conditions

$$\begin{aligned}
\phi_n(0) &= g \sum_r |n - r| \, \rho_r(0) \\
\pi_n(0) &= 0
\end{aligned}$$

where

$$\rho_r(0) = f(r - z(0)) = f(r)$$

Thus, at the start of the simulation, the only field present is the static field of the particle.

The results of the computer simulation are shown in Figure 2.4 and Figure 2.5, which plot the force on the particle as a function of time. The two figures correspond to two limits: a fast moving point particle and a slowly moving extended particle. For a point particle, we would expect the force on the particle to be

$$F_r = -2g^2 v (1 - v^2)^{-1} = -2g^2 a t (1 - (at)^2)^{-1}$$

This limit is shown in Figure 2.4, for which $\sigma = 2$.

The force on a slowly moving extended particle is

$$F_r = -2g^2 v - m_S a = -2g^2 a t - m_S a$$

where

$$m_S \equiv -2g^2 \iint |u-v|\, f(u) f(v)\, du\, dv$$

Note that

$$\iint |u-v|^n\, f(u) f(v)\, du\, dv = \frac{1}{\pi}(2\sigma^2)^{n/2} \int_0^\infty e^{-r^2}\, r^{n+1}\, dr \int_0^{2\pi} |\cos\theta - \sin\theta|^n\, d\theta$$

Thus,

$$m_S = -\frac{4}{\sqrt{\pi}}\, g^2 \sigma$$

This limit is shown in Figure 2.5, for which $\sigma = 10$.

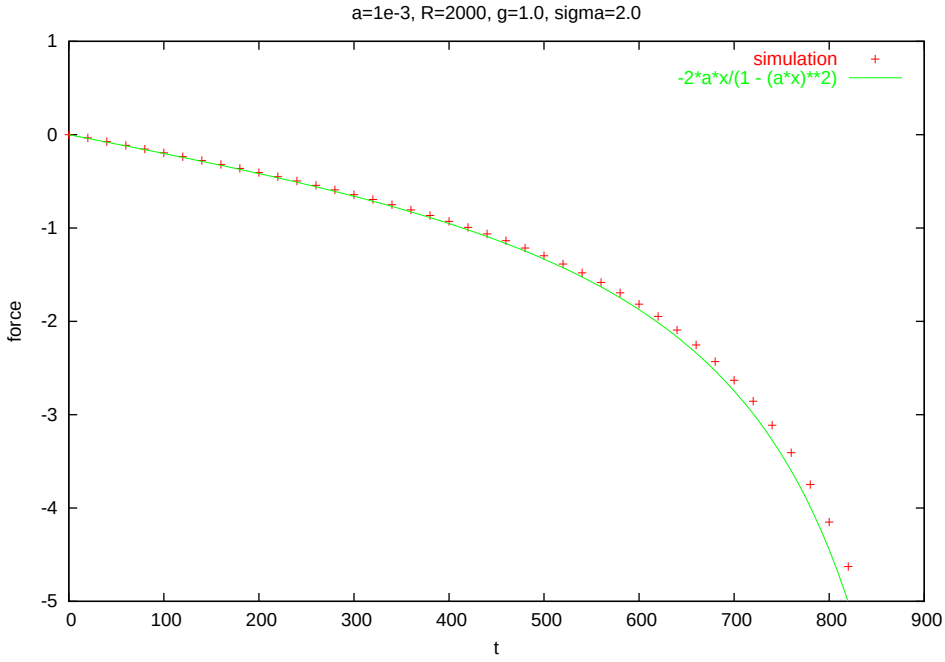

Figure 2.4: Force versus time for a point particle ($\sigma = 2$). The points are from the computer simulation; the curve is the theoretical prediction for a point particle.

2.4.3 Finite lattice with periodic boundary conditions

I now want to consider a variation on the lattice theory from the previous section, where instead of damped boundary conditions we choose periodic boundary conditions. That is, we couple lattice site $+R$ to lattice site $-R$, which is equivalent to making the identifications $\phi_{R+1} \equiv \phi_{-R}$ and $\phi_{-R-1} \equiv \phi_R$. I will assume that there is a single set of field variables (ϕ_n, π_n) that describes the entire lattice. For the continuum theory with periodic boundary conditions that we considered earlier,

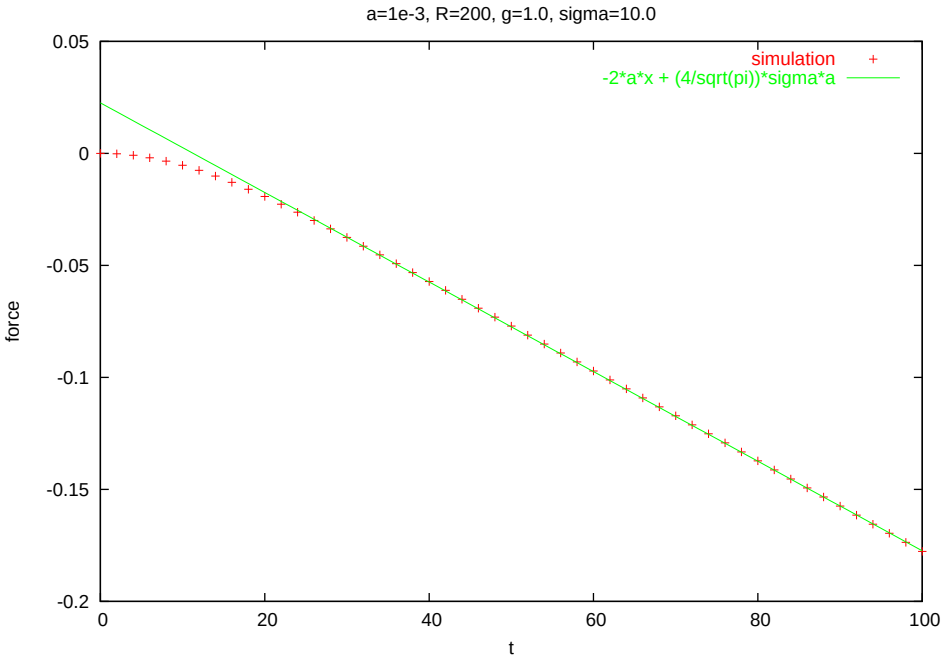

Figure 2.5: Force versus time for an extended particle ($\sigma = 10$). The points are from the computer simulation; the curve is the theoretical prediction.

the equivalent assumption would imply that we are only allowing field configurations for which $E = 0$.

As before, the field and interaction Hamiltonians are

$$\begin{aligned} H_f &= \frac{1}{2}\omega_F \sum_n [\pi_n^2 + (\phi_{n+1} - \phi_n)^2] \\ H_i &= 2g \sum_n \phi_n \rho_n \end{aligned}$$

where ρ_n is the charge at lattice site n. The equations of motion for the field variables are

$$\begin{aligned} \dot{\phi}_n &= \omega_F \pi_n \\ \dot{\pi}_n &= \omega_F \phi_n'' - 2g\rho_n \end{aligned}$$

and the field equation is

$$\ddot{\phi}_n - \omega_F^2 \, \phi_n'' = -2g\omega_F \rho_n$$

I will first calculate the field produced by a static charge distribution, and then generalize to the case of a charge distribution with a specified time dependence. Finally, I'll couple a charged particle to the field and find the equations of motion for the coupled system.

For static charges and fields, the field equation is

$$\phi_n'' = (2g/\omega_F)\rho_n$$

Note that because of the periodic boundary conditions,

$$\sum_n \phi_n'' = \sum_n (\phi_{n+1} + \phi_{n-1} - 2\phi_n) = 0$$

Thus, the total charge must be zero

$$\sum_n \rho_n = 0$$

We can solve for the static field by introducing a Greens function G_n that satisfies

$$G_n'' = 2(\delta_n - 1/N)$$

The static field is then given by

$$\phi_n = (g/\omega_F) \sum_r G_{n-r}\, \rho_r$$

To find G_n, assume it has the form

$$G_n = \sum_{m\neq 0} c_m\, e^{2\pi i m n/N}$$

for some coefficients c_m. Then

$$G_n'' = -\sum_{m\neq 0} (\omega_m/\omega_F)^2 c_m\, e^{2\pi i m n/N}$$

where

$$\omega_m = 2\omega_F |\sin(\pi m/N)|$$

Note that

$$\delta_n = \frac{1}{N}\sum_m e^{2\pi i m n/N} = \frac{1}{N} + \frac{1}{N}\sum_{m\neq 0} e^{2\pi i m n/N}$$

So

$$G_n'' = 2(\delta_n - 1/N) = \frac{2}{N}\sum_{m\neq 0} e^{2\pi i m n/N}$$

Equating the two expressions for G_n'', we find

$$c_m = -\frac{2}{N}(\omega_F/\omega_m)^2$$

Thus,

$$\begin{aligned} G_n &= -\frac{2}{N}\sum_{m\neq 0}(\omega_F/\omega_m)^2\, e^{2\pi imn/N} \\ &= -\frac{2}{N}\sum_{m=1}^{R}\frac{\cos(2\pi mn/N)}{1-\cos(2\pi m/N)} \\ &= |n| - \frac{n^2}{N} - \frac{N^2-1}{6N} \end{aligned}$$

Note that

$$\sum_n G_n = -2\sum_{m\neq 0}(\omega_F/\omega_m)^2\,\delta_m = 0$$

It is convenient to perform a canonical transformation from the field variables (ϕ_n, π_n) to new variables α_m that describe the normal modes of the field:

$$\begin{aligned} \phi_n &= (2N)^{-1/2}\sum_{m\neq 0}(\omega_F/\omega_m)^{1/2}(\alpha_m\, e^{2\pi imn/N} + \alpha_m^*\, e^{-2\pi imn/N}) \\ \pi_n &= -i(2N)^{-1/2}\sum_{m\neq 0}(\omega_m/\omega_F)^{1/2}(\alpha_m\, e^{2\pi imn/N} - \alpha_m^*\, e^{-2\pi imn/N}) \end{aligned}$$

In terms of the new variables, the field Hamiltonian is

$$H_f = \frac{1}{2}\omega_F\sum_n[\pi_n^2 + (\phi_{n+1}-\phi_n)^2] = \sum_{m\neq 0}\omega_m|\alpha_m|^2$$

and the interaction Hamiltonian is

$$H_i = 2g\sum_n \phi_n\rho_n = \sum_{m\neq 0}\omega_m(\alpha_m\lambda_m^* + \alpha_m^*\lambda_m)$$

where

$$\lambda_m = (g_m/\omega_m)\sum_n \rho_n\, e^{-2\pi inm/N}$$

and

$$g_m = (2g)(2N)^{-1/2}(\omega_F/\omega_m)^{1/2}$$

The equations of motion for the normal mode variables are

$$i\dot{\alpha}_m = \omega_m(\alpha_m + \lambda_m)$$

Thus, each field mode acts like a simple harmonic oscillator. The charge distribution displaces each oscillator from its equilibrium value of $\alpha_m = 0$, resulting in a static field $\alpha_m = -\lambda_m$. We can define

new variables by subtracting the contribution from the static field:

$$\bar{\alpha}_m = \alpha_m + \lambda_m$$

Or, in terms of the original field variables,

$$\bar{\phi}_n = \phi_n - (g/\omega_F) \sum_r G_{n-r} \rho_r$$

$$\bar{\pi}_n = \pi_n$$

Then

$$\begin{aligned} H_f + H_i &= \sum_{m \neq 0} \omega_m |\alpha_m|^2 + \sum_{m \neq 0} \omega_m (\alpha_m \lambda_m^* + \alpha_m^* \lambda_m) \\ &= \sum_{m \neq 0} \omega_m |\bar{\alpha}_m|^2 - \sum_{m \neq 0} \omega_m |\lambda_m|^2 \end{aligned}$$

Or, in terms of the original field variables,

$$\begin{aligned} H_f + H_i &= \frac{1}{2} \sum_n (\pi_n^2 + (\phi_{n+1} - \phi_n)^2) + 2g \sum_n \phi_n \rho_n \\ &= \frac{1}{2} \sum_n (\bar{\pi}_n^2 + (\bar{\phi}_{n+1} - \bar{\phi}_n)^2) + g^2 \sum_a \sum_b G_{a-b}\, \rho_a\, \rho_b \end{aligned}$$

where I have used that

$$\sum_{m \neq 0} \omega_m |\lambda_m|^2 = -(g^2/\omega_F) \sum_a \sum_b G_{a-b}\, \rho_a\, \rho_b$$

as can be verified by substituting for λ_m and G_n. Thus, in terms of the transformed variables, the Hamiltonian breaks into a sum of two terms: one that describes a free field, and one that describes the self-interaction of the charge distribution.

Now that we have discussed static charge distributions, let us generalize to the case of a time dependent charge distribution. The equations of motion for the field modes are

$$i\dot{\alpha}_m = \omega_m (\alpha_m + \lambda_m)$$

where now λ_m depends on time. Given the state of the field at time t_0, we can integrate these equations to find the state of the field at a later time t:

$$\alpha_m(t) = \alpha_m(t_0)\, e^{-i\omega_m (t - t_0)} - i\omega_m \int_{t_0}^{t} \lambda_m(t')\, e^{-i\omega_m (t - t')}\, dt'$$

$$= \quad \alpha_m(t_0)\, e^{-i\omega_m(t-t_0)} - ig_m \sum_n e^{-2\pi imr/N} \int_{t_0}^{t} \rho_r(t')\, e^{-i\omega_m(t-t')}\, dt'$$

The first term describes the free evolution of the field, while the second term describes the field generated by the charge distribution. Thus, we can define a retarded field by

$$\alpha_m^{ret}(t) = -ig_m \sum_n e^{-2\pi imr/N} \int \theta(t-t')\, \rho_r(t')\, e^{-i\omega_m(t-t')}\, dt'$$

Or, in terms of the original field variables,

$$\phi_n^{ret}(t) = \sum_r \int G_{n-r}^{ret}(t-t')\, \rho_r(t')\, dt'$$

where

$$G_n^{ret}(t) = (2g/N)\, \theta(t) \sum_{m\neq 0} (\omega_F/\omega_m)\, \sin(2\pi mn/N - \omega_m t)$$

is the retarded Greens function.

So far we have assumed that the charge distribution is an external parameter with a prescribed time dependence, and that only the field is dynamical. Let us now consider what happens when we couple a particle to the field, so the charge distribution depends on the dynamical variables for the particle. Because the total charge must sum to zero for a periodic lattice, I will assume a uniform background charge is present that cancels out the charge of the particle. Thus, the charge distribution can be expressed as

$$\rho_n(t) = f(n - z(t)) - 1/N$$

where $f(x)$ is the charge density profile of the particle and $z(t)$ is the position of the particle at time t. A simple choice for the charge density profile is

$$f(x) = \frac{1}{N} \sum_m \cos(2\pi xm/N) = \frac{1}{N} \sum_m e^{2\pi ixm/N} = \frac{\sin \pi x}{N \sin(\pi x/N)}$$

For simplicity, I have set $\omega_F = 1$, so length and time are dimensionless quantities. Note that for $N \to \infty$

$$f(x) \to (2\pi)^{-1} \int_{-\pi}^{\pi} e^{ix\theta}\, d\theta = \frac{\sin \pi x}{\pi x}$$

For this choice of charge density profile, the charge distribution is

$$\rho_n(t) = \frac{1}{N} \sum_{m\neq 0} e^{2\pi i(n-z(t))m/N}$$

We can check that the total charge is zero:

$$\sum_n \rho_n = \sum_{m\neq 0} \delta_m \, e^{-2\pi i z m/N} = 0$$

The transformed charge distribution is

$$\lambda_m(t) = (g_m/\omega_m) \sum_n \rho_n(t) \, e^{-2\pi i n m/N} = (g_m/\omega_m) \, e^{-ik_m z(t)}$$

where $k_m = 2\pi m/N$. Thus, the interaction Hamiltonian is

$$H_i = \sum_{m\neq 0} g_m(\alpha_m \, e^{ik_m z} + \alpha_m^* \, e^{-ik_m z})$$

If we take the particle Hamiltonian to be

$$H_p = \frac{p^2}{2m} + V(z)$$

then the equations of motion are

$$\begin{aligned} i\dot{\alpha}_m &= \omega_m \alpha_m + g_m \, e^{-ik_m z} \\ \dot{z} &= p/m \\ \dot{p} &= -\frac{dV}{dz} - i \sum_{m\neq 0} g_m k_m (\alpha_m \, e^{ik_m z} - \alpha_m^* \, e^{-ik_m z}) \end{aligned}$$

I will assume that at $t = 0$, only the static field of the charge distribution is present:

$$\alpha_m(0) = -\lambda_m(0) = -(g_m/\omega_m) \, e^{-ik_m z(0)}$$

For later times, part of the field corresponds to the static field, and part corresponds to the radiated field. It is convenient to define a set of variables β_m by subtracting off the instantaneous static field:

$$\beta_m(t) = \alpha_m(t) + (g_m/\omega_m) \, e^{-ik_m z(t)}$$

The equations of motion for the new variables are

$$\begin{aligned} i\dot{\beta}_m &= \omega_m \beta_m + g_m (k_m/\omega_m) v \, e^{-ik_m z} \\ \dot{z} &= p/m \\ \dot{p} &= -\frac{dV}{dz} - i \sum_{m\neq 0} g_m k_m (\beta_m e^{ik_m z} - \beta_m^* \, e^{-ik_m z}) \end{aligned}$$

We can integrate the first equation to solve for $\beta_m(t)$, using that $\beta_m(0) = 0$:

$$\beta_m(t) = -ig_m(k_m/\omega_m) \int_0^t v(t')\, e^{-i[\omega_m(t-t')+k_m z(t')]}\, dt'$$

Thus, the force on the particle is

$$F(t) = -\frac{dV}{dz} - \int_0^t K(t,t')\, v(t')\, dt'$$

where

$$K(t,t') = (4g^2/N) \sum_{m\neq 0} (k_m/\omega_m)^2 \cos(\omega_m(t-t') - k_m(z(t) - z(t')))$$

I will assume that the displacement of the particle is much smaller than the wavelength of the radiation it produces, so we can make the dipole approximation:

$$K(t,t') = (4g^2/N) \sum_{m\neq 0} (k_m/\omega_m)^2 \cos(\omega_m(t-t'))$$

In the dipole approximation, the particle only couples to modes of the field that have wavelengths much longer than the lattice spacing, so we may approximate $\omega_m \simeq |k_m|$ in the sum:

$$K(t,t') \simeq (4g^2/N) \sum_{m\neq 0} \cos(2\pi m(t-t')/N) \simeq 4g^2\, \delta(t-t')$$

Substituting this into the expression for the force, we obtain

$$F = -\frac{dV}{dz} - m\gamma v$$

where

$$\gamma = 2g^2/m$$

This agrees with the force we calculated for the continuum theory, in the limit of a slowly moving point particle.

Chapter 3

Quantum field theory model in $1+1$ dimensions

3.1 Introduction

In this chapter I present a simple quantum field theory model I have developed involving a two level atom coupled to a scalar field on a lattice in $1+1$ dimensions. I begin by discussing a classical scalar field on a lattice, and I then show how the field may be quantized. Finally, I couple the quantized scalar field to a two level atom, and show how properties of the system may be calculated using a number of different methods.

3.2 Classical scalar field on a lattice

Here I consider a simple field theory model: a scalar field on a one dimensional lattice. I write down the Hamiltonian for the model using two different methods; first by considering a mechanical model of the field involving masses coupled by springs, and then by writing down a discrete version of the Lagrangian for a continuum field theory. I find the normal modes of the field and express the Hamiltonian in terms of the normal mode coordinates.

3.2.1 Mechanical model

The Hamiltonian for a scalar field on a lattice can be obtained by considering a mechanical model consisting of a linear chain of masses, where each mass is coupled to its nearest neighbors by springs. We can label the masses by an integer n that runs from $-R$ to R, so the total number of masses in the chain is $N = 2R + 1$. We need boundary conditions for the terminal masses $\pm R$; for simplicity, I will impose periodic boundary conditions, so mass $-R$ is coupled to mass $+R$. If we let $\bar{\phi}_n$ denote the displacement of mass n from its equilibrium position, and let $\bar{\pi}_n$ denote its momentum, then the

Hamiltonian for the system is

$$H = \sum_n [\frac{1}{2m}\bar{\pi}_n^2 + \frac{1}{2}m\omega_F^2(\bar{\phi}_{n+1} - \bar{\phi}_n)^2]$$

where $\bar{\phi}_{R+1} \equiv \bar{\phi}_{-R}$. The Hamiltonian can be simplified by a canonical transformation:

$$\begin{aligned} \phi_n &= (m\omega_F)^{1/2}\,\bar{\phi}_n \\ \pi_n &= (m\omega_F)^{-1/2}\,\bar{\pi}_n \end{aligned}$$

Note that while the original variables $\bar{\phi}_n$ and $\bar{\pi}_n$ have units of length and momentum, respectively, the new field variables both have units of root action. In terms of the new variables, the Hamiltonian is

$$H = \frac{1}{2}\omega_F \sum_n (\pi_n^2 + (\phi_{n+1} - \phi_n)^2)$$

The equations of motion that follow from this Hamiltonian are

$$\begin{aligned} \dot{\phi}_n &= \omega_F \pi_n \\ \dot{\pi}_n &= \omega_F \phi_n'' \end{aligned}$$

where I have defined

$$\phi_n'' = \phi_{n+1} + \phi_{n-1} - 2\phi_n$$

The system is described by $2N$ field variables, but because two degrees of freedom correspond to a uniform translation of the entire lattice, there are only $2N - 2$ physical degrees of freedom. This can be understood as follows. Let us define Φ and Π by

$$\begin{aligned} \Phi &= \sum_n \phi_n \\ \Pi &= \sum_n \pi_n \end{aligned}$$

Using the equations of motion for ϕ_n and π_n, we find

$$\begin{aligned} \dot{\Phi} &= \omega_F \Pi \\ \dot{\Pi} &= 0 \end{aligned}$$

Thus, $\Pi(t) = \Pi_0$ is a constant, and $\Phi(t)$ is given by

$$\Phi(t) = \Phi_0 + \Pi_0\,\omega_F t$$

In what follows, we will assume $\Phi_0 = \Pi_0 = 0$.

3.2.2 Relativistic Lagrangian

A second way to write down the Hamiltonian for the scalar field is to derive it from a relativistic Lagrangian density. The Lagrangian density for a massless scalar field is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) = \frac{1}{2}((\partial_t \phi)^2 - (\partial_x \phi)^2)$$

Since $\mathcal{L}$ has units of energy density, ϕ has units of action. The Lagrangian for the field is

$$L = \int \mathcal{L}\, dx$$

We can put the field on a lattice by making the following replacements:

$$\begin{aligned} \phi(t,x) &\rightarrow \phi_n(t) \\ \partial_t \phi(t,x) &\rightarrow \dot{\phi}_n(t) \\ \partial_x \phi(t,x) &\rightarrow \frac{1}{a}(\phi_{n+1}(t) - \phi_n(t)) \end{aligned}$$

where a is the lattice spacing. Then the Lagrangian density becomes

$$\mathcal{L} \rightarrow \mathcal{L}_n = \frac{1}{2}(\dot{\phi}_n^2 - \frac{1}{a^2}(\phi_{n+1} - \phi_n)^2)$$

and the Lagrangian becomes

$$L \rightarrow a \sum_n \mathcal{L}_n = \frac{1}{2} \sum_n (a\dot{\phi}_n^2 - \frac{1}{a}(\phi_{n+1} - \phi_n)^2)$$

The canonical momentum conjugate to ϕ_n is given by

$$\pi_n = \frac{\partial L}{\partial \dot{\phi}_n} = a\dot{\phi}_n$$

So the Hamiltonian for the field is

$$H = \sum_n \phi_n \pi_n - L = \frac{1}{2}\omega_F \sum_n (\pi_n^2 + (\phi_{n+1} - \phi_n)^2)$$

where $\omega_F = 1/a$. Thus, we obtain the same field Hamiltonian as in the previous section.

3.2.3 Normal modes

I now want to consider a canonical transformation from the field variables $\{\phi_n, \pi_n\}$ to new variables $\{Q_m, P_m\}$ that describe the normal modes of the field. It is convenient to combine Q_m and P_m in a single complex variable α_m:

$$\alpha_m \equiv \frac{1}{\sqrt{2}}(Q_m + iP_m)$$

The transformation $(\phi_n, \pi_n) \to \alpha_m$ is given by

$$\alpha_m = (2N)^{-1/2} \sum_n ((\omega_m/\omega_F)^{1/2}\phi_n + i(\omega_F/\omega_m)^{1/2}\pi_n)\, e^{-2\pi imn/N}$$

where

$$\omega_m = 2\omega_F |\sin(\pi m/N)|$$

and α_m is defined for $m \in \{\pm 1, \cdots, \pm R\}$. Using that

$$\frac{1}{N}\sum_n e^{2\pi imn/N} = \delta_m$$

we can derive the inverse transformation $\alpha_m \to (\phi_n, \pi_n)$:

$$\begin{aligned}
\phi_n &= (2N)^{-1/2} \sum_{m\neq 0} (\omega_F/\omega_m)^{1/2}(\alpha_m\, e^{2\pi imn/N} + \alpha_m^*\, e^{-2\pi imn/N}) \\
&= (2N)^{-1/2} \sum_{m\neq 0} (\omega_F/\omega_m)^{1/2}(\alpha_m + \alpha_{-m}^*)\, e^{2\pi imn/N} \\
\pi_n &= -i(2N)^{-1/2} \sum_{m\neq 0} (\omega_m/\omega_F)^{1/2}(\alpha_m\, e^{2\pi imn/N} - \alpha_m^*\, e^{-2\pi imn/N}) \\
&= -i(2N)^{-1/2} \sum_{m\neq 0} (\omega_m/\omega_F)^{1/2}(\alpha_m - \alpha_{-m}^*)\, e^{2\pi imn/N}
\end{aligned}$$

Note that

$$\phi_{n+1} - \phi_n = (2N)^{-1/2} \sum_{m\neq 0} (\omega_F/\omega_m)^{1/2}(e^{2\pi im/N} - 1)(\alpha_m + \alpha_{-m}^*)\, e^{2\pi imn/N}$$

and

$$|1 - e^{2\pi im/N}|^2 = 4\sin^2(\pi m/N) = (\omega_m/\omega_F)^2$$

Thus,

$$\begin{aligned}
\sum_n (\phi_{n+1} - \phi_n)^2 &= \frac{1}{2}\sum_{m\neq 0} (\omega_m/\omega_F)|\alpha_m + \alpha_{-m}^*|^2 \\
\sum_n \pi_n^2 &= \frac{1}{2}\sum_{m\neq 0} (\omega_m/\omega_F)|\alpha_m - \alpha_{-m}^*|^2
\end{aligned}$$

So the transformed Hamiltonian is

$$H = \frac{1}{2}\omega_F \sum_n (\pi_n^2 + (\phi_{n+1} - \phi_n)^2) = \sum_{m\neq 0} \omega_m |\alpha_m|^2$$

and the equations of motion for the new variables are

$$i\dot{\alpha}_m = \omega_m \alpha_m$$

If we rewrite the above results in terms of the real valued variables Q_m and P_m, we find that the transformation $\{Q_m, P_m\} \rightarrow \{\phi_n, \pi_n\}$ is

$$\begin{aligned}
\phi_n &= N^{-1/2} \sum_{m\neq 0} (\omega_F/\omega_m)^{1/2} [Q_m \cos(2\pi mn/N) - P_m \sin(2\pi mn/N)] \\
\pi_n &= N^{-1/2} \sum_{m\neq 0} (\omega_m/\omega_F)^{1/2} [P_m \cos(2\pi mn/N) + Q_m \sin(2\pi mn/N)]
\end{aligned}$$

the Hamiltonian is

$$H = \frac{1}{2} \sum_{m\neq 0} \omega_m (Q_m^2 + P_m^2)$$

and the equations of motion are

$$\begin{aligned}
\dot{Q}_m &= \omega_m P_m \\
\dot{P}_m &= -\omega_m Q_m
\end{aligned}$$

The variables $\{Q_m, P_m\}$ describe running wave modes; modes with positive m move to the right, and modes with negative m move to the left. We can also describe the system in terms of standing wave modes $\{C_m, c_m, S_m, s_m\}$, which are related to the running wave modes by the canonical transformation

$$\begin{aligned}
Q_{+m} &= \frac{1}{\sqrt{2}}(C_m + s_m) \\
Q_{-m} &= \frac{1}{\sqrt{2}}(C_m - s_m) \\
P_{+m} &= \frac{1}{\sqrt{2}}(c_m - S_m) \\
P_{-m} &= \frac{1}{\sqrt{2}}(c_m + S_m)
\end{aligned}$$

where m is always taken to be positive in the above relations. It is easy to check that $[Q_{m_1}, P_{m_2}] = \delta_{m_1,m_2}$ for all m_1 and m_2; thus the transformation is canonical. The relationship between the

standing wave variables and the variables $\{\phi_n, \pi_n\}$ is

$$\begin{aligned}\phi_n &= (2/N)^{1/2} \sum_{m=1}^{r} (\omega_F/\omega_m)^{1/2}(C_m \cos(2\pi mn/N) + S_m \sin(2\pi mn/N)) \\ \pi_n &= (2/N)^{1/2} \sum_{m=1}^{r} (\omega_m/\omega_F)^{1/2}(c_m \cos(2\pi mn/N) + s_m \sin(2\pi mn/N))\end{aligned}$$

and the transformed Hamiltonian is

$$H = \frac{1}{2} \sum_{m=1}^{r} \omega_m (c_m^2 + C_m^2 + s_m^2 + S_m^2)$$

3.3 Field quantization

We can obtain a model for a quantum field by quantizing the scalar field model from the previous section. I will show that the quantum field obeys the same equations of motion as the classical field in two situations: when the quantum field is in a multi-mode coherent state, and when it is in a single photon state. Also, I will show how a single photon state may be represented in terms of a single particle wavefunction.

In some respects, a classical field is analogous to the quantum wavefunction for a single particle, and a quantized field is analogous to a second quantized wavefunction describing a many particle system. In order to explore the relationships between these systems, I write down the Schrödinger equation on a lattice, second quantize it, and compare it with the quantized scalar field.

3.3.1 Quantized scalar field

The Hamiltonian for the quantized scalar field can be obtained from the Hamiltonian for the classical scalar field via canonical quantization. We have seen that the field variables (ϕ_n, π_n) for the classical field can be expressed in terms of complex normal mode variables α_m:

$$\begin{aligned}\phi_n &= (2N)^{-1/2} \sum_{m \neq 0} (\omega_F/\omega_m)^{1/2}(\alpha_m \, e^{2\pi imn/N} + \alpha_m^* \, e^{-2\pi imn/N}) \\ \pi_n &= -i(2N)^{-1/2} \sum_{m \neq 0} (\omega_m/\omega_F)^{1/2}(\alpha_m \, e^{2\pi imn/N} - \alpha_m^* \, e^{-2\pi imn/N})\end{aligned}$$

In these coordinates, the Hamiltonian describes a set of uncoupled harmonic oscillators:

$$H = \sum_{m \neq 0} \omega_m |\alpha_m|^2$$

We can quantize the system by replacing the amplitudes α_m with annihilation operators $\hat{a}_m$. Thus, the quantized field operators are

$$\begin{aligned}\hat{\phi}_n &= (2N)^{-1/2}\sum_{m\neq 0}(\omega_F/\omega_m)^{1/2}(\hat{a}_m\, e^{2\pi imn/N} + \hat{a}_m^\dagger\, e^{-2\pi imn/N}) \\ \hat{\pi}_n &= -i(2N)^{-1/2}\sum_{m\neq 0}(\omega_m/\omega_F)^{1/2}(\hat{a}_m\, e^{2\pi imn/N} - \hat{a}_m^\dagger\, e^{-2\pi imn/N})\end{aligned}$$

and the quantized Hamiltonian is

$$\hat{H} = \sum_{m\neq 0}\omega_m \hat{a}_m^\dagger \hat{a}_m$$

The Hilbert space for the system may be expressed as a direct sum of Hilbert spaces:

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots$$

where $\mathcal{H}_n$ is the Hilbert space of states containing exactly n photons. The dimensions of the Hilbert spaces for zero, one, and two photons are

$$\begin{aligned}dim(\mathcal{H}_0) &= 1 \\ dim(\mathcal{H}_1) &= N-1 \\ dim(\mathcal{H}_2) &= \frac{1}{2}N(N-1)\end{aligned}$$

Note that the Hamiltonian does not couple Hilbert spaces of different photon number.

3.3.2 Multi-mode coherent state

Let $|\Phi\rangle$ denote the state of the scalar field. Suppose the field is in a multi-mode coherent state, which means that for all m

$$\hat{a}_m|\Phi\rangle = \alpha_m|\Phi\rangle$$

where α_m is a complex number representing the field amplitude for mode m. Note that the field amplitudes may be expressed as

$$\alpha_m = \langle\Phi|\hat{a}_m|\Phi\rangle$$

The state $|\Phi\rangle$ evolves in time according to the Schrödinger equation:

$$i\frac{d}{dt}|\Phi\rangle = \hat{H}|\Phi\rangle$$

So the equations of motion for the field amplitudes are

$$\dot{\alpha}_m = \langle\dot{\Phi}|\hat{a}_m|\Phi\rangle + \langle\Phi|\hat{a}_m|\dot{\Phi}\rangle = -i\langle\Phi|[\hat{a}_m, \hat{H}]|\Phi\rangle = -i\omega_m\langle\Phi|\hat{a}_m|\Phi\rangle = -i\omega_m\alpha_m$$

Thus, the evolution of $|\Phi\rangle$ under the quantum Hamiltonian $\hat{H}$ can be obtained by treating the field amplitudes as normal coordinates for a classical field and evolving them under the classical Hamiltonian H.

3.3.3 Single photon state

Next, suppose the field is in a single photon state $|\Phi\rangle \in \mathcal{H}_1$. We can express $|\Phi\rangle$ as

$$|\Phi\rangle = \sum_{m\neq 0} c_m|1_m\rangle = \sum_{m\neq 0} c_m\hat{a}_m^\dagger|0\rangle$$

for some set of expansion coefficients $\{c_m\}$, which may be thought of as the momentum space wavefunction of the photon. Note that

$$c_m = \langle 0|\hat{a}_m|\Phi\rangle$$

If we substitute our expression for $|\Phi\rangle$ into the Schrödinger equation, we obtain the equations of motion

$$i\dot{c}_m = \omega_m c_m$$

These have the same form as the equations of motion for the normal coordinates of a classical field, so we can formally interpret the single photon wavefunction as a classical field that is given by

$$\begin{aligned}
\phi_n &= (2N)^{-1/2}\sum_{m\neq 0}(\omega_F/\omega_m)^{1/2}(c_m\, e^{2\pi imn/N} + c_m^*\, e^{-2\pi imn/N}) \\
\pi_n &= -i(2N)^{-1/2}\sum_{m\neq 0}(\omega_m/\omega_F)^{1/2}(c_m\, e^{2\pi imn/N} - c_m^*\, e^{-2\pi imn/N})
\end{aligned}$$

These evolve in time according to the classical equations of motion

$$\begin{aligned}
\dot{\phi}_n &= \omega_F\pi_n \\
\dot{\pi}_n &= \omega_F\phi_n''
\end{aligned}$$

We can also represent the single photon state by a position space wavefunction ψ_n, which is a discrete Fourier transform of the momentum space wavefunction c_m:

$$\begin{aligned} \psi_n &= N^{-1/2} \sum_m c_m \, e^{2\pi i m n/N} \\ c_m &= N^{-1/2} \sum_n \psi_n \, e^{-2\pi i m n/N} \end{aligned}$$

The equations of motion for the position space wavefunction are

$$i\dot{\psi}_n = iN^{-1/2} \sum_m \dot{c}_m \, e^{2\pi i m n/N} = N^{-1/2} \sum_m \omega_m c_m \, e^{2\pi i m n/N} = \sum_r H_{n,r} \, \psi_r$$

where

$$H_{n,r} = \frac{1}{N} \sum_{m \neq 0} \omega_m \, e^{2\pi i m (n-r)/N}$$

are the matrix elements of the Hamiltonian in position space. The matrix elements can be expressed as

$$H_{n,r} = \omega_F \, S^1_{n-r}$$

where I have defined

$$S^x_n = \frac{1}{N} \sum_{m \neq 0} (\omega_m/\omega_F)^x \, e^{2\pi i m (n-r)/N}$$

We can evaluate S^1_n as follows:

$$\begin{aligned} S^1_n &= \frac{1}{N} \sum_{m \neq 0} (\omega_m/\omega_F) \cos(2\pi m n/N) \\ &= \frac{2}{N} Im \sum_{m=1}^{R} (e^{\pi i (2n+1) m/N} - e^{\pi i (2n-1) m/N}) \\ &= A_{2n+1} - A_{2n-1} \end{aligned}$$

where

$$A_n = \frac{2}{N} Im \sum_{m=1}^{R} e^{\pi i m n/N} = \frac{2}{N} \frac{\sin(\pi n (R+1)/2N) \sin(\pi n R/2N)}{\sin(\pi n/2N)}$$

Note that for $N \to \infty$,

$$A_n \to \frac{\sin^2(\pi n/4)}{\pi n/4}$$

So

$$S^1_n \to \frac{4}{\pi} \frac{1}{1-4n^2}$$

The equation of motion for the position space wavefunction may be expanded in a lattice Taylor series for some coefficients h_{2r}:

$$i\dot{\psi}_n = \omega_F \sum_r h_{2r}\, \psi_n^{(2r)}$$

where $\psi_n^{(2r)}$ is the lattice derivative of ψ_n of order $2r$, which is defined recursively by

$$\psi_n^{(2s+2)} = \psi_{n+1}^{(2s)} - 2\psi_n^{(2s)} + \psi_{n-1}^{(2s)}$$

and $\psi_n^{(0)} = \psi_n$. For example,

$$\psi_n^{(2)} = \psi_n'' = \psi_{n+1} - 2\psi_n + \psi_{n-1}$$

and

$$\psi_n^{(4)} = \psi_{n+1}'' - \psi_n'' + \psi_{n-1}'' = \psi_{n+2} - 4\psi_{n+1} + 6\psi_n - 4\psi_{n-1} + \psi_{n-2}$$

Note that $H_{n,r}$ is nonlocal, because the time derivative of ψ_n depends on space derivatives of ψ_n of arbitrarily high order. This nonlocality is acceptable, because the position space wavefunction is not observable; the field only couples to matter through the field operator $\hat{\phi}_n$. Thus, the physically relevant representation is the effective classical field ϕ_n, which obeys a local wave equation:

$$\ddot{\phi}_n - \omega_F^2\, \phi_n'' = 0$$

In summary, the equations of motion for the single photon may be obtained from the classical Hamiltonian

$$H = \frac{1}{2}\omega_F \sum_n (\pi_n^2 + (\phi_{n+1} - \phi_n)^2) = \frac{1}{2}\sum_{m\neq 0} \omega_m |c_m|^2$$

or by restricting to the single particle subspace of the quantum Hamiltonian

$$\hat{H} = \sum_n H_{n,r} \hat{\psi}_n^\dagger \hat{\psi}_r = \sum_{m\neq 0} \omega_m \hat{c}_m^\dagger \hat{c}_m$$

The transformation $(\phi_n, \pi_n) \longleftrightarrow c_m$ is canonical, and the transformation $\psi_n \longleftrightarrow c_m$ is unitary. The quantum coordinates are normalized such that

$$\sum_n |\psi_n|^2 = \sum_m |c_m|^2 = 1$$

and the equations of motion preserve this normalization. The different representations of the single photon state $|\Phi\rangle$ are related by

$$\begin{aligned}
\phi_n &= (2N)^{-1/2} \sum_{m\neq 0} (\omega_F/\omega_m)^{1/2} (c_m\, e^{2\pi i m n/N} + c_m^*\, e^{-2\pi i m n/N}) \\
&= \sqrt{2} \sum_r S_{n-r}^{-1/2}\, Re(\psi_r)
\end{aligned}$$

$$\begin{aligned}
\pi_n &= -i(2N)^{-1/2} \sum_{m\neq 0} (\omega_m/\omega_F)^{1/2} (c_m\, e^{2\pi i m n/N} - c_m^*\, e^{-2\pi i m n/N}) \\
&= \sqrt{2} \sum_r S_{n-r}^{1/2}\, Im(\psi_r)
\end{aligned}$$

$$\begin{aligned}
c_m &= (2N)^{-1/2} \sum_n [(\omega_m/\omega_F)^{1/2} \phi_n + i(\omega_F/\omega_m)^{1/2} \pi_n]\, e^{-2\pi i m n/N} \\
&= N^{-1/2} \sum_n \psi_n\, e^{-2\pi i m n/N}
\end{aligned}$$

$$\begin{aligned}
\psi_n &= \frac{1}{\sqrt{2}} \sum_r (S_{n-r}^{1/2}\, \phi_r + i S_{n-r}^{-1/2}\, \pi_r) \\
&= N^{-1/2} \sum_m c_m\, e^{2\pi i m n/N}
\end{aligned}$$

As an example, suppose the position space wavefunction for the photon is a wavepacket with a gaussian envelope:

$$\psi_n(x, \sigma, m) = A\, e^{-(n-x)^2/2\sigma^2}\, e^{2\pi i m n/N}$$

Here x is the position of the packet, σ is the width of the packet, m determines the momentum of the packet, and A is a normalization constant. Graphs of the single photon in each representation are plotted in Figures 3.1, 3.2, and 3.3. For these graphs, $N = 2001$, $x = 0.0$, $\sigma = 200.0$, and $m = 20$.

3.3.4 Second quantized Schrödinger equation

Having discussed the quantum description of fields, I now want to turn to the quantum description of particles. In quantum mechanics, a single particle is described by a wavefunction $\psi(t, x)$, which evolves in time according to the Schrödinger equation:

$$i\partial_t \psi = -\frac{1}{2m} \partial_x^2 \psi + V\psi$$

where $V(x)$ is the potential energy. We can put the theory on a lattice by making the following replacements:

$$V(x) \quad \rightarrow \quad V_n$$

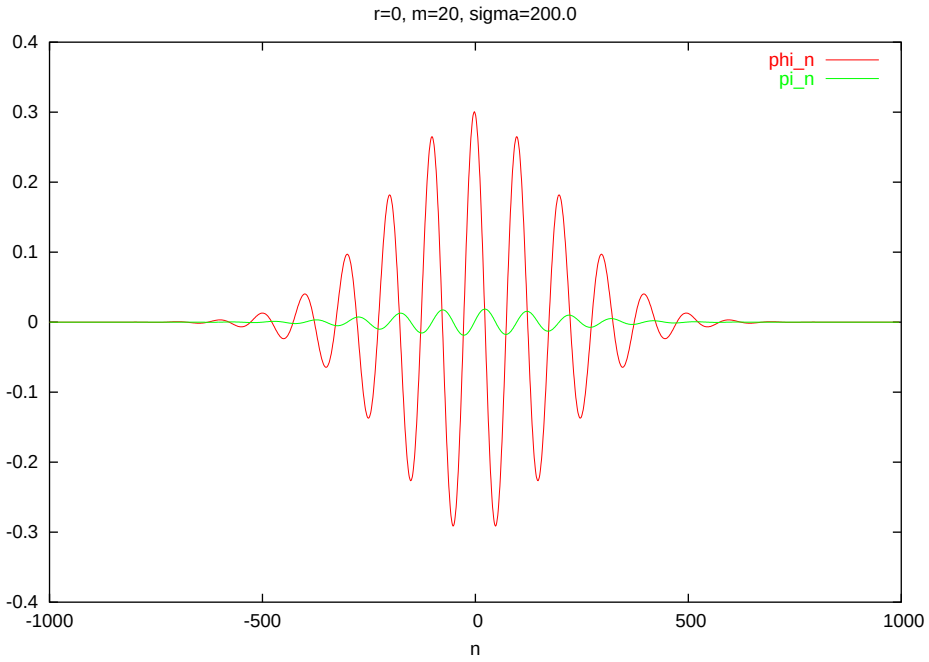

Figure 3.1: Single photon state, ϕ_n and π_n field variables.

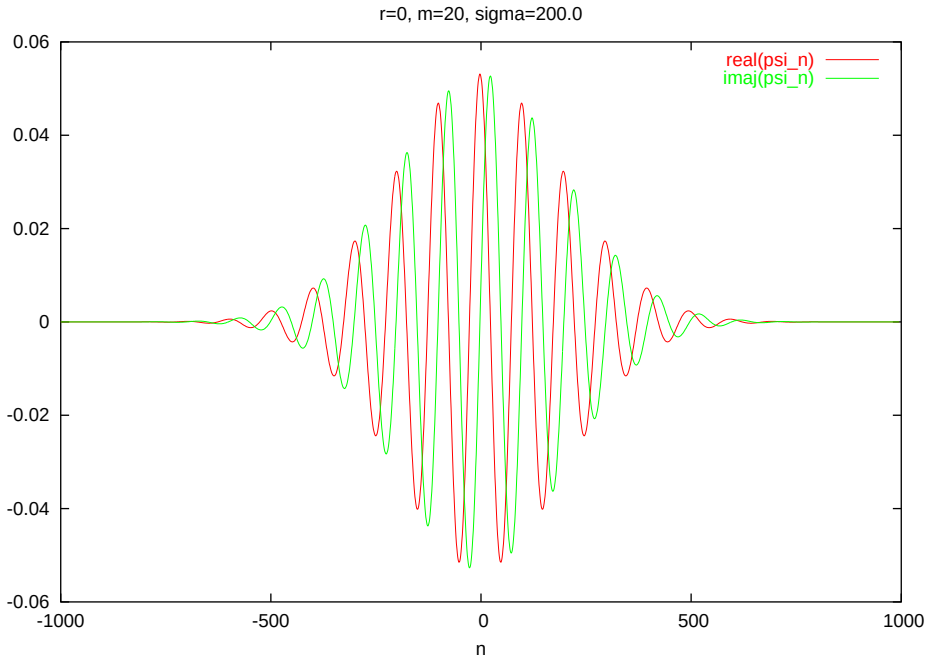

Figure 3.2: Single photon state, position space wavefunction ψ_n.

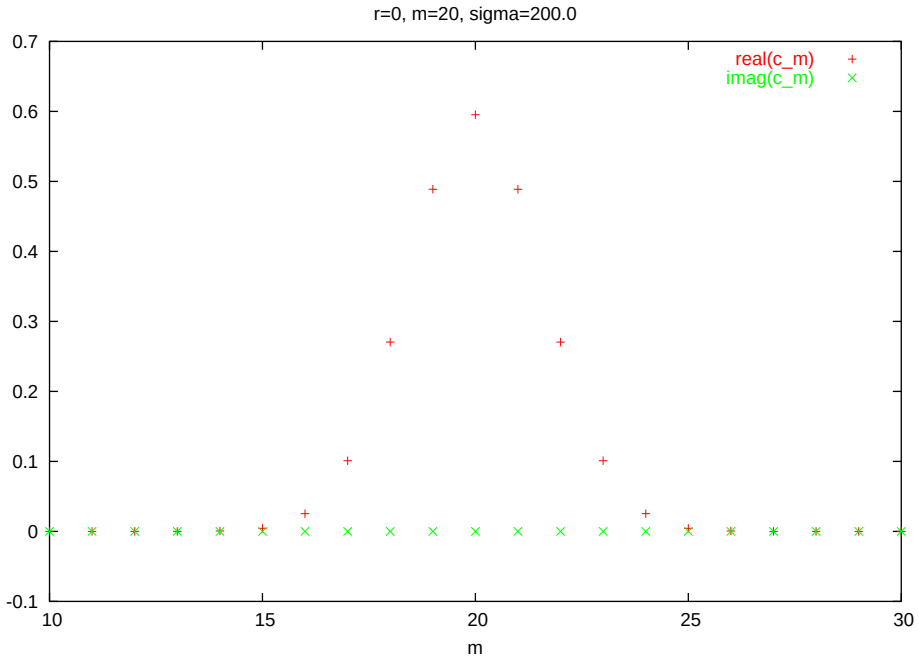

Figure 3.3: Single photon state, momentum space wavefunction c_m.

$$\begin{aligned} \psi(t,x) &\rightarrow \psi_n(t) \\ \partial_t\psi(t,x) &\rightarrow \dot{\psi}_n(t) \\ \partial_x^2\psi(t,x) &\rightarrow \frac{1}{a^2}\,\psi_n''(t) \end{aligned}$$

where

$$\psi_n'' \equiv \psi_{n+1} + \psi_{n-1} - 2\psi_n$$

Thus, the Schrödinger equation for the lattice theory is

$$i\dot{\psi}_n = -\omega_F\,\psi_n'' + V_n\psi_n$$

where

$$\omega_F = (2ma^2)^{-1}$$

In coordinate free form, the wavefunction for the particle is

$$|\psi\rangle = \sum_n \psi_n|n\rangle$$

where $|n\rangle$ represents a state of the system in which the particle is localized at lattice site n. The Schrödinger equation can then be expressed as

$$i\partial_t|\psi\rangle = \hat{H}|\psi\rangle$$

where

$$\hat{H} = -\omega_F \sum_n |n\rangle(\langle n+1| + \langle n-1| - 2\langle n|) + \sum_n V_n|n\rangle\langle n|$$

For a free particle ($V_n = 0$), the eigenstates of $\hat{H}$ are

$$|p_m\rangle = N^{-1/2}\sum_n e^{2\pi imn/N}|n\rangle$$

and the eigenvalues are

$$\omega_m = 4\omega_F \sin^2(\pi m/N) = \frac{2}{ma^2}\sin^2(p_m a/2)$$

where $p_m = 2\pi m/Na$. Note that for $p_m \ll \omega_F$ the eigenvalues can be approximated as

$$\omega_m \sim \frac{p_m^2}{2m}$$

which is the ordinary dispersion relation for a nonrelativistic particle.

In the basis of energy eigenstates, the Hamiltonian is

$$\hat{H} = \sum_m \omega_m |p_m\rangle\langle p_m|$$

Thus, if we expand an arbitrary wavefunction $|\psi\rangle$ in the basis of energy eigenstates

$$|\psi\rangle = \sum_m c_m|p_m\rangle$$

then the equations of motion are

$$i\dot{c}_m = \omega_m c_m$$

I will now show how the single particle wavefunction can be related to a classical field described by the Hamiltonian

$$H = \frac{1}{2}\omega_F \sum_n [(\pi_{n+1} - \pi_n)^2 + (\phi_{n+1} - \phi_n)^2]$$

Consider a canonical transformation from the field variables (ϕ_n, π_n) to new variables (Q_m, P_m) that describe the normal modes of the field. For convenience I will introduce a single complex coordinate

c_m, defined by

$$c_m = \frac{1}{\sqrt{2}}(Q_m + iP_m)$$

The canonical transformation $c_m \to (\phi_n, \pi_n)$ is given by

$$\begin{aligned}
\phi_n &= (2N)^{-1/2} \sum_{m\neq 0} (c_m\, e^{2\pi imn/N} + c_m^*\, e^{-2\pi imn/N}) \\
\pi_n &= -i(2N)^{-1/2} \sum_{m\neq 0} (c_m\, e^{2\pi imn/N} - c_m^*\, e^{-2\pi imn/N})
\end{aligned}$$

Note that

$$|1 - e^{2\pi im/N}|^2 = 4\sin^2(\pi m/N) = \omega_m/\omega_F$$

Thus,

$$\begin{aligned}
\sum_n (\phi_{n+1} - \phi_n)^2 &= \frac{1}{2} \sum_{m\neq 0} (\omega_m/\omega_F)|c_m + c_{-m}^*|^2 \\
\sum_n (\pi_{n+1} - \pi_n)^2 &= \frac{1}{2} \sum_{m\neq 0} (\omega_m/\omega_F)|c_m - c_{-m}^*|^2
\end{aligned}$$

So the transformed Hamiltonian is

$$H = \frac{1}{2}\omega_F \sum_n [(\pi_{n+1} - \pi_n)^2 + (\phi_{n+1} - \phi_n)^2] = \sum_{m\neq 0} \omega_m |c_m|^2$$

Thus, in the new coordinates, the field is described as a set of noninteracting harmonic oscillators. The equations of motion for $\{c_m\}$ that follow from the classical Hamiltonian H are the same as those that follow from the quantum Hamiltonian $\hat{H}$, so we can formally interpret the single particle as a classical field.

Quantizing the classical field is equivalent to second quantizing the quantum wavefunction. This is accomplished by quantizing each of the normal modes ($c_m \to \hat{c}_m$), so the second quantized Hamiltonian is

$$\hat{H} = \sum_m \omega_m \hat{c}_m^\dagger \hat{c}_m = \sum_{nr} h_{nr}\, \hat{\psi}_n^\dagger \hat{\psi}_r$$

The operators $\hat{c}_m^\dagger$ and $\hat{c}_m$ create and destroy a particle in mode m, while $\hat{\psi}_n^\dagger$ and $\hat{\psi}_n$ create and destroy a particle at lattice site n.

The total Hilbert space for the system is a direct sum of n particle Hilbert spaces:

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots$$

where $\mathcal{H}_n$ is the Hilbert space for n particles. The Hamiltonian can describe bosons or fermions, depending on whether we choose commutation or anticommutation relations for the creation and annihilation operators. For bosons, the dimensions of the n-particle Hilbert spaces are

$$\begin{aligned} dim(\mathcal{H}_0) &= 1 \\ dim(\mathcal{H}_1) &= N \\ dim(\mathcal{H}_2) &= \frac{1}{2}N(N+1) \end{aligned}$$

For fermions,

$$\begin{aligned} dim(\mathcal{H}_0) &= 1 \\ dim(\mathcal{H}_1) &= N \\ dim(\mathcal{H}_2) &= \frac{1}{2}N(N-1) \end{aligned}$$

3.3.5 Single particle state

Consider the action of this Hamiltonian on the single particle subspace $\mathcal{H}_1$. On this subspace, we can express the Hamiltonian as

$$\hat{H} = \sum_{m\neq 0} \omega_m |p_m\rangle\langle p_m|$$

Thus, by quantizing the classical field theory, and then restricting to the subspace of single particle states, we obtain the correct quantum Hamiltonian for a free single particle. If we add the following interaction to the classical Hamiltonian:

$$H_i = \frac{1}{2}\sum_n V_n(\pi_n^2 + \phi_n^2)$$

we get the quantum Hamiltonian for a particle in a potential. There is a correspondence between quantum and classical Hamiltonians for even powers of p:

$$\frac{p^2}{2m} \longleftrightarrow -\omega_F \psi_n'' \longleftrightarrow \frac{1}{2}\sum_n [(\pi_{n+1}-\pi_n)^2 + (\phi_{n+1}-\phi_n)^2]$$

$$V(x) \longleftrightarrow \sum_n V_n\psi_n \longleftrightarrow \frac{1}{2}\sum_n V_n(\pi_n^2+\phi_n^2)$$

but there is no correspondence for odd powers of p.

In summary, the equations of motion for the single particle can be obtained from the classical

Hamiltonian

$$H = \frac{1}{2}\omega_F \sum_n [(\pi_{n+1} - \pi_n)^2 + (\phi_{n+1} - \phi_n)^2] = \frac{1}{2}\sum_{m\neq 0} \omega_m (P_m^2 + Q_m^2)$$

or from the quantum Hamiltonian

$$H = \sum_{nr} h_{nr} |n\rangle\langle r| = \sum_m \omega_m |p_m\rangle\langle p_m|$$

The different representations for the single particle are related by

$$\begin{aligned}
\psi_n &= N^{-1/2} \sum_m c_m \, e^{2\pi i m n/N} = \frac{1}{\sqrt{2}}(\phi_n + i\pi_n) \\
c_m &= N^{-1/2} \sum_n \psi_n \, e^{-2\pi i m n/N} = (2N)^{-1/2} \sum_n (\phi_n + i\pi_n) \, e^{-2\pi i m n/N} \\
\phi_n &= (2N)^{-1/2} \sum_{m\neq 0} (c_m \, e^{2\pi i m n/N} + c_m^* \, e^{-2\pi i m n/N}) = \sqrt{2} Re(\psi_n) \\
\pi_n &= -i(2N)^{-1/2} \sum_{m\neq 0} (c_m \, e^{2\pi i m n/N} - c_m^* \, e^{-2\pi i m n/N}) = \sqrt{2} Im(\psi_n)
\end{aligned}$$

3.4 Two-level atom coupled to field

So far I have only discussed quantum fields that are free. In order to study some of the issues that arise when quantum fields interact, I have developed a simple model that describes a scalar field coupled to a two-level atom. Because of the coupling, the atom can decay from its excited state and photons can scatter off of the atom. In the following sections, I write down the Hamiltonian for the model, and then study it from a number of different perspectives. I show several different ways of calculating the spontaneous decay rate of the atom, using both conventional techniques and the full machinery of quantum field theory. I think it is helpful to discuss some of the issues that arise in quantum field theory in the context of a simple model, where they can be more easily understood. Also, because the model is so simple, a full dynamical simulation of scattering and decay processes can be carried out on a computer.

3.4.1 Hamiltonian for the system

A coupled atom-field system can be described by the Hamiltonian

$$H = H_a + H_f + H_i$$

where H_a is the Hamiltonian for the atom, H_f is the Hamiltonian for the field, and H_i describes the atom-field coupling. The Hamiltonian for the atom is given by

$$H_a = \omega_A \sigma_+ \sigma_-$$

where ω_A is the frequency of the atomic transition. The Hamiltonian for the field is given by

$$H_f = \frac{1}{2}\omega_F \sum_n (\pi_n^2 + (\phi_{n+1} - \phi_n)^2) = \sum_{m\neq 0} \omega_m a_m^\dagger a_m$$

where

$$\omega_m = 2\omega_F |\sin(\pi m/N)|$$

is the frequency of mode m. I will take the interaction Hamiltonian to be

$$H_i = \lambda(\sigma_+ + \sigma_-)\phi_0$$

where λ is a constant that determines the strength of the coupling. If we substitute for the field operator, which is given by

$$\phi_n = (2N)^{-1/2} \sum_{m\neq 0} (\omega_F/\omega_m)^{1/2} (a_m\, e^{2\pi i m n/N} + a_m^\dagger\, e^{-2\pi i m n/N})$$

then we find that interaction Hamiltonian can be expressed as

$$H_i = \sum_{m\neq 0} g_m(\sigma_+ a_m + \sigma_- a_m^\dagger)$$

where I have made the rotating wave approximation and defined constants

$$g_m = \lambda(2N)^{-1/2}(\omega_F/\omega_m)^{1/2}$$

which give the coupling strength of the atom to the individual field modes. Putting these Hamiltonians together, we find that the the total Hamiltonian for the coupled atom-field system is

$$H = \omega_A \sigma_+ \sigma_- + \sum_{m\neq 0} \omega_m a_m^\dagger a_m + \sum_{m\neq 0} g_m(\sigma_+ a_m + \sigma_- a_m^\dagger)$$

This Hamiltonian is defined in the Schrödinger picture, but it is often more convenient to work in the interaction picture, in which the time evolution under H_a and H_f is already taken into account.

The interaction picture Hamiltonian $H_I(t)$ is related to the Schrödinger picture Hamiltonian H by

$$H_I(t) = e^{i(H_a+H_f)t}\, H\, e^{-i(H_a+H_f)t} = \sum_{m\neq 0} g_m(\sigma_+ a_m\, e^{-i\delta_m t} + \sigma_- a_m^\dagger\, e^{i\delta_m t})$$

where $\delta_m = \omega_m - \omega_A$ is the detuning of mode m from the atom.

The total Hilbert space of the system is

$$\mathcal{H}^{AF} = \mathcal{H}^A \otimes (\mathcal{H}_0^F \oplus \mathcal{H}_1^F \oplus \mathcal{H}_2^F \oplus \cdots)$$

where $\mathcal{H}^A$ is the Hilbert space for the atom (spanned by $\{|g\rangle, |e\rangle\}$), and $\mathcal{H}_n^F$ is the Hilbert space for n photons. The total Hilbert space decomposes into a direct sum of Hilbert spaces that do not couple to one another:

$$\mathcal{H}^{AF} = \mathcal{H}^A \otimes (\mathcal{H}_0^F \oplus \mathcal{H}_1^F \oplus \mathcal{H}_2^F \oplus \cdots) = \mathcal{H}_0^{AF} \oplus \mathcal{H}_1^{AF} \oplus \mathcal{H}_2^{AF} \oplus \cdots$$

where $\mathcal{H}_0^{AF}$ consists of the single state $|g\rangle|0\rangle$, and $\mathcal{H}_n^{AF}$ for $n > 0$ consists of states of the form

$$|e\rangle|\Phi_{n-1}\rangle + |g\rangle|\Phi_n\rangle$$

where $|\Phi_k\rangle \in \mathcal{H}_k^F$. The subscript n on $\mathcal{H}_n^F$ counts the number of excitations for that Hilbert space, where an excitation is either a photon or an excited atom.

3.4.2 Weisskopf-Wigner approach

Because of the coupling to the field, an atom in the excited state will spontaneously decay to the ground state by emitting a photon. The decay rate can be calculated using a number of different methods. The method I adopt here, which is due to Weisskopf and Wigner [41], involves solving the equations of motion for the coupled atom-field system.

Assume we start the system with the atom in its excited state and all the field modes in their ground states. Because this initial state has a single excitation, and states with different numbers of excitations do not couple to one another, the system is restricted to the subspace $\mathcal{H}_1^{AF}$. Thus, we can express the interaction picture wavefunction as

$$|\Psi_I(t)\rangle = c_e(t)|e,0\rangle + \sum_{m\neq 0} c_{gm}(t)|g,1_m\rangle$$

for some choice of amplitudes c_e and c_{gm}. The wavefunction evolves under the Schrödinger equation:

$$i\partial_t|\Psi_I\rangle = H_I(t)|\Psi_I\rangle$$

So the equations of motion for the amplitudes are

$$\begin{aligned}\dot{c}_{gm} &= -ig_m\, e^{i\delta_m t}\, c_e \\ \dot{c}_e &= -i\sum_{m\neq 0} g_m\, e^{-i\delta_m t}\, c_{gm}\end{aligned}$$

and the initial conditions are

$$\begin{aligned}c_e(0) &= 1 \\ c_{gm}(0) &= 0\end{aligned}$$

If we integrate the first equation subject to the initial conditions $c_{gm}(0) = 0$, we find

$$c_{gm}(t) = -ig_m \int_0^t e^{i\delta_m t'}\, c_e(t')\, dt'$$

We can substitute this into the equation of motion for c_e:

$$\dot{c}_e(t) = -\sum_{m\neq 0} g_m^2 \int_0^t e^{-i\delta_m(t-t')}\, c_e(t')\, dt' = -\sum_{m\neq 0} g_m^2 \int_0^t e^{-i\delta_m \tau}\, c_e(t-\tau)\, d\tau$$

where $\tau = t - t'$. This result is easier to interpret if we express it in terms of the time correlation function of the field. Recall that the interaction picture field operator at lattice site $n = 0$ is

$$\phi_0(t) = (2N)^{-1/2} \sum_{m\neq 0} (\omega_F/\omega_m)^{1/2}(a_m\, e^{-i\omega_m t} + a_m^\dagger\, e^{i\omega_m t})$$

Thus, the time correlation function of the field is (see Figure 3.4)

$$\langle 0|T[\phi_0(t)\phi_0(0)]|0\rangle = (2N)^{-1} \sum_{m\neq 0} (\omega_F/\omega_m)\, e^{-i\omega_m|t|} = \frac{1}{\lambda^2}\sum_{m\neq 0} g_m^2\, e^{-i\omega_m|t|}$$

So the equation of motion for c_e can be expressed as

$$\dot{c}_e(t) = -\lambda^2 \int_0^t \langle 0|\phi_0(\tau)\phi_0(0)|0\rangle\, e^{i\omega_A \tau}\, c_e(t-\tau)\, d\tau$$

If the timescale over which the field fluctuations are correlated is much shorter than the timescale for the evolution of c_e, then we can make the Markoff approximation by replacing $c_e(t-\tau)$ with

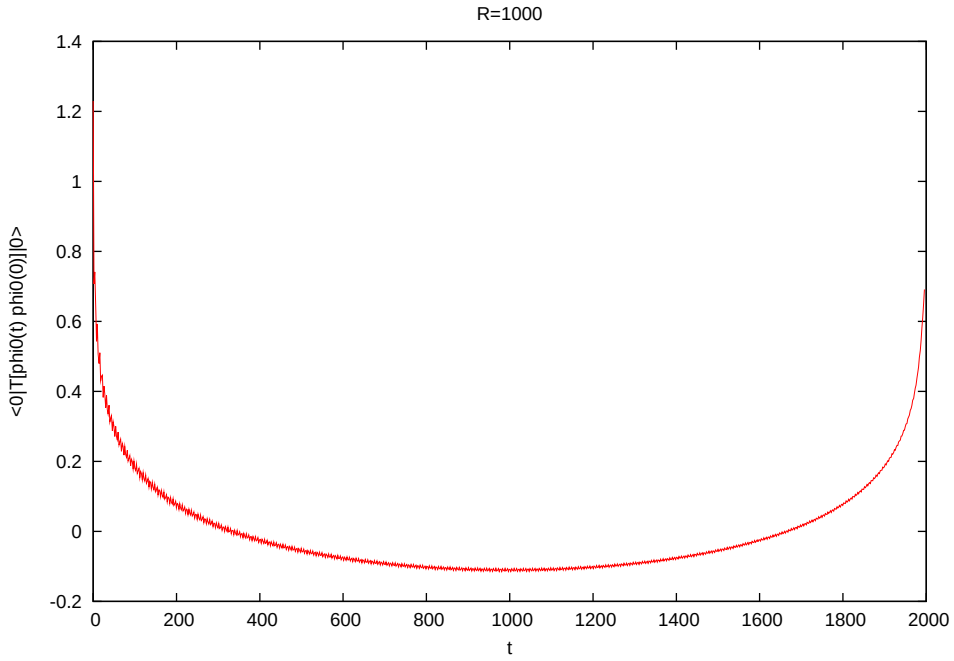

Figure 3.4: Time correlation function for the field ($R = 1000$).

$c_e(t)$ and extending the τ integral to ∞. Then

$$\dot{c}_e(t) = -i\Sigma(\omega_A)\, c_e(t)$$

where

$$\Sigma(\omega) = -i\lambda^2 \int_0^\infty \langle 0|\phi(\tau)\phi(0)|0\rangle\, e^{i\omega\tau}\, d\tau$$

is the spectrum of the field fluctuations. Thus, the vacuum fluctuations of the field drive the atomic decay, and the decay rate is determined the spectral power of the vacuum fluctuations at the atomic transition frequency.

It is straightforward to solve the equation of motion for c_e subject to the initial condition $c_e(0) = 1$:

$$c_e(t) = e^{-i\Sigma(\omega_A)t}$$

If we express $\Sigma(\omega_A)$ in the form

$$\Sigma(\omega_A) = \Delta - i\frac{\gamma}{2}$$

where γ and Δ are real, then

$$c_e(t) = e^{-i\Delta t}\, e^{-\gamma t/2}$$

Thus, γ gives the spontaneous decay rate, and Δ gives a shift in the atomic transition frequency—the Lamb shift. To calculate these quantities, we go back to our expression for $\Sigma(\omega_A)$ and substitute for the field correlation function:

$$\Sigma(\omega_A) = -i \int_0^\infty \sum_{m\neq 0} g_m^2 \, e^{-i(\omega_m - \omega_A - i\epsilon)t} \, dt$$

The sum can be evaluated by taking a continuum limit:

$$\sum_{m\neq 0} \to \int_0^\infty \rho(\omega) \, d\omega$$

where $\rho(\omega)$ is the density of states at frequency ω. In this limit,

$$\Sigma(\omega_A) = -i \int_0^\infty \int_0^\infty g^2(\omega) \, \rho(\omega) \, e^{-i(\omega - \omega_A - i\epsilon)\tau} \, d\tau \, d\omega$$

where

$$g(\omega) = \lambda (2N)^{-1/2} \, (\omega_F/\omega_m)^{1/2}$$

is the coupling strength at frequency ω. Note that

$$\int_0^\infty e^{-i(\omega-\omega_A-i\epsilon)\tau} d\tau = -i(\omega - \omega_A - i\epsilon)^{-1} = \pi\delta(\omega - \omega_A) - iP\frac{1}{\omega - \omega_A}$$

So the spontaneous decay rate is

$$\gamma = 2\pi g^2(\omega_A)\rho(\omega_A)$$

and the Lamb shift is

$$\Delta = P \int_0^\infty \frac{g^2(\omega)\rho(\omega)}{\omega_A - \omega} \, d\omega$$

The density of states at frequency ω can be evaluated by expressing it in terms of the density of states at wavenumber k:

$$\rho(\omega) = (\rho(k) + \rho(-k)) \left| \frac{dk}{d\omega} \right|$$

The wavenumber for mode m is $k_m = 2\pi m/L$, where $L = Na = N/\omega_F$ is the length of the lattice, so the density of states at wavenumber k is

$$\rho(k) = (2\pi)^{-1} L = N/2\pi\omega_F$$

The dispersion relation in the continuum limit is

$$\omega = 2\omega_F |\sin(ka/2)|$$

So

$$\left|\frac{d\omega}{dk}\right| = \cos(ka/2) = (1-(\omega/2\omega_F)^2)^{1/2}$$

Thus, the density of states at frequency ω is

$$\rho(\omega) = (N/\pi\omega_F)(1-(\omega/2\omega_F)^2)^{-1/2}$$

Substituting this into our expression for the spontaneous decay rate, we find

$$\gamma = (\lambda^2/\omega_A)(1-(\omega_A/2\omega_F)^2)^{-1/2}$$

Thus, in terms of the decay rate, the coupling strength is

$$\lambda = (\gamma\omega_A)^{1/2}\,(1-(\omega_A/2\omega_F)^2)^{1/4}$$

and coupling to mode m is

$$g_m = (\gamma\omega_A/2N)^{1/2}\,(\omega_F/\omega_m)^{1/2}\,(1-(\omega_A/2\omega_F)^2)^{1/4}$$

The expression for the Lamb shift can be understood by calculating the energy shifts of the atomic states using second order perturbation theory. There is no energy shift of $|g,0\rangle$, so the Lamb shift is given by the shift of $|e,0\rangle$:

$$\Delta = \sum_{m\neq 0} \frac{\langle g,1_m|H_i|e,0\rangle|^2}{\omega_A-\omega_m} = -\sum_{m\neq 0}\frac{g_m^2}{\delta_m}$$

The coupled atom-field system can be simulated on a computer by integrating the equations of motion starting from a given initial condition. For the simulations presented here, $R = 1000$, $\omega_A = \omega_{300}$, and $\gamma = 0.05$ in units in which $\omega_F = 1$. The initial condition is chosen so the atom is in the excited state and the field is in the vacuum state. Figure 3.5 shows the population in the excited state versus time, while Figures 3.6 and 3.7 show the state of the field at time $t = 500.0$. At this time the atom has completely decayed, and the field is in a single photon state. As discussed in section 3.3.3, a single photon state can be represented in several different ways: Figure 3.6 shows $|c_{gm}|^2$, where c_{gm} is the momentum space wavefunction for the photon, while Figure 3.7 shows $|\psi_n|^2$, where

$$\psi_n = N^{-1/2}\sum_m c_{gm}\,e^{2\pi imn/N}$$

is the the position space wavefunction. Note that because of the Lamb shift, the peak of the momentum space wavefunction is shifted away from $m = 300$. The predicted Lamb shift is $\Delta =$

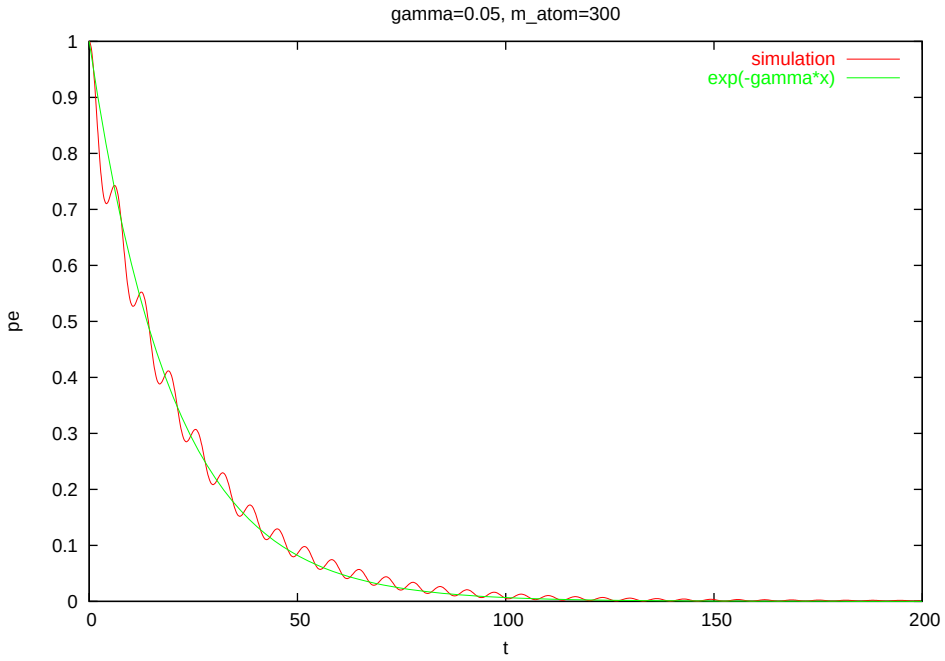

Figure 3.5: Population in the excited state versus time. The red curve is from the simulation; the green curve is the theoretical prediction in the continuum limit.

0.0434 for the parameters used in the simulation, which corresponds to a shift of $\Delta\,\rho(\omega_A) = 15.5$ mode orders, where $\rho(\omega_A)$ is the density of states at the atomic frequency.

3.4.3 Fermi's golden rule

A second way to calculate the spontaneous decay rate is by using Fermi's golden rule. The basic idea is to calculate the probability for the atom to emit into a single mode, and then sum over all the modes to get the total decay rate. If we only consider the coupling of the atom to a single mode m then the Hamiltonian is

$$H_I(t) = g_m(\sigma_+ a_m\, e^{-i\delta_m t} + \sigma_- a_m^\dagger\, e^{i\delta_m t})$$

Since the initial state is $|e,0\rangle$, the state of the system at time t can be expressed as

$$|\Psi(t)\rangle = c_e(t)|e,0\rangle + c_{gm}(t)|g,1_m\rangle$$

where the equations of motion for c_e and c_{gm} are

$$\begin{aligned} \dot{c}_{gm} &= -ig_m e^{i\delta_m t}\, c_e \\ \dot{c}_e &= -ig_m e^{-i\delta_m t}\, c_{gm} \end{aligned}$$

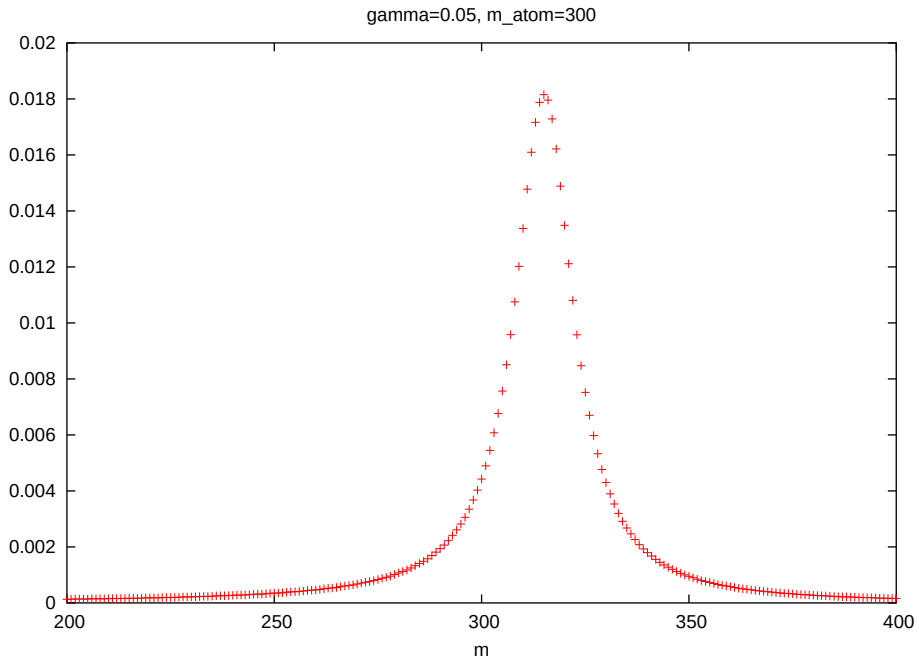

Figure 3.6: $|c_{gm}|^2$, where c_{gm} is the momentum space wavefunction of the photon at $t = 500.0$ (Note the Lamb shift away from $m = 300$).

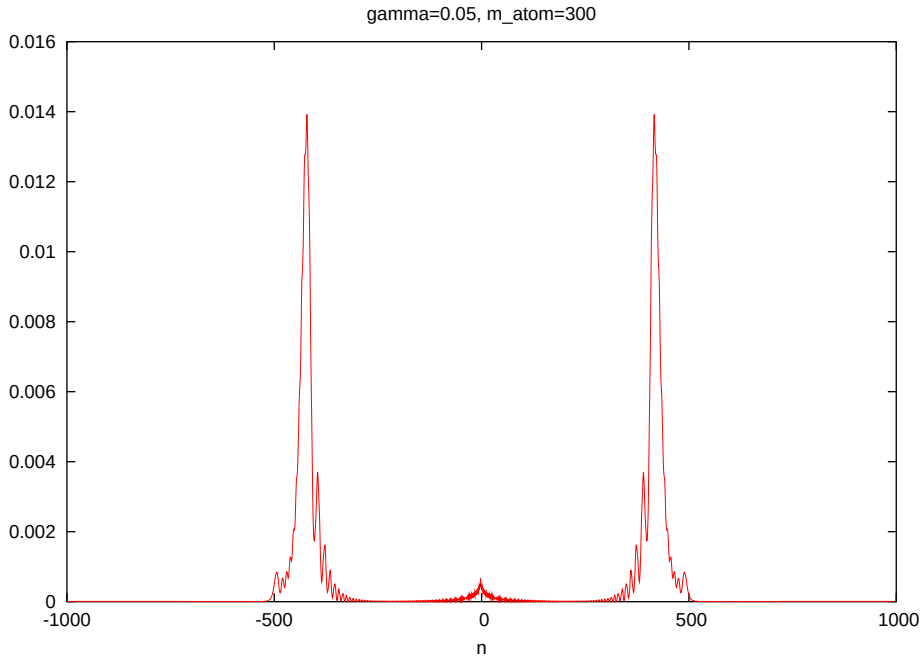

Figure 3.7: $|\psi_n|^2$, where ψ_n is the position space wavefunction of the photon at $t = 500.0$.

At small times $c_e \sim 1$, so the equation of motion for c_{gm} can be approximated by

$$\dot{c}_{gm} = -ig_m\, e^{i\delta_m t}$$

If we integrate this subject to the initial conditions $c_{gm}(0) = 0$, we find

$$c_{gm} = -\frac{g_m}{\delta_m}(e^{i\delta_m t} - 1)$$

Thus, the population in state $|g, 1_m\rangle$ is

$$|c_{gm}|^2 = 4(g_m^2/\delta_m^2)\sin^2(\delta_m t/2) = \pi g_m^2 t^2 D(\delta_m t/2)$$

where

$$D(x) = \frac{\sin^2 x}{\pi x^2}$$

The rate at which population is transfered from $|e, 0\rangle$ to $|g, 1_m\rangle$ is therefore

$$\gamma_m = \frac{|c_{gm}|^2}{t} = \pi g_m^2 t\, D(\delta_m t/2)$$

To get the total decay rate, we sum the decay rates for all the different modes:

$$\gamma = \sum_{m\neq 0} \gamma_m = \pi t \sum_{m\neq 0} g_m^2\, D(\delta_m t/2)$$

As in the previous section, we can evaluate the sum by taking a continuum limit and approximating the sum as an integral:

$$\gamma = \pi t \int_0^\infty g^2(\omega)\, \rho(\omega)\, D((\omega - \omega_A)t/2)\, d\omega$$

The function $D((\omega - \omega_A)t/2)$ is sharply peaked around $\omega = \omega_A$, so we can approximate the decay rate by evaluating $g^2(\omega)$ and $\rho(\omega)$ at the peak value ω_A and extending the lower limit of the integral to $-\infty$:

$$\gamma = \pi t g^2(\omega_A)\rho(\omega_A) \int D((\omega - \omega_A)t/2)\, d\omega = 2\pi g^2(\omega_A)\rho(\omega_A) \int D(x)\, dx$$

But

$$\int D(x)\, dx = \frac{1}{\pi}\int \frac{\sin^2 x}{x^2}\, dx = 1$$

Thus,

$$\gamma = 2\pi g^2(\omega_A)\rho(\omega_A)$$

which agrees with the result derived in the previous section.

The above basically amounts to a derivation of Fermi's golden rule, which states that the decay rate from a state with energy E_i to a state with energy E_f is given by

$$\gamma(E_i \to E_f) = 2\pi |\langle f|H|i\rangle|^2 \, \rho(E_f)$$

where $\rho(E)$ is the density of final states at energy E.

3.4.4 Master equation

Another way to treat the spontaneous decay of the atom is to derive an effective master equation for the atomic density matrix. This is accomplished by writing down the master equation for the coupled atom-field system and then tracing out the degrees of freedom for the field. The derivation of the master equation that I give here is based on a derivation presented by Carmichael in [42]. Since the atom-field system evolves coherently, the master equation is

$$\dot{\rho}_{AF} = -i[H_I(t), \rho_{AF}]$$

where

$$H_I(t) = \sum_{m \neq 0} g_m (\sigma_+ a_m \, e^{-i\delta_m t} + \sigma_- a_m^\dagger \, e^{i\delta_m t})$$

If we integrate the master equation, we obtain

$$\rho_{AF}(t) = \rho_{AF}(0) - i \int_0^t [H_I(t'), \rho_{AF}(t')] \, dt'$$

Now substitute this back in to the original master equation:

$$\dot{\rho}_{AF}(t) = -i[H_I(t), \rho_{AF}(0)] - \int_0^t [H_I(t), [H_I(t'), \rho_{AF}(t')]] \, dt'$$

I will assume that at $t = 0$, the field is in the vacuum state, so the density matrix is

$$\rho_{AR}(0) = \rho(0) \otimes |0\rangle\langle 0|$$

where ρ is the density matrix of the atom alone. Let us assume that the atom is only weakly coupled to the field. Then, because the density matrix factorizes at time 0, it should approximately factorize at time t:

$$\rho_{AR}(t) \simeq \rho(t) \otimes |0\rangle\langle 0|$$

If we substitute this into our expression for $\dot{\rho}_{AF}(t)$ and trace out the field degrees of freedom, we find

$$\dot{\rho}(t) = -iTr_F([H_I(t), \rho(t) \otimes |0\rangle\langle 0|]) - \int_0^t Tr_F([H_I(t), [H_I(t'), \rho(t') \otimes |0\rangle\langle 0|]])\, dt'$$

The first term vanishes. For the second term, note that

$$\begin{aligned} Tr_F([H_I(t), [H_I(t'), \rho(t') \otimes |0\rangle\langle 0|]]) = \\ \langle 0|H_I(t)\, H_I(t')|0\rangle\, \rho(t') - \sum_m \langle 1_m|H_I(t')|0\rangle\, \rho(t')\, \langle 0|H_I(t)|1_m\rangle + \\ \rho(t')\, \langle 0|H_I(t')\, H_I(t)|0\rangle\ - \sum_m \langle 1_m|H_I(t)|0\rangle\, \rho(t')\, \langle 0|H_I(t')|1_m\rangle \end{aligned}$$

The matrix elements are given by

$$\langle 0|H_I(t)\, H_I(t')|0\rangle = \sum_m g_m^2\, e^{-i\delta_m(t-t')}\sigma_+\sigma_-$$

and

$$\sum_m \langle 1_m|H_I(t')|0\rangle\, \rho(t')\, \langle 0|H_I(t)|1_m\rangle = \sum_m g_m^2\, e^{-i\delta_m(t-t')}\sigma_-\rho(t')\sigma_+$$

Thus,

$$\dot{\rho}(t) = \int_0^t \sum_m g_m^2[2\sigma_-\rho(t-\tau)\sigma_+ - \sigma_+\sigma_-\rho(t-\tau) - \rho(t-\tau)\sigma_+\sigma_-]\, e^{-i\delta_m\tau}\, d\tau$$

where $\tau = t - t'$. In the Markoff approximation,

$$\dot{\rho}(t) = \frac{\gamma}{2}(2\sigma_-\rho(t)\sigma_+ - \sigma_+\sigma_-\rho(t) - \rho(t)\sigma_+\sigma_-)$$

where

$$\gamma = 2\int_0^\infty \sum_m g_m^2\, e^{-i\delta_m\tau}\, d\tau$$

We can use the master equation to calculate the spontaneous decay rate of the atom. At $t = 0$ the atom is in the excited state, so

$$\rho(0) = |e\rangle\langle e|$$

The decay is given by

$$\gamma(e \to g) = \langle g|\dot{\rho}|g\rangle = \gamma$$

3.4.5 Propagators

Now I want to show how the spontaneous decay rate of the atom can be calculated using methods from quantum field theory. These methods involve the use of propagators, which describe the time correlation properties of the system. The atomic decay rate can be extracted from the propagator for

a coupled atom, which is obtained by using a Feynman diagram expansion to express the propagator in terms of known propagators for a free atom and a free field.

Consider a theory described by the Hamiltonian

$$H = H_0 + H_i$$

where

$$H_0 = \omega_B b^\dagger b$$

is a free Hamiltonian describing a particle of mass ω_B, and where H_i describes an interaction. The retarded propagator for the particle is defined as

$$S(t) = -i\langle\Omega|T[b_H(t)\, b_H^\dagger(0)]|\Omega\rangle$$

where $|\Omega\rangle$ is the ground state of H, and the subscript on b_H and $b_H^\dagger$ indicates that these are Heisenberg picture operators:

$$b_H(t) = e^{iHt}\, b\, e^{-iHt}$$

In general, the propagator is difficult to calculate for the interacting system but easy to calculate for the free system. We therefore resort to perturbation theory and expand the propagator for the interacting system in a series whose terms consist of propagators for the free system. The retarded propagator of the free system is

$$S_0(t) = -i\langle 0|T[b(t)\, b^\dagger(0)]|0\rangle$$

where $|0\rangle$ is the ground state of H_0, and I have adopted the convention that time-dependent operators without an H subscript are in the interaction picture:

$$b(t) = e^{iH_0 t}\, b\, e^{-iH_0 t} = b\, e^{-i\omega_B t}$$

Thus, the free particle propagator is

$$S_0(t) = -i\theta(t)\, e^{-i\omega_B t}$$

Or, in the frequency domain,

$$S_0(\omega) = \int S_0(t)\, e^{i(\omega+i\epsilon)t}\, dt = (\omega - \omega_B + i\epsilon)^{-1}$$

One can show that the retarded propagator for the full system may be expressed as [43]

$$S(t_a - t_b) = \frac{-i\langle 0|T[\exp(-i\int H_I(t)\,dt)\,b(t_a)\,b^\dagger(t_b)]|0\rangle}{\langle 0|T[\exp(-i\int H_I(t)\,dt)]|0\rangle}$$

where

$$H_I(t) = e^{iH_0 t}\,H_i\,e^{-iH_0 t}$$

is the interaction picture Hamiltonian.

Now I want to specialize to the case of the two-level atom coupled to a scalar field. For this system, the free atom propagator is

$$S_0(\omega) = (\omega - \omega_A + i\epsilon)^{-1}$$

and the free field propagator for a photon in mode m is

$$G_0(m,\omega) = (\omega - \omega_m + i\epsilon)^{-1}$$

The propagator for the interacting atom is given by

$$S(t_a - t_b) = \frac{-i\langle 0|T[\exp(-i\int H_I(t)\,dt)\,\sigma_-(t_a)\,\sigma_+(t_b)]|0\rangle}{\langle 0|T[\exp(-i\int H_I(t)\,dt)]|0\rangle}$$

where

$$H_I(t) = \sum_{m\neq 0} g_m(\sigma_+(t)a_m(t) + \sigma_-(t)a_m^\dagger(t))$$

Note that

$$H_I(t)|g,0\rangle = 0$$

so the ground state of the interacting system is the same as the ground state of the free system ($|\Omega\rangle = |g,0\rangle$). Also,

$$\langle g,0|T[\exp(-i\int H_I(t)\,dt)]|g,0\rangle = 1$$

So the propagator is

$$S(t_a - t_b) = -i\langle g,0|T[\exp(-i\int H_I(t)\,dt)\,\sigma_-(t_a)\,\sigma_+(t_b)]|g,0\rangle$$

The exponential may be expanded in a Taylor series:

$$S(t_a - t_b)$$

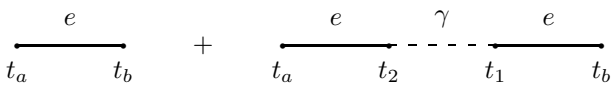

Figure 3.8: Feynman diagrams for the atom propagator to second order. The solid line represents an excited atom; the dashed line represents a photon.

$$= \quad -i \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int dt_1 \cdots \int dt_n \, \langle g, 0 | T[H_I(t_1) \cdots H_I(t_n) \, \sigma_-(t_a) \, \sigma_+(t_b)]| g, 0 \rangle$$

Note that all the odd order terms in the series vanish. The zeroth order term is just the free atom propagator:

$$-i\langle g, 0 | T[\sigma_-(t_a) \, \sigma_+(t_b)]| g, 0 \rangle = S_0(t_a - t_b)$$

The second order term is

$$\frac{i}{2} \iint \langle g, 0 | T[H_I(t_1) H_I(t_2) \sigma_-(t_a) \sigma_+(t_b)]| g, 0 \rangle \, dt_1 \, dt_2$$

If we substitute for $H_I(t)$, we obtain

$$\begin{aligned} &\langle g, 0 | T[H_I(t_1) H_I(t_2) \sigma_-(t_a) \sigma_+(t_b)]| g, 0 \rangle = \\ &\quad \sum_{m \neq 0} g_m^2 [\theta(t_1 - t_2) \, \langle g, 0 | T[\sigma_+(t_1) a_m(t_1) \, \sigma_-(t_2) a_m^\dagger(t_2) \, \sigma_-(t_a) \sigma_+(t_b)]| g, 0 \rangle + (t_1 \leftrightarrow t_2)] \end{aligned}$$

Note that

$$\begin{aligned} &\theta(t_1 - t_2) \, \langle g, 0 | T[\sigma_+(t_1) a_m(t_1) \sigma_-(t_2) a_m^\dagger(t_2) \sigma_-(t_a) \sigma_+(t_b)]| g, 0 \rangle \\ &\quad = \quad \theta(t_a - t_1) \, \theta(t_1 - t_2) \, \theta(t_2 - t_b) \, \langle g, 0 | \sigma_-(t_a) \sigma_+(t_1) \, a_m(t_1) a_m^\dagger(t_2) \, \sigma_-(t_2) \sigma_+(t_b)| g, 0 \rangle \\ &\quad = \quad -i S_0(t_a - t_1) \, G_0(t_1 - t_2) \, S_0(t_2 - t_b) \end{aligned}$$

Thus, the atom propagator is (see Figure 3.8):

$$\begin{aligned} &S(t_a - t_b) = \\ &\qquad S_0(t_a - t_b) + \sum_{m \neq 0} g_m^2 \iint S_0(t_a - t_2) \, G_0(m, t_2 - t_1) \, S_0(t_1 - t_b) \, dt_1 \, dt_2 + \cdots \end{aligned}$$

The series takes on a simpler form if we work in the frequency domain:

$$\begin{aligned} S(\omega) \quad &= \quad S_0(\omega) + S_0(\omega) \, \Sigma(\omega) \, S_0(\omega) + \cdots \\ &= \quad S_0(\omega) \, (1 - \Sigma(\omega) S_0(\omega))^{-1} \\ &= \quad (\omega - \omega_A - \Sigma(\omega) + i\epsilon)^{-1} \end{aligned}$$

where the self-energy is

$$\Sigma(\omega) = \sum_{m\neq 0} g_m^2\, G_0(m,\omega) = -i \int_0^\infty \sum_{m\neq 0} g_m^2\, e^{i(\omega-\omega_m+i\epsilon)\tau}\, d\tau$$

We can check this result for a single mode field. The Hamiltonian for an atom coupled to a single mode field is

$$H = \omega_A \sigma_+ \sigma_- + \omega_F a^\dagger a + H_i$$

where H_i describes the coupling of the atom to the mode:

$$H_i = g(\sigma_- a^\dagger + \sigma_+ a)$$

For simplicity, I will assume that the atom is resonant with the mode ($\omega_A = \omega_F$). The retarded propagator for the atom is

$$S(t) = -i\theta(t)\, \langle g,0|e^{iHt}\sigma_- e^{-iHt}\sigma_+|g,0\rangle = -i\theta(t)\, \langle e,0|e^{-iHt}|e,0\rangle$$

If we define

$$|\psi(t)\rangle = e^{-iHt}|e,0\rangle = e^{-i\omega_A t}(\cos gt|e,0\rangle - i\sin gt|g,1\rangle)$$

then we can express the propagator as

$$S(t) = -i\theta(t)\langle\psi(0)|\psi(t)\rangle = -i\theta(t)\, e^{-i\omega_A t}\, \cos gt$$

In the frequency domain,

$$S(\omega) = \int S(t)\, e^{i(\omega+i\epsilon)t}\, dt = (\omega - \omega_A - \Sigma(\omega) + i\epsilon)^{-1}$$

where the self-energy is

$$\Sigma(\omega) = g^2(\omega - \omega_F + i\epsilon)^{-1}$$

which agrees with the result obtained from the perturbation expansion.

Returning to the full model, recall that the propagator for the atom is

$$S(\omega) = (\omega - \omega_A - \Sigma(\omega) + i\epsilon)^{-1}$$

We can expand $\Sigma(\omega)$ in a power series about ω_A:

$$\Sigma(\omega) = \Sigma(\omega_A) + \Sigma'(\omega_A)\,(\omega - \omega_A) + \cdots$$

Then

$$S(\omega) \simeq Z\,(\omega - \omega_A - \Sigma(\omega_A) + i\epsilon)^{-1}$$

where

$$Z = (1 - \Sigma'(\omega_A))^{-1}$$

Thus, in the Markoff approximation

$$S(\omega) = (\omega - \omega_A - \Sigma(\omega_A) + i\epsilon)^{-1}$$

Or, in the time domain,

$$S(t) = -i\theta(\tau)e^{-i(\omega_A + \Sigma(\omega_A))t}$$

This is what we would expect based on the equations of motion we derived in the Weisskopf-Wigner approach:

$$i\dot{c}_e(t) = \Sigma(\omega_A)\,c_e(t)$$

so for $c_e(0) = 1$,

$$c_e(t) = e^{-i\Sigma(\omega_A)t}$$

Note that the excited atom may be thought of as a quasiparticle with a lifetime given by γ.

3.4.6 Scattering

In this section I consider the scattering of a single photon off of the atom. The initial state of the system is given by

$$|\psi\rangle = \sum_{m\neq 0} c_m |g, 1_m\rangle$$

where this is defined in the interaction picture. We will choose the coefficients $\{c_m\}$ such that the initial state describes an incoming wavepacket that is peaked around a particular mode s. The probability amplitude for the photon to scatter into mode r is given by

$$a = \sum_m \langle g, 1_r | T[\exp(-i \int H_I(t)\,dt)] | g, 1_m\rangle\, c_m$$

If we introduce a large time T, we can express the matrix element in terms of operators that create a photon at time $-T$ (long before the scattering event) and that annihilate the photon at time $+T$

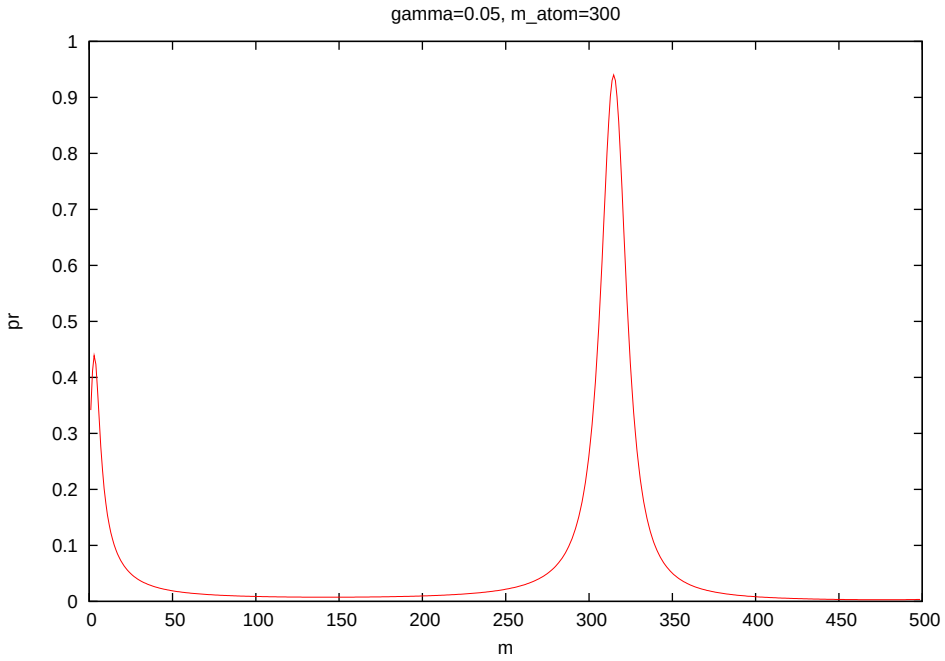

Figure 3.9: Scattering spectrum.

(long after the scattering event):

$$\begin{aligned}
&\langle g,1_r|T[\exp(-i\int H_I(t)\,dt)]|g,1_m\rangle \\
&\quad= e^{i(\omega_r+\omega_m)T}\,\langle g,0|T[\exp(-i\int H_I(t)\,dt)\,a_r(T)\,a_m^\dagger(-T)]|g,0\rangle \\
&\quad= \delta_{r,m}+ig_rg_m\,e^{i(\omega_r+\omega_m)T}\iint G_0(r,T-t_a)\,S(t_a-t_b)\,G_0(m,t_b+T)\,dt_a\,dt_b
\end{aligned}$$

Recall that

$$G_0(s,\tau)=-i\theta(\tau)\,e^{-i\omega_s\tau}$$

and

$$S(\tau)=-i\theta(\tau)\,e^{-i(\omega_A+\Delta-i\gamma/2)\tau}$$

Thus, if we define $\Delta_s=\omega_s-(\omega_A+\Delta-i\gamma/2)$, we find

$$\begin{aligned}
&\langle g,1_r|T[\exp(-i\int H_I(t)\,dt)]|g,1_m\rangle \\
&\quad= \delta_{r,m}-g_rg_m\int_{-T}^{T}dt_a\int_{-T}^{t_a}dt_b\,e^{i\Delta_rt_a}\,e^{-i\Delta_mt_b} \\
&\quad= \delta_{r,m}-i\frac{g_rg_m}{\Delta_m}\int_{-T}^{T}e^{i\Delta_rt_a}\,(e^{-i\Delta_mt_a}-e^{i\Delta_mT})\,dt_a
\end{aligned}$$

For large T, the second term goes away because of the $-\gamma T/2$ term in the exponential, so we are left with

$$\langle g, 1_r | T[\exp(-i \int H_I(t)\, dt)] | g, 1_m \rangle = \delta_{r,m} - i \frac{g_r g_m}{\Delta_m} \int_{-T}^{T} e^{i(\omega_r - \omega_m)t_a}\, dt_a$$

Also, for large T the integral is sharply peaked around $\omega_r \sim \omega_m$, and may be approximated as a delta function:

$$\int_{-T}^{T} e^{i\omega t}\, dt = 2T\, \frac{\sin \omega T}{\omega T} \rightarrow 2\pi\, \delta(\omega)$$

Thus,

$$\langle g, 1_r | T[\exp(-i \int H_I(t)\, dt)] | g, 1_m \rangle = \delta_{r,m} - 2\pi i\, \frac{g_m^2}{\Delta_m}\, \delta(\omega_r - \omega_m)$$

Note that

$$\delta(\omega_r - \omega_s) = \frac{1}{2} \rho(\omega_r)(\delta_{r,m} + \delta_{r,-m})$$

Also,

$$\gamma = 2\pi g^2(\omega_A)\rho(\omega_A)$$

So we find

$$\langle g, 1_r | T[\exp(-i \int H_I(t)\, dt)] | g, 1_m \rangle = \delta_{r,m} - i \frac{\gamma}{2\Delta_m} (\delta_{r,m} + \delta_{r,-m})$$

Thus, the probabilities for the photon to be reflected and transmitted are

$$\begin{aligned} p_r &= |if(\omega)|^2 \\ p_t &= |1 - if(\omega)|^2 \end{aligned}$$

where

$$f(\omega) = (\gamma/2)(\omega - (\omega_A + \Delta) + i\gamma/2)^{-1}$$

Figure 3.9 shows the results of a computer simulation that integrates the equations of motion for a single photon incident on an atom in its ground state. For each value of of m, a gaussian wavepacket is prepared as described in section 3.3.3, where $x = -500.0$ and $\sigma = 100.0$. The system is then evolved in time from $t = 0.0$ to $t = 1000.0$, at which point the reflection and transmission probabilities are computed. The graph shows the reflection probability p_r as a function of the momentum m of the wavepacket.

Chapter 4

Quantum field theory in $0+1$ dimensions

4.1 Introduction

In $0+1$ dimensions, quantum field theory is equivalent to quantum mechanics, as can be understood from the following considerations. Note that a scalar field in $n+1$ dimensions can be viewed as a mapping of spacetime into the real numbers:

$$(t, x_1, \cdots, x_n) \to \phi(t, x_1, \cdots, x_n)$$

The trajectory of a particle in $n+1$ dimensions can be viewed as a mapping of the real numbers into spacetime:

$$t \to (t, x_1(t), \cdots, x_n(t))$$

Thus, a field in $0+1$ dimensions and a particle trajectory in $1+1$ dimensions are described in the same way: the field is described by a mapping $t \to \phi(t)$, and the particle is described by a mapping $t \to x(t)$. The two systems are therefore formally equivalent, and we can translate results back and forth between them.

In the following sections, I present a number of simple field theory models in $0+1$ dimensions, which I solve using both ordinary quantum mechanics and with the formalism of quantum field theory.

4.2 Classical scalar field

4.2.1 Hamiltonian for a scalar field

A classical scalar field in $0+1$ dimensions is described by two dynamical variables ϕ and $\dot{\phi}$, which evolve in time according to the Lagrangian

$$L_0 = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}m^2\phi^2$$

We can couple the field to a source $\rho(t)$ by adding an interaction Lagrangian

$$L_i = \rho(t)\,\phi$$

so the total Lagrangian is

$$L = L_0 + L_i$$

The field equation for ϕ is obtained from L via the Euler-Lagrange equation:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} - \frac{\partial L}{\partial \phi} = 0$$

This gives

$$(\partial_t^2 + m^2)\phi = \rho$$

which is just the Klein-Gordon equation in $0+1$ dimensions.

The canonical momentum π is given by

$$\pi = \frac{\partial L}{\partial \dot{\phi}} = \dot{\phi}$$

Thus, the Hamiltonian is

$$H = \pi\dot{\phi} - L = \frac{1}{2}\pi^2 + \frac{1}{2}m^2\phi^2 - \rho\phi$$

The equations of motion for ϕ and π are

$$\begin{aligned} \dot{\phi} &= \pi \\ \dot{\pi} &= -m^2\phi + \rho \end{aligned}$$

4.2.2 Retarded and advanced fields

We can solve the inhomogeneous field equation by introducing retarded and advanced Greens functions, which satisfy the equations

$$(\partial_t^2 + m^2)G_r(t) = (\partial_t^2 + m^2)G_a(t) = \delta(t)$$

with boundary conditions $G_r(t) = 0$ for $t < 0$ and $G_a(t) = 0$ for $t > 0$. The Greens functions are given by

$$\begin{aligned} G_r(t) &= \frac{1}{m}\theta(t)\sin mt \\ G_a(t) &= -\frac{1}{m}\theta(-t)\sin mt \end{aligned}$$

Using the Greens functions, we can define retarded and advanced fields by

$$\begin{aligned} \phi_r(t) &= \int G_r(t-t')\,\rho(t')\,dt' = \frac{1}{m}\int_{-\infty}^{t} \sin m(t-t')\,\rho(t')\,dt' \\ \phi_a(t) &= \int G_a(t-t')\,\rho(t')\,dt' = \frac{1}{m}\int_{t}^{\infty} \sin m(t'-t)\,\rho(t')\,dt' \end{aligned}$$

From the definitions of the retarded and advanced Greens functions, it follows that retarded and advanced fields are solutions to the inhomogeneous field equation:

$$(\partial_t^2 + m^2)\phi_r(t) = (\partial_t^2 + m^2)\phi_a(t) = \rho(t)$$

4.2.3 *in* and *out* fields

Suppose $\phi(t)$ is a solution to the inhomogeneous field equation. We can define fields $\phi_{in}(t)$ and $\phi_{out}(t)$ by

$$\begin{aligned} \phi_{in}(t) &= \phi(t) - \phi_r(t) \\ \phi_{out}(t) &= \phi(t) - \phi_a(t) \end{aligned}$$

which satisfy the homogeneous field equation

$$(\partial_t^2 + m^2)\phi_{in}(t) = (\partial_t^2 + m^2)\phi_{out}(t) = 0$$

The Greens function $G_-(t)$ for the homogeneous field equation is given by the difference between the retarded and advanced Greens functions:

$$G_-(t) = G_r(t) - G_a(t) = \frac{1}{m}\sin mt$$

$$(\partial_t^2 + m^2)G_-(t) = 0$$

We can use $G_-(t)$ to solve for the time evolution of the *in* and *out* fields as follows. Consider two arbitrary functions $A(\tau)$ and $B(\tau)$, and define a function $C(\tau)$ by

$$C = A\,\partial_\tau B - B\,\partial_\tau A$$

Then

$$\int_0^t (A\,\partial_\tau^2 B - B\,\partial_\tau^2 A)\,d\tau = \int_0^t \partial_\tau C\,d\tau = C(t) - C(0)$$

We will take $A(\tau) = \phi_{in}(\tau)$ and $B(\tau) = G_-(t-\tau)$, so

$$C(\tau) = -\phi_{in}(\tau)\,\partial_t G_-(t-\tau) - \pi_{in}(\tau)\,G_-(t-\tau)$$

Note that

$$(\partial_\tau^2 + m^2)A(\tau) = (\partial_\tau^2 + m^2)B(\tau) = 0$$

so

$$A\,\partial_\tau^2 B - B\,\partial_\tau^2 A = 0$$

Thus, $C(t) = C(0)$. Substituting $G_-(t)$ into our expression for $C(\tau)$, we find that

$$C(0) = -\phi_{in}(0)\,\cos mt - \frac{1}{m}\pi_{in}(0)\,\sin mt$$

and

$$C(t) = -\phi_{in}(t)$$

Thus,

$$\phi_{in}(t) = \phi_{in}(0)\,\cos mt + \frac{1}{m}\pi_{in}(0)\,\sin mt$$

If we take a time derivative, we obtain the corresponding equation for $\pi_{in}(t)$:

$$\pi_{in}(t) = -m\,\phi_{in}(0)\,\sin mt + \pi_{in}(0)\,\cos mt$$

Thus, given the initial conditions $\phi_{in}(0)$ and $\pi_{in}(0)$, we can solve for ϕ_{in} and π_{in} at all times. In the previous section, we showed how to solve for ϕ_r and π_r at all times. By adding the *in* field to

the retarded field, we obtain the total field:

$$\phi(t) = \phi_{in}(t) + \phi_r(t) = \phi_{out}(t) + \phi_a(t)$$

4.2.4 Scattering

Suppose $\rho(t) = 0$ for $t < -T$ and for $t > T$. We can then view the time evolution of the system as a scattering problem: for $t < -T$ the field evolves freely, for $-T < t < T$ the field scatters off the source, and for $t > T$ the scattered field evolves freely again. Note that for $t < -T$, the retarded field vanishes ($\phi_r(t) = 0$) and $\phi(t) = \phi_{in}(t)$, while for $t > T$, the advanced field vanishes ($\phi_a(t) = 0$) and $\phi(t) = \phi_{out}(t)$.

Let us describe the scattering problem in terms of a new set of field coordinates x and p that are given by

$$\begin{aligned} x &= m^{1/2}\,\phi \\ p &= m^{-1/2}\,\pi \end{aligned}$$

They can be combined in a single complex number α:

$$\alpha = \frac{1}{\sqrt{2}}(x + ip)$$

where we consider α and α^* to be independent dynamical variables. Note that

$$\begin{pmatrix} dx \\ dp \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \begin{pmatrix} d\alpha \\ d\alpha^* \end{pmatrix}$$

Thus, using Hamilton's equations for $\dot{x}$ and $\dot{p}$, we find

$$\frac{\partial H}{\partial \alpha^*} = \frac{1}{\sqrt{2}} \left(\frac{\partial H}{\partial x} + i \frac{\partial H}{\partial p} \right) = i\dot{\alpha}$$

So the equation of motion for the field amplitude is

$$\dot{\alpha} = -i \frac{\partial H}{\partial \alpha^*}$$

The free field Hamiltonian is

$$H_0 = \frac{m}{2}(p^2 + x^2) = m|\alpha|^2$$

and the interaction Hamiltonian is

$$H_i = m^{-1/2}\rho(t)x = (2m)^{-1/2}(\alpha + \alpha^*)\rho(t)$$

From these, it follows that the equation of motion for α is

$$\dot{\alpha} = -im\alpha - i(2m)^{-1/2}\rho$$

It is convenient to define a canonical transformation new variables for which the time evolution due to H_0 is already taken into account:

$$\bar{\alpha} = \alpha\, e^{imt}$$

This is the classical analog of the unitary transformation to the interaction picture in quantum mechanics. The transformed Hamiltonians are

$$\bar{H}_0 = 0$$

and

$$\bar{H}_i = (2m)^{-1/2}(\bar{\alpha}\, e^{-imt} + \bar{\alpha}^*\, e^{imt})\rho(t)$$

In this coordinate system, the equation of motion for $\bar{\alpha}$ is

$$\dot{\bar{\alpha}} = -i(2m)^{-1/2}\rho\, e^{imt}$$

We can express the solution as

$$\bar{\alpha}(t) = \bar{\alpha}_{in} + \bar{\alpha}_r(t)$$

where $\bar{\alpha}_{in}$ is a constant, and

$$\bar{\alpha}_r(t) = -i(2m)^{-1/2}\int_{-\infty}^{t} \rho(t')\, e^{imt'}\, dt'$$

Note that for $t > T$, the retarded field amplitude $\bar{\alpha}_r$ is also constant and is given by

$$\bar{\alpha}_r = -i(2m)^{-1/2}\int \rho(t')\, e^{imt'}\, dt'$$

The total energy radiated by the source is

$$E_{rad} = m|\bar{\alpha}_r|^2$$

4.3 Free fields

In this section, I describe several types of free quantum fields in $0+1$ dimensions. Most of the results presented here are obtained by simply transcribing the analogous results for $3+1$ dimensions.

4.3.1 Neutral scalar field

In the previous section, we showed that the Hamiltonian for a free classical scalar field is

$$H_0 = \frac{1}{2}\pi^2 + \frac{1}{2}m^2\phi^2 = m|\alpha|^2$$

where

$$\begin{aligned} \phi &= (2m)^{1/2}(\alpha + \alpha^*) \\ \pi &= -i(m/2)^{1/2}(\alpha - \alpha^*) \end{aligned}$$

To describe a quantum scalar field, we replace the dynamical variables with operators:

$$H_0 = \frac{1}{2}\pi^2 + \frac{1}{2}m^2\phi^2 = m(a^\dagger a + 1/2)$$

where

$$\begin{aligned} \phi &= (2m)^{1/2}(a + a^\dagger) \\ \pi &= -i(m/2)^{1/2}(a - a^\dagger) \end{aligned}$$

Whereas the state of the classical field is given by the dynamical variables ϕ and π, which evolve in time according to Hamilton's equations, the state of the quantum field is given by a wavefunction $|\Phi\rangle$, which evolves in time according to the Schrödinger equation:

$$-i\frac{d}{dt}|\Phi\rangle = H_0|\Phi\rangle$$

The Hamiltonian H_0 has eigenstates $|n\rangle$ and eigenvalues $E_n = m(n + 1/2)$, so we interpret it as describing free particles of mass m, where $|n\rangle$ represents a state in which n particles are present. Because there are no interactions among the particles, the energy of an n-particle state is just the sum of the energies of each particle, plus the vacuum energy $m/2$.

In the interaction picture, the field operator is

$$\phi_I(t) = e^{iH_0 t}\,\phi\,e^{-iH_0 t} = (2m)^{-1/2}\,(a\,e^{-imt} + a^\dagger\,e^{imt})$$

We can define a normal ordering operator that shifts creation operators to the left and annihilation operators to the right. For example,

$$N[a^\dagger a] = N[aa^\dagger] = a^\dagger a$$

Also, we can define a time ordering operator that shifts operators evaluated at later times to the left, and operators evaluated at earlier times to the right. For example,

$$T[\phi_I(t_a)\phi_I(t_b)] = \theta(t_a - t_b)\phi_I(t_a)\phi_I(t_b) + \theta(t_b - t_a)\phi_I(t_b)\phi_I(t_a)$$

The Feynman propagator is

$$D_F(t_a - t_b) = \langle 0|T[\phi_I(t_a)\phi_I(t_b)]|0\rangle = \frac{1}{2m}e^{-i(m-i\epsilon)|t_a - t_b|}$$

where I have added a small imaginary component to the mass so that time integrals over the Feynman propagator converge. Note that the Feynman propagator is the Greens function for the classical Klein-Gordon equation:

$$(\partial_t^2 + m^2)D_F(t) = -i\delta(t)$$

Wick's theorem states that a time ordered product of operators is equal to a normal ordered product of operators, summed over all possible contractions. For two fields, this means that

$$\begin{aligned} T[\phi_I(t_a)\phi_I(t_b)] &= N[\phi_I(t_a)\phi_I(t_b)] + \langle 0|\phi_I(t_a)\phi_I(t_b)|0\rangle \\ &= N[\phi_I(t_a)\phi_I(t_b)] + D_F(t_a - t_b) \end{aligned}$$

For three fields,

$$T[\phi_a\phi_b\phi_c] = N[\phi_a\phi_b\phi_c + \phi_a D_{bc} + \phi_b D_{ca} + \phi_c D_{ab}]$$

where to simplify the notation I have defined $\phi_k = \phi_I(t_k)$ and $D_{ab} = D_F(t_a - t_b)$.

We can introduce a Fourier transformed propagator $\tilde{D}_F(\omega)$, related to $D_F(t)$ by

$$\begin{aligned} D_F(t) &= \frac{1}{2\pi}\int \tilde{D}_F(\omega)\, e^{-i\omega t}\, d\omega = \frac{1}{2m}\, e^{-i(m-i\epsilon)|t|} \\ \tilde{D}_F(\omega) &= \int D_F(t)\, e^{i\omega t}\, dt = i(\omega^2 - m^2 + i\epsilon)^{-1} \end{aligned}$$

4.3.2 Charged scalar field

The Lagrangian for a free charged scalar field of mass m is

$$L_f = |\partial_t \phi|^2 - m^2 |\phi|^2$$

The field can be coupled to an electromagnetic potential A via the substitution

$$\partial_t \rightarrow \partial_t + iA$$

Thus, the Lagrangian for the coupled field is

$$L = |\dot{\phi}|^2 + iA(\phi\,\dot{\phi}^* - \phi^*\,\dot{\phi}) - (m^2 - A^2)|\phi|^2$$

The canonical momentum is

$$\pi = \frac{\partial L}{\partial \dot{\phi}} = \dot{\phi}^* - iA\phi^*$$

The Hamiltonian is given by

$$H = \pi^* \dot{\phi}^* + \pi\dot{\phi} - L = |\pi|^2 + m^2|\phi|^2 - iA(\pi\phi - \pi^*\phi^*)$$

To quantize the system, we introduce operators $a^\dagger$ and a that create and destroy particles, and operators $b^\dagger$ and b that create and destroy antiparticles. They are related to the field operators ϕ and π by

$$\begin{aligned} \phi &= (2m)^{-1/2}(a + b^\dagger) \\ \pi &= -i(m/2)^{1/2}(b - a^\dagger) \end{aligned}$$

The quantum Hamiltonian is

$$H = \pi^\dagger \pi + m^2 \phi^\dagger \phi - iA(\pi\phi - \pi^\dagger \phi^\dagger) = m(a^\dagger a + b^\dagger b) + A(a^\dagger a - b^\dagger b) + m$$

The charge operator is

$$Q = -iN[\pi\phi - \phi^\dagger \pi^\dagger] = a^\dagger a - b^\dagger b$$

4.3.3 Neutral fermion field

We have seen that the quantum harmonic oscillator is equivalent to a neutral boson in $0+1$ QFT. The boson field operator is given by

$$\phi = (2m)^{-1/2}(a + a^\dagger)$$

where the creation and annihilation operators $a^\dagger$ and a obey commutation relations:

$$[a, a] = [a^\dagger, a^\dagger] = 0$$

$$[a, a^\dagger] = 1$$

Similarly, a two-level quantum system is equivalent to a neutral fermion in $0+1$ QFT. The fermion field operator is given by

$$\psi = \sigma_- + \sigma_+ = \sigma_x$$

where the creation and annihilation operators σ_+ and σ_- obey anticommutation relations:

$$\{\sigma_-, \sigma_-\} = \{\sigma_+, \sigma_+\} = 0$$

$$\{\sigma_-, \sigma_+\} = 1$$

The conjugate field operator is given by

$$\bar{\psi} = -\psi\sigma_z = -\sigma_- + \sigma_+ = i\sigma_y$$

The Hamiltonian for the free field is

$$H = \frac{m}{2}\bar{\psi}\psi = \frac{m}{2}(\sigma_+\sigma_- - \sigma_-\sigma_+) = m\sigma_+\sigma_- - m/2$$

Note that the vacuum energy for a neutral fermion has the opposite sign as the vacuum energy for a neutral boson. In the interaction picture, the field operators are

$$\begin{aligned} \psi_I(t) &= \sigma_- e^{-imt} + \sigma_+ e^{imt} \\ \bar{\psi}_I(t) &= -\sigma_- e^{-imt} + \sigma_+ e^{imt} \end{aligned}$$

The normal ordering operator can be generalized to act on fermion fields by including a factor of -1 every time operators are exchanged:

$$N[\sigma_+\sigma_-] = -N[\sigma_-\sigma_+] = \sigma_+\sigma_-$$

and similarly for the time ordering operator:

$$T[\psi_I(t_1)\bar{\psi}_I(t_2)] = \theta(t_1 - t_2)\psi_I(t_1)\bar{\psi}_I(t_2) - \theta(t_2 - t_1)\bar{\psi}_I(t_2)\psi_I(t_1)$$

With these definitions, Wick's theorem has the same form for fermions as it does as for bosons:

$$T[\psi_I(t_1)\bar{\psi}_I(t_2)] = N[\psi_I(t_1)\bar{\psi}_I(t_2)] + \langle 0|T[\psi_I(t_1)\bar{\psi}_I(t_2)]|0\rangle$$

The Feynman propagator is given by

$$S_F(t_1 - t_2) = \langle 0|T[\psi_I(t_1)\bar{\psi}_I(t_2)]|0\rangle = e^{-im|t_1 - t_2|}$$

4.3.4 Charged fermion field

In $3 + 1$ dimensions, the Dirac equation is

$$(i\gamma^\mu\partial_\mu - m)\psi = 0$$

where ψ is a Dirac spinor and γ^μ is a four-vector of 4×4 matrices that satisfies

$$\{\gamma^\mu, \gamma^\nu\} = \eta^{\mu\nu}$$

One possible representation for these matrices is

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

In $0 + 1$ dimensions, the Dirac equation is

$$(i\gamma^0\partial_t - m)\psi = 0$$

where ψ is a spinor and γ^0 is a 2×2 matrix. If we define orthogonal spinors $|\pm\rangle$, we can express ψ as

$$\psi = \psi_+|+\rangle + \psi_-|-\rangle$$

and γ^0 as

$$\gamma^0 = |+\rangle\langle +| - |-\rangle\langle -|$$

The adjoint spinor $\bar{\psi}$ is given by

$$\bar{\psi} \equiv \psi^\dagger \gamma^0 = \psi_+^* \langle +| - \psi_-^* \langle -|$$

The Dirac equation can be obtained from the Lagrangian

$$L = \bar{\psi}(i\gamma^0 \partial_0 - m)\psi$$

or from the Hamiltonian

$$H = m\bar{\psi}\psi = m(|\psi_+|^2 - |\psi_-|^2)$$

We can quantize the system by replacing ψ_+^* and ψ_+ with operators $b^\dagger$ and b that create and destroy fermions, and by replacing ψ_- and ψ_-^* with operators $c^\dagger$ and c that create and destroy anti-fermions:

$$\psi = b|+\rangle + c^\dagger|-\rangle$$

Because the particles obey Fermi statistics, we quantize using anti-commutators rather than commutators:

$$\{b, b^\dagger\} = \{c, c^\dagger\} = 1$$

and all other commutators are equal to zero. In terms of the creation and annihilation operators, the Hamiltonian is

$$H = m(b^\dagger b - cc^\dagger) = m(b^\dagger b + c^\dagger c) - m$$

Note that there is a vacuum energy $-m$; each mode contributes a vacuum energy $-m/2$, and because the modes describe fermions rather than bosons, the sign of the energy is negative rather than positive.

One consequence of the anti-commutation relations is that

$$(b^\dagger)^2 = (c^\dagger)^2 = 0$$

Thus, there are only four states of the system: the vacuum state $|0,0\rangle$, a state with one fermion $b^\dagger|0,0\rangle = |1,0\rangle$, a state with one anti-fermion $c^\dagger|0,0\rangle = |0,1\rangle$, and a state with one fermion and one anti-fermion $b^\dagger c^\dagger|0,0\rangle = |1,1\rangle$.

The Feynman propagator for the Dirac equation is given by

$$S_{F\alpha\beta}(t_a - t_b) = \langle 0|T[\psi_{I\alpha}(t_a)\bar{\psi}_{I\beta}(t_b)]|0\rangle$$

where, as for the neutral fermion field, the time ordering operator is defined such that there is a minus sign every time field operators are exchanged:

$$T[\psi_{I\alpha}(t_a)\bar{\psi}_{I\beta}(t_b)] = \theta(t_a - t_b)\psi_{I\alpha}(t_a)\bar{\psi}_{I\beta}(t_b) - \theta(t_b - t_a)\bar{\psi}_{I\beta}(t_b)\psi_{I\alpha}(t_a)$$

Substituting for the field operators, we find that the Feynman propagator $S_F(t)$ and its Fourier transform $S_F(\omega)$ are given by

$$\begin{aligned} S_F(t) &= \frac{1}{2\pi}\int S_F(\omega)\, e^{-i\omega t}\, d\omega = \frac{1}{2}(1 + \epsilon(t)\,\gamma^0)\, e^{-i(m-i\epsilon)|t|} \\ S_F(\omega) &= \int S_F(t)\, e^{i\omega t}\, dt = i(\omega\gamma^0 - m + i\epsilon)^{-1} \end{aligned}$$

The charge operator is

$$Q = eN[\bar{\psi}\gamma^0\psi] = e(b^\dagger b - c^\dagger c)$$

4.4 Interactions

4.4.1 General results

The Hamiltonian H for a quantum field theory can be expressed as the sum of a free Hamiltonian H_0 and an interaction Hamiltonian H_i:

$$H = H_0 + H_i$$

The free Hamiltonian describes one or more free fields whose quanta are noninteracting particles, while the interaction Hamiltonian couples the fields together and causes the particles to interact. For example, in describing a theory of interacting bosons, we would take H_0 to be the Hamiltonian for a free scalar field:

$$H_0 = \frac{1}{2}\pi^2 + \frac{1}{2}m^2\phi^2 = m(a^\dagger a + 1/2)$$

In the following sections I will give examples of different interaction Hamiltonians H_i, but first I want to discuss some general features of interacting systems.

The interaction Hamiltonian H_i modifies the free particle theory described by H_0 in a number of ways. First, it alters the vacuum state of the theory; in general, the vacuum state of the inter-

acting theory is a superposition of multiparticle states of the free theory. Second, the interaction Hamiltonian not only causes particles to interact with one another, it also causes particles to interact with themselves. This means that the physical mass of the particles (also called the renormalized mass) is not the mass that appears in the free Hamiltonian; rather, it is the sum of the free mass and the mass-energy associated with the particle's self-interaction. Finally, the creation and annihilation operators for the free theory do not create and destroy particles for the interacting theory.

To deal with these modifications, I will introduce creation and annihilation operators $d^\dagger$ and d for the interacting theory. They are defined in terms of the eigenstates $\{|\bar{n}\rangle\}$ of the full Hamiltonian:

$$\begin{aligned} d &= \sum_n \sqrt{n+1}\,|\bar{n}\rangle\langle\bar{n}+1| \\ d^\dagger &= \sum_n \sqrt{n+1}\,|\bar{n}+1\rangle\langle\bar{n}| \end{aligned}$$

where

$$H|\bar{n}\rangle = \bar{E}_n|\bar{n}\rangle$$

From these expressions, it follows that the commutation relation for the new operators is

$$[d, d^\dagger] = 1$$

Thus, d and $d^\dagger$ are related to a and $a^\dagger$ by a canonical transformation. Note that if we define constants m_R and V_k appropriately, we can express eigenvalues of H in the form

$$\bar{E}_n = \bar{E}_0 + m_R\, n + \frac{1}{2!}V_2\, n(n-1) + \frac{1}{3!}V_3\, n(n-1)(n-2) + \cdots$$

For example, $m_R = \bar{E}_1 - \bar{E}_0$ and $V_2 = \bar{E}_2 - 2\bar{E}_1 + \bar{E}_0$. Also, note that

$$(d^\dagger)^k\, d^k|n\rangle = n(n-1)(n-2)\cdots(n-k)|n\rangle$$

Thus, the full Hamiltonian H may be expressed as

$$H = \bar{E}_0 + m_R\, d^\dagger d + \frac{1}{2!}V_2\, d^\dagger d^\dagger dd + \frac{1}{3!}V_3\, d^\dagger d^\dagger d^\dagger ddd + \cdots$$

We can interpret this result by comparing it with the free Hamiltonian H_0:

$$H_0 = \frac{1}{2}m + ma^\dagger a$$

We see that in going from H_0 to H the vacuum energy has changed from $E_0 = m/2$ to $\bar{E}_0$, and the particle mass has changed from $m = E_1 - E_0$, the free mass, to $m_R = \bar{E}_1 - \bar{E}_0$, the renormalized mass. Also, there are n-body interactions among the individual particles, which are described by potentials V_n. In higher dimensions, the interactions lead to particle-particle scattering. In $0+1$ dimensions there is no scattering (there is nowhere for the particles to scatter to); rather, the interactions cause the energy to depend on the number of particles in a nonlinear way.

As an example, suppose we have n particles that interact through a two-body potential. Then the total energy of the system will be given by $\bar{E}_0$ (the vacuum energy), plus the number of particles times m_R (the particle mass), plus the number of pairs of particles times V_2 (the potential energy per pair):

$$\bar{E}_n = \bar{E}_0 + m_R\, n + \frac{1}{2} V_2\, n(n-1)$$

Thus, the two-body potential gives a quadratic contribution to the energy spectrum.

I now want to discuss the correlation functions for the interacting theory. For the free theory, we defined the two-point correlation function as

$$\langle 0|T[\phi_I(t_a)\phi_I(t_b)]|0\rangle$$

where $\phi_I(t)$ is the interaction picture field operator, defined by evolving the Schrödinger picture field operator ϕ under the free Hamiltonian H_0:

$$\phi_I(t) = e^{iH_0 t}\, \phi\, e^{-iH_0 t} = (2m)^{-1/2}(a\, e^{-imt} + a^\dagger\, e^{imt})$$

For the interacting theory, we define the two-point correlation function as

$$\langle \bar{0}|T[\phi(t_a)\phi(t_b)]|\bar{0}\rangle$$

where $\phi(t)$ is the Heisenberg picture field operator, defined by evolving the Schrödinger picture field operator ϕ under the full Hamiltonian H:

$$\phi(t) = e^{iHt}\, \phi\, e^{-iHt}$$

Recall that for the free theory, the two-point correlation function is just the Feynman propagator and may be interpreted as the probability amplitude for a particle that was created at time t_b to propagate to time t_a:

$$\langle 0|[\phi_I(t_a)\phi_I(t_b)]|0\rangle = D_F(t_a - t_b)$$

We can obtain an analogous result for the interacting theory. From the definition of the time ordered product,

$$\langle\bar{0}|T[\phi(t_a)\phi(t_b)]|\bar{0}\rangle = \theta(t_a - t_b)\,\langle\bar{0}|\phi(t_a)\phi(t_b)|\bar{0}\rangle + \theta(t_b - t_a)\,\langle\bar{0}|\phi(t_b)\phi(t_a)|\bar{0}\rangle$$

If we insert a complete set of states, we find

$$\begin{aligned}\langle\bar{0}|\phi(t_a)\phi(t_b)|\bar{0}\rangle &= \sum_n \langle\bar{0}|\phi(t_a)|\bar{n}\rangle\langle\bar{n}|\phi(t_b)|\bar{0}\rangle \\ &= \sum_n e^{-i(\bar{E}_n - \bar{E}_0)(t_a - t_b)}\,\langle\bar{0}|\phi|\bar{n}\rangle\langle\bar{n}|\phi|\bar{0}\rangle\end{aligned}$$

Thus,

$$\begin{aligned}\langle\bar{0}|T[\phi(t_a)\phi(t_b)]|\bar{0}\rangle &= \sum_n e^{-i(\bar{E}_n - \bar{E}_0)|t_a - t_b|}\,|\langle\bar{n}|\phi|\bar{0}\rangle|^2 \\ &= \sum_n \rho_n\, D_F(t_a - t_b, \bar{E}_n - \bar{E}_0)\end{aligned}$$

where

$$\rho_n = 2(\bar{E}_n - \bar{E}_0)|\langle\bar{n}|\phi|\bar{0}\rangle|^2$$

is called the spectral density, and

$$D_F(\tau, \mu) = \frac{1}{2\mu} e^{-i(\mu - i\epsilon)|\tau|}$$

is the Feynman propagator for a particle of mass μ. This result is the 0+1 dimensional equivalent of the Källén-Lehmann spectral representation of the two-point correlation function (see [43] for a discussion of this representation). The quantity ρ_1 deserves special attention; it is called the field strength renormalization and is also denoted by Z:

$$Z = \rho_1 = 2m_R|\langle\bar{1}|\phi|\bar{0}\rangle|^2$$

The field strength renormalization can be interpreted as the probability of creating a single particle state by acting on the interacting vacuum with the free field operator.

I will now show that the spectral density obeys the relation

$$\sum_n \rho_n = 1$$

The proof presented here is adapted from a similar proof for $3+1$ dimensional QFT that is given in [44]. Note that for the free theory,

$$\langle 0|[\phi_I(t),\phi_I(0)]|0\rangle = [\phi_I(t),\phi_I(0)] = -iG_-(t,m)$$

where

$$G_-(t,\mu) = \frac{1}{\mu}\sin\mu t$$

The analogous result for the interacting theory is

$$\begin{aligned}
\langle\bar{0}|[\phi(t),\phi(0)]|\bar{0}\rangle &= \sum_n [\langle\bar{0}|\phi(t)|\bar{n}\rangle\langle\bar{n}|\phi(0)|\bar{0}\rangle - \langle\bar{0}|\phi(0)|\bar{n}\rangle\langle\bar{n}|\phi(t)|\bar{0}\rangle] \\
&= \sum_n (e^{i(\bar{E}_0-\bar{E}_n)t} - e^{i(\bar{E}_n-\bar{E}_0)t})\,|\langle\bar{n}|\phi|\bar{0}\rangle|^2 \\
&= -i\sum_n \rho_n\, G_-(t,E_n-E_0)
\end{aligned}$$

Thus,

$$\partial_t\langle\bar{0}|[\phi(t),\phi(0)]|\bar{0}\rangle\;|_{t=0} = -i\sum_n \rho_n\,\partial_t G_-(t,E_n-E_0)\;|_{t=0} = -i\sum\rho_n$$

But we can also express the time derivative of the correlation function as

$$\partial_t\langle\bar{0}|[\phi(t),\phi(0)]|\bar{0}\rangle\;|_{t=0} = \langle\bar{0}|[\pi(t),\phi(0)]|\bar{0}\rangle\;|_{t=0} = -i$$

where I have used that

$$\partial_t\phi(t) = \pi(t)$$

and

$$[\pi(0),\phi(0)] = -i$$

By comparing these two expressions for the time derivative of the correlation function, we obtain the desired result.

We can interpret the spectral decomposition of the two-point function by comparing the free and interacting theories. For the free theory, ϕ creates single particle states of mass m, which propagate in time according to the Feynman propagator $D_F(t) = D_F(t,m)$. For the interacting theory, ϕ creates both single particle states of mass m_R and multiparticle states of mass $\bar{E}_n - \bar{E}_0$. The probability of creating a single particle state is Z, and the probability of creating an n-particle state is ρ_n. Both single and multiparticle states propagate according to the Feynman propagator, evaluated at their respective masses.

Techniques from quantum field theory can be used to evaluate the two-point correlation function of the interacting theory. One can show that [43]

$$\langle\bar{0}|T[\phi(t_a)\phi(t_b)]|\bar{0}\rangle = \frac{\langle 0|T[\phi_I(t_a)\phi_I(t_b)\exp(-i\int H_I(t)\,dt)]|0\rangle}{\langle 0|T[\exp(-i\int H_I(t)\,dt)]|0\rangle}$$

We can obtain a perturbation series by expanding this result in H_I. If we express H_I in terms of field operators, then the terms in this series take the form of correlation functions of the free theory. Using Wick's theorem, these correlation functions can be expressed in terms of Feynman propagators.

Thus, the two-point correlation function of the interacting theory can be expressed as a series, the terms of which consist of integrals over products of Feynman propagators for the free theory. Each term in the series represents a specific spacetime process and may be depicted as a Feynman diagram: vertices in the diagram represent events, and lines connecting vertices represent the propagation of a particle from one event to another.

4.4.2 ϕ^2 interaction

Consider an interaction of the form

$$H_i = \frac{\lambda}{2}\phi^2 = \frac{\lambda}{4m}(a + a^\dagger)^2$$

The full Hamiltonian is

$$H = H_0 + H_i = \frac{1}{2}\pi^2 + \frac{1}{2}(m^2 + \lambda)\phi^2 = m(a^\dagger a + \frac{1}{2}) + \frac{\lambda}{4m}(a + a^\dagger)^2$$

Thus, the interaction H_i just redefines the mass of the particle. We can immediately write down the solution by introducing operators d and $d^\dagger$, defined by

$$\phi = (2m_R)^{-1/2}(d + d^\dagger) = (2m)^{-1/2}(a + a^\dagger)$$

where

$$m_R = m(1 + \lambda/m^2)^{1/2}$$

is the renormalized mass. In terms of the renormalized mass and the new operators d and $d^\dagger$, the Hamiltonian is

$$H = \frac{1}{2}\pi^2 + \frac{1}{2}m_R^2\phi^2 = m_R(d^\dagger d + 1/2)$$

The canonical transformation from a and $a^\dagger$ to d and $d^\dagger$ can be written down explicitly; it is just the squeezing transformation:

$$\begin{aligned} d &= \cosh r\, a + \sinh r\, a^\dagger \\ d^\dagger &= \sinh r\, a + \cosh r\, a^\dagger \end{aligned}$$

where

$$r = \frac{1}{4}\log(1+\lambda/m^2)$$

By expressing the field operator in terms of d and $d^\dagger$, we can immediately write down the two-point correlation function for the full Hamiltonian:

$$\langle\bar{0}|T[\phi(t_a)\phi(t_b)]|\bar{0}\rangle = \frac{1}{2m_R}e^{-im_R|t_a-t_b|} = D_F(t_a-t_b, m_R)$$

Thus, $Z=1$ and $\rho_n = 0$ for $n>1$.

We can also obtain these results from the Feynman perturbation expansion. Recall that

$$\langle\bar{0}|T[\phi(t_a)\phi(t_b)]|\bar{0}\rangle = \frac{\langle 0|T[\phi_I(t_a)\phi_I(t_b)\exp(-i\int H_I(t)\,dt)]|0\rangle}{\langle 0|T[\exp(-i\int H_I(t)\,dt)]|0\rangle}$$

For the numerator,

$$\begin{aligned} &\langle 0|T[\phi_a\phi_b\exp(-i\lambda\int H_I(t)\,dt)]|0\rangle = \\ &\quad D_{ab} + \\ &\quad (1/1!)(-i\lambda/2)\int (D_{ab}D_{11}+2D_{a1}D_{1b})\,dt_1 + \\ &\quad (1/2!)(-i\lambda/2)^2\iint (D_{ab}D_{11}D_{22}+2D_{ab}D_{12}^2+2D_{a1}D_{1b}D_{22}+ \\ &\quad 2D_{a2}D_{2b}D_{11}+4D_{a1}D_{12}D_{2b}+4D_{a2}D_{21}D_{1b})\,dt_1\,dt_2+\cdots \end{aligned}$$

For the denominator,

$$\begin{aligned} &\langle 0|T[\exp(-i\lambda\int H_I(t)\,dt)]|0\rangle = \\ &\quad 1 + \\ &\quad (1/1!)(-i\lambda/2)\int D_{ab}D_{11}\,dt_1 + \\ &\quad (1/2!)(-i\lambda/2)^2\iint (D_{11}D_{22}+2D_{12}^2)\,dt_1\,dt_2+\cdots \end{aligned}$$

Thus,

$$\langle\bar{0}|T[\phi(t_a)\phi(t_b)]|\bar{0}\rangle = D_{ab} + (-i\lambda)\int D_{a1}D_{1b}\,dt_1 + (-i\lambda)^2\iint D_{a1}D_{12}D_{2b}\,dt_1\,dt_2 + \cdots$$

Note that the denominator cancels out the disconnected diagrams in the numerator.

It is simpler to use the Fourier transformed propagator:

$$\begin{aligned}\int\langle\bar{0}|T[\phi(t)\phi(0)]|\bar{0}\rangle\,e^{i\omega t}\,dt &= D_F(\omega) + (-i\lambda)D_F^2(\omega) + (-i\lambda)^2D_F^3(\omega)\cdots \\ &= (1 + i\lambda D_F(\omega))^{-1}\,D_F(\omega) \\ &= i(\omega^2 - (m^2+\lambda) + i\epsilon)^{-1} \\ &= D_F(\omega, m_R)\end{aligned}$$

Thus, we recover our previous results for m_R and Z.

4.4.3 ϕ^4 interaction

Consider an interaction of the form

$$H_i = \lambda\phi^4$$

Then the total Hamiltonian is

$$H = H_0 + H_i = \frac{1}{2}\pi^2 + \frac{1}{2}m^2\phi^2 + \lambda\phi^4$$

For simplicity, in this section I will set $m = 1$; it can always be restored by using dimensional analysis. The Hamiltonian H cannot be solved in closed form; however, one can perform a perturbative expansion in the parameter λ. To second order, the energy eigenvalues are

$$\bar{E}_n = E_n + \langle n|H_i|n\rangle + \sum_{m\neq n}\frac{|\langle n|H_i|m\rangle|^2}{n-m}$$

The first two energy eigenvalues are

$$\begin{aligned}\bar{E}_0 &= \frac{1}{2} + \frac{3}{4}\lambda - \frac{21}{8}\lambda^2 \\ \bar{E}_1 &= \frac{3}{2} + \frac{15}{4}\lambda - \frac{165}{8}\lambda^2\end{aligned}$$

Thus, the renormalized mass is

$$m_R = 1 + 3\lambda - 18\lambda^2$$

To second order, the wavefunctions are (see [28], for example)

$$\begin{aligned}|\bar{n}\rangle &= \left(1-\frac{1}{2}\sum_{k\neq n}\frac{|\langle k|H_i|n\rangle|^2}{(n-k)^2}\right)|n\rangle + \\ &\sum_{k\neq n}\left(\frac{\langle k|H_i|n\rangle}{n-k}-\frac{\langle n|H_i|n\rangle\langle k|H_i|n\rangle}{(n-k)^2}+\sum_{m\neq n}\frac{\langle k|H_i|m\rangle\langle m|H_i|n\rangle}{(n-k)(n-m)}\right)|k\rangle\end{aligned}$$

Using these wavefunctions, we find that

$$\langle\bar{0}|\phi|\bar{1}\rangle = \frac{1}{\sqrt{2}}(1-\frac{3}{2}\lambda+\frac{189}{16}\lambda^2)$$

Thus, the wavefunction renormalization is

$$Z = 2m_R|\langle\bar{0}|\phi|\bar{1}\rangle|^2 = 1-\frac{9}{8}\lambda^2$$

We can also obtain these results using the Feynman perturbation expansion. Recall that

$$\langle\bar{0}|T[\phi(t_a)\phi(t_b)]|\bar{0}\rangle = \frac{\langle 0|T[\phi_I(t_a)\phi_I(t_b)\exp(-i\int H_I(t)\,dt)]|0\rangle}{\langle 0|T[\exp(-i\int H_I(t)\,dt)]|0\rangle}$$

As we have seen in section 4.4.2, the denominator cancels out the disconnected diagrams in the numerator. Thus, we can obtain the correlation function by expanding the numerator and retaining only the connected diagrams. The following results will be useful:

$$\int D_{a1}D_{1b}\,dt_1 = \frac{1}{2}(|\tau|-i)D_F(\tau)$$

$$\int D_{a1}D_{1b}^3\,dt_1 = \frac{i}{8}D_F^3(\tau)-\frac{3i}{32}D_F(\tau)$$

$$\int |t|\,D_F(t-\tau)\,D_F(t)\,dt = \frac{1}{4}(\tau^2-i|\tau|-1)D_F(\tau)$$

$$\int D_{12}^2\,dt_2 = -i/4$$

where $\tau = t_a - t_b$.

To zeroth order,

$$\langle 0|T[\phi_a\phi_b]|0\rangle = D_{ab}$$

To first order,

$$\langle 0|T[\phi_a\phi_b\phi_1^4]|0\rangle = 3\,D_{ab}\,D_{11}^2 + (4\cdot 3)\,D_{a1}D_{11}D_{1b}$$

Note that since we are contracting six fields, the number of terms is

$$3 + 4 \cdot 3 = 15 = 5 \cdot 3 \cdot 1$$

We see that there is one connected diagram and one disconnected diagram. The connected diagram is given by

$$(4 \cdot 3)\,(1/1!)\,(-i\lambda) \int D_{a1} D_{11} D_{1b}\, dt_1 = -3\lambda(1 + i|\tau|) D_F(\tau)$$

To second order,

$$\begin{aligned}
&\langle 0|T[\phi_a \phi_b \phi_1^4 \phi_2^4]|0\rangle = \\
&\quad D_{ab} \langle 0|T[\phi_1^4 \phi_2^4]|0\rangle + \\
&\quad (4 \cdot 3 \cdot 3)\,(D_{a1} D_{11} D_{1b} D_{22}^2 + D_{a2} D_{22} D_{2b} D_{11}^2) + \\
&\quad (4 \cdot 3)^2\,(D_{a1} D_{b1} D_{12}^2 D_{22} + D_{a2} D_{b2} D_{21}^2 D_{11}) + \\
&\quad (4 \cdot 3)^2\,(D_{a1} D_{11} D_{12} D_{22} D_{2b} + D_{a2} D_{22} D_{21} D_{11} D_{1b}) + \\
&\quad (4^2 \cdot 3 \cdot 2)\,(D_{a1} D_{12}^3 D_{2b} + D_{a2} D_{21}^3 D_{1b})
\end{aligned}$$

The first two lines correspond to disconnected diagrams. Note that the first line involves a contraction of eight fields, which gives $7 \cdot 5 \cdot 3 \cdot 1 = 105$ terms. Thus, the total number of terms is

$$105 + 2((4 \cdot 3 \cdot 3) + (4 \cdot 3)^2 + (4 \cdot 3)^2 + (4^2 \cdot 3 \cdot 2)) = 945 = 9 \cdot 7 \cdot 5 \cdot 3 \cdot 1$$

which is as it should be for a contraction of ten fields. We see that there are three connected diagrams:

The first diagram (Figure 4.1) is

$$\begin{aligned}
&2\,(4 \cdot 3)^2\,(1/2!)\,(-i\lambda)^2 \iint D_{a1} D_{b1} D_{12}^2 D_{22}\, dt_1\, dt_2 \\
&\quad = \quad 18 i \lambda^2 \int D_{a1} D_{b1}\, dt_1 \\
&\quad = \quad 9\lambda^2 (1 + i|\tau|) D_F(\tau))
\end{aligned}$$

The second diagram (Figure 4.2) is

$$\begin{aligned}
&2\,(4 \cdot 3)^2\,(1/2!)\,(-i\lambda)^2 \iint D_{a1} D_{11} D_{12} D_{22} D_{2b}\, dt_1\, dt_2 \\
&\quad = \quad -36\lambda^2 \iint D_{a1} D_{12} D_{2b}\, dt_1\, dt_2 \\
&\quad = \quad \frac{9}{2} \lambda^2 (3 + 3i|\tau| - \tau^2) D_F(\tau)
\end{aligned}$$

The third diagram (Figure 4.3) is

$$\begin{aligned} & 2\,(4^2\cdot 3\cdot 2)\,(1/2!)\,(-i\lambda)^2 \iint D_{a1}D_{12}^3 D_{2b}\,dt_1\,dt_2 \\ &= -96\lambda^2 \iint D_{a1}D_{12}^3 D_{2b}\,dt_1\,dt_2 \\ &= \frac{9}{8}\lambda^2 D_F(\tau,3) + \frac{1}{8}\lambda^2(27+36i|\tau|)D_F(\tau) \end{aligned}$$

Thus, to second order, the two-point correlation function is

$$\begin{aligned} & \langle\bar{0}|T[\phi(t_a)\phi(t_b)]|\bar{0}\rangle \\ &= [1-3\lambda(1+i|\tau|)+\frac{1}{8}\lambda^2(207+216i|\tau|-36\tau^2)]D_F(\tau)+\frac{9}{8}\lambda^2 D_F(\tau,3) \\ &= Z\,D_F(\tau,m_R)+\rho_3\,D_F(\tau,3) \end{aligned}$$

where

$$\begin{aligned} Z &= 1-\frac{9}{8}\lambda^2 \\ m_R &= 1+3\lambda-18\lambda^2 \\ \rho_3 &= \frac{9}{8}\lambda^2 \end{aligned}$$

One can also obtain these results by numerically solving the Schrödinger equation:

$$(-\frac{1}{2}\frac{d^2}{d\phi^2}+\frac{1}{2}\phi^2+\lambda\phi^4)\Phi_n(\phi)=\bar{E}_n\Phi_n(\phi)$$

The mass renormalization is given by

$$m_R=\bar{E}_1-\bar{E}_0$$

and the wavefunction renormalization is given by

$$Z=2m_R\left|\int \Phi_1^*(\phi)\,\phi\,\Phi_0(\phi)\,d\phi\right|^2$$

The results of a numerical calculation of m_R and Z are shown in Figure 4.4 and Figure 4.5.

4.4.4 Yukawa interaction

I now want to consider a theory that describes an interaction between bosons and fermions. The Hamiltonian for a free boson field is

$$H_b=\frac{1}{2}\pi^2+\frac{1}{2}m^2\phi^2=ma^\dagger a$$

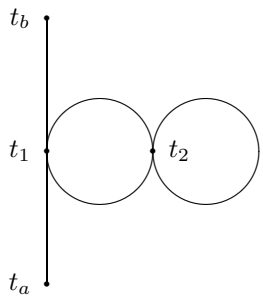

Figure 4.1: First diagram.

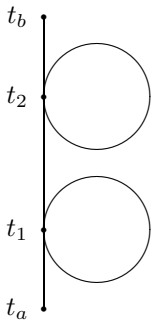

Figure 4.2: Second diagram.

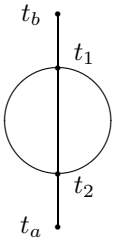

Figure 4.3: Third diagram.

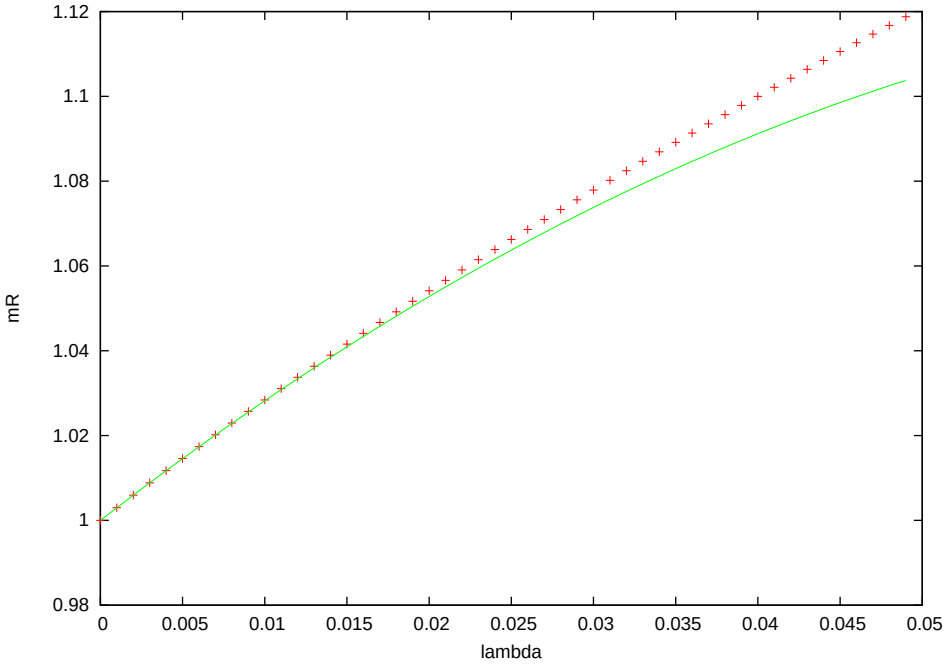

Figure 4.4: Renormalized mass m_R. The points are from a numerical calculation; the curve is the theoretical prediction to second order.

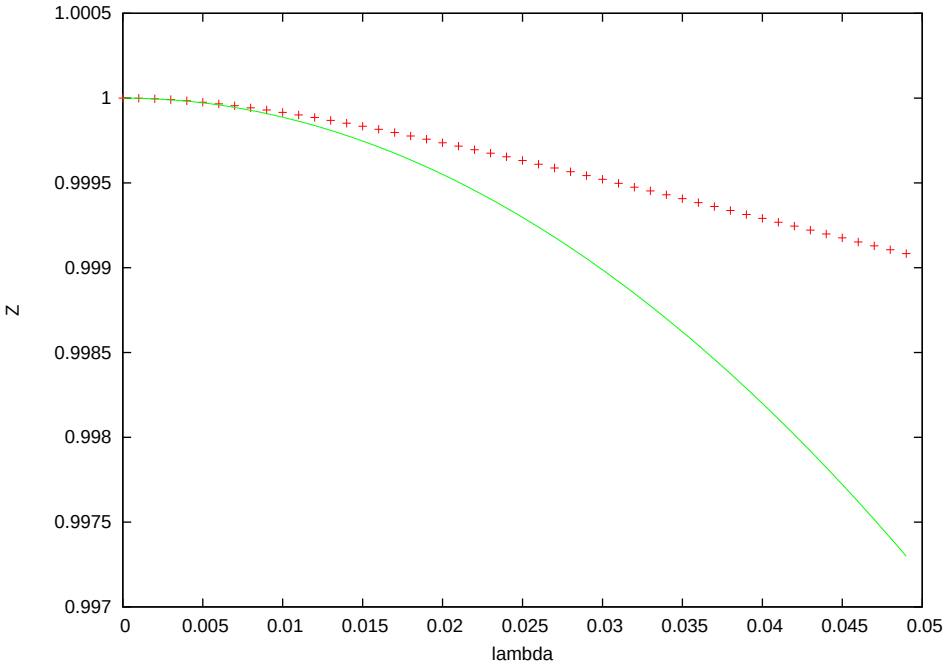

Figure 4.5: Wavefunction renormalization Z. The points are from a numerical calculation; the curve is the theoretical prediction to second order.

The Hamiltonian for a free fermion field is

$$H_f = \mu\bar{\psi}\psi = \mu(b^\dagger b + c^\dagger c)$$

Thus, the free Hamiltonian for a system of bosons and fermions is

$$H_0 = H_b + H_f$$

The eigenstates for H_0 are products of the eigenstates $\{|n_a\rangle\}$ of H_b and the eigenstates $\{|n_b, n_c\rangle\}$ of H_f:

$$H_0|n_a, n_b, n_c\rangle = (mn_a + \mu(n_b + n_c))|n_a, n_b, n_c\rangle$$

where

$$|n_a, n_b, n_c\rangle = |n_a\rangle|n_b, n_c\rangle$$

Consider a Yukawa coupling of the fermion field to the boson field:

$$H_i = \lambda\phi\bar{\psi}\psi = \eta m(a + a^\dagger)(b^\dagger b + c^\dagger c)$$

where I have defined a dimensionless coupling η:

$$\eta = (2m)^{-1/2}\,(\lambda/m)$$

The total Hamiltonian is

$$\begin{aligned} H &= H_b + H_f + H_i \\ &= \frac{1}{2}\pi^2 + \frac{1}{2}m^2\phi^2 + \mu\bar{\psi}\psi + \lambda\phi\bar{\psi}\psi \\ &= ma^\dagger a + \mu(b^\dagger b + c^\dagger c) + \eta m(a + a^\dagger)(b^\dagger b + c^\dagger c) \end{aligned}$$

This Hamiltonian can be solved exactly by introducing an operator d:

$$d = a + \eta(b^\dagger b + c^\dagger c)$$

Note that

$$d^\dagger d = a^\dagger a + \eta(a + a^\dagger)(b^\dagger b + c^\dagger c) + \eta^2(b^\dagger b + c^\dagger c) + 2\eta^2\, b^\dagger b c^\dagger c$$

Thus, we can express the total Hamiltonian as

$$H = md^\dagger d + (\mu - \eta^2 m)(b^\dagger b + c^\dagger c) - 2\eta^2 m b^\dagger b c^\dagger c$$

$$= \quad m_R\, d^\dagger d + \mu_R\, (b^\dagger b + c^\dagger c) + V_2\, b^\dagger b c^\dagger c$$

where

$$m_R = m \qquad \mu_R = \mu - \eta^2 m \qquad V_2 = -2\eta^2 m$$

The eigenstates and eigenvalues of the interacting Hamiltonian are

$$H|\bar{n}_d, n_b, n_c\rangle = (m n_a + \mu_R(n_b + n_c) + V_2\, n_b n_c)|\bar{n}_d, n_b, n_c\rangle$$

We see that the interacting theory can be interpreted as a system of noninteracting bosons of mass m, together with a system of fermions of mass μ_R, where fermions couple to antifermions with strength V_2. We can understand the sign of this coupling by the following argument. In higher dimensions, a Yukawa coupling gives an attractive force between fermions and antifermions [43]. The corresponding result for $0+1$ dimensions is that if we take two systems, one consisting of an isolated fermion and one consisting of an isolated antifermion, and add their energies, then the result is greater than the energy of a single system consisting of both a fermion and an antifermion.

To express the interacting eigenstates in terms of the free eigenstates, first note that

$$d|\bar{0}, n_b, n_c\rangle = (a + \eta(b^\dagger b + c^\dagger c))|\bar{0}, n_b, n_c\rangle = 0$$

Thus,

$$a|\bar{0}, n_b, n_c\rangle = -\eta(b^\dagger b + c^\dagger c)|\bar{0}, n_b, n_c\rangle$$

So $|\bar{0}, n_b, n_c\rangle$ is a coherent state of the operator a. Note that if $|\alpha\rangle$ is a coherent state with amplitude α (that is, $a|\alpha\rangle = \alpha|\alpha\rangle$), then we can express it in terms of the number states as follows:

$$|\alpha\rangle = e^{-|\alpha|^2/2}\, e^{\alpha a^\dagger}|0\rangle = e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}}|n\rangle$$

Thus,

$$|\bar{0}, n_b, n_c\rangle = e^{-\eta^2(n_b+n_c)^2/2} \sum_{n_a} \frac{(-1)^{n_a}}{\sqrt{n_a!}}\, \eta^{n_a}(n_b + n_c)^{n_a}\, |n_a, n_b, n_c\rangle$$

Note that the vacuum states of the free and interacting theories are the same:

$$|\bar{0}, 0, 0\rangle = |0, 0, 0\rangle$$

From the eigenstates of the interacting theory, we can calculate the wavefunction renormalizations. The fermion wavefunction renormalization is

$$Z_f = |\langle \bar{0}, 1, 0 | b^\dagger | \bar{0}, 0, 0 \rangle|^2 = e^{-\eta^2}$$

and the boson wavefunction renormalization is

$$Z_b = 2m_R |\langle \bar{1}, 0, 0 | \phi | \bar{0}, 0, 0 \rangle|^2 = 1$$

Note that we can introduce a new boson field operator:

$$\Phi = (2m)^{-1/2} (d + d^\dagger)$$

Then the Hamiltonian takes the form

$$H = \frac{1}{2}\Pi^2 + \frac{1}{2}m^2\Phi^2 + \mu\bar{\psi}\psi - \eta m(\bar{\psi}\psi)^2$$

This describes a free boson field and a self-interacting fermion field, which is consistent with our previous interpretation.

We can also obtain these results using the Feynman perturbation expansion. To second order in λ,

$$\begin{aligned}
&\langle 0 | T[\psi_{I\alpha}(t_a)\bar{\psi}_{I\beta}(t_b) \exp(-i\lambda \int \phi_I(t)\bar{\psi}_I(t)\psi_I(t)\, dt)] | 0 \rangle \\
&= \langle 0 | T[\psi_{I\alpha}(t_a)\bar{\psi}_{I\beta}(t_b)] | 0 \rangle - \\
&\quad (\lambda^2/2) \iint D_F(t_1 - t_2)\, \langle 0 | T[\psi_{I\alpha}(t_a)\bar{\psi}_{I\beta}(t_b)\psi_{I\gamma}(t_1)\bar{\psi}_{I\gamma}(t_1)\psi_{I\delta}(t_2)\bar{\psi}_{I\delta}(t_2)] | 0 \rangle\, dt_1\, dt_2 \\
&= S_{F\alpha\beta}(t_a - t_b) - \\
&\quad \lambda^2 \iint D_F(t_1 - t_2)\, S_{F\alpha\gamma}(t_a - t_1)\, S_{F\gamma\delta}(t_1 - t_2) S_{F\delta\beta}(t_2 - t_b)\, dt_1\, dt_2 + \\
&\quad \lambda^2 \iint D_F(t_1 - t_2)\, S_{F\alpha\gamma}(t_a - t_1)\, S_{F\gamma\beta}(t_1 - t_b)\, S_{F\delta\delta}(0)\, dt_1\, dt_2 - \\
&\quad (\lambda^2/2)\, S_{F\alpha\beta}(t_a - t_b) \iint D_F(t_1 - t_2)\, \langle 0 | T[\psi_{I\gamma}(t_1)\bar{\psi}_{I\gamma}(t_1)\psi_{I\delta}(t_2)\bar{\psi}_{I\delta}(t_2)] | 0 \rangle\, dt_1\, dt_2
\end{aligned}$$

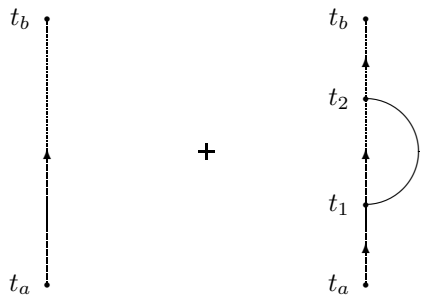

Figure 4.6: Two-point correlation function to second order.

The second line vanishes because $S_F(0) = 0$, and the third line corresponds to disconnected diagrams. Thus, to second order, the fermion two-point correlation function is

$$\langle\bar{0}|T[\psi_\alpha(t_a)\bar{\psi}_\beta(t_b)]|\bar{0}\rangle = \\ [S_F(t_a - t_b) + \iint S_F(t_a - t_1)\,\Sigma(t_1 - t_2)\,S_F(t_2 - t_b)\,dt_1\,dt_2]_{\alpha\beta}$$

where

$$\Sigma(t) = -\lambda^2 S_F(t)\,D_F(t)$$

This expression is represented by the diagrams shown in Figure 4.6. If we Fourier transform the correlation function, we find

$$\int \langle\bar{0}|T[\psi_\alpha(t)\bar{\psi}_\beta(0)]|\bar{0}\rangle\,e^{i\omega t}\,dt = [S_F(\omega) + S_F(E)\,\Sigma(\omega)\,S_F(\omega)]_{\alpha\beta}$$

where

$$\Sigma(\omega) = \int \Sigma(t)\,e^{i\omega t}\,dt = -i\eta^2 m^2(\omega\gamma^0 - (\mu + m))^{-1}$$

Thus,

$$\begin{aligned}
&\int \langle\bar{0}|T[\psi_\alpha(t)\bar{\psi}_\beta(0)]|\bar{0}\rangle\,e^{i\omega t}\,dt \\
&= [i(\omega\gamma^0 - \mu)^{-1} + i\eta^2 m^2(\omega\gamma^0 - \mu)^{-2}(\omega\gamma^0 - (\mu + m))^{-1}]_{\alpha\beta} \\
&= [i(1-\eta^2)(\omega\gamma^0 - (\mu - \eta^2 m))^{-1} + i\eta^2(\omega\gamma^0 - (\mu + m))^{-1}]_{\alpha\beta} \\
&= [(1-\eta^2)\,S_F(\omega, \mu - \eta^2 m) + \eta^2\,S_F(\omega, \mu + m)]_{\alpha\beta}
\end{aligned}$$

To understand why the second and third lines in this expression are equal, it is easiest to simply expand the third line to second order in η and note that it agrees with the second line. From this

result, we can read off the renormalized fermion mass and the fermion wavefunction renormalization:

$$\mu_R = \mu - \eta^2 m \qquad Z_f = 1 - \eta^2$$

4.4.5 Scattering

Consider a field theory in $3+1$ dimensions described by a Hamiltonian

$$H = H_0 + H_i$$

were H_0 is a free Hamiltonian and H_i describes an interaction. We can express the free Hamiltonian as

$$H_0 = \sum_m \omega_m a_m^\dagger a_m$$

where $a_m^\dagger$ and a_m are creation and annihilation operators for mode m. Let us consider scattering in this theory. We start with two widely separated particles A and B, each described by a wavepacket. As the system evolves in time, the particles come together, interact, and then fly apart. In the distant past and the distant future, the particles are too far apart to interact with each other and the evolution of the system can be described by a free Hamiltonian H_f. In general, however, the free Hamiltonian H_f is not the same as the free Hamiltonian H_0, because the free particle states of the interacting theory are not the same as the free particle states of the free theory. For example, the coupling of an electron to the electromagnetic field surrounds it with a Coulomb field, and the energy of this field gives a correction to the mass of the electron. I will call the free particles described by H_0 bare particles, and the free particles described by H_f dressed particles.

In previous sections, we showed that in $0+1$ dimensions one can relate bare particles to dressed particles via a canonical transformation of the particle creation and annihilation operators. Now I want to consider a different method of obtaining the free particle states of the interacting theory, which generalizes to higher dimensions. First, we modify the Hamiltonian by introducing a function $f(t)$:

$$H = H_0 + f(t)\, H_i$$

If we start with a single particle state of H_0 (a bare particle) and evolve it under H by slowly ramping f from 0 to 1, it should adiabatically evolve into a single particle state of H_f (a dressed particle). If we do the same for a two-particle state of H_0, we should end up with a two-particle state of H_f, provided that the two particles start out far away from each other, and that during the adiabatic ramping of f they remain far enough apart that we can neglect their mutual interaction. If we continue to evolve the two-particle state under H, the particles will approach, interact, and

scatter. We can then adiabatically ramp down f, so the dressed particles turn back into bare particles. Because the particles are widely separated when the interaction is ramped on and off, the probability amplitude for producing a given final state from a given initial state is independent of the details of the ramping process. The advantage of the ramping technique is that we don't need to solve explicitly for the asymptotic states of the interacting theory; the adiabatic evolution takes care of this for us (the adiabatic ramping technique is discussed at length in [45]).

The adiabatic ramping technique can be demonstrated in $0+1$ dimensions; however, because particles can never be separated from one another, the result cannot be interpreted as describing a scattering process, and the final states depend on how the ramping is performed. I will demonstrate the technique with a simple model. Consider a free Hamiltonian that describes two types of particles, A particles and B particles, both of which have mass m:

$$H_0 = \frac{1}{2}\pi_a^2 + \frac{m^2}{2}\phi_a^2 + \frac{1}{2}\pi_b^2 + \frac{m^2}{2}\phi_b^2 = ma^\dagger a + mb^\dagger b$$

where

$$\begin{aligned} \phi_a &= (2m)^{1/2}(a + a^\dagger) \\ \phi_b &= (2m)^{1/2}(b + b^\dagger) \end{aligned}$$

Suppose we introduce an interaction

$$H_i = \frac{\lambda}{4}(\phi_a^2 + \phi_b^2)^2 = g((a + a^\dagger)^2 + (b + b^\dagger)^2)^2$$

where

$$g = \frac{\lambda}{16m^2}$$

so the total Hamiltonian is

$$H(t) = H_0 + f(t)\, H_i$$

In the far past and far future $f(t) \to 0$, so the eigenstates of $H(t)$ are just the eigenstates of H_0. The lowest few eigenstates include the ground state $|0,0\rangle$ with energy 0, two single particle states $|1,0\rangle$ and $|0,1\rangle$ with energy m, and three two-particle states $|2,0\rangle$, $|1,1\rangle$, and $|0,2\rangle$ with energy $2m$.

As the interaction is turned on, the ground state $|0,0\rangle$ adiabatically evolves into the interacting ground state $|\Omega\rangle$, and the two single particle states $|1,0\rangle$ and $|0,1\rangle$ adiabatically evolve into dressed

particle states. To first order in g, the energy of the interacting ground state is

$$\langle 0,0|H_0 + H_i|0,0\rangle = 8g$$

and the energy of the dressed particle states is

$$\langle 1,0|H_0 + H_i|1,0\rangle = \langle 0,1|H_0 + H_i|0,1\rangle = m + 24g$$

Thus, the renormalized mass for the particles is $m_R = m + 16g$.

In the far past and far future, the eigenstates $|1,1\rangle$, $|2,0\rangle$, and $|0,2\rangle$ are all degenerate. The state $|1,1\rangle$ remains an eigenstate of H when the interaction is turned on, so there is no AB scattering. However, the interaction Hamiltonian couples $|2,0\rangle$ and $|0,2\rangle$, which leads to $AA \leftrightarrow BB$ scattering. We can calculate the scattering probability as follows. Note that

$$\begin{aligned} \langle 2,0|H_i|2,0\rangle &= \langle 0,2|H_i|0,2\rangle &= 52g \\ \langle 0,2|H_i|2,0\rangle &= \langle 2,0|H_i|0,2\rangle &= 4g \end{aligned}$$

Thus, to first order in g, the eigenstates corresponding to $\{|2,0\rangle, |0,2\rangle\}$ are

$$|\pm\rangle = \frac{1}{\sqrt{2}}(|2,0\rangle \pm |0,2\rangle)$$

and the eigenvalues are

$$E_\pm = 4g(13 \pm 1)$$

Thus, on the subspace spanned by $\{|2,0\rangle, |0,2\rangle\}$, the Hamiltonian can be expressed as

$$\begin{aligned} H &= 2m + f(t)\left(E_+|+\rangle\langle +| + E_-|-\rangle\langle -|\right) \\ &= 2m + \frac{1}{2}f(t)(E_+ + E_-) + \frac{1}{2}f(t)(E_+ - E_-)(|2,0\rangle\langle 0,2| + |0,2\rangle\langle 2,0|) \end{aligned}$$

Thus, if we start with $|2,0\rangle$ in the distant past, then the probability of finding $|0,2\rangle$ in the distant future is

$$p(AA \to BB) = \sin^2\theta$$

where

$$\theta = \frac{1}{2}(E_+ - E_-)\int f(t)\,dt = 4g\int f(t)\,dt$$

4.4.6 Particle production

One type of scattering that can occur in $0+1$ dimensions is particle production from a classical source. The Hamiltonian for the system is:

$$H = H_0 + H_i$$

where H_0 describes the free field:

$$H_0 = \frac{1}{2}\pi^2 + \frac{1}{2}m^2\phi^2$$

and H_i describes the coupling to a classical source $\rho(t)$:

$$H_i = \rho(t)\,\phi$$

I will assume that the system starts in some initial state at time 0 and evolves to time T, and that the $\rho(t)$ is nonzero only for times between t_a and t_b, where $0 < t_a < t_b < T$. We can view the evolution of the system as a scattering process: before t_a and after t_b the field evolves freely, while for times between t_a and t_b the field scatters off the classical source. We would like to solve for the state of the system at time T in terms of the state at time 0. Note that this is the same problem we treated in section 4.2.4, only the classical field has been replaced with a quantum field.

It is convenient to work in the interaction picture. The interaction picture wavefunction is

$$|\Phi_I\rangle = e^{iH_0t}|\Phi_S\rangle$$

and the interaction picture Hamiltonian is

$$H_I = e^{iH_0t}\,H_i\,e^{-iH_0t} = \rho(t)\phi_I(t)$$

The equation of motion is

$$i\partial_t|\Phi_I(t)\rangle = H_I(t)|\Phi_I(t)\rangle$$

Thus, the interaction picture wavefunction at time t is given by

$$|\Phi_I(t)\rangle = U(t)|\Phi_I(0)\rangle$$

where $U(t)$ is defined by

$$i\partial_t U(t) = H_I(t)\,U(t)$$

The solution to this equation, subject to the initial condition $U(0) = 1$, can be expressed as

$$U(t) = T[\exp(-i \int_0^t H_I(t')\, dt')]$$

We are interested in the scattering matrix for the system, which is given by

$$U(T) = T[\exp(-i \int_0^T H_I(t)\, dt)]$$

Since $H_I(t) = 0$ for $t < t_a$ and for $t > t_b$, we can extend the limits of integration to $\pm\infty$:

$$U(T) = T[\exp(-i \int H_I(t)\, dt)]$$

I will now apply Wick's theorem (see following section) to evaluate the scattering matrix. There are only two connected diagrams that occur:

$$A = -i \int H_I(t)\, dt = -i \int \rho(t)\phi_I(t)\, dt$$

with symmetry factor $S(A) = 1$, and

$$B = -\iint \rho(t_1)\rho(t_2)D_F(t_1 - t_2)\, dt_1\, dt_2$$

with symmetry factor $S(B) = 2$. Thus, using Wick's theorem,

$$U(T) = N[\exp(A/S(A) + B/S(B))] = N[\exp(A + B/2)]$$

We can evaluate the diagrams by introducing the Fourier transforms of the classical source and the propagator:

$$\begin{aligned} \rho(t) &= (2\pi)^{-1} \int \tilde{\rho}(\omega)\, e^{-i\omega t}\, d\omega \\ D_F(t) &= (2\pi)^{-1} \int \tilde{D}_F(\omega)\, e^{-i\omega t}\, d\omega \end{aligned}$$

Since $\rho(t)$ is real,

$$\tilde{\rho}(-\omega) = \tilde{\rho}^*(\omega)$$

Diagram A is

$$A = -i(2m)^{-1/2} \int \rho(t)(a\, e^{-imt} + a^\dagger\, e^{imt}) = \alpha a^\dagger - \alpha^* a$$

where I have defined

$$\alpha = -i(2m)^{-1/2} \int \rho(t)\, e^{imt}\, dt = -i(2m)^{-1/2}\, \tilde{\rho}(m)$$

Diagram B is

$$B = -(2\pi)^{-1} \int |\tilde{\rho}(\omega)|^2\, \tilde{D}_F(\omega)\, d\omega$$

Note that

$$\tilde{D}_F(\omega) = i(\omega^2 - m^2 + i\epsilon)^{-1} = i(2m)^{-1}\left((\omega - m + i\epsilon)^{-1} - (\omega + m - i\epsilon)^{-1}\right)$$

and

$$(x + i\epsilon)^{-1} = P\frac{1}{x} - i\pi\, \delta(x)$$

Thus, the real and imaginary parts of $\tilde{D}_F(\omega)$ are

$$\begin{aligned} Re\, \tilde{D}_F(\omega) &= \pi(2m)^{-1}(\delta(\omega - m) + \delta(\omega + m)) \\ Im\, \tilde{D}_F(\omega) &= (2m)^{-1} P((\omega - m)^{-1} + (\omega + m)^{-1}) \end{aligned}$$

So the real part of B is

$$Re\, B = -(4m)^{-1}(|\tilde{\rho}(m)|^2 + |\tilde{\rho}(-m)|^2) = -|\alpha|^2$$

Thus,

$$B = -|\alpha|^2 - i\theta$$

where

$$\theta = (2\pi)^{-1} \int |\tilde{\rho}(\omega)|^2\, Im\, \tilde{D}_F(\omega)\, d\omega$$

Thus, the scattering matrix is

$$U(T) = e^{-i\theta/2}\, D(\alpha)$$

where $D(\alpha)$, the displacement operator, is

$$D(\alpha) = e^{-|\alpha|^2/2}\, e^{\alpha a^\dagger}\, e^{-\alpha^* a}$$

One can show that $D(\alpha)$ can also be written in the form

$$D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a)$$

4.4.7 Wick's theorem

We want to evaluate the scattering matrix

$$U(T) = T[\exp(-i \int_0^T H_I(t)\, dt)]$$

If we expand the exponential in a power series, we obtain

$$U(T) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_0^T dt_1 \cdots \int_0^T dt_n\, T[H_I(t_1) \cdots H_I(t_n)]$$

Using Wick's theorem, we can express the time ordered products as normally ordered sums over all possible contractions:

$$(-i)^n\, T[H_I(t_1) \cdots H_I(t_n)] = \sum_{\alpha} N[A_\alpha^n(t_1, \cdots, t_n)]$$

where the A_α^n's are the individual terms in the sum. Thus,

$$U(T) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^T dt_1 \cdots \int_0^T dt_n \sum_{\alpha} N[A_\alpha^n(t_1, \cdots, t_n)]$$

If we integrate a term $A_\alpha^n(t_1, \cdots, t_n)$ over all the times $t_1, \cdots, t_n$, we obtain a result $\Delta(A_\alpha^n)$, which I will call the diagram corresponding to that term:

$$\Delta(A_\alpha^n) = \int_0^T dt_1 \cdots \int_0^T dt_n\, A_\alpha^n(t_1, \cdots, t_n)$$

Thus,

$$U(T) = \sum_{n=0}^{\infty} \sum_{\alpha} \frac{1}{n!} N[\Delta(A_\alpha^n)]$$

We can define an action of the symmetric group S_n on the set of terms $\{A_\alpha^n\}$ as follows. For $\pi \in S_n$, define a mapping $\pi(A_\alpha^n) = A_\beta^n$ such that A_β^n is obtained from A_α^n by permuting the times $t_1, \cdots, t_n$ according to the permutation π. The orbit of a term is the set of terms that can be obtained from it via the group action:

$$Orbit(A_\alpha^n) = \{\pi(A_\alpha^n) : \pi \in S_n\}$$

and the stabilizer of a term is the set of permutations that leave the term invariant:

$$Stab(A_\alpha^n) = \{\pi \in S_n : \pi(A_\alpha^n) = A_\alpha^n\}$$

Note that

$$|Orbit(A_\alpha^n)|\, |Stab(A_\alpha^n)| = |S_n| = n!$$

Now, all the terms in $Orbit(A_\alpha^n)$ correspond to the same diagram $\Delta(A_\alpha^n)$. Thus, if we define

$$s(\Delta(A_\alpha^n)) = |Stab(A_\alpha^n)| = \frac{n!}{|Orbit(A_\alpha^n)|}$$

we can express the sum over diagrams as

$$\sum_\alpha \Delta(A_\alpha^n) = \sum_{\{D^n\}} \frac{n!}{s(D^n)}\, D^n$$

where $\{D^n\}$ denotes the set of distinct diagrams of order n. Thus,

$$U(T) = \sum_{n=0}^{\infty} \sum_{\{D^n\}} \frac{N[D^n]}{s(D^n)} = \sum_{\{D\}} \frac{N[D]}{s(D)}$$

As an example, suppose

$$H_I = \phi_I$$

Then the sum over the third order terms is

$$\sum_\alpha A_\alpha^3 = (-i)^3\, T[\phi_1\phi_2\phi_3] = iN[\phi_1\phi_2\phi_3 + D_{12}\phi_3 + D_{23}\phi_2 + D_{31}\phi_1]$$

Note that

$$Orbit(i\phi_1\phi_2\phi_3) = \{i\phi_1\phi_2\phi_3\}$$

$$Stab(i\phi_1\phi_2\phi_3) = S_3$$

and

$$Orbit(iD_{12}\phi_3) = \{iD_{12}\phi_3, iD_{23}\phi_1, iD_{31}\phi_2\}$$

$$Stab(iD_{12}\phi_3) = \{(), (12)\}$$

Thus,

$$\sum_\alpha \Delta(A_\alpha^3) = \Delta(i\phi_1\phi_2\phi_3) + 3\Delta(iD_{12}\phi_3)$$

Suppose a diagram D factors into disconnected diagrams:

$$D = \prod_k D_k^{n_k}$$

Then

$$s(D) = \prod_k s(D_k)^{n_k}\, n_k!$$

Thus,

$$U(T) = N[\exp(\sum_{\{D_k\}} D_k/s(D_k))]$$

where the sum is taken over all connected diagrams $\{D_k\}$.

4.5 Path integrals

4.5.1 QFT at finite temperature

Consider a neutral scalar field ϕ and a neutral fermion field ψ. At zero temperature, the Feynman propagators for the fields are given by

$$G(t_1 - t_2) = \begin{cases} -i\langle 0|T[\phi(t_1)\phi(t_2)]|0\rangle & \text{(for bosons)} \\ \\ -i\langle 0|T[\psi(t_1)\bar{\psi}(t_2)]|0\rangle & \text{(for fermions)} \end{cases}$$

One can also define retarded propagators, which are given by

$$G_R(t_1 - t_2) = \begin{cases} -i\theta(t)\,\langle 0|[\phi(t_1)\phi(t_2)]|0\rangle & \text{(for bosons)} \\ \\ -i\theta(t)\,\langle 0|\{\psi(t_1)\bar{\psi}(t_2)\}|0\rangle & \text{(for fermions)} \end{cases}$$

At finite temperature, the Feynman propagators generalize to

$$G(t_1 - t_2) = \begin{cases} -iTr[\rho_{ss}T[\phi(t_1)\phi(t_2)]] & \text{(for bosons)} \\ \\ -iTr[\rho_{ss}T[\psi(t_1)\bar{\psi}(t_2)]] & \text{(for fermions)} \end{cases}$$

and the retarded propagators generalize to

$$G_R(t_1 - t_2) = \begin{cases} -i\theta(t)\,Tr[\rho_{ss}[\phi(t_1)\phi(t_2)]] & \text{(for bosons)} \\ \\ -i\theta(t)\,Tr[\rho_{ss}\{\psi(t_1)\bar{\psi}(t_2)\}] & \text{(for fermions)} \end{cases}$$

where

$$\rho_{ss} = Z^{-1}\,e^{-\beta H}$$

is the steady state density matrix, and

$$Z = Tr[e^{-\beta H}]$$

is the partition function. The retarded propagators are related to the Feynman propagators by [46]

$$\tilde{G}_R(\omega) = \begin{cases} Re\,\tilde{G}(\omega) + i\tanh(\beta\omega/2)\,Im\,\tilde{G}(\omega) & \text{(for bosons)} \\ \\ Re\,\tilde{G}(\omega) + i\coth(\beta\omega/2)\,Im\,\tilde{G}(\omega) & \text{(for fermions)} \end{cases}$$

where

$$\begin{aligned} \tilde{G}(\omega) &= \int G(t)\,e^{i\omega t}\,dt \\ \tilde{G}_R(\omega) &= \int G_R(t)\,e^{i\omega t}\,dt \end{aligned}$$

At finite temperatures, it is convenient to define temperature Greens functions by

$$D(\tau_t - \tau_2) = \begin{cases} Tr[\rho_{ss}T[\phi(\tau_1)\phi(\tau_2)]] & \text{(for bosons)} \\ \\ Tr[\rho_{ss}T[\psi(\tau_1)\bar{\psi}(\tau_2)]] & \text{(for fermions)} \end{cases}$$

where for an arbitrary operator A, we define $A(\tau)$ by

$$A(\tau) = e^{H\tau}\,A\,e^{-H\tau}$$

The temperature Greens functions are related to the retarded Greens functions by

$$D(\omega_n) = -G_R(i\omega_n)$$

One can show that for $-\beta < \tau < 0$, $D(\tau+\beta) = \pm D(\tau)$, where the plus sign is for bosons and the minus sign is for fermions. Also, for neutral fields, $D(\tau) = D(-\tau)$ (the proof of these results can be found in [46], for example).

Since $D(\tau)$ is periodic on $[-\beta, \beta]$, its Fourier transform is defined at discrete frequencies ω_s, where

$$\omega_s = \frac{\pi}{\beta}s$$

so we can express $D(\tau)$ as

$$D(\tau) = \frac{1}{2\beta}\sum_s \tilde{D}_s\,e^{-i\omega_s\tau}$$

which can be inverted to give

$$\tilde{D}_s = \int_{-\beta}^{\beta} D(\tau)\,e^{i\omega_s\tau}\,d\tau$$

For bosons, $\tilde{D}_s = 0$ for odd s, while for fermions, $\tilde{D}_s = 0$ for even s.

For free bosons the causal Greens function is

$$G(t) = -i(2m)^{-1}((\bar{n}+1)\,e^{-im|t|} + \bar{n}\,e^{im|t|})$$

$$\begin{aligned}\tilde{G}(\omega) &= (\bar{n}+1)(2m)^{-1}((\omega - m + i\epsilon)^{-1} - (\omega + m - i\epsilon)^{-1}) + \\ &\quad \bar{n}(2m)^{-1}((\omega + m + i\epsilon)^{-1} - (\omega - m - i\epsilon)^{-1})\end{aligned}$$

where the mean photon number $\bar{n}$ is

$$\bar{n} = \frac{e^{-\beta m/2}}{2\sinh(\beta m/2)}$$

The retarded Greens function is

$$\begin{aligned}G_R(t) &= -\frac{1}{m}\theta(t)\sin mt \\ \tilde{G}_R(\omega) &= (2m)^{-1}((\omega - m + i\epsilon)^{-1} - (\omega + m + i\epsilon)^{-1})\end{aligned}$$

and the temperature Greens function is (see Figure 4.7)

$$\begin{aligned}D(\tau) &= (2m)^{-1}\left((\bar{n}+1)\,e^{-m|\tau|} + \bar{n}\,e^{m|\tau|}\right) \\ \tilde{D}(\omega_s) &= (\omega_s^2 + m^2)^{-1}\end{aligned}$$

For free fermions, the causal Greens function at finite temperature is

$$G(t) = -i(p_+\,e^{-im|t|} - p_-\,e^{im|t|})$$

$$\begin{aligned}\tilde{G}(\omega) &= p_+((\omega - m + i\epsilon)^{-1} - (\omega + m - i\epsilon)^{-1}) - \\ &\quad p_-((\omega + m + i\epsilon)^{-1} - (\omega - m - i\epsilon)^{-1})\end{aligned}$$

where

$$p_\pm = \frac{e^{\pm\beta m/2}}{2\cosh(\beta m/2)}$$

The retarded Greens function is

$$\begin{aligned}G_R(t) &= -2\theta(t)\sin mt \\ \tilde{G}_R(\omega) &= (\omega - m + i\epsilon)^{-1} - (\omega + m + i\epsilon)^{-1}\end{aligned}$$

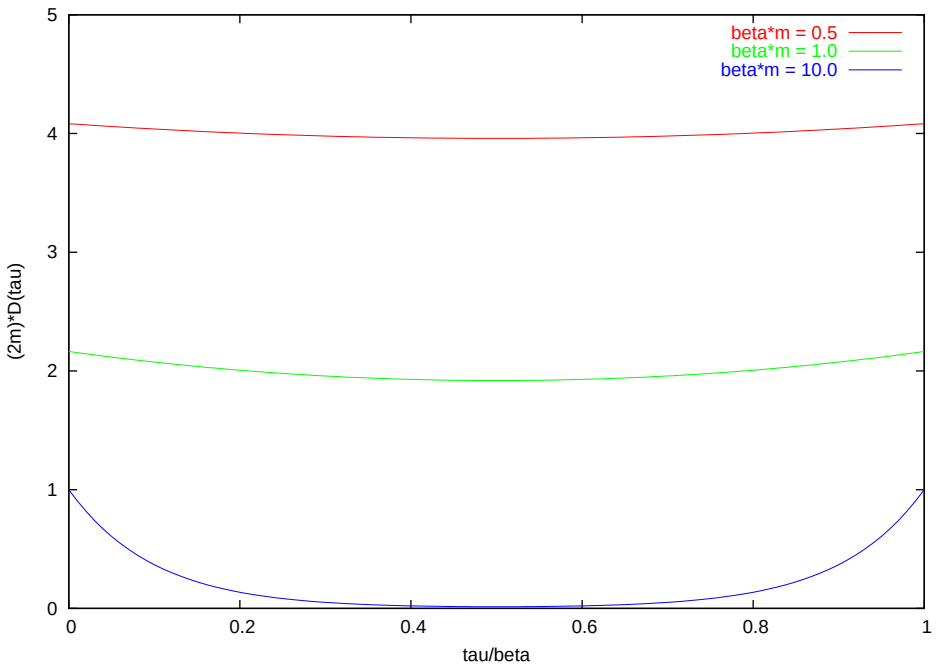

Figure 4.7: Temperature Greens function $D(\tau)$ for bosons. Curves are shown for $\beta m = 0.5, 1, 10$.

and the temperature Greens function is (see Figure 4.8):

$$\begin{aligned} D(\tau) &= p_+ \, e^{-m|\tau|} - p_- \, e^{m|\tau|} \\ \tilde{D}(\omega_s) &= 2m(\omega_s^2 + m^2)^{-1} \end{aligned}$$

4.5.2 Boson lattice path integral

The temperature Greens function for neutral bosons is

$$D(\tau) = Tr[\rho_{ss} T[\phi(\tau)\phi(0)]] = Z^{-1} \, Tr[e^{-(\beta-\tau)H} \, \phi \, e^{-\tau H} \, \phi]$$

where Z, the partition function, is

$$Z = Tr[e^{-\beta H}] = \int \langle \phi | e^{-\beta H} | \phi \rangle \, d\phi$$

Because $D(\tau + \beta) = D(\tau)$ for $\tau < 0$, we only need to consider $\tau \in [0, \beta]$. We can chop the interval $[0, \beta]$ into N segments of length $\Delta = \beta/N$, and insert a complete set of states after each segment. Then the temperature Greens function becomes

$$D(\tau) = Z^{-1} \left(\prod_{n=0}^{N-1} \int d\phi_n \right) \phi_r \, \phi_0 \left(\prod_{n=0}^{N-1} T(\phi_{n+1}, \phi_n) \right)$$

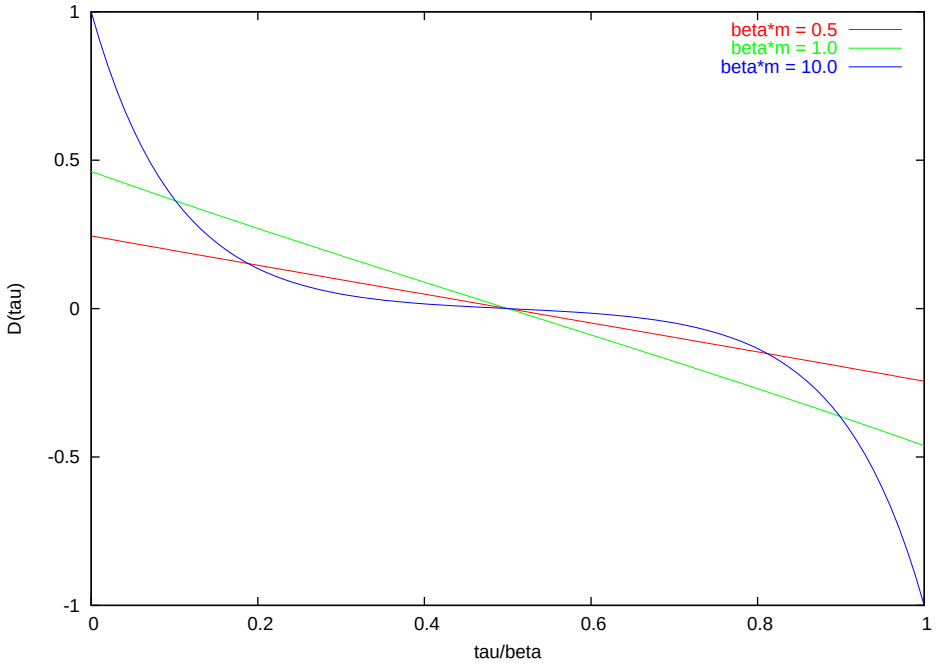

Figure 4.8: Temperature Greens function $D(\tau)$ for fermions. Curves are shown for $\beta m = 0.5, 1, 10$.

and the partition function becomes

$$Z = \left(\prod_{n=0}^{N-1} \int d\phi_n \right) \left(\prod_{n=0}^{N-1} T(\phi_{n+1}, \phi_n) \right)$$

where $\phi_N \equiv \phi_0$, $r/N = \tau/\beta$, and T, the transfer matrix, is

$$T(\phi_a, \phi_b) = \langle \phi_a | e^{-\Delta H} | \phi_b \rangle = \langle \phi_a | 1 - \Delta H | \phi_b \rangle$$

Assume H has the form

$$H = \frac{1}{2}\pi^2 + V(\phi)$$

Substituting H into the transfer matrix, we obtain

$$T(\phi_a, \phi_b) = \langle \phi_a | 1 - \Delta(\frac{1}{2}\pi^2 + V((\phi_a + \phi_b)/2)) | \phi_b \rangle$$

The operator π has eigenstates $\{|p\rangle\}$, which are given by

$$\langle \phi | p \rangle = (2\pi)^{-1/2} \, e^{ip\phi}$$

By inserting a complete set of states $\{|p\rangle\}$, we can express the transfer matrix as

$$\begin{aligned} T(\phi_a,\phi_b) &= \int \langle\phi_a|p\rangle\,[1-\Delta\,(\frac{1}{2}p^2+V((\phi_a+\phi_b)/2))]\,\langle p|\phi_b\rangle\,dp \\ &= (2\pi)^{-1}\int \exp[-\frac{\Delta}{2}p^2+ip(\phi_a-\phi_b)-\Delta V((\phi_a+\phi_b)/2)]\,dp \\ &= (2\pi\Delta)^{-1/2}\exp(-\frac{1}{2\Delta}(\phi_a-\phi_b)^2-\Delta\,V((\phi_a+\phi_b)/2)) \end{aligned}$$

Substituting this expression for the transfer matrix into the partition function, we find

$$Z=(2\pi\Delta)^{-N/2}\left(\prod_{n=0}^{N-1}\int d\phi_n\right)\exp(-\Delta\sum_{n=0}^{N-1}(\frac{1}{2\Delta^2}(\phi_{n+1}-\phi_n)^2+V((\phi_{n+1}+\phi_n)/2)))$$

Thus, we may view the partition function as a weighted sum over all possible field configurations. In the continuum limit, the sum over field configurations becomes a path integral: the partition function is given by

$$Z=\int D[\phi]\,\exp(\int_0^\beta L[\phi(\tau)]\,d\tau)$$

where L is the Lagrangian in complex time $\tau=it$:

$$L[\phi(\tau)]=-\frac{1}{2}(\partial_\tau\phi)^2-V(\phi)$$

I now want to return to the discrete case and work out explicitly the case of a free field, for which

$$V(\phi)=\frac{1}{2}m^2\phi^2$$

The partition function for the free field is

$$Z=(2\pi\Delta)^{-N/2}\left(\prod_{n=0}^{N-1}\int d\phi_n\right)\exp(-\Delta\sum_{n=0}^{N-1}(\frac{1}{2\Delta^2}(\phi_{n+1}-\phi_n)^2+\frac{m^2}{8}(\phi_{n+1}+\phi_n)^2))$$

It is convenient to introduce dimensionless variables $q_n=m^{1/2}\,\phi_n$, $\epsilon=m\Delta=\beta m/N$. We can then express the partition function as

$$Z=(2\pi\epsilon)^{-N/2}\left(\prod_{n=0}^{N-1}\int dq_n\right)\exp(-F[\{q_n\}])$$

where

$$F[\{q_n\}]=\frac{1}{4}\sum_n[\frac{2}{\epsilon}(q_{n+1}-q_n)^2+\frac{\epsilon}{2}(q_{n+1}+q_n)^2]$$

Similarly, the temperature Greens function is

$$D(\tau) = \frac{1}{m} \langle q_r q_0 \rangle$$

where I have defined a correlation function

$$\langle q_r q_0 \rangle = Z^{-1} (2\pi\epsilon)^{-N/2} \left(\prod_{n=0}^{N-1} \int dq_n \right) q_r \, q_0 \, \exp(-F[\{q_n\}])$$

I will introduce a parameter x, related to ϵ by

$$\epsilon = 2 \tanh x/2$$

Then

$$\begin{aligned} F[\{q_n\}] &= \frac{1}{4} \sum_n [(\coth x/2) \, (q_{n+1} - q_n)^2 + (\tanh x/2) \, (q_{n+1} + q_n)^2] \\ &= \frac{1}{4} \sum_n [2(\coth x/2 + \tanh x/2) \, q_n^2 + (\coth x/2 - \tanh x/2)(q_{n+1} + q_{n-1}) \, q_n] \\ &= \frac{1}{2} \, q \cdot A \cdot q \end{aligned}$$

where

$$A_{ij} = (\sinh x)^{-1} (2 \cosh x \, \delta_{ij} - \delta_{i,j+1} - \delta_{i,j-1})$$

Thus, the partition function is

$$Z = (2\pi)^{-N/2} \, (2 \tanh x/2)^{-N/2} \left(\prod_{n=0}^{N-1} \int dq_n \right) \exp(-\frac{1}{2} \, q \cdot A \cdot q)$$

The integrals can be performed by using the following result, which holds for any symmetric matrix B_{ij} and any vector J_i (see [47], for example):

$$\left(\prod_{n=0}^{N-1} \int dq_n \right) \exp(-\frac{1}{2} \, q \cdot B \cdot q + J \cdot q) = (2\pi)^{N/2} \, (\det B)^{-1/2} \, \exp(-\frac{1}{2} \, J \cdot B^{-1} \cdot J)$$

Using this result, we find that the partition function is

$$Z = (2 \tanh x/2)^{-N/2} \, (\det A)^{-1/2}$$

and the correlation function is

$$\langle q_i q_j \rangle = (A^{-1})_{ij}$$

The determinant and inverse of A_{ij} can by calculated by Fourier transforming the field variables q_i. For simplicity, I will assume that $N = 2R + 1$ is odd. Then

$$q_n = N^{-1/2} \sum_{m=-R}^{R} e^{2\pi i n m/N} u_m$$

Because q_n is real, $u_{-m} = u_m^*$, so we can express u_m in terms of real valued variables c, $a_1, \cdots, a_R$, $b_1, \cdots, b_R$ as follows:

$$\begin{aligned} u_{+m} &= (1/\sqrt{2})(a_m - i b_m) \\ u_0 &= c \\ u_{-m} &= (1/\sqrt{2})(a_m + i b_m) \end{aligned}$$

where m is always taken to be positive in the above relations. We obtain an orthogonal transformation from the variables q_n to the variables c, a_m, b_m:

$$q_n = N^{-1/2}[c + \sqrt{2} \sum_{m=1}^{R} (\cos(2\pi n m/N)\, a_m + \sin(2\pi n m/N)\, b_m)]$$

Because the transformation is orthogonal, it preserves the normalization:

$$\sum_{n=0}^{N-1} q_n^2 = \sum_{m=-R}^{R} |u_m|^2 = c^2 + \sum_{m=1}^{R} (a_m^2 + b_m^2)$$

The transformation diagonalizes A_{ij}:

$$q \cdot A \cdot q = \sum_{m=-R}^{R} (2/\sinh x)(\cosh x - \cos(2\pi m/N))|u_m|^2$$

Thus,

$$\det A = \prod_{m=-R}^{R} (2/\sinh x)(\cosh x - \cos(2\pi m/N))$$

Also,

$$q \cdot A^{-1} \cdot q = \sum_{m=-R}^{R} \frac{|u_m|^2}{(2/\sinh x)(\cosh x - \cos(2\pi m/N))}$$

Thus, the inverse of A is given by

$$(A^{-1})_{ij} = \frac{1}{N} \sum_{m=-R}^{R} \frac{\cos(2\pi m(i-j)/N)}{(2/\sinh x)(\cosh x - \cos(2\pi m/N))}$$

Using the identity (see [48] for a proof of this result)

$$\prod_{m=0}^{N-1} 2(\cosh x - \cos(2\pi m/N)) = 4\sinh^2(Nx/2)$$

we can express the partition function as

$$Z = (2\tanh x/2)^{-N/2}\,(\det A)^{-1/2} = \frac{\cosh^N(x/2)}{2\sinh(Nx/2)}$$

We can check that this gives the correct continuum limit by taking $N \to \infty$:

$$Z \to (2\sinh(\beta m/2))^{-1}$$

To evaluate the correlation function, we can use the identity (see [48] for a proof of this result)

$$\frac{1}{N}\sum_{m=-R}^{R} \frac{\cos(2\pi n m/N)}{\cosh x - \cos(2\pi m/N)} = \frac{\cosh((N/2-n)x)}{\sinh x\,\sinh(Nx/2)}$$

Thus, the correlation function is

$$\langle q_i q_j\rangle = (A^{-1})_{ij} = \frac{\cosh((N/2-|i-j|)x)}{2\sinh(Nx/2)}$$

We can check that this gives the correct temperature Greens function in the continuum limit by taking $N \to \infty$:

$$D(\tau) = \frac{1}{m}\,\langle q(\tau)q(0)\rangle = \frac{1}{2m}((\bar{n}+1)\,e^{-m\tau} + \bar{n}\,e^{m\tau})$$

where

$$\bar{n} = \frac{e^{-\beta m/2}}{2\sinh(\beta m/2)}$$

I now want to show how a renormalization group transformation can be carried out analytically for the boson path integral. Note that we can express the partition function as

$$Z = \left(\prod_{n=0}^{N-1}\int dq_n\right)\left(\prod_{n=0}^{N-1} T_\epsilon(q_{n+1},q_n)\right)$$

where the transfer matrix is

$$T_\epsilon(q_a,q_b) = (2\pi\epsilon)^{-1/2}\exp[-\frac{1}{2\epsilon}(q_a-q_b)^2 - \frac{\epsilon}{8}(q_a+q_b)^2]$$

Or, in terms of x,

$$\begin{aligned} & T_x(q_a, q_b) \\ &= (2\pi)^{-1/2}(2\tanh x/2)^{-1/2}\exp(-\frac{1}{4}((\coth x/2)\,(q_a-q_b)^2+(\tanh x/2)\,(q_a+q_b)^2)) \\ &= (2\pi)^{-1/2}(2\tanh x/2)^{-1/2}\exp(-(\sinh x)^{-1}(\frac{1}{2}(\cosh x)(q_a^2+q_b^2)-q_aq_b)) \end{aligned}$$

It is straightforward to check that

$$\int T_x(q_a,q)\,T_x(q,q_b)\,dq = \frac{\tanh x}{2\tanh x/2}\,T_{2x}(q_a,q_b)$$

Thus, if we start with a lattice of $2N$ sites and perform the q_n integrals for odd n, we get

$$\begin{aligned} & Z[2N,x] \\ &= \left(\prod_{n=0}^{2N-1}\int dq_n\right) T_x(q_0,q_1)\,T_x(q_1,q_2)\cdots T_x(q_{N-1},q_0) \\ &= \left(\frac{\tanh x}{2\tanh x/2}\right)^N \left(\prod_{n\ \text{even}}^{2N-1}\int dq_n\right) T_{2x}(q_0,q_2)\,T_{2x}(q_2,q_4)\cdots T_{2x}(q_{N-2},q_0) \\ &= \left(\frac{\tanh x}{2\tanh x/2}\right)^N Z[N,2x] \end{aligned}$$

Thus, we can interpret a scale transformation $\Delta \to 2\Delta$ as a redefinition $x \to 2x$. Since we have the exact solution for Z, we can check that this works. Properties of the system are invariant under this transformation; for example, the mean squared field fluctuation is

$$\langle q_n^2\rangle = \frac{1}{2}\coth(Nx/2)$$

4.5.3 Fermion lattice path integral

The temperature Greens function for neutral fermions is

$$D(\tau) = Tr[\rho_{ss}T[\psi(\tau)\bar{\psi}(0)]] = Tr[e^{-(\beta-\tau)H}\,\psi\,e^{-\tau H}\,\bar{\psi}]/Tr[e^{-\beta H}]$$

Because $D(\tau+\beta) = -D(\tau)$ for $\tau < 0$, we only need to consider $\tau \in [0,\beta]$. Suppose we chop the interval $[0,\beta]$ into N segments, and suppose that $\tau/\beta = r/N$ for some r. Then

$$D(\tau) = Tr[U^{N-r}\psi U^r\bar{\psi}]/Tr[U^N] = Tr[U^{N-r}\sigma_x U^r\sigma_z\sigma_x]/Tr[U^N]$$

where

$$U = e^{-\beta H/N} = \mathbb{1} - \beta H/N = \mathbb{1} - (\beta m/2N)\,\sigma_z = \mathbb{1} - e^{-2J}\,\sigma_z$$

and

$$J = \frac{1}{2}\log(2N/\beta m)$$

We can relate $D(\tau)$ to the correlation function for the 1D Ising model with antiperiodic boundary conditions. The partition function for the Ising model with periodic boundary conditions ($s_N = s_0$) is

$$Z_P = Tr[T^N]$$

where

$$T = e^J + e^{-J}\sigma_x$$

The spin correlation function is

$$\langle s_i s_j \rangle_P = Z_P^{-1}\, Tr[\sigma_z T^r \sigma_z T^{N-r}]$$

For the case of antiperiodic boundary conditions ($s_N = -s_0$), the partition function is

$$Z_A = Tr[T^N \sigma_x]$$

The spin correlation function is

$$\langle s_i s_j \rangle_A = Z_A^{-1}\, Tr[\sigma_z T^r \sigma_z T^{N-r} \sigma_x]$$

Note that we can relate the unitary transformation U for the fermion path integral to the transfer matrix T for the Ising model by

$$U = e^{-J} R T R^{-1}$$

where R is a unitary transformation corresponding to a rotation of $\pi/2$ about the y-axis. The temperature Greens function is therefore

$$\begin{aligned} D(\tau) &= Tr[U^{N-r}\sigma_x U^r \sigma_z \sigma_x]/Tr[U^N] \\ &= -Tr[T^{N-r}\,(R^{-1}\sigma_x R)\,T^r\,(R^{-1}\sigma_z\sigma_x R)]/Tr[T^N] \\ &= -Tr[T^{N-r}\sigma_z T^r \sigma_x \sigma_z]/Tr[T^N] \\ &= -(Z_A/Z_P)\,\langle s_r s_0 \rangle_A \end{aligned}$$

4.5.4 Diffusion

Another way to think about path integrals is in terms of diffusion.

Consider a particle on a one dimensional lattice that randomly hops from site to site in discrete timesteps, such that if it is at lattice site n for timestep N then it will have equal chances of landing at sites $n+1$ and $n-1$ for timestep $N+1$. We can define a propagator $K(n_a, n_b; N)$ that gives the probability for the particle to move from site n_b to site n_a after N timesteps:

$$K(n_a, n_b, N) = 2^{-N}\, c(n_a - n_b, N)$$

where $c(n, N)$ is the number of different paths the particle could take that would displace it n sites in N timesteps. Note that $c(n, N)$ can be obtained from the generating function

$$(x + x^{-1})^N = \sum_{n=-N}^{N} c(n, N)\, x^n$$

Thus,

$$c(n, N) = \frac{N!}{((N-n)/2)!\,((N+n)/2)!} \simeq 2^N\, (2\pi N)^{-1/2}\, e^{-n^2/2N}$$

where I have applied Stirling's approximation:

$$\log n! \sim n \log n - n + \frac{1}{2} \log(2\pi n)$$

Thus,

$$K(n_a, n_b; N) \simeq (2\pi N)^{-1/2}\, e^{-|n_a - n_b|^2/2N}$$

Now consider an ensemble of independently random walking particles, and let $\rho_n(N)$ be the number of particles at lattice site n after N timesteps. Note that

$$\rho_n(N+1) = \frac{1}{2}(\rho_{n+1}(N) + \rho_{n-1}(N))$$

Thus,

$$\rho_n(N+1) - \rho_n(N) = \frac{1}{2}\, \rho''_n(N)$$

where

$$\rho''_n(N) = \rho_{n+1}(N) - 2\rho_n(N) + \rho_{n-1}(N)$$

This can be thought of as a discrete field equation for $\rho_n(N)$. The propagator is the Greens function for this field equation:

$$\rho_n(N) = \sum_r K(n, r; N)\, \rho_r(0)$$

Suppose the distance between adjacent lattice sites is a, and the duration of a single timestep is Δ. If we view the system on distance scales that are large compared to a and on timescales that are

long compared to Δ, then we can approximate the lattice by a continuum. The continuum theory describes the evolution of the particle density $\rho(\tau, x)$, which is related to $\rho_n(k)$ by

$$\begin{aligned} \rho_n(N) &\rightarrow \rho(\tau, x) \\ \rho_n(N+1) - \rho_n(N) &\rightarrow \Delta\, \partial_\tau \rho(\tau, x) \\ \rho_n''(N) &\rightarrow a^2\, \partial_x^2 \rho(\tau, x) \end{aligned}$$

Thus, we obtain a field equation for ρ:

$$\partial_\tau \rho(\tau, x) = D\, \partial_x^2 \rho(\tau, x)$$

This is the diffusion equation, where $D = a^2/2\Delta$ is the diffusion constant. The propagator for the continuum theory is

$$K(x, y; \tau) = (4\pi D\tau)^{-1/2}\, e^{-(x-y)^2/4D\tau}$$

and

$$\rho(\tau, x) = \int K(x, y; \tau)\, \rho(0, y)\, dy$$

We can relate the diffusion equation to the Schrödinger equation by setting $\tau = it$:

$$i\partial_t \psi(t, x) = -\frac{1}{2m}\, \partial_x^2 \psi(t, x) \quad \rightarrow \quad \partial_\tau \psi(t, x) = D\, \partial_x^2 \psi(t, x)$$

where $D = 1/2m$ is the diffusion constant. We can define a propagator for the quantum particle, which gives the amplitude for the particle to propagate from point x_b to point x_a in time t:

$$K(x_a, x_b; t) = \langle x_a | e^{-iHt} | x_b \rangle$$

So

$$\langle x_a | \psi(t) \rangle = \int K(x_a, x_b; t)\, \langle x_b | \psi(0) \rangle\, dx_b$$

For a free particle, the propagator is

$$K(x_a, x_b; t) = (m/2\pi it)^{1/2}\, e^{imx^2/2t}$$

This can be obtained from the propagator for the diffusion equation by substituting $D = 1/2m$, $\tau = it$.

Diffusion can be viewed as a sum over all possible trajectories that an individual particle could follow; similarly, the time evolution of a quantum particle can be viewed as a sum over all possible

quantum trajectories. If we divide the time interval t into N segments of duration Δ, and proceed as for the boson path integral, we can express the propagator as

$$K(x_a, x_b; t) = (m/2\pi i\Delta)^{N/2} \left(\prod_{n=1}^{N-1} \int dx_n \right) e^{iS[\{x_n\}]}$$

where $x_0 = x_b$, $x_N = x_a$, and

$$S[\{x_n\}] = \Delta \sum_n \left(\frac{m}{2\Delta^2} (x_{n+1} - x_n)^2 - V((x_{n+1} + x_n)/2) \right)$$

As $N \to \infty$,

$$K(x_a, x_b; t) = \int D[x(t)]\, e^{iS[x(t)]}$$

where

$$S[x(t)] = \int_0^t L(\dot{x}, x)\, dt$$

is the classical action, and

$$L(\dot{x}, x) = \frac{m}{2}\dot{x}^2 - V(x)$$

is the classical Lagrangian. The functional integral is over all paths such that $x(0) = x_0$ and $x(t) = x_N$. It is defined by

$$\int D[x(t)] = \lim_{N\to\infty} \left(\prod_{n=1}^{N-1} \int dx_n \right)$$

4.6 1D Ising model

We have seen that there is a correspondence between the fermion lattice path integral and the 1D Ising model. In this section I review the 1D Ising model and calculate the partition function, the magnetization, and the correlation function. Most of these results are well known (see [49], for example). I also show that the Ising model is equivalent to a two-level quantum system.

4.6.1 Description of model

The 1D Ising model is a lattice of N spins $s_n = \pm 1$ with periodic boundary conditions ($s_N \equiv s_0$) described by the Hamiltonian

$$-\beta H[\{s_n\}] = J \sum_n s_n s_{n+1} + B \sum_n s_n$$

The first term couples spins to their nearest neighbors; the second term describes an interaction with an external magnetic field. We can express the Hamiltonian as

$$-\beta H[\{s_n\}] = \sum_n t(s_{n+1}, s_n)$$

where

$$t(s_a, s_b) = J s_a s_b + \frac{1}{2} B(s_a + s_b)$$

The partition function is given by

$$Z = \sum_{\{s_n\}} e^{-\beta H[\{s_n\}]} = \sum_{\{s_n\}} \prod_n e^{t(s_{n+1}, s_n)} = \sum_{\{s_n\}} \prod_n \langle s_{n+1}|T|s_n\rangle = Tr[T^N]$$

where $|\pm 1\rangle$ are orthogonal unit vectors, and the transfer matrix T is defined by

$$\langle s_a|T|s_b\rangle = e^{t(s_a, s_b)}$$

If we expand T in the Pauli spin matrices, we find

$$T = \begin{pmatrix} e^{J+B} & e^{-J} \\ e^{-J} & e^{J-B} \end{pmatrix} = e^J \cosh B + e^{-J}\,\sigma_x + e^J \sinh B\,\sigma_z$$

From this expression, we see that T can be diagonalized via a rotation about the y axis. A rotation of angle θ about the y axis is described by a unitary transformation U that has the property

$$\begin{aligned} U\sigma_x U^{-1} &= \sigma_x \cos\theta + \sigma_z \sin\theta \\ U\sigma_z U^{-1} &= \sigma_z \cos\theta - \sigma_x \sin\theta \end{aligned}$$

Thus,

$$UTU^{-1} = e^J \cosh B + (e^{-J}\cos\theta - e^J \sinh B \sin\theta)\,\sigma_x + (e^{-J}\sin\theta + e^J \sinh B \cos\theta)\,\sigma_z$$

We will choose θ so as to eliminate the σ_x term:

$$\tan\theta = (e^{2J}\sinh B)^{-1}$$

Note that

$$\begin{aligned}\cos\theta &= (\sinh^2 B + e^{-4J})^{-1/2}\sinh B \\ \sin\theta &= (\sinh^2 B + e^{-4J})^{-1/2} e^{-2J}\end{aligned}$$

So we are left with

$$UTU^{-1} = e^J \cosh B + e^J(\sinh^2 B + e^{-4J})^{1/2}\sigma_z = \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix}$$

where

$$\lambda_\pm = e^J \cosh B \pm (e^{2J}\sinh^2 B + e^{-2J})^{1/2}$$

Thus, the partition function is

$$Z = Tr[T^N] = Tr[(UTU^{-1})^N] = \lambda_+^N + \lambda_-^N$$

The magnetization per spin is given by

$$m = \langle s_i \rangle = \frac{1}{N}\frac{\partial}{\partial B}\log Z = (\lambda_+^N + \lambda_-^N)^{-1}\left(\lambda_+^{N-1}\frac{\partial\lambda_+}{\partial B} + \lambda_-^{N-1}\frac{\partial\lambda_-}{\partial B}\right)$$

Note that

$$\frac{1}{\lambda_+}\frac{\partial\lambda_+}{\partial B} = -\frac{1}{\lambda_-}\frac{\partial\lambda_-}{\partial B} = \frac{\sinh B}{(\sinh^2 B + e^{-4J})^{1/2}} = \cos\theta$$

Thus,

$$m = \left(\frac{\lambda_+^N - \lambda_-^N}{\lambda_+^N + \lambda_-^N}\right)\cos\theta$$

If J and B are finite, then in the limit of an infinite lattice ($N \to \infty$) we have that $\lambda_-/\lambda_+ \to 0$, so

$$\begin{aligned} Z &\to \lambda_+^N \\ m &\to \cos\theta = (\sinh^2 B + e^{-4J})^{-1/2}\sinh B\end{aligned}$$

The magnetization $m(J, B)$ is shown in Figure 4.9.

The susceptibility at $B = 0$ is

$$\chi = \frac{\partial}{\partial B}m(J, B)\mid_{B=0} = e^{2J}$$

Thus, although the susceptibility diverges as $J \to \infty$, it remains finite for any finite value of J, and therefore there is no phase transition.

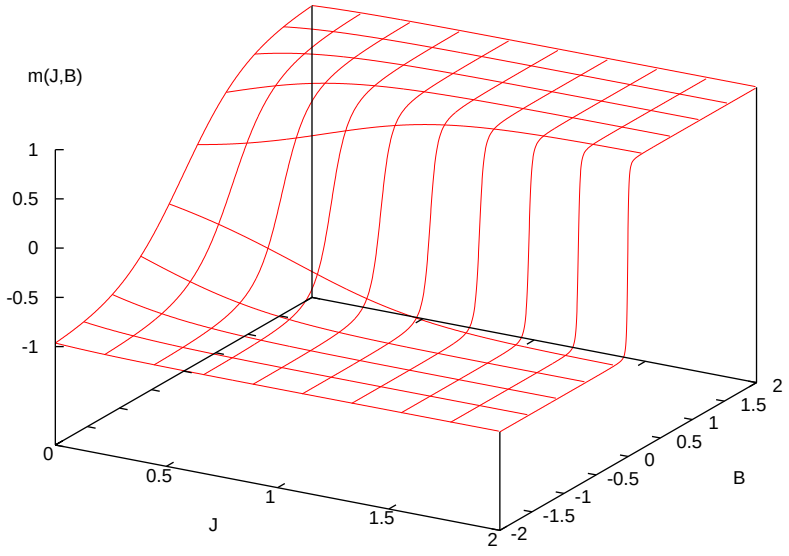

Figure 4.9: Magnetization $m(J, B)$.

4.6.2 Correlation function

The spin correlation function is

$$\langle s_i s_j \rangle = Z^{-1} \sum_{\{s_n\}} s_i s_j \, e^{-\beta H[\{s_n\}]} = Z^{-1} \sum_{\{s_n\}} s_i s_j \prod_n \langle s_{n+1}|T|s_n\rangle$$

Note that

$$\sum_{s_1} \langle s_a|T|s_1\rangle s_1 \langle s_1|T|s_b\rangle = \sum_{s_1}\sum_{s_2} \langle s_a|T|s_2\rangle \langle s_2|\sigma_z|s_1\rangle \langle s_1|T|s_b\rangle$$

Thus, we can express the correlation function as

$$\langle s_i s_j \rangle = Z^{-1} \, Tr[\sigma_z T^r \sigma_z T^{N-r}]$$

where $r = |i - j|$. We can simplify the expression for the correlation function by transforming the matrices using the matrix U from the previous section:

$$\langle s_i s_j \rangle = Z^{-1} \, Tr[(U\sigma_z U^{-1})(UTU^{-1})^r (U\sigma_z U^{-1})(UTU^{-1})^{N-r}]$$

Recall that

$$UTU^{-1} = \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix}$$

and

$$U\sigma_z U^{-1} = \sigma_z \cos\theta - \sigma_x \sin\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & -\cos\theta \end{pmatrix}$$

Thus,

$$\langle s_i s_j \rangle = \cos^2\theta + \left(\frac{\lambda_+^{N-r}\,\lambda_-^r + \lambda_-^{N-r}\,\lambda_+^r}{\lambda_+^N + \lambda_-^N} \right) \sin^2\theta$$

In the limit $N \to \infty$,

$$\langle s_i s_j \rangle = \cos^2\theta + (\lambda_-/\lambda_+)^r \sin^2\theta = m^2 + (1-m^2)\, e^{-|i-j|/r_c}$$

where r_c, the correlation length, is given by

$$r_c^{-1} = \log(\lambda_+/\lambda_-) = \log\left(\frac{\cosh B + (\sinh^2 B + e^{-4J})^{1/2}}{\cosh B - (\sinh^2 B + e^{-4J})^{1/2}} \right)$$

4.6.3 Renormalization group

I now want to consider renormalization group transformations for the Ising model. Recall that the transfer matrix is

$$T(J,B) = \begin{pmatrix} e^{J+B} & e^{-J} \\ e^{-J} & e^{J-B} \end{pmatrix}$$

Thus,

$$T^2(J,B) = 2 \begin{pmatrix} e^B \cosh(2J+B) & \cosh B \\ \cosh B & e^{-B}\cosh(2J-B) \end{pmatrix} = e^\lambda\, T(J',B')$$

where

$$\begin{aligned} J' &= \frac{1}{4}\log\left(\frac{\cosh(2J+B)\cosh(2J-B)}{\cosh^2 B} \right) \\ B' &= B + \frac{1}{2}\log\left(\frac{\cosh(2J+B)}{\cosh(2J-B)} \right) \\ e^{2\lambda} &= 16\cosh(2J+B)\cosh(2J-B)\cosh^2 B \end{aligned}$$

Thus, we may view a scale transformation from a N site Ising model to a $N/2$ site Ising model as a transformation $(J,B) \to (J',B')$ of the coupling constants. Physical properties of the system are invariant under this transformation; for example, consider the magnetization m:

$$m(J,B) = (\sinh^2 B + e^{-4J})^{-1/2} \sinh B$$

It is straightforward to check that $m(J,B) = m(J',B')$.

4.6.4 Two-level system

In this section I show that a two-level quantum system is equivalent to the 1D Ising model with an effective coupling and magnetic field.

Let us consider a finite dimensional quantum system with a basis of states $\{|s_n\rangle\}$, which is described by a Hamiltonian H_Q. The partition function for the system is

$$Z_Q = Tr[e^{-\beta H_Q}] = Tr[T^N] = \sum_{\{s_n\}} \langle s_0|T|s_{N-1}\rangle \cdots \langle s_2|T|s_1\rangle\langle s_1|T|s_0\rangle$$

where T, the transfer matrix, is given by

$$T = e^{-\beta H_Q/N} = 1 - (\beta/N)H_Q$$

If we define a matrix t by

$$\langle s_a|t|s_b\rangle = \log(\langle s_a|T|s_b\rangle)$$

then we can express the partition function as

$$Z_Q = \sum_{\{s_n\}} \exp(\sum_n \langle s_{n+1}|t|s_n\rangle)$$

We can define a classical Hamiltonian H_C for a linear chain of spins $\{s_n\}$ by

$$-\beta H_C[\{s_n\}] = -NC + \sum_n \langle s_{n+1}|t|s_n\rangle$$

where C is a constant chosen for later convenience. The Hamiltonian describes nearest neighbor couplings between the spins that are determined by the matrix elements of t. The partition function for the lattice of spins is

$$Z_C = \sum_{\{s_n\}} \exp(-\beta H_C[\{s_n\}]) = e^{-NC}\, Z_Q$$

As a specific example, consider a quantum system with two states $\{|\pm 1\rangle\}$ described by the Hamiltonian

$$-\beta H_Q = \alpha \hat{n} \cdot \vec{\sigma}$$

where $\hat{n}$ is an arbitrary unit vector. In spherical coordinates,

$$\hat{n} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$$

The transfer matrix is

$$T = 1 + \epsilon \hat{n} \cdot \vec{\sigma} = \begin{pmatrix} 1 + \epsilon \cos\theta & \epsilon \sin\theta \, e^{-i\phi} \\ \epsilon \sin\theta \, e^{i\phi} & 1 - \epsilon \cos\theta \end{pmatrix}$$

where $\epsilon = \alpha/N$. The matrix t is given by

$$\begin{aligned} t &= \begin{pmatrix} \log(1 + \epsilon \cos\theta) & \log(\epsilon \sin\theta) - i\phi \\ \log(\epsilon \sin\theta) + i\phi & \log(1 - \epsilon \cos\theta) \end{pmatrix} \\ &= \frac{1}{2}(\log(1 + \epsilon \cos\theta) + \log(1 - \epsilon \cos\theta)) + \log(\epsilon \sin\theta)\, \sigma_x + \phi\, \sigma_y + \\ &\quad \frac{1}{2}(\log(1 + \epsilon \cos\theta) - \log(1 - \epsilon \cos\theta))\, \sigma_z \end{aligned}$$

To first order in ϵ,

$$t = \log(\epsilon \sin\theta)\, \sigma_x + \phi\, \sigma_y + \epsilon \cos\theta\, \sigma_z$$

Note that

$$\begin{aligned} \langle s_a | \mathbb{1} | s_b \rangle &= \frac{1}{2}(1 + s_a s_b) \\ \langle s_a | \sigma_x | s_b \rangle &= \frac{1}{2}(1 - s_a s_b) \\ \langle s_a | \sigma_y | s_b \rangle &= -\frac{i}{2}(s_a - s_b) \\ \langle s_a | \sigma_z | s_b \rangle &= \frac{1}{2}(s_a + s_b) \end{aligned}$$

Thus,

$$\langle s_a | t | s_b \rangle = C + J s_a s_b + \frac{1}{2} B (s_a + s_b) - \frac{i}{2} \phi (s_a - s_b)$$

where

$$C = -J = \frac{1}{2} \log(\epsilon \sin\theta)$$

$$B = \epsilon \cos\theta$$

Thus, the classical Hamiltonian H_Q corresponding to the quantum Hamiltonian H_Q is

$$-\beta H_C[\{s_n\}] = J \sum_n s_n s_{n+1} + B \sum_n s_n$$

This is just the 1D Ising model with nearest neighbor coupling J and magnetic field coupling B. Recall that the magnetization per spin is

$$m = \left(\frac{\lambda_+^N - \lambda_-^N}{\lambda_+^N + \lambda_-^N} \right) \cos\theta$$

where

$$\lambda_{\pm} = e^{J} \cosh B \pm (e^{2J} \sinh^2 B + e^{-2J})^{1/2} = (1 \pm \epsilon)\, e^{-C}$$

If we substitute for B and J in terms of ϵ and θ, we find

$$m = \left(\frac{(1+\epsilon)^N - (1-\epsilon)^N}{(1+\epsilon)^N + (1-\epsilon)^N} \right) \cos\theta = \tanh\alpha \cos\theta$$

Chapter 5

Relativistic field theory in $3+1$ and $1+1$ dimensions

5.1 Introduction

Here I consider relativistic field theory in both $3+1$ and $1+1$ dimensions, and for both vector and scalar fields. I discuss the coupling of the fields to point particles and calculate the fields radiated by a moving particle. Also, I show that if the coupling to matter is chosen properly, scalar field theory in $1+1$ dimensions can be viewed as the $1+1$ dimensional analog of general relativity.

5.2 Useful results

In this section I discuss some results that are useful in working with relativistic field theories. I first discuss point particles, and then present two different ways of obtaining a conserved energy-momentum tensor from a given field Lagrangian.

5.2.1 Point particles

We can describe the motion of a point particle by specifying its trajectory $z^\mu(\lambda)$. The parameter λ is arbitrary; a transformation $\lambda \to \bar{\lambda}$ changes our description of the trajectory, but not any physical property of the particle. It is convenient to parameterize in terms of the proper time τ, which is defined by the condition

$$\eta_{\mu\nu} w^\mu w^\nu = 1$$

where

$$w^\mu \equiv \frac{dz^\mu}{d\tau}$$

If a and b are two points along the particle's trajectory, then the proper time elapsed between point a and point b is

$$\tau_a - \tau_b = \int_{\lambda_b}^{\lambda_a} (\eta_{\mu\nu} v^\mu v^\nu)^{1/2} \, d\lambda$$

where

$$v^\mu = \frac{dz^\mu}{d\lambda}$$

This gives the transformation from an arbitrary parameterization λ to a parameterization in terms of τ.

Given a trajectory $z^\mu(\tau)$ parameterized in terms of its proper time τ, we can define a number of Lorentz invariant densities. There is a scalar density

$$\rho(x) = m \int \delta^{(D)}(x - z(\tau)) \, d\tau$$

which gives the mass density in the rest frame of the particle, a vector density

$$J^\mu(x) = e \int w^\mu \, \delta^{(D)}(x - z(\tau)) \, d\tau$$

which gives the charge current, and a tensor density

$$T_p^{\mu\nu}(x) = m \int w^\mu w^\nu \, \delta^{(D)}(x - z(\tau)) \, d\tau$$

which is the energy-momentum tensor for a free particle. Note that

$$w^\mu \, \partial_\mu \delta^{(D)}(x - z(\tau)) = -\frac{d}{d\tau} \delta^{(D)}(x - z(\tau))$$

Thus, the charge current is conserved:

$$\partial_\mu J^\mu(x) = 0$$

Also,

$$\partial_\mu T_p^{\mu\nu}(x) = m \int \frac{dw^\nu}{d\tau} \, \delta^{(D)}(x - z(\tau)) \, d\tau$$

Note that $\partial_\mu T_p^{\mu\nu}(x) = 0$ for a free particle.

The trajectory for a free point particle can be obtained from the action

$$S_p = \int \mathcal{L}_p(x) \, d^2x = \int L_p \, d\lambda$$

where $\mathcal{L}_p$ is a Lagrangian density and L_p is a relativistic Lagrangian. There are several ways to define a free particle Lagrangian. One possibility is

$$L_p(z^\mu, v^\mu) = -m(\eta_{\alpha\beta} v^\alpha v^\beta)^{1/2}$$

This corresponds to a Lagrangian density

$$\mathcal{L}_p(x) = -\rho(x)$$

A second possibility is

$$L_p = -\frac{1}{2} m \eta_{\alpha\beta} w^\alpha w^\beta$$

Note that w^0 and w^1 are not independent, since they are related by the constraint

$$\eta_{\mu\nu} w^\mu w^\nu = 1$$

However, the correct equations of motion result if we ignore the constraint and vary them independently.

We can also derive the equation of motion from the action

$$S = \frac{1}{2} \int (e^{-1} g_{\mu\nu} v^\mu v^\nu + m^2) \sqrt{e}\, d\lambda = \int L_p\, d\lambda$$

where

$$e = e_{\lambda\lambda} = det(e_{\lambda\lambda}) = (e^{\lambda\lambda})^{-1}$$

can be thought of as a one-dimensional metric for the trajectory. The Lagrangian that follows from this action is

$$L_p(z^\mu, v^\mu, e) = \frac{1}{2}(e^{-1/2} g_{\mu\nu} v^\mu v^\nu + e^{1/2} m^2)$$

The equation of motion for e is

$$\frac{\partial L_p}{\partial e} = 0$$

so

$$e = m^{-2} g_{\mu\nu} v^\mu v^\nu$$

If we substitute this into the original Lagrangian, we obtain

$$L_p(z^\mu, v^\mu) = m(g_{\mu\nu} v^\mu v^\nu)^{1/2}$$

which reproduces our first Lagrangian.

5.2.2 Energy-momentum tensor

We will often want to construct the energy-momentum tensor corresponding to a given field Lagrangian $\mathcal{L}$. One way to obtain the energy-momentum tensor is to apply Noether's theorem to spacetime translations. According to Noether's theorem, we can define a canonical stress tensor $\theta^{\alpha\beta}$ that has the property

$$\partial_\alpha \theta^{\alpha\beta} = -\partial^\beta \mathcal{L} \mid_{expl}$$

Here $\partial^\beta \mathcal{L} \mid_{expl}$ indicates that only explicit spatial dependence of $\mathcal{L}$ is included in the derivative; implicit spatial dependence, due to the dependence of $\mathcal{L}$ on spatially dependent dynamical variables, is not included. For a scalar field ϕ, the canonical stress tensor is given by

$$\theta^{\alpha\beta} = \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi)} (\partial^\beta \phi) - \eta^{\alpha\beta} \mathcal{L}$$

while for a vector field A^μ, it is given by

$$\theta^{\alpha\beta} = \frac{\partial \mathcal{L}}{\partial(\partial_\alpha A_\mu)} (\partial^\beta A_\mu) - \eta^{\alpha\beta} \mathcal{L}$$

The energy-momentum tensor $T^{\alpha\beta}$ can usually be obtained in a straightforward way from the canonical stress tensor $\theta^{\alpha\beta}$.

A second way to obtain the energy-momentum tensor involves writing down the action S for the field in a form that is invariant under general coordinate transformations. We can then find the energy-momentum tensor $T^{\alpha\beta}$ by varying the action with respect to the metric tensor (see [50] for a proof of this result):

$$\delta S = -\frac{1}{2} \int T^{\alpha\beta} \, \delta g_{\alpha\beta} \, g^{1/2} \, d^2x$$

where g is minus the determinant of $g_{\mu\nu}$. For this method to be applicable, all the fields in the action must be dynamical; that is, there can be no explicit spacetime dependence of the Lagrangian.

There are several results that are useful in performing the variations. First, the variation of the inverse metric tensor is given by

$$\delta g^{\mu\nu} = -g^{\mu\alpha} g^{\nu\beta} \delta g_{\alpha\beta}$$

This follows from

$$\delta(g^{\alpha\mu} \, g_{\mu\beta}) = g^{\alpha\mu} \, \delta g_{\mu\beta} + g_{\mu\beta} \, \delta g^{\alpha\mu} = \delta(\delta^\alpha{}_\beta) = 0$$

Second, the variation of g is given by

$$\delta g = g g^{\mu\nu}\, \delta g_{\mu\nu}$$

This is easy to see in $1+1$ dimensions. In $1+1$ dimensions, g is

$$g = g_{10}\, g_{01} - g_{11}\, g_{00} = -\frac{1}{2}\epsilon^{\alpha\mu}\epsilon^{\beta\nu} g_{\alpha\beta} g_{\mu\nu}$$

and the inverse metric tensor is

$$g^{\mu\nu} = -\frac{1}{g}\, \epsilon^{\alpha\mu}\epsilon^{\beta\nu} g_{\alpha\beta}$$

Thus,

$$\delta g = -\epsilon^{\alpha\mu}\epsilon^{\beta\nu} g_{\alpha\beta}\, \delta g_{\mu\nu} = g g^{\mu\nu}\, \delta g_{\mu\nu}$$

I now consider some examples.

The action for a free point particle is

$$S = -m \int (\eta_{\alpha\beta} v^\alpha v^\beta)^{1/2}\, d\lambda$$

In covariant form this becomes

$$S = -m \int (g_{\alpha\beta} v^\alpha v^\beta)^{1/2}\, d\lambda = -m \iint (g_{\alpha\beta} v^\alpha v^\beta)^{1/2}\, \delta^{(D)}(x - z(\lambda))\, d\lambda\, d^D x$$

Thus, the energy-momentum tensor is

$$T^{\alpha\beta} = m g^{-1/2} \int (g_{\alpha\beta} v^\alpha v^\beta)^{-1/2}\, v^\alpha v^\beta\, \delta^{(D)}(x - z(\lambda))\, d\lambda$$

In Lorentz coordinates, this becomes

$$T^{\alpha\beta} = m \int w^\alpha w^\beta\, \delta^{(D)}(x - z(\tau))\, d\tau$$

For a free scalar field, the action is

$$S = \frac{1}{2} \int (\partial_\mu \phi)(\partial^\mu \phi)\, dx^2$$

In covariant form, this becomes

$$S = \frac{1}{2} \int g^{\mu\nu} (\partial_\mu \phi)(\partial_\nu \phi)\, g^{1/2}\, dx^2$$

Thus, the energy-momentum tensor is

$$T^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} (\partial_\alpha \phi)(\partial_\beta \phi) - \frac{1}{2} g^{\mu\nu} g^{\alpha\beta} (\partial_\alpha \phi)(\partial_\beta \phi)$$

For a free vector field, the action is

$$S_f = -\frac{1}{4} \int F_{\mu\nu} F^{\mu\nu} \, d^2x$$

In covariant form, this becomes

$$S_f = -\frac{1}{4} \int g^{\alpha\beta} g^{\mu\nu} F_{\alpha\mu} F_{\beta\nu} \, g^{1/2} \, d^2x$$

Thus, the energy-momentum tensor is

$$T^{\alpha\beta} = -g^{\alpha\mu} g^{\beta\nu} g^{\sigma\tau} F_{\mu\sigma} F_{\nu\tau} + \frac{1}{4} g^{\alpha\beta} g^{\mu\nu} g^{\sigma\tau} F_{\mu\sigma} F_{\nu\tau}$$

5.3 Vector field

As an example of a relativistic field theory, let us consider a point particle coupled to a vector field. We will first discuss a dynamical field coupled to a non-dynamical source, and then a dynamical particle coupled to a non-dynamical field. Finally, we will consider a system where both the particle and the field are dynamical.

5.3.1 Dynamical field

The Lagrangian for a free vector field A^μ is

$$\mathcal{L}_f = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

where $F^{\mu\nu}$, the field strength tensor, is related to A^μ by

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

Note that

$$\frac{\partial \mathcal{L}_f}{\partial(\partial_\mu A_\nu)} = -F^{\mu\nu}$$

So the field equations are

$$\partial_\mu F^{\mu\nu} = 0$$

The canonical stress tensor corresponding to $\mathcal{L}_f$ is

$$\theta_f^{\alpha\beta} = -F^{\alpha\mu}\,\partial^\beta A_\mu + \frac{1}{4}\eta^{\alpha\beta}F_{\mu\nu}F^{\mu\nu} = T_f^{\alpha\beta} - F^{\alpha\mu}\,\partial_\mu A^\beta$$

where the energy-momentum tensor $T_f^{\mu\nu}$ is

$$T_f^{\mu\nu} = F^{\mu\alpha}F_\alpha{}^\nu + \frac{1}{4}\eta^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}$$

Using the field equations, we can express $\theta_f^{\alpha\beta}$ as

$$\theta_f^{\alpha\beta} = T_f^{\alpha\beta} - \partial_\mu(F^{\alpha\mu}A^\beta)$$

Noether's theorem states that

$$\partial_\alpha\theta_f^{\alpha\beta} = -\partial^\beta\mathcal{L}_f\ |_{expl} = 0$$

so

$$\partial_\alpha T_f^{\alpha\beta} = 0$$

as can be verified by using the field equations.

Now, suppose we add an interaction Lagrangian $\mathcal{L}_i$:

$$\mathcal{L}_i = -J^\mu A_\mu$$

so the total Lagrangian is $\mathcal{L}_{fi} = \mathcal{L}_f + \mathcal{L}_i$. Since the source J^μ will in general be spatially dependent, the total Lagrangian is no longer invariant under translations. The field equations that follow from $\mathcal{L}_{fi}$ are

$$\partial_\mu F^{\mu\nu} = J^\nu$$

The canonical stress tensor corresponding to $\mathcal{L}_{fi}$ is

$$\theta_{fi}^{\alpha\beta} = T_f^{\alpha\beta} - F^{\alpha\mu}\,\partial_\mu A^\beta + \eta^{\alpha\beta}J^\mu A_\mu$$

Or, using the field equations,

$$\theta_{fi}^{\alpha\beta} = T_f^{\alpha\beta} - \partial_\mu(F^{\alpha\mu}A^\beta) - J^\alpha A^\beta + \eta^{\alpha\beta}J^\mu A_\mu$$

Note that

$$\partial_\alpha\theta_{fi}^{\alpha\beta} \quad = \quad \partial_\alpha T^{\alpha\beta} - J^\alpha(\partial_\alpha A^\beta) + (\partial^\beta J^\mu)A_\mu + J^\mu(\partial^\beta A_\mu)$$

$$= \quad \partial_\alpha T_f^{\alpha\beta} - F^{\alpha\beta} J_\alpha + (\partial^\beta J^\mu) A_\mu$$

where I have used that the current is conserved ($\partial_\mu J^\mu = 0$). From Noether's theorem, we have that

$$\partial_\alpha \theta_{fi}^{\alpha\beta} = -\partial^\beta \mathcal{L}_{fi} \mid_{expl} = (\partial^\beta J^\mu) A_\mu$$

Thus, equating these two expressions for $\partial_\alpha \theta_{fi}^{\alpha\beta}$, we find

$$\partial_\alpha T_f^{\alpha\beta} = F^{\alpha\beta} J_\alpha$$

as can be verified using the field equations.

I now want to specialize to the case of a vector field in 1+1 dimensions. It is useful to introduce a tensor $\epsilon_{\mu\nu}$, which is defined by $\epsilon_{01} = -\epsilon_{10} = 1$, $\epsilon_{00} = \epsilon_{11} = 0$. It follows that $\epsilon^{01} = -\epsilon^{10} = -1$, $\epsilon^{00} = \epsilon^{11} = 0$. The following results hold:

$$\begin{aligned} \epsilon^{\mu\nu}\epsilon_{\alpha\beta} &= \delta^\mu{}_\beta \, \delta^\nu{}_\alpha - \delta^\mu{}_\alpha \, \delta^\nu{}_\beta \\ \epsilon^{\mu\nu}\epsilon_{\nu\alpha} &= \delta^\mu{}_\alpha \\ \epsilon^{\mu\nu}\epsilon_{\mu\nu} &= -2 \end{aligned}$$

The field strength tensor is

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \epsilon^{\mu\nu} E$$

where the electric field E is

$$E = -\partial_x A^0 - \partial_t A^1 = F^{10}$$

The Lagrangian for a free vector field is

$$\mathcal{L}_f = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2} E^2$$

The energy-momentum tensor $T_f^{\mu\nu}$ is

$$T_f^{\mu\nu} = F^{\mu\alpha} F_\alpha{}^\nu + \frac{1}{4} \eta^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} = -\frac{1}{4} F^{\alpha\beta} F_{\alpha\beta} \eta^{\mu\nu} = \frac{1}{2} E^2 \, \eta^{\mu\nu}$$

The field equations are

$$\partial_\mu E = -\epsilon_{\mu\nu} J^\nu$$

Or, in component notation,

$$\begin{aligned} \partial_x E &= J^0 \\ \partial_t E &= -J^1 \end{aligned}$$

These are just Maxwell's equations in 1+1 dimensions.

5.3.2 Dynamical particle

Now let us consider a dynamical point particle coupled to a non-dynamical vector field A^μ. The Lagrangian density for the system is

$$\mathcal{L}_{pi} = -\rho - J_\mu A^\mu$$

where

$$\rho(x) = m \int \delta^{(D)}(x - z(\tau))\, d\tau$$

is the mass density as measured in the instantaneous rest frame of the particle, and

$$J^\mu(x) = e \int w^\mu\, \delta^{(D)}(x - z(\tau))\, d\tau$$

is the current. The action is

$$S = \int \mathcal{L}_{pi}\, dx^2 = -\int L_{pi}\, d\lambda$$

where

$$L_{pi}(z^\mu, v^\mu) = m(\eta_{\alpha\beta} v^\alpha v^\beta)^{1/2} + e v^\mu A_\mu$$

The equation of motion is given by

$$\frac{d}{d\lambda}\frac{\partial L_{pi}}{\partial v^\mu} - \frac{\partial L_{pi}}{\partial z^\mu} = 0$$

Note that

$$\begin{aligned} \frac{\partial L_{pi}}{\partial v^\mu} &= m\,\eta_{\mu\nu} v^\nu\, (\eta_{\alpha\beta} v^\alpha v^\beta)^{-1/2} + e A_\mu = m \eta_{\mu\nu} w^\nu + e A_\mu \\ \frac{\partial L_{pi}}{\partial z^\mu} &= e v^\nu\, \partial_\mu A_\nu \end{aligned}$$

Thus,

$$\frac{d}{d\tau} w^\mu = \frac{e}{m} F^{\mu\nu} w_\nu$$

Recall that the energy-momentum tensor for a free particle is

$$T_p^{\mu\nu}(x) = m \int w^\mu w^\nu \, \delta^{(D)}(x - z(\tau)) \, d\tau$$

Thus,

$$\partial_\mu T_p^{\mu\nu}(x) = m \int \frac{dw^\nu}{d\tau} \, \delta^{(D)}(x - z(\tau)) \, d\tau$$

If we substitute the equation of motion for the particle, we find

$$\partial_\mu T_p^{\mu\nu} = F^{\mu\nu} J_\nu$$

5.3.3 Dynamical particle and field

Now let us consider a point particle coupled to a vector field A^μ, where both the particle and the field are dynamical. The total Lagrangian for the system is

$$\mathcal{L} = \mathcal{L}_f + \mathcal{L}_p + \mathcal{L}_i = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \rho - J_\mu A^\mu$$

where J^μ is related to the particle dynamical variables by

$$J^\mu(x) = e \int w^\mu \, \delta^{(D)}(x - z(\tau)) \, d\tau$$

The total energy-momentum tensor is

$$T^{\mu\nu} = T_f^{\mu\nu} + T_p^{\mu\nu}$$

where

$$T_p^{\mu\nu}(x) = m \int w^\mu w^\nu \, \delta^{(D)}(x - z(\tau)) \, d\tau$$

and

$$T_f^{\mu\nu} = F^{\mu\alpha} F_\alpha{}^\nu + \frac{1}{4} \eta^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta}$$

The field equation is

$$\partial_\mu F^{\mu\nu} = J^\nu$$

The equation of motion for the particle is

$$\frac{d}{d\tau} w^\mu = \frac{e}{m} F^{\mu\nu} w_\nu$$

From these, it follows that

$$\partial_\mu T_p^{\mu\nu} = -\partial_\mu T_f^{\mu\nu} = F^{\mu\nu} J_\nu$$

So the total energy-momentum tensor is conserved:

$$\partial^\mu T^{\mu\nu} = 0$$

Because the only spatial dependence of the total Lagrangian is the implicit spatial dependence from the dynamical variables, we can also obtain the energy-momentum tensor by varying the covariant form of the action with respect to the metric tensor:

$$\begin{aligned} S &= -\frac{1}{4}\int g^{\alpha\beta} g^{\mu\nu} F_{\alpha\mu} F_{\beta\nu}\, g^{1/2}\, d^2x - \\ &\quad m \iint (g_{\mu\nu} v^\mu v^\nu)^{1/2}\, \delta^{(D)}(x - z(\lambda))\, d\lambda\, d^D x \end{aligned}$$

5.4 Scalar field

As a second example of a relativistic field theory, let us consider a point particle coupled to a scalar field. We will first discuss a dynamical field coupled to a non-dynamical source, and then a dynamical particle coupled to a non-dynamical field. Finally, we will consider a system where both the particle and the field are dynamical.

5.4.1 Dynamical field

The Lagrangian for a free scalar field is

$$\mathcal{L}_f = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi)$$

The field equation that follows from $\mathcal{L}_f$ is

$$\Box \phi = 0$$

The energy-momentum tensor $T_f^{\alpha\beta}$ for the field is just given by the canonical stress tensor $\theta_f^{\alpha\beta}$ corresponding to $\mathcal{L}_f$:

$$T_f^{\alpha\beta} = \theta_f^{\alpha\beta} = (\partial^\alpha \phi)(\partial^\beta \phi) - \frac{1}{2}\eta^{\alpha\beta}(\partial_\mu \phi)(\partial^\mu \phi)$$

Thus, from Noether's theorem,

$$\partial_\alpha T_f^{\alpha\beta} = \partial_\alpha \theta_f^{\alpha\beta} = -\partial^\beta \mathcal{L}_f \mid_{expl} = 0$$

as can be verified using the equation of motion.

Now suppose we add an interaction of the form

$$\mathcal{L}_i = -f(\phi)\,\rho$$

where ρ describes a source. Later we will let ρ depend on other dynamical variables in the Lagrangian, but for now it is just a spacetime-dependent parameter. The equation of motion that follows from $\mathcal{L}_{fi} = \mathcal{L}_f + \mathcal{L}_i$ is

$$\Box\phi + \frac{df}{d\phi}\,\rho = 0$$

The canonical stress tensor corresponding to $\mathcal{L}_{fi}$ is

$$\theta^{\alpha\beta}_{fi} = (\partial^\alpha\phi)(\partial^\beta\phi) - \frac{1}{2}\eta^{\alpha\beta}(\partial_\mu\phi)(\partial^\mu\phi) + \eta^{\alpha\beta}f\rho = T^{\alpha\beta}_f + \eta^{\alpha\beta}f\rho$$

Thus, from Noether's theorem,

$$\partial_\alpha\theta^{\alpha\beta}_{fi} = -\partial^\beta\mathcal{L}_{fi}\mid_{expl} = f\,\partial^\beta\rho$$

as can be verified using the equations of motion. Note that because ρ is a parameter rather than a function of dynamical variables, its spatial dependence is included in $\partial^\beta\mathcal{L}_{fi}\mid_{expl}$. We find that

$$\partial^\alpha T^{\alpha\beta}_f = -(\partial^\beta f)\rho$$

5.4.2 Dynamical particle

The Lagrangian density for a particle coupled to a scalar field is

$$\mathcal{L}_{pi} = \mathcal{L}_p + \mathcal{L}_i = -(1 + f(\phi))\,\rho$$

where $f(\phi)$ is a function that describes the form of the coupling, and for now the field ϕ is taken to be a spacetime-dependent parameter rather than a dynamical variable. The action is

$$S = \int \mathcal{L}_{pi}\,d^2x = -\int L_{pi}\,d\lambda$$

where the Lagrangian is

$$L_{pi}(z^\mu, v^\mu) = m(1+f)(\eta_{\mu\nu}v^\mu v^\nu)^{1/2}$$

The equation of motion for the particle is given by

$$\frac{d}{d\lambda}\frac{\partial L_{pi}}{\partial v^\gamma} - \frac{\partial L_{pi}}{\partial z^\gamma} = 0$$

Note that

$$\begin{aligned}
\frac{\partial L_{pi}}{\partial v^\gamma} &= m(1+f)\,\eta_{\gamma\mu}v^\mu\,(\eta_{\alpha\beta}v^\alpha v^\beta)^{-1/2} = m(1+f)w_\gamma = \pi_\gamma \\
\frac{\partial L_{pi}}{\partial z^\gamma} &= m(\partial_\gamma f)(\eta_{\alpha\beta}v^\alpha v^\beta)^{1/2}
\end{aligned}$$

Thus, the equation of motion is

$$\frac{d}{d\tau}w^\mu + (1+f)^{-1}(\partial_\nu f)(w^\mu w^\nu - \eta^{\mu\nu}) = 0$$

Or

$$\frac{d}{d\tau}\pi_\mu = \partial_\mu f$$

Recall that the energy-momentum tensor for a free particle is

$$T_p^{\mu\nu}(x) = m\int w^\mu w^\nu\,\delta^{(D)}(x - z(\tau))\,d\tau$$

We can define an energy-momentum tensor for a coupled particle that includes the interaction energy of the particle with the field:

$$T_{pi}^{\mu\nu}(x) = m(1+f)\int w^\mu w^\nu\,\delta^{(D)}(x - z(\tau))\,d\tau$$

Note that

$$\begin{aligned}
\partial_\mu T_{pi}^{\mu\nu}(x) &= m(\partial_\mu f)\int w^\mu w^\nu\,\delta^{(D)}(x - z(\tau))\,d\tau + \\
&\quad m(1+f)\int \frac{dw^\nu}{d\tau}\,\delta^{(D)}(x - z(\tau))\,d\tau
\end{aligned}$$

If we substitute the equation of motion for the particle, we find that

$$\partial_\mu T_{pi}^{\mu\nu}(x) = (\partial^\nu f)\rho$$

where

$$\rho(x) = m\int \delta^{(D)}(x - z(\tau))\,d\tau$$

5.4.3 Geometric interpretation

In the previous section, we showed that for the Lagrangian density

$$\mathcal{L}_{pi} = \mathcal{L}_p + \mathcal{L}_i = -(1 + f(\phi))\,\rho$$

the equation of motion for the particle is

$$\frac{d}{d\tau}w^\mu + (1+f)^{-1}(\partial_\nu f)(w^\mu w^\nu - \eta^{\mu\nu}) = 0$$

I now want to show that we can reinterpret this result: instead of viewing the particle as moving in a flat spacetime and being accelerated by the field, we can view the particle as falling freely in a curved spacetime, with a metric tensor given by

$$g_{\mu\nu} = (1+f)^2\,\eta_{\mu\nu}$$

Note that given a particle trajectory $z^\mu(\lambda)$, where λ is an arbitrary parameter, we can define two different proper times: a proper time τ relative to the metric $\eta_{\mu\nu}$, and a proper time s relative to the metric $g_{\mu\nu}$:

$$\begin{aligned} d\tau &= (\eta_{\mu\nu}\,dz^\mu dz^\nu)^{1/2} \\ ds &= (g_{\mu\nu}\,dz^\mu dz^\nu)^{1/2} \end{aligned}$$

Since we are dealing with two metric tensors, to avoid ambiguity I will be careful not to raise or lower indices. We can define a velocity v^μ relative to the arbitrary parameter λ, and velocities w^μ and u^μ relative to the proper times λ and τ:

$$\begin{aligned} v^\mu &= \frac{dz^\mu}{d\lambda} \\ w^\mu &= \frac{dz^\mu}{d\tau} \\ u^\mu &= \frac{dz^\mu}{ds} \end{aligned}$$

Note that

$$\eta_{\mu\nu}w^\mu w^\nu = g_{\mu\nu}u^\mu u^\nu = 1$$

Also,

$$\frac{d\tau}{d\lambda} = (\eta_{\mu\nu}v^\mu v^\nu)^{1/2}$$

$$\frac{ds}{d\lambda} = (g_{\mu\nu}v^\mu v^\nu)^{1/2}$$

We can express the particle Lagrangian as

$$L_{pi}(z^\mu, v^\mu) = m(1+f)(\eta_{\mu\nu}v^\mu v^\nu)^{1/2} = m(g_{\mu\nu}v^\mu v^\nu)^{1/2}$$

The equation of motion for the particle is given by the Euler-Lagrange equations:

$$\frac{d}{d\lambda}\frac{\partial L_{pi}}{\partial v^\gamma} - \frac{\partial L_{pi}}{\partial z^\gamma} = 0$$

Note that

$$\begin{aligned}\frac{\partial L_{pi}}{\partial v^\gamma} &= m\, g_{\gamma\nu}v^\nu\, (g_{\alpha\beta}v^\alpha v^\beta)^{-1/2} = m g_{\gamma\nu}u^\nu \\ \frac{\partial L_{pi}}{\partial z^\gamma} &= \frac{1}{2}m\, g_{\mu\nu,\gamma}v^\mu v^\nu\, (g_{\alpha\beta}v^\alpha v^\beta)^{-1/2}\end{aligned}$$

Thus, the equation of motion is

$$\frac{d}{ds}(g_{\gamma\nu}u^\nu) - \frac{1}{2}g_{\mu\nu,\gamma}\, u^\mu u^\nu = 0$$

Or

$$\frac{d}{ds}u^\alpha + \frac{1}{2}g^{\alpha\gamma}(g_{\gamma\nu,\mu} + g_{\mu\gamma,\nu} - g_{\mu\nu,\gamma})\, u^\mu u^\nu = 0$$

The Christoffel symbols are given by

$$\Gamma^\alpha{}_{\mu\nu} = \frac{1}{2}g^{\alpha\gamma}(g_{\gamma\nu,\mu} + g_{\mu\gamma,\nu} - g_{\mu\nu,\gamma})$$

Thus, we can express the equation of motion as

$$\frac{d}{ds}u^\gamma + \Gamma^\gamma{}_{\alpha\beta}u^\alpha u^\beta = 0$$

so the particles move along geodesics of the metric $g_{\mu\nu}$.

5.4.4 Dynamical particle and field

Now let us consider a point particle coupled to a scalar field ϕ, where both the particle and the field are dynamical. The Lagrangian for the system is

$$\mathcal{L} = \mathcal{L}_f + \mathcal{L}_p + \mathcal{L}_i = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - (1+f)\rho$$

where ρ is related to the particle dynamical variables by

$$\rho(x) = m \int \delta^{(D)}(x - z(\tau))\, d\tau$$

The total energy-momentum tensor is

$$T^{\alpha\beta} = T_f^{\alpha\beta} + T_{pi}^{\alpha\beta}$$

where

$$T_{pi}^{\mu\nu}(x) = m(1+f) \int w^\mu w^\nu\, \delta^{(D)}(x - z(\tau))\, d\tau$$

and

$$T_f^{\alpha\beta} = (\partial^\alpha \phi)(\partial^\beta \phi) - \frac{1}{2}\eta^{\alpha\beta}(\partial_\mu \phi)(\partial^\mu \phi)$$

The field equation is

$$\Box\phi + \frac{df}{d\phi}\,\rho = 0$$

The equation of motion for the particle is

$$\frac{dw^\mu}{d\tau} + (1+f)^{-1}(\partial_\nu f)(w^\mu w^\nu - \eta^{\mu\nu}) = 0$$

From these, it follows that

$$\partial_\alpha T_{pi}^{\alpha\beta} = -\partial_\alpha T_f^{\alpha\beta} = (\partial^\beta f)\rho$$

So the total energy-momentum tensor is conserved:

$$\partial_\alpha T^{\alpha\beta} = 0$$

Note that while for a vector field the total energy-momentum tensor of the system is just the sum of the individual energy-momentum tensors for the particle and field, for a scalar field the total energy-momentum tensor also includes a contribution due to the particle/field interaction.

Because the only spatial dependence of the total Lagrangian is the implicit spatial dependence from the dynamical variables, we can also obtain the energy-momentum tensor by varying the covariant form of the action with respect to the metric tensor:

$$\begin{aligned} S &= \frac{1}{2}\int g^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi)\, g^{1/2}\, d^2x - \\ &\quad m \iint (1+f)\, (g_{\mu\nu} v^\mu v^\nu)^{1/2}\, \delta^{(D)}(x - z(\lambda))\, d\lambda\, d^D x \end{aligned}$$

5.5 Greens function for the wave equation

In the next section, I will calculate the fields radiated by a moving particle for several different field theories. For linear systems (that is, systems that obey the superposition principle), this is most easily accomplished by the use of Greens functions, which are solutions to the field equation for a point source. By integrating the Greens function against the source distribution function, we obtain the fields produced by that source. Here I calculate retarded Greens functions for the wave equation in various dimensions and show how the Greens function in D spacetime dimensions may be related to the Greens function in $D+1$ spacetime dimensions.

5.5.1 General expression for the Greens function

The Greens function for the wave equation satisfies

$$\Box G(x) = \Omega_{D-2}\, \delta^{(D)}(x)$$

where D is the spacetime dimension and Ω_n is the surface area of S^n:

$$\Omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$$

The Greens function $G(x)$ is related to its Fourier transform $\tilde{G}(q)$ by

$$G(x) = (2\pi)^{-D} \int e^{-iq\cdot x}\, \tilde{G}(q)\, d^D q$$

Thus,

$$\Box G(x) = -(2\pi)^{-D} \int (q\cdot q)\, e^{-iq\cdot x}\, \tilde{G}(q)\, d^D q$$

The delta function can be expressed as

$$\delta^{(D)}(x) = (2\pi)^{-D} \int e^{-iq\cdot x}\, dq$$

So from the definition of the Greens function, we find that

$$\tilde{G}(q) = -\frac{\Omega_{D-2}}{q\cdot q}$$

Substituting this result into the expression for $G(x)$, we obtain

$$G(x) = -\Omega_{D-2}\, (2\pi)^{-D} \int (q\cdot q)^{-1}\, e^{-iq\cdot x}\, d^D q$$

Note that

$$\int (q\cdot q)^{-1}\, e^{-iq\cdot x}\, d^D q = \int\int (\omega^2 - k^2)^{-1}\, e^{-i(\omega t - \vec{k}\cdot\vec{r})}\, d\omega\, d^{D-1}k$$

The ω integral is given by

$$\begin{aligned}\int (\omega^2 - k^2)^{-1}\, e^{-i\omega t}\, d\omega &= \int (\omega + k + i\epsilon)^{-1}\, (\omega - k + i\epsilon)^{-1}\, e^{-i\omega t}\, d\omega \\ &= -2\pi\, \theta(t)\, \frac{\sin kt}{k}\end{aligned}$$

where I have chosen the poles in such a way that we get a retarded Greens function (other choices of poles result in Greens functions that satisfy other boundary conditions). Thus,

$$G_r(x) = \Omega_{D-2}\, (2\pi)^{-(D-1)}\, \theta(t) \int \frac{\sin kt}{k}\, e^{i\vec{k}\cdot\vec{r}}\, d^{D-1}k$$

A D dimensional spacetime may be viewed as a $D+1$ dimensional spacetime that is symmetric under translations along one of the spatial dimensions. This observation can be used to relate the Greens function in D spacetime dimensions to the Greens function in $D+1$ spacetime dimensions:

$$G_r^D(t, x_1, \cdots, x_{D-1}) = \frac{\Omega_{D-2}}{\Omega_{D-1}} \int G_r^{D+1}(t, x_1, \cdots, x_D)\, dx_D$$

5.5.2 Greens function in $3+1$ dimensions

For $3+1$ dimensions, the retarded Greens function is

$$\begin{aligned}G_r(\vec{r}, t) &= (4\pi)(2\pi)^{-3}\, \theta(t) \int \frac{\sin kt}{k}\, e^{i\vec{k}\cdot\vec{r}}\, d^3k \\ &= -i(2\pi)^{-1}\, \theta(t) \int_0^\infty \int_{-1}^1 e^{ikr\cos\theta}\, (e^{ikt} - e^{-ikt})\, d(\cos\theta)\, k\, dk \\ &= -(2\pi)^{-1}\, \frac{\theta(t)}{r} \int_0^\infty (e^{ikr} - e^{-ikr})(e^{ikt} - e^{-ikt})\, dk \\ &= -\frac{\theta(t)}{r}\, (\delta(t+r) - \delta(t-r)) \\ &= \frac{\delta(t-r)}{r}\end{aligned}$$

Note that

$$\delta(t^2 - r^2) = \frac{1}{2r}(\delta(t+r) + \delta(t-r))$$

Thus, we can also express the Greens function in a manifestly Lorentz invariant form:

$$G_r(x) = 2\theta(t)\, \delta(x\cdot x)$$

5.5.3 Greens function in $2+1$ dimensions

For $2+1$ dimensions, the retarded Greens function is

$$\begin{aligned} G_r(\vec{r},t) &= (2\pi)(2\pi)^{-2}\,\theta(t)\int \frac{\sin kt}{k}\,e^{i\vec{k}\cdot\vec{r}}\,d^2k \\ &= (2\pi)^{-1}\,\theta(t)\int_0^\infty\int_0^{2\pi} e^{ikr\cos\theta}\,\sin kt\,d\theta\,dk \end{aligned}$$

Note that [51]

$$J_0(x) = (2\pi)^{-1}\int_0^{2\pi} e^{ix\cos\theta}\,d\theta$$

and [51]

$$\int_0^\infty J_0(kr)\,\sin kt\,dk = \theta(t-r)\,(t^2-r^2)^{-1/2}$$

Thus,

$$G_r(\vec{r},t) = \theta(t-r)\,(t^2-r^2)^{-1/2}$$

Or, in manifestly Lorentz invariant form,

$$G_r(x) = \theta(x\cdot x)\,(x\cdot x)^{-1/2}$$

We can also obtain the Greens function for $2+1$ dimensions from the Greens function for $3+1$ dimensions:

$$G_r(t,\vec{r}) = \frac{1}{2}\,\theta(t)\int (r^2+z^2)^{-1/2}\,\delta(t-(r^2+z^2)^{1/2})\,dz$$

where $\vec{r} = x\,\hat{x} + y\,\hat{y}$. We can express the delta function as $\delta(f(z))$, where

$$f(z) = t - (r^2+z^2)^{1/2}$$

If $t > r$, then $f(z)$ has zeros at $z = \pm z_0$, where

$$z_0 = (t^2-r^2)^{1/2}$$

Note that

$$|f'(\pm z_0)| = z_0\,(r^2+z_0^2)^{-1/2}$$

If $t < r$, then $f(z)$ is negative and the delta function vanishes. Thus,

$$\delta(t-(r^2+z^2)^{1/2}) = \frac{1}{z_0}\,\theta(t-r)\,(r^2+z_0^2)^{1/2}\,(\delta(z+z_0)+\delta(z-z_0))$$

If we substitute this result into our expression for the Greens function, we obtain

$$G_r(t,\vec{r}) = \frac{1}{z_0}\,\theta(t-r) = \theta(t-r)\,(t^2-r^2)^{-1/2}$$

which agrees with our previous result.

Note that in $3+1$ dimensions, waves only propagate on the light cone, while in $2+1$ dimensions, waves propagate both on and inside the light cone. This can be understood by noting that a point source in $2+1$ dimensions is equivalent to a line source in $3+1$ dimensions.

5.5.4 Greens function in $1+1$ dimensions

For $1+1$ dimensions, the retarded Greens function is

$$G_r(x,t) = 2(2\pi)^{-1}\,\theta(t)\int \frac{\sin kt}{k}\,e^{ikx}\,dk = \theta(t)\,I(0)$$

where

$$I(\alpha) = \frac{2}{\pi}\int_0^\infty \frac{e^{-\alpha k}}{k}\,\sin kt\,\cos kx\,dk$$

Note that $I(\infty)=0$. If we differentiate with respect to α, we can perform the integral:

$$\frac{dI}{d\alpha} = -\frac{1}{\pi}\left(\frac{t+x}{\alpha^2+(t+x)^2} + \frac{t-x}{\alpha^2+(t-x)^2}\right)$$

Now integrate over α to get

$$I(\alpha) = I(0) - \frac{1}{\pi}\tan^{-1}\left(\frac{\alpha}{t+x}\right) - \frac{1}{\pi}\tan^{-1}\left(\frac{\alpha}{t-x}\right)$$

Using this expression, together with the fact that $I(\infty)=0$, we find

$$I(0) = \theta(t-|x|)$$

Thus,

$$G_r(x,t) = \theta(t-|x|)$$

Or, in manifestly Lorentz invariant form,

$$G_r(x) = \theta(x\cdot x)$$

We can also obtain the Greens function for $1+1$ dimensions from the Greens function in $2+1$ dimensions:

$$G_r(t,x) = \frac{1}{\pi}\int \theta(t-(x^2+y^2)^{1/2})\,(t^2-x^2-y^2)^{-1/2}\,dy$$

Note that, because of the θ function, the integral is nonzero only if $t > |x|$. Thus, we can express the Greens function as

$$G_r(t,x) = \frac{1}{\pi}\,\theta(t-|x|)\int_{-y_0}^{y_0}(y_0^2-y^2)^{-1/2}\,dy$$

where

$$y_0 = (t^2-x^2)^{1/2}$$

If we perform the integral, we find

$$G_r(t,x) = \theta(t-|x|)$$

which agrees with our previous result.

5.6 Fields radiated by a moving particle

I now want to evaluate the fields radiated by a moving particle, in both $1+1$ and $3+1$ dimensions, and for both scalar and vector fields.

5.6.1 General results

Here I collect a number of results that are useful for calculating the fields and field gradients produced by a moving particle. To calculate the field at x, we need to evaluate the retarded Greens function $G_r(r(x,\tau))$, where

$$r(x,\tau) = x - z(\tau)$$

and $z(\tau)$ is the location of the particle at proper time τ. Note that

$$\frac{dr^\mu}{d\tau} = -w^\mu$$

where

$$w^\mu = \frac{dz^\mu}{d\tau}$$

is the velocity of the particle.

First I want to relate the gradient of $G_r(r)$ to the derivative of $G_r(r)$ with respect to the proper

time τ. Because $G_r(r)$ is Lorentz invariant, we can express the gradient in the form

$$\partial_\mu G_r(r) = h(r)\, r_\mu$$

for some function $h(r)$. The derivative of $G_r(r)$ with respect to the proper time is

$$\frac{d}{d\tau} G_r(r) = \frac{dr^\mu}{d\tau}\, \partial_\mu G_r(r) = -w^\mu\, \partial_\mu G_r(r) = -(r\cdot w)\, h(r)$$

So

$$\partial_\mu G_r(r) = -(r\cdot w)^{-1}\, r_\mu\, \frac{d}{d\tau} G_r(r)$$

Next, I want to evaluate $\delta(r\cdot r)$. Note that for a point x that does not lie on the world line of the particle, there are two proper times for which $r\cdot r$ vanishes: the retarded proper time $\tau_r(x)$, and the advanced proper time $\tau_a(x)$. They are defined implicitly by

$$r(x,\tau_r)\cdot r(x,\tau_r) = r(x,\tau_a)\cdot r(x,\tau_a) = 0$$

and

$$z^0(\tau_r) < x^0 < z^0(\tau_a)$$

Thus,

$$\begin{aligned}
\delta(r\cdot r) &= |\frac{d}{d\tau}(r\cdot r)|^{-1}\, (\delta(\tau-\tau_r) + \delta(\tau-\tau_a)) \\
&= \frac{1}{2}\, (r\cdot w)^{-1}\, (\delta(\tau-\tau_r) - \delta(\tau-\tau_a))
\end{aligned}$$

In what follows I will assume that r and w are evaluated at the retarded proper time $\tau_r(x)$; that is, $r \equiv r(x,\tau_r(x))$ and $w \equiv w(\tau_r(x))$.

Note that r may be expressed as

$$r = (r\cdot w)(w+n)$$

where $n \equiv n(x)$ is a vector field given by

$$n = (r\cdot w)^{-1}\, r - w$$

The vectors n and w obey the following orthonormality conditions:

$$w\cdot w = -n\cdot n = 1$$

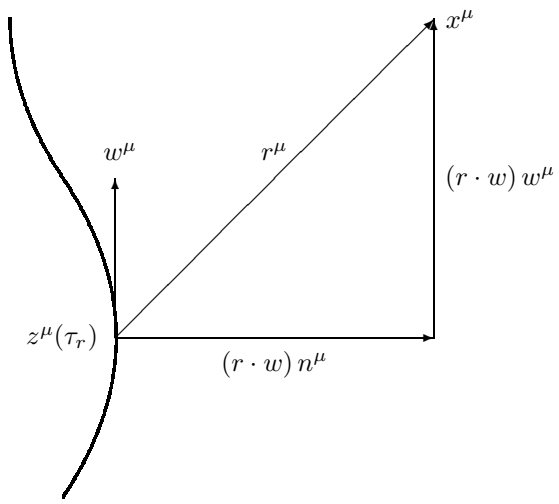

Figure 5.1: Diagram of vectors, drawn in the instantaneous rest frame of the particle at the retarded proper time $\tau_r(x)$ of point x^μ.

$$w \cdot n = 0$$

Note that $r \cdot w$ may be interpreted as the distance from the particle to the point x, as measured in the instantaneous rest frame of the particle at the retarded proper time $\tau_r(x)$.

In summary, given a particle trajectory $z(\tau)$, we have defined vector fields r, w, and n by

$$\begin{aligned} r &= r(x, \tau_r(x)) = x - z(\tau_r(x)) \\ w &= w(\tau_r(x)) \\ n &= (r(x, \tau_r(x)) \cdot w(\tau_r))^{-1}\, r(x, \tau_r(x)) - w(\tau_r(x)) \end{aligned}$$

Note that r is everywhere lightlike, w is everywhere timelike, and n is everywhere spacelike. A diagram of the vectors r, n, and w for a specific point x is shown in Figure 5.1.

I now want to calculate the gradient of r. Note that

$$\partial_\alpha r^\beta = \partial_\alpha (x^\beta - z^\beta(\tau_r)) = \delta^\beta{}_\alpha - w^\beta\, \partial_\alpha \tau_r$$

Since $r \cdot r = 0$, it follows that

$$\frac{1}{2}\, \partial_\alpha (r \cdot r) = r_\beta\, \partial_\alpha r^\beta = r_\alpha - (r \cdot w)\, \partial_\alpha \tau_r = 0$$

Thus, the gradient of the proper time is

$$\partial_\alpha \tau_r = (r \cdot w)^{-1} r_\alpha = w_\alpha + n_\alpha$$

So the gradient of r is

$$\partial_\alpha r^\beta = \delta^\beta{}_\alpha - (r \cdot w)^{-1} r_\alpha w^\beta = \delta^\beta{}_\alpha - (w_\alpha + n_\alpha) w^\beta$$

Note that

$$\partial_\alpha r^\alpha = D - 1$$

where D is the spacetime dimension.

5.6.2 General results for a vector field

Consider a vector field A^μ coupled to a current J^μ in D spacetime dimensions. The Lagrangian is

$$\mathcal{L} = -\frac{1}{4\Omega_{D-2}} F^{\mu\nu} F_{\mu\nu} - \rho - J_\mu A^\mu$$

The field equation is

$$\partial_\mu F^{\mu\nu} = \Omega_{D-2} J^\nu$$

In the Lorentz gauge ($\partial_\mu A^\mu = 0$), this becomes

$$\Box A^\mu(x) = \Omega_{D-2}\, J^\mu(x)$$

For a point particle with trajectory $z(\tau)$, the current is

$$J^\mu(x) = e \int w^\mu\, \delta^{(D)}(x - z(\tau))\, d\tau$$

Thus, the retarded vector potential for the particle is

$$A^\mu_r(x) = \int G_r(x - y)\, J^\mu(y)\, d^D y = e \int w^\mu\, G_r(r)\, d\tau$$

The gradients are

$$\begin{aligned} \partial_\nu A^\mu_r(x) &= e \int w^\mu\, \partial_\nu G_r(r)\, d\tau \\ &= -e \int (r \cdot w)^{-1} r_\nu w^\mu\, \frac{d}{d\tau} G_r(r)\, d\tau \\ &= e \int G_r(r)\, \frac{d}{d\tau} ((r \cdot w)^{-1} r_\nu w^\mu)\, d\tau \end{aligned}$$

Note that

$$\frac{d}{d\tau}((r\cdot w)^{-1}\, r^\nu w^\mu) = -(r\cdot w)^{-1}\, w^\nu w^\mu + (r\cdot w)^{-2}\, r^\nu w^\mu + (r\cdot w)^{-1}\, r^\nu s^\mu$$

where I have used that $w\cdot w = 1$, and I have defined

$$s = a - (r\cdot w)^{-1}\,(r\cdot a)\, w = a - (a\cdot n)\, w$$

Note that

$$r\cdot s = 0$$

and

$$s\cdot s = a\cdot a + (a\cdot n)^2$$

The retarded field strength tensor is

$$F_r^{\mu\nu} = \partial^\mu A_r^\nu - \partial^\nu A_r^\mu = e\int G_r(r)\,((r\cdot w)^{-2}\, r^{[\mu}\, w^{\nu]} + (r\cdot w)^{-1}\, r^{[\mu}\, s^{\nu]})\, d\tau$$

I now want to discuss the conservation of energy and momentum. Recall that we can define an energy-momentum tensor for the field

$$T_f^{\mu\nu} = \Omega_{D-2}^{-1}(\eta_{\alpha\beta}\, F^{\mu\alpha}\, F^{\beta\nu} + \frac{1}{4}\eta^{\mu\nu}\,\eta_{\alpha\beta}\,\eta_{\gamma\sigma}\, F^{\alpha\gamma}\, F^{\beta\sigma})$$

such that

$$\partial_\mu T_f^{\mu\nu} = F^{\mu\nu} J_\mu$$

Using that $F^{\mu\nu} = F_r^{\mu\nu} + F_{in}^{\mu\nu}$, we can decompose the energy-momentum tensor as follows:

$$T_f^{\alpha\beta} = T_{in}^{\alpha\beta} + T_r^{\alpha\beta} + T_{cross}^{\alpha\beta}$$

where

$$T_{in}^{\mu\nu} = \Omega_{D-2}^{-1}(\eta_{\alpha\beta}\, F_{in}^{\mu\alpha}\, F_{in}^{\beta\nu} + \frac{1}{4}\eta^{\mu\nu}\,\eta_{\alpha\beta}\,\eta_{\gamma\sigma}\, F_{in}^{\alpha\gamma}\, F_{in}^{\beta\sigma})$$

is the energy-momentum tensor for the *in* field,

$$T_r^{\mu\nu} = \Omega_{D-2}^{-1}(\eta_{\alpha\beta}\, F_r^{\mu\alpha}\, F_r^{\beta\nu} + \frac{1}{4}\eta^{\mu\nu}\,\eta_{\alpha\beta}\,\eta_{\gamma\sigma}\, F_r^{\alpha\gamma}\, F_r^{\beta\sigma})$$

is the energy-momentum tensor for the retarded field, and

$$T_{cross}^{\mu\nu} = \Omega_{D-2}^{-1}(\eta_{\alpha\beta}\, F_r^{\mu\alpha}\, F_{in}^{\beta\nu} + \eta_{\alpha\beta}\, F_{in}^{\mu\alpha}\, F_r^{\beta\nu} + \frac{1}{2}\eta^{\mu\nu}\,\eta_{\alpha\beta}\,\eta_{\gamma\sigma}\, F_r^{\alpha\gamma}\, F_{in}^{\beta\sigma})$$

is the energy-momentum tensor associated with the interference between the *in* field and the retarded field. Note that

$$\partial_\mu F_r^{\mu\nu} = \Omega_{D-2} J^\nu$$

and

$$\partial_\mu F_{in}^{\mu\nu} = 0$$

from which it follows that $T_{in}^{\alpha\beta}$, $T_{cross}^{\alpha\beta}$, and $T_r^{\alpha\beta}$ obey the individual conservation laws

$$\begin{aligned} \partial_\alpha T_{in}^{\alpha\beta} &= 0 \\ \partial_\alpha T_{cross}^{\alpha\beta} &= F_{in}^{\mu\nu} J_\mu \\ \partial_\alpha T_r^{\alpha\beta} &= F_r^{\mu\nu} J_\mu \end{aligned}$$

5.6.3 General results for a scalar field

Consider a scalar field ϕ coupled to a source ρ in D spacetime dimensions. The Lagrangian is

$$\mathcal{L} = \frac{1}{2\Omega_{D-2}} (\partial_\mu \phi)(\partial^\mu \phi) - (1+\phi)\rho$$

The field equation is

$$\Box \phi = -\Omega_{D-2}\, \rho$$

For a point particle with trajectory $z(\tau)$,

$$\rho(x) = \int \delta^{(D)}(x - z(\tau))\, d\tau$$

Thus, the retarded field radiated by the source is

$$\phi_r(x) = -\int G_r(x-y)\, \rho(y)\, d^D y = -\int G_r(r)\, d\tau$$

The gradient of the retarded field is

$$\begin{aligned} \partial_\mu \phi_r(x) &= -\int \partial_\mu G_r(r)\, d\tau \\ &= \int (r\cdot w)^{-1}\, r_\mu\, \frac{d}{d\tau} G_r(r)\, d\tau \\ &= -\int G_r(r)\, \frac{d}{d\tau}((r\cdot w)^{-1}\, r_\mu)\, d\tau \end{aligned}$$

Note that

$$\frac{d}{d\tau}((r\cdot w)^{-1}\, r_\mu) = (r\cdot w)^{-2}\, (r_\mu - (r\cdot w)\, w_\mu - (r\cdot a)\, r_\mu)$$

Thus, the gradient of the retarded field is

$$\partial_\mu \phi_r(x) = -\int G_r(r)\,(r\cdot w)^{-2}\,(r_\mu - (r\cdot w)\,w_\mu - (r\cdot a)\,r_\mu)\,d\tau$$

I now want to discuss the conservation of energy and momentum. Recall that we can define an energy-momentum tensor for the field

$$T_f^{\alpha\beta} = \Omega_{D-2}^{-1}\,((\partial^\alpha\phi)(\partial^\beta\phi) - \frac{1}{2}\eta^{\alpha\beta}(\partial_\mu\phi)(\partial^\mu\phi))$$

such that

$$\partial_\alpha T_f^{\alpha\beta} = -(\partial^\beta\phi)\rho$$

Using that $\phi = \phi_r + \phi_{in}$, we can decompose the energy-momentum tensor as follows:

$$T_f^{\alpha\beta} = T_{in}^{\alpha\beta} + T_r^{\alpha\beta} + T_{cross}^{\alpha\beta}$$

where

$$T_{in}^{\alpha\beta} = \Omega_{D-2}^{-1}\,((\partial^\alpha\phi_{in})(\partial^\beta\phi_{in}) - \frac{1}{2}\eta^{\alpha\beta}(\partial_\mu\phi_{in})(\partial^\mu\phi_{in}))$$

is the energy-momentum tensor for the *in* field,

$$T_r^{\alpha\beta} = \Omega_{D-2}^{-1}\,((\partial^\alpha\phi_r)(\partial^\beta\phi_r) - \frac{1}{2}\eta^{\alpha\beta}(\partial_\mu\phi_r)(\partial^\mu\phi_r))$$

is the energy-momentum tensor for the retarded field, and

$$T_{cross}^{\alpha\beta} = \Omega_{D-2}^{-1}\,((\partial^\alpha\phi_r)(\partial^\beta\phi_{in}) + (\partial^\beta\phi_r)(\partial^\alpha\phi_{in}) - \eta^{\alpha\beta}(\partial_\mu\phi_r)(\partial^\mu\phi_{in}))$$

is the energy-momentum tensor associated with interference between the *in* field and the retarded field. Note that

$$\Box\phi_r = -\Omega_{D-2}\,\rho$$

and

$$\Box\phi_{in} = 0$$

from which it follows that $T_{in}^{\alpha\beta}$, $T_{cross}^{\alpha\beta}$, and $T_r^{\alpha\beta}$ obey the individual conservation laws

$$\begin{aligned}
\partial_\alpha T_{in}^{\alpha\beta} &= 0 \\
\partial_\alpha T_{cross}^{\alpha\beta} &= -\rho\,\partial^\beta\phi_{in} \\
\partial_\alpha T_r^{\alpha\beta} &= -\rho\,\partial^\beta\phi_r
\end{aligned}$$

5.6.4 Vector field in $3+1$ dimensions

Let us begin our study of vector fields in $3+1$ dimensions by calculating the retarded field produced by a moving particle. We will start with points x that do not lie on the world line of the particle. In section 5.6.2, we showed that the retarded field strength tensor is given by

$$F_r^{\mu\nu} = e \int G_r(r) \left((r \cdot w)^{-2} \, r^{[\mu} \, w^{\nu]} + (r \cdot w)^{-1} \, r^{[\mu} \, s^{\nu]} \right) d\tau$$

The retarded and advanced Greens functions in $3+1$ dimensions are

$$\begin{aligned} G_r(r) &= 2\theta(r^0) \, \delta(r \cdot r) = (r \cdot w)^{-1} \, \delta(\tau - \tau_r) \\ G_a(r) &= 2\theta(-r^0) \, \delta(r \cdot r) = -(r \cdot w)^{-1} \, \delta(\tau - \tau_a) \end{aligned}$$

Thus,

$$F_r^{\mu\nu} = F_{vel}^{\mu\nu} + F_{rad}^{\mu\nu}$$

where

$$\begin{aligned} F_{vel}^{\mu\nu} &= e(r \cdot w)^{-3} \left(r^\mu w^\nu - r^\nu w^\mu \right) \\ F_{rad}^{\mu\nu} &= e(r \cdot w)^{-2} \left(r^\mu s^\nu - r^\nu s^\mu \right) \end{aligned}$$

and where r, w, and s are all evaluated at the retarded proper time $\tau_r(x)$. We see that the retarded field strength tensor is the sum of a velocity field $F_{vel}^{\mu\nu}$, which falls off like $1/R^2$, and a radiation field $F_{rad}^{\mu\nu}$, which falls off like $1/R$.

Now let us calculate the retarded field for points $x = z(\tau')$ that do lie on the particle world line. This requires care, because the field diverges as we approach the particle. We can isolate the divergent part of the field by taking linear combinations of the retarded and advanced field strength tensors (I believe this observation was first made by Dirac):

$$F_\pm^{\mu\nu} = F_r^{\mu\nu} \pm F_a^{\mu\nu}$$

Then

$$F_\pm^{\mu\nu}(x) \mid_{x=z(\tau')} = e \int G_\pm(r(x,\tau)) \, \frac{d}{d\tau} \left(\frac{r^\mu(x,\tau) \, w^\nu(\tau) - r^\nu(x,\tau) \, w^\mu(\tau)}{r(x,\tau) \cdot w(\tau)} \right) d\tau$$

where

$$\begin{aligned} G_+(r) &= G_r(r) + G_a(r) = 2\delta(r \cdot r) \\ G_-(r) &= G_r(r) - G_a(r) = 2\epsilon(r^0) \, \delta(r \cdot r) \end{aligned}$$

We can calculate these fields using a method presented in [38], which involves expanding the integrand in the parameter $u = \tau - \tau'$. Let us define $z \equiv z(\tau')$, $w \equiv w(\tau')$, $a \equiv a(\tau')$, $\dot{a} \equiv \dot{a}(\tau')$. If we expand $z(\tau)$ and $w(\tau)$ in u, we find

$$\begin{aligned} z(\tau) &= z + uw + \frac{1}{2}u^2 a + \frac{1}{6}u^3 \dot{a} + O(u^4) \\ w(\tau) &= w + ua + \frac{1}{2}u^2 \dot{a} + O(u^3) \end{aligned}$$

Thus,

$$r(z,\tau) = z - z(\tau) = -u(w + \frac{1}{2}ua + \frac{1}{6}u^2\dot{a}) + O(u^4)$$

and

$$r(z,\tau) \cdot w(\tau) = -u(1 - \frac{1}{6}u^2\,(a \cdot a)) + O(u^4)$$

where I have used that $a \cdot a + w \cdot \dot{a} = 0$ (this follows from $w \cdot a = 0$). From these results, we find that

$$r^\mu(x,\tau)\, w^\nu(\tau) - r^\nu(x,\tau)\, w^\mu(\tau) = -\frac{u^2}{2} w^{[\mu}\, a^{\nu]} - \frac{u^3}{3} w^{[\mu}\, \dot{a}^{\nu]} + O(u^4)$$

and

$$\frac{r^\mu(x,\tau)\, w^\nu(\tau) - r^\nu(x,\tau)\, w^\mu(\tau)}{r(x,\tau)\cdot w(\tau)} = \frac{u}{2}\, w^{[\mu}\, a^{\nu]} + \frac{u^2}{3}\, w^{[\mu}\, \dot{a}^{\nu]} + O(u^3)$$

If we substitute these into these expressions into the field strength tensor, we obtain

$$F_\pm(z(\tau')) = e \int G_\pm(-uw)\, (\frac{1}{2} w^{[\mu}\, a^{\nu]} + \frac{2}{3} u\, w^{[\mu}\, \dot{a}^{\nu]} + O(u^2))\, du$$

where

$$\begin{aligned} G_+(-uw) &= 2\delta(u^2) \\ G_-(-uw) &= -2\epsilon(u)\, \delta(u^2) = 2\delta'(u) \end{aligned}$$

Thus,

$$\begin{aligned} F_-^{\mu\nu} &= -\frac{4}{3} e\, w^{[\mu}\, \dot{a}^{\nu]} \\ F_+^{\mu\nu} &= e\, w^{[\mu}\, a^{\nu]} \int \delta(u^2)\, du = 2(m_S/e) w^{[\mu}\, a^{\nu]} \end{aligned}$$

where I have defined a divergent self-energy m_S:

$$m_S = \frac{e^2}{2} \int \delta(u^2)\, du$$

The retarded field strength tensor is therefore

$$F_r^{\mu\nu} = \frac{1}{2}(F_+^{\mu\nu} + F_-^{\mu\nu}) = -\frac{2}{3}e\, w^{[\mu}\, \dot{a}^{\nu]} + (m_S/e)w^{[\mu}\, a^{\nu]}$$

The equation of motion for the particle is

$$ma^\mu = eF^{\mu\nu}w_\nu = e(F_{in}^{\mu\nu} + F_r^{\mu\nu})w_\nu$$

Substituting for the retarded field, we find

$$ma^\mu = eF_{in}^{\mu\nu}w_\nu + \frac{2}{3}e^2\,(\dot{a}^\mu + (a\cdot a)\, w^\mu) - m_S a^\mu$$

Or

$$m_R a^\mu = eF_{in}^{\mu\nu}w_\nu + \frac{2}{3}e^2\,(\dot{a}^\mu + (a\cdot a)\, w^\mu)$$

where $m_R = m + m_S$ is the renormalized mass.

Now let us calculate the energy-momentum tensor for the retarded field:

$$T_r^{\mu\nu} = (4\pi)^{-1}(\eta_{\alpha\beta}\, F_r^{\mu\alpha}\, F_r^{\beta\nu} + \frac{1}{4}\eta^{\mu\nu}\, \eta_{\alpha\beta}\, \eta_{\gamma\sigma}\, F_r^{\alpha\gamma}\, F_r^{\beta\sigma})$$

If we substitute $F_r^{\mu\nu} = F_{vel}^{\mu\nu} + F_{rad}^{\mu\nu}$, we find that

$$\eta_{\alpha\mu}\, F_{rad}^{\alpha\beta}\, F_{vel}^{\mu\nu} = 0$$

Thus, the energy-momentum tensor for the retarded field can be expressed as the sum of energy-momentum tensors for the velocity field and for the acceleration field:

$$T_r^{\mu\nu} = T_{vel}^{\mu\nu} + T_{rad}^{\mu\nu}$$

where

$$\begin{aligned} T_{vel}^{\mu\nu} &= (4\pi)^{-1}(\eta_{\alpha\beta}\, F_{vel}^{\mu\alpha}\, F_{vel}^{\beta\nu} + \frac{1}{4}\eta^{\mu\nu}\, \eta_{\alpha\beta}\, \eta_{\gamma\sigma}\, F_{vel}^{\alpha\gamma}\, F_{vel}^{\beta\sigma}) \\ &= (e^2/4\pi)(r\cdot w)^{-5}\,(r^\mu w^\nu + r^\nu w^\mu - \frac{1}{2}(r\cdot w)\,\eta^{\mu\nu}) - (e^2/4\pi)(r\cdot w)^{-6}\, r^\mu r^\nu \end{aligned}$$

and

$$T_{rad}^{\mu\nu} = (4\pi)^{-1}(\eta_{\alpha\beta}\, F_{rad}^{\mu\alpha}\, F_{rad}^{\beta\nu} + \frac{1}{4}\eta^{\mu\nu}\, \eta_{\alpha\beta}\, \eta_{\gamma\sigma}\, F_{rad}^{\alpha\gamma}\, F_{rad}^{\beta\sigma})$$

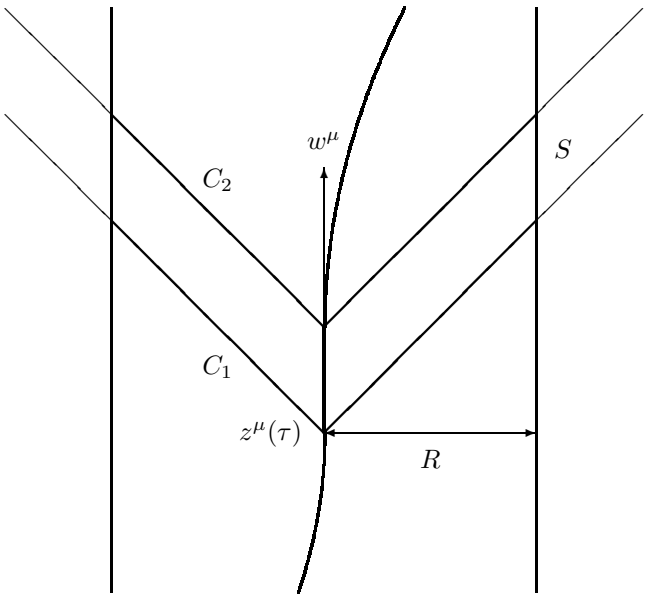

Figure 5.2: The region V is defined by the intersection of surfaces S, C_1, and C_2. This diagram is drawn in the instantaneous rest frame of the particle at proper time τ.

$$\begin{aligned} &= -(e^2/4\pi)\,(r\cdot w)^{-4}\,(s\cdot s)\,r^\mu r^\nu \\ &= -(e^2/4\pi)\,(r\cdot w)^{-2}\,(a\cdot a+(a\cdot n)^2)\,(w^\mu+n^\mu)\,(w^\nu+n^\nu) \end{aligned}$$

Here I have used that

$$\begin{aligned} \eta_{\alpha\mu}\,\eta_{\beta\nu}\,F^{\alpha\beta}_{vel}\,F^{\mu\nu}_{vel} &= -2e^2\,(r\cdot w)^{-4} \\ \eta_{\alpha\mu}\,\eta_{\beta\nu}\,F^{\alpha\beta}_{rad}\,F^{\mu\nu}_{rad} &= 0 \end{aligned}$$

Using these results, we can calculate the momentum that the particle exchanges with the field during a proper time interval $[\tau,\tau+d\tau]$. Let C_1 and C_2 denote the lightcones for points $z(\tau)$ and $z(\tau+d\tau)$. Working in the instantaneous rest frame of the particle at proper time τ, we can define a sphere of radius R that is centered on $z(\tau)$. Let S denote the surface swept out by the spacetime trajectory of this sphere. The total momentum P^α flowing out of the region V that is enclosed by surfaces S, C_1, and C_2 is given by (see Figure 5.2)

$$\Delta P^\alpha = \int_{\partial V} T^{\alpha\beta}_r\,d\sigma_\beta$$

where $d\sigma_\beta(x)$ is the outward-pointing surface element at point x. For points $x \in S$, $d\sigma^\mu$ can be expressed as

$$d\sigma^\mu = -R^2\, n^\mu\, d\Omega\, d\tau$$

where the minus sign is required so that $d\sigma^\mu$ points outward ($n \cdot d\sigma > 0$). For points $x \in C_2$, $d\sigma^\mu$ can be expressed as

$$d\sigma^\mu = \rho^2\, (n^\mu + w^\mu)\, d\Omega\, d\rho = \rho\, r^\mu\, d\Omega\, d\rho$$

while for points $x \in C_1$, $d\sigma^\mu$ can be expressed as

$$d\sigma^\mu = -\rho^2\, (n^\mu + w^\mu)\, d\Omega\, d\rho = -\rho\, r^\mu\, d\Omega\, d\rho$$

where $\rho(x) = r^0(x)$ is a radial coordinate.

Part of the momentum flux is due to radiation, and part is due to momentum corresponding to the velocity field; that is, $P^\alpha = P^\alpha_{rad} + P^\alpha_{vel}$, where

$$\begin{aligned} \Delta P^\alpha_{rad} &= \int_{\partial V} T^{\alpha\beta}_{rad}\, d\sigma_\beta \\ \Delta P^\alpha_{vel} &= \int_{\partial V} T^{\alpha\beta}_{vel}\, d\sigma_\beta \end{aligned}$$

Note that

$$\begin{aligned} r_\mu\, T^{\mu\nu}_{rad} &= 0 \\ r_\mu\, T^{\mu\nu}_{vel} &= \frac{e^2}{8\pi}\, (r \cdot w)^{-4}\, r^\nu \end{aligned}$$

Thus, momentum from the radiation field can only flow through surface S:

$$\Delta P^\alpha_{rad} = \int_S T^{\alpha\beta}_{rad}\, d\sigma_\beta = -(e^2/4\pi)\, w^\alpha\, d\tau \int_S (a \cdot a + (a \cdot n)^2)\, d\Omega$$

In the instantaneous rest frame of the particle at τ,

$$\begin{aligned} a^\mu &= (0, \vec{a}) \\ n^\mu &= (0, \hat{n}) \end{aligned}$$

Thus,

$$(a \cdot n)^2 = (\vec{a} \cdot \hat{n})^2 = |\vec{a}|^2\, \cos^2\theta = -(a \cdot a)\, \cos^2\theta$$

where θ is the angle between $\vec{a}$ and $\hat{n}$. Substituting this result into the integral, we find that

$$\Delta P^\alpha_{rad} = -\frac{2}{3}e^2(a\cdot a)\,w^\alpha\,d\tau$$

Now let us consider the momentum flux corresponding to the velocity field. Momentum from the velocity field can flow through S, C_1, and C_2; however, because the energy-momentum tensor for the velocity field falls off like $1/R^4$, for large R the momentum flow through S is negligible. Thus, momentum corresponding to the velocity field only flows through C_1 and C_2:

$$\begin{aligned}\Delta P^\alpha_{vel} &= \int_{C_2} T^{\alpha\beta}_{vel}\,d\sigma_\beta + \int_{C_1} T^{\alpha\beta}_{vel}\,d\sigma_\beta \\ &= (e^2/2)(w^\alpha(\tau+d\tau) - w^\alpha(\tau))\int_0^R \frac{1}{\rho^2}\,d\rho \\ &= m_S\,a^\alpha(\tau)\,d\tau\end{aligned}$$

where

$$m_S = \frac{e^2}{2}\int_0^R \frac{1}{\rho^2}\,d\rho$$

Thus, the total momentum flux is

$$\frac{\Delta P^\alpha}{d\tau} = \frac{\Delta P^\alpha_{rad}}{d\tau} + \frac{\Delta P^\alpha_{vel}}{d\tau} = -\frac{2}{3}e^2(a\cdot a)\,w^\alpha + m_S\,a^\alpha$$

We obtained this expression for the total momentum flux by evaluating a surface integral over the boundary of the region V. But if we apply Gauss's theorem, we can also express ΔP^α in terms of a volume integral:

$$\Delta P^\alpha = \int_{\partial V} T^{\alpha\beta}_r\,d\sigma_\beta = \int_V \partial_\beta T^{\alpha\beta}_r\,d^4x = -\int_V F^{\alpha\beta}_r J_\beta\,d^4x = -eF^{\alpha\beta}_r(z(\tau_r))\,w_\beta\,d\tau$$

Thus, substituting the expression for $F^{\alpha\beta}_r(z(\tau_r))$ that we derived earlier, we find

$$\frac{\Delta P^\alpha}{d\tau} = -\frac{2}{3}e^2\left(\dot{a}^\alpha + (a\cdot a)\,w^\alpha\right) + m_S a^\alpha$$

This is the same as the expression we found by performing the surface integral, except that there is an additional $\dot{a}$ term. I'm not sure why the surface integral failed to produce this term; I assume it is somehow due to the contribution of the surface integral at the points $z(\tau)$ and $z(\tau+d\tau)$ where the particle intersects the surface. At these points the energy-momentum tensor is singular, and a more careful treatment of the surface integral is probably needed.

5.6.5 Scalar field in $3+1$ dimensions

Let us now consider a scalar field in $3+1$ dimensions and calculate the retarded field produced by a moving particle. We will start with points x that do not lie on the world line of the particle. In section 5.6.3, we showed that the retarded field is given by

$$\partial_\mu \phi_r(x) = -\int G_r(r)\,(r\cdot w)^{-2}\,(r_\mu - (r\cdot w)\,w_\mu - (r\cdot a)\,r_\mu)\,d\tau$$

Substituting for the Greens function, we find

$$\partial_\mu \phi_r = \partial_\mu \phi_{rad} + \partial_\mu \phi_{vel}$$

where

$$\begin{aligned}
\partial_\mu \phi_{rad} &= (r\cdot w)^{-3}\,(r\cdot a)\,r_\mu = (r\cdot w)^{-1}\,(a\cdot n)\,(w_\mu + n_\mu) \\
\partial_\mu \phi_{vel} &= -(r\cdot w)^{-3}\,(r_\mu - (r\cdot w)\,w_\mu) = -(r\cdot w)^{-2}\,n_\mu
\end{aligned}$$

Now consider points $x = z(\tau')$ that do lie on the world line of the particle. The gradients of the fields at x are

$$\partial_\mu \phi_\pm(x) = -\int G_\pm(r(x,\tau))\,\frac{d}{d\tau}\left(\frac{r_\mu(x,\tau)}{r(x,\tau)\cdot w(\tau)}\right)\,d\tau$$

By expanding in $u = \tau - \tau'$, we can show that (see section 5.6.4)

$$\frac{r_\mu(x,\tau)}{r(x,\tau)\cdot w(\tau)} = w_\mu + \frac{1}{2}u\,a_\mu + \frac{1}{6}u^2\,(\dot{a}_\mu + (a\cdot a)\,w_\mu) + O(u^3)$$

So

$$\partial_\mu \phi_\pm(x) = -\int G_\pm(-uw)\,(\frac{1}{2}\,a_\mu + \frac{1}{3}u\,(\dot{a}_\mu + (a\cdot a)\,w_\mu) + O(u^2))\,du$$

Thus,

$$\begin{aligned}
\partial_\mu \phi_- &= \frac{2}{3}(\dot{a}_\mu + (a\cdot a)\,w_\mu) \\
\partial_\mu \phi_+ &= -a_\mu \int \delta(u^2)\,du = -2m_S a_\mu
\end{aligned}$$

where

$$m_S = \frac{1}{2}\int \delta(u^2)\,du$$

is the self-energy. The gradient of the retarded field is therefore

$$\partial_\mu \phi_r = \frac{1}{2}(\partial_\mu \phi_+ + \partial_\mu \phi_-) = \frac{1}{3}(\dot{a}_\mu + (a\cdot a)\,w_\mu) - m_S a_\mu$$

The energy-momentum tensor for the radiation field is

$$\begin{aligned} T^{\alpha\beta}_{rad} &= (4\pi)^{-1}[(\partial^\alpha\phi)(\partial^\beta\phi) - \frac{1}{2}\eta^{\alpha\beta}(\partial_\mu\phi)(\partial^\mu\phi)] \\ &= (4\pi)^{-1}\,(r\cdot w)^{-2}\,(a\cdot n)^2\,(w^\alpha + n^\alpha)\,(w^\beta + n^\beta) \end{aligned}$$

Using this expression, we can calculate ΔP^α_{rad} as defined in the previous section:

$$\begin{aligned} \Delta P^\alpha_{rad} &= \int_S T^{\alpha\beta}_{rad}\, d\sigma_\beta \\ &= (4\pi)^{-1}\, d\tau \int_S (n\cdot a)^2\,(w^\alpha + n^\alpha)\, d\Omega \\ &= (4\pi)^{-1}\, d\tau\, w^\alpha \int_S (n\cdot a)^2\, d\Omega \\ &= -\frac{1}{3}\,(a\cdot a)\, w^\alpha\, d\tau \end{aligned}$$

where I have used that

$$(n\cdot a)^2 = -(a\cdot a)\,\cos^2\theta$$

as was shown in section 5.6.4.

5.6.6 Vector field in $1+1$ dimensions

Consider a particle coupled to a vector field in $1+1$ dimensions. The retarded Greens function in $1+1$ dimensions is

$$G_r(r) = \theta(r^0)\,\theta(r\cdot r)$$

Note that

$$\frac{d}{d\tau}G_r(r) = -2(r\cdot w)\,\theta(r^0)\,\delta(r\cdot r) = -\delta(\tau-\tau_r)$$

Thus, the gradient of the retarded field produced by the particle is

$$\partial_\nu A^\mu_r = -e\int (r\cdot w)^{-1}\, r_\nu w^\mu\, \frac{d}{d\tau}G_r(r)\, d\tau = e\,(r\cdot w)^{-1}\, r_\nu w^\mu = e\,(w_\nu + n_\nu)\, w^\mu$$

where r and w are evaluated at the retarded proper time $\tau_r(x)$. The retarded field strength tensor is

$$F^{\mu\nu}_r = \partial^\mu A^\nu_r - \partial^\nu A^\mu_r = e\,(n^\mu w^\nu - n^\nu w^\mu) = E\,\epsilon^{\mu\nu}$$

Thus, the electric field is

$$E = e\,\epsilon_{\mu\nu}\, w^\mu\, n^\nu = e\,\epsilon(x^1 - z(x^0))$$

Here I have used that in $1+1$ dimensions, n can be expressed in terms of w:

$$n_\nu(x) = -\epsilon(x^1 - z^1(\tau_r))\,\epsilon_{\mu\nu}\,w^\mu(\tau_r)$$

so

$$\epsilon_{\mu\nu}\,w^\mu n^\nu = \epsilon(x^1 - z^1(\tau_r)) = \epsilon(x^1 - z^1(x^0))$$

5.6.7 Scalar field in $1+1$ dimensions

Now consider a particle coupled to a scalar field in $1+1$ dimensions. The retarded and advanced Greens functions in $1+1$ dimensions are

$$\begin{aligned} G_r(r) &= \theta(r^0)\,\theta(r\cdot r) \\ G_a(r) &= \theta(-r^0)\,\theta(r\cdot r) \end{aligned}$$

Note that

$$\frac{d}{d\tau}G_r(r) = -2(r\cdot w)\,\theta(r^0)\,\delta(r\cdot r) = -\delta(\tau-\tau_r)$$

Using the results of section 5.6.3, we find that for points x that do not lie on the world line of the particle, the gradient of the retarded field is

$$\partial_\mu\phi_r(x) = \int (r\cdot w)^{-1}\,r_\mu\,\frac{d}{d\tau}G_r(r)\,d\tau = -(r\cdot w)^{-1}\,r_\mu = -w_\mu - n_\mu$$

where r and w are evaluated at the retarded proper time $\tau_r(x)$.

For points $x = z(\tau')$ that do lie on the world line of the particle, a more careful treatment is needed. The field gradients are

$$\partial_\mu\phi_{r,a}(x) = \int (r\cdot w)^{-1}\,r_\mu\,\frac{d}{d\tau}G_{r,a}(r)\,d\tau$$

As in section 5.6.4, we will define $z \equiv z(\tau')$, $w \equiv w(\tau')$, and expand the integrand in the parameter $u = \tau - \tau'$. We find

$$(r\cdot w)^{-1}\,r_\mu = w_\mu + O(u)$$

Thus, the field gradients are

$$\partial_\mu\phi_{r,a}\mid_{x=z(\tau')} = w_\mu \int \frac{d}{du}G_{r,a}(-uw)\,du$$

Note that

$$\begin{aligned} G_r(-uw) &= \theta(-u) \\ G_a(-uw) &= \theta(u) \end{aligned}$$

So

$$\begin{aligned} \partial_\mu \phi_r \mid_{x=z(\tau)} &= -w_\mu(\tau) \\ \partial_\mu \phi_a \mid_{x=z(\tau)} &= w_\mu(\tau) \end{aligned}$$

In summary, for points x that do not lie on the particle trajectory, the gradient of the retarded field is

$$\partial_\mu \phi_r(x) = -w_\mu(\tau_r(x)) - n_\mu(\tau_r(x)).$$

For points $x = z(\tau)$ that do lie on the world line of the particle, the gradient of the retarded field is

$$\partial_\mu \phi_r(z(\tau)) = -w_\mu(\tau)$$

The equation of motion for the particle is

$$\frac{d}{d\tau} w^\mu + (1+\phi)^{-1} (\partial_\nu \phi)(w^\mu w^\nu - \eta^{\mu\nu}) = 0$$

Or, using that $\phi = \phi_{in} + \phi_r$,

$$\frac{d}{d\tau} w^\mu + (1+\phi_r+\phi_{in})^{-1} (\partial_\nu \phi_{in})(w^\mu w^\nu - \eta^{\mu\nu}) = 0$$

We can evaluate ϕ_r using the general expression we found in section 5.6.3:

$$\phi_r(z(\tau')) = -\int G_r(z(\tau') - z(\tau))\, d\tau = -\int \theta(z^0(\tau') - z^0(\tau))\, d\tau = \tau_0 - \tau'$$

where τ_0 is an integration constant. Thus,

$$\frac{d}{d\tau} w^\mu + (1+\tau_0-\tau+\phi_{in})^{-1} (\partial_\nu \phi_{in})(w^\mu w^\nu - \eta^{\mu\nu}) = 0$$

The dependence of the equation of motion on the absolute proper time τ is strange, and I'm not sure how to understand this result. It seems to indicate that as $|\tau - \tau_0| \to \pm\infty$, the force on the particle goes to zero.

Let us now calculate the energy-momentum tensor for the retarded field. For points that do not lie on the world line of the particle, the energy-momentum tensor is

$$T_r^{\alpha\beta} = (1/2)(r \cdot w)^{-2}\, r^\alpha r^\beta = (1/2)(w^\alpha + n^\alpha)(w^\beta + n^\beta)$$

We can check that this satisfies the energy conservation equation

$$\partial_\alpha T_r^{\alpha\beta} = -\rho\, \partial^\beta \phi_r$$

For points that do not lie on the world line of the particle $\rho = 0$, so the energy conservation equation becomes

$$\partial_\alpha T_r^{\alpha\beta} = 0$$

We can check this result by evaluating the partial derivative

$$\begin{aligned} \partial_\alpha T_r^{\alpha\beta} &= (1/2)\, \partial_\alpha ((r \cdot w)^{-2}\, r^\alpha r^\beta) \\ &= (1/2)\, (r \cdot w)^{-2}\, (r^\beta\, \partial_\alpha r^\alpha + r^\alpha\, \partial_\alpha r^\beta) - (r \cdot w)^{-3}\, r^\beta r^\alpha\, \partial_\alpha (r \cdot w) \end{aligned}$$

In section 5.6.1 we showed that $r^\alpha\, \partial_\alpha r^\beta = r^\beta$, $r^\alpha\, \partial_\alpha \tau_r = 0$, and $\partial_\alpha r^\alpha = 1$. Thus,

$$r^\alpha\, \partial_\alpha (r \cdot w) = r^\alpha (\partial_\alpha r^\beta) w_\beta = r \cdot w$$

So we find that $\partial_\alpha T_r^{\alpha\beta} = 0$, as expected.

We can use our expression for the energy-momentum tensor of the retarded field to calculate the momentum that the particle exchanges with the field during a proper time interval $[\tau, \tau + d\tau]$. Let us consider a spacetime region V defined as in section 5.6.4. The total momentum flowing out of this region is

$$\Delta P^\alpha = \int_{\partial V} T_r^{\alpha\beta}\, d\sigma_\beta$$

where for points $x \in S$, the surface element $d\sigma^\mu$ is

$$d\sigma^\mu = -n^\mu\, d\tau$$

for points $x \in C_2$, the surface element $d\sigma^\mu$ is

$$d\sigma^\mu = (n^\mu + w^\mu)\, d\rho = \frac{1}{\rho} r^\mu\, d\rho$$

and for points $x \in C_1$, the surface element $d\sigma^\mu$ is

$$d\sigma^\mu = -(n^\mu + w^\mu)\, d\rho = -\frac{1}{\rho} r^\mu \, d\rho$$

where $\rho = r^0$. Since $r_\mu T_r^{\mu\nu} = 0$, only the surface S contributes to the surface integral:

$$\Delta P^\alpha = \int_S T_r^{\alpha\beta}\, d\sigma_\beta = w^\alpha \, d\tau$$

where I have substituted $T_r^{\alpha\beta} = (1/2)(w^\alpha + n^\alpha)(w^\beta + n^\beta)$.

We can check this result by using Gauss's theorem to convert the surface integral to a volume integral:

$$\Delta P^\alpha = \int_{\partial V} T_r^{\alpha\beta}\, d\sigma_\beta = \int_V \partial_\beta T_r^{\alpha\beta}\, d^2x$$

If we substitute the energy conservation equation

$$\partial_\beta T_r^{\alpha\beta} = -\rho\, \partial^\alpha \phi_r$$

we obtain

$$\Delta P^\alpha = -\int_V \rho\, \partial^\alpha \phi_r \, d^2x = -\partial^\alpha \phi_r \mid_{x=z(\tau)} \, d\tau = w^\alpha \, d\tau$$

which agrees with our previous result.

5.7 Gravity

It is straightforward to write down the 1+1 dimensional analog of Newton's theory of gravity. The field equation is

$$\partial_x^2 \phi = 2G\rho$$

where ρ is the mass density. For a point particle following a trajectory $z(t)$, the mass density is

$$\rho(t, x) = m \int \delta(x - z(t))\, dt$$

The equation of motion for the particle is

$$\ddot{z} = -\partial_x \phi$$

It is not so straightforward to write down the analog of general relativity in 1+1 dimensions. Since the Einstein field equations involve tensors that are well defined in any number of dimensions, the

most obvious approach is to simply apply them to the 1+1 dimensional case. This doesn't work, however, for the following reason. In 1+1 dimensions, the Riemann tensor can be expressed in terms of the curvature scalar [52]:

$$R^{\gamma}{}_{\nu\alpha\beta} = \frac{R}{2}\, g^{\gamma\mu}\left(g_{\mu\alpha}\, g_{\nu\beta} - g_{\mu\beta}\, g_{\nu\alpha}\right)$$

From this result, it follows that the Ricci tensor is

$$R_{\nu\beta} = R^{\gamma}{}_{\nu\gamma\beta} = \frac{R}{2}\, g_{\nu\beta}$$

which implies that in 1+1 dimensions the Einstein tensor vanishes:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{R}{2}\, g_{\mu\nu} = 0$$

and the Einstein field equations are

$$G_{\mu\nu} = 8\pi G\, T_{\mu\nu} = 0$$

Thus, the Einstein field equations do not constrain the metric tensor and simply state that the energy-momentum tensor vanishes.

In the following sections, I consider an alternative theory of gravity in 1+1 dimensions, which is described by the field equation

$$R = 4G g_{\alpha\beta} T^{\alpha\beta}$$

This is the direct 1+1 dimensional analog of a theory of gravity in 3+1 dimensions that was proposed by Nordstrøm in 1913. His field equations were (see [53] and the discussion of prior geometry in [54])

$$R = 24\pi G g_{\alpha\beta} T^{\alpha\beta}$$

$$C^{\alpha\beta}{}_{\mu\nu} = 0$$

where $C^{\alpha\beta}{}_{\mu\nu}$ is the Weyl tensor. In 1+1 dimensions, the Weyl tensor vanishes identically, so the Nordstrøm equations are equivalent to the proposed field equation for 1+1 dimensional gravity. This theory of $1+1$ dimensional gravity was first written down by Robert Mann, who has investigated the theory in depth (see [55] and [56], for example). Here I derive the field equation for 1+1 dimensional gravity using two different approaches, and then I present the 1+1 dimensional analog of the Schwarzschild solution and the solution for a static star of uniform density.

5.7.1 Field equation: first approach

The first approach is to consider gravitational tidal forces. In Newtonian gravity, two nearby test particles that are falling freely can experience a relative acceleration due to local variations in the strength of the gravitational field. We can attempt to interpret this effect geometrically: we assume that the particles move along geodesics in a curved spacetime, and that the relative acceleration is the geodesic deviation caused by the curvature. By comparing the Newtonian theory with the geometric interpretation we can relate the mass density to the curvature tensor, and thereby obtain the 1+1 dimensional analog of the Einstein field equations.

We will begin by using the Newtonian theory to derive an equation for the tidal acceleration of two nearby freely falling particles A and B. Let $z(t)$ denote the position of particle A, and let $z(t) + \Delta(t)$ denote the position of particle B. The equation of motion for particle A is

$$\ddot{z} = -\partial_x \phi$$

and the equation of motion for particle B is

$$\ddot{z} + \ddot{\Delta} = -\partial_x \phi + \Delta\, \partial_x^2 \phi$$

where I have assumed that $\Delta(t)$, the separation between the two particles, is small relative to the spatial variation of the gravitational field. If we subtract the first equation from the second, we find that

$$\ddot{\Delta} + (\partial_x^2 \phi)\, \Delta = 0$$

We can substitute the Newtonian field equation to obtain

$$\ddot{\Delta} + 2G\rho\, \Delta = 0$$

We would like to interpret this as the nonrelativistic limit of the equation for geodesic deviation:

$$\frac{d^2\Delta}{ds^2} + \frac{R}{2}\, \Delta = 0$$

Thus, we are led to identify

$$R = 4G\rho$$

This should be the nonrelativistic limit of our new field equation. This cannot be the correct relativistic field equation, since R is a scalar under general coordinate transformations, while ρ is the time-time component of a rank two tensor (the energy-momentum tensor). However, note that

the trace of the energy-momentum tensor is a scalar and reduces to ρ in the nonrelativistic limit:

$$g_{\alpha\beta}T^{\alpha\beta} = \rho - p \to \rho$$

Thus, we take

$$R = 4Gg_{\alpha\beta}T^{\alpha\beta}$$

as our gravitational field equation.

5.7.2 Field equation: second approach

The second approach is to start with a relativistic field theory in flat spacetime and attempt to modify it so as to incorporate the principle of equivalence. This approach is meant to parallel Feynman's derivation of the Einstein field equations starting from a spin-2 field in flat spacetime [57]. The principle of equivalence states that gravity couples to all forms of energy-momentum, including the energy-momentum of the gravitational field itself. Thus, I will consider a system consisting of a point particle interacting with a scalar field, where the form of the interaction is determined by requiring that the field couple to the trace of the total energy-momentum tensor for the system. This gives a nonlinear field theory in flat spacetime, which can then be reinterpreted as a field theory in curved spacetime with metric tensor $g_{\alpha\beta} = e^{2\phi}\,\eta_{\alpha\beta}$.

Consider a theory in flat spacetime that describes a point particle coupled to a scalar field ϕ. The Lagrangian for the system is

$$\mathcal{L} = \frac{1}{4G}(\partial_\mu\phi)(\partial^\mu\phi) - (1 + f(\phi))\rho$$

where $f(\phi)$ is a function that describes the coupling, G is the coupling strength, and

$$\rho(x) = m\int \delta^{(2)}(x - z(\tau))\,d\tau$$

is the mass density in the rest frame of the particle. We will determine $f(\phi)$ by requiring that the field couple to the trace of the total energy-momentum tensor for the system. From the Lagrangian, it follows that the total energy-momentum tensor is

$$\begin{aligned} T^{\alpha\beta}_{(\eta)} &= \frac{1}{2G}((\partial^\alpha\phi)(\partial^\beta\phi) - \frac{1}{2}\eta^{\alpha\beta}(\partial_\mu\phi)(\partial^\mu\phi)) + \\ &\quad m(1+f)\int w^\alpha w^\beta\,\delta^{(2)}(x - z(\tau))\,d\tau \end{aligned}$$

The subscript indicates that the energy-momentum tensor is defined relative to the point of view in which the particle is moving in flat spacetime with metric $\eta_{\mu\nu}$; later, when we consider the particle to be moving in a curved spacetime with metric $g_{\mu\nu}$, we will need to distinguish the flat space energy-momentum tensor $T^{\alpha\beta}_{(\eta)}$ from the curved space energy-momentum tensor $T^{\alpha\beta}_{(g)}$.

The field equation is

$$\Box\phi = -2G\,\frac{df}{d\phi}\,\rho$$

The equation of motion for the particle is

$$\frac{d}{d\tau}w^\mu + (1+f)^{-1}(\partial_\nu f)(w^\mu w^\nu - \eta^{\mu\nu}) = 0$$

From these it follows that the energy-momentum tensor is conserved:

$$\partial_\alpha T^{\alpha\beta}_{(\eta)} = 0$$

We want to choose $f(\phi)$ such that the field couples to the trace of the total energy-momentum tensor:

$$\Box\phi = -2G\,\eta_{\alpha\beta}T^{\alpha\beta}_{(\eta)}$$

If we substitute the field equation and the expression for $T^{\alpha\beta}_{(\eta)}$, we see that this condition implies

$$\frac{df}{d\phi} = 1 + f$$

So

$$f(\phi) = Ae^\phi - 1$$

for some constant A. If the field vanishes, the particle should be free, so we require that $f(0) = 0$, which implies $A = 1$:

$$f(\phi) = e^\phi - 1$$

Thus, the Lagrangian is

$$\mathcal{L} = \frac{1}{4G}(\partial_\mu\phi)(\partial^\mu\phi) - e^\phi\rho$$

the energy-momentum tensor is

$$\begin{aligned} T^{\alpha\beta}_{(\eta)} &= \frac{1}{2G}((\partial^\alpha\phi)(\partial^\beta\phi) - \frac{1}{2}\eta^{\alpha\beta}(\partial_\mu\phi)(\partial^\mu\phi)) + \\ &\quad me^\phi\int w^\alpha w^\beta\,\delta^{(2)}(x - z(\tau))\,d\tau \end{aligned}$$

the field equation is

$$\Box\phi = -2Ge^{\phi}\rho$$

and the equation of motion is

$$\frac{d}{d\tau}w^{\mu} + (\partial_{\nu}\phi)(w^{\mu}w^{\nu} - \eta^{\mu\nu}) = 0$$

As we have seen in section 5.4.3, we can view the particle as moving freely in a curved spacetime with metric tensor

$$g_{\mu\nu} = (1+f)^2\,\eta_{\mu\nu} = e^{2\phi}\eta_{\mu\nu}$$

From this point of view, the energy-momentum tensor of the particle is

$$T^{\alpha\beta}_{(g)}(x) = mg^{-1/2}\int u^{\alpha}u^{\beta}\,\delta^{(2)}(x - z(s))\,ds$$

Thus,

$$g_{\alpha\beta}T^{\alpha\beta}_{(g)} = mg^{-1/2}\int \delta^{(2)}(x - z(s))\,ds = me^{-2\phi}\int \delta^{(2)}(x - z(s))\,ds$$

Note that

$$ds = e^{\phi}\,d\tau$$

Thus,

$$\eta_{\alpha\beta}T^{\alpha\beta}_{(\eta)} = me^{\phi}\int \delta^{(2)}(x - z(\tau))\,d\tau = m\int \delta^{(2)}(x - z(s))\,ds = e^{2\phi}\,g_{\alpha\beta}T^{\alpha\beta}_{(g)}$$

The curvature scalar corresponding to $g_{\mu\nu}$ is (see the following section)

$$R = -2e^{-2\phi}\,\eta^{\mu\nu}\partial_{\mu}\partial_{\nu}\phi = -2e^{-2\phi}\,\Box\phi$$

So we can express the flat spacetime field equation as

$$R = 4G\,g_{\alpha\beta}T^{\alpha\beta}_{(g)}$$

In summary, from one point of view the spacetime is flat, and the particle interacts with a scalar field ϕ. The field equation is

$$\Box\phi = -2Ge^{\phi}\rho$$

and the equation of motion for the particle is

$$\frac{d}{d\tau}w^{\mu} + (\partial_{\nu}\phi)(w^{\mu}w^{\nu} - \eta^{\mu\nu}) = 0$$

From another point of view, the spacetime is curved, and the particle falls freely. The metric tensor satisfies

$$R = 4G\, g_{\alpha\beta} T^{\alpha\beta}_{(g)}$$

and the equation of motion for the particle is

$$\frac{d}{ds} u^\gamma + \Gamma^\gamma{}_{\alpha\beta} u^\alpha u^\beta = 0$$

5.7.3 Geometry in conformal coordinates

Here I calculate the Christoffel symbols and curvature scalar for the metric

$$g_{\mu\nu} = e^{2\phi} \eta_{\mu\nu}$$

Note that

$$g = -\det g_{\mu\nu} = e^{4\phi}$$

The Christoffel symbols are

$$\begin{aligned} \Gamma^\mu{}_{\alpha\beta} &= \frac{1}{2} g^{\mu\nu} (g_{\nu\beta,\alpha} + g_{\alpha\nu,\beta} - g_{\alpha\beta,\nu}) \\ &= \eta^{\mu\nu} (\eta_{\nu\beta}\, \partial_\alpha \phi + \eta_{\alpha\nu}\, \partial_\beta \phi - \eta_{\alpha\beta}\, \partial_\nu \phi) \\ &= \delta^\mu{}_\beta\, \partial_\alpha \phi + \delta^\mu{}_\alpha\, \partial_\beta \phi - g_{\alpha\beta} g^{\mu\nu}\, \partial_\nu \phi \end{aligned}$$

Thus, the geodesic equation

$$\frac{d}{ds} u^\mu + \Gamma^\mu{}_{\alpha\beta} u^\alpha u^\beta = 0$$

can be expressed as

$$\frac{d}{ds} u^\mu + (\partial_\nu \phi)(2u^\mu u^\nu - g^{\mu\nu}) = 0$$

The Riemann curvature tensor is

$$R^\mu{}_{\alpha\nu\beta} = \partial_\nu \Gamma^\mu{}_{\alpha\beta} - \partial_\beta \Gamma^\mu{}_{\alpha\nu} + \Gamma^\gamma{}_{\alpha\beta} \Gamma^\mu{}_{\gamma\nu} - \Gamma^\gamma{}_{\alpha\nu} \Gamma^\mu{}_{\gamma\beta}$$

The scalar curvature is

$$R = g^{\alpha\beta} R^\mu{}_{\alpha\mu\beta} = g^{\alpha\beta} (\partial_\mu \Gamma^\mu{}_{\alpha\beta} - \partial_\beta \Gamma^\mu{}_{\alpha\mu} + \Gamma^\gamma{}_{\alpha\beta} \Gamma^\mu{}_{\gamma\mu} - \Gamma^\gamma{}_{\alpha\mu} \Gamma^\mu{}_{\gamma\beta})$$

For the first term, note that

$$\partial_\mu \Gamma^\mu{}_{\alpha\beta} = 2\partial_\alpha \partial_\beta \phi - g_{\alpha\beta} g^{\mu\nu} \partial_\mu \partial_\nu \phi$$

Thus,

$$g^{\alpha\beta}(\partial_\mu \Gamma^\mu{}_{\alpha\beta}) = 0$$

For the second term, note that

$$\Gamma^\mu{}_{\alpha\mu} = 2\partial_\alpha \phi$$

Thus,

$$g^{\alpha\beta}(\partial_\beta \Gamma^\mu{}_{\alpha\mu}) = 2g^{\alpha\beta}\partial_\alpha \partial_\beta \phi$$

For the third term, note that

$$g^{\alpha\beta}\Gamma^\mu{}_{\alpha\beta} = 0$$

Thus,

$$g^{\alpha\beta}\Gamma^\mu{}_{\alpha\beta}\Gamma^\nu{}_{\mu\nu} = 0$$

For the last term,

$$g^{\alpha\beta}\Gamma^\mu{}_{\alpha\nu}\Gamma^\nu{}_{\mu\beta} = 0$$

This can be seen by substituting for the Christoffel symbols. Thus, the scalar curvature is

$$R = -2g^{\mu\nu}\partial_\mu \partial_\nu \phi = -2e^{-2\phi}\eta^{\mu\nu}\partial_\mu \partial_\nu \phi$$

5.7.4 Schwarzschild solution

I now want to present some solutions for the theory of gravity in 1+1 dimensions. We will begin with the 1+1 dimensional analog of the Schwarzschild solution, which was first written down in [58]. The derivation I give here parallels the derivation of the Schwarzschild solution that is presented in [59].

For a static solution, the metric has the form

$$ds^2 = A(y)\, dt^2 - B(y)\, dy^2$$

Note that because the solution is static, there is no $dt\, dx$ term. If $A(y)$ and $B(y)$ are positive and nonsingular, we can define a coordinate transformation

$$\bar{y} = \int_0^y (A(u)B(u))^{1/2}\, du$$

If we perform this transformation and drop the bars, the metric takes the form

$$ds^2 = A(y)\, dt^2 - A^{-1}(y)\, dy^2$$

The scalar curvature for this metric is

$$R = \frac{d^2 A}{dy^2}$$

I will assume that $g_{\alpha\beta} T^{\alpha\beta} = 0$ for $y \neq 0$, and that the space is symmetric under $y \to -y$. Then the solution to the field equation is

$$A(y) = 2a|y| + b$$

for some constants a and b. Thus, the metric tensor is

$$ds^2 = (2a|y| + b)\, dt^2 - (2a|y| + b)^{-1}\, dy^2$$

Note that a must be positive so that as $y \to \infty$ the metric has the correct signature, but b may have either sign. For $b < 0$, there are event horizons at $x = \pm b/2a$, which are analogous to the event horizons of the Schwarzschild solution. For $b > 0$, there are no horizons.

To gain some insight into the $b > 0$ class of solutions, let us look at the solutions in several different coordinate systems. We will call the original (t, x) coordinates Schwarzschild coordinates. Let us consider a transformation from the Schwarzchild coordinates (t, x) to a new set of coordinates (u, v) that are given by

$$\begin{aligned} u &= a^{-1} (2a|x| + b)^{1/2} \sinh at \\ v &= a^{-1} \epsilon(x) (2a|x| + b)^{1/2} \cosh at \end{aligned}$$

The (u, v) coordinates will be called Kruskal coordinates, after an analogous coordinate system used to describe the Schwarzschild solution [60]. Note that

$$\begin{pmatrix} du \\ dv \end{pmatrix} = \begin{pmatrix} (2ax + b)^{1/2} \cosh at & \epsilon(x) (2ax + b)^{-1/2} \sinh at \\ \epsilon(x) (2ax + b)^{1/2} \sinh at & (2ax + b)^{-1/2} \cosh at \end{pmatrix} \begin{pmatrix} dt \\ dx \end{pmatrix}$$

If we express the metric in terms of the Kruskal coordinates, we find

$$ds^2 = (2ax + b)\, dt^2 - (2ax + b)^{-1}\, dx^2 = du^2 - dv^2$$

Thus, the Kruskal coordinates describe a flat spacetime in Lorentz coordinates. However, only the regions to the left and to the right of the hyperbolas given by $v^2 - u^2 = b/a^2$ describe physical events. These two regions are glued together along the hyperbolas; that is, if an event has coordinates (u, v) such that $v^2 - u^2 = b/a^2$, then (u, v) and $(u, -v)$ are two coordinates for the same event. We can therefore consider the left and right hyperbolic regions as two separate coordinate patches. Instead

of describing both regions with the single coordinate system (u, v), let us describe the left region with coordinates (u_-, v_-), defined by

$$\begin{aligned} u_- &= a^{-1}(b-2ax)^{1/2}\sinh at \\ v_- &= -a^{-1}(b-2ax)^{1/2}\cosh at \end{aligned}$$

and the right region with coordinates (u_+, v_+), defined by

$$\begin{aligned} u_+ &= a^{-1}(2ax+b)^{1/2}\sinh at \\ v_+ &= a^{-1}(2ax+b)^{1/2}\cosh at \end{aligned}$$

An event with Schwarzschild coordinates (t, x) is described by the (u_+, v_+) coordinate system if $x > 0$, and by the (u_-, v_-) coordinate system if $x < 0$. Points with $x = 0$ (that is, points on the singularity) are described by both coordinate systems, which are related by

$$\begin{aligned} u_+ &= u_- \\ v_+ &= -v_- \end{aligned}$$

In order to glue the coordinate systems together, we need to relate the coordinate descriptions of vectors in the tangent space of the overlap region $x = 0$. Note that for $x > 0$,

$$\begin{pmatrix} du_+ \\ dv_+ \end{pmatrix} = \begin{pmatrix} (2ax+b)^{1/2}\cosh at & (2ax+b)^{-1/2}\sinh at \\ (2ax+b)^{1/2}\sinh at & (2ax+b)^{-1/2}\cosh at \end{pmatrix} \begin{pmatrix} dt \\ dx \end{pmatrix}$$

So as $x \to 0$ from above,

$$\begin{pmatrix} du_+ \\ dv_+ \end{pmatrix} = \begin{pmatrix} b^{1/2}\cosh at & b^{-1/2}\sinh at \\ b^{1/2}\sinh at & b^{-1/2}\cosh at \end{pmatrix} \begin{pmatrix} dt \\ dx \end{pmatrix}$$

For $x < 0$,

$$\begin{pmatrix} du_- \\ dv_- \end{pmatrix} = \begin{pmatrix} (b-2ax)^{1/2}\cosh at & -(b-2ax)^{-1/2}\sinh at \\ -(b-2ax)^{1/2}\sinh at & (b-2ax)^{-1/2}\cosh at \end{pmatrix} \begin{pmatrix} dt \\ dx \end{pmatrix}$$

So as $x \to 0$ from below,

$$\begin{pmatrix} du_- \\ dv_- \end{pmatrix} = \begin{pmatrix} b^{1/2}\cosh at & -b^{-1/2}\sinh at \\ -b^{1/2}\sinh at & b^{-1/2}\cosh at \end{pmatrix} \begin{pmatrix} dt \\ dx \end{pmatrix}$$

Thus, at $x = 0$, the tangent spaces for the two coordinate systems are related by

$$\begin{pmatrix} du_- \\ dv_- \end{pmatrix} = \begin{pmatrix} \cosh 2at & -\sinh 2at \\ -\sinh 2at & \cosh 2at \end{pmatrix} \begin{pmatrix} du_+ \\ dv_+ \end{pmatrix}$$

This is just a Lorentz transformation with $\gamma = \tanh 2at$.

In summary, the Kruskal coordinates are convenient because in this system the metric tensor is the Minkowski metric. The cost of this simplification is that instead of one global coordinate patch (t, y) we need two separate coordinate patches (u_+, v_+) and (u_-, v_-), which are glued together in a nontrivial way.

Let us now consider a second coordinate transformation, which will allow us to express the solution in terms of a scalar field in flat spacetime. Consider a transformation from the Schwarzschild coordinates (t, y) to a new set of coordinates (t, x) that are defined by

$$y = \frac{1}{2a}\epsilon(x)(e^{2\phi(x)} - e^{2\phi_0})$$

where

$$\phi(x) = a|x| + \phi_0$$

and

$$\phi_0 = \frac{1}{2}\log b$$

Note that

$$A(y) = 2a|y| + b = e^{2\phi(x)}$$

and

$$dy = e^{2\phi(x)}\, dx$$

Thus, the metric in the new coordinate system is

$$ds^2 = e^{2\phi(x)}\,(dt^2 - dx^2)$$

This is just the Lorentz metric multiplied by a conformal factor, so I will call the (t, x) coordinates conformal coordinates. Using the results of section 5.4.3, we can interpret the system as describing a flat spacetime with a scalar field $\phi(x)$.

5.7.5 Solution for a static star of uniform density

Here I present a solution to the field equation for a static star of uniform density. I will assume the star is a perfect fluid, so its energy-momentum tensor is given by

$$T^{\alpha\beta} = (\rho + p)\, u^\alpha u^\beta - p\, g^{\alpha\beta}$$

where $\rho(x)$, $p(x)$, and $u^\alpha(x)$ are the density, pressure, and velocity of the fluid at point x. I assume the metric tensor has the form

$$g_{\mu\nu} = e^{2\phi}\eta_{\mu\nu}$$

Since the fluid is static, the components of the velocity vector u^α are

$$\begin{aligned} u^0 &= e^{-\phi} \\ u^1 &= 0 \end{aligned}$$

Thus, the components of the energy-momentum tensor are

$$\begin{aligned} T^{00} &= e^{-2\phi}\,\rho \\ T^{11} &= e^{-2\phi}\,p \\ T^{01} &= 0 \\ T^{10} &= 0 \end{aligned}$$

The energy-momentum tensor obeys the conservation law

$$T^{\alpha\beta}{}_{;\beta} = \partial_\beta T^{\alpha\beta} + \Gamma^\alpha{}_{\mu\beta}\, T^{\mu\beta} + \Gamma^\beta{}_{\mu\beta}\, T^{\alpha\mu} = 0$$

If we substitute for Christoffel symbols (see section 5.7.3), we find

$$\begin{aligned} T^{1\beta}{}_{;\beta} &= \partial_\beta T^{1\beta} + \Gamma^1{}_{\mu\beta}\, T^{\mu\beta} + \Gamma^\beta{}_{\mu\beta}\, T^{1\mu} \\ &= \partial_x T^{11} + (\delta^1{}_\beta\, \partial_\mu\phi + \delta^1{}_\mu\, \partial_\beta\phi - g_{\mu\beta}\, g^{1\nu}\, \partial_\nu\phi)\, T^{\mu\beta} + 2\partial_\mu\phi\, T^{1\mu} \\ &= e^{-2\phi}\,(\partial_x p + (\rho + p)\,\partial_x\phi) \end{aligned}$$

Thus, we obtain an equation of hydrostatic equilibrium:

$$\partial_x p = -(\rho + p)\,\partial_x\phi$$

The gravitational field equation is

$$R = -2e^{-2\phi}\,\Box\phi = 4G\,g_{\alpha\beta}T^{\alpha\beta} = 4G(\rho - p)$$

Since the field is static, this reduces to

$$\partial_x^2\phi = 2G(\rho - p)\,e^{2\phi}$$

Thus, the pressure $p(x)$ and the gravitational potential $\phi(x)$ obey the coupled set of equations

$$\begin{aligned}\partial_x^2\phi &= 2G(\rho - p)\,e^{2\phi} \\ \partial_x p &= -(\rho + p)\,\partial_x\phi\end{aligned}$$

We can express the solution to these equations in terms of the pressure and gravitational potential at the center of the star ($p_0 = p(0)$, $\phi_0 = \phi(0)$).

The equations can be simplified by defining a dimensionless spatial coordinate u and a dimensionless pressure $P(u)$:

$$u = (4Gp_0)^{1/2}\,e^{\phi_0}\,x$$

$$P(u) = p(u)/p_0$$

In terms of $\phi(u)$ and $P(u)$, the equations are

$$\begin{aligned}\phi'' &= \frac{1}{2}(\alpha - P)\,e^{2(\phi-\phi_0)} \\ P' &= -(\alpha + P)\,\phi'\end{aligned}$$

where the primes denote derivatives with respect to u, and where $\alpha = \rho/p_0$ is the ratio of the density of the star to the pressure at the center of the star.

If we integrate the second equation, we find

$$\phi(u) = \phi_0 - \log\left(\frac{\alpha + P(u)}{\alpha + 1}\right)$$

Or, solving for the pressure,

$$P = (\alpha + 1)\,e^{-(\phi-\phi_0)} - \alpha$$

Substituting this result into the first equation, we obtain

$$\phi'' = \alpha\, e^{2(\phi-\phi_0)} - \frac{1}{2}(\alpha+1)\, e^{(\phi-\phi_0)}$$

I will assume the star is symmetric about $u = 0$, so $\phi'(0) = 0$. Using this boundary condition, we find

$$\phi(u) = \phi_0 - \log(\frac{1}{2}(\alpha + 1 - (\alpha-1)\cosh u))$$

Thus, the gradient of the potential is

$$\phi'(u) = \frac{1}{2}(\alpha-1)\, e^{(\phi-\phi_0)}\, \sinh u$$

Note that

$$e^{-(\phi(u)-\phi_0)} = \frac{1}{2}(\alpha + 1 - (\alpha-1)\cosh u)$$

If we substitute this result into the equation for the pressure, we find

$$P(u) = \frac{1}{2}(\alpha^2 + 1 - (\alpha^2-1)\cosh u)$$

By definition, the pressure vanishes at the surface r of the star ($P(r) = 0$), so

$$\cosh r = \frac{\alpha^2+1}{\alpha^2-1}$$

Note that

$$\sinh r = \frac{2\alpha}{\alpha^2-1}$$

Thus, we can express r in terms of α:

$$r = \log\left(\frac{\alpha+1}{\alpha-1}\right)$$

Note that

$$\phi(r) - \phi_0 = \log(1 + 1/\alpha)$$

and

$$\phi'(r) = 1$$

The corresponding equations for Newtonian gravity are

$$\begin{aligned} \phi'' &= \frac{1}{2}\alpha \\ P' &= -\alpha\phi' \end{aligned}$$

The solutions are

$$\phi(u) = \frac{1}{4}\alpha u^2$$

$$P(u) = 1 - \frac{1}{4}\alpha^2 u^2$$

Chapter 6

Other

6.1 Introduction

Here I discuss a few random results that didn't fit in elsewhere.

6.2 Spacetime lattice

Here I present a 1+1 dimensional spacetime lattice whose symmetry group is a discrete subgroup of the Poincare group. The lattice consists of vectors of the form

$$\vec{R}_{m,n} = m\,\vec{e}_0 + n\,\vec{e}_1$$

where n and m are integers, and the basis vectors $\vec{e}_0$ and $\vec{e}_1$ are

$$\vec{e}_0 = \frac{1}{2}\begin{pmatrix} \sqrt{5} \\ 1 \end{pmatrix}$$

and

$$\vec{e}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The Minkowski metric is

$$\eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

So the magnitudes of the basis vectors are

$$\eta(\vec{e}_0, \vec{e}_0) = -\eta(\vec{e}_1, \vec{e}_1) = 1$$

and their scalar product is

$$\eta(\vec{e}_0, \vec{e}_1) = -1/2$$

Consider the Lorentz transformation for velocity $\beta = \sqrt{5}/3$:

$$\Lambda = \frac{1}{2} \begin{pmatrix} 3 & \sqrt{5} \\ \sqrt{5} & 3 \end{pmatrix}$$

Note that

$$\begin{aligned} \Lambda\vec{e}_0 &= 2\vec{e}_0 + \vec{e}_1 \\ \Lambda\vec{e}_1 &= \vec{e}_0 + \vec{e}_1 \end{aligned}$$

Thus, Λ maps the lattice onto itself:

$$\Lambda\vec{R}_{m,n} = \vec{R}_{2m+n,m+n}$$

We can also define translation operators T_k by

$$T_k\,\vec{r} = \vec{r} + \vec{e}_k$$

where $\vec{r}$ is an arbitrary vector. The translation operators also map the lattice onto itself:

$$\begin{aligned} T_0\vec{R}_{m,n} &= \vec{R}_{m+1,n} \\ T_1\vec{R}_{m,n} &= \vec{R}_{m,n+1} \end{aligned}$$

The symmetry group of the lattice is generated by the operators Λ, T_0, and T_1, and forms a discrete subgroup of the Poincare group.

I was hoping to propagate waves on the lattice (that is, to define initial conditions on a space-like line of nodes, and then use a discretized wave equation to evolve the wavefront in time), but I don't think this is possible, since no two nodes are lightlike separated:

$$\begin{aligned} \eta(\vec{R}_{m,n}, \vec{R}_{m,n}) &= m^2\,\eta(\vec{e}_0, \vec{e}_0) + 2nm\,\eta(\vec{e}_0, \vec{e}_1) + n^2\,\eta(\vec{e}_1, \vec{e}_1) \\ &= m^2 - nm - n^2 \end{aligned}$$

This is nonzero unless

$$m = (1/2)(1 \pm \sqrt{5})n$$

which is impossible if m and n are integers. Thus, if a node emits a pulse of light, the pulse will never hit another node in the lattice.

6.3 Dynamical model for a 1D ideal gas

Here I present a model for a one-dimensional ideal gas. The model consists of a collection of N atoms that bounce back and forth between a piston and an infinitely massive wall, where a constant force is applied to the piston so that if it wasn't supported by collisions with the atoms it would accelerate uniformly. I will assume that the atoms only collide with the wall and the piston, and not with one another.

Given this specification of the model, we can calculate how the system evolves in time. For simplicity, we will first consider the case of a single atom. Let (x, v) denote the position and velocity of the atom, and let (y, w) denote the position and velocity of the piston, where the coordinate system is chosen such that the wall lies at the origin. Suppose we start the system in some initial state (x_0, v_0, y_0, w_0) at time t_0 and let it evolve to time $t_1 = t_0 + \tau$. If the atom and piston do not collide during the interval $[t_0, t_1]$, then the state at time t_1 is

$$\begin{aligned} y_1(\tau) &= y_0 + w_0\tau - \frac{1}{2}\tau^2 \\ w_1(\tau) &= w_0 - \tau \\ x_1(\tau) &= x_0 + v_0\tau \\ v_1(\tau) &= v_0 \end{aligned}$$

where I have chosen units such that the acceleration of the piston is 1.

Let us assume that at time t_0 the atom had just collided with the piston, so $x_0 = y_0$. We want to evolve the system in time until the next atom/piston collision occurs. There are two cases to consider: either the atom bounces off the wall and hits the piston, or the atom hits the piston directly. In the first case, which I will call case A, the atom collides with the wall after a time $-x_0/v_0$, and then collides with the piston after a time τ_A given by

$$y_1(\tau_A) = -x_1(\tau_A)$$

Thus,

$$\tau_A = v_0 + w_0 + ((v_0 + w_0)^2 + 4x_0)^{1/2}$$

and the configuration at $t_1 = t_0 + \tau_A$ is

$$\begin{aligned} y_1 &= x_1 \\ w_1 &= w_0 - \tau_A \\ x_1 &= -(x_0 + v_0\tau_A) \\ v_1 &= -v_0 \end{aligned}$$

In the second case, which I will call case B, the atom collides with the piston after a time τ_B given by

$$y_1(\tau_B) = x_1(\tau_B)$$

Thus,

$$\tau_B = 2(w_0 - v_0)$$

and the configuration at $t_1 = t_0 + \tau_B$ is

$$\begin{aligned} y_1 &= x_1 \\ w_1 &= w_0 - \tau_B \\ x_1 &= x_0 + v_0\tau_B \\ v_1 &= v_0 \end{aligned}$$

We can determine which case applies by looking at the sign of y_1 as computed for case B: if the sign is positive then case B applies, and if the sign is negative then case A applies.

The above procedure allows us to evolve the system from just after a collision (time t_0) to just before the next collision (time t_1). To evolve the system from just before the next collision (time t_1) to just after the next collision (time t_2), we apply the conservation laws for energy and momentum. Let r be the ratio of the mass of the piston to the mass of the atom. From the conservation of momentum we have

$$v_1 + rw_1 = v_2 + rw_2$$

and from conservation of energy we have

$$v_1^2 + rw_1^2 = v_2^2 + rw_2^2$$

Solving for v_2 and w_2, we find

$$w_2 = (r+1)^{-1}\left((r-1)w_1 + 2v_1\right)$$

$$v_2 \quad = \quad (r+1)^{-1}\,(2rw_1 - (r-1)v_1)$$

Note that by evolving the system from t_0 to t_1, and then from t_1 to t_2, we obtain a mapping of the state of the system immediately after a collision to the state of the system immediately after the next collision:

$$(x_0, y_0, v_0, w_0) \rightarrow (x_2, y_2, v_2, w_2)$$

If we iterate this mapping we can evolve the system in time from one collision to the next.

It is straightforward to generalize to the N atom case: we just search through all the atoms and select the one that collides with the piston first, and then proceed as for the one atom case (note that the positions of the $N-1$ unselected atoms must also be updated). For large N, the piston/atom mass ratio r sets the equilibrium pressure p of the gas: $p = rm$, where m is the mass of an atom, since in equilibrium the pressure p must balance the force rm that is exerted on the piston.

The model can easily be simulated on a computer. For the simulations I present here, I start the system in a non-equilibrium state: the initial state of the piston is taken to be $y_0 = N/r$, $w_0 = 0$, while the initial state of the atoms is chosen such that the atom positions are uniformly distributed in $[0, y_0]$ and the atom velocities are uniformly distributed in $[-v_0, v_0]$, where v_0 is chosen such that $\langle v^2 \rangle = 1$. Starting from this initial state, I evolve system and plot velocity histograms at different times (see Figure 6.1).

The simulation can also be used to calculate the entropy of the system as a function of time. The entropy is given by

$$S = -\iint f(x,p)\,\log(\Lambda f(x,p))\,dx\,dp$$

where $f(x,p)$ is the phase space distribution of the atoms and Λ is an arbitrary constant with units of action. In equilibrium, the phase space distribution is just the Maxwell-Boltzmann distribution:

$$f(x,p) = n\,(2\pi mT)^{-1/2}\,e^{-p^2/2mT}$$

where n is the density and T is the temperature. If we substitute this result into our expression for the entropy, we find that the maximum possible entropy is

$$S_{max} = N(\log((2\pi mT)^{1/2}/\Lambda n)) + \frac{1}{2})$$

Note that

$$T = m\langle v^2 \rangle = m$$

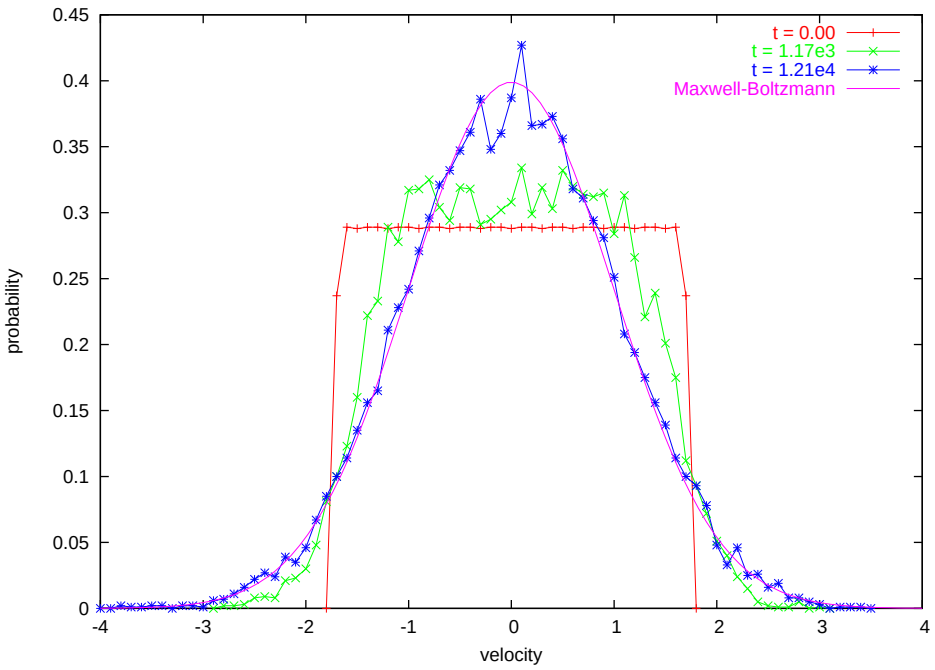

Figure 6.1: Histogram of velocities at different times ($t = 0.00$, $t = 1.17 \times 10^3$, $t = 1.21 \times 10^4$). For this simulation, $N = 10^4$ and $r = 10^3$. The solid curve is the Maxwell-Boltzmann distribution.

From the ideal gas law, and our previous observation that $p = rm$, we find that

$$p = rm = nT = nm$$

so $n = r$. Thus, if we choose $\Lambda = m$,

$$S_{max} = N(\frac{1}{2}(1 + \log(2\pi)) - \log r)$$

A graph of the entropy versus time is shown in Figure 6.2, together with the predicted maximum entropy.

Let us return to the case of a single atom. The state of the system is given by coordinates (x, v, y, w), which are restricted to the constant energy surface defined by

$$E = \frac{1}{2}v^2 + \frac{1}{2}rw^2 + ry$$

where for simplicity I have set $m = 1$. Thus, the maximum atom velocity is

$$v_m = (2E)^{1/2}$$

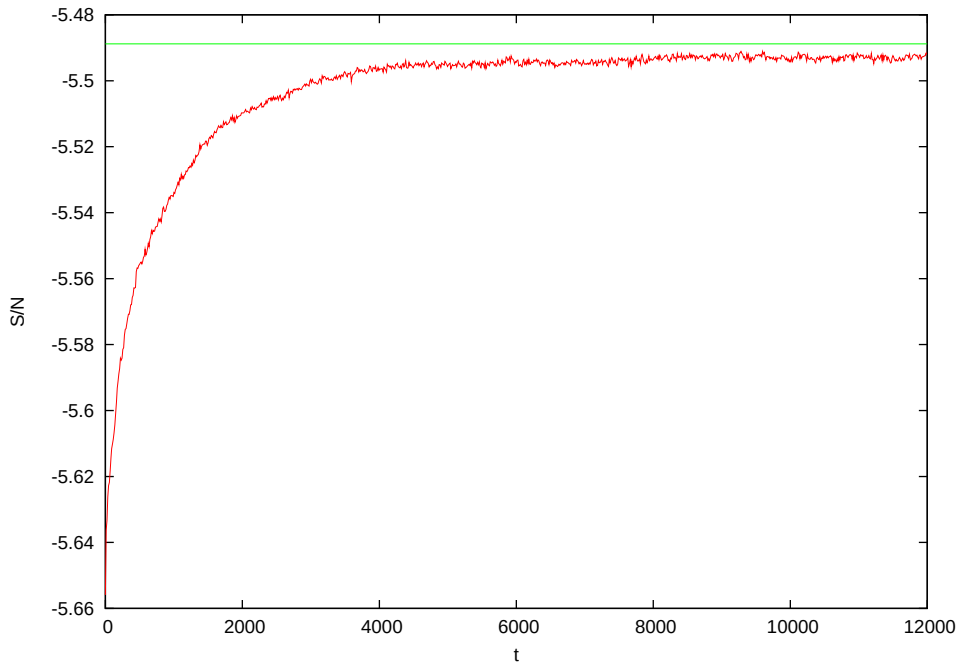

Figure 6.2: Entropy per atom versus time. For this simulation, $N = 10^4$ and $r = 10^3$. The horizontal line is the predicted maximum entropy.

and the maximum piston velocity is

$$w_m = (2E/r)^{1/2}$$

It is convenient to introduce new coordinates α and β that are defined by

$$\begin{aligned} \alpha &= \frac{v}{v_m}\cos\theta + \frac{w}{w_m}\sin\theta \\ \beta &= -\frac{v}{v_m}\sin\theta + \frac{w}{w_m}\cos\theta \end{aligned}$$

Let us choose $\tan\theta = r^{1/2}$, so

$$\begin{aligned} \cos\theta &= (1+r)^{-1/2} \\ \sin\theta &= r^{1/2}(1+r)^{-1/2} \end{aligned}$$

Then

$$\begin{aligned} \alpha &= (2E)^{-1/2}\,(1+r)^{-1/2}\,(v+rw) \\ \beta &= (2E)^{-1/2}\,(1+r)^{-1/2}\,r^{1/2}\,(v-w) \end{aligned}$$

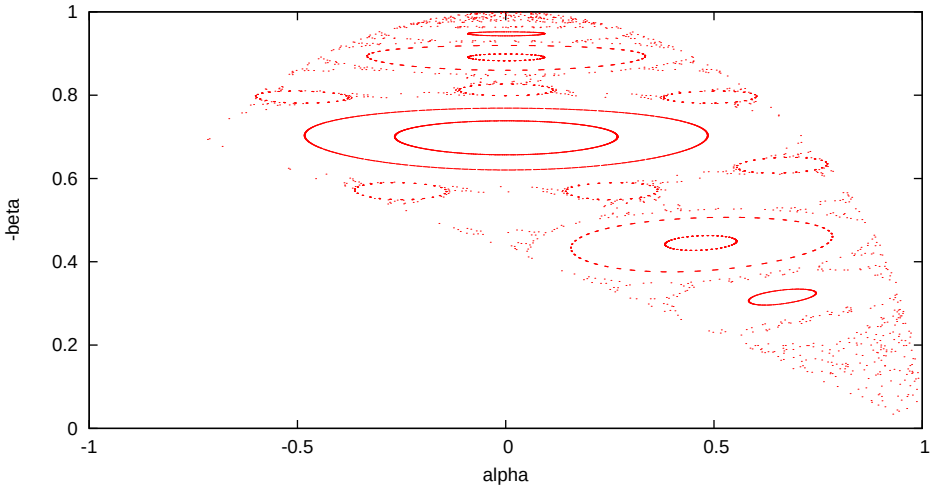

Figure 6.3: Sample configuration space trajectories ($r = 1.1$).

In the new coordinates, the equation for the constant energy surface is

$$\alpha^2 + \beta^2 + (r/E)y = 1$$

We can simulate the system on a computer. The configuration space is three dimensional, but we can plot a two dimensional slice through the space by imposing the requirement $x = y$, that is, that a collision just occurred. Figure 6.3 shows some configuration space trajectories of the system.

6.4 Chaotic mappings and the renormalization group

By applying renormalization group techniques to chaotic mappings, Feigenbaum was able to understand universality in the period doubling route to chaos for iterated 1D mappings [61]. Here I present a highly simplified renormalization group approach to understand mappings of the form

$$f_\lambda(x) = \lambda x(1 - x)$$

where $\lambda \in [0, 4]$ is a control parameter. Before explaining this approach, I will briefly review the behavior of this type of mapping.

Given an initial point $x_0 \in [0, 1]$, we can generate a series of points $\{x_n\}$ by defining $x_{n+1} = f_\lambda(x_n)$.

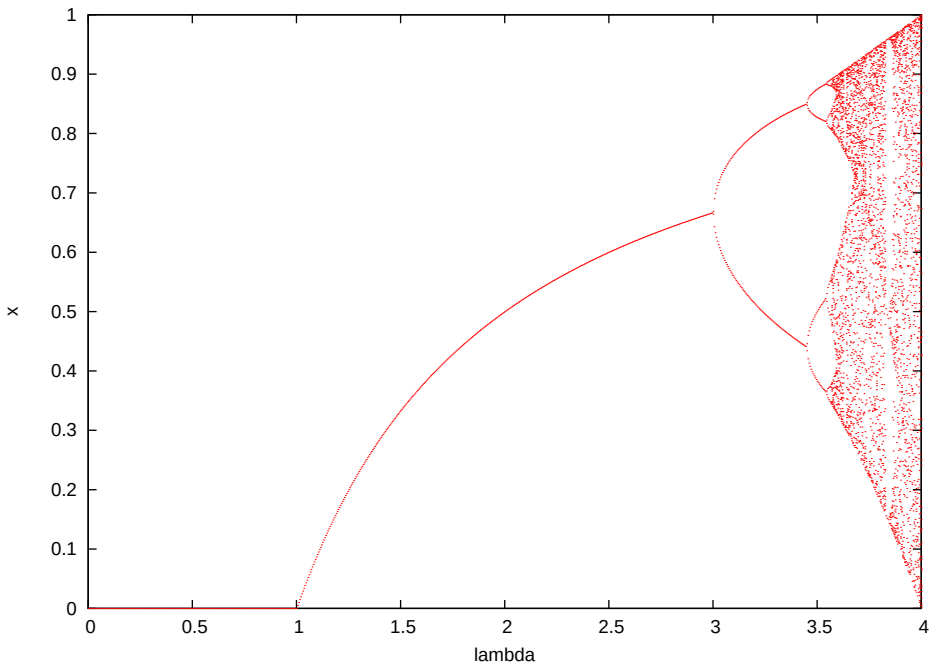

Figure 6.4: Asymptotic behavior of $\{x_n\}$ for different values of λ.

The asymptotic behavior of this series depends on the value of the control parameter λ: for small values of λ the series rapidly approaches a fixed point; for larger values of λ the series bounces around among a finite set of fixed points; and for still larger values of λ the series becomes chaotic. A graph of the asymptotic behavior of the series as a function of λ is shown in Figure 6.4.

As a first step toward understanding this behavior, I want to show how we can calculate the values and stability criteria for the first and second order fixed points. The first order fixed points of f_λ are given by

$$f_\lambda(x) = x$$

which implies $x = 0$ or $x = a$, where $a = 1 - 1/\lambda$. The stability of a first order fixed point is determined by the slope of f_λ: if the slope lies in the range $(-1, 1)$ then the fixed point is stable; otherwise it is unstable. Note that

$$\frac{d}{dx} f_\lambda(x) \mid_{x=0} = \lambda$$

and

$$\frac{d}{dx} f_\lambda(x) \mid_{x=a} = 2 - \lambda$$

Thus, the fixed point $x = 0$ is stable for $\lambda \in [0, 1)$, and the fixed point $x = a$ is stable for $\lambda \in (1, 3)$.

For $\lambda > 3$, there are two second order fixed points $b_\pm$, defined by $f_\lambda(b_\pm) = b_\mp$. We can solve

for the fixed points by writing them in the form $b_\pm = \alpha \pm \beta$ for some constants α and β. Then

$$\begin{aligned} f(\alpha+\beta) &= \lambda(\alpha+\beta-\alpha^2-2\alpha\beta-\beta^2) = \alpha-\beta \\ f(\alpha-\beta) &= \lambda(\alpha-\beta-\alpha^2+2\alpha\beta-\beta^2) = \alpha+\beta \end{aligned}$$

If we add and subtract these equations, we find

$$\begin{aligned} \alpha &= \lambda(\alpha-\alpha^2-\beta^2) \\ \beta &= \lambda(2\alpha\beta-\beta) \end{aligned}$$

From the second of these, we obtain

$$\alpha = \frac{1}{2}(1+1/\lambda)$$

Substitute this into the first equation:

$$\beta^2 = \alpha(1-1/\lambda-\alpha) = \frac{1}{4}(1+1/\lambda)(1-3/\lambda)$$

Thus, the second order fixed points are given by

$$b_\pm = \frac{1}{2}(1+1/\lambda) \pm \frac{1}{2}(1+1/\lambda)^{1/2}\,(1-3/\lambda)^{1/2}$$

The stability of the second order fixed points is determined by the slope of $f_\lambda(f_\lambda(x))$. Note that

$$\frac{d}{dx} f_\lambda(f_\lambda(x))\mid_{x=b_\pm} = f'_\lambda(f_\lambda(x))\, f'_\lambda(x)\mid_{x=b_\pm} = f'_\lambda(b_+)\, f'_\lambda(b_-) = 4+2\lambda-\lambda^2$$

where

$$f'_\lambda(x) \equiv \frac{d}{dx} f_\lambda(x)$$

Thus, the second order fixed points are stable for $\lambda \in [3, \lambda_2]$, where $\lambda_2 \equiv 1+\sqrt{6} = 3.449$.

We can use our understanding of first and second order fixed points to define a renormalization group transformation that maps second order fixed points at one value control parameter to first order fixed points at a smaller value of the control parameter. Note that we can think of the second order fixed point b_+ of $f_\lambda(x)$ as a first order fixed point of $f_\lambda(f_\lambda(x))$, and that near b_+ the function $f_\lambda(f_\lambda(x))$ looks similar to a smaller version of $f_{\bar\lambda}$, for some choice of control parameter $\bar\lambda$ (see Figure 6.5).

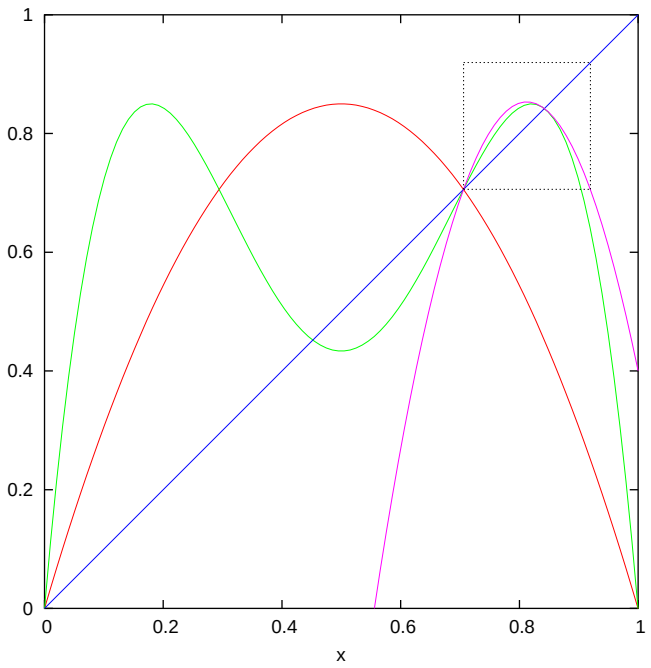

Figure 6.5: Renormalization group transformation. The red curve is $f_\lambda(x)$ and the green curve is $f_\lambda(f_\lambda(x))$. The lower left and upper right corners of the square have coordinates (a, a) and (y, y); if we zoom in on this region, we see that $f_\lambda(f_\lambda(x))$ looks like a smaller version of $f_{\bar{\lambda}}$, which is shown in the pink curve. For this graph $\lambda = 3.4$, which gives $\bar{\lambda} = 2.76$, $a = 0.706$, $b_+ = 0.852$, $y = 0.920$.

To make this correspondence explicit, define a coordinate transformation

$$\bar{x} = \frac{x - a}{y - a}$$

where y is yet to be determined. We want to choose $\bar{\lambda}$ and y such that

$$f_{\bar{\lambda}}(\bar{x}) \simeq \overline{f_\lambda(f_\lambda(x))}$$

for $x \simeq b_+$. Since we are solving for two unknowns ($\bar{\lambda}$ and y), we need two equations. For the first equation, I will require that the second order fixed point b_+ of $f_\lambda(f_\lambda(x))$ be mapped into a first order fixed point of $f_{\bar{\lambda}}$:

$$f_{\bar{\lambda}}(\bar{b}_+) = \bar{b}_+$$

Since the stability of a fixed point is determined by the slope, I will take the second equation to be

$$\frac{d}{d\bar{x}} f_{\bar{\lambda}}(\bar{x}) \mid_{\bar{x}=\bar{b}_+} = \frac{d}{dx} f_\lambda(f_\lambda(x)) \mid_{x=b_+}$$

The first equation says that $\bar{b}_+$ is a first order fixed point of $f_{\bar{\lambda}}$, so

$$\bar{b}_+ = 1 - 1/\bar{\lambda}$$

The second equation gives

$$\bar{\lambda}(1 - 2\bar{b}_+) = 4 + 2\lambda - \lambda^2$$

Combining these results, we obtain an equation for the coupling constant flow:

$$\bar{\lambda} = \lambda^2 - 2\lambda - 2$$

This has a fixed point $\bar{\lambda}_c = \lambda_c$, where

$$\lambda_c = \lambda_c^2 - 2\lambda_c - 2$$

Thus,

$$\lambda_c = \frac{1}{2}(3 + \sqrt{17}) = 3.562$$

We can solve for y by using that

$$\bar{b}_+ = 1 - 1/\bar{\lambda} = \frac{b_+ - a}{y - a}$$

Thus,

$$y = a + \frac{b_+ - a}{1 - 1/\bar{\lambda}}$$

where (summarizing our previous results)

$$\begin{aligned} \bar{\lambda} &= \lambda^2 - 2\lambda - 2 \\ a &= 1 - 1/\lambda \\ b_+ &= \frac{1}{2}(1 + 1/\lambda) + \frac{1}{2}(1 + 1/\lambda)^{1/2}\,(1 - 3/\lambda)^{1/2} \end{aligned}$$

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