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Gauge Theory and Supergravity Duality in the PP-Wave Background

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To My Family

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Abstract

We test the matrix theory conjecture in the pp-wave by studying two-body interactions between gravitons and membranes. We compute the one-loop effective potential of matrix theory and compare it to the light cone Lagrangian of linearized supergravity. We have exact agreement in the absence of M-momentum transfer. We also find the effective potential that smoothly interpolates between the spherical membrane result and the graviton result. We also collect here partial results from our investigation of interactions with M-momentum transfer.

Contents

Chapter 1

Introduction

The matrix theory [1] is conjectured to be a non-perturbative description of the mysterious M-theory, which is believed to underlie the various string theories [2, 3]. While there are several possible definitions of the term "M-theory", in this thesis we will use it to refer to the eleven-dimensional quantum theory that arises in the strong coupling limit of IIA string theory. The low energy limit of M-theory is conjectured to be the eleven-dimensional supergravity, so via supergravity we can indirectly study the properties of M-theory. This thesis is devoted to the comparison between matrix theory and supergravity, thereby providing tests of the matrix theory conjecture.

M-theory contains gravitons, membranes and five-branes. As evidence for the matrix theory conjecture, the authors of [1] computed graviton scattering in flat space using matrix theory and found exact agreement with eleven-dimensional supergravity. Since then more detailed investigations have been performed in flat space [15, 17, 16].

The matrix theory action in the maximally supersymmetric pp-wave background was proposed in [4]. One is naturally led to the question of whether this new matrix theory will give the same predictions as supergravity in the pp-wave background. In this thesis we provide positive evidence for the matrix theory conjecture in pp-wave by studying the interactions between gravitons and membranes using matrix theory and supergravity and confirm that the two sides agree.

The method we use to compare the two sides is similar to the one in flat space. We begin with graviton-graviton interaction [5]. On the supergravity side, we have one graviton at the origin (the source) and another graviton (the probe) moving in the vicinity. The Einstein equation gives the gravitational field generated by the source graviton, which we can then use to compute the light cone Lagrangian $\mathcal{L}_{lc}$ of the probe graviton. On the matrix theory side, we first identify the matrix field configuration (the background field) that represents the two gravitons in the eleven-dimensional picture. We then expand about this background field and integrate out the fluctuations to compute the effective action V_{eff}. Since both the light cone Lagrangian and the effective potential represent the interactions between the two gravitons, if matrix theory indeed describes M-theory we will have the equality $\mathcal{L}_{lc} = V_{eff}$. With a slight abuse in terminology, we will often refer to both $\mathcal{L}_{lc}$ and V_{eff} as the effective potentials. The methods to compare graviton-membrane and membrane-membrane interactions are similar [5].

On the supergravity side, we will use linear approximation when solving the field equations. This means all the metric components computed this way will be proportional to no higher than the first power of gravitational coupling κ_{11}^2, higher order effects such as recoiling and other back reactions can then be neglected. On the matrix theory side, the interactions begin at one loop, and for the purpose of comparing to linearized supergravity, only the one-loop effective potential is needed.

There is in fact one extra subtlety in the above comparison. A careful analysis shows that the regimes of validity of the two sides do not overlap so the two effective potentials may not match even if the matrix theory conjecture is correct. This issue is the same as in flat space, where a non-renormalization theorem [14] ensures that the results are compatible despite the subtlety. Since the pp-wave background preserves the same number of supersymmetries as in flat space, a similar non-renormalization theorem may be at work to allow for a meaningful comparison of the two sides. Since such a theorem has not yet been proven, we may take the agreement of the effective potentials as evidence for the existence of the non-renormalization theorem in pp-wave. In this thesis we will not focus on this issue further.

We will first consider the interactions when there is no M-momentum[1] transfer. The relation $\mathcal{L}_{lc} = V_{eff}$ is shown to hold for two-body interactions of gravitons and

[1]The term M-momentum refers to the momentum in the x^- direction.

membranes, hence providing evidence for the matrix theory conjecture in pp-wave. The last part of the thesis concerns membrane interactions with M-momentum transfer, however we cannot compare the results directly because we are only able to derive partial results on both sides.

This thesis is organized as follows. Chapter 2 briefly reviews the pp-wave geometry and the matrix theory action in this background. It also explains how the gravitons and the membranes are represented in matrix theory. Chapter 3 describes the effective potentials of both sides, how they are computed and the approximations involved. Chapter 4 presents the detailed computations of the two-graviton case without M-momentum transfer. The effective potentials from both sides are compared and exact agreement is found. The level of difficulty increases substantially when membrane interaction without M-momentum transfer is considered in Chapter 5. On the supergravity side, we systematically diagonalize the Einstein equations to solve for the metric and the three-form field in the near membrane limit. On the matrix theory side we expand the field fluctuations in spherical harmonics to evaluate the effective potential. Due to the complexity of the field equations of eleven-dimensional supergravity, we will only compute the effective potential of the supergravity side in the near membrane limit ($z \ll r_0$) and the graviton limit ($z \gg r_0$), where z denotes the separation of the two membranes and r_0 is the radius. On the matrix theory side, it is possible to compute the expression for general z and r_0, and by taking the appropriate limits, we are able to find perfect agreement with supergravity. In other words, the matrix theory result provides a smooth interpolation between the near membrane limit and the graviton limit. We also compare our results with the those of Shin and Yoshida [31, 32] and where there is overlap we again have perfect agreement. Chapter 6 considers membrane interaction with M-momentum transfer. We begin by constructing a three-dimensional action on a sphere that represents the two membranes. Just as in flat space, we expect M-momentum transfer to be represented by instanton solutions to the field equations. One simple instanton solution is presented. However, we are not able to write down the higher instanton solutions explicitly. To obtain a partial result without the instanton solutions, we consider a circular probe trajectory. Using supersymmetry alone we are able to show that the effective action vanishes up to order $(\frac{r}{M_W})^4$, where M_W is the radial separation of the two membranes in the x^1 to x^3 directions and r is the radius of the probe trajectory in the x^4 to x^9 directions. On the supergravity side the computation is similar to earlier chapters, but the presence of x^- dependence greatly complicates the equations. The metric is computed order by order in curvature corrections and the results are presented up to the singular terms. In order to compare with the prediction of the gauge theory side directly we need even higher curvature corrections from the supergravity solutions. This work is still in progress. We conclude this thesis by a discussion in Chapter 7. Our notations can be found in Appendix A

Chapter 2

A Brief Review of the PP-wave Geometry

In this chapter we will briefly review the maximally supersymmetric eleven-dimensional pp-wave background. The term "pp-wave" stands for "plane-fronted gravitational waves with parallel rays" and a pp-wave metric has the general form:

$$ds^2 = 2dudv + H(u, x^B)du^2 + 2K_A(u, x^B)dudx^A + (dx^A)^2 \tag{2.1}$$

The term "plane-fronted" refers to the fact that the wave fronts $u = constant$ are planar (flat) and "parallel rays" refers to the existence of a covariantly constant null vector ∂_v.

We are interested in a special case of the above pp-wave metric that preserves all 32 supersymmetries in eleven dimensions. The maximally supersymmetric eleven-dimensional pp-wave metric and the four-form field strength are given by:

$$ds^2 = 2dx^+dx^- - \mu^2\left(\frac{1}{3^2}(x^i)^2 + \frac{1}{6^2}(x^a)^2\right)(dx^+)^2 + (dx^i)^2 + (dx^a)^2 \tag{2.2}$$

$$F_{123+} = \mu \tag{2.3}$$

All the other components of the field strength are identically zero. The index convention throughout this thesis is: $\mu, \nu, \rho, \ldots$ take the values $+, -, 1, \ldots, 9$; $A, B, C, \ldots$ take the values $1, \ldots, 9$; $i, j, k, \ldots$ take the values $1, \ldots, 3$; and $a, b, c, \ldots$ take the values $4, \ldots, 9$. This supergravity solution was discovered by Kowalski-Glikman [11] and is also known as KG space. This solution has 32 supersymmetries and the Killing spinors are worked out in detail in [10]. In this thesis we will simply refer to this solution as the pp-wave. In the following sections we will discuss how this metric arises from the Penrose limit and the M-theory object that appears in this background.

2.1 The Penrose Limit

There are only four maximally supersymmetric solutions of eleven-dimensional supergravity. They are Minkowski space, $AdS_4 \times S_7$, $AdS_7 \times S_4$ and the pp-wave solution above. As it turns out, the pp-wave solution can be obtained by taking the Penrose limit of either $AdS_4 \times S_7$ or $AdS_7 \times S_4$. We will show how this is achieved in this section.

Roughly speaking, the Penrose limit is an expansion of the metric about an almost null geodesic. We begin with the metric of $AdS_{p+2} \times S_{p'+2}$:

$$\begin{aligned} ds^2 = R^2 \Big\{ & \lambda^2(-\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\, d\Omega_p^2) \\ & + (\cos^2\theta\, d\psi^2 + d\theta^2 + \sin^2\theta\, d\tilde{\Omega}_{p'}^2) \Big\} \end{aligned} \tag{2.4}$$

where $\lambda = \frac{p+1}{p'+1}$.

Next we define the new coordinates:

$$\rho = \frac{x}{R} \tag{2.5}$$

$$\theta = \frac{y}{R} \tag{2.6}$$

$$\frac{-\lambda dt + d\psi}{\sqrt{2}} = \frac{1}{\mu R^2} dx^- \tag{2.7}$$

$$\frac{\lambda dt + d\psi}{\sqrt{2}} = \mu dx^+ \tag{2.8}$$

The Penrose limit is achieved by taking $R \to \infty$. This implies $\lambda dt \approx d\psi$ and hence can be interpreted as the trajectory of a particle traveling along an almost null geodesic. Note also that a mass parameter μ is introduced so that x^μ has the dimension of length.

After this limit, we get the metric:

$$ds^2 = 2dx^+dx^- - \frac{\mu^2}{2}(\frac{1}{\lambda^2}x^2 + y^2)(dx^+)^2 + d\vec{x}^2_{p+1} + d\vec{y}^2_{p'+1} \tag{2.9}$$

For instance, for $AdS_4 \times S_7$, we have $p = 2$, $p' = 5$ and $\lambda = 1/2$. Then by rescaling $\mu \to \frac{\mu}{\sqrt{18}}$ we get the pp-wave metric in eqn(2.4).

2.2 The Matrix Theory in the PP-wave Background

The matrix theory action in pp-wave was first derived in [4]:

$$\begin{aligned}\mathcal{S} = \int dt Tr\Bigg\{ &\sum_{A=1}^{9} \frac{1}{2R}(D_0 X^A)^2 + i\psi^T D_0 \psi + \frac{(M^3R)^2}{4R} \sum_{A,B=1}^{9} [X^A, X^B]^2 \\ &+ (M^3R)\sum_{B=1}^{9} \psi^T \gamma^B [X^B, \psi] + \frac{1}{2R}\left(-(\frac{\mu}{3})^2 \sum_{i=1}^{3} (X^i)^2 - (\frac{\mu}{6})^2 \sum_{a=4}^{9} (X^a)^2 \right) - i\frac{\mu}{4}\psi^T \gamma_{123}\psi \\ &- \frac{(M^3R)\mu}{3R} i \sum_{i,j,k=1}^{3} \epsilon_{ijk} X^i X^j X^k \Bigg\}\end{aligned} \tag{2.10}$$

where $D_t X = \partial_t X^A - i[X_0, X^A]$. All the variables are $N \times N$ Hermitian matrices. M is the eleven-dimensional Planck mass and R is the radius of compactification in the x^- direction in the DLCQ formalism [8]. Putting $\mu = 0$ reduces the above action to the matrix theory action in flat space. For finite N the action describes the sector with momentum $P_- = \frac{N}{R}$.

The potential energy term in the action can be written as:

$$\frac{1}{2}\left(\frac{\mu}{3}X_i + i\frac{1}{2}\epsilon_{ijk}[X_j, X_k]\right)^2 + \frac{1}{2}\frac{\mu^2}{6^2}X_a^2 + (\text{cross terms of } X_i \text{ and } X_a) \tag{2.11}$$

The vacuum states are therefore given by:

$$X_a = 0 \qquad (2.12)$$

$$X_i = \frac{\mu}{3} J_i \qquad (2.13)$$

where J_i are N dimensional $SU(2)$ generators that satisfy $[J_i, J_j] = i\epsilon_{ijk} J_k$. The vacua are labeled by the ways of dividing J_i into irreducible representations. The vacuum solutions above can be understood as D0 branes blowing up into fuzzy sphere due to the Myers effect [12]. As $N \to \infty$ these solutions can be interpreted as spherical membranes of M-theory centered at the origin. For example, for an N dimensional irreducible solution, the radius is given by:

$$r_0 = \sqrt{\frac{1}{N} Tr X_i^2} = \frac{\mu}{6}\sqrt{N^2 - 1} \approx \frac{\mu N}{6}. \qquad (2.14)$$

where we have assumed $N \gg 1$.

The computation for a membrane with momentum $P_- = \frac{N}{R}$ on the supergravity side gives the same radius after the identification $M^3 = 2\pi T$. This interpretation of the vacuum solutions as membranes is very natural. Just like the case in flat space, the above matrix theory action can be derived by discretization of the action of the spherical membrane in the pp-wave background [13]. This identification with membranes of M-theory allows us to compute their interactions with matrix theory, which is the subject of section 5.2. The result will then be compared with a direct calculation using supergravity in section 5.1.

2.3 Interactions in the PP-wave Background

In this thesis we will study the two-body interactions of gravitons and membranes. Roughly speaking we will calculate the interactions using matrix theory and supergravity independently and see if the results match. In chapter 3 we will explain in detail how the comparison between matrix theory and supergravity is made. In this section, we will briefly describe how the two bodies interacting are described in matrix theory.

As stated in the section 2.2, each N dimensional irreducible component in the matrix X corresponds to a spherical membrane of radius $r_0 = \frac{\mu}{6}\sqrt{N^2-1}$. Therefore, to represent two concentric spherical membranes of radii $\frac{\mu}{6}\sqrt{N_1^2-1}$ and $\frac{\mu}{6}\sqrt{N_2^2-1}$ we can choose X_i to be:

$$X_i = \left(\begin{array}{c|c} J_i^{(N_1)} & 0 \\ \hline 0 & J_i^{(N_2)} \end{array}\right) \tag{2.15}$$

where N_1, N_2 indicate the dimensions of the corresponding irreducible representations. We have also assumed $X_a = 0$.

It is easily checked that such a configuration preserves 16 of the supersymmetries and has vanishing interaction amplitude. To introduce non-trivial interactions, we will add to the above configuration a small perturbation:

$$\delta X_A = \left(\begin{array}{c|c} x_A I^{(N_1)} & 0 \\ \hline 0 & 0 \end{array}\right) \tag{2.16}$$

where $A = 1, 2, \cdots, 9$ and $I^{(N_1)}$ is the N_1 dimensional identity matrix.

This perturbation can be interpreted as shifting the center of one of the spheres to the position x_A. In the following chapters we will call this membrane the probe membrane and the one at the origin the source membrane. In general this perturbation will break all the supersymmetries and leads to non-vanishing interactions. By allowing time dependence in x_A, we can study the interactions with different probe trajectories.

So far we have been talking only about membranes, so a natural question is how gravitons are represented in the matrix theory. We know that in flat space, a graviton with $P_- = N/R$ at the origin is represented by $X_A = 0^{(N)}$, where $0^{(N)}$ denotes a $N \times N$ null matrix. In the pp-wave, $0^{(N)}$ can be regarded as N one-dimensional irreducible representation of the $SU(2)$ algebra. Applying the statements made about spherical membranes earlier, $0^{(N)}$ thus represents N membranes with radii $r_0 = 0$, each carries one unit of momentum $P_- = 1/R$. This collection of N point-like particles at the origin taken together is simply a graviton with momentum $P_- = N/R$. In other words, a graviton is a collection of spherical membrane with zero radius.

For example, a configuration with a graviton with momentum $P_- = N_1/R$ and a spherical membrane with momentum $P_- = N_2/R$ both centered at the origin is given by:

$$X_i = \left(\begin{array}{c|c} 0^{(N_1)} & 0 \\ \hline 0 & J_i^{(N_2)} \end{array}\right). \tag{2.17}$$

Again we have put $X_a = 0$. We can also displace the graviton by adding the same perturbation from eqn(2.16), therefore a spherical membrane at the origin and a graviton at position x_A is represented by:

$$X_i = \left(\begin{array}{c|c} x_i I^{(N_1)} & 0 \\ \hline 0 & J_i^{(N_2)} \end{array}\right) \tag{2.18}$$

$$X_a = \left(\begin{array}{c|c} x_a I^{(N_1)} & 0 \\ \hline 0 & 0^{(N_2)} \end{array}\right) \tag{2.19}$$

Similarly, the configuration of a graviton at origin and a graviton at the position x_A is represented by:

$$X_A = \left(\begin{array}{c|c} x_A I^{(N_1)} & 0 \\ \hline 0 & 0^{(N_2)} \end{array}\right). \tag{2.20}$$

Later in this thesis we will expand the matrix theory action about the above configurations. The field fluctuations are then integrated out to give the effective action which is compared with the computation on the supergravity side.

Chapter 3

The Effective Potential of Matrix Theory and Supergravity

3.1 The Two Sides of the Duality

In this thesis we will compute the effective potential of the matrix theory for various objects up to one-loop. This effective potential, denoted as V_{eff}, will then be compared to the light cone Lagrangian density $\mathcal{L}_{lc}$ on the supergravity side. With a slight abuse in terminology, we will often refer to both as the effective potential. One can intuitively understand why these two objects should be compared with each other despite their different origins in the following way. On the matrix theory side, the effective action is defined by integrating out all the higher loops quantum effects. This action can then be used as if it is a tree level action, which automatically takes into account all the higher loops effects. Therefore, one should compare the effective potential of the matrix theory side with a tree level Lagrangian that describes the same physics. The light cone Lagrangian of supergravity is precisely such an object. It is used only at tree level, and describes the interactions of objects in eleven dimensions, which is the conjectured arena of the matrix theory. In this chapter we will describe in detail how the effective potentials are computed on both sides.

The computation of the effective potential is carried out in the DLCQ formalism. This formalism was proposed in Susskind's finite N conjecture [7], and further elucidated by [8, 9]. In this formalism x^- and $x^- + 2\pi R$ are identified. P_- is therefore quantized in units of $1/R$.

The implications of such a light-like compactification, however, are far from trivial [22]. One such complication arises from the longitudinal zero modes, which appear to cause perturbative amplitudes to diverge. In addition, there are concerns that the DLCQ of M-theory in the low energy limit is not necessarily the DLCQ of 11-dimensional supergravity because some exotic degrees of freedom such as membranes wrapped around the lightlike direction may contribute.

Here we are going to take the viewpoint in [23]. Essentially, the presence of a source exerts a pressure that decompactifies the region surrounding it, rendering x^- effectively spacelike by providing a nonzero g_{--} component in the metric. In the limit of large N, this bubble of 11-dimensional space expands, and the approximation of supergravity as a low energy description is thus justified. This view is further elucidated in [24], and we do not expect new issues to arise in the pp-wave background.

On the matrix theory side, the effective potential is computed up to one-loop. As in flat space, it corresponds to terms of order κ_{11}^2 on the supergravity side. The relation $\kappa_{11}^2 = 16\pi^5/M^9$ [16] means only terms of order $1/M^9$ are relevant on the matrix theory side for the purpose of such comparison. A natural length scale that arises on the matrix theory side is $1/(M^3R)^{1/2}$, which for convenience we will denote[1] as $(\alpha)^{1/2}$.

3.2 The Membrane Limit and the Graviton Limit

In this thesis the two types of M-theory objects we will be looking at are gravitons and membranes. We will argue in this section that certain limits of the effective potential has to be taken before a comparison between matrix theory and supergravity is possible. A naive comparison of the effective potential on both sides will not work because the effective potential V_{eff} includes matrix theory corrections to supergravity. Here we should point out the term "corrections" is a slight misnomer in the sense that the terms in the effective potential corresponding to such "corrections" could in fact

[1]This α should not be confused with the string scale α'.

be larger than the terms corresponding to supergravity in extreme short distances. Therefore, when we use the term "corrections to supergravity," we are by no means implying the "corrections" are automatically a small perturbation to the supergravity results. These matrix theory corrections, while interesting in their own rights, are not our focus here. Here we are not interested in how matrix theory modifies supergravity predictions, rather we are interested in whether matrix theory is capable of reproducing known results from supergravity. Therefore before a comparison is made, such corrections must first be removed. In other words, we must make sure our parameters are chosen such that the matrix theory corrections do not dominate.

Besides having to take care of the matrix theory corrections, we also have to make sure the equations are indeed describing the correct M-theory objects. In the pp-wave there is an interesting connection we could make between a graviton and an M2 brane. Under the influence of the 3-form background whose strength is proportional to a parameter μ, each stable M2 brane curls up into a sphere, with its radius r_0 proportional to μ and its total momentum in the x^- direction, i.e., we have $r_0 \sim \mu P_-$. If one takes the limit of $\mu \to 0$ while keeping the total momentum P_- fixed, then the radius of the sphere goes to zero and we get a point-like graviton. If on the other hand, we increase P_- simultaneously, keeping the momentum density $p_- \sim P_-/r_0^2$ fixed so that the radius goes to infinity, then the end product will be a flat membrane instead. Thus the gravitons and the flat membranes of flat space could both be regarded as different limits of spherical membranes in pp-wave. One could make another observation by considering two spherical membranes separated by a distance z. If $z \gg r_0$, then one expects the membranes to interact like two point-like gravitons. If $z \ll r_0$ then the interaction should be akin to that between flat membranes. Therefore by computing the interactions between membranes of arbitrary radii and separation, we could then take different limits to understand the interactions of both gravitons and membranes.

With this picture in mind, we will now state the two limits we are interested in:

The membrane limit:

$$\begin{aligned} \frac{z}{\alpha\mu} &\gg 1 \qquad (3.1) \\ \frac{z}{\alpha\mu} &\ll N \qquad (3.2) \end{aligned}$$

The graviton limit:

$$\begin{aligned} \frac{z}{\alpha\mu} &\gg 1 \qquad (3.3) \\ \frac{z}{\alpha\mu} &\gg N \qquad (3.4) \end{aligned}$$

where we used z to denote the separation of the two spherical membranes in the x^4 to x^9 directions. $\alpha = \frac{1}{M^3 R}$ as before.

The membrane limit is derived from the condition:

$$\frac{1}{N} \ll \frac{z}{r_0} \ll 1 \qquad (3.5)$$

The first inequality ensures that the effect of non-zero z is greater than any matrix theory corrections to supergravity, which we are not interested in. The second inequality ensures we are at the near membrane limit. Using $r_0 = \frac{\alpha\mu N}{6}$, we arrive at the limit as stated.

The graviton limit is when z is much greater than r_0, so that the two spheres interact approximately like two point-like gravitons. We still enforce the condition $\frac{1}{N} \ll \frac{z}{r_0}$ for comparison with supergravity but reverse the second inequality in the membrane limit to $\frac{z}{r_0} \gg 1$ to arrive at the graviton limit stated above. An equivalent point of view is that the point-like gravitons polarize into membranes in the presence of the 3-form potential $A_{\mu\nu\rho}$ through Myers' dielectric effect. In order to treat the polarized gravitons as (almost) point-like objects, we must require that the two gravitons be separated by a distance substantially greater than the fuzzy radius of each graviton.

Later on in section 5.3 we will compute an effective potential that smoothly interpolates between these two limits. In the next two sections we will describe how the supergravity light cone Lagrangian $\mathcal{L}_{lc}$ and the matrix theory effective potential V_{eff} are computed.

3.3 The Light Cone Lagrangian of Supergravity

In this section, we will derive the light cone Lagrangian of a graviton and also that of a membrane in an arbitrary background. This Lagrangian will be compared to the effective potential on the matrix theory side in later chapters.

We will begin with the simple case of a graviton, and then move on to the case of a membrane in the pp-wave background, and eventually we will show how to obtain the light cone Lagrangian of a membrane in an arbitrary background. The Lagrangian will be derived using two approaches. The first is the quicker method of explicit gauge fixing, the second is using the more rigorous method of constrained Hamiltonian.

3.3.1 The Light Cone Lagrangian of the Graviton

A graviton is a massless point particle. To derive the light cone Lagrangian, we will first assume it has a mass m and in the end takes the mass to zero. With this assumption, we have as our first step:

$$\mathcal{L} = -m\sqrt{-G_{\mu\nu}\dot{X}^\mu \dot{X}^\nu} \tag{3.6}$$

Next we are going to do a Legendre transform, removing any explicit $\dot{X}^-$ in favor of P_-. We first define P_-:

$$P_- = \frac{\partial \mathcal{L}}{\partial \dot{X}^-} = \frac{m}{\sqrt{-G_{\mu\nu}\dot{X}^\mu \dot{X}^\nu}} G_{-\mu}\dot{X}^\mu \tag{3.7}$$

Now we take the limit $m \to 0$ keeping P_- fixed. This implies $G_{\mu\nu}\dot{X}^\mu \dot{X}^\nu \to 0$. In the light cone gauge we fix $X^+ = \tau$. Writing $G_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$, where $g_{\mu\nu}$ denotes the background pp-wave metric and $h_{\mu\nu}$ denote the perturbation on the background, we could solve for $\dot{X}^-$ from the equation $G_{\mu\nu}\dot{X}^\mu\dot{X}^\nu = 0$:

$$\dot{X}^- = -\frac{1}{2}(\dot{X}^2 + g_{++} + h_{\mu\nu}\dot{X}^\mu\dot{X}^\nu) \tag{3.8}$$

We have written $(\dot{X}^A)^2$ as $\dot{X}^2$ for simplicity. In the last line one sees that $\dot{X}^-$ also appears on the right-hand side, but it always appears in the company of $h_{\mu\nu}$, so in first-order perturbation theory, the $\dot{X}^-$ on the right-hand side is understood as the zeroth-order approximation $\dot{X}^- \approx -\frac{1}{2}(\dot{X}^2 + g_{++})$. Effectively we treat the last equation as an iterative formula for $\dot{X}^-$.

We are now ready to write down the light cone Lagrangian for the graviton:

$$\mathcal{L}_{lc} = \mathcal{L} - P_-\dot{X}^- \tag{3.9}$$

Taking the limit $m \to 0$, we have:

$$\mathcal{L}_{lc} = -P_-\dot{X}^- = \frac{1}{2}P_-(\dot{X}^2 + g_{++} + h_{\mu\nu}\dot{X}^\mu\dot{X}^\nu) \tag{3.10}$$

Again, the $\dot{X}^-$ on the right-hand side is understood as the zeroth-order approximation $\dot{X}^- \approx -\frac{1}{2}(\dot{X}^2 + g_{++})$.

Explicitly, we have the light cone Lagrangian:

$$\begin{aligned}\mathcal{L}_{lc} = \frac{1}{2}P_-\Big\{ & \dot{X}^2 + g_{++} + \frac{1}{4}h_{--}(\dot{X}^2 + g_{++})^2 - h_{-A}(\dot{X}^2 + g_{++})\dot{X}^A \\ & -h_{-+}(\dot{X}^2 + g_{++}) + h_{AB}\dot{X}^A\dot{X}^B + 2h_{+A}\dot{X}^A + h_{++}\Big\} + \mathcal{O}[h^2]\end{aligned} \tag{3.11}$$

3.3.2 The Light Cone Lagrangian for the Membrane in the PP-wave Background

In this section we will illustrate how to find the light cone Lagrangian for a membrane in the pp-wave background. The purpose of this section is to provide a simple warm-up exercise for the next section, which involves a membrane in a general background. A second reason for this section is that some of the gauge choices are best explained in the simple case of a pure pp-wave background. Because the most general results will be presented in the next section, in the current section we will make the simplifying assumption $A_{\mu\nu\rho} = 0$. This of course is a deviation from the actual pp-wave background, but it is sufficient for our purpose of illustrating the techniques. The 3-form will be restored in the next section. We will also put the tension of the membrane T to one. T can be restored by dimensional analysis.

Without the 3-form field, the Lagrangian density for the membrane in a pp-wave background is given by:

$$\mathcal{L} = -\sqrt{-g} \tag{3.12}$$

where $g_{\alpha\beta} = G_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu$ and g denotes its determinant. Here $\alpha, \beta = 0, 1, 2$ label the world volume coordinates of the membrane, while $\mu, \nu = +, -, 1, 2, \cdots, 9$ denote the target space coordinates.

We choose the light cone gauge $X^+ = \sigma^0 = \tau$. Using $r, s = 1, 2$ to label the spatial world volume coordinates, we have:

$$g_{00} = 2\dot{X}^- + \dot{X}^2 + g_{++} \tag{3.13}$$

$$g_{0r} \equiv u_r = \partial_r X^- + \dot{X}^A \partial_r X^A \tag{3.14}$$

$$g_{rs} \equiv \bar{g}_{rs} = \partial_r X \partial_s X \tag{3.15}$$

Again we have written $(\dot{X}^A)^2$ as $\dot{X}^2$ for simplicity.

The Lagrangian density can now be rewritten as:

$$\mathcal{L} = -\sqrt{\Delta \bar{g}} \tag{3.16}$$

where

$$\Delta = -g_{00} + u_r \bar{g}^{rs} u_s \tag{3.17}$$

$$\bar{g} = \det \bar{g}_{rs} = \frac{1}{2}\{X^A, X^B\}^2 = \frac{1}{2}\left(\partial_1 X^A \partial_2 X^B - \partial_2 X^A \partial_1 X^B\right)^2 \tag{3.18}$$

The momentum density p_- is found to be:

$$p_- = \frac{\partial \mathcal{L}}{\partial \dot{X}^-} = \sqrt{\frac{\bar{g}}{\Delta}} \tag{3.19}$$

Next we fix the gauge explicitly by choosing $u_r = 0$. This gauge choice is fixed using the reparametrization freedom on the spatial world volume coordinates of σ^r. We will comment more on this gauge choice at the end of the section.

After u_r is set to zero, we could solve $\dot{X}^-$ in terms of p_- using eqn(3.19):

$$\dot{X}^- = -\frac{1}{2}\left(\frac{1}{p_-^2}\bar{g} + \dot{X}^2 + g_{++}\right) \tag{3.20}$$

Noting that $\mathcal{L} = -\sqrt{\Delta \bar{g}} = -\bar{g}\sqrt{\frac{\Delta}{\bar{g}}} = -\frac{1}{p_-}\bar{g}$, we can now compute the light cone Lagrangian density:

$$\begin{aligned}
\mathcal{L}_{lc} &= \mathcal{L} - p_- \dot{X}^- && (3.21)\\
&= -\frac{1}{p_-}\bar{g} + \frac{1}{2}p_-\left(\frac{1}{p_-^2}\bar{g} + \dot{X}^2 + g_{++}\right) && (3.22)\\
&= \frac{1}{2}p_-\left(\dot{X}^2 + g_{++} - \frac{1}{p_-^2}\bar{g}\right) && (3.23)
\end{aligned}$$

Similarly, the light cone Hamiltonian can also be computed. The result is:

$$\mathcal{H}_{lc} = \frac{1}{2p_-}(p^2 + \bar{g}) - \frac{1}{2}p_- g_{++} \tag{3.24}$$

where $p^2 = (p_A)^2$.

From the Hamiltonian, we could see that $\dot{p}_- = 0$, which was the motivation behind choosing the gauge $u_r = 0$. In the language of constrained Hamiltonians, this is equivalent to setting to zero the Lagrange multipliers c^r in the "total" Hamiltonian $\mathcal{H}_T = \mathcal{H}_{lc} + c^r \phi_r$, where ϕ_r are the primary constraints. This more rigorous approach will be discussed in section 3.3.4.

3.3.3 The Light Cone Lagrangian of the Membrane in a General Background

In this section, we will compute the light cone Lagrangian of a membrane in a background $G_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$ to the first-order in perturbation on the pp-wave background using the techniques developed in the last two sections. First we would like to specify a gauge choice. As usual we will choose $X^+ = \tau$. As for the gauge choice of u_r, we wish to choose a gauge such that $\dot{p}_- = 0$. The precise details of this choice turns out not to be a concern for the following reasons. First, from the last section, we saw that $u_r = 0$ is an appropriate gauge choice that achieved $\dot{p}_- = 0$ in the pp-wave background[2]. This means that when a perturbation $h_{\mu\nu}$ is present, u_r will be at least first-order in h. However, u_r enters only in $\Delta = -G_{00} + u_r \bar{G}^{rs} u_s$, which is second order in u_r. Therefore the second term can be ignored in first-order perturbation theory even if u_r is non-zero. Effectively, we can approximate $\Delta \approx -G_{00}$ throughout.

The perturbation from the pp-wave background on the metric and the 3-form field will be denoted as $h_{\mu\nu}$ and $a_{\mu\nu\rho}$ respectively. The 3-form field in the exact pp-wave background is taken to be $A^{(pp)}_{+ij} = \frac{\mu}{3}\epsilon_{ijk}X^k$ so that $F_{123+} = \mu$. $G_{\mu\nu}$ and $A_{\mu\nu\rho}$ denotes the general background while $g_{\mu\nu}$ is the pp-wave metric as before.
The Lagrangian density of a membrane in a general background is given by:

$$\mathcal{L} = -\sqrt{\Delta\bar{G}} + A_{\mu\nu\rho}\partial_0 X^\mu \partial_1 X^\nu \partial_2 X^\rho \tag{3.25}$$

where $\bar{G} = \det(g_{rs} + h_{rs})$ The momentum density p_- is given by:

$$p_- = \sqrt{\frac{\bar{G}}{\Delta}}(1 + h_{-\mu}\dot{X}^\mu) + a_{-\mu\nu}\partial_1 X^\mu \partial_2 X^\nu \tag{3.26}$$

Using $\Delta \approx -G_{00}$, we have:

$$-\Delta = 2\dot{X}^- + g_{++} + \dot{X}^2 + h_{\mu\nu}\dot{X}^\mu \dot{X}^\nu \tag{3.27}$$

[2]One can show easily that this statement remains true even when the 3-form field is restored.

Solving for $\dot{X}^-$ in terms of p_-, we have:

$$\begin{aligned}\dot{X}^- &= -\frac{1}{2}\Big\{\dot{X}^2 + g_{++} + \frac{1}{p_-^2}\bar{G} \\ &\quad +h_{\mu\nu}\dot{X}^\mu\dot{X}^\nu + \frac{2}{p_-^2}\bar{g}h_{-\mu}\dot{X}^\mu + \frac{2}{p_-^3}\bar{g}a_{-\mu\nu}\partial_1 X^\mu\partial_2 X^\nu\Big\} \qquad (3.28)\end{aligned}$$

Same as in eqn(3.8), whenever $\dot{X}^-$ appears on the right-hand side, it is understood as the zeroth-order approximation $\dot{X}^- \approx -\frac{1}{2}\left(\dot{X}^2 + g_{++} + \frac{1}{p_-^2}\bar{g}\right)$.

Next we use the equation of p_- to rewrite the Lagrangian density:

$$\begin{aligned}\mathcal{L} &= -\bar{G}\sqrt{\frac{\Delta}{\bar{G}}} + A_{\mu\nu\rho}\partial_0 X^\mu\partial_1 X^\nu\partial_2 X^\rho &(3.29)\\ &= -\frac{1}{p_- - a_{-\mu\nu}\partial_1 X^\mu\partial_2 X^\nu}(1+h_{-\mu}\dot{X}^\mu)\bar{G} + A_{\mu\nu\rho}\partial_0 X^\mu\partial_1 X^\nu\partial_2 X^\rho &(3.30)\\ &= -\frac{1}{p_-}\bar{G} - \frac{1}{p_-}\bar{g}h_{-\mu}\dot{X}^\mu + A_{\mu\nu\rho}\partial_0 X^\mu\partial_1 X^\nu\partial_2 X^\rho \\ &\qquad -\frac{1}{p_-^2}\bar{g}a_{-\mu\nu}\partial_1 X^\mu\partial_2 X^\nu &(3.31)\end{aligned}$$

Now we can construct the light cone Lagrangian density:

$$\begin{aligned}\mathcal{L}_{lc} &= \mathcal{L} - p_-\dot{X}^- &(3.32)\\ &= \frac{1}{2}p_-\left\{\dot{X}^2 + g_{++} - \frac{1}{p_-^2}\bar{G}\right\} + \frac{1}{2}p_- h_{\mu\nu}\dot{X}^\mu\dot{X}^\nu + A_{\mu\nu\rho}\partial_0 X^\mu\partial_1 X^\nu\partial_2 X^\rho &(3.33)\end{aligned}$$

As before, $\dot{X}^-$ on the right-hand side is understood as $\dot{X}^- \approx -\frac{1}{2}\left(\dot{X}^2 + g_{++} + \frac{1}{p_-^2}\bar{g}\right)$. Separating the Lagrangian density by $\mathcal{L}_{lc} = \mathcal{L}_{lc}^{(pp)} + \delta\mathcal{L}_{lc}$, we have:

$$\begin{aligned}\mathcal{L}_{lc}^{(pp)} &= \frac{1}{2}p_-\left\{\dot{X}^2 + g_{++} - \frac{1}{p_-^2}\bar{g}\right\} + \frac{\mu}{3}\epsilon_{ijk}\partial_1 X^i\partial_2 X^j X^k &(3.34)\\ \delta\mathcal{L}_{lc} &= \frac{1}{2}p_- h_{\mu\nu}\dot{X}^\mu\dot{X}^\nu + a_{\mu\nu\rho}\partial_0 X^\mu\partial_1 X^\nu\partial_2 X^\rho - \frac{1}{2p_-}\delta\bar{G} &(3.35)\end{aligned}$$

where $\delta\bar{G} = \delta\det\bar{G}_{rs} = \bar{G} - \bar{g}$.

Because the light cone Lagrangian is the primary object we use to compare to the matrix theory, in the next section we will use the more rigorous method of constrained

Hamiltonian to derive the light cone Lagrangian. Although the intermediate steps differ, the final result is the same as the one we found in this section.

3.3.4 The Light Cone Hamiltonian with Constraints

In this section, we will re-derive the results of the previous sections using the rigorous method of constrained Hamiltonian. Basically, in this approach the light cone Hamiltonian is defined as the momentum in the "+" light cone direction. The light cone Lagrangian is obtained by a Legendre transform on the light cone Hamiltonian. Unlike the previous approach, here the gauge choice is left open until the last stage of the derivation. In this section we will denote the momentum density[3] by Π_μ.

Once again, we begin with the Lagrangian density:

$$\mathcal{L} = -\sqrt{-\det G_{\alpha\beta}} + A_{\mu\nu\rho}\partial_0 X^\mu \partial_1 X^\nu \partial_2 X^\rho \tag{3.36}$$

The momentum density is given by:

$$\Pi_\lambda \equiv \frac{\partial\mathcal{L}}{\partial(\partial_0 X^\lambda)} = -\sqrt{-G}\; G^{0\alpha}(\partial_\alpha X^\mu)G_{\lambda\mu} + A_{\lambda\nu\rho}\partial_1 X^\nu \partial_2 X^\rho \tag{3.37}$$

Defining $\tilde{\Pi}_\lambda = \Pi_\lambda - A_{\lambda\nu\rho}\partial_1 X^\nu \partial_2 X^\rho = -\sqrt{-G}\; G^{0\alpha}(\partial_\alpha X^\mu)G_{\lambda\mu}$, then we have the primary constraints:

$$\phi_0 = G^{\mu\nu}\tilde{\Pi}_\mu\tilde{\Pi}_\nu + \bar{G} = 0 \tag{3.38}$$

$$\phi_r = \tilde{\Pi}_\mu \partial_r X^\mu = 0 \tag{3.39}$$

where $r, s = 1, 2$ and $\bar{G} = \det G_{rs}$ as before.

[3]Both Π_μ and p_μ denote momentum density. In our convention, we usually reserve p_μ to denote momentum *density* with respect to the physical distance on the membrane (by choosing σ_r to have the dimension of length), hence giving it the dimension of $(mass)^3$. In this chapter, however, the distinction between the two is not important because we have not yet specified the explicit choice for the coordinates σ_r. The two different notations are used to conform to the conventions commonly used in the two approaches.

The "total" Hamiltonian density is given by:

$$\mathcal{H}_T = \mathcal{H} + c^\alpha \phi_\alpha \tag{3.40}$$

The equations of motions are evaluated with $\mathcal{H}_T$. The original Hamiltonian $\mathcal{H}$ in the above equation is defined by $\mathcal{H} = \Pi_\mu \dot{X}^\mu - \mathcal{L}$ as usual. However, due to the reparametrization degrees of freedom, $\mathcal{H} = 0$, therefore we simply have:

$$\mathcal{H}_T = c^\alpha \phi_\alpha \tag{3.41}$$

The equations of motion are deduced from the total Hamiltonian $H_T = \int d^2\sigma \mathcal{H}_T$:

$$\dot{\Pi}_\mu = -\frac{\delta H_T}{\delta X^\mu} = -\frac{\partial \mathcal{H}_T}{\partial X^\mu} + \partial_r \frac{\partial \mathcal{H}_T}{\partial(\partial_r X^\mu)} \tag{3.42}$$

$$\dot{X}^\mu = \frac{\delta H_T}{\delta \Pi_\mu} = \frac{\partial \mathcal{H}_T}{\partial \Pi_\mu} \tag{3.43}$$

The symbol δ denotes functional derivative. In the second line, we have used the fact that the Hamiltonian is independent of $\partial_\nu \Pi_\mu$. Using the constraints, we get the following equations of motions:

$$\dot{\Pi}_\mu = \partial_r \left(c^\alpha \frac{\partial \phi_\alpha}{\partial(\partial_r X^\mu)} \right) - c^\alpha \frac{\partial \phi_\alpha}{\partial X^\mu} \tag{3.44}$$

$$\dot{X}^\mu = 2c^0 G^{\mu\nu} \tilde{\Pi}_\nu + c^r \partial_r X^\mu \tag{3.45}$$

The right-hand side of $\dot{\Pi}_\mu$ can be evaluated with the help of the following equations:

$$\frac{\partial \phi_0}{\partial(\partial_r X^\mu)} = 2G^{\lambda\xi} \frac{\partial \tilde{\Pi}_\lambda}{\partial(\partial_r X^\mu)} \tilde{\Pi}_\xi + \frac{\partial \bar{G}}{\partial(\partial_r X^\mu)} \tag{3.46}$$

with

$$\frac{\partial \tilde{\Pi}_\lambda}{\partial(\partial_r X^\mu)} = -A_{\lambda\mu\rho}(\delta_1^r \partial_2 X^\rho - \delta_2^r \partial_1 X^\rho)$$
$$\frac{\partial \bar{G}}{\partial(\partial_r X^\mu)} = G_{\mu\alpha}\left\{2\delta_1^r(\partial_1 X^\alpha)G_{22} + 2\delta_2^r(\partial_2 X^\alpha)G_{11} - 2G_{12}(\delta_1^r \partial_2 X^\alpha + \delta_2^r \partial_1 X^\alpha)\right\} \tag{3.47}$$

$$\frac{\partial \phi_0}{\partial X^\mu} = \frac{\partial G^{\lambda\xi}}{\partial X^\mu}\tilde{\Pi}_\lambda \tilde{\Pi}_\xi + 2G^{\lambda\xi}\frac{\partial \tilde{\Pi}_\lambda}{\partial X^\mu}\tilde{\Pi}_\xi + \frac{\partial \bar{G}}{\partial X^\mu} \tag{3.48}$$

with

$$\frac{\partial \tilde{\Pi}_\lambda}{\partial X^\mu} = -\frac{\partial A_{\lambda\nu\rho}}{\partial X^\mu}\partial_1 X^\nu \partial_2 X^\rho$$
$$\frac{\partial \bar{G}}{\partial X^\mu} = \frac{\partial G_{\alpha\beta}}{\partial X^\mu}(\partial_1 X^\alpha \partial_1 X^\beta G_{22} + G_{11}\partial_2 X^\alpha \partial_2 X^\beta - 2G_{12}\partial_1 X^\alpha \partial_2 X^\beta) \tag{3.49}$$

and

$$\frac{\partial \phi_s}{\partial(\partial_r X^\mu)} = \delta_s^r \Pi_\mu, \quad \frac{\partial \phi_s}{\partial X^\mu} = 0 \tag{3.50}$$

Using the light cone gauge $X^+ = \tau$, we get from eqn(3.45):

$$1 = 2c^0 G^{+\nu}\tilde{\Pi}_\nu \tag{3.51}$$

which implies $c^0 = \frac{1}{2G^{+\nu}\tilde{\Pi}_\nu}$. As for the gauge choice of c^r, we first look at the exact pp-wave geometry. In this case, eqn(3.44) gives:

$$\dot{\Pi}_- = \partial_r(c^r \Pi_-) \tag{3.52}$$

This motivates a gauge choice of $c^r = 0$ so that $\dot{\Pi}_- = 0$.

In the case of a background geometry perturbed from the pp-wave, c^r can differ from zero, but it turns out in first-order perturbation theory the details on c^r does not matter because it will cancel out in the end. We have of course encountered this very same statement in section 3.3.3, where the detailed choice of u_r turned out not to affect the final light cone Lagrangian.

Now we are ready to construct the light cone Lagrangian. First we define the light cone Hamiltonian density:

$$\mathcal{H}_{lc} = -\Pi_+ \tag{3.53}$$

Next we define the light cone Lagrangian density via the above light cone Hamiltonian density:

$$\mathcal{L}_{lc} = \Pi_A \dot{X}^A - \mathcal{H}_{lc} = \Pi_A \dot{X}^A + \Pi_+ \tag{3.54}$$

To compute the light cone Lagrangian in this approach, we first have to find Π_+. This could be done by solving for Π_+ from the constraint equation $\phi_0 = 0$. Next we need to write Π_A in terms of $\dot{X}^A$. This can be done using eqn(3.45).

To illustrates the techniques, we apply the above steps to the exact pp-wave background. In this case, the constraint ϕ_0 gives:

$$2\tilde{\Pi}_+\tilde{\Pi}_- + g^{--}\tilde{\Pi}_-^2 + \tilde{\Pi}^2 + \bar{g} = 0 \tag{3.55}$$

where $\tilde{\Pi}^2 = (\tilde{\Pi}_A)^2$. Solving this constraint for Π_+ gives:

$$\begin{aligned} \Pi_+ &= \tilde{\Pi}_+ + A_{+IJ}\partial_1 X^I \partial_2 X^J && (3.56) \\ &= -\frac{1}{2\tilde{\Pi}_-}\left\{\tilde{\Pi}^2 + g^{--}\tilde{\Pi}_-^2 + \bar{g}\right\} + A_{+IJ}\partial_1 X^I \partial_2 X^J && (3.57) \end{aligned}$$

In the pp-wave background, one sees that $\tilde{\Pi}_- = \Pi_-$ and $\tilde{\Pi}_I = \Pi_I$ because $A_{-\mu\nu} = 0$ and $\partial_r X^+ = 0$. Eqn(3.45) gives us:

$$\dot{X}^A = 2c^0\tilde{\Pi}_A = \frac{1}{\Pi_-}\Pi_A \tag{3.58}$$

which implies $\Pi_A = \Pi_- \dot{X}^A$. The light cone Lagrangian is therefore given by:

$$\begin{aligned} \mathcal{L}_{lc}^{(pp)} &= \Pi_A \dot{X}^A + \Pi_+ && (3.59)\\ &= \Pi_- \dot{X}^2 - \frac{1}{2\Pi_-}\left\{\Pi^2 + g^{--}\Pi_-^2 + \bar{g}\right\} + A_{+AB}\partial_1 X^A \partial_2 X^B && (3.60)\\ &= \frac{1}{2}\Pi_- \left\{\dot{X}^2 + g_{++} - \frac{1}{\Pi_-^2}\bar{g}\right\} + \frac{\mu}{3}\epsilon_{ijk}\partial_1 X^i \partial_2 X^j X^k && (3.61) \end{aligned}$$

where we have used $g^{--} = -g_{++}$. This light cone Lagrangian is of course identical to the one we found in eqn(3.61).

The above techniques can be applied to the perturbed pp-wave background, although the calculation is a lot more involved. Writing the light cone Lagrangian as $\mathcal{L}_{lc} = \mathcal{L}_{lc}^{(pp)} + \delta\mathcal{L}_{lc}$, we have $\mathcal{L}_{lc}^{(pp)}$ as before, while $\delta\mathcal{L}_{lc}$ is given by:

$$\begin{aligned} \delta\mathcal{L}_{lc} = \frac{-1}{2\Pi_-}\Big\{ & - h_{--}(\tilde{\Pi}_+)_{pp}^2 - 2\Pi_-(h_{+-} - g_{++}h_{--})(\tilde{\Pi}_+)_{pp} \\ & - 2T(\tilde{\Pi}_+)_{pp} a_{-\nu\rho}\partial_1 X^\nu \partial_2 X^\rho - 2T\Pi_- a_{+\nu\rho}\partial_1 X^\nu \partial_2 X^\rho - 2h_{-A}\Pi_A(\tilde{\Pi}_+)_{pp} \\ & - \Pi_-^2\left(h_{++} + (g_{++})^2 h_{--} - 2g_{++}h_{+-}\right) - 2\Pi_-(h_{+A} - g_{++}h_{-A})\Pi_A \\ & + 2T\Pi_- g_{++} a_{-\nu\rho}\partial_1 X^\nu \partial_2 X^\rho - h_{AB}\Pi_A\Pi_B - 2T\Pi_A a_{A\nu\rho}\partial_1 X^\nu \partial_2 X^\rho + T^2\delta\bar{G}\Big\} \end{aligned} \quad (3.62)$$

where $(\tilde{\Pi}_+)_{pp} = -\frac{1}{2\Pi_-}\left\{\Pi^2 - g_{++}\Pi_-^2 + \bar{g}\right\}$ and the momentum Π_A appearing in this equation can be approximated by $(\Pi_A)_{pp} = \Pi_- \dot{X}^I$ in first-order perturbation theory. The Lagrangian is found to be independent of the gauge choice c^r. After appropriate substitutions, this light cone Lagrangian is found to be identical to the result in eqn(3.35).

3.4 The Effective Potential of Matrix Theory

In this section we will describe how to construct the one-loop effective potential of matrix theory. We will use the background field method, which begins with expanding the action about a certain background to quadratic order in fluctuations. The fluctuations are then integrated out to produce the effective action. We will take a short cut at the second step by using the "sum over mass formula," which will be explained in detail below.

3.4.1 The Background Field Method

We will follow the background field method as reviewed in [25]. X is expanded into the background field B and the field fluctuation Y, i.e., $X = B + Y$. Only the part of the action that is quadratic in Y will be of interest below.

First recall again the matrix theory action in the maximally supersymmetric pp-wave background [4]:

$$\mathcal{S} = \int dtTr\Bigg\{\sum_{A=1}^{9}\frac{1}{2R}(D_0X^A)^2 + i\psi^T D_0\psi + \frac{(M^3R)^2}{4R}\sum_{A,B=1}^{9}[X^A, X^B]^2 \\ + (M^3R)\sum_{B=1}^{9}\psi^T\gamma^B[X^B,\psi] + \frac{1}{2R}\left(-(\frac{\mu}{3})^2\sum_{i=1}^{3}(X^i)^2 - (\frac{\mu}{6})^2\sum_{a=4}^{9}(X^a)^2\right) - i\frac{\mu}{4}\psi^T\gamma_{123}\psi \\ - \frac{(M^3R)\mu}{3R}i\sum_{i,j,k=1}^{3}\epsilon_{ijk}X^iX^jX^k\Bigg\} \quad (3.63)$$

where $D_0X = \partial_t X^A - i[X_0, X^A]$. Here and in the rest of this thesis, unless stated otherwise, we will always assume the indices i goes from 1 to 3, a goes from 4 to 9, and A goes from 1 to 9.

Assume X is of the order of a parameter z, then taking the ratios of any of the μ-dependent terms to the μ-independent non-derivative terms gives the quantity $(\frac{\alpha\mu}{z})^2$. In other words, the assumption $\frac{z}{\alpha\mu} \gg 1$ in section 3.2 when enforcing the membrane and graviton limits is identical to treating the new terms arising from the pp-wave background as a perturbation to flat space. Note that this is exactly the opposite of the approximation made in [13], where the μ-independent terms are treated as perturbations to the μ-dependent terms. As explained earlier, this limit ensures we are not in the extreme short distance regime where matrix theory corrections dominate. While the computation of the matrix theory one-loop effective potential is possible no matter what the value of $\frac{z}{\alpha\mu}$ is, an agreement with supergravity is expected only when $\frac{z}{\alpha\mu} \gg 1$.

In addition to the action above, there are terms arising from the ghosts and gauge fixing, which we simply state below:

$$S_{gf} = \int dtTr\left\{ -\frac{1}{2R}(\partial_t X_0 + i[B_A, X_A])^2 \right\} \tag{3.64}$$

$$S_{ghost} = \int dtTr\left\{ \bar{c}\partial_t^2 c - \partial_t \bar{c}[X_0, c] + \bar{c}[B^A, [X^A, c]] \right\} \tag{3.65}$$

The complete matrix theory action is:

$$S_M = S + S_{gf} + S_{ghost} \tag{3.66}$$

To simplify the notation, we will often put $M^3R = 1/\alpha = 1$. This factor can be restored by dimensonal analysis.

3.4.2 Expansion about the Background

To illustrate the background field method, we now look at the case of two-graviton interactions in detail. The membrane calculation can be carried out similarly.

The fields X, ψ, and c are expanded about a purely bosonic background. Here we set $N_p = N_s = 1$, i.e., we set all the matrices to dimension of 2×2. We will restore N_p and N_s in the end:

$$X_\mu = B_\mu + \sqrt{R}Y_\mu \quad ; \quad \mu = 0, 1, 2, ..., 9$$

$$B_A = \begin{pmatrix} x_A & 0 \\ 0 & 0 \end{pmatrix} \quad ; \quad Y_A = \begin{pmatrix} \zeta_A & z_A \\ \bar{z}_A & \widetilde{\zeta}_A \end{pmatrix}$$

$$B_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad ; \quad Y_0 = \begin{pmatrix} \zeta_0 & z_0 \\ \bar{z}_0 & \widetilde{\zeta}_0 \end{pmatrix}$$

$$\psi = \begin{pmatrix} \eta & \theta \\ \bar{\theta} & \widetilde{\eta} \end{pmatrix} \quad ; \quad c = \begin{pmatrix} \epsilon & c_1 \\ c_2 & \widetilde{\epsilon} \end{pmatrix}$$

The above background has the interpretation of one graviton (the source) sitting at the origin[4], while another graviton (the probe) approaches from the position given by x^A in the matrix B. We will use the shorthand $r^2 = \sum_{A=1}^{9}(x^A)^2$.

After a Wick rotation, where we define $S = iS^{(E)}$ and $\tau = it$, and at the same time rotating X_0 to $iX_0^{(E)}$, the quadratic part of the action is: [5]

$$
\begin{aligned}
\mathcal{S}^{(E)}_{boson} = \int d\tau \Bigg\{ \quad & -\frac{1}{2}\zeta_0\partial_\tau^2\zeta_0 - \frac{1}{2}\widetilde{\zeta}_0\partial_\tau^2\widetilde{\zeta}_0 + \frac{1}{2}\zeta_i(-\partial_\tau^2 + (\mu/3)^2)\zeta_i + \frac{1}{2}\zeta_a(-\partial_\tau^2 + (\mu/6)^2)\zeta_a \\
& + \frac{1}{2}\widetilde{\zeta}_i(-\partial_\tau^2 + (\mu/3)^2)\widetilde{\zeta}_i + \widetilde{\zeta}_a(-\partial_\tau^2 + (\mu/6)^2)\widetilde{\zeta}_a \\
& + \overline{z}_0(-\partial_\tau^2 + r^2)z_0 - 2i\partial_\tau x_I(\overline{z}_I z_0 - \overline{z}_0 z_I) \\
& + \overline{z}_i(-\partial_\tau^2 + r^2 + (\mu/3)^2)z_i + \overline{z}_a(-\partial_\tau^2 + r^2 + (\mu/6)^2)z_a - i\mu\epsilon_{ijk}x_i\overline{z}_j z_k \Bigg\}
\end{aligned}
\tag{3.67}
$$

$$
\mathcal{S}^{(E)}_{fermion} = \int d\tau \left\{ \eta(\partial_\tau + i\frac{\mu}{4}\gamma_{123})\eta + \widetilde{\eta}(\partial_\tau + i\frac{\mu}{4}\gamma_{123})\widetilde{\eta} + 2\overline{\theta}(\partial_\tau - x_A\gamma_A + i\frac{\mu}{4}\gamma_{123})\theta \right\}
\tag{3.68}
$$

$$
\mathcal{S}^{(E)}_{ghost} = \int d\tau \left\{ \overline{\epsilon}\partial_\tau^2\epsilon + \overline{\widetilde{\epsilon}}\partial_\tau^2\widetilde{\epsilon} + \overline{c}_1(\partial_\tau^2 - r^2)c_2 + \overline{c_2}(\partial_\tau^2 - r^2)c_1 \right\}
\tag{3.69}
$$

3.4.3 The Sum Over Mass

The partition function, $\mathcal{Z}$ of the above action can be computed as a product of functional determinants. The 1-loop effective action Γ is then simply related to $\mathcal{Z}$ via:

$$
exp(-\Gamma) = \mathcal{Z}
\tag{3.70}
$$

[4]Another possible interpretation is a transverse five brane at the origin [28].

[5]For simplicity, all subsequent superscripts of (E) on the Euclideanized fluctuation fields will be omitted.

The 1-loop effective potential is defined as:

$$\Gamma = -\int d\tau \ V_{eff} \tag{3.71}$$

The minus sign in front of the integral is slightly unconventional, but it was put there for the convenience of comparison with supergravity. It was chosen such that the tree level part of the effective potential is simply the light cone Lagrangian $(\mathcal{L}_{lc})_{pp}$ rather than $-(\mathcal{L}_{lc})_{pp}$. After V_{eff} is computed, the result could then be analytically continued back into Minkowski signature by replacing $v_E \to iv_M$.

To first approximation, however, it is not necessary to compute the functional determinants. As was suggested by Talfjord and Periwal [27] and Taylor [26], one could deduce the effective potential by simply evaluating the mass spectrum of the fluctuating fields. From the masses, the 1-loop contribution to V_{eff} could be easily deduced using the formula:

$$V_{eff}^{1-loop} = -\frac{1}{2}\left(\sum_{real\ bosons} m_b - \sum_{real\ fermions} m_f - \sum_{real\ ghosts} m_g\right) \tag{3.72}$$

The physical reasons for this is that at large distances, i.e., the limit where supergravity is valid, all the string stretching between the D0-branes can be assumed to lie in their ground state. This result can also be verified using the complete expression for V_{eff} in terms of functional determinants. We provide an argument for this in Appendix A. In what follows, we will omit the superscript "1-loop," assuming this is understood. The contribution from tree level, which does not concern us here, is simply the Lagrangian with X replaced by B. Both contributions will be put back together at the end in eqn(4.7).

One important point to note is that this method is valid only up to the lowest powers of v, as is already known in the flat space case. In flat space, the above formula reproduces every term predicted by a supergravity computation with the right coefficients, but the matrix theory corrections to supergravity, i.e., terms with even higher powers of v and $1/r$ which would not be found in supergravity, will not come out with the correct coefficients. In fact, the parameter α can be treated as the counting parameter for this purpose. All terms of order α^3, which is basically κ_{11}^2 in the supergravity language, will be found on the supergravity side, but terms on the matrix theory side with higher powers of α, which represent short distance effects, should be treated as corrections. To compute them correctly, one needs to make use of the complete expression in terms of functional determinants.

For our purpose, however, the above approach is sufficient. We are not interested in computating the correction to supergravity, rather we would like to check whether the terms already predicted by supergravity in the pp-wave background can be reproduced by a matrix theory calculation.

Chapter 4

Two-Graviton Interaction without M-momentum Transfer

4.1 A Simple Case

In the next section we will work out a more efficient method to compute V_{eff} without explicitly diagonalizing the mass matrix. Nevertheless, it is instructive to work out the simplest case in a direct approach to get the basic idea of the computation.

In this simple case, we put $x^8 = b$ and $x^9 = v\tau$, while all the other x^A are set to zero[1]. Here b is a constant, which can be interpreted as the impact parameter of the approaching probe graviton towards the source sitting at the origin. In this thesis we often use z to denote the separation in the x^4 to x^9 directions, but we use r in this chapter to avoid confusion with field fluctuations z^A. In this case, the mass matrix constructed from eqn(3.67), (3.68) and (3.69) is easily diagonalized to give the mass spectrum listed in Table 4.1. It should be noted that the velocity in the table above is measured in Euclidean time τ, i.e., $v = \frac{\partial x}{\partial \tau}$. In a comparison with supergravity, a Wick rotation back into Minkowski time $t = -i\tau$ is required, which introduces extra minus signs in V_{eff}.

With the mass spectrum at hand, V_{eff} can be evaluated using eqn(3.72):

[1]Note that by putting all x^i to zero for $i = 1, 2, 3$, we made sure that in this case the Myers term will not contribute to the mass matrix.

$\mathbf{m^2}$	**Fields**
0	ζ^0
$\mu^2/3^2$	$\zeta^i \quad ; i = 1, 2, 3$
$\mu^2/6^2$	$\zeta^a \quad ; a = 4, ..., 9$
0	$\widetilde{\zeta}^0$
$\mu^2/3^2$	$\widetilde{\zeta}^i \quad ; i = 1, 2, 3$
$\mu^2/6^2$	$\widetilde{\zeta}^a \quad ; a = 4, ..., 9$
$r^2 + \mu^2/3^2$	$\overline{z}^i, z^i \quad ; i = 1, 2, 3$
$r^2 + \mu^2/6^2$	$\overline{z}^a, z^a \quad ; a = 4, ..., 8$
$r^2 + \eta_+$	$\overline{z}^0 + \overline{z}^9, z^0 + z^9$
$r^2 + \eta_-$	$\overline{z}^0 - \overline{z}^9, z^0 - z^9$
$\mu^2/4^2$	$\eta \quad (8)$
$\mu^2/4^2$	$\widetilde{\eta} \quad (8)$
$r^2 + \mu^2/4^2 + v$	$\theta \quad (8)$
$r^2 + \mu^2/4^2 - v$	$\overline{\theta} \quad (8)$
0	$\overline{\epsilon}, \epsilon$
0	$\overline{\widetilde{\epsilon}}, \widetilde{\epsilon}$
r^2	$\overline{c}_I, c_I \quad ; I = 1, 2$

Table 4.1: The Mass Spectrum for a Simple Case. r is the separation of the gravitons. The numbers inside the round brackets indicate the number of physical degrees of freedom of the fermions with the given mass. $\eta_\pm$ is given by $\frac{1}{2}[\frac{\mu^2}{6^2} \pm \sqrt{(\frac{\mu^2}{6^2})^2 + 16v^2}]$.

$$V_{eff} = -\frac{1}{2}(2)(3\frac{\mu}{3} + 6\frac{\mu}{6} - 8\frac{\mu}{4}) - \frac{1}{2}\Bigg\{6\sqrt{r^2 + \mu^2/3^2} + 10\sqrt{r^2 + \mu^2/6^2} + 2\sqrt{r^2 + \eta_+}$$
$$+ 2\sqrt{r^2 + \eta_-} - 8\sqrt{r^2 + \mu^2/4^2 + v} - 8\sqrt{r^2 + \mu^2/4^2 - v} - 4r\Bigg\} \quad (4.1)$$

At this point it is useful to restore the factors of M^3R, which we denote as $1/\alpha$. For instance, the first square root term in the about equation becomes:

$$\sqrt{\frac{r^2}{\alpha^2} + \frac{\mu^2}{3^2}} \quad (4.2)$$

This can in turn be written as:

$$\frac{r}{\alpha}\sqrt{1 + \frac{1}{3^2}(\frac{\alpha}{r^2})(\alpha\mu^2)}$$

The expression for V_{eff} given above, being a matrix theory result, is only expected to match with supergravity in the large r limit (if it does at all!). Defining the large r limit by eqn(3.3), we can then expand the 1-loop effective potential in powers of $\alpha^2\mu^2/r^2$. Thus, expanding V_{eff} gives:

$$V_{eff} = \alpha^3(\frac{15}{16}\frac{v^4}{r^7} + \frac{7}{96}\frac{\mu^2 v^2}{r^5} + \frac{1}{768}\frac{\mu^4}{r^3}) + O[\alpha^5] \quad (4.3)$$

Wick rotating v, and restoring N_p, N_s gives:

$$V_{eff} = \frac{N_p N_s}{M^9 R^3}(\frac{15}{16}\frac{v^4}{r^7} - \frac{7}{96}\frac{\mu^2 v^2}{r^5} + \frac{1}{768}\frac{\mu^4}{r^3}) + O[\alpha^5] \quad (4.4)$$

The α^3 terms give the factor $1/M^9$, which translates into κ_{11}^2 in the supergravity language. This is the order we are interested in. We throw away the higher powers of α (which are always accompanied by powers of $1/r$) because they correspond to short distance corrections to supergravity, just as in flat space.

Here the first term is just the flat space result. The second and the third term are the interesting ones, with new $\mu^2 v^2$ and μ^4 dependence created by the pp-wave background. A comparison of their coefficients with supergravity will show exact agreement.

4.2 Mass Matrix Computation

In the more general cases, when the velocity and the impact parameter point in arbitrary directions, calculating the effective potential V_{eff} by finding the entire m^2 spectrum, then taking their square roots and expanding them in powers of μ and v becomes inefficient, since in the most general case this involves finding the eigenvalues of mass matrices of very high dimension.

Instead, it is possible to make use of the sum over mass formula in eqn(3.72) without explicitly diagonalizing the mass matrix. Let us denote the square of the mass matrix as $W = M^2$. Since there is never any mixing between the bosons, the fermions and the ghosts, we can study their mass matrices separately.

In terms of W, the sum over mass formula becomes:

$$V_{eff}^{1-loop} = -\frac{1}{2}tr(\sqrt{W_b} - \sqrt{W_f} - \sqrt{W_g}) \tag{4.5}$$

The square root of W can be defined unambigously by its expansion in powers of α/r^2 in the supergravity limit, as was discussed in Section 4.1. Note that M_b is defined to be the mass matrix for *real* bosons. If it is taken to be the mass matrix for the complex bosons, then there will be an extra factor of two in front of $\sqrt{W_b}$.

4.2.1 Simple Recipe for Mass Matrix

In this subsection we will give a simple recipe for writing out M^2 for both the bosons and the fermions. The mass for the ghosts is exactly the same as in the simple case of Section 4.1.

First of all, we should note that the mass of ζ^i and ζ^a are always $\mu/3$ and $\mu/6$ respectively for $i = 1, 2, 3$ and $a = 4, ..., 9$. The mass of all eight physical degrees in η is always $\mu/4$. These are independent of the background B. Mixing occurs only among the z^A and among the θ and $\overline{\theta}$. Hence in what follows, we will denote the component arising from say $\overline{z}^A z^B$ in the bosonic Lagrangian simply as $(M^2)_{AB}$ without mentioning z explicitly. Note also that M^2 is symmetric.

4.2.1.1 Rules for Bosons

1. $(M^2)_{00} = r^2$; $\quad (M^2)_{ii} = r^2 + \mu^2/3^2$; $\quad (M^2)_{aa} = r^2 + \mu^2/6^2$;
2. $\dot{x}^A = v_A$ mixes z^0 and $z^A \Rightarrow \ (M^2)_{0A} = -2v_A$
3. $x^1 = b_1$ mixes z^2 and z^3... etc $\Rightarrow \ (M^2)_{jk} = i\mu\epsilon_{ijk}b_i$

Note that Rule 3 applies only to z^i but not z^a. Such mixing is the effect of the Myers term in the matrix theory action.

4.2.1.2 Rules for Fermions

The mass matrix for the fermions can be written in a closed form:

$$M^2 = r^2 + \mu^2/4^2 + \sum_{A=1}^{9} v_A\gamma_A + \sum_{i=1}^{3} \frac{i\mu x^i}{4}\{\gamma_i, \gamma_{123}\} \tag{4.6}$$

4.3 The General Case

Once the mass matrix squared $W = M^2$ is known, eqn(4.5) can then be used to compute the 1-loop effective potential explicitly. In accordance with our earlier discussions, only terms up to order $\alpha^3 \sim 16\pi^5/M^9 = \kappa_{11}^2$ are kept. After restoring all factors of M^3R, N_p, and N_s, the 0 and 1-loop effective potential is given by:

$$V_{eff}^{0,1-loop} = \frac{N_p}{2R}(\sum_{A=1}^{9} v_A^2 + g_{++}) + \frac{N_p N_s}{M^9 R^3}\Bigg\{ \frac{15(\sum_{A=1}^{9} v_A^2)^2}{16r^7} - \frac{\mu^2 \sum_{i=1}^{3} v_i^2}{96r^5} - \frac{7\mu^2 \sum_{a=4}^{9} v_a^2}{96r^5}$$
$$+ \frac{15\mu^2}{32r^7}\left[\sum_{i=1}^{3} x_i^2 \left(-\sum_{i=1}^{3} v_i^2 + \sum_{a=4}^{9} v_a^2\right) + 2(\sum_{i=1}^{3} x_i v_i)^2\right]\Bigg\}$$
$$+ \frac{\mu^4 N_p N_s}{R^3 M^9}\frac{1}{768r^7}\left\{32\left[\sum_{i=1}^{3}(x^i)^2\right]^2 + \left[\sum_{a=4}^{9}(x^a)^2\right]^2 - 12\sum_{i=1}^{3}(x^i)^2 \cdot \sum_{a=4}^{9}(x^a)^2\right\} \quad (4.7)$$

This is the equation to be compared with the supergravity result. Notice the effective potential has manifest $SO(3) \times SO(6)$ symmetry, as should be expected from the symmetry of the original matrix theory action. Just as in flat space [18], one should be able to recast this 1-loop effective potential in the form $T^{\mu\nu}G_{\mu\nu}$. A comparison with the supergravity side will indeed confirm this, as this is precisely the form of the effective potential on the supergravity side as derived in Appendix B.

Having computed the effective potential on the matrix theory side, the next step will be to compare it with the result from a supergravity calculation. Before this could be done, the issue of gauge choice has to be addressed.

It is necessary to make a gauge choice when solving the Einstein equations. A gauge choice corresponds to a choice of the coordinate system one uses to describe the physics. On the matrix theory side, such a choice of coordinates was made right from the very beginning: The action in eqn(2.10) was written in coordinates that made the $SO(3) \times SO(6)$ symmetry manifest. Before a comparison is possible, a corresponding choice of coordinates (i.e., a choice of gauge) has to be made on the supergravity side.

A comparison of the above equation with the general expression for V_{eff} in eqn(4.44) will in the end determine the correct gauge choice for the supergravity computation. There will be a further discussion about gauge choice in the supergravity section.

4.4 The Supergravity Light Cone Lagrangian

To find the two-body effective action, one only needs to solve for the metric perturbation caused by the source graviton at the linear order ($\sim \kappa_{11}^2$).

The action is given by:

$$S = S_G + S_A + S_P \tag{4.8}$$

S_G is the Einstein action for the metric:

$$S_G = \frac{1}{\kappa_{11}^2} \int d^{11}x \sqrt{|g|} R \tag{4.9}$$

S_A is the action for the three-form:

$$S_A = -\frac{2}{\kappa_{11}^2} \int d^{11}x \left\{ \frac{\sqrt{|g|}}{2 \cdot 2 \cdot 4!} F^{\mu\nu\lambda\xi} F_{\mu\nu\lambda\xi} + \frac{1}{12} \frac{1}{3!(4!)^2} \epsilon^{\mu_1 \ldots \mu_{11}} A_{\mu_1\mu_2\mu_3} F_{\mu_4 \ldots \mu_7} F_{\mu_8 \ldots \mu_{11}} \right\} \tag{4.10}$$

S_P is the action for the source graviton (the subscript P means "particle"):

$$S_P = C_P \frac{1}{2} \int_{-\infty}^{+\infty} d\xi \left(\frac{1}{\beta(\xi)} g_{\mu\nu}(y) \frac{dy^\mu}{d\xi} \frac{dy^\nu}{d\xi} - \beta(\xi) m^2 \right) \tag{4.11}$$

with C_P being some constant.

The above action gives the equations of motion for the metric, the 3-form field, and the source graviton, all listed below.

The Einstein equation:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa_{11}^2 \left([T_{\mu\nu}]_A + [T_{\mu\nu}]_P \right) \tag{4.12}$$

The Maxwell equation:

$$\partial_\mu \left(\sqrt{|g|} F^{\mu\nu\lambda\xi} \right) - \frac{1}{1152} \epsilon^{\nu\lambda\xi\rho_1 \ldots \rho_8} F_{\rho_1 \ldots \rho_4} F_{\rho_5 \ldots \rho_8} = 0 \tag{4.13}$$

The geodesic equation (with the gauge choice $\beta = constant$):

$$\frac{d^2 y^\mu}{d\xi^2} + \Gamma^\mu_{\rho\nu}(y)\frac{dy^\rho}{d\xi}\frac{dy^\nu}{d\xi} = 0 \tag{4.14}$$

$[T_{\mu\nu}]_A$ and $[T_{\mu\nu}]_P$ are the stress tensors obtained by varying S_A and S_P w.r.t. the metric, given below:

$$[T_{\mu\nu}]_A = \frac{1}{12\kappa_{11}^2}\left(F_{\mu\lambda\xi\rho}F_\nu^{\lambda\xi\rho} - \frac{1}{8}g_{\mu\nu}F^{\rho\sigma\lambda\xi}F_{\rho\sigma\lambda\xi}\right) \tag{4.15}$$

$$[T_{\mu\nu}]_P(x) = C_P\frac{1}{\sqrt{|g(x)|}}g_{\mu\rho}(x)g_{\nu\lambda}(x)\int_{-\infty}^{+\infty}d\xi\frac{1}{\beta}\frac{dy^\rho(\xi)}{d\xi}\frac{dy^\lambda(\xi)}{d\xi}\delta^{(11)}(x - y(\xi)) \tag{4.16}$$

Setting C_P to zero means the absence of the source graviton. In this case, a solution to the above equations of motion is the pp-wave background. The metric $g_{\mu\nu}$ and the 4-form field strength are given by:

$$g_{+-} = 1,\ g_{++} = -\mu^2\left[\frac{1}{9}\sum_{i=1}^{3}(x^i)^2 + \frac{1}{36}\sum_{a=4}^{9}(x^a)^2\right],\ g_{AB} = \delta_{AB} \tag{4.17}$$

$$F_{123+} = \mu \tag{4.18}$$

In our convention, $\mu, \nu, \rho, \ldots$ take the values $+, -, 1, \ldots, 9$; $A, B, C, \ldots$ take the values $1, \ldots, 9$; $i, j, k, \ldots$ take the values $1, \ldots, 3$; and $a, b, c, \ldots$ take the values $4, \ldots, 9$

The introduction of a source graviton, i.e., a non-zero C_P, perturbs the above pp-wave solution:

$$g_{\mu\nu} \longrightarrow g_{\mu\nu} + h_{\mu\nu} \equiv G_{\mu\nu};\ F_{\mu\nu\rho\sigma} \longrightarrow F_{\mu\nu\rho\sigma} + f_{\mu\nu\rho\sigma}$$

It suffices to solve the geodesic equation at the zeroth-order of C_P, which gives a solution

$$x^+ = \xi,\ x^- = 0,\ x^A = 0$$

and the corresponding stress tensor of the source graviton is then:

$$[T_{\mu\nu}]_P(x) = P_- g_{\mu+} g_{\nu+} \delta(x^-) \prod_{A=1}^{9} \delta(x^A) \tag{4.19}$$

where $P_- \equiv \frac{C_P}{\beta}$ is a constant and in what follows we will use P_- instead of C_P. Note that the order of κ_{11}^2 is the same as the order of P_-. Also note that the only non-vanishing component of $[T_{\mu\nu}]_P$ is $[T_{--}]_P = P_- \delta(x^-) \prod_{A=1}^{9} \delta(x^A)$.

In what follows we will integrate everything over the x^- direction, thus getting rid of $\delta(x^-)$ and derivatives w.r.t x^-. On the matrix theory side, the effective potential was only computed up to 1-loop. In supergravity language, that means we are only looking at order κ_{11}^2. To find the effective potential on the supergravity side up to this order, we need only the linearized (i.e., to the linear order of P_-) Einstein equation and Maxwell equation.

We consider static solutions which has no x^+ dependence. Also we restrict our attention to metric and gauge field perturbations that go to zero at infinity. The linearized Einstein equation in 11 dimension is:

$$\delta R_{\mu\nu} = \kappa_{11}^2 \left[\delta T_{\mu\nu} + \frac{1}{9} g_{\mu\nu} \left(T^{\alpha\beta} h_{\alpha\beta} - g^{\alpha\beta} \delta T_{\alpha\beta} \right) \right] \equiv \mathcal{T}_{\mu\nu} \tag{4.20}$$

where the perturbation to the total stress tensor is given by

$$\delta T_{\alpha\beta} = [\delta T_{\alpha\beta}]_A + [T_{\alpha\beta}]_P \tag{4.21}$$

$[\delta T_{\alpha\beta}]_A$ is the perturbation to the stress tensor of the gauge field, which is to be expressed in terms of the perturbation to the field strength.

First look at the $(--)$ component of the Einstein equation:

$$\delta R_{--} = -\frac{1}{2} \sum_{A=1}^{9} \frac{\partial^2 h_{--}}{\partial x^A \partial x^A} \tag{4.22}$$

and

$$\mathcal{T}_{--} = \kappa_{11}^2 \delta T_{--} = \kappa_{11}^2 \left[T_{--}\right]_P = \kappa_{11}^2 P_- \prod_{A=1}^{9} \delta(x^A) \tag{4.23}$$

where $\left[\delta T_{--}\right]_A = 0$ (as can be readily verified) has been used.

This gives

$$h_{--} = \frac{\kappa_{11}^2 P_-}{\pi^4} \frac{15}{16} \frac{1}{|\vec{x}|^7} \tag{4.24}$$

where we use $\vec{x}$ to denote the 9-dimensional vector in the transverse directions.

The $(-A)$ component of the Einstein equation is,

$$\delta R_{-A} = -\frac{1}{2} \sum_{B=1}^{9} \frac{\partial^2 h_{-A}}{\partial x^B \partial x^B} + \frac{1}{2} \sum_{B=1}^{9} \frac{\partial^2 h_{-B}}{\partial x^A \partial x^B} \tag{4.25}$$

and

$$\mathcal{T}_{-A} = 0 \tag{4.26}$$

which gives

$$h_{-A} = 0 \tag{4.27}$$

Now we look at the linearized Maxwell equation, in terms of the gauge potential perturbation $a_{\mu\nu\rho}$ (note $f_{\lambda\mu\nu\rho} = \partial_\lambda a_{\mu\nu\rho} - \partial_\mu a_{\nu\rho\lambda} + \partial_\nu a_{\rho\lambda\mu} - \partial_\rho a_{\lambda\mu\nu}$). We choose to work in the "Lorentz gauge" where $\sum_{D=1}^{9} \partial_D a_{\mu\nu D} = 0$. The upper $(AB+)$ component of the Maxwell equation gives:

$$\sum_{D=1}^{9} \partial_D^2 a_{AB-} - \sum_{D=1}^{9} \partial_D \left[h_{--} F_{DAB+}\right] = 0 \tag{4.28}$$

Using the expression for h_{--} that we just found, we have:

$$a_{ij-} = \frac{\mu\kappa_{11}^2 P_-}{\pi^4}\ \frac{15}{32}\sum_{k=1}^{3}\epsilon_{ijk}\frac{x^k}{|\vec{x}|^7} \tag{4.29}$$

while all other a_{AB-}'s vanish. This gives the field strength:

$$\begin{aligned} f_{-ijk} &= \frac{\mu\kappa_{11}^2 P_-}{\pi^4}\ \frac{15}{32}\epsilon_{ijk}\left[7\frac{\sum_{i=1}^{3}(x^i)^2}{|\vec{x}|^9} - 3\frac{1}{|\vec{x}|^7}\right] \\ f_{-ijb} &= \frac{\mu\kappa_{11}^2 P_-}{\pi^4}\ \frac{15}{32}\sum_{k=1}^{3}\epsilon_{ijk}\left[7\frac{x^k x^b}{|\vec{x}|^9}\right] \end{aligned} \tag{4.30}$$

Next consider the upper (ABC) component of the Maxwell equation. Using the fact that $h_{-A} = 0$ and $a_{AB-} = 0$ except for a_{ij-}, we have:

$$\sum_{D=1}^{9}\partial_D^2 a_{ABC} = 0 \tag{4.31}$$

hence, all $a_{ABC} = 0$. Now the $(A+-)$ component. Using $h_{-A} = 0$ we get

$$\sum_{D=1}^{9}\partial_D^2 a_{A-+} = 0 \tag{4.32}$$

thus $a_{A-+} = 0$. Now we go back to look at the $(+A)$ component of the Einstein equation. Using $h_{-A} = 0$, we get

$$\delta R_{+A} = -\frac{1}{2}\sum_{B=1}^{9}\frac{\partial^2 h_{+A}}{\partial x^B \partial x^B} + \frac{1}{2}\sum_{B=1}^{9}\frac{\partial^2 h_{+B}}{\partial x^A \partial x^B} \tag{4.33}$$

Using $a_{A-+} = 0$, $a_{ABC} = 0$, and $h_{-A} = 0$, we get

$$\mathcal{T}_{+A} = 0 \tag{4.34}$$

So we conclude that

$$h_{+A} = 0 \tag{4.35}$$

Now consider the $(+-)$ component of the Einstein equation

$$\delta R_{+-} = -\frac{1}{2}\sum_{A=1}^{9}\frac{\partial^2 h_{+-}}{\partial x^A \partial x^A} + \frac{1}{2}\sum_{A=1}^{9}\frac{\partial g_{++}}{\partial x^A}\frac{\partial h_{--}}{\partial x^A} \tag{4.36}$$

and

$$\mathcal{T}_{+-} = \frac{1}{6}\left(\mu^2 h_{--} - \mu f_{-123}\right) \tag{4.37}$$

In writing $\mathcal{T}_{+-}$, we made use of the following equations:

$$\begin{aligned}
[\delta T_{+-}]_A &= \frac{\mu^2}{4\kappa_{11}^2} h_{--} \\
[\delta T_{ij}]_A &= \frac{1}{4\kappa_{11}^2}\delta_{ij}\left(-2\mu f_{-123} - \mu^2 h_{--}\right) \\
[\delta T_{bc}]_A &= \frac{1}{4\kappa_{11}^2}\delta_{bc}\left(2\mu f_{-123} + \mu^2 h_{--}\right) \\
[\delta T_{ib}]_A &= -\frac{\mu}{4\kappa_{11}^2}\sum_{j,k=1}^{3}\epsilon_{ijk} f_{-jkb}
\end{aligned} \tag{4.38}$$

Solving this Einstein equation we have:

$$h_{+-} = -\frac{\mu^2\kappa_{11}^2 P_-}{\pi^4}\left[\frac{5}{64}\frac{\sum_{i=1}^{3}(x^i)^2}{|\vec{x}|^7} + \frac{1}{192}\frac{1}{|\vec{x}|^5}\right] \tag{4.39}$$

The (AB) component of the Einstein equation reads:

$$\begin{aligned}
\delta R_{AB} = &-\frac{1}{2}\left[\sum_{C=1}^{9}\frac{\partial^2 h_{AB}}{\partial x^C \partial x^C} - \sum_{C=1}^{9}\frac{\partial^2 h_{AC}}{\partial x^B \partial x^C} - \sum_{C=1}^{9}\frac{\partial^2 h_{BC}}{\partial x^A \partial x^C} + \sum_{C=1}^{9}\frac{\partial^2 h_{CC}}{\partial x^A \partial x^B} + 2\frac{\partial^2 h_{+-}}{\partial x^A \partial x^B}\right] \\
&+ \frac{1}{4}\left[2h_{--}\frac{\partial^2 g_{++}}{\partial x^A \partial x^B} + 2g_{++}\frac{\partial^2 h_{--}}{\partial x^A \partial x^B} + \frac{\partial g_{++}}{\partial x^A}\frac{\partial h_{--}}{\partial x^B} + \frac{\partial g_{++}}{\partial x^B}\frac{\partial h_{--}}{\partial x^A}\right]
\end{aligned} \tag{4.40}$$

and

$$
\begin{aligned}
\mathcal{T}_{ij} &= -\frac{1}{3}\delta_{ij}\left(2\mu f_{-123}+\mu^2 h_{--}\right)\\
\mathcal{T}_{bc} &= \frac{1}{6}\delta_{bc}\left(2\mu f_{-123}+\mu^2 h_{--}\right)\\
\mathcal{T}_{ib} &= -\frac{\mu}{4}\sum_{j,k=1}^{3}\epsilon_{ijk} f_{-jkb}
\end{aligned}
\tag{4.41}
$$

So far the need to make a gauge choice for the metric has not arisen. Now to solve for h_{AB} we must make a gauge choice for the metric. Let $G^{\rho\sigma}$ and $\Gamma^{\mu}_{\rho\sigma}$ denote the complete inverse metric and Christoffel symbol respectively (by "complete," we mean they include both the unperturbed and perturbed part). We shall fix the gauge by specifying $G^{\rho\sigma}\Gamma^{\mu}_{\rho\sigma}$.

As can be easily verified,

$$
\begin{aligned}
G^{\rho\sigma}\Gamma^{+}_{\rho\sigma} &= \sum_{C=1}^{9}\partial_C h_{-C}=0\\
G^{\rho\sigma}\Gamma^{-}_{\rho\sigma} &= \sum_{C=1}^{9}\left(-h_{-C}\partial_C g_{++}+\partial_C h_{+C}-g_{++}\partial_C h_{-C}\right)=0\\
G^{\rho\sigma}\Gamma^{A}_{\rho\sigma} &= \sum_{C=1}^{9}\partial_C h_{AC}-\frac{1}{2}\partial_A\left(\sum_{C=1}^{9}h_{CC}+2h_{+-}-g_{++}h_{--}\right)
\end{aligned}
\tag{4.42}
$$

so we need to specify $G^{\rho\sigma}\Gamma^{A}_{\rho\sigma}$ to fix the gauge.

Using the above expressions for $G^{\rho\sigma}\Gamma^{A}_{\rho\sigma}$, we can rewrite δR_{AB} as

$$
\begin{aligned}
\delta R_{AB} = -\frac{1}{2}\Bigg\{ &\sum_{C=1}^{9}\frac{\partial^2 h_{AB}}{\partial x^C\partial x^C}-\frac{\partial\left(G^{\rho\sigma}\Gamma^{A}_{\rho\sigma}\right)}{\partial x^B}-\frac{\partial\left(G^{\rho\sigma}\Gamma^{B}_{\rho\sigma}\right)}{\partial x^A}\\
&+\frac{1}{2}\left(\frac{\partial g_{++}}{\partial x^A}\frac{\partial h_{--}}{\partial x^B}+\frac{\partial g_{++}}{\partial x^B}\frac{\partial h_{--}}{\partial x^A}\right)\Bigg\}
\end{aligned}
\tag{4.43}
$$

In general relativity we often use the "harmonic gauge" where we set $G^{\rho\sigma}\Gamma^{A}_{\rho\sigma}=0$ (which is satisfied by the unperturbed pp-wave background). Here, however, we shall opt for a different gauge.

As derived in the section 3.3.1, the effective potential is given by:

$$\mathcal{L}_{lc} = \frac{1}{2}P_-\Big\{\dot{X}^2 + g_{++} + \frac{1}{4}h_{--}(\dot{X}^2 + g_{++})^2 - h_{-I}(\dot{X}^2 + g_{++})\dot{X}^I \\ -h_{-+}(\dot{X}^2 + g_{++}) + h_{IJ}\dot{X}^I\dot{X}^J + 2h_{+I}\dot{X}^I + h_{++}\Big\} \tag{4.44}$$

where $P_- = N_p/R$ and $\dot{X}^2 = (\dot{X}^I)^2 = v^2$. As h_{+A}, h_{-A} all vanish, they simply drop out of the effective potential.

The computation on matrix theory side in section 4.3 tells us that in the effective potential there are no terms of the form $v^a v^b$ for $a \neq b$, nor are there terms of the form $v^i v^a$. This suggests we choose the gauge such that $h_{ab} \propto \delta_{ab}$, and $h_{ia} = 0$. To make $h_{ab} \propto \delta_{ab}$, we set:

$$G^{\rho\sigma}\Gamma^a_{\rho\sigma} = \frac{1}{2}h_{--}\partial_a g_{++} \tag{4.45}$$

then, to make $h_{ia} = 0$, we set:

$$\partial_b\left(G^{\rho\sigma}\Gamma^i_{\rho\sigma}\right) = \frac{1}{2}\partial_i g_{++}\partial_b h_{--} - \frac{\mu}{2}\epsilon_{ijk}f_{-jkb}$$

which implies

$$G^{\rho\sigma}\Gamma^i_{\rho\sigma} = \frac{35}{96}\frac{\mu^2\kappa_{11}^2 P_-}{\pi^4}\frac{x^i}{|\vec{x}|^7} \tag{4.46}$$

In this gauge, the Einstein equation gives:

$$h_{ab} = \delta_{ab}\frac{\mu^2\kappa_{11}^2 P_-}{\pi^4}\frac{1}{96}\left[\frac{15}{2}\frac{\sum_{k=1}^3 (x^k)^2}{|\vec{x}|^7} - \frac{1}{|\vec{x}|^5}\right] \tag{4.47}$$

$$h_{ij} = \delta_{ij}\frac{\mu^2\kappa_{11}^2 P_-}{\pi^4}\frac{1}{96}\left[-15\frac{\sum_{k=1}^3 (x^k)^2}{|\vec{x}|^7} + \frac{1}{2}\frac{1}{|\vec{x}|^5}\right] + \frac{\mu^2\kappa_{11}^2 P_-}{\pi^4}\frac{15}{64}\frac{x^i x^j}{|\vec{x}|^7} \tag{4.48}$$

Now let us look at the upper $(AB-)$ component of the Maxwell equation. It gives the following equations:

$$\sum_{D=1}^{9} \partial_D^2 a_{ij+} - g_{++} \sum_{D=1}^{9} \partial_D^2 a_{ij-} - \sum_{D=1}^{9} \partial_D g_{++} \left(\partial_D a_{ij-} + \partial_i a_{jD-} + \partial_j a_{Di-}\right)$$
$$+ \mu \sum_{k=1}^{3} \epsilon_{ijk} \Bigg\{ - \sum_{D=1}^{9} \partial_D h_{Dk} + \sum_{m=1}^{3} \left(\partial_m h_{mk} - \partial_k h_{mm}\right)$$
$$+ \partial_k \left[\frac{1}{2} \left(g_{++} h_{--} + \sum_{D=1}^{9} h_{DD} \right) \right] \Bigg\} = 0 \quad (4.49)$$

$$\sum_{D=1}^{9} \partial_D^2 a_{bc+} = 0 \quad (4.50)$$

$$\sum_{D=1}^{9} \partial_D^2 a_{ib+} = 0 \quad (4.51)$$

Solving them gives:

$$a_{ij+} = \frac{\mu^3 \kappa_{11}^2 P_-}{\pi^4} \left(\sum_{k=1}^{3} \epsilon_{ijk} x^k \right) \frac{1}{384\, |\vec{x}|^7} \left[-29 \sum_{m=1}^{3} (x^m)^2 + \sum_{a=4}^{9} (x^a)^2 \right] \quad (4.52)$$

$$a_{bc+} = 0 \quad (4.53)$$

$$a_{ib+} = 0 \quad (4.54)$$

They give the field strength:

$$\begin{aligned} f_{+ijk} &= \frac{\mu^3 \kappa_{11}^2 P_-}{\pi^4} \epsilon_{ijk} \frac{1}{384\, |\vec{x}|^9} \left[-58 \sum_{m=1}^{3} (x^m)^2 - 3 \sum_{a=4}^{9} (x^a)^2 + 149 \sum_{m=1}^{3} (x^m)^2 \cdot \sum_{a=4}^{9} (x^a)^2 \right] \\ f_{+ijb} &= \frac{\mu^3 \kappa_{11}^2 P_-}{\pi^4} \left(\sum_{k=1}^{3} \epsilon_{ijk} x^k \right) \frac{5}{384} \frac{x^b}{|\vec{x}|^9} \left[-41 \sum_{m=1}^{3} (x^m)^2 + \sum_{a=4}^{9} (x^a)^2 \right] \end{aligned} \quad (4.55)$$

As can be easily checked, all the $a_{\mu\nu\rho}$ we have found indeed satisfy the Lorentz gauge.

Finally, we consider the (++) component of the Einstein equation:

$$\begin{aligned}\delta R_{++} \;=\;& -\frac{1}{2}\sum_{A=1}^{9}\partial_A^2 h_{++} + \frac{1}{2}\sum_{A,B=1}^{9}\partial_A g_{++}\partial_B h_{AB} - \frac{1}{4}\sum_{A,B=1}^{9}\partial_A g_{++}\partial_A h_{BB} \\ & +\frac{1}{2}\sum_{A,B=1}^{9} h_{AB}\partial_A\partial_B g_{++} + \frac{1}{2}\sum_{A=1}^{9}\partial_A g_{++}\partial_A h_{+-} + \frac{1}{4}\sum_{A=1}^{9} g_{++}\partial_A g_{++}\partial_A h_{--} \\ & -\frac{1}{4}\sum_{A=1}^{9} h_{--}\left(\partial_A g_{++}\right)^2 \end{aligned} \tag{4.56}$$

and

$$\mathcal{T}_{++} = -\frac{\mu}{2}\left(2f_{+123} + \mu\sum_{i=1}^{3} h_{ii}\right) + \frac{\mu}{6} g_{++}\left(2f_{-123} + \mu h_{--}\right) \tag{4.57}$$

From this we find

$$h_{++} = \frac{\mu^4\kappa_{11}^2 P_-}{\pi^4}\,\frac{1}{6912\,|\vec{x}|^7}\left\{116\left[\sum_{i=1}^{3}(x^i)^2\right]^2 + 2\left[\sum_{a=4}^{9}(x^a)^2\right]^2 - 17\sum_{i=1}^{3}(x^i)^2\cdot\sum_{a=4}^{9}(x^a)^2\right\} \tag{4.58}$$

To summarize, the nonzero components of the metric perturbation are : h_{--} [eqn(4.24)], h_{+-} [eqn(4.39)], h_{ab} [eqn(4.47)], h_{ij} [eqn(4.48)], h_{++} [eqn(4.58)]; and the nonzero components of the field strength perturbation are: f_{-ijk}, f_{-ijb} [eqn(4.30)] and f_{+ijk}, f_{+ijb} [eqn(4.55)].

Substituting the expressions for the metric into our formula for V_{eff} in eqn(4.44), averaging $h_{\mu\nu}$ over x^-(i.e., dividing by $2\pi R$), and noting that $\kappa_{11}^2 = \frac{16\pi^5}{M^9}$, $P_- = \frac{N_s}{R}$, we find

$$
\begin{aligned}
V_{eff} &= \frac{N_p}{2R}(v^2+g_{++}) + \frac{15}{16}\frac{N_pN_s}{M^9R^3}\frac{v^4}{|\vec{x}|^7} \\
&+ \frac{\mu^2 N_pN_s}{R^3M^9}\left\{\left[-\frac{1}{96}\frac{1}{|\vec{x}|^5} - \frac{15}{32}\frac{(x^1)^2+(x^2)^2+(x^3)^2}{|\vec{x}|^7}\right]\sum_{i=1}^{3}(v^i)^2\right. \\
&\quad \left. + \frac{15}{16}\frac{\sum_{i,j=1}^{3} x^ix^jv^iv^j}{|\vec{x}|^7} + \left[-\frac{7}{96}\frac{1}{|\vec{x}|^5} + \frac{15}{32}\frac{(x^1)^2+(x^2)^2+(x^3)^2}{|\vec{x}|^7}\right]\sum_{a=4}^{9}(v^a)^2\right\} \\
&+ \frac{\mu^4 N_pN_s}{R^3M^9}\frac{1}{768\,|\vec{x}|^7}\left\{32\left[\sum_{i=1}^{3}(x^i)^2\right]^2 + \left[\sum_{a=4}^{9}(x^a)^2\right]^2 - 12\sum_{i=1}^{3}(x^i)^2\cdot\sum_{a=4}^{9}(x^a)^2\right\}
\end{aligned}
\tag{4.59}
$$

Comparison of the above formula with eqn(4.7) on the matrix theory side shows exact agreement.

We would like to emphasize the approximations involved once again. We treated the source graviton as a perturbation to the exact pp-wave background, and the calculation was performed only to first-order in P_-. However the solution that we found for these linearized equations is exact in μ.

Chapter 5

Two-membrane Interactions without M-momentum Transfer

5.1 Supergravity Computation

5.1.1 Diagonalizing the Field Equations for Arbitrary Static Source

In this subsection we present the diagonalization of the linearized supergravity equations of motions for arbitrary static sources. There is, of course, no highbrow knowledge involved here: we are just solving the linearized Einstein equations and Maxwell equations, which are coupled; and by "diagonalization" we basically just mean the prescription by which we get a decoupled Laplace equation for each component of the metric and three-form perturbations. The unperturbed background is the 11-D pp-wave, and we only consider static, i.e., x^+-independent, field configurations, thanks to the fact that the sources considered are taken to be static, i.e., with x^+-independent stress tensor and three-form current.

Since we leave the source arbitrary, what we will present here are the left-hand side of the linearized equations. These are tensors whose computation is straightforward though a bit tedious: the reason we present them here is because they are necessary when solving the field equations, and to the best of our knowledge have not been explicitly given elsewhere.

A somewhat related problem is the diagonalization of the equations of motion when the source is absent. This requires field configurations with x^+-dependence. One work along this line is [36]. Roughly speaking, borrowing the language of electromagnetism, what's considered in [36] are electromagnetic waves in vacuum, while what we are considering here are electrostatics and magnetostatics for arbitrary static sources.

The nonzero components up to (anti)symmetry of the Christoffel symbol, Riemann tensor, and Ricci tensor of the 11-D pp-wave are

$$\begin{aligned}\Gamma^A_{++} &= -\frac{1}{2}\partial_A g_{++}, \quad \Gamma^-_{+A} = \frac{1}{2}\partial_A g_{++} \\ R_{+A+B} &= -\frac{1}{2}\partial_A\partial_B g_{++}, \quad R_{++} = -\frac{1}{2}\partial_C\partial_C g_{++}\end{aligned} \tag{5.1}$$

Now we perturb the pp-wave background by adding a source. Denote the metric perturbation $\delta g_{\mu\nu}$ by $h_{\mu\nu}$, and the gauge potential perturbation by $\delta A_{\mu\nu\rho} = a_{\mu\nu\rho}$. $h_{\mu\nu}, a_{\mu\nu\rho}$ are treated as rank-two and rank-three tensors, respectively, the covariant derivative ∇ acting on them is defined using the connection coefficient of the unperturbed pp-wave background, and indices are raised/lowered, traces taken using the background metric $g_{\mu\nu}$.

We will deal with the Einstein equations first. Define $\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}g_{\mu\nu}h$, where $h \equiv g^{\mu\nu}h_{\mu\nu}$. Without the source, the Einstein equation is

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - \kappa_{11}^2[T_{\mu\nu}]_A = 0 \tag{5.2}$$

Recall that the stress tensor of the gauge field is

$$[T_{\mu\nu}]_A = \frac{1}{12\kappa_{11}^2}\left(F_{\mu\lambda\xi\rho}F_\nu{}^{\lambda\xi\rho} - \frac{1}{8}g_{\mu\nu}F^{\rho\sigma\lambda\xi}F_{\rho\sigma\lambda\xi}\right) \tag{5.3}$$

The source perturbs the Einstein equation to

$$\delta\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right) - \kappa_{11}^2\delta[T_{\mu\nu}]_A = \kappa_{11}^2[T_{\mu\nu}]_S \tag{5.4}$$

with $[T_{\mu\nu}]_S$ standing for the stress tensor of the source.

As usual, it helps to proceed in an organized manner, grouping different terms in the above perturbed Einstein equations. One finds, $\delta\left(R_{\mu\rho}-\frac{1}{2}Rg_{\mu\rho}\right)=-\frac{1}{2}\nabla^\sigma\nabla_\sigma\bar{h}_{\mu\rho}+K_{\mu\rho}+Q_{\mu\rho}$, and $\kappa_{11}^2\delta[T_{\mu\nu}]_A=N_{\mu\nu}+L_{\mu\nu}$, where the explicit expressions of the symmetric tensors $\nabla^\sigma\nabla_\sigma\bar{h}_{\mu\nu}, K_{\mu\nu}, Q_{\mu\nu}, N_{\mu\nu}$, and $L_{\mu\nu}$ can be obtained after some work. Their definitions and components are given below [1]

- $\nabla^\sigma\nabla_\sigma\bar{h}_{\mu\nu}$

$$\begin{aligned}\nabla^\sigma\nabla_\sigma\bar{h}_{++}= \quad & g^{\mu\nu}\partial_\mu\partial_\nu\bar{h}_{++}+\left[-(\partial_A\partial_A g_{++})\bar{h}_{+-}+\frac{1}{2}(\partial_A g_{++}\partial_A g_{++})\bar{h}_{--}\right]\\ &+2\left[\partial_A g_{++}\partial_-\bar{h}_{+A}-\partial_A g_{++}\partial_A\bar{h}_{+-}\right]\end{aligned}\tag{5.5}$$

$$\nabla^\sigma\nabla_\sigma\bar{h}_{+-}=g^{\mu\nu}\partial_\mu\partial_\nu\bar{h}_{+-}-\frac{1}{2}(\partial_A\partial_A g_{++})\bar{h}_{--}+\partial_A g_{++}\partial_-\bar{h}_{-A}-\partial_A g_{++}\partial_A\bar{h}_{--}\tag{5.6}$$

$$\nabla^\sigma\nabla_\sigma\bar{h}_{+C}=g^{\mu\nu}\partial_\mu\partial_\nu\bar{h}_{+C}-\frac{1}{2}(\partial_A\partial_A g_{++})\bar{h}_{-C}+\partial_A g_{++}\partial_-\bar{h}_{AC}-\partial_C g_{++}\partial_-\bar{h}_{+-}-\partial_A g_{++}\partial_A\bar{h}_{-C}\tag{5.7}$$

$$\nabla^\sigma\nabla_\sigma\bar{h}_{--}=g^{\mu\nu}\partial_\mu\partial_\nu\bar{h}_{--}\tag{5.8}$$

$$\nabla^\sigma\nabla_\sigma\bar{h}_{-C}=g^{\mu\nu}\partial_\mu\partial_\nu\bar{h}_{-C}-\partial_C g_{++}\partial_-\bar{h}_{--}\tag{5.9}$$

$$\nabla^\sigma\nabla_\sigma\bar{h}_{CD}=g^{\mu\nu}\partial_\mu\partial_\nu\bar{h}_{CD}-\partial_C g_{++}\partial_-\bar{h}_{-D}-\partial_D g_{++}\partial_-\bar{h}_{-C}\tag{5.10}$$

- $K_{\mu\nu}$ Its definition is

$$K_{\mu\rho}\equiv\frac{1}{2}\left(R_\mu{}^\xi\bar{h}_{\xi\rho}+R_\rho{}^\xi\bar{h}_{\xi\mu}\right)+R^\sigma{}_{\mu\rho}{}^\xi\bar{h}_{\sigma\xi}+\frac{1}{2}g_{\mu\rho}R^{\xi\sigma}\bar{h}_{\xi\sigma}-\frac{1}{2}R\bar{h}_{\mu\rho}\tag{5.11}$$

[1] Notice that ∂_+ will never appear because we only consider the static case; also note $g^{\mu\nu}\partial_\mu\partial_\nu=-g_{++}\partial_-^2+\partial_A\partial_A$ for static configurations.

Its components are given by

$$K_{++} = \left(-\frac{1}{2}\partial_A\partial_A g_{++}\right)\left(\bar{h}_{+-} + \frac{1}{2}g_{++}\bar{h}_{--}\right) + \frac{1}{2}\left(\partial_A\partial_B g_{++}\right)\bar{h}_{AB} \tag{5.12}$$

$$K_{+-} = \left(-\frac{1}{2}\partial_A\partial_A g_{++}\right)\bar{h}_{--} \tag{5.13}$$

$$K_{+A} = \left(-\frac{1}{4}\partial_C\partial_C g_{++}\right)\bar{h}_{-A} + \left(-\frac{1}{2}\partial_A\partial_B g_{++}\right)\bar{h}_{-B} \tag{5.14}$$

$$K_{--} = 0 \tag{5.15}$$

$$K_{-A} = 0 \tag{5.16}$$

$$K_{AB} = \frac{1}{2}\left[\partial_A\partial_B g_{++} - \frac{1}{2}\delta_{AB}\partial_C\partial_C g_{++}\right]\bar{h}_{--} \tag{5.17}$$

- $Q_{\mu\nu}$ Its definition is $Q_{\mu\rho} \equiv \frac{1}{2}(\nabla_\mu q_\rho + \nabla_\rho q_\mu) - \frac{1}{2}g_{\mu\rho}\nabla^\alpha q_\alpha$, where $q_\alpha \equiv \nabla^\beta \bar{h}_{\beta\alpha}$.

As one can recognize, $Q_{\mu\rho}$ contains the arbitrariness of making different gauge choices when solving the Einstein equation, where one makes a gauge choice by specifying the q_μ's. The components of $Q_{\mu\rho}$ are

$$\begin{aligned}
&Q_{--} = \partial_- q_-, \quad Q_{-A} = \frac{1}{2}(\partial_- q_A + \partial_A q_-), \quad Q_{-+} = \frac{1}{2}(g_{++}\partial_- q_- - \partial_A q_A) \\
&Q_{AB} = \frac{1}{2}(\partial_A q_B + \partial_B q_A) - \frac{1}{2}\delta_{AB}(\partial_- q_+ - g_{++}\partial_- q_- + \partial_A q_A) \\
&Q_{+A} = \frac{1}{2}\left[\partial_A q_+ - (\partial_A g_{++})q_-\right] \\
&Q_{++} = \frac{1}{2}(\partial_A g_{++})q_A - \frac{1}{2}g_{++}(\partial_- q_+ - g_{++}\partial_- q_- + \partial_A q_A)
\end{aligned} \tag{5.18}$$

- $N_{\mu\nu}$ It is defined to be the part of $\kappa_{11}^2\delta[T_{\mu\nu}]_A$ that contains only the metric perturbation, but not the three-form gauge potential perturbation. Its components are given by

$$
\begin{aligned}
&N_{++} = \mu^2 \left(\frac{1}{3}\bar{h}_{+-} + \frac{1}{12} g_{++}\bar{h}_{--} - \frac{1}{3}\sum_{i=1}^{3}\bar{h}_{ii} + \frac{1}{6}\sum_{a=4}^{9}\bar{h}_{aa} \right) \\
&N_{+-} = \frac{\mu^2}{4}\bar{h}_{--}, \quad N_{+i} = \frac{\mu^2}{2}\bar{h}_{-i}, \quad N_{+b} = 0 \\
&N_{--} = 0, \quad N_{-i} = 0, \quad N_{-b} = 0 \\
&N_{ij} = -\frac{\mu^2}{4}\delta_{ij}\bar{h}_{--}, \quad N_{ib} = 0, \quad N_{ab} = \frac{\mu^2}{4}\delta_{ab}\bar{h}_{--}
\end{aligned} \tag{5.19}
$$

- $L_{\mu\nu}$ This is defined to be the part of $\kappa_{11}^2 \delta[T_{\mu\nu}]_A$ that contains only the three-form perturbation, but not the metric perturbation. Its components are given by

$$
\begin{aligned}
&L_{++} = \mu\left(\delta F_{123+} - \frac{1}{2} g_{++}\delta F_{123-}\right), \; L_{+-} = 0, \; L_{+i} = \frac{\mu}{4}\epsilon_{ijk}\delta F_{+jk-}, \; L_{+b} = \frac{\mu}{2}\delta F_{123b} \\
&L_{--} = 0, \quad L_{-i} = 0, \quad L_{-b} = 0 \\
&L_{ij} = \frac{\mu}{2}\delta_{ij}\delta F_{123-}, \; L_{ib} = \frac{\mu}{4}\epsilon_{ijk}\delta F_{bjk-}, \; L_{bd} = -\frac{\mu}{2}\delta_{bd}\delta F_{123-}
\end{aligned} \tag{5.20}
$$

Next let us deal with the Maxwell equation. In the absence of the source, it is

$$
\frac{1}{\sqrt{-g}}\partial_\lambda\left(\sqrt{-g}\; F^{\lambda\mu_1\mu_2\mu_3}\right) - \frac{\tilde{\eta}}{1152}\frac{\epsilon^{\mu_1\cdots\mu_{11}}}{\sqrt{-g}} F_{\mu_4\ldots\mu_7}F_{\mu_8\ldots\mu_{11}} = 0 \tag{5.21}
$$

where $\tilde{\eta}$ is either $+1$ or -1 depending on the choice of convention, which one can fix later by requiring the consistency of the conventions for the equations and the solutions under consideration. When the source is present, we add its current $J^{\mu_1\mu_2\mu_3}$ to the right-hand side of the above equation, and get

$$
\delta\left[\frac{1}{\sqrt{-g}}\partial_\lambda\left(\sqrt{-g}\; F^{\lambda\mu_1\mu_2\mu_3}\right) - \frac{\tilde{\eta}}{1152}\frac{\epsilon^{\mu_1\cdots\mu_{11}}}{\sqrt{-g}} F_{\mu_4\ldots\mu_7}F_{\mu_8\ldots\mu_{11}}\right] = J^{\mu_1\mu_2\mu_3} \tag{5.22}
$$

Here δ denotes the perturbation to the left-hand side due to the introduction of a source.

We can write the left-hand side of the above equation as the sum of two totally antisymmetric tensors $Z^{\mu_1\mu_2\mu_3} + S^{\mu_1\mu_2\mu_3}$, where $Z^{\mu_1\mu_2\mu_3}$ is defined to be the part that contains the metric perturbation only, and $S^{\mu_1\mu_2\mu_3}$ is defined to be the part that contains the three-form perturbation only. One finds

$$
\begin{aligned}
&Z^{+-i} = \mu\epsilon_{ijk}\partial_j\bar{h}_{-k}, \quad Z^{+-b} = 0, \quad Z^{+ij} = \mu\epsilon_{ijk}(\partial_-\bar{h}_{-k} - \partial_k\bar{h}_{--}), \quad Z^{+ib} = 0, \quad Z^{+bc} = 0 \\
&Z^{-ij} = \mu\epsilon_{ijk}\left[\partial_k\left(\frac{1}{3}\bar{h} - \bar{h}_-{}^- - \sum_{i=1}^{3}\bar{h}_{ii}\right) - \partial_b\bar{h}_{kb}\right], \quad Z^{-ib} = \mu\epsilon_{ijk}\partial_j\bar{h}_{kb}, \quad Z^{-bc} = 0 \\
&Z^{ijk} = -\mu\epsilon_{ijk}\left[\partial_-\left(\frac{1}{3}\bar{h} - \bar{h}_-{}^- - \sum_{i=1}^{3}\bar{h}_{ii}\right) - \partial_b\bar{h}_{-b}\right], \quad Z^{ijb} = \mu\epsilon_{ijk}(\partial_-\bar{h}_{kb} - \partial_k\bar{h}_{-b}) \\
&Z^{ibc} = 0, \quad Z^{bce} = 0 \qquad (5.23)
\end{aligned}
$$

and

$$
S^{+-A} = g^{\mu\nu}\partial_\mu\partial_\nu a_{-+A} + \partial_B g_{++}\partial_- a_{BA-} - \partial_-(\nabla^\mu a_{\mu+A}) + \partial_A(\nabla^\mu a_{\mu+-}) \qquad (5.24)
$$

$$
S^{+AB} = g^{\mu\nu}\partial_\mu\partial_\nu a_{-AB} - \partial_-(\nabla^\mu a_{\mu AB}) + \partial_A(\nabla^\mu a_{\mu-B}) - \partial_B(\nabla^\mu a_{\mu-A}) \qquad (5.25)
$$

$$
\begin{aligned}
S^{-AB} = \quad & g^{\mu\nu}\partial_\mu\partial_\nu a_{+AB} - g_{++}S^{+AB} \\
& + \{[(\partial_A g_{++})(\partial_- a_{-+B}) + \partial_A(\nabla^\mu a_{\mu+B}) - \partial_A\left(a_{EB-}\partial_E g_{++}\right)] - [A \leftrightarrow B]\} \\
& -(\partial_D g_{++})\delta F_{D-AB} - \mu\frac{\tilde{\eta}}{24}\epsilon^{-AB\mu_4\ldots\mu_7 123+}\delta F_{\mu_4\ldots\mu_7} \qquad (5.26)
\end{aligned}
$$

$$
\begin{aligned}
S^{ABE} = \quad & g^{\mu\nu}\partial_\mu\partial_\nu a_{ABE} - (\partial_A g_{++})(\partial_- a_{-BE}) - (\partial_B g_{++})(\partial_- a_{-EA}) - (\partial_E g_{++})(\partial_- a_{-AB}) \\
& -\partial_A(\nabla^\mu a_{\mu BE}) - \partial_B(\nabla^\mu a_{\mu EA}) - \partial_E(\nabla^\mu a_{\mu AB}) - \mu\frac{\tilde{\eta}}{24}\epsilon^{ABE\mu_4\ldots\mu_7 123+}\delta F_{\mu_4\ldots\mu_7} \\
& \qquad (5.27)
\end{aligned}
$$

Notice that $S^{\mu_1\mu_2\mu_3}$ contains $\nabla^\mu a_{\mu\rho\lambda}$ and its derivatives. Those terms correspond to the gauge freedom for the three-form gauge potential.

Now that we have collected the expressions for the various tensors, we are ready to diagonalize the field equations. Recall that the Einstein equation is

$$-\frac{1}{2}\nabla^\sigma\nabla_\sigma\bar{h}_{\mu\nu} + K_{\mu\nu} + Q_{\mu\nu} - N_{\mu\nu} - L_{\mu\nu} = \kappa_{11}^2[T_{\mu\nu}]_S \tag{5.28}$$

and the Maxwell equation is

$$Z^{\mu_1\mu_2\mu_3} + S^{\mu_1\mu_2\mu_3} = J^{\mu_1\mu_2\mu_3} \tag{5.29}$$

The right-hand side of these equations is given by specifying the source that we consider (recall that the three-form current J is of order κ_{11}^2), hence we only need to concentrate on diagonalizing the left-hand sides.

As will be seen shortly, it is useful to define "level" for tensors: lower $+$/upper $-$ indices contribute $+1$ to the level; lower $-$/upper $+$ indices contribute -1 to level; and the upper A/lower A indices contribute zero to the level. We shall see that the field equations should be solved in ascending order of their levels. The following is the detailed prescription of the diagonalization procedure. Let us use the shorthand notation $(E.E.)_{\mu\nu}$ for the lower $(\mu\nu)$ component of the Einstein equation, and $(M.E.)^{\mu_1\mu_2\mu_3}$ for the upper $(\mu_1\mu_2\mu_3)$ component of the Maxwell equation.

- at level -2

The only field equation at this level is $(E.E.)_{--}$, which reads, upon using the expressions of the various tensors $\nabla^\sigma\nabla_\sigma\bar{h}_{\mu\nu}, K_{\mu\nu}, Q_{\mu\nu}$... etc., that we have given above

$$-\frac{1}{2}g^{\mu\nu}\partial_\mu\partial_\nu\bar{h}_{--} + Q_{--} = \kappa_{11}^2[T_{--}]_S \tag{5.30}$$

This equation can be immediately solved for $\bar{h}_{--}$ after specifying the source term and the gauge choice term Q_{--}.

- at level -1

We have $(E.E.)_{-A}$, which reads

$$-\frac{1}{2}\left[g^{\mu\nu}\partial_\mu\partial_\nu\bar{h}_{-A} - (\partial_A g_{++})(\partial_-\bar{h}_{--})\right] + Q_{-A} = \kappa_{11}^2[T_{-A}]_S \tag{5.31}$$

which can now be solved for $\bar{h}_{-A}$, using the $\bar{h}_{--}$ found previously. Also at this level is $(M.E.)^{+AB}$, which reads,

$$\begin{aligned} g^{\mu\nu}\partial_\mu\partial_\nu a_{-ij} - \partial_-(\nabla^\mu a_{\mu ij}) + \partial_i(\nabla^\mu a_{\mu -j}) - \partial_j(\nabla^\mu a_{\mu -i}) & \\ +\mu\epsilon_{ijk}(\partial_-\bar{h}_{-k} - \partial_k\bar{h}_{--}) &= J^{+ij} \\ g^{\mu\nu}\partial_\mu\partial_\nu a_{-ib} - \partial_-(\nabla^\mu a_{\mu ib}) + \partial_i(\nabla^\mu a_{\mu -b}) - \partial_b(\nabla^\mu a_{\mu -i}) &= J^{+ib} \\ g^{\mu\nu}\partial_\mu\partial_\nu a_{-bc} - \partial_-(\nabla^\mu a_{\mu bc}) + \partial_b(\nabla^\mu a_{\mu -c}) - \partial_c(\nabla^\mu a_{\mu -b}) &= J^{+bc} \end{aligned} \tag{5.32}$$

from which we can find a_{-AB}, upon specifying the gauge choice $\nabla^\mu a_{\mu\rho\lambda}$ for the three-form and using the $\bar{h}_{-A}$ and $\bar{h}_{--}$ found previously.

- at level 0

At this level we have $(E.E.)_{+-}$, $(M.E.)^{+-A}$, $(E.E.)_{AB}$, and $(M.E.)^{ABE}$.

$(E.E.)_{+-}$ is of the form

$$-\frac{1}{2}g^{\mu\nu}\partial_\mu\partial_\nu\bar{h}_{+-} = \text{known terms} \tag{5.33}$$

From now on, we will not write down the detailed equations; "known terms" refers to the gauge choice terms $Q_{\mu\nu}$, $\nabla^\mu a_{\mu\rho\lambda}$, source terms, and terms containing previously found $\bar{h}_{\mu\nu}$'s and $a_{\mu\nu\rho}$'s, one can write those down by looking up the expressions given earlier for the various tensors. Solving the above equation we get $\bar{h}_{+-}$. Solving $(M.E.)^{+-A}$ gives a_{-+A}.

$(E.E.)_{AB}$ and $(M.E.)^{ABE}$ are coupled, so a little more work is needed. The following are the details. First notice that the only unknown in $(M.E.)^{bce}$ is a_{bce}, hence solving this equation we find a_{bce} ($(M.E.)^{bce}$ contains the usual term $g^{\mu\nu}\partial_\mu\partial_\nu a_{bce}$ and also a term of the form $\partial_- a_{dfg}$ which comes from the $F \wedge F$ in the Maxwell equation, hence it is not quite a Laplace equation. But, that being said, one shouldn't have any difficulty solving it.)

$(M.E.)^{ibc}$ is of the form $g^{\mu\nu}\partial_\mu\partial_\nu a_{ibc}$ = known terms, solving which gives a_{ibc}.

$(M.E.)^{ijb}$ and $(E.E.)_{kb}$ are coupled in the following manner

$$
\begin{aligned}
g^{\mu\nu}\partial_\mu\partial_\nu a_{ijb} + \mu\epsilon_{ijk}\partial_-\bar{h}_{kb} &= \text{known terms} \\
-\frac{1}{2}g^{\mu\nu}\partial_\mu\partial_\nu\bar{h}_{kb} + \frac{1}{4}\mu\epsilon_{klm}\partial_- a_{lmb} &= \text{known terms}
\end{aligned} \tag{5.34}
$$

Decoupling these two equations is quite easy. Let us take a_{12b} and $\bar{h}_{3b}$ as the representative case. One sees that these two equations can be recombined to give

$$
\begin{aligned}
(g^{\mu\nu}\partial_\mu\partial_\nu + i\mu\partial_-)(\bar{h}_{3b} + ia_{12b}) &= \text{known terms} \\
(g^{\mu\nu}\partial_\mu\partial_\nu - i\mu\partial_-)(\bar{h}_{3b} - ia_{12b}) &= \text{known terms}
\end{aligned} \tag{5.35}
$$

Solving these equations gives $(\bar{h}_{3b} + ia_{12b})$ and $(\bar{h}_{3b} - ia_{12b})$, and in turn $\bar{h}_{3b}$ and a_{12b}.

$(M.E.)^{ijk}$ is coupled to $(E.E.)_{ij}$ and $(E.E.)_{bd}$ through the quantity $H \equiv \frac{2}{3}\sum_{i=1}^{3}\bar{h}_{ii} - \frac{1}{3}\sum_{a=4}^{9}\bar{h}_{aa}$ in the following manner

$$
\begin{aligned}
g^{\mu\nu}\partial_\mu\partial_\nu a_{123} + \mu\partial_- H &= \text{known terms} \\
-\frac{1}{2}g^{\mu\nu}\partial_\mu\partial_\nu\bar{h}_{ij} + \frac{1}{2}\mu\delta_{ij}\partial_- a_{123} &= \text{known terms} \\
-\frac{1}{2}g^{\mu\nu}\partial_\mu\partial_\nu\bar{h}_{bd} - \frac{1}{2}\mu\delta_{bd}\partial_- a_{123} &= \text{known terms}
\end{aligned} \tag{5.36}
$$

Combining the last two equations gives

$$
-g^{\mu\nu}\partial_\mu\partial_\nu H + 4\mu\partial_- a_{123} = \text{known terms} \tag{5.37}
$$

Recombining this with first equation, we get

$$
\begin{aligned}
(g^{\mu\nu}\partial_\mu\partial_\nu + 2i\mu\partial_-)(H + 2ia_{123}) &= \text{known terms} \\
(g^{\mu\nu}\partial_\mu\partial_\nu - 2i\mu\partial_-)(H - 2ia_{123}) &= \text{known terms}
\end{aligned} \tag{5.38}
$$

solving which individually gives H and a_{123}. Using the resulting expression for a_{123} one can then find $\bar{h}_{ij}$ and $\bar{h}_{bd}$. Thus we are done with $(E.E.)_{AB}$ and $(M.E.)^{ABE}$.

- at level 1

$(M.E.)^{-AB}$ is of the form $g^{\mu\nu}\partial_\mu\partial_\nu a_{+AB}$ = known terms, solving which gives a_{+AB}.

$(E.E.)_{+A}$ is of the form $-\frac{1}{2}g^{\mu\nu}\partial_\mu\partial_\nu\bar{h}_{+A}$ = known terms, solving which gives $\bar{h}_{+A}$.

- at level 2

$(E.E.)_{++}$ is of the form $-\frac{1}{2}g^{\mu\nu}\partial_\mu\partial_\nu\bar{h}_{++}$ = known terms, solving which gives $\bar{h}_{++}$.

Thus we have diagonalized the whole set of Einstein equations and Maxwell equations.

5.1.2 Application to a Spherical Membrane Source using the Near-Membrane Expansion

Now let us apply the general formalism of the previous subsection to the case of interest, with the source being a spherical membrane sitting at the origin of the transverse directions, i.e., having $(X^1)^2 + (X^2)^2 + (X^3)^2 = r_0^2$, $X^4 = 0, ..., X^9 = 0$, and $X^+ = t, X^- = 0$. The gauge choice we shall take is: $q_\alpha = 0$ (hence all the $Q_{\alpha\beta}$'s vanish); and $\nabla^\mu a_{\mu\rho\lambda} = 0$. The nonzero components of the stress tensor and three-form current for this source are given by

$$\begin{aligned}
&[T_{--}]_s = T\delta(x^-)\delta(r-r_0)\delta(x^4)...\delta(x^9)\left(\frac{\mu r_0}{3}\right)^{-1} \\
&[T_{+-}]_s = -\left(\frac{\mu r_0}{3}\right)^2 [T_{--}]_s, \quad [T_{ij}]_s = \left(\frac{\mu r_0}{3}\right)^2\left(\frac{x^i x^j}{r_0^2} - \delta^{ij}\right)[T_{--}]_s, \\
&[T_{++}]_s = \left(\frac{\mu r_0}{3}\right)^4 [T_{--}]_s
\end{aligned} \tag{5.39}$$

and

$$J^{+ij} = \kappa_{11}^2(-2)\left(\frac{\mu r_0}{3}\right)\epsilon_{ijk}\frac{x^k}{r_0}[T_{--}]_s \tag{5.40}$$

where $r \equiv \sqrt{x^i x^i}$.

Now let us explain what we mean by "near-membrane expansion." Define $w \equiv r - r_0$, $z \equiv \sqrt{x^a x^a}$, and $\xi \equiv \sqrt{w^2 + z^2}$, which parameterize the distance away from the source membrane. We shall assume that w, z, ξ are of the same order of magnitude. The near-membrane expansion is an expansion in ξ/r_0. When one sits very close to membrane, one just sees a flat membrane, which is the zeroth-order of the expansion. As one moves away from the membrane, one begins to feel the curvature of membrane, which gives the higher order corrections in the expansion. One should also note that the zeroth-order of this expansion is just the flat space limit: $\mu \to 0$, $r_0 \to \infty$, with μr_0 kept finite.

It is instructive to see how the zeroth-order works. At this order, in the Einstein equations and Maxwell equations, the effective source terms (which are of the forms $(\partial g_{++})\bar{h}_{\mu\nu}$, etc.) arising from the various tensors $K_{\mu\nu}$, $N_{\mu\nu}$, $L_{\mu\nu}$, $Z^{\mu\nu\rho}$ etc. are less singular then the delta-functional sources $[T_{\mu\nu}]_s$ and $J^{\mu\nu\rho}$, and can thus be thrown away. Then the resulting equations are trivially decoupled. Also, at this order, we can treat the x^i's in $[T_{\mu\nu}]_s$ and $J^{\mu\nu\rho}$ as constant vectors. One finds (using the subscript 0 to denote "zeroth order")

$$\begin{aligned}
&[\bar{h}_{-A}]_0 = 0, \quad [a_{-ij}]_0 = \left(\frac{\mu r_0}{3}\right)\epsilon_{ijk}\frac{x^k}{r_0}[\bar{h}_{--}]_0, \quad [a_{-ib}]_0 = 0, \quad [a_{-bc}]_0 = 0 \\
&[\bar{h}_{+-}]_0 = -\left(\frac{\mu r_0}{3}\right)^2 [\bar{h}_{--}]_0, \quad [a_{-+A}]_0 = 0, \quad [\bar{h}_{ij}]_0 = \left(\frac{\mu r_0}{3}\right)^2\left(\frac{x^i x^j}{r_0^2} - \delta^{ij}\right)[\bar{h}_{--}]_0 \\
&[\bar{h}_{ib}]_0 = 0, \quad [\bar{h}_{bc}]_0 = 0, \quad [a_{ABE}]_0 = 0 \\
&[a_{+ij}]_0 = -\left(\frac{\mu r_0}{3}\right)^3\epsilon_{ijk}\frac{x^k}{r_0}[\bar{h}_{--}]_0, \quad [a_{+ib}]_0 = 0, \quad [a_{+bc}]_0 = 0, \quad [\bar{h}_{+A}]_0 = 0 \\
&[\bar{h}_{++}]_0 = \left(\frac{\mu r_0}{3}\right)^4 [\bar{h}_{--}]_0
\end{aligned} \tag{5.41}$$

where $[\bar{h}_{--}]_0$ satisfies

$$\begin{aligned}
&\left[-\left(\frac{\mu r_0}{3}\right)^2 k_-^2 + \frac{\partial^2}{(\partial w)^2} + \frac{\partial^2}{(\partial x^4)^2} + \dots\frac{\partial^2}{(\partial x^9)^2}\right][\bar{h}_{--}]_0 \\
&\qquad = -\frac{1}{\pi R}\left(\frac{\mu r_0}{3}\right)^{-1}\kappa_{11}^2 T\delta(w)\delta(x^4)\dots\delta(x^9)
\end{aligned} \tag{5.42}$$

(where we have multiplied the right-hand side of the equation by $\frac{1}{2\pi R}$ due to the Fourier transform along the x^- direction), and is given by

$$[\bar{h}_{--}]_0 = \Delta\frac{\exp\left(-\frac{\mu r_0}{3}k_-\xi\right)}{\xi^5}\left[3 + 3\left(\frac{\mu r_0}{3}k_-\xi\right) + \left(\frac{\mu r_0}{3}k_-\xi\right)^2\right] \tag{5.43}$$

with $\Delta \equiv \frac{\kappa_{11}^2 T}{16\pi^4 R\left(\frac{\mu r_0}{3}\right)}$.

For the zeroth-order we can put $r_0' = r_0$. From eqn(3.61) we have $r_0 = \frac{\mu \Pi_-}{3T\sin\theta}$ if we choose the world volume coordinates to be $\sigma^1 = \theta$ and $\sigma^2 = \phi$. We can use then eliminate r_0 in favor of Π_-. Putting the above zeroth-order solution of $\bar{h}_{--}$ into the light cone Lagrangian $\delta\mathcal{L}_{lc}$ given in eqn (3.35), for a spherical probe membrane with radius r_0', sitting at rest in the $1, 2, 3$ directions, and moving about in the 4 through 9 directions: $(X^1)^2 + (X^2)^2 + (X^3)^2 = r_0'^2$, $X^4(t), ..., X^9(t)$, and $X^+ = t, X^- = 0$, one finds:

$$\delta\mathcal{L}_{lc} = \frac{1}{8}\Pi_-[\bar{h}_{--}]_0(\dot{X}^a\dot{X}^a)^2 \tag{5.44}$$

It is worth noting that, keeping only the leading order term in k_- in $[\bar{h}_{--}]_0$, eqn (5.44) becomes the v^4 Lagrangian for the case of longitudinal momentum transfer between two membranes in the flat space, given in [37].

Now let us go on to consider higher orders in the near-membrane expansion. Since in this thesis we do not consider longitudinal momentum transfer, we shall set $k_- = 0$ (which makes many fields equations decouple).

Denote

$$\begin{aligned}\Box &\equiv \partial_A\partial_A \\ &\quad \text{(when acting on functions of } (w, z)\text{)} \\ &= \Box_0 + \delta\Box\end{aligned} \tag{5.45}$$

with $\Box_0 \equiv \frac{\partial^2}{\partial w^2} + \frac{\partial^2}{\partial z^2} + \frac{5}{z}\frac{\partial}{\partial z} = \partial_w^2 + \partial_a\partial_a$ being the zeroth-order Laplace operator, and $\delta\Box \equiv \frac{2}{r_0+w}\frac{\partial}{\partial w}$ being curvature correction to it.

At level -2

$(E.E.)_{--}$

$$\Box\bar{h}_{--} = -\frac{1}{\pi R}\kappa_{11}^2 T\delta(w)\delta(x^4)...\delta(x^9)\left(\frac{\mu r_0}{3}\right)^{-1} \tag{5.46}$$

Let

$$\bar{h}_{--} = [\bar{h}_{--}]_0 + \delta\bar{h}_{--} \tag{5.47}$$

with $[\bar{h}_{--}]_0 \equiv \Delta\frac{3}{\xi^5}$, which satisfies

$$\Box_0[\bar{h}_{--}]_0 = -\frac{1}{\pi R}\kappa_{11}^2 T\delta(w)\delta(x^4)...\delta(x^9)\left(\frac{\mu r_0}{3}\right)^{-1} \tag{5.48}$$

Then $(E.E.)_{--}$ becomes

$$\Box_0\delta\bar{h}_{--} + \delta\Box[\bar{h}_{--}]_0 + \delta\Box\delta\bar{h}_{--} = 0 \tag{5.49}$$

Now we look at the order of magnitude of each term in the above equation. Notice that $\Box_0 \sim \frac{1}{\xi^2}$, $\delta\Box \sim \frac{1}{r_0\xi}$. The second term is thus $\sim \Delta\frac{1}{r_0\xi^6}$, which tells us $\delta\bar{h}_{--} \sim \Delta\frac{1}{r_0\xi^4}$. Solving the equation iteratively we find

$$\delta\bar{h}_{--} = [\bar{h}_{--}]_0\left(-\frac{w}{r_0} + \frac{w^2}{r_0^2} - \frac{w^3}{r_0^3} + \frac{w^4}{r_0^4}\right) \tag{5.50}$$

and thus

$$\bar{h}_{--} = 3\Delta\frac{1}{\xi^5}\left[\frac{r_0}{r_0+w} + O\left(\frac{\xi^5}{r_0^5}\right)\right] \tag{5.51}$$

We did not compute the $O\left(\frac{\xi^5}{r_0^5}\right)$ terms, because we are only interested in the part of the solution that is singular as $\xi \to 0$. Solving the other field equations is similar, so we just present the results below, omitting the $O\left(\frac{\xi^5}{r_0^5}\right)$ symbol.

At level -1

$$\begin{aligned}
\bar{h}_{-A} &= 0 \\
a_{-bc} &= 0 \\
a_{-ib} &= 0 \\
a_{-ij} &= \epsilon_{ijk}x^k\mu\Delta\frac{1}{\xi^5}\left[1 - \frac{1}{2}\frac{w}{r_0} + \frac{1}{6}\frac{w^2+z^2}{r_0^2} - \frac{1}{2}\frac{wz^2}{r_0^3} + \frac{w^2z^2}{r_0^4}\right]
\end{aligned} \tag{5.52}$$

At level 0

$$
\begin{aligned}
\bar{h}_{+-} &= -\left(\frac{\mu r_0}{3}\right)^2 \frac{3\Delta}{\xi^5}\left[1+\left(\frac{5}{4}\frac{w^2}{r_0^2}+\frac{7}{8}\frac{z^2}{r_0^2}\right)\frac{r_0}{r_0+w}\right] \\
a_{-+A} &= 0 \\
a_{ABD} &= 0
\end{aligned} \tag{5.53}
$$

$$
\begin{aligned}
\bar{h}_{kb} = \quad & x^k x^b \mu^2 \Delta \frac{1}{\xi^5}\left(-\frac{1}{2}\right)\left\{1-\frac{5}{4}\frac{w}{r_0}+\frac{17}{12}\frac{w^2}{r_0^2}-\frac{1}{12}\frac{z^2}{r_0^2}-\frac{3}{2}\frac{w^3}{r_0^3}\right. \\
& \left. +\frac{1}{4}\frac{wz^2}{r_0^3}+\frac{3}{2}\frac{w^4}{r_0^4}-\frac{1}{2}\frac{w^2z^2}{r_0^4}\right\} \qquad (5.54) \\
\bar{h}_{ij} = \quad & \frac{x^i x^j}{r^2}\left(\frac{\mu r_0}{3}\right)^2 \Delta \frac{3}{\xi^5}\left\{1-\frac{w}{r_0}-\frac{z^2}{r_0^2}+\frac{w^3}{r_0^3}+2\frac{wz^2}{r_0^3}-\frac{w^4}{r_0^4}-\frac{w^2z^2}{r_0^4}+\frac{z^4}{r_0^4}\right\} \\
& -\delta^{ij}\left(\frac{\mu r_0}{3}\right)^2 \Delta \frac{3}{\xi^5}\left\{1+\frac{1}{2}\frac{w}{r_0}+\frac{7}{12}\frac{w^2}{r_0^2}-\frac{1}{24}\frac{z^2}{r_0^2}-\frac{1}{4}\frac{w^3}{r_0^3}+\frac{3}{8}\frac{wz^2}{r_0^3}\right. \\
& \left. +\frac{1}{4}\frac{w^4}{r_0^4}-\frac{1}{24}\frac{w^2z^2}{r_0^4}+\frac{1}{3}\frac{z^4}{r_0^4}\right\} \qquad (5.55) \\
\bar{h}_{bd} = \quad & \delta_{bd}(\mu r_0)^2 \Delta \frac{1}{\xi^5}\left\{\frac{1}{2}\frac{w}{r_0}-\frac{7}{18}\frac{w^2}{r_0^2}-\frac{19}{72}\frac{z^2}{r_0^2}+\frac{7}{18}\frac{w^3}{r_0^3}+\frac{19}{72}\frac{wz^2}{r_0^3}\right. \\
& \left. -\frac{7}{18}\frac{w^4}{r_0^4}-\frac{19}{72}\frac{w^2z^2}{r_0^4}\right\} \qquad (5.56)
\end{aligned}
$$

At level +1

$$
\begin{aligned}
& a_{+bc} = 0 \\
& a_{+ib} = 0 \\
& a_{+ij} = -\epsilon_{ijk} x^k \left(\frac{\mu r_0}{3}\right)^2 \mu \Delta \frac{1}{\xi^5}\left\{1+2\frac{w}{r_0}-\frac{w^2}{r_0^2}-\frac{5}{4}\frac{z^2}{r_0^2}+\frac{3}{2}\frac{w^3}{r_0^3}+\frac{7}{4}\frac{wz^2}{r_0^3}\right. \\
& \qquad \left. -\frac{3}{2}\frac{w^4}{r_0^4}-\frac{5}{4}\frac{w^2z^2}{r_0^4}+\frac{1}{2}\frac{z^4}{r_0^4}\right\} \qquad (5.57) \\
& \bar{h}_{+A} = 0 \qquad (5.58)
\end{aligned}
$$

At level +2

$$\bar{h}_{++} = \left(\frac{\mu r_0}{3}\right)^4 \Delta \frac{3}{\xi^5}\Big\{ \quad 1 \quad + \frac{5}{2}\frac{w}{r_0} + \frac{31}{12}\frac{w^2}{r_0^2} - \frac{1}{24}\frac{z^2}{r_0^2} + \frac{17}{12}\frac{w^3}{r_0^3} - \frac{1}{12}\frac{wz^2}{r_0^3} + \frac{1}{3}\frac{w^4}{r_0^4} + \frac{23}{24}\frac{w^2z^2}{r_0^4} + \frac{17}{32}\frac{z^4}{r_0^4}\Big\} \tag{5.59}$$

Again, let the probe membrane have a radius $r_0' = r_0 + w$, with the trajectory $(X^1)^2 + (X^2)^2 + (X^3)^2 = r_0'^2$, $X^4(t), ..., X^9(t)$, and $X^+ = t, X^- = 0$. We shall take $v \equiv \sqrt{\dot{X}^a \dot{X}^a}$ to be of order μz (recall that the supersymmetric circular orbit has $v = \frac{1}{6}\mu z$; so here we are considering generically nonsupersymmetric orbits that can be regarded as deformations of the supersymmetric circular one). Putting in the supergravity solution given above into eqn (3.35), using $T = \frac{\mu \Pi_-}{3 r_0' \sin\theta}$, and in the end keeping only the part of $\delta\mathcal{L}_{lc}$ that is singular as $\xi \to 0$, we find the probe's $\mathcal{L}_{lc}$ to be

$$\mathcal{L}_{lc} = (\mathcal{L}_{lc})_{pp} + \delta\mathcal{L}_{lc} \tag{5.60}$$

with $(\mathcal{L}_{lc})_{pp}$ being the action in the unperturbed pp-wave background, and

$$\delta\mathcal{L}_{lc} = \Pi_- \Delta \frac{\mu^4(4w^2z^2 + 7z^4) - 72\mu^2 v^2(2w^2 + 5z^2) + 3888v^4}{10368\xi^5} \tag{5.61}$$

Notice that the above $\delta\mathcal{L}_{lc}$'s singular behavior as $\xi \to 0$ is homogeneous: $\sim \frac{1}{\xi}$ (since v is of order μz). The expression in the numerator: $\mu^4(4w^2z^2 + 7z^4) - 72\mu^2v^2(2w^2 + 5z^2) + 3888v^4$ nicely factorizes into $(36v^2 - z^2\mu^2)[108v^2 - (4w^2 + 7z^2)\mu^2]$, which shows that for the special case of the supersymmetric circular orbit $v = \frac{1}{6}\mu z$ considered in [31], $\delta\mathcal{L}_{lc}$ vanishes as expected.

To be more precise, so far we have been talking about Lagrangian density. Since the membrane worldvolume is taken to be a unit sphere, the Lagragian is given by

$$\begin{aligned} \delta L_{lc} = \int d\theta d\phi \delta\mathcal{L}_{lc} &= \frac{\kappa_{11}^2 T^2}{4\pi^3 R\mu^2} \frac{(36v^2 - z^2\mu^2)[108v^2 - (4w^2 + 7z^2)\mu^2]}{1152\xi^5} \\ &= \frac{\alpha}{\mu^2} \frac{(36v^2 - z^2\mu^2)[108v^2 - (4w^2 + 7z^2)\mu^2]}{1152\xi^5} \end{aligned} \tag{5.62}$$

where to get the first line we used $T = \frac{\mu \Pi_-}{3r_0' \sin\theta}$ to eliminate Π_- in terms of T and r_0', and set $r_0' \approx r_0$ in the end to remove higher $\frac{w}{r_0}$ order curvature correction. To get the second line we used the expressions for κ_{11}^2, T, and α in terms of M and R given at the end of subsection 3.

Here we would like to make a brief comparison of the above membrane result with the graviton result given in [5].

First of all, the membrane result contains the variable w (the difference in radius between the probe membrane and the source membrane), which has no counterpart in the graviton case. Secondly, in terms of the x^1 through x^3 directions, the two membranes are sitting at rest at the origin; this corresponds to setting $x^i = 0$ and $v_i = 0$ in the graviton case.

If we set $w = 0$ in eqn (5.62), i.e., consider two membranes of the same size, then $\xi = z$, and

$$(\delta L_{lc})_{\text{membrane}} = \frac{\alpha}{1152\mu^2 z^5}(36v^2 - z^2\mu^2)(108v^2 - 7z^2\mu^2) \tag{5.63}$$

while for the gravitons, upon setting $N_p = N_s = N$, $x^i = 0$, and $v_i = 0$ in eqn (19) of [5], we have

$$(\delta L_{lc})_{\text{graviton}} = \frac{\alpha^3 N^2}{5376 z^7}(36v^2 - z^2\mu^2)(140v^2 - 7z^2\mu^2) \tag{5.64}$$

When comparing the above two expressions, note the difference between the numerators: $(108v^2 - 7z^2\mu^2)$ for membrane, and $(140v^2 - 7z^2\mu^2)$ for graviton. Also notice their different power law dependence on z: $\frac{1}{z^5}$ for membrane and $\frac{1}{z^7}$ for graviton, which cannot be undone by integrating over the membrane (r_0, the radius of the membrane, has nothing to do with z, the separation of the membranes in the X^4 to X^9 directions).

5.2 Matrix Theory Computation - The Membrane Limit

Shin and Yoshida have previously calculated the one-loop effective action for membrane fuzzy spheres extended in the first three directions and having periodic motion in a sub-plane of the remaining six transverse directions. In reference [31] they considered the case of supersymmetric circular motion for an arbitrary radius and angular frequency $\frac{\mu}{6}$ (this orbit preserves eight supersymmetries and was first found by [4]). Here we generalize that analysis to orbits which are not supersymmetric. The procedures are: expanding the action to quadratic order in fluctuations, writing the fields in terms of the matrix spherical harmonics introduced in [13] [2], diagonalizing the mass matrices of the bosons, fermions, and ghosts, and finally summing up the masses to get the one-loop effective action. In doing so, we shall adopt the notations of [31]. We shall consider the background

$$B^I = \begin{pmatrix} B^I_{(1)} & 0 \\ 0 & B^I_{(2)} \end{pmatrix} \tag{5.65}$$

where

$$\begin{aligned} B^i_{(1)} &= \frac{\mu}{3} J^i_{(1)N_1\times N_1} & B^a_{(1)} &= 0\cdot\ 1_{N_1\times N_1} \\ B^i_{(2)} &= \frac{\mu}{3} J^i_{(2)N_2\times N_2} + x^i(t) 1_{N_2\times N_2} & B^a_{(2)} &= x^a(t) 1_{N_2\times N_2} \end{aligned} \tag{5.66}$$

with the J^i's being $su(2)$ generators. The above background has the interpretation of one spherical membrane (labeled by the subscript (1)) sitting at the origin, and the other spherical membrane (labeled by the subscript (2)) moving along the arbitrary orbit given by $\{x^i(t), x^a(t)\}$.

[2] For the computation below, one only needs the transformation of the matrix spherical harmonics under $SU(2)$, however, for detailed construction of the matrix spherical harmonics, see, e.g., Appendix A of [49].

The fluctuations of the bosonic fields are given by

$$A = \begin{pmatrix} Z^0_{(1)} & \Phi^0 \\ (\Phi^0)^\dagger & Z^0_{(2)} \end{pmatrix}$$
$$Y^I = \begin{pmatrix} Z^I_{(1)} & \Phi^I \\ (\Phi^I)^\dagger & Z^I_{(2)} \end{pmatrix} \tag{5.67}$$

As it turns out, the part of the bosonic action containing the diagonal fluctuations Z^0, Z^I does not contain any new terms in addition to those given in [31]. So we do not write it out here. (It shall be the same situation for the fermionic and ghost parts of the action; it is the off-diagonal fluctuations that give the one-loop interaction potential between the two membranes.) The action for the off-diagonal fluctuations is

$$\begin{aligned} S_{OD} = \int dt\, Tr\, [& - |\dot{\Phi^0}|^2 + x^2|\Phi^0|^2 + \left(\frac{\mu}{3}\right)^2 |J^i \circ \Phi^0|^2 - 2\left(\frac{\mu}{3}\right) x^i Re((J^i \circ \Phi^0)(\Phi^0)^\dagger) \\ & + |\dot{\Phi^i}|^2 - x^2|\Phi^i|^2 - \left(\frac{\mu}{3}\right)^2 |\Phi^i + i\epsilon^{ijk} J^j \circ \Phi^k|^2 + \left(\frac{\mu}{3}\right)^2 |J^i \circ \Phi^i|^2 \\ & + 2\left(\frac{\mu}{3}\right) x^i Re((J^i \circ \Phi^0)(\Phi^0)^\dagger) - 2i\dot{x^i}((\Phi^0)^\dagger \Phi^i - (\Phi^i)^\dagger \Phi^0) \\ & + |\dot{\Phi^a}|^2 - \left(x^2 + \frac{\mu^2}{6^2}\right) |\Phi^a|^2 - \left(\frac{\mu}{3}\right)^2 |J^i \circ \Phi^a|^2 \\ & + 2\left(\frac{\mu}{3}\right) x^i Re((J^i \circ \Phi^a)(\Phi^a)^\dagger) - 2i\dot{x^a}((\Phi^0)^\dagger \Phi^a - (\Phi^a)^\dagger \Phi^0) \Big] \end{aligned} \tag{5.68}$$

where $x^2 \equiv x^i x^i + x^a x^a$, and dots mean time-derivatives.

We specialize to the case where, $x^i = 0, x^8 = b, x^9 = vt$, with $(x^8)^2 + (x^9)^2$ denoted by z^2. The effective potential will be computed by summing over the mass of the fermionic and ghost fluctuations and then subtracting the mass of the bosonic fluctuations. This method, which we will refer to as the sum over mass method, is the same as the one used in [5]. Although the above trajectory has the form of a straight line with constant velocity in the (x^8, x^9) plane, the final expression of V_{eff} in terms of $z \equiv \sqrt{(x^a)^2}$ and $v \equiv \sqrt{(\dot{x^a})^2}$ should suffice for the purpose of comparing with supergravity for arbitrary orbits $x^a(t)$. One may ask whether the sum over mass formula is valid when the masses of the fluctuations are time-dependent (one origin for such a time-dependence is the acceleration of the trajectory). In section 4.2 and Appendix A of [5], it was carefully shown that, in the case of two-graviton interaction in the pp-wave background, the sum over mass formula was sufficient in computing the terms that could occur on the supergravity side. The time-dependence in the masses of the fluctuations will give terms of the form of matrix theory corrections to supergravity (i.e., terms that dominate at extreme short distances and cannot be observed in supergravity), which does not concern us since we are only interested in a comparison with supergravity. Here we expect a similar argument to hold in the case of two-membrane interaction. The rather non-trivial agreement with supergravity presented at the end of this section and also the agreement with the work of Shin and Yoshida [32] in section 5.3.2 confirm the validity of the sum over mass method.

Expand the fields in terms of matrix spherical harmonics

$$\Phi^{0,I} = \sum_{j=\frac{1}{2}|N_1-N_2|}^{\frac{1}{2}(N_1+N_2)-1} \sum_{m=-j}^{j} \phi^{0,I}_{jm} Y^{N_1\times N_2}_{jm}. \tag{5.69}$$

For our choice of background the masses of modes in the $i = 1, 2, 3$ and $a = 4, 5, 6, 7, 8$ directions are the same as those in [31]. For the gauge field and $a = 9$ direction we have

$$\begin{aligned} S = \int dt \sum_{j=\frac{1}{2}|N_1-N_2|}^{\frac{1}{2}(N_1+N_2)-1} & \left[-|\dot{\phi}^0_{jm}|^2 + \left(z^2 + \left(\frac{\mu}{3}\right)^2 j(j+1)\right) |\phi^0_{jm}|^2 \right. \\ & + |\dot{\phi}^9_{jm}|^2 - \left(z^2 + \left(\frac{\mu}{6}\right)^2 + \left(\frac{\mu}{3}\right)^2 j(j+1)\right) |\phi^0_{jm}|^2 \\ & \left. - 2iv\left(\left(\phi^0_{jm}\right)^* \phi^9_{jm} - \left(\phi^9_{jm}\right)^* \phi^0_{jm}\right)\right] \end{aligned} \tag{5.70}$$

where there is an implicit sum over $-j \leq m \leq j$. It is straightforward to diagonalize the mass matrix for these modes. Combining all contributions from bosonic fluctuations we get an bosonic effective potential[3] given by,

$$
\begin{aligned}
V^B_{eff} = & -\sum_{j=\frac{1}{2}|N_1-N_2|-1}^{\frac{1}{2}(N_1+N_2)-2}(2j+1)\sqrt{z^2+\left(\frac{\mu}{3}\right)^2(j+1)^2} \\
& -\sum_{j=\frac{1}{2}|N_1-N_2|+1}^{\frac{1}{2}(N_1+N_2)}(2j+1)\sqrt{z^2+\left(\frac{\mu}{3}\right)^2 j^2} \\
& -\sum_{j=\frac{1}{2}|N_1-N_2|}^{\frac{1}{2}(N_1+N_2)-1}(2j+1)\sqrt{z^2+\left(\frac{\mu}{3}\right)^2 j(j+1)} \\
& -\sum_{j=\frac{1}{2}|N_1-N_2|}^{\frac{1}{2}(N_1+N_2)-1}5(2j+1)\sqrt{z^2+\left(\frac{\mu}{3}\right)^2 (j+\frac{1}{2})^2} \\
& -\sum_{j=\frac{1}{2}|N_1-N_2|}^{\frac{1}{2}(N_1+N_2)-1}(2j+1)\left(\sqrt{z^2+\left(\frac{\mu}{3}\right)^2(j(j+1)+\frac{1}{8})+\frac{1}{2}\sqrt{\left(\frac{\mu}{6}\right)^4+16v^2}}\right. \\
& \qquad\qquad \left. +\sqrt{z^2+\left(\frac{\mu}{3}\right)^2(j(j+1)+\frac{1}{8})-\frac{1}{2}\sqrt{\left(\frac{\mu}{6}\right)^4+16v^2}}\right)
\end{aligned}
\tag{5.71}
$$

Now for the fermions we start with the action given in [31]

$$
L_F = Tr(i\Psi^\dagger\dot{\Psi} - \Psi^\dagger\gamma^I[\Psi, B^I] - i\frac{\mu}{4}\Psi^\dagger\gamma^{123}\Psi) \tag{5.72}
$$

with

$$
\Psi = \begin{pmatrix} \Psi_{(1)} & \chi \\ \chi^\dagger & \Psi_{(2)} \end{pmatrix}. \tag{5.73}
$$

Again the action for the diagonal fluctuations has no new terms, and for the off-diagonal part we decompose the $SO(9)$ spinor χ using the subgroup $SO(3)\times SO(6) \sim$

[3]Note that our convention differs from that of [31] by an overall minus sign. See section 3.1.

$SU(2) \times SU(4)$ preserved by PP-wave, $\mathbf{16} \to (\mathbf{2},\mathbf{4}) + (\bar{\mathbf{2}},\bar{\mathbf{4}})$

$$\chi = \frac{1}{\sqrt{2}} \begin{pmatrix} \chi_{A\alpha} \\ \hat{\chi}^{A_\alpha} \end{pmatrix}. \tag{5.74}$$

Substituting this into L_F,

$$\begin{aligned} L_F = Tr \Big[& i(\chi^\dagger)^{A\alpha}\dot{\chi}_{A\alpha} - \frac{1}{4}(\chi^\dagger)^{A\alpha}\chi_{A\alpha} - \frac{\mu}{3}(\chi^\dagger)^{A\alpha}(\sigma^i)^\beta_\alpha J^i \circ \chi_{A\beta} \\ & + i(\hat{\chi}^\dagger)^{A\alpha}\dot{\hat{\chi}}_{A\alpha} + \frac{1}{4}(\hat{\chi}^\dagger)^{A\alpha}\hat{\chi}_{A\alpha} + \frac{\mu}{3}(\hat{\chi}^\dagger)^{A\alpha}(\sigma^i)^\beta_\alpha J^i \circ \hat{\chi}_{A\beta} \\ & + x^i(-(\chi^\dagger)^{A\alpha}(\sigma^i)^\beta_\alpha \chi_{A\beta} + (\hat{\chi}^\dagger)^{A\alpha}(\sigma^i)^\beta_\alpha \hat{\chi}_{A\beta}) \\ & + x^a((\chi^\dagger)^{A\alpha}\rho^a_{AB}\hat{\chi}^B_\alpha + (\hat{\chi}^\dagger)^\alpha_A((\rho^a)^{AB})^\dagger \chi_{B\alpha}) \Big] \end{aligned} \tag{5.75}$$

with the σ^i's and ρ^a's being the gamma matrices for $SO(3)$ and $SO(6)$ respectively. We now substitute our specific background and expand in modes,

$$\begin{aligned} L_F \;=\; & \sum_{j=\frac{1}{2}|N_1-N_2|-1/2}^{\frac{1}{2}(N_1+N_2)-3/2} \Big[i\pi^\dagger_{jm}\dot{\pi}_{jm} + i\hat{\pi}^\dagger_{jm}\dot{\hat{\pi}}_{jm} - \frac{1}{3}\left(j+\frac{3}{4}\right)(\pi^\dagger_{jm}\pi_{jm} - \hat{\pi}^\dagger_{jm}\hat{\pi}_{jm}) \\ & \qquad + x^a(\pi^\dagger_{jm}\rho^a\hat{\pi}_{jm} + \hat{\pi}^\dagger_{jm}(\rho^a)^\dagger\pi_{jm}) \Big] \\ & + \sum_{j=\frac{1}{2}|N_1-N_2|+1/2}^{\frac{1}{2}(N_1+N_2)-1/2} \Big[i\eta^\dagger_{jm}\dot{\eta}_{jm} + i\hat{\eta}^\dagger_{jm}\dot{\hat{\eta}}_{jm} - \frac{1}{3}\left(j+\frac{1}{4}\right)(\eta^\dagger_{jm}\eta_{jm} - \hat{\eta}^\dagger_{jm}\hat{\eta}_{jm}) \\ & \qquad + x^a(\eta^\dagger_{jm}\rho^a\hat{\eta}_{jm} + \hat{\eta}^\dagger_{jm}(\rho^a)^\dagger\eta_{jm}) \Big] \end{aligned} \tag{5.76}$$

where again there is an implicit sum over m. This can be diagonalized and contributes to the effective action

$$
\begin{aligned}
V^F_{eff} = \quad & 2\sum_{j=\frac{1}{2}|N_1-N_2|-1/2}^{\frac{1}{2}(N_1+N_2)-3/2}(2j+1)\left(\sqrt{z^2+\left(\frac{\mu}{3}\right)^2(j+\frac{3}{4})^2+v}\right. \\
& \left.+\sqrt{z^2+\left(\frac{\mu}{3}\right)^2(j+\frac{3}{4})^2-v}\right) \\
& +2\sum_{j=\frac{1}{2}|N_1-N_2|+1/2}^{\frac{1}{2}(N_1+N_2)-1/2}(2j+1)\left(\sqrt{z^2+\left(\frac{\mu}{3}\right)^2(j+\frac{1}{4})^2+v}\right. \\
& \left.+\sqrt{z^2+\left(\frac{\mu}{3}\right)^2(j+\frac{1}{4})^2-v}\right)
\end{aligned} \tag{5.77}
$$

There is also the contribution from the ghosts, however, this has no new terms for our choice of background,

$$
V^G_{eff} = 2\sum_{j=\frac{1}{2}|N_1-N_2|}^{\frac{1}{2}(N_1+N_2)-1}(2j+1)\sqrt{z^2+\left(\frac{\mu}{3}\right)^2 j(j+1)} \tag{5.78}
$$

and the total effective action is the sum of the three pieces

$$
V_{eff} = V^B_{eff} + V^F_{eff} + V^G_{eff}. \tag{5.79}
$$

We introduce the variables N and u,

$$
N_1 = N + 2u \qquad N_2 = N, \tag{5.80}
$$

where u will be related to the difference in radii of the two spheres and from now on we will restore α using dimensional analysis. Define

$$
\begin{aligned}
\eta^2_\pm &= z^2 \pm \frac{1}{2}\sqrt{\left(\frac{\alpha\mu}{6}\right)^4 + 16\alpha^2 v^2} \\
\nu^2_\pm &= z^2 \pm \alpha v
\end{aligned} \tag{5.81}
$$

and assuming that $u \geq 1$ (rather than assuming $u \geq 0$ because the lower limit of the first summation in eqn (5.71) has to be non-negative) we can write the effective action so that j always starts from 0 and finishes at $N-1$.

$$
\begin{aligned}
V_{eff} = -\frac{1}{\alpha}\sum_{j=0}^{N-1}\Bigg\{ & (2j+2u-1)\left[z^2+\left(\frac{\alpha\mu}{3}\right)^2(j+u)^2\right]^{\frac{1}{2}} \\
& +(2j+2u+3)\left[z^2+\left(\frac{\alpha\mu}{3}\right)^2(j+u+1)^2\right]^{\frac{1}{2}} \\
& +(2j+2u+1)\left[z^2+\left(\frac{\alpha\mu}{3}\right)^2(j+u)(j+u+1)\right]^{\frac{1}{2}} \\
& +5(2j+2u+1)\left[z^2+\left(\frac{\alpha\mu}{3}\right)^2(j+u+1/2)^2\right]^{\frac{1}{2}} \\
& +(2j+2u+1)\Bigg(\left[\eta_+^2+\left(\frac{\alpha\mu}{3}\right)^2((j+u)(j+u+1)+\frac{1}{8})\right]^{\frac{1}{2}} \\
& \qquad +\left[\eta_-^2+\left(\frac{\alpha\mu}{3}\right)^2((j+u)(j+u+1)+\frac{1}{8})\right]^{\frac{1}{2}}\Bigg) \\
& -2(2j+2u)\Bigg(\left[\nu_+^2+\left(\frac{\alpha\mu}{3}\right)^2(j+u+1/4)^2\right]^{\frac{1}{2}} \\
& \qquad +\left[\nu_-^2+\left(\frac{\alpha\mu}{3}\right)^2(j+u+1/4)^2\right]^{\frac{1}{2}}\Bigg) \\
& -2(2j+2u+2)\Bigg(\left[\nu_+^2+\left(\frac{\alpha\mu}{3}\right)^2(j+u+3/4)^2\right]^{\frac{1}{2}} \\
& \qquad +\left[\nu_-^2+\left(\frac{\alpha\mu}{3}\right)^2(j+u+3/4)^2\right]^{\frac{1}{2}}\Bigg) \\
& -2(2j+2u+1)\left[z^2+\left(\frac{\alpha\mu}{3}\right)^2(j+u)(j+u+1)\right]^{\frac{1}{2}}\Bigg\}
\end{aligned} \tag{5.82}
$$

Let us write the above summation as

$$
V_{eff} = \sum_{j=0}^{N-1}\mathcal{V}(j) \tag{5.83}
$$

and use Euler-Maclaurin sum formula to convert it into integrals

$$V_{eff} = \int_0^N \mathcal{V}(j)dj - \frac{1}{2}[\mathcal{V}(N) + \mathcal{V}(0)] + \frac{1}{12}[\mathcal{V}'(N) - \mathcal{V}'(0)]$$
$$-\frac{1}{720}[\mathcal{V}'''(N) - \mathcal{V}'''(0)] + \cdots \tag{5.84}$$

It is useful to first see what we are expecting from such an integral. From eqn(5.82), we see that a typical term of $\mathcal{V}(j)$ is roughly of the form:

$$\begin{aligned}\mathcal{V}(j) &= \frac{1}{\alpha} j\sqrt{z^2 + \alpha v + \alpha^2\mu^2 j^2} \\ &= \mu N^2 \frac{j}{N}\sqrt{(\frac{j}{N})^2 + \frac{1}{N^2}((\frac{z}{\alpha\mu})^2 + \frac{v}{\alpha\mu^2})}\end{aligned} \tag{5.85}$$

Putting the above form of $\mathcal{V}$ into eqn (5.84) and defining new variables $\gamma = j/N$ and $\zeta = \sqrt{(\frac{z}{\alpha\mu})^2 + \frac{v}{\alpha\mu^2}}$, not keeping track of the exact coefficients, we have:

$$\begin{aligned}V_{eff} &= \mu N^3 \int_0^1 d\gamma\ \gamma\sqrt{\gamma^2 + (\frac{\zeta}{N})^2} + \mu N^2 \left(\sqrt{1 + (\frac{\zeta}{N})^2}\right) \\ &\quad + \mu N \left(\partial_\gamma \left(\gamma\sqrt{\gamma^2 + (\frac{\zeta}{N})^2}\right)\right)\Bigg|_0^1 + \frac{\mu}{N}\left(\partial_\gamma^3 \left(\gamma\sqrt{\gamma^2 + (\frac{\zeta}{N})^2}\right)\right)\Bigg|_0^1 + \cdots \\ &= \mu N^3 \left\{ F_0(\frac{\zeta}{N}) + \frac{1}{N} F_1(\frac{\zeta}{N}) + \frac{1}{N^2} F_2(\frac{\zeta}{N}) + \cdots \right\}\end{aligned} \tag{5.86}$$

where F_n are functions of $\frac{\zeta}{N} = \frac{1}{N}\sqrt{((\frac{z}{\alpha\mu})^2 + \frac{v}{\alpha\mu^2})}$ originating from the n-th derivative of γ (we shall see in the next paragraph why we use $\frac{\zeta}{N}$ as the argument for F_n). Note that each F_n is weighted by a factor $\frac{1}{N^n}$.

First we look at the term F_0. Assuming in a power expansion of $F_0(x)$ there exists a term x^3, after expanding in α it would contribute to V_{eff} a v^4 term of the form $\mu N^3 (\frac{\zeta}{N})^3 \sim \frac{\alpha v^4}{\mu^2 z^5}$, which has the correct form of membrane interactions and has the correct order of N (see section 3.1). If in the power expansion of $F_0(x)$ there also exists a term x^1, then it would contribute to V_{eff} a v^4 term of the form $\mu N^3 (\frac{\zeta}{N}) \sim \frac{N^2\alpha^3 v^4}{z^7}$. This is the same expression as graviton interactions. The details of the coefficients and whether such terms exist in the power expansion of course has to be seen by actually performing the integration, however as we will see later, the integrals do produce terms of the correct interactions, in both the membrane and the graviton limit.

Next we look at F_n for $n > 0$. The whole argument in the last paragraph goes through, except now every term is weighted by an extra factor of $\frac{1}{N^n}$. For example, the membrane-like interaction produced by F_n looks like $\frac{1}{N^n}\frac{\alpha v^4}{\mu^2 z^5}$. This factor of $\frac{1}{N^n}$ means that this term is in fact a matrix theory correction to supergravity, because it vanishes as $N \to \infty$. Therefore, we could see that in converting the summation into a series of integrals, only the first one is needed for comparison with supergravity. All the other F_n with $n > 0$ produces only matrix theory corrections which is not the subject of interest here.

Now we go back to eqn (5.82). As discussed above, we only need the integral part F_0, which we calculate using Mathematica. After calculating the integrals, u is replaced by the supergravity variable $w = \frac{\alpha\mu}{3}u$, and the answer is expanded first in large N, keeping only the leading order (which is order N^0), and then expanded in small α, keeping only up to the $(\alpha)^1$ order (which is the appropriate order for a comparison with supergravity in the membrane limit). Finally the answer is converted back into Minkowski signature by sending $v^2 \to -v^2$. We obtain the following one-loop potential in the membrane limit[4]:

$$V_{eff} = \alpha\left(\frac{(36v^2 - z^2w^2)(108v^2 - (4w^2 + 7z^2)\mu^2)}{1152(w^2 + z^2)^{5/2}\mu^2}\right). \tag{5.87}$$

We find exact agreement when this expression is compared with the supergravity light cone Lagrangian given in eqn (5.62).

[4]This is only the effective potential in the membrane limit because when we expand in large N and keeping only the lowest order we essentially send the radius of the spheres $r_0 \sim \alpha\mu N$ to infinity, thus compare with z we have $\frac{z}{r_0} \ll 1$. Note also that the order the limits are taken is important. If the small α limit is taken before the large N limit, implicitly we would be assuming $\frac{z}{\alpha\mu} \gg N$ which is the graviton limit.

5.3 The Interpolation of the Effective Potential

5.3.1 The Interpolation

The purpose of this section is to find the effective potential that interpolates between the membrane limit and the graviton limit. Due to the complexity of the field equations on the supergravity side, this problem could only be attacked on the matrix theory side. On the matrix theory side, however, there is a subtlety that needs to be taken into account before such a potential could be found. Here we will analyse only the v^4 term, the $\mu^2 v^2$ term as well as μ^4 term can be studied in exactly the same way.

From the supergravity side, what we wish to find is more or less clear. Near the membrane, when $\frac{z}{r_0} \ll 1$, we expect an expansion like:

$$V_{eff} = \frac{\alpha v^4}{\mu^2 z^5}(1 + \frac{z}{r_0} + (\frac{z}{r_0})^2 + (\frac{z}{r_0})^3 + (\frac{z}{r_0})^4 + \cdots) \tag{5.88}$$

Far away from the membrane, when $\frac{z}{r_0} \gg 1$, we expect an expansion:

$$V_{eff} = \frac{\alpha v^4}{\mu^2 z^5}\frac{r_0^2}{z^2}(1 + \frac{r_0}{z} + (\frac{r_0}{z})^2 + (\frac{r_0}{z})^3 + (\frac{r_0}{z})^4 + \cdots) \tag{5.89}$$

We have used $\frac{N^2\alpha^3 v^4}{z^7} = \frac{\alpha v^4}{\mu^2 z^5}\frac{r_0^2}{z^2}$ to rewrite the graviton result so that it looks more similar to the membrane effective potential.

Therefore, we are basically looking for a function $\mathcal{C}(x)$ which appears in the effective potential in the following way:

$$V_{eff} = \frac{\alpha v^4}{\mu^2 z^5}\mathcal{C}(\frac{z}{r_0}) \tag{5.90}$$

As one could see, both the graviton and the membrane action is in this form. $\mathcal{C}(x)$ should have the appropriate limit at $x \to 0$ and $x \to \infty$ to give the correct potential at the membrane and the graviton limit respectively. Physically it represents curvature corrections due to the finite size of the spherical membrane.

So now we go to the matrix theory side to try to find $\mathcal{C}(x)$. The subtlety is that the effective potential on the matrix theory side not only includes curvature corrections but also the matrix theory corrections to supergravity which we are not interested in, not to mention that the sum over mass formula is incapable of deducing such matrix theory corrections exactly [5].

For our purpose of comparing with supergravity, therefore, all matrix theory corrections to supergravity should be thrown away. Such corrections appear on the matrix theory side as $1/N$ corrections, mixed together with the curvature corrections, and it looks something like:

$$V_{eff} = \frac{\alpha v^4}{\mu^2 z^5}\left(\mathcal{C}_0(\frac{z}{r_0}) + \frac{1}{N}\mathcal{C}_1(\frac{z}{r_0}) + \frac{1}{N^2}\mathcal{C}_2(\frac{z}{r_0}) + \cdots\right) \tag{5.91}$$

In such an expansion, only the $\mathcal{C}_0$ term should be kept. The readers are cautioned that naively sending N to infinity will not give us the correct interpolating potential because r_0 also depends on N and such limit would only result in the effective potential in the membrane limit.

There are many ways such matrix corrections could appear. For example, in a typical matrix theory computation we may get terms of the form:

$$V_{eff} \sim \frac{\alpha v^4}{\mu^2 z^7}(z^2 + \alpha^2\mu^2) \tag{5.92}$$

Rewriting the above gives:

$$V_{eff} \sim \frac{\alpha v^4}{\mu^2 z^5}(1 + \frac{\alpha^2\mu^2 N^2}{z^2}\frac{1}{N^2}) \tag{5.93}$$

$$\sim \frac{\alpha v^4}{\mu^2 z^5}(1 + \frac{r_0^2}{z^2}\frac{1}{N^2}) \tag{5.94}$$

The second term could now be identified as a matrix theory correction and is irrelevant to us.

To isolate the curvature corrections (which we want) from the matrix theory corrections (which we do not want), we look at eqn(5.91) more carefully. We could see that since $\frac{1}{N} = \frac{\alpha\mu}{6r_0}$, all the matrix theory corrections will appear in higher order in α.

Therefore to get the interpolating effective potential from the matrix theory side, we could follow the steps below:

1. Change the summation over j in eqn(5.82) into an integral over j from 0 to N;
2. Replace N by $\frac{6r_0}{\alpha\mu}$, and u by $\frac{3w}{\alpha\mu}$;
3. Expand in small α and keeping only the lowest order (which shall turn out to be order α^1, with all lower orders vanishing). This is the interpolating effective potential.

With Mathematica, the interpolating effective potential for two spheres of the same radius ($w = 0$) in Minkowski signature is found to be:

$$\begin{aligned}V_{eff} = {} & \frac{\alpha}{\mu^2 z^5}\frac{36v^2 - z^2\mu^2}{1152(4r_0^2 + z^2)^{5/2}} \\ & \times \Big\{ 108v^2\Big(- z^5 + 16r_0^4\sqrt{4r_0^2 + z^2} + 8r_0^2 z^2\sqrt{4r_0^2 + z^2} + z^4\sqrt{4r_0^2 + z^2}\Big) \\ & - z^2\mu^2\Big(112r_0^4\sqrt{4r_0^2 + z^2} + 7z^4(-z + \sqrt{4r_0^2 + z^2}) + 8r_0^2 z^2(-2z + 7\sqrt{4r_0^2 + z^2})\Big)\Big\}\end{aligned} \tag{5.95}$$

We see that V_{eff} always carries a factor of $36v^2 - z^2\mu^2$, meaning the effective potential vanishes whenever $v = \frac{\mu z}{6}$. This is expected because such configurations correspond to circular orbits which preserve half of the supersymmetries.

Expanding this potential in the membrane limit of large r_0 we get:

$$V_{eff} = \frac{\alpha(3888v^4 - 360v^2 z^2\mu^2 + 7z^4\mu^4)}{1152\mu^2 z^5} = \frac{\alpha(36v^2 - z^2\mu^2)(108v^2 - 7z^2\mu^2)}{1152\mu^2 z^5} \tag{5.96}$$

This result is of course identical to the matrix theory result (with $w = 0$) in section 5.2 where the membrane limit was taken in advance.

In the graviton limit of small r_0, after replacing r_0 by $\alpha\mu N/6$, we have:

$$V_{eff} = \frac{N^2\alpha^3(720v^4 - 56v^2 z^2\mu^2 + z^4\mu^4)}{768z^7} = \frac{N^2\alpha^3(20v^2 - z^2\mu^2)(36v^2 - z^2\mu^2)}{768z^7} \tag{5.97}$$

The two limits of the effective action could then be compared with that of the supergravity side. Indeed from the expressions (5.63) and (5.64) we see that we have perfect agreement.

In above we have only given the expression of the interpolating potential for two membranes with the same radius ($w = 0$). We have also found the interpolating potential with w included, using the same steps given above. However we choose to omit the rather lengthy expression here for brevity.

5.3.2 Comparison with Shin and Yoshida

As mentioned in section 5, ref. [31], considered the case of supersymmetric circular motion with angular frequency $\frac{\mu}{6}$ and found a flat potential, which agrees with what we have found (see the comment under eqn(5.95)).

In a subsequent paper, [32], the authors considered the case of a slightly elliptical orbit with separation, z,

$$\begin{aligned} z &= \sqrt{(r_2+\epsilon)^2\cos^2\left(\frac{\mu t}{6}\right)+(r_2-\epsilon)^2\sin^2\left(\frac{\mu t}{6}\right)} \\ &= \sqrt{r_2^2+\epsilon^2+2r_2\epsilon\cos\left(\frac{\mu t}{3}\right)} \end{aligned} \tag{5.98}$$

and velocity, v,

$$\begin{aligned} v &= \frac{\mu}{6}\sqrt{\left((r_2+\epsilon)^2\sin^2(\frac{\mu t}{6})+(r_2-\epsilon)^2\cos^2\left(\frac{\mu t}{6}\right)\right)} \\ &= \frac{\mu}{6}\sqrt{\left((r_2^2+\epsilon^2-2r_2\epsilon\cos\left(\frac{\mu t}{3}\right)\right)}. \end{aligned} \tag{5.99}$$

where ϵ is the small expansion parameter for the eccentricity of the orbit. They considered the large separation limit, $r_2 \gg 0$, and found an effective action, eqn(1.2) of [32],

$$\begin{aligned}\Gamma_{eff} &= \epsilon^4 \int dt\, (\alpha^3\mu^4 N^2)\left(\frac{35}{2^7\cdot 3}\frac{1}{r_2^7} - \frac{385(\alpha\mu N)^2}{2^{11}\cdot 3^3}(4-\frac{1}{N^2})\frac{1}{r_2^9}\right)\\ &= \frac{35}{384}\epsilon^4 \int dt\, (\alpha^3\mu^4 N^2)\left(\frac{1}{r_2^7} - \frac{11(\alpha\mu N)^2}{36}(1-\frac{1}{4N^2})\frac{1}{r_2^9}\right) \qquad (5.100)\end{aligned}$$

after expanding to $O(\epsilon^4)$ and $O(1/r_2^9)$. Note that in the equation above we have restored μ and α, and set $N_1 = N_2 = N$, $r_1 = 0$. To compare this with our result we substitute the above expressions of $z(t)$ and $v(t)$ into our interpolating potential eqn (5.95) and expand in the parameters $1/r_2$ and ϵ. As we are only comparing effective actions we average our potential over one period of oscillation. We find for our time-averaged potential,

$$\begin{aligned}V_{eff} &= \alpha\epsilon^4\mu^2\frac{105}{32}r_0^2\left(\frac{1}{r_2^7} - 11\frac{r_0^2}{r_2^9}\right)\\ &= \frac{35}{384}\alpha^3\epsilon^4\mu^4 N^2\left(\frac{1}{r_2^7} - \frac{11(\alpha\mu N)^2}{36}\frac{1}{r_2^9}\right) \qquad (5.101)\end{aligned}$$

(where to reach the final line we used $r_0 = \frac{\alpha\mu N}{6}$) which agrees with eqn (5.100) after throwing away the $\frac{1}{N^2}$ matrix theory corrections in the latter. In calculating the matrix theory effective potential in section 5.2 we assumed a constant velocity. However, as we can see by this comparison, as long as we ignore matrix theory corrections, it leads to the correct result. Thus we see that we can consistently neglect acceleration terms in the effective potential as discussed earlier.

Chapter 6

Membrane Interaction in PP-wave with M-momentum Transfer

In this chapter, we are interested in comparing the effective actions of matrix theory and supergravity arising from interactions between two membranes in the eleven-dimensional pp-wave geometry. In particular, we would like to allow for M-momentum transfer, and check the agreement on both sides. This computation is similar in spirit to the previous chapters, though the details and techniques involved on the matrix theory side are very different.

Unfortunately, the complexity of the equations involved has so far prevented us from obtaining a direct comparison between matrix theory and supergravity. Nevertheless, limited predictions can be made on both sides under certain approximations, this chapter is a collection of the results so far.

In section 6.1 we will give a description of the theories on both sides. The gauge theory will be presented in two equivalent descriptions we call the A-formalism and the Y-formalism. In section 6.2 one can find the vacuum solutions of the gauge theory. A precise correspondence of the two theories is given, mapping variables on the gauge theory side to those on the supergravity side. The magnetic flux on the gauge theory side is identified with the total momentum of the membranes on the supergravity side. The $\mu \to 0$ limit will also be discussed. As before it is necessary to carefully distinguish the graviton limit and the flat membrane limit. Section 6.3 gives the simplest instanton solution of the Euclideanized gauge theory. Connections with matrix theory are also discussed. Section 6.4 describes our attempts in finding higher instantons of the theory. A particularly interesting point is how the self dual equations in pp-wave can be mapped to those in a conformally flat space. Section 6.5 looks at the supersymmetries of the Lagrangian and examines the BPS conditions. In section 6.7 the interaction between a source membrane at the origin and a probe membrane in circular orbit is studied. Although the precise computations cannot be carried out because of the lack of the general instanton solution, a limited prediction can be made under certain approximations, and awaits confirmation by the supergravity side. In section 6.8 we present the results from the supergravity side so far. This is followed by brief comments on the limitations of our results and the current status of the project in section 6.9. Our notations and some frequently used equations are collected in Appendix A.

6.1 The Gauge Theory

Now we will describe the gauge theory we use to compute the effective action. There are many approaches to get the non-Abelian three-dimensional theory for our instanton computations. One is to take the Abelian supermembrane action in pp-wave [13] for a single membrane and generalize to a non-Abelian theory using the constraints from supersymmetry and gauge symmetry. Here we take a different route worked out in [28] by taking the continuum limit of the matrix theory action. A small advantage of this method is that the coupling constant of the resulting three-dimensional gauge theory is readily identified with M-theory parameters as we will see below.

Once again we begin with the matrix theory action in pp-wave:

$$\mathcal{S} = \int dt Tr \Bigg\{ \sum_{A=1}^{9} \frac{1}{2R}(D_0 X^A)^2 + i\psi^T D_0 \psi + \frac{(M^3R)^2}{4R} \sum_{A,B=1}^{9} [X^A, X^B]^2$$
$$+ (M^3 R) \sum_{B=1}^{9} \psi^T \gamma^B [X^B, \psi] + \frac{1}{2R}\left(-(\frac{\mu}{3})^2 \sum_{i=1}^{3} (X^i)^2 - (\frac{\mu}{6})^2 \sum_{a=4}^{9} (X^a)^2 \right) - i\frac{\mu}{4}\psi^T \gamma_{123}\psi$$
$$- \frac{(M^3R)\mu}{3R} i \sum_{i,j,k=1}^{3} \epsilon_{ijk} X^i X^j X^k \Bigg\} \tag{6.1}$$

where $D_t X = \partial_t X^A - i[X_0, X^A]$. From this point onwards we will put $M = R = 1$ unless stated otherwise.

A vacuum solution of this matrix theory action is $X_i = \frac{\mu}{3} J_i$, where J_i is a representation of the $SU(2)$ algebra. The three-dimensional gauge theory can be obtained by expanding about this vacuum. First we define:

$$\tau = \frac{\mu}{3} t \tag{6.2}$$
$$X_i = \frac{\mu}{3} J_i + Y_i \tag{6.3}$$

Next we do the following replacement:

$$[J_i, Z] \quad \rightarrow i\{x_i, Z\} = -i\epsilon_{ijk} x_j \partial_k Z \tag{6.4}$$
$$Tr \quad \rightarrow N \int d^2\Omega \tag{6.5}$$
$$Y_i \quad \rightarrow \sqrt{\frac{3}{\mu N}} Y_i = \sqrt{\frac{1}{2r_0}} Y_i = \sqrt{\frac{\pi \mu p}{3}} Y_i \tag{6.6}$$
$$X_a \quad \rightarrow \sqrt{\frac{3}{\mu N}} X_a = \sqrt{\frac{1}{2r_0}} X_a = \sqrt{\frac{\pi \mu p}{3}} X_a \tag{6.7}$$
$$\psi \quad \rightarrow \frac{1}{\sqrt{N}} \psi \tag{6.8}$$

The last three rescaling of the variables are done for later convenience. The Poisson bracket is defined in Appendix A, and so is x_i, which is the Cartesian coordinates parametrizing a unit sphere in R^3. The variable Z that appears in one of the above relations denote a general matrix function, such as X and Y. Note also that the momentum along the X^- direction (the M-momentum) $P_- = N/R = N$ once we put $R = 1$, and from now on we will use P exclusively in this chapter instead of N. We will often omit the subscript "-" when referring to the M-momentum.

After all these steps, one obtains the following three-dimensional action:

$$\mathcal{S} = \int d\tau d^2\Omega \; Tr\left\{ \frac{1}{2}(\partial_\tau Y_i)^2 + \frac{1}{2}(\partial_\tau X_a)^2 - \frac{1}{8}X_a^2 - \frac{1}{2}(Y_i + i\epsilon_{ijk}L_jY_k + \frac{i}{2}g\epsilon_{ijk}[Y_j, Y_k])^2 \right.$$
$$\left. + \frac{1}{2}(\mathcal{L}_iX_a)^2 + \frac{1}{4}g^2[X_a, X_b]^2 + i\psi^T\partial_\tau\psi - i\frac{3}{4}\psi^T\gamma_{123}\psi + g\psi^T\gamma_a[X_a, \psi] + \psi^T\gamma_i\mathcal{L}_i\psi \right\} \tag{6.9}$$

Here $d^2\Omega$ is the solid angle element on the sphere. The operators L_i and $\mathcal{L}_i$ are defined by $L_iZ = i\{x_i, Z\} = -i\epsilon_{ijk}x_j\partial_kZ$ and $\mathcal{L}_iZ = L_iZ + g[Y_i, Z]$ and g is given in terms of M-theory parameters as $g = \frac{3^{3/2}}{\mu^{3/2}P^{1/2}} = 2^{1/2}\pi p\sqrt{r_0} = \frac{3}{2^{1/2}\mu\sqrt{r_0}} = 3^{1/2}\pi^{1/2}\sqrt{\frac{p}{\mu}}$. Here $r_0 \equiv \frac{\mu}{6}P$ and $p \equiv \frac{P}{4\pi r_0^2}$ are the radius and the momentum density of the spherical membrane respectively. From now on we will omit the symbol Tr for trace, which should be clear in the context.

Since we are interested in two-membrane interactions, from now on we will restrict our attention to the gauge group $U(2)$. All the fields appearing in the above action are treated as 2×2 matrices, or equivalently they can be expanded in terms of the generators of $SU(2)$, appearing below as $\frac{\sigma_m}{2}$ in addition to the $U(1)$ component. Just as it was in our previous graviton computation, we keep the $U(1)$ part so as to allow for configurations where the center of mass of the system is not located at the origin.

It is convenient to define $\mathcal{B}_i = Y_i + i\epsilon_{ijk}L_jY_k + \frac{i}{2}g\epsilon_{ijk}[Y_j, Y_k]$. As we will see later, the right-hand side becomes the magnetic field $B_i = \frac{1}{2}\epsilon_{ijk}F_{jk}$ in the flat membrane limit.

This action has 16 supersymmetries and a gauge symmetry under the following transformation:

$$Y \to U^{-1}YU + \frac{1}{g}U^{-1}LU \tag{6.10}$$
$$X \to U^{-1}XU \tag{6.11}$$
$$\psi \to U^{-1}\psi U \tag{6.12}$$

The above gauge theory has an unconventional form, which we shall refer to as the Y-formalism. We shall note in passing that restoring A_τ is an easy task, but we would not go into it at this point. It will be restored below in the A-Formalism. Although the fields Y do not transform exactly as gauge fields do, it is possible to make a field redefinition to make it look like a proper gauge theory. To this end we define:

$$Y_i = x_i\Phi + \epsilon_{ijk}x_jA_k \tag{6.13}$$

Direct substitution gives:

$$i\mathcal{L}_iX = ix_ig[\Phi, X] + \epsilon_{ijk}x_jD_kX \tag{6.14}$$
$$\mathcal{B}_i = x_i(\frac{1}{2}\epsilon_{jkl}x_jF_{kl} - \Phi) + D_i\Phi = x_i(\csc\theta F_{\theta\phi} - \Phi) + D_i\Phi \tag{6.15}$$

where $D_iZ = \partial_iZ + ig[A_i, Z]$ and $F_{ij} = \partial_iA_j - \partial_jA_i - ig[A_i, A_j]$

After restoring A_τ, the bosonic action in this A-formalism is now given by:

$$\begin{aligned}\mathcal{S} = \int d\tau d^2\Omega\frac{1}{2}\Big\{ & F_{\tau\theta}^2 + \csc^2\theta F_{\tau\phi}^2 - (\csc\theta F_{\theta\phi} - \Phi)^2 \\ & + (D_\tau\Phi)^2 - (D_\theta\Phi)^2 - \csc^2\theta(D_\phi\Phi)^2 \\ & + (D_\tau X_a)^2 - (D_\theta X_a)^2 - \csc^2\theta(D_\phi X_a)^2 \\ & - \frac{1}{4}X_a^2 + \frac{1}{2}g^2[X_a, X_b]^2 + g^2[\Phi, X_a]^2\Big\}\end{aligned} \tag{6.16}$$

If one takes away the mass term for X as well as the Φ term in $(\csc\theta F_{\theta\phi} - \Phi)^2$, this is the action one would naively expect on $R \times S^2$, with various factors of $csc\theta$

coming from the metric on S^2. The inclusion of these terms, however, as we will see later when discussing the instanton equations, actually makes it possible to map the problem to an equivalent one in flat space. The term $(\csc\theta F_{\theta\phi} - \Phi)^2$ is an interesting one. It contributes a factor of $F^2_{\theta\phi}$ to the action just as for a flat space gauge theory, but at the same time plays the role of a Higgs potential fixing the field Φ at infinity. There are no free parameters as those in a usual Higgs potential, and the theory is supersymmetric. It may be of interest to consider the effect of this term in greater details.

In finding the instanton solutions below, we will first put $X_a = 0$. Ignoring the terms dependent on X for now, the above bosonic action can be rewritten once more to a more suggestive form in one higher dimension, similar to how a three-dimensional instanton can be written as a solution of a four-dimensional theory. To do this we first introduce a fictitious coordinate x^Φ on which none of the fields depends. We define the following *four-dimensional* metric on $R^{1,1} \times S^2$:

$$ds^2 = -d\tau^2 + d\theta^2 + sin^2\theta d\phi^2 + (dx^\Phi)^2 \equiv g_{mn} dx^m dx^n \tag{6.17}$$

m, n labels the coordinates of $R^{1,1} \times S^2$.

Defining $A_\Phi = \Phi$ and $\mathcal{F}_{mn} = F_{mn} - \varepsilon_{\tau\Phi mn}\Phi$, with F_{mn} being the usual field strength $F_{mn} = \partial_m A_n - \partial_n A_m + ig[A_m, A_n]$ and $\varepsilon_{\tau\Phi\theta\phi} = +\sqrt{|g|} = sin\theta$, we can rewrite the X-independent part of the bosonic action simply as [1]:

$$\mathcal{S} = \int d\tau d^2\Omega \left\{ -\frac{1}{4}\mathcal{F}_{mn}\mathcal{F}^{mn} \right\} \tag{6.18}$$

6.2 The Vacuum Solutions

We have obtained the three-dimensional gauge theory in both the A-formalism and the Y-formalism. In this section we will study its vacuum solutions and give their interpretations on the eleven-dimensional supergravity picture.

[1] In particular, we have $F_{\Phi m} = ig[\Phi, A_m]$.

We first look at the Y-formalism. From the action in eqn(6.9) we see that the stationary vacuum solution must satisfy the following conditions:

$$X_a = 0 \tag{6.19}$$
$$\mathcal{B}_i = Y_i + i\epsilon_{ijk}L_jY_k + \frac{i}{2}g\epsilon_{ijk}[Y_j, Y_k] = 0 \tag{6.20}$$

In the absence of A_τ, of course all the fields have to be time independent. Putting $Y = 0$ gives you the trivial vacuum, but as we will see below, the general vacuum is labeled by an integer n.

6.2.1 The $n = 1$ Hedgehog Vacuum

We will begin with a simple ansatz, $Y_i \propto \sigma_i$. Direct substitution gives:

$$Y_i = \frac{1}{g}(\frac{\sigma_i}{2}) \tag{6.21}$$

It is instructive to look at Φ in the A-formalism:

$$\Phi = x_iY_i = \frac{1}{g}x_i(\frac{\sigma_i}{2}) \tag{6.22}$$

We shall call this the hedgehog vacuum for the obvious reasons.

6.2.2 The general n Vacua

The general solutions are found by means of an ansatz. I will merely state the results here:

$$\begin{aligned}
Y_1 &= \frac{1}{g}[n\cos\phi\cos n\phi + \sin\phi\sin n\phi](\frac{\sigma_1}{2}) + \frac{1}{g}[n\cos\phi\sin n\phi - \sin\phi\cos n\phi](\frac{\sigma_2}{2}) \\
Y_2 &= \frac{1}{g}[n\sin\phi\cos n\phi - \cos\phi\sin n\phi](\frac{\sigma_1}{2}) + \frac{1}{g}[n\sin\phi\sin n\phi + \cos\phi\cos n\phi](\frac{\sigma_2}{2}) \\
Y_3 &= \frac{n}{g}(\frac{\sigma_3}{2})
\end{aligned} \tag{6.23}$$

Through a finite gauge transformation $U = \cos(\theta/2) - i\sin(\theta/2)(\sin(n\phi)\sigma_1 -$

$\cos(n\phi)\sigma_2)$, the above solution can be transformed into a *singular gauge* where all fields point along the σ_3 direction in $SU(2)$:

$$Y_1 = \frac{n}{g}\frac{1-\cos\theta}{\sin\theta}\cos\phi(\frac{\sigma_3}{2}) \tag{6.24}$$

$$Y_2 = \frac{n}{g}\frac{1-\cos\theta}{\sin\theta}\sin\phi(\frac{\sigma_3}{2}) \tag{6.25}$$

$$Y_3 = \frac{n}{g}(\frac{\sigma_3}{2}) \tag{6.26}$$

This gauge is evidently singular at $\theta \to \pi$, so in fact requires a second cover on the lower hemisphere with $\frac{1-\cos\theta}{\sin\theta}$ replaced by $\frac{1+\cos\theta}{\sin\theta}$.

In the A-formalism, we have:

$$\Phi = \frac{n}{g}(\frac{\sigma_3}{2}) \tag{6.27}$$

In addition, for the nth vacuum we can see immediately that in this gauge, A is also proportional to σ_3 everywhere (except at the south pole).

As we have seen in the previous section, the fields Y can be mapped to the gauge fields A, which we will not do explicitly here. One should note for the sake of our considerations on instantons later, although it is always possible to go to a singular gauge where Φ always points in the σ_3 direction, it is in general impossible to make both Φ and A to be in the σ_3 direction at all times.

6.2.3 The Interpretations of the Vacua

From the definition $X = \frac{\mu}{3}J + Y$ and $\Phi = x_i Y_i$, one sees that Φ is related to the radial separation of the membrane. It is easiest to see this in the singular gauge in eqn(6.27) where Φ is diagonal. There the two diagonal elements can be interpreted as the radial perturbation of the two membranes away from the mean radius r_0, meaning that we have one membrane at $r_0 - \frac{n}{2g\sqrt{2r_0}}$ and the second one at $r_0 + \frac{n}{2g\sqrt{2r_0}}$. The extra factor of $g\sqrt{2r_0}$ here comes from the rescaling from eqn(6.6) to eqn(6.8).

Hence we find the radial separation of the two membranes to be $\frac{n}{g\sqrt{2r_0}}$. On the supergravity side, one can show easily that two spherical membranes carrying total M-momentum P_1 and P_2 have radii $\frac{\mu}{6}P_1$ and $\frac{\mu}{6}P_2$ respectively and therefore a radial separation of $\frac{\mu}{6}\Delta P$, where $\Delta P = P_1 - P_2$. Equating the two, we have:

$$\frac{\mu}{6}\Delta P = \frac{n}{g\sqrt{2r_0}} \tag{6.28}$$

But $g\sqrt{2r_0} = 3/\mu$, so we arrive at the important relation:

$$2n = \Delta P \tag{6.29}$$

In other words, in the supergravity picture, $2n$ denotes the difference in total momentum of the two membranes[2].

Through the relation $0 = x_i \mathcal{B}_i = \Phi - F_{\theta\phi}$, we can also relate n to the field strength and hence the total flux on the sphere. Putting it together, the integer n that labels the vacua represents on the gauge theory side the magnetic flux on the sphere, while on the supergravity side carries the meaning of $\Delta P/2$, the averaged difference in total momentum of the two membranes. Therefore, the process of M-momentum transfer between two membranes in eleven dimension is represented on the gauge theory side by an instanton process through which the magnetic flux changes. For example, an instanton that takes us from the n_- vacuum to the n_+ vacuum represents the process where the radial separation of the two membranes changes from $\frac{\mu}{3}n_-$ to $\frac{\mu}{3}n_+$.

6.2.4 The Flat Space Limit

Now that we have the correct interpretations of the various fields and parameters, this is a good place to discuss the flat space limit $\mu \to 0$. As stated in the previous subsection, the radial separation of the two spherical membranes for finite μ is given

[2]The reason that it is $2n$ instead of n is that we implicitly fixed the total momentum $\Sigma P \equiv P_1 + P_2$ of the system to be even when we expand the theory around a configuration of $X^i = \left(\begin{array}{c|c} J^i & 0 \\ \hline 0 & J^i \end{array}\right)$, implying that ΣP, which equals the dimension of the matrix, is even. Therefore $\Delta P = P_1 - P_2 = 2P_1 - \Sigma P$ must be even too. Another way to say this is to begin with two membranes of the same momentum. Each time one unit of momentum is transfered from one to the other, the difference between the momentum increases by two units.

by:

$$\Delta r = \frac{\mu}{6}\Delta P = \frac{\mu}{3}n \tag{6.30}$$

Therefore, to get to the limit where the two flat membranes are separated by a finite distance, we have to take the limit $\mu \to 0$ while keeping μn fixed. We will call this the *flat membrane limit*.

It is useful to understand this another way. Suppose the two membranes carry total momentum P_1 and P_2, then the radii are $r_1 = \frac{\mu}{6}P_1$ and $r_2 = \frac{\mu}{6}P_2$ respectively. The above limit is taken such that $\Delta r = r_1 - r_2$ is fixed. Of course a flat membrane is simply a spherical membrane of infinite radius, so both P_1 and P_2 have to go to infinity to make r_1 and r_2 infinite. For a flat membrane, a useful parameter is the *momentum density*:

$$p \equiv \frac{P}{4\pi r^2} = \frac{3}{2\pi\mu r} \tag{6.31}$$

The difference in p of the two membranes to the lowest order is simply:

$$\Delta p = p\frac{\Delta r}{r} \tag{6.32}$$

So we see that Δp goes to zero as $r \to \infty$ in the flat membrane limit. In other words, the two membranes with finite separation in flat space carry identical momentum density p. This is expected for two membranes at rest in flat space, and of course we can change p by boosting one of the membranes.

To summarize, the flat membrane limit takes $\mu \to 0$ and $r, P, \Delta P, n \to \infty$ while keeping $\Delta r, \mu n, \mu\Delta P, p$ fixed.

Another interesting flat space limit is taking $\mu \to 0$ while keeping the total momentum P fixed. In this case, we can see that each spherical membrane collapses into a point particle sitting at the origin. The momentum density becomes infinite. These point particles can be interpreted as the usual point gravitons, or D0 branes in the IIA perspective. We call this the *graviton limit*. It would be interesting to see if the results of the three-dimensional gauge theory would reduce to those for point-like particle in this limit. For instance, it may be possible to use this limit to find graviton scattering in flat space with M-momentum transfer, which as far as I can tell has not yet been done. In this thesis, however, we will be concerned with the flat membrane limit only.

6.3 The Instanton

6.3.1 The Instanton Equations

We begin by Euclideanization of the action. For now we will only be interested in configurations with $X_a = 0$, so we will retain in this section only the relevant terms. We begin with the Y-formalism.

Replacing $\tau \to -i\tau$ and defining $\mathcal{S}_E = -i\mathcal{S}$, we have:

$$\mathcal{S}_E = \int d\tau d^2\Omega \frac{1}{2}\{(\partial_\tau Y_i)^2 + \mathcal{B}_i^2\} \tag{6.33}$$

As before, $\mathcal{B}_i = Y_i + i\epsilon_{ijk}L_jY_k + \frac{i}{2}g\epsilon_{ijk}[Y_j, Y_k]$.
Rearranging the terms gives:

$$\mathcal{S}_E = \int d\tau d^2\Omega \{\frac{1}{2}(\pm\partial_\tau Y_i + \mathcal{B}_i)^2 \mp \mathcal{B}_i\partial_\tau Y_i\} \tag{6.34}$$

$\mathcal{B}_i\partial_\tau Y_i$ can be written as a total time derivative:

$$\mathcal{B}_i\partial_\tau Y_i = \partial_\tau\{\frac{1}{2}Y_i^2 + \frac{i}{2}\epsilon_{ijk}Y_iL_jY_k + \frac{i}{3}g\epsilon_{ijk}Y_iY_jY_k\} \tag{6.35}$$

Defining $K(\tau) = \int d^2\Omega \left\{\frac{1}{2}Y_i^2 + \frac{i}{2}\epsilon_{ijk}Y_iL_jY_k + \frac{i}{3}g\epsilon_{ijk}Y_iY_jY_k\right\}$, we have:

$$\mathcal{S}_E = \frac{1}{2}\int d\tau d^2\Omega(\pm\partial_\tau Y_i + \mathcal{B}_i)^2 \mp [K(+\infty) - K(-\infty)] \tag{6.36}$$

The instanton equation is therefore given by:

$$\pm\partial_\tau Y_i + \mathcal{B}_i = 0 \tag{6.37}$$

In the A-formalism, Euclideanization gives:

$$\mathcal{S}_E = \frac{1}{2}\int d\tau d^2\Omega\{F_{\tau\theta}^2 + \csc^2\theta F_{\tau\phi}^2 + (\csc\theta F_{\theta\phi} - \Phi)^2 + (D_\tau\Phi)^2 + (D_\theta\Phi)^2 + \csc^2\theta(D_\phi\Phi)^2\} \tag{6.38}$$

Alternatively, Euclideanizing the metric on $R^{1,1} \times S^2$ in eqn(6.17) gives a four-dimensional metric on $R^2 \times S^2$:

$$ds^2 = d\tau^2 + d\theta^2 + sin^2\theta d\phi^2 + dx^{\Phi^2} \equiv g_{mn}dx^m dx^n \tag{6.39}$$

Having $\mathcal{F}_{mn} = F_{mn} - \varepsilon_{\tau\Phi mn}\Phi$ as before [3], the Euclidean action becomes:

$$\mathcal{S}_E = \frac{1}{4}\int d\tau d^2\Omega \mathcal{F}_{mn}\mathcal{F}^{mn} \tag{6.40}$$

Defining $\tilde{\mathcal{F}}_{mn} = \frac{1}{2}\varepsilon_{mnpq}\mathcal{F}^{pq}$, the action can be rewritten as:

$$\mathcal{S}_E = \frac{1}{8}\int d\tau d^2\Omega\left\{(\pm\mathcal{F}_{mn} + \tilde{\mathcal{F}}_{mn})(\pm\mathcal{F}^{mn} + \tilde{\mathcal{F}}^{mn}) \mp \varepsilon_{mnpq}\mathcal{F}^{mn}\mathcal{F}^{pq}\right\} \tag{6.41}$$

Simplifying the second term:

$$\begin{aligned} \mp\frac{1}{8}\int d\tau d^2\Omega\ \varepsilon_{mnpq}\mathcal{F}^{mn}\mathcal{F}^{pq} &= \mp\frac{1}{2}\int d\tau d\theta d\phi\{D_\tau\Phi(F_{\theta\phi} - \sin\theta\Phi) + F_{\tau\theta}D_\phi\Phi + F_{\phi\tau}D_\theta\Phi\} \\ &= \mp\int d\tau d^2\Omega D_\tau(\frac{1}{4}\Phi^2) \end{aligned} \tag{6.42}$$

Note that in the first line we canceled the factor of $\sin\theta$ in $d^2\Omega$ against the $1/\sin\theta$ from the integrand to give $d\theta d\phi$. Integration by part and the Bianchi identity for F_{mn} was used to arrive at the second line, while keeping track of all the boundary conditions carefully. In the end the $\sin\theta$ in front of Φ restore the measure to $d^2\Omega$.

In analogy with the Y-formalism, we define $K(\tau) = \frac{1}{4}\int d^2\Omega\ Tr\Phi^2$. The action

[3] Remember that ε_{mnpq} contains a factor of $\sqrt{|g|}$.

can now be written as:

$$\mathcal{S}_E = \int d\tau d^2\Omega \left\{ \frac{1}{2}(\pm\mathcal{F}_{mn} + \tilde{\mathcal{F}}_{mn})(\pm\mathcal{F}^{mn} + \tilde{\mathcal{F}}^{mn}) \right\} \mp [K(+\infty) - K(-\infty)] \tag{6.43}$$

Defining $\Phi_\pm$ by the relation $\Phi(\tau \to \pm\infty) = \Phi_\pm(\frac{\sigma}{2})$, the instanton that takes the n_- vacuum to the n_+ vacuum has an action:

$$\begin{aligned} S_0 &= \mp[K(+\infty) - K(-\infty)] &(6.44)\\ &= \frac{\pi}{2}|\Phi_+^2 - \Phi_-^2| &(6.45) \end{aligned}$$

Putting in $\Phi_\pm = \frac{n_\pm}{g}$, we have:

$$S_0 = \frac{\pi}{2g^2}|n_+^2 - n_-^2| \tag{6.46}$$

The instanton equations follow:

$$\pm\mathcal{F}_{mn} + \tilde{\mathcal{F}}_{mn} = 0 \tag{6.47}$$

In terms of Φ and F_{mn} (picking the upper sign for convenience):

$$\begin{aligned} \csc\theta F_{\theta\phi} - \Phi &= -D_\tau\Phi &(6.48)\\ F_{\tau\theta} &= -\csc\theta D_\phi\Phi &(6.49)\\ \csc\theta F_{\phi\tau} &= -D_\theta\Phi &(6.50) \end{aligned}$$

6.3.2 The $\mathbf{n = 1 \to n = 0}$ Instanton

A particularly simple instanton can be found by the ansatz:

$$Y_i(\tau) = f(\tau)(\frac{1}{g}\frac{\sigma_i}{2}) \tag{6.51}$$

Choosing the upper sign in the instanton equation, we have:

$$f(\tau) = \frac{1}{\exp(\tau - \tau_0) + 1} \tag{6.52}$$

τ_0 is an integration constant.

This instanton takes us from the $n = 1$ vacuum to the $n = 0$ vacuum. We call this the $(1, 0)$ instanton (see Appendix A for notations). In the eleven-dimensional point of view, it represents two spherical membranes of different radius exchanging one unit of M-momentum resulting in two overlapping membranes with the same radius.

6.3.3 Connection with Matrix Theory

First we write out the instanton equations of our three-dimensional theory in the Y-formalism in full:

$$\pm\partial_\tau Y_i + Y_i + i\epsilon_{ijk} L_j Y_k + \frac{i}{2} g\epsilon_{ijk}[Y_j, Y_k] = 0 \tag{6.53}$$

Compare this with the instanton equation for matrix theory in pp-wave [43] (rewritten in our convention after the field rescaling in eqn(6.6)):

$$\pm\partial_\tau X_i + X_i + \frac{i}{2} g\epsilon_{ijk}[X_j, X_k] = 0 \tag{6.54}$$

One can hardly fail to notice the resemblance and this is of course no accident. Expanding X about its vacuum solution as $X = \frac{1}{g} J + Y$, eqn(6.54) gives[4]:

$$\pm\partial_\tau Y_i + Y_i + i\epsilon_{ijk} J_j Y_k + \frac{i}{2} g\epsilon_{ijk}[Y_j, Y_k] = 0 \tag{6.55}$$

This is identical with the three-dimensional instanton equation after the replacement:

$$[J_j, Y_k] \to i\{x_j, Y_k\} = L_j Y_k \tag{6.56}$$

[4]The factor of $\frac{1}{g}$ in front of J is due to the field rescaling in eqn(6.6).

This is of course the same procedure as taking the continuum limit. It is worthwhile to understand this procedure in more details. We will see shortly that it gives us a new matrix theory instanton that is related to our three-dimensional (1,0) instanton.

Let us begin with a $k \times k$ matrix Y in the three-dimensional theory. Each element of this matrix can be expanded in terms of spherical harmonics $\mathcal{Y}^{[lm]}(\theta, \phi)$:

$$Y_{\alpha\beta}(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} c_{\alpha\beta}^{[lm]} \mathcal{Y}^{[lm]}(\theta, \phi) \tag{6.57}$$

$\alpha, \beta = 1, 2, \cdots, k$ labels the elements of the matrix Y.

This Y of the $U(k)$ three-dimensional theory can be mapped to a $U(Nk)$ theory in one dimension the usual way, by replacing each of the spherical harmonics by their $N \times N$ matrix representation:

$$\mathcal{Y}^{[lm]}(\theta, \phi) \to \mathcal{Y}_{\mu\nu}^{[lm]} \tag{6.58}$$

$\mu, \nu = 1, 2, \cdots, N$ labels the elements of the $N \times N$ matrix $\mathcal{Y}^{[lm]}$. Note also that l now only goes up to $N - 1$, in other words $\mathcal{Y}^{[lm]}$ for $l = 0, 1, \cdots, N - 1$ forms a complete basis for all $N \times N$ matrices.

The matrices $\mathcal{Y}_{\mu\nu}^{[lm]}$ can be constructed by writing the spherical harmonic $\mathcal{Y}^{[lm]}(\theta, \phi)$ in terms of x_i on a sphere in R^3 and then replacing every x_i by the $N \times N$ representation of $J_{\mu\nu}^i$.

Y in the $U(Nk)$ theory is now:

$$Y_{(\alpha\mu)(\beta\nu)} = \sum_{l=0}^{N-1} \sum_{m=-l}^{+l} c_{\alpha\beta}^{[lm]} \mathcal{Y}_{\mu\nu}^{[lm]} \tag{6.59}$$

From this we can see immediately that the differential operator L_j of the $U(k)$ theory should be replaced by the matrix $I_{\alpha\beta} J_{\mu\nu}^j$ in the $U(Nk)$ theory. Here I is the identity matrix while J^j is the generator of $SU(2)$. The index j will be suppressed in some of the following equations for simplicity.

Therefore, given an instanton solution $Y(\theta, \phi)$ of the $U(k)$ three-dimensional theory, we can translate it back into the one-dimensional $U(Nk)$ theory where the solution X is represented by [5]:

$$
\begin{aligned}
X_{(\alpha\mu)(\beta\nu)} &= \frac{1}{g} I_{\alpha\beta} J^j_{\mu\nu} + Y_{(\alpha\mu)(\beta\nu)} \\
&= \frac{1}{g} I_{\alpha\beta} J^j_{\mu\nu} + \sum_{l=0}^{N-1} \sum_{m=-l}^{+l} c^{[lm]}_{\alpha\beta} \mathcal{Y}^{[lm]}_{\mu\nu}
\end{aligned}
\tag{6.60}
$$

Let us now apply this to translate our (1,0) three-dimensional $U(2)$ instanton into a matrix theory instanton. From eqn(6.51) we see that Y is independent of θ and ϕ, so only the $l = 0, m = 0$ spherical harmonic component appears in the expansion, and this $\mathcal{Y}^{[00]}$ is mapped simply to the $N \times N$ identity matrix I_N:

$$
\mathcal{Y}^{[00]} = 1 \rightarrow I_N \tag{6.61}
$$

Therefore, the matrix theory X that corresponds to this (1,0) instanton is given by:

$$
X_i = \frac{1}{g} \{ I_2 \otimes J^i_N + f(\tau) J^i_2 \otimes I_N \} \tag{6.62}
$$

The symbol $\otimes$ denotes direct product, and J^i_2 is simply $\frac{\sigma_i}{2}$.

To be a little bit more general, if Y^i_k is a $k \times k$ instanton solution of the three-dimensional $U(k)$ theory that is independent of θ and ϕ, it can be transformed into X in the $U(Nk)$ theory by:

$$
X_i = \frac{1}{g} I_k \otimes J^i_N + Y^i_k \otimes I_N \tag{6.63}
$$

This can be verified by a direct substitution.

[5] This is only strictly true in the $N \rightarrow \infty$ limit.

6.4 The Higher Instantons

In the previous section we gave the explicit expression for the (1,0) instanton, i.e., one that tunnels from the $n = 1$ vacuum to the $n = 0$ vacuum. It is natural is ask whether it is possible to construct other instantons. We will denote an instanton that tunnels from n_- vacuum to n_+ vacuum as the (n_-, n_+) instanton.

A difference with flat space can already be seen from the results that we have. In flat space, an instanton is labeled by a single integer k, which represents a tunneling process between a $n + k$ vacuum and a n vacuum [44, 45]. The k-instanton action is proportional to $\frac{k(\Delta r)}{e^2}$ and is independent of n.

However, in our case, an (n_-, n_+) instanton gives an action of $S_0 \sim \frac{|n_-^2 - n_+^2|}{g^2}$, which depends not only on the difference $k = n_+ - n_-$ but on both numbers. In the flat membrane limit, where $n \to \infty$ while k is kept finite, this action becomes $S_0 \sim \frac{nk}{g^2} \sim \frac{\mu n k}{p} \sim \frac{k(\Delta r)}{e^2}$. We used the identification $p \sim e^2$ in flat space [37]. Indeed our instanton gives the flat space result in the appropriate limit. Nevertheless, the fact that the instanton depends on both n_- and n_+ also tells us that it is inherently a more involved problem.

6.4.1 The Conformally Flat Picture

In the A-formalism, finding an instanton means solving eqn(6.47). Even without the extra terms coming from our pp-wave problem, explicit solutions for the equivalent flat space problem is difficult to write down. Finding solutions for these equations appears to be a rather formidable task. However, a simple rearrangement of the terms simplifies the equations such that they become identical to those in flat space.

Define $r = e^\tau$ and $A_4 = \frac{1}{r}\Phi$, the equations (choosing the upper sign) becomes:

$$\frac{1}{r^2 \sin\theta} F_{\theta\phi} = -D_r A_4 \tag{6.64}$$

$$\frac{1}{r} F_{r\theta} = -\frac{1}{r\sin\theta} D_\phi A_4 \tag{6.65}$$

$$\frac{1}{r\sin\theta} F_{\phi r} = -\frac{1}{r} D_\theta A_4 \tag{6.66}$$

This is of course the self dual equations of a four-dimensional space with a metric:

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2 + (dx^4)^2 = \sum_{m=1}^{4}(x^i)^2 \tag{6.67}$$

In other words these are simply the self dual equations of flat space.

To understand this mapping better, we can go back to the $R^2 \times S^2$ picture in eqn(6.39). The metric we had there was:

$$ds^2 = d\tau^2 + d\theta^2 + sin^2\theta d\phi^2 + dx^{\Phi^2} \tag{6.68}$$

By $r = e^\tau$, and defining $rdx^\Phi = dx^4$, the metric becomes:

$$ds^2 = \frac{1}{r^2}(dr^2 + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2 + dx^{4^2}) = \frac{1}{r^2}\{\sum_{m=1}^{4}(x^i)^2\} \tag{6.69}$$

This metric is flat up to a conformal transformation. One can quickly verify that the conformal factor in front does not affect the self dual equations, meaning the self dual equations in all conformally flat spaces are in fact identical to those in flat space, and this is the underlying reason of the simplification we obtain above.

Writing the one form $\Phi dx^\Phi \equiv A_4 dx^4 = A_4 r dx^\Phi$ gives the relation $A_4 = \frac{1}{r}\Phi$ that we used above. From now on, we shall call this the *conformally flat picture*.

In terms of Cartesian coordinates, the self dual equations are simply:

$$F_{ij} = -\epsilon_{ijk} D_k A_4 \tag{6.70}$$

where $\epsilon_{123} = +1$.

Although we managed to map our instanton problem into one in flat space, our trouble is far from over. Because of the relation $A_4 = \frac{1}{r}\Phi$, all solutions that have Φ finite at $\tau \to -\infty$ must have a pole in A_4 at the origin in the conformally flat picture. Furthermore, for Φ to settle at a stable vacuum at $\tau \to +\infty$, A_4 must falls off as $\frac{1}{r}$ as $r \to +\infty$. In other words, we are looking for a very specific type of solution in the conformally flat space that contains a singularity at the origin. As far as we can tell, such solutions are not well studied, as they correspond to monopoles that carries infinite energy in flat space.

6.5 The BPS Condition

The supersymmetry transformation of the action in eqn (6.9) is given by:

$$\begin{aligned} \delta Y_i &= -i\varepsilon^T \gamma_i \psi \\ \delta X_a &= -i\varepsilon^T \gamma_a \psi \\ \delta\psi &= \left\{ \frac{1}{2}\dot{Y}_i\gamma_i + \frac{1}{2}\mathcal{B}_i\gamma_i\gamma_{123} + \frac{1}{2}\dot{X}_a\gamma_a - \frac{1}{2}i\mathcal{L}_i X_a\gamma_i\gamma_a - i\frac{1}{4}g[X_a, X_b]\gamma_{ab} - \frac{1}{4}X_a\gamma_a\gamma_{123} \right\}\varepsilon \end{aligned} \tag{6.71}$$

where $\varepsilon = \exp(-\frac{1}{4}\gamma_{123}\tau)\epsilon_0$ with ϵ_0 a constant spinor.

6.5.1 The $r \neq 0$ Case: Circular Orbits

Although most of what we will do in the rest of this chapter has to do with $X_a = 0$, in this section we will consider a slightly more general case where the membranes move in a *circular orbit* around the origin in the 45-plane. As we will see shortly, this configuration preserves 8 supersymmetries [47]. When we return to the discussions of the $X_a = 0$ case, all we have to do is to put the radius of the orbit r to zero. With this in mind, we choose:

$$X_4 = r\cos(\frac{1}{2}\tau) \tag{6.72}$$

$$X_5 = r\sin(\frac{1}{2}\tau) \tag{6.73}$$

Here r is a $U(2)$ matrix which may be time dependent[6]. The readers should recall $\tau = \frac{\mu}{3}t$ due to earlier rescaling, so this is a rotation of angular frequency $\frac{\mu}{6}$.

[6]It can be time dependent even for a circular orbit because only the eigen values of r is related to the physical radius, but not the r *matrix* itself.

The supersymmetry transformation for ψ can then be simplified to:

$$\delta\psi = \frac{1}{2}\{(\dot{Y}_i + \mathcal{B}_i\gamma_{123})\gamma_i + i\mathcal{L}_i r\big(\cos(\frac{1}{2}\tau)\gamma_4 + \sin(\frac{1}{2}\tau)\gamma_5\big)\gamma_i + \big(\cos(\frac{1}{2}\tau)\gamma_4 + \sin(\frac{1}{2}\tau)\gamma_5\big)[\frac{1}{2}r\gamma_{45}(1+\gamma_{12345}) + \dot{r}]\}\varepsilon \quad (6.74)$$

Noting that $\cos(\frac{1}{2}\tau)\gamma_4 + \sin(\frac{1}{2}\tau)\gamma_5 = \exp(-\frac{1}{2}\gamma_{45}\tau)\gamma_4$, $\delta\psi$ can be rewritten after pulling out all the explicit time dependence:

$$\delta\psi = \frac{1}{2}\Big\{\exp\big(-\frac{1}{4}\gamma_{123}\tau\big)(\dot{Y}_i + \mathcal{B}_i\gamma_{123})\gamma_i + \exp\big(-\frac{1}{4}(2\gamma_{45} - \gamma_{123})\tau\big)i\mathcal{L}_i r\gamma_4\gamma_i + \exp\big(-\frac{1}{4}(2\gamma_{45} - \gamma_{123})\tau\big)\gamma_4[\frac{1}{2}r\gamma_{45}(1+\gamma_{12345}) + \dot{r}]\Big\}\epsilon_0 \quad (6.75)$$

Before we rewrite this equation in another yet simpler form, consider first the case of a stationary configuration for $r \neq 0$. In this case $\dot{Y} = \mathcal{B} = \dot{r} = 0$, and $\delta\psi$ becomes:

$$\delta\psi = \frac{1}{2}\Big\{\exp\big(-\frac{1}{4}(2\gamma_{45}-\gamma_{123})\tau\big)i\mathcal{L}_i r\gamma_4\gamma_i + \exp\big(-\frac{1}{4}(2\gamma_{45}-\gamma_{123})\tau\big)\gamma_4[\frac{1}{2}r\gamma_{45}(1+\gamma_{12345})]\Big\}\epsilon_0 \quad (6.76)$$

From this we can see that 8 supersymmetries can be preserved by having $\mathcal{L}_i r = 0$ and $\gamma_{12345}\epsilon_0 = -\epsilon_0$. First we consider the $U(1)$ part of r, which satisfies the first condition if it is constant over the sphere. To this we can add an $SU(2)$ part, for which the first condition can be satisfied by having $r \propto \Phi$ if $\Phi \neq 0$ and $r = constant$ if $\Phi = 0$. Of course if $r = 0$ then the condition $\gamma_{12345}\epsilon_0 = -\epsilon_0$ does not apply and all supersymmetries are unbroken.

Now we return to the general case in eqn(6.75), but instead of assuming it is a stationary state, we assume only that the membranes tends to a stationary configuration at the infinite past with $r \neq 0$, i.e., $\dot{Y} = \mathcal{B} = \dot{r} = 0$ as $\tau \to -\infty$. Then the arguments in the last paragraph tell us that the unbroken supersymmetries in the infinite past is given by the condition $\gamma_{12345}\epsilon_0 = -\epsilon_0$. Recall that ϵ_0 is time independent, it implies the unbroken supersymmetries at finite time (if there are any) must satisfy this condition also. This gets rid of the term $\frac{1}{2}r\gamma_{45}(1+\gamma_{12345})$ for all finite time. It also tells us $\gamma_{45}\epsilon_0 = +\gamma_{123}\epsilon_0$ and this means we can do the following replacement:

$$\exp\big(-\frac{1}{4}(2\gamma_{45}-\gamma_{123})\tau\big) \to \exp\big(-\frac{1}{4}\gamma_{123}\tau\big) \tag{6.77}$$

Therefore under the condition $\gamma_{12345}\epsilon_0 = -\epsilon_0$ the supersymmetry transformation for $\delta\psi$ can be written as:

$$\delta\psi = \frac{1}{2}\exp\big(-\frac{1}{4}\gamma_{123}\tau\big)\Big\{(\dot{Y}_i + \mathcal{B}_i\gamma_{123})\gamma_i + i\mathcal{L}_i r\gamma_4\gamma_i + \gamma_4\dot{r}\Big\}\epsilon_0 \tag{6.78}$$

Out of the eight ϵ_0 that satisfy $\gamma_{12345}\epsilon_0 = -\epsilon_0$, the fraction of them that gives $\delta\psi = 0$ at all times are the unbroken supersymmetries.

Since we are interested in instanton solutions, we need to Euclideanize the above equation by $\tau_E = i\tau$. Omitting the subscript $_E$, we have:

$$\delta\psi = \frac{1}{2}i\exp\big(i\frac{1}{4}\gamma_{123}\tau\big)\Big\{(\partial_\tau Y_i - \mathcal{B}_i i\gamma_{123})\gamma_i + \mathcal{L}_i r\gamma_4\gamma_i + \gamma_4\partial_\tau r\Big\}\epsilon_0 \tag{6.79}$$

where $\gamma_{12345}\epsilon_0 = -\epsilon_0$.

At this point it is tempting to say $\pm\partial_\tau Y_i - \mathcal{B}_i = \mathcal{L}_i r = \partial_\tau r = 0$ and impose $i\gamma_{123}\epsilon_0 = \pm\epsilon_0$ to get 4 unbroken supersymmetries. Once again, we first consider the $U(1)$ component of r. Indeed all these conditions can be satisfied if $r = \mathit{constant} \times I_2$. This configuration can be interpreted as two membranes overlapping one another in the 45-plane (remember that each membrane is a single point in the 45-plane) and circling the origin together. Therefore the instanton process involving two such membranes preserves 4 supersymmetries.

It is natural to ask if it is possible to add a $SU(2)$ component to the $U(1)$ part of r above and still preserves 4 supersymmetries. However, for a non-trivial instanton solution $\mathcal{L}_i r = 0$ and $\partial_\tau r = 0$ cannot be satisfied at the same time once r contains a $SU(2)$ component. This can be seen by differentiating $\mathcal{L}_i r = 0$ with respect to τ. Since $\mathcal{L}_i$ contains Y (see Appendix A for the notation), it implies $\partial_\tau Y = 0$, which cannot be true for the non-trivial instanton solutions.

6.5.2 The $r = 0$ Case

When $r = 0$, the Euclideanized $\delta\psi$ is given by:

$$\delta\psi = \frac{1}{2} i \exp\left(i\frac{1}{4}\gamma_{123}\tau\right)\Big\{(\partial_\tau Y_i - \mathcal{B}_i i\gamma_{123})\gamma_i\Big\}\epsilon_0 = \frac{1}{2}\Big\{(\partial_\tau Y_i - \mathcal{B}_i i\gamma_{123})\gamma_i\Big\}\varepsilon \qquad (6.80)$$

Note that we no longer require $\gamma_{12345}\epsilon_0 = -\epsilon_0$ in the $r = 0$ case.

Requiring $\delta\psi = 0$ gives the instanton equations as well as the condition for unbroken supersymmetries:

$$\pm\partial_\tau Y_i - \mathcal{B}_i = 0 \qquad (6.81)$$

$$i\gamma_{123}\varepsilon = \pm\varepsilon \qquad (6.82)$$

Therefore the instantons given in section 6.3 and 6.4 preserves half of the supersymmetries.

The broken supersymmetries give us the fermionic zero modes λ. For simplicity, we will pick the lower sign from now on. Hence by putting $-\partial_\tau Y_i - \mathcal{B}_i = 0$ and $i\gamma_{123}\varepsilon = +\varepsilon$ (note that the broken supersymmetries have the opposite sign to the unbroken supersymmetries) into $\delta\psi$, we have:

$$\lambda = -\mathcal{B}_i\gamma_i\varepsilon = \partial_\tau Y_i\gamma_i\varepsilon \qquad (6.83)$$

Now let us look at the condition for ϵ_0. If we take the complex conjugate, using the fact that the $SO(9)$ gamma matrices in our convention are real and symmetric, we get:

$$i\gamma_{123}\epsilon_0 = +\epsilon_0 \qquad (6.84)$$

$$i\gamma_{123}\epsilon_0^* = -\epsilon_0^* \qquad (6.85)$$

This means that ϵ_0 cannot be real. This in turns implies that the fermionic zero modes λ must also be complex. The reason we began with real $SO(9)$ spinors of dimension 16, and yet arrived at eight complex zero modes lies in the Euclideanization of the fermions, after which the fermions are doubled. In terms of Majorana spinors, it means the reality condition is no longer imposed after Euclideanization. Understanding this point will be the subject of the next section.

6.6 The Euclideanization of Fermions

It is not a surprise that when we Euclideanize a theory, the fermions have to be treated carefully. In our case, the spinors representing the fermions originate from the group $SO(9,1)$, which has a minimum representation of 16 real dimension[7]. After Euclideanization, the group becomes $SO(10)$, which has a minimum representation of 32 real dimension. One can of course take the Euclideanized action as the starting point, in which case one deals with spinors of 32 real dimension right from the beginning. Such an approach was taken in [37]. An equivalent approach is to use the "doubling trick" on fermions after Euclideanization, in effect bringing the real dimension of the spinors to 32.

Let us first review some of the ideas in flat space. In this section, we will denote the Minkowski space-time indices by $\mu, \nu, \cdots$, and their Euclidean counterparts by $m, n, \cdots$. Here we pick a representation such that $\Gamma^{\mu\dagger} = -\Gamma^0\Gamma^\mu(\Gamma^0)^{-1}$. Under a Lorentz rotation, a spinor transforms as follows:

$$\Psi' = \exp(\Gamma^{\mu\nu})\Psi \tag{6.86}$$

Defining $\bar{\Psi} = \Psi^\dagger\Gamma^0$, one can show it transforms opposite to Ψ:

$$\bar{\Psi}' = \bar{\Psi}\exp(-\Gamma^{\mu\nu}) \tag{6.87}$$

Therefore Lorentz invariants that appear in the action can be created by combining $\bar{\Psi}$ with Ψ.

In Euclidean space, we define $\tau = ix^0$. This gives $\Gamma^\tau = i\Gamma^0$. Therefore all the

[7]The term real dimension is the number of real components in the spinor representation.

gamma matrices are now hermitian:

$$\Gamma^{m\dagger} = \Gamma^{m} \tag{6.88}$$

This gives:

$$\Psi'^{\dagger} = \Psi^{\dagger} \exp(-\Gamma^{\mu\nu}) \tag{6.89}$$

but one should note that $\bar{\Psi}' \neq \bar{\Psi} \exp(-\Gamma^{\mu\nu})$ after Euclideanization.

To construct the Euclidean action for the fermions, we will follow the procedure in [48]. During Euclideanization in four dimensions, the spinors are rotated by the matrix $S = \exp(\Gamma^{45}\pi/4)$:

$$\Psi = S\Psi_E \tag{6.90}$$

$$\Psi^{\dagger} = \Psi_E^{\dagger} S \tag{6.91}$$

$$\Gamma_E^{\mu} = S^{-1}\Gamma^{\mu} S \tag{6.92}$$

For Majorana spinors, the result of Euclideanization is particularly simple. In Minkowski space, a reality condition $\bar{\Psi} = \Psi^T C$ is imposed on the Majorana spinors. After Euclideanization, the net effect of the spinor rotation is simply the replacement of $\bar{\Psi}$ by $\Psi^T C$, where C is the charge conjugation matrix[8] satisfying the following conditions:

$$C^T = -C \tag{6.93}$$

$$C\Gamma^m C^{-1} = -(\Gamma^m)^T \tag{6.94}$$

It is easy to see that under Euclidean rotation, $\Psi'^T C = \Psi^T C \exp(-\Gamma^{\mu\nu})$. The invariance of the action follows. One very important aspect of the Euclideanized action is that the reality condition is no longer imposed on the spinor Ψ. This is the origin of the "fermion doubling" that was mentioned earlier. To understand this point bet-

[8]Strictly speaking, we should replace C by $C_E = S^T CS$, but in fact the Minkowski C will suffice and we need not rotate the gamma matrices either in the Majorana formulation. See [48] for details.

ter, we will look at the degrees of freedom being integrated over in the path integral formalism in four dimensions.

6.6.1 Euclideanization in Four Dimensions

A Dirac spinor in four dimensions can be written in the Weyl basis[9] as $\Psi = \begin{pmatrix} \psi \\ \bar{\chi} \end{pmatrix}$, where ψ and χ are each two component spinors.

- For a Dirac spinor in Minkowski space, the path integral involves integration over $\psi, \bar{\psi}, \chi, \bar{\chi}$, a total of 8 real degrees of freedom.
- For a Majorana spinor in Minkowski space, the reality condition gives $\psi = \chi$, and the path integral is now over $\psi, \bar{\psi}$, a total of 4 degrees of freedom.
- After Euclideanization the Majorana spinor no longer have a reality condition, hence we do not require $\psi = \chi$. However, by the construction above using $\Psi^T C$, $\bar{\psi}$ and χ do not appear in the action, so the path integral is only over $\psi, \bar{\chi}$, a total of 4 degrees of freedom.

It should be clear from this simple counting that even though the fermions are "doubled" in the sense that the reality conditions is removed, the total degrees of freedom being integrated over in fact remains unchanged. The distinction between the second and the third case above is more than just formal. Take the condition for unbroken supersymmetry:

$$\Gamma^{\tau 123}\epsilon = i\Gamma^{0123}\epsilon = -\epsilon \tag{6.95}$$

In the Weyl basis, it implies we must have $\epsilon = \begin{pmatrix} \epsilon_0 \\ 0 \end{pmatrix}$. This condition can only be satisfied after the reality condition is removed, and is therefore made possible only by Euclideanization.

[9]The Weyl basis is the representation with a diagonal Γ_5.

To make this last point clearer, one can look at the kinetic term of the fermions in four dimensions. Restoring the spinor indices for the Dirac spinor, we have:

$$\Psi = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} \tag{6.96}$$

We can pick the representation:

$$\Gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ -\bar{\sigma}^\mu & 0 \end{pmatrix} \tag{6.97}$$

$$C = \begin{pmatrix} \epsilon^{\alpha\beta} & 0 \\ 0 & -\epsilon_{\dot{\alpha}\dot{\beta}} \end{pmatrix} \tag{6.98}$$

Writing $\sigma^\mu D_\mu = D$ and $\bar{\sigma}^\mu D_\mu = \bar{D}$, we have:

$$i\bar{\Psi}\Gamma^\mu D_\mu \Psi = -i\bar{\chi}\bar{D}\chi - i\bar{\psi}\bar{D}\psi \tag{6.99}$$

$$i\Psi^T C\Gamma^\mu D_\mu \Psi = -2i\bar{\chi}\bar{D}\psi \tag{6.100}$$

The two equations are of course the same if the reality condition is imposed such that $\psi = \chi$. The second equation is the one that should be used in constructing the Euclidean action, and it can be seen here that unlike the first equation, only ψ and $\bar{\chi}$ appears, while $\bar{\psi}$ and χ are absent, as pointed out earlier in the counting of degrees of freedom.

Everything that was said about Majorana spinors in four dimensions can be re-derived using Weyl spinors as well, but we will not go into the details here. In the following we will simply Euclideanize the fermionic action of Majorana spinors by replacing $\bar{\Psi}$ by $\Psi^T C$ and removing the reality condition. In this procedure the total degrees of freedom in the path integral is unchanged.

6.6.2 Euclideanization of the Membrane Action

The fermionic action before Euclideanization was given in eqn(6.9):

$$\mathcal{S}_F = \int d\tau d^2\Omega \left\{ i\psi^T D_\tau \psi - i\frac{3}{4}\psi^T \gamma_{123}\psi + g\psi^T\gamma_a[X_a,\psi] + \psi^T\gamma_i\mathcal{L}_i\psi \right\} \tag{6.101}$$

Here ψ are real 16-component spinors of $SO(9)$ and $\mathcal{L}_i\Psi = -i(\epsilon_{ijk}x_j\partial_k x^\Omega D_\Omega\Psi + x_i D_\Phi\Psi)$.

To prepare for Euclideanization, we restore Γ^0 and expand ψ to real 32-component spinors of $SO(9,1)$ denoted as Ψ:

$$\mathcal{S}_F = \int d\tau d^2\Omega \Big\{ - i\bar{\Psi}\Gamma^0 D_0\Psi + \bar{\Psi}\Gamma^i\mathcal{L}_i\Psi - i\frac{3}{4}\bar{\Psi}\Gamma_{123}\Psi + g\bar{\Psi}\Gamma^a[X_a,\Psi] \Big\} \tag{6.102}$$

We defined the Dirac conjugate as $\bar{\Psi} = \Psi^\dagger\Gamma^0$. It can be seen to reduce to the previous equation most easily by putting $\chi = 0$ in the following basis:

$$\Gamma^0 = I_{16}\otimes i\sigma_2 = \begin{pmatrix} 0 & I_{16} \\ -I_{16} & 0 \end{pmatrix} \tag{6.103}$$

$$\Gamma^A = \gamma^A \otimes \sigma_1 = \begin{pmatrix} 0 & \gamma^A \\ \gamma^A & 0 \end{pmatrix} \tag{6.104}$$

$$\Psi = \begin{pmatrix} \psi \\ \chi \end{pmatrix} \tag{6.105}$$

This basis, however, is not the most convenient one for our purpose. The pp-wave theory has $SO(3)\times SO(6)$ symmetry and at the same time, as we saw in section 6.4.1, the action can be expressed most compactly in four-dimensional notation just as in flat space. Therefore, we choose a basis that makes the $SO(3,1)\times SO(6) \sim SL(2,C)\times SU(4) \subset SO(9,1)$ subgroup explicit:

$$\Gamma^m = \begin{pmatrix} 0 & \sigma^m_{\alpha\dot{\alpha}} \\ -\bar{\sigma}^{m\dot{\alpha}\alpha} & 0 \end{pmatrix} \otimes \hat{\gamma} \tag{6.106}$$

$$\Gamma^a = I_4 \otimes \gamma^a \tag{6.107}$$

where $\hat{\gamma} = i\gamma^4 \cdots \gamma^9 = \begin{pmatrix} I_4 & 0 \\ 0 & -I_4 \end{pmatrix}$ and $\gamma^a = \begin{pmatrix} 0 & \rho^a_{AB} \\ \bar{\rho}^{aAB} & 0 \end{pmatrix}$, while $m = 0,1,2,3$; $a = 4, 5, \cdots, 9$ and $\mu, \nu = 0, 1, 2, \cdots, 9$ just as before. $A, B = 1, 2, 3, 4$ now denote the $SU(4)$ indices and should not be confused with the target space indices which have the same notation. ρ and $\bar{\rho}$ are 4×4 antisymmetric matrices that are related by $\rho^\dagger = \bar{\rho}$ so that Γ^a are hermitian. They satisfy the relation:

$$\rho^a \bar{\rho}^b + \rho^b \bar{\rho}^b = 2\delta^{ab} \tag{6.108}$$

As for the σ matrices we follow the Wess and Bagger notation:

$$\sigma^m = (1, \vec{\sigma}) \tag{6.109}$$

$$\bar{\sigma}^m = (1, -\vec{\sigma}) \tag{6.110}$$

The gamma matrices satisfy $\Gamma^{\mu\dagger} = -\Gamma^0 \Gamma^\mu (\Gamma^0)^{-1}$ as well as the Clifford algebra:

$$\frac{1}{2}\{\Gamma^\mu, \Gamma^\nu\} = \eta^{\mu\nu} = diag(-, +, +, \cdots, +) \tag{6.111}$$

For completeness we give also an explicit form for the ρ matrices:

$$\begin{aligned} \rho^4 &= \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix}, \quad \rho^5 = \begin{pmatrix} i\epsilon & 0 \\ 0 & -i\epsilon^{-1} \end{pmatrix}, \quad \rho^6 = \begin{pmatrix} 0 & i\sigma^3 \\ -i(\sigma^3)^T & 0 \end{pmatrix}, \\ \rho^7 &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \rho^8 = \begin{pmatrix} 0 & i\sigma^1 \\ -i(\sigma^1)^T & 0 \end{pmatrix}, \quad \rho^9 = \begin{pmatrix} 0 & i\sigma^2 \\ -i(\sigma^2)^T & 0 \end{pmatrix}. \end{aligned} \tag{6.112}$$

Expanding the 32×32 gamma matrices further, we have:

$$\Gamma^m = \begin{pmatrix} 0 & 0 & \sigma^m & 0 \\ 0 & 0 & 0 & -\sigma^m \\ -\bar{\sigma}^m & 0 & 0 & 0 \\ 0 & \bar{\sigma}^m & 0 & 0 \end{pmatrix}, \quad \Gamma^a = \begin{pmatrix} 0 & \rho^a & 0 & 0 \\ \bar{\rho}^a & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho^a \\ 0 & 0 & \bar{\rho}^a & 0 \end{pmatrix}, \quad \Psi = \begin{pmatrix} \psi_{\alpha A} \\ \tilde{\psi}_\alpha^A \\ \bar{\tilde{\chi}}^{\dot{\alpha}}_A \\ \bar{\chi}^{\dot{\alpha}A} \end{pmatrix} \tag{6.113}$$

Due to the property of the unitary group, complex conjugation takes an $SU(4)$ superscript ^{A} to a subscript $_A$ and vice versa. With this in mind one can check that all the upper and lower indices match perfectly in the action, and any $SL(2,C) \times SU(4)$ invariant quantities should be constructed accordingly. The $SU(4)$ index can be understood from the four-dimensional point of view as a label for the different "species" of fermions related by R-symmetry.

The charge conjugation matrix is given by:

$$C = \begin{pmatrix} \epsilon^{\alpha\beta} & 0 \\ 0 & \epsilon_{\dot{\alpha}\dot{\beta}} \end{pmatrix} \otimes \begin{pmatrix} 0 & I_4 \\ I_4 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \epsilon^{\alpha\beta} & 0 & 0 \\ \epsilon^{\alpha\beta} & 0 & 0 & 0 \\ 0 & 0 & 0 & \epsilon_{\dot{\alpha}\dot{\beta}} \\ 0 & 0 & \epsilon_{\dot{\alpha}\dot{\beta}} & 0 \end{pmatrix} \tag{6.114}$$

$$C^{-1} = \begin{pmatrix} \epsilon_{\alpha\beta} & 0 \\ 0 & \epsilon^{\dot{\alpha}\dot{\beta}} \end{pmatrix} \otimes \begin{pmatrix} 0 & I_4 \\ I_4 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \epsilon_{\alpha\beta} & 0 & 0 \\ \epsilon_{\alpha\beta} & 0 & 0 & 0 \\ 0 & 0 & 0 & \epsilon^{\dot{\alpha}\dot{\beta}} \\ 0 & 0 & \epsilon^{\dot{\alpha}\dot{\beta}} & 0 \end{pmatrix} \tag{6.115}$$

Using the relations $\epsilon\sigma^T\epsilon = -\bar{\sigma}$ and $\epsilon\bar{\sigma}^T\epsilon = -\sigma$ it is not difficult to check the usual properties of the charge conjugation matrix:

$$C^T = -C \tag{6.116}$$

$$C\Gamma^\mu C^{-1} = -(\Gamma^\mu)^T \tag{6.117}$$

Two other matrices we will use frequently are the chirality operator:

$$\Gamma^{\tau 123} = i\Gamma^{0123} = \begin{pmatrix} -I_{16} & 0 \\ 0 & +I_{16} \end{pmatrix} \tag{6.118}$$

$$\Gamma^{01\cdots 9} = \begin{pmatrix} I_8 & 0 & 0 & 0 \\ 0 & -I_8 & 0 & 0 \\ 0 & 0 & -I_8 & 0 \\ 0 & 0 & 0 & I_8 \end{pmatrix} \tag{6.119}$$

For later convenience we also write out explicitly the following expressions:

$$\Psi^T = \begin{pmatrix} \psi_{\alpha A} & \tilde{\psi}^A_\alpha & \bar{\tilde{\chi}}^{\dot\alpha}_A & \bar{\chi}^{\dot\alpha A} \end{pmatrix} \tag{6.120}$$

$$\Psi^\dagger = \begin{pmatrix} \bar{\psi}^A_{\dot\alpha} & \bar{\tilde{\psi}}_{\dot\alpha A} & \tilde{\chi}^{\alpha A} & \chi^\alpha_A \end{pmatrix} \tag{6.121}$$

$$\bar{\Psi} = \begin{pmatrix} -\tilde{\chi}^{\alpha A} & \chi^\alpha_A & \bar{\psi}^A_{\dot\alpha} & -\bar{\tilde{\psi}}_{\dot\alpha A} \end{pmatrix} \tag{6.122}$$

$$\Psi^T C = \begin{pmatrix} -\tilde{\psi}^{\alpha A} & -\psi^\alpha_A & -\bar{\chi}^A_{\dot\alpha} & -\bar{\tilde{\chi}}_{\dot\alpha A} \end{pmatrix} \tag{6.123}$$

$$\tag{6.124}$$

If the chirality condition $\Gamma^{01\cdots 9}\Psi = +\Psi$ is imposed, the 32-component spinor will be reduced to:

$$\Psi = \begin{pmatrix} \psi^A_\alpha \\ 0 \\ 0 \\ \bar{\chi}^{\dot\alpha A} \end{pmatrix} \tag{6.125}$$

We can also impose the reality condition, reducing the degrees of freedom by half:

$$\Psi^T C = \bar{\Psi} \tag{6.126}$$

This gives the following relations between the component fields (and their complex conjugate):

$$\tilde{\chi}^{\alpha A} = \tilde{\psi}^{\alpha A} \tag{6.127}$$

$$\chi^{\alpha}_{A} = -\psi^{\alpha}_{A} \tag{6.128}$$

The reality condition and the chirality projection are compatible in Minkowski space but not in Euclidean space.

We will Euclideanize by keeping the chirality condition while abandoning the reality condition. To relate the Minkowski action to the Euclidean action we define $\tau_E = i\tau_M$, $S_E = -iS_M$ and $\Gamma^{\tau}_E = i\Gamma^0_M$. All $\bar{\Psi}$ in the action are replaced by $\Psi^T C$. Omitting the subscript $_E$ from now on, the fermionic action becomes:

$$\mathcal{S}_F = \int d\tau d^2\Omega \Big\{ i\Psi^T C\Gamma^{\tau} D_{\tau}\Psi - \Psi^T C\Gamma^i \mathcal{L}_i \Psi + i\frac{3}{4}\Psi^T C\Gamma_{123}\Psi - g\Psi^T C\Gamma^a [X_a, \Psi] \Big\} \tag{6.129}$$

Ψ is now given by the eqn(6.125).

Expanding in component form, the fermionic action is:

$$\mathcal{S}_F = \int d\tau d^2\Omega \; 2\left\{ i\bar{\chi}\bar{\sigma}^{\tau} D_{\tau}\psi - \bar{\chi}\bar{\sigma}^i \mathcal{L}_i \psi - i\frac{3}{4}\bar{\chi}\bar{\sigma}^{\tau}\psi + \frac{1}{2}g\psi\bar{\rho}^a [X_a, \psi] + \frac{1}{2}g\bar{\chi}\rho^a [X_a, \bar{\chi}] \right\} \tag{6.130}$$

We have defined $\sigma^{\tau} = i\sigma^0$ and $\bar{\sigma}^{\tau} = i\bar{\sigma}^0$. Just as in the four-dimensional example we looked at, the total degrees of freedom to be integrated over in the path integral formalism remains 16, the same as its Minkowski counterpart. This is because only ψ and $\bar{\chi}$ appears while $\bar{\psi}$ and χ are absent. Once again the readers are reminded of the fact that this would not be the case had we not used $\Psi^T C$ instead of $\bar{\Psi}$. Since the reality condition is removed, there is now no constraints between ψ and $\bar{\chi}$.

From now on we will often switch between the 32-component formalism in terms of Γ^{μ} and Ψ and the component form of ψ and $\bar{\chi}$.

6.6.3 The BPS Condition Revisited

After Euclideanization, the supersymmetric transformation of the fermion when $X_a = 0$ is:

$$\delta\Psi = \frac{1}{2}\Gamma^{\tau i}(F_{\tau i} - \mathcal{B}_i\Gamma^{\tau 123})\varepsilon \tag{6.131}$$

where $\varepsilon = \exp(\frac{1}{4}\Gamma^{\tau 123}\tau)\epsilon_0$
Choose the sign so that $F_{\tau i} + \mathcal{B}_i = 0$, then we have:

$$\delta\Psi = F_{\tau i}\frac{1}{2}\Gamma^{\tau i}(1 + \Gamma^{\tau 123})\varepsilon \tag{6.132}$$

The unbroken supersymmetry is now given by the condition $\Gamma^{\tau 123}\varepsilon = -\varepsilon$. The constant spinor ϵ_0 written in component form is now:

$$\epsilon_0 = \begin{pmatrix} \epsilon_{\alpha A} \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{6.133}$$

The broken supersymmetry satisfies $\Gamma^{\tau 123}\varepsilon = +\varepsilon$, giving:

$$\epsilon_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \bar{\epsilon}^{\dot{\alpha} A} \end{pmatrix} \tag{6.134}$$

This in turn gives the fermionic zero modes:

$$\lambda = \delta\Psi = F_{\tau i} e^{\frac{1}{4}\tau} \Gamma^{\tau i} \epsilon_0 \tag{6.135}$$

$$\equiv \begin{pmatrix} 0 \\ 0 \\ 0 \\ \bar{\lambda}^{\dot{\alpha}A} \end{pmatrix} \tag{6.136}$$

where $\bar{\lambda}^{\dot{\alpha}A} = -e^{\frac{1}{4}\tau} F_{\tau i} (\bar{\sigma}^\tau \sigma^i)^{\dot{\alpha}}{}_{\dot{\beta}} \bar{\epsilon}^{\dot{\beta}A}$. Labeling each zero mode by the indices ${}_{[\dot{\beta}B]}$, the 8 zero modes are now given by:

$$\bar{\lambda}^{\dot{\alpha}A}_{[\dot{\beta}B]} = -e^{\frac{1}{4}\tau} F_{\tau i} (\bar{\sigma}^\tau \sigma^i)^{\dot{\alpha}}{}_{\dot{\gamma}} \bar{\epsilon}^{\dot{\gamma}A}_{[\dot{\beta}B]} \tag{6.137}$$

In order to perform the path integral for the fermions, for each of the above zero modes we multiply by a Grassmann number $\xi^{[\dot{\beta}B]}$:

$$\bar{\lambda}^{\dot{\alpha}A} = \xi^{[\dot{\beta}B]} \bar{\lambda}^{\dot{\alpha}A}_{[\dot{\beta}B]} = -e^{\frac{1}{4}\tau} F_{\tau i} (\bar{\sigma}^\tau \sigma^i)^{\dot{\alpha}}{}_{\dot{\gamma}} \bar{\epsilon}^{\dot{\gamma}A}_{[\dot{\beta}B]} \xi^{[\dot{\beta}B]} \tag{6.138}$$

Now we choose the basis of the zero modes such that $\bar{\epsilon}^{\dot{\gamma}A}_{[\dot{\beta}B]} = \delta^A_B \delta^{\dot{\gamma}}_{\dot{\beta}}$. Removing the square bracket on ξ for simplicity, we arrive at the eight zero modes written in terms of Grassmann numbers:

$$\bar{\lambda}^{\dot{\alpha}A} = -e^{\frac{1}{4}\tau} F_{\tau i} (\bar{\sigma}^\tau \sigma^i)^{\dot{\alpha}}{}_{\dot{\beta}} \bar{\xi}^{\dot{\beta}A} \tag{6.139}$$

The indices ${}^{\dot{\alpha}A}$ on ξ labels the different fermionic zero modes, but can also be treated as proper spinor and $SU(4)$ indices.

6.7 The Membrane Interaction

6.7.1 The Interaction Term

In this section, the indices run as follows:

$$\begin{aligned} i,j &= 1,2,3 \\ m,n &= \tau,\Phi,\theta,\phi \\ \alpha,\beta &= \Phi,\theta,\phi \\ \mu,\nu &= \tau,\theta,\phi \\ \Omega &= \theta,\phi \end{aligned}$$

Defining:

$$\begin{cases} \Gamma_\Omega = \epsilon_{ijk}\Gamma^i x^j \partial_\Omega x^k \\ \Gamma_\Phi = x_i \Gamma^i \end{cases} \Leftrightarrow \begin{cases} \Gamma^\Omega = \epsilon_{ijk}\Gamma^i x^j \partial_k x^\Omega \\ \Gamma^\Phi = x_i \Gamma^i \end{cases} \tag{6.140}$$

then we have $\Gamma_i = x_i\Gamma^\Phi + \epsilon_{ijk}x^j\partial_\Omega x^k\Gamma^\Omega$.

In terms of these gamma matrices, the action becomes:

$$\mathcal{S}_F = \int d\tau d^2\Omega \Big\{ i\Psi^T C\Gamma^m D_m\Psi + i\frac{3}{4}\Psi^T C\Gamma_{123}\Psi - g\Psi^T C\Gamma^a[X_a,\Psi] \Big\} \tag{6.141}$$

Now we want to look at the interaction of the membranes from the term:

$$\mathcal{L}_{INT} = -g\Psi^T C\Gamma^a[X_a,\Psi] \tag{6.142}$$

Put $X_a = \mathsf{X}_a\Phi$, where X_a is just a c-function (as opposed to a matrix), we have[10]:

$$\mathcal{L}_{INT} = i\Psi^T C\mathsf{X}_a\Gamma^a D_\Phi\Psi \tag{6.143}$$

As before, $D_\Phi\Psi \equiv ig[\Phi,\Psi]$.

[10]The relation $X_a = \mathsf{X}_a\Phi$ is a condition that should be checked carefully. In fact it is only approximately true when time dependent perturbation is present. For circular orbit this is good enough under certain limit, see the next subsection for details.

In first-order perturbation theory, Ψ can be assumed to be the zero modes when X_a is absent, hence they satisfy:

$$\Gamma^m D_m \Psi + \frac{3}{4}\Gamma_{123}\Psi = 0 \tag{6.144}$$

This gives:

$$D_\Phi \Psi = -\Gamma^{\Phi\mu} D_\mu \Psi - \frac{3}{4}\Gamma^{\Phi}\Gamma_{123}\Psi \tag{6.145}$$

Using this equation we could simplify the interaction term (not keeping total derivatives):

$$\begin{aligned}
\mathcal{L}_{INT} &= \frac{i}{2}\Psi^T C \mathsf{X}_a \Gamma^a D_\Phi \Psi - \frac{i}{2} D_\Phi \Psi^T C \mathsf{X}_a \Gamma^a \Psi \\
&= -\frac{i}{2}\Psi^T C \mathsf{X}_a \Gamma^a \Gamma^{\Phi\mu} D_\mu \Psi - \frac{i}{2} D_\mu \Psi^T C \mathsf{X}_a \Gamma^a \Gamma^{\Phi\mu}\Psi - \frac{3}{4} i \Psi^T C \mathsf{X}_a \Gamma^a \Gamma^\Phi \Gamma_{123}\Psi \\
&= \frac{i}{2}\Psi^T C \mathcal{D}_\mu (\mathsf{X}_a \Gamma^a \Gamma^{\Phi\mu})\Psi - \frac{3}{4} i \Psi^T C \mathsf{X}_a \Gamma^a \Gamma^\Phi \Gamma_{123}\Psi \\
&= \frac{i}{2}\Psi^T C \partial_\mu (\mathsf{X}_a)\Gamma^a \Gamma^{\Phi\mu}\Psi - \frac{1}{4} i \Psi^T C \mathsf{X}_a \Gamma^a \Gamma^\Phi \Gamma_{123}\Psi \\
&= i\Psi^T C \Gamma^a \Gamma^{\Phi\tau}(\frac{1}{2}\partial_\tau \mathsf{X}_a + \frac{1}{4}\mathsf{X}_a \Gamma_{\tau 123})\Psi
\end{aligned} \tag{6.146}$$

We assumed in the above calculation that X_a was dependent on τ only and used the following formulae:

$$\Gamma_{\Phi\theta\phi} = +\sqrt{g}\Gamma_{123} \tag{6.147}$$

$$\Gamma_{mn} = -\frac{1}{2}\varepsilon_{mnpq}\Gamma^{pq}\Gamma_{\tau 123} \tag{6.148}$$

$$\mathcal{D}_\mu \Gamma^{\Phi\mu} = \nabla_\mu \Gamma^{\Phi\mu} = 2\Gamma^\Phi \Gamma_{123} = 2\Gamma^{\Phi\tau}\Gamma_{\tau 123} \tag{6.149}$$

The first term in $\mathcal{L}_{INT}$ is the velocity term that appeared in flat space, while the second arises from the mass term of the fermions and vanishes in the flat space limit. For zero modes that satisfy $\Gamma_{\tau 123}\Psi = +\Psi$, $\mathcal{L}_{INT}$ simplifies further. Written in component form, we finally get:

$$\begin{aligned} \mathcal{L}_{INT} &= -\frac{1}{2}i(\partial_\tau \mathsf{X}_a + \frac{1}{2}\mathsf{X}_a)\bar{\lambda}^{\dot{\alpha}A}(\epsilon\bar{\sigma}^\Phi\sigma^\tau)_{\dot{\alpha}\dot{\beta}}\rho^a_{AB}\bar{\lambda}^{\dot{\beta}B} \qquad (6.150) \\ &= \lambda^T \triangle \lambda \qquad (6.151) \end{aligned}$$

where $\triangle \equiv -\frac{1}{2}i(\partial_\tau \mathsf{X}_a + \frac{1}{2}\mathsf{X}_a)(\epsilon\bar{\sigma}^\Phi\sigma^\tau)_{\dot{\alpha}\dot{\beta}}\rho^a_{AB}$.

Naively, the Grassmann path integral with a properly defined measure should give a contribution of $\sqrt{\det \triangle}$ which is now very easy to evaluate, but in fact this is not quite true. While $\triangle$ is an 8×8 matrix, the determinant produced by the path integral is a determinant in both the matrix space and the functional space. The eight fermionic zero modes λ is a complete basis in the 8×8 matrix space but not the functional space. Evaluating its contribution to the path integral requires using the explicit expression of λ and doing the eight Grassmann integrations. However, without the expressions for a general (n_-, n_+) instanton this integration could not be carried out explicitly.

6.7.2 Interaction for Circular Orbit

Computing the interaction amplitude for a general trajectory under a general instanton involves computing integrals that require the explicit form of the (n_i, n_f) instanton solution. Since the explicit solution is currently unknown, and is likely to take a very complicated form even if found, we will be content to investigate the special case of a circular orbit, which as we will argue below gives a zero amplitude under certain approximations and is thus independent of the precise form of the measure.

First, we would like to revisit the assumption $X_a = \mathsf{X}_a\Phi$ used in the previous subsection, where X_a is a c-function. We begin by inspecting the Euclideanized equations of motion for a circular orbit in the 45-plane, written in terms of $Z = X_4 + iX_5$ and

$\bar{Z} = X_4 - iX_5$:

$$\begin{aligned} -\mathcal{D}^2 Z + \frac{1}{4}Z + g^2[\Phi,[\Phi,Z]] - \frac{1}{2}g^2[Z,[Z,\bar{Z}]] &= 0 \\ -\mathcal{D}^2 \bar{Z} + \frac{1}{4}\bar{Z} + g^2[\Phi,[\Phi,\bar{Z}]] - \frac{1}{2}g^2[\bar{Z},[\bar{Z},Z]] &= 0 \end{aligned} \tag{6.152}$$

where $\mathcal{D}$ again denotes the covariant derivative on the sphere.

For a vacuum configuration, $D_\mu \Phi = 0$, and it is clear that $Z = \frac{1}{M_W} e^{\tau/2} r\Phi$ and $\bar{Z} = \frac{1}{M_W} e^{-\tau/2} r\Phi$ with r a constant and $M_W = |\Phi|$, satisfy the above equation of motion[11], so indeed the form $X_a = \mathsf{X}_a \Phi$ is justified.

The situation is different, however, when an instanton is present. First of all, $|\Phi|$ no longer stays constant because of the transition between distinct vacua. Defining the constants $\Phi_\pm$ by the following equation in the singular gauge:

$$\Phi(\tau \to \pm\infty) = \Phi_\pm(\frac{\sigma_3}{2}) \tag{6.153}$$

Then in the past infinity, when the membranes tends to the Φ_- vacuum, Z should be of the form $Z = \frac{1}{\Phi_-} e^{\tau/2} r\Phi$ and likewise $Z = \frac{1}{\Phi_+} e^{\tau/2} r\Phi$ in the future infinity. When the instanton is non-trivial, $\Phi_- \neq \Phi_+$, so this prefactor itself must change with time.

To negate this complication, we make an assumption that $|\Phi_+ - \Phi_-| \ll \Phi_+$. In this case, the change in this prefactor is negligible, and could simply be replaced by $M_W = (\Phi_+ - \Phi_-)/2$. For an $(n, n-1)$ instanton, this is the same as requiring $n \gg 1$, which is the limit that is visible on the supergravity side. This limit is therefore quite natural for a comparison to supergravity. However, such a requirement also means the $(1, 0)$ instanton could not be used for a direct comparison with supergravity.

Under the assumption[12] $n \gg 1$, we know that Z should take the form $Z = \frac{1}{M_W} e^{\tau/2} r\Phi$ in past and future infinity. However, whether this is true during the intermediate time is not for certain. One thing we know about such a transition

[11] $|\Phi|$ could be defined in the singular gauge by $\Phi = |\Phi|\frac{\sigma_3}{2}$. Hence $M_W = |\Phi|$ is a constant for a vacuum configuration.

[12] The readers are also reminded that we are doing first-order perturbation theory, which is an expansion on the small parameter r/M_W, meaning that the separation in the 4 to 9 directions is assumed to be small compared with the separation between the two membranes.

is that the action should be minimized, meaning that Z should take up the form that minimizes the action while interpolating between the two vacua. This means we should look for Z that satisfies the equations of motion. To find out what Z should look like, we define:

$$\begin{aligned} Z &= \frac{r}{M_W} e^{\tau/2}(\Phi + W) \\ \bar{Z} &= \frac{r}{M_W} e^{-\tau/2}(\Phi + \bar{W}) \end{aligned} \tag{6.154}$$

where W and $\bar{W}$ are unrelated matrix-valued functions and r is a constant just as before.

In order that Z gives the correct vacua in the past and in the future, W and $\bar{W}$ must go to zero at past and future infinity. Substitution into the equations of motion, and using the fact $\mathcal{D}^2\Phi = D_\tau\Phi$, we have:

$$\begin{aligned} \mathcal{D}^2 W + D_\tau W - g^2[\Phi,[\Phi,W]] &= -2D_\tau\Phi + \mathcal{O}[(\frac{r}{M_W})^2] \\ \mathcal{D}^2 \bar{W} - D_\tau \bar{W} - g^2[\Phi,[\Phi,\bar{W}]] &= 0 + \mathcal{O}[(\frac{r}{M_W})^2] \end{aligned} \tag{6.155}$$

Note that the last term in eqn(6.152) was dropped because it is of order $(r/M_W)^2$ smaller compared with the rest of the equation.

Treating the right-hand side as the driving term, we could see that $\bar{W} = 0$ to the lowest order, while W will be some non-trivial function. In other words, while $\bar{Z}$ could be approximated by $\bar{Z} = \frac{r}{M_W} e^{-\tau/2}\Phi$ in the first-order perturbation in r/M_W, Z must be modified by this unknown function W. Even though we do not know how to solve for W, the relation $\bar{W} = 0$ is enough to ensure that the amplitude of such transition be zero. To see this we note that in eqn(6.150), X_a always appears with ρ^a, and using the explicit form for the ρ matrices given in eqn(6.112), we have:

$$\mathsf{X}_a \rho^a = \begin{pmatrix} \epsilon(\mathsf{X}_4 + i\mathsf{X}_5) & 0 \\ 0 & \epsilon^{-1}(\mathsf{X}_4 - i\mathsf{X}_5) \end{pmatrix} \tag{6.156}$$

Since eqn(6.150) was derived assuming $X = \mathsf{X}\Phi$, which as we just see is only valid for $\bar{Z} = X_4 - iX_5$ but not for $Z = X_4 + iX_5$, the upper left matrix element is not to be trusted. The lower right element, namely $\epsilon^{-1}(\mathsf{X}_4 - i\mathsf{X}_5)$, however, is correct up to the order we are interested in. Therefore we could put $(\mathsf{X}_4 - i\mathsf{X}_5) = e^{-\tau/2}r$. Since the interaction term $\mathcal{L}_{INT}$ in eqn(6.150) is proportional to $(\partial_\tau \mathsf{X}_a + \frac{1}{2}\mathsf{X}_a)\rho^a$, this gives zero in the lower right matrix element. This zero means the term in $\mathcal{L}_{INT}$ that looks like $\bar{\lambda}^{\dot{\alpha}A}(\epsilon\bar{\sigma}^\Phi\sigma^\tau)_{\dot{\alpha}\dot{\beta}}\bar{\lambda}^{\dot{\beta}B}$ will be absent when $A = 3, B = 4$. In other words, of the eight fermionic zero modes that need to be saturated by the fermions in $\mathcal{L}_{INT}$, four of them are absent at the lowest order of r/M_W because of this cancellation, and thus the amplitude remains zero for a circular orbit in an instanton background.

The circular orbit in an instanton background is not supersymmetric, but the above calculation shows that when the separation r in the 4 to 9 directions is sufficiently small compared to M_W, the separation of the membranes in the 1 to 3 directions, the amplitude for quantum tunneling between such vacua is zero. A different way to put it is that such configuration is almost supersymmetric such that the amplitude is suppressed.

Recall that $V_{eff} \sim \sqrt{\det \Delta}$, if it were not for the cancellations above, we would have an interaction of order $(\frac{r}{M_W})^4$, with each of the eight fermion zero modes contributing a factor of $(\frac{r}{M_W})^{1/2}$. However, due to the fact that the circular orbit is "almost supersymmetric," the interaction term for such a trajectory begins at least at order $(\frac{r}{M_W})^5$. In fact from eqn(6.155), we see that $\bar{W}$ begins only at order $(\frac{r}{M_W})^2$. Putting it in $\sqrt{\det \Delta}$, it implies that the interaction term in fact appears only at order $(\frac{r}{M_W})^8$ and higher.

Rephrasing in the eleven-dimensional picture, we begin with two concentric spherical membranes in the X^1 to X^3 directions whose radii differ by $\Delta r_0 = M_W$. Now we allow one of the membranes to move in a circular orbit of radius r in the X^4 to X^9 directions around the membrane fixed at the origin. If M-momentum transfer takes place, the interaction amplitude should be non-zero. However, if we assume $\frac{r}{M_W} \ll 1$, then the amplitude expanded in this small parameter should occur only above order $(\frac{r}{M_W})^4$. This is our limited prediction on the matrix theory side.

6.8 The Supergravity Side

The computation of the supergravity light cone Lagrangian proceeds in similar fashion as in section 5.1. The Einstein equations are diagonalized and solved order by order in curvature corrections expanded in the small parameters $(\frac{\xi}{r_0})$, $(\frac{w}{r_0})$ and $(\frac{z}{r_0})$ where $\xi = \sqrt{w^2+z^2}$. For simplicity we will often denote these small parameters as $(\frac{\xi}{r_0})$ below but it should be clear in the context. The major difference in this section is that we no longer assume the M-momentum transfer k to be zero. This makes the equations much more involved. In this thesis the metric is determined up to the singular terms, i.e., terms that go to infinity as $\xi \to 0$. However, as we will elaborate later, yet higher order in curvature corrections is necessary for a comparison to the matrix theory result on circular trajectory. In this chapter we will briefly describe the results we have obtained so far.

The linearized Einstein equations are in general Laplace equations with singular sources. The simplest of these is the equation of $\bar{h}_{--}$:

$$\Box \bar{h}_{--} = \kappa_{11}^2 T \delta(w)\delta(x^4)...\delta(x^9)\left(\frac{\mu r_0}{3}\right)^{-1} \tag{6.157}$$

where $\Box = g_{++}k^2 + \partial_A\partial_A = \Box_0 + \delta\Box$ with

$$\Box_0 = -\left(\frac{\mu r_0}{3}k\right)^2 + \left(\frac{\partial^2}{\partial w^2} + \frac{\partial^2}{\partial z^2} + \frac{5}{z}\frac{\partial}{\partial z}\right) \tag{6.158}$$

$$\delta\Box = (\frac{\mu r_0 k}{3})^2(2\frac{w}{r_0} + \frac{w^2}{r_0^2} + \frac{z^2}{4r_0^2}) + \frac{2}{r_0+w}\frac{\partial}{\partial w} \tag{6.159}$$

The other equations can be written using the systematic approach described in section 5.1. The metric and the three-form potential up to third order in curvature corrections are:

$$\begin{aligned}\bar{h}_{--} = \Delta\frac{\exp\left(-\frac{\mu r_0}{3}k\xi\right)}{\xi^5}&\left[3\left(\frac{r_0}{r_0+w}\right) + 3\left(\frac{\mu r_0}{3}k\xi\right)\left(\frac{r_0}{r_0+w}\right)\right.\\ &\left. + \left(\frac{\mu r_0}{3}k\xi\right)^2\left(1 - \frac{3}{2}\frac{w}{r_0} + \frac{9}{8}\frac{w^2}{r_0^2} - \frac{1}{4}\frac{z^2}{r_0^2}\right) + \left(\frac{\mu r_0}{3}k\xi\right)^3\left(\frac{-1}{2}\right)\frac{w}{r_0} + O(\xi^5)\right]\end{aligned} \tag{6.160}$$

At level -1, we find

$$\bar{h}_{-i} = x^i \Delta \frac{\exp\left(-\frac{\mu r_0}{3} k\xi\right)}{9\xi^3} ik\mu^2 \left[\left(1 - \frac{3}{2}\frac{w}{r_0} + \frac{3}{2}\frac{w^2}{r_0^2} - \frac{1}{2}\frac{z^2}{r_0^2}\right)\right.$$
$$\left. + \left(\frac{\mu r_0}{3} k\xi\right)\left(1 - \frac{3}{2}\frac{w}{r_0}\right) + O(\xi^3)\right] \quad (6.161)$$

$$\bar{h}_{-a} = x^a \Delta \frac{\exp\left(-\frac{\mu r_0}{3} k\xi\right)}{72\xi^3} ik\mu^2 \left[\left(1 - \frac{w}{r_0} + \frac{w^2}{r_0^2}\right) + \left(\frac{\mu r_0}{3} k\xi\right)\left(1 - \frac{w}{r_0}\right) + O(\xi^3)\right] \quad (6.162)$$

$$\begin{aligned} a_{-ij} = \quad & \epsilon_{ijk} x^k \frac{\mu}{3} \Delta \frac{\exp\left(-\frac{\mu r_0}{3} k\xi\right)}{\xi^5} \left[3\left(1 - \frac{1}{2}\frac{w}{r_0} + \frac{1}{6}\frac{w^2+z^2}{r_0^2} - \frac{1}{2}\frac{wz^2}{r_0^3} + \frac{w^2z^2}{r_0^4}\right)\right. \\ & +3\left(\frac{\mu r_0}{3} k\xi\right)\left(1 - \frac{1}{2}\frac{w}{r_0} + \frac{1}{6}\frac{w^2+z^2}{r_0^2} - \frac{1}{2}\frac{wz^2}{r_0^3}\right) \\ & \left. + \left(\frac{\mu r_0}{3} k\xi\right)^2\left(1 - \frac{w}{r_0} - \frac{7}{8}\frac{w^2}{r_0^2} - \frac{z^2}{r_0^2}\right) + \left(\frac{\mu r_0}{3} k\xi\right)^3\left(\frac{-1}{2}\right)\frac{w}{r_0} + O(\xi^5)\right] \end{aligned}$$

$$a_{-ib} = 0, \quad a_{-bc} = 0 \quad (6.163)$$

At level 0,

$$\bar{h}_{+-} = -\left(\frac{\mu r_0}{3}\right)^2 \Delta \frac{\exp\left(-\frac{\mu r_0}{3} k\xi\right)}{\xi^5}\left[3\left(1 + \left(\frac{5}{4}\frac{w^2}{r_0^2} + \frac{7}{8}\frac{z^2}{r_0^2}\right)\left(\frac{r_0}{r_0 + w}\right)\right)\right.$$
$$+ 3\left(\frac{\mu r_0}{3} k\xi\right)\left(1 + \left(\frac{5}{4}\frac{w^2}{r_0^2} + \frac{7}{8}\frac{z^2}{r_0^2}\right)\left(\frac{r_0}{r_0 + w}\right)\right)$$
$$\left. + \left(\frac{\mu r_0}{3} k\xi\right)^2\left(1 - \frac{1}{2}\frac{w}{r_0} - \frac{3}{8}\frac{w^2}{r_0^2} - \frac{5}{8}\frac{z^2}{r_0^2}\right) + \left(\frac{\mu r_0}{3} k\xi\right)^3\left(\frac{-1}{2}\right)\frac{w}{r_0} + O(\xi^5)\right] \quad (6.164)$$

$$a_{-+A} = 0, \quad a_{bce} = 0, \quad a_{ibc} = 0 \quad (6.165)$$

$$a_{ijb} = \epsilon_{ijk} x^k x^b \Delta \frac{\exp\left(-\frac{\mu r_0}{3} k\xi\right)}{2\xi^5} i\mu^2 \left[\left(\frac{\mu r_0}{3} k\xi\right)\left(-\frac{2}{9}\frac{\xi}{r_0} + \frac{13}{36}\frac{w\xi}{r_0^2} - \frac{1}{2}\frac{w^2\xi}{r_0^3} + \frac{5}{36}\frac{\xi^3}{r_0^3}\right)\right.$$
$$\left. + \frac{1}{3}\left(\frac{\mu r_0}{3} k\xi\right)^2 \left(-\frac{2}{3}\frac{\xi}{r_0} + \frac{13}{12}\frac{w\xi}{r_0^2}\right) + O(\xi^5)\right] \quad (6.166)$$

$$\bar{h}_{kb} = -x^k x^b \Delta \frac{\exp\left(-\frac{\mu r_0}{3} k\xi\right)}{2\xi^5} \mu^2 \left[\left(1 - \frac{5}{4}\frac{w}{r_0} + \frac{17}{12}\frac{w^2}{r_0^2} - \frac{1}{12}\frac{z^2}{r_0^2} - \frac{3}{2}\frac{w^3}{r_0^3}\right.\right.$$
$$\left. + \frac{1}{4}\frac{wz^2}{r_0^3} + \frac{3}{2}\frac{w^4}{r_0^4} - \frac{1}{2}\frac{w^2 z^2}{r_0^4}\right)$$
$$+ \left(\frac{\mu r_0}{3} k\xi\right)\left(1 - \frac{5}{4}\frac{w}{r_0} + \frac{17}{12}\frac{w^2}{r_0^2} - \frac{1}{12}\frac{z^2}{r_0^2} - \frac{3}{2}\frac{w^3}{r_0^3} + \frac{1}{4}\frac{wz^2}{r_0^3}\right)$$
$$\left. + \frac{1}{3}\left(\frac{\mu r_0}{3} k\xi\right)^2 \left(1 - \frac{7}{4}\frac{w}{r_0} + \frac{43}{24}\frac{w^2}{r_0^2} - \frac{5}{24}\frac{z^2}{r_0^2}\right) - \frac{1}{6}\left(\frac{\mu r_0}{3} k\xi\right)^3 \frac{w}{r_0} + O(\xi^5)\right] \quad (6.167)$$

and

$$\bar{h}_{ij} = \frac{x^i x^j}{r^2} G + \delta^{ij} K \quad (6.168)$$

with

$$G = \left(\frac{\mu r_0}{3}\right)^2 \Delta \frac{\exp\left(-\frac{\mu r_0}{3} k\xi\right)}{\xi^5} \left[3\left(1 - \frac{w}{r_0} - \frac{z^2}{r_0^2} + \frac{w^3}{r_0^3} + 2\frac{wz^2}{r_0^3} - \frac{w^4}{r_0^4} - \frac{w^2 z^2}{r_0^4} + \frac{z^4}{r_0^4}\right)\right.$$
$$+3\left(\frac{\mu r_0}{3} k\xi\right)\left(1 - \frac{w}{r_0} - \frac{z^2}{r_0^2} + \frac{w^3}{r_0^3} + 2\frac{wz^2}{r_0^3}\right)$$
$$\left. + \left(\frac{\mu r_0}{3} k\xi\right)^2 \left(1 - \frac{3}{2}\frac{w}{r_0} + \frac{1}{8}\frac{w^2}{r_0^2} - \frac{5}{4}\frac{z^2}{r_0^2}\right) + \left(\frac{\mu r_0}{3} k\xi\right)^3 \left(-\frac{1}{2}\right)\frac{w}{r_0} + O(\xi^5)\right]$$

$$K = -\left(\frac{\mu r_0}{3}\right)^2 \Delta \frac{\exp\left(-\frac{\mu r_0}{3} k\xi\right)}{\xi^5} \cdot$$
$$\left[3\left(1 + \frac{1}{2}\frac{w}{r_0} + \frac{7}{12}\frac{w^2}{r_0^2} - \frac{1}{24}\frac{z^2}{r_0^2} - \frac{1}{4}\frac{w^3}{r_0^3} + \frac{3}{8}\frac{wz^2}{r_0^3} + \frac{1}{4}\frac{w^4}{r_0^4} - \frac{1}{24}\frac{w^2 z^2}{r_0^4} + \frac{1}{3}\frac{z^4}{r_0^4}\right)\right.$$
$$+3\left(\frac{\mu r_0}{3} k\xi\right)\left(1 + \frac{1}{2}\frac{w}{r_0} + \frac{7}{12}\frac{w^2}{r_0^2} - \frac{1}{24}\frac{z^2}{r_0^2} - \frac{1}{4}\frac{w^3}{r_0^3} + \frac{3}{8}\frac{wz^2}{r_0^3}\right)$$
$$\left. + \left(\frac{\mu r_0}{3} k\xi\right)^2 \left(1 + \frac{3}{8}\frac{w^2}{r_0^2} + \frac{1}{8}\frac{z^2}{r_0^2}\right) + \left(\frac{\mu r_0}{3} k\xi\right)^3 \left(-\frac{1}{2}\right)\frac{w}{r_0} + O(\xi^5)\right] \quad (6.169)$$

$$\bar{h}_{bd} = \frac{x^b x^d}{z^2} Q + \delta^{bd} S \tag{6.170}$$

with

$$Q = 0 \tag{6.171}$$

$$S = (\mu r_0)^2 \Delta \frac{\exp\left(-\frac{\mu r_0}{3} k\xi\right)}{\xi^5} \cdot$$
$$\left[\left(\frac{1}{2}\frac{w}{r_0} - \frac{7}{18}\frac{w^2}{r_0^2} - \frac{19}{72}\frac{z^2}{r_0^2} + \frac{7}{18}\frac{w^3}{r_0^3} + \frac{19}{72}\frac{wz^2}{r_0^3} - \frac{7}{18}\frac{w^4}{r_0^4} - \frac{19}{72}\frac{w^2z^2}{r_0^4}\right)\right.$$
$$+ \left(\frac{\mu r_0}{3} k\xi\right)\left(\frac{1}{2}\frac{w}{r_0} - \frac{7}{18}\frac{w^2}{r_0^2} - \frac{19}{72}\frac{z^2}{r_0^2} + \frac{7}{18}\frac{w^3}{r_0^3} + \frac{19}{72}\frac{wz^2}{r_0^3}\right)$$
$$\left. + \left(\frac{\mu r_0}{3} k\xi\right)^2 \left(\frac{1}{6}\frac{w}{r_0} - \frac{1}{12}\frac{w^2}{r_0^2} + \frac{1}{24}\frac{z^2}{r_0^2}\right) + O(\xi^5)\right] \tag{6.172}$$

$$a_{ijk} = i\epsilon_{ijk} \left(\frac{\mu r_0}{3}\right)^2 \Delta \frac{\exp\left(-\frac{\mu r_0}{3} k\xi\right)}{\xi^5} \quad \left[3\left(\frac{\mu r_0}{3} k\xi\right)\left(-\frac{1}{6}\frac{\xi}{r_0} - \frac{2}{3}\frac{w\xi}{r_0^2} + \frac{1}{6}\frac{w^2\xi}{r_0^3} + \frac{7}{12}\frac{\xi^3}{r_0^3}\right)\right.$$
$$\left. + \left(\frac{\mu r_0}{3} k\xi\right)^2 \left(-\frac{1}{2}\frac{\xi}{r_0} - 2\frac{w\xi}{r_0^2}\right) + O(\xi^5)\right] \tag{6.173}$$

At level +1

$$a_{+bc} = \quad 0 \tag{6.174}$$

$$a_{+ib} = \quad 0 \tag{6.175}$$

$$a_{+ij} = \quad -\epsilon_{ijk} x^k \left(\frac{\mu r_0}{3}\right)^3 \frac{1}{r_0} \Delta \frac{\exp\left(-\frac{\mu r_0}{3} k\xi\right)}{\xi^5} \cdot$$
$$\left[3\left(1 + 2\frac{w}{r_0} - \frac{w^2}{r_0^2} - \frac{5}{4}\frac{z^2}{r_0^2} + \frac{3}{2}\frac{w^3}{r_0^3} + \frac{7}{4}\frac{wz^2}{r_0^3} - \frac{3}{2}\frac{w^4}{r_0^4} - \frac{5}{4}\frac{w^2z^2}{r_0^4} + \frac{1}{2}\frac{z^4}{r_0^4}\right)\right.$$
$$+3\left(\frac{\mu r_0}{3} k\xi\right)\left(1 + 2\frac{w}{r_0} - \frac{w^2}{r_0^2} - \frac{5}{4}\frac{z^2}{r_0^2} + \frac{3}{2}\frac{w^3}{r_0^3} + \frac{7}{4}\frac{wz^2}{r_0^3}\right)$$
$$\left. + \left(\frac{\mu r_0}{3} k\xi\right)^2 \left(1 + \frac{3}{2}\frac{w}{r_0} - \frac{3}{8}\frac{w^2}{r_0^2} + \frac{1}{2}\frac{z^2}{r_0^2}\right) + \left(\frac{\mu r_0}{3} k\xi\right)^3 \left(-\frac{1}{2}\right)\frac{w}{r_0} + O(\xi^5)\right] \tag{6.176}$$

$$\bar{h}_{+i} = x^i \Delta \frac{\exp\left(-\frac{\mu r_0}{3} k\xi\right)}{162\xi^3} ik\mu^4 r_0^2 \left[\left(1 + \frac{5}{2}\frac{w}{r_0} - \frac{w^2}{r_0^2}\right) + \left(\frac{\mu r_0}{3} k\xi\right)\left(1 + \frac{5}{2}\frac{w}{r_0}\right) + O(\xi^3)\right] \quad (6.177)$$

$$\bar{h}_{+a} = x^a \Delta \frac{\exp\left(-\frac{\mu r_0}{3} k\xi\right)}{324\xi^3} ik\mu^4 r_0^2 \left[\left(1 - \frac{3}{2}\frac{w}{r_0} - \frac{15}{32}\frac{z^2}{r_0^2}\right) + \left(\frac{\mu r_0}{3} k\xi\right)\left(1 - \frac{3}{2}\frac{w}{r_0}\right) + O(\xi^3)\right] \quad (6.178)$$

At level +2

$$\begin{aligned}\bar{h}_{++} = {} & \left(\frac{\mu r_0}{3}\right)^4 \Delta \frac{\exp\left(-\frac{\mu r_0}{3} k\xi\right)}{\xi^5} \cdot \\ & \left[3\left(1 + \frac{5}{2}\frac{w}{r_0} + \frac{31}{12}\frac{w^2}{r_0^2} - \frac{1}{24}\frac{z^2}{r_0^2} + \frac{17}{12}\frac{w^3}{r_0^3} - \frac{1}{12}\frac{wz^2}{r_0^3} + \frac{1}{3}\frac{w^4}{r_0^4} + \frac{23}{24}\frac{w^2z^2}{r_0^4} + \frac{17}{32}\frac{z^4}{r_0^4}\right)\right. \\ & + 3\left(\frac{\mu r_0}{3} k\xi\right)\left(1 + \frac{5}{2}\frac{w}{r_0} + \frac{31}{12}\frac{w^2}{r_0^2} - \frac{1}{24}\frac{z^2}{r_0^2} + \frac{17}{12}\frac{w^3}{r_0^3} - \frac{1}{12}\frac{wz^2}{r_0^3}\right) \\ & \left. + \left(\frac{\mu r_0}{3} k\xi\right)^2 \left(1 + 2\frac{w}{r_0} + \frac{11}{8}\frac{w^2}{r_0^2} + \frac{1}{8}\frac{z^2}{r_0^2}\right) + \left(\frac{\mu r_0}{3} k\xi\right)^3 \left(-\frac{1}{2}\right)\frac{w}{r_0} + O(\xi^5)\right] \end{aligned} \quad (6.179)$$

The light cone Lagrangian can be computed using the metric and three-form potential we obtained above. Using the subscript and superscript to denote the power of velocity v and M-momentum transfer k in the Lagrangian respectively, we have:

$$\delta\mathcal{L}_4^{(3)} = \frac{1}{8}\Pi_- v^4 \Delta \frac{\exp\left(-\frac{\mu r_0}{3} k\xi\right)}{\xi^5} \left(\frac{\mu r_0}{3} k\xi\right)^3 \left(\frac{-1}{2}\right)\frac{w}{r_0} \quad (6.180)$$

$$\delta\mathcal{L}_3^{(3)} = 0,\ \delta\mathcal{L}_2^{(3)} = 0,\ \delta\mathcal{L}_1^{(3)} = 0,\ \delta\mathcal{L}_0^{(3)} = 0 \quad (6.181)$$

$$\delta\mathcal{L}_4^{(2)} = \frac{1}{8}\Pi_- v^4 \Delta \frac{\exp\left(-\frac{\mu r_0}{3} k\xi\right)}{\xi^5} \left(\frac{\mu r_0}{3} k\xi\right)^2 \left(1 - \frac{3}{2}\frac{w}{r_0} + \frac{9}{8}\frac{w^2}{r_0^2} - \frac{1}{4}\frac{z^2}{r_0^2}\right) \quad (6.182)$$

$$\delta\mathcal{L}_3^{(2)} = -\frac{1}{2}\Pi_- v^2 (X^b\partial_0 X^b)\Delta \frac{\exp\left(-\frac{\mu r_0}{3}k\xi\right)}{72\xi^3} ik\mu^2 \left(\frac{\mu r_0}{3}k\xi\right) \tag{6.183}$$

$$\delta\mathcal{L}_2^{(2)} = -\frac{1}{144}\Pi_- v^2 \Delta \frac{\exp\left(-\frac{\mu r_0}{3}k\xi\right)}{\xi^5} \left(\frac{\mu r_0}{3}k\xi\right)^2 (-\mu^2)(6w^2+5z^2) \tag{6.184}$$

$$\delta\mathcal{L}_1^{(2)} = \frac{1}{72}\Pi_- (X^b\partial_0 X^b)\Delta \frac{\exp\left(-\frac{\mu r_0}{3}k\xi\right)}{\xi^2} (i\mu^5 r_0^3 k^2)\left(-\frac{2}{9}\right) \tag{6.185}$$

$$\delta\mathcal{L}_0^{(2)} = 0 \tag{6.186}$$

$$\delta\mathcal{L}_4^{(1)} = \frac{1}{8}\Pi_- v^4 \Delta \frac{\exp\left(-\frac{\mu r_0}{3}k\xi\right)}{\xi^5} 3\left(\frac{\mu r_0}{3}k\xi\right)\left(\frac{r_0}{r_0+w}\right) \tag{6.187}$$

$$\delta\mathcal{L}_3^{(1)} = -\frac{1}{2}\Pi_- v^2 (X^b\partial_0 X^b)\Delta \frac{\exp\left(-\frac{\mu r_0}{3}k\xi\right)}{72\xi^3} ik\mu^2 \left(1-\frac{w}{r_0}\right) \tag{6.188}$$

$$\delta\mathcal{L}_2^{(1)} = \Pi_- v^2 k\mu^3 \Delta \frac{\exp\left(-\frac{\mu r_0}{3}k\xi\right)}{432\xi^4} (w-r_0)(2w^2+5z^2) \tag{6.189}$$

$$\delta\mathcal{L}_1^{(1)} = -\frac{1}{108}\Pi_- (X^b\partial_0 X^b)\Delta \frac{\exp\left(-\frac{\mu r_0}{3}k\xi\right)}{\xi^3} ik\mu^4 r_0(r_0+w) \tag{6.190}$$

$$\delta\mathcal{L}_0^{(1)} = 0 \tag{6.191}$$

Putting $k = 0$ gives the same result as chapter 5 for zero momentum transfer. In the above expressions for $\delta\mathcal{L}$, we only kept the singular terms, i.e., terms that go to infinity as $\xi \to 0$. In general when even higher curvature corrections are included, there will be regular terms in the Lagrangian. However, for any fixed value of v, the singular terms will dominate if we take the membrane limit such that $\xi \to 0$. Therefore, the above expression is the supergravity prediction for the interaction amplitude for the near membrane limit with finite velocity.

6.9 The Limitation of the Computation

At this point we have not yet achieved a direct comparison between matrix theory and supergravity. On the matrix theory side, we have a very limited prediction about the vanishing of the interaction amplitude for circular trajectory up to certain order in the small parameter $\frac{r}{M_W}$. To compute the effective action for a general trajectory will require carefully calculating the measure of the path integral which we have not carried out in this thesis. On the supergravity side, the curvature expansion is carried out only up to the singular terms. We can see that these terms are insufficient for the purpose of verifying the matrix theory claim for circular trajectory by studying a typical term in $\delta\mathcal{L}$:

$$\delta\mathcal{L} \sim \Pi_{-}\Delta e^{-\beta\xi}\frac{1}{\xi^5}(\beta\xi)^2(v^4 + v^2\mu^2\xi^2 + \mu^4\xi^4) \tag{6.192}$$

where $\beta = \frac{\mu r_0 k}{3}$ and we did not keep track of the exact coefficients in the equation above. The first term which is proportional to v^4 is singular in the limit $\xi \to 0$ with v fixed. This term has already been included in the Lagrangian in eqn(6.183). The last term, however, is regular as $\xi \to 0$ with v fixed. This term can come from a metric component of the form:

$$\bar{h}_{++} = \Delta e^{-\beta\xi}\frac{1}{\xi^5}(\beta\xi)^2\mu^4\xi^4 = (\mu r_0)^4\Delta e^{-\beta\xi}(\beta\xi)^2\frac{1}{\xi^5}(\frac{\xi}{r_0})^4 \tag{6.193}$$

where the last equality emphasize that it belongs to the fourth-order curvature correction. Note that $\bar{h}_{++}$ is regular in the limit $\xi \to 0$.

For a circular orbit, $v \sim \mu\xi$, matrix theory predicts that the three terms in eqn(6.192) (with the correct coefficients) will cancel out each other. However, since the last term is not yet computed on the supergravity side, we are unable at this point to make the final comparison, which will require finding even higher curvature correction terms for the metric. One also has to deal with the complementary solution of the Laplace equation carefully, but we will leave that as possible future work.

Chapter 7

Discussion

In this thesis we compared interactions of gravitons and membranes computed using matrix theory and supergravity. We found agreement in the absence of M-momentum transfer. This can be viewed as evidence for the matrix theory conjecture in the pp-wave background. It also points to the existence of a non-renormalization theorem similar to the one in flat space [14].

We can extend the above results by pushing the matrix computations to higher loops. On the supergravity side, this corresponds to terms of higher order in κ_{11}^2. This means we have to take into account recoil and other back reactions carefully. We can also generalize to three-body interactions, for which interesting terms will begin to appear at two loops. The configurations considered in this thesis all have a source located at the origin, we can instead allow both the source and the probe to take up more general trajectories and get a more general result for the effective potentials. We can also push our computation on the supergravity side beyond the near-membrane expansion, finding the solution to the field equations which will then give the $\delta\mathcal{L}_{lc}$ to be compared with the fully interpolating potential (5.95) found on the matrix theory side. The major obstacle in such pursuits is the large amount of algebra involved.

For interactions with M-momentum transfer, we need to push the supergravity calculation to even higher order to get a limited comparison with matrix theory. The instanton equation of the three-dimensional theory is also quite interesting in its own right and deserves deeper study.

One other generalization is to consider more complicated membrane configurations. We have restricted our attention to a probe membrane which is spherical and has no velocity in the x^1 through x^3 directions. For example, we could consider deforming the probe membrane so that it is no longer a perfect sphere. It means that the coordinates of the membrane X^A will now be some general function of θ and ϕ, and in particular it no longer has to appear only as a point in the 4 through 9 directions. We can also give the probe membrane a nonzero velocity in the x^1 through x^3 direction. These generalizations give more interesting dynamics and can be fairly easily carried out. On the supergravity side, this just requires putting the relevant probe configuration into the light cone Lagrangian; on the matrix theory side, this requires replacing the background configurations B^A in this thesis with some more general configurations.

There are also M-theory pp-wave backgrounds with fewer supersymmetries, and matrix theories in these pp-waves backgrounds have been proposed [54, 55]. It will be interesting to investigate the gauge/gravity duality in these less supersymmetric settings.

Our final remark concerns the non-renormalization theorem. Although it is not emphasized in this thesis, the non-renormalization theorem is a necessary ingredient for a meaningful comparison between matrix theory and supergravity. Our results give evidence for its existence, but it is obviously desirable to derive it directly using the symmetries of the pp-wave background. In flat space the derivation relies on the $SO(9)$ symmetry of the transverse space, but in pp-wave this is broken to $SO(3) \times SO(6)$ by the mass parameter μ. It remains to be seen how the flat space result can be generalized.

Appendix A

Notations

Notations and Some Frequently Used Equations

The Indices:

Target space indices:

$\mu, \nu, ... = +, -, 1, 2, ..., 9$

$A, B, ... = 1, 2, ..., 9$

$i, j, k, ... = 1, 2, 3$

$a, b, ... = 4, 5, ..., 9$

World volume indices for a membrane (the three-dimensional gauge theory):

$\alpha, \beta, ... = 1, 2, 3$

$SU(2)$ group indices in the adjoint representation:

$m, n, p = 1, 2, 3$

Exceptions:

Occasionally we use $m, n = 1, 2, 3, 4$ to label the coordinates of the four-dimensional gauge theory on $R^{1,1} \times S^2$, and $i, j, k = 1, 2, 3$ as a label for x_i which parametrizes a unit two sphere in R^3. Superscript ^{A,B} are used to denote $SU(4)$ indices as well.

In section 6.7 we use the following indices:

$$\begin{aligned} i, j &= 1, 2, 3 \\ m, n &= \tau, \Phi, \theta, \phi \\ \alpha, \beta &= \Phi, \theta, \phi \\ \mu, \nu &= \tau, \theta, \phi \\ \Omega &= \theta, \phi \end{aligned}$$

The interpretations of variables:

M: The eleven-dimensional Planck constant.

R: The light-like compactification radius in the DLCQ formalism of matrix theory.

μ: The pp-wave parameter. In our convention the four form field strength $F_{+123} = \mu$.

r_0: The physical radius of the spherical membrane in the eleven-dimensional picture.

P: The *total* M-momentum carried by the membrane.

p: The M-momentum *density* of the membrane. This is basically the total momentum above divided by the area of the membrane.

g: The coupling constant in the three-dimensional gauge theory.

x_i: The Cartesian coordinates parametrizing a unit two sphere in R^3.

The operators:

$\{f, g\} = \frac{1}{sin\theta}(\partial_\theta f \partial_\phi g - \partial_\theta g \partial_\phi f)$

$L_i Z = i\{x_i, Z\} = -i\epsilon_{ijk} x_j \partial_k Z$

$\mathcal{L}_i Z = L_i Z + g[Y_i, Z]$

$D_i Z = \partial_i Z - ig[A_i, Z]$

$\mathcal{D}_m a_n = \nabla_m a_n - ig[A^{cl}_m, a_n] = \partial_m a_n - \Gamma^p_{mn} a_p - ig[A^{cl}_m, a_n] = D_m a_n - \Gamma^p_{mn} a_p$

$F_{ij} = \partial_i A_j - \partial_j A_i - ig[A_i, A_j]$

The four-dimensional theory on $R^2 \times S^2$ (after Euclideanization):

$ds^2 = d\tau^2 + d\theta^2 + sin^2\theta d\phi^2 + (dx^\Phi)^2 \equiv g_{mn} dx^m dx^n$

$A_\Phi = \Phi$

$\mathcal{F}_{mn} = F_{mn} - \varepsilon_{\tau\Phi mn}\Phi$

$F_{mn} = \partial_m A_n - \partial_n A_m - ig[A_m, A_n]$

$\varepsilon_{\tau\Phi\theta\phi} = +\sqrt{|g|}\epsilon_{\tau\Phi\theta\phi} = \sin\theta$

$\mathcal{D}_m a_n = \nabla_m a_n - ig[A^{cl}_m, a_n] = \partial_m a_n - \Gamma^p_{mn} a_p - ig[A^{cl}_m, a_n] = D_m a_n - \Gamma^p_{mn} a_p$

$\langle a|a\rangle = \int d\tau d^2\Omega g^{mn} a_m a_n$

Euclideanization

$\tau_E = i\tau_M$

$S_E = -iS_M$

$\Gamma^\tau = i\Gamma^0$

$\sigma^\tau = i\sigma^0$

$\bar{\sigma}^\tau = i\bar{\sigma}^0$

Gamma Matrices

$\Gamma_\Omega = \epsilon_{ijk}\Gamma^i x^j \partial_\Omega x^k$

$\Gamma_\Phi = x_i \Gamma^i$

$\Gamma_i = x_i \Gamma^\Phi + \epsilon_{ijk} x^j \partial_\Omega x^k \Gamma^\Omega$

$\Gamma_{\Phi\theta\phi} = +\sqrt{g}\Gamma_{123}$

$\Gamma_{mn} = -\frac{1}{2}\varepsilon_{mnpq}\Gamma^{pq}\Gamma_{\tau 123}$

$\mathcal{D}_\mu \Gamma^{\Phi\mu} = \nabla_\mu \Gamma^{\Phi\mu} = 2\Gamma^\Phi \Gamma_{123} = 2\Gamma^{\Phi\tau}\Gamma_{\tau 123}$

Other notations:

$\alpha = \frac{1}{M^3 R}$

$\tau = \mu t$

$d^2\Omega = d\theta d\phi sin\theta$

$F_{ij} = \partial_i A_j - \partial_j A_i - ig[A_i, A_j]$

$\mathcal{B}_i = Y_i + i\epsilon_{ijk} L_j Y_k + \frac{i}{2} g\epsilon_{ijk}[Y_j, Y_k] = Y_i + \frac{i}{2}\epsilon_{ijk}(\mathcal{L}_j Y_k - \mathcal{L}_k Y_j)$

$\sigma^m = (1, \vec{\sigma})$

$\bar{\sigma}^m = (1, -\vec{\sigma})$

An (n_i, n_f) instanton is an instanton that takes us from the n_i vacuum to the n_f vacuum.

The $SO(9)$ gamma matrices γ are chosen to be real and symmetric with dimension 16×16.

The $SO(9,1)$ gamma matrices Γ are complex with dimension 32×32.

Some Frequently Used Equations:

$$
\begin{aligned}
P &= N/R \\
r_0 &= \frac{\mu}{6}P \\
P &= 4\pi r_0^2 p \\
\mu p r_0 &= \frac{3}{2\pi} \\
\mu^2 p P &= \frac{9}{\pi} \\
g &= \frac{3^{3/2}}{\mu^{3/2}P^{1/2}} = 2^{1/2}\pi p\sqrt{r_0} = \frac{3}{2^{1/2}\mu\sqrt{r_0}} = 3^{1/2}\pi^{1/2}\sqrt{\frac{p}{\mu}} \\
\{x_i, x_j\} &= \epsilon_{ijk}x_k \\
\mathcal{B}_i &= Y_i + i\epsilon_{ijk}L_jY_k + \frac{i}{2}g\epsilon_{ijk}[Y_j, Y_k] = x_i(\Phi - F_{\theta\phi}) - D_i\Phi \\
K(\tau) &= \int d^2\Omega\{-\frac{1}{2}Y_i^2 + \frac{i}{2}\epsilon_{ijk}Y_iL_jY_k + \frac{i}{3}g\epsilon_{ijk}Y_iY_jY_k\} = -\frac{1}{2}\int d^2\Omega\ \Phi^2 \\
\mathcal{D}^2\Phi &= D_\tau\Phi
\end{aligned}
$$

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