Appendix B. Using Unit Quaternions to Represent Rotations: How a Unit Quaterion Applied to a Reference Vector Can Create Our Fault Parameters, Strike, Dip, and Rake

Two alternative representations of a 3D rotation are quaternions and Euler angle rotations. Typically, Euler angle rotations are applied to a reference vector to describe an earthquake slip vector in terms of the fault parameters, strike, dip, and rake (Θ,δ,λ). This appendix utilizes the relationships between unit quaternions and Euler angles to derive the fault parameters (Θ,δ,λ) given a unit quaternion applied to a reference slip vector.

Figure B.1 starts with an arbitrary slip vector on a fault plane. The local, rotated coordinates for this slip vector are defined as $\hat{E}'''$, $\hat{N}'''$, and $\hat{Up}'''$ where $\hat{E}'''$ and $\hat{N}'''$ are on the fault plane and $\hat{Up}'''$ is perpendicular to the fault plane. $\hat{N}'''$ is aligned with the slip vector. The slip vector represents the motion of the footwall; therefore, this is primarily a normal fault with a small amount of right-lateral strike-slip motion. To represent the slip vector in terms of the coordinates, $\hat{E}$, $\hat{N}$, and $\hat{Up}$, we rotate the coordinates with Euler angle rotation matrices. The first rotation matrix is

$$R_\lambda = \begin{pmatrix} \cos\lambda & -\sin\lambda & 0 \\ \sin\lambda & \cos\lambda & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \text{(B.1)}$$

which is a clockwise rotation of the coordinate system by the rake angle, λ. The second one is

$$R_\delta = \begin{pmatrix} \cos\delta & 0 & \sin\delta \\ 0 & 1 & 0 \\ -\sin\delta & 0 & \cos\delta \end{pmatrix}, \qquad \text{(B.2)}$$

which is a clockwise rotation of the coordinate system by the dip angle, δ. The last and final rotation matrix is

$$R_{\Theta} = \begin{pmatrix} \cos\Theta & \sin\Theta & 0 \\ -\sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{B.3}$$

which is a counter-clockwise rotation of the coordinates by the strike angle, Θ. The three consecutive rotations are shown in Figure B.1, where the triple primed coordinates, $\hat{E}'''$, $\hat{N}'''$, and $\hat{U}p'''$ are transformed into the unprimed coordinates, $\hat{E}$, $\hat{N}$, and $\hat{U}p$.

Indeed, if we begin with the slip vector $\vec{l}''' = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ in the triple primed coordinates, aligned with the $\hat{N}'''$ direction, and apply these three rotation matrices to transform the coordinates, we derive

$$\vec{l} = \begin{pmatrix} \sin\Theta\cos\lambda - \cos\Theta\cos\delta\sin\lambda \\ \cos\Theta\cos\lambda + \sin\Theta\cos\delta\sin\lambda \\ \sin\delta\sin\lambda \end{pmatrix},$$

which is the same as Equation A.9 from Appendix A.

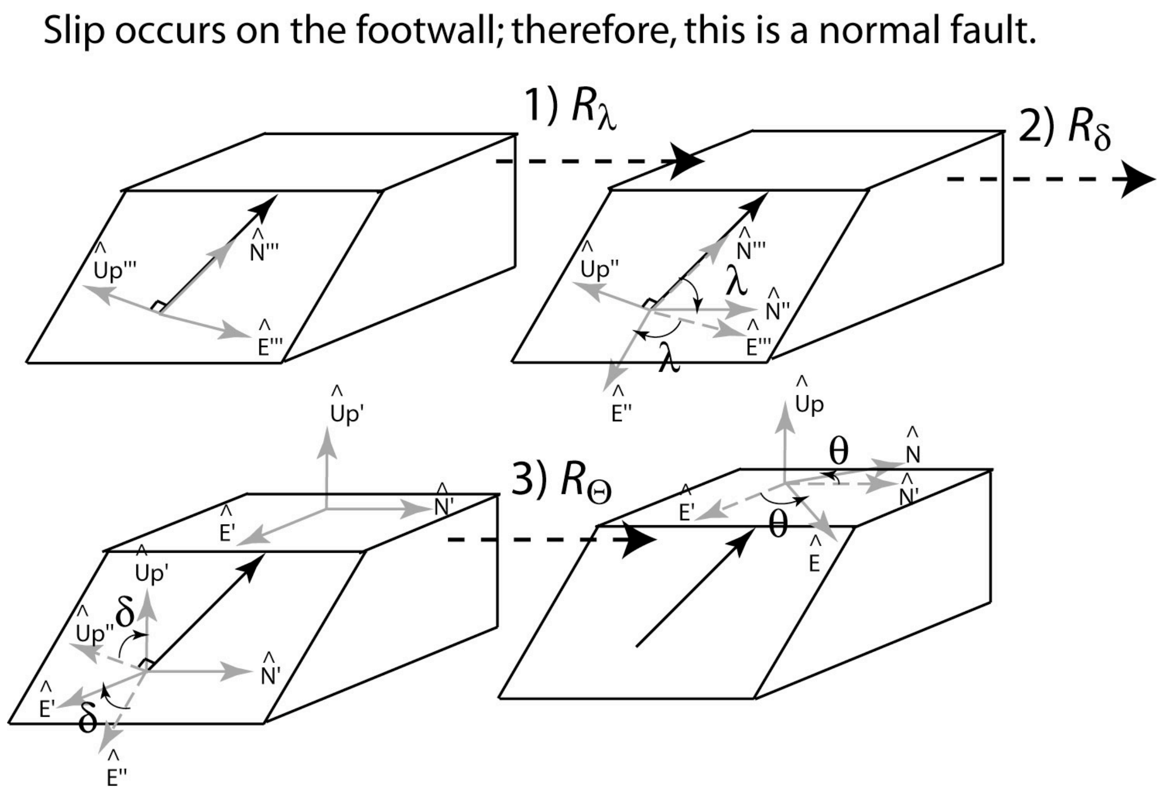

Figure B.1. *Our triple primed coordinates are oriented so that our slip vector is aligned with $\hat{N}'''$. We rotate the coordinate system with three rotations: 1) A clockwise rotation of angle λ about the $\hat{U}p'''$ axis. 2) A clockwise rotation of angle δ about the $\hat{N}''$ axis. 3) A counter-clockwise rotation of angle Θ about the $\hat{U}p'$ axis.*

This demonstration of fault parameters strike, dip, and rake $(\Theta, \delta, \lambda)$ using Euler angle rotations on a reference slip vector, will help us explain how to apply a unit quaternion to a reference slip vector to produce $(\Theta, \delta, \lambda)$.

To work with unit quaternions, we start with the reference slip vector, $\vec{l}''' = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ in the local, triple primed coordinate system, and measure other slip vectors in terms of their rotations relative to $\vec{l}'''$. The unit quaternion, $\vec{q}''' = [q_0''', q_1''', q_2''', q_3'''] = [1,0,0,0]$, produces a rotation of zero degrees; therefore, when applied to $\vec{l}'''$, simply reproduces $\vec{l}'''$. We then translate our Euler angle rotations $(\Theta, \delta, \lambda)$ into unit quaternions, and rotate $\vec{l}'''$ into the global, unprimed coordinate system. From Chapter 3, we can rewrite Equation 3.12 in terms of a Cartesian coordinate system for our Euler angle rotations:

$$
\begin{aligned}
q_0 &= \cos(\omega/2) \\
q_1 &= \sin(\omega/2)u_1 \\
q_2 &= \sin(\omega/2)u_2 \\
q_3 &= \sin(\omega/2)u_3
\end{aligned}
\tag{B.4}
$$

where $\vec{u} = [u_1, u_2, u_3]$ is the rotation axis for the Euler rotations, and $|\vec{u}| = \sqrt{u_1^2 + u_2^2 + u_3^2} = 1.0$.

If $u_1 = \hat{E}'''$, $u_2 = \hat{N}'''$, and $u_3 = \hat{U}p'''$, for the first Euler rotation of angle λ, then the corresponding unit quaternion would be

$$
\begin{aligned}
q_0^\lambda &= \cos\frac{\lambda}{2} \\
q_1^\lambda &= 0 \\
q_2^\lambda &= 0 \\
q_3^\lambda &= \sin\frac{\lambda}{2}.
\end{aligned}
\tag{B.5}
$$

If $u_1 = \hat{E}''$, $u_2 = \hat{N}''$, and $u_3 = \hat{U}p''$, for the second Euler rotation of angle δ, then the corresponding unit quaternion would be

$$
\begin{aligned}
q_0^\delta &= \cos\frac{\delta}{2} \\
q_1^\delta &= 0 \\
q_2^\delta &= \sin\frac{\delta}{2} \\
q_3^\delta &= 0.
\end{aligned}
\tag{B.6}
$$

Last, if $u_1 = \hat{E}'$, $u_2 = \hat{N}'$, and $u_3 = \hat{U}p'$, for the third Euler rotation of angle Θ, then the corresponding unit quaternion would be

$$
\begin{aligned}
q_0^\Theta &= \cos\frac{\Theta}{2} \\
q_1^\Theta &= 0 \\
q_2^\Theta &= 0 \\
q_3^\Theta &= \sin\frac{-\Theta}{2} = -\sin\frac{\Theta}{2}.
\end{aligned}
\tag{B.7}
$$

Since a quaternion is a hyper complex four vector,

$$
\vec{q} = q_0 + \boldsymbol{i}q_1 + \boldsymbol{j}q_2 + \boldsymbol{k}q_3, \tag{B.8}
$$

with the rules that

$$
\begin{aligned}
\boldsymbol{i}^2 &= \boldsymbol{j}^2 = \boldsymbol{k}^2 = -1 \\
\boldsymbol{ij} &= -\boldsymbol{ji} = \boldsymbol{k} \\
\boldsymbol{jk} &= -\boldsymbol{kj} = \boldsymbol{i} \\
\boldsymbol{ki} &= -\boldsymbol{ik} = \boldsymbol{j},
\end{aligned}
\tag{B.9}
$$

we can rewrite our three unit quaternions as

$$
\begin{aligned}
\vec{q}^{\,\lambda} &= \cos\frac{\lambda}{2} + \boldsymbol{k}\sin\frac{\lambda}{2} \\
\vec{q}^{\,\delta} &= \cos\frac{\delta}{2} + \boldsymbol{j}\sin\frac{\delta}{2} \\
\vec{q}^{\,\Theta} &= \cos\frac{\Theta}{2} - \boldsymbol{k}\sin\frac{\Theta}{2}
\end{aligned}
\tag{B.10}
$$

and simply multiply these three unit quaternions in Equation (B.10), using the rules in Equation (B.9) to produce the unit quaternion that rotates our reference slip vector, $\vec{l}'''$, into the unprimed, global coordinate system, $\hat{E}$, $\hat{N}$, and $\hat{U}p$.

$$\vec{q}^{\Theta\delta\lambda} = \vec{q}^{\Theta}\vec{q}^{\delta}\vec{q}^{\lambda} = \left(\cos\frac{\Theta}{2} - \boldsymbol{k}\sin\frac{\Theta}{2}\right)\left(\cos\frac{\delta}{2} + \boldsymbol{j}\sin\frac{\delta}{2}\right)\left(\cos\frac{\lambda}{2} + \boldsymbol{k}\sin\frac{\lambda}{2}\right)$$

$$\begin{aligned}
q_0^{\Theta\delta\lambda} &= \cos\frac{\Theta}{2}\cos\frac{\delta}{2}\cos\frac{\lambda}{2} + \sin\frac{\Theta}{2}\cos\frac{\delta}{2}\sin\frac{\lambda}{2} \\
q_1^{\Theta\delta\lambda} &= \cos\frac{\Theta}{2}\sin\frac{\delta}{2}\sin\frac{\lambda}{2} + \sin\frac{\Theta}{2}\sin\frac{\delta}{2}\cos\frac{\lambda}{2} \\
q_2^{\Theta\delta\lambda} &= \cos\frac{\Theta}{2}\sin\frac{\delta}{2}\cos\frac{\lambda}{2} - \sin\frac{\Theta}{2}\sin\frac{\delta}{2}\sin\frac{\lambda}{2} \\
q_3^{\Theta\delta\lambda} &= \cos\frac{\Theta}{2}\cos\frac{\delta}{2}\sin\frac{\lambda}{2} - \sin\frac{\Theta}{2}\cos\frac{\delta}{2}\cos\frac{\lambda}{2}
\end{aligned} \tag{B.11}$$

Rewriting equation (B.11), we can also represent our fault parameters, $(\Theta, \delta, \lambda)$ in terms of unit quaterion components, given our reference slip vector.

$$\begin{aligned}
\lambda &= \tan^{-1}\left(\frac{q_0^{\Theta\delta\lambda}q_1^{\Theta\delta\lambda} + q_2^{\Theta\delta\lambda}q_3^{\Theta\delta\lambda}}{q_0^{\Theta\delta\lambda}q_2^{\Theta\delta\lambda} - q_1^{\Theta\delta\lambda}q_3^{\Theta\delta\lambda}}\right) \\
\Theta &= \tan^{-1}\left(\frac{q_0^{\Theta\delta\lambda}q_1^{\Theta\delta\lambda} - q_2^{\Theta\delta\lambda}q_3^{\Theta\delta\lambda}}{q_0^{\Theta\delta\lambda}q_2^{\Theta\delta\lambda} + q_1^{\Theta\delta\lambda}q_3^{\Theta\delta\lambda}}\right) \\
\delta &= \tan^{-1}\left(\frac{2\left(q_0^{\Theta\delta\lambda}q_1^{\Theta\delta\lambda} + q_2^{\Theta\delta\lambda}q_3^{\Theta\delta\lambda}\right)/\sin\lambda}{q_0^{\Theta\delta\lambda}q_0^{\Theta\delta\lambda} - q_1^{\Theta\delta\lambda}q_1^{\Theta\delta\lambda} - q_2^{\Theta\delta\lambda}q_2^{\Theta\delta\lambda} + q_3^{\Theta\delta\lambda}q_3^{\Theta\delta\lambda}}\right)
\end{aligned} \tag{B.12}$$

For a unit quaternion, $\vec{q} = q_0 + \boldsymbol{i}q_1 + \boldsymbol{j}q_2 + \boldsymbol{k}q_3$, the rotation matrix one would apply to rotate the coordinates of our vector $\vec{l}''' = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ into another reference frame, is

$$\vec{l} = \begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix} = \begin{pmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2q_1q_2 - 2q_0q_3 & 2q_0q_2 + 2q_1q_3 \\ 2q_0q_3 + 2q_1q_2 & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2q_2q_3 - 2q_0q_1 \\ 2q_1q_3 - 2q_0q_2 & 2q_0q_1 + 2q_2q_3 & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}. \quad \text{(B.13)}$$

In our case, we want to use the unit quaternion described in equation (B.11)

Therefore, we can write the components of $\vec{l}$ as

$$\begin{aligned} l_1 &= 2q_1^{\Theta\delta\lambda}q_2^{\Theta\delta\lambda} - 2q_0^{\Theta\delta\lambda}q_3^{\Theta\delta\lambda} \\ l_2 &= \left(q_0^{\Theta\delta\lambda}\right)^2 - \left(q_1^{\Theta\delta\lambda}\right)^2 + \left(q_2^{\Theta\delta\lambda}\right)^2 - \left(q_3^{\Theta\delta\lambda}\right)^2 \\ l_3 &= 2q_0^{\Theta\delta\lambda}q_1^{\Theta\delta\lambda} + 2q_2^{\Theta\delta\lambda}q_3^{\Theta\delta\lambda} \end{aligned} \quad \text{(B.14)}$$

or as

$$\begin{aligned} &l_1 \\ &= 2\left(\left(\cos\frac{\Theta}{2}\sin\frac{\delta}{2}\sin\frac{\lambda}{2} + \sin\frac{\Theta}{2}\sin\frac{\delta}{2}\cos\frac{\lambda}{2}\right)\left(\cos\frac{\Theta}{2}\sin\frac{\delta}{2}\cos\frac{\lambda}{2} - \sin\frac{\Theta}{2}\sin\frac{\delta}{2}\sin\frac{\lambda}{2}\right)\right) \\ &-2\left(\left(\cos\frac{\Theta}{2}\cos\frac{\delta}{2}\cos\frac{\lambda}{2} + \sin\frac{\Theta}{2}\cos\frac{\delta}{2}\sin\frac{\lambda}{2}\right)\left(\cos\frac{\Theta}{2}\cos\frac{\delta}{2}\sin\frac{\lambda}{2} - \sin\frac{\Theta}{2}\cos\frac{\delta}{2}\cos\frac{\lambda}{2}\right)\right) \\ &l_1 = \cos^2\frac{\Theta}{2}\sin^2\frac{\delta}{2}\sin\lambda - \sin^2\frac{\delta}{2}\sin^2\frac{\lambda}{2}\sin\Theta + \sin^2\frac{\delta}{2}\cos^2\frac{\lambda}{2}\sin\Theta - \sin^2\frac{\Theta}{2}\sin^2\frac{\delta}{2}\sin\lambda \quad \text{(B.15)} \\ &-\cos^2\frac{\Theta}{2}\cos^2\frac{\delta}{2}\sin\lambda + \cos^2\frac{\delta}{2}\cos^2\frac{\lambda}{2}\sin\Theta - \cos^2\frac{\delta}{2}\sin^2\frac{\lambda}{2}\sin\Theta + \sin^2\frac{\Theta}{2}\cos^2\frac{\delta}{2}\sin\lambda \\ &l_1 = \sin^2\frac{\delta}{2}\cos\lambda\sin\Theta + \sin^2\frac{\delta}{2}\cos\Theta\sin\lambda + \cos^2\frac{\delta}{2}\cos\lambda\sin\Theta - \cos^2\frac{\delta}{2}\cos\Theta\sin\lambda \\ &l_1 = \sin\Theta\cos\lambda - \cos\Theta\cos\delta\sin\lambda \end{aligned}$$

$$
\begin{aligned}
l_2 &= \\
&= \left(\cos\frac{\Theta}{2}\cos\frac{\delta}{2}\cos\frac{\lambda}{2}+\sin\frac{\Theta}{2}\cos\frac{\delta}{2}\sin\frac{\lambda}{2}\right)^2-\left(\cos\frac{\Theta}{2}\sin\frac{\delta}{2}\sin\frac{\lambda}{2}+\sin\frac{\Theta}{2}\sin\frac{\delta}{2}\cos\frac{\lambda}{2}\right)^2 \\
&+\left(\cos\frac{\Theta}{2}\sin\frac{\delta}{2}\cos\frac{\lambda}{2}-\sin\frac{\Theta}{2}\sin\frac{\delta}{2}\sin\frac{\lambda}{2}\right)^2-\left(\cos\frac{\Theta}{2}\cos\frac{\delta}{2}\sin\frac{\lambda}{2}-\sin\frac{\Theta}{2}\cos\frac{\delta}{2}\cos\frac{\lambda}{2}\right)^2 \\
l_2 &= \cos^2\frac{\Theta}{2}\cos^2\frac{\lambda}{2}\left(\cos^2\frac{\delta}{2}+\sin^2\frac{\delta}{2}\right)+\sin^2\frac{\Theta}{2}\sin^2\frac{\lambda}{2}\left(\cos^2\frac{\delta}{2}+\sin^2\frac{\delta}{2}\right) \\
&-\cos^2\frac{\Theta}{2}\sin^2\frac{\lambda}{2}\left(\cos^2\frac{\delta}{2}+\sin^2\frac{\delta}{2}\right)-\sin^2\frac{\Theta}{2}\cos^2\frac{\lambda}{2}\left(\cos^2\frac{\delta}{2}+\sin^2\frac{\delta}{2}\right) \\
&+\sin\Theta\sin\lambda\cos^2\frac{\delta}{2}-\sin\Theta\sin\lambda\sin^2\frac{\delta}{2} \\
l_2 &= \left(\cos^2\frac{\Theta}{2}-\sin^2\frac{\Theta}{2}\right)\cos\lambda+\sin\Theta\sin\lambda\cos\delta \\
l_2 &= \cos\Theta\cos\lambda+\sin\Theta\cos\delta\sin\lambda
\end{aligned}
\tag{B.16}
$$

$$
\begin{aligned}
l_3 & \\
&= 2\left(\left(\cos\frac{\Theta}{2}\cos\frac{\delta}{2}\cos\frac{\lambda}{2}+\sin\frac{\Theta}{2}\cos\frac{\delta}{2}\sin\frac{\lambda}{2}\right)\left(\cos\frac{\Theta}{2}\sin\frac{\delta}{2}\sin\frac{\lambda}{2}+\sin\frac{\Theta}{2}\sin\frac{\delta}{2}\cos\frac{\lambda}{2}\right)\right) \\
&+2\left(\left(\cos\frac{\Theta}{2}\sin\frac{\delta}{2}\cos\frac{\lambda}{2}-\sin\frac{\Theta}{2}\sin\frac{\delta}{2}\sin\frac{\lambda}{2}\right)\left(\cos\frac{\Theta}{2}\cos\frac{\delta}{2}\sin\frac{\lambda}{2}-\sin\frac{\Theta}{2}\cos\frac{\delta}{2}\cos\frac{\lambda}{2}\right)\right) \\
l_3 &= \cos^2\frac{\Theta}{2}\sin\delta\sin\lambda+\sin^2\frac{\Theta}{2}\sin\delta\sin\lambda \\
l_3 &= \sin\delta\sin\lambda.
\end{aligned}
\tag{B.17}
$$

So again, now with unit quaternions, we have calculated that

$$
\vec{l} = \begin{pmatrix} \sin\Theta\cos\lambda-\cos\Theta\cos\delta\sin\lambda \\ \cos\Theta\cos\lambda+\sin\Theta\cos\delta\sin\lambda \\ \sin\delta\sin\lambda \end{pmatrix}.
$$