Topological Sigma Models and Generalized Geometries

Thesis by

Yi Li

In Partial Fulfillment of the Requirements
for the Degree of
Doctor of Philosophy

California Institute of Technology
Pasadena, California

2005
(Submitted June 1, 2005)

Acknowledgements

I am deeply grateful to my advisor, Anton Kapustin, for teaching me many things in physics and mathematics, and for generously sharing his ideas and insights with me. I have benefited greatly from numerous discussions with Anton during the past three and a half years. I am also thankful to John Schwarz, for his continued support during my entire stay in the Caltech Theory Group, and for intellectual conversations on issues in both physics and the real world. I would also like to thank Hirosi Ooguri for kindly letting me attend his group meetings on topological strings, and Jaume Gomis for collaboration and discussions.

I am indebted to many colleagues, notably Ke Zhu, Marco Gualtieri, Manfred Herbst, and Alessandro Tomasiello, for answering my many questions and for patiently explaining their work to me. I have also benefited from discussions with Sujay Ashok, Kentaro Hori, and Eric Sharpe.

Special thanks also go to Takuya Okuda, with whom I have enjoyed ambitious, well-planned study sessions; to Jongwon Park and Peter Lee, for their passionate participation in my first (unsuccessful) project; to Arkadas Özakin, for intelligent conversations on just anything in life: from politics to photography, from caving to cooking; and to my officemate Vadim Borokhov, for tolerating my ever-presence in our crowded living space. I thank all the members of the Caltech Theory Group for providing a stimulating environment.

Finally, I would like to thank my wife, Min, for her unfailing love and unconditional support even during the most difficult times, and for sharing my happiness, sorrow, and wildest dreams.

I dedicate this thesis to my parents.

Abstract

We study the relation between topological sigma models and generalized geometries. The existence conditions for the most general type of topological sigma models obtained from twisting the $N = (2, 2)$ supersymmetric sigma model are investigated, and are found to be related to twisted generalized Calabi-Yau structures. The properties of these topological sigma models are analyzed in detail. The observables are shown to be described by the cohomology of a Lie algebroid, which is intrinsically associated with the twisted generalized Calabi-Yau structure. The Frobenius structure on the space of states and the effects of instantons are analyzed. We also study D-branes in these topological sigma models, and demonstrate that they also admit descriptions in terms of generalized geometries.

Contents

Chapter 1

Introduction

Since their discovery in the 1970s [35] and 80s [38], topological field theories (TFT) have proven to be a remarkably fruitful area of research for both physics and mathematics. Physically, many TFTs admit exact treatment and yet share important properties of their more "physical" siblings. The study of TFTs has had a great impact on our understanding of quantum field theories and string theories in general. A good example in this regard is the realization that certain quantities in the physical model can be computed exactly in the much-simpler topological model [3]. From the mathematical point of view, TFTs are typically closely related to questions of interest in geometry and topology. The methods employed to solve the TFTs have led to new developments in these subjects. One example in this regard is E. Witten's solution of the Chern-Simons theory, which has resulted in surprising advances in knot theory.

A particularly important class of TFTs are the two-dimensional topological sigma models [39]. Loosely speaking, a topological sigma model is a TFT whose fields are constructed from maps from a Riemann surface Σ (the worldsheet) to a manifold M (the target). It was first discovered by Witten that the topological sigma model can be defined on any almost Kähler manifold [39]. The correlation functions in such a theory are the celebrated Gromov-Witten invariants of J-holomorphic curves. If the target manifold is actually Kähler, then the topological sigma model can be constructed alternatively by "twisting" an $N = (2, 2)$ supersymmetric sigma model. In fact, there are two inequivalent twistings, and the resulting topological field theories are called the A-model and the B-model, respectively[40].

The A-model and B-model are topological field theories in the sense that they do not depend on the metric on the worldsheet Σ. In addition, the A-model only depends on the Kähler moduli of M, while the B-model only depends on the complex moduli of M. An important fact concerning the B-model is that, due to an analogue of the familiar chiral anomaly in four dimensions, the B-model is well-defined as a quantum theory if and only if M is a Calabi-Yau manifold. There is no such restriction for the A-model. Mirror symmetry exchanges the B-model on a Calabi-Yau manifold M with the A-model on the mirror Calabi-Yau $\tilde{M}$.

Although the above story has become a well-established part of our knowledge, there are certain unsatisfactory aspects in this picture. Perhaps the most glaring one is the fact that the A-model and the B-model are not defined on the same category of manifolds. As already mentioned, while the A-model can be defined on any Kähler manifold, the B-model only exists on Calabi-Yau manifolds. In fact, the A-model only depends on the symplectic structure on M and, when it is formulated on a symplectic manifold, it is precisely the topological sigma model originally discovered by Witten. Given the stark differences between the A-model and B-model, it is natural to ask whether there is a more general class of topological sigma models, which includes the A/B models as special examples, and which is defined on a new type of geometry that include both the symplectic geometry and the Calabi-Yau geometry?

One of the main goals of this thesis is to investigate such a possibility. It turns out that there is indeed a more general type of topological sigma model that includes the A-model and the B-model as two extreme examples. It is defined on a new type of manifolds recently introduced by N. Hitchin–the generalized Calabi-Yau manifolds [15]. As the name suggests, the usual Calabi-Yau manifolds are special examples of the generalized Calabi-Yau manifolds.

We begin with a brief introduction to generalized geometries in Chapter 2. This chapter is aimed at providing the necessary mathematical background that is needed in later chapters. This is a review of known mathematical facts, mainly developed by N. Hitchin and M. Gualtieri. Given a smooth manifold M, let us denote its tangent bundle by T and cotangent bundle by T^*. In the realm of "generalized" geometries, the geometric properties of M are studied from the viewpoint of the direct sum $T \oplus T^*$ rather than of T or T^* alone. Just like on the tangent bundle T, there exists a natural bracket structure on $T \oplus T^*$. This is the so-called Courant bracket. As we shall see, the Courant bracket plays a crucial role in defining the generalized complex structure. Other types of geometric structures also have their "generalized" counterparts; important examples for our purpose are generalized Kähler and generalized Calabi-Yau geometries. It should be noted that all generalized geometries mentioned above have a twisted version, whenever a closed 3-form is present.

In Chapter 3, we begin by briefly reviewing our physical setup: the $N = (2, 2)$ supersymmetric sigma model in two dimensions. The most general $(2, 2)$ sigma model was first analyzed by Gates, Hull, and Rocek [9] in 1984. The main novelty in their findings is that the $(2, 2)$ theory actually allows two independent complex structures as well as a Neveu-Schwarz three-form H. Each of the complex structures is Hermitian with respect to the metric and is covariantly constant with respect to a specific connection with torsion. The torsion is closely related to the three-form H. It is a remarkable fact, proved by M. Gualtieri [12], that the rather intricate geometric structure associated with $(2, 2)$ sigma models is equivalent to the twisted generalized Kähler structure. Generalized geometry enters the scene of $(2, 2)$ sigma models in a natural but nontrivial manner.

In the same chapter, we describe in detail how to obtain a topological field theory from twisting the generic $(2, 2)$ sigma model. In particular, we construct a scalar odd operator Q_B from the twisted $(2, 2)$ supersymmetry operators, which satisfies $Q_B^2 = 0$. It is regarded as a BRST operator in the twisted theory, and observables of the twisted theory are identified with the Q_B-cohomology. Such field theories are usually called cohomological field theories. For reasons that will be explained in detail later, we shall call this twisted theory the generalized B-model. Due to quantum anomalies, the twisted theory makes sense as a cohomological field theory at the quantum level if and only if certain geometric and topological conditions are satisfied. This can already be seen in the case of the ordinary B-model, which is well-defined if and only if its canonical bundle admits a nowhere zero holomorphic section. For the generalized B-model, the analogous condition turns out to be precisely the one that defines the so-called twisted generalized Calabi-Yau (TGCY) structure introduced by N. Hitchin in [15]. When the TGCY condition is met, we prove that the twisted theory is a topological field theory, in the sense that its energy-momentum tensor is BRST exact.

We analyze the fundamental properties of the topological theory in Chapter 4. A main result is the identification of local BRST invariant observables of the generalized B-model with the Lie algebroid cohomology of an intrinsically associated complex Lie algebroid. The space of observables not only has a ring structure naturally derived from the Lie algebroid cohomology ring, but also has a compatible product, defined by tree-level two-point correlation functions. This is usually referred to as a Frobenius structure. We also comment on the role that instantons play in the quantum theory.

Another goal of the thesis is to initiate an investigation of topological D-branes in the general topological sigma models discussed above. The rank-one objects were first described by A. Kapustin in geometric terms [21]. Again, generalized geometry enters the scene: a D-brane preserves the topological algebra of the generalized B-model if and only if it defines a so-called generalized complex submanifold. We call such a brane a GC brane. In Chapter 5, we analyze the open string BRST cohomology for rank-1 GC branes. This material is based on [23]. It turns out that, just like in the closed string case, the open string BRST cohomology is computed by a Lie algebroid cohomology. This viewpoint sheds new light even on the rank-1 ordinary A-branes and B-branes. For example, the Lie algebroid cohomology for ordinary B-branes naturally leads to the spectral sequence of S. Katz and E. Sharpe [26], while for rank-one coisotropic A-branes it leads to a proof of a conjecture of Kapustin and Orlov [25]. Finally, we indicate how to extend the analysis to higher-rank topological D-branes.

The majority of the thesis is based on recently published joint work with A. Kapustin [22, 23], although some content has not appeared in the literature before. One new result is the proof of topological invariance of the generalized B-model in Sec. 3.4. Another is the treatment of higher-rank topological D-branes in Sec. 5.3, which is based on ongoing work with A. Kapustin. Finally, in the Summary, the discussion of the deformation space of twisted generalized Calabi-Yau manifolds is based on still-unpublished material, and the space-time effective description of the generalized B-model is based on ongoing work with J. Gomis.

Chapter 2

Geometric Background

2.1 The geometry of $T \oplus T^*$

The essence of generalized geometries is to regard various geometric structures as defined on the direct sum of the tangent bundle and the cotangent bundle $T \oplus T^*$, instead of structures on the tangent bundle or cotangent bundle alone. We will introduce various types of generalized geometries in the following. In this section we will briefly review the geometry of $T \oplus T^*$. We will follow refs. [15, 12] closely.

A notable difference between $T \oplus T^*$ and T is that the former comes with a natural non-degenerate inner product. Given any $X, Y \in \Gamma(T)$[1] and $\xi, \eta \in \Gamma(T^*)$, one defines

$$\langle X + \xi, Y + \eta \rangle = \frac{1}{2}\big(\eta(X) + \xi(Y)\big).$$

The pairing $\langle , \rangle$ is a nondegenerate symmetric bilinear form and so can naturally be regarded as an inner produce on $T \oplus T^*$. Note that $\langle , \rangle$ has signature (n, n), where $n = \dim T$.

Just like there is a Lie bracket on T, there is also a natural bracket structure on $T \oplus T^*$, the Courant bracket. It is defined by

$$[X + \xi, Y + \eta] = [X, Y] + L_X\eta - L_Y\xi - \frac{1}{2}d(\iota_X\eta - \iota_Y\xi)$$

[1]In this thesis, $\Gamma(E)$ always denote the space of C^∞ sections of vector bundle E.

for any $X+\xi, Y+\eta \in \Gamma(T\oplus T^*)$. Here on the right-hand side, $[,]$ denotes the usual Lie bracket on T, L_X is the Lie derivative along X, and ι_X the interior product with X.

Although the Courant bracket $[,]$ is antisymmetric by definition, it is not a Lie bracket, because it does not satisfy the Jacobi identity. Given three sections A, B, C, the failure of the Jacobi identity of their brackets is measured by $d\langle A, [B,C]\rangle +$cyclic. From this it follows that when restricted to an isotropic subbundle of $T\oplus T^*$, the Courant bracket becomes a Lie bracket. This fact is crucial for us, as we will frequently work with eigenbundles of generalized complex structures (see next section), and these eigenbundles are maximal isotropic subbundles of the complexification of $T\oplus T^*$. The (complex) Lie bracket induced from the Courant bracket endows these eigenbundles with the structure of *Lie algebroids*. The definition of a Lie algebroid is given in the Appendix.

If $H\in\Omega^3(M)$ is a closed 3-form, then one can define the *twisted* Courant bracket, denoted by $[,]_H$. For any $X+\xi$, $Y+\eta$, $X, Y\in\Gamma(T)$, $\xi,\eta\in\Gamma(T^*))$, one defines

$$[X+\xi, Y+\eta]_H = [X+\xi, Y+\eta] + \iota_Y\iota_X H\ .$$

Most properties of the Courant bracket extend to its twisted cousin. In particular, when restricted to an isotropic subbundle, it also becomes a true Lie bracket (here it is important that $dH=0$).

2.2 Generalized complex structures

Before we venture into the land of "generalized" geometries, it is a good idea to first recall how an ordinary complex structure is defined in differential geometry. Given an even-dimensional manifold M, an almost complex structure is a smooth section J of the endomorphism bundle of TM, $\mathrm{End}\,TM$, such that $J^2=-\mathrm{id}$. [2] A complex structure is an almost complex structure that is integrable. There are several equivalent ways to formulate the integrability condition.

[2] Of course an almost complex structure does not always exist on an even-dimensional manifold, and there are topological obstructions in general. The question of finding the sufficient and necessary conditions for the existence of almost complex structures is an open problem in differential geometry.

One approach that is particularly convenient for us uses the natural Lie bracket $[,]$ on TM. Let $TM_{\mathbb{C}}$ be the complexified tangent bundle. Extending by $\mathbb{C}$-linearity, $[,]$ also gives a Lie bracket on $TM_{\mathbb{C}}$. Let $TM^{1,0}$ (resp. $TM^{0,1}$) denote the $+i$-eigenbundle (resp. $-i$-eigenbundle) with respect to the almost complex structure J. J is said to be integrable if and only if $TM^{1,0}$ (or $TM^{0,1}$) is closed under the Lie bracket $[,]$.

The definition of generalized complex structure (GC structure) is similar. It also involves two parts: defining the almost structure, and specifying a suitable integrability condition. On a generic even-dimensional[3] manifold, the structure group of $T \oplus T^*$ is $O(2n, 2n)$. A generalized almost complex (almost GC) structure is a reduction of the structure group from $O(2n, 2n)$ to $U(n, n)$. In geometric terms, a generalized almost GC structure $\mathcal{J}$ is a section of the endomorphism bundle $\mathrm{End}\,(T \oplus T^*)$ such that the following two conditions are satisfied: 1) $\mathcal{J}$ is orthogonal with respect to the canonical inner product $\langle , \rangle$ on $T \oplus T^*$; 2) $\mathcal{J}$ defines an almost complex structure on $T \oplus T^*$, namely $\mathcal{J}^2 = -1$.

It can be shown that almost GC structures only exist on even-dimensional manifolds. Also, as in the case of almost complex structure, there are in general obstructions to their existence. Thanks to a result of M. Gualtieri, topological obstructions to almost GC structures are the same as the ones to almost complex structures. In other words, an even-dimensional real manifold admits an almost GC structure if and only if it admits an ordinary almost complex structure.

A generalized complex structure (GC structure) is an almost GC structure satisfying a further integrability condition. In the definition of ordinary complex structures, the natural Lie bracket on $T_{\mathbb{C}}$ is used to define the integrability condition. In the generalized case, it should be rather evident that one should make use of the natural bracket structure on $T_{\mathbb{C}} \oplus T^*_{\mathbb{C}}$, the Courant bracket. Let $E_{\mathcal{J}}$ be the $+i$ (or $-i$) eigenbundle of $\mathcal{J}$ in $T_{\mathbb{C}} \oplus T^*_{\mathbb{C}}$. The almost GC structure $\mathcal{J}$ is said to be *integrable* if $E_{\mathcal{J}}$ is closed under the Courant bracket on $T_{\mathbb{C}} \oplus T^*_{\mathbb{C}}$. An integrable almost GC structure is called a generalized complex (GC) structure.

Before we move to more specialized generalized geometries, let us first look at some

[3] Like in the case of complex geometry, the existence of generalized almost complex structure requires the dimension of the manifold to be even.

standard examples of generalized complex manifolds. Not surprisingly, ordinary complex manifolds are GC manifolds. This can be seen as follows. Let (M, I) be a complex manifold with complex structure I. Let us construct an almost GC structure $\mathcal{J}_I$ as

$$\mathcal{J}_I = \begin{pmatrix} I & 0 \\ 0 & -I^* \end{pmatrix}. \tag{2.2.1}$$

Here we write $\mathcal{J}_I$ in the matrix form with respect to the splitting $TM \oplus TM^*$, and $I^* : T^* \to T^*$ is the dual of I (i.e., $I^*\xi(X) = \xi(IX)$ for any $X \in T, \xi \in T^*$). Obviously $\mathcal{J}_I$ specifies an almost complex structure on $TM \oplus TM^*$ because I is an almost complex structure. It is also easy to see that $\mathcal{J}_I$ is integrable if and only if I is an integrable almost complex structure.

Perhaps more surprizing is the fact that symplectic manifolds provide another class of examples of GC manifolds. If M is a symplectic manifold with symplectic form ω, one can define

$$\mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}. \tag{2.2.2}$$

Here the inverse of the symplectic 2-form ω^{-1} is a nondegenerate Poisson bivector. Clearly $\mathcal{J}_\omega^2 = -\mathrm{id}$, so it is an almost GC structure. The $+i$ eigenbundle is $E_{\mathcal{J}_\omega} = \{X - i\omega(X) | X \in TM_{\mathbb{C}}\}$. It is easy to show that $E_{\mathcal{J}_\omega}$ is closed under the Courant bracket if and only if $d\omega = 0$, which is precisely the condition for the presymplectic manifold (M, ω) to be symplectic.

From the above examples, one can see that generalized complex geometry unifies ordinary complex geometry and symplectic geometry. In fact, complex geometry and symplectic geometry can be regarded as two extreme cases of generalized geometry. To state this in a more concrete way, we recall the definition of the *type* of a generalized complex manifold. Let $(M, \mathcal{J})$ be a GC manifold of real dimension $2n$, and L be the image of the projection of $E_{\mathcal{J}}$ to $TM \otimes \mathbb{C}$. The type of the generalized complex structure $\mathcal{J}$ at $p \in M$ is defined to be the codimension of $(E_{\mathcal{J}})_x$ in $T_x \otimes \mathbb{C}$. It is easy to see that complex manifolds are examples of type-n GC manifolds while symplectic manifolds are examples of type-0 GC manifolds. There is a large middle ground in between. In addition, the type of a GC manifold is not required to be fixed over the whole manifold. More exotic examples of GC manifolds of varying types are discussed in M. Gualtieri's thesis [12].

When there is a closed 3-form H, the Courant bracket can be twisted. One can define the *twisted* generalized complex (TGC) structure correspondingly. The definition of the almost structure is the same as before, but the integrability is defined via the twisted Courant bracket. In other words, one requires $E_{\mathcal{J}}$ to be closed under $[,]_H$. As in the untwisted case, the twisted Courant bracket $[,]_H$ restricts to a Lie bracket on $E_{\mathcal{J}}$.

2.3 Generalized Kähler structures

A generalized Kähler (GK) structure is a pair of *commuting* GC structures $(\mathcal{J}_1, \mathcal{J}_2)$ such that $G = -\mathcal{J}_1\mathcal{J}_2$ defines a positive definite metric on $T \oplus T^*$. Here we have used the intrinsic inner product $\langle,\rangle$ to identify a metric with an endomorphism. Similarly, in the presence of a closed 3-form H, one can correspondingly define the *twisted* generalized Kähler (TGK) structure, by demanding $(\mathcal{J}_1, \mathcal{J}_2)$ to be a pair of commuting TGC structures.

From a more algebraic viewpoint, a (twisted) generalized Kähler structure is an integrable reduction of the structure group of $T \oplus T^*$ to $U(n) \times U(n)$. We briefly recall the construction, following [12]. The original structure group of $T \oplus T^*$ is $O(2n, 2n)$. A choice of a maximal subbundle C_+ that is positive definite with respect to $\langle,\rangle$ induces a positive definite metric on $T \oplus T^*$, and reduces the structure group to $O(2n) \times O(2n)$. On the other hand, an almost generalized complex structure reduces the structure group to $U(n, n)$. By analogy with Kähler geometry, one requires the positive metric G and the almost generalized complex structure $\mathcal{J}$ to be compatible. If we regard G as an endomorphism on $T \oplus T^*$, this simply means that G commutes with $\mathcal{J}$. One can also show that $G\mathcal{J}$ is another almost generalized complex structure, and it commutes with $\mathcal{J}$. By *integrable* reduction to $U(n) \times U(n)$ we simply demand that both almost generalized complex structures be Courant integrable. Obviously this construction extends to the twisted case with $H \neq 0$.

To see that generalized Kähler structure is really a generalization of the ordinary Kähler structure, let us take a Kähler manifold M, with a complex structure I and a compatible symplectic form ω. We define the following GC structures

$$\mathcal{J}_I = \begin{pmatrix} I & 0 \\ 0 & -I^* \end{pmatrix}, \qquad \mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}.$$

That $\mathcal{J}_I$ commutes with $\mathcal{J}_\omega$ is equivalent to the fact that ω is compatible with I (i.e. ω is of type-$(1,1)$ with respect to I). That $G = -\mathcal{J}_I \mathcal{J}_\omega$ is a positive definite metric on $TM \oplus TM^*$ is automatic, as

$$G = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix},$$

where $g = -\omega I$ is the Hermitian metric on M. Obviously $\langle GA, A \rangle > 0$ for any nonzero $A \in T \oplus T^*$.

The importance of TGK geometry lies in the fact that it is precisely the geometry of general $(2,2)$ supersymmetric sigma models, as we will see in Chapter 3.

2.4 Differential forms and generalized Calabi-Yau manifolds

There are close relationships between the properties of differential operators acting on differential forms and the geometric structure of the manifold. A well-known example is the Dolbeault operator on an almost complex manifold. Given an almost complex structure I on an even dimensional manifold M, the complexified tangent (or cotangent) bundle is decomposed into $\pm i$ eigenbundles of I. Therefore, I induces a second grading on $\Omega^\bullet(M, \mathbb{C})$, the space of complex differential forms on M:

$$\Omega^r(M, \mathbb{C}) = \bigoplus_{p+q=r} \Omega^{p,q}.$$

Let $\bar{\partial}$ be the operator whose action on $\Omega^{p,q}$ is the composition of exterior differential d followed by projection to $\Omega^{p,q+1}$. Similarly define ∂ to be the operator whose action on $\Omega^{p,q}$ is the composition of d followed by projection to $\Omega^{p+1,q}$. It is a well-known theorem in complex geometry that the almost complex structure I is integrable if and only if $d = \bar{\partial} + \partial$.

In the case of twisted generalized complex geometry, there is an analogous, but less familiar story. In order to explain this, let us recall that on an almost TGC manifold $(M, \mathcal{J})$ of dimension $2n$, the canonical bundle is defined to be $\Lambda^{2n} E_{\mathcal{J}}$. Alternatively, one can define the canonical bundle K using differential forms:

$$K = \left\{ \rho \in \Omega^{\bullet}(M, \mathbb{C}) \,\middle|\, (X + \xi) \cdot \rho = 0, \forall\, X + \xi \in \Gamma(E_{\mathcal{J}}) \right\},$$

where the action of $X + \xi$ on $\Omega^{\bullet}(M, \mathbb{C})$ is the Clifford multiplication:

$$(X + \xi) \cdot \rho = \xi \wedge \rho + \iota_X \rho, \qquad \forall X \in T_{\mathbb{C}}, \xi \in T^*_{\mathbb{C}} \,.$$

The canonical bundle K is a uniquely determined line bundle once $\mathcal{J}$ is given.

Using the fact that $(T \oplus T^*)_{\mathbb{C}} \simeq E \oplus E^* \simeq E \oplus \bar{E}$, and that E annihilates K, we can give an alternative grading to the space of complex differential forms

$$\Omega^{\bullet}(M, \mathbb{C}) \simeq U_0 \oplus U_1 \oplus \cdots \oplus U_{2n} \,, \tag{2.4.1}$$

where $U_0 = K$ is the canonical bundle, and $U_k \equiv \Lambda^k \bar{E} \cdot U_0$.[4] Let us also define the twisted differential $d_H = d - H\wedge$. Similar to the case of ordinary complex geometry, we introduce the "generalized Dolbeault" operators. It is enough to specify their action on U_k. We define $\bar{\partial}_H : \Gamma(U_k) \to \Gamma(U_{k+1})$ by the composition of d_H followed by projection to $\Gamma(U_{k+1})$. Similarly, we define $\partial_H : \Gamma(U_k) \to \Gamma(U_{k-1})$ by the composition of d_H and projection to $\Gamma(U_{k-1})$. We have the following theorem:

[4]Intuitively, one can regard elements in E to be annihilation operators, those in $\bar{E}$ creation operators, and $K = U_0$ the "vacuum" bundle. That one only needs to consider antisymmetric products of elements in $\bar{E}$ is because when acting on $\Omega^{\bullet}(M)$, they can always be expressed as operators with less degrees via Clifford algebra.

Theorem. *The almost TGC structure* $\mathcal{J}$ *is integrable if and only if* $d_H = \partial_H + \bar{\partial}_H$.

The proof of the theorem in the special case of $H = 0$ is given by Gualtieri [12], while the more general form as stated above is proven in [22]. Here we shall give a proof of this important theorem following [22].

Let ρ be an arbitrary differential form, and let $A = X + \xi$, $B = Y + \eta$ be arbitrary sections of E. It is straightforward to show (using Eq. (4.1.5)) and the Cartan identities $\mathcal{L}_X = \iota_X \circ d + d \circ \iota_X$, $\iota_{[X,Y]} = [\mathcal{L}_X, \iota_Y]$) that

$$A \cdot B \cdot d\rho = d(BA\rho) + B \cdot d(A\rho) - A \cdot d(B\rho) + [A, B]\rho \tag{2.4.2}$$

$$\begin{aligned} A \cdot B \cdot (H \wedge \rho) &= -\iota_Y \iota_X H \wedge \rho + \iota_Y H \wedge (A\rho) \\ &\quad -\iota_X H \wedge (B\rho) + H \wedge (AB\rho). \end{aligned} \tag{2.4.3}$$

Subtracting (2.4.3) from (2.4.2), one obtains

$$A \cdot B \cdot d_H\rho = d_H(BA\rho) + B \cdot d_H(A\rho) - A \cdot d_H(B\rho) + [A, B]_H \cdot \rho. \tag{2.4.4}$$

The rest of the proof now follows exactly as in Ref. [12]. First let us assume $\mathcal{J}$ is integrable. For $\rho \in \Gamma(U_0)$, (2.4.4) reduces to $AB \cdot d_H\rho = [A, B]_H \cdot \rho = 0$. Since $d_H\rho$ has no component in U_0, it follows that $d(\Gamma(U_0)) \subset \Gamma(U_1)$ and thus $d_H = \partial_H + \bar{\partial}_H$ holds for $\rho \in \Gamma(U_0)$. Now assume $d_H = \partial_H + \bar{\partial}_H$ holds for all $U_k, 0 \leq k < i$, and let $\rho \in \Gamma(U_i)$ and $A, B \in \Gamma(E)$ as before. Equation (2.4.4) now shows that $AB \cdot d_H\rho \in \Gamma(U_{i-3} \oplus U_{i-1})$, which in turn implies $d_H\rho \in \Gamma(U_{i-1} \oplus U_{i+1})$. By induction, one concludes that $d_H = \partial_H + \bar{\partial}_H$ on $\Lambda^\bullet TM^* \otimes \mathbb{C}$. The converse is also true by a similar argument.

Now we can give a definition for twisted generalized Calabi-Yau (TGCY) manifolds. A twisted generalized Calabi-Yau structure is a twisted generalized complex structure that satisfies the following two conditions:

- There exists a nowhere zero section ρ of the canonical bundle.

- ρ is d_H-closed, i.e., $d_H\rho = 0$.

The first condition simply means that the canonical bundle K is topologically trivial. The second condition is more interesting. Recall that ∂_H always decreases the grading by one. Since $\rho \in \Gamma(K)$ already has the lowest grading, $\partial_H \rho = 0$. Therefore $d_H \rho = 0$ is equivalent to $\bar{\partial}_H \rho = 0$, in viewing of the above theorem. This means ρ is a nowhere zero generalized holomorphic section. Alternatively, one can rephrase the TGCY condintion as follows: *a twisted generalized complex manifold is twisted generalized Calabi-Yau if and only if its canonical bundle has a nowhere vanishing generalized holomorphic section.* As we will see in Chapter 3, the condition for TGCY manifold is precisely the condition that a $(2,2)$ sigma model can be twisted into a topological quantum field theory.

An ordinary Calabi-Yau manifold is clearly a GCY manifold. Interestingly, symplectic manifolds provide another class of examples of GCY manifolds. If (M, ω) is a symplectic manifold, then the associated GCY structure is the one corresponding to $\rho = e^{i\omega}$. Obviously the GCY condition is satisfied and (M, ω) defines a GCY structure.

In [12], M. Gualtieri also introduced a much stronger version of GCY, the generalized Calabi-Yau *metric*. Roughly speaking, a generalized Calabi-Yau metric is a generalized Kähler structure $(\mathcal{J}_1, \mathcal{J}_2)$ such that each of the generalized complex structures defines a generalized Calabi-Yau structure. The (twisted) generalized Calabi-Yau metrics are also important for us. Although the weaker form of (twisted) GC structure is all one needs in order to extract a topological field theory from the $(2,2)$ sigma model, the topological theory has certain "nice" properties if the stronger form of (twisted) GCY metric holds. For example, it can be related to $N = 2$ space-time supersymmetry if the stronger form is satisfied.

Chapter 3

Topological Theories from $N = (2,2)$ Sigma Models

3.1 $N = (2,2)$ sigma models in two dimensions

The topological sigma models that we will be studying are constructed from $(2,2)$ supersymmetric sigma models by twisting. In this section, we review the basic facts about the general $(2,2)$ supersymmetric sigma models and set up the notation. Let Σ be a two dimensional Riemann surface (the worldsheet) and M a Riemannian manifold (the target). We shall take a Lorentz metric on Σ for the moment. Let $\phi : \Sigma \to M$ be a map from the worldsheet to the target. It gives the fields of the bosonic sigma model. It is well-known that the bosonic sigma model in $1+1$ dimensions with any Riemannian target manifold always admits a $(1,1)$ supersymmetric extension. The action of the $(1,1)$-extended theory can be conveniently written in $(1,1)$ superspace. let us introduce $(1,1)$ superfields

$$\Phi^a = \phi^a + \theta^+ \psi_+^a + \theta^- \psi_-^a + \theta^- \theta^+ F^a.$$

Here $\psi_\pm$ are worldsheet fermions of left/right chirality. In mathematical terms, they are sections of $\phi^*(TM) \otimes S_\pm$, $S_\pm$ being the two inequivalent spinor bundles on Σ. F is an auxiliary field that will be integrated out eventually. We also introduce the standard supercovariant derivatives

$$D_+ = \frac{\partial}{\partial\theta^+} + i\theta^+\partial_+, \qquad D_- = \frac{\partial}{\partial\theta^-} + i\theta^-\partial_-, \qquad \partial_\pm \equiv \partial_0 \pm \partial_1.$$

We will take the manifest $(1,1)$ supersymmetric action to be

$$S \;=\; \frac{1}{2}\int \mathrm{d}^2\sigma \mathrm{d}^2\theta \big(g_{ab}(\Phi) + B_{ab}(\Phi)\big) D_+\Phi^a D_-\Phi^b. \tag{3.1.1}$$

Here g is a Riemannian metric and B a 2-form potential for the Neuve-Schwarz 3-form H. Strictly speaking B is not a globally defined 2-form if H is a nontrivial class in $H^3(M)$. At the quantum level, H is quantized so that $[H/2\pi]$ must define a class in $H^3(M,\mathbb{Z})$ for the path integral to be well-defined.

The action is invariant under the manifest $(1,1)$ supersymmetry generated by

$$Q_+ = \frac{\partial}{\partial\theta^+} - i\theta^+\partial_+, \qquad Q_- = \frac{\partial}{\partial\theta^-} - i\theta^-\partial_-.$$

When the target manifold M possesses additional structure, the theory may have a larger supersymmetry. For example, it is well-known that when (M, g) is Kähler and $H = 0$, the theory has $(2,2)$ supersymmetry. A natural question is whether $(2,2)$ supersymmetry implies Kähler geometry. This has been answered in the negative by Gates, Hull, and Rocek [9]. We shall summarize their results below.

In order to get $(2,2)$ supersymmetry, one needs to find one more supersymmetry generator for each chirality, $\tilde{Q}_\pm$, such that $(Q_\pm, \tilde{Q}_\pm)$ generate $(2,2)$ supersymmetry algebra. They must take the following form by dimensional reasons:

$$\tilde{\delta}\Phi^a = (\tilde{\epsilon}^+\tilde{Q}_+ + \tilde{\epsilon}^-\tilde{Q}_-)\Phi^a = \tilde{\epsilon}^+ I_+(\Phi)^a{}_b D_+\Phi^b + \tilde{\epsilon}^- I_-(\Phi)^a{}_b D_+\Phi^b,$$

where $I_\pm$ are certain sections of the endomorphism bundle $\mathrm{End}\,(TM)$. The requirement that $(Q_\pm, \tilde{Q}_\pm)$ generate the $(2,2)$ algebra, and that the action be invariant under this larger supersymmetry, boils down to the following conditions [9]:

- (I_+, I_-) are a pair of integrable almost complex structures on TM.

- The metric g is Hermitian with respect to both I_+ and I_-.

- I_+ and I_- are covariantly constant with respect to a pair of connections $\nabla^{(\pm)} = \nabla \pm \frac{1}{2}g^{-1}H$, respectively:

$$\nabla^{(+)} I_+ = 0, \qquad \nabla^{(-)} I_- = 0.$$

  Here ∇ denotes the Levi-Civita connection on TM. The connections $\nabla^{(\pm)}$ are torsionful, with the torsion given by

$$T = \pm\frac{1}{2}g^{-1}H.$$

In addition to having two independent complex structures, the difference from the usual Kähler geometry also lies in the fact that in general (g, I_+) (or (g, I_-)) do not define a Kähler structure. This means that the two forms $\omega_\pm = gI_\pm$ are not closed in general due to the presence of torsion.

An interesting special case is when $[I_+, I_-] = 0$. In this case, I_+ and I_- can be simultaneously diagonalized and one can decompose $TM_{\mathbb{C}} = \ker(I_+ - I_-) \oplus \ker(I_+ + I_-)$. It can be shown [9] that $\ker(I_+ - I_-)$ is an integrable distribution and it can be parametrized by $N = 2$ chiral superfields. On the other hand, $\ker(I_+ + I_-)$ is an integrable distribution that can be parametrized by twisted chiral superfields. A manifold with the property $[I_+, I_-] = 0$ is said to admit a product structure. The name comes from the fact that locally, such a manifold is a product of two Kähler manifolds, although globally it is not Kähler in general. Another interesting class of examples is provided by hyper-Kähler manifolds, which admit a family of complex structures parametrized by $\vec{x} \in S^2$. One may take I_+ and I_- to be any two complex structures parametrized by two points $\vec{x}_\pm \in S^2$. With a generic choice, one has $\ker[I_+, I_-] = 0$. This is in a sense the opposite to the case with product structure. We refer to refs. [9, 33, 20, 30] for more details on related issues.

It is a remarkable fact that the geometric structure (g, I_+, I_-, H) defined by the $(2, 2)$ supersymmetric sigma model is equivalent precisely to the twisted generalized Kähler structure

[12]. The explicit map between the two descriptions is

$$\mathcal{J}_1 = \begin{pmatrix} \tilde{I} & -\alpha \\ \delta\omega & -\tilde{I}^t \end{pmatrix}, \qquad \mathcal{J}_2 = \begin{pmatrix} \delta I & -\beta \\ \tilde{\omega} & -\delta I^t \end{pmatrix}, \tag{3.1.2}$$

with

$$\begin{aligned} \tilde{I} &= (I_+ + I_-)/2, & \delta I &= (I_+ - I_-)/2, \\ \beta &= (\omega_+^{-1} + \omega_-^{-1})/2, & \alpha &= (\omega_+^{-1} - \omega_-^{-1})/2, \\ \tilde{\omega} &= (\omega_+ + \omega_-)/2, & \delta\omega &= (\omega_+ - \omega_-)/2. \end{aligned}$$

Both α and β can be shown to be Poisson bivectors [30].

For later use, we give the explicit expression for the action of the $(2,2)$ sigma model, after integrating out the auxiliary fields F^a:

$$\begin{aligned} S \;=\; \int_M \frac{1}{2}(g_{ab} + B_{ab})\partial_+\phi^a\partial_-\phi^b \;&+\; \frac{i}{2} g_{ab}\left(\psi_+^a \nabla_-^{(+)} \psi_+^b + \psi_-^a \nabla_+^{(-)} \psi_-^b\right) \\ &+\; \frac{1}{4} R^{(+)}_{abcd} \psi_+^a \psi_+^b \psi_-^c \psi_-^d, \end{aligned} \tag{3.1.3}$$

where $R^{(+)}_{abcd}$ is the Riemann tensor with respect to the connection $\nabla^{(+)}$:

$$R^{(+)}{}_{ab}{}^{c}{}_{d} = \partial_a \Gamma_+{}^c{}_{bd} - \partial_b \Gamma_+{}^c{}_{ad} + \Gamma_+{}^c{}_{ae} \Gamma_+{}^e{}_{bd} - \Gamma_+{}^c{}_{be} \Gamma_+{}^e{}_{ad}.$$

Here $\Gamma_\pm$ are the Christoffel symbols for the connections $\nabla^{(\pm)}$, respectively. When H vanishes, they are simply the familiar Christoffel symbols for the Levi-Civita connection.

3.2 Twisting the $N = (2,2)$ sigma model

In this section, we construct a twisted version of the $(2,2)$ sigma model. This section is largely based on [22]. In the original construction of A-model and B-model by Witten [40], the target manifold is an ordinary Kähler manifold, with vanishing Neveu-Schwarz 3-form. We know from previous discussion that, with a possibly nontrivial NS 3-form, the natural geometric structure for the $(2,2)$ sigma models is the twisted generalized Kähler structure. In addition to the 3-form flux, it also allows for two independent complex structures. We will see below that this more general class of sigma models can also be twisted in the sense of [40], if extra topological and geometrical conditions are satisfied.

The general procedure of twisting is to shift the worldsheet spin of all the fields by the charges of a conserved $U(1)_R$ symmetry. Since R-symmetry leaves bosons intact, only the spin of the fermions is changed. The purpose of doing this is to make all fields in the theory to have integral spin on the worldsheet. Of particular importance is that, after the twisting, certain supersymmetry generators will become worldsheet scalars and can be used to construct the BRST operator. Obviously for this procedure to work (i.e. to yield a theory all of whose fields have integral worldsheet spin), an integrality condition must be satisfied: for any field φ of spin $s(\varphi)$ and $U(1)_R$ charge $q_R(\varphi)$,

$$s(\varphi) + \frac{1}{2}\, q_R(\varphi) \in \mathbb{Z}.$$

In such a case, we simply shift the spin of any field by one-half of its $U(1)_R$ charge.

In the case of $(2,2)$ sigma models discussed above, one can indeed find such a $U(1)$ R-symmetry. In fact, as in the more special case considered in [40], there are two inequivalent $U(1)_R$ symmetries. The two complex structures induce two different decompositions of the complexified tangent bundle

$$TM_{\mathbb{C}} \simeq T_+^{1,0} \oplus T_+^{0,1} \simeq T_-^{1,0} \oplus T_-^{0,1}.$$

With such decompositions, the fermionic fields $\psi_\pm$ splits accordingly into the holomorphic and anti-holomorphic components:

$$\psi_+ = \frac{1}{2}(1 - iI_+)\psi_+ + \frac{1}{2}(1 + iI_+)\psi_+, \quad \psi_- = \frac{1}{2}(1 - iI_-)\psi_- + \frac{1}{2}(1 + iI_-)\psi_-.$$

At the classical level, there are two inequivalent ways to assign $U(1)$ R-charges to fermions (the bosons have zero charge):

$$\begin{aligned} U(1)_V: \quad & q_V\Big(\frac{1}{2}(1-iI_+)\psi_+\Big) = -1, \qquad q_V\Big(\frac{1}{2}(1-iI_-)\psi_-\Big) = -1; \\ U(1)_A: \quad & q_A\Big(\frac{1}{2}(1-iI_+)\psi_+\Big) = -1, \qquad q_A\Big(\frac{1}{2}(1-iI_-)\psi_-\Big) = 1. \end{aligned}$$

Obviously both $U(1)_R$ symmetries satisfy the above integrability condition, and therefore can be used to twist the theory: we simply shift the spin of the fermions by half of the $U(1)_V$ or $U(1)_A$ charges. We call the operation that shifts the spin by $U(1)_V$ (resp. $U(1)_A$) the A-twist (resp. the B-twist). Note that flipping the sign of I_- interchanges the A-twist and the B-twist. The following table summarizes the worldsheet spin of fermions as well as the supercharges after the twist. Note that we use the shorthand notation $\psi_\pm^{1,0} = \frac{1}{2}(1-iI_\pm)\psi_\pm$, $\psi_\pm^{0,1} = \frac{1}{2}(1+iI_\pm)\psi_\pm$.

	$Q_+ + i\tilde{Q}_+$	$Q_+ - i\tilde{Q}_+$	$Q_- + i\tilde{Q}_-$	$Q_- - i\tilde{Q}_-$	$\psi_+^{0,1}$	$\psi_+^{1,0}$	$\psi_-^{0,1}$	$\psi_-^{1,0}$
A-twist	0	1	1	0	0	1	1	0
B-twist	0	1	0	1	0	1	0	1

An important fact to note is that the spin-0 fields can always be expressed as sections of an eigenbundle of $\mathcal{J}_1$ of $\mathcal{J}_2$, depending on the twist. For concreteness, let us consider the B-twist. The spin-0 fermions are $\chi \equiv \psi_+^{0,1}$ and $\lambda \equiv \psi_-^{0,1}$. Alternatively, one can take the independent spin-0 fermions to be $\chi+\lambda$ and $g(\chi-\lambda)$. These fields can be regarded as living in the bundle E_1, the $-i$-eigenbundle of $\mathcal{J}_1$ in $(T \oplus T^*) \otimes \mathbb{C}$. To see that, let us introduce the following fields:

$$\psi = \psi_+ + \psi_-, \qquad \rho = g(\psi_+ - \psi_-).$$

We regard $\psi \in \Gamma(TM_{\mathbb{C}})$ and $\rho \in \Gamma(TM^*_{\mathbb{C}})$, and their direct sum $(\psi,\rho)^T$ as a section of $(TM \oplus TM^*) \otimes \mathbb{C}$. It is easy to show, recalling the defining equation (3.1.2) for $\mathcal{J}_1$, that

$$\begin{pmatrix} \chi+\lambda \\ g(\chi-\lambda) \end{pmatrix} = (1+i\mathcal{J}_1)\begin{pmatrix} \psi \\ \rho \end{pmatrix}. \tag{3.2.1}$$

The importance of this relation will become apparent when we construct the local observables of the twisted theory in the next chapter.

So far our analysis has been classical. For the A-twist (resp. the B-twist) to make sense at the quantum level, one must make sure that $U(1)_V$ (resp. $U(1)_A$) is anomaly-free. Mathematically, the anomalies are computed by the Atiyah-Singer index theorem, and the resulting anomaly-free conditions are

$$\begin{aligned} U(1)_V: &\qquad c_1(T_-^{1,0}) - c_1(T_+^{1,0}) = 0, \\ U(1)_A: &\qquad c_1(T_-^{1,0}) + c_1(T_+^{1,0}) = 0. \end{aligned} \tag{3.2.2}$$

It is possible to express the anomaly conditions in terms of twisted generalized complex structures. Recall that $(2,2)$ sigma models require the target space M to be a twisted generalized Kähler manifold, with a pair of commuting twisted generalized complex structures $(\mathcal{J}_1, \mathcal{J}_2)$ and a positive definite metric $\mathcal{G} = -\mathcal{J}_1\mathcal{J}_2$ on $TM \oplus TM^*$. As mentioned in Chap. 2, we use the non-degenerate inner product $\langle,\rangle$ to identify $TM \oplus TM^*$ with its dual, so $\mathcal{G} = -\mathcal{J}_1\mathcal{J}_2$ can be regarded as an endomorphism on $TM \oplus TM^*$. Obviously $\mathcal{G}^2 = 1$. Let $C_\pm$ be the ± 1 eigenbundles of $\mathcal{G}$. The natural projection from $TM \oplus TM^*$ to TM induces bundle isomorphisms

$$\pi_+ : C_+ \simeq TM, \qquad \pi_- : C_- \simeq TM.$$

The twisted generalized complex structure $\mathcal{J}_1$ naturally induces a pair of complex structures (I_+, I_-) on TM via $\pi_\pm$. More specifically, the action of I_+ on a section X of TM is obtained from first lifting X to a section $\tilde{X} = \pi_+^{-1}(X)$ on C_+, acting on it by $\mathcal{J}_1$, and finally projecting back to TM via π_+. This action is well-defined since $\mathcal{J}_1$ preserves C_+. Obviously I_+ so defined is an almost complex structure. It is in fact integrable because $\mathcal{J}_1$ is twisted Courant integrable. The other complex structure I_- is defined similarly, except that one uses the isomorphism π_-. One can show that these are precisely the complex structures $I_\pm$ that appeared in the $(2,2)$ sigma model.

Let E_1 and E_2 denote the i-eigenbundles of $\mathcal{J}_1$ and $\mathcal{J}_2$, respectively. Since $\mathcal{J}_1$ and $\mathcal{J}_2$

commute, one has the further decompositions $E_1 = E_1^+ \oplus E_1^-$ and $E_2 = E_2^+ \oplus E_2^-$, where the superscripts $\pm$ label the eigenvalues of the other twisted generalized complex structure. For example, E_1^- has $\mathcal{J}_1 = i, \mathcal{J}_2 = -i$. It follows that

$$C_\pm \otimes \mathbb{C} = E_1^\pm \oplus (E_1^\pm)^* = E_2^\pm \oplus (E_2^\pm)^*$$

and from the definition of $\pi_\pm$, we obtain the following bundle isomorphisms:

$$E_1^+ \simeq T_+^{1,0}, \qquad E_1^- \simeq T_-^{1,0}.$$

Now we can rewrite Eq. (3.2.2) in terms of bundles E_1 and E_2:

$$\begin{aligned} U(1)_V : \qquad & c_1(E_2) = 0, \\ U(1)_A : \qquad & c_1(E_1) = 0. \end{aligned} \tag{3.2.3}$$

From this equation, we immediately see that $U(1)_A$ (resp. $U(1)_V$) is anomaly-free if and only if the canonical bundle for $\mathcal{J}_1$ (resp. $\mathcal{J}_2$) is topologically trivial. As already discussed in Chapter 2, this is precisely the first requirement for $(M, \mathcal{J}_1)$ (resp. $(M, \mathcal{J}_2)$) to be a twisted generalized Calabi-Yau manifold. Of course, for $(M, \mathcal{J}_{1,2})$ to really define twisted generalized Calabi-Yau structures, the second requirement that the canonical bundle be holomorphically trivial must also be satisfied. In the next section, we will show that this second requirement guarantees the absence of BRST anomaly.

As already mentioned, flipping the relative sign of $I_\pm$ exchanges the A-twist and B-twist. This symmetry is also reflected at the level of anomaly-free conditions, since flipping the relative sign of $I_\pm$ is equivalent to exchanging $\mathcal{J}_1$ and $\mathcal{J}_2$. Due to this symmetry, we will only consider the B-twist explicitly in the following.

3.3 The BRST anomaly

The ordinary B-model is a cohomological field theory; namely, it is equipped with a BRST operator, which is an odd operator that squares to zero, and its physical observables are classified by the BRST cohomology. The B-twisted theory obtained above also shares this important property. Its BRST operator, denoted by Q_B, is constructed from a linear combination of the spin-0 components of the twisted $(2,2)$ supercharges, and $Q_B^2 = 0$ follows directly from the $(2,2)$ algebra. We shall describe the operator Q_B and its cohomology in more detail in Chapter 4. Classically, Q_B is a BRST symmetry of the B-twisted model, as it preserves the action. In this section, we address the question of when Q_B generates a BRST symmetry even at the quantum level. In other words, we look for possible anomalies of the BRST symmetry. From the path integral point of view, the BRST anomaly shows up whenever the path integral measure is not BRST-invariant. As is usual, only the fermionic zero modes contribute to the anomaly, and the anomaly can be computed by an index theorem.

Let us consider tree-level amplitudes in the B-twisted theory. It follows from the Riemann-Roch theorem that the number of excessive $\psi_+^{0,1}$ and $\psi_-^{0,1}$ zero modes is given by

$$2\text{dim}_{\mathbb{C}}X + c_1(\phi^* T_+^{0,1}) + c_1(\phi^* T_-^{0,1}) = 2n + \phi^* c_1(E_1),$$

where n is the complex dimension of M. As mentioned in the previous section, the B-twist makes sense only if $c_1(E_1) = 0$, so the second term above vanishes and the total number of excessive $\psi_\pm^{0,1}$ zero modes is simply $2\text{dim}_C X$. As argued in the previous section, functions of these zero modes can be interpreted as $C^\infty(\Pi E_1) \simeq \Lambda^\bullet(E_1^*)$. The path integral measure for the zero modes is simply a choice of a nowhere zero section Ω of $\Lambda^{\text{top}}(E_1) = \Lambda^{2n}(E_1)$. The latter is precisely the canonical bundle for the complex Lie algebroid E_1. As the canonical bundle is topologically trivial, such a nowhere zero global section always exists. The BRST invariance of the quantum theory imposes a further condition that this section should be annihilated by Q_B. As we will show in Chapter 4, Q_B is identified as the Lie algebroid differential d_{E_1}. When represented on the Lie algebroid module $\Omega^\bullet(M, \mathbb{C})$, the measure for the zero modes is a nowhere vanishing section Ω of U_0 (see Eq. (2.4.1)), and Q_B is identified with the generalized Dolbeault operator $\bar{\partial}_H$. The requirement that Q_B annihilate the measure for the zero modes is precisely the second condition in the definition of twisted generalized Calabi-Yau structures that the canonical bundle should be generalized holomorphically trivial.

3.4 Topological invariance of the twisted theory

The geometric picture emerging so far is rather pleasing. We have found that the general $N = (2,2)$ sigma model can be twisted into a very special quantum field theory, if and only if the target manifold is endowed with a (twisted) generalized Calabi-Yau structure. This resulting quantum field theory carries a nilpotent fermionic operator Q_B, and it has the unusual spin-statistics property that all its fields, bosonic or not, have integral spin on the worldsheet. In this section, we show that the twisted theory is in fact a topological field theory. As always, we shall only consider the B-twist explicitly as the A-twist can be obtained by simply exchanging the two twisted GC structures.

Let us start by renaming the fermionic fields:

$$\chi = \frac{1}{2}(1 + iI_+)\psi_+, \qquad \lambda = \frac{1}{2}(1 + iI_-)\psi_-, \tag{3.4.1}$$

$$\rho_z = \frac{1}{2}(1 - iI_+)\psi_+, \qquad \eta_{\bar{z}} = \frac{1}{2}(1 - iI_-)\psi_-. \tag{3.4.2}$$

Note that the subscripts on ρ and η denotes their worldsheet spin. In order to write down a worldsheet covariant action, let us introduce a worldsheet metric $h_{\mu\nu}$ and the epsilon symbol $\epsilon^{\mu\nu}$ such that $\epsilon^{\bar{z}z} = -\epsilon^{z\bar{z}} = i$. We promote ρ_z and $\eta_{\bar{z}}$ to covariant spin-1 fields

$$\rho_z \to \rho_\mu, \qquad \eta_{\bar{z}} \to \eta_\mu,$$

which satisfy the following (anti)self-dual conditions:

$$\rho_\mu = -i\varepsilon_\mu{}^\nu \rho_\nu, \qquad \eta_\mu = i\varepsilon_\mu{}^\nu \eta_\nu\,. \tag{3.4.3}$$

Here $\varepsilon_\mu{}^\nu$ is the tensor defined by

$$\varepsilon_\mu{}^\nu = \frac{\epsilon_\mu{}^\nu}{\sqrt{h}}.$$

Obviously $\varepsilon_z{}^z = -\varepsilon_{\bar z}{}^{\bar z} = i$, and it can be regarded as a complex structure on the worldsheet.

From this point on in this chapter, we will set $H = 0$ for simplicity, although the results should hold even when $H \neq 0$. In this case, we can set $B = 0$, since a non-zero B-field only contributes a topological term to the action, so it has no effect on our analysis. In the covariantized form, the action reads

$$\begin{aligned} S &= \int_\Sigma \mathrm{d}^2\sigma \sqrt{h} h^{\mu\nu} \Big\{ \frac{1}{4} g_{ab} \partial_\mu \phi^a \partial_\nu \phi^b + i g_{ab} \rho^a_\mu D_\nu \chi^b \\ &\qquad + i g_{ab} \eta^a_\mu D_\nu \lambda^b + R_{abcd} \rho^a_\mu \chi^b \eta^c_\nu \lambda^d \Big\}. \end{aligned} \tag{3.4.4}$$

Note that in the case of $H = 0$, there is no need to distinguish $\nabla^{(+)}$ and $\nabla^{(-)}$, and we write D_μ for the covariant derivative on the fermions.

The next step is to write down the covariantized BRST transformations for the fields. They turn out to be:

$$\{Q_B, \phi^a\} = \chi^a + \lambda^a, \tag{3.4.5}$$

$$\{Q_B, \chi^a\} = \Gamma^a{}_{bc} \chi^b \eta^c, \tag{3.4.6}$$

$$\{Q_B, \lambda^a\} = \Gamma^a{}_{bc} \lambda^b \chi^c, \tag{3.4.7}$$

$$\{Q_B, \rho^a_\mu\} = \frac{i}{2}\Big((\delta_\mu{}^\nu - i\varepsilon_\mu{}^\nu) \frac{1}{2} (1 - iI_+)^a_b \partial_\nu \phi^b \Big) + \Gamma^a{}_{bc} \rho^b_\mu (\chi + \lambda)^c, \tag{3.4.8}$$

$$\{Q_B, \eta^a_\mu\} = \frac{i}{2}\Big((\delta_\mu{}^\nu + i\varepsilon_\mu{}^\nu) \frac{1}{2} (1 - iI_-)^a_b \partial_\nu \phi^b \Big) + \Gamma^a{}_{bc} \eta^b_\mu (\chi + \lambda)^c. \tag{3.4.9}$$

It is easy to check that the (anti)self-dual equation (3.4.3) is preserved by the covariant BRST transformation.

Let us define

$$V = -\frac{i}{2}\sqrt{h}h^{\mu\nu}g_{ab}\big(\rho_\mu^a\partial_\nu\phi^b + \eta_\mu^a\partial_\nu\phi^b\big).$$

It is straightforward to show that the covariant action in (3.4.4) can be written in the following form:

$$\begin{aligned} S &= \{Q_B, V\} + \frac{1}{4}\int_\Sigma \mathrm{d}^2\sigma\, \epsilon^{\mu\nu}(\delta\omega)_{ab}\partial_\mu\phi^a\partial_\nu\phi^b \\ &+ \int_\Sigma \mathrm{d}^2\sigma\sqrt{h}h^{\mu\nu}\Big(\frac{i}{2}g_{ab}(\rho_\mu^a - \eta_\mu^a)D_\nu(\chi-\lambda)^b + R_{abcd}\rho_\mu^a\chi^b\eta_\nu^c\lambda^d\Big). \end{aligned} \tag{3.4.10}$$

What do we learn from Eq. (3.4.10)? Were the second line not there, we would have shown the topological invariance of the generalized B-model. This is because all the dependence on the worldsheet metric would then be contained in $\{Q_B, V\}$.[1] This implies that variation of the worldsheet metric only changes the action by Q_B-exact terms. In other words, the stress-energy tensor is Q_B-exact. It is well known that perturbing the action by BRST-exact terms does not change the theory.

The second line in Eq.(3.4.10) does appear to explicitly depend on the worldsheet metric. However we will show in the following that it vanishes on shell. More precisely, it vanishes after we impose the equations of motion for ρ_μ and η_μ. The slightly unsatisfactory situation that the Q_B-exactness of the stree-energy tensor only holds on shell is perhaps not surprizing, in view of the fact that not only did we integrate out the auxilliary fields F^a, but also the $N = (2,2)$ algebra itself used to construct Q_B holds only on-shell. This being said, the on shell Q_B-exactness of the stress-energy tensor is all we need to guarantee the topological invariance of the generalized B-model.

From the covariant action (3.4.4) it is easy to read off the following equations of motion:

$$\begin{aligned} ig_{ab}D_\nu\chi^b + R_{abcd}\chi^b\eta_\nu^c\lambda^d &= 0, \\ ig_{ab}D_\nu\lambda^b + R_{abcd}\lambda^b\rho_\nu^c\chi^d &= 0. \end{aligned}$$

[1]Note that the second term in the first line in (3.4.10) can be expressed as an integral of differential forms on Σ, so it does not depend on the worldsheet metric.

It follows that

$$\begin{aligned}\frac{i}{2}g_{ab}\big((\rho^a_\mu - \eta^a_\mu)D_\nu(\chi^b - \lambda^b)\big) &= -\frac{1}{2}R_{abcd}(\rho^a_\mu\chi^b\eta^c_\nu\lambda^d + \eta^a_\mu\lambda^b\rho^c_\nu\chi^d) \\ &\quad +\frac{1}{2}R_{abcd}(\rho^a_\mu\lambda^b\rho^c_\nu\chi^d + \eta^a_\mu\chi^b\eta^c_\nu\lambda^d).\end{aligned}$$

The first line on the RHS in (3.4.11) can be shown to give $-R_{abcd}\rho^a_\mu\chi^b\eta^c_\nu\lambda^d$, using the first Bianchi identity

$$R_{abcd} + R_{acdb} + R_{adbc} \;=\; 0.$$

In addition, the contraction of the second line in (3.4.11) with the inverse metric $h^{\mu\nu}$ is zero, a direct consequence of the (anti)self-dual property of the spin-1 fields ρ and η. Therefore we see that the second line in Eq. (3.4.10) precisely cancels out after imposing the EOMs for χ and λ. Introducing $\omega_\pm = gI_\pm$ and $\delta\omega = (\omega_+ - \omega_-)/2$ as in Chapter 2, one can rewrite (3.4.10) concisely as

$$S \;=\; \int_\Sigma \mathrm{d}^2\sigma\{Q_B, V\} + \frac{1}{2}\int_\Sigma \phi^*(\delta\omega), \qquad \text{on shell.} \tag{3.4.11}$$

From this key equation it is obvious that the stress-energy tensor is Q_B-exact; namely $T^{\mu\nu} = \{Q_B, b^{\mu\nu}\}$, where $b^{\mu\nu} = \delta V/\delta h_{\mu\nu}$. This confirms that the B-twisted theory is indeed a topological field theory. Extending the usual terminology, we shall call it the generalized B-model, although our construction makes the A-twist and B-twist completely symmetric. In particular, ordinary A-model and B-model are simply two extreme limits of the generalized B-model.

We will come back to Eq.(3.4.11) in Chapter 4 when we study the instanton corrections in the generalized B-model.

Chapter 4

Properties of the Generalized B-model

4.1 Topological observables

4.1.1 A differential bicomplex

Let us recall that the fermionic fields of the B-twisted theory are χ, λ, ρ_z, and $\eta_{\bar{z}}$. As discussed in detail in Chap. 3, the first two are spin-0 fields on the worldsheet, while ρ_z (resp. $\eta_{\bar{z}}$) is a $(1,0)$-form (resp. $(0,1)$-form) on the worldsheet. There is a hierarchy of observables in the topological sigma model. The most fundamental ones are 0-forms on the worldsheet. There are also 1-form and 2-form observables in the theory; in particular the 2-form observables correspond to deformations of the original Lagrangian. In this thesis, we shall only consider the fundamental 0-form observables, since other observables can be constructed from the fundamental ones by descent equations [40, 7].

The fundamental 0-form observables of the genealized B-model are constructed from spin-0 fields, namely ϕ, χ, and λ. We call smooth functions of these spin-0 fields preobservables. A preobservable is not an observable yet. Like in the case of gauge theory, a preobservable is admissible if and only if it is BRST invariant. Also a standard argument shows that adding BRST-exact pieces to an admissible preobservable does not change the associated observable. This means the space of 0-form observables is identified with the BRST cohomology.

We recall that the BRST operator of the generalized B-model is given by

$$Q_B = Q_L + Q_R, \qquad Q_L \equiv \frac{1}{2}(Q_+ + i\tilde{Q}_+), \qquad Q_R \equiv \frac{1}{2}(Q_- + i\tilde{Q}_-).$$

As we saw in Table 3.2, both Q_L and Q_R are worldsheet scalars. Furthermore, they satisfy $Q_L^2 = 0$, $Q_R^2 = 0$, and $\{Q_L, Q_R\} = 0$. First let us write down the explicit transformations of scalar fields under Q_L and Q_R:

$$\begin{aligned}
\{Q_L, \phi^a\} &= \chi^a, \\
\{Q_L, \chi^a\} &= 0, \\
\{Q_L, \lambda^a\} &= -\Gamma_-{}^a{}_{bc}\chi^b\lambda^c, \\
\{Q_R, \phi^a\} &= \lambda^a, \\
\{Q_R, \lambda^a\} &= 0, \\
\{Q_R, \chi^a\} &= -\Gamma_+{}^a{}_{bc}\lambda^b\chi^c.
\end{aligned} \tag{4.1.1}$$

The preobservables of the theory must take the following form

$$\mathcal{O}_f \;=\; f_{a_1\cdots a_p;b_1\cdots b_q}\chi^{a_1}\cdots\chi^{a_p}\lambda^{b_1}\cdots\lambda^{b_q},$$

where f is totally anti-symmetric in a's as well as in b's. Recall that $\chi \in \Gamma(T_+^{0,1})$, $\lambda \in \Gamma(T_-^{0,1})$, so one can regard f as a section of $\Omega_+^{0,p}(M)\otimes\Omega_-^{0,q}(M)$. Here the subscripts $\pm$ remind us with respect to which complex structure the differential forms are decomposed. Next we must require that $\mathcal{O}_f$ be annihilated by the BRST operator $Q_L + Q_R$.To write down the action of Q_L on $\mathcal{O}_f$, it is convenient to regard f as a $(0,p)$ form for the complex structure I_+, with values in $\Omega_-^{0,q}(M)$. A straightforward calculation gives

$$\{Q_L, \mathcal{O}_f\} \;=\; \mathcal{O}_{\bar{D}_{(+)}f}.$$

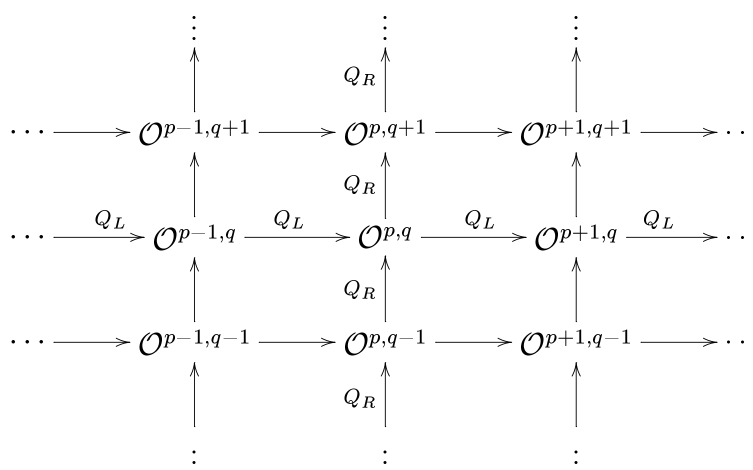
Here $\bar{D}_{(+)}$ is a covariantization of the ordinary Dolbeault operator $\bar{\partial}$ corresponding to I_+.

The covariantization uses the connection on $\Omega_-^{0,q}(M)$ coming from the connection Γ_- on TM. On the other hand, one can regard f as a $(0,q)$ form for I_-, taking values in $\Omega_+^{0,p}(M)$. One gets

$$\{Q_R, \mathcal{O}_f\} \;=\; \mathcal{O}_{\bar{D}_{(-)}f},$$

where $\bar{D}_{(-)}$ now stands for a covariantization of Dolbeault operator $\bar{\partial}$ for I_- using the connection Γ_+ on TM.

The space of local observables has a natural bigrading by the left and right moving R-charges. With respect to this bigrading, Q_L (resp. Q_R) has degree $(1,0)$ (resp. $(0,1)$). The local observables fit into the following bicomplex:



The total cohomology of this bicomplex is the space of "physical" observables in our topological theory. As usual, this means that there are two spectral sequences which converge to the BRST cohomology $H^\bullet_{Q_B}$. For references on spectral sequences, see [5, 11, 32]. In practice, the computation is usually quite involved, unless the spectral sequence degenerates at very early stage. In addition, the above formulation is somewhat awkward as it makes the underlying twisted generalized Kähler structure completely hidden. In the following we will proceed from a different route by working directly with the twisted generalized complex structures $(\mathcal{J}_1, \mathcal{J}_2)$.

4.1.2 Ramond-Ramond ground states and differential forms

Our strategy is to make use of the state-operator correspondence to translate the problem of computing the BRST cohomology of operators into the equivalent problem of computing the BRST cohomology of states. In a topological field theory, there is a state-operator isomorphism, so the two cohomologies are identical. If the topological field theory is obtained from twisting a physical theory, then this isomorphism should also be seen at the level of the physical theory. In the special case that the physical theory is an $N = 2$ superconformal field theory, the cohomology of operators is reinterpreted as the chiral ring, while the cohomology of states is reinterpreted as the space of zero-energy states in the Ramond-Ramond sector. In CFT terms, the isomorphism between these two spaces is given by the spectral flow.

There are several reasons to adopt this alternative point of view. The first is mainly a technical one. It turns out that when Ramond-Ramond ground states are concerned, one can obtain exact results by going to the point-particle limit. In this limit, the 2-D sigma model reduces to the supersymmetric quantum mechanics, whose Hilbert space can be naturally identified with the space of differential forms on X. The supercharges of the original sigma model all become differential operators on forms, and can be easily computed. We will see that this operator is exactly the differential associated to the twisted generalized complex structure $\mathcal{J}_1$ in Ref. [12]. From this one can infer without any computations that the chiral ring coincides with the Lie algebroid cohomology associated to $\mathcal{J}_1$. This is the result claimed at the end of the previous section, except that we do not need to assume the existence of the topological twist.

Secondly, it may be interesting to consider the $N = (2, 2)$ sigma model with H-flux even when the $U(1)_A$ charge is anomalous, i.e., the condition Eq. (3.2.2) is not fulfilled, or more generally, when the twisted generalized Calabi-Yau condition is not fulfilled. Such theories cannot be topologically twisted, but both the chiral ring and the space of RR ground states are perfectly well-defined and in general non-isomorphic. In the Kähler case (i.e., $H = 0$ and $I_+ = I_-$), this is a familiar situation: the chiral ring is given by $H^\bullet(\Lambda^\bullet TX)$, while the space of RR ground states is $H^\bullet(\Omega^\bullet_X)$. Only in the Calabi-Yau case are the two spaces naturally isomorphic.

Before going into detailed analysis of Ramond-Ramond ground states, we would like to comment on a subtlety that involves the direction of "time" on the worldsheet. One might be puzzled why this question is relevant, because in the topologically twisted theory the worldsheet is a Riemann surface and there is no preferred time direction at all. To understand the underlying reason for this, let us consider in more detail the operator-state correspondence in a TFT. To physically realize this correspondence, one needs to introduce a time direction explicitly on the worldsheet. This is illustrated in Fig. 4.1. An insertion of an operator (regarded as an element in the chiral ring in our setup) at the tip of the hemisphere gives rise to a state in the RR sector at the "out-going" boundary circle of the cigar.[1] In this formulation, the direction of the cigar is the time direction. If one stretches the cigar to infinite length, the resulting state on the boundary circle becomes a true ground state of zero energy. It should be mentioned that, although this physical realization operator-state correspondence picks a "time" direction on the worldsheet explicitly, the choice of the time direction has no physical consequence: the BRST cohomologies are all isomorphic for different choices of worldsheet time direction.

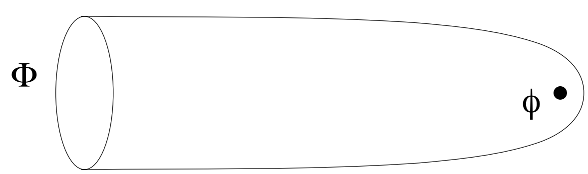


Figure 4.1: An insertion of an operator (an element in the chiral ring) ϕ on the tip of the cigar produces a state Φ in the Ramond-Ramond sector, after coupling the R-current to the spin connection on the cigar.

However, things become more subtle when we include the open string sector, as we shall do in Chapter 5. It then becomes important that we adopt a consistent choice. To explain this point, let us consider a disk correlation function with both bulk and boundary insertions.

[1] To ensure the resulting state to live in the Ramond-Ramond sector, one also has to turn on a gauge field, which equals the spin connection on the worldsheet, that couples to the R-current. This is the physical realization of "twisting" that we used in Chap. 3 to construct the generalized B-model.

The setup is illustrated in Fig. 4.2, with the boundary of the cigar now ending on a D-brane. Now it becomes apparent that the usual choice of time direction for open strings, denoted by t_o in the figure, is incompatible with the direction of time (denoted by t_c) picked by the operator-state map above. In fact, this discrepancy is a well-known fact in the open-closed string duality. The times t_o and t_c are usually referred to as the open-string channel time and the closed-string channel time, respectively. In this terminology, all the formulas that we have written down so far should really be regarded as formulated in the *open-string channel*, if we allow it to couple to time-like boundaries (i.e., boundaries that are specified by $\sigma = 0, \pi$ on the worldsheet). On the other hand, the time appearing in the state-operator map is the *closed* string channel time.

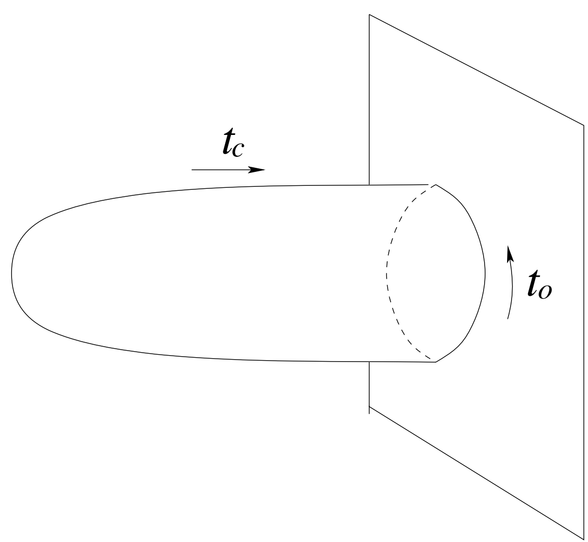


Figure 4.2: It is important to distinguish the open-string channel time t_o and the closed-string channel time t_c when studying bulk-boundary correlation functions.

The essence of this rather lengthy discussion is that one should choose a compatible convention for the closed strings and open strings, if they are allowed to couple to each other. This is what we will do here, as eventually we shall couple the theory to open strings. Note that the open/closed channel times are related to each other by a 90-degree rotation on the worldsheet (see Fig. 4.2). Among other things, this rotation induces a nontrivial relative phase factor i for fermions with different worldsheet chiralities. Without loss of generality, we assume that the transformation from open-string channel to closed-string channel is given by

$$\psi_+ \to \psi_+, \qquad \psi_- \to i\psi_-.$$

The fields ψ and ρ introduced earlier now become

$$\psi = \psi_+ + i\psi_-, \qquad \rho = g(\psi_+ - i\psi_-).$$

The important relation (3.2.1) that associates the spin-0 fermions to sections of E_1 in the twisted theory now becomes

$$\begin{pmatrix} \chi + i\lambda \\ g(\chi - i\lambda) \end{pmatrix} = (1 + \sqrt{-1}\mathcal{J}_1) \begin{pmatrix} \psi \\ \rho \end{pmatrix}.$$

Finally, since the supercharges are spinors, they also transform nontrivially:

$$Q_+ \to Q_+, \qquad \tilde{Q}_+ \to \tilde{Q}_+, \qquad Q_- \to iQ_-, \qquad \tilde{Q}_- \to i\tilde{Q}_-.$$

In particular, the supercharge that corresponds to the BRST operator Q_B is represented in the closed string channel as

$$Q_B = Q_L + iQ_R = \frac{1}{2}\big(Q_+ + i\tilde{Q}_+ + iQ_- - \tilde{Q}_-\big).$$

Let us start by writing down the Noether charges associated with Q_+ and Q_- in the point-particle approximation[2]:

$$\begin{aligned} Q_+ &= \psi_+^a g_{ab}\dot{\phi}^b - \frac{i}{6}H_{abc}\psi_+^a\psi_+^b\psi_+^c, \\ Q_- &= \psi_-^a g_{ab}\dot{\phi}^b + \frac{i}{6}H_{abc}\psi_-^a\psi_-^b\psi_-^c. \end{aligned} \tag{4.1.2}$$

[2]We use the same symbols for the generators and their associated charges.

Let $Q = Q_+ + iQ_-$ and $Q^* = Q_+ - iQ_-$. The supersymmetry algebra implies that

$$Q^2 = Q^{*2} = 0, \qquad \{Q, Q^*\} = 4\mathcal{H}.$$

where $\mathcal{H}$ is the Hamiltonian of the supersymmetric quantum mechanics:

$$\mathcal{H} = \frac{1}{2} g_{ab} \dot{\phi}^a \dot{\phi}^b - \frac{1}{4} R^{(+)}_{abcd} \psi^a_+ \psi^b_+ \psi^c_- \psi^d_- .$$

By the standard Hodge–de Rham argument, the supersymmetric ground states are in one-to-one correspondence with the elements of Q-cohomology.

The charge Q can be thought of as an operator on differential forms via canonical quantization. The classical phase space of the supersymmetric quantum mechanical system is $TM \oplus \Pi(TM \oplus TM)$, where $\Pi(TM \oplus TM)$ is the parity reversal of $TM \oplus TM$. The two "fermionic" copies of TM come from ψ_+ and ψ_-. The symplectic form on TM is the standard one, while the symplectic form in the odd directions (which is actually symmetric) is given by the Riemannian metric g:

$$\{\psi^a_\pm, \psi^b_\pm\}_{P.B.} = -ig^{ab}, \quad \{\psi^a_\pm, \psi^b_\mp\}_{P.B.} = 0.$$

Canonical quantization identifies the Hilbert space with $L^2(S)$, the space of square-integrable sections of the spin bundle $S = S(TM \oplus TM)$. In the case at hand, $TM \oplus TM$ has a natural complex polarization, with respect to which the spin bundle S can be identified with $\Lambda^\bullet(TM^*)$. In other words, instead of $\psi_\pm$ we use the coordinates ψ and ρ, which can be quantized by letting ψ^a be a wedge product with dx^a, and letting ρ_b be a contraction with the vector field $\frac{\partial}{\partial x^b}$:

$$\psi^a \to dx^a \wedge, \qquad \rho_a \to \iota_a \equiv \iota_{\partial/\partial_a}. \tag{4.1.3}$$

Now we discuss how $N = 1$ supercharges Q and Q^* act on the Hilbert space. Let us first consider the case $H = 0$. Following the standard quantization procedure, one can easily show that $Q = -i\sqrt{2}\psi^a \nabla_a$, with ∇ being the covariant derivative on the sections of the spin bundle $S(TM \oplus TM)$ that is induced from the Levi-Civita connection on TM, and with ψ^a acting as a Clifford multiplication. Under the isomorphism $S(TM \oplus TM) \simeq \Lambda^\bullet(TM^*)$, Q is identified with the de Rham differential d, up to a factor $-i\sqrt{2}$. This is the familiar statement that the space of ground states in an $N = 1$ supersymmetric quantum mechanics is isomorphic to the de Rham cohomology of the target space. Now let us consider the case $H \neq 0$. Using (4.1.2) and the identification (4.1.3), one can show that

$$
\begin{aligned}
Q &= -\sqrt{2}i\psi^a\nabla_a + \frac{\sqrt{2}i}{6}H_{abc}\psi^a\psi^b\psi^c, \\
Q^* &= -\sqrt{2}ig^{ab}\rho_b\nabla_a + \frac{\sqrt{2}i}{6}H^{abc}\rho_a\rho_b\rho_c.
\end{aligned}
$$

Up to a numerical factor $-\sqrt{2}i$, Q is identified with a twisted de Rham operator

$$d_H = d - H\wedge,$$

while Q^* is identified with its adjoint. Therefore the supersymmetric ground states are in one-to-one correspondence with the d_H-cohomology. This statement is also well-known [34].

It remains to identify the BRST operator, Q_B, in this context. The Ramond-Ramond ground states are graded by the $U(1)_V$ R-symmetry. The corresponding R-current is given by

$$J = -\frac{i}{2}\Big(\omega_+(\psi_+, \psi_+) + \omega_-(\psi_-, \psi_-)\Big),$$

under which $(1 + iI_+)\psi_+$ and $(1 + iI_-)\psi_-$ have charge $+1$ by canonical anticommutation relations. For our purpose, it will be more convenient to express J in terms of the fermions ψ, ρ:

$$J = -\frac{i}{2}\big(\delta\omega(\psi, \psi) - \alpha(\rho, \rho) - 2\langle \tilde{I}\psi, \rho\rangle\big).$$

As discussed above, quantization amounts to substitutions:

$$\psi^a \leftrightarrow dx^a\wedge, \qquad \rho_a \leftrightarrow \iota_{\partial/\partial x^a} \equiv \iota_a.$$

Then the R-current is identified with the following operator on differential forms:

$$J \;=\; -i\big(\delta\omega\wedge - \iota_\alpha - \iota_{\tilde I}\big)$$

where ι_α is the contraction with the Poisson bivector α, and $\iota_{\tilde I}$ is defined in a local coordinate basis as

$$\iota_{\tilde I} \;=\; \tilde I^a_b (dx^b\wedge)\circ\iota_a .$$

Note that $\delta\omega, \alpha$, and $\tilde I$ can be read off $\mathcal{J}_1$ (see Eq. (3.1.2)), and therefore the operator J depends only on the TGC structure $\mathcal{J}_1$.

It is not difficult to see that the BRST operator can be expressed as

$$Q_B = \frac{1}{2}\big(Q + [J, Q]\big).$$

Since $Q = d_H$, it is clear that Q_B depends only on the 3-form H and the twisted generalized complex structure $\mathcal{J}_1$. In the following subsection, we will relate Q_B to the generalized Dolbeault operator defined in Section 2.4.

4.1.3 BRST cohomology of states and operators

To express the the BRST operator on the Ramond-Ramond states as an operator on differential forms, we first recall the definition of generalized Dolbeault operators $(\bar\partial_H, \partial_H)$ associated with an almost TGC structure $\mathcal{J}$: $\bar\partial_H$ is the degree-1 part of $d_H = d - H\wedge$ associated with an alternative grading induced by $\mathcal{J}_1$

$$\Omega^\bullet(M, \mathbb{C}) \simeq U_0 \oplus U_1 \oplus \cdots \oplus U_{2n}\,. \tag{4.1.4}$$

Here U_0 is the canonical bundle, $U_k = \Lambda^k \bar E_1 \cdot U_0$, and E_1 the $-i$-eigenbundle of $\mathcal{J}_1$. See the discussion in Section 2.4.

We are going to show that the BRST cohomology of states is isomorphic to the cohomology of $\bar{\partial}_H$ on differential forms on the TGC manifold $(M, \mathcal{J}_1)$. As a preliminary step, let us obtain a convenient explicit formula for the grading operator on $\Omega^\bullet(M)$, defined in the previous subsection, in terms of the twisted generalized complex structure $\mathcal{J}_1$. Let $A = X + \xi \in \Gamma(TM_{\mathbb{C}} \oplus TM^*_{\mathbb{C}})$. It can be regarded as an endomorphism of $\Lambda^\bullet TM^*_{\mathbb{C}}$:

$$A \cdot \rho = \iota_X \rho + \xi \wedge \rho. \tag{4.1.5}$$

On the other hand, $\mathcal{J}_1$ is an endomorphism of $TM_{\mathbb{C}} \oplus TM^*_{\mathbb{C}}$ with eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$. By definition, the grading operator $R(\mathcal{J}_1)$ must satisfy

$$[R(\mathcal{J}_1), A] = -i\mathcal{J}_1 A, \quad \forall A \in \Gamma(TM_{\mathbb{C}} \oplus TM^*_{\mathbb{C}}).$$

Obviously, this condition determines $R(\mathcal{J}_1)$ up to a constant. Using the explicit matrix form (3.1.2) of $\mathcal{J}_1$, one gets

$$\mathcal{J}_1 A \cdot \rho = \iota_{\tilde{I}X} + \iota_{\alpha(\xi)} - \iota_X \delta\omega \wedge -\tilde{I}^t(\xi) \wedge .$$

Then it is easy to check that the following is a solution to the above equation:

$$R(\mathcal{J}_1) = -\sqrt{-1}(\delta\omega \wedge -\iota_\alpha - \iota_{\tilde{I}}).$$

The general solution may differ from this only by a constant.

From the result of the last section, one immediately sees that under the identification of $Q \leftrightarrow d_H$ and the identification of the Hilbert space as the space of differential forms, the R-current generator J is identified as

$$J \leftrightarrow R(\mathcal{J}_1) + \text{const}.$$

The BRST operator Q_B then becomes

$$\begin{aligned} Q_B &= \frac{1}{2}\big(Q + [J, Q]\big) \\ &= \frac{1}{2}\big(d_H + [R(\mathcal{J}_1), d_H]\big) \\ &= \bar{\partial}_H. \end{aligned}$$

This is the desired result: the BRST complex of Ramond-Ramond ground states is identified with the generalized Dolbeault complex on the twisted GCY manifold $(M, \mathcal{J}_1, H)$.

A direct consequence of this analysis is that the BRST cohomology of operators in the twisted theory is isomorphic to the Lie algebroid cohomology of E_1. Recall that the generalized Dolbeault complex $(\Omega^\bullet(M, \mathbb{C}), \bar{\partial}_H)$ is a Lie algebroid module over the Lie algebroid complex $(\Lambda^\bullet \bar{E}_1, d_{E_1})$. As discussed in the Appendix, this means for any section s of $\Lambda^\bullet \bar{E}_1$ and any differential form ρ:

$$\bar{\partial}_H(s \cdot \rho) = (d_{E_1} s) \cdot \rho + (-1)^{|s|} s \cdot \bar{\partial}_H \rho.$$

This is equivalent to the following operator equation:

$$\{Q_B, s\} = d_{E_1} s,$$

with $\{,\}$ the graded commutator of operators. Since we identified the space of sections of $\Lambda^\bullet \bar{E}_1$ with the space of operators, the space of forms with the Hilbert space of the supersymmetric quantum mechanics, and $\bar{\partial}_H$ with the representation of the BRST charge on the Hilbert space, this simply implies that the cohomology of d_L is isomorphic to the BRST cohomology of operators, as claimed.

4.2 The Frobenius structure

For any $N = 2$, 2-D field theory we may consider the chiral ring, as well as the cohomology of the supercharge Q_B on the states in the Ramond sector. As already mentioned above, the latter is a module over the former. The two spaces are not isomorphic in general. But if the theory admits a topological B-twist, the two spaces are always isomorphic, by virtue of the state-operator correspondence in a topological field theory. More precisely, the space of states of a 2-D TFT is an algebra with a nondegenerate scalar product $(\cdot, \cdot)$ such that

$$(a, bc) = (ab, c).$$

Such algebras are called Frobenius. All topological correlators can be expressed in terms of the Frobenius structure on the space of states. For example, genus-zero correlators are given by

$$\langle a_1 \dots a_n \rangle_{g=0} = (a_1, a_2 \dots a_n).$$

Consider now an $N = 2$ sigma model defined on a twisted generalized Calabi-Yau manifold, and the $U(1)_A$ R-charge is nonanomalous. One expects that the theory admits a topological B-twist, and therefore the chiral ring, which is known to be isomorphic to the Lie algebroid cohomology of E_1, is a supercommutative Frobenius algebra. Note that the Frobenius scalar product $(\cdot, \cdot)$ can be recovered from the "trace" function:

$$\mathrm{Tr}\,(a) = (1, a)$$

by letting $(a, b) = \mathrm{Tr}\,(ab)$. The name "trace" is used because Tr vanishes on commutators (in the graded case, on graded commutators). Let Ω be a $\bar{\partial}_H$-closed differential form that sits in the component U_0. For a twisted generalized Calabi-Yau such a form exists and is unique up to a constant factor. Note that Ω is also d_H-closed. Consider now a bundle automorphism

$p : TM \oplus TM^* \to TM \oplus TM^*$ that looks like

$$p : (v, \xi) \mapsto (v, -\xi), \quad \forall v \in \Gamma(TM), \forall \xi \in \Gamma(TM^*). \tag{4.2.1}$$

This automorphism takes the form q to $-q$, and maps the Courant bracket twisted by H to the Courant bracket twisted by $-H$. It follows from this that for any twisted generalized complex structure $\mathcal{J}$ the bundle map $\mathcal{J}' = p^{-1}\mathcal{J}p$ is also a twisted generalized complex structure, for the opposite H-field $H' = -H$. Moreover, it is easy to see that $(M, -H, \mathcal{J}')$ is a twisted generalized Calabi-Yau if and only if $(M, H, \mathcal{J})$ is one. (From the physical viewpoint, p corresponds to worldsheet parity transformation, and the above facts are obvious). In particular, we have a decomposition

$$\Lambda^\bullet TM^* \otimes \mathbb{C} = U_0' \oplus U_1' \oplus \cdots \oplus U_{2n}' .$$

Let Ω' be the $\bar{\partial}_{H'}$-closed differential form that sits in the component U_0'. We claim that the trace function on the Lie algebroid cohomology is given by

$$\mathrm{Tr}\,(\alpha) \sim \int_M \Omega' \wedge \alpha \cdot \Omega, \tag{4.2.2}$$

where α is a d_L-closed section of $\Lambda^\bullet(E_1^*)$.

To derive this formula, we recall that the Frobenius trace is computed by the path-integral on a Riemann sphere with an insertion of the operator corresponding to α. Since we are dealing with a topological theory, we must also turn on a $U(1)$ gauge field coupled to the R-current participating in the twisting. This gauge field must be equal to the spin connection, which means that the total flux through the sphere is 2π. Let us stretch the sphere into a long and thin cigar, so that the insertion point of α is somewhere in the middle portion. The value of the path-integral does not change, of course, but it may now be evaluated more easily by reducing the computation to the supersymmetric quantum mechanics. The path-integral on each hemisphere gives a state in the Ramond-Ramond sector, which may be approximated in the point-particle limit by a function of the zero-modes. Bosonic zero-modes are simply coordinates on M, while fermionic zero-modes are ψ^i, taking values in $TM_{\mathbb{C}}$. Thus the Ramond-Ramond state is represented by a function on $\Pi TM_{\mathbb{C}}$, i.e. by a (complex-valued) differential form. We have described above how α acts on differential forms.

It remains to identify the particular RR states arising from performing the path-integral over each hemisphere, and then integrate over the zero-modes. Since we are not inserting any operators on the hemispheres, the RR ground state in question is the spectral flow of the unique vacuum state in the NS sector, and therefore in the point-particle approximation is represented by the form Ω defined above. However, there is a subtlety related to the choice of orientation. This subtlety arises because our identification of RR states with differential forms depends on orientation: exchanging left-moving and right-moving fermions is equivalent to performing a Hodge duality on forms. In the physical language, Hodge duality is simply the Fourier transform of fermionic zero-modes. If we induce the orientations of both hemispheres from a global orientation of the Riemann sphere, then the wave-function coming from one hemisphere will be given by a function of the fermionic zero-modes ψ^i, while the wave-function from the other hemisphere will be a function of the Fourier-dual zero-modes. In order to evaluate the path-integral one first has to Fourier transform the second state, and only then multiply the wave-functions and integrate over the zero-modes. Alternatively, we can choose the opposite orientation for the second hemisphere, so that there is no need for Fourier transform. We can always choose the worldsheet parity transform to act as

$$\psi_+ \to i\psi_-, \qquad \psi_- \to -i\psi_+.$$

Apparently this worldsheet parity transformation leaves ψ intact but flips the sign of ρ. This is reflected on the sections of $TM \oplus TM^*$ by the automorphism p described in (4.2.1). The sign of H is also reversed, since it is an odd form. We conclude that the wave-function from the second hemisphere is given by Ω', and the path-integral in question is given by

$$\mathrm{Tr}\,(\alpha) \sim \int_M \Omega' \wedge \alpha \cdot \Omega.$$

Let us check that this formula is indeed consistent, i.e., that it vanishes if $\alpha = d_{E_1}\beta$ for some β. Indeed, we have

$$\begin{aligned} \mathrm{Tr}\,(d_L\beta) &= \int_M \Omega' \wedge (d_{E_1}\beta) \cdot \Omega \\ &= \int_M \Omega' \wedge \bar{\partial}_H(\beta \cdot \Omega) \\ &= \int_M \Omega' \wedge (d_H + [R(\mathcal{J}_1), d_H])(\beta \cdot \Omega). \end{aligned} \tag{4.2.3}$$

Next we make use of the following two identities valid for any two forms γ, η:

$$\int_M \gamma \wedge d_H\eta = -(-1)^{|\gamma|} \int_M (d_{H'}\gamma) \wedge \eta, \tag{4.2.4}$$

$$\int_M \gamma \wedge R(\mathcal{J}_1)\eta = -\int_M (R(\mathcal{J}_1')\gamma) \wedge \eta, \tag{4.2.5}$$

where $H' = -H$, and $\mathcal{J}' = p^{-1}\mathcal{J}p$. The first identity is obvious. To prove (4.2.5), we note that $R(\mathcal{J}_1) = -i(\delta\omega \wedge -\iota_\alpha - \iota_{\tilde{I}})$. It is not difficult to see that when pulling $R(\mathcal{J}_1)$ to the left of γ in (4.2.5), $\delta\omega$ and ι_α do not change, but $\iota_{\tilde{I}}$ picks up a minus sign; namely

$$\int_M \gamma \wedge (\delta\omega \wedge -\iota_\alpha - \iota_{\tilde{I}})\eta = \int_M (\delta\omega - \iota_\alpha + \iota_{\tilde{I}})\gamma \wedge \eta.$$

Now recall that $\mathcal{J}_1'$ is given by the same matrix form (3.1.2) as $\mathcal{J}_1$, except that $\delta\omega$ and α are replaced by $-\delta\omega$ and $-\alpha$, respectively. This immediately gives (4.2.5).

Using the identities (4.2.5) and (4.2.5), we get

$$\begin{aligned}\mathrm{Tr}\,(d_L\beta) &= -(-1)^{|\Omega'|}\int_M ((d_{H'}+[R(\mathcal{J}_1'),d_{H'}])\Omega')\wedge\beta\cdot\Omega \\ &= -(-1)^{|\Omega'|}\int_M \bar\partial_{H'}\Omega'\wedge\beta\cdot\Omega \\ &= 0. \end{aligned} \tag{4.2.6}$$

Let us also check that the formula for the trace (4.2.2) reduces to the known expressions in the case of the ordinary A- and B-models, by setting $H=0$ and $I_+=\pm I_-=I$. For the ordinary B-model, we have $I_+=I_-=I$, so $\mathcal{J}_1=\mathcal{J}_I$ is simply the generalized complex structure corresponding to the ordinary complex structure I. In this case, $\mathcal{J}_1'=\mathcal{J}_1$, $\Omega'=\Omega$, and the form Ω is simply the top holomorphic form on M. It is obvious that our formula for the trace function reduces to the standard formula for the B-model [40]:

$$\mathrm{Tr}\,(\alpha)=\int_M \Omega\wedge(\alpha\lrcorner\Omega).$$

For the A-model, the situation is more interesting. We have $I_+=-I_-=I$, and the relevant generalized complex structure is $\mathcal{J}_1=\mathcal{J}_\omega$ is the standard one associated with the symplectic structure $\omega=gI$. In this case, $\mathcal{J}_1'=-\mathcal{J}_1$, and the forms Ω and Ω' are given by

$$\Omega=e^{i\omega},\quad \Omega'=e^{-i\omega}.$$

The complex Lie algebroid for the A-model is isomorphic to $TM_{\mathbb{C}}$, thus the Lie algebroid cohomology is isomorphic to the complex de Rham cohomology. The usual formula for the Frobenius trace on $H^\bullet(M)$ is

$$\mathrm{Tr}\,(\beta)=\int_M\beta,\quad \beta\in\Omega^\bullet(M), d\beta=0.$$

This does not seem to agree with our formula. But one should keep in mind that the identification between the Lie algebroid cohomology and de Rham cohomology is nontrivial, and as a result, although the bundle $\Lambda^\bullet E_2^*$ is isomorphic to $\Omega^\bullet(M)$, the action of $\Omega^\bullet(M)$ on itself coming from the action of $\Lambda^\bullet E_2^*$ on $\Omega^\bullet(M)$ is not given by the wedge product. To describe this action, let us identify the space of sections of $\Omega^\bullet(M)$ with the graded supermanifold ΠTM. Let $\alpha \in \Omega^k(M)$ be given by

$$\alpha = \frac{1}{k!}\alpha_{a_1\ldots a_k} dx^{a_1} \wedge \ldots \wedge dx^{a_k}.$$

The action we are after is obtained by associating to α the following differential operator on ΠTM:

$$\frac{1}{k!}\alpha_{a_1\ldots a_k} D^{a_1} \ldots D^{a_k},$$

where

$$D^a = \theta^a + i\left(\omega^{-1}\right)^{ab}\frac{\partial}{\partial\theta^b}.$$

The operators D^a anticommute, so this is well-defined. On the other hand, in the usual description of the A-model, the action of $\Omega^\bullet(M)$ on itself is given by the ordinary wedge product (plus quantum corrections, which we neglect all together).

The difference between our description of the A-model and the usual one is due to a different identification of the fermionic fields with operators on forms. While we identified ψ^a with "creation" operators dx^a and ρ_a with "annihilation" operators, the usual identification is different:

$$\psi_+^{\bar{i}} \mapsto dx^{\bar{i}}, \quad \psi_-^i \mapsto dx^i, \quad g_{\bar{i}j}\psi_+^j \mapsto \iota_{\frac{\partial}{\partial x^{\bar{i}}}}, \quad g_{j\bar{i}}\psi_+^{\bar{i}} \mapsto \iota_{\frac{\partial}{\partial x^j}}.$$

This choice is related to ours by a Bogolyubov transformation. In the usual description the vacuum state with the lowest R-charge $J_L - J_R$ is given by the constant 0-form on M. It is easy to see that the Bogolyubov transformation maps it to the inhomogeneous form $e^{i\omega}$. The same transformation also maps the ordinary degree of a differential form to the nonstandard grading on $\Omega^\bullet(M)$ defined in Ref. [12] and explained above. Thus our formula agrees with the standard one after a Bogolyubov transformation (and if one neglects quantum corrections).

4.3 Instanton corrections and the generalized quantum cohomology ring

So far we have only discussed the classical ring structure on the space of topological observables. In general, the actual ring structure is deformed by quantum effects. A well-known example is the ordinary A-model, whose ring of BRST-invariant observables (the quantum cohomology ring) is a deformation of the de Rham cohomology ring $H^\bullet(M,\mathbb{C})$ induced by worldsheet instantons. In this section, we carry out the analysis for generic twisted topological sigma model with H-flux, and identify worldsheet instantons that can contribute to the deformation of the ring structure. This section is essentially an extension of the analysis of Section 8.2 of Ref. [21] to the case $H \neq 0$.

To identify the instantons that extremize the bosonic part of the action is easy. From Eq.(3.4.11) and the transformation formulas for ρ_μ and η_μ, it is rather obvious that the bosonic action[3] is minimized around the configurations

$$(\delta_\mu{}^\nu - i\varepsilon_\mu{}^\nu)\frac{1}{2}(1-iI_+)^a_b\partial_\nu\phi^b = 0, \qquad (\delta_\mu{}^\nu + i\varepsilon_\mu{}^\nu)\frac{1}{2}(1-iI_-)^a_b\partial_\nu\phi^b = 0. \tag{4.3.1}$$

The above equations, together with their conjugates, can be written equivalently as

$$\partial_\mu\phi = \varepsilon_\mu{}^\nu I_+\partial_\nu\phi, \qquad \partial_\mu\phi = -\varepsilon_\mu{}^\nu I_-\partial_\nu\phi. \tag{4.3.2}$$

The instanton equation can be written in a form that renders the generalized complex structure $\mathcal{J}_2$ apparent. Let $\iota : TM \to TM \oplus TM^*$ be the natural inclusion of TM into $TM \oplus TM^*$, and $d\phi : T\Sigma \to TM$ be the derivative of $\phi : \Sigma \to M$. Recalling that the other generalized complex structure $\mathcal{J}_2$ is

$$\mathcal{J}_2 = \begin{pmatrix} \delta I & -\beta \\ \tilde{\omega} & -\delta I^* \end{pmatrix},$$

[3]The fact that Eq.(3.4.11) only holds on shell is irrelevant to the analysis in the following, since the on shell conditions used only affect terms involving fermions.

we can write the instanton equation equivalently as

$$\mathcal{J}_2 \circ (\iota \circ d\phi) = \iota \circ (d\phi \circ \varepsilon). \tag{4.3.3}$$

Note that ε acts as a complex structure on the worldsheet. Equation (4.3.3) shows that the instanton configurations of the generalized B-model are those maps ϕ that intertwine with the worldsheet complex structure ε and the generalized complex structure $\mathcal{J}_2$ on M. We call such maps generalized holomorphic curves.

It is easy to see that the equation of generalized holomorphic curves reproduces the well-known results in the special cases of ordinary A/B-models. In the case of A-model, $\mathcal{J}_2$ is the generalized complex structure corresponding to an ordinary complex structure

$$\mathcal{J}_2 = \begin{pmatrix} I & 0 \\ 0 & -I^* \end{pmatrix}.$$

The instanton equation (4.3.3) reduces to $I \circ d\phi = d\phi \circ \varepsilon$, which is the equation for ordinary holomorphic curves. It is well-known that in the ordinary A-model, the instantons are precisely given by holomorphic curves [40].

In the case of ordinary B-model, $\mathcal{J}_2$ is the generalized complex structure corresponding to the symplectic structure ω:

$$\mathcal{J}_2 = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}.$$

The instanton equation (4.3.3) obviously reduces to $d\phi = 0$ in this case. Therefore there are no nontrivial instantons in the ordinary B-model–again a well-known fact.

The path-integral of the generalized B-model can be localized around the generalized holomorphic curves, and the semiclassical calculation is exact. To see this, let us consider a one-parameter family of theories whose Lagrangians are defined to be $t\{Q_B, V\} + \phi^*(\delta\omega)/2$. When $t = 1$, this is just the Lagrangian of our generalized B-model, as given in (3.4.11). On the other hand, as variation of t changes the Lagrangian by Q_B-exact terms, the generalized

B-model is equivalent to the theory defined by any t. In particular, it is equivalent to the theory in the limit of $t \to \infty$. In such a limit, the path integral can obviously be computed exactly using semiclassical methods near the minima given by (4.3.1).

There is another way to see that the path integral can be localized near the instantons (4.3.1). As is well-known, the path integral of a (cohomological) TFT, with BRST operator Q, can be localized around the Q-invariant field configurations [40]. In the case of the generalized B-model, the BRST variations of the fields are given in eqs. (3.4.5)–(3.4.9).[4] It is easy to see that the Q_B-invariant solutions are precisely the instantons found in (4.3.1). Therefore, the path integral of the generalized B-model can be localized near the generalized holomorphic curves.

As we know from Section 4.1, the classical ring structure of topological observables (i.e., BRST-invariant observables) is the natural one associated with a Lie algebroid cohomology. On general grounds, this ring structure may be subject to quantum corrections coming from worldsheet instantons. In other words, the Lie algebroid cohomology ring can be deformed due to the fact that tree-level correlation functions of topological observables may have nontrivial contributions coming from rational generalized holomorphic curves. We shall call this deformed ring structure the quantum Lie algebroid cohomology ring. Like in the special case of ordinary A-model, the quantum Lie algebroid cohomology ring should be related to an intersection theory on the moduli space $\mathcal{M}$ of generalized rational curves on generalized Calabi-Yau manifolds.

Unfortunately, very little is known of the moduli space $\mathcal{M}$. Therefore we shall limit ourselves to making some speculations on the generic structures of the quantum Lie algebroid cohomology ring. First we note that, although the equation of generalized holomorphic curves specifically depends on $\mathcal{J}_2$, the quantum Lie algebroid ring structure does not. In fact, more is true: the generalized B-model does not depend on $\mathcal{J}_2$ at all. This can be seen from Eq.(3.4.11). Suppose we fix $\mathcal{J}_1$ and varies $\mathcal{J}_2$. Since Q_B is identified with the Lie algebroid differential associated with $\mathcal{J}_1$, it is unchanged. The topological term $\int_\Sigma \phi^*(\delta\omega)$ is

[4] These transformations ignore the H-field. However, it can be shown that including the H-field does not change the Q_B-invariant field configurations.

also unchanged, because of Eq.(3.1.2) and the fact that $\mathcal{J}_1$ is kept fixed. Therefore, variations of $\mathcal{J}_2$ only perturb the action by Q_B-exact terms and so leave the theory invariant.

We will not try to describe these corrections more precisely here. But note that for a generic TGC structure $\mathcal{J}_2$ the TG-holomorphic instanton equation is much more restrictive than the ordinary holomorphic instanton equation. Indeed, it requires the image of $T\Sigma$ under $d\phi$ to lie in the kernel of the map $\tilde{\omega}$. For a generic $\mathcal{J}_2$ and at a generic point of M the 2-form $\tilde{\omega}$ is nondegenerate, and so this condition does not allow nonconstant instantons. In other words, all nontrivial instantons must be contained in the subvariety where $\tilde{\omega}$ is degenerate. The extreme cases are the ordinary B-model, where $\tilde{\omega}$ is a symplectic form and there are no nontrivial instantons, and the ordinary A-model, where $\tilde{\omega}$ vanishes identically. It would be of some interest to better understand the moduli space of these generalized holomorphic curves.

Chapter 5

Topological D-branes

To any topologically twisted sigma model one can associate a category of topological D-branes. The objects of this category are topological D-branes themselves, and the space of morphisms is the open-string BRST cohomology, which can alternatively be represented by the open-string Ramond ground states. In this work, we will largely forget this categorical structure but rather focus on its building blocks. In the first two sections we shall focus on the geometric description of rank-1 topological D-branes in the generalized B-model, and compute their open-string BRST cohomology. In Sec. 5.3 we will take the initial steps to analyze topological D-branes of higher rank.

5.1 Rank-1 topological D-branes in the generalized B-model

In geometric terms, a D-brane of rank 1 is a submanifold Y together with a Hermitian line bundle $\mathcal{E}$ equipped with a unitary connection ∇. Its curvature $F = -i\nabla^2$ is a real closed 2-form on Y whose periods are integral multiples of 2π. For the purpose of classifying the topological D-branes, only the curvature of the connection ∇ will be important; for this reason we will regard a rank-1 brane as a pair (Y, F).

If $H \neq 0$, then there is an additional constraint of Y: the restriction of H to Y must be exact. That is, while the B-field on X is not a globally well-defined 2-form, its restriction to

Y is. The set of B-fields on X is acted upon by 1-form gauge transformations:

$$B \mapsto B + d\lambda, \quad \lambda \in \Omega^1(X).$$

Under this gauge transformation, the connection on $\mathcal{E}$ transforms as follows:

$$\nabla \mapsto \nabla - i\lambda|_Y.$$

The curvature of ∇ is not invariant under these transformations; the gauge-invariant combination is

$$\mathcal{F} = B|_Y + F.$$

In the following, we shall ignore the torsion H, and always work with $B = 0$ for simplicity. The results we obtain, however, are also applicable to the case of nonzero B-field, except that one should replace every F in sight by $\mathcal{F}$, and the gauge field A by $A - \lambda$.

Topological D-branes of the generalized B-model are D-branes that preserve the topological algebra. In other words, they must preserve the $N = 1$ supersymmetry Q and the $U(1)_V$ R-current J. As mentioned before, the $N = 1$ supersymmetry takes the form of $Q = Q_+ + Q_-$ in the open-string channel, which is natural for the purpose of analyzing D-brane boundary conditions. One might think that the boundary conditions that preserve Q are those that guarantee the invariance of the the action in Eq. (3.1.1). This is a little naive, since the presence of boundary in general introduces corrections to the action due to the bulk-boundary coupling. In the path integral formulation, every boundary component, which is diffeomorphic to an S^1, incorporates a Wilson loop operator in the path integral; namely [41],

$$\int [D\phi] e^{-S[\phi]} \rightarrow \int [D\phi] e^{-S[\phi]} \cdot \mathrm{Tr}\, P \exp(\oint A),$$

where A is the gauge field supported on the D-brane, and the loop integral is over the boundary component S^1. If more than one boundary components are present, one simply insert multiple Wilson loop operators inside the path integral.

In the case of rank-1 branes, which are the object of our study in this section, the gauge field is abelian, and the contribution of a Wilson-loop operator simplifies: it gives an overall factor

$$\exp\Big(\int_{\partial\Sigma} A\Big).$$

This overall factor inside the path integral can be interpreted as an effective shifting of the original bulk action $S_{\rm bulk}$ by a boundary contribution: $S_{\rm bulk} \to S_{\rm bulk} + S_{\rm boundary}$, where

$$S_{\rm boundary} = -\int_{\partial\Sigma} A = -\int_{\partial\Sigma} A_\mu \dot{\phi}^\mu \, d\tau.$$

Boundary conditions preserving Q are those that keep the total action $S = S_{\rm bulk} + S_{\rm boundary}$ Q-invariant. Note that in general the original bulk action $S_{\rm bulk}$ is no longer Q-invariant, contrary to the case without boundary.

Now let us work out the Q-preserving boundary conditions. We take the following ansatz for the fermions: when restricted to $\partial\Sigma$,

$$\psi \in TY, \tag{5.1.1}$$

$$\rho_\mu\big|_{TY} = F_{\mu\nu}\psi^\nu. \tag{5.1.2}$$

This choice is not as arbitrary as it may appear. To see why, let us write down the corresponding boundary condition for the bosons obtained from the Q-variation of (5.1.1) and (5.1.2):

$$\dot{\phi} \in TY, \tag{5.1.3}$$

$$g_{\mu\nu}\partial_1\phi^\nu = F_{\mu\nu}\dot{\phi}^\nu - \frac{i}{2}\left(D_\lambda F_{\mu\nu} - \Gamma^\rho{}_{\mu\lambda}F_{\rho\nu}\right)\psi^\lambda\psi^\nu. \tag{5.1.4}$$

Eq. (5.1.3) simply says that the D-brane wraps on the submanifold Y. In the limit of flat metric and uniform field strength, Eq. (5.1.4) has a simple physical interpretation: it says that the tension at the ends of the string is balanced by the Lorentz force as its charged ends move on Y.

It is not difficult to check that these boundary conditions indeed preserve the total action S. The bulk variation is

$$\begin{aligned}\delta S_{\text{bulk}} &= \int d^2\sigma d\theta^1 d\theta^2 \epsilon (Q_+ + Q_-) L \\ &= -\frac{\epsilon}{2}\int_{\partial\Sigma} d\tau \left(\rho_\mu \dot{\phi}^\mu - g_{\mu\nu}\partial_1\phi^\mu\psi^\nu\right).\end{aligned}$$

Using the boundary conditions (5.1.2) and (5.1.4), it comes out to be

$$\delta S_{\text{bulk}} = -\epsilon \int_{\partial\Sigma} d\tau\ F_{\mu\nu}\dot{\phi}^\mu\psi^\nu.$$

A simple calculation shows that $\delta S_{\text{boundary}}$ precisely cancels δS_{bulk}; thus the total action is Q-invariant.

It turns out that the boundary conditions (5.1.1) and (5.1.2) for the fermions have a natural interpretation in generalized geometry. The relevant concept is that of the *generalized tangent bundle* [12]. Given a pair (Y, F),where Y is a submanifold of M and F a closed 2-form on Y, its generalized tangent bundle, denoted by $\mathcal{T}Y_F$, is a subbundle of $TM \oplus TM^*|_Y$ defined by

$$Z + \xi \in \mathcal{T}Y_F \Longleftrightarrow Z \in TY,\ \xi + \iota_Z F \in NY^*.$$

As is clear from the discussion above, the boundary condition for fermions that preserves the $N = 1$ supersymmetry is precisely that when restricted on Y, the fermion field

$$\Psi = \begin{pmatrix} \psi \\ \rho \end{pmatrix}$$

lives in the generalized tangent bundle $\mathcal{T}Y_F$.

For the brane to be a topological D-brane in the generalized B-model, it must also preserve the $U(1)_V$ R-symmetry. Note that the boundary action S_{boundary} is invariant under the R-symmetry, so the bulk action S_{bulk} itself must also be invariant. It follows that the

R-current J_V must be tangent to Y. From the worldsheet point of view, this means

$$J_V^1 = \omega_+(\psi_+, \psi_+) - \omega_-(\psi_-, \psi_-) = 0.$$

To make contact with generalized geometry, it is convenient to write the above equation in terms of ψ and ρ:

$$\delta\omega(\psi, \psi) + 2\rho(\tilde{I}\psi) - \alpha(\rho, \rho) = 0, \tag{5.1.5}$$

where $\delta\omega, \alpha, \tilde{I}$ are as defined in (3.1.2). To see the meaning of (5.1.5) more clearly, let us introduce a new field $\rho' = \rho - F\psi$. The usefulness of ρ' lies in the fact that it lives in the conormal bundle NY^*. Thus we may treat ρ' and ψ as the independent unconstrained variables. It is not difficult to check that Eq. (5.1.5) breaks into two separate conditions:

$$\begin{aligned} (\tilde{I} - \alpha \cdot F)\psi - \alpha(\rho') &\in TY, \\ (\delta\omega - \tilde{I}^t \cdot F - F \cdot \tilde{I} + F \cdot \alpha \cdot F)\psi - (\tilde{I}^t - F \cdot \alpha)\rho' &\in NY^*. \end{aligned}$$

This is same as the statement that the B-transformed generalized complex structure $e^B \mathcal{J}_1 e^{-B}$, with $B = -F$, preserves the subbundle $TY \oplus NY^*$ in $TM \oplus TM^*$. Equivalently, the boundary condition of the fermions preserves the R-current if and only if $\mathcal{J}_1$ preserves the generalized tangent bundle $\mathcal{T}Y_F$.

This condition also corresponds to a construction in generalized geometry. It is the concept of the *generalized complex submanifolds*, also due to Gualtieri. A pair (Y, F), with Y a submanifold of a generalized complex manifold $(M, \mathcal{J})$ and F a closed 2-form on Y is called a generalized complex submanifold if its generalized tangent bundle $\mathcal{T}Y_F$ is stable with respect to $\mathcal{J}$. Our analysis above shows that a rank-1 brane specified by (Y, F) is a topological D-brane in the generalized B-model if and only if (Y, F) is a generalized complex submanifold. For this reason, we call topological D-branes in the generalized B-model generalized complex branes, or simply GC branes. The geometric description of the GC branes was first given in [21]. See also [43].

Remarkably, the GC branes reproduce all known examples of rank-1 topological D-branes in the ordinary A- and B-model. In the case of B-model, the generalized complex structure is the one associated with an ordinary complex structure I, given in (2.2.1). The generalized complex submanifold condition is the same as that $TY \oplus NY^*$ is preserved by

$$e^{-F}\mathcal{J}_I e^F = \begin{pmatrix} I & 0 \\ -(FI + I^*F) & -I^* \end{pmatrix}.$$

This is equivalent to two conditions. The first is that I preserves TY; this shows that Y is a complex submanifold of M. The second is that $FI + I^*F = 0$; this means the curvature 2-form F is of type $(1,1)$. In other words, the line bundle is holomorphic. This is indeed the well-known result for ordinary B-branes [18].

The case of A-brane is slightly more involved. To use the general formalism discussed above, the generalized complex structure is now the one associated with a symplectic structure, as given in (2.2.2). The generalized complex manifold condition is that $TY \oplus NY^*$ be stable under

$$e^{-F}\mathcal{J}_\omega e^F = \begin{pmatrix} -\omega^{-1}F & -\omega^{-1} \\ \omega + F\omega^{-1}F & F\omega^{-1} \end{pmatrix}.$$

In the special case of flat bundle (i.e., $F = 0$), the above equation reduces to two simple relations: $\omega(TY) \subset NY^*$, $\omega^{-1}(NY^*) \subset TY$. The first condition shows that Y is isotropic, while the second shows it is coisotropic. It follows that Y is a Lagrangian submanifold. That flat bundles supported on Lagrangian submanifolds provide consistent boundary conditions for the A-model was first discovered by Witten more than a decade ago [41].

When $F \neq 0$, the conditions are more complicated, and we do not analyze them in detail here but refer the reader to Ref. [12, 21]. What is remarkable is that these GC branes turn out to be precisely the coisotropic A-branes, discovered by Kapustin and Orlov rather recently [25]. In short, they are certain special coisotropic manifolds carrying transverse holomorphic structure. We will come back to these interesting objects in the next section when we compute the open string spectra on them.

We end this section by making a few comments. First, the boundary condition analyzed above is only valid for rank-1 branes. Higher-rank D-branes carry non-abelian gauge fields on their worldvolume, and the corresponding boundary conditions are much more complicated. This will require somewhat different methods, as we will do in Sec. 5.3.

Secondly, our analysis so far (and indeed in this whole chapter) is classical. At the quantum level, anomalies may render an otherwise perfectly well-defined classical topological D-brane an unacceptable object. For example, although the boundary conditions defined by a GC brane preserve the $U(1)_V$ R-current classically, the presence of the boundary may induce a quantum anomaly and therefore break the topological algebra. This boundary-induced anomaly is already present in the case of the ordinary A-model, whose branes we call A-branes. It is known that, although a Lagrangian submanifold defines a classical A-brane, it does not define a quantum A-brane unless its Maslov class vanishes [17]. A Lagrangian submanifold with trivial Maslov class is called gradable. For the case of coisotropic A-branes, there is a similar story. A classical coisotropic A-brane can be lifted to a quantum A-brane if and only if its so-called generalized Maslov class vanishes [28, 24]. When its generalized Maslov class vanishes, a coisotropic A-brane is called a *graded* coisotropic brane. Gradability is a generic feature of quantum topological D-branes. In the case of GC branes in generalized B-model, a form of gradability proposed by Hitchin [16] might play a role of ensuring the absence of anomalies.

5.2 Open-string BRST cohomology

We described the condition of (classical) rank-1 topological D-branes in the generalized B-model in terms of generalized geometries. These branes must wrap on generalized complex submanifolds and they form elementary objects in the category of topological D-branes. The natural question to ask next is what the morphisms between a pair of such objects are. By definition, the space of morphisms between a pair of topological D-branes $\mathcal{E}$ and $\mathcal{E}'$ is the BRST cohomology of the space of open strings, with boundary conditions given by $\mathcal{E}$

and $\mathcal{E}'$. From the physical viewpoint, these are simply open string states in the Ramond sector which have zero energy. Open-string BRST cohomology for topological D-branes in A- and B-models has been intensively studied during the last decade both for physical and mathematical reasons. In this section, we compute the BRST cohomology for open strings with ends on the same GC brane. The treatment here mainly follows Ref. [23]. See [44] for a similar analysis in a somewhat different context.

As in the bulk generalized B-model, we show that to any GC brane wrapped on a submanifold Y of a GC manifold X one can associate a Lie algebroid E_Y, whose cohomology computes the open-string BRST cohomology. It turns out that this result has interesting implications even for the relatively well-understood rank-1 A-branes and B-branes (i.e., rank-1 topological D-branes of the ordinary A- and B-model). Here we briefly comment on the case of B-branes. It is believed that the category of B-branes is equivalent to the bounded derived category of coherent sheaves on X, and the rank-1 objects are relatively simple to describe: they are holomorphic line bundles supported on complex submanifolds, as we have seen in the previous section. However, even in this relatively simple case, the mathematical procedure for computing endomorphisms of the corresponding object in $D^b(X)$ is rather complicated and involves some arbitrariness. From the derived category point of view, the correct spaces of endomorphisms are the Ext-groups. The most explicit way to relate to this result is to say that there is a spectral sequence converging to the desired space of endomorphisms (i.e., the Ext-groups) whose E_2 terms are given by the sheaf cohomology

$$\bigoplus_{p,q} H^p(\Lambda^q NY^{1,0}). \tag{5.2.1}$$

The differential d_2 can also be described completely explicitly [26]. It is the composition of the cohomology class $\beta_Y \in H^1(TY \otimes NY^*)$ corresponding to the exact sequence

$$0 \longrightarrow TY \longrightarrow TX|_Y \longrightarrow NY \longrightarrow 0$$

and the cohomology class $[F] \in H^1(TY^*)$ represented by the curvature of the line bundle

$\mathcal{E}$. The class $[F]$ is known as the Atiyah class of the holomorphic line bundle $\mathcal{E}$; it is the obstruction to the existence of a holomorphic connection on $\mathcal{E}$. The class β_Y measures the extent to which $TX|_Y$ fails to split holomorphically as $TY \oplus NY$. Their composition $[F]\lrcorner\beta_Y$ is a class in $H^2(NY^*)$.

Back-of-the-envelope estimates of the open-string BRST cohomology give E_2 as the physical result, but a more careful computation shows that the whole spectral sequence arises [26]. This serves as an important check that the category of B-branes is indeed equivalent to $D^b(X)$. Our result shows that one can dispense with the spectral sequence and write down an explicit graded vector bundle on Y and a differential Q_Y on its space of sections, such that Q_Y-cohomology computes the space of endomorphisms of the B-brane. Specifically, the graded bundle is isomorphic to

$$\bigoplus_{p,q} \Omega^{0,p} \otimes \Lambda^q NY^{1,0} \tag{5.2.2}$$

(the grading being $p + q$), and the differential Q_Y is mapped by this isomorphism to a deformation of the Dolbeault differential

$$\bar{\partial} + \delta(Y, F).$$

The correction term $\delta(Y, F)$ has bidegree $(2, -1)$ and depends both on the way Y sits in X and the curvature of the line bundle on Y. The correction term is by itself a differential. One can write down a spectral sequence which converges to Q_Y-cohomology and whose E_2 terms are given by Eq. (5.2.1). It also turns out that the differential d_2 is equal to $[F]\lrcorner\beta_Y$. This confirms that our result by using Lie algebroid cohomology matches the known results.

It should be mentioned that the isomorphism of our graded bundle on Y with the graded bundle Eq. (5.2.2) is not canonical, and as a result the form $\delta(Y, F)$ is not completely canonical either. However, the construction of the original graded bundle and the differential Q_Y is completely canonical and involves no arbitrariness.

We note that in the case of B-branes of higher rank we do not have analogous results. The spectral sequence computing endomorphisms still exists, but we do not know how to get it from a complex of vector bundles on Y. Hopefully, an extension of the computations in this paper will enable one to find such a complex.[1]

5.2.1 The Lie algebroid of a GC brane

Let (Y, F) be a GC brane in $(X, \mathcal{I})$. As we saw in Sec. 5.1, the generalized tangent bundle $\mathcal{T}Y_F$ is stable under $\mathcal{I}$. Therefore $\mathcal{I}$ restricts to an almost complex structure on $\mathcal{T}Y_F$. Let E_Y be the $-i$ eigenbundle of this almost complex structure in the complexification of $\mathcal{T}Y_F$. It turns out there is a natural Lie algebroid structure on E_Y [23, 14]. Let us explain it. To define the Lie algebroid, it suffices to specify its anchor map and the Lie bracket on its sections. The anchor map is taken to be the obvious projection to $TY_{\mathbb{C}}$. The Lie bracket is defined as follows. Recall that E is the $-i$ eigenbundle of $\mathcal{I}$ in $(T \oplus T^*) \otimes \mathbb{C}$. Given any two sections of E_Y, we can regard them as sections of $E|_Y$, because E_Y is a subbundle of $E|_Y$. Extend them off Y, compute the twisted Courant bracket, and restrict back to Y. One can easily check that the resulting section lies in E_Y, and does not depend on how we extend sections off Y. We define it to be the Lie bracket of the two sections that we start with.

5.2.2 Open-string BRST cohomology for a GC brane

In this section we show that the cohomology of the Lie algebroid E_Y is isomorphic (classically, i.e., if one neglects instantons) to the BRST cohomology of the open string space of states, where both ends of the open string are on the GC brane (Y, F).

The proof is a combination of two tautological lemmas. The first one is that open-string BRST cohomology is isomorphic to the cohomology of a degree-1 vector field Q_Y on a certain graded supermanifold of the form $L[1]$, where L is some complex vector bundle over Y. Indeed, in the zero-mode approximation (which is sufficient for computing the BRST cohomology) open string preobservables are functions of both bosonic coordinates on Y and fermionic coordinates taking values in some vector bundle over Y. Fermionic coordinates can have R-charge 1 or -1. In order to compute the BRST cohomology, it is sufficient to

[1] We would like to emphasize that the results of Ref. [26] apply only to rank-1 bundles.

consider fermionic coordinates with R-charge 1, since the BRST-variation of the ones with R-charge -1 contains spatial derivatives of bosonic coordinates. Let L be the vector bundle over Y where fermionic coordinates of charge 1 take values. Then the space of observables is the space of functions on the graded supermanifold $L[1]$. The generator of the BRST transformation is a degree-1 vector field on $L[1]$ that squares to zero.

We can be more specific about the bundle L. As always, it is convenient to work with the linear combinations of $\psi_\pm$

$$\psi = \psi_+ + \psi_-, \qquad \rho = g\,(\psi_+ - \psi_-)\,,$$

in order to make contact with generalized geometry. Open-string boundary conditions (5.1.1) and (5.1.2) put a linear constraint on the fermionic zero-modes (ψ, ρ), which requires them to be in the fibers of the generalized tangent bundle $\mathcal{T}Y_F$. Finally, the requirement that the fermions have $U(1)_V$ R-charge one is equivalent to the requirement that the fermions be in the subbundle E_Y. It then follows that L, as a vector bundle, is precisely the vector bundle E_Y associated with the GC brane.

The vector field Q_Y on $L[1] = E_Y[1]$ can be thought of as follows. In the closed-string case, bosonic zero-modes take values in the whole X, while fermionic zero-modes with R-charge 1 take values in the bundle E. The closed-string BRST operator Q_B can be thought of as a degree-1 vector field on $E[1]$. Open-string boundary conditions select a submanifold $E_Y[1]$ of $E[1]$. In these terms, compatibility of the boundary condition with the BRST symmetry means that Q_B is tangent to $E_Y[1]$. Therefore Q_B induces a degree-1 vector field Q_Y on functions of $E_Y[1]$. Obviously, this vector field generates BRST transformations of open string preobservables.

The second tautological statement is that the Lie algebroid cohomology of E_Y is isomorphic to the cohomology of Q_Y acting on functions on $E_Y[1]$. This is fairly obvious from the way the Lie bracket on E_Y was defined. Suppose f is a function on $E_Y[1]$. To compute $Q_Y(f)$, we must extend f to a function on the ambient supermanifold $E[1]$, apply Q_B and restrict back to $E_Y[1]$. We can think of the extension as a two-step procedure. First we extend in the fermionic directions. This means that if we regard f as a section of $\Lambda^\bullet E_Y^*$, we must lift it to a section $\tilde{f}$ of $\Lambda^\bullet E^*|_Y$ (the former vector bundle is quotient of the latter). Second, we extend in the bosonic directions. This means that we extend the section $\tilde{f}$ of $\Lambda^\bullet E^*|_Y$ off Y. Then we apply the vector field Q_B, restrict back to Y, and project to $\Lambda^\bullet E_Y^*$. One can easily see that these are precisely the manipulations one has to do to compute the action of the Lie-algebroid differential for E_Y on a section f of $\Lambda^\bullet E_Y^*$.

5.2.3 Three examples

5.2.3.1 Flat bundles

It is easy to verify that for flat line bundles over GC submanifolds (i.e. for $F = 0$) the application of the results of the previous section gives familiar results. For a GC structure $\mathcal{I}$ coming from a complex structure I on X (and note that we always work with $B = 0$), a GC submanifold is simply a complex submanifold. For such a submanifold, $E_Y = TY^{0,1} \oplus (NY^*)^{1,0}$. The Lie bracket is the obvious one: $TY^{0,1}$ has the standard Lie bracket, the conormal part is an abelian subalgebra, and $TY^{0,1}$ acts on sections of $(NY^*)^{1,0}$ via the ordinary $\bar{\partial}$ operator. Lie algebroid cohomology of E_Y is therefore isomorphic to

$$\bigoplus_{p,q} H^p\left(\Lambda^q NY^{1,0}\right).$$

We note that this is precisely the naive answer that we mentioned a while ago. That the BRST cohomology in this case is actually given by the naive answer is a direct result of the flatness of the line bundle, as it will become clear in the next subsection. More generally, for B-branes supporting non-flat gauge fields, the above sheaf cohomology is only the E_2 term of a whole spectral sequence.

If $\mathcal{I}$ comes from a symplectic structure on X, then as we saw in Sec. 5.1 a GC submanifold is simply a Lagrangian submanifold. In this case E_Y is isomorphic to $TY_{\mathbb{C}}$ as a Lie algebroid. Hence Lie algebroid cohomology is isomorphic to the de Rham cohomology $H^\bullet(Y, \mathbb{C})$. This also confirms the well-known result [41].

5.2.3.2 Rank-1 B-branes

Now let $B = 0$ and let $\mathcal{I}$ come from a complex structure on X. Let (Y, F) be an arbitrary GC brane. As discussed in the previous section, Y must be a complex submanifold of X, and the curvature F of the connection ∇ is of type $(1,1)$ (i.e., the line bundle $\mathcal{E}$ is holomorphic). Let us compute the Lie algebroid structure of E_Y. If the line bundle is flat, then from the definition of the generalized tangent bundle $\mathcal{T}Y_F$, one sees immediately that $E_Y \simeq TY^{0,1} \oplus (NY^*)^{1,0}$. We will see below that when $F \neq 0$, the Lie algebroid structure of E_Y is deformed from the naive direct sum

$$E_Y^0 \equiv TY^{0,1} \oplus (NY^*)^{1,0}\,.$$

To see why the Lie algebroid E_Y is a deformation of E_Y^0 in general, let us do a simple local calculation. Let (z^p, u^i) be local holomorphic coordinates on X such that Y is locally given by the equations $z^p = 0$. Their complex conjugates will be denoted $\bar{z}^{\bar{p}}, \bar{u}^{\bar{i}}$. We want to choose a local basis of sections for E_Y. The most obvious choice is

$$e_{\bar{i}} = \frac{\partial}{\partial \bar{u}^{\bar{i}}} - F_{\bar{i}j} du^j, \qquad e^p = dz^p.$$

It is easy to see that this is a local trivialization of E_Y. Moreover, it is easy to check that all Lie brackets vanish (it is important here that $dF = 0$). On the other hand, the obvious Lie algebroid $E_Y^0 = TY^{0,1} \oplus (NY^*)^{1,0}$ has the following obvious local trivialization:

$$f_{\bar{i}} = \frac{\partial}{\partial \bar{u}^{\bar{i}}}, \quad f^p = dz^p.$$

Obviously, all Lie brackets vanish as well. It seems at this stage that we have proved that the two Lie algebroids are isomorphic. However, this conclusion is premature, because the transition functions in the two cases are different. Namely, as one goes from one chart to another, the covectors du^j mix up with dz^p, and so $e_{\bar{i}}$ mix with e^p; on the other hand, $f_{\bar{i}}$ does not mix with f^p.

To compare the two Lie algebroids, it is convenient to adopt a more complicated local trivialization for E_Y, so that the transition functions are the same as for E_Y^0. This will prove that E_Y and E_Y^0 are isomorphic as smooth vector bundles. However, we will see that the Lie bracket on E_Y is deformed from that on E_Y^0 in general, so they are not isomorphic as Lie algebroids.

It is possible to give a precise mathematical description of the deformation of the Lie bracket on E_Y, using the language of Cech cohomology. Let $\{U_\alpha\}$ be an open cover of Y. On each U_α[2], we choose local holomorphic coordinates (z_α^p, u_α^i), so that Y is defined over U_α by $z_\alpha = 0$. On double overlaps $U_{\alpha\beta} = U_\alpha \cap U_\beta$ one has

$$\frac{\partial}{\partial \bar{u}_\alpha^{\bar{i}}} = (A_{\alpha\beta})_{\bar{i}}^{\bar{j}} \frac{\partial}{\partial \bar{u}_\beta^{\bar{j}}}, \qquad du_\alpha^j = (B_{\alpha\beta})_i^j du_\beta^i + (C_{\alpha\beta})_p^j dz_\beta^p,$$

where $B_{\alpha\beta} = (\partial u_\alpha/\partial u_\beta)$ and $A_{\alpha\beta} = (\partial \bar{u}_\beta/\partial \bar{u}_\alpha)$ are the gluing cocycles for $(TY^*)^{1,0}$ and $TY^{0,1}$. The entries of the rectangular matrices $C_{\alpha\beta} = (\partial u_\alpha/\partial z_\beta)$ are local holomorphic functions, and they measure the mixing of covectors du_α and dz_β. Now consider a local holomorphic section of $TY^{1,0} \otimes (NY^*)^{1,0}$ that is defined on $U_{\alpha\beta}$ by

$$c_{\alpha\beta} = (C_{\alpha\beta})_p^j \frac{\partial}{\partial u_\alpha^j} \otimes dz_\beta^p.$$

By definition, $c_{\alpha\beta}$ is a Cech 1-cochain with values in the coherent sheaf $TY^{1,0} \otimes (NY^*)^{1,0}$. What is less obvious, but true, is that $c_{\alpha\beta}$ is actually a 1-cocycle. To see this, let $U_{\alpha\beta\gamma}$ be a non-empty triple overlap. Over it we have (we suppress the matrix indices to simplify the notation):

$$\begin{aligned} du_\alpha &= B_{\alpha\beta} du_\beta + C_{\alpha\beta} dz_\beta \\ &= B_{\alpha\beta}\big(B_{\beta\gamma} du_\gamma + C_{\beta\gamma} dz_\gamma\big) + C_{\alpha\beta}\left(\frac{\partial z_\beta}{\partial z_\gamma}\right) dz_\gamma \\ &= B_{\alpha\gamma} du_\gamma + \left[B_{\alpha\beta} C_\beta \gamma + \left(\frac{\partial z_\beta}{\partial z_\gamma}\right) C_{\alpha\beta}\right] dz_\gamma\,. \end{aligned}$$

[2]Actually, on an open neighborhood of U_α in X, but we shall not elaborate on this technical issue here.

We immediately read off

$$C_{\alpha\gamma} = B_{\alpha\beta} C_{\beta\gamma} + \Big(\frac{\partial z_\beta}{\partial z_\gamma}\Big) C_{\alpha\beta},$$

and it easily follows that

$$c_{\alpha\gamma} = C_{\alpha\gamma} \frac{\partial}{\partial u_\alpha} \otimes dz_\gamma = C_{\alpha\beta} \frac{\partial}{\partial u_\alpha} \otimes dz_\beta + C_{\beta\gamma} \frac{\partial}{\partial u_\beta} \otimes dz_\gamma = c_{\alpha\beta} + c_{\beta\gamma}.$$

This shows that $c_{\alpha\beta}$ indeed defines a Cech 1-cocycle with respect to the open cover $\{U_\alpha\}$. We can of course choose $\{U_\alpha\}$ to be a good open cover; in this case $c_{\alpha\beta}$ defines a class in the first Cech cohomology $H^1(TY^{1,0} \otimes (NY^*)^{1,0})$, which measures the failure of $TX|_Y$ to split holomorphically as $TY \oplus NY$. This Cech cohomology class is precisely the class β_Y that appeared earlier in this section.

One can actually write the cocycle $c_{\alpha\beta}$ as a coboundary of a *smooth* 0-cochain. The existence of such a cochain is guaranteed by the existence of partition of unity for smooth sections. In sheaf theoretical terminology, this reflects the fact that the sheaf of smooth sections of $TY^{1,0} \otimes (NY^*)^{1,0}$ is a fine sheaf, so its first sheaf coholomogy H^1 vanishes[3]. Since we know $c_{\alpha\beta}$ is a 1-cocycle, it must be a coboundary.

To explicitly construct such a 0-cochain, it suffices to define it locally (i.e., on each open set U_α). Locally such a 0-cochain is defined in terms of matrices C_α of smooth functions:

$$c_\alpha = (C_\alpha)^j_p \frac{\partial}{\partial u^j_\alpha} \otimes dz^p_\alpha.$$

That $c_{\alpha\beta}$ is the coboundary of the 0-cochain defined by c_α is the statement that one can choose the matrices C_α such that on double overlaps

$$c_{\alpha\beta} = (C_\alpha)^j_p \frac{\partial}{\partial u^j_\alpha} \otimes dz^p_\alpha - (C_\beta)^j_p \frac{\partial}{\partial u^j_\beta} \otimes dz^p_\beta.$$

As already mentioned, such a choice always exists, and is a consequence of the existence of partition of unity.

[3] In fact, all higher sheaf cohomology of a fine sheaf vanishes.

Using the cochain c_α, we can define a modified local trivialization of E_Y. Namely, over U_α we choose the local basis

$$(e_\alpha)_{\bar{i}} = \frac{\partial}{\partial \bar{u}_\alpha^{\bar{i}}} - (F_\alpha)_{\bar{i}j}\left(du_\alpha^j - (C_\alpha)_p^j dz_\alpha^p\right), \qquad (e_\alpha)^p = dz_\alpha^p.$$

Here $(F_\alpha)_{i\bar{j}}$ are the components of the curvature 2-form F in local coordinates $u_\alpha, \bar{u}_\alpha$. They glue on double overlaps by $(F_\alpha)_{\bar{i}j} = (\bar{B}_{\alpha\beta})_{\bar{i}}^{\bar{j}}(B_{\alpha\beta})_j^i(F_\beta)_{\bar{j}i}$. It is easy to check that the gluing cocycle for the modified local bases is exactly the same as that for the standard basis

$$(f_\alpha)_{\bar{i}} = \frac{\partial}{\partial \bar{u}_\alpha^{\bar{i}}}, \qquad (f_\alpha)^p = dz_\alpha$$

of $TY^{0,1} \oplus (NY^*)^{1,0}$. Namely, the gluing cocycle is given by

$$(e_\alpha)_{\bar{i}} = (\partial \bar{u}_\beta / \partial \bar{u}_\alpha)_{\bar{i}}^{\bar{j}}(e_\beta)_{\bar{j}}, \qquad (e_\alpha)^p = (\partial z_\alpha / \partial z_\beta)_q^p (e_\beta)^q.$$

This shows that E_Y is isomorphic to E_Y^0 as a vector bundle.

Now let us compute the Lie bracket of elements of the modified local basis e_α. Locally, these come out to be

$$[(e_\alpha)_{\bar{i}}, (e_\alpha)_{\bar{j}}] = \left((F_\alpha)_{\bar{j}k}\frac{\partial (C_\alpha)_p^k}{\partial \bar{u}_\alpha^{\bar{i}}} - (F_\alpha)_{\bar{i}k}\frac{\partial (C_\alpha)_p^k}{\partial_{\bar{u}} \alpha^{\bar{j}}}\right)(e_\alpha)^p, \quad [(e_\alpha)^p, (e_\alpha)^q] = 0, \quad [(e_\alpha)_{\bar{i}}, (e_\alpha)^p] = 0.$$

Unlike the local basis f_α of E_Y^0, the commutators of the local basis e_α of E_Y have a non-vanishing part

$$\delta(Y, F)\left(\frac{\partial}{\partial \bar{u}_\alpha^{\bar{i}}}, \frac{\partial}{\partial \bar{u}_\alpha^{\bar{j}}}\right),$$

where $\delta(Y, F)$ is a local section of $\Omega^{0,2}(Y) \otimes (NY^*)^{1,0}$ given in U_α by

$$\delta(Y, F) = \left((F_\alpha)_{\bar{j}k}\frac{\partial (C_\alpha)_p^k}{\partial \bar{u}_\alpha^{\bar{i}}} - (F_\alpha)_{\bar{i}k}\frac{\partial (C_\alpha)_p^k}{\partial \bar{u}_\alpha^{\bar{j}}}\right) dz_\alpha^p \otimes d\bar{z}_\alpha^{\bar{i}} \wedge d\bar{z}_\alpha^{\bar{j}}.$$

It follows from the definition of C_α and the fact that the cocycle $c_{\alpha\beta}$ is holomorphic that

$\delta(Y,F)$ is in fact globally defined. To see this more clearly, note that on the double overlap $U_{\alpha\beta}$,

$$\frac{\partial (C_\alpha)^k_p}{\partial \bar{u}^{\bar{i}}_\alpha}\frac{\partial}{\partial u^k_\alpha}\otimes dz^p_\alpha = \frac{\partial (C_\beta)^k_p}{\partial \bar{u}^{\bar{i}}_\alpha}\frac{\partial}{\partial u^k_\beta}\otimes dz^p_\beta.$$

This expression defines a global section d_Y of $\Omega^{0,1}(Y)\otimes TY^{1,0}\otimes (NY^*)^{1,0}$ whose local expression is

$$d_Y = \frac{\partial (C_\alpha)^k_p}{\partial \bar{u}^{\bar{i}}_\alpha}\frac{\partial}{\partial {u_\alpha}^k}\otimes dz^p_\alpha \otimes d\bar{u}^{\bar{i}}_\alpha.$$

It is easy to see that d_Y is $\bar{\partial}$-closed and therefore represents a class $\beta_Y \in H^1(TY\otimes NY^*)$. This is simply the Dolbeault representative of the cohomology class whose Cech representative was denoted $c_{\alpha\beta}$. The form $\delta(Y,F)$ is obtained by taking the wedge product of d_Y and $F\in\Omega^{0,1}\otimes (TY^*)^{1,0}$ and contracting $TY^{1,0}$ with $(TY^*)^{1,0}$. Since both F and d_Y are $\bar{\partial}$-closed, so is $\delta(Y,F)$.

The Lie algebroid differential for E_Y is now easily computed. Since $E_Y \simeq TY^{0,1}\oplus (NY^*)^{1,0}$, it is a degree-1 differential Q_Y acting on smooth sections of the graded bundle

$$\bigoplus_{r,s}\Omega^{0,r}(Y)\otimes\Lambda^s NY^{1,0},$$

where the grading is given by $r+s$. One easily sees that if ζ is a section of this graded bundle, then

$$Q_Y(\zeta) = \bar{\partial}\zeta + \delta(Y,F)\lrcorner\,\zeta.$$

Here $\lrcorner$ means contraction of NY^* and $\Lambda^s NY$. We conclude that the Lie algebroids E_Y and E^0_Y are not isomorphic in general: the former is a deformation of the latter.

Note that the sheaf cohomology class represented by $\delta(Y,F)$ is exactly the product of the class $\beta_Y\in H^1(TY\otimes NY^*)$ and a class in $H^1(TY^*)$ represented by the $(1,1)$ form F. The latter class is the Atiyah class of the line bundle on the brane Y. There is a standard filtration on the doubly-graded complex $K^{r,s}=\oplus_{r,s}\Omega^{0,r}(Y)\otimes\Lambda^s NY^{1,0}$, which gives rise to a spectral sequence that converges to the Q_Y-cohomology. More concretely, we get a spectral sequence whose E_0 and E_1 terms are the graded complex K itself, and whose whose E_2 term is simply the now familiar $\bar{\partial}$-cohomology:

$$\bigoplus_{p,q} H^p(\Lambda^q NY^{1,0}).$$

The second differential d_2 is the product of $[d_Y] = \beta_Y$ and the Atiyah class. This is precisely the E_2 term in the spectral sequence expected to compute the Ext-groups of the object of $D^b(X)$ corresponding to our brane (Y, F) we saw earlier [26]. (The object is the push-forward of the locally free sheaf $\mathcal{E}$ on Y to the ambient manifold X). This matches the previously known results and provides some evidence that the Lie algebroid cohomology computes the Ext-groups.

5.2.3.3 Coisotropic A-branes

Since the geometry of coisotropic A-branes is somewhat more complicated than that of B-branes, we start with a brief review of the data involved (see Ref. [25] for more details). A coisotropic A-brane is a triple (Y, ∇, F) such that $Y \subset X$ is a coisotropic submanifold, and ∇ is a unitary connection on a line bundle on Y with curvature F. By definition, $\mathcal{L}Y \equiv \ker(\omega|_Y)$ forms an integrable distribution of constant rank, which is the codimension of Y. In addition, the curvature form F, regarded as a bundle map $F : TY \to TY^*$, must annihilate $\mathcal{L}Y$. So if we denote the quotient bundle $TY/\mathcal{L}Y$ by $\mathcal{F}Y$, F descends to a section of $\Lambda^2\mathcal{F}Y^*$. Finally, the restriction of $\omega^{-1}F$ to $\mathcal{F}Y$ defines a transverse almost-complex structure on Y with respect to the foliation $\mathcal{L}Y$. (This transverse almost complex structure is automatically integrable.) It follows from these conditions that the complex dimension of $\mathcal{F}Y$ is even. Furthermore, both F and $\omega|_{\mathcal{F}Y}$ are of type $(2,0)+(0,2)$ with respect to the transverse complex structure $J = \omega^{-1}F|_{\mathcal{F}Y}$.

The Lie algebroid associated with the brane (Y, F) is $E_Y = \ker(\mathcal{I}_Y + i)$, where $\mathcal{I}_Y$ is the restriction to Y of the generalized complex structure associated to the symplectic structure

ω on X:

$$\mathcal{I} = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}.$$

It is easy to see that topologically E_Y is isomorphic to $\mathcal{L}_{\mathbb{C}}Y \oplus \mathcal{F}Y^{1,0}$. However, we will show that $E_Y \simeq \mathcal{L}Y_{\mathbb{C}} \oplus \mathcal{F}Y^{1,0}$ also as Lie algebroids. To this end, we perform the same kind of calculation as in the B-brane case. As the calculation is similar, we shall be brief. Let us choose a local system of coordinates $(x^a, z^i, \bar{z}^{\bar{i}}, y^\mu)$ on X such that the submanifold Y is locally defined by $x^a = 0$, y^μ parametrize the leaves of the foliation defined by $\mathcal{L}Y$, while the z's are holomorphic coordinates in the transverse directions. Note that the splitting of transverse coordinates into holomorphic and anti-holomorphic ones is done with respect to the complex structure J on $\mathcal{F}Y$. A local trivialization for E_Y is given by

$$e_i = \frac{\partial}{\partial z^i} - i\omega_{ij}dz^j - i\omega_{ia}dx^a, \qquad e_\mu = \frac{\partial}{\partial y^\mu} - i\omega_{\mu a}dx^a.$$

To find the gluing cocycle in this basis, let us take another local system of coordinates (u, w, v) that overlaps with the old one. As equations $x^a = 0$ and $u^a = 0$ define the same submanifold Y locally, one must have $u = u(x)$ on the overlap. In addition, from $\omega(\mathcal{L}Y, TY) = 0$ one deduces that $w = w(x, z), \bar{w} = \bar{w}(x, \bar{z})$. In other words, the Jacobian for the coordinate change takes the following "upper triangular" form[4]

$$\frac{\partial(x, z, y)}{\partial(u, w, v)} = \begin{pmatrix} \partial x/\partial u & \partial z/\partial u & \partial y/\partial u \\ 0 & \partial z/\partial w & \partial y/\partial w \\ 0 & 0 & \partial y/\partial v \end{pmatrix}.$$

It immediately follows that the e_μ's transform among themselves in a simple way:

$$e'_\mu = \frac{\partial y^\nu}{\partial v^\mu} e_\nu.$$

[4]Here we write z to denote both z and $\bar{z}$ coordinates to simplify the notation. The same applies to w.

The transformation law for the e_i's is slightly more complicated. In the new chart, we have

$$e'_i = \frac{\partial}{\partial w^i} - i\omega'_{ij} dw^j - i\omega'_{ia} du^a.$$

The form of the Jacobian implies that the components of ω transform according to

$$\begin{aligned} \omega'_{ij} &= \frac{\partial z^k}{\partial w^i}\frac{\partial z^\ell}{\partial w^j}\,\omega_{k\ell} \\ \omega'_{ia} &= \frac{\partial z^j}{\partial w^i}\frac{\partial x^b}{\partial u^a}\,\omega_{jb} + \frac{\partial z^j}{\partial w^i}\frac{\partial z^k}{\partial u^a}\,\omega_{jk} + \frac{\partial y^\mu}{\partial w^i}\frac{\partial x^b}{\partial u^a}\,\omega_{\mu b} \end{aligned}$$

Combining this with the transformation law for the coordinate basis, one can show that

$$e'_i = \frac{\partial z^j}{\partial w^i}\, e_j + \frac{\partial y^\mu}{\partial w^i}\, e_\mu.$$

This is the same gluing cocycle if we take

$$\tilde{e}_i = \frac{\partial}{\partial z^i}, \qquad \tilde{e}_\mu = \frac{\partial}{\partial y^\mu} \tag{5.2.3}$$

as the obvious local trivialization of $\mathcal{L}Y_{\mathbb{C}} \oplus \mathcal{F}Y^{1,0}$. Therefore our choice of basis establishes an isomorphism between E_Y and $\mathcal{L}Y_{\mathbb{C}} \oplus \mathcal{F}Y^{1,0}$ as vector bundles.

One can further show that these basis sections of E_Y commute under the Lie bracket derived from the Dorfman bracket on $TX \oplus TX^*$. For instance, we have

$$\begin{aligned} [e_i, e_j] &= -i\mathcal{L}_{\partial_i}(\omega_{j\alpha} dq^\alpha) + i\mathcal{L}_{\partial_j}(\omega_{i\alpha} dq^\alpha) - d\iota_{\partial_j}(i\omega_{i\alpha} dq^\alpha) \\ &= -i(\partial_i \omega_{j\alpha} - \partial_j \omega_{i\alpha} + \partial_\alpha \omega_{ij}) dq^\alpha \\ &= 0, \end{aligned} \tag{5.2.4}$$

where q^α's denote all of $x^a, z^i, \bar{z}^{\bar{i}}, y^\mu$. The last step follows directly from $d\omega = 0$. By the same token, we have $[e_\mu, e_\nu] = 0$, $[e_i, e_\mu] = 0$. Namely, our basis sections for E_Y have the same (vanishing) Lie brackets among themselves, just as the basis sections Eq. (5.2.3) of

$\mathcal{L}Y_{\mathbb{C}} \oplus \mathcal{F}Y^{1,0}$.

This shows that $E_Y \simeq \mathcal{L}Y_{\mathbb{C}} \oplus \mathcal{F}Y^{1,0}$ not only as vector bundles but also as Lie algebroids. Since Lie algebroid structures on $E \to X$ are in one-to-one correspondence with degree-1 homological vector fields on $E[1]$ (i.e., BRST operators in the jargon of TFT), we conclude that one can use the obvious Lie algebroid structure on $\mathcal{L}Y_{\mathbb{C}} \oplus \mathcal{F}Y^{1,0}$ to compute the open string ground states for a coisotropic A-brane. Namely one can use the simplified BRST operator[5]

$$Q_Y = d_{\mathcal{L}Y} + \bar{\partial}_{\mathcal{F}Y},$$

where $d_{\mathcal{L}Y}$ is the de Rham differential in the leaf direction, and $\bar{\partial}_{\mathcal{F}Y}$ is the Dolbeault operator in the directions transverse to the foliation. This proves the claim by A. Kapustin and D. Orlov [25] that the open-string BRST cohomology for a coisotropic A-brane is isomorphic to the cohomology of the sheaf of functions locally constant along the leaves of the characteristic foliation of Y and holomorphic in the transverse directions.

5.3 Towards a theory of higher-rank generalized complex branes

So far our discussion of topological D-branes has been focused on the relatively simple case of rank 1. To generalize our analysis to higher-rank cases, there are several obstacles to overcome. First, the Wilson-loop operator

$$\mathrm{Tr}\, P \exp\left(-\oint_{\partial\Sigma} A\right)$$

does not yield a local correction to the bulk action without introducing additional boundary degrees of freedom. Secondly, the Lie algebroid structure of the BRST cohomology is lost. In our description of the open string BRST cohomology in terms of Lie algebroid cohomology, it is crucial that functions of the fermionic zero modes form a graded commutative alge-

[5] We have changed ∂ to $\bar{\partial}$ to conform with the usual conventions.

bra $C^\infty(\Pi E)$. For higher-rank D-branes, the gauge fields are non-abelian, and this graded commutative algebra must be replaced by some non-commutative algebra that is presently unknown. One expects the open-string BRST cohomology to be some module over this non-commutative algebra. This non-commutativity is unavoidable even when one considers open strings stretched between two different rank-1 branes, as recent calculation of open string spectra between pairs of coisotropic A-branes confirms [1].

In this final section, we shall only focus on the first aspect; namely to try to provide a geometric description of higher-rank topological D-branes in generalized B-model. The question of computing the BRST cohomology is much more involved and is left to future work. As in the rank-1 case, we restrict to the case of $H = 0$ and in fact set $B = 0$ for simplicity. The sigma model bulk action is still given by

$$S_{\text{bulk}} = \frac{1}{2}\int d^2\sigma \int d\theta^- d\theta^+ g_{\mu\nu}(\Phi) D_+\Phi^\mu D_-\Phi^\nu.$$

and the $N = 1$ supersymmetry for the bulk fields are

$$\begin{aligned} \delta\phi^\mu &= \epsilon\psi^\mu \\ \delta\psi^\mu &= 2i\epsilon\dot{\phi}^\mu \\ \delta\rho_\mu &= 2i\epsilon g_{\mu\nu}\partial_1\phi^\nu + \epsilon\Gamma^\nu{}_{\mu\lambda}\psi^\lambda\rho_\nu. \end{aligned}$$

As before, the variation of the bulk action under the $N = 1$ supersymmetry is

$$\delta S_{\text{bulk}} = -\frac{\epsilon}{2}\int_{\partial\Sigma} d\tau \Big(\rho_\mu\dot{\phi}^\mu - g_{\mu\nu}\partial_1\phi^\mu\psi^\nu\Big).$$

In the rank-1 case, the variation of the bulk action is compensated by the variation of the boundary action that comes from the Wilson-loop operator. In the higher-rank case at hand, the Wilson-loop operator is non-local, and it certainly does not give a local boundary action in the usual sense. Is there a way around this conumdrum?

The answer to this question is that there *is* a local boundary action that one can write down, but to achieve that one needs to introduce extra fields. These extra fields can be taken as fermions living on the boundary only. The specific construction runs as follows. Let k be the rank of the brane. One introduces k boundary fermions[6] η_a, $a = 1, \ldots, k$. These fermions are charged under the non-abelian gauge fields A and they live in the fundamental representation. The claim is that inserting the non-local Wilson-loop operator in the path integral is equivalent to enlarge the system by including the boundary fermions η_a, with their action given by

$$S_{\text{boundary}} = i \int d\tau \Big(\bar{\eta}^a \dot{\eta}_a + i A_\mu{}^b{}_a \bar{\eta}^a \eta_b \dot{\phi}^\mu \Big). \tag{5.3.1}$$

There is a further constraint to be satisfied for the boundary fermions: their Hilbert space must be projected to the subspace of fermion number one. This last condition guarantees that one gets precisely the path-ordered phase factor. For more details on this matter, see Ref. [19, 2, 8].

Mimicking the ansatz we used in Sec. 5.1, we speculate the following form of boundary condition for the bulk fermions:

$$\psi \in TY, \qquad \rho_\mu|_{TY} = F_{\mu\nu}{}^a{}_b \bar{\eta}^b \eta_a \psi^\nu. \tag{5.3.2}$$

We will come to the correct interpretation of it in a moment. To write down the boundary condition for bosons, which is obtained from the Q-variation of (5.3.2), we need to extend the $N = 1$ supersymmetry to the boundary fermions as well. It turns out that a consistent choice is

$$\delta\eta_a = i\epsilon A_\mu{}^b{}_a \eta_b \psi^\mu, \qquad \delta\bar{\eta}^a = -i\epsilon A_\mu{}^a{}_b \bar{\eta}^b \psi^\mu. \tag{5.3.3}$$

To see that (5.3.3) indeed closes under $N = 1$ supersymmetry, we note

$$\begin{aligned} \{Q, \{Q, \eta_a\}\} &= i\{Q, A_\mu{}^b{}_a \eta_b \psi^\mu\} \\ &= -\frac{i}{2} (F_{\mu\nu})^b{}_a \psi^\mu \psi^\nu \eta_b + 2 (A_\mu)^b{}_a \dot{\phi}^\mu \eta^b. \end{aligned}$$

[6]Do not confuse the η_a's with the spin-1 bulk fermion in Sec. 3.4.

We claim that the right-hand side is actually equal to $2i\dot{\eta}_a$ on-shell, as required by the $N=1$ supersymmetry. To write down the equation of motion for η_a, it is crucial to note that ρ and η_a are constrained by (5.3.2). It comes out that the e.o.m. for η_a takes the following form

$$\dot{\eta}_a = -i(A_\mu)^b{}_a\eta_b\dot{\phi}^\mu - \frac{1}{4}(F_{\mu\nu})^b{}_a\eta_b\psi^\mu\psi^\nu.$$

From this it follows directly that $Q^2\eta_a = 2i\dot{\eta}_a$, and the $N=1$ supersymmetry closes.

The boundary condition for the bosons can now be determined from (5.3.2) from the extended $N=1$ supersymmetry: besides the constraint $\phi \in Y$, the bosons must also satisfy

$$g_{\mu\nu}\partial_1\phi^\nu = F_{\mu\nu}{}^a{}_b\bar{\eta}^b\eta_a\dot{\phi}^\nu - \frac{i}{2}\big(D_\lambda F_{\mu\nu} - \Gamma^\rho{}_{\mu\lambda}F_{\rho\nu}\big)^a{}_b\bar{\eta}^b\eta_a\psi^\lambda\psi^\nu. \qquad (5.3.4)$$

Using (5.3.2) and (5.3.4), the variation of the bulk action can be shown to be

$$\delta S_{\text{bulk}} = -\epsilon\int d\tau\ F_{\mu\nu}{}^a{}_b\bar{\eta}^b\eta_a\psi^\nu\dot{\phi}^\mu.$$

It is remarkable that the bulk variation δS_{bulk} is precisely cancelled by the variation of the boundary action (5.3.1) that we wrote down before. This is easy to see because S_{boundary} varies under Q as

$$\delta S_{\text{boundary}} = \epsilon\int d\tau\ F_{\mu\nu}{}^a{}_b\bar{\eta}^b\eta_a\psi^\nu\dot{\phi}^\mu,$$

which precisely cancels the bulk variation. It follows that the boundary condition (5.3.2) preserves the $N=1$ supersymmetry.

Next we consider the condition of unbroken $U(1)_V$ R-symmetry. For the boundary fields, we take the ansatz

$$\delta_R\eta = 0, \qquad \delta_R\bar{\eta} = 0.$$

As before, the variations for the bulk fermions are

$$\begin{aligned} \delta_R \rho &= i\delta\omega(\psi) - i\tilde{I}^t \rho, \\ \delta_R \psi &= i\tilde{I}\psi - i\alpha(\rho). \end{aligned}$$

Again, the various symbols appearing above are defined in (3.1.2). For topological D-branes, the R-current J_V^1 must vanish at the boundary as in the rank-1 case:

$$\delta\omega(\psi, \psi) + 2\rho \cdot \tilde{I}\psi - \alpha(\rho, \rho) = 0.$$

The only difference is that one has to replace $F_{\mu\nu}$ by $(F_{\mu\nu})^a{}_b \bar{\eta}^b \eta_a$. We write the latter simply as $F\bar{\eta}\eta$ in the following. As before, introduce $\rho' = \rho - F\bar{\eta}\eta\psi$, which is interpreted as living in NY^*. We have

$$\begin{aligned} &\big(\delta\omega - \tilde{I}^t F\bar{\eta}\eta - F\bar{\eta}\eta\tilde{I} + F\bar{\eta}\eta \cdot \alpha \cdot F\bar{\eta}\eta\big)(\psi, \psi) \\ &\qquad -\alpha(\rho', \rho') + 2\rho' \cdot \tilde{I}\psi - 2\rho' \cdot \alpha \cdot F\bar{\eta}\eta\psi = 0. \end{aligned} \tag{5.3.5}$$

The crucial point now is that one must regard equations (5.3.2) and (5.3.5) as *operator* equations in the sub-sector of the boundary Hilbert space with fermion number one, $\mathcal{H}_1$. To find a solution, let's first consider equation (5.3.2). Let $v = v_a \bar{\eta}^a |0\rangle \in \mathcal{H}_1$, where $|0\rangle$ is the vacuum state. One can regard v as a section of a vector bundle E over Y, and (5.3.2) can be rewritten as

$$\rho \otimes v - F(\psi \otimes v) \in NY^* \otimes E.$$

Here F is regarded as a section of $\mathrm{Hom}(TY, TY^*) \otimes \mathrm{End}\,(E)$:

$$F : e_\mu \otimes \bar{\eta}^a \mapsto -(F_{\mu\nu})^a_b \theta^\nu \otimes \bar{\eta}^b,$$

with $\{e_\mu\}$ a basis for TY and $\{\theta^\nu\}$ the dual basis.

Now let us consider the meaning of (5.3.5). Acting on $v \in \mathcal{H}_1$, it yields

$$\big(\delta\omega(\psi,\psi) + 2\rho' \cdot \tilde{I}\psi - \alpha(\rho',\rho')\big)v_a - 2\rho' \cdot \alpha \cdot F^b_{\ a}\psi v_b$$
$$-2\psi \cdot \tilde{I}^t F^b_{\ a}\psi v_b + F^b_{\ c}\alpha F^c_{\ a}(\psi,\psi)v_b = 0.$$

The invariant way to express the above result is that the subbundle $(TY \oplus NY^*) \otimes E \subset (TX \oplus TX^*) \otimes E$ is preserved by the following endomorphism of $TX \oplus TX^*) \otimes E$:

$$\mathcal{I}_F = \begin{pmatrix} 1\otimes 1 & 0 \\ -F & 1\otimes 1 \end{pmatrix} \begin{pmatrix} \tilde{I}\otimes 1 & -\alpha\otimes 1 \\ \delta\omega\otimes 1 & -\tilde{I}^t\otimes 1 \end{pmatrix} \begin{pmatrix} 1\otimes 1 & 0 \\ F & 1\otimes 1 \end{pmatrix}.$$

Let $\mathcal{T}Y_F(E)$ denote the subbundle of $(TY \oplus TX^*|_Y) \otimes E$ that is defined by

$$\big\{X \in TY \otimes E, \xi \in TX^*|_Y \otimes E \big| \xi - F(X) \in NY^* \otimes E\big\}.$$

It is the higher-rank version of the generalized tangent bundle $\mathcal{T}Y_F$. We see that the boundary condition (5.3.2) defines a (classical) topological D-brane if $\mathcal{T}Y_F(E)$ is stable under the action $\mathcal{I} \otimes \mathrm{id}_E$. This is a generalization of the results obtained in Sec. 5.1 for rank-1 branes.

Chapter 6

Summary and Outlook

In this work we have studied the topological sector of the generic $(2, 2)$ sigma model, with possible H-flux, as well as (classical) topological D-branes in them. There is an intimate relation between the newlyfound mathematical theory of generalized geometries, and the properties of these topological sigma models. For example, the ring of local observables in the topological theory is isomorphic, on the semiclassical level, to the cohomology of a certain Lie algebroid that controls the deformation theory of a twisted generalized complex structure. On the quantum level, the two rings are isomorphic as vector spaces, but the ring structures may be different due to worldsheet instantons. It would be very interesting to further explore this direction. In Sec. 4.3 we already took an initial step by showing that the quantum ring structure depends only on one of the two twisted generalized complex structures present. This is the analogue of the statement that the ordinary quantum cohomology ring is independent of the choice of the complex structure [40]. To fully unravel the structure this quantum ring structure, one needs to understand the moduli space of generalized holomorphic curves better.

It is expected on general grounds that the moduli space of $N = 2$ SCFTs is a product of two spaces, corresponding to deformations by elements of the (c,c) and (a,c) rings. It follows from our work that for $N = 2$ sigma models with H-flux these two moduli spaces are identified with the moduli spaces of two independent twisted generalized complex structures $\mathcal{J}_1$ and $\mathcal{J}_2$. The mathematical implication of this observation is that the deformation theory of twisted generalized Calabi-Yau manifolds is unobstructed. One can prove this statement rigorously [29, 14]. Here we briefly outline the main idea behind the proof. As is shown in [12], deformations of a twisted GC structure $\mathcal{J}$ are represented by sections $\epsilon \in \Gamma(\Lambda^2 E^*)$, which must satisfy the following Maurer-Cartan equation:

$$d_E\epsilon + \frac{1}{2}[\epsilon, \epsilon]_{E^*} = 0. \qquad (6.0.1)$$

Here as before E is the $+i$ (or $-i$) eigenbundle of $\mathcal{J}$. Infinitesimal deformations are in one-to-one correspondence with elements of the second Lie algebroid cohomology $H^2_E(M)$. For a generic twisted generalized Kähler manifold, the deformation space is obstructed in general; namely given an infinitesimal deformation, there is no guarantee that it can be integrated into a true, finite deformation. However, in the case that $(M, \mathcal{J})$ is twisted generalized Calabi-Yau, things are different. It can be shown that in this case any infinitesimal deformation of $\mathcal{J}$ can be made into a true deformation. The proof of this fact uses the generalized Hodge decomposition discovered in [13]. We have already seen that the Lie algebroid cohomology $H^2_E(M)$ is isomorphic to the generalized Dolbeault cohomology group $H^2_{\bar{\partial}_H}(M)$, which in turn is isomorphic to the space of generalized harmonic forms $\mathcal{H}^2(M)$, by Gualtieri's result. Let $\epsilon_1 \in \Gamma(\Lambda^2 E^*)$ be a representative of a class in $H^2_E(M))$ corresponding to an infinitesimal deformation of $\mathcal{J}$. Using the isomorphism $' : \Lambda^\bullet E^* \to \Omega^\bullet(M, \mathbb{C})$ induced by the spin representation

$$' : \alpha \mapsto \alpha' = \alpha \cdot \Omega,$$

one can assume ϵ_1' to be a generalized harmonic form. In particular, one may assume $\partial_H \epsilon_1' = 0$. It is shown in [29] that starting with such a generalized harmonic form ϵ_1', one can construct a power series solution

$$\epsilon = \sum_{i \geq 1} \epsilon_i \, t^i, \qquad \epsilon_i \in \Gamma(\Lambda^2 E^*) \; \forall \, i$$

such that it solves the Maurer-Cartan equation (6.0.1). The power series solution can be shown to converge in a neighborhood of $t = 0$, following the line of argument in [27].

Just as the Kodaira-Spencer theory of complex deformation on an ordinary Calabi-Yau manifold yields an effective field theory that is the string field theory of the B-model [3], the above solution of the Maurer-Cartan equation (6.0.1) for twisted GCY manifolds also has an interpretation of an effective field theory. The recursive solutions for ϵ_i found in [29] reveal the perturbative structure of this field theory. In fact, this field theory can be identified with the semiclassical limit of the string field theory of the generalized B-model [10]. The Maurer-Cartan equation, when written in the equivalent form

$$\bar{\partial}_H \epsilon' - \frac{1}{2}\partial_H(\epsilon \wedge \epsilon)' \; = \; 0,$$

is interpreted as the classical equation of motion of the string field.

If there are no nontrivial instantons in the theory (see our discussion of worldsheet instantons in generalized B-model in Sec. 4.3) this field theoretical description of the generalized B-model is exact. When nontrivial instantons exist, however, this effective field theory only describes the generalized B-model up to the semiclassical level.[1] semiclassically, the general construction in Ref. [10] reduces to the known results in the limit of the ordinary A-model and B-model. In the case of the B-model, it reduces to the familiar Kodaira-Spencer theory of gravity[2] proposed in [3]. In the case of the A-model, the resulting theory after reduction turns out to be what is called the Kähler gravity, first analyzed in [4]. Here the identification is not as straightforward as in the case of B-model, due to the fact that the natural identification of fermions with operators on differential forms in the generalized B-model is different from that of the conventional choice for the A-model. This is already discussed in Sec. 4.2. One needs to perform a Bogolyubov transformation to relate to the Kähler gravity

[1] The string field theory of the generalized B-model is a field theory on the loop space LM. When nontrivial worldsheet instantons are present, it cannot be reduced to a field theory on M except in the limit of large volumes, where the worldsheet instantons are suppressed.

[2] The reduction of the string field theory of generalized B-model in this case actually include more fields than in [3]. The extra fields correspond to deformations parametrized by a holomorphic Poisson bivector and a holomorphic $(0, 2)$-form. The former can be interpreted as turning on certain noncommutativity on M, while the latter can be described as deforming the category of coherent sheaves on M. Both deformations are natural from the viewpoint of generalized B-model.

formulated in [4].

Another very interesting direction for future study, which we have not discussed at all in this work, is to understand the mirror symmetry for generalized Calabi-Yau manifolds. Up to now, most results are from supergravity solutions of flux compactifications, and are largely confined to special backgrounds (for a recent discussion, see [36]). To make further progress for more general twisted GCY backgrounds, it is likely that more powerful worldsheet methods must be developed. Hopefully, such new tools will also enable us to construct examples of generalized Kähler and generalized Calabi-Yau manifolds in a systematic manner. In the ordinary Kähler case, Witten's gauged linear sigma model (GLSM) is the physical realization of Kähler quotient, and one can obtain toric backgrounds[3] as the IR limit of the GLSM [42]. In the case of (twisted) generalized Kähler manifolds, such a worldsheet theory has not been found so far. Compounding the matter even more, it is not even clear what is the counterpart of the Kähler quotient for generalized Kähler geometry, although recently there is progress in this direction [6]. We leave this interesting topic to the future.

[3]More general backgrounds are possible, if one allows non-abelian gauge groups.

Appendix A

Lie algebroids and Lie algebroid cohomology

Certain vector bundles have the special property that they carry natural Lie bracket structures on the sections. The tangent bundle of a differential manifold and a bundle of Lie algebras are the first examples to come to mind. The former carries a natural Lie bracket of vector fields, while in the latter the Lie bracket structure is provided by the Lie bracket on the Lie algebra. Lie algebroids can be regarded as generalizations of the tangent bundle as well as a bundle of Lie algebras. For a thorough discussion of the topic, see [31].

At the more formal level, a Lie algebroid on a differential manifold M is a vector bundle $E \to M$ with a Lie bracket $[,]$ on its sections, together with a bundle map (called the anchor) $a : E \to TM$ such that

- The anchor induces a Lie algebra homomorphism, i.e. $a[X,Y] = [a(X), a(Y)]$, $\forall X, Y \in \Gamma(E)$.
- The Leibniz rule is satisfied:

$$[X, fY] = f[X,Y] + (a(X)f)Y, \qquad \forall X, Y \in \Gamma(E), f \in C^\infty(M).$$

The tangent bundle is a trivial example of Lie algebroid, with the anchor map being the identity map. Another trivial example is the cotangent bundle. Here both the Lie bracket $[,]_{T^*}$ and the anchor map are zero. A less trivial example is the cotangent bundle on a symplectic manifold (M, ω). Let $\alpha = \omega^{-1}$. It is a nondegenerate Poisson bivector and defines a nondegenerate map $a : T^*M \to TM$ via $\xi \mapsto \alpha(\xi)$. There is a bracket on T^*M given by

$$[\xi, \eta]_{T^*M} \equiv \omega([\alpha(\xi), \alpha(\eta)]).$$

That this gives a Lie bracket is the consequence of the symplectic condition $d\omega = 0$. Regarding a to be the anchor map, it is easy to see that a induces a Lie algebra homomorphism. Therefore the cotangent bundle with the anchor map and Lie bracket defined above is a Lie algebroid.

On an arbitrary differential manifold M, there is a canonically associated differential complex $(\Lambda^\bullet T^*M, d)$, the de Rham complex. There is similar construction for a generic Lie algebroid $E \to M$. Consider the derived bundles constructed from exterior products of E^*. We claim there is actually a differential complex, $(\Lambda^\bullet E^*, d_E)$, for some bundle map $d_E : \Lambda^p E^* \to \Lambda^{p+1} E^*, \forall p$. For any given $\omega \in \Gamma(\Lambda^p E^*)$, the map d_E is defined via

$$\begin{aligned} d_E\omega(X_0, X_1, \ldots, X_p) &= \sum_{i=0}^{p} (-1)^i a(X_i)\omega(X_0, \ldots, \hat{X}_i, \ldots, X_p) \\ &\quad + \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_p) \end{aligned}$$

for arbitrary $X_0, X_1, \ldots, X_p \in \Gamma(E)$. One can show that $d_E^2 = 0$, so $(\Lambda^\bullet E^*, d_E)$ does form a differential complex. Its cohomology is called the Lie algebroid cohomology of E.

Let L be a Lie algebroid, and E a vector bundle over the same base manifold M. A *Lie algebroid connection* on E is a linear operator $D : \Gamma(E) \to \Gamma(L^* \otimes E)$ such that

$$D(fs) = d_L f \otimes s + f Ds$$

for any $s \in \Gamma(E)$, $f \in C^\infty(M)$. It can be extended to $D : \Lambda^\bullet L \otimes E \to \Lambda^{\bullet+1} \otimes E$ via

$$D(\alpha s) = d_L\alpha \otimes s + (-1)^{|\alpha|}\alpha \wedge Ds.$$

Just like ordinary connections on vector bundles, one can show D^2 is a section of $\Lambda^2 L^* \otimes \operatorname{End} E$. It is called the curvature of D. When the curvature vanishes, D itself becomes a differential operator of the associated complex. In this case, we say $(\Lambda^\bullet \otimes E, D)$ form a Lie algebroid module over $(\Lambda^\bullet L^*, d_L)$.

There is an alternative, and perhaps a more intuitive, way to think about Lie algebroids. For any vector bundle E we may consider a graded supermanifold ΠE, i.e., the total space of the bundle E with the fiber directions regarded as odd and having degree 1. It turns out that there is a one-to-one correspondence between Lie-algebroid structures on E and degree-1 odd vector fields Q on ΠE satisfying $\{Q, Q\} = 2Q^2 = 0$ [37]. The correspondence goes as follows. Let (x^b, ξ^μ) be local coordinates on ΠE, where x^b are local coordinates on M, and ξ^μ are linear coordinates on the fiber. Any degree-1 odd vector field on ΠE has the form

$$Q = a^b_\mu \xi^\mu \frac{\partial}{\partial x^b} + \frac{1}{2} c^\mu_{\nu\rho} \xi^\nu \xi^\rho \frac{\partial}{\partial \xi^\mu},$$

where a^b_μ and $c^\mu_{\nu\rho}$ are locally-defined functions on M. Let e_μ be the basis of sections of E dual to the coordinates ξ^μ. Define a map $a : E \to TM$ by

$$a(e_\mu) = a^b_\mu \frac{\partial}{\partial x^b},$$

and a bracket by

$$[e_\nu, e_\rho] = c^\mu_{\nu\rho} e_\mu.$$

One can show that these data define on E the structure of a Lie algebroid if and only if $Q^2 = 0$.

Bibliography

[1] M. Aldi and E. Zaslow, "Coisotropic Branes, Noncommutativity, and the Mirror Correspondence," `hep-th/0501247`.

[2] L. Alvarez-Gaume, "Supersymmetry and the Atiyah-Singer Index Theorem,". Presented at 7th Int. Conf. on Mathematical Physics, Boulder, CO, Aug. 1–10, 1983.

[3] M. Bershadsky, S. Cecotti, H. Ooguri, and C. Vafa, "Kodaira-Spencer Theory of Gravity and Exact Results for Quantum String Amplitudes," *Commun. Math. Phys.* **165** (1994) 311–428, `hep-th/9309140`.

[4] M. Bershadsky and V. Sadov, "Theory of Kahler Gravity," *Int. J. Mod. Phys.* **A11** (1996) 4689–4730, `hep-th/9410011`.

[5] R. Bott and L. Tu, *Differential Forms in Algebraic Topology*, Springer-Verlag, 1983.

[6] H. Bursztyn and M. Gualtieri, work in progress.

[7] R. Dijkgraaf, H. Verlinde, and E. Verlinde, "Topological Strings in $d < 1$," *Nucl. Phys.* **B352** (1991) 59–86.

[8] D. Friedan and P. Windey, "Supersymmetric Derivation of the Atiyah-Singer Index and the Chiral Anomaly," *Nucl. Phys.* **B235** (1984) 395.

[9] J. Gates, S. J., C. M. Hull, and M. Rocek, "Twisted Multiplets and New Supersymmetric Nonlinear Sigma Models," *Nucl. Phys.* **B248** (1984) 157.

[10] J. Gomis and Y. Li, in preparation.

[11] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, Wiley-Interscience, 1978.

[12] M. Gualtieri, "Generalized Complex Geometry," Oxford D.Phil. Thesis (2004) `math.DG/0401221`.

[13] M. Gualtieri, "Generalized Geometry and the Hodge Decomposition," `math.DG/0409093`.

[14] M. Gualtieri, in preparation.

[15] N. Hitchin, "Generalized Calabi-Yau manifolds," *Quart. J. Math. Oxford Ser.* **54** (2003) 281–308, `math.dg/0209099`.

[16] N. Hitchin, unpublished.

[17] K. Hori, in *Mirror Symmetry*, ed. K. Hori etc., Clay Mathematics Monographs, vol. 1, 2003.

[18] K. Hori, A. Iqbal, and C. Vafa, "D-branes and Mirror Symmetry," `hep-th/0005247`.

[19] J. Ishida and A. Hosoya, "Path Integral for a Color Spin and Path Ordered Phase Factor," *Prog. Theor. Phys.* **62** (1979) 544.

[20] I. T. Ivanov, B.-b. Kim, and M. Rocek, "Complex Structures, Duality and WZW Models in Extended Superspace," *Phys. Lett.* **B343** (1995) 133–143, `hep-th/9406063`.

[21] A. Kapustin, "Topological Strings on Noncommutative Manifolds," *Int. J. Geom. Meth. Mod. Phys.* **1** (2004) 49–81, `hep-th/0310057`.

[22] A. Kapustin and Y. Li, "Topological Sigma-Models with H-flux and Twisted Generalized Complex Manifolds," `hep-th/0407249`.

[23] A. Kapustin and Y. Li, "Open String BRST Cohomology for Generalized Complex Branes," `hep-th/0501071`.

[24] A. Kapustin and Y. Li, "Stability Conditions for Topological D-branes: A Worldsheet Approach," `hep-th/0311101`.

[25] A. Kapustin and D. Orlov, "Remarks on A-branes, Mirror Symmetry, and the Fukaya Category," *J. Geom. Phys.* **48** (2003) `hep-th/0109098`.

[26] S. Katz and E. Sharpe, "D-branes, Open String Vertex Operators, and Ext Groups," *Adv. Theor. Math. Phys.* **6** (2003) 979–1030, `hep-th/0208104`.

[27] K. Kodaira, L. Nirenberg and D.C. Spencer, "On the Existence of Deformations of Complex Analytic Structures," *Annals of Math.* **68** (1958), 450–459.

[28] Y. Li, "Anomalies and Graded Coisotropic Branes," `hep-th/0405280`.

[29] Y. Li, "On the Unobstructedness of Deformation Space of Twisted Generalized Calabi-Yau Manifolds," in preparation.

[30] S. Lyakhovich and M. Zabzine, "Poisson Geometry of Sigma Models with Extended Supersymmetry," *Phys. Lett.* **B548** (2002) 243–251, `hep-th/0210043`.

[31] K. Mackenzie, *Lie Groupoids and Lie Algebroids in Differential Geometry*, Cambridge University Press, 1987.

[32] J. McCleary, *A User's Guide to Spectral Sequences*, 2nd ed., Cambridge University Press, 2000.

[33] M. Rocek, "Modified Calabi-Yau Manifolds with Torsion," *Proc. of Mirror Symmetry Workshop,* MSRI, Berkeley, CA, May 1991.

[34] R. Rohm and E. Witten, "The Antisymmetric Tensor Field in Superstring Theory," *Ann. Phys.* **170** (1986) 454.

[35] A. S. Schwarz, "The Partition Function of Degenerate Quadratic Functional and Ray-Singer Invariants," *Lett. Math. Phys.* **2** (1978) 247–252.

[36] A. Tomasiello, “Topological mirror symmetry with fluxes,” `hep-th/0502148`.

[37] A. Yu. Vaintrob, ”Lie Algebroids and Homological Vector Fields,” Uspekhi Mat. Nauk **52** (1997) 161–162.

[38] E. Witten, “Topological Quantum Field Theory,” *Commun. Math. Phys.* **117** (1988) 353.

[39] E. Witten, “Topological Sigma Models,” *Commun. Math. Phys.* **118** (1988) 411.

[40] E. Witten, “Mirror Manifolds and Topological Field Theory,” `hep-th/9112056`.

[41] E. Witten, “Chern-Simons Gauge Theory as a String Theory,” *Prog. Math.* **133** (1995) 637–678, `hep-th/9207094`.

[42] E. Witten, “Phases of N = 2 Theories in Two Dimensions,” *Nucl. Phys.* **B403** (1993) 159–222, `hep-th/9301042`.

[43] M. Zabzine, “Geometry of D-branes for General $N = (2, 2)$ Sigma Models,” `hep-th/0405240.`

[44] R. Zucchini, ”Generalized Complex Geometry, Generalized Branes and the Hitchin Sigma Model,” `hep-th/0501062`.