Four-dimensional Galois representations of solvable type and automorphic forms

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Abstract

Let F be a number field and $\rho : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\mathbb{C})$ be a continuous, irreducible representation. Artin conjectured that if ρ is non-trivial, then the associated L-function $L(s, \rho)$ is entire. Langlands generalized this conjecture by asserting that there should be a cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_F)$ such that $L(s, \rho)$ and $L(s, \pi)$ agree at almost all places. If such a π exists, ρ is said to be *modular*. Langlands's conjecture does indeed imply Artin's conjecture.

We consider in the thesis the case where ρ is a four-dimensional representation of *solvable type*, i.e., the image of ρ is solvable. We study what is known about Artin's and Langlands's conjectures for ρ. Artin's conjecture is already known in the imprimitive cases, but not in the primitive ones. We show in two new cases, one primitive and one monomial, that ρ is modular; the former case yields new instances of Artin's conjecture. We show that there are only two other primitive cases where one does not know Langlands's conjecture for ρ, and that these cases are symplectic and would follow from certain instances of non-normal quintic base change for GL_4. Our new monomial case is non-essentially-self-dual. In fact we show that if ρ is monomial and essentially self-dual, then it is modular.

We have two other small results for representations in other dimensions. First, if ρ is primitive and three dimensional, then in certain cases we show that the associated eight-dimensional representation $\mathrm{Ad}(\rho)$ is modular. Second, we show that ρ of dimension n having supersolvable image is modular if $n = 2^j$ or $n = 2^j \cdot 3$ for some j.

Lastly, we include in an appendix a proof of Ramakrishnan that if ρ corresponds to π as above, then the complete L-functions for ρ and π are equal as Euler products over $\mathbb{Q}$. More precisely, $L(s, \rho_v) = L(s, \pi_v)$ at every finite place, and $\prod_{v|\infty} L(s, \rho_v) = \prod_{v|\infty} L(s, \pi_v)$.

Contents

Chapter 1

Introduction

Let F be a number field and let ρ be an irreducible continuous representation of the absolute Galois group G_F of F into $\mathrm{GL}_n(\mathbb{C})$. Recall that there is associated to ρ an Artin L-function $L(s, \rho)$ which is meromorphic on the whole complex plane. When ρ is the trivial representation, $L(s, \rho) = \zeta_F(s)$ is the Dedekind zeta function of F, which has a simple pole at $s = 1$ and is analytic elsewhere. Artin conjectured that if ρ is not the trivial representation, then $L(s, \rho)$ is entire. This conjecture was proved for $n = 1$ by associating continuous characters of G_F to Hecke characters over F, which were known to have entire L-functions.

Langlands generalized this idea and further conjectured that, for any n, ρ corresponds to a cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_F)$. If there exists such a π, then ρ is said to be *modular*. This conjecture is known as the *strong Artin conjecture* or *Langlands's modularity conjecture*. Because $L(s, \pi)$ is entire, the strong Artin conjecture does indeed imply the Artin conjecture. Examples of when Artin's conjecture is known but the strong Artin conjecture is not will be given below.

In Section 3.3.4, we summarize several results about these conjectures. Progress has primarily been made for representations of *solvable type*, i.e., when the image of ρ is solvable. Some cases where ρ of solvable type is known to be modular are: when $n = 2$ ([La2], [Tu]); when $n = 3$ and ρ is induced ([JPSS]); when $n = 3$ and ρ is the twist of a symmetric square ([GeJ]); or when $n = 4$ and the image of ρ is orthogonal ([Ra2]). Also in the non-solvable case, certain two-dimensional icosahedral representations are known to be modular (for example, see [BDST], [Ta], [Goi], [Bu], [KW], [JM], [BS]).

We consider irreducible Galois representations $\rho : G_F \to \mathrm{GL}_4(\mathbb{C})$ of *solvable type.* The main goal of the thesis is to classify such representations ρ by their projective images into three categories:

(i) representations ρ which can be proved to be modular;

(ii) representations ρ for which we can prove the Artin but not the strong Artin conjecture;

(iii) representations ρ for which we cannot prove Artin's conjecture.

Modularity in case (iii) will follow if certain instances of functoriality can be proved.

Let $\overline{\rho}$ denote the corresponding projective representation and $\overline{G} = \mathrm{Im}(\overline{\rho})$. Then $\overline{G}$ is a finite subgroup of $\mathrm{PGL}_4(\mathbb{C})$.

Artin's conjecture is known for ρ of solvable type unless ρ is primitive, and for the imprimitive cases modularity is known unless ρ is induced by a character of a quartic extension K/F with no intermediate subfields. The classification for projective images of primitive four-dimensional representations is known. A consequence of this classification is that for primitive ρ with solvable image, ρ maps into either $\mathrm{GO}_4(\mathbb{C})$ or $\mathrm{GSp}_4(\mathbb{C})$. In the former case, ρ is known to be modular by [Ra2]. In the latter, there are three possibilities for $\overline{G}$: $E_{2^4} \cdot C_5$, $E_{2^4} \cdot D_{10}$ and $E_{2^4} \cdot F_{20}$. Here $A \cdot B$ denotes a group extension of B by A (which in these cases are actually semidirect products), E_{2^4} denotes the elementary abelian group of order 2^4, C_5 is cyclic, D_{10} is dihedral and F_{20} is Frobenius of order 20. We give a proof of these facts in Chapter 5.

Our first result is the following

Theorem 6.1. *Suppose* $\rho : G_F \to \mathrm{GSp}_4(\mathbb{C}) \subseteq \mathrm{GL}_4(\mathbb{C})$ *has projective image* $\overline{G} = E_{2^4} \cdot C_5$. *Then* ρ *is modular.*

This appeared in [Ma1]. Consequently, we obtain new instances of Artin's conjecture. We also show that in the other two cases, the modularity of ρ would follow from certain instances of (solvable) non-normal quintic base change for GL_4. Thus the two cases where $\overline{G} \simeq E_{2^4} \cdot D_{10}$ and $\overline{G} \simeq E_{2^4} \cdot F_{20}$ are the only ones in category (iii) above.

In Chapter 7, we consider the case where ρ is imprimitive (of solvable type still). Then ρ is either in category (i) or (ii). If ρ is induced from a normal extension, then it is modular. So it suffices to assume ρ is induced from a character of a non-normal quartic extension with no intermediate subfields. We give some criteria about the projective images of such representations and show that if ρ is essentially self-dual, then it is modular. One possible non-essentially-self-dual case occurs when $\overline{G}$ is an extension of A_4 by the Klein group V_4. In this case, we call ρ *hypertetrahedral* and we are able to prove the following

Theorem 7.1. *Let F be a number field and ρ a hypertetrahedral representation of G_F. Then ρ is modular. There are infinitely many such representations with projective image $V_4 \cdot A_4$ which are not essentially self-dual.*

This gives new examples of non-normal induction and new examples of modular representations. These results are also contained in [Ma2].

Let us say a few words about how we prove these specific cases of modularity. First we briefly recall the ideas of Langlands's proof of the tetrahedral case. See Chapter 3 for definitions of additional notation. Let $\sigma : G_F \to \mathrm{GL}_2(\mathbb{C})$ be a tetrahedral representation. Then there is a normal cubic extension K/F such that σ_K is modular. Say $\sigma_K \leftrightarrow \Pi$. There are three representations π_0, π_1, π_2 of $\mathrm{GL}_2(\mathbb{A}_F)$ whose base change $\pi_{i,K}$ to K is Π. One of these should actually correspond to σ. There is a unique $\pi = \pi_i$ whose central character matches with the determinant of σ. Then one proves $\mathrm{Sym}^2(\sigma) \leftrightarrow \mathrm{Sym}^2(\pi)$. This combined with the correspondence $\sigma_K \leftrightarrow \pi_K$ allows one to conclude that, at any unramified place v, either $\sigma_v \leftrightarrow \pi_v$ or $\overline{\sigma}(\mathrm{Fr}_v) \in A_4$ has order divisible by 6. But A_4 has no elements of order 6, so in fact $\sigma \leftrightarrow \pi$.

In the proofs of Theorems 6.1 and 7.1 we will similarly have an normal cyclic extension K/F, which will either be cubic or quintic, such that ρ_K is modular. Then there will be either three or five π_i's such that $\rho_K \leftrightarrow \pi_{i,K}$. Among these π_i's there is a unique one, π, such that $\Lambda^2(\rho) \leftrightarrow \Lambda^2(\pi)$. Then one concludes that at any unramified place v either $\rho_v \leftrightarrow \pi_v$ or $\overline{\rho}(\mathrm{Fr}_v) \in \overline{G}$ has order divisible by 6 or 10. But our projective group does not have any elements of order 6 or 10, so indeed $\rho \leftrightarrow \pi$. Chapter 4 presents more formally our method of proof.

We also remark that the cases $\overline{G} \simeq E_{2^4} \cdot D_{10}$ and $\overline{G} \simeq E_{2^4} \cdot F_{20}$ are analogous to Tunnell's octahedral case. Tunnell matched up an octahedral Galois representation σ to an automorphic represenation π such that $\sigma_K \leftrightarrow \pi_K$ for two extensions K, one normal and quadratic, the other non-normal and cubic. The latter part relied on the non-normal cubic base change of Jacquet, Piatetski-Shapiro and Shalika [JPSS]. Then, using the fact that S_4 has no elements of order 6, Tunnell concluded that $\sigma \leftrightarrow \pi$.

If ρ is such that $\overline{G} \simeq E_{2^4} \cdot D_{10}$ or $\overline{G} \simeq E_{2^4} \cdot F_{20}$, then knowing certain instances of non-normal quintic base change would allow us to demonstrate the existence of an automorphic representation π such that $\rho_K \leftrightarrow \pi_K$ for both a normal quadratic extension K/F and a non-normal quintic extension K/F. As $\overline{G}$ has no elements of order 10, we could conclude that $\rho \leftrightarrow \pi$.

One can also ask what can be said if ρ is reducible (of solvable type). The interesting cases are when $\rho = \chi \oplus \tau$, where χ is a character and τ is a primitive, irreducible three-dimensional representation. In Chapter 8.1, we recall the classification for such three-dimensional representations τ. There are essentially three cases, all of which are non-essentially-self-dual. Neither the Artin nor the strong Artin conjecture are known for these τ, but we show in two of these cases that the associated eight-dimensional self-dual representation $\mathrm{Ad}(\tau)$ is modular.

In Chapter 9, using a direct application of automorphic induction, we deduce that if ρ is of dimension 2^j or $2^j \cdot 3$ and ρ has supersolvable image, then ρ is modular.

Chapter 2 reviews some definitions and results on groups and representations which we will need. Chapter 3 covers relevant material on Galois and automorphic representations.

In Appendix α, we give a proof of a result related to the strong Artin conjecture: if $L(s, \rho_v) = L(s, \pi_v)$ at almost all places v, then $L(s, \rho_v) = L(s, \pi_v)$ for all finite places v and $\prod_{v|\infty} L(s, \rho_v) = \prod_{v|\infty} L(s, \pi_v)$. This result, shown to us by Ramakrishnan, is not (to our knowledge) in the current literature.

Many elementary properties of specific finite groups were computed in the computer algebra package [GAP]. In Appendix B, we give the GAP notation for many of the finite groups appearing in the text.

Chapter 2

Preliminaries on Linear and Finite Groups

2.1 Linear Groups

2.1.1 Classical Terminology

Definition 2.1. *Let* $x \in \mathrm{GL}_n(\mathbb{C})$*. If* $\det x = 1$*, we will say* x *is* unimodular*. Moreover, any subgroup of* $\mathrm{GL}_n(\mathbb{C})$ *comprising only unimodular elements will be called* unimodular.[1] *The group of all unimodular elements of* $\mathrm{GL}_n(\mathbb{C})$ *is denoted* $\mathrm{SL}_n(\mathbb{C})$*.*

For a matrix group G, $\mathrm{Diag}(G)$ will denote the subgroup of diagonal matrices contained in G.

Definition 2.2. *Let* $G \leq \mathrm{GL}_n(\mathbb{C})$*. If* G *is isomorphic to a subgroup of* $\mathrm{GL}_r(\mathbb{C}) \times \mathrm{GL}_{n-r}(\mathbb{C})$ *for some* $1 \leq r \leq n-1$*, then we will say* G *is* reducible*. Otherwise,* G *is said to be* irreducible.

Definition 2.3. *Let* $G \leq \mathrm{GL}_n(\mathbb{C})$ *be irreducible. Suppose* $\mathbb{C}^n$ *decomposes into subspaces* $V_1 \oplus \cdots \oplus V_m$*,* $m > 1$*, such that* G *acts (transitively) on set of subspaces* $\{V_i\}$*. Then* G *is said to be* imprimitive*. Otherwise,* G *is* primitive.

Observe that the condition that G is irreducible automatically ensures that, if G acts on a set of subspaces $\{V_i\}$ where $\mathbb{C}^n = V_1 \oplus \cdots \oplus V_m$, G acts transitively of $\{V_i\}$.

Let ρ be the standard representation of a finite group $G \leq \mathrm{GL}_n(\mathbb{C})$. Then G is irreducible (resp. primitive) if and only if ρ is irreducible (resp. primitive).

[1] Note our usage of the term "unimodular group" differs from that in the study of topological groups, i.e., we do not mean that it has a bi-invariant Haar measure.

2.1.2 Orthogonal and Symplectic Groups

Let V be an n-dimensional vector space over $\mathbb{C}$ and let $f : V \times V \to \mathbb{C}$ be a non-degenerate bilinear form on V. Recall that, after fixing a basis for V, every bilinear form f can be associated to a matrix B by the defining equation $f(x, y) = {}^t xBy$. We say f is *orthogonal* if f is symmetric, i.e., $f(x, y) = f(y, x)$ for all $x, y \in V$; and we say f is *symplectic* if f is skew symmetric, i.e., $f(x, y) = -f(y, x)$ for all $x, y \in V$. Let $O(f)$ be the automorphism group of this space (V, f), i.e.,

$$O(f) = \{g \in \mathrm{GL}(V) : f(g(x), g(y)) = f(x, y) \text{ for all } x, y \in V\}. \tag{2.1}$$

Similarly we can define the similitude group $GO(V, f)$ by

$$GO(f) = \{g \in \mathrm{GL}(V) : f(g(x), g(y)) = \lambda(g)f(x, y) \text{ for all } x, y \in V \text{ and some } \lambda(g) \in \mathbb{C}\}. \tag{2.2}$$

The orthogonal and symplectic (similitude) groups we consider are subgroups of $\mathrm{GL}_n(\mathbb{C})$ which respectively preserve orthogonal and symplectic forms. Since all orthogonal forms are equivalent over $\mathbb{C}$, and similarly for symplectic forms (which only occur when n is even), we will make an explicit choice of orthogonal and symplectic forms and define the orthogonal and symplectic groups in terms of corresponding matrices.

Definition 2.4. *The* orthogonal group $\mathrm{O}_n(\mathbb{C}) = \left\{x \in \mathrm{GL}_n(\mathbb{C}) : {}^t xx = I\right\}$. *The* orthogonal similitude group $\mathrm{GO}_n(\mathbb{C}) = \left\{x \in \mathrm{GL}_n(\mathbb{C}) : {}^t xx = \lambda I, \lambda \in \mathbb{C}\right\}$.

Here our orthogonal form is just the usual one, which in matrix representation is just the identity.

Consider the symplectic form given by the $2m \times 2m$ matrix,

$$J = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}. \tag{2.3}$$

Definition 2.5. *The* symplectic group $\mathrm{Sp}_{2m}(\mathbb{C}) = \left\{x \in \mathrm{GL}_{2m}(\mathbb{C}) : {}^t xJx = J\right\}$. *The* symplectic similitude group $\mathrm{GSp}_{2m}(\mathbb{C}) = \left\{x \in \mathrm{GL}_{2m}(\mathbb{C}) : {}^t xJx = \lambda J, \lambda \in \mathbb{C}\right\}$.

Note that if we had chosen different forms, the corresponding orthogonal and symplectic groups would just be conjugates of the ones we defined.

Definition 2.6. *Let $G \leq \mathrm{GL}_n(\mathbb{C})$. We will say that G is* orthogonal *(resp.,* symplectic*) if G is conjugate to a subgroup of $\mathrm{O}_n(\mathbb{C})$ (resp., $\mathrm{Sp}_n(\mathbb{C})$), and that G is of* orthogonal *(resp.,* symplectic*)* type *if G is conjugate to a subgroup of $\mathrm{GO}_n(\mathbb{C})$ (resp., $\mathrm{GSp}_n(\mathbb{C})$).*

2.2 Finite-Dimensional Representations

By *character*, we will always mean a linear character, i.e., a one-dimensional representation, unless we explicitly say the character of a representation.

Let G be a group and $\rho : G \to \mathrm{GL}_n(\mathbb{C})$ be a linear representation. Denote the image of ρ by $\mathrm{Im}(\rho)$. Let $\overline{\rho}$ be the composition of ρ with the canonical projection from $\mathrm{GL}_n(\mathbb{C})$ to $\mathrm{PGL}_n(\mathbb{C})$.

$$\begin{array}{ccc} G & \xrightarrow{\rho} & \mathrm{GL}_n(\mathbb{C}) \\ & \searrow^{\overline{\rho}} & \downarrow \\ & & \mathrm{PGL}_n(\mathbb{C}) \end{array}$$

Then $\overline{\rho}$ is the *projective representation associated to* ρ and the image $\mathrm{Im}(\overline{\rho})$ of $\overline{\rho}$ is called the *projective image* of ρ. We will say ρ is of *solvable type* if the image of ρ is solvable. This is evidently equivalent to the image of $\overline{\rho}$ being solvable.

Let $H \subseteq G$. Then ρ_H denotes the restriction of ρ to H. If σ is a representation of H, then $\mathrm{Ind}_H^G \sigma$ denotes the induction of σ from H to G.

2.2.1 Self-Dual Representations

Let $\langle\ ,\ \rangle$ be a (non-degenerate) orthogonal or symplectic form on $V = \mathbb{C}^n$. The *contragredient* of ρ with respect to $\langle\ ,\ \rangle$ is the unique representation which satisfies

$$\langle \rho(g)v, \check{\rho}(g)w \rangle = \langle v, w \rangle \ \forall g \in G, \ \forall v, w \in V. \tag{2.4}$$

While the contragredient depends upon the form chosen, it is independent of the choice of form up to equivalence. For instance, $\check{\rho}$ is equivalent to the representation of G given by $g \mapsto {}^t\rho(g^{-1})$, which is the contragredient of ρ with respect to the standard orthogonal form on $\mathbb{C}^n$. Hence we will simply speak of *the* contragredient representation, which is well-defined up to equivalence.

We say that ρ is *self-dual* if $\check{\rho} \simeq \rho$ and that ρ is *essentially self-dual* if $\check{\rho} \simeq \rho \otimes \chi$ for some character χ of G. In particular, if $\rho \simeq \check{\rho} \otimes \chi$, then there is an orthogonal or symplectic form $\langle\ ,\ \rangle$ (depending on the isomorphism between ρ and $\check{\rho} \otimes \chi$) such that

$$\langle \rho(g)v, \rho(g)w \rangle = \chi(g)\langle v, w \rangle \ \forall g \in G,\ \ \forall v, w \in V. \tag{2.5}$$

In addition, ρ is said to be *orthogonal* (resp. *symplectic*, *of orthogonal type*, *of symplectic type*) if $\mathrm{Im}(\rho)$ is.

Recall that $\rho \otimes \rho = Sym^2(\rho) \otimes \Lambda^2(\rho)$. We can relate the notions of being self-dual, orthogonal and symplectic as below.

Proposition 2.1. *The representation ρ is self-dual if and only if $\rho \otimes \rho$ contains the trivial character, and is essentially self-dual if and only if $\rho \otimes \rho$ contains a character. Further, ρ is of orthogonal (resp. symplectic) type if and only if $Sym^2(\rho)$ (resp. $\Lambda^2(\rho)$) contains a character. If $Sym^2(\rho)$ (resp., $\Lambda^2(\rho)$) contains the trivial character, then ρ is actually orthogonal (resp. symplectic).*

The endomorphism ring $\mathrm{End}(\rho)$ is isomorphic to the (self-dual) representation $\rho \otimes \check{\rho}$. In particular, $\rho \otimes \check{\rho}$ always contains the trivial representation. By Schur's lemma, we have the following useful characterization.

Proposition 2.2. *The representation $\rho \otimes \check{\rho}$ contains the trivial representation with multiplicity one if and only if ρ is irreducible.*

2.2.2 Clifford's Theorem

Let V be a representation of a group G. We will say a subspace $W \subseteq V$ is an *eigenspace for G with character λ* if there exists a character $\lambda : G \to \mathbb{C}^\times$ such that $g(w) = \lambda(g)w$ for all $w \in W$ and all $g \in G$.

Theorem 2.1. (Clifford) *Let (ρ, V) be a finite-dimensional irreducible representation of G and $H \trianglelefteq G$.*

(i) ρ_H decomposes into irreducible representations of equal dimension, all of which are isomorphic if ρ is primitive.

(ii) Suppose that $V = \oplus W_i$, where the W_i are eigenspaces for H with distinct characters. Then G/H permutes the W_i's.

Note that what is typically referred to as "Clifford's theorem" is a more general and detailed statement about the decomposition of representations restricted to subgroups and does not usually include the specific result in (i) for when ρ is primitive. However this is a corollary of the usual Clifford's theorem and just what we want. See, for example, [As] or [Gor].

2.3 A Little Finite Group Theory

Now we recall some standard definitions and results on finite groups. For proofs, see [As] or [Gor].

Let G be a finite group. Denote by $Z(G)$ its center, $\mathrm{Aut}(G)$ its automorphism group, and by $[G,G]$ its commutator. For $S \subseteq G$, let $C_G(S)$ denote the centralizer of S in G. For a prime number p, we write $\mathrm{Syl}_p(G)$ for some Sylow p-subgroup of G.

Definition 2.7. *Suppose $H \trianglelefteq G$ and $Q = G/H$, i.e., we have a short exact sequence*

$$1 \longrightarrow H \longrightarrow G \longrightarrow Q \longrightarrow 1. \tag{2.6}$$

Then we say that G is an extension of Q by H *and write $G = H \cdot Q$.*

Definition 2.8. *Let G and H be finite groups with an injective homomorphism of centers $\phi : Z(G) \hookrightarrow Z(H)$. Let $M = \langle (z, \phi(z)) : z \in Z(G) \rangle$. The group $(G \times H)/M$, denoted by $G \circ H$ or $H \circ G$, is called a* central product of G and H with identified centers.

We remark that in general, given two groups G and H, there may be many central products $G \circ H$. In the case that $Z = Z(G) = Z(H)$ with $\mathrm{Aut}(Z)$ contained in both $\mathrm{Aut}(G)$ and $\mathrm{Aut}(H)$, the central product $G \circ H$ is uniquely defined. More generally, one can uniquely define a central product $G_1 \circ G_2 \circ \cdots \circ G_r$ if $Z(G_1) = Z(G_i)$ and $\mathrm{Aut}(Z(G_i)) \subseteq \mathrm{Aut}(G_i)$ for all i. In this case, taking central products is associative.

We also need to define the following 2-groups of order 2^m, each with a cyclic subgroup of index 2:

—the *dihedral group* $D_{2^m} = \langle x, y | x^{2^{m-1}} = y^2 = 1, yxy = x^{-1} \rangle$ for $m \geq 2$;

—the *semidihedral group* $SD_{2^m} = \langle x, y | x^{2^{m-1}} = y^2 = 1, yxy = x^{2^{m-2}-1} \rangle$ for $m \geq 4$; and

—the *quaternion group* $Q_{2^m} = \langle x, y | x^{2^{m-1}} = y^4 = 1, yxy = x^{-1} \rangle / \langle x^{2^{m-2}} y^2 \rangle$ for $m \geq 4$.

2.3.1 The Fitting and Frattini Subgroups

Recall that a subgroup H of G is called *characteristic* if H is invariant under $\mathrm{Aut}(G)$. Note that if $H \trianglelefteq G$ and K is a characteristic subgroup of H, then $K \trianglelefteq G$.

Definition 2.9. *The* Fitting subgroup *of G is the maximal normal nilpotent subgroup of G, and is denoted by $F(G)$.*

We remark that such a unique subgroup exists because the product of two normal nilpotent subgroups is again normal and nilpotent. The uniqueness evidently makes $F(G)$ a characteristic subgroup of G.

Proposition 2.3. *If G is solvable, then $C_G(F(G)) \leq F(G)$.*

Definition 2.10. *The* Frattini subgroup $\Phi(G)$ *of G is the intersection of all maximal subgroups of G.*

Note that $\Phi(G)$ is also a characteristic subgroup of G.

Proposition 2.4. *Let P be a p-group. Then $\Phi(P)$ is the smallest normal subgroup of P such that $P/\Phi(P)$ is elementary abelian.*

2.3.2 Extraspecial Groups

Definition 2.11. *Let P be a p-group. We say P is* extraspecial *if $[P, P] = \Phi(P) = Z(P)$ and $Z(P)$ is cyclic.*

For an extraspecial p-group P, it follows that $|Z(P)| = p$ and $|P| = p^{2r+1}$ for some r.

Let $Q_8^r D_8^s$ denote the central product of r copies of Q_8 and s copies of D_8 with all centers identified. There is a unique such group for given r, s. This is an extraspecial 2-group of order $2^{2(r+s)+1}$. As $D_8 \circ D_8 = Q_8 \circ Q_8$, we see that $Q_8^r D_8^s = Q_8^{r+s-1} D_8$ or Q_8^{r+s} depending on whether s is odd or even. In fact any extraspecial 2-group of order 2^{2r+1} is isomorphic to either $Q_8^{r-1} D_8$ or Q_8^r for some r.

For any p-group P, the irreducible representations of P are monomial (i.e., induced from characters), and therefore have p-power dimension. For extraspecial p-groups, we have the following

Proposition 2.5. *Let P be an extraspecial group of order p^{2r+1}. Then all faithful irreducible complex representations of P are of degree p^r.*

Proposition 2.6. (Hall) *A p-group P with no noncyclic characteristic abelian subgroups is a central product $E \circ R$ of two groups E and R, where E is trivial or extraspecial and either R is cyclic or $R = D_{2^m}$, SD_{2^m} or Q_{2^m} with $m \geq 4$.*

Chapter 3

Preliminaries on Galois and Automorphic Representations

This chapter reviews theory of Galois and automorphic representations, and in particular the Artin and strong Artin (Langlands's modularity) conjectures.

3.1 Galois Representations and Artin's Conjecture

By a *Galois representation*, we will always mean a continuous, complex, finite-dimensional representation of the absolute Galois group of a number field.

Let F be a number field and $G_F = \mathrm{Gal}(\overline{F}/F)$ be its absolute Galois group. Let

$$\rho : G_F \to \mathrm{GL}_n(\mathbb{C})$$

be a Galois representation, continuous for the profinite topology on G_F. Such a representation factors through a finite quotient $\mathrm{Gal}(L/F)$. The field L *cut out* by ρ is $L = \overline{F}^{\ker \rho}$, i.e., the fixed field of $\ker \rho$. Recall that $\overline{\rho}$ denotes the corresponding projective representation into $\mathrm{PGL}_n(\mathbb{C})$. Then the field L_0 *cut out* by $\overline{\rho}$ is $L_0 = \overline{F}^{\ker \overline{\rho}}$.

For a Galois representation ρ, we denote induction and restriction by the corresponding fields. Specifically if $K \subseteq F \subseteq E$ are number fields, then $\rho_E = \rho_{G_E}$ denotes the restriction of ρ to G_E and $\mathrm{Ind}_F^K \rho = \mathrm{Ind}_{G_F}^{G_K} \rho$ denotes the induction of ρ to G_K.

When $n = 1$, then ρ is just a character χ. In fact, take any (possibly infinite order) character χ of G_F. Then χ factors through the abelianization $G_F^{\mathrm{ab}} = G_F/[G_F, G_F]$. Let $C_F = F^\times \backslash \mathbb{A}_F^\times$ be the idèle class group of F. By class field theory, there is an isomorphism called the *Artin map* from C_F to G_F^{ab}. There is a one-to-one correspondence between characters χ of G_F and idèle class characters (or *Hecke characters*) $\widetilde{\chi}$ of C_F. Hence we may identify χ with $\widetilde{\chi}$. We will sometimes simply say that χ or $\widetilde{\chi}$ is a *character of* F. In the case where χ is finite order and L is the field cut out by χ, then $\widetilde{\chi}$ factors through $\mathfrak{N}_{L/F}(C_L)\backslash C_F$, where $\mathfrak{N}_{L/F}$ is the norm map from L to F. If $\chi = \widetilde{\chi}$ is non-trivial, we say it is an *(idèle class) character of* L/F.

Now let us return to the general setting. Artin associated to ρ an L-function $L(s,\rho)$ to study the primes of F. The Artin L-function generalizes the Dirichlet L-series and the Riemann zeta function. Specifically, for $\Re(s) > 1$ we can define $L(s,\rho)$ by an Euler product,

$$L(s,\rho) = \prod_{v<\infty} L_v(s,\rho), \tag{3.1}$$

where the product is over all finite places v of F and the *local* L*-factors* $L_v(s,\rho)$ are given as follows.

Let L be the field cut out by ρ and let w be a place of L above a finite place v. Denote the localization of F at v by F_v and its ring of integers by $\mathcal{O}_{F_v}$. Let $\mathrm{Fr}_v \in \mathrm{Gal}(L/F)$ denote an element of the Frobenius conjugacy class at v, I_v the inertia group, and q_v the size of the residue field $|\mathcal{O}_{L_w}/\mathcal{O}_{F_v}|$. Let V be the representation space of ρ. Denote by V^{I_v} the space of inertial invariants, i.e., the subspace on which ρ_{I_v} acts trivially. Then we may define the local L-factors by

$$L_v(s,\rho) = \frac{1}{\det\left(I - \rho(\mathrm{Fr}_v)q_v^{-s}|V^{I_v}\right)}. \tag{3.2}$$

For almost all v, we have $V^{I_v} = V$, in which case we say that ρ is *unramified* at v, or simply that v is *unramified*. At these places we just have

$$L_v(s,\rho) = \frac{1}{\det\left(I - \rho(\mathrm{Fr}_v)q_v^{-s}\right)}. \tag{3.3}$$

So defined, one can show that the Euler product (3.1) does indeed converge (in fact, absolutely and uniformly in compact sets) in $\Re(s) > 1$.

Observe that if χ is a Dirichlet character, then $L(s,\chi)$ is in fact the Dirichlet L-series $\sum_n \chi(n)n^{-s} = \prod_p (1-\chi(p)p^{-s})^{-1}$ associated to χ. If 1 is the trivial representation on G_F, then $L(s,1)$ is the Dedekind zeta function $\sum_{\mathfrak{a}} \mathfrak{N}(\mathfrak{a})^{-s} = \prod_{\mathfrak{p}}(1-\mathfrak{N}(\mathfrak{p})^{-s})^{-1}$ attached to F. When $F = \mathbb{Q}$, $L(s,1)$ is just the Riemann zeta function $\sum_n n^{-s} = \prod_p (1-p^{-s})^{-1}$.

Presently it will be convenient to view the local L-factors $L_v(s,\rho)$ as L-functions $L(s,\rho_v)$

of local representations. For any global or local field K, let W_K denote the *Weil group* of K. Recall that this is a dense subgroup of the absolute Galois group of K. Now take $K = F_v$ for some place v. Then $W_{F_v} \subseteq \mathrm{Gal}(\overline{F_v}/F_v)$ embeds in $G_F = \mathrm{Gal}(\overline{F}/F)$ non-canonically, but the embeddings are conjugate. Fix one such embedding

$$i : W_{F_v} \hookrightarrow G_F.$$

We can define the local representation

$$\rho_v : W_{F_v} \overset{i}{\hookrightarrow} G_F \overset{\rho}{\to} \mathrm{GL}_n(\mathbb{C})$$

by $\rho_v = \rho \circ i$. As the Frobenius Fr_v is contained in W_{F_v}, it makes sense to define the L-function $L(s, \rho_v)$ for ρ_v by

$$L(s, \rho_v) = L_v(s, \rho). \tag{3.4}$$

Note that this local L-function depends only on ρ and v, not on the choice of the embedding i. Hence we will hereafter suppress the choice of i.

One can also define L-factors at the infinite places. Let v be an infinite place of F. Either $F_v = \mathbb{R}$ or $F_v = \mathbb{C}$. Define $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$ and $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$. If $F_v = \mathbb{R}$ let $n^+ = \frac{1}{2}(n + \mathrm{tr}\,\rho(\mathrm{Fr}_v))$ and $n^- = \frac{1}{2}(n - \mathrm{tr}\,\rho(\mathrm{Fr}_v))$. Then set

$$L(s, \rho_v) = \begin{cases} \Gamma_{\mathbb{R}}(s)^{n^+}\Gamma_{\mathbb{R}}(s)^{n^-}, & \text{if } F_v = \mathbb{R} \\ \Gamma_{\mathbb{C}}(s)^n, & \text{if } F_v = \mathbb{C}. \end{cases} \tag{3.5}$$

One defines the *completed Artin L-function* to be

$$L^*(s, \rho) = \prod_v L(s, \rho_v), \tag{3.6}$$

where the product runs over *all* places v of F. We write $L_\infty(s, \rho)$ for the product $\prod_{v|\infty} L(s, \rho_v)$ of local factors at the infinite places. Then

$$L^*(s, \rho) = L(s, \rho)L_\infty(s, \rho). \tag{3.7}$$

Artin proved that L-functions are *inductive*, i.e., if ρ is induced from some representation

σ of a subgroup, then $L(s,\rho) = L(s,\sigma)$. For any one-dimensional representation ρ, there exists by class field theory a Hecke character χ such that $L(s,\rho) = L(s,\chi)$. The Hecke L-function $L(s,\chi)$ has a functional equation relating $L(s,\chi)$ to $L(1-s,\overline{\chi})$ of the form

$$\gamma(s,\chi)L(s,\chi) = WN^{\frac{1}{2}-s}\gamma(1-s,\chi)L(1-s,\overline{\chi}), \tag{3.8}$$

where $|W| = 1$, $N \in \mathbb{Z}^+$ and $\gamma(s,\chi)$ is a product of Γ-functions. More precisely, W is the "root number," $\gamma(s,\chi) = L_\infty(s,\rho)$ and $N = D^n\mathfrak{N}(\mathfrak{m})$ where D is the absolute value of the discriminant of F, $\mathfrak{N}$ is the absolute norm and $\mathfrak{m}$ is the module of definition of χ (i.e., χ is a primitive Größencharacter mod $\mathfrak{m}$). Hecke had proved that $L(s,\chi)$ is a meromorphic function on all of $\mathbb{C}$, which is in fact analytic for $s \in \mathbb{C} - \{1\}$. Moreover, $L(s,\chi)$ has a simple pole at $s = 1$ if χ is trivial and $L(s,\chi)$ is entire if χ is non-trivial.

This Artin-Hecke correspondence implied that, for *monomial* representations ρ (i.e., those induced from one-dimensional representations), $L(s,\rho)$ is meromorphic on all of $\mathbb{C}$ and entire unless ρ contains the trivial character. The functional equation (3.8) for $L(s,\chi)$ gives a functional equation for $L(s,\rho)$ of the form

$$L_\infty(s,\rho)L(s,\rho) = W(D^n\mathfrak{N}(\mathfrak{f}))^{\frac{1}{2}-s}L_\infty(1-s,\check{\rho})L(1-s,\check{\rho}), \tag{3.9}$$

i.e.,

$$L^*(s,\rho) = W(D^n\mathfrak{N}(\mathfrak{f}))^{\frac{1}{2}-s}L^*(1-s,\check{\rho}), \tag{3.10}$$

where $W \in S^1$ is the root number and $\mathfrak{f}$ is the Artin conductor of ρ. Further, L-functions are *additive*: $L(s,\rho\oplus\sigma) = L(s,\rho)L(s,\sigma)$. So we also get the above results for direct sums of monomial representations.

Brauer then showed that the character of any Galois representation ρ is a $\mathbb{Z}$-linear combination of characters of monomial representations. Let us write $\rho = \sum m_i \mathrm{Ind}_{K_i}^F \chi_i$, where $m_i \in \mathbb{Z}$. Then the L-function $L(s,\rho) = \prod L(s,\chi_i)^{m_i}$ admits a meromorphic continuation to the entire complex plane with a functional equation of the form (3.10) and has a pole at $s = 1$ whose order is the number of times the trivial representation occurs in ρ. Whence we arrive at

Conjecture 3.1. (Artin) *Let ρ be a Galois representation. Then $L(s, \rho)$ is analytic in $\mathbb{C} - \{1\}$. In particular, $L(s, \rho)$ is entire unless ρ contains the trivial character.*

We remark that, by the additivity of L-functions, to prove Artin's conjecture for ρ, it suffices to prove it for all the irreducible constituents of ρ. By Proposition 2.2, we see that $L(s, \rho \otimes \check{\rho})$ has a simple pole at $s = 1$ if and only if ρ is irreducible.

Knowing Artin's conjecture for all ρ would give us information about the primes of F. For instance, Artin's conjecture has applications to bounding the error term in the Chebotarev density theorem (see Section 8 of Chapter 2 in [MM] for more information).

Now suppose $\rho : G_F \to \mathrm{GL}_2(\mathbb{C})$. Let $\overline{\rho} : G_F \to \mathrm{PGL}_2(\mathbb{C})$ be the composition of ρ with the canonical projection from $\mathrm{GL}_2(\mathbb{C})$ to $\mathrm{PGL}_2(\mathbb{C})$. Let $\overline{G} = \mathrm{Im}(\overline{\rho})$. Then $\overline{G}$ is a finite subgroup of $\mathrm{PGL}_2(\mathbb{C})$. The finite subgroups of $\mathrm{PGL}_2(\mathbb{C}) \simeq \mathrm{SO}_3(\mathbb{C})$ have been classified by Klein, and there are five possibilties:

(1) $\overline{G}$ is cyclic;

(2) $\overline{G}$ is dihedral;

(3) $\overline{G}$ is tetrahedral (isomorphic to A_4);

(4) $\overline{G}$ is octahedral (isomorphic to S_4); or

(5) $\overline{G}$ is icosahedral (isomorphic to A_5).

Accordingly, ρ is said to be *cyclic, dihedral, tetrahedral, octahedral* or *icosahedral*. In the cyclic case ρ is reducible, and in the dihedral case ρ is monomial. Thus the first two cases follow from the work of Artin, Brauer and Hecke.

The next major step in this case was accomplished by Robert Langlands in [La2]. Langlands's philosophy involves a vast generalization of abelian class field theory and Artin's conjecture. We will not delve deeply into the so-called Langlands program, but only discuss Langlands's modularity conjecture, which implies Artin's conjecture.

3.2 Automorphic Representations and Langlands's Modularity Conjecture

Let F be a number field and $\mathbb{A}_F$ be its ring of adèles, which is the restricted direct product $\prod'_v F_v$ over all places v of F with respect to $\{\mathcal{O}_{F_v}\}$. Let G be an algebraic group over F. Then $G(\mathbb{A}_F)$ is the restricted direct product $\prod'_v G(F_v)$ with respect to $\{G(\mathcal{O}_{F_v})\}$. One can consider automorphic representations of $G(\mathbb{A}_F)$. We restrict ourselves to the case where $G = \mathrm{GL}_n$. In this case, the *L-group* of G is just

$$^LG = {}^LG^0 \times W_F, \tag{3.11}$$

where the connected component of the L-group is

$$^LG^0 = \mathrm{GL}_n(\mathbb{C}). \tag{3.12}$$

The Borel subgroup B is the subgroup of G comprising upper-triangular matrices. If v is non-archimedean then the maximal compact subgroup K_v of $G(F_v) = \mathrm{GL}_n(F_v)$ is $\mathrm{GL}_n(\mathcal{O}_{F_v})$; otherwise $K_v = \mathrm{O}(n)$ or $\mathrm{U}(n)$ according to whether F_v is $\mathbb{R}$ or $\mathbb{C}$.

Let π be an *automorphic representation* of $G(\mathbb{A}_F)$, by which we mean an irreducible unitary representation of $G(\mathbb{A}_F)$ occurring in the space of automorphic forms on $G(\mathbb{A}_F)$. (Note that for $n = 1$, π is just a Hecke character.) Then π can be written as a restricted tensor product $\otimes_v' \pi_v$ over all places v. Each π_v is a local irreducible admissible representation of $G(F_v)$ such that, for almost all v, π_v is *unramified*, i.e., the restriction of π_v to K_v contains the trivial representation. If π_v is unramified, then sometimes we will just say call v *unramified (for π)*. At such a place, π_v is induced from the Borel subgroup $B(F_v)$ of some tensor product $\mu_1 \otimes \cdots \otimes \mu_n$, where each μ_i is an unramified character of $F_v^\times$. If v is finite, let ϖ denote the uniformizer for F_v, i.e., a generator for the maximal prime ideal in F_v. Then one can associate to an unramified π_v the semisimple conjugacy class in $^LG^0$ represented by

$$A(\pi_v) = \mathrm{diag}(\mu_1(\varpi), \ldots, \mu_n(\varpi)). \tag{3.13}$$

The eigenvalues of $A(\pi_v)$ are called the *Satake parameters* for π_v. Here we may define the local L-function

$$L(s, \pi_v) = \frac{1}{\det(I - A(\pi_v)q_v^{-s})}. \tag{3.14}$$

Let S be a finite set of places of F containing the archimedean places and the places at which π_v is ramified. Then we may formally define the incomplete automorphic L-function by an Euler product,

$$L^S(s, \pi) = \prod_{v \notin S} L(s, \pi_v) \tag{3.15}$$

More generally, $L(s, \pi_v)$ can be defined at any place and then we can formally define the complete automorphic L-function by

$$L(s, \pi) = \prod_v L(s, \pi_v). \tag{3.16}$$

Denote the contragredient of π by $\check{\pi}$. Let D be the absolute value of the discriminant of F. For cuspidal π, let $\mathfrak{f}_\pi$ be the conductor of π and $W(\pi)$ the root number. Godemont and Jacquet showed the following

Theorem 3.1. ([GoJ] or [Ja]) *Let π be an automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$. Then the Euler product $L(s, \pi)$ converges in some right half-plane and $L(s, \pi)$ meromorphically continues to the whole complex plane with a functional equation of the form*

$$L(s, \pi) = \varepsilon(s, \pi) L(1 - s, \check{\pi}). \tag{3.17}$$

The epsilon factor $\varepsilon(s, \pi)$ is non-vanishing entire function, which is of the form

$$\varepsilon(s, \pi) = (D^n \mathfrak{N}(\mathfrak{f}_\pi))^{\frac{1}{2} - s} W(\pi) \tag{3.18}$$

when π is cuspidal. Moreover, if π is non-trivial and cuspidal, then $L(s, \pi)$ is entire.

Suppose π and π' are automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$. Then by the work of many ([JS], [JPSS2], [Shah], [GS], [JS3], [CPS]) the Rankin-Selberg product L-function can be defined by

$$L(s, \pi \times \pi') = \prod_v L(s, \pi_v \times \pi'_v), \tag{3.19}$$

where at unramified finite places v, the local factor is defined by

$$L(s, \pi_v \times \pi'_v) = \frac{1}{\det(I - (A(\pi_v) \otimes A(\pi'_v)) q_v^{-s})}. \tag{3.20}$$

This Euler product for $L(s, \pi \times \pi')$ converges absolutely in the right half-plane $\Re(s) > 1$, meromorphically continues to the entire complex plane, is bounded in vertical strips, and has a functional equation of the form

$$L(s, \pi_v \times \pi'_v) = \varepsilon(s, \pi \times \pi') L(1 - s, \check{\pi} \times \check{\pi}'). \tag{3.21}$$

The global ε-factor is given by an Euler product expansion

$$\varepsilon(s, \pi \times \pi') = \prod_v \varepsilon(s, \pi_v \times \pi'_v, \psi_v), \tag{3.22}$$

where ψ is a non-trivial continuous additive character of $\mathbb{A}_F$ which is trivial on F, and the local ε-factors are those defined in [JPSS2], [JS3] and [CPS]; the nonarchimedean local ε-factors are trivial whenever π_v, π'_v and ψ_v are unramified. Using Rankin-Selberg products, Jacquet and Shalika [JS] showed that the Euler product for $L(s, \pi)$ converges absolutely for $\Re(s) > 1$.

Theorem 3.2. ([JS], [JPSS2], Appendix to [MW]) *Let π (resp. π') be a cuspidal automorphic representation of $\mathrm{GL}_m(\mathbb{A}_F)$ (resp. $\mathrm{GL}_n(\mathbb{A}_F)$). Then $L(s, \pi \times \pi')$ is entire if and only if $\pi' \not\simeq \breve{\pi}$. The only poles of $L(s, \pi \times \breve{\pi})$ are simple poles at $s = 0$ and $s = 1$.*

As a corollary to this theorem, we see that an isobaric representation π is cuspidal if and only if $L(s, \pi \times \breve{\pi})$ has a simple pole at $s = 1$.

Theorem 3.3. ([La], [JS]) *Let π_i be an automorphic cuspidal representation of $\mathrm{GL}_{n_i}(\mathbb{A}_F)$ for $1 \leq i \leq r$ and set $n = n_1 + n_2 + \cdots + n_r$. Then there exists an automorphic representation $\pi = \pi_1 \boxplus \pi_2 \boxplus \cdots \boxplus \pi_r$ of $\mathrm{GL}_n(\mathbb{A}_F)$, unique up to equivalence, such that*

$$L(s, \pi_v) = L(s, \pi_{1,v}) L(s, \pi_{2,v}) \cdots L(s, \pi_{r,v}), \tag{3.23}$$

for all places v. Consequently, $L(s, \pi) = L(s, \pi_1) L(s, \pi_2) \cdots L(s, \pi_r)$.

Such a sum $\pi_1 \boxplus \pi_2 \boxplus \cdots \boxplus \pi_r$ is called an *isobaric sum.* An automorphic representation is called *isobaric* if it can be written as a (finite) isobaric sum of cuspidal representations. The decomposition of an isobaric representation into cuspidal components is unique.

Theorem 3.4. (Strong Multiplicity One [JS]) *Let π and π' be isobaric automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$. If $\pi_v \simeq \pi'_v$ at almost all places, then in fact $\pi \simeq \pi'$.*

Now we state the *Langlands modularity conjecture*, also called the *strong Artin conjecture.*

Conjecture 3.2. *Let $\rho : G_F \to \mathrm{GL}_n(\mathbb{C})$ be a continuous Galois representation. Then there exists an isobaric automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_F)$ such that $L(s, \rho_v) = L(s, \pi_v)$ for almost all places v of F.*

If such a π exists, we will say that ρ is *modular*, or that ρ *corresponds to* π and write $\rho \leftrightarrow \pi$. Note that π is unique by Strong Multiplicity One. Whenever $L(s, \rho_v) = L(s, \pi_v)$, we say the *local representations* ρ_v and π_v *correspond*, and we write $\rho_v \leftrightarrow \pi_v$. The aforementioned corollary to Theorem 3.2 tells us that if ρ is irreducible, then π is cuspidal (cf. Proposition 2.2).

The isobaric sum is the automorphic analogue of the direct sum of Galois representations in the sense that if $\rho_i \leftrightarrow \pi_i$, $1 \leq i \leq r$, then $\rho_1 \oplus \rho_2 \oplus \cdots \oplus \rho_r \leftrightarrow \pi_1 \boxplus \pi_2 \boxplus \cdots \boxplus \pi_r$ by Theorem 3.3. Thus it suffices to prove modularity for irreducible ρ. Now suppose that ρ is an irreducible representation and $\rho \leftrightarrow \pi$. Then π must be cuspidal. Also, any local factor $L(s, \rho_v)$ or $L(s, \pi_v)$ is an entire function which is never zero. Hence $L(s, \rho)$ differs from $L(s, \pi)$ by a product of non-vanishing entire functions. Therefore $L(s, \rho)$ is entire by Theorem 3.1. Thus the strong Artin conjecture for ρ indeed implies the Artin conjecture for ρ.

Remarks.

(1) If $\rho \leftrightarrow \pi$, then it is actually true that $L(s, \rho_v) = L(s, \pi_v)$ at all *finite* places v. In other words, $L(s, \rho) = L(s, \pi_f)$ as Euler products (over F), where π_f denotes the restricted tensor product over all finite places $\otimes'_{v<\infty}\pi_v$ and $L(s, \pi_f) = \prod_{v<\infty} L(s, \pi_v)$. Moreover, the complete L-functions $L^*(s, \rho)$ and $L(s, \pi)$ are equal as Euler products over $\mathbb{Q}$. We give a proof of this, due to Ramakrishnan, in Appendix α.

(2) The strong Artin conjecture holds for $n = 1$, and this is essentially just abelian class field theory as worked out by Artin.

(3) Automorphic forms are generalizations of modular forms and Maass forms. For $n = 2$ and $F = \mathbb{Q}$, the modularity of ρ would mean that ρ corresponds, by the work of Deligne and Serre [DS], to either a modular form of weight 1 (if ρ is odd) or a Maass form with Laplacian eigenvalue $\frac{1}{4}$ (if ρ is even).

(4) The strong Artin conjecture really is stronger (given what one knows today) than the Artin conjecture. For example, monomial representations are not known to be modular in general. However, for a fixed $n = 2, 3$, the Artin and strong Artin conjectures are *globally* equivalent. That is, knowing Artin's conjecture for all n-dimensional representations is equivalent to knowing the strong Artin conjecture for all n-dimensional representations. Recently, Booker [Bo] showed for two-dimensional representations over $\mathbb{Q}$ that $L(s, \rho)$ is

entire if and only if ρ is modular. Whether these two conjectures are equivalent in higher dimensons (either globally or for a specific representation) is not currently known.
(5) The local Langlands correspondence for $\mathrm{GL}_n(\mathbb{C})$ was proven recently by Harris and Taylor [HT] and subsequently by Henniart [He]. This means that, if $\rho \leftrightarrow \pi$, for each place v we can attach a local irreducible admissible representation π_v to ρ_v. Thus ρ should correspond to $\pi = \otimes' \pi_v$. We know π is an irreducible, admissible representation but the difficulty lies is proving π is actually automorphic.

The first (post-Artinian) cases of the strong Artin conjecture are stated in the celebrated

Theorem 3.5. (Langlands [La2], Tunnell [Tu]) *Let F be a number field and let ρ be a two-dimensional complex Galois representation of G_F. If the image of ρ is solvable, then ρ is modular.*

The two main ingredients in the proof of this theorem, base change and the symmetric square lift, will be introduced in the next section.

3.3 Functoriality Results

The general philosophy of Langlands is captured in the idea of functoriality, which we will not describe in general but only in specific cases. Consider $H = \prod_i \mathrm{GL}_{n_i}$, $G = \mathrm{GL}_n$ as algebraic groups over F. Then ${}^L H^0 = \prod_i \mathrm{GL}_{n_i}(\mathbb{C})$ and ${}^L G^0 = \mathrm{GL}_n(\mathbb{C})$. Let r be an algebraic homomorphism (an *L-homomorphism*)

$$r : {}^L H = {}^L H^0 \times W_F \;\rightarrow\; {}^L G = {}^L G^0 \times W_F. \tag{3.24}$$

For an automorphic representation π of H unramified at a finite place v, there is an associated semisimple conjugacy class $A(\pi_v)$. We say r is *functorial* if, for any automorphic representation π of H, we can *transfer* π to an automorphic representation π' of G such that $A(\pi'_v) = r(A(\pi_v))$ at all finite unramified places. Langlands's *functoriality conjecture* states that all such r are functorial. We remark that by the local Langlands correspondence ([HT], [He]), if $\rho \leftrightarrow \pi$ then a functorial map r will specify $r(\pi)$ at all places by the correspondence $r(\rho)_v \leftrightarrow r(\pi)_v$.

In our special cases, we will consider a homomorphism $r : \prod_i \mathrm{GL}_{n_i}(\mathbb{C}) \rightarrow \mathrm{GL}_n(\mathbb{C})$ of the connected components of the L-groups. In general, the L-group is a semidirect product of

${}^LG^0$ with W_F, but here it is just a direct product. So r can be extended (in many ways) to a homomorphism of L-groups. In all of our examples below, excepting automorphic induction, we will extend r to ${}^LH$ by acting trivially on W_F. The idea of functoriality and why it is important should be made clear through the following examples.

3.3.1 Base Change

Key ideas in the proof of Theorem 3.5 relied on certain instances of *base change* for GL_2 (cf. Introduction). Let ρ be a Galois representation of G_F and let E/F be a finite extension. The idea is that if $\rho \leftrightarrow \pi$ then there should be an automorphic representation π_E of $\mathrm{GL}_n(\mathbb{A}_E)$, called the *base change of π to E*, such that $\rho_E \leftrightarrow \pi_E$. In terms of functoriality, we mean that the restriction of scalars map from $\mathrm{GL}_n(\mathbb{A}_F)$ to $\mathrm{GL}_n(\mathbb{A}_E) \simeq \mathrm{GL}_{n[E:F]}(\mathbb{A}_F)$ should be functorial. Langlands [La2] proved normal cyclic base change for GL_2, which was subsequently generalized to GL_n by Arthur and Clozel as described below.

Let $\mathfrak{N}_{E/F}$ denote the norm map from E to F.

Theorem 3.6. ([AC]) *Let E/F be a normal cyclic extension of prime degree. For each isobaric representation π of $\mathrm{GL}_n(\mathbb{A}_F)$, there exists a unique isobaric automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$ called the* base change *of π to E and denoted by π_E such that*

(i) (descent) a cuspidal representation Π of $\mathrm{GL}_n(\mathbb{A}_E)$ is the base change π_E of some π if and only if Π is Galois invariant (in particular, if $\rho_E \leftrightarrow \Pi$ where ρ is some representation of $\mathrm{Gal}(L/F)$);

(ii) if π' is also an isobaric representation of $\mathrm{GL}_n(\mathbb{A}_F)$ then $\pi_E = \pi'_E$ if and only if $\pi' = \pi \otimes \delta$ for some idèle class character δ of $\mathfrak{N}_{E/F}(C_E)\backslash C_F \simeq \mathrm{Gal}(E/F)$;

(iii) (compatibility with reciprocity) if ρ is a representation of $\mathrm{Gal}(L/F)$ such that $\rho \leftrightarrow \pi$ and ρ_E is modular, then $\rho_E \leftrightarrow \pi_E$; and

(iv) (compatibility with twisting) if χ is an idèle class character of F and the restriction $\chi_E = \chi \circ \mathfrak{N}_{E/F}$, then

$$(\pi \otimes \chi)_E = \pi_E \otimes \chi_E.$$

Note that, with the hypotheses of this theorem, if $p = [E : F]$, v is finite and unramified, and $\{a_1, a_2, \dots, a_n\}$ are the Satake parameters for π_v, then the Satake parameters for the local base change $\pi_{E,v}$ are $\{a_1^p, a_2^p, \ldots, a_n^p\}$.

The adjoint construction to base change is *automorphic induction* ([AC],[HH]), which corresponds to induction of Galois representations.

Theorem 3.7. ([AC], [HH]) *Let E/F be a normal cyclic extension of prime degree p and π an automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$. Then there exists an automorphic representation $\mathrm{I}_{\mathrm{E}}^{\mathrm{F}}\pi$ of $\mathrm{GL}_{np}(\mathbb{A}_F)$, called the* automorphic induction *of π, such that $L(s,\pi) = L(s, \mathrm{I}_{\mathrm{E}}^{\mathrm{F}}\pi)$ as Euler products over F. In particular, if ρ is a Galois representation of G_F such that $\rho \leftrightarrow \pi$, then $\mathrm{Ind}_E^F\rho \leftrightarrow \mathrm{I}_{\mathrm{E}}^{\mathrm{F}}\pi$.*

Jacquet, Piatetski-Shapiro and Shalika proved the following non-normal case of base change and automorphic induction.

Theorem 3.8. ([JPSS]) *Let E/F be a non-normal cubic extension of number fields. Let χ be an idèle class character of E and π an automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$. Then the automorphic induction $\mathrm{I}_{\mathrm{E}}^{\mathrm{F}}\chi$ exists as an automorphic representation of $\mathrm{GL}_3(\mathbb{A}_F)$ and the base change π_E exists as an automorphic representation of $\mathrm{GL}_2(\mathbb{A}_E)$.*

3.3.2 Symmetric and Exterior Powers

Let $\rho : G_F \to \mathrm{GL}_n(\mathbb{C})$ be a continuous representation and let $r : \mathrm{GL}_n \to \mathrm{GL}_m$ be a symmetric or exterior power lifting. Then the local L-factors for the representation $r(\rho) : G_F \to \mathrm{GL}_m(\mathbb{C})$ will just be $L(s, r(\rho)_v) = \det(I - r(\rho(\mathrm{Fr}_v))q_v^{-s})^{-1}$ for finite unramified v. Analogously, at finite unramified v, one defines local automorphic L-factors

$$L(s, \pi_v; r) = \frac{1}{\det(I - r(A(\pi_v))q_v^{-s})} \tag{3.25}$$

for automorphic representations π of $\mathrm{GL}_n(\mathbb{A}_F)$. In this setting, the functoriality conjecture says that one should be able to lift each π to an automorphic representation $r(\pi)$ of $\mathrm{GL}_m(\mathbb{A}_F)$ such that $L(s, r(\pi)_v) = L(s, \pi_v; r)$ at all unramified places v. In particular if $\rho \leftrightarrow \pi$, then $r(\rho) \leftrightarrow r(\pi)$.

Theorem 3.9. *The following maps are functorial:*

(1) $\mathrm{Sym}^2 : \mathrm{GL}_2 \to \mathrm{GL}_3$ ([GeJ]),

(2) $\mathrm{Sym}^3 : \mathrm{GL}_2 \to \mathrm{GL}_4$ ([KS]),

(3) $\mathrm{Sym}^4 : \mathrm{GL}_2 \to \mathrm{GL}_5$ ([Ki]),

(4) $\Lambda^2 : \mathrm{GL}_4 \to \mathrm{GL}_6$ ([Ki], [He2]).

We remark that Kim [Ki] proved the exterior square $\Lambda^2(\pi)$ equals an automorphic representation of $\mathrm{GL}_6(\mathbb{A}_F)$ at all places, except possibly those above 2 and 3. This is actually enough for our purposes, since we only consider a correspondence at almost all places. However, we note that Henniart [He2] indicated how he can prove equality at the remaining places in question in a letter to Kim and Shahidi.

3.3.3 Tensor Products

Fix positive integers m and n. Let π_1 (resp. π_2) be an automorphic representation of $\mathrm{GL}_m(\mathbb{A}_F)$ (resp. $\mathrm{GL}_n(\mathbb{A}_F)$). Let r be the tensor product map $\otimes : \mathrm{GL}_m \times \mathrm{GL}_n \to \mathrm{GL}_{mn}$ of F-groups. Functoriality of r means that there is an automorphic representation $\pi_1 \otimes \pi_2$ of $\mathrm{GL}_{mn}(\mathbb{A}_F)$ such that $L(s, (\pi_1 \otimes \pi_2)_v) = L(s, \pi_{1,v} \times \pi_{2,v})$ for all v. In particular if ρ_1 and ρ_2 are Galois representations such that $\rho_i \leftrightarrow \pi_i$, then $\rho_1 \otimes \rho_2 \leftrightarrow \pi_1 \otimes \pi_2$. When $m = 1$, functoriality is known because if π is automorphic, then so is $\pi \otimes \chi$ for any (unitary) character χ of $F^\times \backslash \mathbb{A}_F^\times$.

Theorem 3.10. *The following maps are functorial:*

(1) $\otimes : \mathrm{GL}_2 \times \mathrm{GL}_2 \to \mathrm{GL}_4$ ([Ra])

(2) $\otimes : \mathrm{GL}_2 \times \mathrm{GL}_3 \to \mathrm{GL}_6$ ([KS]).

3.3.4 Applications to the Modularity Conjecture

Modularity holds for the following kinds of Galois representations:

(1) the direct sum of modular representations

(2) the induction of a modular representation from a normal subgroup

(3) the induction of a one-dimensional representation from a non-normal subgroup of index 3

(4) $\mathrm{Sym}^m(\rho)$ for two-dimensional modular ρ, $m = 2, 3, 4$

(5) $\Lambda^2(\rho)$ for four-dimensional modular ρ

(6) the tensor product of two modular representations of dimensions 2 and 2, or 2 and 3

(7) two-dimensional representations with solvable image

(8) two-dimensional icosahedral representations over $\mathbb{Q}$ satisfying certain ramification criteria

(9) the Asai lift of a modular two-dimensional representation

(10) irreducible representations into $\mathrm{GO}_4(\mathbb{C})$ with solvable image

(11) representations with nilpotent images.

Note that (1)–(6) are direct applications of functoriality mentioned in the preceding sections. Case (7) is the aforementioned result of Langlands and Tunnell. Under case (8), various kinds of icosahedral representations have been shown to be modular. The works of Buzzard, Dickinson, Sheppard-Barron and Taylor [BDST]; Taylor [Ta]; and Goins [Goi] prove modularity for infinite families of icosahedral representations. For other approaches and examples, see also the works of Buhler [Bu]; Kiming and Wang [KW]; Jehanne and Müller [JM]; and Buzzard and Stein [BS]. Cases (9) and (10) were done by Ramakrishnan in [Ra2]. Case (11), which we state precisely in Proposition 9.1, was deduced from base change results by Arthur and Clozel [AC]. We emphasize that in all these instances, except perhaps (8), the proof relies heavily on some type of functoriality.

Combining cases (8) and (4) gives examples of modular representations with non-solvable image in dimensions 3, 4 and 5. For primitive three-dimensional examples, see Section 8.1. Primitive four-dimensional examples are constructed using Sym^3 in Section 10 of [KS]. A five-dimensional example induced from a non-normal quintic extension is constructed using Sym^4 in [Ki2].

We remark that essentially the only cases where the Artin conjecture is known but the strong Artin conjecture is not are for the following types of Galois representations:

(i) the induction of a modular representation from a non-normal subgroup

(ii) the tensor product of two modular representations.

Case (ii) follows from what is known about the Rankin-Selberg product of L-functions. In particular, if ρ_1, ρ_2 and ρ_3 are non-trivial, irreducible, modular representations of dimensions n_1, n_2 and n_3, where n_1 is 2 and n_2 is 2 or 3, then the triple product $\rho_1 \otimes \rho_2 \otimes \rho_3$ has an entire L-function by case (ii) and Theorem 3.10 above.

Chapter 4

The Method

Let $\rho : G_F \to \mathrm{GL}_4(\mathbb{C})$ be a continuous representation with solvable image. We are concerned with the following question: How can we show ρ is modular? Here we will describe our basic approach to this problem.

First recall that ρ has finite image by continuity, so $L = \overline{F}^{\ker \rho}$ is a finite Galois extension of F. Thus ρ factors through a representation of $\mathrm{Gal}(L/F)$ with the same L-function. Hence sometimes we may view ρ as a faithful representation of $\mathrm{Gal}(L/F)$ into $\mathrm{GL}_4(\mathbb{C})$.

Let $\overline{\rho} : G_F \to \mathrm{PGL}_4(\mathbb{C})$ be the composition of ρ with the standard projection from $\mathrm{GL}_4(\mathbb{C})$ to $\mathrm{PGL}_4(\mathbb{C})$.

$$\begin{array}{ccc} G_{\mathbb{Q}} & \xrightarrow{\rho} & \mathrm{GL}_4(\mathbb{C}) \\ & \overset{\overline{\rho}}{\searrow} & \downarrow \\ & & \mathrm{PGL}_4(\mathbb{C}) \end{array}$$

Let G denote the image of ρ and $\overline{G}$ denote the image of $\overline{\rho}$. As in the GL_2 case, we will classify ρ by its projective image $\overline{G}$. To determine the possible cases for $\overline{G}$, we will use the classification for certain subgroups of $\mathrm{GL}_4(\mathbb{C})$ and the group theory recalled in Chapter 2.

Note that, conversely, for each finite solvable subgroup $\overline{G}$ of $\mathrm{PGL}_4(\mathbb{C})$, there is a representation $\rho : G_F \to \mathrm{GL}_4(\mathbb{C})$ with projective image $\overline{G}$ for some number field F. In fact, we may even take $F = \mathbb{Q}$. This is because $\overline{G}$ is the image of some finite solvable subgroup G of $\mathrm{GL}_4(\mathbb{C})$. A (non-constructive) theorem of Shafarevich (Theorem 9.5.1 of [NSW]) says that if G is solvable, then there is a finite extension $L/\mathbb{Q}$ such that $G = \mathrm{Gal}(L/\mathbb{Q})$. Thus $\mathrm{Gal}(L/\mathbb{Q})$ (and hence $G_{\mathbb{Q}}$) has a representation ρ into $\mathrm{GL}_4(\mathbb{C})$ with projective image $\overline{G}$.

We may restrict the problem of classifying subgroups of $\mathrm{GL}_4(\mathbb{C})$ to the corresponding problem for $\mathrm{SL}_4(\mathbb{C})$. The reason for this is as follows. Consider a subgroup G of $\mathrm{GL}_4(\mathbb{C})$ of order n. Write $\zeta_m = e^{2\pi i/m}$. Let $Z = \langle \zeta_{4n} I \rangle$ be the cyclic group of scalar matrices of order $4n$ inside $\mathrm{GL}_4(\mathbb{C})$. Set $G' = ZG$. Since we have only added scalar matrices to get G', then $\overline{G'} = \overline{G}$. Given any $g \in G$, $\det g = \zeta_n^k$ for some k. Let $z = \zeta_{4n}^{-k}$. Then

$$\det zg = \left(\zeta_{4n}^{-k}\right)^4 \det g = \zeta_n^{-k} \zeta_n^k = 1, \tag{4.1}$$

i.e., $zg \in \mathrm{SL}_n(\mathbb{C})$. Now let $H = G' \cap \mathrm{SL}_4(\mathbb{C})$. Then any $g \in G'$ can be written as zh where $z \in Z$ and $h \in H$. Therefore $\overline{H} = \overline{G'} = \overline{G}$, and $G \subseteq ZH = G'$. Note the G is irreducible if and only if G' is irreducible, which is if and only if H is irreducible. By the same logic, we also have that G is primitive if and only if H is primitive. So to determine the possibilities for the projective image of irreducible or primitive G, it suffices to assume that G is unimodular, i.e., that G is contained in $\mathrm{SL}_4(\mathbb{C})$.

As an aside which we shall not have cause to use later (but we will morally use for the A_5 case in Section 8.1), we observe that in fact G' is both a central product of Z with G and of Z with H. For example, let $C = Z(G)$ be the center of G. Then $C \subseteq Z$ and $G' = Z \times_C G$. Let ρ be the standard representation of G. Then the standard representation of G' is some twist $\rho' = \chi \otimes \rho$, where χ is a character of Z. (More precisely, the tensored representation $\chi \otimes \rho$ of $Z \times G$ factors through the representation ρ' of $G' = Z \times_C G$.) Hence the representation ρ of G transfers to a representation ρ'_H of H into $\mathrm{SL}_4(\mathbb{C})$ which is irreducible (primitive) if ρ is.

4.1 The Reducible Case

Before moving on to the primary (irreducible) case, let us first remark on modularity of reducible four-dimensional representations of solvable type. Consider a reducible Galois representation $\rho : G_F \to \mathrm{GL}_4(\mathbb{C})$ where F is some number field. Say ρ is a direct sum of r irreducible representations $\tau_1, \ldots, \tau_r$. Then ρ is modular if and only if each τ_i is. Any representation τ_i which is one-dimensional is modular. If we require that ρ have solvable image, then so does each τ_i. So in this case any two-dimensional τ_i is modular by the work of Langlands and Tunnell (Theorem 3.5). Thus the only interesting reducible case of solvable type is when ρ decomposes into a character χ and an irreducible three-dimensional representation τ. Then we are reduced to the modularity problem for τ.

Let $\tau : G_F \to \mathrm{GL}_3(\mathbb{C})$ be an irreducible Galois representation of solvable type. Recall that if τ is monomial, then τ is modular by Theorem 3.8. Note that if τ is essentially self-dual, then the image of τ lies in $\mathrm{GO}_3(\mathbb{C})$. So Proposition 1.3 of [Ra2] tell us that τ must be monomial. Hence we may further assume that τ is a primitive three-dimensional representation (which, for solvable type, is necessarily non-essentially-self-dual). In Chapter 8, we examine the possible cases of such three-dimensional representations. There are no primitive three-dimensional τ of solvable type which are known to be modular (or to satisfy Artin's conjecture). However, in certain cases, we are able to show that the associated eight-dimensional (self-dual) adjoint representation $\mathrm{Ad}(\tau)$ is modular.

4.2 The Irreducible Case

We will now suppose that ρ is irreducible. Any scalar matrix in G lies in the center $Z(G)$ of G. On the other hand, Schur's lemma implies that the center of G is contained inside the scalar matrices. Therefore $\overline{G} = G/Z(G)$.

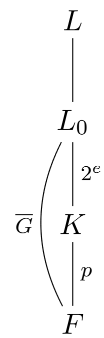
As outlined in the introduction, our proofs of modularity will be similar to Langlands's proof of the tetrahedral case. However whereas Langlands used the symmetric square lift $\mathrm{Sym}^2 : \mathrm{GL}_2 \to \mathrm{GL}_3$ of Gelbart and Jacquet, we use the exterior square lift $\Lambda^2 : \mathrm{GL}_4 \to \mathrm{GL}_6$ of Kim and Henniart. Specifically, we can show the following.

Proposition 4.1. *Let* $\rho : \mathrm{Gal}(L/F) \to \mathrm{GL}_4(\mathbb{C})$ *be an irreducible continuous representation with projective image* $\overline{G}$*. Suppose that* $|\overline{G}| = 2^e p$ *for some* $e \geq 0$ *and an odd prime* p*, and that* $\overline{G}$ *has no elements of order* $2p$*. Further suppose there is an automorphic representation* π *on* $\mathrm{GL}_4(\mathbb{A}_F)$ *such that:*

(1) $\Lambda^2(\rho) \leftrightarrow \Lambda^2(\pi)$*; and*

(2) for some degree p *subextension* K/F *of the field* L_0 *cut out by* $\overline{\rho}$*, the base change* π_K *exists and* $\rho_K \leftrightarrow \pi_K$*.*

Then ρ *is modular.*



Remark. As all the degree p subextensions K/F of L_0 correspond to 2-Sylow subgroups of $\overline{G}$, they are all conjugate.

Proof. It suffices to show $L(s,\rho_v) = L(s,\pi_v)$ for almost all places v. Hence we will assume v is a finite place such that both ρ_v and π_v are unramified. Then we can write down Frobenius eigenvalues $\{a,b,c,d\}$ for ρ_v and Satake parameters $\{e,f,g,h\}$ for π_v. We want to show

$$\{a,b,c,d\} = \{e,f,g,h\}\,. \tag{4.2}$$

As $\Lambda^2(D) = 1$ implies that $D = \pm I$ for a diagonal matrix D, condition (1) implies that

$$\{a,b,c,d\} = \pm\{e,f,g,h\}\,. \tag{4.3}$$

If these sets are equal, then we are done. So we may as well assume that

$$\{a,b,c,d\} = -\{e,f,g,h\}\,. \tag{4.4}$$

Now we claim that the p-th powers of these sets are also equal. To see this, consider the p-th power of the Frobenius, $\mathrm{Fr}_v^p \in \mathrm{Gal}(L/F)$. For any $x \in \overline{G}$, $x^{2^e p} = 1$ so x^p is of 2-power order. Thus x^p lies inside some 2-Sylow subgroup of $\overline{G}$. By taking an appropriate conjugate of Fr_v, we may assume that $\mathrm{Fr}_v^p \in \mathrm{Gal}(L/K)$. Then $\rho(\mathrm{Fr}_v^p) = \rho_K(\mathrm{Fr}_v^p)$. By Theorem 3.6(iii), we know $\rho_K \leftrightarrow \pi_K$, which implies

$$\{a^p,b^p,c^p,d^p\} = \{e^p,f^p,g^p,h^p\}\,. \tag{4.5}$$

Without losing generality we may assume that $a = \zeta_p e$ and $e = -b$. Then $a = -\zeta_p b$ and

$$\rho(Fr_v) \sim \begin{pmatrix} -\zeta_p b & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix}. \tag{4.6}$$

Therefore

$$\overline{\rho}(Fr_v) \sim \begin{pmatrix} -\zeta_p & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{c}{b} & 0 \\ 0 & 0 & 0 & \frac{c}{a} \end{pmatrix} \tag{4.7}$$

is an element of order divisible by $2p$ inside $\overline{G}$. Contradiction! □

In the case that the extension K/F of degree p is *normal*, things are a bit simpler as we will now show. (In fact, this is the only case where we know condition (2) of the above proposition is satisfied.)

Proposition 4.2. *Let $\rho : G_F \to \mathrm{GL}_4(\mathbb{C})$ be an irreducible continuous representation with projective image $\overline{G}$. Suppose that $|\overline{G}| = 2^e p$ for some $e \geq 0$ and an odd prime p, and that $\overline{G}$ has no elements of order $2p$. Further suppose $\overline{G}$ has a normal 2-Sylow subgroup $\overline{H}$. Let H be the pre-image of $\overline{H}$ in G_F. Then ρ is modular if $\Lambda^2(\rho)$ and ρ_H are modular.*

Proof. Let K/F be the subextension of L corresponding to H. Appealing to the previous proposition, we only need to show $\Lambda^2(\rho) \leftrightarrow \pi$ and $\rho_K \leftrightarrow \pi_K$ for some automorphic representation π of $\mathrm{GL}_4(\mathbb{A}_F)$. We know ρ_K corresponds to some automorphic representation Π of $\mathrm{GL}_4(\mathbb{A}_K)$.

Now we claim that Π is cuspidal, i.e., that ρ_K is irreducible. Suppose instead that ρ_K is a sum of irreducibles $\tau_1 \oplus \cdots \oplus \tau_r$ with $r > 1$. Then K/F acts transitively on the τ_i's, so there must be $r = p$ of them. On the other hand, Clifford's theorem says they all have the same dimension m; so $mp = 4$. But p is assumed to be prime to 2. Contradiction.

The extension K/F is normal and Π is cuspidal. By parts (i) and (ii) of Theorem 3.6, there are p automorphic representations $\pi_0, \pi_1, \ldots, \pi_{p-1}$ of $\mathrm{GL}_4(\mathbb{A}_F)$ such that $\pi_{i,K} \simeq \Pi$.

Furthermore, up to reordering, we may write

$$\pi_i \simeq \pi_0 \otimes \delta^i, \tag{4.8}$$

where δ is an idèle class character of K/F.

Note that δ is a character of order p. Therefore the representations

$$\Lambda^2(\pi_i) \simeq \Lambda^2(\pi_0 \otimes \delta^i) \simeq \Lambda^2(\pi_0) \otimes \delta^{2i} \tag{4.9}$$

are all distinct because they have distinct central characters

$$\omega_{\Lambda^2(\pi_i)} = \omega_{\Lambda^2(\pi_0)}\delta^{2i}. \tag{4.10}$$

As δ_K is trivial, they all base change to $\Lambda^2(\pi_0)_K$ by Theorem 3.6(iv). Let β be the automorphic representation over F which is associated to $\Lambda^2(\rho)$. Because $\Lambda^2(\rho)_K \leftrightarrow \Lambda^2(\pi_0)_K$, we also know that

$$\beta_K \simeq \Lambda^2(\pi_0)_K. \tag{4.11}$$

However there are only p distinct automorphic representations which base change to $\Lambda^2(\pi_0)_K$, so we must have

$$\beta \simeq \Lambda^2(\pi_i) \tag{4.12}$$

for some i. This π_i is our desired π. $\square$

Another way we can demonstrate that ρ is modular is to replace the hypothesis that $\Lambda^2(\rho) \leftrightarrow \Lambda^2(\pi)$ in Proposition 4.1 with the condition that $\rho_M \leftrightarrow \pi_M$ for a quadratic extension M/F. This kind of argument is given in Section 6.2.

Chapter 5

Solvable Primitive Subgroups of $\mathrm{GL}_4(\mathbb{C})$

We are interested in (irreducible) primitive, solvable, finite subgroups G of $\mathrm{SL}_4(\mathbb{C})$. Such groups, including the non-solvable ones, were classified by Blichfeldt ([Bl]) in the early 1900's. However, Blichfeldt's list is presented in terms of generating matrices and geometrical invariants, which is not the most convenient form for us. We remark that Feit ([Fe]) presented a partial listing of such groups in a more modern form, but Feit's list is also not quite sufficient for us.

Thus we will rework, in large part, the classification of such solvable groups. Following a suggestion of D. Wales, we will study G by determining the possibilities for its Fitting subgroup. Then we follow methods of Blichfeldt to finish the classification we want. In particular we will show that G is either orthogonal or symplectic, and in the latter case there are precisely five possibilities for G.

5.1 The Fitting Subgroup

Let G be a primitive, solvable, finite subgroup of $\mathrm{SL}_4(\mathbb{C})$. Let ρ be the standard representation of G on $\mathbb{C}^4$.

Proposition 5.1. *Any normal abelian subgroup of G is cyclic of order 1, 2 or 4, and is contained in $Z(G)$. In particular, $|Z(G)|$ is 1, 2 or 4.*

Proof. Let A be a normal abelian subgroup of G. Then ρ_A is a sum of isomorphic characters $\chi \oplus \chi \oplus \chi \oplus \chi$ by Clifford's theorem. Then $A = \rho(A)$ is a subgroup of scalar matrices in G, whence central. The only unimodular scalar 4×4 matrices are $\pm I$ and $\pm iI$, so A must be cyclic of order 1, 2 or 4. □

Corollary 5.1. *Any characteristic abelian subgroup of $F(G)$ is cyclic of order dividing 4 and contained in $Z(G)$.*

Proof. Let A be an abelian characteristic subgroup of $F(G)$. Then $A \trianglelefteq G$, whereby Proposition 5.1 applies. □

Proposition 5.2. *The Fitting subgroup $F(G)$ is a 2-group.*

Proof. Since $F(G)$ is nilpotent, its Sylow subgroups $\mathrm{Syl}_p(F(G))$ are normal and $F(G)$ is a direct product of them. Let p be any odd prime and $P = \mathrm{Syl}_p(F(G))$. Any automorphism of $F(G)$ must leave P invariant, i.e., P is characteristic in $F(G)$, whence normal in G. Then by Clifford's theorem, ρ_P is a direct sum of irreducibles of the same degree. The degrees of the irreducible representations of P are powers of p. Thus ρ_P is a direct sum of characters. As ρ is faithful, P must be abelian. Hence P is trivial by Corollary 5.1. □

Proposition 5.3. *The restricted representation $\rho_{F(G)}$ is irreducible.*

Proof. Suppose otherwise, i.e., suppose that $\rho_{F(G)} = \tau_1 \oplus \cdots \oplus \tau_r$ for $r > 1$. By Clifford's theorem, either $r = 2$ and each τ_i is two dimensional, or $r = 4$ and each τ_i is one dimensional. Let $Q = G/F(G)$. Because ρ is irreducible, Q acts transitively on $\{\tau_i\}$. Let Q_0 be the stabilizer in Q of τ_1 and let G_0 be the pre-image of Q_0 in G. Note $[G : G_0] = [Q : Q_0] = r$. Also, by definition of Q_0, ρ_{G_0} decomposes as a direct sum of r representations, each of dimension $\frac{4}{r}$. Therefore, ρ is induced from G_0 by one of the components of ρ_{G_0}, contradicting primitivity. □

Proposition 5.4. *Either $F(G) = E$ or $F(G) = E \circ C_4$, where E is extraspecial of order 32.*

Proof. Write $F(G)$ as a central product of two groups E and R as in Proposition 2.6. First we show that R must be cyclic. Suppose otherwise, i.e., that $R = D_{2^m}$, SD_{2^m} or Q_{2^m} for some $m \geq 4$. Then R contains a cyclic subgroup $X = \langle x \rangle$ of order 2^{m-1}. By Proposition 2.4, every element of $R/\Phi(R)$ has order 2, so $\Phi(R)$ must contain the group $\langle x^2 \rangle$ of order 2^{m-2}. On the other hand, $\Phi(R)$ is contained in X because X is maximal, and this containment must be strict as $\Phi(R)$ is the intersection of all maximal subgroups of R. Thus $\Phi(R) = \langle x^2 \rangle$.

Clearly, $\Phi(R) \subseteq \Phi(F(G))$. Since $Z(E) = \Phi(E) = C_2$, the quotient E/C_2 is elementary abelian. Therefore $(E \times R)/(C_2 \times \Phi(R))$ is elementary abelian, from which $(E \circ R)/\Phi(R)$ is also elementary abelian. Hence Proposition 2.4 yields that $\Phi(R) = \Phi(F(G))$. So $\Phi(R)$ is a characteristic abelian subgroup of $F(G)$ of order 2^{m-2}. By Corollary 5.1, we must have that $m = 4$ and $\Phi(R)$ is a central subgroup of G (and R) of order 4. In particular $|Z(R)| \geq 4$. But it is apparent from the definitions of D_{2^m}, SD_{2^m} and Q_{2^m} that $|Z(R)| = 2$. Contradiction! Wherefore R is cyclic.

If R is cyclic of order $2^m > 4$, then it would be the only such subgroup of $F(G)$, whence characteristic in $F(G)$. Again Corollary 5.1 contradicts this. Thus R is cyclic of order 1, 2 or 4. Note that $E \circ C_2 = E$, so we may assume $|R| = 1$ or 4. It now suffices to show $|E| = 32$. By Proposition 5.3, we know that $\rho_{F(G)}$ is irreducible. Thus ρ_E is also an irreducible four-dimensional representation. Proposition 2.5 then requires E to be of order 32. □

There are two extraspecial groups of order 32, which we denote by Q_8D_8 and Q_8^2. The first is the central product of Q_8 by D_8, and the second is the central product of Q_8 by itself.

Proposition 5.5. $Q_8^2 \circ C_4 = Q_8D_8 \circ C_4$.

Proof. Observe that for any finite group H and a cyclic group C, the central product $H \circ C$ is uniquely defined up to isomorphism.

We first claim that $Q_8 \circ C_4 = D_8 \circ C_4$. Write

$$Q_8 = \langle i, j : i^4 = j^4 = (ij)^4 = 1, iji^{-1} = j^{-1} \rangle \tag{5.1}$$

and

$$D_8 = \langle \sigma, \tau : \sigma^4 = \tau^2 = 1, \tau\sigma\tau = \sigma^{-1} \rangle. \tag{5.2}$$

Let $C_4 = \langle z \rangle$ where $z^4 = 1$ and z commutes with Q_8 and D_8. Define a map

$$\phi : Q_8 \circ C_4 \to D_8 \circ C_4 \tag{5.3}$$

by first setting

$$\phi(z) = z, \;\; \phi(i) = \sigma, \;\; \phi(j) = z\tau. \tag{5.4}$$

Looking at the relations for Q_8 and D_8, one sees that ϕ extends to a homomorphism of $Q_8 \circ C_4$ into $D_8 \circ C_4$, which is actually an isomorphism.

Then

$$Q_8^2 \circ C_4 = Q_8 \circ (Q_8 \circ C_4) = Q_8 \circ (D_8 \circ C_4) = Q_8 D_8 \circ C_4. \tag{5.5}$$

(Note the central product is associative here because C_4 is cyclic.) □

This implies there are only three cases for $F(G)$: Q_8D_8, Q_8^2 and $Q_8^2 \circ C_4$. We will write $Q_8^2 \circ C_4$ simply as $Q_8^2C_4$.

5.2 The Structure of G

Now we will be able to use this information about the Fitting subgroup to determine the structure of G. Specifically, G acts by conjugation on $F(G)$, which gives us a homomorphism

$$\phi : G \to \mathrm{Aut}(F(G)) \tag{5.6}$$

with kernel $Z(F(G))$. The image of $F(G)$ under this map ϕ will then be the inner automorphism group $\mathrm{Inn}(F(G))$. Thus $G/F(G)$ is isomorphic to a subgroup of the outer automorphism group $\mathrm{Out}(F(G)) = \mathrm{Aut}(F(G))/\mathrm{Inn}(F(G))$. The outer automorphism groups of extraspecial groups are known. Specifically, $\mathrm{Out}(Q_8D_8) \simeq A_5$, $\mathrm{Out}(Q_8^2) \simeq \mathrm{O}_4^+(\mathbb{F}_2)$ is the group of order 72 contained in S_6, and $\mathrm{Out}(Q_8^2C_4) \simeq \mathrm{Sp}_4(\mathbb{F}_2) \times C_2$ is of order 1440. However we will only use the precise structure of $\mathrm{Out}(F(G))$ at the end to simplify the proof of Proposition 5.7. Now, following the methods of Blichfeldt ([Bl]), we will embed $G/F(G)$ inside $S_6 \simeq \mathrm{Sp}_4(\mathbb{F}_2)$.

As an extraspecial group of order 32 has a unique irreducible four-dimensional representation, we can write down a specific matrix representation for it, up to a change of basis.

Let

$$B_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$B_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad B_4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

and $B_5 = iI$ be linear transformations of $V = \mathbb{C}^4$ with respect to a basis $\{v_1, v_2, v_3, v_4\}$. Then

$$Q_8^2 \simeq \langle B_1, B_2, B_3, B_4 \rangle \tag{5.7}$$

and

$$Q_8^2 C_4 \simeq \langle B_1, B_2, B_3, \ldots, B_5 \rangle. \tag{5.8}$$

Let $X = \langle B_1, B_2, B_3, \ldots, B_5 \rangle$. Since $C_4 \subseteq Z(Q_8^2 C_4)$ and $Q_8^2 C_4$ is irreducible, there is only one way to extend Q_8^2 to $Q_8^2 C_4$, and that is by adjoining B_5. Thus we may assume $Q_8^2 C_4$ is represented by X. (Note that $Q_8^2 C_4$ actually has two inequivalent irreducible four-dimensional representations, but one may be obtained from the other by interchanging B_5 with $-B_5$.) By Proposition 5.5, $Q_8 D_8$ is also contained in $X = Q_8^2 C_4$. So in any case, we may assume that $F(G) \subseteq X$.

Note that one matrix representation of Q_8D_8 is given by the group generated by

$$C_1 = B_1B_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad C_2 = B_1B_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

$$C_3 = B_3B_5 = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{pmatrix}, \quad C_4 = B_4B_5 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}.$$

Let $\{v_{ij} = v_i \otimes v_j - v_j \otimes v_i\}_{1 \leq i < j \leq 4}$ be a basis for the six-dimensional vector space $\Lambda^2(V)$. Then $\Lambda^2(\rho)$ provides an action of G on $\Lambda^2(V)$. In particular, the B_i's act as follows:

$\cdot$	v_{12}	v_{13}	v_{14}	v_{23}	v_{24}	v_{34}
$\Lambda^2(B_1)$	v_{12}	$-v_{13}$	$-v_{14}$	$-v_{23}$	$-v_{24}$	v_{34}
$\Lambda^2(B_2)$	$-v_{12}$	$-v_{13}$	v_{14}	v_{23}	$-v_{24}$	$-v_{34}$
$\Lambda^2(B_3)$	$-v_{12}$	v_{24}	v_{23}	v_{14}	v_{13}	$-v_{34}$
$\Lambda^2(B_4)$	v_{34}	$-v_{13}$	$-v_{23}$	$-v_{14}$	$-v_{24}$	v_{12}
$\Lambda^2(B_5)$	$-v_{12}$	$-v_{13}$	$-v_{14}$	$-v_{23}$	$-v_{24}$	$-v_{34}$

Consider the basis $\{w_i\}$ for $\Lambda^2(V)$, where

$$w_1 = v_{12} + v_{34},\ \ w_2 = v_{12} - v_{34},\ \ w_3 = v_{13} - v_{24},$$

$$w_4 = v_{13} + v_{24},\ \ w_5 = v_{14} + v_{23},\ \ w_6 = v_{14} - v_{23}.$$

Let σ denote the composition of $\Lambda^2(\rho)$ with the change of basis from $\{v_i\}$ to $\{w_i\}$. Then the B_i's act on the w_j's as follows:

$\cdot$	w_1	w_2	w_3	w_4	w_5	w_6
$\sigma(B_1)$	w_1	w_2	$-w_3$	$-w_4$	$-w_5$	$-w_6$
$\sigma(B_2)$	$-w_1$	$-w_2$	$-w_3$	$-w_4$	w_5	w_6
$\sigma(B_3)$	$-w_1$	$-w_2$	$-w_3$	w_4	w_5	$-w_6$
$\sigma(B_4)$	w_1	$-w_2$	$-w_3$	$-w_4$	$-w_5$	w_6
$\sigma(B_5)$	$-w_1$	$-w_2$	$-w_3$	$-w_4$	$-w_5$	$-w_6$

Thus the basis $\{w_i\}$ diagonalizes X, proving that

$$\sigma(F(G)) \subseteq \mathrm{Diag}(\mathrm{GL}_6(\mathbb{C})). \tag{5.9}$$

Using the table above, we can also compute how the C_i's act on the w_j's:

$\cdot$	w_1	w_2	w_3	w_4	w_5	w_6
$\sigma(C_1)$	$-w_1$	$-w_2$	w_3	w_4	$-w_5$	$-w_6$
$\sigma(C_2)$	$-w_1$	$-w_2$	w_3	$-w_4$	$-w_5$	w_6
$\sigma(C_3)$	w_1	w_2	w_3	$-w_4$	$-w_5$	w_6
$\sigma(C_4)$	$-w_1$	w_2	w_3	w_4	w_5	$-w_6$

Proposition 5.6. *$G/F(G)$ embeds into S_6.*

Proof. Looking at the columns in the multiplication tables for the $\sigma(B_i)$'s and the $\sigma(C_i)$'s above, we observe that $\sigma_{F(G)}$ decomposes into six distinct characters. Thus $\Lambda^2(V) = \bigoplus W_i$, where each $W_i = \langle w_i \rangle$ is an eigenspace for $F(G)$ with a distinct character. By Clifford's theorem, $G/F(G)$, acting by σ, permutes the six W_i. This permutation induces a map $\sigma' : G \to S_6$. It remains to show that $\ker \sigma' = F(G)$. Let $H = \ker \sigma'$. Clearly $F(G) \subseteq H$. Note that $\sigma(H) \subseteq \mathrm{Diag}(\mathrm{GL}_6(\mathbb{C}))$, whence $\sigma(H)$ is abelian. Since $\ker \sigma = \{\pm I\}$, we have that $H/\{\pm I\}$ is abelian with $\{\pm I\} \subseteq Z(H)$.

Recall that a finite group is nilpotent if and only if each Sylow subgroup is normal. Hence if C is a central subgroup of a finite group N and N/C is nilpotent, then N is nilpotent. In particular, H is nilpotent. As H is a normal nilpotent subgroup of G, we must have $H \subseteq F(G)$. Thus $F(G) = H = \ker \sigma'$. $\square$

Now define

$$S = \frac{1+i}{\sqrt{2}} \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad T = \frac{1+i}{2} \begin{pmatrix} -i & 0 & 0 & i \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & -i & i & 0 \end{pmatrix},$$

and $J = \langle F(G), S, T\rangle$. We may extend σ to J so that $\sigma(S)$ and $\sigma(T)$ act as the permutations $(w_1 w_2)$ and $(w_2 w_4 w_6 w_3 w_5)$ respectively. Because $\langle \sigma(S), \sigma(T)\rangle = S_6$, Proposition 5.6 implies that J is the largest finite subgroup of $\mathrm{SL}_4(\mathbb{C})$ which normalizes $F(G)$. Thus $G \leq J$.

Let F_{20} be the group of order 20 contained in S_5, and let

$$E = Q_8 D_8 \ , \ \ EC_4 = Q_8^2 C_4. \tag{5.10}$$

Note that F_{20} is the Frobenius group of order 20. (Recall that a group F is a Frobenius group if F acts transitively on a set S such that (i) each $x \in F - \{1\}$ fixes at most one element of S and (ii) at least one $x \in F - \{1\}$ fixes an element of S.) It contains D_{10} as a normal subgroup.

Proposition 5.7. *If $G/F(G)$ has an element of order 5, then G is primitive and of symplectic type. In this case, there are precisely five possibilites for G up to isomorphism:*

(1) When $F(G) = E$, either $G = E \cdot C_5$ or $G = E \cdot D_{10}$;

(2) When $F(G) = EC_4$, either $G = EC_4 \cdot C_5$, $G = EC_4 \cdot D_{10}$ or $G = EC_4 \cdot F_{20}$.

Note that this gives exactly three possibilities for the projective image $\overline{G}$: $E_{2^4} \cdot C_5$, $E_{2^4} \cdot D_{10}$ and $E_{2^4} \cdot F_{20}$, where E_{2^4} denotes the elementary abelian group of order 2^4. The fact that $G \leq J$ implies that these groups (both the projective ones and their pre-images) are in fact semidirect products.

Proof. Suppose $G/F(G)$ does contain an element of order 5. Then G contains an element α of order 5. We first show that G is primitive. As $F(G)$ is a 2-group, it acts imprimitively on V. So we may suppose $V = V_1 \oplus V_2 \oplus \cdots \oplus V_m$, where $m = 2$ or 4, and $F(G)$ acts transitively on the V_i's. If G is imprimitive, then G, and in particular α, must also permute the V_i's. Here α must act with order 1 or 5. Since there are less than five V_i's, the latter is impossible. Thus α must fix each V_i, which contradicts α not commuting with $F(G)$. Hence G is primitive.

The only solvable subgroups of S_6 containing an element of order 5 are C_5, D_{10} and F_{20}. Recall that G is of symplectic type if and only if $\Lambda^2(G)$ contains an invariant line. So it suffices to show that $\sigma'(G)$ acts intransitively on the lines $W_i = \langle w_i \rangle$. Clearly C_5 cannot act transitively on 6 points. Since $C_5 \trianglelefteq D_{10}$ of index 2, D_{10} cannot act transitively on 6 points either. As $D_{10} \trianglelefteq F_{20}$ of index 2, we see that F_{20} is also intransitive.

To see that G is one of the five possible groups listed above, note that $F(G)$ cannot be Q_8^2 because $\mathrm{Out}(Q_8^2) = G_{72}$ does not contain an element of order 5. Also $Q_8 D_8 \cdot F_{20}$ cannot occur because F_{20} is not contained in $\mathrm{Out}(Q_8 D_8) = A_5$. Note that $Q_8 D_8 \cdot C_5$ and $Q_8 D_8 \cdot D_{10}$ occur as C_5 and D_{10} are contained in $\mathrm{Out}(Q_8 D_8)$. The three possibilities for $F(G) = Q_8^2 C_4$ all occur as they are contained in J.

For each of the extensions listed, there is only one possible isomorphism type as all subgroups of S_6 which are isomorphic to C_5, D_{10}, or F_{20} are respectively conjugate. □

Proposition 5.8. *If $G/F(G)$ does not have an element of order 5, then G is of orthogonal type.*

Proof. See [Bl]. □

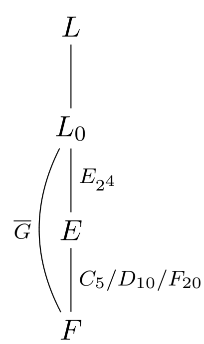
We remark that Blichfeldt's classification (for solvable and non-solvable G) gives 30 types of projective images of primitive groups (up to conjugacy, not isomorphism) in $\mathrm{GL}_4(\mathbb{C})$: six simple, two with projective image S_5, one with projective image S_6, twelve of orthogonal type, and nine which are extensions of a subgroup H of S_6 by E_{2^4} where H contains an element of order 5.

Chapter 6

The Primitive Cases

Let F be a number field and $\rho : G_F \to \mathrm{GL}_4(\mathbb{C})$ be a continuous primitive Galois representation with image G and projective image $\overline{G} \subseteq \mathrm{PGL}_4(\mathbb{C})$. Denote the projective representation by $\overline{\rho}$. By Chapter 5, ρ is of either orthogonal or symplectic type. If ρ is of orthogonal type, then it is modular by [Ra2]. Let us suppose ρ is of symplectic type. Then by Proposition 5.7, either $\overline{G} = E_{2^4} \cdot C_5$, $\overline{G} = E_{2^4} \cdot D_{10}$ or $\overline{G} = E_{2^4} \cdot F_{20}$.

Let L (resp. L_0) be the field cut out by ρ (resp. $\overline{\rho}$). Let E be the subextension of L/F such that $\mathrm{Gal}(L_0/E) = E_{2^4}$. Note that E/F is a normal extension.



6.1 The Case $E_{2^4} \cdot C_5$

First we need to know the following about our group.

Lemma 6.1. *Every element $g \in E_{2^4} \cdot C_5$ has order 5, except for the elements in the normal subgroup E_{2^4} which have order 1 or 2.*

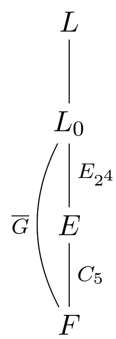
Proof. By Sylow's theorem the number n_5 of Sylow 5-groups in $E_{2^4} \cdot C_5$ is congruent to 1 mod 5 and also divides 16. Since $E_{2^4} \cdot C_5$ is not a direct product of E_{2^4} by C_5, a Sylow

5-subgroup is non-normal, i.e., $n_5 = 16$. This accounts for $16 \cdot 4$ elements of order 5 in $E_{2^4} \cdot C_5$. Hence the remaining 16 elements must comprise E_{2^4}. □

Now we can prove the main result of this section.

Theorem 6.1. *Suppose* $\rho : G_F \to \mathrm{GSp}_4(\mathbb{C}) \subseteq \mathrm{GL}_4(\mathbb{C})$ *has projective image* $\overline{G} = E_{2^4} \cdot C_5$. *Then* ρ *is modular.*

Proof. As above, $L = \overline{F}^{\ker \rho}$, $L_0 = \overline{F}^{\ker \overline{\rho}}$ and now $E \subseteq L_0$ is the normal quintic subextension of F such that $\mathrm{Gal}(L_0/E) = E_{2^4}$.



First we will show that ρ_E is modular. As $\mathrm{Gal}(L/E)$ is a cyclic central extension of a 2-group, it is a direct product of a 2-group P_2 with a cyclic group C of odd order. Therefore $\mathrm{Gal}(L/E)$ is nilpotent. A theorem of Arthur and Clozel (Proposition 9.1) states that all Galois representations with nilpotent image are modular. In particular ρ_E is modular.

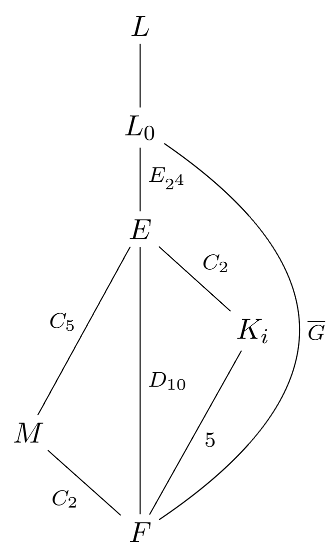
Now we claim that $\Lambda^2(\rho)$ is also modular. Since ρ is of symplectic type, $\Lambda^2(\rho)$ has an invariant line. Write $\Lambda^2(\rho) = \nu \oplus r$ where ν is one dimensional and r is five dimensional. Note r is irreducible because it factors through $E_{2^4} \cdot C_5$, which only has one- and five-dimensional irreducible representations. We claim r is induced from E. As P_2 is a 2-group, each irreducible representation of P_2 has dimension 2^j for some j. Therefore the same is true for $\mathrm{Gal}(L/E) \simeq P_2 \times C$. Hence in the decomposition of r_E into its irreducible components, we must have a one-dimensional representation λ. In particular, $r = \mathrm{Ind}_E^F \lambda$. Since E is a normal subextension, r is modular by automorphic induction (Theorem 3.7), whence $\Lambda^2(\rho)$ is also.

Hence Proposition 4.2 implies the theorem. □

Remark. We know that examples of $E_{2^5} \cdot C_5$ extensions of $\mathbb{Q}$ exist by Shafarevich's Theorem (Theorem 9.5.1 of [NSW]) because $E_{2^5} \cdot C_5$ is solvable. Though Shafarevich's proof is non-constructive, we can illustrate how to construct such an extension in our case. Let ζ_{11} be a primitive 11-th root of unity and set $\alpha_i = \zeta_{11}^i + \zeta_{11}^{-i}$. Let E be the cyclic Galois extension $\mathbb{Q}(\alpha_1)$ of $\mathbb{Q}$ of degree 5. It is known how to construct Q_8 and D_8 extensions of a number field ([JLY]). Let $K = E(\sqrt{1+A+B+AB})$ and $M = E(\sqrt{\alpha_1 + i\alpha_2}, \sqrt{\alpha_2+\alpha_4+4})$, where $A = (3+\alpha_5)^{-1/2}$ and $B = (1+\alpha_1^2+\alpha_1^2\alpha_3^2)^{-1/2}$. Then $\mathrm{Gal}(K/E) \simeq Q_8$ and $\mathrm{Gal}(M/E) \simeq D_8$. The compositum KM has three normal (over E) subextensions of index 2. Let L/E be the one corresponding to the central product of Q_8 with D_8. Then $L/\mathbb{Q}$ is Galois with Galois group $E_{2^5} \cdot C_5$.

6.2 The Cases $E_{2^4} \cdot D_{10}$ and $E_{2^4} \cdot F_{20}$

We keep the notation from the start of the chapter. First suppose $\mathrm{Gal}(L_0/F) = E_{2^4} \cdot D_{10}$. Then there exists a (normal) quadratic extension M of F contained in L_0 such that $\mathrm{Gal}(L_0/M) = E_{2^4} \cdot C_5$ and there are five non-normal quintic subextensions $K_1, K_2, K_3, \dots, K_5$ of L_0/F.



Definition 6.1. *We will say that* condition (NNQ) is satisfied for L/F *if:*

> *(NNQ) For any automorphic representation π of $\mathrm{GL}_4(\mathbb{A}_F)$ such that π_M is associated to a representation of* $\mathrm{Gal}(L/M)$*, the base change π_{K_i} exists as an automorphic representation for some $i = 1, 2, 3, \dots, 5$.*

Theorem 6.2. *Suppose $\rho : G_F \to \mathrm{GSp}_4(\mathbb{C}) \subseteq \mathrm{GL}_4(\mathbb{C})$ has projective image $\overline{G} = E_{2^4} \cdot D_{10}$. Further suppose that condition (NNQ) holds for L/F. Then ρ is modular.*

Proof. By Theorem 6.1, there are two automorphic representations π_1, π_2 of $\mathrm{GL}_4(\mathbb{A}_F)$ such that $\rho_M \leftrightarrow \pi_{i,M}$. Let $\delta = \delta_{M/F}$ be the idèle class character of M/F. Then

$$\pi_1 \simeq \pi_2 \otimes \delta. \tag{6.1}$$

Let $K \subseteq L$ be a non-normal quintic extension of F such that the base change π_K exists by (NNQ). We claim that

$$\rho_K \leftrightarrow \pi_{i,K} \tag{6.2}$$

for precisely one of the π_i's. First note that $\mathrm{Gal}(L_0/K)$ is a 2-group. Thus $\mathrm{Gal}(L/K)$, being a cyclic central extension of $\mathrm{Gal}(L_0/K)$, is the direct product of a 2-group and a cyclic group of odd order. In particular, $\mathrm{Gal}(L/K)$ is nilpotent. Hence ρ_K is modular. Let us say Π is the automorphic representation of $\mathrm{GL}_4(\mathbb{A}_K)$ such that

$$\rho_K \leftrightarrow \Pi. \tag{6.3}$$

We want to know that Π is cuspidal, i.e., that ρ_K is irreducible. Let H be the Fitting subgroup $F(\mathrm{Gal}(L/F))$. Because $\mathrm{Gal}(L/K)$ is a maximal nilpotent subgroup, we have

$$H = F(\mathrm{Gal}(L/F)) \subseteq \mathrm{Gal}(L/K). \tag{6.4}$$

By Proposition 5.3, $\rho_H = (\rho_K)_H$ is irreducible, wherefore ρ_K is also. (Though all of our groups were unimodular in Chapter 5, the proof of Proposition 5.3 still works without the unimodularity assumption.) Similarly, ρ_E is irreducible because $\mathrm{Gal}(L/E) = F(\mathrm{Gal}(L/M))$ is. Thus Π and Π_E are cuspidal.

By Theorem 3.6(i),(ii), there are precisely two automorphic representations of $\mathrm{GL}_4(\mathbb{A}_K)$ whose base change to E is Π_E. They are Π and $\Pi \otimes \delta_{E/K}$, where $\delta_{E/K}$ is the idèle class character of E/K. As $\pi_{1,E}, \pi_{2,E}$ and Π_E all correspond to ρ_E, they must be equal by Strong Multiplicity One. So it must be that

$$\left\{\Pi, \Pi \otimes \delta_{E/K}\right\} = \left\{\pi_{1,K}, \pi_{2,K} \simeq \pi_{1,K} \otimes \delta_{E/K}\right\}. \tag{6.5}$$

Thus $\Pi \simeq \pi_{i,K}$ for some i. Denote this π_i simply by π, so then

$$\rho_M \leftrightarrow \pi_M \ , \ \rho_K \leftrightarrow \pi_K. \tag{6.6}$$

We want to show that $\rho_v \leftrightarrow \pi_v$ for almost all places v. Let v be a finite place for which both ρ_v and π_v are unramified. Let $\{a_1, a_2, \ldots, a_4\}$ be the eigenvalues of $\rho(\mathrm{Fr}_v)$ and $\{b_1, b_2, \ldots, b_4\}$ those of $A(\pi_v)$. We want to show these sets are equal. As any element of D_{10} squared lies inside C_5, we have that $\mathrm{Fr}_v^2 \in \mathrm{Gal}(L/M)$. Thus

$$\rho(\mathrm{Fr}_v^2) = \rho_M(\mathrm{Fr}_v^2) \sim \left\{a_1^2, a_2^2, \ldots, a_4^2\right\}. \tag{6.7}$$

Since $\rho_{M,v} \leftrightarrow \pi_{M,v}$, we have that

$$\left\{a_1^2, \ldots, a_4^2\right\} = \left\{b_1^2, \ldots, b_4^2\right\}. \tag{6.8}$$

We may further assume that

$$a_i^2 = b_i^2 \tag{6.9}$$

for each $i = 1, 2, \ldots, 4$. On the other hand, $\mathrm{Fr}_v^5 \in \mathrm{Gal}(L/K')$ for some conjugate $K' \in \{K_i\}$ of K. We may choose an appropriate element from the Frobenius class to assume that K' is in fact K. Then

$$\rho(\mathrm{Fr}_v^5) = \rho_K(\mathrm{Fr}_v^5) \sim \left\{a_1^5, a_2^5, \ldots, a_4^5\right\}. \tag{6.10}$$

Hence

$$\left\{a_1^5, \ldots, a_4^5\right\} = \left\{b_1^5, \ldots, b_4^5\right\}. \tag{6.11}$$

If $\{a_i\} = \{b_i\}$ then we are done. So we may assume that they are not. Without loss of generality, suppose that

$$b_1 = -a_1 \ , \ b_r = \zeta_5 a_s, \tag{6.12}$$

for some r, s and a primitive fifth root of unity ζ_5. We want to deduce that $\overline{\rho}(\mathrm{Fr}_v)$ has order divisible by 10, which would contradict the fact that $\overline{G}$ has no elements of order 10. Note that we may assume

$$a_i \neq -\zeta a_j \tag{6.13}$$

for any primitive fifth root of unity ζ. For otherwise we would have that

$$\overline{\rho}(\mathrm{Fr}_v) \sim \mathrm{diag}\left(1, -\zeta, \frac{a_k}{a_i}, \frac{a_l}{a_i}\right) \tag{6.14}$$

is an element of order 10 in $\overline{G}$, where k and l are appropriate indices.

First suppose $s = 1$. Then we may assume $r = 2$ so that $b_2 = \zeta_5 a_1$. If $b_2 = -a_2$, then $a_2 = -\zeta_5 a_1$, opposing assumption (6.13). Hence $b_2 = a_2 = \zeta_5 a_1$. Note $b_1 = -a_1 = -\zeta_5^{-1} a_2$ so $b_1^5 \neq a_2^5$. Interchanging a_3 and a_4 if necessary, we then have $a_3^5 = b_1^5 = -a_1^5$. By assumption (6.13) again, this means that $a_3 = -a_1$. Thus

$$\overline{\rho}(\mathrm{Fr}_v) \sim \mathrm{diag}\left(a_1, \zeta_5 a_1, -a_1, a_4\right) \sim \mathrm{diag}\left(1, \zeta_5, -1, \frac{a_4}{a_1}\right) \tag{6.15}$$

is an element of order divisible by 10. Contradiction!

Now we may assume $s = 2$ and either $r = 1$ or $r = 3$. If $r = 1$, then $a_1 = -b_1 = -\zeta_5 a_2$ contradicting (6.13). Thus $r = 3$ and $b_3 = \zeta_5 a_2$. By (6.13), we cannot have $b_3 = -a_3$, so $b_3 = a_3$. Then (6.11) reads

$$\left\{a_1^5, a_2^5, a_2^5, a_4^5\right\} = \left\{-a_1^5, b_2^5, a_2^5, b_4^5\right\}. \tag{6.16}$$

Either $-a_1^5 = a_2^5$ or $-a_1^5 = a_4^5$. Consider first $-a_1^5 = a_2^5$. Then $a_1 = -a_2$ and

$$\overline{\rho}(\mathrm{Fr}_v) \sim \mathrm{diag}\left(-a_2, a_2\zeta_5, a_2, a_4\right) \sim \mathrm{diag}\left(-1, 1, \zeta_5, \frac{a_4}{a_2}\right) \tag{6.17}$$

has order divisible by 10.

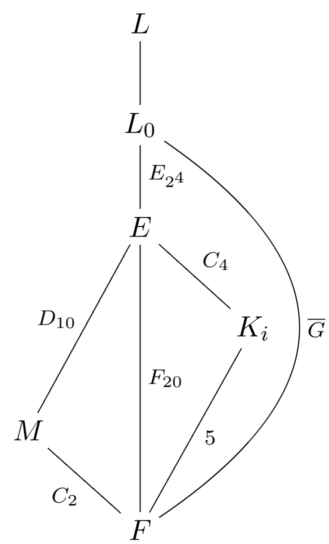
Thus we can consider the (final!) case $-a_1^5 = a_4^5$, i.e., $a_1 = -a_4$. Here we also have that

$$\overline{\rho}(\mathrm{Fr}_v) \sim \mathrm{diag}\left(a_1, a_2, \zeta_5 a_2, -a_1\right) \sim \mathrm{diag}\left(\frac{a_1}{a_2}, 1, \zeta_5, -\frac{a_1}{a_2}\right) \tag{6.18}$$

is an element of $\overline{G}$ of order divisible by 10. In any case, we get a contradiction, so we are done! □

Now consider $\mathrm{Gal}(L_0/F) = E_{2^4} \cdot F_{20}$. Then there exists a (normal) quadratic extension M of F inside L_0 such that $\mathrm{Gal}(L_0/M) = E_{2^4} \cdot D_{10}$ and there are five non-normal quintic subextensions $K_1, K_2, K_3, \ldots, K_5$ of L_0/F. (See the diagram below.) The definition of condition (NNQ) for L/F also makes sense in this case, and by exactly the same method of the proof of Theorem 6.2 we can also prove the following.

Theorem 6.3. *Suppose $\rho : G_F \to \mathrm{GSp}_4(\mathbb{C}) \subseteq \mathrm{GL}_4(\mathbb{C})$ has projective image $\overline{G} = E_{2^4} \cdot F_{20}$. Further suppose that condition (NNQ) holds for L/F. Then ρ is modular.* □



6.3 Transfer to GSp_4

Consider any irreducible four-dimensional Galois representation $\rho : G_F \to \mathrm{GSp}_4(\mathbb{C})$. The Langlands philosophy predicts that ρ should correspond not only to an automorphic representation of $\mathrm{GL}_4(\mathbb{A}_F)$, but also to an automorphic representation of the group $\mathrm{GSp}_4(\mathbb{A}_F)$ (since ${}^L\mathrm{GSp}_4^0 = \mathrm{GSp}_4(\mathbb{C})$). Suppose that ρ is modular, i.e., that ρ corresponds to some cuspidal representation π of $\mathrm{GL}_4(\mathbb{A}_F)$. The fact that $\mathrm{Im}(\rho) \subseteq \mathrm{GSp}_4(\mathbb{C})$ implies that $L(s, \Lambda^2(\rho) \otimes \nu^{-1})$ has a simple pole at $s = 1$ for a suitable one-dimensional representation ν of G (the "polarization").

This implies that the corresponding automorphic L-function $L(s, \pi; \Lambda^2 \otimes \nu^{-1})$ has a pole at $s = 1$. An unpublished theorem of Jacquet, Piatetski-Shapiro and Shalika says that, because of this pole, π *transfers* to a generic irreducible cuspidal automorphic representation Π of $\mathrm{GSp}_4(\mathbb{A}_F)$ with central character ν such that,

$$L^S(s, \Pi) = L^S(s, \pi),$$

for any finite set of primes S outside of which π is unramified. Here the L-function on the left is the degree four L-function of Π studied in [PS]; and if $L(s) = \prod_v L_v(s)$ is an Euler product, then $L^S(s)$ denotes the incomplete L-function $\prod_{v \notin S} L_v(s)$. Thus ρ in fact corresponds to the cuspidal representation Π of $\mathrm{GSp}_4(\mathbb{A}_F)$, i.e., ρ is *modular of symplectic type* as predicted by Langlands.

In particular, the representations in Theorem 6.1 should actually correspond to automorphic representations on $\mathrm{GSp}_4(\mathbb{A}_F)$. However we are not stressing this here because this theorem of Jacquet, Piatetski-Shapiro and Shalika remains unpublished. We hope to work this out in the future, which should not be too difficult using some recent results. The key point is that GL_4 maps into the connected component of GO_6 and π gives rise to a cuspidal automorphic representation π' of $\mathrm{GO}_6(\mathbb{A}_F)^0$. The desired Π is obtained by the theta correspondence. The obstruction to this transfer is the residue of the pole of $L(s, \pi; \Lambda^2 \otimes \nu^{-1})$ at $s = 1$ (see [JS2]). Finally the ongoing work of J. Arthur will give another proof, using the trace formula, of the existence of Π and other members of its packet (see [Ar] for his program).

Chapter 7

Monomial Representations

Suppose now $\rho : G_F \to \mathrm{GL}_4(\mathbb{C})$ is an irreducible, continuous, imprimitive representation with solvable image, where as before F denotes a number field. Either ρ is induced from a character or a two-dimensional representation (or both). If ρ is induced from a two-dimensional representation σ, then the fact that σ is modular (Theorem 3.5), combined with automorphic induction (Theorem 3.7), yields that ρ is also modular. Thus we will assume that ρ is not induced from a two-dimensional representation. Hence ρ is monomial, i.e., there is a quartic extension K of F and a character χ of K such that

$$\rho = \mathrm{Ind}_K^F \chi. \tag{7.1}$$

If K/F is normal, then ρ is modular by Theorem 3.7. So assume K/F is non-normal. If E is an intermediate subfield of K/F, then ρ is induced by the two-dimensional representation $\sigma = \mathrm{Ind}_K^E \chi$ of G_E, going against our assumption above. Hence K/F has no intermediate subfields.

In this case, Artin's conjecture is known, but the strong Artin conjecture is not.

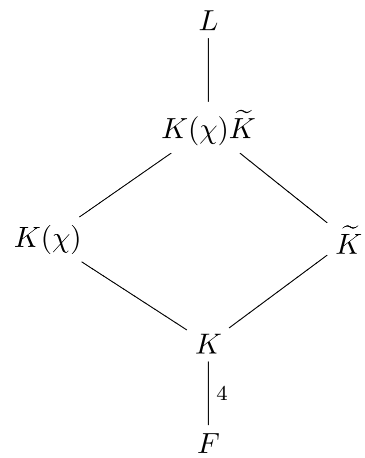
7.1 The Structure of $\mathrm{Im}(\rho)$

We keep the assumptions above, i.e., that

(i) $\rho = \mathrm{Ind}_K^F \chi$ for some character χ, and

(ii) K/F is a non-normal quartic extension with no intermediate subfields.

Let $K(\chi)$ be the fixed field of $\ker \chi$, $\widetilde{K}$ the Galois closure of K/F, and L the Galois closure of $K(\chi)/F$.



Then ρ factors through G_L. In fact,

$$\rho : \mathrm{Gal}(L/F) \to \mathrm{GL}_4(\mathbb{C}) \tag{7.2}$$

is faithful, for $L = \overline{F}^{\ker \rho}$ is a normal (Galois) extension of F.

Proposition 7.1. *Suppose $\rho : G_F \to \mathrm{GL}_4(\mathbb{C})$ is monomial, induced from quartic extension K/F, and ρ is not induced from any quadratic extensions. Then $\mathrm{Gal}(\widetilde{K}/F)$ is isomorphic to either A_4 or S_4, where $\widetilde{K}/F$ is the Galois closure of K/F.*

Proof. We know that $\mathrm{Gal}(\widetilde{K}/F)$ is a subgroup of S_4 with order divisible by 4 but not equal to 4. Thus $|\mathrm{Gal}(\widetilde{K}/F)|$ must be 8, 12 or 24. If $|\mathrm{Gal}(\widetilde{K}/F)| = 8$, then $\mathrm{Gal}(\widetilde{K}/K)$ is contained in a maximal subgroup M of $\mathrm{Gal}(\widetilde{K}/F)$ of index 2. However, M would correspond to an intermediate subfield of K/F, giving a contradiction. So $|\mathrm{Gal}(\widetilde{K}/F)| = 12$ or 24, in which case $\mathrm{Gal}(\widetilde{K}/F) \simeq A_4$ or S_4. □

Note that as $\rho_{\widetilde{K}}$ contains a character, it must be the direct sum of four characters by Clifford's theorem. Therefore $\mathrm{Gal}(L/\widetilde{K})$ is an abelian group A of rank at most 4.

Let $G = \mathrm{Gal}(L/F)$. Then G is of the form $A \cdot A_4$ or $A \cdot S_4$. In fact, we can say a bit more.

Lemma 7.1. *With ρ, G and A as above, the Fitting subgroup $F(G) = P_2 \times B$ where P_2 is a 2-group and B is an abelian group of odd order.*

Proof. The Fitting subgroup $F(G)$ is a product $\prod \mathrm{Syl}_p(F(G))$ of Sylow p-subgroups. Let p be an odd prime and $P = \mathrm{Syl}_p(F(G))$. We want to show that P is abelian. As P

is characteristic in $F(G)$, it must be normal in G. Therefore, ρ_P decomposes a sum of representations whose dimensions are all equal by Clifford's theorem. On the other hand, any irreducible representation of P has p-power dimension. Thus the only possibility is that ρ_P decomposes as a sum of characters. Since ρ is faithful, this implies that P is abelian.

□

Lemma 7.2. *Let $\rho : G_F \to \mathrm{GL}_4(\mathbb{C})$ be an irreducible representation. If $\Lambda^2(\rho)$ contains two one-dimensionals, then ρ is induced by a two-dimensional representation τ along a quadratic extension.*

Proof. First note that $\Lambda^4(\rho)$ is the determinant $\det(\rho)$. Let V be the representation space for ρ. We have a natural pairing,

$$\Lambda^2(\rho) \times \Lambda^2(\rho) \longrightarrow \Lambda^4(\rho) = \det(\rho), \tag{7.3}$$

i.e.,

$$\langle \Lambda^2(\rho)(g)v, \Lambda^2(\rho)(g)w \rangle = \det(\rho)(g)\langle v, w \rangle \ \forall g \in G_F, \ \forall v, w \in \Lambda^2(V), \tag{7.4}$$

where $\langle\ ,\ \rangle$ denotes an orthogonal form on $\Lambda^2(V) \simeq \mathbb{C}^6$. Thus $\Lambda^2(\rho)$ an essentially self-dual representation for any four-dimensional ρ, whether or not ρ is itself essentially self-dual. Moreover, if ν is any polarization, i.e., a one-dimensional summand of $\Lambda^2(\rho)$, then $\nu^2 = \det(\rho)$. This can be seen by restricting (7.4) to the line $\langle v_0 \rangle \subseteq \Lambda^2(V)$ which is ν-invariant, i.e.,

$$\nu(g)^2\langle v_0, v_0 \rangle = \langle \nu(g)v_0, \nu(g)v_0 \rangle = \langle \Lambda^2(\rho)(g)v_0, \Lambda^2(\rho)(g)v_0 \rangle = \det(\rho)(g)\langle v_0, v_0 \rangle \ \forall g \in G_F. \tag{7.5}$$

Suppose ν' is another character occurring in $\Lambda^2(\rho)$ and let $\mu = \nu^{-1}\nu'$. Then

$$\mu^2 = \nu^{-2}(\nu')^2 = \det(\rho)^{-1}\det(\rho) = 1. \tag{7.6}$$

We claim that $\mu \neq 1$, i.e., $\nu \not\simeq \nu'$. First observe that, because ν is contained in $\Lambda^2(\rho)$, ρ is of symplectic type for some symplectic form $\langle\ ,\ \rangle_\nu$ such that

$$\langle \rho(g)v, \rho(g)w \rangle_\nu = \nu(g)\langle v, w \rangle_\nu \ \forall g \in G_F, \ \forall v, w \in V, \tag{7.7}$$

i.e., $\check{\rho} \simeq \rho \otimes \nu^{-1}$. Therefore $\mathrm{End}(\rho) = \rho \otimes \check{\rho} \simeq \rho \otimes \rho \otimes \nu^{-1}$ contains $\Lambda^2(\rho) \otimes \nu^{-1}$. If $\nu' \simeq \nu$, then $\Lambda^2(\rho) \otimes \nu^{-1}$, and hence $\rho \otimes \check{\rho}$, contains the trivial representation with multiplicity at least two, contradicting the irreducibility of ρ (Proposition 2.2). Hence $\mu \neq 1$.

Now let M be the quadratic extension of F which is cut out by μ. Then $\Lambda^2(\rho_M)$ contains ν_M with multiplicity 2. The argument in the previous paragraph applies here to yield that ρ_M is reducible. Hence ρ is induced from a two-dimensional representation defined over M. □

Proposition 7.2. *Suppose that $\rho : G_F \to \mathrm{GL}_4(\mathbb{C})$ a Galois representation of solvable type which is imprimitive and essentially self-dual. Then ρ is modular.*

Proof. An essentially self-dual, reducible, four-dimensional representation of solvable type is always modular (Section 4.1). Therefore, assume that ρ is irreducible. Either ρ is of orthogonal or symplectic type. If ρ is of orthogonal type, then it is modular by the recent work of Ramakrishnan [Ra2]. So let us further assume that ρ is of symplectic type. Thus $\Lambda^2(\rho)$ decomposes into

$$\Lambda^2(\rho) = \nu \oplus \tau, \tag{7.8}$$

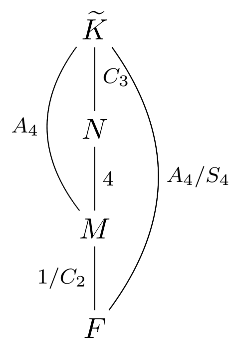
where ν is the polarization character and τ is a five-dimensional representation.

By earlier remarks, we may assume that $\rho = \mathrm{Ind}_K^F \chi$ for some non-normal quartic extension K/F and a character χ of G_K. We claim that ρ is induced from a two-dimensional representation. Suppose it is not. Then $\mathrm{Gal}(\widetilde{K}/F) \simeq A_4$ or $\mathrm{Gal}(\widetilde{K}/F) \simeq S_4$. By Lemma 7.2, $\Lambda^2(\rho)$ cannot contain two characters, i.e., τ cannot contain a character. Because the restriction $\rho_{\widetilde{K}}$ decomposes as a sum of characters, so does $\tau_{\widetilde{K}}$. If τ were irreducible, $\mathrm{Gal}(\widetilde{K}/F)$ would act transitively on the *five* characters contained in τ_K, which is impossible. Hence τ is reducible and must decompose as

$$\tau = \alpha \oplus \beta, \tag{7.9}$$

where α is an irreducible two-dimensional representation and β is an irreducible three-dimensional representation.

Now we essentially reduce to the case where $\mathrm{Gal}(\widetilde{K}/F) \simeq A_4$. Let M/F be the subextension of $\widetilde{K}/F$ corresponding to the subgroup A_4 of $\mathrm{Gal}(\widetilde{K}/F)$. So if $\mathrm{Gal}(\widetilde{K}/F) \simeq A_4$ then $M = F$ and if $\mathrm{Gal}(\widetilde{K}/F) \simeq S_4$ then M/F is quadratic. In either case β_M must be irreducible, using the fact that $\mathrm{Gal}(M/F)$ acts transitively on the irreducible components of β_M. Let N be the subextension of the A_4-extension $\widetilde{K}/M$ corresponding to a cyclic subgroup of order 3.



Thus N/M is a (non-normal) quartic extension and we can also show that β_N is irreducible. To see this, first note that $\beta_{\widetilde{K}}$ is a sum of characters because $\rho_{\widetilde{K}}$ is. The Galois action of $\mathrm{Gal}(\widetilde{K}/N)$ on the characters in $\beta_{\widetilde{K}}$ implies that either β_N is irreducible or it is a sum of characters $\mu_1 \oplus \mu_2 \oplus \mu_3$. Suppose the latter. Then $\mathrm{Gal}(\widetilde{K}/N)$ stabilizes each component $\mu_{i,\widetilde{K}}$ of $\beta_{\widetilde{K}}$. But $\mathrm{Gal}(\widetilde{K}/M)$ acts transitively on $\left\{\mu_{i,\widetilde{K}}\right\}$ because β_M is irreducible. In fact the action of $\mathrm{Gal}(\widetilde{K}/M)$ must factor through C_3 since S_3 is not a factor of A_4. Now as $\mathrm{Gal}(\widetilde{K}/N)$ is a maximal subgroup of $\mathrm{Gal}(\widetilde{K}/M)$, it must be the stabilizer of each $\mu_{i,\widetilde{K}}$. Hence we have a short exact sequence

$$1 \longrightarrow \overbrace{\mathrm{Gal}(\widetilde{K}/N)}^{C_3} \longrightarrow \overbrace{\mathrm{Gal}(\widetilde{K}/M)}^{A_4} \longrightarrow C_3 \longrightarrow 1, \tag{7.10}$$

which is just wrong in so many ways! (well, at least two). Thus β_N is indeed irreducible.

Note that as $\rho_{\widetilde{K}}$ is a sum of characters, we can write $\rho_N = \lambda \oplus \sigma$, where σ is a (possibly reducible) three-dimensional representation. Then $\sigma_{\widetilde{K}}$ is also a sum of characters. As the Galois group of $\widetilde{K}/N$ acts on the summands of $\sigma_{\widetilde{K}}$, the representation σ must either be irreducible or itself a sum of characters. If σ is a sum of characters, then so are ρ_N and $\Lambda^2(\rho_N)$, contradicting the irreducibility of β_N. Thus σ is irreducible.

Now on one hand we have

$$\Lambda^2(\rho_N) = \Lambda^2(\chi \oplus \sigma) = \Lambda^2(\sigma) \oplus \chi\sigma. \tag{7.11}$$

On the other hand,

$$\Lambda^2(\rho_N) = \Lambda^2(\rho)_N = \nu_N \oplus \alpha_N \oplus \beta_N. \tag{7.12}$$

Since $\chi\sigma$ is irreducible, dimensions require that $\chi\sigma = \beta_N$ and therefore

$$\Lambda^2(\sigma) = \nu_N \oplus \alpha_N \tag{7.13}$$

is reducible. But the fact that we have a natural pairing

$$\Lambda^2(\sigma) \times \sigma \simeq \Lambda^2(\sigma) \times \Lambda(\sigma) \longrightarrow \Lambda^3(\sigma) \simeq \det(\sigma) \tag{7.14}$$

implies that $\Lambda^2(\sigma)$ is a twist of σ, whence irreducible, contradicting (7.13).

Thus ρ is induced from two-dimensional representation, and therefore modular. □

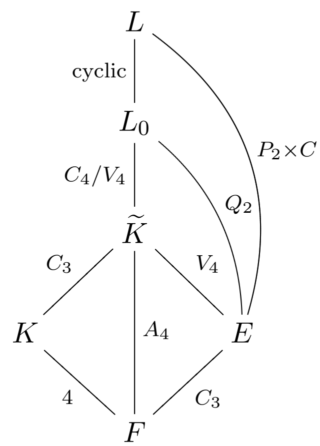
7.2 Hypertetrahedral Representations

In this section, we consider the case where $\overline{G}$ is an extension of A_4 by a group of order 4. Such groups can be easily computed in [GAP] (or even rather easily, but more tediously, by hand). There are six possibilities for $\overline{G}$: $C_4 \times A_4$, $V_4 \times A_4$, $\mathrm{SL}_2(\mathbb{F}_3) \times C_2$, $\mathrm{SL}_2(\mathbb{F}_3) \rtimes C_2$, $V_4 \rtimes A_4$ and $V_4 \cdot A_4$. Here V_4 is the Klein 4-group, and $V_4 \cdot A_4$ is the unique group of order 48 containing both V_4 and A_4 as subgroups which is not a semidirect product of the two. In all of these cases, ρ is modular. However, the first four cases are the less interesting ones, as we will see anon. First we need the

Proposition 7.3. *Let ρ be an irreducible four-dimensional complex representation of G_F such that $\overline{G}$ is one of the six possible extensions of A_4 by C_4 or V_4. Then ρ is monomial. More precisely, let L be the fixed field of* $\ker(\rho)$*, and let $\widetilde{K}/F$ be a subextension of L/F corresponding to the quotient group A_4. Let K/F be a non-normal quartic subextension of $\widetilde{K}/F$. Then ρ is induced from a character of G_K.*

Proof. Let E be the intermediate field of $\widetilde{K}/F$ corresponding to the subgroup V_4. Then E/F is a normal extension of degree 3. Let L_0 be the field cut out by $\overline{\rho}$. (See the diagram below.)

We claim that ρ is induced from K, i.e., that ρ_K contains a character. Assume otherwise. Since $\mathrm{Gal}(L_0/\widetilde{K}) = C_4$ or V_4, any irreducible representation of $\mathrm{Gal}(L/\widetilde{K})$ has dimension 1 or 2. Thus ρ_K cannot be irreducible since the restriction $\rho_{\widetilde{K}}$ to a normal cubic extension is not. So we may assume that ρ_K is a sum of two irreducible two-dimensional representations. Restricting, we see that $\rho_{\widetilde{K}}$ is also sum of two irreducible two-dimensional representations, say $\rho_{\widetilde{K}} = \sigma \oplus \tau$. As ρ is irreducible, $\mathrm{Gal}(\widetilde{K}/F) = A_4$ acts transitively on $\{\sigma, \tau\}$. So the stabilizer of σ in A_4 is a subgroup of index 2. But A_4 has no subgroups of index 2, a contradiction. □



Proposition 7.4. *Suppose $\rho : G_F \to \mathrm{GL}_4(\mathbb{C})$ is a Galois representation whose projective image $\overline{G}$ is an extension of A_4 by a group of order 4. If ρ is reducible, then ρ is modular. If $\overline{G} \simeq C_4 \times A_4$, $V_4 \times A_4$, $\mathrm{SL}_2(\mathbb{F}_3) \times C_2$, or $\mathrm{SL}_2(\mathbb{F}_3) \rtimes C_2$, then ρ is reducible.*

Proof. Note that ρ is of solvable type. Suppose that ρ is reducible. Then ρ is modular except possibly when ρ decomposes as a character χ plus a three-dimensional τ (see Section 4.1). However A_4 has a normal subgroup of index 3, which means that $\overline{G}$ does also. Denote the corresponding cubic extension of F by E. Let L be the field cut out by ρ and L_0 be the field cut out by $\overline{\rho}$. Then $\mathrm{Gal}(L_0/E)$ is a 2-group Q_2, so the cyclic central extension $\mathrm{Gal}(L/E)$ is the direct product of a 2-group P_2 with a cyclic group C of odd order. (See diagram above.) In particular, any irreducible representation of $\mathrm{Gal}(L/E)$ has dimension a power of 2. Thus ρ_E must contain a linear character χ. Hence, ρ is induced by χ along the normal cubic extension E/F. Theorem 3.7 then implies that ρ is modular.

Now suppose that ρ is irreducible. The Galois group $\mathrm{Gal}(\widetilde{K}/F) = A_4$ acts transitively on the four distinct characters occuring in $\rho_{\widetilde{K}}$. This implies that $\mathrm{Gal}(\widetilde{K}/F)$ cannot fix $\mathrm{Gal}(L_0/\widetilde{K})$ pointwise. However, as may be checked in [GAP] for example, each of the four groups $C_4 \times A_4$, $V_4 \times A_4$, $\mathrm{SL}_2(\mathbb{F}_3) \times C_2$ and $\mathrm{SL}_2(\mathbb{F}_3) \rtimes C_2$ has a unique normal subgroup of order four whose quotient is A_4. But in all of these cases, this normal subgroup is the center of $\overline{G}$, i.e., every element of $\mathrm{Gal}(L_0/F)$ fixes $\mathrm{Gal}(L_0/\widetilde{K})$ pointwise. This shows that $\overline{G} = V_4 \rtimes A_4$ or $V_4 \cdot A_4$. □

If ρ is irreducible and $\overline{G}$ is an extension of A_4 by V_4, then we will say that ρ is *hypertetrahedral*. Hypertetrahedral representations occur for both of the remaining cases $\overline{G} \simeq V_4 \rtimes A_4$ and $\overline{G} \simeq V_4 \cdot A_4$. We remark however that such a projective image does not guarantee irreducibility. For example, $V_4 \cdot A_4$ has a faithful irreducible three-dimensional representation τ. Since $V_4 \cdot A_4$ is centerless, $1 \oplus \tau$ is a reducible four-dimensional representation with projective image $V_4 \cdot A_4$.

Now we come to the main result of this chapter:

Theorem 7.1. *Let F be a number field and ρ a hypertetrahedral representation of G_F. Then ρ is modular. There are infinitely many such representations with projective image $V_4 \cdot A_4$ which are not essentially self-dual.*

Remarks. The case where $\overline{G} = V_4 \rtimes A_4$ yields examples of irreducible monomial four-dimensional representations of orthogonal type, which are modular by [Ra2]. However in the case where $\overline{G} = V_4 \cdot A_4$, we obtain below irreducible monomial representations ρ which are not of orthogonal type, whence not essentially self-dual. Then ρ is not a tensor product of two two-dimensional representations since its image does not lie in $\mathrm{GO}_4(\mathbb{C})$. Nor is ρ a symmetric cube lift of a two-dimensional representation because $\overline{G}$ is not a subgroup of $\mathrm{PGL}_2(\mathbb{C})$.

Proof. Let L be the field cut out by ρ and let E be the subfield corresponding to a normal subgroup of $\mathrm{Gal}(L/F)$ of index 3. By definition, ρ is irreducible. So Proposition 7.4 implies that either $\overline{G} = V_4 \rtimes A_4$ or $V_4 \cdot A_4$. Both of these groups have no elements of order 6, hence by Proposition 4.2 we only need to show that ρ_E and $\Lambda^2(\rho)$ are modular.

As remarked in the proof of Proposition 7.4, $\mathrm{Gal}(L/E)$ is a direct product of a 2-group P_2 with a cyclic group C of odd order. Therefore $\mathrm{Gal}(L/E)$ is nilpotent. By a theorem of Arthur and Clozel (Proposition 9.1), this implies that ρ_E is modular.

We now show $\Lambda^2(\rho)$ is modular. We may assume that $\Lambda^2(\rho)$ does not contain any characters by Proposition 7.2. Thus $\Lambda^2(\rho)$ cannot contain an irreducible five-dimensional representation either. Any two-dimensional representation contained in $\Lambda^2(\rho)$ is modular by Theorem 3.5.

Now suppose $\Lambda^2(\rho)$ contains an irreducible representation τ of dimension 3 or 6. We know that all irreducible representations of $\mathrm{Gal}(L/E)$ have dimension a power of two because $\mathrm{Gal}(L/E) = P_2 \times C$. Thus τ_E must be reducible, whence τ is induced from the normal extension E and therefore modular.

Finally, consider the case where $\Lambda^2(\rho)$ contains an irreducible four-dimensional representation σ. As noted in the proof of Lemma 7.2, there is a natural symmetric pairing $\Lambda^2(\rho) \times \Lambda^2(\rho) \to \Lambda^4(\rho) = \det(\rho)$. Hence $\Lambda^2(\rho)$, and also σ, maps into $\mathrm{GO}_6(\mathbb{C})$. The dimension of σ implies that its image in fact lies in $\mathrm{GO}_4(\mathbb{C})$. Hence σ is modular by [Ra2].

Thus all irreducible components of $\Lambda^2(\rho)$ must be modular, so $\Lambda^2(\rho)$ is also, so ρ is also.

To see that there are infinitely many non-essentially-self-dual hypertetrahedral representations, take the group G_{192} of order 192 generated by

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \text{ and } \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

As this is solvable, it occurs as a Galois group over $\mathbb{Q}$ by a theorem of Shafarevich ([NSW]). The group G_{192} has center $\langle iI \rangle$ of order 4, and $\overline{G_{192}} = G_{192}/\langle iI \rangle \simeq V_4 \cdot A_4$. This group (realized as a Galois group) has hypertetrahedral representation ρ which is not essentially self-dual and not induced from a normal extension. Such examples exist of orders $192 \cdot k$, $k = 1, 2, 3, \ldots$. This can easily be seen by taking central products of G_{192} with cyclic groups. $\square$

Chapter 8

Three-Dimensional Representations

Let $\rho : G_F \to \mathrm{GL}_3(\mathbb{C})$ be a Galois representation. By Theorems 3.7 and 3.8, any imprimitive (and hence monomial) three-dimensional Galois representation is modular. In particular, if ρ is essentially self-dual of solvable type then ρ is modular (cf. Proposition 1.3 of [Ra2] or Section 8.1 below). Here we will consider the case when ρ is primitive, about which little is known regarding modularity. No primitive examples of solvable type are known to be modular. In fact, the only primitive cases known are when ρ is a twist of the symmetric square of a modular two-dimensional icosahedral representation. However, for two primitive cases of solvable type, we show that the adjoint $\mathrm{Ad}(\rho)$ is modular.

8.1 Primitive Three-Dimensional Representations

Let $\rho : G_F \to \mathrm{GL}_3(\mathbb{C})$ be a continuous irreducible primitive representation with projective image $\overline{G}$. Then $\overline{G}$ is a finite subgroup of $\mathrm{PGL}_3(\mathbb{C})$. The finite subgroups of $\mathrm{PGL}_3(\mathbb{C})$ have been classified by Blichfeldt (see [MBD] or [Bl]). The primitivity condition leaves us with six possibilities for $\overline{G}$.

Fix a primitive ninth root of unity ζ_9 and set $\omega = \zeta_9^6$. Consider the following elements of $\mathrm{GL}_3(\mathbb{C})$:

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \ , \ T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

$$U = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon\omega \end{pmatrix}, \quad V = \frac{1}{\omega - \omega^2} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}.$$

Let

$$\begin{aligned} G_{36} &= \overline{\langle S, T, V \rangle} = \langle S, T, V \rangle / \sim \ \leq \mathrm{PGL}_3(\mathbb{C}), \\ G_{72} &= \overline{\langle S, T, V, UVU^{-1} \rangle} \leq \mathrm{PGL}_3(\mathbb{C}), \\ G_{216} &= \overline{\langle S, T, V, U \rangle} \leq \mathrm{PGL}_3(\mathbb{C}). \end{aligned}$$

As is suggested from the notation G_{36} has order 36, G_{72} has order 72, and so on and so forth. Then for a primitive (irreducible) subgroup G of $\mathrm{GL}_3(\mathbb{C})$, the projective image $\overline{G}$ in $\mathrm{PGL}_3(\mathbb{C})$ falls into one of the following six cases:

(1) $\overline{G} \simeq G_{36}$;

(2) $\overline{G} \simeq G_{72}$;

(3) $\overline{G} \simeq G_{216}$;

(4) $\overline{G} \simeq A_5 \simeq \mathrm{PSL}_2(\mathbb{F}_3)$;

(5) $\overline{G} \simeq A_6$;

(6) $\overline{G} \simeq \mathrm{PSL}_2(\mathbb{F}_7)$.

The first three cases are solvable and the latter three are simple and non-solvable. Observe that case (4) is always essentially self-dual, which we shall justify presently. This means if G is a finite primitive subgroup of $\mathrm{GL}_3(\mathbb{C})$ such that $\overline{G} \simeq A_5$, then $G \subseteq \mathrm{GO}_3(\mathbb{C})$. Let ρ be the standard representation of G. Then by Section 3 of [Ra2], $\rho \simeq \mathrm{Sym}^2(\sigma) \otimes \chi$ for some two-dimensional representation σ and some character χ. By the symmetric square lift of Gelbart and Jacquet [GeJ], the question of modularity in case (4) reduces to the modularity of two-dimensional icosahedral representations σ. However, none of the cases (1) – (3), (5) or (6) are essentially self-dual, and modularity is not known in these cases.

Let G be a finite primitive subgroup of $\mathrm{GL}_3(\mathbb{C})$ with $\overline{G} \simeq A_5$. Let us briefly explain why $G \subseteq \mathrm{GO}_3(\mathbb{C})$. Put $\zeta_m = e^{2\pi i/m}$. Let n be the order of G and $Z = \langle \zeta_{3n} \rangle$. Set $G' = ZG$. Given any $g \in G$, $\det g = \zeta_n^k$ for some k. Let $z = \zeta_{3n}^{-k}$. Then

$$\det zg = \left(\zeta_{3n}^{-k}\right)^3 \det g = \zeta_n^{-k} \zeta_n^k = 1, \tag{8.1}$$

i.e., $zg \in \mathrm{SL}_3(\mathbb{C})$. Let $H = G' \cap \mathrm{SL}_3(\mathbb{C})$. Note that $\overline{H} = \overline{G'} = \overline{G} \simeq A_5$. Then any $g \in G$ can be written as $g = zh$ where $z \in Z$ and $h \in H$. Suppose now that $H \subseteq \mathrm{O}_3(\mathbb{C})$. Let f denote the standard orthogonal form on $\mathbb{C}^3$. Write any $g \in G$ as $g = zh$ as above. Then for $x, y \in \mathbb{C}^3$,

$$f(g(x), g(y)) = f(zh(x), zh(y)) = z^2 f(h(x), h(y)) = z^2 f(x, y), \tag{8.2}$$

i.e., $g \in \mathrm{GO}_3(\mathbb{C})$.

So it suffices to show that indeed $H \subseteq \mathrm{O}_3(\mathbb{C})$. We know H is a subgroup of $\mathrm{SL}_3(\mathbb{C})$ with projective image A_5. As the center of $\mathrm{SL}_3(\mathbb{C})$ is C_3, H must either be A_5 or an cyclic central extension of A_5 by C_3. However the only extension of A_5 by C_3 is the trivial one, $C_3 \times A_5$, where C_3 is embedded as the group of scalar matrices of order 3 inside $\mathrm{SL}_3(\mathbb{C})$. Thus $C_3 \subseteq \mathrm{O}_3(\mathbb{C})$. So it suffices to check that A_5 get embedded into $\mathrm{O}_3(\mathbb{C})$. But A_5 has only two three-dimensional representations, both of which are orthogonal. *Quod erat demonstrandum.*

8.2 $\mathrm{Ad} : \mathrm{GL}_3 \to \mathrm{GL}_8$

Let $\rho : G_F \to \mathrm{GL}_3(\mathbb{C})$ be an irreducible representation with image G and projective image $\overline{G}$. Then $\rho \otimes \check{\rho}$ is a nine-dimensional representation which contains the trivial representation. Write

$$\rho \otimes \check{\rho} = 1 \oplus \mathrm{Ad}(\rho), \tag{8.3}$$

where $\mathrm{Ad}(\rho)$ is an eight-dimensional representation called the *adjoint representation* of ρ. Observe that $\mathrm{Ad}(\rho)$ is self-dual because $\rho \otimes \check{\rho}$ is.

Note that if $\rho(g) = zI_3$ for some $z \in \mathbb{C}$, then $(\rho \otimes \check{\rho})(g) = zI_3 \otimes z^{-1}I_3 = I_9$. (Here I_m is the $m \times m$ identity matrix.) Hence $\mathrm{Ad}(\rho)$ is an eight-dimensional representation which factors through $\overline{G} \subseteq \mathrm{PGL}_3(\mathbb{C})$. This will be a key point in studying $\mathrm{Ad}(\rho)$.

Proposition 8.1. *Let $\rho : G_F \to \mathrm{GL}_3(\mathbb{C})$ be irreducible with projective image $\overline{G} \simeq G_{36}$ or G_{72}. In the former case, $\mathrm{Ad}(\rho)$ is a sum of two irreducible four-dimensionals representations; in the latter, $\mathrm{Ad}(\rho)$ is irreducible. In both cases, $\mathrm{Ad}(\rho)$ is induced by characters along a series of quadratic extensions. Hence $\mathrm{Ad}(\rho)$ is modular.*

Proof. First consider the case $\overline{G} \simeq G_{36}$. Looking at the character table for G_{36}, we see that G_{36} has four one-dimensional representations and two irreducible four-dimensional representations. Because ρ is irreducible, $\mathrm{Ad}(\rho)$ cannot contain any one-dimensional representations by Proposition 2.2. Hence $\mathrm{Ad}(\rho)$ decomposes as the sum of two four-dimensional representations.

Let σ be either of the four-dimensional components of $\mathrm{Ad}(\rho)$. Note that G_{36} has a normal subgroup G_{18} of index 2, which itself has a normal subgroup C_3^2 of index 2. In fact, $C_3^2 \trianglelefteq G_{36}$. Restricting the four-dimensional representation σ to a subgroup of order 9 must yield a sum of characters. Hence σ is induced from the normal subgroup C_3^2 (and also from G_{36}). By cyclic automorphic induction (Theorem 3.7), σ is modular. Therefore $\mathrm{Ad}(\rho)$ is also modular.

Now consider the case $\overline{G} \simeq G_{72}$. Looking at the character table for G_{72}, we see that G_{72} has four one-dimensional representations, one irreducible two-dimensional and one irreducible eight-dimensional. It is immediate from the definitions that G_{72} contains G_{36}. By the previous case, when we restrict $\mathrm{Ad}(\rho)$ to G_{72} we must get a sum of two irreducible four-dimensional representations. Thus $\mathrm{Ad}(\rho)$ must be irreducible and induced from the four-dimensional representations in the G_{36} case. Theorem 3.7 again implies that $\mathrm{Ad}(\rho)$ is modular. □

We remark that in the case $\overline{G} \simeq G_{216}$, $\mathrm{Ad}(\rho)$ is irreducible but monomial. However it is only induced from non-normal extensions of degrees 4 and 8. In fact, G_{216} has no normal subgroups of index 2, 4 or 8.

Suppose now that $\overline{G} \simeq A_5$. Then $\rho \simeq \mathrm{Sym}^2(\sigma)\otimes\chi$ for some two-dimensional icosahedral representation σ and a character χ (cf. Section 3 of [Ra2]). Because the representation $\mathrm{Ad} : \mathrm{GL}_3(\mathbb{C}) \to \mathrm{GL}_8(\mathbb{C})$ factors through $\mathrm{PGL}_3(\mathbb{C})$, $\mathrm{Ad}(\rho)$ factors through $\mathrm{Ad}(\overline{\rho})$. Thus, to study $\mathrm{Ad}(\rho)$ it suffices to consider when ρ is actually a representation of A_5. In this case, one easily checks that

$$\mathrm{Ad}(\rho) \simeq \rho \oplus \mathrm{Sym}^4(\sigma) \simeq (\mathrm{Sym}^2(\sigma) \otimes \chi) \oplus \mathrm{Sym}^4(\sigma). \tag{8.4}$$

Many cases of icosahedral representations are known to be modular. If σ is modular, then $\mathrm{Ad}(\rho)$ also is by the functoriality of Sym^2 and Sym^4 on GL_2.

If $\overline{G} \simeq A_6$, then $\mathrm{Ad}(\rho)$ is irreducible and primitive. Irreducibility can be seen just by looking at dimensions of the irreducible representations for A_6, which has 1 one-dimensional, 2 five-dimensional, 2 eight-dimensional, 1 nine-dimensional and 1 ten-dimensional irreducible representations. As $\mathrm{Ad}(\rho)$ is eight-dimensional and does not contain the trivial representation, it must be irreducible. Primitivity follows because A_6 has no subgroups of index 2, 4 or 8.

If $\overline{G} \simeq \mathrm{PSL}_2(\mathbb{F}_7)$, then $\mathrm{Ad}(\rho)$ is irreducible and monomial. Here also, irreducibility follows immediately from the character table for $\mathrm{PSL}_2(\mathbb{F}_7)$, which has 1 one-dimensional, 2 three-dimensional, 1 six-dimensional, 1 seven-dimensional and 1 eight-dimensional irreducible representations. The group $\mathrm{PSL}_2(\mathbb{F}_7)$ has no subgroups of index 2 or 4, but it does have a maximal subgroup of index 4, being the non-abelian group G_{21} of order 21. Restricting the eight-dimensional representation of $\mathrm{PSL}_2(\mathbb{F}_7)$ to G_{21} we see $\mathrm{Ad}(\rho)$ is monomial.

On a final note, even though knowing modularity of $\mathrm{Ad}(\sigma)$ combined with a base change is enough to prove modularity for two-dimensional tetrahedral σ, it will not suffice in the three-dimensional case. One issue is that no subgroups (normal or non-normal) of index 3 exist in G_{36}. In fact, every maximal subgroup of G_{36} has index either 2 or 9.

Chapter 9

Supersolvable Representations

A group is *supersolvable* if it has a finite normal series with each factor cyclic. Recall we have the following inclusions of classes of finite groups:

$$\{\text{nilpotent groups}\} \subseteq \{\text{supersolvable groups}\} \subseteq \{\text{monomial groups}\} \subseteq \{\text{solvable groups}\}.$$

Artin's conjecture is known for all representations whose image is monomial and the strong Artin conjecture is known for all representations whose image is nilpotent. It is natural to ask what can be said in the case of representations with supersolvable image. We refer the reader to [Ha] for the basics on supersolvable groups. The key result we need from group theory is that any maximal subgroup of a supersolvable group has prime index. (In fact, Huppert proved that a finite group G is supersolvable if and only if all maximal subgroups are of prime index.)

Let us first elaborate on the nilpotent case because somewhat more is known. Here we will use the term *character* for the character of any finite-dimensional representation, not just a linear character.

Let G be a finite group. Recall that H is a *subnormal* subgroup of G if there exist subgroups $H_1, H_2, \ldots, H_r$ of G such that $H \trianglelefteq H_1 \trianglelefteq H_2 \trianglelefteq \cdots \trianglelefteq H_r \trianglelefteq G$. An *accessible character* of G is a $\mathbb{Z}$-linear combination of characters induced from one-dimensional representations of subnormal subgroups of G. Call a representation *accessible* if its character is. An immediate consequence of the base change theory for GL_n developed by Arthur and Clozel is

Proposition 9.1. (Proposition 7.2 of [AC]) *Let ρ be a continuous Galois representation over $\mathbb{C}$ of solvable type. If ρ is accessible, then it is modular.*

In particular, when ρ has nilpotent image, ρ is modular. Dade [Da] showed that, if G is solvable and ρ is an irreducible accessible representation, then ρ is monomial and induced from a one-dimensional representation of a subnormal subgroup. Hence Proposition 9.1 gives us no new instances of Artin's conjecture, though modularity is stronger and was previously unknown.

Call a finite solvable group G *accessible* if every irreducible representation of G has an accessible character. Then the class of accessible groups also lies strictly between the nilpotent groups and the solvable groups, and has a non-trivial intersection with the class of supersolvable groups, without either class contained in the other. (For example, S_3 is accessible and supersolvable but not nilpotent; A_4 is accessible but not supersolvable; the group G_{108} below is supersolvable but not accessible.) We show modularity for certain representations with supersolvable image, some of which are accessible and some of which are not.

Proposition 9.2. *Let $\rho : G_F \to \mathrm{GL}_n(\mathbb{C})$ be a continuous representation with supersolvable image. If $n = 2^j$ or $n = 2^j \cdot 3$ for some $j \geq 0$, then ρ is modular.*

We need to know the following fact.

Lemma 9.1. *Let G be a supersolvable subgroup of S_6 and $H \leq G$ of index 6. Then H is contained in a subgroup of G of index 2.*

Proof. Let G be such a group. Then $|G| = 6m$. The proof is to simply look at all possibilities for G and check they satisfy the desired property. We will omit the details, which can easily be checked in [GAP] for instance. If $m = 1$, $G = C_6$ or $G = S_3$. In both cases the lemma obviously holds. If $m = 2$, $G = D_{12}$ and it is easy to see the lemma holds in this case also. If $m = 3$, $G = C_3 \times S_3$ or $G = C_3 \cdot S_3 = \langle (1,2,3), (4,5,6), (1,2)(4,5) \rangle$. If $m = 6$, then $G = S_3 \times S_3$. There are no other supersolvable subgroups of S_6 of order divisible by 6. □

Proof of Proposition. Let L be the field cut out by ρ and $G = \mathrm{Gal}(L/F)$. We view ρ as a representation of G. As G is supersolvable, ρ is monomial. Say ρ is induced from a character χ of $\mathrm{Gal}(L/K)$ for some extension K/F of degree n.

When $j = 0$ this is either the well-known one-dimensional case or the case of cubic induction from GL_1 to GL_3 (Theorems 3.7 and 3.8). Suppose $j \geq 1$.

First consider the case $n = 2^j$. Let M be the field corresponding to a maximal subgroup of G containing $\mathrm{Gal}(L/K)$. Since G is supersolvable, $[M : F] = p$ for some prime p. Let $\sigma = \mathrm{Ind}_K^M \chi$. So σ is $\frac{n}{p}$-dimensional and $\rho = \mathrm{Ind}_M^F \sigma$. In particular, $p|n$. If M is normal and σ is modular, then so is ρ by automorphic induction (Theorem 3.7). As $n = 2$, we know $p = 2$ and M is normal. Hence, by Fermat's method of descent, we can reduce to the $j = 0$ case and obtain that ρ is modular.

Now assume $n = 2^j \cdot 3$. Let $\widetilde{K}$ be the Galois closure of K/F. Then the factor group $\mathrm{Gal}(\widetilde{K}/F)$ of G is supersolvable, so there exists a maximal subgroup H_1 of prime index in $\mathrm{Gal}(\widetilde{K}/F)$ which contains $\mathrm{Gal}(\widetilde{K}/K)$. Similarly there exists a subgroup H_2 of prime index in H_1 which contains $\mathrm{Gal}(\widetilde{K}/K)$. Repeating this, we get a maximal series of groups

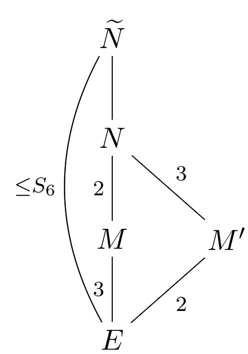
$$\mathrm{Gal}(\widetilde{K}/K) = H_{j+1} \leq H_j \leq \cdots \leq H_1 \leq H_0 = \mathrm{Gal}(\widetilde{K}/F), \tag{9.1}$$

each of prime index in the next. Then we have a corresponding tower of fields

$$K = K_{j+1} \supseteq K_j \supseteq \cdots \supseteq K_1 \supseteq K_0 = F. \tag{9.2}$$

One of the terms, say $K_{i+1} \supseteq K_i$, must be index 3 and the rest are index 2.

We want to reduce to the case where $i = j$, so assume that $i < j$. For convenience, let us write $N = K_{i+2}$, $M = K_{i+1}$, and $E = K_i$. Let $\widetilde{N}$ be the Galois closure of N/K. Therefore $\mathrm{Gal}(\widetilde{N}/E)$ is a subgroup of S_6. Note that $\mathrm{Gal}(\widetilde{N}/E)$ is supersolvable because $\mathrm{Gal}(L/E)$ is. By the lemma above, $\mathrm{Gal}(\widetilde{N}/N)$ is contained in a subgroup of index 2. Let M' be the corresponding field extension so $[N : M'] = 3$ and $[M' : E] = 2$. Pictorially we have the following diagram.



Thus we can reduce to the case where $[K_{i+2} : K_{i+1}] = 3$ and $[K_{i+1} : K_i] = 2$. Repeating this process allows us to assume that $i = j$ as desired. In other words, we have a tower of fields

$$K = K_{j+1} \overset{3}{\supseteq} K_j \overset{2}{\supseteq} \cdots \overset{2}{\supseteq} K_1 \overset{2}{\supseteq} K_0 = F. \tag{9.3}$$

Then $\mathrm{Ind}_K^{K_j}\chi$ is modular by the $j=0$ case. As all of the subsequent extensions K_i/K_{i-1} are normal for $j \geq i \geq 1$, we can apply normal (quadratic) automorphic induction j times to say that $\rho = \mathrm{Ind}_K^F\chi = \mathrm{Ind}_{K_j}^F\left(\mathrm{Ind}_K^{K_j}\chi\right)$ is modular. □

Remarks. Suppose $\rho : G_F \to \mathrm{GL}_n(\mathbb{C})$ has finite supersolvable image.

(1) If $n = 2^j$, then ρ is accessible and modularity was known by Arthur and Clozel. However, ρ need not have nilpotent image, e.g., $n = 2$ and $\mathrm{Im}(\rho) = S_3$. Note that even for $n = 2$ there exist non-accessible representations of solvable type, namely the ones which are primitive, i.e., the tetrahedral and octahedral representations.

(2) If $n = 2^j \cdot 3$, then whether ρ is accessible will depend upon whether the index 3 extension K_{i+1}/K_i can be chosen to be normal. For example, there exists a supersolvable group G_{108} of order 108 which is a semidirect product $G_{108} = C_3^2 \rtimes D_{12}$ and has an irreducible six-dimensional representation ρ. The group G_{108} has no normal subgroups of index 3. It has three subgroups H of index 2, and ρ_H is only reducible for one of them, $G_{54} = C_3^2 \rtimes S_3$. By the proof of the proposition above, ρ is induced from a subgroup of index 3 of G_{54}, but none of these subgroups are normal. Hence ρ is not accessible. (Note: the semidirect product notation does not determine G_{54} or G_{108} uniquely, but the GAP notation given in Appendix B does.)

Appendix α

Here we give a proof of the following result, shown to us by Ramakrishnan. It is very similar to the proof of Proposition 3.2.1 in [Ra], and the ideas for GL_2 go back to [DS].

Proposition A.1. *Let F be a number field and let $\sigma : G_F \to \mathrm{GL}_n(\mathbb{C})$ be an irreducible continuous Galois representation. Let π be a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ such that $L(s, \pi_v) = L(s, \sigma_v)$ for almost all v. Then $L(s, \pi_v) = L(s, \sigma_v)$ for all finite places v. Moreover, $\prod_{v|\infty} L(s, \pi_v) = \prod_{v|\infty} L(s, \sigma_v)$. In other words, we have the equalities $L(s, \pi) = L^*(s, \sigma)$ of Euler products over $\mathbb{Q}$ and $L(s, \pi_f) = L(s, \sigma)$ of Euler products over F, where $\pi_f = \otimes'_{v<\infty} \pi_v$.*

Note that when $F = \mathbb{Q}$, this proposition says all of the local L-factors are equal. When F is CM or totally real, a similar result at the finite places is contained in Section 6 of [He].

Proof. Let S designate a finite set of places containing the ramified places such that $L(s, \pi_v) = L(s, \sigma_v)$ for all $v \notin S$.

We first do the case of infinite primes. By [JPSS2], we may choose a character χ of F such that $\chi_\infty = 1$ and χ_u is sufficiently ramified so that

$$L(s, \pi_u \otimes \chi_u) = L(s, \check{\pi}_u \otimes \chi_u^{-1}) = L(s, \sigma_u \otimes \chi_u) = L(s, \check{\sigma}_u \otimes \chi_u^{-1}) = 1 \tag{A.1}$$

for all $u \in S$. Then $L(s, \pi_f \otimes \chi_f) = L(s, \sigma \otimes \chi)$ and $L(s, \check{\pi}_f \otimes \chi_f) = L(s, \check{\sigma} \otimes \chi)$, where $\chi = \chi_f \chi_\infty$.

Dividing the functional equation for $L(s, \pi \otimes \chi)$ by that for $L^*(s, \sigma \otimes \chi)$ gives

$$\frac{L(1-s, \check{\pi} \otimes \chi^{-1})}{L^*(1-s, \check{\sigma} \otimes \chi^{-1})} = \frac{\varepsilon(s, \pi \otimes \chi)}{\varepsilon(s, \sigma \otimes \chi)} \frac{L(s, \pi \otimes \chi)}{L^*(s, \sigma \otimes \chi)}. \tag{A.2}$$

Set $L_\infty(s,\pi) = \prod_{v|\infty} L(s,\pi_v)$ and $L_\infty(s,\sigma) = \prod_{v|\infty} L(s,\sigma_v)$. We have

$$L(s,\pi\otimes\chi) = L_\infty(s,\pi\otimes\chi)L(s,\pi_f\otimes\chi_f) = L_\infty(s,\pi)L(s,\pi_f\otimes\chi_f), \tag{A.3}$$

$$L^*(s,\sigma\otimes\chi) = L_\infty(s,\sigma\otimes\chi)L(s,\sigma\otimes\chi) = L_\infty(s,\sigma)L(s,\sigma\otimes\chi), \tag{A.4}$$

and similarly for $\check{\pi}$ and $\check{\sigma}$. Using this, (A.2) becomes

$$\frac{L_\infty(1-s,\check{\pi})}{L_\infty(1-s,\check{\sigma})} = \varepsilon(s)\frac{L_\infty(s,\pi)}{L_\infty(s,\sigma)}, \tag{A.5}$$

where $\varepsilon(s) = \frac{\varepsilon(s,\pi\otimes\chi)}{\varepsilon(s,\sigma\otimes\chi)}$ is a non-vanishing entire function. Because π is cuspidal, a result of Shalika [Shal] tells us that π_v is generic for all v. Then by Proposition 2.1 of [BR], $L(s,\pi_v)$ and $L(s,\check{\pi}_v)$ have no poles in $\Re(s) > \frac{1}{2} - \delta$ for some δ. Also $L_\infty(s,\sigma)$ and $L_\infty(s,\check{\sigma})$ have no poles to the right of $\Re(s) = 0$; this is because σ has finite image by assumption, thus making the Frobenius eigenvalues roots of unity. In other words, no pole of $L_\infty(s,\pi)$ is also a pole of $L_\infty(1-s,\check{\pi})$ and no pole of $L_\infty(s,\sigma)$ is a pole of $L_\infty(1-s,\check{\sigma})$.

Since the local factors are never zero, the poles of $L_\infty(s,\pi)$ must coincide with the poles of $L_\infty(s,\sigma)$, with the right multiplicities. We can write

$$L_\infty(s,\pi) = \prod_1^{n[F:\mathbb{Q}]} \Gamma_\mathbb{R}(s+a_j), \;\; L_\infty(s,\sigma) = \prod_1^{n[F:\mathbb{Q}]} \Gamma_\mathbb{R}(s+b_j), \tag{A.6}$$

where $\Gamma_\mathbb{R}(s) = \pi^{-s/2}\Gamma(s/2)$. Because their poles agree, we must have $\{a_j\} = \{b_j\}$, i.e., $L_\infty(s,\pi) = L_\infty(s,\sigma)$.

Now fix a finite place $v \in S$. Choose a character χ such that $\chi_v = 1$ and (A.1) holds for each $u \in S - \{v\}$. By the method above, we get

$$\frac{L(1-s,\check{\pi}_v)}{L(1-s,\check{\sigma}_v)} = \varepsilon(s)\frac{L(s,\pi_v)}{L(s,\sigma_v)}, \tag{A.7}$$

where $\varepsilon(s) = \frac{\varepsilon(s,\pi\otimes\chi)}{\varepsilon(s,\sigma\otimes\chi)}$ is a non-vanishing entire function. As above, using Proposition 2.1 of [BR], no pole of $L(s,\pi_v)$ is also a pole of $L(1-s,\check{\pi}_v)$. Similarly no pole of $L(s,\sigma_v)$ is a pole of $L(1-s,\check{\sigma}_v)$. Hence the poles of $L(s,\pi_v)$ must coincide with the poles of $L(s,\sigma_v)$, and therefore $L(s,\pi_v) = L(s,\sigma_v)$. □

Appendix B

GAP Notation

The following table provides GAP Notation for many groups appearing in the text. The group $[a, b]$ is the group of order a accessed by the command `SmallGroup`(a, b) in [GAP].

G	$[a,b]$
F_{20}	[20,3]
Q_8^2	[32, 49]
Q_8D_8	[32, 50]
$Q_8^2C_4$	[64, 266]
$\mathrm{Out}(Q_8^2) = \mathrm{O}_4^+(\mathbb{F}_2)$	[72, 40]
$E_{2^4} \cdot C_5$	[80, 49]
$E_{2^4} \cdot D_{10}$	[160, 234]
$Q_8D_8 \cdot C_5$	[160, 199]
$Q_8D_8 \cdot D_{10}$	[320, 1581]
$\mathrm{Out}(Q_8^2C_4) = \mathrm{Sp}_4(\mathbb{F}_2) \times C_2$	[1440, 5842]
$V_4 \cdot A_4$	[48, 3]
$C_4 \times A_4$	[48, 31]
$\mathrm{SL}_2(\mathbb{F}_3) \times C_2$	[48, 32]
$\mathrm{SL}_2(\mathbb{F}_3) \rtimes C_2$	[48, 33]
$V_4 \times A_4$	[48, 49]
$V_4 \rtimes A_4$	[48, 50]
G_{192}	[192, 4]
G_{36}	[36, 9]
G_{72}	[72, 41]
G_{216}	[216, 153]
G_{54}	[54, 8]
G_{108}	[108, 17]

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