# Zeros of Random Orthogonal Polynomials on the Unit Circle

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# Abstract


We consider polynomials on the unit circle defined by the recurrence relation

$$\Phi_{k+1}(z) = z\Phi_k(z) - \overline{\alpha}_k \Phi_k^*(z) \qquad k \geq 0, \quad \Phi_0 = 1$$

For each $n$ we take $\alpha_0, \alpha_1, \dots, \alpha_{n-2}$ to be independent identically distributed random variables uniformly distributed in a disk of radius $r < 1$ and $\alpha_{n-1}$ to be another random variable independent of the previous ones and distributed uniformly on the unit circle. The previous recurrence relation gives a sequence of random paraorthogonal polynomials $\{\Phi_n\}_{n\geq 0}$. For any $n$, the zeros of $\Phi_n$ are $n$ random points on the unit circle.

We prove that, for any $e^{i\theta} \in \partial\mathbb{D}$, the distribution of the zeros of $\Phi_n$ in intervals of size $O(\frac{1}{n})$ near $e^{i\theta}$ is the same as the distribution of $n$ independent random points uniformly distributed on the unit circle (i.e., Poisson).

This means that for any fixed $a_1 < b_1 \leq a_2 < b_2 \leq \cdots \leq a_m < b_m$ and any nonnegative integers $k_1, k_2, \dots, k_m$, we have

$$\mathbb{P}\left(\zeta^{(n)}\left(e^{i(\theta_0 + \frac{2\pi a_1}{n})}, e^{i(\theta_0 + \frac{2\pi b_1}{n})}\right) = k_1, \dots,\ \zeta^{(n)}\left(e^{i(\theta_0 + \frac{2\pi a_m}{n})}, e^{i(\theta_0 + \frac{2\pi b_m}{n})}\right) = k_m\right) \longrightarrow e^{-(b_1 - a_1)} \frac{(b_1 - a_1)^{k_1}}{k_1!} \dots e^{-(b_m - a_m)} \frac{(b_m - a_m)^{k_m}}{k_m!}$$

as $n \to \infty$, where by $\zeta^{(n)}(I)$ we denote the number of zeros of the polynomial $\Phi_n$ situated in the interval $I$.

Therefore, for large $n$, there is no local correlation between the zeros of the considered random paraorthogonal polynomials.

The same result holds when we take $\alpha_0, \alpha_1, \ldots, \alpha_{n-2}$ to be independent identically distributed random variables uniformly distributed in a circle of radius $r < 1$ and $\alpha_{n-1}$ to be another random variable independent of the previous ones and distributed uniformly on the unit circle.

# Contents

# Chapter 1

# Introduction

In this thesis we study orthogonal polynomials on the unit circle given by random Verblunsky coefficients. More precisely, we will consider random paraorthogonal polynomials (which have their zeros on the unit circle) and we will describe the statistical distribution of these zeros.

For any sequence of complex numbers $\{\alpha_n\}_{n\geq 0}$ we can consider the sequence of polynomials $\{\Phi_n\}_{n\geq 0}$ defined by $\Phi_0 = 1$ and by the recurrence relation

$$\Phi_{k+1}(z) = z\Phi_k(z) - \overline{\alpha}_k \Phi_k^*(z) \qquad k \geq 0, \tag{1.0.1}$$

where for any $k > 0$, $\Phi_k^*(z) = z^k\, \overline{\Phi_k(\frac{1}{\overline{z}})}$ (or, equivalently, if $\Phi_k(z) = \sum_{j=0}^{k} a_j z^j$, then $\Phi_k^*(z) = \sum_{j=0}^{k} \overline{a}_{k-j}\, z^j$).

Henceforth, we will denote by $\mathbb{D}$ the unit disk and by $\partial\mathbb{D}$ the unit circle.

Consider a nontrivial (i.e., not supported on a finite set) probability measure on the unit circle. Then the sequence of polynomials $1, z, z^2, \ldots$ is in $L^2(\partial\mathbb{D}, d\mu)$. Since the measure $\mu$ is nontrivial, the polynomials $1, z, z^2, \ldots$ are linearly independent. We can apply the Gram-Schmidt process to this sequence and get a sequence of orthogonal monic polynomials $\Phi_0, \Phi_1, \ldots$. It is easy to see that these polynomials obey the recurrence relation (1.0.1), where the complex numbers $\{\alpha_n\}_{n\geq 0}$ satisfy $|\alpha_n| < 1$ for any $n$.

**Theorem 1.0.1 (Verblunsky).** *There is a bijection between nontrivial (i.e., not supported on a finite set) probability measures on the unit circle and sequences of complex numbers* $\{\alpha_n\}_{n\geq 0}$ *with* $|\alpha_n| < 1$ *for any* $n$.

The previous theorem shows that we can parameterize the set of nontrivial probability measures on the unit circle using sequences $\{\alpha_n\}_{n\geq 0} \in \times_{k=0}^{\infty}\mathbb{D}$.

Let us consider the unitary operator $T : L^2(\partial\mathbb{D}, d\mu) \to L^2(\partial\mathbb{D}, d\mu)$ defined by

$$Tf(z) = zf(z) \tag{1.0.2}$$

The family $\mathfrak{B}_0 = \{1, z, z^{-1}, z^2, z^{-2}, \ldots\}$ is a basis of the space $L^2(\partial\mathbb{D}, d\mu)$ (i.e., finite linear combintions of elements in $\mathfrak{B}_0$ are dense in $L^2(\partial\mathbb{D}, d\mu)$). By applying the Gram-Schmidt process to this family, we get an orthonormal basis $\mathfrak{B} = \{\chi_0, \chi_1, \chi_2, \ldots\}$. The matrix representation of the operator $T$ in the basis $\mathfrak{B}$ is

$$\mathcal{C} = \begin{pmatrix} \bar{\alpha}_0 & \bar{\alpha}_1\rho_0 & \rho_1\rho_0 & 0 & 0 & \ldots \\ \rho_0 & -\bar{\alpha}_1\alpha_0 & -\rho_1\alpha_0 & 0 & 0 & \ldots \\ 0 & \bar{\alpha}_2\rho_1 & -\bar{\alpha}_2\alpha_1 & \bar{\alpha}_3\rho_2 & \rho_3\rho_2 & \ldots \\ 0 & \rho_2\rho_1 & -\rho_2\alpha_1 & -\bar{\alpha}_3\alpha_2 & -\rho_3\alpha_2 & \ldots \\ 0 & 0 & 0 & \bar{\alpha}_4\rho_3 & -\bar{\alpha}_4\alpha_3 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{1.0.3}$$

where $\rho_k = \sqrt{1 - |\alpha_k|^2}$. The matrix $\mathcal{C}$ is called the CMV matrix associated to the measure $\mu$. This representation is a recent discovery of Cantero, Moral, and Velázquez [CMV]. The CMV matrices can be seen as unitary analogues of the (self-adjoint) Jacobi matrices, which are extensively studied in the mathematical physics literature (see Cycon et al. [CFKS], Teschl [Tes]). This point of view gives powerful tools for the study of the spectral properties of the CMV matrices.

It is interesting to see how the CMV matrices decouple. Consider the matrix $\mathcal{C}$ defined by (1.0.3). If $|\alpha_n| = 1$, then $\rho_n = 0$, and therefore the matrix $\mathcal{C}$ decouples between $(n-1)$ and $n$ (we start numbering the rows and the columns at 0). We can write

$$\mathcal{C} = \mathcal{C}^{(n)} \bigoplus \tilde{\mathcal{C}}^{(n)} \tag{1.0.4}$$

where both $\mathcal{C}^{(n)}$ and $\tilde{\mathcal{C}}^{(n)}$ are unitary matrices and $\mathcal{C}^{(n)}$ is the upper left corner of the matrix

$\mathcal{C}$. The $n \times n$ matrix $\mathcal{C}^{(n)}$ will be called the truncated CMV matrix.

Recall that the orthogonal polynomials $\Phi_0, \Phi_1, \ldots$ were obtained by applying the Gram-Schmidt process to the sequence $1, z, \ldots$ in the space $L^2(\partial\mathbb{D}, d\mu)$ ($\mu$ is a nontrivial measure, i.e., $\operatorname{supp}\mu$ is infinite). In this situation we have $|\alpha_k| < 1$ for any $k$. On the other hand, the case $\alpha_0, \alpha_1, \ldots \alpha_{n-1} \in \mathbb{D}$ and $|\alpha_n| = 1$ corresponds to the situation when the Gram-Schmidt process stops. In this situation, the support of the measure $\mu$ is finite and the space $L^2(\partial\mathbb{D}, d\mu)$ is finite-dimensional. For any $k \in \{1, \ldots, n-1\}$, the zeros of $\Phi_k$ are in $\mathbb{D}$. The zeros of the polynomial $\Phi_n$ are on $\partial\mathbb{D}$. It is not hard to see that the zeros of $\Phi_n$ are exactly the eigenvalues of the matrix $\mathcal{C}^{(n)}$ defined in (1.0.4).

The polynomials $\Phi_n$ obtained by the recurrence relation (1.0.1) with $\alpha_0, \alpha_1, \ldots, \alpha_{n-2} \in \mathbb{D}$ and with $\alpha_{n-1} \in \partial\mathbb{D}$ are called by some authors paraorthogonal polynomials. Their zeros are used in the quadrature theorem of Jones, Njåstad, and Thron [JNT].

Before we state this theorem we should define the Christoffel coefficients associated to a measure $\mu$ at a point $z \in \mathbb{C}$ to be

$$\lambda_n(z,\mu) = \min\left\{ \int_{\partial\mathbb{D}} |\pi(e^{i\theta})|^2 d\mu(\theta) \;\middle|\; \pi \text{ polynomial },\ \deg\pi \le n,\ \pi(z) = 1 \right\} \qquad (1.0.5)$$

**Theorem 1.0.2 (Jones, Njåstad, Thron).** *Let $\mu$ be a nontrivial probability measure on the unit circle and $\{\alpha_n\}_{n\ge0}$ the set of corresponding Verblunsky coefficients. Let $\beta_1, \beta_2, \ldots$ be a sequence of points on the unit circle and $\Phi_{n+1}$ the paraorthogonal polynomial obtained from the coefficients $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}, \beta_n$. Let $z_1^{(n+1)}, z_2^{(n+1)}, \ldots, z_{n+1}^{(n+1)}$ be the zeros of the polynomial $\Phi_{n+1}$ and*

$$\mu_{n+1}(\beta_n) = \sum_{k=1}^{n+1} \lambda_n(z_k^{(n+1)}, \mu)\, \delta_{z_k^{(n+1)}} \qquad (1.0.6)$$

*where the $\lambda_n(z_k^{(n+1)}, \mu)$ are the Christoffel coefficients corresponding to the measure $\mu$ and the points $\lambda_n(z_k^{(n+1)}, \mu)$.*

*Then, for any choice of $\beta_1, \beta_2, \ldots$, the sequence of measures $\mu_{n+1}(\beta_n)$ converges weakly to $\mu$.*

We will sometimes use the abbreviations OPUC for orthogonal polynomials on the unit circle and POPUC for paraorthogonal polynomials on the unit circle.

We will consider random paraorthogonal polynomials (i.e., paraorthogonal polynomials defined by random recurrence coefficients $\alpha_0, \alpha_1, \ldots, \alpha_{n-2} \in \mathbb{D}$ and $\beta \in \partial\mathbb{D}$). We will study the distribution of the zeros of the paraorthogonal polynomials as $n \to \infty$. As we have seen before, this is equivalent to the study of the distribution of the eigenvalues of the truncated CMV matrices.

It will be useful to view the random points on the unit circle as a point process (for a general introduction to the theory of point processes, see [DVJ03]). Let $X$ be a fixed topological space ($X$ will be $\mathbb{R}$ or $\partial\mathbb{D}$). We define by $\mathcal{M}(X)$ the space of all nonnegative Radon measures on $X$. Let also $\mathcal{M}_p(X)$ be the space of all integer-valued Radon measures on $X$. Clearly any measure $\zeta \in \mathcal{M}_p(X)$ can be written as

$$\zeta = \sum_k \delta_{\zeta_k} \tag{1.0.7}$$

where the sum is finite or countable, $\zeta_k \in X$ for any $k$, and the set $\{\zeta_k\}$ has no accumulation point in $X$. As usual, we endow the space $\mathcal{M}(X)$ with the vague topology and $\mathcal{M}_p(X)$ with the topology induced from $\mathcal{M}(X)$. Let's also fix a probability space $(\Omega, \mathbb{P})$.

By definition, a point process is a measurable function

$$\zeta = \zeta(\omega) : \Omega \to \mathcal{M}_p(X) \tag{1.0.8}$$

It is clear that for any Borel set $A \subset X$, we can define a random variable

$$\zeta(A) : \Omega \to \mathbb{Z} \tag{1.0.9}$$

defined by $\zeta(A)(\omega) = \zeta(\omega)(A)$ (i.e., the number of points $\zeta(\omega)$ has in the set $A$).

We say that the point process $\zeta$ has the intensity measure $\mu$ if and only if, for any Borel set $A$, we have

$$\mu(A) = \mathbb{E}\left(\zeta(A)\right) \tag{1.0.10}$$

It is clear that the study of any point process is equivalent to the study of the integer-valued random variables $\zeta(A)$ for Borel sets $A \subset X$.

One of the most important point processes is the Poisson point process. Let's fix a measure $\mu$ on the real line. By definition, the Poisson point process with intensity measure $\mu$ is the point process $\zeta : \Omega \to \mathcal{M}_p(X)$, which obeys the following:

i) For any Borel set $A \subset X$, the random variable $\zeta(A)$ has a Poisson distribution with intensity measure $\mu(A)$. This means that for any nonnegative integer $k$,

$$\mathbb{P}(\zeta(A) = k) = e^{-\mu(A)} \frac{\mu(A)^k}{k!} \tag{1.0.11}$$

ii) If the $A_1, A_2, \ldots, A_m$ are disjoint Borel subsets of $X$, then $\zeta(A_1), \zeta(A_2), \ldots, \zeta(A_m)$ are independent random variables.

The Poisson distribution for a collection of random points indicates the fact that the points are completely independent in the space $X$ (i.e., they don't "see" each other). This means that there is no correlation (attraction or repulsion) between these points. An elementary introduction to the theory of Poisson processes is [Kin].

We will want to prove that a sequence of point processes $\{\zeta^{(n)}\}$ on the probability space $(\Omega, \mathbb{P})$ converges to the Poisson point process and therefore to conclude that there is no correlation between points as $n \to \infty$. In order to do this, we should first define the notion of limit of a sequence of point processes.

Consider another point process $\zeta$ on a probability space $(\tilde{\Omega}, \tilde{\mathbb{P}})$. By definition, we say that $\zeta_n$ converges (weakly) to $\zeta$ if and only if, for any bounded continuous function $F : \mathcal{M}_p(X) \to \mathbb{C}$, we have

$$\lim_{n \to \infty} \int_\Omega F(\zeta(\omega)) \mathbb{P}(\omega) = \int_{\tilde{\Omega}} F(\zeta(\omega)) \tilde{\mathbb{P}}(\omega) \tag{1.0.12}$$

A very important problem in mathematical physics is the study of the statistical distribution of the eigenvalues of some classes of $(n \times n)$ random matrices (self-adjoint or unitary). For any $n$ we will denote by $\zeta^{(n)}$ the point process obtained by taking a Dirac measure of mass 1 at any eigenvalue of the $(n \times n)$ matrix considered. For this situation it would be meaningless to study the random variables $\zeta^{(n)}(A)$. This is because for any fixed set $A$ with $\mathbb{P}(A) > 0$, the expectation of the random variable $\zeta^{(n)}(A)$ will converge to $\infty$ as $n \to \infty$.

Instead, we should rescale the set $A$ by a factor of $n$ for every value of $n$. We will do this in the following way: We will first fix a point $x \in X$ and a set $A \subset X$. Then, for any $n$, we will rescale the set $A$ near the point $x$ by a factor of $n$ and get a set $A^{(n)}$ of measure $O(\frac{1}{n})$. We will then study the random variables $\zeta^{(n)}(A^{(n)})$ as $n \to \infty$.

For example, if $X = \mathbb{R}$, then, for a point $r \in \mathbb{R}$ and an interval $A = (a, b) \subset \mathbb{R}$, we will have $A^{(n)} = (r + \frac{a}{n}, r + \frac{b}{n})$. In the case $X = \partial\mathbb{D}$, for a point $e^{i\theta} \in \partial\mathbb{D}$ and an interval (arc) $A = (e^{ia}, e^{ib})$, we have $A^{(n)} = (e^{i(\theta + \frac{2\pi a}{n})}, e^{i(\theta + \frac{2\pi b}{n})})$.

We will say that the rescaled (by the factor $n$) point process $\zeta^{(n)}$ converges locally to the Poisson point process with the intensity measure $\mu$ near the point $p \in X$ (which is $r \in \mathbb{R}$ in the first case and $e^{i\theta} \in \partial\mathbb{D}$ in the second case) if and only if

i) For any disjoint $A_1, A_2, \ldots, A_m \in X$, the random variables $\zeta^{(n)}(A_1^{(n)})$, $\zeta^{(n)}(A_2^{(n)})$, …, $\zeta^{(n)}(A_m^{(n)})$ are independent;

ii) For any fixed $a_1 < b_1 \leq a_2 < b_2 \leq \cdots \leq a_m < b_m$ and any nonnegative integers $k_1, k_2, \ldots, k_m$, we have, in the first case ($X = \mathbb{R}, r \in \mathbb{R}$):

$$\mathbb{P}\left(\zeta^{(n)}\left(r + \frac{a_1}{n}, r + \frac{b_1}{n}\right) = k_1, \ldots,\ \zeta^{(n)}\left(r + \frac{a_m}{n}, r + \frac{b_m}{n}\right) = k_m\right)$$
$$\longrightarrow e^{-\mu((a_1,b_1))}\frac{\mu((a_1,b_1))^{k_1}}{k_1!}\ldots e^{-\mu((a_m,b_m))}\frac{\mu((a_m,b_m))^{k_m}}{k_m!} \tag{1.0.13}$$

as $n \to \infty$.

In the second case ($X = \partial\mathbb{D}, e^{i\theta} \in \partial\mathbb{D}$):

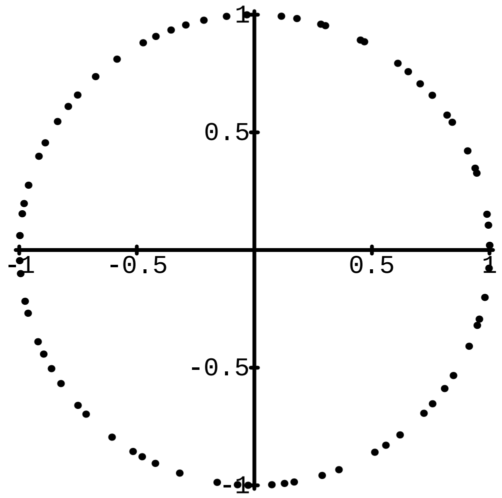
$$\mathbb{P}\left(\zeta^{(n)}\left(e^{i(\theta + \frac{2\pi a_1}{n})}, e^{i(\theta + \frac{2\pi b_1}{n})}\right) = k_1, \ldots,\ \zeta^{(n)}\left(e^{i(\theta + \frac{2\pi a_m}{n})}, e^{i(\theta + \frac{2\pi b_m}{n})}\right) = k_m\right)$$
$$\longrightarrow e^{-\mu((a_1,b_1))}\frac{\mu((a_1,b_1))^{k_1}}{k_1!}\ldots e^{-\mu((a_m,b_m))}\frac{\mu((a_m,b_m))^{k_m}}{k_m!} \tag{1.0.14}$$

as $n \to \infty$.

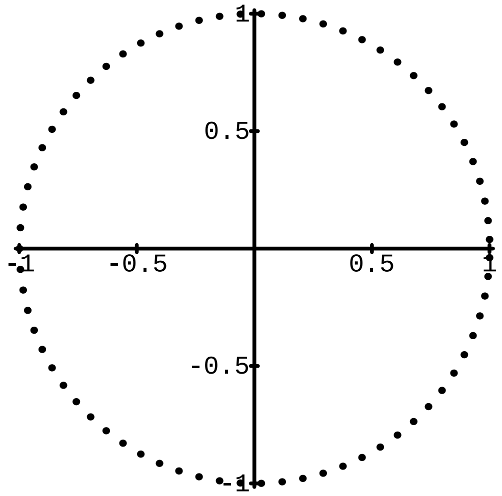
As mentioned earlier, we will consider random $n \times n$ truncated CMV matrices. For any $n$, we define, as before, the point process $\zeta^{(n)}$. Since the truncated CMV matrices are unitary, $\zeta^{(n)}$ is a point process on the unit circle. We will prove that for any $e^{i\theta} \in \partial\mathbb{D}$, the sequence $\{\zeta^{(n)}\}$ converges locally (on intervals of size $O(\frac{1}{n})$ near $e^{i\theta}$) to a Poisson point process.

We will now set the stage for this result.

Let's consider random paraorthogonal polynomials defined by i.i.d. Verblunsky coefficients $\alpha_0, \alpha_1, \ldots, \alpha_{n-2}$ uniformly distributed in a disk of radius $r < 1$ and the random variable $\beta$, which is independent of the previous ones and uniformly distributed on the unit circle. A generic plot of the zeros of these polynomials is




It is interesting to compare this plot with the plot of the zeros of paraorthogonal polynomials defined by recurrence coefficients $\alpha_n = Cb^n + O((b\tau)^n)$, where $b \in (0,1)$, $C \in \mathbb{C}$ and $\tau \in (0,1)$:




In the second situation (which, for the case of orthogonal polynomials, was studied by

Simon in [Sim3]), we observe that the zeros are repelling each other. Since they are confined to the unit circle, this gives a "clock behavior."

The situation is completely different in the first case, where we observe clumps and gaps. This suggests that there is no correlation between the points, and therefore we have a "Poisson behavior."

The main result of this thesis is to prove that for random paraorthogonal polynomials defined before (obtained with random i.i.d. $\alpha_0, \alpha_1, \ldots, \alpha_{n-2}$ distributed uniformly in $D(0, r)$ and $\beta$ distributed uniformly on the unit circle), the statistical distribution of the zeros is locally Poisson.

Similar results have appeared in the mathematics literature for the case of random Schrödinger operators; see Molchanov [Mo2] and Minami [Mi]. The study of the spectrum of random Schrödinger operators and the distribution of the eigenvalues was initiated by the very important paper of Anderson [And], who showed that certain random lattices exhibit absence of diffusion.

Rigorous mathematical proofs of the Anderson localization were given by Goldsheid-Molchanov-Pastur [GMP] for one-dimensional models and by Fröhlich-Spencer [FS] for multidimensional Schrödinger operators. We should also mention here that these papers use different models to derive the Anderson localization. The first paper considered the continuous one-dimensional Schrödinger operator with the random potential given by a Brownian motion on a compact Riemannian manifold, while the second paper studied the finite-difference Laplacian on $\mathbb{Z}^d$ with the random potential $v = \{v(j)\}$ consisting of independent identically distributed random variables $\{v(j)\}, j \in \mathbb{Z}^d$. The second model is now considered to be the standard Anderson model (the Anderson tight binding model).

Several other proofs of the Anderson localization, containing improvements and simplifications, were published later. Here we will only mention Aizenman-Molchanov [AM] and Simon-Wolff [SW], which are relevant for our approach.

In the case of the unit circle, similar localization results were obtained by Teplyaev [Tep].

**Theorem 1.0.3 (Teplyaev).** *Let $\mu$ be a probability measure on the unit disk, which is* *absolutely continuous with respect to the Lebesgue measure, and*

$$\int_{\mathbb{D}} \log(1-|x|)\, d\mu(x) > -\infty$$

*Consider a sequence of independent identically distributed random variables $a_0, a_1, \ldots$ with the probability distribution $\mu$. Let $\sigma = \sigma(a_0, a_1, \ldots)$ be the random probability measure on the unit circle given by the Verblunsky's theorem. Then, with probability $1$, $\sigma$ is a pure point measure.*

A recent important development in the theory of orthogonal polynomials on the unit circle was obtained by Golinskii and Nevai in [GN]. In this paper the authors use transfer matrices and the theory of subordinate solutions (some of the key ingredients in the modern theory of Schrödinger operators) to investigate the spectral measures corresponding to orthogonal polynomials on the unit circle.

In addition to the phenomenon of localization, one can also analyze the local structure of the spectrum. It turns out that, for the case of the Schrödinger operator, there is no repulsion between the energy levels. This was shown by Molchanov [Mo2] for the model of the one-dimensional Schrödinger operator studied by the Russian school.

**Theorem 1.0.4 (Molchanov).** *For any $V > 0$, consider the one-dimensional Schrödinger operator on $L^2((-V, V))$,*

$$H_V = -\frac{d^2}{dt^2} + q(t, \omega)$$

*with Dirichlet boundary conditions. The random potential $q(t, \omega)$ has the form $q(t, \omega) = F(x_t)$ where $x_t$ is a Brownian motion on a compact Riemannian manifold $K$ and $F : K \to \mathbb{R}$ is a smooth Morse function with $\min_{x \in K} F(x) = 0$. Denote by $N_V(I)$ the number of eigenvalues of the operator $H_V$ situated in the interval $I$. Let $E_0 > 0$ and $n(E_0)$ the limit density of states of the operator $H_V$ as $V \to \infty$. Then:*

$$\lim_{V \to \infty} \mathbb{P}\left(N_V\left(E_0 - \frac{a}{2V}, E_0 + \frac{a}{2V}\right)\right) \longrightarrow e^{-a\, n(E_0)} \frac{(a\, n(E_0))^k}{k!}$$

Using the terminology of point processes presented before, this means that the local statistical distribution of the eigenvalues of the operator $H_V$ (rescaled near $E_0$) converges, as $V \to \infty$, to the Poisson point process with intensity measure $n(E_0)dx$ (here $dx$ denotes the Lebesgue measure).

The case of the multidimensional discrete Schrödinger operator was analyzed by Minami in [Mi].

**Theorem 1.0.5 (Minami).** *Consider the Anderson tight binding model*

$$H = -\Delta + V(\omega)$$

*and denote by $H^\Lambda$ its restriction to hypercubes $\Lambda \subset \mathbb{Z}^d$. Let $E \in \mathbb{R}$ be an energy for which the Aizenman-Molchanov bounds* (3.1.17) *hold* (*see also Theorem* 3.1.9) *and denote by $n(E)$ the density of states at $E$. For any hypercube $\Lambda$, let $\{E_j(\Lambda)\}_j$, $j \geq 1$ be the eigenvalues of the operator $H^\Lambda$ and let*

$$\zeta_j(\Lambda, E) = |\Lambda|(E_j(\Lambda) - E)$$

*be its rescaled eigenvalues. Then the point process $\zeta(\Lambda) = \{\zeta_j(\Lambda, E)\}_j$ converges to the Poisson point process with intensity measure $n(E)dx$ as the box $\Lambda$ gets large ($\Lambda \uparrow \mathbb{Z}^d$).*

Both in Theorems 1.0.4 and 1.0.5, Molchanov and Minami proved that the statistical distribution of the rescaled eigenvalues converges locally to a stationary Poisson point process. This means that there is no correlation between eigenvalues.

We will use some techniques from the spectral theory of discrete Schrödinger operators to study the distribution of the zeros of the random paraorthogonal polynomials, especially ideas and methods developed in [AM]. However, our model on the unit circle has many different features compared to the discrete Schrödinger operator (perhaps the most important one is that we have to consider unitary operators on the unit circle instead of self-adjoint operators on the real line). Therefore, we will have to use new ideas and techniques that work for this situation (see [Sto]).

The final goal is the following:

**Theorem 1.0.6 (Main Theorem).** *Consider the random polynomials on the unit circle* *given by the following recurrence relations:*

$$\Phi_{k+1}(z) = z\Phi_k(z) - \overline{\alpha}_k \Phi_k^*(z) \qquad k \geq 0, \qquad \Phi_0 = 1 \tag{1.0.15}$$

*where* $\alpha_0, \alpha_1, \ldots, \alpha_{n-2}$ *are i.i.d. random variables distributed uniformly in a disk of radius* $r < 1$, *and* $\alpha_{n-1}$ *is another random variable independent of the previous ones and uniformly distributed on the unit circle.*

*Consider the space* $\Omega = \{\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_{n-2}, \alpha_{n-1}) \in D(0,r) \times D(0,r) \times \cdots \times D(0,r) \times \partial\mathbb{D}\}$ *with the probability measure* $\mathbb{P}$ *obtained by taking the product of the uniform (Lebesgue) measures on each* $D(0,r)$ *and on* $\partial\mathbb{D}$. *Fix a point* $e^{i\theta_0} \in \mathbb{D}$ *and let* $\zeta^{(n)}$ *be the point process defined previously (by taking a Dirac measure of mass* $1$ *at any eigenvalue of the matrix* $\mathcal{C}^{(n)}$*).*

*Then, on a fine scale (of order* $\frac{1}{n}$*) near* $e^{i\theta_0}$, *the point process* $\zeta^{(n)}$ *converges to the Poisson point process with intensity measure* $n\,\frac{d\theta}{2\pi}$ *(where* $\frac{d\theta}{2\pi}$ *is the normalized Lebesgue measure). This means that for any fixed* $a_1 < b_1 \leq a_2 < b_2 \leq \cdots \leq a_m < b_m$ *and any nonnegative integers* $k_1, k_2, \ldots, k_m$, *we have*

$$\mathbb{P}\left(\zeta^{(n)}\left(e^{i(\theta_0 + \frac{2\pi a_1}{n})}, e^{i(\theta_0 + \frac{2\pi b_1}{n})}\right) = k_1, \ldots,\ \zeta^{(n)}\left(e^{i(\theta_0 + \frac{2\pi a_m}{n})}, e^{i(\theta_0 + \frac{2\pi b_m}{n})}\right) = k_m\right)$$
$$\longrightarrow e^{-(b_1 - a_1)}\frac{(b_1 - a_1)^{k_1}}{k_1!} \ldots e^{-(b_m - a_m)}\frac{(b_m - a_m)^{k_m}}{k_m!} \tag{1.0.16}$$

*as* $n \to \infty$.

## 1.1 Outline of the Proof of the Main Theorem

From now on we will work under the hypotheses of Theorem 1.0.6. We will study the statistical distribution of the eigenvalues of the random CMV matrices

$$\mathcal{C}^{(n)} = \mathcal{C}_\alpha^{(n)} \tag{1.1.1}$$

for $\alpha \in \Omega$ (with the space $\Omega$ defined in Theorem 1.0.6).

A first step in the study of the spectrum of random CMV matrix is proving the exponential decay of the fractional moments of the resolvent of the CMV matrix. These ideas were developed in the case of Anderson models by Aizenman-Molchanov [AM] and by Aizenman et al. [ASFH]; they provide a powerful method for proving spectral localization, dynamical localization, and the absence of level repulsion.

Before we state the Aizenman-Molchanov bounds, we have to make a few remarks on the boundary behavior of the matrix elements of the resolvent of the CMV matrix. For any $z \in \mathbb{D}$ and any $0 \leq k, l \leq (n-1)$, we will use the following notation:

$$F_{kl}(z, \mathcal{C}_\alpha^{(n)}) = \left[ \frac{\mathcal{C}_\alpha^{(n)} + z}{\mathcal{C}_\alpha^{(n)} - z} \right]_{kl} \tag{1.1.2}$$

As we will see in the next section, using properties of Carathéodory functions, we will get that for any $\alpha \in \Omega$, the radial limit

$$F_{kl}(e^{i\theta}, \mathcal{C}_\alpha^{(n)}) = \lim_{r \uparrow 1} F_{kl}(re^{i\theta}, \mathcal{C}_\alpha^{(n)}) \tag{1.1.3}$$

exists for Lebesgue almost every $e^{i\theta} \in \partial\mathbb{D}$ and $F_{kl}(\,\cdot\,, \mathcal{C}_\alpha^{(n)}) \in L^s(\partial\mathbb{D})$ for any $s \in (0,1)$. Since the distributions of $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$ are rotationally invariant, we obtain that for any fixed $e^{i\theta} \in \partial\mathbb{D}$, the radial limit $F_{kl}(e^{i\theta}, \mathcal{C}_\alpha^{(n)})$ exists for almost every $\alpha \in \Omega$. We can also define

$$G_{kl}(z, \mathcal{C}_\alpha^{(n)}) = \left[ \frac{1}{\mathcal{C}_\alpha^{(n)} - z} \right]_{kl} \tag{1.1.4}$$

and

$$G_{kl}(e^{i\theta}, \mathcal{C}_\alpha^{(n)}) = \lim_{r \uparrow 1} G_{kl}(re^{i\theta}, \mathcal{C}_\alpha^{(n)}) \tag{1.1.5}$$

Using the previous notation, we have

**Theorem 1.1.1 (Aizenman-Molchanov Bounds for the Resolvent of the CMV Matrix).** *For the model considered in Theorem 1.0.6 and for any $s \in (0,1)$, there exist constants $C_1, D_1 > 0$ such that for any $n > 0$, any $k, l$, $0 \leq k, l \leq n-1$, and any $e^{i\theta} \in \partial\mathbb{D}$,* *we have*

$$\mathbb{E}\left(\left|F_{kl}(e^{i\theta}, \mathcal{C}_\alpha^{(n)})\right|^s\right) \leq C_1\, e^{-D_1|k-l|} \tag{1.1.6}$$

*where* $\mathcal{C}^{(n)}$ *is the* $(n \times n)$ *CMV matrix obtained for* $\alpha_0, \alpha_1, \ldots, \alpha_{n-2}$ *uniformly distributed in* $D(0,r)$ *and* $\alpha_{n-1}$ *uniformly distributed in* $\partial\mathbb{D}$*.*

Using Theorem 1.1.1, we will then be able to control the structure of the eigenfunctions of the matrix $\mathcal{C}^{(n)}$.

**Theorem 1.1.2 (The Localized Structure of the Eigenfunctions).** *For the model considered in Theorem 1.0.6, the eigenfunctions of the random matrices* $\mathcal{C}^{(n)} = \mathcal{C}_\alpha^{(n)}$ *are exponentially localized with probability* 1*, that is, exponentially small outside sets of size proportional to* $(\ln n)$*. Hence there exists a constant* $D_2 > 0$*, and for almost every* $\alpha \in \Omega$*, there exists a constant* $C_\alpha > 0$ *such that for any unitary eigenfunction* $\varphi_\alpha^{(n)}$*, there exists a point* $m(\varphi_\alpha^{(n)})$ $(1 \leq m(\varphi_\alpha^{(n)}) \leq n)$ *with the property that for any* $m, |m - m(\varphi_\alpha^{(n)})| \geq D_2 \ln(n+1)$*, we have*

$$|\varphi_\alpha^{(n)}(m)| \leq C_\alpha\, e^{-(4/D_2)\,|m - m(\varphi_\alpha^{(n)})|} \tag{1.1.7}$$

The point $m(\varphi_\alpha^{(n)})$ will be taken to be the smallest integer where the eigenfunction $\varphi_\alpha^{(n)}(m)$ attains its maximum absolute value.

In order to obtain a Poisson distribution in the limit as $n \to \infty$, we will use the approach of Molchanov [Mo2] and Minami [Mi]. The first step is to decouple the point process $\zeta^{(n)}$ into the direct sum of smaller point processes. We will do the decoupling process in the following way: For any positive integer $n$, let $\tilde{\mathcal{C}}^{(n)}$ be the CMV matrix obtained for the coefficients $\alpha_0, \alpha_1, \ldots, \alpha_n$ with the additional restrictions $\alpha_{\left[\frac{n}{\ln n}\right]} = e^{i\eta_1}$, $\alpha_{2\left[\frac{n}{\ln n}\right]} = e^{i\eta_2}$, $\ldots$, $\alpha_n = e^{i\eta_{[\ln n]}}$, where $e^{i\eta_1}, e^{i\eta_2}, \ldots, e^{i\eta_{[\ln n]}}$ are independent random points uniformly distributed on the unit circle. Note that the matrix $\tilde{\mathcal{C}}^{(n)}$ decouples into the direct sum of $\approx [\ln n]$ unitary matrices $\tilde{\mathcal{C}}_1^{(n)}$, $\tilde{\mathcal{C}}_2^{(n)}$, $\ldots$, $\tilde{\mathcal{C}}_{[\ln n]}^{(n)}$. We should note here that the actual number of blocks $\tilde{\mathcal{C}}_i^{(n)}$ is slightly larger than $[\ln n]$ and that the dimension of one of the blocks (the last one) could be smaller than $\left[\frac{n}{\ln n}\right]$.

However, since we are only interested in the asymptotic behavior of the distribution of the eigenvalues, we can, without loss of generality, work with matrices of size $N = [\ln n]\left[\frac{n}{\ln n}\right]$. The matrix $\tilde{\mathcal{C}}^{(N)}$ is the direct sum of exactly $[\ln n]$ smaller blocks $\tilde{\mathcal{C}}_1^{(N)}$, $\tilde{\mathcal{C}}_2^{(N)}$, $\ldots$, $\tilde{\mathcal{C}}_{[\ln n]}^{(N)}$. Let's define $\zeta^{(N,p)} = \sum_{k=1}^{[n/\ln n]} \delta_{z_k^{(p)}}$ where $z_1^{(p)}, z_2^{(p)}, \ldots, z_{[n/\ln n]}^{(p)}$ are the eigenvalues of the matrix $\tilde{\mathcal{C}}_p^{(N)}$. The decoupling result is formulated in the following theorem:

**Theorem 1.1.3.** *The point process $\zeta^{(N)}$ can be asymptotically approximated by the direct sum of point processes $\sum_{p=1}^{[\ln n]} \zeta^{(N,p)}$. In other words, the distribution of the eigenvalues of the matrix $\mathcal{C}^{(N)}$ can be asymptotically approximated by the distribution of the eigenvalues of the direct sum of the matrices $\tilde{\mathcal{C}}_1^{(N)}, \tilde{\mathcal{C}}_2^{(N)}, \ldots, \tilde{\mathcal{C}}_{[\ln n]}^{(N)}$.*

The decoupling property is the first step in proving that the statistical distribution of the eigenvalues of $\mathcal{C}^{(N)}$ is Poisson. In the theory of point processes (see, e.g., Daley and Vere-Jones [DVJ88]), a point process obeying this decoupling property is called an infinitely divisible point process. In order to show that this distribution is Poisson on a scale of order $O(\frac{1}{n})$ near a point $e^{i\theta}$, we need to check two conditions:

$$\text{i)} \quad \sum_{p=1}^{[\ln n]} \mathbb{P}\left(\zeta^{(N,p)}\left(A\left(N,\theta\right)\right) \geq 1\right) \to |A| \quad \text{as} \quad n \to \infty \tag{1.1.8}$$

$$\text{ii)} \quad \sum_{p=1}^{[\ln n]} \mathbb{P}\left(\zeta^{(N,p)}\left(A\left(N,\theta\right)\right) \geq 2\right) \to 0 \quad \text{as} \quad n \to \infty \tag{1.1.9}$$

where for an interval $A = [a, b]$ we define $A(N, \theta) = (e^{i(\theta + \frac{2\pi a}{N})}, e^{i(\theta + \frac{2\pi b}{N})})$ and $|\cdot|$ is the Lebesgue measure (and we extend this definition to unions of intervals). The second condition shows that it is asymptotically impossible that any of the matrices $\tilde{\mathcal{C}}_1^{(N)}, \tilde{\mathcal{C}}_2^{(N)}, \ldots, \tilde{\mathcal{C}}_{[\ln n]}^{(N)}$ has two or more eigenvalues situated in an interval of size $\frac{1}{N}$. Therefore, each of the matrices $\tilde{\mathcal{C}}_1^{(N)}$, $\tilde{\mathcal{C}}_2^{(N)}$, $\ldots$, $\tilde{\mathcal{C}}_{[\ln n]}^{(N)}$ contributes at most one eigenvalue in an interval of size $\frac{1}{N}$. But the matrices $\tilde{\mathcal{C}}_1^{(N)}$, $\tilde{\mathcal{C}}_2^{(N)}$, $\ldots$, $\tilde{\mathcal{C}}_{[\ln n]}^{(N)}$ are decoupled, hence independent, and therefore we get a Poisson distribution. Condition i) now gives Theorem 1.0.6.

# Chapter 2

# Background on OPUC

In this chapter we will present a few basic results in the theory of orthogonal polynomials on the unit circle. Special attention will be devoted to the zeros of these polynomials and to the CMV matrices. We will also introduce some of the main tools used in the study of these mathematical objects. Most of the results and proofs of the theorems stated in this chapter can be found in [Sim4] and [Sim5].

## 2.1 Definition, Basic Properties, Examples

As explained in Chapter 1, for any nontrivial probability measure $\mu$ on $\partial\mathbb{D}$, defining the infinite-dimensional Hilbert space $L^2(\partial\mathbb{D}, d\mu)$, the orthonormal polynomials $\varphi_0(z,\mu)$, $\varphi_1(z,\mu)$, $\varphi_2(z,\mu)$, ... are obtained by applying the Gram-Schmidt process to the sequence of polynomials $1, z, z^2, \ldots$. For any $n \geq 0$, we can define the corresponding monic polynomial

$$\Phi_n(z,\mu) = \frac{1}{\kappa_n}\, \varphi_n(z,\mu) \tag{2.1.1}$$

(where by $\kappa_n$ we denote the leading coefficient of the polynomial $\varphi_n(z,\mu)$).

The monic polynomials $\{\Phi_n(z,\mu)\}_{n\geq 0}$ obey the recurrence relation

$$\Phi_{n+1}(z) = z\,\Phi_n(z) - \overline{\alpha}_n\,\Phi_n^*(z) \qquad n \geq 0 \tag{2.1.2}$$

where for any $n \geq 0$, we have $\alpha_n \in \mathbb{D}$. The complex numbers $\alpha_n$ are called the Verblunsky coefficients associated to the measure $\mu$.

Verblunsky's theorem (Theorem 1.0.1) states that there is a bijection between nontrivial measures $\mu$ on the unit circle and sequences of complex numbers $\{\alpha_n\}_{n\geq 0} \subset \mathbb{D}$. One of the most important questions in the theory of orthogonal polynomials on the unit circle is how properties of the measure $\mu$ correspond to properties of the Verblunsky coefficients $\{\alpha_n\}_{n\geq 0}$ and vice-versa.

Two very important tools in the study of orthogonal polynomials on the unit circle are the Carathéodory and the Schur functions associated to the spectral measure $\mu$. The Carathéodory function is defined for $z \in \mathbb{D}$ by the relation:

$$F(z) = \int_{\partial\mathbb{D}} \frac{e^{i\theta} + z}{e^{i\theta} - z}\, d\mu(\theta) \tag{2.1.3}$$

Note that $F$ is analytic on $\mathbb{D}$, $F(0) = 1$, and $\operatorname{Re} F(z) \geq 0$ for any $z \in \mathbb{D}$.

The Schur function $f : \mathbb{D} \to \mathbb{C}$, is corresponding to the measure $\mu$, can be obtained from the Carathéodory function $F$ using the relation

$$f(z) = \frac{1}{z}\frac{F(z) - 1}{F(z) + 1} \tag{2.1.4}$$

Note that the function $f$ is analytic and $|f(z)| \leq 1$ for any $z \in \mathbb{D}$.

A few important examples of orthogonal polynomials on the unit circle are

1. The Free Case: $\mu = \frac{d\theta}{2\pi}$ (the normalized Lebesgue measure). In this case we have $\alpha_n = 0$ for any $n$, $F(z) = 1$, and $f(z) = 0$.

2. Bernstein-Szegö Polynomials with Parameter $\zeta = re^{i\theta} \in \mathbb{D}$: $\mu = P_r(\theta, \varphi)\, \frac{d\theta}{2\pi}$, where

$$P_r(\theta, \varphi) = \frac{1 - r^2}{1 + r^2 - 2r\cos(\theta - \varphi)}$$

is the Poisson kernel. In this case we have $\alpha_0 = re^{i\theta}$ and $\alpha_n = 0$ for any $n \geq 1$. Also $F(z) = \frac{1+z\zeta}{1-z\zeta}$ and $f(z) = \zeta$.

3. Single Inserted Mass Point with Parameter $\gamma \in (0, 1)$: $\mu = (1 - \gamma)\frac{d\theta}{2\pi} + \gamma\delta(\theta - 0)$. In this case $\alpha_n = \frac{\gamma}{1+n\gamma}$, $F(z) = \frac{1-(1-2\gamma)z}{1-z}$, and $f(z) = \frac{\gamma}{1-(1-\gamma)z}$.

4. Geronimus Polynomials with Parameter $\alpha \in \mathbb{D}$: $\mu = w(\theta)\frac{d\theta}{2\pi} + \mu_s$, where for the

auxiliary parameter $\beta$ defined by $1+\bar{\alpha} = |1+\bar{\alpha}|e^{i\beta/2}$, we have

$$w(\theta) = \begin{cases} \frac{1}{|1+\alpha|}\frac{\sqrt{\cos^2(\arcsin(|\alpha|))-\cos^2(\theta/2)}}{\sin((\theta-\beta)/2)}, & \theta \in (2\arcsin(|\alpha|, 2\pi - 2\arcsin(|\alpha|)) \\ 0, & \theta \in [-2\arcsin(|\alpha|), 2\arcsin(|\alpha|)] \end{cases} \tag{2.1.5}$$

and

$$\mu_s = \begin{cases} 0, & \text{if } |\alpha + \frac{1}{2}| \le \frac{1}{2}; \\ \frac{2}{|1+\alpha|^2}(|\alpha+\frac{1}{2}|^2 - \frac{1}{4})\delta(\theta-\beta), & \text{if } |\alpha+\frac{1}{2}| > \frac{1}{2} \end{cases} \tag{2.1.6}$$

In this case we have $\alpha_n = \alpha$ for any $n \ge 0$. The formulae for the Carathéodory and Schur functions are more complicated:

$$F(z) = 1 + \frac{2\left(z+1+\sqrt{(z-e^{2i\arcsin(|\alpha|)})(z-e^{-2i\arcsin(|\alpha|)})}+\bar{\alpha}z-1\right)}{1+\bar{\alpha}-(1-\alpha)z} \tag{2.1.7}$$

and

$$f(z) = \frac{z+1+\sqrt{(z-e^{2i\arcsin(|\alpha|)})(z-e^{-2i\arcsin(|\alpha|)})}-2}{2\bar{\alpha}z} \tag{2.1.8}$$

More examples of orthogonal polynomials on the unit circle are presented in Section 1.6 of [Sim4].

## 2.2 Zeros of OPUC and POPUC

One of the central questions in the theory of orthogonal polynomials on the unit circle is to understand the location of the zeros of $\Phi_n(z,\mu)$.

We will start with the basic theorems:

**Theorem 2.2.1 (Zeros Theorem for OPUC).** *Let $\Phi_n(z,\mu)$ be the n-th degree monic orthogonal polynomial corresponding to the measure $\mu$. Then all the zeros of $\Phi_n(z,\mu)$ lie in $\mathbb{D}$.*

As explained in Chapter 1, for any nontrivial probability measure $\mu$ with the Verblunsky coefficients $\{\alpha_k\}_{k\ge0}$ and the orthogonal polynomials $\{\Phi_k(z,\mu)\}_{k\ge0}$, any $n \ge 0$ and any

$\beta \in \partial\mathbb{D}$, we can define the paraorthogonal polynomials

$$\Phi_{n+1}(z, \mu, \beta) = z\Phi_n(z, \mu) + \beta\Phi_n^*(z, \mu) \tag{2.2.1}$$

For the zeros of paraorthogonal polynomials we have

**Theorem 2.2.2 (Zeros Theorem for POPUC).** *Let $\Phi_n(z, \mu, \beta)$ be the n-th degree monic paraorthogonal polynomial defined before. Then all the zeros of $\Phi_n(z, \mu, \beta)$ lie in $\partial\mathbb{D}$.*

The proofs of Theorems 2.2.1 and 2.2.2 can be found in [Sim4].

We will now present a few results about the zeros of OPUC and later turn our attention to zeros of POPUC. One of the first questions that can be asked is whether any countable set of points in $\mathbb{D}$ can be the zeros of orthogonal polynomials. The answer is positive and is given by the following:

**Theorem 2.2.3 (Alfaro-Vigil [AlVi]).** *Let $\{z_n\}_{n\geq 1}$ be a sequence of complex numbers of absolute value strictly less than 1. Then there exists a unique nontrivial probability measure $\mu$ with $\Phi_n(z_n, \mu) = 0$.*

A clear limitation of the previous theorem is that we can only prescribe one zero for each orthogonal polynomial. It is natural to ask whether we can prescribe more zeros for the polynomial $\Phi_n$. The answer is again positive and was recently discovered by Simon and Totik:

**Theorem 2.2.4 (Simon-Totik [ST]).** *Let $\{z_n\}_{n\geq 1}$ be a sequence of complex numbers of absolute value strictly less than 1 and let $0 < m_1 < m_2 < m_3 < \cdots$ be any increasing sequence of positive numbers. Then there exists a nontrivial probability measure $\mu$ on $\partial\mathbb{D}$ with*

$$\Phi_{m_j}(z_k, \mu) = 0 \qquad k = m_{j-1} + 1, \ldots, m_j \tag{2.2.2}$$

Another question that one can ask is whether the particular properties of the measure $\mu$ can influence the location of the zeros of the orthogonal polynomials. It turns out that the zeros of the orthogonal polynomials can only be located in the convex hull of $\operatorname{supp}(\mu)$.

**Theorem 2.2.5 (Fejér).** *Let $\mu$ be a nontrivial measure on $\partial\mathbb{D}$ and let $\Phi_n(z,\mu)$ be the corresponding monic polynomials associated to this measure. Then all the zeros of $\Phi_n(z,\mu)$ lie in the interior of the convex hull of* $\mathrm{supp}(\mu)$.

It is worth mentioning that the previous theorem holds for arbitrary nontrivial measures in $\mathbb{C}$.

The zeros of paraorthogonal polynomials on the unit circle were studied by Golinskii in [Gol]. We will list here the main results obtained in this paper. As before, we fix a nontrivial probability measure $\mu$ on $\partial\mathbb{D}$ with Verblunsky coefficients $\{\alpha_n\}_{n\geq 0}$ and orthogonal polynomials $\{\Phi_n(z,\mu)\}_{n\geq 0}$. For a fixed complex number $\beta \in \partial\mathbb{D}$, we consider the paraorthogonal polynomials $\{\Phi_n(z,\mu,\beta)\}_{n\geq 0}$.

For any $n$, denote by $\{z_{n,k}\}_{0\leq k\leq(n-1)}$ the zeros of the polynomial $\Phi_n(z,\mu,\beta)$ (they are situated on the unit circle). We order the numbers $z_{n,k}$ such that for any $k$, the point $z_{n,k}$ is between $z_{n,(k-1)}$ and $z_{n,(k+1)}$ (with the convention $z_{n,n} = z_{n,0}$). Note that the ordering is unique once we fix the starting point, $z_{n,0}$.

**Theorem 2.2.6.** *For any integer $n \geq 2$, there exists particular orderings of the zeros of the polynomials $\Phi_n(z,\mu,\beta)$ and $\Phi_{n+1}(z,\mu,\beta)$ such that the sets $S_1 = \{z_{n,k}\}_{0\leq k\leq(n-1)}$ and $S_2 = \{z_{n+1,k}\}_{1\leq k\leq n}$ are alternating* (*i.e., between each two adjacent points of either of them, there is exactly one point of the other*).

It is also interesting to control the distance between the zeros of the polynomials $\Phi_n(z,\mu,\beta)$ in terms of the measure $\mu$. Suppose that the Lebesgue decomposition of the spectral measure $\mu$ is $\mu = f(\theta)\,\frac{d\theta}{2\pi} + \mu_{\mathrm{s}}$. Then we have the following:

**Theorem 2.2.7.** *If $\log f \in L^1(\partial\mathbb{D})$, then there exists a constant $C = C(\mu)$ such that for any $0 \leq k \leq (n-1)$ we have*

$$|z_{n,k+1} - z_{n,k}| \leq \frac{C}{\sqrt{n}} \tag{2.2.3}$$

Lower bounds and better upper bounds can be obtained if the measure $\mu$ has no singular part and the function $f$ is bounded from above and from below. Thus:

**Theorem 2.2.8.** *If the spectral measure $\mu$ is absolutely continuous on $\partial\mathbb{D}$* ($\mu = f(\theta)\frac{d\theta}{2\pi}$)

*and for two positive constants $A$ and $B$ we have*

$$0 < A \leq f \leq B, \tag{2.2.4}$$

*then*

$$\frac{4}{n}\sqrt{\frac{A}{B}} \leq |z_{n,k+1} - z_{n,k}| \leq \frac{4\pi B}{A(n+1)} \tag{2.2.5}$$

In both Theorems 2.2.7 and 2.2.8, we used the convention $z_{n,n} = z_{n,0}$.

We will finish this brief review of the results known about the zeros of OPUC and POPUC by mentioning another result of Golinskii, which gives necessary and sufficient conditions for the distribution of the zeros of POPUC to be uniformly distributed on the unit circle.

By definition, we say that the sequence $\{z_{n,k}\}_{0 \leq k \leq n}$ is uniformly distributed on $\partial\mathbb{D}$ if and only if, for any $f \in C(\partial\mathbb{D})$, we have

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} f(z_{n,k}) = \int_{\partial\mathbb{D}} f(\theta)\, \frac{d\theta}{2\pi} \tag{2.2.6}$$

This definition can be extended to indicator functions on $\partial\mathbb{D}$ and gives (for $f = 1_\Gamma$, with $\Gamma$ an arc in $\partial\mathbb{D}$) the very natural definition of the uniform distribution

$$\lim_{n\to\infty} \frac{\#\{k \mid z_{n,k} \in \Gamma\}}{n+1} = |\Gamma| \tag{2.2.7}$$

where by $|\Gamma|$ we denote the Lebesgue measure of the arc $\Gamma$.

With this definition we can now state:

**Theorem 2.2.9.** *The sequence $\{z_{n,k}\}_{0 \leq k \leq n}$ of zeros of paraorthogonal polynomials is uniformly distributed on $\partial\mathbb{D}$ if and only if*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} |\varphi_k(z)|^2 d\,\mu = \frac{d\theta}{2\pi} \tag{2.2.8}$$

*where the limit on the left-hand side is taken in the space of all finite Borel measures on $\partial\mathbb{D}$.*

Theorems 2.2.6–2.2.9 were discovered by Golinskii and published in [Gol].

We should mention here that all the results presented in this section are deterministic. We will study the zeros of POPUC from a different point of view: We will consider random Verblunsky coefficients, which will define random spectral measures. We will obtain results that hold with probability 1. Also, in the random setting, it is impossible to obtain precise estimates like the ones in Theorems 2.2.7 and 2.2.8. Instead, we will study the local statistical distribution of these zeros.

## 2.3 The CMV Matrix and Its Resolvent, Dynamical Properties of the CMV Matrices

In this section we will discuss one of the main tools used in the study of the zeros of orthogonal polynomials on the unit circle: the CMV matrix (see [CMV]). For any nontrivial measure $\mu$ on $\partial\mathbb{D}$, the CMV matrix is a matrix representation of the unitary operator $T : L^2(\partial\mathbb{D}, d\mu) \to L^2(\partial\mathbb{D}, d\mu)$,

$$Tf(z) = zf(z) \tag{2.3.1}$$

Consider the sequence $\mathfrak{B}_0 = \{1, z, z^{-1}, z^2, z^{-2}, \ldots\}$. It is a basis of the Hilbert space $L^2(\partial\mathbb{D}, d\mu)$ (i.e., the finite linear combinations of elements in $\mathfrak{B}_0$ are dense in $L^2(\partial\mathbb{D}, d\mu)$). By applying the Gram-Schmidt process to this family, we get an orthonormal basis $\mathfrak{B} = \{\chi_0, \chi_1, \chi_2, \ldots\}$. The matrix representation of the operator $T$ in the basis $\mathfrak{B}$ (called the CMV basis) is

$$\mathcal{C} = \begin{pmatrix} \bar\alpha_0 & \bar\alpha_1\rho_0 & \rho_1\rho_0 & 0 & 0 & \ldots \\ \rho_0 & -\bar\alpha_1\alpha_0 & -\rho_1\alpha_0 & 0 & 0 & \ldots \\ 0 & \bar\alpha_2\rho_1 & -\bar\alpha_2\alpha_1 & \bar\alpha_3\rho_2 & \rho_3\rho_2 & \ldots \\ 0 & \rho_2\rho_1 & -\rho_2\alpha_1 & -\bar\alpha_3\alpha_2 & -\rho_3\alpha_2 & \ldots \\ 0 & 0 & 0 & \bar\alpha_4\rho_3 & -\bar\alpha_4\alpha_3 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{2.3.2}$$

where $\rho_k = \sqrt{1 - |\alpha_k|^2}$.

If we consider the basis $\tilde{\mathfrak{B}}_0 = \{1, z^{-1}, z, z^{-2}, z^2, \ldots\}$ for the Hilbert space $L^2(\partial\mathbb{D}, d\mu)$, we can apply the Gram-Schmidt process and get another orthonormal basis $\tilde{\mathfrak{B}} = \{x_0, x_1, x_2, \dots\}$. The matrix representation of the operator $T$ in the basis $\tilde{\mathfrak{B}}$ (called the alternate CMV basis) is

$$\tilde{\mathcal{C}} = \begin{pmatrix} \bar{\alpha}_0 & \rho_0 & 0 & 0 & 0 & \dots \\ \bar{\alpha}_1\rho_0 & -\bar{\alpha}_1\alpha_0 & \alpha_2\rho_1 & \rho_2\rho_1 & 0 & \dots \\ \rho_1\rho_0 & -\rho_1\alpha_0 & -\bar{\alpha}_2\alpha_1 & -\rho_2\alpha_1 & 0 & \dots \\ 0 & 0 & \alpha_3\rho_2 & -\bar{\alpha}_3\alpha_2 & \bar{\alpha}_4\rho_3 & \dots \\ 0 & 0 & \rho_3\rho_2 & -\rho_3\alpha_2 & -\bar{\alpha}_4\alpha_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{2.3.3}$$

We will list here a few identities which will be useful in later computations (the proofs can be found in Section 4.2 of [Sim4]).

**Theorem 2.3.1.** *With the previous notations we have, for any $n$,*

*(a)* $\chi_{2n-1}(z) = z^{-n+1}\varphi_{2n-1}(z)$

*(b)* $\chi_{2n}(z) = z^{-n}\varphi^*_{2n}(z)$

*(c)* $x_{2n-1}(z) = z^{-n}\varphi^*_{2n-1}(z)$

*(d)* $x_{2n}(z) = z^{-n}\varphi_{2n}(z)$

*(e)* $x_n(z) = \overline{\chi_n(1/\bar{z})}$

*where $\varphi^*_n(z)$ is the normalized reversed orthogonal polynomial.*

Both the CMV matrix $\mathcal{C}$ and the alternate CMV matrix $\tilde{\mathcal{C}}$ can be written as the product of two tridiagonal unitary matrices. Thus, if we define, for any $j$,

$$\Theta_j = \begin{pmatrix} \bar{\alpha}_j & \rho_j \\ \rho_j & -\alpha_j \end{pmatrix} \tag{2.3.4}$$

and

$$\mathcal{M} = \begin{pmatrix} 1 & 0 & 0 & \dots \\ 0 & \Theta_1 & 0 & \dots \\ 0 & 0 & \Theta_3 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{and} \quad \mathcal{L} = \begin{pmatrix} \Theta_0 & 0 & 0 & \dots \\ 0 & \Theta_2 & 0 & \dots \\ 0 & 0 & \Theta_4 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{2.3.5}$$

then we have

**Theorem 2.3.2 (The Θ-factorization).** *With the previous notation,*

*(a)* $\mathcal{C} = \mathcal{L}\mathcal{M}$

*(b)* $\tilde{\mathcal{C}} = \mathcal{M}\mathcal{L}$

For any nontrivial probability measure $\mu$ on $\partial\mathbb{D}$ with Verblunsky coefficients $\{\alpha_n\}_{n\geq 0}$ and any $\lambda = e^{i\theta} \in \partial\mathbb{D}$, we define the Alexandrov measure $\mu_\lambda$ to be the unique nontrivial measure on $\partial\mathbb{D}$ with Verblunsky coefficients $\{\lambda\alpha_n\}_{n\geq 0}$. Therefore,

$$\alpha_n(\mu_\lambda) = \lambda\,\alpha_n(\mu) \tag{2.3.6}$$

The orthogonal polynomials corresponding to the case $\lambda = -1$ are called the second kind polynomials associated to the measure $\mu$ and are denoted by

$$\Psi_n(z,\mu) = \Phi_n(z,\mu_\lambda) \tag{2.3.7}$$

As before, we can define the normalized polynomials $\psi_n$, the reversed polynomials $\Psi_n^*$, and the normalized reversed polynomials $\psi_n^*$.

The Carathéodory function $F(z)$ associated to the measure $\mu$ (see (2.1.3)) has a property analogous to a defining property of the Weyl function in the case of differential equations.

**Theorem 2.3.3 (Golinskii-Nevai [GN]).** *For a fixed point $z \in \mathbb{D}$, the unique complex number $r$ for which*

$$\begin{pmatrix} \psi_n(z) \\ -\psi_n^*(z) \end{pmatrix} + r \begin{pmatrix} \varphi_n(z) \\ \varphi_n^*(z) \end{pmatrix} \in l^2(\mathbb{Z}^+, \mathbb{C}^2) \tag{2.3.8}$$

*is* $r = F(z)$.

Therefore we obtain the Weyl solutions

$$w_1(z) = \psi_n(z) + F(z)\varphi_n(z) \tag{2.3.9}$$

and

$$w_2(z) = -\varphi_n^*(z) + F(z)\varphi_n^*(z) \tag{2.3.10}$$

We can now define the CMV analogues of the Weyl solutions (see Theorem 2.3.1):

$$y_n(z) = \begin{cases} z^{-l}\psi_{2l}(z), & n = 2l \\ -z^{-l}\psi_{2l-1}^*(z), & n = 2l-1 \end{cases} \tag{2.3.11}$$

and

$$\Upsilon_n = \begin{cases} -z^{-l}\psi_{2l}^*, & n = 2l \\ z^{-l+1}\psi_{2l-1}, & n = 2l-1 \end{cases} \tag{2.3.12}$$

We now get the second kind analogues of the Weyl solutions

$$p_n = y_n + F(z)x_n \tag{2.3.13}$$

$$\pi_n = \Upsilon_n + F(z)\chi_n \tag{2.3.14}$$

These functions will allow us to give a precise formula for the resolvent of the CMV matrix.

**Theorem 2.3.4.** *For any* $z \in \mathbb{D}$, *we have*

$$\left[(\mathcal{C} - z)^{-1}\right]_{kl} = \begin{cases} (2z)^{-1}\chi_l(z)p_k(z), & k > l \quad \text{or} \quad k = l = 2n-1 \\ (2z)^{-1}\pi_l(z)x_k(z), & l > k \quad \text{or} \quad k = l = 2n \end{cases} \tag{2.3.15}$$

A detailed proof of this theorem is given in Section 4.4 of [Sim4]. As we explained in (1.1.4), we will use the notation $G(z) = (\mathcal{C} - z)^{-1}$ (Green's function of the unitary matrix $\mathcal{C}$).

We will be interested in the behavior of the matrix elements of $(\mathcal{C}-z)^{-1}$ for $z \in \partial\mathbb{D}$. It is clear that whenever $e^{i\theta} \in \partial\mathbb{D}$ and the CMV matrix $\mathcal{C}$ are such that $\lim_{r\uparrow 1}\left[(\mathcal{C}-re^{i\theta})^{-1}\right]_{kl}$ exists, then the formula (2.3.15) holds for $z = e^{i\theta}$.

A useful criterion for the existence of the boundary values of the Green functions is:

**Theorem 2.3.5.** *If for a $e^{i\theta} \in \partial\mathbb{D}$ we have that $\lim_{r\uparrow 1} F(re^{i\theta})$ exists and is in $i\mathbb{R}$, then, for all $k, l$,*

$$\lim_{r\uparrow 1} G_{kl}(re^{i\theta}) = G_{kl}(e^{i\theta}) \tag{2.3.16}$$

*exists and obeys*

$$\lim_{L\to\infty} \frac{\sum_{l=1}^{L} |G_{kl}(e^{i\theta})|^2}{\sum_{l=1}^{L} |\varphi_l(e^{i\theta})|^2} = 0 \tag{2.3.17}$$

The previous theorem is a part of Theorem 10.9.2 in [Sim5]. We will also need Theorem 10.9.3 from [Sim5]:

**Theorem 2.3.6.** *For each $z = e^{i\theta} \in \partial\mathbb{D}$, the following are equivalent:*

*i) The limit in* (2.3.16) *exists, and for each fixed $k$,*

$$\lim_{l\to\infty} G_{kl}(e^{i\theta}) = 0 \tag{2.3.18}$$

*ii) There is a $\lambda \neq 1$ so that*

$$\lim_{l\to\infty} T_l(e^{i\theta}) \begin{pmatrix} 1 \\ \lambda \end{pmatrix} = 0 \tag{2.3.19}$$

Consider as before a CMV matrix with Verblunsky coefficients $\{\alpha_n\}_{n\geq 0}$. The transfer matrices associated to the sequence $\{\alpha_n\}_{n\geq 0}$ are

$$T_n(z) = A(\alpha_n, z)\ldots A(\alpha_0, z) \tag{2.3.20}$$

where

$$A(\alpha, z) = (1-|\alpha|^2)^{-1/2} \left(\begin{smallmatrix} z & -\overline{\alpha} \\ -\alpha z & 1 \end{smallmatrix}\right) \tag{2.3.21}$$

The Lyapunov exponent associated to the sequence $\{\alpha_n\}_{n\geq 0}$ is

$$\gamma(z) = \gamma(z, \{\alpha_n\}) = \lim_{n\to\infty} \frac{1}{n} \ln \|T_n(z, \{\alpha_n\})\| \tag{2.3.22}$$

(provided this limit exists).

A detailed analysis of the Lyapunov exponent, as well as necessary and sufficient conditions for its existence, can be found in Section 10.5 of [Sim5].

We will finish this section with two results that describe the dependence of the Schur function on the Verblunsky coefficients.

**Theorem 2.3.7 (Geronimus [Ger]).** *Let $\{\alpha_n\}_{n\geq 0}$ be a sequence of complex numbers situated in the unit disk. Let $\mu$ be the unique nontrivial probability measure on $\partial\mathbb{D}$ with Verblunsky coefficients $\{\alpha_0, \alpha_1, \ldots\}$ and $\mu_1$ the measure with Verblunsky coefficients $\{\alpha_1, \alpha_2, \ldots\}$. Let $f(z)$ and $f_1(z)$ be the Schur functions associated to the measures $\mu$ and $\mu_1$. Then*

$$f(z) = \frac{\alpha_0 + z f_1(z)}{1 + \bar{\alpha}_0 z f_1(z)} \tag{2.3.23}$$

Also, if we denote by $f(z; S)$ the Schur function associated to the family of Verblunsky coefficients $S$, we have

**Theorem 2.3.8 (Khrushchev [Khr]).** *Let $\mu$ be a nontrivial measure with Verblunsky coefficients $\{\alpha_n\}_{n\geq 0}$. Then the Schur function associated to the measure $d\mu_n = |\varphi_n(z,\mu)|^2 d\mu$ is*

$$f(z; \alpha_n, \alpha_{n+1}, \ldots)\, f(z; -\bar{\alpha}_{k-1}, -\bar{\alpha}_{k-2}, \ldots, -\bar{\alpha}_0, 1) \tag{2.3.24}$$

Proofs for theorems 2.3.7 and 2.3.8 can be found in Section 4.5 of [Sim4]. We will use these theorems in spectral averaging.

## 2.4 Properties of Random CMV Matrices

In this section we will present a few results on random CMV matrices that will be useful in the next chapters. Detailed proofs and more related results can be found in [Sim5].

We will consider sequences $\{\alpha_n\}_{n\geq 0}$ of Verblunsky coefficients that are independent identically distributed random variables. Let $\beta_0$ be the probability distribution of any random variable $\alpha_n$. Clearly, $\beta_0$ is a probability measure on $\mathbb{D}$. Any sequence $\alpha = (\alpha_n)_{n\geq 0}$ is an element of the probability space $\Omega = \times_{n=0}^{\infty} \bar{\mathbb{D}}$ (endowed with the probability measure $\beta(\alpha) = \times_{n=0}^{\infty} \beta_0(\alpha_n)$). Let's denote by $\mu_\alpha$ the probability measure on $\partial\mathbb{D}$ with Verblunsky coefficients $\alpha = \{\alpha_n\}_{n\geq 0}$.

In this thesis we will consider only random Verblunsky coefficients for which the measure $\beta_0$ is rotation invariant. It is worth mentioning that most of the results presented in this section hold for more general probability distributions. The general results can be found in Chapters 10 and 12 of [Sim5].

For any $n \in \mathbb{Z}$, let

$$\Theta_n = \begin{pmatrix} \bar{\alpha}_n & \sqrt{1-\alpha_n^2} \\ \sqrt{1-\alpha_n^2} & -\alpha_n \end{pmatrix} \tag{2.4.1}$$

and

$$\tilde{\mathcal{M}} = \bigoplus_{j \text{ odd}} \Theta_j \qquad\qquad \tilde{\mathcal{L}} = \bigoplus_{j \text{ even}} \Theta_j \tag{2.4.2}$$

We will now define the extended matrix $\mathcal{E} = \tilde{\mathcal{L}}\tilde{\mathcal{M}}$. Suppose that we have, as before, $\{\alpha_n\}_{n\in\mathbb{Z}}$ independent identically distributed random variables with the common probability distribution $\beta_0$. Any sequence $\tilde{\alpha} = (\alpha_n)_{n\in\mathbb{Z}}$ is an element of the probability space $\tilde{\Omega} = \times_{n=-\infty}^{\infty} \bar{\mathbb{D}}$ (endowed with the probability measure $\tilde{\beta}(\alpha) = \times_{n=-\infty}^{\infty} \beta_0(\alpha_n)$). Clearly, $\mathcal{E} = \mathcal{E}(\tilde{\alpha})$ is a random unitary operator.

Let $\eta_{\tilde{\alpha},0}$ be the spectral measure for $\mathcal{E}(\tilde{\alpha})$ and vector $\delta_0$. It is the unique measure with the property that for any $g \in C(\partial\mathbb{D})$, we have

$$\int_{\partial\mathbb{D}} g(\theta)\, d\eta_{\tilde{\alpha},0}(\theta) = \int_{\partial\mathbb{D}} \left(\delta_0, g(\mathcal{E}(\tilde{\alpha}))\delta_0\right)\, d\tilde{\beta}(\tilde{\alpha}) \tag{2.4.3}$$

We can now define now the density of states measure $d\nu$ on $\partial\mathbb{D}$ by

$$d\nu(\theta) = \int_{\partial\mathbb{D}} [\eta_{\tilde{\alpha},0}(\theta)] d\tilde{\beta}(\tilde{\alpha}) \tag{2.4.4}$$

**Theorem 2.4.1.** *Suppose that*

$$\int_{\mathbb{D}} \log(1-|\alpha_0|)\, d\beta_0(\alpha_0) > -\infty \tag{2.4.5}$$

*Then the Lyapunov exponent* $\gamma(z,\alpha)$ *(see* (2.3.22)*) exists for almost every* $\alpha \in \Omega$ *and is a.e.* $\alpha$ *independent.*

*If we also have*

$$\int_{\mathbb{D}} \log(|\alpha_0|)\, d\beta_0(\alpha_0) > -\infty \tag{2.4.6}$$

*then*

$$\gamma(z) = -\tfrac{1}{2}\int_{\mathbb{D}} \log(1-|\alpha_0|^2)\, d\beta_0(\alpha_0) + \int_{\mathbb{D}} \log|z-e^{i\theta}|\, d\nu(\theta) \tag{2.4.7}$$

*Remark.* (2.4.7) is called the Thouless formula.

Let's observe that if the measure $\beta_0$ is rotation invariant, then $\nu(\theta) = \frac{d\theta}{2\pi}$, and a simple computation shows that the second integral in (2.4.7) vanishes. The Thouless formula can be written in this case as

$$\gamma(z) = -\tfrac{1}{2}\int_{\mathbb{D}} \log(1-|\alpha_0|^2)\, d\beta_0(\alpha_0) \tag{2.4.8}$$

This result is Theorem 12.6.2 in [Sim5].

We will denote by $\mathcal{C}_\alpha$ the CMV matrix obtained from the Verblunsky coefficients $\alpha = \{\alpha_n\}_{n\geq 0}$. Then:

**Theorem 2.4.2.** *If the condition* (2.4.5) *holds and* $\beta_0$ *is rotation invariant, then the spectrum of the unitary operator* $\mathcal{C}_\alpha$ *is pure point for almost every* $\alpha \in \Omega$*.*

This phenomenon is called localization and was observed for the first time for Schrödinger operators. We will discuss this topic in more detail in Chapter 3.

We should mention here that another class of random unitary operators was recently studied by Joye et al. In a series of papers, they obtained localization results similar to the ones presented here (see [BHJ], [Joye1], and [Joye2]).

Another useful theorem is

**Theorem 2.4.3 (Ruelle-Osceledec).** *Suppose that we have a CMV matrix with Verblunsky coefficients $\{\alpha_n\}_{n\geq 0}$ and such that for a $z_0 \in \partial\mathbb{D}$, the Lyapunov exponent $\gamma(z_0) = \gamma(z_0, \{\alpha_n\})$ defined by (2.3.22) exists and is positive. Then there exists a one-dimensional subspace $P_\infty \in l^2(\mathbb{Z}^+)$ such that*

*a) For any $u \in P_\infty$, $u \neq 0$, we have*

$$\lim_{n\to\infty} \frac{1}{n} \log \|T_n u\| = -\gamma \tag{2.4.9}$$

*(i.e., $\|T_n u\|$ has exponential decay).*

*b) For any $u \notin P_\infty$, we have*

$$\lim_{n\to\infty} \frac{1}{n} \log \|T_n u\| = \gamma \tag{2.4.10}$$

*(i.e., $\|T_n u\|$ has exponential growth).*

In the analysis of the random matrix $\mathcal{C}_\alpha$, it is very important to understand the structure of the eigenfunctions. Of particular interest will be the the powers $\mathcal{C}_\alpha^n$ for $n \in \mathbb{Z}$.

For any $z \in \mathbb{D}$ and any nonnegative integers $k, l$, define the complex function

$$F_{kl}(z) = \left[\frac{\mathcal{C}_\alpha + z}{\mathcal{C}_\alpha - z}\right]_{kl} \tag{2.4.11}$$

One can check, using Kolmogorov's theorem ([Dur]) that for any $p \in (0,1)$, we have $F_{kl} \in H^p(\mathbb{D})$. Therefore, the function $F_{kl}$ has boundary values almost everywhere on $\partial\mathbb{D}$, and if we define $F_{kl}(e^{i\theta}) = \lim_{r\uparrow 1} F_{kl}(re^{i\theta})$, then $F_{kl}(e^{i\theta}) \in L^p(\partial\mathbb{D})$.

**Theorem 2.4.4 (Simon [Sim1]).** *Suppose that the Verblunsky coefficients $\{\alpha_n\}_{n\geq 0}$ are independent identically distributed random variables with a common probability distribution that is invariant under translations. Suppose that there exist $p \in (0,1)$ and two constants $C_1, \kappa_1 > 0$ such that*

$$\mathbb{E}\left(\int_{\partial\mathbb{D}} |F_{kl}(e^{i\theta})|^p \frac{d\theta}{2\pi}\right) \leq C_1 e^{-\kappa_1 |k-l|} \tag{2.4.12}$$

*Then there exist two constants $C_2, \kappa_2 > 0$ such that*

$$\mathbb{E}\left(\sup_{n\in\mathbb{Z}} |(\mathcal{C}_\alpha^{(n)})_{kl}|\right) \leq C_2 e^{-\kappa_2 |k-l|} \tag{2.4.13}$$

Aizenman obtained a similar theorem for Schrödinger operators [Aiz2]. The CMV version was discovered by Simon [Sim1].

This theorem is useful because we will be able to check that for some classes of random CMV matrices, the condition (2.4.12) holds. The conclusion (2.4.13) will allow us to get information on the decay of the eigenfunctions of the matrix $\mathcal{C}_\alpha$.

# Chapter 3

# Background on Random Schrödinger Operators

The development of the mathematical theory of random Schrödinger operators was largely motivated by the work of the physicist P. W. Anderson. In his seminal paper [And], he explained why certain random lattices exhibit lack of diffusion, a phenomenon that was later called Anderson localization.

From a mathematical point of view, localization means that the Schrödinger operator has a pure point spectrum, and the corresponding eigenfunctions are exponentially localized. Several proofs of this result (for various classes of random Schrödinger operators) are known. A question related to localization is the study of the local distribution of the eigenvalues. In this chapter we will give a brief overview of these results.

## 3.1 The Anderson Model

The first model for the random Schrödinger operator was studied by the Russian school (see [GMP] and [Mo1]). The model considered in these papers is the one-dimensional continuous Schrödinger operator defined on $L^2(\mathbb{R})$ by

$$H = -\frac{d^2}{dt^2} + q(t,\omega), \qquad t \in \mathbb{R}, \quad \omega \in \Omega \tag{3.1.1}$$

where $q(t,\omega)$ is a stationary random potential.

In order to define $q(t,\omega)$, we consider a $\nu$-dimensional compact Riemannian manifold $K$

with the metric $ds^2 = g_{ij}dx^i dx^j$ in the local coordinates $x^i, i = 1, \dots, \nu$. Let

$$\Delta = \frac{1}{\sqrt{\det g}} \left( \frac{\partial}{\partial x_i} \left( g^{ij} \sqrt{\det g} \frac{\partial}{\partial x_i} \right) \right) \tag{3.1.2}$$

be the Laplace-Beltrami operator of the metric $ds^2$.

We can now consider $x_t(\omega)$, the Brownian motion on the manifold $K$ with the generating operator $\Delta$ (note that the invariant measure of $x_t$ is the natural Riemannian measure). We take $\Omega$ to be the probability space of all the realizations of the process $x_t$ with the probability measure $\mathbb{P}$ induced by the stationary Markov process $x_t$ (see [McK]).

Let also $F : K \to \mathbb{R}$ be a smooth ($C^\infty$) nonflat Morse function (i.e., there exists a number $N$ such that for any point $x_0 \in K$, we can find a $k \leq N$ such that $d^k F(x_0) \neq 0$). In addition to these properties, we will also assume

$$\min_{x \in K} F(x) = 0 \qquad \text{and} \qquad \max_{x \in K} F(x) = 1 \tag{3.1.3}$$

The random potentials considered in (3.1.1) will be of the form

$$q(t, \omega) = F(x_t(\omega)) \tag{3.1.4}$$

The conditions (3.1.3) imply that the operator $H$ has a unique self-adjoint extension to an operator on $L^2(\mathbb{R})$. We will also consider the operator $H_V$, which is the restriction of $H$ to the space $L^2(-V, V)$ with Dirichlet boundary conditions.

We can now state the main result in [GMP]:

**Theorem 3.1.1 (Goldsheid-Molchanov-Pastur).** *The operator $H = H(\omega)$ defined by* (3.1.1) *has pure point spectrum for almost all $\omega \in \Omega$ (i.e., a complete system of eigenvalues in $L^2(\mathbb{R})$).*

In addition to this result, one can also analyze the decay of the eigenvalues. This was done in [Mo1]:

**Theorem 3.1.2 (Molchanov).** *With probability* 1*, each eigenfunction of the operator $H = H(\omega)$ defined by* (3.1.1) *decreases exponentially.*

In addition to the decay of the eigenfunctions of the randon Schrödinger operator on the whole line, Molchanov also proved in [Mo1] that with high probability a "majority" of the eigenvalues of the operator $H_V$ are well-localized. In order to state the precise result, let's consider, for any $\varepsilon, \delta, V > 0$ the event $A_V^{\varepsilon,\delta} = \{$For any eigenfunction $\psi_E(s)$, $s \in [-V, V]$ (corresponding to the operator $H_V$ and the energy E), there exists a point $\tau = \tau(\psi_E)$ so that $r(\psi_E) = \sqrt{\psi_E^2 + (\psi_E')^2} \leq e^{-\delta|s-\tau|}$ for any $s \notin \left(\tau - \ln^{1+\varepsilon} V, \tau + \ln^{1+\varepsilon} V\right)\}$. The point $\tau(\psi_E)$ (which may not be unique) is called the center of localization of the eigenfunction $\psi_E$. We can now state the localization result for the eigenfunctions of the operator $H_V$:

**Theorem 3.1.3.** *With the notation presented before, for any $\varepsilon > 0$, there exists a $\delta > 0$ such that*

$$\lim_{V \to \infty} \mathbb{P}(A_V^{\varepsilon,\delta}) = 1 \tag{3.1.5}$$

As we will see later, this result will be a key ingredient in the study of the local statistical distribution of the eigenvalues of $H_V$.

The next natural step was to consider multidimensional Schrödinger operators. In the early eighties, a new model for the Anderson model became common in the mathematical physics literature: the Anderson tight binding model, which is the discrete Schrödinger operator with random independent identically distributed potentials.

We will briefly describe the model and present the main localization results obtained in this setting.

We will first define the finite-difference Laplacian on $\mathbb{Z}^\nu$. For any $x \in \mathbb{Z}^\nu$, let $\delta_x$ be the vector $\delta_x : \mathbb{Z}^\nu \to \mathbb{C}$ which satisfies $\delta_x(y) = \delta_{x,y}$ for any $y \in \mathbb{Z}^\nu$. We consider the norm $|\cdot|$ on $\mathbb{Z}^\nu$ defined by $|x| = \sum_{i \in \mathbb{Z}^\nu} |x_i|$ for any $x = (x_i)_{i \in \mathbb{Z}^\nu}$, $x_i \in \mathbb{C}$. The finite-difference Laplacian on $\mathbb{Z}^\nu$ is the bounded operator $\Delta : l^2(\mathbb{Z}^\nu) \to l^2(\mathbb{Z}^\nu)$ with the matrix representation

$$(\delta_x, \Delta\delta_y) = \begin{cases} 1, & \text{if } |x - y| = 1 \\ 0, & \text{otherwise.} \end{cases} \tag{3.1.6}$$

The $\nu$-dimensional Anderson tight binding model is given by the random Hamiltonian

$$H(\omega) = -\Delta + v(\omega) \tag{3.1.7}$$

on $l^2(\mathbb{Z}^\nu)$ where $\Delta$ is the finite-difference Laplace operator defined before and $v = v(\omega) = (v_j(\omega))_{j\in\mathbb{Z}^\nu}$ is a random potential defined by independent identically distributed random variables $v_j = v_j(\omega), j \in \mathbb{Z}^\nu$.

The random potential $v(\omega)$ is the diagonal operator and acts on $l^2(\mathbb{Z}^\nu)$ by

$$(v(\omega)\ w)_j = v_j(\omega)\ w_j \tag{3.1.8}$$

for any $w = (w_j)_{j\in\mathbb{Z}^\nu} \in l^2(\mathbb{Z}^\nu)$.

Therefore, the action of the operator $H(\omega)$ on vectors $w = (w_j)_{j\in\mathbb{Z}^\nu} \in l^2(\mathbb{Z}^\nu)$ is

$$(H(\omega)\, w)_j = \sum_{|k|=1} w_{j+k} \ + \ v_j(\omega)\ w_j \tag{3.1.9}$$

Let $(\tilde{\Omega}, \tilde{\mathbb{P}})$ be a probability space and suppose that for each $j \in \mathbb{Z}^\nu$, $v_j : \tilde{\Omega} \to \mathbb{R}$. Then the total probability space is $\Omega = \times_{j\in\mathbb{Z}^\nu}\tilde{\Omega}$ with the probability measure $\mathbb{P}$ obtained by taking the product of the probability measures on each space $\tilde{\Omega}$. Suppose also that the common probability distribution of the random variables $v(j)$ is the measure $\mu$.

Anderson localization for the random Hamiltonian $H(\omega)$ means that the spectrum of this operator is pure point for almost every $\omega \in \Omega$ and the corresponding eigenvalues are exponentially localized. We should mention here that a number of subtle issues related to localization are discussed in [dRJLS].

It is clear that the methods used for proving the Anderson localization depend on the probability distribution $\mu$. There are two very different and, in a way, opposite situations: (i) $\mu$ is absolutely continuous with respect to the Lebesgue measure and (ii) $\mu$ is supported on only two points (i.e., $\mu$ is the sum of two weighted Dirac measures).

The general strategy for obtaining the Anderson localization is to study the Green function associated to the operator $H(\omega)$:

$$G_\omega(z) = (H(\omega) - z)^{-1} \tag{3.1.10}$$

for $z \in \mathbb{C} \setminus \mathbb{R}$. If the matrix elements of the Green function decay exponentially (along the rows and columns), one can show that the eigenfunctions of $H(\omega)$ decay exponentially.

The exponential decay of the Green function (with very high probability) was obtained by Fröhlich and Spencer in [FS] for probability distributions $\mu$ absolutely continuous with respect to the Lebesgue measure. For such a measure, let $\mu' \in L^1(\mathbb{R})$ be its Radon-Nikodym derivative with respect to the Lebesgue measure. Then

**Theorem 3.1.4 (Fröhlich-Spencer [FS]).** *Consider the operator* $H(\omega)$ *(defined by* (3.1.7)*) in any dimension* $\nu$*. Suppose that either*

a) $\mu$ *is absolutely continuous with respect to the Lebesgue measure with* $\|\mu'\|_\infty$ *sufficiently small and* $E$ *is arbitrary, or*

b) $\mu$ *is Gaussian and* $|E|$ *is large.*

*Then, for constants* $C$ *and* $m$ *depending only on* $\|\mu'\|_\infty$ *(or* $E$*) and* $p$*, one has*

$$\sup_{0<\varepsilon<1} |(\delta_0, G_\omega(E+i\varepsilon)\delta_n)| \leq e^{m(N-|n|)} \tag{3.1.11}$$

*with probability at least* $1 - CN^{-p}$*. Moreover, as* $\|\mu'\|_\infty$ *or* $1/E$ *tends to 0, the constant* $m$ *tends to* $\infty$*.*

In order to conclude that we have Anderson localization in the sense described before, we can use the following criterion:

**Theorem 3.1.5 (Simon-Wolff [SW]).** *Consider the operator* $H(\omega)$ *(defined by* 3.1.7*) in dimension* $\nu$*. Suppose that* $\int_{\mathbb{R}} (\log|x|) d\mu(x) < \infty$ *and let* $(a, b) \subset \mathbb{R}$*. Consider the following two statements*

a) *For almost every* $\omega \in \Omega$*,* $H_\omega$ *has only point spectrum in* $(a, b)$

b) *For almost every* $E \in (a, b)$ *and almost every* $\omega$*,*

$$\lim_{\varepsilon \downarrow 0} \left[ \sum_{n \in \mathbb{Z}^\nu} |(\delta_n, G_\omega(E+i\varepsilon)\delta_0)|^2 \right] < \infty \tag{3.1.12}$$

*Then,*

i) *If* $\mu$ *is purely absolutely continuous, then b) implies a) (in any dimension).*

ii) *If* $\nu = 1$ *and the measure* $\mu$ *has a nonzero absolutely continuous component, then b) implies a).*

iii) *If the essential support of the measure* $\mu$ *is* $\mathbb{R}$, *then* a) *implies* b).

We can easily see that the conclusion of Theorem 3.1.4 implies that condition b) of Theorem 3.1.5 is satisfied and therefore we get:

**Theorem 3.1.6 (Localization in Arbitrary Dimension).** *Under the hypotheses of Theorem 3.1.4, for almost every* $\omega \in \Omega$, *the random Hamiltonian* $H(\omega)$ *has only dense point spectrum (for all* $E$ *if condition* a) *holds and for* $|E|$ *large if condition* b) *holds) with exponentially decaying eigenfunctions.*

A stronger result can be obtained for one-dimensional random Schrödinger operators:

**Theorem 3.1.7.** *Consider the operator* $H(\omega)$ *(defined by* (3.1.7)) *in dimension* $\nu = 1$. *Suppose that* $\int_{\mathbb{R}} (\log_+ |x|) d\mu(x) < \infty$ *and the measure* $\mu$ *has an absolutely continuous component. Then the operator* $H(\omega)$ *has only pure point spectrum for almost every* $\omega \in \Omega$.

Theorems 3.1.6 and 3.1.7 were discovered by Simon and Wolff in [SW]. The condition that the measure $\mu$ contains an absolutely continuous component was critical for their approach. For the case of very singular distributions (e.g., Bernoulli: $\mu = p\delta_a + (1-p)\delta_b$, with $a, b \in \mathbb{R}$, $p \in (0,1)$), the Anderson localization in dimension $\nu = 1$ was obtained by Carmona, Klein, and Martinelli in [CKM]:

**Theorem 3.1.8 (Carmona-Klein-Martinelli).** *Consider the operator* $H(\omega)$ *(defined by* (3.1.7)) *in dimension* $\nu = 1$. *Suppose that*

$$\int_{\mathbb{R}} |x|^{\eta} d\mu(x) < \infty \tag{3.1.13}$$

*for some* $\eta > 0$ *and the support of* $\mu$ *has at least two points. Then the spectrum of* $H(\omega)$ *is pure point for almost every* $\omega \in \Omega$ *and the corresponding eigenfunctions are exponentially localized.*

Note that the condition (3.1.13) covers the case of Bernoulli distributions.

At the end of this section, we will present a more recent result on localization, obtained by Aizenman and Molchanov in [AM].

**Theorem 3.1.9 (Aizenman-Molchanov).** *Consider random Hamiltonians in any dimension $\nu$,*

$$H(\omega) = -\Delta + \lambda\, v(\omega) \tag{3.1.14}$$

*where $\lambda > 0$ and the common probability distribution for the random variables $v_j(\omega)$ is $d\mu(x) = \frac{1}{2}\chi_{[-1,1]}(x)$ ($\lambda$ is called the coupling constant). Then, for every $s \in (0,1)$, there exists a constant $\kappa_s$ and a function $\zeta_s$ on $\mathbb{R}$ which is strictly positive on $(\kappa_s, \infty)$, identically $-\infty$ on $(-\infty, \kappa_s]$, and obeys $\lim_{z\to\infty} \frac{\zeta_s(z)}{z} = 1$, and such that*

i) (*High disorder*) *For any $s \in (0,1)$ and any coupling constant $\lambda$ with*

$$|\lambda| > \frac{(2d)^{1/s}}{\kappa_s} \tag{3.1.15}$$

*the operator $H(\omega)$ has only pure point spectrum for almost every $\omega \in \Omega$, and the corresponding eigenfunctions are exponentially localized.*

ii) (*Extreme energies*) *With no restriction on $\lambda$, in the energy range*

$$\left\{ E \in \mathbb{R} :\ |E - 2\nu| > |\lambda|\ \zeta_s \left( \frac{(2d)^{1/s}}{|\lambda|} \right) \right\} \tag{3.1.16}$$

*the operator $H(\omega)$ has only pure point spectrum for almost every $\omega \in \Omega$, and the corresponding eigenfunctions are exponentially localized.*

The main new idea of the localization proof in [AM] is to show that for the random Schrödinger operator defined in (3.1.14) and for any $s \in (0,1)$, there exist a constant $\kappa_s$ and a function $\zeta_s$ as described in Theorem 3.1.9 such that at high disorder (condition (3.1.15)) or at extreme energies (condition (3.1.16)),

$$\mathbb{E}(|(\delta_x, G^\Lambda(E)\delta_y)|^s) \le De^{-m|x-y|} \tag{3.1.17}$$

where $D < \infty$, $m > 0$, and the above estimate holds uniformly in any finite volume $\Lambda \subset \mathbb{Z}^\nu$. Here $G^\Lambda(E) = (H^\Lambda - E)^{-1}$, where $H^\Lambda$ is the restriction of the random Hamiltonian $H$ to a finite box $\Lambda$. Such bounds also hold for fractional moments of infinite-volume quantity $(\delta_x, G(E + i\,0)\delta_y)$ at Lebesgue almost every energy in the corresponding range. Bounds of type (3.1.17) on the resolvent of a random operator are called Aizenman-Molchanov bounds.

From the exponential decay of the $s$-fractional moments of the matrix elements of the Green function, one can conclude that the Anderson localization holds, using the Simon-Wolff criterion (Theorem 3.1.5). A very concise and clear presentation of the Aizenman-Molchanov methods can be found in [Sim2].

## 3.2 The Statistical Distribution of the Eigenvalues for the Anderson Model: The Results of Molchanov and Minami

In addition to the phenomenon of localization, one can also study the statistical distribution of the truncated Schrödinger operator near energies at which we expect to have localization (e.g., in the case of the Anderson tight binding model, in the two regimes described in Theorem 3.1.9: high disorder/any energy or any disorder/extreme energies).

The first result in this direction was obtained by Molchanov in [Mo2]. He studied the one-dimensional Anderson model considered by the Russian school and described in the previous section.

For a random one-dimensional Hamiltonian defined by (3.1.1), we consider $H_V$ its restriction to the space $L^2(-V, V)$ with Dirichlet boundary conditions. Let $E_1^{(V)} \leq E_2^{(V)} \leq \cdots$ be the eigenvalues of this operator (they are random points on the real line). We also denote by $E_1 \leq E_2 \leq \cdots$ the eigenvalues of the random operator $H$.

We will start with a heuristic description of the possible situations that can occur when we analyze the statistical distribution of these eigenvalues. We will consider two consecutive eigenvalues $E_n^{(V)}$ and $E_{n+1}^{(V)}$ for $n$ proportional to $V$ (say $n = kV$ for a constant $k > 0$, which implies that as $V \to \infty$, $E_n^{(V)} \to E_k$). More precisely, we will want to understand the behavior of the random variable "spectral split":

$$\Delta_n^{(V)} = \frac{E_{n+1}^{(V)} - E_n^{(V)}}{\mathbb{E}\left(E_{n+1}^{(V)} - E_n^{(V)}\right)} \tag{3.2.1}$$

Let's suppose that the limit distribution function of $\Delta_n^{(V)}$ exists as $V \to \infty$ and denote it by

$$\varphi(x) = \lim_{V \to \infty} \mathbb{P}\left(\Delta_n^{(V)} \leq x\right) \tag{3.2.2}$$

Depending on the behavior of the function $\varphi(x)$ near 0, we can have:

i) If $\varphi(x) = o(x)$ as $x \to 0$, then the probability that the eigenvalues are close (i.e., closer than the expected distance) as $x \to 0$ is very small. This means that there is repulsion between the eigenvalues $E_n^{(V)}$ and $E_{n+1}^{(V)}$.

ii) If $\varphi(x)/x \to \infty$ as $x \to 0$, then the probability that the consecutive eigenvalues $E_n^{(V)}$ and $E_{n+1}^{(V)}$ are close (i.e., closer than the expected distance) is big, so there is attraction between the eigenvalues (i.e., they will tend to form clusters).

iii) The case $\varphi(x)/x \to c$ as $x \to 0$, where $c > 0$, corresponds to the situation when there is no interaction between the points $E_n^{(V)}$ (which, for $n \sim kV$, are situated near $E_k$).

Molchanov proved that for the one-dimensional random Schrödinger operator (3.1.1) we are in the third situation (the eigenvalues are not correlated). This result clarified some existing confusion in the mathematical physics literature at that time that was caused by a series of physics papers initiated by [Pok], which asserted that there is repulsion between the eigenvalues of the random Schrödinger operator.

In order to state Molchanov's result, we will have to define the density of states corresponding to the random Schrödinger operator. Thus, if we consider the Dirichlet spectral problem

$$\begin{aligned} &H_V \psi = -\frac{d^2\psi}{dt^2} + F(x_t(\omega))\psi = E\psi, \qquad t \in (-V, V), \\ &\psi(-V) = \psi(V) = 0 \end{aligned} \tag{3.2.3}$$

with the eigenvalues $0 < E_1(V, \omega) < E_2(V, \omega) < \cdots$, then the integrated density of states

$$N(E) = \lim_{V \to \infty} \frac{\#\{i \mid E_i(V, \omega) \leq E\}}{2V} \tag{3.2.4}$$

exists for any $E > 0$ and is independent of $\omega \in \Omega$. Furthermore, the function $N = N(E)$ is absolutely continuous, so its derivative (called the density of states)

$$n(E) = \frac{dN(E)}{dE} \tag{3.2.5}$$

exists for any $E > 0$ and is positive. The proof of the existence of the density of states for this model (and also for more general models) can be found in [Pas].

We can now state

**Theorem 3.2.1 (Molchanov [Mo2]).** *Let $H_V$ be the random one-dimensional Schrödinger operator defined before. Denote by $N_V(I)$ the number of eigenvalues of the operator $H_V$ situated in the interval $I$. Then, for any fixed $a_1 < b_1 \le a_2 < b_2 \le \cdots \le a_n < b_n$ and any nonnegative integers $k_1, k_2, \ldots, k_n$,*

$$\lim_{V\to\infty} \mathbb{P}\left( N_V\left(E_0 + \frac{a_1}{2V}, E_0 + \frac{b_1}{2V}\right) = k_1, \ \ldots, \ N_V\left(E_0 + \frac{a_n}{2V}, E_0 + \frac{b_n}{2V}\right) = k_n \right) \tag{3.2.6}$$

$$= e^{-(b_1-a_1)\, n(E_0)} \frac{((b_1 - a_1)\, n(E_0))^{k_1}}{k_1!} \ \cdots \ e^{-(b_n-a_n)\, n(E_0)} \frac{((b_n - a_n)\, n(E_0))^{k_n}}{k_n!} \tag{3.2.7}$$

This means (using the terminology of point processes presented in Chapter 1) that the local statistical distribution of the eigenvalues of the operator $H_V$ (rescaled near $E_0$) converges, as $V \to \infty$, to the Poisson point process with intensity measure $n(E_0)\, dx$, where $dx$ denotes the Lebesgue measure.

It is natural to ask if the same statistical distribution holds for the eigenvalues of the multidimensional Schrödinger operator. This was an open problem for more than a decade and was solved by Minami in [Mi].

Minami considered the Anderson tight binding model in $\nu$ dimensions (3.1.7) with the common probability distribution $\mu$ absolutely continuous with respect to the Lebesgue measure and with $\|\mu'\|_\infty < \infty$.

As in the one-dimensional case, we consider the restriction $H^\Lambda$ of the operator $H$ to the finite box $\Lambda \subset \mathbb{Z}^\nu$. Let

$$E_1(\Lambda, \omega) \le \cdots \le E_{|\Lambda|}(\Lambda, \omega) \tag{3.2.8}$$

be the eigenvalues of $H^\Lambda$.

Under these hypotheses (see [CL]), the integrated density of states

$$N(E) = \lim_{\Lambda \uparrow \mathbb{Z}^\nu} \frac{\# \{ j \mid E_j(\Lambda, \omega) \leq E \}}{|\Lambda|} \tag{3.2.9}$$

exists for almost every $\omega \in \Omega$ and is independent of $\omega$. Furthermore, its derivative (the density of states) $n(E) = \frac{dN(E)}{dE}$ exists at all $E \in \mathbb{R}$.

Let's consider $E \in \mathbb{R}$ with $n(E) > 0$. For technical purposes, we also need to take $E$ in one of the two regimes (high disorder or extreme energies) where the Aizenman-Molchanov bounds (3.1.17) hold. We will want to study the statistical distribution of the eigenvalues of the operator $H^\Lambda$ that are situated near $E$. The condition $n(E) > 0$ implies that the average spacing between the eigenvalues of $H^\Lambda$ situated near $E$ is of order $|\Lambda|^{-1}$. It is therefore interesting to rescale the spectrum near $E$ and consider the points

$$\zeta_j(\Lambda, E) = |\Lambda|(E_j(\Lambda, \omega) - E) \tag{3.2.10}$$

We want to study the limit of the point process $\zeta(\Lambda) = \{\zeta_j(\Lambda, E)\}_j$ as $\Lambda \uparrow \mathbb{Z}^\nu$. We can now state Minami's result:

**Theorem 3.2.2 (Minami [Mi]).** *Consider the Anderson tight binding model*

$$H = -\Delta + V(\omega)$$

*and denote by $H^\Lambda$ its restriction to hypercubes $\Lambda \subset \mathbb{Z}^d$. Let $E \in \mathbb{R}$ be an energy for which the Aizenman-Molchanov bounds* (3.1.17) *hold* (*see also Theorem 3.1.9*) *and such that $n(E) > 0$. Then the point process $\zeta(\Lambda) = \{\zeta_j(\Lambda, E)\}_j$ converges to the Poisson point process with intensity measure $n(E)dx$ as the box $\Lambda$ gets large* ($\Lambda \uparrow \mathbb{Z}^d$).

We will finish this chapter by making a few remarks about the proofs of Theorems 3.2.1 and 3.2.2. Both proofs follow a standard road map towards proving Poisson statistics: the decoupling of the point process into the direct sum of smaller point processes and the proof that the ratio of the probability that each of the small processes contributes two or more eigenvalues in each rescaled interval and the probability that it contributes one eigenvalue is small.

In Molchanov's work, the point process is decoupled by writing the interval $(-V, V)$ as the union of $[\ln^{1+\varepsilon} V]$ smaller intervals and considering the same spectral problem (3.2.3) on each small interval. The asymptotic local statistical distribution of the eigenfunctions remains unchanged because most of the eigenfunctions of $H_V$ are well localized (Theorem 3.1.3).

In Minami's work, the exponential localization of the eigenfunctions is derived from the Aizenman-Molchanov bounds. The decoupling is done using a standard method for Schrödinger operators: by turning "off" some bonds. The Aizenman-Molchanov bounds are used again to show that, as before, the asymptotical local statistical distribution of the eigenfunctions remains unchanged.

The second part is to show that each one of the small point processes contributes at most one eigenvalue in each rescaled (i.e., of size $O(\frac{1}{N})$) interval. While Molchanov's proof is very involved from a technical point of view, Minami's proof is short and concise, hence very tempting for mathematicians who want to prove Poisson statistics for other models.

Minami reduces the proof to finding a uniform bound for the expectations of $2 \times 2$ determinants included in the imaginary part of the matrix representation of the resolvent of $H^\Lambda$. He uses a rank two perturbation to express this resolvent in terms of the resolvent of a Hamiltonian $\tilde{H}^\Lambda$ with two sites turned "off" and then gets the uniform bound through spectral averaging. The cancellations that appear in the spectral averaging are very efficient and spectacular—they give exactly the desired result. Experts call this step "the Minami trick" and agree that the cancellations are somehow mysterious and might hide a deep undiscovered mathematical fact. A proof of the local Poisson statistics for the eigenvalues of the $n$-dimensional random Schrödinger operator that does not make use of the "Minami trick" is yet to be discovered.

In this thesis we prove that the local statistical distribution of the zeros of some classes of random paraorthogonal polynomials. We follow the general strategy presented before and we avoid the "Minami trick" (see Section 6.1). It seems that our method should work for one-dimensional random Schrödinger operators, but will most likely fail in higher dimensions.

# Chapter 4

# Aizenman-Molchanov Bounds for the Resolvent of the CMV Matrix

We will study the random CMV matrices defined in (1.1.1). We will analyze the matrix elements of the resolvent $(\mathcal{C}^{(n)} - z)^{-1}$ of the CMV matrix, or, what is equivalent, the matrix elements of

$$F(z, \mathcal{C}^{(n)}) = (\mathcal{C}^{(n)} + z)(\mathcal{C}^{(n)} - z)^{-1} = I + 2z\,(\mathcal{C}^{(n)} - z)^{-1} \tag{4.0.1}$$

(we consider $z \in \mathbb{D}$). More precisely, we will be interested in the expectations of the fractional moments of the matrix elements of the resolvent. This method (sometimes called the fractional moments method) is useful in the study of the eigenvalues and of the eigenfunctions and was introduced by Aizenman and Molchanov in [AM].

We will prove that the expected value of the fractional moment of the matrix elements of the resolvent decays exponentially (see (1.1.6)). The proof of this result is rather involved; the main steps will be:

Step 1. The fractional moments $\mathbb{E}(|F_{kl}(z, \mathcal{C}_\alpha^{(n)})|^s)$ are uniformly bounded (Lemma 4.1.1).

Step 2. The fractional moments $\mathbb{E}(|F_{kl}(z, \mathcal{C}_\alpha^{(n)})|^s)$ converge to 0 uniformly along the rows (Lemma 4.2.5).

Step 3. The fractional moments $\mathbb{E}(|F_{kl}(z, \mathcal{C}_\alpha^{(n)})|^s)$ decay exponentially (Theorem 1.1.1).

## 4.1 Uniform Bounds for the Fractional Moments

We will now begin the analysis of $\mathbb{E}(|F_{kl}(z, \mathcal{C}_\alpha^{(n)})|^s)$.

It is not hard to see that $\mathrm{Re}\left[(\mathcal{C}^{(n)} + z)(\mathcal{C}^{(n)} - z)^{-1}\right]$ is a positive operator. This will help us prove

**Lemma 4.1.1.** *For any $s \in (0,1)$, any $k, l$, $1 \le k, l \le n$, and any $z \in \overline{\mathbb{D}}$, we have*

$$\mathbb{E}(|F_{kl}(z, \mathcal{C}_\alpha^{(n)})|^s) \le C \tag{4.1.1}$$

*where $C = \frac{2^{2-s}}{\cos\frac{\pi s}{2}}$.*

*Proof.* Let $F_\varphi(z) = (\varphi, (\mathcal{C}_\alpha^{(n)} + z)(\mathcal{C}_\alpha^{(n)} - z)^{-1}\varphi)$. Since $\operatorname{Re} F_\varphi \ge 0$, the function $F_\varphi$ is a Carathéodory function for any unit vector $\varphi$. Fix $\rho \in (0,1)$. Then, by a version of Kolmogorov's theorem (see Duren [Dur] or Khodakovsky [Kho]),

$$\int_0^{2\pi} \left| (\varphi, (\mathcal{C}_\alpha^{(n)} + \rho e^{i\theta})(\mathcal{C}_\alpha^{(n)} - \rho e^{i\theta})^{-1}\varphi) \right|^s \frac{d\theta}{2\pi} \le C_1 \tag{4.1.2}$$

where $C_1 = \frac{1}{\cos\frac{\pi s}{2}}$.

The polarization identity gives (assuming that our scalar product is antilinear in the first variable and linear in the second variable)

$$F_{kl}(\rho e^{i\theta}, \mathcal{C}_\alpha^{(n)}) = \frac{1}{4} \sum_{m=0}^{3} (-i)^m \Big( (\delta_k + i^m \delta_l),\, F(\rho e^{i\theta}, \mathcal{C}_\alpha^{(n)})(\delta_k + i^m \delta_l) \Big) \tag{4.1.3}$$

which, using the fact that $|a+b|^s \le |a|^s + |b|^s$, implies

$$\left| F_{kl}(\rho e^{i\theta}, \mathcal{C}_\alpha^{(n)}) \right|^s \le \frac{1}{2^s} \sum_{m=0}^{3} \left| \left( \frac{(\delta_k + i^m \delta_l)}{\sqrt{2}},\, F(\rho e^{i\theta}, \mathcal{C}_\alpha^{(n)}) \frac{(\delta_k + i^m \delta_l)}{\sqrt{2}} \right) \right|^s \tag{4.1.4}$$

Using (4.1.2) and (4.1.4), we get, for any $\mathcal{C}_\alpha^{(n)}$,

$$\int_0^{2\pi} \left| F_{kl}(\rho e^{i\theta}, \mathcal{C}_\alpha^{(n)}) \right|^s \frac{d\theta}{2\pi} \le C \tag{4.1.5}$$

where $C = \frac{2^{2-s}}{\cos\frac{\pi s}{2}}$.

Therefore, after taking expectations and using Fubini's theorem,

$$\int_0^{2\pi} \mathbb{E}\left( \left| F_{kl}(\rho e^{i\theta}, \mathcal{C}_\alpha^{(n)}) \right|^s \right) \frac{d\theta}{2\pi} \le C \tag{4.1.6}$$

The coefficients $\alpha_0$, $\alpha_1$, $\dots$, $\alpha_{n-1}$ define a measure $d\mu$ on $\partial\mathbb{D}$. Let us consider another measure $d\mu_{R\theta}(e^{i\tau}) = d\mu(e^{i(\tau-\theta)})$. This measure defines Verblunsky coefficients $\alpha_{0,\theta}$, $\alpha_{1,\theta}$, $\dots$, $\alpha_{n-1,\theta}$, a CMV matrix $\mathcal{C}^{(n)}_{\alpha,\theta}$, and unitary orthogonal polynomials $\varphi_{0,\theta}$, $\varphi_{1,\theta}$, $\dots$, $\varphi_{n-1,\theta}$. Using the results presented in Section 1.6 of Simon [Sim4], for any $k$, $0 \le k \le n-1$,

$$\alpha_{k,\theta} = e^{-i(k+1)\theta}\alpha_k \tag{4.1.7}$$

$$\varphi_{k,\theta}(z) = e^{ik\theta}\varphi_k(e^{-i\theta}z) \tag{4.1.8}$$

The relation (4.1.8) shows that for any $k$ and $\theta$, $\chi_{k,\theta}(z) = \lambda_{k,\theta}\,\chi_k(e^{-i\theta}z)$ where $|\lambda_{k,\theta}| = 1$.

Since $\alpha_0, \alpha_1, \dots, \alpha_{n-1}$ are independent and the distribution of each one of them is rotationally invariant, we have

$$\mathbb{E}\left(\left|F_{kl}(\rho e^{i\theta}, \mathcal{C}^{(n)}_{\alpha})\right|^s\right) = \mathbb{E}\left(\left|F_{kl}(\rho e^{i\theta}, \mathcal{C}^{(n)}_{\alpha,\theta})\right|^s\right) \tag{4.1.9}$$

But, using (4.1.7) and (4.1.8),

$$\begin{aligned}
F_{kl}(\rho e^{i\theta}, \mathcal{C}^{(n)}_{\alpha,\theta}) &= \int_{\partial\mathbb{D}} \frac{e^{i\tau}+\rho e^{i\theta}}{e^{i\tau}-\rho e^{i\theta}}\,\chi_{l,\theta}(e^{i\tau})\,\overline{\chi_{k,\theta}(e^{i\tau})}\,d\mu_{R\theta}(e^{i\tau}) \\
&= \int_{\partial\mathbb{D}} \frac{e^{i\tau}+\rho e^{i\theta}}{e^{i\tau}-\rho e^{i\theta}}\,\chi_{l,\theta}(e^{i\tau})\,\overline{\chi_{k,\theta}(e^{i\tau})}\,d\mu(e^{i(\tau-\theta)}) \\
&= \int_{\partial\mathbb{D}} \frac{e^{i(\tau+\theta)}+\rho e^{i\theta}}{e^{i(\tau+\theta)}-\rho e^{i\theta}}\,\chi_{l,\theta}(e^{i(\tau+\theta)})\,\overline{\chi_{k,\theta}(e^{i(\tau+\theta)})}\,d\mu(e^{i\tau}) \\
&= \lambda_{l,\theta}\overline{\lambda}_{k,\theta} \int_{\partial\mathbb{D}} \frac{e^{i\tau}+\rho}{e^{i\tau}-\rho}\,\chi_{l}(e^{i\tau})\,\overline{\chi_{k}(e^{i\tau})}\,d\mu(e^{i\tau}) \\
&= \lambda_{l,\theta}\,\overline{\lambda}_{k,\theta}\,F_{kl}(\rho, \mathcal{C}^{(n)}_{\alpha})
\end{aligned}$$

where $|\lambda_{l,\theta}\overline{\lambda}_{k,\theta}| = 1$.

Therefore, the function $\theta \to \mathbb{E}\left(\left|F_{kl}(\rho e^{i\theta}, \mathcal{C}^{(n)}_{\alpha})\right|^s\right)$ is constant, so, using (4.1.6), we get

$$\mathbb{E}\left(\left|F_{kl}(\rho e^{i\theta}, \mathcal{C}^{(n)}_{\alpha,\theta})\right|^s\right) \le C \tag{4.1.10}$$

Since $\rho$ and $\theta$ are arbitrary, we now get the desired conclusion for any $z \in \mathbb{D}$.

Observe that, by (4.1.3), $F_{kl}$ is a linear combination of Carathéodory functions. Using a classical result in harmonic analysis (see, e.g., [Dur]), any Carathéodory function is in $H^s(\mathbb{D})$ $(0 < s < 1)$ and therefore it has boundary values almost everywhere on $\partial\mathbb{D}$. Thus we get that, for any fixed $\alpha \in \Omega$ and for Lebesgue almost any $z = e^{i\theta} \in \partial\mathbb{D}$, the radial limit $F_{kl}(e^{i\theta}, \mathcal{C}_\alpha^{(n)})$ exists, where

$$F_{kl}(e^{i\theta}, \mathcal{C}_\alpha^{(n)}) = \lim_{\rho \uparrow 1} F_{kl}(\rho e^{i\theta}, \mathcal{C}_\alpha^{(n)}) \tag{4.1.11}$$

Also, by the properties of Hardy spaces, $F_{kl}(\,\cdot\,, \mathcal{C}_\alpha^{(n)}) \in L^s(\partial\mathbb{D})$ for any $s \in (0,1)$. Since the distributions of $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$ are rotationally invariant, we obtain that for any fixed $e^{i\theta} \in \partial\mathbb{D}$, the radial limit $F_{kl}(e^{i\theta}, \mathcal{C}_\alpha^{(n)})$ exists for almost every $\alpha \in \Omega$.

The relation (4.1.10) gives

$$\sup_{\rho \in (0,1)} \mathbb{E}\left( \left| F_{kl}(\rho e^{i\theta}, \mathcal{C}_\alpha^{(n)}) \right|^s \right) \leq C \tag{4.1.12}$$

By taking $\rho \uparrow 1$ and using Fatou's lemma we get

$$\mathbb{E}\left( \left| F_{kl}(e^{i\theta}, \mathcal{C}_\alpha^{(n)}) \right|^s \right) \leq C \tag{4.1.13}$$

$\square$

Note that the argument from Lemma 4.1.1 works in the same way when we replace the unitary matrix $\mathcal{C}_\alpha^{(n)}$ with the unitary operator $\mathcal{C}_\alpha$ (corresponding to random Verblunsky coefficients uniformly distributed in $D(0,r)$), so we also have

$$\mathbb{E}\left( \left| F_{kl}(e^{i\theta}, \mathcal{C}_\alpha) \right|^s \right) \leq C \tag{4.1.14}$$

for any nonnegative integers $k, l$ and for any $e^{i\theta} \in \partial\mathbb{D}$.

## 4.2 The Uniform Decay of the Expectations of the Fractional Moments

The next step is to prove that the expectations of the fractional moments of the resolvent of $\mathcal{C}^{(n)}$ tend to zero on the rows. We will start with the following lemma suggested to us by Aizenman [Aiz1]:

**Lemma 4.2.1.** *Let $\{X_n = X_n(\omega)\}_{n\geq 0}$, $\omega \in \Omega$ be a family of positive random variables such that there exists a constant $C > 0$ such that $\mathbb{E}(X_n) < C$ and, for almost any $\omega \in \Omega$, $\lim_{n\to\infty} X_n(\omega) = 0$. Then, for any $s \in (0,1)$,*

$$\lim_{n\to\infty} \mathbb{E}(X_n^s) = 0 \tag{4.2.1}$$

*Proof.* Let $\varepsilon > 0$ and $M > 0$ such that $M^{s-1} < \varepsilon$. Observe that if $X_n(\omega) > M$, then $X_n^s(\omega) < M^{s-1} X_n(\omega)$. Therefore

$$X_n^s(\omega) \leq X_n^s(\omega)\, \chi_{\{\omega; X_n(\omega)\leq M\}}(\omega) + M^{s-1} X_n(\omega) \tag{4.2.2}$$

Clearly, $\mathbb{E}(M^{s-1}X_n) \leq \varepsilon C$ and, using dominated convergence,

$$\mathbb{E}(X_n^s\, \chi_{\{\omega; X_n(\omega)\leq M\}}) \to 0 \qquad \text{as} \quad n \to \infty \tag{4.2.3}$$

We immediately get that for any $\varepsilon > 0$, we have

$$\limsup_{n\to\infty} \mathbb{E}(X_n^s) \leq \varepsilon C \tag{4.2.4}$$

so we can conclude that (4.2.1) holds. $\square$

We will use Lemma 4.2.1 to prove that for any fixed $j$, the quantities $\mathbb{E}\left(\left|F_{j,j+k}(e^{i\theta}, \mathcal{C}_\alpha)\right|^s\right)$ and $\mathbb{E}\left(\left|F_{j,j+k}(e^{i\theta}, \mathcal{C}_\alpha^n)\right|^s\right)$ converge to 0 as $k \to \infty$. From now on, it will be more convenient to work with the resolvent $G$ instead of the Carathéodory function $F$.

**Lemma 4.2.2.** *Let $\mathcal{C} = \mathcal{C}_\alpha$ be the random CMV matrix associated to a family of Verblunsky* *coefficients* $\{\alpha_n\}_{n\geq 0}$ *with* $\alpha_n$ *i.i.d. random variables uniformly distributed in a disk* $D(0,r)$, $0<r<1$. *Let* $s \in (0,1)$, $z \in \overline{\mathbb{D}}$, *and* $j$ *a positive integer. Then we have*

$$\lim_{k\to\infty} \mathbb{E}\left(|G_{j,j+k}(z,\mathcal{C})|^s\right) = 0 \tag{4.2.5}$$

*Proof.* For any fixed $z \in \mathbb{D}$, the rows and columns of $G(z,\mathcal{C})$ are $l^2$ at infinity, hence converge to 0. Let $s' \in (s,1)$. Then we get (4.2.5) by applying Lemma 4.2.1 to the random variables $X_k = |G_{j,j+k}(z,\mathcal{C})|^{s'}$ and using the power $\frac{s}{s'} < 1$.

We will now prove (4.2.5) for $z = e^{i\theta} \in \partial\mathbb{D}$. In order to do this, we will have to apply the heavy machinery of transfer matrices and Lyapunov exponents developed in [Sim5]. Thus, the transfer matrices corresponding to the sequence of Verblunsky coefficients $\{\alpha_n\}_{n\geq 0}$ are

$$T_n(z) = A(\alpha_n, z)\ldots A(\alpha_0, z) \tag{4.2.6}$$

where $A(\alpha,z) = (1-|\alpha|^2)^{-1/2}\left(\begin{smallmatrix} z & -\overline{\alpha} \\ -\alpha z & 1\end{smallmatrix}\right)$ and the Lyapunov exponent is

$$\gamma(z) = \lim_{n\to\infty} \frac{1}{n}\log\|T_n(z,\{\alpha_n\})\| \tag{4.2.7}$$

(provided this limit exists).

Observe that the common distribution $d\mu_\alpha$ of the Verblunsky coefficients $\alpha_n$ is rotationally invariant and

$$\int_{D(0,1)} -\log(1-\omega)\, d\mu_\alpha(\omega) < \infty \tag{4.2.8}$$

and

$$\int_{D(0,1)} -\log|\omega|\, d\mu_\alpha(\omega) < \infty \tag{4.2.9}$$

For every $z = e^{i\theta} \in \partial\mathbb{D}$, the Lyapunov exponent exists and the Thouless formula for rotation invariant distributions (see (2.4.8)) gives

$$\gamma(z) = -\tfrac{1}{2}\int_{D(0,1)} \log(1-|\omega|^2)\, d\mu_\alpha(\omega) \tag{4.2.10}$$

By an immediate computation we get $\gamma(z) = \frac{r^2+(1-r^2)\log(1-r^2)}{2r^2} > 0$.

The positivity of the Lyapunov exponent $\gamma(e^{i\theta})$ implies, using Theorem 2.4.3 (Ruelle-Osceledec), that there exists a constant $\lambda \neq 1$ (defining a boundary condition) for which

$$\lim_{n\to\infty} T_n(e^{i\theta}) \begin{pmatrix} 1 \\ \lambda \end{pmatrix} = 0 \tag{4.2.11}$$

From here we immediately get (using Theorem 2.3.6) that for any $j$ and almost every $e^{i\theta} \in \partial\mathbb{D}$,

$$\lim_{k\to\infty} G_{j,j+k}(e^{i\theta}, \mathcal{C}) = 0 \tag{4.2.12}$$

We can now use (4.1.14) and (4.2.12) to verify the hypothesis of Lemma 4.2.1 for the random variables

$$X_k = \left| G_{j,j+k}(e^{i\theta}, \mathcal{C}) \right|^{s'} \tag{4.2.13}$$

where $s' \in (s, 1)$. We therefore get

$$\lim_{k\to\infty} \mathbb{E}\left( \left| G_{j,j+k}(e^{i\theta}, \mathcal{C}) \right|^{s} \right) = 0 \tag{4.2.14}$$

□

The next step is to get the same result for the finite volume case (i.e., when we replace the matrix $\mathcal{C} = \mathcal{C}_\alpha$ by the matrix $\mathcal{C}_\alpha^{(n)}$).

**Lemma 4.2.3.** *For any fixed $j$, any $s \in (0, \frac{1}{2})$, and any $z \in \overline{\mathbb{D}}$,*

$$\lim_{k\to\infty} \sup_{n\geq k} \mathbb{E}\left( \left| G_{j,j+k}(z, \mathcal{C}_\alpha^{(n)}) \right|^{s} \right) = 0 \tag{4.2.15}$$

*Proof.* Let $\mathcal{C}$ be the CMV matrix corresponding to a family of Verblunsky coefficients $\{\alpha_n\}_{n\geq 0}$, with $|\alpha_n| < r$ for any $n$. Since $\mathbb{E}(|G_{j,j+k}(z,\mathcal{C})|^s) \to 0$ and $\mathbb{E}(|G_{j,j+k}(z,\mathcal{C})|^{2s}) \to 0$ as $k \to \infty$, we can take $k_\varepsilon \geq 0$ such that for any $k \geq k_\varepsilon$, $\mathbb{E}(|G_{j,j+k}(z,\mathcal{C})|^s) \leq \varepsilon$ and $\mathbb{E}(|G_{j,j+k}(z,\mathcal{C})|^{2s}) \leq \varepsilon$.

For $n \geq (k_\varepsilon + 2)$, let $\mathcal{C}^{(n)}$ be the CMV matrix obtained with the same $\alpha_0$, $\alpha_1$, $\dots$, $\alpha_{n-2}$, $\alpha_n$, $\dots$ and with $\alpha_{n-1} \in \partial\mathbb{D}$. From now on, we will use $G(z,\mathcal{C}) = (\mathcal{C} - z)^{-1}$ and

$G(z, \mathcal{C}_\alpha^{(n)}) = (\mathcal{C}_\alpha^{(n)} - z)^{-1}$. Then

$$(\mathcal{C}_\alpha^{(n)} - z)^{-1} - (\mathcal{C} - z)^{-1} = (\mathcal{C} - z)^{-1}(\mathcal{C} - \mathcal{C}_\alpha^{(n)})(\mathcal{C}_\alpha^{(n)} - z)^{-1} \tag{4.2.16}$$

Note that the matrix $(\mathcal{C} - \mathcal{C}^{(n)})$ has at most eight nonzero terms, each of absolute value of at most 2. These nonzero terms are situated at positions $(m, m')$ and $|m - n| \leq 2$, $|m' - n| \leq 2$. Then

$$\begin{aligned} \mathbb{E}\left(|(\mathcal{C}_\alpha^{(n)} - z)_{j,j+k}^{-1}|^s\right) \leq \mathbb{E}&\left(|(\mathcal{C} - z)_{j,j+k}^{-1}|^s\right) \\ &+ 2^s \sum_{8\,\text{terms}} \mathbb{E}\left(|(\mathcal{C} - z)_{j,m}^{-1}|^s\, |(\mathcal{C}_\alpha^{(n)} - z)_{m',j+k}^{-1}|^s\right) \end{aligned} \tag{4.2.17}$$

Using Schwarz's inequality,

$$\mathbb{E}\left(|\mathcal{C} - z)_{j,m}^{-1}|^s\, |(\mathcal{C}_\alpha^{(n)} - z)_{m',j+k}^{-1}|^s\right) \leq \mathbb{E}\left(|\mathcal{C} - z)_{j,m}^{-1}|^{2s}\right)^{1/2} \mathbb{E}\left(|\mathcal{C}_\alpha^{(n)} - z)_{m',j+k}^{-1}|^{2s}\right)^{1/2} \tag{4.2.18}$$

We clearly have $m \geq k_\varepsilon$ and therefore $\mathbb{E}(|\mathcal{C} - z)_{j,m}^{-1}|^{2s}) \leq \varepsilon$. Also, from Lemma 4.1.1, there exists a constant $C$ depending only on $s$ such that $\mathbb{E}\left(|\mathcal{C}_\alpha^{(n)} - z)_{m',j+k}^{-1}|^{2s}\right) \leq C$.

Therefore, for any $k \geq k_\varepsilon$, $\mathbb{E}\left(|(\mathcal{C}_\alpha^{(n)} - z)_{j,j+k}^{-1}|^s\right) \leq \varepsilon + \varepsilon^{1/2} C$.

Since $\varepsilon$ is arbitrary, we obtain (4.2.15). □

Note that Lemma 4.2.3 holds for any $s \in (0, \frac{1}{2})$. The result can be improved using a standard method:

**Lemma 4.2.4.** *For any fixed $j$, any $s \in (0,1)$, and any $z \in \mathbb{D}$,*

$$\lim_{k\to\infty,\, k\leq n} \mathbb{E}\left(\left|G_{j,j+k}(z, \mathcal{C}_\alpha^{(n)})\right|^s\right) = 0 \tag{4.2.19}$$

*Proof.* Let $s \in [\frac{1}{2}, 1), t \in (s, 1), r \in (0, \frac{1}{2})$. Then using the Hölder inequality for $p = \frac{t-r}{t-s}$ and for $q = \frac{t-r}{s-r}$, we get

$$\begin{aligned} \mathbb{E}\left(|(\mathcal{C}_\alpha^{(n)} - z)_{j,j+k}^{-1}|^s\right) &= \mathbb{E}\left(|(\mathcal{C}_\alpha^{(n)} - z)_{j,j+k}^{-1}|^{\frac{r(t-s)}{t-r}}\; |(\mathcal{C}_\alpha^{(n)} - z)_{j,j+k}^{-1}|^{\frac{t(s-r)}{t-r}}\right) \\ &\leq \left(\mathbb{E}\left(|(\mathcal{C}_\alpha^{(n)} - z)_{j,j+k}^{-1}|^r\right)\right)^{\frac{t-s}{t-r}} \left(\mathbb{E}\left(|(\mathcal{C}_\alpha^{(n)} - z)_{j,j+k}^{-1}|^t\right)\right)^{\frac{s-r}{t-r}} \end{aligned} \tag{4.2.20}$$

From Lemma 4.1.1, $\mathbb{E}(|(\mathcal{C}_\alpha^{(n)} - z)^{-1}_{j,j+k}|^t)$ is bounded by a constant depending only on $t$ and from Lemma 4.2.3, $\mathbb{E}(|(\mathcal{C}_\alpha^{(n)} - z)^{-1}_{j,j+k}|^r)$ tends to 0 as $k \to \infty$. We immediately get (4.2.19). $\square$

We can improve the previous lemma to get that the convergence to 0 of $\mathbb{E}(|(\mathcal{C}_\alpha^{(n)} - z)^{-1}_{j,j+k}|^s)$ is uniform in row $j$.

**Lemma 4.2.5.** *For any $\varepsilon > 0$, there exists a $k_\varepsilon \geq 0$ such that, for any $s, k, j, n$, $s \in (0,1), k > k_\varepsilon, n > 0, 0 \leq j \leq (n-1)$, and for any $z \in \overline{\mathbb{D}}$, we have*

$$\mathbb{E}\left(\left|G_{j,j+k}\left(z, \mathcal{C}_\alpha^{(n)}\right)\right|^s\right) < \varepsilon \tag{4.2.21}$$

*Proof.* As in the previous lemma, it is enough to prove the result for all $z \in \mathbb{D}$. Suppose the matrix $\mathcal{C}^{(n)}$ is obtained from the Verblunsky coefficients $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$. Let's consider the matrix $\mathcal{C}_{\text{dec}}^{(n)}$ obtained from the same Verblunsky coefficients with the additional restriction $\alpha_m = e^{i\theta}$ where $m$ is chosen to be bigger but close to $j$ (for example $m = j+3$). We will now compare $(\mathcal{C}^{(n)} - z)^{-1}_{j,j+k}$ and $(\mathcal{C}_{\text{dec}}^{(n)} - z)^{-1}_{j,j+k}$. By the resolvent identity,

$$|(\mathcal{C}^{(n)} - z)^{-1}_{j,j+k}| = \left|(\mathcal{C}^{(n)} - z)^{-1}_{j,j+k} - (\mathcal{C}_{\text{dec}}^{(n)} - z)^{-1}_{j,j+k}\right| \tag{4.2.22}$$

$$\leq 2 \sum_{|l-m|\leq 2, |l'-m|\leq 2} \left|(\mathcal{C}^{(n)} - z)^{-1}_{j,l}\right| \left|(\mathcal{C}_{\text{dec}}^{(n)} - z)^{-1}_{l',j+k}\right| \tag{4.2.23}$$

The matrix $(\mathcal{C}_{\text{dec}}^{(n)} - z)^{-1}$ decouples between $m-1$ and $m$. Also, since $|l'-m| \leq 2$, we get by Lemma 4.2.5 that for any fixed $\varepsilon > 0$, we can pick a $k_\varepsilon$ such that for any $k \geq k_\varepsilon$ and any $l', |l'-m| \leq 2$, we have

$$\mathbb{E}\left(\left|(\mathcal{C}_{\text{dec}}^{(n)} - z)^{-1}_{l',j+k}\right|^s\right) \leq \varepsilon \tag{4.2.24}$$

(In other words, the decay is uniform on the five rows $m-2$, $m-1$, $m$, $m+1$, and $m+2$ situated at distance of at most 2 from the place where the matrix $\mathcal{C}_{\text{dec}}^{(n)}$ decouples.)

As in Lemma 4.2.3, we can now use Schwarz's inequality to get that for any $\varepsilon > 0$ and for any $s \in (0, \frac{1}{2})$, there exists a $k_\varepsilon$ such that for any $j$ and any $k \geq k_\varepsilon$,

$$\mathbb{E}\left(\left|(\mathcal{C}^{(n)} - z)^{-1}_{j,j+k}\right|^s\right) < \varepsilon \tag{4.2.25}$$

Using the same method as in Lemma 4.2.4, we get (4.2.21) for any $s \in (0,1)$. □

## 4.3 The Exponential Decay of the Fractional Moments

We are heading towards proving the exponential decay of the fractional moments of the matrix elements of the resolvent of the CMV matrix. We will first prove a lemma about the behavior of the entries in the resolvent of the CMV matrix.

**Lemma 4.3.1.** *Suppose the random CMV matrix $\mathcal{C}^{(n)} = \mathcal{C}_\alpha^{(n)}$ is given as before (i.e., $\alpha_0, \alpha_1, \ldots, \alpha_{n-2}, \alpha_{n-1}$ are independent random variables, the first $(n-1)$ uniformly distributed inside a disk of radius $r$ and the last uniformly distributed on the unit circle). Then, for any point $e^{i\theta} \in \partial\mathbb{D}$ and for any $\alpha \in \Omega$ where $G(e^{i\theta}, \mathcal{C}_\alpha^{(n)}) = (\mathcal{C}_\alpha^{(n)} - e^{i\theta})^{-1}$ exists, we have*

$$\frac{|G_{kl}(e^{i\theta}, \mathcal{C}_\alpha^{(n)})|}{|G_{ij}(e^{i\theta}, \mathcal{C}_\alpha^{(n)})|} \leq \left(\frac{2}{\sqrt{1-r^2}}\right)^{|k-i|+|l-j|} \tag{4.3.1}$$

*Proof.* Using Theorem 2.3.4, the matrix elements of the resolvent of the CMV matrix are given by the following formulae:

$$\left[(\mathcal{C}-z)^{-1}\right]_{kl} = \begin{cases} (2z)^{-1}\chi_l(z)p_k(z), & k > l \quad \text{or} \quad k = l = 2n-1 \\ (2z)^{-1}\pi_l(z)x_k(z), & l > k \quad \text{or} \quad k = l = 2n \end{cases} \tag{4.3.2}$$

where the polynomials $\chi_l(z)$ are obtained by the Gram-Schmidt process applied to $\{1, z, z^{-1}, \ldots\}$ in $L^2(\partial\mathbb{D}, d\mu)$ and the polynomials $x_k(z)$ are obtained by the Gram-Schmidt process applied to $\{1, z^{-1}, z, \ldots\}$ in $L^2(\partial\mathbb{D}, d\mu)$. Also, $p_n$ and $\pi_n$ are the analogues of the Weyl solutions of Golinskii-Nevai [GN] and are defined by (2.3.13) and (2.3.14). (See also (2.3.11) and (2.3.12) for the definition of $y_n$ and $\Upsilon_n$, the second kind analogues of the CMV bases).

We will be interested in the values of the resolvent on the unit circle (we know they exist a.e. for the random matrices considered here). For any $z \in \partial\mathbb{D}$, the values of $F(z)$ are purely imaginary and also $\overline{\chi_n(z)} = x_n(z)$ and $\overline{\Upsilon_n(z)} = -y_n(z)$. In particular, $|\overline{\chi_n(z)}| = |x_n(z)|$ for any $z \in \partial\mathbb{D}$.

Therefore, $\overline{\pi_n(z)} = \overline{\Upsilon_n(z)} + \overline{F(z)\chi_n(z)} = -p_n(z)$, so $|\overline{\pi_n(z)}| = |p_n(z)|$ for any $z \in$

$\partial\mathbb{D}$. We will also use $|\chi_{2n+1}(z)| = |\varphi_{2n+1}(z)|, |\chi_{2n}(z)| = |\varphi^*_{2n}(z)|, |x_{2n}(z)| = |\varphi_{2n}(z)|$, and $|x_{2n-1}(z)| = |\varphi^*_{2n-1}(z)|$ for any $z \in \partial\mathbb{D}$. Also, from Section 1.5 in [Sim4], we have

$$\left|\frac{\varphi_{n\pm1}(z)}{\varphi_n(z)}\right| \le C \tag{4.3.3}$$

for any $z \in \partial\mathbb{D}$, where $C = 2/\sqrt{1-r^2}$.

The key fact for proving (4.3.3) is that the orthogonal polynomials $\varphi_n$ satisfy a recurrence relation

$$\varphi_{n+1}(z) = \rho_n^{-1}(z\varphi_n(z) - \overline{\alpha}_n\varphi_n^*(z)) \tag{4.3.4}$$

This immediately gives the corresponding recurrence relation for the second order polynomials

$$\psi_{n+1}(z) = \rho_n^{-1}(z\psi_n(z) + \overline{\alpha}_n\psi_n^*(z)) \tag{4.3.5}$$

Using (4.3.4) and (4.3.5), we will now prove a similar recurrence relation for the polynomials $\pi_n$. For any $z \in \partial\mathbb{D}$, we have

$$\begin{aligned}\pi_{2l+1}(z) &= \Upsilon_{2l+1}(z) + F(z)\chi_{2l+1}(z)\\ &= z^{-l}(\psi_{2l+1}(z) + F(z)\varphi_{2l+1}(z))\\ &= -\rho_{2l}^{-1}\, z\,\overline{\pi_{2l}(z)} + \rho_{2l}^{-1}\,\overline{\alpha}_{2l}\,\pi_{2l}(z)\end{aligned} \tag{4.3.6}$$

and similarly, we get

$$\pi_{2l}(z) = -\rho_{2l-1}^{-1}\overline{\pi_{2l-1}(z)} - \alpha_{2l-1}\rho_{2l-1}^{-1}\pi_{2l-1}(z) \tag{4.3.7}$$

where we used the fact that for any $z \in \mathbb{D}$, $F(z)$ is purely imaginary, hence $\overline{F(z)} = -F(z)$.

Since $\rho_n^{-1} \le \frac{1}{\sqrt{1-r^2}}$, the equations (4.3.6) and (4.3.7) will give that for any integer $n$ and any $z \in \mathbb{D}$,

$$\left|\frac{\pi_{n\pm1}(z)}{\pi_n(z)}\right| \le C \tag{4.3.8}$$

where $C = 2/\sqrt{1-r^2}$.

Using these observations and (4.3.2) we get, for any $z \in \partial\mathbb{D}$,

$$\left|\left[(\mathcal{C}^{(n)} - z)^{-1}\right]_{k,l}\right| \leq C \left|\left[(\mathcal{C}^{(n)} - z)^{-1}\right]_{k,l\pm 1}\right| \tag{4.3.9}$$

and also

$$\left|\left[(\mathcal{C}^{(n)} - z)^{-1}\right]_{k,l}\right| \leq C \left|\left[(\mathcal{C}^{(n)} - z)^{-1}\right]_{k\pm 1,l}\right| \tag{4.3.10}$$

We can now combine (4.3.9) and (4.3.10) to get (4.3.1). ☐

We will now prove a simple lemma that will be useful in computations.

**Lemma 4.3.2.** *For any $s \in (0,1)$ and any constant $\beta \in \mathbb{C}$, we have*

$$\int_{-1}^{1} \frac{1}{|x-\beta|^s}\, dx \leq \int_{-1}^{1} \frac{1}{|x|^s}\, dx \tag{4.3.11}$$

*Proof.* Let $\beta = \beta_1 + i\beta_2$ with $\beta_1, \beta_2 \in \mathbb{R}$. Then

$$\int_{-1}^{1} \frac{1}{|x-\beta|^s}\, dx = \int_{-1}^{1} \frac{1}{|(x-\beta_1)^2 + \beta_2^2|^{s/2}}\, dx \leq \int_{-1}^{1} \frac{1}{|x-\beta_1|^s}\, dx \tag{4.3.12}$$

But $1/|x|^s$ is the symmetric decreasing rearrangement of $1/|x-\beta_1|^s$, so we get

$$\int_{-1}^{1} \frac{1}{|x-\beta_1|^s}\, dx \leq \int_{-1}^{1} \frac{1}{|x|^s}\, dx \tag{4.3.13}$$

and therefore we immediately obtain (4.3.11). ☐

The following lemma shows that we can control conditional expectations of the diagonal elements of the matrix $\mathcal{C}^{(n)}$.

**Lemma 4.3.3.** *For any $s \in (0,1)$, any $k$, $1 \leq k \leq n$, and any choice of $\alpha_0$, $\alpha_1$, …, $\alpha_{k-1}$, $\alpha_{k+1}$, …, $\alpha_{n-2}$, $\alpha_{n-1}$,*

$$\mathbb{E}\left(\left|F_{kk}(z, \mathcal{C}_{\alpha}^{(n)})\right|^s \;\middle|\; \{\alpha_i\}_{i\neq k}\right) \leq K \tag{4.3.14}$$

*where a possible value for the constant is $K = K(r,s) = \max\left\{\frac{4\,r^{-s}}{\pi(1-s)}\, 32^s, 2^{s+2}\left(\frac{64}{31}\right)^s\right\}$.*

*Proof.* For a fixed family of Verblunsky coefficients $\{\alpha_n\}_{n\in\mathbb{N}}$, the diagonal elements of the resolvent of the CMV matrix $\mathcal{C}$ can be obtained using the formula

$$(\delta_k, (\mathcal{C}+z)(\mathcal{C}-z)^{-1}\delta_k) = \int_{\partial\mathbb{D}} \frac{e^{i\theta}+z}{e^{i\theta}-z} \, |\varphi_k(e^{i\theta})|^2 \, d\mu(e^{i\theta}) \tag{4.3.15}$$

where $\mu$ is the measure on $\partial\mathbb{D}$ associated with the Verblunsky coefficients $\{\alpha_n\}_{n\in\mathbb{N}}$ and $\{\varphi_n\}_{n\in\mathbb{N}}$ are the corresponding normalized orthogonal polynomials.

Using Theorem 2.3.7, the Schur function of the measure $|\varphi_k(e^{i\theta})|^2 \, d\mu(e^{i\theta})$ is

$$g_k(z) = f(z; \alpha_k, \alpha_{k+1}, \ldots)\, f(z; -\overline{\alpha}_{k-1}, -\overline{\alpha}_{k-2}, \ldots, -\overline{\alpha}_0, 1) \tag{4.3.16}$$

Since, by Theorem 2.3.8, the dependence of $f(z; \alpha_k, \alpha_{k+1}, \ldots)$ on $\alpha_k$ is given by

$$f(z; \alpha_k, \alpha_{k+1}, \ldots) = \frac{\alpha_k + z f(z; \alpha_{k+1}, \alpha_{k+2}, \ldots)}{1 + \overline{\alpha}_k z f(z; \alpha_{k+1}, \alpha_{k+2}, \ldots)} \tag{4.3.17}$$

we get that the dependence of $g_k(z)$ on $\alpha_k$ is given by

$$g_k(z) = C_1 \frac{\alpha_k + C_2}{1 + \overline{\alpha}_k C_2} \tag{4.3.18}$$

where

$$C_1 = f(z; -\overline{\alpha}_{k-1}, -\overline{\alpha}_{k-2}, \ldots, -\overline{\alpha}_0, 1) \tag{4.3.19}$$

$$C_2 = z f(z; \alpha_{k+1}, \alpha_{k+2}, \ldots) \tag{4.3.20}$$

Note that the numbers $C_1$ and $C_2$ do not depend on $\alpha_k$, $|C_1|, |C_2| \le 1$.

We now evaluate the Carathéodory function $F(z; |\varphi_k(e^{i\theta})|^2 \, d\mu(e^{i\theta}))$ associated to the measure $|\varphi_k(e^{i\theta})|^2 \, d\mu(e^{i\theta})$. By definition,

$$F(z; |\varphi_k(e^{i\theta})|^2 \, d\mu(e^{i\theta})) = \int_{\partial\mathbb{D}} \frac{e^{i\theta}+z}{e^{i\theta}-z} \, |\varphi_k(e^{i\theta})|^2 \, d\mu(e^{i\theta}) \tag{4.3.21}$$

$$= (\delta_k, (\mathcal{C}+z)(\mathcal{C}-z)^{-1}\delta_k) \tag{4.3.22}$$

We now have

$$\left|F(z;|\varphi_k(e^{i\theta})|^2\,d\mu(e^{i\theta}))\right| = \left|\frac{1+zg_k(z)}{1-zg_k(z)}\right| \le \left|\frac{2}{1-z\,C_1\,\frac{\alpha_k+C_2}{1+\overline{\alpha}_k C_2}}\right| \tag{4.3.23}$$

It suffices to prove that

$$\sup_{w_1,w_2\in\mathbb{D}} \int_{D(0,r)} \left|\frac{2}{1-w_1\,\frac{\alpha_k+w_2}{1+\overline{\alpha}_k w_2}}\right|^s \frac{d\alpha_k}{\pi r^2} \;<\; \infty \tag{4.3.24}$$

Clearly,

$$\begin{aligned}\left|\frac{2}{1-w_1\,\frac{\alpha_k+w_2}{1+\overline{\alpha}_k w_2}}\right| &= \left|\frac{2(1+\overline{\alpha}_k w_2)}{1+\overline{\alpha}_k w_2 - w_1(\alpha_k+w_2)}\right| \\ &\le \left|\frac{4}{1+\overline{\alpha}_k w_2 - w_1(\alpha_k+w_2)}\right| \end{aligned}\tag{4.3.25}$$

For $\alpha_k = x+iy$, $1+\overline{\alpha}_k w_2 - w_1(\alpha_k+w_2) = x(-w_1+w_2)+y(-iw_1-iw_2)+(1-w_1w_2)$. Since for $w_1,w_2\in\mathbb{D}$, $(-w_1+w_2)$, $(-iw_1-iw_2)$, and $(1-w_1w_2)$ cannot all be small, we will be able to prove (4.3.24).

If $|-w_1+w_2|\ge\varepsilon$,

$$\frac{1}{\pi r^2}\int_{D(0,r)} \left|\frac{2}{1-w_1\,\frac{\alpha_k+w_2}{1+\overline{\alpha}_k w_2}}\right|^s d\alpha_k \le \frac{1}{\pi r^2}\left(\frac{4}{\varepsilon}\right)^s \int_{-r}^{r}\int_{-r}^{r} \frac{1}{|x+yD+E|^s}\,dx\,dy \tag{4.3.26}$$

$$\le \frac{2}{\pi r}\left(\frac{4}{\varepsilon}\right)^s \int_{-r}^{r}\frac{1}{|x|^s}\,dx = \frac{4\,r^{-s}}{\pi(1-s)}\left(\frac{4}{\varepsilon}\right)^s \tag{4.3.27}$$

(where for the last inequality we used Lemma 4.3.2).

The same bound can be obtained for $|w_1+w_2|\ge\varepsilon$.

If $|-w_1+w_2|\le\varepsilon$ and $|w_1+w_2|\le\varepsilon$, then

$$|x(-w_1+w_2)+y(-iw_1-iw_2)+(1-w_1w_2)| \ge (1-\varepsilon^2-4\varepsilon) \tag{4.3.28}$$

so

$$\int_{D(0,r)} \left|\frac{2}{1-w_1\,\frac{\alpha_k+w_2}{1+\overline{\alpha}_k w_2}}\right|^s \frac{d\alpha_k}{\pi r^2} \;\le\; 2^{s+2}\left(\frac{1}{1-\varepsilon^2-4\varepsilon}\right)^s \tag{4.3.29}$$

Therefore for any small $\varepsilon$, we get (4.3.14) with

$$K = \max\left\{\frac{4\,r^{-s}}{\pi(1-s)}\left(\frac{4}{\varepsilon}\right)^{s}, 2^{s+2}\left(\frac{1}{1-\varepsilon^2-4\varepsilon}\right)^{s}\right\} \tag{4.3.30}$$

For example, for $\varepsilon = \frac{1}{8}$, we get

$$K = K(r,s) = \max\left\{\frac{4\,r^{-s}}{\pi(1-s)}\,32^{s}, 2^{s+2}\left(\frac{64}{31}\right)^{s}\right\} \tag{4.3.31}$$

which is exactly (4.3.14). □

We will now be able to prove Theorem 1.1.1.

*Proof of Theorem 1.1.1.* We will use the method developed by Aizenman et al. [ASFH] for Schrödinger operators. The basic idea is to use the uniform decay of the expectations of the fractional moments of the matrix elements of $\mathcal{C}^{(n)}$ (Lemma 4.2.5) to derive the exponential decay.

We consider the matrix $\mathcal{C}^{(n)}$ obtained for the Verblunsky coefficients $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$. Fix $k$, with $0 \leq k \leq (n-1)$. Let $\mathcal{C}_1^{(n)}$ be the matrix obtained for the Verblunsky coefficients $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$ with the additional condition $\alpha_{k+m} = 1$, and $\mathcal{C}_2^{(n)}$ the matrix obtained from $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$ with the additional restriction $\alpha_{k+m+3} = e^{i\theta}$ ($m$ is an integer $\geq 3$ which will be specified later, and $e^{i\theta}$ is a random point uniformly distributed on $\partial\mathbb{D}$).

Using the resolvent identity, we have

$$(\mathcal{C}^{(n)} - z)^{-1} - (\mathcal{C}_1^{(n)} - z)^{-1} = (\mathcal{C}_1^{(n)} - z)^{-1}\,(\mathcal{C}_1^{(n)} - \mathcal{C}^{(n)})\,(\mathcal{C}^{(n)} - z)^{-1} \tag{4.3.32}$$

and

$$(\mathcal{C}^{(n)} - z)^{-1} - (\mathcal{C}_2^{(n)} - z)^{-1} = (\mathcal{C}^{(n)} - z)^{-1}\,(\mathcal{C}_2^{(n)} - \mathcal{C}^{(n)})\,(\mathcal{C}_2^{(n)} - z)^{-1} \tag{4.3.33}$$

Combining (4.3.32) and (4.3.33), we get

$$(\mathcal{C}^{(n)} - z)^{-1} = (\mathcal{C}_1^{(n)} - z)^{-1} + (\mathcal{C}_1^{(n)} - z)^{-1}\,(\mathcal{C}_1^{(n)} - \mathcal{C}^{(n)})\,(\mathcal{C}_2^{(n)} - z)^{-1}$$

$$+(\mathcal{C}_1^{(n)} - z)^{-1}\,(\mathcal{C}_1^{(n)} - \mathcal{C}^{(n)})\,(\mathcal{C}^{(n)} - z)^{-1}\,(\mathcal{C}_2^{(n)} - \mathcal{C}^{(n)})\,(\mathcal{C}_2^{(n)} - z)^{-1} \tag{4.3.34}$$

For any $k, l$ with $l \geq (k+m)$, we have

$$\left[(\mathcal{C}_1^{(n)} - z)^{-1}\right]_{kl} = 0 \tag{4.3.35}$$

and

$$\left[(\mathcal{C}_1^{(n)} - z)^{-1}\,(\mathcal{C}_1^{(n)} - \mathcal{C}^{(n)})\,(\mathcal{C}_2^{(n)} - z)^{-1}\right]_{kl} = 0 \tag{4.3.36}$$

Therefore, since each of the matrices $(\mathcal{C}_1^{(n)} - \mathcal{C}^{(n)})$ and $(\mathcal{C}_2^{(n)} - \mathcal{C})$ has at most eight nonzero entries, we get that

$$\begin{aligned}&\left[(\mathcal{C}^{(n)} - z)^{-1}\right]_{kl} \\ &\quad = \sum_{64 \text{ terms}} (\mathcal{C}_1^{(n)} - z)^{-1}_{ks_1}\,(\mathcal{C}_1^{(n)} - \mathcal{C}^{(n)})_{s_1 s_2}\,(\mathcal{C}^{(n)} - z)^{-1}_{s_2 s_3}\,(\mathcal{C}_2^{(n)} - \mathcal{C}^{(n)})_{s_3 s_4}\,(\mathcal{C}_2^{(n)} - z)^{-1}_{s_4 l}\end{aligned} \tag{4.3.37}$$

which gives

$$\mathbb{E}\left(\left|(\mathcal{C}^{(n)} - z)^{-1}_{kl}\right|^s\right) \leq 4^s \sum_{64 \text{ terms}} \mathbb{E}\left(\left|(\mathcal{C}_1^{(n)} - z)^{-1}_{ks_1}\,(\mathcal{C}^{(n)} - z)^{-1}_{s_2 s_3}\,(\mathcal{C}_2^{(n)} - z)^{-1}_{s_4 l}\right|^s\right) \tag{4.3.38}$$

where, since the matrix $\mathcal{C}_1^{(n)}$ decouples at $(k+m)$, we have $|s_2 - (k+m)| \leq 2$ and, since the matrix $\mathcal{C}_1^{(n)}$ decouples at $(k+m+3)$, we have $|s_3 - (k+m+3)| \leq 2$.

By Lemma 4.3.1, we have for any $e^{i\theta} \in \partial\mathbb{D}$,

$$\frac{\left|(\mathcal{C}^{(n)} - e^{i\theta})^{-1}_{s_2 s_3}\right|}{\left|(\mathcal{C}^{(n)} - e^{i\theta})^{-1}_{k+m+1,k+m+1}\right|} \leq \left(\frac{2}{\sqrt{1-r^2}}\right)^7 \tag{4.3.39}$$

Observe that $(\mathcal{C}_1^{(n)} - z)^{-1}_{ks_1}$ and $(\mathcal{C}_2^{(n)} - z)^{-1}_{s_4 l}$ do not depend on $\alpha_{k+m+1}$, and therefore using Lemma 4.3.3, we get

$$\mathbb{E}\left(\left|(\mathcal{C}_1^{(n)}-z)^{-1}_{ks_1}\,(\mathcal{C}^{(n)}-z)^{-1}_{s_2s_3}\,(\mathcal{C}_2^{(n)}-z)^{-1}_{s_4l}\right|^s\;\Big|\;\{\alpha_i\}_{i\neq(k+m+1)}\right)$$
$$\leq K(r,s)\left(\frac{2}{\sqrt{1-r^2}}\right)^7\left|(\mathcal{C}_1^{(n)}-z)^{-1}_{ks_1}\right|^s\left|(\mathcal{C}_2^{(n)}-z)^{-1}_{s_4l}\right|^s \tag{4.3.40}$$

Since the random variables $(\mathcal{C}_1^{(n)}-z)^{-1}_{ks_1}$ and $(\mathcal{C}_2^{(n)}-z)^{-1}_{s_4l}$ are independent (they depend on different sets of Verblunsky coefficients), we get

$$\mathbb{E}\left(\left|(\mathcal{C}_1^{(n)}-z)^{-1}_{ks_1}\,(\mathcal{C}^{(n)}-z)^{-1}_{s_2s_3}\,(\mathcal{C}_2^{(n)}-z)^{-1}_{s_4l}\right|^s\right)$$
$$\leq C(r,s)\;\mathbb{E}\left(\left|(\mathcal{C}_1^{(n)}-z)^{-1}_{ks_1}\right|^s\right)\mathbb{E}\left(\left|(\mathcal{C}_1^{(n)}-z)^{-1}_{s_4l}\right|^s\right) \tag{4.3.41}$$

where $C(r,s)=K(r,s)\left(\frac{2}{\sqrt{1-r^2}}\right)^7$ and the constant $K(r,s)$ is defined by (4.3.31).

The idea for obtaining exponential decay is to use the terms $\mathbb{E}(|(\mathcal{C}_1^{(n)}-z)^{-1}_{ks_1}|^s)$ to get smallness and the terms $\mathbb{E}(|(\mathcal{C}_1^{(n)}-z)^{-1}_{s_4l}|^s)$ to repeat the process. Thus, using Lemma 4.2.5, we get that for any $\beta<1$, there exists a fixed constant $m\geq 0$ such that, for any $s_1$, $|s_1-(k+m)|\leq 2$, we have

$$4^s\cdot 64\cdot C(r,s)\cdot\mathbb{E}\left(\left|(\mathcal{C}_1^{(n)}-z)^{-1}_{ks_1}\right|^s\right)<\beta \tag{4.3.42}$$

We can now repeat the same procedure for each term $\mathbb{E}(|(\mathcal{C}_1^{(n)}-z)^{-1}_{s_4l}|^s)$ and we gain one more coefficient $\beta$. At each step, we move $(m+3)$ spots to the right from $k$ to $l$. We can repeat this procedure $\left[\frac{l-k}{m+3}\right]$ times and we get

$$\mathbb{E}\left(\left|(\mathcal{C}^{(n)}-z)^{-1}_{kl}\right|^s\right)\leq C\beta^{(l-k)/(m+3)} \tag{4.3.43}$$

which immediately gives (1.1.6). $\square$

# Chapter 5

# Spectral Properties of Random CMV Matrices

## 5.1 The Localized Structure of the Eigenfunctions

In this section we will study the eigenfunctions of the random CMV matrices considered in (1.1.1). We will prove that, with probability 1, each eigenfunction of these matrices will be exponentially localized about a certain point, called the center of localization. We will follow ideas from del Rio et al. [dRJLS] and Aizenman [Aiz2].

Theorem 1.1.1 will give that, for any $z \in \partial\mathbb{D}$, any integer $n$, and any $s \in (0,1)$,

$$\mathbb{E}\left(\left|F_{kl}(z, \mathcal{C}_\alpha^{(n)})\right|^s\right) \leq C\, e^{-D|k-l|} \tag{5.1.1}$$

Aizenman's theorem for CMV matrices (see Theorem 2.4.4) shows that (5.1.1) implies that for some positive constants $C_0$ and $D_0$ depending on $s$, we have

$$\mathbb{E}\left(\sup_{j\in\mathbb{Z}}\left|\left(\delta_k, (\mathcal{C}_\alpha^{(n)})^j \delta_l\right)\right|\right) \leq C_0\, e^{-D_0|k-l|} \tag{5.1.2}$$

This will allow us to conclude that the eigenfunctions of the CMV matrix are exponentially localized. The first step will be

**Lemma 5.1.1.** *For almost every $\alpha \in \Omega$, there exists a constant $D_\alpha > 0$ such that for any*

*$n$, any $k, l$, with $1 \leq k, l \leq n$, we have*

$$\sup_{j\in\mathbb{Z}} |(\delta_k, (\mathcal{C}_\alpha^{(n)})^j \delta_l)| \leq D_\alpha \, (1+n)^6 \, e^{-D_0|k-l|} \tag{5.1.3}$$

*Proof.* From (5.1.2) we get that

$$\int_\Omega \left( \sup_{j\in\mathbb{Z}} \left| \left( \delta_k, (\mathcal{C}_\alpha^{(n)})^j \delta_l \right) \right| \right) d\mathbb{P}(\alpha) \;\leq\; C_0 \, e^{-D_0|k-l|} \tag{5.1.4}$$

and therefore there exists a constant $C_1 > 0$ such that

$$\sum_{n,k,l=1}^{\infty} \sum_{k,l\leq n} \int_\Omega \frac{1}{(1+n)^2(1+k)^2(1+l)^2} \left( \sup_{j\in\mathbb{Z}} \left| \left( \delta_k, (\mathcal{C}_\alpha^{(n)})^j \delta_l \right) \right| \right) e^{D_0|k-l|} \; d\mathbb{P}(\alpha) \;\leq\; C_1 \tag{5.1.5}$$

It is clear that for any $k, l$, with $1 \leq k, l \leq n$, the function

$$\alpha \longrightarrow \frac{1}{(1+n)^2(1+k)^2(1+l)^2} \left( \sup_{j\in\mathbb{Z}} \left| \left( \delta_k, (\mathcal{C}_\alpha^{(n)})^j \delta_l \right) \right| \right) e^{D_0|k-l|} \tag{5.1.6}$$

is integrable.

Hence, for almost every $\alpha \in \Omega$, there exists a constant $D_\alpha > 0$ such that for any $n, k, l$, with $1 \leq k, l \leq n$,

$$\sup_{j\in\mathbb{Z}} \left| \left( \delta_k, (\mathcal{C}_\alpha^{(n)})^j \delta_l \right) \right| \;\leq\; D_\alpha \, (1+n)^6 \, e^{-D_0|k-l|} \tag{5.1.7}$$

□

A useful version of the previous lemma is

**Lemma 5.1.2.** *For almost every $\alpha \in \Omega$, there exists a constant $C_\alpha > 0$ such that for any $n$, any $k, l$, with $1 \leq k, l \leq n$, and $|k-l| \geq \frac{12}{D_0} \ln(n+1)$, we have*

$$\sup_{j\in\mathbb{Z}} |(\delta_k, (\mathcal{C}_\alpha^{(n)})^j \delta_l)| \leq C_\alpha \, e^{-\frac{D_0}{2}|k-l|} \tag{5.1.8}$$

*Proof.* It is clear that for any $n, k, l$, with $1 \leq k, l \leq n$ and $|k-l| \geq \frac{12}{D_0} \ln(n+1)$,

$$\frac{1}{(1+n^2)(1+k^2)(1+l^2)} \, e^{\frac{D_0}{2}|k-l|} \geq 1 \tag{5.1.9}$$

In particular, for any $n, k, l$ with $|k-l| \geq \frac{12}{D_0}\ln(n+1)$, the function

$$\Omega \ni \alpha \longrightarrow \left(\sup_{j\in\mathbb{Z}} \left|\left(\delta_k, (\mathcal{C}_\alpha^{(n)})^j \delta_l\right)\right|\right) e^{\frac{D_0}{2}|k-l|} \tag{5.1.10}$$

is integrable, so it is finite for almost every $\alpha$.

Hence, for almost every $\alpha \in \Omega$, there exists a constant $C_\alpha > 0$ such that for any $k, l$, $|k-l| \geq \frac{12}{D_0}\ln(n+1)$,

$$\sup_{j\in\mathbb{Z}} \left|\left(\delta_k, (\mathcal{C}_\alpha^{(n)})^j \delta_l\right)\right| \leq C_\alpha \, e^{-\frac{D_0}{2}|k-l|} \tag{5.1.11}$$

$\square$

*Proof of Theorem 1.1.2.* Let us start with a CMV matrix $\mathcal{C}^{(n)} = \mathcal{C}_\alpha^{(n)}$ corresponding to the Verblunsky coefficients $\alpha_0, \alpha_1, \ldots, \alpha_{n-2}, \alpha_{n-1}$. As mentioned before, the spectrum of $\mathcal{C}^{(n)}$ is simple. Let $e^{i\theta_\alpha}$ be an eigenvalue of the matrix $\mathcal{C}_\alpha^{(n)}$ and $\varphi_\alpha^{(n)}$ a corresponding eigenfunction.

We see that, on the unit circle, the sequence of functions

$$f_M(e^{i\theta}) = \frac{1}{2M+1} \sum_{j=-M}^{M} e^{ij(\theta-\theta_\alpha)} \tag{5.1.12}$$

is uniformly bounded (by 1) and converge pointwise (as $M \to \infty$) to the characteristic function of the point $e^{i\theta_\alpha}$. Let $P_{\{e^{i\theta_\alpha}\}} = \chi_{\{e^{i\theta_\alpha}\}}(\mathcal{C}_\alpha^{(n)})$.

By Lemma 5.1.2, we have, for any $k, l$, with $|k-l| \geq \frac{12}{D_0}\ln(n+1)$,

$$\left|\left(\delta_k, f_M(\mathcal{C}_\alpha^{(n)})\, \delta_l\right)\right| = \frac{1}{2M+1} \left| \sum_{j=-M}^{M} \left(\delta_k, e^{-ij\,\theta_\alpha} (\mathcal{C}_\alpha^{(n)})^j\, \delta_l\right)\right| \tag{5.1.13}$$

$$\leq \frac{1}{2M+1} \sum_{j=-M}^{M} \left|\left(\delta_k, (\mathcal{C}_\alpha^{(n)})^j\, \delta_l\right)\right| \leq C_\alpha \, e^{-\frac{D_0}{2}|k-l|} \tag{5.1.14}$$

where for the last inequality we used (5.1.1).

By taking $M \to \infty$ in the previous inequality, we get

$$\left|\left(\delta_k, P_{\{e^{i\theta_\alpha}\}}\, \delta_l\right)\right| \leq C_\alpha \, e^{-\frac{D_0}{2}|k-l|} \tag{5.1.15}$$

and therefore

$$\left|\varphi_\alpha^{(n)}(k)\,\varphi_\alpha^{(n)}(l)\right| \leq \; C_\alpha\, e^{-\frac{D_0}{2}|k-l|} \tag{5.1.16}$$

We can now pick as the center of localization the smallest integer $m(\varphi_\alpha^{(n)})$ such that

$$|\varphi_\alpha^{(n)}(m(\varphi_\alpha^{(n)}))| = \max_m |\varphi_\alpha^{(n)}(m)| \tag{5.1.17}$$

We clearly have $|\varphi_\alpha^{(n)}(m(\varphi_\alpha^{(n)}))| \geq \frac{1}{\sqrt{n+1}}$.

Using the inequality (5.1.16) with $k = m$ and $l = m(\varphi_\alpha^{(n)})$ we get, for any $m$ with $|m - m(\varphi_\alpha^{(n)})| \geq \frac{12}{D_0}\ln(n+1)$,

$$\left|\varphi_\alpha^{(n)}(m)\right| \leq \; C_\alpha\, e^{-\frac{D_0}{2}|m-m(\varphi_\alpha^{(n)})|}\sqrt{n+1} \tag{5.1.18}$$

Since for large $n$, $e^{-\frac{D_0}{2}|k-l|}\sqrt{n+1} \leq e^{-\frac{D_0}{3}|k-l|}$ for any $k, l$, $|k-l| \geq \frac{12}{D_0}\ln(n+1)$, we get the desired conclusion (we can take $D_2 = \frac{12}{D_0}$). □

For any eigenfunction $\varphi_\alpha^{(n)}$, the point $m(\varphi_\alpha^{(n)})$ is called its center of localization. The eigenfunction is concentrated (has its large values) near the point $m(\varphi_\alpha^{(n)})$ and is tiny at sites that are far from $m(\varphi_\alpha^{(n)})$. This structure of the eigenfunctions will allow us to prove a decoupling property of the CMV matrix.

Note that we used Lemma 5.1.2 in the proof of Theorem 1.1.2. We can get a stronger result by using Lemma 5.1.1 (we replace (5.1.11) by (5.1.7)). Thus, for any $n$ and any $m \leq n$, we have

$$\left|\varphi_\alpha^{(n)}(m)\right| \leq \; D_\alpha\,(1+n)^6\, e^{-\frac{D_0}{2}\,|m-m(\varphi_\alpha^{(n)})|}\sqrt{n+1} \tag{5.1.19}$$

where $m(\varphi_\alpha^{(n)})$ is the center of localization of the eigenfunction $\varphi_\alpha^{(n)}$.

## 5.2 Decoupling the CMV Matrices

We will now show that the distribution of the eigenvalues of the CMV matrix $\mathcal{C}^{(n)}$ can be approximated (as $n \to \infty$) by the distribution of the eigenvalues of another matrix CMV matrix $\tilde{\mathcal{C}}^{(n)}$, which decouples into the direct sum of smaller matrices.

As explained in the introduction, for the CMV matrix $\mathcal{C}^{(n)}$ obtained with the Verblunsky coefficients $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_{n-1}) \in \Omega$, we consider $\tilde{\mathcal{C}}^{(n)}$ the CMV matrix obtained from the same Verblunsky coefficients with the additional restrictions $\alpha_{\left[\frac{n}{\ln n}\right]} = e^{i\eta_1}$, $\alpha_{2\left[\frac{n}{\ln n}\right]} = e^{i\eta_2}$, $\ldots$, $\alpha_{n-1} = e^{i\eta_{[\ln n]}}$, where $e^{i\eta_1}, e^{i\eta_2}, \ldots, e^{i\eta_{[\ln n]}}$ are independent random points uniformly distributed on the unit circle. The matrix $\tilde{\mathcal{C}}^{(n)}$ decouples into the direct sum of approximately $[\ln n]$ unitary matrices $\tilde{\mathcal{C}}_1^{(n)}$, $\tilde{\mathcal{C}}_2^{(n)}$, $\ldots$, $\tilde{\mathcal{C}}_{[\ln n]}^{(n)}$. Since we are interested in the asymptotic distribution of the eigenvalues, it will be enough to study the distribution (as $n \to \infty$) of the eigenvalues of the matrices $\mathcal{C}^{(N)}$ of size $N = [\ln n]\left[\frac{n}{\ln n}\right]$. Note that in this situation, the corresponding truncated matrix $\tilde{\mathcal{C}}^{(N)}$ will decouple into the direct sum of exactly $[\ln n]$ identical blocks of size $\left[\frac{n}{\ln n}\right]$.

We will begin by comparing the matrices $\mathcal{C}^{(N)}$ and $\tilde{\mathcal{C}}^{(N)}$.

**Lemma 5.2.1.** *For $N = [\ln n]\left[\frac{n}{\ln n}\right]$, the matrix $\mathcal{C}^{(N)} - \tilde{\mathcal{C}}^{(N)}$ has at most $4[\ln n]$ nonzero rows.*

*Proof.* In our analysis, we will start counting the rows of the CMV matrix with row 0. A simple inspection of the CMV matrix shows that for even Verblunsky coefficients $\alpha_{2k}$, only the rows $2k$ and $2k+1$ depend on $\alpha_{2k}$. For odd Verblunsky coefficients $\alpha_{2k+1}$, only the rows $2k, 2k+1, 2k+2, 2k+3$ depend on $\alpha_{2k+1}$.

Since in order to obtain the matrix $\tilde{\mathcal{C}}^{(N)}$ from $\mathcal{C}^{(N)}$ we modify $[\ln n]$ Verblunsky coefficients $\alpha_{\left[\frac{n}{\ln n}\right]}, \alpha_{2\left[\frac{n}{\ln n}\right]}, \ldots, \alpha_{[\ln n]\left[\frac{n}{\ln n}\right]}$, and we immediately see that at most $4[\ln n]$ rows of $\mathcal{C}^{(N)}$ are modified.

Therefore $\mathcal{C}^{(N)} - \tilde{\mathcal{C}}^{(N)}$ has at most $4[\ln n]$ nonzero rows (and, by the same argument, at most four columns around each point where the matrix $\tilde{\mathcal{C}}^{(N)}$ decouples). $\square$

Since we are interested in the points situated near the places where the matrix $\tilde{\mathcal{C}}^{(N)}$ decouples, a useful notation will be

$$S_N(K) = S^{(1)}(K) \cup S^{(2)}(K) \cup \cdots \cup S^{([\ln n])}(K) \tag{5.2.1}$$

where $S^{(k)}(K)$ is a set of $K$ integers centered at $k\left[\frac{n}{\ln n}\right]$ (e.g., for $K = 2p$, $S^{(k)}(K) = \left\{\, k\left[\frac{n}{\ln n}\right] - p + 1,\ k\left[\frac{n}{\ln n}\right] - p + 2,\ \ldots,\ k\left[\frac{n}{\ln n}\right] + p\right\}$). Using this notation, we also have

$$S_N(1) = \left\{ \left[\frac{n}{\ln n}\right], 2\left[\frac{n}{\ln n}\right], \dots, [\ln n]\left[\frac{n}{\ln n}\right] \right\} \tag{5.2.2}$$

Consider the intervals $I_{N,k}$, $1 \le k \le m$, of size $\frac{1}{N}$ near the point $e^{i\alpha}$ on the unit circle (e.g., $I_{N,k} = (e^{i(\alpha+\frac{a_k}{N})}, e^{i(\alpha+\frac{b_k}{N})})$), where $a_1 < b_1 \le a_2 < b_2 \le \cdots \le a_m < b_m$. We will denote by $\mathcal{N}_N(I)$ the number of eigenvalues of $\mathcal{C}^{(N)}$ situated in the interval $I$, and by $\tilde{\mathcal{N}}_N(I)$ the number of eigenvalues of $\tilde{\mathcal{C}}^{(N)}$ situated in $I$. We will prove that, for large $N$, $\mathcal{N}_N(I_{N,k})$ can be approximated by $\tilde{\mathcal{N}}_N(I_{N,k})$, that is, for any integers $k_1, k_2, \dots, k_m \ge 0$, we have, for $N \to \infty$,

$$\begin{aligned} &\big|\ \mathbb{P}(\mathcal{N}_N(I_{N,1}) = k_1, \mathcal{N}_N(I_{N,2}) = k_2, \dots, \mathcal{N}_N(I_{N,m}) = k_m) \\ &- \mathbb{P}(\tilde{\mathcal{N}}_N(I_{N,1}) = k_1, \tilde{\mathcal{N}}_N(I_{N,2}) = k_2, \dots, \tilde{\mathcal{N}}_N(I_{N,m}) = k_m)\big| \longrightarrow 0 \end{aligned} \tag{5.2.3}$$

Since, by the results in Section 5.1, the eigenfunctions of the matrix $\mathcal{C}^{(N)}$ are exponentially localized (supported on a set of size $2T[\ln(n+1)]$, where, from now on, $T = \frac{14}{D_0}$), some of them will have the center of localization near $S_N(1)$ (the set of points where the matrix $\tilde{\mathcal{C}}^{(N)}$ decouples) and others will have centers of localization away from this set (i.e., because of exponential localization, inside an interval $\left(k\left[\frac{n}{\ln n}\right], (k+1)\left[\frac{n}{\ln n}\right]\right)$).

Roughly speaking, each eigenfunction of the second type will produce an "almost" eigenfunction for one of the blocks of the decoupled matrix $\tilde{\mathcal{C}}^{(N)}$. These eigenfunctions will allow us to compare $\mathcal{N}_N(I_{N,k})$ and $\tilde{\mathcal{N}}_N(I_{N,k})$.

We see that any eigenfunction with the center of localization outside the set $S_N(4T[\ln n])$ will be tiny on the set $S_N(1)$. Therefore, if we want to estimate the number of eigenfunctions that are supported close to $S_N(1)$, it will be enough to analyze the number $b_{N,\alpha}$, where $b_{N,\alpha}$ equals the number of eigenfunctions of $\mathcal{C}_\alpha^{(N)}$ with the center of localization inside $S_N(4T[\ln n])$ (we will call these eigenfunctions "bad eigenfunctions"). We will now prove that the number $b_{N,\alpha}$ is small compared to $N$.

A technical complication is generated by the fact that in the exponential localization of eigenfunctions given by (5.1.3), the constant $D_\alpha$ depends on $\alpha \in \Omega$. We define

$$\mathcal{M}_K = \left\{ \alpha \in \Omega,\ \sup_{j \in \mathbb{Z}} |(\delta_k, (\mathcal{C}^{(N)})^j \delta_l)| \leq K\,(1+N)^6\, e^{-D_0|k-l|} \right\} \tag{5.2.4}$$

Note that for any $K > 0$, the set $\mathcal{M}_K \subset \Omega$ is invariant under rotation. Also, we can immediately see that the sets $\mathcal{M}_K$ grow with $K$ and

$$\lim_{K \to \infty} \mathbb{P}(\mathcal{M}_K) = 1 \tag{5.2.5}$$

We will be able to control the number of "bad eigenfunctions" for $\alpha \in \mathcal{M}_K$ using the following lemma:

**Lemma 5.2.2.** *For any $K > 0$ and any $\alpha \in \mathcal{M}_K$, there exists a constant $C_K > 0$ such that*

$$b_{N,\alpha} \leq C_K \left(\ln(1+N)\right)^2 \tag{5.2.6}$$

*Proof.* For any $K > 0$, any $\alpha \in \mathcal{M}_K$, and any eigenfunction $\varphi_\alpha^N$ that is exponentially localized about a point $m(\varphi_\alpha^N)$, we have, using (5.1.19),

$$\left|\varphi_\alpha^{(N)}(m)\right| \leq\ K\, e^{-\frac{D_0}{2}\,|m - m(\varphi_\alpha^{(N)})|}\,(1+N)^6\,\sqrt{1+N} \tag{5.2.7}$$

Therefore, for any $m$ such that $|m - m(\varphi_\alpha^N)| \geq \left[\frac{14}{D_0}\ln(1+N)\right]$, we have

$$\begin{aligned} \sum_{|m - m(\varphi_\alpha^N)| \geq \left[\frac{14}{D_0}\ln(1+N)\right]} \left|\varphi_\alpha^{(N)}(m)\right|^2 &\leq 2(1+N)^{-14}\,(1+N)^{13}\,\sum_{k=0}^{\infty} K^2\, e^{-D_0 k} \\ &\leq (1+N)^{-1} K^2 \frac{2e^{D_0}}{e^{D_0}-1} \end{aligned} \tag{5.2.8}$$

Therefore, for any fixed $K$ and $s$, we can find an $N_0 = N_0(k,s)$ such that for any $N \geq N_0$,

$$\sum_{|m - m(\varphi_\alpha^N)| \leq \left[\frac{14}{D_0}\ln(1+N)\right]} \left|\varphi_\alpha^{(N)}(m)\right|^2 \geq \tfrac{1}{2} \tag{5.2.9}$$

We will consider eigenfunctions $\varphi_\alpha^N$ with the center of localization in $S_N\left(4T[\ln N]\right)$. For a fixed $\alpha \in \mathcal{M}_K$, we denote the number of these eigenfunctions by $b_{N,\alpha}$. We denote by $\{\psi_1, \psi_2, \dots, \psi_{b_{N,\alpha}}\}$ the set of these eigenfunctions. Since the spectrum of $\mathcal{C}^{(N)}$ is simple, this is an orthonormal set.

Therefore, if we denote by card$(A)$ the number of elements of the set $A$, we get

$$\begin{aligned}\sum_{m\in S\left(4T[\ln N]+\left[\frac{14}{D_0}\ln(1+N)\right]\right)} \sum_{i=1}^{b_{N,\alpha}} |\psi_i(m)|^2 \\ &\leq \text{card}\left\{S\left(4T[\ln N]+\left[\frac{14}{D_0}\ln(1+N)\right]\right)\right\} \\ &\leq \left(4T+\frac{14}{D_0}\right)(\ln(1+N))^2\end{aligned}$$

Also, from (5.2.9), for any $N \geq N_0(K,s)$,

$$\sum_{m\in S\left(4T[\ln N]+\left[\frac{14}{D_0}\ln(1+N)\right]\right)} \sum_{i=1}^{b_{N,\alpha}} |\psi_i(m)|^2 \geq \tfrac{1}{2}\, b_{N,\alpha} \tag{5.2.10}$$

Therefore, for any $K > 0$ and any $\alpha \in \mathcal{M}_K$, we have, for $N \geq N_0(K,s)$,

$$b_{N,\alpha} \leq 2\left(4T+\frac{14}{D_0}\right)(\ln(1+N))^2 \tag{5.2.11}$$

and we can now conclude that (5.2.6) holds. $\square$

Lemma 5.2.2 shows that for any $K \geq 0$, the number of "bad eigenfunctions" corresponding to $\alpha \in \mathcal{M}_K$ is of the order $(\ln N)^2$ (hence small compared to $N$).

Since the distributions for our Verblunsky coefficients are taken to be rotationally invariant, the distribution of the eigenvalues is rotationally invariant. Therefore, for any interval $I_N$ of size $\frac{1}{N}$ on the unit circle, and for any fixed set $\mathcal{M}_K \subset \Omega$, the expected number of "bad eigenfunctions" corresponding to eigenvalues in $I_N$ is of size $\frac{(\ln N)^2}{N}$. We then get that the probability of the event "there are bad eigenfunctions corresponding to eigenvalues in the interval $I_N$" converges to 0. This fact will allow us to prove

**Lemma 5.2.3.** *For any $K > 0$, any disjoint intervals $I_{N,1}, I_{N,2}, \dots, I_{N,m}$ (each one of size*

$\frac{1}{N}$ *and situated near the point* $e^{i\alpha}$) *and any positive integers* $k_1, k_2, \ldots, k_m$, *we have*

$$\begin{aligned}&\Big|\ \mathbb{P}(\{\mathcal{N}_N(I_{N,1}) = k_1,\, \mathcal{N}_N(I_{N,2}) = k_2, \ldots, \mathcal{N}_N(I_{N,m}) = k_m\} \cap \mathcal{M}_K) \\ &- \mathbb{P}(\{\tilde{\mathcal{N}}_N(I_{N,1}) = k_1,\, \tilde{\mathcal{N}}_N(I_{N,2}) = k_2,\, \ldots,\, \tilde{\mathcal{N}}_N(I_{N,m}) = k_m\} \cap \mathcal{M}_K)\Big| \longrightarrow 0 \qquad (5.2.12)\end{aligned}$$

*as* $N \to \infty$.

*Proof.* We will work with $\alpha \in \mathcal{M}_K$. We first observe that any "good eigenfunction" (i.e., an eigenfunction with the center of localization outside $S_N\left(4T[\ln N]\right)$) is tiny on $S_N(1)$.

Indeed, from (5.1.19), for any eigenfunction $\varphi_\alpha^{(N)}$ with the center of localization $m(\varphi_\alpha^{(N)})$ and for any $m$ with $|m - m(\varphi_\alpha^{(N)})| \geq \frac{18}{D_0}[\ln(N+1)]$,

$$|\varphi_\alpha^{(N)}(m)| \leq K e^{-\frac{D_0}{2}|m - m(\varphi_\alpha^{(N)})|}(1+N)^6\sqrt{1+N} \qquad (5.2.13)$$

In particular, if the center of localization of $\varphi_\alpha^{(N)}$ is outside $S_N\left(4T[\ln N]\right)$, then for all $m \in S_N(1)$, we have

$$|\varphi_\alpha^{(N)}(m)| \leq K(1+N)^{-2} \qquad (5.2.14)$$

We will use the fact that if $N$ is a normal matrix, $z_0 \in \mathbb{C}$, $\varepsilon > 0$, and $\varphi$ is a unit vector with

$$\|\,(N - z_0)\varphi\,\| < \varepsilon \qquad (5.2.15)$$

then $N$ has an eigenvalue in $\{z \mid |z - z_0| < \varepsilon\}$.

For any "good eigenfunction" $\varphi_\alpha^{(N)}$, we have $\mathcal{C}_\alpha^{(N)}\varphi_\alpha^{(N)} = 0$ and therefore, using Lemma 5.2.1,

$$\|\tilde{\mathcal{C}}_\alpha^{(N)}\varphi_\alpha^{(N)}\| \leq 2K[\ln N](1+N)^{-2} \qquad (5.2.16)$$

Therefore, for any interval $I_N$ of size $\frac{1}{N}$, we have

$$\mathcal{N}_N(I_N) \leq \tilde{\mathcal{N}}_N(\tilde{I}_N) \qquad (5.2.17)$$

where $\tilde{I}_N$ is the interval $I_N$ augmented by $2K[\ln N](1+N)^{-2}$.

Since $2K[\ln N](1+N)^{-2} = o(\frac{1}{N})$, we can now conclude that

$$\mathbb{P}\big((\mathcal{N}_N(I_N) \leq \tilde{\mathcal{N}}_N(I_N)) \cap \mathcal{M}_K\big) \to 1 \quad \text{as} \quad n \to \infty \tag{5.2.18}$$

We can use the same argument (starting from the eigenfunctions of $\tilde{\mathcal{C}}_\alpha^{(N)}$, which are also exponentially localized) to show that

$$\mathbb{P}\big((\mathcal{N}_N(I_N) \geq \tilde{\mathcal{N}}_N(I_N)) \cap \mathcal{M}_K\big) \to 1 \quad \text{as} \quad n \to \infty \tag{5.2.19}$$

so we can now conclude that

$$\mathbb{P}\big((\mathcal{N}_N(I_N) = \tilde{\mathcal{N}}_N(I_N)) \cap \mathcal{M}_K\big) \to 1 \quad \text{as} \quad n \to \infty \tag{5.2.20}$$

Instead of one interval $I_N$, we can take $m$ intervals $I_{N,1}, I_{N,2}, \ldots, I_{N,m}$ so we get (5.2.12). $\square$

*Proof of Theorem 1.1.3.* Lemma 5.2.3 shows that for any $K > 0$, the distribution of the eigenvalues of the matrix $\mathcal{C}^{(N)}$ can be approximated by the distribution of the eigenvalues of the matrix $\tilde{\mathcal{C}}^{(N)}$ when we restrict to the set $\mathcal{M}_K \subset \Omega$. Since by (5.2.5) the sets $\mathcal{M}_K$ grow with $K$ and $\lim_{K\to\infty} \mathbb{P}(\mathcal{M}_K) = 1$, we get the desired result. $\square$

# Chapter 6

# The Local Statistical Distribution of the Zeros of Random Paraorthogonal Polynomials

## 6.1 Estimating the Probability of Having Two or More Eigenvalues in an Interval

The results from the previous chapter show that the local distribution of the eigenvalues of the matrix $\mathcal{C}^{(N)}$ can be approximated by the direct sum of the local distribution of $[\ln n]$ matrices of size $\left[\frac{n}{\ln n}\right]$, $\mathcal{C}_1^{(N)}, \mathcal{C}_2^{(N)}, \dots, \mathcal{C}_{[\ln n]}^{(N)}$. These matrices are decoupled and depend on independent sets of Verblunsky coefficients; hence, they are independent.

For a fixed point $e^{i\theta_0} \in \partial\mathbb{D}$, and an interval $I_N = (e^{i(\theta_0 + \frac{2\pi a}{N})}, e^{i(\theta_0 + \frac{2\pi b}{N})})$, we will now want to control the probability of the event "$\mathcal{C}^{(N)}$ has $k$ eigenvalues in $I_N$." We will analyze the distribution of the eigenvalues of the direct sum of the matrices $\mathcal{C}_1^{(N)}, \mathcal{C}_2^{(N)}, \dots, \mathcal{C}_{[\ln n]}^{(N)}$. We will prove that, as $n \to \infty$, each of the decoupled matrices $\mathcal{C}_k^{(N)}$ contributes (up to a negligible error) at most one eigenvalue in the interval $I_N$.

For any nonnegative integer $m$, denote by $A(m, \mathcal{C}, I)$ the event

$$A(m, \mathcal{C}, I) = \text{"}\mathcal{C} \text{ has at least } m \text{ eigenvalues in the interval } I\text{"} \tag{6.1.1}$$

and by $B(m, \mathcal{C}, I)$ the event

$$B(m, \mathcal{C}, I) = \text{"}\mathcal{C} \text{ has exactly } m \text{ eigenvalues in the interval } I\text{"} \tag{6.1.2}$$

In order to simplify future notation, for any point $e^{i\theta} \in \partial\mathbb{D}$, we also define the event $M(e^{i\theta})$ to be

$$M(e^{i\theta}) = \text{“}e^{i\theta} \text{ is an eigenvalue of } \mathcal{C}^{(N)}\text{”} \tag{6.1.3}$$

We can begin by observing that the eigenvalues of the matrix $\mathcal{C}^{(N)}$ are the zeros of the $N$-th paraorthogonal polynomial,

$$\Phi_N(z, d\mu, \beta) = z\Phi_{N-1}(z, d\mu) - \overline{\beta}\,\Phi_{N-1}^*(z, d\mu) \tag{6.1.4}$$

Therefore, we can consider the complex function

$$B_N(z) = \frac{\beta\, z\, \Phi_{N-1}(z)}{\Phi_{N-1}^*(z)} \tag{6.1.5}$$

which has the property that $\Phi_N(e^{i\theta}) = 0$ if and only if $B_N(e^{i\theta}) = 1$.

By writing the polynomials $\Phi_{N-1}$ and $\Phi_{N-1}^*$ as products of their zeros, we can see that the function $B_N$ is a Blaschke product.

Let $\eta_N : [0, 2\pi) \to \mathbb{R}$ be a continuous function such that

$$B_N(e^{i\theta}) = e^{i\,\eta_N(\theta)} \tag{6.1.6}$$

(we will only be interested in the values of the function $\eta_N$ near a fixed point $e^{i\theta_0} \in \partial\mathbb{D}$). We should remark here that the relation (6.1.6) defines the function $\eta_N$ up to a $2\pi$ factor. We will pick $\eta_N(\theta_0) \in [0, 2\pi)$ and thus the function $\eta_N$ will be uniquely defined. Note that for any fixed $\theta \in \partial\mathbb{D}$, we have that $\eta(\theta)$ is a random variable depending on $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_{N-2}, \alpha_{N-1} = \beta) \in \Omega$.

We will now study the properties of the random variable $\eta_N(\theta) = \eta_N(\theta, \alpha_0, \alpha_1, \ldots, \alpha_{N-2}, \beta)$. Thus

**Lemma 6.1.1.** *For any $\theta_1$ and $\theta_2$, the random variables $\frac{\partial \eta_N}{\partial \theta}(\theta_1)$ and $\eta_N(\theta_2)$ are independent. Also, for any fixed value $w \in \mathbb{R}$,*

$$\mathbb{E}\left( \frac{\partial \eta_N}{\partial \theta}(\theta_1) \,\middle|\, \eta_N(\theta_2) = w \right) = N \tag{6.1.7}$$

*Proof.* The equation (6.1.5) gives

$$\eta_N(\theta) = \gamma + \tau(\theta) \tag{6.1.8}$$

where $e^{i\gamma} = \beta$ and $e^{i\tau(\theta)} = \frac{e^{i\theta}\,\Phi_{N-1}(e^{i\theta})}{\Phi^*_{N-1}(e^{i\theta})}$. Since the distribution of each of the random variables $\alpha_0, \alpha_1, \ldots, \alpha_{N-2}$ and $\beta$ is rotationally invariant, for any $\theta \in [0, 2\pi)$, $\gamma$ and $\tau(\theta)$ are random variables uniformly distributed. We can immediately see that $\gamma$ and $\tau(\theta)$ are independent. Since $\gamma$ does not depend on $\theta$, for any fixed $\theta_1, \theta_2 \in [0, 2\pi)$, we have that the random variables $\frac{\partial \eta_N}{\partial \theta}(\theta_1)$ and $\eta_N(\theta_2)$ are independent.

We see now that for any Blaschke factor $B_a(z) = \frac{z-a}{1-\bar{a}z}$, we can define a real-valued function $\eta_a$ on $\partial\mathbb{D}$ such that

$$e^{i\eta_a(\theta)} = B_a(e^{i\theta}) \tag{6.1.9}$$

A straightforward computation gives

$$\frac{\partial \eta_a}{\partial \theta}(\theta) = \frac{1 - |a|^2}{|e^{i\theta} - a|^2} > 0 \tag{6.1.10}$$

Since $B_N$ is a Blaschke product, we now get that for any fixed $\alpha \in \Omega$, $\frac{\partial \eta_N}{\partial \theta}$ has a constant sign (positive). This implies that the function $\eta_N$ is strictly increasing. The function $B_N(z)$ is analytic and has exactly $N$ zeros in $\mathbb{D}$ and therefore, using the argument principle, we get that

$$\int_0^{2\pi} \frac{\partial \eta_N}{\partial \theta}(\theta)\, d\theta = 2\pi N \tag{6.1.11}$$

Note that $\frac{\partial \eta_N}{\partial \theta}$ does not depend on $\beta$ (it depends only on $\alpha_0, \alpha_1, \ldots, \alpha_{N-2}$). Also, using the same argument as in Lemma 4.1.1, we have that for any angles $\theta$ and $\varphi$,

$$\frac{\partial \eta_N}{\partial \theta}(\theta) = \frac{\partial \tilde{\eta}_N}{\partial \theta}(\theta - \varphi) \tag{6.1.12}$$

where $\tilde{\eta}$ is the function $\eta$ that corresponds to the Verblunsky coefficients

$$\alpha_{k,\varphi} = e^{-i(k+1)\varphi}\alpha_k \qquad k = 0, 1, \ldots, (N-2) \tag{6.1.13}$$

Since the distribution of $\alpha_0, \alpha_1, \ldots, \alpha_{N-2}$ is rotationally invariant, we get from (6.1.12) that the function $\theta \to \mathbb{E}\left(\frac{\partial \eta_N}{\partial \theta}(\theta)\right)$ is constant.

Taking expectations and using Fubini's theorem (as we also did in Lemma 4.1.1), we get, for any angle $\theta_0$,

$$2\pi N = \mathbb{E}\left(\int_0^{2\pi} \frac{\partial \eta_N}{\partial \theta}(\theta)\, d\theta\right) = \int_0^{2\pi} \mathbb{E}\left(\frac{\partial \eta_N}{\partial \theta}(\theta)\right) d\theta = 2\pi\, \mathbb{E}\left(\frac{\partial \eta_N}{\partial \theta}(\theta_0)\right) \tag{6.1.14}$$

and therefore

$$\mathbb{E}\left(\frac{\partial \eta_N}{\partial \theta}(\theta_0)\right) = N \tag{6.1.15}$$

Since for any $\theta_1, \theta_2 \in [0, 2\pi)$, we have that $\frac{\partial \eta_N}{\partial \theta}(\theta_1)$ and $\eta_N(\theta_2)$ are independent, (6.1.15) implies that for any fixed value $w \in \mathbb{R}$,

$$\mathbb{E}\left(\frac{\partial \eta_N}{\partial \theta}(\theta_1) \,\middle|\, \eta_N(\theta_2) = w\right) = N \tag{6.1.16}$$

□

We will now control the probability of having at least two eigenvalues in $I_N$ conditioned by the event that we already have an eigenvalue at one fixed point $e^{i\theta_1} \in I_N$. This will be shown in the following lemma:

**Lemma 6.1.2.** *With $\mathcal{C}^{(N)}, I_N$, and the events $A(m, \mathcal{C}, I)$ and $M(e^{i\theta})$ defined before, and for any $e^{i\theta_1} \in I_N$, we have*

$$\mathbb{P}\left(A(2, \mathcal{C}^{(N)}, I_N) \,\middle|\, M(e^{i\theta_1})\right) \leq (b-a) \tag{6.1.17}$$

*Proof.* Using the relation (6.1.16) and the fact that the function $\theta \to \mathbb{E}\left(\frac{\partial \eta_N}{\partial \theta}(\theta)\right)$ is constant, we get that

$$\mathbb{E}\left(\int_{\theta_0 + \frac{2\pi a}{N}}^{\theta_0 + \frac{2\pi b}{N}} \frac{\partial \eta_N}{\partial \theta}(\theta_1)\; d\theta_1 \,\middle|\, \eta_N(\theta_2) = w\right) = 2\pi\,(b-a) \tag{6.1.18}$$

We see that

$$\Phi_N(e^{i\theta}) = 0 \iff B_N(e^{i\theta}) = 1 \iff \eta_N(\theta) = 0 \pmod{2\pi} \tag{6.1.19}$$

Therefore, if the event $A(2, \mathcal{C}^{(N)}, I_N)$ takes place (i.e., if the polynomial $\Phi_N$ vanishes at least twice in the interval $I_N$), then the function $\eta_N$ changes by at least $2\pi$ in the interval $I_N$, and therefore we have that

$$\int_{\theta_0+\frac{2\pi a}{N}}^{\theta_0+\frac{2\pi b}{N}} \frac{\partial \eta_N}{\partial \theta}(\theta)\, d\theta \geq 2\pi \tag{6.1.20}$$

whenever the event $A(2, \mathcal{C}^{(N)}, I_N)$ takes place.

For any $\theta_1 \in I_N$ we have, using the independence of the random variables $\frac{\partial \eta_N}{\partial \theta}(\theta_1)$ and $\eta_N(\theta_2)$ for the first inequality and Chebyshev's inequality for the second inequality,

$$\begin{aligned} \mathbb{P}\left(A(2, \mathcal{C}^{(N)}, I_N) \,\middle|\, M(e^{i\theta_1})\right) &\leq \mathbb{P}\left(\int_{\theta_0+\frac{2\pi a}{N}}^{\theta_0+\frac{2\pi b}{N}} \frac{\partial \eta_N}{\partial \theta}(\theta)\, d\theta \geq 2\pi \,\middle|\, M(e^{i\theta_1})\right) \\ &\leq \frac{1}{2\pi}\, \mathbb{E}\left(\int_{\theta_0+\frac{2\pi a}{N}}^{\theta_0+\frac{2\pi b}{N}} \frac{\partial \eta_N}{\partial \theta}(\theta)\, d\theta \,\middle|\, M(e^{i\theta_1})\right) \end{aligned} \tag{6.1.21}$$

The formula (6.1.21) shows that we can control the probability of having more than two eigenvalues in the interval $I_N$ conditioned by the event that a fixed $e^{i\theta_1}$ is an eigenvalue. We now obtain, using (6.1.18) with $w = 2\pi m$, $m \in \mathbb{Z}$,

$$\mathbb{P}\Big(A(2, \mathcal{C}^{(N)}, I_N) \;\Big|\; M(e^{i\theta_1})\Big) \leq (b-a) \tag{6.1.22}$$

□

We can now control the probability of having two or more eigenvalues in $I_N$.

**Theorem 6.1.3.** *With $\mathcal{C}^{(N)}, I_N$, and the event $A(m, \mathcal{C}, I)$ defined before, we have*

$$\mathbb{P}\left(A(2, \mathcal{C}^{(N)}, I_N)\right) \leq \frac{(b-a)^2}{2} \tag{6.1.23}$$

*Proof.* For any positive integer $k$, we have

$$\mathbb{P}\left(B(k, \mathcal{C}^{(N)}, I_N)\right) = \frac{1}{k} \int_{\theta_0+\frac{2\pi a}{N}}^{\theta_0+\frac{2\pi b}{N}} \mathbb{P}\left(B(k, \mathcal{C}^{(N)}, I_N) \mid M(e^{i\theta})\right) N\, d\nu_N(\theta) \tag{6.1.24}$$

where the measure $\nu_N$ is the density of eigenvalues.

Note that the factor $\frac{1}{k}$ appears because the selected point $e^{i\theta}$ where we take the conditional probability can be any one of the $k$ points.

We will now use the fact that the distribution of the Verblunsky coefficients is rotationally invariant, and therefore for any $N$, we have $d\nu_N = \frac{d\theta}{2\pi}$, where $\frac{d\theta}{2\pi}$ is the normalized Lebesgue measure on the unit circle.

Since for any $k \geq 2$, we have $\frac{1}{k} \leq \frac{1}{2}$, we get that for any integer $k \geq 2$ and for large $N$,

$$\mathbb{P}\left(B(k, \mathcal{C}^{(N)}, I_N)\right) \leq \frac{N}{2} \int_{\theta_0 + \frac{2\pi a}{N}}^{\theta_0 + \frac{2\pi b}{N}} \mathbb{P}\left(B(k, \mathcal{C}^{(N)}, I_N) \mid M(e^{i\theta})\right) \frac{d\theta}{2\pi} \tag{6.1.25}$$

and therefore,

$$\mathbb{P}\left(A(2, \mathcal{C}^{(N)}, I_N)\right) \leq \frac{N}{2} \int_{\theta_0 + \frac{2\pi a}{N}}^{\theta_0 + \frac{2\pi b}{N}} \mathbb{P}\left(A(2, \mathcal{C}^{(N)}, I_N) \mid M(e^{i\theta})\right) \frac{d\theta}{2\pi} \tag{6.1.26}$$

Using Lemma 6.1.2, we get

$$\mathbb{P}(A(2, \mathcal{C}^{(N)}, I_N)) \leq \frac{N}{2} \frac{(b-a)}{N} (b-a) = \frac{(b-a)^2}{2} \tag{6.1.27}$$

□

**Theorem 6.1.4.** *With $\mathcal{C}^{(N)}, \mathcal{C}_1^{(N)}, \mathcal{C}_2^{(N)}, \ldots, \mathcal{C}_{[\ln n]}^{(N)}, I_N$, and the event $A(m, \mathcal{C}, I)$ defined previously, we have, for any $k$, $1 \leq k \leq [\ln n]$,*

$$\mathbb{P}\left(A(2, \mathcal{C}_k^{(N)}, I_N)\right) = O\left(([\ln n])^{-2}\right) \qquad \text{as} \quad n \to \infty \tag{6.1.28}$$

*Proof.* We will use the previous theorems for the CMV matrix $\mathcal{C}_k^{(N)}$. Recall that $N = [\ln n]\left[\frac{n}{\ln n}\right]$. Since this matrix has $\left[\frac{n}{\ln n}\right]$ eigenvalues, we can use the proof of Lemma 6.1.2 to obtain that for any $e^{i\theta} \in I_N$,

$$\mathbb{P}\left(A(2, \mathcal{C}_k^{(N)}, I_N) \,|\, M(e^{i\theta})\right) \leq \frac{1}{2\pi} \frac{2\pi(b-a)}{N} \left[\frac{n}{\ln n}\right] = \frac{b-a}{[\ln n]} \tag{6.1.29}$$

The proof of Theorem 6.1.3 now gives

$$\mathbb{P}\left(A(2, \mathcal{C}_k^{(N)}, I_N)\right) \leq \frac{(b-a)^2}{2\,[\ln n]^2} \tag{6.1.30}$$

and hence (6.1.28) follows. $\square$

This theorem shows that as $N \to \infty$, any of the decoupled matrices contributes at most one eigenvalue in each interval of size $\frac{1}{N}$.

## 6.2 Proof of the Main Theorem

We will now use the results of the previous sections to conclude that the statistical distribution of the zeros of the random paraorthogonal polynomials is Poisson.

*Proof of Theorem 1.0.6.* It is enough to study the statistical distribution of the zeros of polynomials of degree $N = [\ln n]\left[\frac{n}{\ln n}\right]$. These zeros are exactly the eigenvalues of the CMV matrix $\mathcal{C}^{(N)}$, so, by the results in Section 5.2, the distribution of these zeros can be approximated by the distribution of the direct sum of the eigenvalues of $[\ln n]$ matrices $\mathcal{C}_1^{(N)}, \mathcal{C}_2^{(N)}, \ldots, \mathcal{C}_{[\ln n]}^{(N)}$.

In Section 6.1 (Theorem 6.1.4), we showed that the probability that any of the matrices $\mathcal{C}_1^{(N)}$, $\mathcal{C}_2^{(N)}$, …, $\mathcal{C}_{[\ln n]}^{(N)}$ contributes two or more eigenvalues in each interval of size $\frac{1}{N}$ situated near a fixed point $e^{i\theta} \in \partial\mathbb{D}$ is of order $O([\ln n]^{-2})$. Since the matrices $\mathcal{C}_1^{(N)}, \mathcal{C}_2^{(N)}, \ldots, \mathcal{C}_{[\ln n]}^{(N)}$ are identically distributed and independent, we immediately get that the probability that the direct sum of these matrices has two or more eigenvalues in an interval of size $\frac{1}{N}$ situated near $e^{i\theta}$ is $[\ln n]\, O([\ln n]^{-2})$ and therefore converges to 0 as $n \to \infty$.

We can now conclude that as $n \to \infty$, the local distribution of the eigenvalues converges to a Poisson process with intensity measure $n\, \frac{d\theta}{2\pi}$, using a standard technique in probability theory. We first fix an interval $I_N = (e^{i(\theta_0 + \frac{2\pi a}{N})}, e^{i(\theta_0 + \frac{2\pi b}{N})})$ near the point $e^{i\theta_0}$ (as before, we take $N = [\ln n]\left[\frac{n}{\ln n}\right]$). Let us consider $[\ln n]$ random variables $X_1, X_2, \ldots, X_{[\ln n]}$ where $X_k$ = number of the eigenvalues of the matrix $\mathcal{C}_k^{(N)}$ situated in the interval $I_N$, and let $S_n(I_N) = X_1 + X_2 + \cdots + X_{[\ln n]}$. Note that $S_n(I_N)$ = the number of eigenvalues of the matrix $\tilde{\mathcal{C}}^{(N)}$ situated in the interval $I_N$. We want to prove that

$$\lim_{n\to\infty} \mathbb{P}(S_n(I_N) = k) = e^{-(b-a)} \frac{(b-a)^k}{k!} \tag{6.2.1}$$

Theorem 6.1.4 shows that we can assume without loss of generality that for any $k, 1 \leq k \leq [\ln n]$, we have $X_k \in \{0, 1\}$. Also, because of rotation invariance, we can assume, for large $n$, that

$$\mathbb{P}(X_k = 1) = \frac{(b-a)}{[\ln n]} \tag{6.2.2}$$

$$\mathbb{P}(X_k = 0) = 1 - \frac{(b-a)}{[\ln n]} \tag{6.2.3}$$

The random variable $S_n(I_N)$ can now be viewed as the sum of $[\ln n]$ Bernoulli trials, each with the probability of success $\frac{(b-a)}{[\ln n]}$ and

$$\mathbb{P}(S_n(I_N) = k) = \binom{[\ln n]}{k} \left(\frac{(b-a)}{[\ln n]}\right)^k \left(1 - \frac{(b-a)}{[\ln n]}\right)^{[\ln n]-k} \tag{6.2.4}$$

which converges to $e^{-\lambda} \frac{\lambda^k}{k!}$, where $\lambda = [\ln n] \frac{(b-a)}{[\ln n]} = (b-a)$. Therefore, we get (6.2.1).

Since for any disjoint intervals $I_{N,k}, 1 \leq k \leq [\ln n]$ situated near $e^{i\theta_0}$, the random variables $S_n(I_{N,k})$ are independent, (6.2.1) will now give (1.0.16) and therefore the proof of the main theorem is complete. □

## 6.3 The Case of Random Verblunsky Coefficients Uniformly Distributed on a Circle

In the Theorem 1.0.6, the distribution of the Verblunsky coefficients was taken to be uniform on a disk of radius $r < 1$. This distribution is rotation invariant and absolutely continuous with respect to the Lebesgue measure. We will now consider random Verblunsky coefficients uniformly distributed on a circle of radius $r < 1$ (this circle will be denoted by $C(0, r)$).

**Theorem 6.3.1.** *Consider the random polynomials on the unit circle given by the following* *recurrence relations:*

$$\Phi_{k+1}(z) = z\Phi_k(z) - \overline{\alpha}_k \Phi_k^*(z) \qquad k \geq 0, \qquad \Phi_0 = 1 \tag{6.3.1}$$

*where* $\alpha_0, \alpha_1, \ldots, \alpha_{n-2}$ *are i.i.d. random variables distributed uniformly in a circle of radius* $r < 1$ *and* $\alpha_{n-1}$ *is another random variable independent of the previous ones and uniformly distributed on the unit circle.*

*Consider the space* $\Omega = \{\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_{n-2}, \alpha_{n-1}) \in C(0,r) \times C(0,r) \times \cdots \times C(0,r) \times \partial\mathbb{D}\}$ *with the probability measure* $\mathbb{P}$ *obtained by taking the product of the uniform (one-dimensional Lebesgue) measures on each* $C(0,r)$ *and on* $\partial\mathbb{D}$*. Fix a point* $e^{i\theta_0} \in \mathbb{D}$ *and let* $\zeta^{(n)}$ *be the point process obtained from the eigenvalues of the truncated CMV matrix* $\mathcal{C}^{(n)}$*.*

*Then, on a fine scale (of order* $\frac{1}{n}$*) near* $e^{i\theta_0}$*, the point process* $\zeta^{(n)}$ *converges to the Poisson point process with intensity measure* $n\,\frac{d\theta}{2\pi}$ *(where* $\frac{d\theta}{2\pi}$ *is the normalized Lebesgue measure). This means that for any fixed* $a_1 < b_1 \leq a_2 < b_2 \leq \cdots \leq a_m < b_m$ *and any nonnegative integers* $k_1, k_2, \ldots, k_m$*, we have*

$$\mathbb{P}\left(\zeta^{(n)}\left(e^{i(\theta_0 + \frac{2\pi a_1}{n})}, e^{i(\theta_0 + \frac{2\pi b_1}{n})}\right) = k_1, \ldots, \ \zeta^{(n)}\left(e^{i(\theta_0 + \frac{2\pi a_m}{n})}, e^{i(\theta_0 + \frac{2\pi b_m}{n})}\right) = k_m\right)$$
$$\longrightarrow e^{-(b_1 - a_1)} \frac{(b_1 - a_1)^{k_1}}{k_1!} \cdots e^{-(b_m - a_m)} \frac{(b_m - a_m)^{k_m}}{k_m!} \tag{6.3.2}$$

*as* $n \longrightarrow \infty$*.*

*Proof.* The proof is similar to the proof of the Main Theorem, with a few changes. These modifications are required by the fact that we use a different probability distribution for the random Verblunsky coefficients and therefore some estimates need to be changed.

Let's denote by $\mu_r$ the Lebesgue measure $\frac{d\theta}{2\pi}$ on the circle $\{z \,|\, z = re^{i\theta}\}$. Since $\mu_r$ is rotation invariant, Lemma 4.1.1 holds, hence we get that the fractional moments of the matrix elements of the Carathéodory function associated to $\mathcal{C}_\alpha^{(n)}$ are uniformly bounded.

Lemma 6.3.2 (see below) shows that the conditional moments of order $s$ with $s \in (0, \frac{1}{2})$ are uniformly bounded. Using a standard trick (an application of Hölder's inequality, as in Lemma 4.2.4), the result in Lemma 6.3.2 holds for all $s \in (0, 1)$.

We should also mention that Aizenman's theorem for orthogonal polynomials on the unit circle (see [Sim1]) holds for the distribution $\mu_r$. Also, simple computations show that

$$\int_{\mathbb{D}} \log(1-|\omega|)\, d\mu_r(\omega) > -\infty \tag{6.3.3}$$

$$\int_{\mathbb{D}} \log(|\omega|)\, d\mu_r(\omega) > -\infty \tag{6.3.4}$$

We can now derive the Aizenman-Molchanov bounds for the resolvent of $\mathcal{C}_\alpha^{(n)}$ using exactly the same methods in Chapter 4. Once we have the exponential decay of the fractional moments of the matrix elements of the resolvent of $\mathcal{C}_\alpha^{(n)}$, we can follow the same route as in the proof of Theorem 1.0.6 (the eigenfunctions are exponentially localized, the matrix $\mathcal{C}_\alpha^{(n)}$ can be decoupled, and the decoupled matrices contribute at most one eigenvalue in each interval of size $O(\frac{1}{n})$) to conclude that (6.3.2) holds, that is, the local statistical distribution of the eigenvalues of $\mathcal{C}_\alpha^{(n)}$ is Poisson.

$\square$

We will now give the analog of the spectral averaging lemma (Lemma 4.3.3).

**Lemma 6.3.2.** *For any $s \in (0, \frac{1}{2})$, any $k$, $1 \le k \le n$, and any choice of $\alpha = \alpha_0, \alpha_1, \ldots, \alpha_{n-1}$ as in Theorem 6.3.1,*

$$\mathbb{E}\left( \left| F_{kk}(z, \mathcal{C}_\alpha^{(n)}) \right|^s \;\middle|\; \{\alpha_i\}_{i \neq k} \right) \le K(s,r) \tag{6.3.5}$$

*where a possible value for the constant is* $K(s,r) = 4 \cdot 64^s \cdot \frac{\pi^{1-2s}}{(1-2s)r^s}$.

*Proof.* Using exactly the same method as in the proof of Lemma 4.3.3, we find that it is enough to find a uniform bound for

$$I = \int_{C(0,r)} \left| \frac{4}{1 + \overline{\alpha}_k w_2 - w_1(\alpha_k + w_2)} \right|^s d\mu_r(\alpha_k) \tag{6.3.6}$$

for any $w_1, w_2 \in \mathbb{D}$.

Let $\alpha_k = x + iy$, with $x, y \in \mathbb{R}$. Then

$$1 + \overline{\alpha}_k w_2 - w_1(\alpha_k + w_2) = x(-w_1 + w_2) + y(-iw_1 - iw_2) + (1 - w_1 w_2) \tag{6.3.7}$$

Let's denote by $M(x, y, w_1, w_2)$ the right-hand side of the previous equation.

Then

$$\int_{C(0,r)} \left| \frac{4}{M(x,y,w_1,w_2)} \right|^s d\mu_r(\alpha_k) \leq \min\{I_1, I_2\} \tag{6.3.8}$$

where

$$I_1 = \int_{C(0,r)} \left| \frac{4}{\operatorname{Re} M(x,y,w_1,w_2)} \right|^s d\mu_r(\alpha_k) \quad \text{and} \quad I_2 = \int_{C(0,r)} \left| \frac{4}{\operatorname{Im} M(x,y,w_1,w_2)} \right|^s d\mu_r(\alpha_k)$$

Let $\varepsilon > 0$ (we will choose $\varepsilon$ later). If $|w_2 - w_1| < \varepsilon$ and $|w_2 + w_1| < \varepsilon$, then $|M(x,y,w_1,w_2)| \geq 1 - 2\varepsilon - \varepsilon^2$. We immediately get

$$\int_{C(0,r)} \left| \frac{4}{M(x,y,w_1,w_2)} \right|^s d\mu_r(\alpha_k) \leq \left( \frac{4}{1 - 2\varepsilon - \varepsilon^2} \right)^s \tag{6.3.9}$$

If at least one of the complex numbers $(w_2 - w_1)$ and $(w_2 + w_1)$ is greater or equal in absolute value than $\varepsilon$, then at least one of $M_1(w_1,w_2) = \sqrt{(\operatorname{Re}(w_2 - w_1))^2 + (\operatorname{Re}(-iw_1 - iw_2))^2}$ and $M_2(w_1,w_2) = \sqrt{(\operatorname{Im}(w_2 - w_1))^2 + (\operatorname{Im}(-iw_1 - iw_2))^2}$ is greater than $\varepsilon/2$. Without loss of generality, we can assume that $M_1(w_1,w_2) \geq \varepsilon/2$.

Then

$$I_1 = \int_{C(0,r)} \left| \frac{4}{x \operatorname{Re}(-w_1 + w_2) + y \operatorname{Re}(-iw_1 - iw_2) + \operatorname{Re}(1 - w_1 w_2)} \right|^s d\mu_r(\alpha_k) \tag{6.3.10}$$

$$= \int_{C(0,r)} \left| \frac{4}{x \frac{\operatorname{Re}(-w_1+w_2)}{M_1(w_1,w_2)} + y \frac{\operatorname{Re}(-iw_1-iw_2)}{M_1(w_1,w_2)} + \frac{\operatorname{Re}(1-w_1w_2)}{M_1(w_1,w_2)}} \right|^s \left| \frac{1}{M_1(w_1,w_2)} \right|^s d\mu_r(\alpha_k) \tag{6.3.11}$$

$$= \int_0^{2\pi} \frac{1}{r^s} \left| \frac{4}{\cos\theta \frac{\operatorname{Re}(-w_1+w_2)}{M_1(w_1,w_2)} + \sin\theta \frac{\operatorname{Re}(-iw_1-iw_2)}{M_1(w_1,w_2)} + \frac{\operatorname{Re}(1-w_1w_2)}{rM_1(w_1,w_2)}} \right|^s \left| \frac{1}{M_1(w_1,w_2)} \right|^s \frac{d\theta}{2\pi} \tag{6.3.12}$$

Let $\theta_0 = \theta_0(w_1,w_2)$ be an angle such that $\sin\theta_0 = \frac{\operatorname{Re}(-w_1+w_2)}{M_1(w_1,w_2)}$ and $\cos\theta_0 = \frac{\operatorname{Re}(-iw_1-iw_2)}{M_1(w_1,w_2)}$ and let $K_0 = K_0(w_1,w_2,r)$ be defined by $K_0 = \frac{\operatorname{Re}(1-w_1w_2)}{rM_1(w_1,w_2)}$.

Using (6.3.12), we get

$$I_1 \leq \frac{8^s}{\varepsilon^s r^s} \int_0^{2\pi} \left| \frac{1}{\sin(\theta + \theta_0) + K_0} \right|^s \frac{d\theta}{2\pi} = \frac{8^s}{\varepsilon^s r^s} \int_0^{2\pi} \left| \frac{1}{\sin\theta + K_0} \right|^s \frac{d\theta}{2\pi} \tag{6.3.13}$$

Without loss of generality, we can assume that in the previous relation we have $K_0 \in [-1,1]$ and therefore $K_0 = \sin\kappa$, for a $\kappa \in [0, 2\pi)$. Therefore,

$$I_1 \leq \frac{8^s}{\varepsilon^s r^s} \int_0^{2\pi} \left| \frac{1}{2\left(\sin(\frac{\theta+\kappa}{2})\cos(\frac{\theta-\kappa}{2})\right)} \right|^s \frac{d\theta}{2\pi} = \frac{8^s}{\varepsilon^s r^s} \int_0^{2\pi} \left| \frac{1}{2\left(\sin(\frac{\theta+\kappa}{2})\sin(\frac{\pi-\theta+\kappa}{2})\right)} \right|^s \frac{d\theta}{2\pi} \tag{6.3.14}$$

The function $\sin x$ vanishes as fast as $x$ at each zero. We actually have

$$|\sin x| \geq \frac{2}{\pi}|x| \qquad \text{for each } x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \tag{6.3.15}$$

so using (6.3.14) and the fact that the function sin vanishes twice in an interval of length $2\pi$, we get

$$I_1 \leq 4 \cdot \frac{4^s}{\varepsilon^s r^s} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{|\theta|^{2s}} \frac{d\theta}{2\pi} = \frac{2^{2s+3}}{\varepsilon^s r^s} \frac{\left(\frac{\pi}{2}\right)^{1-2s}}{1-2s} = 2^{4s+2} \frac{\pi^{1-2s}}{\varepsilon^s r^s} \frac{1}{1-2s} \tag{6.3.16}$$

We can therefore conclude that for $I$ defined in (6.3.6), we have

$$I \leq \max\left\{ \left(\frac{4}{1-2\varepsilon-\varepsilon^2}\right)^s, 2^{4s+2} \frac{\pi^{1-2s}}{\varepsilon^s r^s} \frac{1}{1-2s} \right\} \tag{6.3.17}$$

For $\varepsilon = \frac{1}{4}$ we get

$$I \leq 4 \cdot 64^s \cdot \frac{\pi^{1-2s}}{r^s(1-2s)} \tag{6.3.18}$$

which is exactly (6.3.5). $\square$

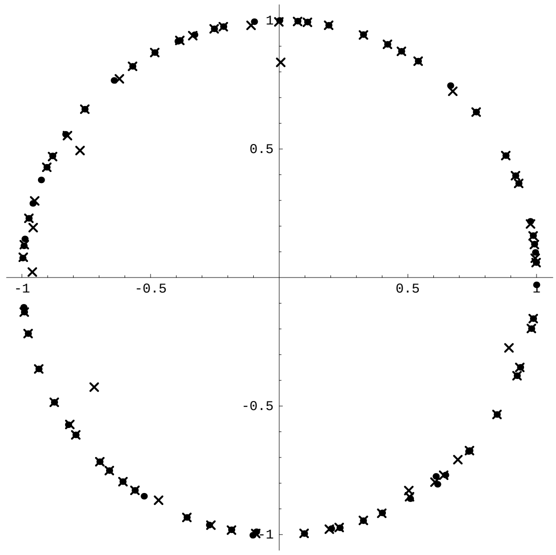
# Chapter 7

# Appendices

In this section we present a few numerical simulations and describe a few possible extensions of the results presented in this thesis.

### Appendix 1.

In this thesis we study the statistical distribution of the zeros of paraorthogonal polynomials. It would be interesting to understand the statistical distribution of the zeros of orthogonal polynomials. A generic plot of the zeros of paraorthogonal polynomials versus the zeros of orthogonal polynomials is

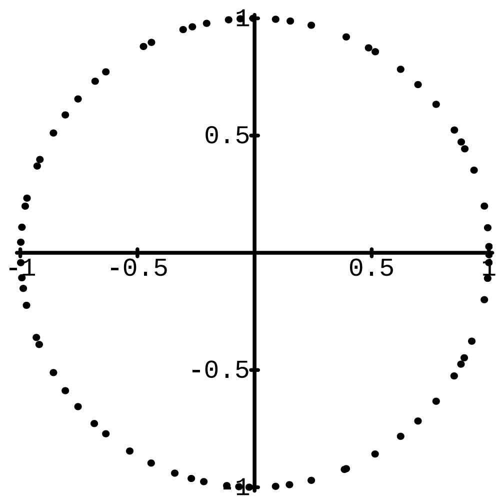



In this Mathematica plot, the points represent the zeros of paraorthogonal polynomials obtained by randomly choosing $\alpha_0, \alpha_1, \ldots, \alpha_{69}$ from the uniform distribution on $D(0, \frac{1}{2})$ and $\alpha_{70}$ from the uniform distribution on $\partial\mathbb{D}$. The crosses represent the zeros of the orthogonal polynomials obtained from the same $\alpha_0, \alpha_1, \ldots, \alpha_{69}$ and an $\alpha_{70}$ randomly chosen from the uniform distribution on $D(0, \frac{1}{2})$.

We observe that, with the exception of a few points (probably corresponding to "bad eigenfunctions"), the zeros of paraorthogonal polynomials and those of orthogonal polynomials are very close. We conjecture that these zeros are pairwise exponentially close, with the exception of $O((\ln N)^2)$ of them. We expect that the distribution of the arguments of the zeros of orthogonal polynomials on the unit circle is also Poisson.

## Appendix 2.

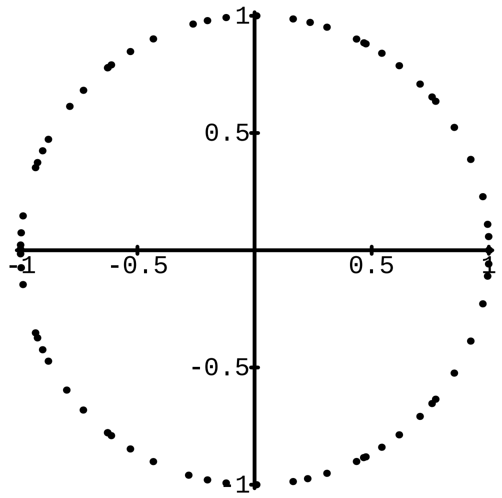
Numerical simulations show that the Poisson distribution in expected for other distributions, which are not rotationally invariant. For example, if the Verblunsky coefficients are uniformly distributed in the interval $[-\frac{1}{2}, \frac{1}{2}]$, then a typical plot of the zeros of the paraorthogonal polynomials is




This indicates that the assumption that the distribution of the Verblunsky coefficients is invariant under rotations might not be essential for getting the local Poisson distribution.

Another interesting distribution for the Verblunsky coefficients (which is, in a way, opposite to the distribution considered in the Main Theorem) is the Bernoulli distribution $\mu = p\,\delta_z + (1-p)\,\delta_w$, where $z, w \in \mathbb{D}$, $z \neq w$, and $p \in (0, 1)$. We can consider paraorthogonal polynomials defined by random Verblunsky coefficients with Bernoulli distributions (i.e., $\alpha_n = z$ with probability $p$ and $\alpha_n = w$ with probability $(1-p)$).

The numerical simulations for the case of Bernoulli distributions also suggest a local Poisson distribution for the zeros of the paraorthogonal polynomials at most points $z \in \partial\mathbb{D}$. For example, if $z = \frac{1}{2}$, $w = -\frac{1}{2}$, and $p = \frac{1}{2}$, we get the following typical plot:

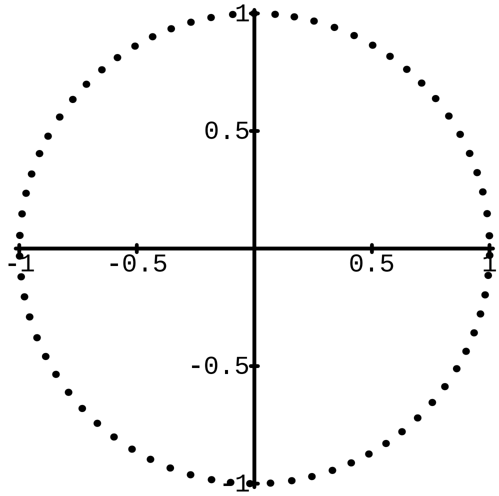



Note that in the two cases presented earlier we do not have local Poisson distribution at the points $\pm 1$.

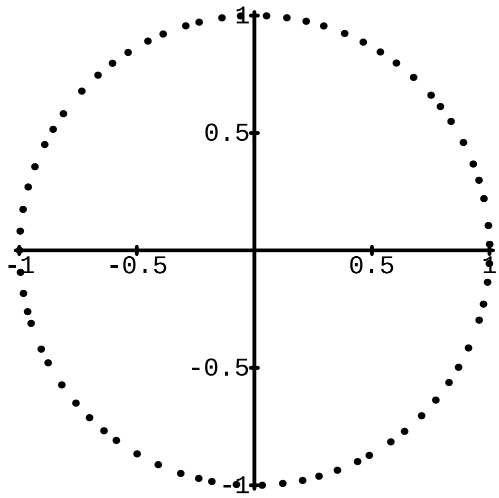
As we observed before, if $|\alpha_n|$ are uniformly distributed in a disk of radius $r < 1$, the Main Theorem shows that we have a Poisson behavior (i.e., noncorrelation); on the other hand, Simon [Sim3] proved that if $|\alpha_n|$ decay sufficiently fast (more precisely, if $\{\alpha_n\}_{n\geq 0} \in l^1(\mathbb{Z}^+)$), then we have a clock behavior (i.e., repulsion). These are two opposite situations. It would be interesting to study and understand what happens in between these two situations (i.e., when the Verblunsky coefficients $|\alpha_n|$ decay at a polynomial rate).

A few numerical simulations indicate that there is a transition from Poisson behavior to clock behavior:

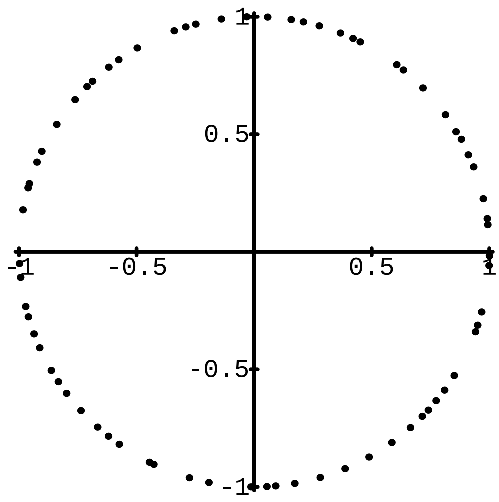
Case 1: $\alpha_n$ uniformly distributed in the disk $D(0, \frac{1}{n})$ (fast decay; the plot suggests clock behavior):




Case 2: $\alpha_n$ uniformly distributed in the disk $D(0, \frac{1}{n^{1/2}})$ (intermediate decay):



Case 3: $\alpha_n$ uniformly distributed in the disk $D(0, \frac{1}{n^{1/10}})$ (slow decay; the plot suggests Poisson behavior):




A reasonable conjecture would be that the transition point between "clock" and "Poisson" is for $\alpha_n$ uniformly distributed in the disk $D(0, \frac{1}{n^{1/2}})$. This is motivated by facts known for random Schrödinger operators with decaying potentials and by the results from Section 12.7 in [Sim5].

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