The Near-Penrose Limit of *AdS/CFT*

Thesis by

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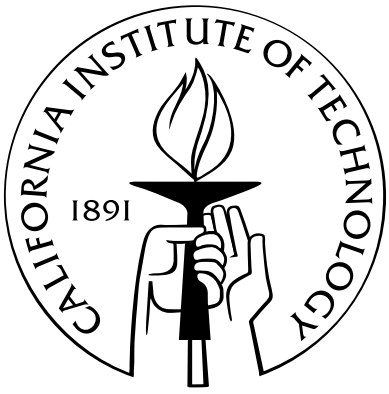
In Partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy



California Institute of Technology

Pasadena, California

2005

(Submitted May 24, 2005)

Acknowledgements

I am fortunate to owe a great many thanks to many people. Firstly, to my advisor John Schwarz who has graciously supported and guided me throughout my graduate career. To my collaborators Curt Callan, Jonathan Heckman, Hok Kong Lee, Ian Swanson and Xinkai Wu who deserve whatever credit there is and none of the blame. I have benefited from the insight and knowledge of many of the physics graduate students, post-docs and faculty at Caltech and for that I thank them. In different but equally appreciated ways, my time in graduate school has been brightened by the friends I have found. Too many to name but hopefully you know who you are. I must further thank my roommates of the last five years for all their efforts, Claudine for her patience, her time and her suggestions (particularly the ones I disagreed with) and finally my Mum, Dad and brother for always being there for me.

Abstract

The conjectured duality between type IIB string theory on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ $SU(N_c)$ super Yang-Mills theory in four dimensions simplifies in the Penrose limit or, in other words, when the string's angular momentum is large. As the string action in this limit is solvable, it is possible to go beyond the supergravity approximation and compare exact string energies with the anomalous dimensions of a sector of large $\mathcal{R}$-charge operators. This equivalence should of course extend to the full $AdS_5 \times S^5$ space and to operators of finite $\mathcal{R}$-charge. We take some modest steps in this direction by expanding the full string action in inverse powers of the angular momentum and finding the first order perturbative corrections to the energy spectrum. These corrections reproduce the gauge theory anomalous dimensions for a range of different operators to two-loops in the 't Hooft parameter but disagree at three-loops. Furthermore, these near-plane wave results are useful in studying the recently discovered integrability in this AdS/CFT system and can be used to motivate the form of quantum string scattering matrices.

Contents

Chapter 1

Introduction

The originals are not original. There is imitation, model, and suggestion, to the very archangels, if we knew their history. –R. W. Emerson, Quotation and Originality

String theory originated in attempts, called dual models, to explain the interactions of hadrons and initially had several successes. Large numbers of hadron resonances were being discovered and it was noted that the mass of the lightest hadrons with a given spin J roughly lay on Regge trajectories, that is they satisfied

$$m^2 \sim \frac{J}{\alpha'} + \text{constant}, \quad \alpha' \sim (1\text{GeV})^{-2} \ . \tag{1.0.1}$$

Furthermore, based on experimental evidence, Dolen, Horn and Schmid [1] proposed that scattering amplitudes possessed a duality between different momentum channels. If two hadrons have a scattering amplitude $A(s,t)$, where $s = -(p_1 + p_2)^2$ and $t = -(p_1+p_3)$ are the Mandelstam variables, then the duality implies that $A(s,t) = A(t,s)$ which requires that summing poles in one channel is equivalent to summing poles in the other channel. Veneziano [2] constructed an explicit amplitude which satisfied these properties

$$A(s,t) = \frac{\Gamma(-\alpha(s))\Gamma(-\alpha(t))}{\Gamma(-\alpha(s)-\alpha(t))} \tag{1.0.2}$$

with $\alpha(s) = \alpha(0) + \alpha' s$. This result was generalized to a variety of other scattering processes [3,4,5], N particle scattering [6,7,8] and processes including isospin quantum numbers [9]. It was realized that these results, which agreed with the phenomenology to a certain degree, could be derived from a fundamental physical model of a relativistic one-dimensional extended object, a string, whose length was $\sim \sqrt{\alpha'}$. In order to construct more realistic string theories it was necessary to include fermions [10,11] and these RNS models were shown to possess a two-dimensional supersymmetry relating the bosonic and fermionic degrees of freedom [12].

However, further scattering experiments, at high energies and fixed angle, established that amplitudes had a power law decay rather than exponential as predicted by string theory. There were several other difficulties with dual models as a theory of hadrons. Most notably they seemed to predict that space-time was either twenty six or ten dimensional and all models seemed to include massless particles in their spectra. Remarkably several of these problems could be seen as virtues by a change in perspective. It was shown [13, 14] that in the $\alpha' \to 0$ limit the massless spin two particle of the Virasoro-Shapiro model has the correct gauge properties to be interpreted as a graviton, and its interactions reproduce those of Einstein-Hilbert gravity coupled to a massless scalar. As string theories are well behaved at high energies they could give a consistent theory of quantum gravity and, as they include all the appropriate fields, may provide a unified theory of forces. This new interpretation of the strings required that the string length be given by $\alpha' \sim 10^{-34}(GeV^{-2})$ [13] so that to a low energy observer these strings look point-like. It was also now possible, along the lines of earlier Kaluza-Klein constructions in gravitational theories, to interpret the extra dimensions as small and compact. However all of these theories still possessed a tachyon in their spectra until it was shown by Gliozzi, Scherk and Olive [15] that it is possible to consistently truncate the RNS string model to remove the tachyon and furthermore show that these theories are spacetime supersymmetric.

Eventually it was discovered that hadrons are made of quarks which are described by QCD, a gauge theory based on the group $SU(3)$, rather than large hadronic

strings. This theory is asymptotically free and so the coupling constant decreases as the energy increases. The successes of this theory hardly need to be described here and the experimental verification of the standard model, an $SU(3) \times SU(2) \times U(1)$ gauge theory describing in addition to the strong force, the weak and electromagnetic forces, is impressive. That being said, Yang-Mills gauge theories are in many regards poorly understood and at present, the best available approach to studying their strongly coupled behavior is to use numerical simulations on the lattice. Given the origins of string theory, it is perhaps unsurprising that there are conjectures that strings still play a role in the confining phase of a gauge theory (for example [16, 17]). It was suggested by 't Hooft [18] that $U(N_c)$ gauge theory simplifies when N_c, the numbers of colors, is large. Furthermore, the diagrammatic expansion of the gauge theory suggests that the large N_c theory is a string theory with coupling constant $1/N_c$ (similar, though less well-formed, ideas had arisen in the interpretation of dual graphs as sums of Feynman diagrams [19, 20]). As 't Hooft's arguments are quite general they should apply to almost any gauge theory. Of particular relevance to us is $SU(N_c)$ gauge theory in four dimensions with $\mathcal{N} = 4$ supersymmetry (for a total of sixteen supersymmetries) [21]. $SO(4, 2)$ is an exact symmetry group since this theory is classically conformal as the coupling constant is dimensionless in four dimensions and quantum mechanically as the beta-function vanishes. These symmetries should be apparent in the dual string theory and this can be realized by having the background geometry be AdS_5 which has $SO(4, 2)$ as its isometry group. The gauge theory also contains six scalar fields and has an $SO(6)$ global symmetry so it is natural that the string background include a five dimensional space with $SO(6)$ isometry group, an S^5. We conclude that $\mathcal{N} = 4$ $SU(N_c)$ Yang-Mills could be dual to string theory on $AdS_5 \times S^5$. Finally as the gauge theory is supersymmetric we consider supersymmetric strings which are consistent in ten dimensions.

This specific string/gauge duality was first conjectured by Maldacena in [22] where it was motivated by considering the low-energy description of stacks of $D3$-branes (see [23] for a review and a extensive list of references). Dp-branes are $(p + 1)$-

dimensional solitons of string theory and were originally discovered as solutions of the classical supergravity equations of motion. All string theories have a low energy description containing gravity and in addition the type IIB string theory low energy spectrum contains Ramond-Ramond fields described by $(p+1)$-forms, $A_{(p+1)}$, where p is odd and the field strength is a $p+2$ form denoted by $F_{(p+2)}$. It is possible to find black p-brane solutions to the equations of motion of type IIB supergravity that are electrically charged with respect to A_{p+1}. The black 3-brane solution is of the form

$$\begin{aligned} ds^2 &= f^{-1/2}(-dt^2+dx_1^2+dx_2^2+dx_3^2)+f^{1/2}(dr^2+r^2d\Omega_5^2), \\ F_5 &= dt\wedge dx_1\wedge dx_2\wedge dx_3\wedge d(f^{-1})+*dt\wedge dx_1\wedge dx_2\wedge dx_3\wedge d(f^{-1}), \\ & f=1+\frac{R^4}{r^4}, \quad R^4\equiv 4\pi g_s\alpha' N\ , \end{aligned} \tag{1.0.3}$$

which has a horizon at $r=0$. If we consider only low-energy excitations in this background we find two distinct types: massless excitations far from the horizon, $r >> R$, and any type of excitation near the horizon which appear to an observer at infinity to have been red-shifted. Excitations in this near-horizon, $r << R$ region move in a background

$$ds^2=\frac{r^2}{R^2}(-dt^2+dx_1^2+dx_2^2+dx_3^2)+R^2\frac{dr^2}{r^2}+R^2d\Omega_5^2, \tag{1.0.4}$$

which is $AdS_5\times S^5$. Furthermore it is possible to show that these two systems, arbitrary excitations in the near-horizon region and supergravity in flat space, do not interact. In perturbative string theory D-branes are hypersurfaces upon which open strings end. Open-string fluctuations then describe oscillations of the branes and excitations of gauge theories living on the branes. In fact, for N_c coincident $D3$-branes, the low-energy effective theory for open strings is $SU(N_c)$ gauge theory with $\mathcal{N}=4$ supersymmetry. In the low-energy approximation it is possible to show that these N_c $D3$-branes sitting in ten dimensional flat Minkowski space decouple from the bulk supergravity theory. We thus have two descriptions for the low energy

excitations of $D3$-branes, both of which include supergravity in the bulk. Equating the remaining two components we are led to the conjecture that type IIB string theory on $AdS_5 \times S^5$ is dual to $\mathcal{N} = 4$ $SU(N_c)$ Yang-Mills. The parameters on both both sides of the duality can be related; the flux of the five-form field strength on the S^5 is equal to the number of colors in the gauge theory and from the physics of D-branes [24] we know that the Yang-Mills coupling is related to the string coupling by $g^2_{YM} = g_s$. The correspondence was given a more precise formulation and elaborated upon in [25, 26], where the generating function for correlators of gauge theory operators was related to the string partition function with specified asymptotic conditions. Consider ϕ a massless scalar field on AdS_5 with a value ϕ_0 on the boundary of the AdS_5 and which couples to an operator in the conformal gauge theory, $\mathcal{O}$, by means of a coupling $\int \phi_0 \mathcal{O}$. The ansatz for the relationship between conformal gauge theory on the boundary to fields on the AdS space is

$$\langle e^{\int d^4x \ \phi_0(\vec{x})\mathcal{O}(\vec{x})} \rangle_{CFT} = Z_{\text{string}} \left[\phi(\vec{x}, r)|_{r=\text{boundary}} = \phi_0 \right], \tag{1.0.5}$$

where Z_{string} is the supergravity partition function with the boundary condition $\phi = \phi_0$. It is possible to generalize this relation to include different massless and massive fields. For example, the graviton couples to the energy-momentum tensor. In particular, dimensions of operators in the conformal field theory are given by the masses of excitations in supergravity. A large amount of evidence in support of this duality was discovered though, at least initially, most relied upon the supergravity approximation as the full superstring action in $AdS_5 \times S^5$ is highly non-linear and so difficult to quantize.

Berenstein, Maldacena and Nastase (BMN) [27] took a significant step beyond the supergravity approximation by considering strings moving in a certain pp-wave background. This background is a maximally supersymmetric solution of type IIB supergravity which can be obtained as a Penrose limit of $AdS_5 \times S^5$ [28, 29]. The string action in this background simplifies dramatically in lightcone gauge and it is

possible to find the exact string spectrum. BMN were able to identify a subsector of gauge operators dual to this string theory and match the operator dimensions with the string energies. Taking the Penrose limit corresponds to boosting a string to large angular momentum and studying its fluctuations in a semi-classical expansion. In fact, the BMN limit corresponds to studying quadratic fluctuations about a pointlike closed string classical solution with large angular momentum, J, along a geodesic of the S^5 [30, 31]. The semi-classical expansion may be regarded as more general than Penrose/BMN limit in that one may start with any stable classical string solution with a large quantum number and expand about this point in solution space (see [32] for a review). It is of interest to extend this analysis to higher orders in fluctuations or, in other words, to study the finite J effects on the string spectrum and capture more information on the full AdS geometry. These higher order corrections will be our main object of study.

In Chapter Two we review certain relevant aspects of the AdS/CFT correspondence and the BMN limit. In particular we will introduce the Green-Schwarz (GS) form of the superstring $AdS_5 \times S^5$ action and expand it to quadratic order in fluctuations where we find agreement with the string action in the pp-wave. In Chapter Three we review $\mathcal{N} = 4$ $SU(N_c)$ gauge theory and using the structure of the superconformal group $PSU(2,2|4)$ we study the anomalous dimensions of so-called two impurity gauge theory operators at one-loop order in the 't Hooft coupling $\lambda = g_s N$. In order to calculate the anomalous dimensions of higher impurity operators at higher loops we take a virial approach to the dimension operator and use numerical methods. In Chapter Four we expand the string action to quartic order in fields and using perturbation theory find the curvature corrections to the energies of string states dual to two-impurity operators. These results are extended to three impurities for the full string theory and arbitrary number of impurities for certain sub-sectors in Chapter Five. In Chapter Six we compare our results to those found by means of the Bethe ansatz for various sectors of the gauge theory and study how our results relate to those of the conjectured quantum string Bethe ansatz. We then extend our

results and those of the quantum Bethe ansatz to open string theories which descend from the closed string. In Appendix A we list our notations and our conventions for spinors.

Chapter 2

AdS/CFT and the BMN limit

2.1 Strings in $AdS_5 \times S^5$

2.1.1 General considerations

We begin with a brief review of some pertinent aspects of the AdS/CFT correspondence and in particular the description of supersymmetric strings on $AdS_5 \times S^5$. In global coordinates, the $AdS_5 \times S^5$ metric can be written in the form

$$ds^2 = R^2(-\cosh^2\rho\ dt^2 + d\rho^2 + \sinh^2\rho\ d\Omega_3^2 + \cos^2\theta\ d\phi^2 + d\theta^2 + \sin^2\theta\ d\tilde{\Omega}_3^2)\ , \quad (2.1.1)$$

where R denotes the radius of both the sphere and the AdS space, and $d\Omega_3^2$, $d\tilde{\Omega}_3^2$ denote separate three-spheres. The coordinate ϕ is periodic with period 2π and, strictly speaking, so is the time coordinate t, though in order to accommodate string dynamics, it is necessary to pass to the covering space where time is not taken to be periodic. This geometry, accompanied by an RR field with N_c units of flux on the sphere, is a consistent, maximally supersymmetric type IIB superstring background, provided that

$$R^4 = g_s N_c (\alpha')^2 \quad \text{with } g_s \text{ the string coupling.} \quad (2.1.2)$$

The AdS/CFT correspondence asserts that type IIB string theory in this back-

ground is equivalent to $\mathcal{N} = 4$ super Yang-Mills theory in four dimensions with an $SU(N_c)$ gauge group and coupling constant $g^2_{YM} = g_s$. This holographically dual theory is defined on the boundary of AdS_5, which is $R \times S^3$, and the theory is conformal. AdS_5 contains the conformal symmetry group of Minkowski space as part of its isometry group and this identification is a basic feature of the conjectured duality. In fact the full symmetry group of the gauge theory is a supersymmetric extension of the conformal group, $PSU(2,2|4)$, and this indeed matches the full isometry group of the $AdS_5 \times S^5$ background. Duality thus demands that operator dimensions in the gauge theory must be equal to energies of states in the string theory. A large amount of evidence in favor of this conjecture has been discovered. In its initial stages much of the attention focused on the supergravity approximation to the string theory in $AdS_5 \times S^5$. Recently, attention has turned to the problem of evaluating truly stringy physics in this background and studying its match to gauge theory physics. Quantizing strings in curved geometries is in general difficult as the action becomes non-linear. It has been shown that the sigma model in $AdS_5 \times S^5$ is integrable [33,34] and so there may be some hope that it is exactly solvable. We will pursue a perturbative approach studying the dynamics of a string that has been boosted to lightlike momentum along some direction, or, equivalently, by quantizing the string in the background obtained by taking the Penrose limit of the original geometry using the lightlike geodesic corresponding to the boosted trajectory.

We will be interested in matching gauge theory anomalous dimensions with string energies and in particular we will use light-cone gauge to find the spectrum of the superstrings physical Hamiltonian. It is perhaps useful to outline the derivation of the bosonic light-cone Hamiltonian as this will allow us to highlight some general issues in a more transparent setting (see [35] for a similar treatment though in a different coordinate system).

After introducing light-cone coordinates $x^{\pm}$ according to,

$$t \to x^+ \qquad \phi \to x^+ + \frac{x^-}{R^2} \qquad \rho \to \frac{z}{R} \qquad \theta \to \frac{y}{R} \tag{2.1.3}$$

the general $AdS_5 \times S^5$ metric can be cast in the form

$$ds^2 = 2G_{+-}dx^+dx^- + G_{++}dx^+dx^+ + G_{--}dx^-dx^- + G_{AB}dx^Adx^B \ , \qquad (2.1.4)$$

where x^A $(A = 1, \ldots, 8)$ labels the eight transverse directions, the metric components are functions of the x^A only, and the components G_{+A} and G_{-A} are not present.

The general bosonic Lagrangian density has a simple expression in terms of the target space metric

$$\mathcal{L} = \frac{1}{2} h^{ab} G_{\mu\nu} \partial_a x^\mu \partial_b x^\nu \ , \qquad (2.1.5)$$

where h is built out of the worldsheet metric γ according to $h^{ab} = \sqrt{-\det \gamma}\gamma^{ab}$ and the indices a, b label the worldsheet coordinates σ, τ. Since $\det h = -1$, there are only two independent components of h. The canonical momenta (and their inversion in terms of velocities) are

$$p_\mu = h^{\tau a} G_{\mu\nu} \partial_a x^\nu \ , \qquad \dot{x}^\mu = \frac{1}{h^{\tau\tau}} G^{\mu\nu} p_\nu - \frac{h^{\tau\sigma}}{h^{\tau\tau}} x'^\mu \ . \qquad (2.1.6)$$

The Hamiltonian density $\mathcal{H} = p_\mu \dot{x}^\mu - \mathcal{L}$ is

$$\mathcal{H} = \frac{1}{2h^{\tau\tau}} (p_\mu G^{\mu\nu} p_\nu + x'^\mu G_{\mu\nu} x'^\nu) - \frac{h^{\tau\sigma}}{h^{\tau\tau}} (x'^\mu p_\mu) \ . \qquad (2.1.7)$$

As is usual in theories with general coordinate invariance (on the worldsheet in this case), the Hamiltonian is a sum of constraints times Lagrange multipliers built out of metric coefficients ($1/h^{\tau\tau}$ and $h^{\tau\sigma}/h^{\tau\tau}$).

One can think of the dynamical system we wish to solve as being defined by $\mathcal{L} = p_\mu \dot{x}^\mu - \mathcal{H}$ (a phase space Lagrangian) regarded as a function of the coordinates x^μ, the momenta p_μ and the components h^{ab} of the worldsheet metric. To compute the quantum path integral, the exponential of the action constructed from this Lagrangian is functionally integrated over each of these variables. For a spacetime geometry like (2.1.4), one finds that with a suitable gauge choice for the worldsheet coordinates (τ, σ), the functional integrations over all but the transverse (physical) coordinates

and momenta can be performed, leaving an effective path integral for these physical variables. This is the essence of the light-cone approach to quantization.

The first step is to eliminate integrations over x^+ and p_- by imposing the light-cone gauge conditions $x^+ = p_- \tau$ with $p_- = \text{const}$. (At this level of analysis, which is essentially classical, we will not be concerned with ghost determinants arising from this gauge choice.) As noted above, integrations over the worldsheet metric cause the coefficients $1/h^{\tau\tau}$ and $h^{\tau\sigma}/h^{\tau\tau}$ to act as Lagrange multipliers, generating delta functions that impose two constraints:

$$\begin{aligned} x'^- p_- + x'^A p_A &= 0 \\ G^{++} p_+^2 + 2G^{+-} p_+ p_- + G^{--} p_-^2 + p_A G^{AB} p_B + x'^A G_{AB} x'^B + G_{--} \frac{(x'^A p_A)^2}{p_-^2} &= 0. \end{aligned} \tag{2.1.8}$$

When integrations over x^- and p_+ are performed, the constraint delta functions serve to evaluate x^- and p_+ in terms of the dynamical transverse variables (and the constant p_-). The first constraint is linear in x^- and yields $x'^- = -x'^A p_A / p_-$. Integrating this over σ and using the periodicity of x^- yields the standard level-matching constraint, without any modifications. The second constraint is quadratic in p_+ and can be solved explicitly for $p_+ = -\mathcal{H}_{lc}(x^A, p_A)/p_-$. The remaining transverse coordinates and momenta have dynamics which follow from the phase space Lagrangian

$$\mathcal{L}_{\text{ps}} = p_+ p_- + p_- \dot{x}^- + p_A \dot{x}^A \sim p_A \dot{x}^A - \mathcal{H}_{lc}(x^A, p_A) \ , \tag{2.1.9}$$

where we have eliminated the $\dot{x}^-$ term by integrating by parts in time and using the fact that p_- is constant. The essential result is that $-p_+ = \mathcal{H}_{lc}/p_-$ is the Hamiltonian that generates evolution of the physical variables $x^A,\ p_A$ in worldsheet time τ. This is, of course, dynamically consistent with the light-cone gauge identification $x^+ = p_- \tau$ (which requires worldsheet and target space time translation to be the same).

We can solve the quadratic constraint equation (2.1.8) for $p_+ = -\mathcal{H}_{\text{lc}}/p_-$ explicitly,

obtaining the uninspiring result

$$\mathcal{H}_{\rm lc} = -\frac{p_-^2 G_{+-}}{G_{--}} - \frac{p_-^2\sqrt{G}}{G_{--}}\sqrt{1+\frac{G_{--}}{p_-^2}(p_A G^{AB} p_B + x'^A G_{AB} x'^B) + \frac{G_{--}^2}{p_-^4}(x'^A p_A)^2}\ , \tag{2.1.10}$$

where

$$G \equiv G_{+-}^2 - G_{++}G_{--}\ . \tag{2.1.11}$$

This is not very useful as it stands but following the ideas of BMN, which we describe in greater detail below, we can put it in a more manageable form by expanding in powers of $1/R^2$ and calculating the energies perturbatively.

2.1.2 Exact solution for zero-modes

We will mostly pursue the perturbative approach to finding the effects of the $AdS_5\times S^5$ background on the string spectrum. That being said, it is instructive to study a different limit in which the kinematics are unrestricted (no large-J limit is taken) but only modes of the string that are independent of the worldsheet coordinate (the zero-modes of the string) are kept in the Hamiltonian. This is the problem of quantizing the superparticle of the underlying supergravity in the $AdS_5 \times S^5$ background, a problem which has been solved many times (for references, see [23]). A remarkable fact, which seems not to have been explicitly observed before, is that the spectrum of the zero-mode Hamiltonian is *exactly* a sum of harmonic oscillators: the curvature corrections we propose to compute actually vanish on this special subspace. This fact is important to an understanding of the full problem, so we will make a brief digression to explain the solution to this toy problem.

The quantization of the superparticle in a supergravity background is equivalent to finding the eigensolutions of certain Laplacians, one for each spin that occurs in the superparticle massless multiplet. The point of interest to us can be made by analyzing the dynamics of the scalar particle and its associated scalar Laplacian, which only depends on the background metric. We adopt here another version of the $AdS_5 \times S^5$

metric, chosen because the scalar Laplacian is very simple in these coordinates:

$$ds^2 = -dt^2(R^2+z^2) + d\phi^2(R^2-y^2) + dz^j\left(\delta_{jk} - \frac{z^j z^k}{R^2+z^2}\right)dz^k + dy^{j'}\left(\delta_{j'k'} + \frac{y^{j'}y^{k'}}{R^2-y^2}\right)dy^{k'} \ . \tag{2.1.12}$$

The coordinates z^k and $y^{k'}$ parameterize two $SO(4)$ subspaces, and the indices j,k and j',k' run over $j,k = 1,\ldots,4$, and $j',k' = 5,\ldots,8$. This is a natural metric for analyzing fluctuations of a particle (or string) around the lightlike trajectory $\phi = t$ and $\vec{z} = \vec{y} = 0$. Because the metric components depend neither on t nor on ϕ, and because the problem is clearly separable in $\vec{z}$ and $\vec{y}$, it makes sense to look for solutions of the form $\Phi = e^{-i\omega t}e^{iJ\phi}F(\vec{z})G(\vec{y})$. The scalar Laplacian for ϕ in the above metric then reduces to

$$\Big[-\frac{\omega^2}{R^2+\vec{z}^2} + \frac{J^2}{R^2-\vec{y}^2} - \frac{\partial}{\partial z^j}\Big(\delta^{jk} + \frac{z^j z^k}{R^2}\Big)\frac{\partial}{\partial z^k} - \frac{\partial}{\partial y^{j'}}\Big(\delta^{j'k'} - \frac{y^{j'}y^{k'}}{R^2}\Big)\frac{\partial}{\partial y^{k'}}\Big]F(z)G(y) = 0 \ . \tag{2.1.13}$$

The radius R disappears from the equation upon rescaling the transverse coordinates by $z \to z/R$ and $y \to y/R$, so we can set $R = 1$ in what follows. The scalar Laplacian is essentially the light-cone Hamiltonian constraint (2.1.8) for string coordinates $z^k, y^{k'}$ and string momenta $p_z^k = -i\frac{\partial}{\partial z^k}$ and $p_y^{k'} = -i\frac{\partial}{\partial y^{k'}}$ (projected onto their zero modes). This implies that we can use the structure of the Laplacian to correctly order operators in the string Hamiltonian.

The periodicity $\phi \equiv \phi + 2\pi$ means that the angular momentum J is integrally quantized. The allowed values of ω then follow from the solution of the eigenvalue problem posed by (2.1.13). As the trial function Φ indicates, (2.1.13) breaks into separate problems for $\vec{z}$ and $\vec{y}$:

$$\begin{aligned}\mathcal{H}_{AdS_5}F(\vec{z}) &= \left[p_j^z(\delta^{jk} + z^j z^k)p_k^z + \omega^2\frac{z_k z^k}{1+(z_k z^k)^2}\right]F(\vec{z}) = A(\omega)F(\vec{z}) \\ \mathcal{H}_{S^5}G(\vec{y}) &= \left[p_{j'}^y(\delta^{j'k'} - y^{j'}y^{k'})p_{k'}^y + J^2\frac{y_{k'}y^{k'}}{1-(y_{k'}y^{k'})^2}\right]G(\vec{y}) = B(J)G(\vec{y}) \ ,\end{aligned} \tag{2.1.14}$$

where $\omega^2 - J^2 = A + B$. The separation eigenvalues A, B depend on their respective parameters ω, J, and we determine the energy eigenvalues ω by finding the roots of the potentially complicated equation $\omega^2 - J^2 - A - B = 0$. The scalar Laplacian (2.1.13) is equivalent to the constraint equation (2.1.8) projected onto string zero-modes, and we see once again that the constraint doesn't directly give the Hamiltonian but rather an equation (quadratic or worse) to be solved for the Hamiltonian.

The $\mathcal{H}_{S^5}$ equation is just a repackaging of the problem of finding the eigenvalues of the $SO(6)$ Casimir invariant (another name for the scalar Laplacian on S^5) and $\mathcal{H}_{AdS_5}$ poses the corresponding problem for $SO(4,2)$. The $SO(6)$ eigenvalues are obviously discrete, and the $SO(4,2)$ problem also turns out to be discrete when one imposes the condition of finiteness at $z^2 \to \infty$ on the eigenfunctions (this is a natural restriction in the context of the AdS/CFT correspondence; for a detailed discussion see [23]). Thus we expect ω to have a purely discrete spectrum, with eigenvalues labeled by a set of integers. The simplest way to solve for the spectrum is to expand $F(\vec{z})$ and $G(\vec{y})$ in $SO(4)$ harmonics (since this symmetry is explicit), recognize that the radial equation is, in both cases, an example of Riemann's differential equation and then use known properties of the hypergeometric function to find the eigenvalues and eigenfunctions of (2.1.14). Since it takes three integers to specify an $SO(4)$ harmonic and one to specify a radial quantum number, we expect each of the two separated equations to have a spectrum labeled by four integers. The exact results for the separation eigenvalues turn out to be remarkably simple:

$$\begin{aligned} A &= 2\omega \sum_1^4 \left(n_i + \frac{1}{2}\right) - \left[\sum_1^4 \left(n_i + \frac{1}{2}\right)\right]^2 + 4 \qquad n_i = 0, 1, 2, \ldots \\ B &= 2J \sum_1^4 \left(m_i + \frac{1}{2}\right) + \left[\sum_1^4 \left(m_i + \frac{1}{2}\right)\right]^2 + 4 \qquad m_i = 0, 1, 2, \ldots \end{aligned} \tag{2.1.15}$$

Different eigenfunctions correspond to different choices of the collection of eight integers $\{n_i, m_i\}$, and the fact that the energies depend only on Σn_i and Σm_i correctly accounts for the degeneracy of eigenvalues. The special form of A and B means that

the equation for the energy eigenvalue, $\omega^2 - J^2 - A - B = 0$, can be factored as

$$\left[\omega - J - \sum_1^4 \left(n_i + \frac{1}{2}\right) - \sum_1^4 \left(m_i + \frac{1}{2}\right)\right] \times \left[\omega + J - \sum_1^4 \left(n_i + \frac{1}{2}\right) + \sum_1^4 \left(m_i + \frac{1}{2}\right)\right] = 0 \ . \tag{2.1.16}$$

For obvious reasons, we retain the root that assigns only positive values to ω, the energy conjugate to the global time t:

$$\omega - J \ = \ \sum_1^4 \left(n_i + \frac{1}{2}\right) \ + \ \sum_1^4 \left(m_i + \frac{1}{2}\right) \ . \tag{2.1.17}$$

From the string point of view, ω catalogs the eigenvalues of the string worldsheet Hamiltonian restricted to the zero-mode subspace. Quite remarkably, it is an exact 'sum of harmonic oscillators,' independent of whether J (and ω) are large or not. This is simply to say that the eigenvalues of the string Hamiltonian restricted to the zero-mode sector receive no curvature corrections. We have only shown this for the massless scalars of the theory, but we expect the same thing to be true for all the massless fields of type IIB supergravity. The implication for a perturbative account of the string spectrum is that states created using only zero-mode oscillators (of any type) will receive no curvature corrections. This feature will turn out to be a useful consistency check on our quantization procedure. It is of course not true for a general classical background and is yet another manifestation of the special nature of the $AdS_5 \times S^5$ geometry.

2.2 GS superstring action on $AdS_5 \times S^5$

The presence of Ramond-Ramond background fields suggests that we use the GS formalism to describe our superstrings, though there is of course an alternative for-

malism developed by Berkovits [36, 37] which can also incorporate RR-backgrounds and has many advantages. In general the problem of finding the explicit superstring action for curved backgrounds is quite difficult as we need to know the components of the supergravity superfields, fortunately the $AdS_5 \times S^5$ target space can be realized as the coset superspace

$$G/H = \frac{SU(2,2|4)}{SO(4,1) \times SO(5)} \tag{2.2.1}$$

which makes the problem tractable. The bosonic reduction of this coset is precisely $SO(4,2) \times SO(6)/SO(4,1) \times SO(5) \equiv AdS_5 \times S^5$. There is a general strategy for constructing a non-linear sigma model on a super-coset space in terms of the Cartan one-forms [1] and superconnections of the super-coset manifold. In such a construction, the symmetries of the stabilizer subgroup, H, remain manifest in the action while the remaining symmetries are nonlinearly realized (see, e.g., [39, 40, 41, 42, 43, 44]). Metsaev and Tseytlin [40] carried out this construction for the $AdS_5 \times S^5$ geometry, producing a type IIB superstring action possessing the full $PSU(2,2|4)$ supersymmetry of $AdS_5 \times S^5$, which is furthermore κ-symmetric. This κ-symmetry is a local fermionic symmetry which is crucial for the consistency of the superstring action. For example, κ-symmetry is preserved at a classical level if the background satisfies the supergravity equations of motion and it is κ-symmetry symmetry which ensures that there are the correct number of physical fermionic degrees of freedom. The of action Metsaev and Tseytlin is conceptually simple, comprising a kinetic term and a Wess–Zumino term built out of Cartan (super)one-forms on the super-coset manifold in the following way (this form was first presented in [45]):

$$\mathcal{S} \;=\; -\frac{1}{2}\int_{\partial M_3} d^2\sigma \; h^{ab} L_a^\mu L_b^\mu + i\int_{M_3} s^{IJ} L^\mu \wedge \bar{L}^I \Gamma^\mu \wedge L^J \; . \tag{2.2.2}$$

Repeated upper indices are summed with a Minkowskian inner product. The indices a, b are used to indicate the worldsheet coordinates (τ, σ), and we use the values

[1] See [38] for an introduction to the formalism of Cartan forms on super-coset spaces.

$a, b = 0$ to indicate the worldsheet time direction τ, and $a, b = 1$ to specify the σ direction. The matrix s^{IJ} is defined by $s^{IJ} \equiv \mathrm{diag}(1, -1)$, where $I, J = 1, 2$. The Wess-Zumino term appears as an integral over a 3-manifold M_3, while the kinetic term is integrated over the two-dimensional boundary ∂M_3. The left-invariant Cartan forms are defined in terms of the coset space representative G by

$$G^{-1}dG = L^{\mu}P^{\mu} + L^{\alpha}\,\bar{Q}_{\alpha} + \bar{L}^{\alpha}Q_{\alpha} + \frac{1}{2}L^{\mu\nu}\,J^{\mu\nu}$$
$$L^N = dX^M L^N_M \qquad L^N_a = L^N_M\,\partial_a X^M \qquad X^M = (x^m, \theta^{\alpha}, \bar{\theta}^{\alpha})\ . \tag{2.2.3}$$

The explicit expansion of this action in terms of independent fermionic degrees of freedom is rather complicated and highly non-linear. We briefly outline our conventions here though see also Appendix A: we start with two 32-component Majorana-Weyl spinors in 10 dimensions, θ^I, where $I = 1, 2$ which can be combined into a single complex spinor, θ, as in the equation above. In a suitably-chosen representation for the 32×32 ten-dimensional gamma matrices Γ^{μ}, the Weyl projection reduces to picking out the upper 16 components of θ and the surviving spinors can be combined into one complex 16-component spinor ψ. We use the convention that the indices $\mu, \nu, \rho = 0, \ldots, 9$ denote tangent $SO(9,1)$ vectors, $m, n, l = 0, \ldots, 9$ spacetime indices, and we will denote the corresponding spinor indices by $\alpha, \beta, \gamma, \delta = 1, \ldots, 16$ (we also use the convention that upper-case indices $A, B, C, D = 1, \ldots, 8$ indicate vectors of $SO(8)$, while $i, j, k = 1, \ldots, 4$ ($i', j', k' = 5, \ldots, 8$) indicate vectors from the $SO(4)$ subspaces associated with AdS_5 and S^5 respectively. A representation of the 16×16 γ^A matrices which will be convenient for explicit calculation is given in Appendix A. We also fix the κ-symmetry by imposing the condition $\bar{\gamma}^9\psi = \psi$. This restricts the worldsheet fermions to lie in the 8_s representation of $SO(8)$ (and projects out the 8_c spinor), thus reducing the number of independent components of the worldsheet spinor from 32 to 16 (this is further halved by the fermionic equations of motion so that we end up with 8 physical degrees of freedom).

Kallosh, Rahmfeld and Rajaraman presented in [39] a general solution to the

supergravity constraints (Maurer-Cartan equations) for coset spaces with an isometry algebra of the form

$$\begin{aligned} [B_\mu, B_\nu] &= f_{\mu\nu}^{\rho} B_\rho \\ [F_\alpha, B_\nu] &= f_{\alpha\nu}^{\beta} F_\beta \\ \{F_\alpha, F_\beta\} &= f_{\alpha\beta}^{\mu} B_\mu \ , \end{aligned} \tag{2.2.4}$$

with B_μ and F_α representing bosonic and fermionic generators, respectively. In terms of these generators, the Cartan forms L^μ and superconnections L^α, satisfying the Maurer-Cartan equations,

$$dL^\mu + L^\nu \wedge L^\rho f_{\nu\rho}^{\mu} - L^\alpha \wedge L^\beta f_{\alpha\beta}^{\mu} = 0 \tag{2.2.5}$$

$$dL^\alpha + L^\mu \wedge L^\beta f_{\mu\nu}^{\alpha} = 0 \tag{2.2.6}$$

are completely determined by the structure constants $f_{\alpha\mu}^{J}$ and $f_{\alpha\beta}^{\mu}$:

$$L_{at}^{\alpha} = \left(\frac{\sinh t\mathcal{M}}{\mathcal{M}} \right)_{\beta}^{\alpha} (\mathcal{D}_a \theta)^\beta \tag{2.2.7}$$

$$L_{at}^{\mu} = e^{\mu}{}_{m} \partial_a x^m + 2\theta^\alpha f_{\alpha\beta}^{\mu} \left(\frac{\sinh^2(t\mathcal{M}/2)}{\mathcal{M}^2} \right)_{\gamma}^{\beta} (\mathcal{D}_a \theta)^\gamma \tag{2.2.8}$$

$$\text{where} \qquad (\mathcal{M}^2)_{\beta}^{\alpha} = -\theta^\gamma f_{\gamma\mu}^{\alpha} \theta^\delta f_{\delta\beta}^{\mu} \ .$$

The dimensionless parameter t is used here to define "shifted" Cartan forms and superconnections where, for example, $L_a^\mu = L_{at}^\mu|_{t=1}$. In the case of $AdS_5 \times S^5$, the Lagrangian density takes the form

$$\mathcal{L}_{\text{Kin}} = -\frac{1}{2} h^{ab} L_a^\mu L_b^\mu \tag{2.2.9}$$

$$\mathcal{L}_{\text{WZ}} = -2i\epsilon^{ab} \int_0^1 dt \, L_{at}^\mu s^{IJ} \bar{\theta}^I \Gamma^\mu L_{bt}^J \ . \tag{2.2.10}$$

and the superconformal algebra can be written explicitly as:

$$\begin{aligned}
\left[P^+, P^k\right] &= J^{+k} & \left[P^+, P^{k'}\right] &= -J^{+k'} \\
\left[P^+, J^{+k}\right] &= -P^k & \left[P^+, J^{+k'}\right] &= P^{k'} \\
\left[P^-, P^A\right] &= J^{+A} & \left[P^-, J^{+A}\right] &= P^A \\
\left[P^j, P^k\right] &= J^{jk} & \left[P^{j'}, P^{k'}\right] &= -J^{j'k'} \\
\left[J^{+j}, J^{+k}\right] &= J^{jk} & \left[J^{+j'}, J^{+k'}\right] &= -J^{j'k'}
\end{aligned} \tag{2.2.11}$$

and

$$\begin{aligned}
\left[P^j, J^{+k}\right] &= -\delta^{jk}(P^+ - P^-) & \left[P^r, J^{+s}\right] &= -\delta^{rs}(P^+ + P^-) \\
\left[P^i, J^{jk}\right] &= \delta^{ij}P^k - \delta^{ik}P^j & \left[P^{i'}, J^{j'k'}\right] &= \delta^{i'j'}P^{k'} - \delta^{i'j'}P^{k'} \\
\left[J^{+i}, J^{jk}\right] &= \delta^{ij}J^{+k} - \delta^{ik}J^{+j} & \left[J^{+i'}, J^{j'k'}\right] &= \delta^{i'j'}J^{+k'} - \delta^{i'j'}J^{+k'} \\
\left[J^{ij}, J^{kl}\right] &= \delta^{jk}J^{il} + 3 \text{ terms} & \left[J^{i'j'}, J^{k'l'}\right] &= \delta^{j'k'}J^{i'l'} + 3 \text{ terms} .
\end{aligned} \tag{2.2.12}$$

The bosonic-fermionic sector of the algebra is

$$\begin{aligned}
\left[J^{ij}, Q_\alpha\right] &= \frac{1}{2}Q_\beta(\gamma^{ij})^\beta{}_\alpha \\
\left[J^{i'j'}, Q_\alpha\right] &= \frac{1}{2}Q_\beta(\gamma^{i'j'})^\beta{}_\alpha \\
\left[J^{+i}, Q_\alpha\right] &= \frac{1}{2}Q_\beta(\gamma^{+i} - \gamma^{-i})^\beta{}_\alpha \\
\left[J^{+i'}, Q_\alpha\right] &= \frac{1}{2}Q_\beta(\gamma^{+i'} + \gamma^{-i'})^\beta{}_\alpha \\
\left[P^\mu, Q_\alpha\right] &= \frac{i}{2}Q_\beta(\Pi\gamma^+\bar{\gamma}^\mu)^\beta{}_\alpha - \frac{i}{2}Q_\beta(\Pi\gamma^-\bar{\gamma}^\mu)^\beta{}_\alpha .
\end{aligned} \tag{2.2.13}$$

The fermionic-fermionic anticommutation relations are

$$\begin{aligned}
\{Q_\alpha, \bar{Q}_\beta\} &= -2i\gamma^\mu_{\alpha\beta}P^\mu - 2(\bar{\gamma}^k\Pi)_{\alpha\beta}J^{+k} - 2(\bar{\gamma}^{k'}\Pi)_{\alpha\beta}J^{+k'} \\
&\quad +(\bar{\gamma}^+\gamma^{jk}\Pi)_{\alpha\beta}J^{jk} + (\bar{\gamma}^+\gamma^{j'k'}\Pi)_{\alpha\beta}J^{j'k'} \\
&\quad -(\bar{\gamma}^-\gamma^{jk}\Pi)_{\alpha\beta}J^{jk} + (\bar{\gamma}^-\gamma^{j'k'}\Pi)_{\alpha\beta}J^{j'k'} .
\end{aligned} \tag{2.2.14}$$

Here we have used the 16-component notation for the fermions. In the 32 component

notation we have

$$\begin{aligned}
[Q_I, P^\mu] &= \frac{i}{2}\epsilon^{IJ} Q_J \Gamma_* \Gamma^\mu \\
[Q_I, J^{\mu\nu}] &= -\frac{1}{2} Q_I \Gamma^{\mu\nu} \\
\{(Q_I)^\mu, (Q_J)_\mu\} &= -2i\delta_{IJ}\Gamma^0\Gamma^\rho P_\rho + \epsilon^{IJ}\left(-\Gamma^0\Gamma^{jk}\Gamma_* J_{jk} + \Gamma^0\Gamma^{j'k'}\Gamma'_* J_{j'k'}\right) ,
\end{aligned} \tag{2.2.15}$$

where

$$\Gamma_* \equiv i\Gamma_{01234} \qquad \Gamma'_* \equiv i\Gamma_{56789} \ . \tag{2.2.16}$$

One now chooses a parameterization of the coset representative

$$G(x,\theta) = f(x)g(\theta) \qquad g(\theta) = \exp(\theta^I Q_I) \tag{2.2.17}$$

and shifts $\theta \to t\theta$ giving rise to a set of coupled differential equations. These can be solved and with the appropriate choice of boundary conditions give

$$\begin{aligned}
L^J_{bt} &= \frac{\sinh t\mathcal{M}}{\mathcal{M}} \mathcal{D}_b \theta^J \\
L^\mu_{at} &= e^\mu{}_\rho \partial_a x^\rho - 4i\bar\theta^I \Gamma^\mu \left(\frac{\sinh^2(t\mathcal{M}/2)}{\mathcal{M}^2}\right) \mathcal{D}_a \theta^I \ ,
\end{aligned} \tag{2.2.18}$$

where the covariant derivative is given by

$$(\mathcal{D}_a\theta)^I = \left(\partial_a\theta + \frac{1}{4}\left(\omega^{\mu\,\nu}{}_\rho\, \partial_a x^\rho\right)\Gamma^{\mu\nu}\theta\right)^I - \frac{i}{2}\epsilon^{IJ} e^\mu{}_\rho\, \partial_a x^\rho \Gamma_* \Gamma^\mu \theta^J \ . \tag{2.2.19}$$

The object $\mathcal{M}$ is a 2×2 matrix which, for convenience, is defined in terms of its square:

$$(\mathcal{M}^2)^{IL} = -\epsilon^{IJ}(\Gamma_*\Gamma^\mu\theta^J\bar\theta^L\Gamma^\mu) + \frac{1}{2}\epsilon^{KL}(-\Gamma^{jk}\theta^I\bar\theta^K\Gamma^{jk}\Gamma_* + \Gamma^{j'k'}\theta^I\bar\theta^K\Gamma^{j'k'}\Gamma'_*) \ . \tag{2.2.20}$$

At this point, the GS action on $AdS_5 \times S^5$ (2.2.9,2.2.10) may be expanded to arbitrary order in fermionic and bosonic fields.

2.3 BMN limit

The existence of a new maximally supersymmetric IIB string background was discovered in [28]. This geometry corresponds to a symmetric space G/H with a homogeneous five-form flux and due to its resemblance to a similar 11 dimensional supergravity solution was called Hpp-wave geometry. This geometry was shown to be the Penrose limit of $AdS_5 \times S^5$ in [29]. Metsaev [41] constructed the covariant GS superstring action for this background using Cartan forms on a super-coset space and fixing the kappa and gauge symmetries found the light-cone Hamiltonian. In this background the string equations of motion are solvable and so the string energies could be explicitly found. According to the AdS/CFT duality these string energies should match anomalous dimensions of certain gauge theory operators. Berenstein, Maldacena and Nastase [27] were able to identify this subset of gauge theory operators. They demonstrated that a perturbative calculation of the dimensions of these operators was reliable in this limit and matched the free string energies. It is helpful to make the reparametrizations

$$\cosh\rho = \frac{1+z^2/4}{1-z^2/4} \qquad \cos\theta = \frac{1-y^2/4}{1+y^2/4}\ , \tag{2.3.1}$$

and work with the metric

$$ds^2 = R^2\Big[-\left(\frac{1+\frac{1}{4}z^2}{1-\frac{1}{4}z^2}\right)^2 dt^2 + \left(\frac{1-\frac{1}{4}y^2}{1+\frac{1}{4}y^2}\right)^2 d\phi^2 + \frac{dz_k dz_k}{(1-\frac{1}{4}z^2)^2} + \frac{dy_{k'}dy_{k'}}{(1+\frac{1}{4}y^2)^2}\Big]\ , \tag{2.3.2}$$

where $y^2 = y_{k'}y^{k'}$ with $k' = 5,\ldots,8$ and $z^2 = z_k z^k$ with $k = 1,\ldots,4$ define eight 'Cartesian' transverse coordinates. This form of the metric is well-suited to the present calculation; the spin connection, which will be important for the superstring action, turns out to have a simple form, and the AdS_5 and S^5 subspaces appear nearly

symmetrically. This metric is invariant under the full $SO(4,2)\times SO(6)$ symmetry, but only translation invariance in t and ϕ and the $SO(4)\times SO(4)$ symmetry of the transverse coordinates remain manifest in this form. The translation symmetries mean that string states have a conserved energy ω, conjugate to t, and a conserved (integer) angular momentum J, conjugate to ϕ. Boosting along the equatorial geodesic is equivalent to studying states with large J and the light-cone Hamiltonian will give the (finite) allowed values for $\omega - J$ in that limit. The S^5 isometry group is dual to an $SO(6)$ R-symmetry group, and J corresponds to the eigenvalue of an $SO(2)$ R-symmetry generator. The AdS/CFT correspondence implies that string energies in the large-J limit should match operator dimensions in the limit of large R-charge.

On dimensional grounds, taking the $J\to\infty$ limit on string states is equivalent to taking the $R\to\infty$ limit of the geometry (in properly chosen coordinates). The coordinate redefinitions

$$t\to x^+ \qquad \phi\to x^+ + \frac{x^-}{R^2} \qquad z_k\to\frac{z_k}{R} \qquad y_{k'}\to\frac{y_{k'}}{R} \tag{2.3.3}$$

make it possible to take a smooth $R\to\infty$ limit. (The light-cone coordinates $x^\pm$ are a bit unusual, but have been chosen for future convenience in quantizing the worldsheet Hamiltonian). Expressing the metric (2.3.2) in these new coordinates, we obtain the following expansion in powers of $1/R^2$:

$$\begin{aligned} ds^2 \;\approx\;& 2\,dx^+dx^- + dz^2 + dy^2 - \left(z^2+y^2\right)(dx^+)^2 + \\ & \frac{1}{R^2}\left[-2y^2dx^-dx^+ + \frac{1}{2}\left(y^4-z^4\right)(dx^+)^2 + \left(dx^-\right)^2 + \frac{1}{2}z^2dz^2 - \frac{1}{2}y^2dy^2\right] \\ & +\mathcal{O}(1/R^4)\ . \end{aligned} \tag{2.3.4}$$

The leading R-independent part is the Penrose limit, or pp-wave geometry: it describes the geometry seen by the infinitely boosted string. For future reference, we define this limiting metric as

$$ds^2_{pp} = 2\,dx^+dx^- + dz^2 + dy^2 - \left(z^2+y^2\right)(dx^+)^2\ . \tag{2.3.5}$$

The x^+ coordinate is dimensionless, x^- has dimensions of length squared, and the transverse coordinates now have dimensions of length.

As we described above, in light-cone gauge quantization, one identifies worldsheet time τ with the x^+ coordinate, so that the worldsheet Hamiltonian corresponds to the conjugate space-time momentum $P_+ = \omega - J$, additionally, one sets the worldsheet momentum density p_- =constant. Once $x^\pm$ are eliminated, the quadratic dependence of ds^2_{pp} on the remaining eight transverse bosonic coordinates leads to a a simple bosonic light-cone Hamiltonian which can be solved. Things are less simple when $1/R^2$ corrections to the metric are taken into account: they add quartic interactions to the light-cone Hamiltonian.

To include fermions we start with the action for the IIB superstring in the $AdS_5 \times S^5$ background [44], we pass to light-cone gauge and then take the Penrose limit. The latter step reduces the otherwise extremely complicated action to a worldsheet theory of free, equally massive transverse bosons and fermions. We could of course start with the pp-wave isometry algebra and construct the appropriate Cartan forms and the κ-symmetric action; this was done in [41], but as we are also interested in the next order corrections the perturbative expansion of the action is appropriate. Light-cone quantization has several disadvantages, not the least of which is the lack of manifest covariance which is particularly troublesome in the plane wave background as there is no J^{+-} isometry. The Berkovits formalism avoids these disadvantages and leads to quantizable actions for quite general backgrounds however the GS light-cone formalism is sufficient for our purposes.

We give here a concise summary of the construction and properties of the light-cone Hamiltonian H^{GS}_{pp} that describes the superstring in this limit. This will be a helpful preliminary to our principal goal of evaluating the corrections to the Penrose limit of the GS action and will allow us to fix some notation and conventions.

Gauge fixing eliminates both light-cone coordinates $x^\pm$, leaving eight transverse coordinates x^I as bosonic dynamical variables. As in the $AdS_5 \times S^5$ the IIB supergravity has two ten-dimensional supersymmetries that are described by two sixteen-

component Majorana–Weyl spinors of the same ten-dimensional chirality. As we described, in the course of light-cone gauge fixing, half of these fermi fields are set to zero, leaving behind a complex eight-component worldsheet fermion ψ. In a sixteen-component notation the restriction of the worldsheet fermions to the $\mathbf{8}_s$ representation is implemented by the condition $\gamma^9\psi = +\psi$. Another quantity, which proves to be important in what follows, is $\Pi \equiv \gamma^1\gamma^2\gamma^3\gamma^4$. One could also define $\tilde{\Pi} = \gamma^5\gamma^6\gamma^7\gamma^8$, but $\Pi\psi = \tilde{\Pi}\psi$ for an $\mathbf{8}_s$ spinor.

In the Penrose limit, the light-cone GS superstring action takes the form (with the appropriate choice of α')

$$S_{pp} = \frac{1}{2\pi}\int d\tau \int_0^{2\pi} d\sigma(\mathcal{L}_B + \mathcal{L}_F) \ , \qquad \text{where}$$
$$\mathcal{L}_B = \frac{1}{2}\left[(\dot{x}^A)^2 - (x'^A)^2 - p_-^2(x^A)^2\right] \ , \mathcal{L}_F = i\psi^\dagger\dot{\psi} + p_-\psi^\dagger\Pi\psi + \frac{i}{2}(\psi\psi' + \psi^\dagger\psi'^\dagger). \quad (2.3.6)$$

The fermion mass term $p_-\psi^\dagger\Pi\psi$ arises from the coupling to the background RR 5-form field strength, and matches the bosonic mass term (as required by supersymmetry). It is important that the quantization procedure preserve supersymmetry. However, as is typical in light-cone quantization, some of the conserved generators are linearly realized on the x^A and ψ^α while others have a more complicated non-linear realization.

The eight bosonic transverse string coordinates obey the equation

$$\ddot{x}^A - x''^A + p_-^2 x^A = 0 \ , \quad (2.3.7)$$

which is solved by the usual expansion in terms of Fourier modes

$$\begin{aligned} x^A(\sigma,\tau) &= \sum_{n=-\infty}^{\infty} x_n^A(\tau)e^{-ik_n\sigma} \\ x_n^A(\tau) &= \frac{i}{\sqrt{2\omega_n}}(a_n^A e^{-i\omega_n\tau} - a_{-n}^{A\dagger}e^{i\omega_n\tau}) \ , \end{aligned} \quad (2.3.8)$$

where $k_n = n \in \mathbb{Z}$, $\omega_n = \sqrt{p_-^2 + k_n^2}$, and the raising and lowering operators obey

the commutation relation $[a_m^A, a_n^{B\dagger}] = \delta_{mn}\delta^{AB}$. The bosonic piece of the pp-wave Hamiltonian takes the form

$$\mathcal{H}^B_{\rm pp} = \frac{1}{p_-}\sum_{n=-\infty}^{\infty}\omega_n\left(a_n^{A\dagger}a_n^A + 4\right) . \tag{2.3.9}$$

The fermionic equations of motion are

$$(\dot{\psi}^\dagger + \psi') + ip_-\Pi\psi^\dagger = 0 \tag{2.3.10}$$

$$(\dot{\psi} + \psi'^\dagger) - ip_-\Pi\psi = 0 , \tag{2.3.11}$$

and are solved by

$$\psi = \sum_{n=-\infty}^{\infty}\psi_n(\tau)e^{-ik_n\sigma} \tag{2.3.12}$$

$$\psi_n(\tau) = \frac{1}{2\sqrt{p_-}}\left(A_n b_n e^{-i\omega_n\tau} + B_n b_{-n}^\dagger e^{i\omega_n\tau}\right)e^{-ik_n\sigma} \tag{2.3.13}$$

$$\psi_n^\dagger(\tau) = \frac{1}{2\sqrt{p_-}}\left(\Pi B_n b_n e^{-i\omega_n\tau} - \Pi A_n b_{-n}^\dagger e^{i\omega_n\tau}\right)e^{-ik_n\sigma} , \tag{2.3.14}$$

where we have defined

$$A_n \equiv \frac{1}{\sqrt{\omega_n}}\left(\sqrt{\omega_n - k_n} - \sqrt{\omega_n + k_n}\Pi\right) \tag{2.3.15}$$

$$B_n \equiv \frac{1}{\sqrt{\omega_n}}\left(\sqrt{\omega_n + k_n} + \sqrt{\omega_n - k_n}\Pi\right) . \tag{2.3.16}$$

The fermionic canonical momentum is $\rho = ip_-\psi^\dagger$, which implies that the fermionic creation and annihilation operators obey the anticommutation rule $\{b_m^\alpha, b_n^{\beta\dagger}\} = \delta^{\alpha\beta}\delta_{mn}$. The fermionic piece of the pp-wave Hamiltonian can be written in terms of these operators as

$$\mathcal{H}^F_{\rm pp} = \frac{1}{p_-}\sum_{n=-\infty}^{\infty}\omega_n\left(b_n^{\alpha\dagger}b_n^\alpha - 4\right) . \tag{2.3.17}$$

Given our earlier conventions, it was necessary to invoke the coordinate reflection $x^\mu \to -x^\mu$ (Metsaev uses a similar operation on the pp-wave Hamiltonian in [41]). Such a transformation is, at this stage, equivalent to sending $x^A \to -x^A$, $p_- \to -p_-$, and $\mathcal{H} \to -\mathcal{H}$. In essence, this operation allows us to choose the positive-energy solutions to the fermionic equations of motion while maintaining our convention that $b^{\alpha^\dagger}$ represent a creation operator and b^α denote an annihilation operator. The total pp-wave Hamiltonian

$$\mathcal{H}_{\rm pp} = \frac{1}{p_-} \sum_{n=-\infty}^{\infty} \omega_n \left(a_n^{A\dagger} a_n^A + b_n^{\alpha\dagger} b_n^\alpha \right) \tag{2.3.18}$$

is just a collection of free, equal mass fermionic and bosonic oscillators.

$$\left[a_m^A, a_n^{B\dagger}\right] = \delta_{mn}\delta^{AB} \;\Rightarrow\; \left[x^A(\sigma), p^B(\sigma')\right] = i2\pi\alpha'\delta(\sigma-\sigma')\delta^{AB}\ . \tag{2.3.19}$$

The harmonic oscillator zero-point energies nicely cancel between bosons and fermions for each mode n. We can absorb the overall factor of p_- into the frequencies ω_n and express them in terms of the single parameter $\lambda' = 1/p_-^2$

$$\lambda' = g_{YM}^2 N_c / J^2 \qquad \omega_n = \sqrt{1+\lambda' n^2}, \tag{2.3.20}$$

so that one can take J *and* $g_{YM}^2 N_c$ to be simultaneously large while keeping λ' fixed. If λ' is kept fixed and small, ω_n may be expanded in powers of λ', suggesting that contact with perturbative Yang-Mills gauge theory is possible.

The spectrum is generated by $8+8$ transverse oscillators acting on ground states labeled by an $SO(2)$ angular momentum taking integer values $-\infty < J < \infty$ (note that the oscillators themselves carry zero $SO(2)$ charge). Any combination of oscillators may be applied to a ground state, subject to the constraint that the sum of the oscillator mode numbers must vanish, this is the level-matching constraint. The energies of these states are the sum of the individual oscillator energies, and the

spectrum is very degenerate.[2] For example, the 256 states of the form $A_n^\dagger B_{-n}^\dagger|J\rangle$ for a given mode number n (where $A^\dagger$ and $B^\dagger$ each can be any of the 8+8 bosonic and fermionic oscillators) all have the energy

$$P_+ = \omega - J = 2\sqrt{1 + (g_{YM}^2 N_c n^2/J^2)} \sim 2 + (g_{YM}^2 N_c n^2/J^2) + \dots . \tag{2.3.21}$$

In the weak coupling limit ($\lambda' \to 0$) the degeneracy is even larger because the dependence on the oscillator mode number n goes away! This actually makes sense from the dual gauge theory point of view where $P_+ \to D - \mathcal{R}$ (D is the dimension and $\mathcal{R}$ is the $\mathcal{R}$-charge carried by gauge-invariant operators of large $\mathcal{R}$); at zero coupling, operators have integer dimensions and the number of operators with $D - \mathcal{R} = 2$, for example, grows with $\mathcal{R}$, providing a basis on which string multiplicities are reproduced. Even more remarkably, BMN were able to show [27] that subleading terms in a λ' expansion of the string energies match the first perturbative corrections to the gauge theory operator dimensions in the large $\mathcal{R}$-charge limit.

More generally, we expect exact string energies in the $AdS_5 \times S^5$ background to have a joint expansion in the parameters λ', defined above, and $1/J$. We also expect the degeneracies found in the $J \to \infty$ limit (for fixed λ') to be lifted by the interaction terms that arise in the worldsheet Hamiltonian describing string physics at large but finite J. Large degeneracies must nevertheless remain in order for the spectrum to be consistent with the $PSU(2,2|4)$ global supergroup that should characterize the exact string dynamics. The specific pattern of degeneracies should also match that of operator dimensions in the $\mathcal{N} = 4$ super Yang-Mills theory.

[2] Note that the $n = 0$ oscillators raise and lower the string energy by a protected amount $\delta P_+ = 1$, independent of the variable parameters. These oscillators play a special role, enlarging the degeneracy of the string states in a crucial way, and we will call them 'zero-modes' for short.

Chapter 3

$\mathcal{N} = 4$ Super Yang-Mills theory

3.1 Introduction

The action for $\mathcal{N} = 4$ $SU(N_c)$ super Yang-Mills [15,21] can be found by dimensionally reducing ten-dimensional super Yang-Mills on a six-dimensional torus giving

$$S = \frac{1}{g_{YM}^2}\int d^4x \ \mathrm{tr}\left\{-\frac{1}{4}F^2 - \frac{1}{2}(\mathcal{D}_\mu\phi^I)^2 + \bar{\psi}\gamma^\mu\mathcal{D}_\mu\psi + \frac{1}{4}\left[\phi^I,\phi^J\right]^2 - \frac{i}{2}\psi\Gamma^I\left[\phi^I,\psi\right] - \frac{i}{2}\bar{\psi}\Gamma^I\left[\phi^I,\bar{\psi}\right]\right\}, \quad (3.1.1)$$

where, here, $\mu,\nu = 0,\ldots,3$; $I,J = 4,\ldots,9$ and we have used ten-dimensional gamma matrices $\Gamma^A = (\gamma^\mu,\Gamma^I)$. The gluinos are written as a sixteen component Majorana spinor, however we will find it convenient to rewrite them as two-component Weyl spacetime spinors. The gauge field, gluinos and scalars are all in the adjoint of the gauge group. This theory has an exact global $SO(6) \simeq SU(4)$ $\mathcal{R}$-symmetry under which the gluinos transform as a $\mathbf{4}$ and $\bar{\mathbf{4}}$, the scalars as a $\mathbf{6}$ and as we will see it is helpful to classify operators according to their $SU(4)$ representation. Irreducible tensor representations of $SU(4)$ are indexed by Young diagrams describing their symmetries under permutations of the tensor indices. Such diagrams contain up to three rows of boxes with non-increasing numbers of boxes per row and are denoted by a set of three integers (n_1,n_2,n_3) giving the differences in length of successive rows. The total number of boxes in the diagram is the total number of $SU(4)$ indices in

the tensor. The boxes are filled in with tensor indices in some canonical order and the representations are antisymmetric under the exchange of any pair of indices in the same column. More specifically, the scalars are in the six-dimensional $(0,1,0)$ representation of $SU(4)$, the gluinos are in the four-dimensional fundamental $(1,0,0)$ plus an adjoint field in the four-dimensional anti-fundamental $(0,0,1)$:

$$\text{Scalars}: \ \phi^{\begin{smallmatrix}\square\\\square\end{smallmatrix}} \qquad \text{Gluinos}: \ \chi^{\square}{}_{\alpha}, \ \overline{\chi}^{\begin{smallmatrix}\square\\\square\end{smallmatrix}}{}_{\dot{\alpha}} \ . \tag{3.1.2}$$

The α (resp. $\dot{\alpha}$) indices on the gluinos indicate that they transform in the $(\mathbf{2},\mathbf{1})$ (resp. $(\mathbf{1},\mathbf{2})$) representations of the $SL(2,C)$ covering group of the spacetime Lorentz group. The Young diagram superscript is a shorthand for indicating the $SU(4)$ tensor character of the fields (viz. ϕ is a rank-two antisymmetric tensor, χ_a is a rank-one tensor and so on).

Of course this $SU(4)$ $\mathcal{R}$-symmetry is not the full symmetry of $\mathcal{N}=4$ SYM, it also exhibits conformal symmetry i.e., the group of transformations which preserve the form of the metric up to an overall factor. The generators of this group are the Lorentz transformations $M_{\mu\nu}$, translations P_μ, special conformal transformations $K_{\mu\nu}$ and the scaling transformation D. When these are combined with the supersymmetry generators Q we form the superconformal algebra, which further includes the special fermionic generators S and the generators of the $\mathcal{R}$-symmetry mentioned above (for a more detailed discussion of this algebra and a list of references see [23]). In our case the full symmetry group is $PSU(2,2|4)$, the same as isometry group in the string theory and whose algebra we wrote down (using a different set of conventions) in (2.2.11). One key point is that the dimension D commutes with the full $\mathcal{R}$-symmetry group. As we have said, the AdS/CFT correspondence conjectures that this gauge theory is equivalent to type IIB superstring theory on $AdS_5 \times S^5$ with the identification $\left(\frac{R^2}{\alpha'}\right)^2 = g_{YM}^2 N_c$ and $g_{YM}^2 = g_s$. The gauge theory can be studied in the limit where $g_{YM} \to 0$ and $N_c \to 0$ with the 't Hooft coupling $\lambda = g_{YM}^2 N_c$ kept fixed. This corresponds to considering only planar diagrams in the gauge theory and taking the classical ($g_s \to 0$ limit of the string theory). However for reasons previously outlined

we will be interested in a different limit that corresponds to taking the Penrose limit on the string theory side. In this BMN limit, [27], we consider single-trace operators with a charge $\mathcal{R}$ under an $SO(2)$ subgroup of the $SO(6) \cong SU(4)$ $\mathcal{R}$-symmetry and take the limit

$$N_c, \mathcal{R} \to \infty \qquad \text{with} \qquad \lambda' = \frac{g_{YM}^2 N_c}{\mathcal{R}} \qquad \text{and} \qquad g_2 = \frac{\mathcal{R}^2}{N_c} \qquad \text{fixed.} \quad (3.1.3)$$

In fact it was noticed [46] and [47] that in this limit it is not sufficient to consider only planar amplitudes but that it also admits all-genus amplitudes. In our work however we will further restrict our interest to the case where the effective genus counting parameter, g_2, is zero. Thus we need only consider monomials involving a single gauge trace and we now consider the dimensions of these operators. This discussion follows [48] which in turn relied in part on [49] however it allows us to present the results in a form suitable for comparison with the string theory. For the moment, we limit our attention to operators that are spacetime scalars. The $SO(2)$ scalar $\mathcal{R}$-charge that will eventually be taken to infinity (to match the $J \to \infty$ limit of the string spectrum) is defined by the decomposition $SU(4) \supset SU(2) \times SU(2) \times U(1)_R$ (equivalently $SO(6) \supset SO(4) \times SO(2)$). The scalar $\mathcal{R}$-charge of the various components of the gauge theory fields is assessed by distributing indices in the boxes of the Young diagram superscripts, subject to the rule of column antisymmetry and assigning $\mathcal{R} = \frac{1}{2}(-\frac{1}{2})$ to $SU(4)$ indices $1, 2$ $(3, 4)$ respectively. The result is as follows:

$$\mathcal{R} = 1: \ \phi^{\begin{smallmatrix}1\\2\end{smallmatrix}}(Z), \qquad \mathcal{R} = 0: \ \phi^{\begin{smallmatrix}1\\3\end{smallmatrix}}, \phi^{\begin{smallmatrix}1\\4\end{smallmatrix}}, \phi^{\begin{smallmatrix}2\\3\end{smallmatrix}}, \phi^{\begin{smallmatrix}2\\4\end{smallmatrix}}(\phi^{i'}), \qquad \mathcal{R} = -1: \ \phi^{\begin{smallmatrix}3\\4\end{smallmatrix}}(\bar{Z})$$
$$\mathcal{R} = 1/2: \ \chi^{\boxed{1}}, \chi^{\boxed{2}}, \bar{\chi}^{\begin{smallmatrix}1\\3\end{smallmatrix}}, \bar{\chi}^{\begin{smallmatrix}1\\4\end{smallmatrix}}, \qquad \mathcal{R} = -1/2: \ \chi^{\boxed{3}}, \chi^{\boxed{4}}, \bar{\chi}^{\begin{smallmatrix}1\\3\\4\end{smallmatrix}}, \bar{\chi}^{\begin{smallmatrix}2\\3\\4\end{smallmatrix}}. \quad (3.1.4)$$

We can introduce an alternate notation for the scalars (to be used later) $(Z, \bar{Z}, \phi^{i'}, i' = 1, .., 4)$ that emphasizes their $SO(4)$ content. As discussed earlier, we need a basis of operators with large naive dimension which we call D_0, large scalar $\mathcal{R}$-charge and fixed $\Delta_0 = D_0 - \mathcal{R}$. BMN showed that, in this limit, such operators correspond to string states created by a fixed finite number (Δ_0) of string oscillators acting on the

pp-wave ground state of angular momentum $\mathcal{R}$. Operators with $\Delta_0 = 0$ are BPS, and their dimensions are protected by supersymmetry. In what follows, we will, for simplicity, restrict the discussion to $\Delta_0 = 2$ operators, corresponding to string states created by two oscillators acting on the vacuum (so-called 'two-impurity' states).

3.2 Two impurity states

The list of *all* single-trace spacetime scalar operators of naive dimension D_0 which can have $\Delta_0 \leq 2$ is as follows:

$$\begin{array}{cccc}
 & & \mathrm{tr}\left((\phi^{\square\square})^{D_0}\right), & (\mathcal{R}_{\max} = D_0) \\
\mathrm{tr}\left((\chi^{\square}\sigma_2\chi^{\square})(\phi^{\square\square})^{D_0-3}\right), & \mathrm{tr}\left((\chi^{\square}\phi^{\square\square}\sigma_2\chi^{\square})(\phi^{\square\square})^{D_0-4}\right), & \ldots & (\mathcal{R}_{\max} = D_0 - 2) \\
\mathrm{tr}\left((\overline{\chi}^{\square\square}\sigma_2\overline{\chi}^{\square\square})(\phi^{\square\square})^{D_0-3}\right), & \mathrm{tr}\left((\overline{\chi}^{\square\square}\phi^{\square\square}\sigma_2\overline{\chi}^{\square\square})(\phi^{\square\square})^{D_0-4}\right), & \ldots & (\mathcal{R}_{\max} = D_0 - 2) \\
 & & \mathrm{tr}\left(\mathcal{D}_\mu\phi^{\square\square}\mathcal{D}^\mu\phi^{\square\square}(\phi^{\square\square})^{D_0-4}\right), & (\mathcal{R}_{\max} = D_0 - 2)\ . \quad (3.2.1)
\end{array}$$

The fields inside the operators are $SU(N_c)$ adjoint matrices and the trace is taken over gauge indices; spacetime spinor indices on the χ are contracted to produce a spacetime scalar (note that a product of a $\chi^{\square}$ and a $\overline{\chi}^{\square\square}$ cannot make a scalar because they transform under inequivalent irreps of spacetime $SL(2,C)$); $\mathcal{D}$ is the spacetime gauge-covariant derivative. There are multiple versions of operators involving gluinos and spacetime derivatives arising from the different ways that scalars may be distributed among them (and the cyclic symmetry of the gauge trace reduces the number of independent operators one can construct). These operators provide a basis for a reducible representation of the global $SU(4)$ R-symmetry group. Since the anomalous dimension operator commutes with this $SU(4)$, it will have no matrix elements between different $SU(4)$ irreps, and our first task is to find linear combinations of the above operators that provide a basis for these irreps (and find the multiplicities of inequivalent occurrences of the same irrep). The group theory analysis helps us obtain precise control of the subleading corrections in $1/D_0$ to the structure of the

operators and their anomalous dimensions.

For the bosonic operators with no derivatives, we have a reducible $SU(4)$ tensor of rank $2D_0$, which we must decompose into irreducible $SU(4)$ tensors of rank $2D_0$. These irreps are symbolized by Young diagrams with $2D_0$ boxes; the main problem is to determine the multiplicity with which each such diagram appears. The standard algorithm for projecting a reducible character onto irreducible characters [50] cannot be implemented because of the cyclic symmetry of single-trace monomials. The algorithm, however, can be adapted with some effort to the case at hand to compute the desired multiplicities. Although the total number of irreducible tensors in the expansion grows rapidly with D_0, only a few can have $\Delta_0 = D_0 - \mathcal{R} = 0, 2$ and we report only the multiplicities of that limited set of irreps. The results are slightly different for odd and even D_0, but we will eventually see that this even/odd difference is harmless. For D_0 odd we have

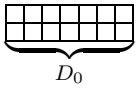
$$\mathrm{tr}\left(\phi^{\square\,D_0}\right) \to\ 1 \times \underbrace{\square\cdots\square}_{D_0} \oplus \left(\frac{D_0-1}{2}\right) \times \underbrace{\square\cdots\square}_{D_0-1} \oplus \left(\frac{D_0-1}{2}\right) \times \underbrace{\square\cdots\square}_{D_0-2} \oplus \left(\frac{D_0-1}{2}\right) \times \underbrace{\begin{array}{l}\square\cdots\square\\ \square\end{array}}_{D_0-1} \oplus \left(\frac{D_0-3}{2}\right) \times \underbrace{\begin{array}{l}\square\cdots\square\\ \square\square\end{array}}_{D_0} \oplus \ \ldots\ , \tag{3.2.2}$$

while for D_0 even we have

$$\mathrm{tr}\left(\phi^{\square\,D_0}\right) \to\ 1 \times \underbrace{\square\cdots\square}_{D_0} \oplus \left(\frac{D_0-2}{2}\right) \times \underbrace{\square\cdots\square}_{D_0-1} \oplus \left(\frac{D_0}{2}\right) \times \underbrace{\square\cdots\square}_{D_0-2} \oplus \left(\frac{D_0-2}{2}\right) \times \underbrace{\begin{array}{l}\square\cdots\square\\ \square\end{array}}_{D_0-1} \oplus \left(\frac{D_0-2}{2}\right) \times \underbrace{\begin{array}{l}\square\cdots\square\\ \square\square\end{array}}_{D_0} \oplus \ \ldots \tag{3.2.3}$$

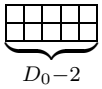
These irrep expansions could equally well have been done using the bosonic $\mathcal{R}$-symmetry group $SO(6)$: this is what was done, with the same results, by Beisert in [49]. As the algorithm for determining the multiplicities in general is quite complicated it is perhaps useful to explain some of the above results by repeating Beiserts analysis and restricting to scalars. We are interested in the product of of D_0 scalars

which in general will be a sum of irreps with Young tableau of $2D_0, 2D_0-4, \ldots$ boxes. We are interested in operators with at most two charge defects and thus in general we need only consider the irreps which are listed in (3.2.2, 3.2.3) plus one that is inconsistent with the cyclic symmetry of the trace. By examining the Young tableau we can find the maximum $\mathcal{R}$-charge in a given representation and how the weights with maximum $\mathcal{R}$-charge transform under the remaining $SO(4)$. For example the representation



(3.2.4)

obviously has an operator with maximum R-charge D_0, which transforms as a singlet under the unbroken $SO(4)$. This representation also contains operators with one scalar impurity having $\mathcal{R}$-charge $D_0 - 1$, which transform as a $\mathbf{4}$, and operators with two impurities having $\mathcal{R}$-charge $D_0 - 2$, which transform as a $\mathbf{1} + \mathbf{9}$. Similarly,



(3.2.5)

has a maximum $\mathcal{R}$-charge $D_0 - 2$ operator which transform as a singlet under the $SO(4)$. The remaining irreps contain operators also having a maximum $\mathcal{R}$-charge $D_0 - 2$ but which transform as a $\mathbf{3}$,$\bar{\mathbf{3}}$ and a $\mathbf{9}$. We can now count the number of operators made of D_0 scalars in a single trace with the $\mathcal{R}$-charge $\geq D_0 - 2$. Most of the scalars will be Z's with at most two scalar impurities. The four scalars impurities comprise a $\mathbf{4}$ of the $SO(4)$ and the tensor product is $\mathbf{4} \otimes \mathbf{4} = \mathbf{3} \oplus \bar{\mathbf{3}} \oplus \mathbf{9}$. We must consider all possible positions of the defects in the cyclic trace and we note that the maximum separation for symmetric combinations of impurities is $[(D_0 - 2)/2]$ and $[(D_0 - 3)/2]$ for antisymmetric combinations. We must also include the additional $SO(4)$ singlet due to a single $\bar{Z}$ impurity. In addition, there is the singlet operator at $\mathcal{R}$-charge D_0 and a single $\mathbf{4}$ at $\mathcal{R}$-charge $D_0 - 1$. It is now possible to match this counting with the $SU(4)$ irreps. For example, there is one singlet at $\mathcal{R}$-charge D_0

and this must lie in the $\underbrace{\square\cdots\square}_{D_0}$ irrep which then has multiplicity one. This irrep also contains a $\mathbf{4}$ at $\mathcal{R}$-charge $D_0 - 1$ and a $\mathbf{1} + \mathbf{9}$ at $\mathcal{R}$-charge $D_0 - 2$; we can now associate the other operators with the remaining irreps and find the multiplicities in (3.2.2, 3.2.3).

Returning to our general $SU(4)$ analysis we note that the irreps with larger minimal values of $\Delta_0 = D_0 - R$ have multiplicities that grow as higher powers of D_0. This is very significant for the eventual string theory interpretation of the anomalous dimensions, but we will not expand on this point here.

Other spacetime scalar operators that can have $\Delta_0 = D_0 - R = 2$ are the 'bifermions,' or products of two gluinos and $D_0 - 3$ scalars. Including only the irreps that can actually have $\Delta_0 = 2$, their expansions are as follows:

$$\mathrm{tr}\left(\chi^{\square}\,\sigma_2\,\chi^{\square}\,(\phi^{\square})^{D_0-3}\right) \to\ 1\times\underbrace{\square\cdots\square}_{D_0-2}\ \oplus\ \ 1\times\underbrace{\square\cdots\square\,\square}_{D_0-1}\ \oplus\ \ \ldots \tag{3.2.6}$$

$$\mathrm{tr}\left(\overline{\chi}^{\begin{smallmatrix}\square\\\square\end{smallmatrix}}\,\sigma_2\,\overline{\chi}^{\begin{smallmatrix}\square\\\square\end{smallmatrix}}\,(\phi^{\begin{smallmatrix}\square\\\square\end{smallmatrix}})^{D_0-3}\right) \to\ 1\times\underbrace{\begin{smallmatrix}\square\cdots\square\\\square\end{smallmatrix}}_{D_0-1}\ \oplus\ \ 1\times\underbrace{\square\cdots\square}_{D_0-2}\ \oplus\ \ \ldots\ . \tag{3.2.7}$$

There are identical expansions for operators arising from different placements of the fermions with respect to the bosons. Because of cyclicity of the gauge trace and the fermi statistics of the gluino fields, these operators are not all independent. The counting of independent operators depends, once again, on whether D_0 is even or odd. Using an obvious shorthand notation, the multiplicities of bifermion irreps are as follows for D_0 odd:

$$\mathrm{tr}\left(\chi^{\square}\,\sigma_2\,\chi^{\square}\,(\phi^{\square})^{D_0-3}\right) \to \left(\frac{D_0-3}{2}\right)\times\underbrace{\square\cdots\square}_{D_0-2}\ \oplus\left(\frac{D_0-1}{2}\right)\times\underbrace{\square\cdots\square\,\square}_{D_0-1}\ \oplus\ldots \tag{3.2.8}$$

$$\mathrm{tr}\left(\overline{\chi}^{\begin{smallmatrix}\square\\\square\end{smallmatrix}}\,\sigma_2\,\overline{\chi}^{\begin{smallmatrix}\square\\\square\end{smallmatrix}}\,(\phi^{\begin{smallmatrix}\square\\\square\end{smallmatrix}})^{D_0-3}\right) \to \left(\frac{D_0-3}{2}\right)\times\underbrace{\square\cdots\square}_{D_0-2}\ \oplus\left(\frac{D_0-1}{2}\right)\times\underbrace{\begin{smallmatrix}\square\cdots\square\\\square\end{smallmatrix}}_{D_0-1}\ \oplus\ldots. \tag{3.2.9}$$

The results for D_0 even are, once again, slightly different:

$$\mathrm{tr}\left(\chi^{\square}\,\sigma_2\,\chi^{\square}\,(\phi^{\square\!\square})^{D_0-3}\right) \to \left(\frac{D_0-2}{2}\right)\times\underbrace{\square\!\square\!\square\!\square\!\square}_{D_0-2}\ \oplus\left(\frac{D_0-2}{2}\right)\times\underbrace{\square\!\square\!\square\!\square\!\square\!\square}_{D_0-1}\ \oplus\ldots \quad (3.2.10)$$

$$\mathrm{tr}\left(\overline{\chi}^{\square\!\square}\,\sigma_2\,\overline{\chi}^{\square\!\square}\,(\phi^{\square\!\square})^{D_0-3}\right) \to \left(\frac{D_0-2}{2}\right)\times\underbrace{\square\!\square\!\square\!\square\!\square}_{D_0-2}\ \oplus\left(\frac{D_0-2}{2}\right)\times\underbrace{\begin{array}{l}\square\!\square\!\square\!\square\!\square\\ \square\end{array}}_{D_0-1}\ \oplus\ldots. \quad (3.2.11)$$

The point of all this is that the dimension operator can only have matrix elements between operators belonging to the same $SU(4)$ irrep. There is a unique irrep, $(0, D_0, 0)$ (i.e., two rows of D_0 boxes), which contains 'top' states with dimension equal to $\mathcal{R}$-charge (or $\Delta_0 = \Delta = 0$). The latter are known to be BPS states and get no correction to their dimension. Thus the dimension of the whole irrep, including all its components with $\Delta_0 > 0$, is unmodified by interactions. The other irreps displayed above have multiplicities that grow roughly as $D_0/2$ for large D_0. The irreps we have not displayed have higher values of Δ_0 *and* multiplicities that grow as higher powers of D_0. The dimension operator will, in general, have matrix elements between all the operators belonging to a given representation. We therefore have to diagonalize a matrix of size roughly $D_0/2 \times D_0/2$ and will find $O(D_0/2)$ eigenvalues. The key question will then be the evolution of the spectrum as $D_0 \to \infty$. From the work of BMN, we expect to find a spectrum that can be interpreted, at large $D_0 = \mathcal{R} + 2$ and fixed $\Delta_0 = D_0 - \mathcal{R} = 2$, as due to the action of two string modes on a string ground state of angular momentum $J = \mathcal{R}$. Our goal is to evaluate and compare the $1/\mathcal{R}$ corrections on both sides of this correspondence. One benefit of the group theory analysis is immediately apparent: the irrep $(2, D_0 - 4, 2)$ appears *only* in the reduction of the purely bosonic operator. For this irrep, the anomalous dimension matrix must act purely within the space of bosonic operators, a welcome simplification. By contrast, the irrep $(0, D_0 - 3, 2)$ appears both in the purely bosonic operators and in one of the two-fermion operators (with the same multiplicity in both cases). Thus, there can be matrix elements of the dimension operator between boson

and fermion states and the diagonalization problem will be more complicated. In fact, the results of the diagonalization will test the fermionic structure of the string Hamiltonian, which makes this a particularly important computation to carry out.

Having calculated the multiplicity of specific irreps, we turn to the perturbative diagonalization of the dimension operator. A simple approach begins with the two-point function between elements of the operator basis $\{O_a(x)\}$, calculated to first non-trivial order in perturbation theory. The typical result is

$$\langle O_a(x) O_b(0)\rangle \sim (x)^{-2d_0}(\delta_{ab} + \ln(x^2)\ d_1^{ab})\ , \tag{3.2.12}$$

where d_0 is the naive dimension. The leading Kronecker δ_{ab} implies that the operator basis is orthonormal in the free theory (in the large-N_c limit, this is enforced by multiplying the operator basis by a common overall normalization constant). The anomalous dimensions are then the eigenvalues of the mixing matrix d_1^{ab}, and the eigenoperators of definite dimension are linear combinations of basis operators defined by the eigenvectors. One should be careful to pick out conformal primary operators, but this subtlety is not too troublesome for one-loop perturbative calculations.

Group theory tells us that the dimension operator D block-diagonalizes under the different $SU(4)$ irreps, and it is not too hard to show in concrete detail how it works in the purely bosonic sector. Consider a basis of $D_0 - 1$ bosonic operator monomials of dimension D_0 and $\Delta_0 = D_0 - \mathcal{R} = 2$:

$$\begin{aligned}\{O^{AB}_{D_0,1}, \ldots, O^{AB}_{D_0,D_0-1}\} = \{&\mathrm{tr}(ABZ^{D_0-2}),\ \ \mathrm{tr}(AZBZ^{D_0-3}), \ldots, \\ &\mathrm{tr}(AZ^{D_0-3}BZ),\ \ \mathrm{tr}(AZ^{D_0-2}B)\}\ ,\end{aligned} \tag{3.2.13}$$

where Z stands for $\phi^{\boxed{1}\boxed{2}}$ and has $\mathcal{R} = 1$, while A, B stand for any of the four $\phi^{i'}$ ($i' = 1, \ldots, 4$) and have $\mathcal{R} = 0$. The overall constant needed to orthonormalize this basis (in the large-N_c limit) is easy to compute, but not needed for present purposes. In the $SO(2) \times SO(4)$ decomposition of $SU(4)$, A, B are $SO(4)$ vectors so that the

operators of this basis are rank-two $SO(4)$ tensors. In particular, the symmetric traceless tensor belongs to the $(2, D_0-4, 2)$ irrep of $SU(4)$, the antisymmetric tensor belongs to the pair $(0, D_0-3, 2)+(2, D_0-3, 0)$, and the $SO(4)$ trace (when completed to a full $SO(6)$ trace) belongs to the $(0, D_0-2, 0)$ irrep. In what follows, we refer to these three classes of operator as $\overline{T}^{(+)}_{D_0}$, $\overline{T}^{(-)}_{D_0}$ and $\overline{T}^{(0)}_{D_0}$, respectively. If we take $A \neq B$, the trace part drops out and the $\overline{T}^{(\pm)}_{D_0}$ operators are isolated by symmetrizing and antisymmetrizing on A, B.

A simple extension of the BMN argument can be used to give the $O(g^2_{YM}N_c)$ action of the anomalous dimension operator on the basis (3.2.13), correct to all orders in $1/D_0$. In the leading large-N_c limit and leading order in g^2_{YM}, the gauge theory interaction term $\mathrm{tr}([\phi^a, \phi^b][\phi^a, \phi^b])$ has a very simple action on single-trace monomials in the ϕ's: it produces a sum of interchanges of all nearest-neighbors in the trace. Diagrams that lead to exchanges at greater distances are non-planar and suppressed by powers of $1/N_c$. For the restricted case $A \neq B$, the leading action of the anomalous dimension on the $D_0 - 1$ bosonic monomials of (3.2.13) has the following detailed structure:

$$\begin{aligned}
(ABZ^{D_0-2}) &\to (BAZ^{D_0-2}) + 2(AZBZ^{D_0-3}) + (D_0-3)(ABZ^{D_0-2}) \\
(AZBZ^{D_0-3}) &\to 2(ABZ^{D_0-2}) + 2(AZ^2BZ^{D_0-4}) + (D_0-4)(AZBZ^{D_0-3}) \ \ldots, \\
(AZ^{D_0-2}B) &\to 2(AZ^{D_0-3}BZ) + (D_0-3)(BAZ^{D_0-2}) + (ABZ^{D_0-2}) \ , \qquad (3.2.14)
\end{aligned}$$

(omitting the overall factor coming from the details of the Feynman diagram). The action on the trace parts when $A = B$ is more complicated, and we will omit the detailed argument for that case. In an obvious matrix notation, we have

$$\left[\begin{array}{cc} \text{Anom} & \text{Dim} \end{array} \right]_{(D_0-1)\times(D_0-1)} \sim \begin{pmatrix} D_0-3 & 2 & 0 & \ldots & 1 \\ 2 & D_0-4 & 2 & \ldots & 0 \\ & & & & \\ 0 & \ldots & 2 & D_0-4 & 2 \\ 1 & \ldots & 0 & 2 & D_0-3 \end{pmatrix} . \qquad (3.2.15)$$

The logic of renormalization theory allows for a subtraction on the diagonal of this matrix, and in fact one is needed. The vector $\vec{X}_0 = (1, \ldots, 1)$, corresponding to the operator in which all operators in (3.2.13) are summed over with equal weight, is an eigenvector with eigenvalue D_0. This particular operator actually belongs to the special representation $(0, D_0, 0)$, whose anomalous dimensions must vanish because it contains the chiral primary BPS operator $\mathrm{tr}(Z_0^D)$ (whose dimension is equal to the $\mathcal{R}$-charge). To properly normalize (3.2.15) and ensure that this eigenvector has eigenvalue zero, we subtract D_0 times the unit matrix and drop the zero eigenvector of the anomalous dimension matrix on the grounds that it belongs to the 'uninteresting' $(0, D_0, 0)$ representation. The anomalous dimensions we seek are therefore the non-zero eigenvalues of the matrix

$$\left[\text{ Anom } \text{ Dim } \right]_{(D_0-1)\times(D_0-1)} \sim \begin{pmatrix} -3 & +2 & 0 & \ldots & 1 \\ +2 & -4 & +2 & \ldots & 0 \\ & & & & \\ 0 & \ldots & +2 & -4 & +2 \\ +1 & \ldots & 0 & +2 & -3 \end{pmatrix} . \tag{3.2.16}$$

This looks very much like the lattice Laplacian for a particle hopping from site to site on a periodic lattice. The special structure of the first and last rows assigns an extra energy to the particle when it hops past the origin. This breaks strict lattice translation invariance but makes sense as a picture of the dynamics involving two-impurity states: the impurities propagate freely when they are on different sites and have a contact interaction when they collide. This picture has lead people to map the problem of finding operator dimensions onto the technically much simpler one of finding the spectrum of an equivalent quantum-mechanical Hamiltonian [51]. In one version, the map is to a spin-chain system with integrable dynamics [52], suggesting that exact results for many quantities of interest may be possible.

Before diagonalizing (3.2.16), we note a useful symmetry of the problem: the op-

erator monomials in the basis (3.2.13) go into each other pairwise under $A \leftrightarrow B$ and, at the same time, the vector $\vec{C} = (C_1, \ldots, C_{D_0-1})$ representing a linear combination of monomials transforms as $C_i \to C_{D_0-i}$. Since (3.2.16) is invariant under this transformation, its eigenvectors will be either even ($C_i = C_{D_0-i}$) or odd ($C_i = -C_{D_0-i}$) under it. Since the two options (even or odd under $A \leftrightarrow B$) correspond to different $SU(4)$ irreps, assessing the $SU(4)$ assignment of the different eigenvalues will be easy. The two classes of eigenvalues and normalized eigenvectors are as follows:

$$\lambda_n^{(D_0+)} = 8\sin^2\left(\frac{n\pi}{D_0-1}\right) \qquad n = (0), 1, 2, \ldots, n_{max} = \begin{cases} (D_0-3)/2 & D_0 \text{ odd} \\ (D_0-2)/2 & D_0 \text{ even} \end{cases} ,$$
$$C_{n,i}^{(D_0+)} = \frac{2}{\sqrt{D_0-1}} \cos\left[\frac{2\pi n}{D_0-1}(i-\frac{1}{2})\right] \qquad i = 1, \ldots, D_0-1 \ , \quad (3.2.17)$$

$$\lambda_n^{(D_0-)} = 8\sin^2\left(\frac{n\pi}{D_0}\right) \qquad n = 1, 2, \ldots, n_{max} = \begin{cases} (D_0-1)/2 & D_0 \text{ odd} \\ (D_0-2)/2 & D_0 \text{ even} \end{cases} ,$$
$$C_{n,i}^{(D_0-)} = \frac{2}{\sqrt{D_0}} \sin\left[\frac{2\pi n}{D_0}(i)\right] \qquad i = 1, \ldots, D_0-1 \ . \quad (3.2.18)$$

For the case of $\lambda_n^{(D_0+)}$, we indicate that $n = 0$ is a possible eigenvalue, but we must remember that it belongs to the $(0, D_0, 0)$ irrep when we count irrep multiplicities. The eigenoperators corresponding to the various dimensions are constructed from the eigenvectors according to

$$\overline{T}_{D_0,n}^{(\pm)}(x) = \sum_{i=1}^{D_0-1} C_{n,i}^{(D_0\pm)} O_{D_0,i}^{AB}(x) \ . \quad (3.2.19)$$

The subscript n will not be displayed in the following.

To get $\Delta = D - \mathcal{R}$, we multiply these eigenvalues by the appropriate overall normalization factor and add the zeroth order value $\Delta_0 = 2$. The results for $\overline{T}_{D_0}^{(+)}$ (symmetric traceless, belonging to the $(2, D_0-4, 2)$ irrep), $\overline{T}_{D_0}^{(-)}$ (antisymmetric, belonging to the $(0, D_0-3, 2) + (2, D_0-3, 0)$ irreps) and $\overline{T}_{D_0}^{(0)}$ (trace, belonging to the

$(0, D_0 - 2, 0)$ irrep) are

$$\Delta(\overline{T}^{(+)}_{D_0}) = 2 + \frac{g^2_{YM}N_c}{\pi^2}\sin^2\left(\frac{n\pi}{D_0-1}\right) \qquad n = 1,2,\ldots,n_{max} = \begin{cases} (D_0-3)/2 & D_0 \text{ odd} \\ (D_0-2)/2 & D_0 \text{ even} \end{cases} ,$$

$$\Delta(\overline{T}^{(-)}_{D_0}) = 2 + \frac{g^2_{YM}N_c}{\pi^2}\sin^2\left(\frac{n\pi}{D_0}\right) \qquad n = 1,2,\ldots,n_{max} = \begin{cases} (D_0-1)/2 & D_0 \text{ odd} \\ (D_0-2)/2 & D_0 \text{ even} \end{cases} ,$$

$$\Delta(\overline{T}^{(0)}_{D_0}) = 2 + \frac{g^2_{YM}N_c}{\pi^2}\sin^2\left(\frac{n\pi}{D_0+1}\right) \qquad n = 1,2,\ldots,n_{max} = \begin{cases} (D_0-1)/2 & D_0 \text{ odd} \\ (D_0/2) & D_0 \text{ even} \end{cases} . \tag{3.2.20}$$

Note that the counting of eigenvalues corresponds exactly to the multiplicities of these irreps as reported in (3.2.2) and (3.2.3). The above results on dimensions and eigenoperators can all be found in [49] and, piecemeal, in earlier discussions of the one-loop operator dimension problem.

The expressions in (3.2.20) are the first terms in a perturbative expansion. Since we must work in the limit of large $g^2_{YM}N_c$, this expansion is not guaranteed to be reliable. The string theory discussion will show that the eigenvalue index n is to be interpreted as the mode number of an excited string oscillator. This implies a limiting procedure in which n is held fixed while $\mathcal{R}$ and $g^2_{YM}N_c$ are taken to infinity such that there are two controlled, small parameters, $g^2_{YM}N_c/\mathcal{R}^2$ and $1/\mathcal{R}$. We will assume, as proposed by BMN, that the smallness of $g^2_{YM}N_c/\mathcal{R}^2$ makes perturbation theory reliable, at least for fixed-n eigenvalues (without this assumption, there is little one can calculate on the gauge theory side). At the same time, the smallness of $1/\mathcal{R}$ controls the size of interaction corrections to the Penrose limit string worldsheet Hamiltonian. If we express the dimension formulae (3.2.20) in terms of $\mathcal{R}$-charge $\mathcal{R}$, rather than naive dimension D_0 (using $D_0 = \mathcal{R} + 2$) and take the limit in this way, we find

$$\Delta(\overline{T}^{(+)}_{\mathcal{R}+2}) \to 2 + \frac{g^2_{YM}N_c}{\mathcal{R}^2}\ n^2\ \left(1 - \frac{2}{\mathcal{R}} + O(\mathcal{R}^{-2})\right) ,$$

$$\Delta(\overline{T}^{(-)}_{\mathcal{R}+2}) \to 2 + \frac{g^2_{YM}N_c}{\mathcal{R}^2}\ n^2\ \left(1 - \frac{4}{\mathcal{R}} + O(\mathcal{R}^{-2})\right)\ ,$$

$$\Delta(\overline{T}^{(0)}_{\mathcal{R}+2}) \to 2 + \frac{g^2_{YM}N_c}{\mathcal{R}^2}\ n^2\ \left(1 - \frac{6}{\mathcal{R}} + O(\mathcal{R}^{-2})\right)\ . \tag{3.2.21}$$

To leading order in $1/\mathcal{R}$, the dimensions of these operator multiplets are degenerate and agree with the corresponding expression in the Penrose limit (2.3.21). The degeneracy is lifted at subleading order in $1/\mathcal{R}$, just as the Penrose limit degeneracy of string worldsheet energies should be lifted by string worldsheet interactions. Our goal is show that the two approaches to the lifting of operator dimension (string energy) degeneracy give equivalent results on each side of the duality.

The AdS/CFT interpretation of the operator dimensions displayed in (3.2.21) is that they are dual to the energies of string states built out of two bosonic mode creation operators: $(a^A_n)^\dagger(a^B_{-n})^\dagger|\mathcal{R}\rangle$. It is important to note that these anomalous dimensions are valid for *all* operators in the representations in question, not just those for which $\Delta_0 = D_0 - \mathcal{R} = 2$; this is a simple consequence of the global $SU(4)$ $\mathcal{R}$-symmetry. We believe that this translates on the string theory side into the existence of exact zero-mode oscillators a^A_0, which augment the P_+ eigenvalue of a state by unity, independent of $g^2_{YM}N_c/J^2$ and $1/J$. This is true in the Penrose limit, as we can infer from (2.3.21), and we expect it to continue to be true to all orders in $1/J$. If so, the string states

$$(a^A_n)^\dagger(a^B_{-n})^\dagger(a^{C_1}_0)^\dagger \ldots (a^{C_s}_0)^\dagger|J-2-s\rangle \tag{3.2.22}$$

should all have the same energy and correspond to the $\Delta_0 > 2$ components of the $(2, J-4, 2)$ irrep (if we project onto operators symmetric and traceless on A, B, for example). This suggests that the interaction terms in the string worldsheet Hamiltonian should not involve zero-mode oscillators at all. We will eventually see that this is the case, at least to the order we are able to study.

We have given a rather detailed treatment of the calculation of the anomalous dimensions of two specific operator multiplets. To fully address the issues that will arise in string theory, we need expressions like (3.2.21) for *all* operator multiplets (not just spacetime scalars) that contain components with $\Delta_0 = 2$. It is possible to carry out some version of the above lattice Laplacian argument for all the relevant operator classes, but we can use supersymmetry to circumvent this tedious task. The extended superconformal symmetry of the gauge theory means that conformal primary operators are organized into multiplets obtained from a lowest-dimension primary $\mathcal{O}_D$ of dimension D by anticommutation with the supercharges Q_i^α (α is an $SL(2,C)$ Lorentz spinor index and i is an $SU(4)$ index). We need only concern ourselves here with the case in which $\mathcal{O}_D$ is a spacetime scalar (of dimension D and $\mathcal{R}$-charge $\mathcal{R}$). There are sixteen supercharges and we can choose eight of them to be raising operators; there are $2^8 = 256$ operators we can reach by 'raising' the lowest one. Since the raising operators increase the dimension and $\mathcal{R}$-charge by $1/2$ each time they act, the operators at level L_V, obtained by acting with L_V supercharges, all have the same dimension and $\mathcal{R}$-charge. The corresponding decomposition of the 256-dimensional multiplet is shown in Table 3.1.

Level	0	1	2	3	4	5	6	7	8
Multiplicity	1	8	28	56	70	56	28	8	1
Dimension	D	$D+1/2$	$D+1$	$D+3/2$	$D+2$	$D+5/2$	$D+3$	$D+7/2$	$D+4$
$\mathcal{R}-$charge	$\mathcal{R}$	$\mathcal{R}+1/2$	$\mathcal{R}+1$	$\mathcal{R}+3/2$	$\mathcal{R}+2$	$\mathcal{R}+5/2$	$\mathcal{R}+3$	$\mathcal{R}+7/2$	$\mathcal{R}+4$

Table 3.1: $\mathcal{R}$-charge content of a supermultiplet

The states at each level can be classified under the Lorentz group and the $SO(4) \sim SU(2) \times SU(2)$ subgroup of the $\mathcal{R}$-symmetry group, which is unbroken after we have fixed the $SO(2)$ $\mathcal{R}$-charge. For instance, the 28 states at level 2 decompose under $SO(4)_{Lor} \times SO(4)_{\mathcal{R}}$ as $(6,1)+(1,6)+(4,4)$. For the present, the most important point is that, given the dimension of one operator at one level, we can infer the dimensions of all other operators in the supermultiplet.

We can use this logic to get a complete accounting of the dimensions of the $\Delta_0 = 2$ BMN operators. Here we summarize work by Beisert [49], recasting his results to fit our needs. The supermultiplet of interest is based on the set of scalars $\Sigma_A \,\mathrm{tr}\left(\phi^A Z^p \phi^A Z^{\mathcal{R}-p}\right)$, the operator class we have denoted by $\overline{T}^{(0)}_{\mathcal{R}+2}$. According to (3.2.20), the spectrum of $\Delta = D - \mathcal{R}$ eigenvalues associated with this operator basis is

$$\Delta(\overline{T}^{(0)}_{\mathcal{R}+2}) = 2 + \frac{g^2_{YM} N_c}{\pi^2} \sin^2\left(\frac{n\pi}{\mathcal{R}+3}\right) \to 2 + \frac{g^2_{YM} N_c}{\mathcal{R}^2}\, n^2 \left(1 - \frac{6}{\mathcal{R}} + O(\mathcal{R}^{-2})\right). \quad (3.2.23)$$

The other spacetime scalar operators $\overline{T}^{(\pm)}_{\mathcal{R}+2}$ displayed in (3.2.20) have dimension formulae that appear to differ from this. However, when they are put into the context of a supermultiplet and the dimension formulae are expressed in terms of the $\mathcal{R}$-charge of the lowest-dimension member of the supermultiplet, it turns out that (3.2.23) governs *all* the operators at *all* levels in the supermultiplet. We summarize the situation for the spacetime scalar members of the multiplet in Table 3.2. The last

L_V	$\mathcal{R}$	$SU(4)$ Irreps	Operator	$\Delta - 2$	Multiplicity
0	R_0	$(0, R_0, 0)$	$\Sigma_A \,\mathrm{tr}\left(\phi^A Z^p \phi^A Z^{R_0-p}\right)$	$\frac{g^2_{YM}N_c}{\pi^2}\sin^2(\frac{n\pi}{(R_0)+3})$	$n = 1, ., \frac{R_0+1}{2}$
2	R_0+1	$(0, R_0, 2) + c.c.$	$\mathrm{tr}\left(\phi^{[i} Z^p \phi^{j]} Z^{R_0+1-p}\right)$	$\frac{g^2_{YM}N_c}{\pi^2}\sin^2(\frac{n\pi}{(R_0+1)+2})$	$n = 1, ., \frac{R_0+1}{2}$
4	R_0+2	$(2, R_0, 2)$	$\mathrm{tr}\left(\phi^{(i} Z^p \phi^{j)} Z^{R_0+2-p}\right)$	$\frac{g^2_{YM}N_c}{\pi^2}\sin^2(\frac{n\pi}{(R_0+2)+1})$	$n = 1, ., \frac{R_0+1}{2}$
4	R_0+2	$(0, R_0+2, 0) \times 2$	$\mathrm{tr}\left(\chi^{[\alpha} Z^p \chi^{\beta]} Z^{R_0+1-p}\right)$	$\frac{g^2_{YM}N_c}{\pi^2}\sin^2(\frac{n\pi}{(R_0+2)+1})$	$n = 1, ., \frac{R_0+1}{2}$
6	R_0+3	$(0, R_0+2, 2) + c.c.$	$\mathrm{tr}\left(\chi^{(\alpha} Z^p \chi^{\beta)} Z^{R_0+2-p}\right)$	$\frac{g^2_{YM}N_c}{\pi^2}\sin^2(\frac{n\pi}{(R_0+3)+0})$	$n = 1, ., \frac{R_0+1}{2}$
8	R_0+4	$(0, R_0, 0)$	$\mathrm{tr}\left(\mathcal{D}_\mu Z Z^p \mathcal{D}^\mu Z Z^{R_0+2-p}\right)$	$\frac{g^2_{YM}N_c}{\pi^2}\sin^2(\frac{n\pi}{(R_0+4)-1})$	$n = 1, ., \frac{R_0+1}{2}$

Table 3.2: Dimensions and multiplicities of spacetime scalar operators

column displays the allowed range of the eigenvalue index n at each level (for R_0 odd only, just to save space) computed from our results for $SU(4)$ irrep multiplicities. It is non-trivial that the result is the same at each level; were it not so, the levels could not be assembled into a single supermultiplet. The universal dimension formula is written at each level in such a way as to emphasize the dependence on the $\mathcal{R}$-charge of the particular level. This shows how the different results (3.2.20) and (3.2.23) are reconciled in the supermultiplet.

The supermultiplet contains operators that are not spacetime scalars (i.e., that transform non-trivially under the $SU(2,2)$ conformal group) and group theory determines at what levels in the supermultiplet they must lie. A representative sampling of data on such operators (extracted from Beisert's paper) is collected in Table 3.3. We have worked out neither the $SU(4)$ representations to which these lowest-Δ operators belong nor their precise multiplicities. The ellipses indicate that the operators in question contain further monomials involving fermion fields (so that they are not uniquely specified by their bosonic content). This information will be useful in consistency checks to be carried out below.

L_V	$\mathcal{R}$	Operator	$\Delta - 2$	$\Delta - 2 \to$
2	R_0+1	$\mathrm{tr}\left(\phi^i Z^p \mathcal{D}_\mu Z Z^{R_0-p}\right) + \ldots$	$\frac{g_{YM}^2 N_c}{\pi^2} \sin^2(\frac{n\pi}{(R_0+1)+2})$	$\frac{g_{YM}^2 N_c}{R_0^2} n^2 (1 - \frac{4}{R_0})$
4	R_0+2	$\mathrm{tr}\left(\phi^i Z^p \mathcal{D}_\mu Z Z^{R_0+1-p}\right)$	$\frac{g_{YM}^2 N_c}{\pi^2} \sin^2(\frac{n\pi}{(R_0+2)+1})$	$\frac{g_{YM}^2 N_c}{R_0^2} n^2 (1 - \frac{2}{R_0})$
4	R_0+2	$\mathrm{tr}\left(\mathcal{D}_{(\mu} Z Z^p \mathcal{D}_{\nu)} Z Z^{R_0-p}\right)$	$\frac{g_{YM}^2 N_c}{\pi^2} \sin^2(\frac{n\pi}{(R_0+2)+1})$	$\frac{g_{YM}^2 N_c}{R_0^2} n^2 (1 - \frac{2}{R_0})$
6	R_0+3	$\mathrm{tr}\left(\phi^i Z^p \mathcal{D}_\mu Z Z^{R_0+2-p}\right) + \ldots$	$\frac{g_{YM}^2 N_c}{\pi^2} \sin^2(\frac{n\pi}{R_0+3})$	$\frac{g_{YM}^2 N_c}{R_0^2} n^2 (1 - \frac{0}{R_0})$
6	R_0+3	$\mathrm{tr}\left(\mathcal{D}_{[\mu} Z Z^p \mathcal{D}_{\nu]} Z Z^{R_0+1-p}\right)$	$\frac{g_{YM}^2 N_c}{\pi^2} \sin^2(\frac{n\pi}{R_0+3})$	$\frac{g_{YM}^2 N_c}{R_0^2} n^2 (1 - \frac{0}{R_0})$

Table 3.3: Anomalous dimensions of some operators that are not scalars

As far as dimensions are concerned, all of the above can be summarized by saying that the dimensions of the operators of $\mathcal{R}$-charge $\mathcal{R}$ at level L_V in the supermultiplet are given by the general formula (valid for large $\mathcal{R}$ and fixed n):

$$\begin{aligned} \Delta_n^{\mathcal{R},L_V} &= 2 + \frac{g_{YM}^2 N_c}{\pi^2} \sin^2\left(\frac{n\pi}{\mathcal{R}+3-L_V/2}\right) \\ &\to 2 + \frac{g_{YM}^2 N_c}{\mathcal{R}^2}\, n^2 \left(1 - \frac{6-L_V}{\mathcal{R}} + O(\mathcal{R}^{-2})\right) . \end{aligned} \tag{3.2.24}$$

This amounts to a gauge theory prediction for the way in which worldsheet interactions lift the degeneracy of the two-impurity string multiplet. The 256 states of the form $A_n^\dagger B_{-n}^\dagger |\mathcal{R}\rangle$, for a given mode number n, (where $A^\dagger$ and $B^\dagger$ each can be any of the 8+8 bosonic and fermionic oscillators) should break up as shown in table 3.4. It

should be emphasized that, for fixed $\mathcal{R}$, the operators associated with different levels are actually coming from *different* supermultiplets; this is why they have different dimensions! As mentioned before, we can also precisely identify transformation properties under the Lorentz group and under the rest of the $\mathcal{R}$-symmetry group of the degenerate states at each level. This again leads to useful consistency checks, and we will elaborate on this when we analyze the eigenstates of the string worldsheet Hamiltonian.

Level	0	1	2	3	4	5	6	7	8
Multiplicity	1	8	28	56	70	56	28	8	1
$\delta E \times (\mathcal{R}^2/g_{YM}^2 N_c n^2)$	$-6/\mathcal{R}$	$-5/\mathcal{R}$	$-4/\mathcal{R}$	$-3/\mathcal{R}$	$-2/\mathcal{R}$	$-1/\mathcal{R}$	0	$1/\mathcal{R}$	$2/\mathcal{R}$

Table 3.4: Predicted energy shifts of two-impurity string states

3.3 Higher loops and more impurities

In this section we will extend the calculation of anomalous dimensions to higher numbers of impurities and to higher loops though only for a small number of closed sectors i.e., subsets of operators which mix only amongst themselves. It is convenient to shift attention away from two-point functions (3.2.12) and focus on the dilation operator acting on states which can then be identified with the Hamiltonian of a quantum spin chain. This was initially done at one-loop, for planar diagrams and for the scalars in [52] and for the BMN limit in [51]. The complete one-loop dilation operator was derived by Beisert in [53] and extended to two- and three-loops for the $\mathfrak{su}(2|3)$ sector consisting of three bosons, ϕ, and two fermions χ in [54]. For the $\mathfrak{su}(2)$ sector consisting of two scalars, motivated by integrability and BMN scaling, a five-loop dilation operator was conjectured in [55]. For a comprehensive review of the $\mathcal{N} = 4$ dilation operator see [56]. Finding the spectra of the dilatation operator is greatly facilitated by the fact that it can be identified with a Hamiltonian that is integrable. Indeed in many cases a Bethe ansatz which diagonalizes the dilatation operator can be found and we will discuss this in a subsequent chapter. However, to improve on the current limitations of Bethe ansatz techniques, we have developed a virial approach to the spin chain systems. The generic spin-chain Hamiltonian acts on single-impurity pseudoparticles as a lattice Laplacian and the higher N-body interactions amongst pseudoparticles are suppressed by inverse powers of the lattice length L. Surprisingly, this expansion of the spin-chain Hamiltonian is truncated at $O(L^{-3})$ in certain subsectors of the theory, allowing straightforward eigenvalue calculations that are exact in the chain length for operators with more than two $\mathcal{R}$-charge impurities. Furthermore, since the goal is to eventually compare anomalous dimensions with $1/J$ energy corrections to corresponding string states near the pp-wave limit of $AdS_5 \times S^5$, and because the string angular momentum J is related to the lattice length L, this virial expansion is precisely what is needed to devise a practical method for testing the AdS/CFT correspondence at any order in the gauge theory

loop expansion for an arbitrary number of $\mathcal{R}$-charge impurities.

3.3.1 The $\mathfrak{su}(2)$ sector

Single-trace operators in the closed $\mathfrak{su}(2)$ sector are constructed from the two complex scalar fields Z and ϕ. In the $SO(4)$ notation they consist of operators with two types of impurities $\phi^{i'}, \phi^{j'}$ and which are symmetric and traceless in all indices. Generalizing from the previous section, the basis of length-L operators in the planar limit is constructed from single-trace monomials with I impurities and total $\mathcal{R}$-charge equal to $L - I$:

$$\mathrm{tr}(\phi^I Z^{L-I}) \ , \qquad \mathrm{tr}(\phi^{I-1} Z \phi Z^{L-I-1}) \ , \qquad \mathrm{tr}(\phi^{I-2} Z \phi^2 Z^{L-I-1}) \ , \qquad \ldots \quad (3.3.1)$$

The statement that this sector of operators is "closed" means simply that the anomalous dimension operator only mixes these states amongst themselves, at least to leading order in large N_c [57, 53].

The heart of the spin-chain approach is the proposition that there exists a one-dimensional spin system whose Hamiltonian can be identified with the large-N_c limit of the anomalous dimension operator acting on this closed subspace of operators [52]. Since the anomalous dimensions are perturbative in the 't Hooft coupling λ, it is natural to expand the $\mathfrak{su}(2)$ spin chain Hamiltonian in powers of λ as well:

$$H_{\mathfrak{su}(2)} = I + \sum_n \left(\frac{\lambda}{8\pi^2} \right)^n H^{(2n)}_{\mathfrak{su}(2)} \ . \quad (3.3.2)$$

Comparison with the gauge theory has shown that successive terms in the expansion of the Hamiltonian have a remarkably simple structure: the one-loop-order Hamiltonian $H^{(2)}_{\mathfrak{su}(2)}$ is built out of permutations of pairs of nearest-neighbor fields and, at n-th order, the Hamiltonian permutes among themselves fields which are at most n lattice sites apart. This is a universal structure which leads to remarkable simplifications in the various closed sectors of the theory [58].

Beisert, Kristjansen and Staudacher [57] have introduced the following useful notation for products of permutations acting on operators separated by an arbitrary number of lattice sites:

$$\{n_1, n_2, \dots\} = \sum_{k=1}^{L} P_{k+n_1,k+n_1+1} P_{k+n_2,k+n_2+1} \cdots \ , \tag{3.3.3}$$

where $P_{i,j}$ simply exchanges fields on the i^{th} and j^{th} lattice sites on the chain. The spin-chain Hamiltonian for the $\mathfrak{su}(2)$ sector can be written in a rather compact form in terms of this notation. The result, correct to three-loops, is (see [57])

$$H^{(2)}_{\mathfrak{su}(2)} = 2\left(\{\} - \{0\}\right) \tag{3.3.4}$$

$$H^{(4)}_{\mathfrak{su}(2)} = 2\big(-4\{\} + 6\{0\} - (\{0,1\} + \{1,0\})\big) \tag{3.3.5}$$

$$H^{(6)}_{\mathfrak{su}(2)} = 4\big(15\{\} - 26\{0\} + 6\left(\{0,1\} + \{1,0\}\right) + \{0,2\} - (\{0,1,2\} + \{2,1,0\})\big) \ . \tag{3.3.6}$$

This form of the three-loop term $H^{(6)}_{\mathfrak{su}(2)}$ was first conjectured in [57] based on integrability restrictions and BMN scaling; this conjecture was later corroborated by direct field-theoretic methods in [54] (see also [59] for relevant discussion on this point). Our goal is to develop practical methods for finding the eigenvalue spectrum of the spin-chain Hamiltonian for various interesting cases.

3.3.1.1 One-loop order

We start at one-loop with $H^{(2)}_{\mathfrak{su}(2)}$ in equation (3.3.4), which provides a natural 'position-space' prescription for constructing matrix elements in an I-impurity basis of operators. We are primarily interested in systems with few impurities compared to the length of the spin chain and we expect that impurity interaction terms in the Hamiltonian will be suppressed by powers of the impurity density (i.e., inverse powers of the lattice length). This suggests that we develop a virial expansion of the spin-chain Hamiltonian in which the leading-order term gives the energy of free pseudoparticle

states on the lattice and higher $1/L$ corrections come from N-body interactions described by vertices V_N. A reasonable guess about how the N-body interactions should scale with $1/L$ suggests that we can write the one-loop-order energy for I impurities in the form

$$E(\{n_i\}) = I + \frac{\lambda}{2\pi^2}\sum_{i=1}^{I}\sin^2\frac{n_i\pi}{L} + \sum_{N=2}^{2I}\frac{\lambda}{L^{2N-1}}V_{N-\text{body}}(n_1,\ldots,n_I) + \cdots \ , \quad (3.3.7)$$

where the leading-order contribution, I, measures the naive dimension minus $\mathcal{R}$-charge, the next term is the lattice Laplacian energy of I non-interacting pseudoparticles and the $1/L$ corrections account for interactions between pseudoparticles (which may depend on the lattice momenta mode numbers n_i). In the many-body approach, one would try to derive such energy expressions by rewriting the Hamiltonian in terms of creation/annihilation operators b_{n_i}, $b^\dagger_{n_i}$ for the pseudoparticles (commuting or anticommuting as appropriate). The N-body interaction vertex would generically be written in terms of the $b, b^\dagger$ as

$$V_N = \sum_{n_i,m_i}\delta_{n_1+\cdots+n_N,m_1+\cdots+m_N}f_N(\{n_i\},\{m_i\})\prod_{i=1}^{N}b^\dagger_{n_i}\prod_{i=1}^{N}b_{m_i} \ , \quad (3.3.8)$$

where $f_N(\{n_i\},\{m_i\})$ is some function of the lattice momenta and the Kronecker delta enforces lattice momentum conservation. One has to determine the functions f_N by matching the many-body form of the Hamiltonian to exact spin-chain expressions such as equation (3.3.4).

The discussion so far has been in the context of one-loop gauge theory physics, but the logic of the virial expansion should be applicable to the general case. To include higher-loop order physics we must do two things: a) generalize the functions $f_N(\{n_i\},\{m_i\})$ defining the multi-particle interaction vertices to power series in λ and b) allow the free pseudoparticle kinetic energies themselves to become power series in λ. We will be able to carry out the detailed construction of the higher-loop virial Hamiltonian in a few well-chosen cases. To match this expansion at n-loop order in

λ to the corresponding loop order (in the modified 't Hooft coupling $\lambda' = g_{\rm YM}^2 N_c/J^2$) in the string theory, we need to determine the Hamiltonian to $O(L^{-(2n+1)})$ in this virial expansion. (The first curvature correction to the pp-wave string theory at one loop, for example, appears at $O(\lambda'/J)$ or, in terms of gauge theory parameters, at $O(\lambda/L^3)$.) Auspiciously, it will turn out that this virial expansion in the $\mathfrak{su}(2)$ sector is truncated at small orders in $1/L$, allowing for simple eigenvalue calculations that are exact in L.

The first step toward obtaining the desired virial expansion is to recast the spin chain Hamiltonian $H_{\mathfrak{su}(2)}$, which is initially expressed in terms of permutation operators, in terms of a creation and annihilation operator algebra. We begin by introducing the spin operators

$$S^{\pm} = \frac{1}{2}\left(\sigma_x \pm i\sigma_y\right) \qquad S^z = \frac{1}{2}\sigma_z \ , \tag{3.3.9}$$

where $\vec{\sigma}$ are the Pauli matrices and $S_j^{\pm},\ S_j^z$ act on a two-dimensional spinor space at the j^{th} lattice site in the chain. In this setting the Z and ϕ fields are understood to be modeled by up and down spins on the lattice. The nearest-neighbor permutation operator $P_{i,i+1}$ can be written in terms of spin operators as

$$P_{i,i+1} = S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+ + 2S_i^z S_{i+1}^z + \frac{1}{2} \ , \tag{3.3.10}$$

and the one-loop Hamiltonian in equation (3.3.4) can be written as

$$H_{\mathfrak{su}(2)}^{(2)} = -\sum_{j=1}^{L}\left(S_j^+ S_{j+1}^- + S_j^- S_{j+1}^+\right) - 2\sum_{j=1}^{L} S_j^z S_{j+1}^z + \frac{1}{2} \ . \tag{3.3.11}$$

A Jordan-Wigner transformation can now be used to express the spin generators in terms of anti-commuting creation and annihilation operators (anti-commuting because each site can be either unoccupied (Z) or occupied once (ϕ)). A pedagogical

introduction to this technique can be found in [60]. The explicit transformation is

$$\begin{aligned} S_j^+ &= b_j^\dagger K(j) = K(j) b_j^\dagger \\ S_j^- &= K(j) b_j = b_j K(j) \\ S_j^z &= b_j^\dagger b_j - 1/2 \ , \end{aligned} \tag{3.3.12}$$

where the Klein factors

$$K(j) = \exp\left(i\pi \sum_{k=1}^{j-1} b_k^\dagger b_k \right) \tag{3.3.13}$$

serve to ensure that spin operators on different sites commute, despite the anticommuting nature of the b_j. The functions $K(j)$ are real, Abelian and, for $j \leq k$,

$$[K(j), \mathbf{S}_k] = 0 \ . \tag{3.3.14}$$

The operators b_j and $b_j^\dagger$ can therefore be written as

$$b_j^\dagger = S_j^+ K(j) \qquad b_j = S_j^- K(j) \ , \tag{3.3.15}$$

and we easily verify that they satisfy the standard anticommutation relations

$$\{b_j, b_k^\dagger\} = \delta_{jk} \qquad \{b_j^\dagger, b_k^\dagger\} = \{b_j, b_k\} = 0 \ . \tag{3.3.16}$$

Cyclicity on the lattice requires that $\mathbf{S}_{L+1} = \mathbf{S}_1$, a condition which can be enforced by the following boundary condition on the creation and annihilation operators

$$b_{L+1} = (-1)^{I+1} b_1 \qquad I \equiv \sum_{j=1}^{L} b_j^\dagger b_j \ , \tag{3.3.17}$$

where the integer I counts the number of spin-chain impurities. Since we are primarily interested in the three-impurity problem, we will henceforth impose the boundary

conditions in equation (3.3.17) for odd impurity number only. We can use all of this to re-express equation (3.3.11) in creation and annihilation operator language, with the result

$$H^{(2)}_{\mathfrak{su}(2)} = \sum_{j=1}^{L} \left(b_j^\dagger b_j + b_{j+1}^\dagger b_{j+1} - b_{j+1}^\dagger b_j - b_j^\dagger b_{j+1} + 2\, b_j^\dagger b_{j+1}^\dagger b_j b_{j+1} \right) \ . \tag{3.3.18}$$

Converting to momentum space via the usual Fourier transform

$$b_j = \frac{1}{\sqrt{L}} \sum_{p=0}^{L-1} e^{-\frac{2\pi i j}{L} p}\ \tilde{b}_p \tag{3.3.19}$$

yields

$$H^{(2)}_{\mathfrak{su}(2)} \ = \ 4 \sum_{p=0}^{L-1} \sin^2\left(\frac{\pi p}{L}\right) \tilde{b}_p^\dagger \tilde{b}_p + \frac{2}{L} \sum_{p,q,r,s=0}^{L-1} e^{\frac{2\pi i (q-s)}{L}} \tilde{b}_p^\dagger \tilde{b}_q^\dagger \tilde{b}_r \tilde{b}_s\, \delta_{p+q,r+s} \ . \tag{3.3.20}$$

This is a rather standard many-body Hamiltonian: it acts on a Fock space of momentum eigenstate pseudoparticles, contains a one-body pseudoparticle kinetic energy term and a two-body pseudoparticle interaction (the latter having the critical property that it conserves the number of pseudoparticles). Note that the Hamiltonian terminates at two-body interactions, a fact which will simplify the virial expansion of the energy spectrum.

Because the pseudoparticle (or impurity) number is conserved by the interaction, three-impurity eigenstates of the Hamiltonian must lie in the space spanned by

$$\tilde{b}_{k_1}^\dagger \tilde{b}_{k_2}^\dagger \tilde{b}_{k_3}^\dagger \, |L\rangle \qquad k_1 + k_2 + k_3 = 0 \mod L \ , \tag{3.3.21}$$

where the ground state $|L\rangle$ is identified with the zero-impurity operator $\mathrm{tr}(Z^L)$ and the condition of vanishing net lattice momentum arises from translation invariance on the spin-chain (which in turn arises from the cyclicity of the single-trace operators in the operator basis).

The construction and diagonalization of the Hamiltonian matrix on the degenerate basis of three-impurity operators can easily be carried out for a given L. According to equation (3.3.7), we expect the eigenvalues of $H^{(2)}_{\mathfrak{su}(2)}$ to scale for large L according to

$$E_L(\{k_i\}) = \frac{\lambda}{L^2}E^{(1,2)}(\{k_i\}) + \frac{\lambda}{L^3}E^{(1,3)}(\{k_i\}) + O(\lambda L^{-4}) \ . \tag{3.3.22}$$

The scaling coefficients $E^{(1,2)}_{\mathfrak{su}(2)}$ and $E^{(1,3)}_{\mathfrak{su}(2)}$ can easily be extracted from the data by fitting the spectral curves to large-order polynomials in $1/L$ (a similar treatment was used in [61]). The results of this procedure are recorded for several low-lying levels in the spectrum (excluding zero eigenvalues) in Table 3.5. As we will see the string

$E^{(1,2)}_{\mathfrak{su}(2)}$	$E^{(1,3)}_{\mathfrak{su}(2)}$	$E^{(1,3)}_{\mathfrak{su}(2)}/E^{(1,2)}_{\mathfrak{su}(2)}$	Lattice Momenta (k_1, k_2, k_3)
$1 + 2.6 \times 10^{-9}$	$2 - 4.9 \times 10^{-7}$	$2 - 5.0 \times 10^{-7}$	$(1, 0, -1)$
$3 + 4.6 \times 10^{-9}$	$7 - 8.8 \times 10^{-7}$	$7/3 - 3.0 \times 10^{-7}$	$(1, 1, -2)$
$3 + 4.6 \times 10^{-9}$	$7 - 8.8 \times 10^{-7}$	$7/3 - 3.0 \times 10^{-7}$	$(-1, -1, 2)$
$4 + 6.0 \times 10^{-9}$	$8 - 1.1 \times 10^{-6}$	$2 - 2.9 \times 10^{-7}$	$(2, 0, -2)$
$7 + 3.2 \times 10^{-8}$	$14 - 7.1 \times 10^{-6}$	$2 - 1.0 \times 10^{-6}$	$(1, 2, -3)$
$7 + 3.2 \times 10^{-8}$	$14 - 7.1 \times 10^{-6}$	$2 - 1.0 \times 10^{-6}$	$(-1, -2, 3)$
$9 + 2.2 \times 10^{-7}$	$18 - 5.1 \times 10^{-5}$	$2 - 5.7 \times 10^{-6}$	$(3, 0, -3)$
$12 + 5.7 \times 10^{-5}$	$28 + 3.8 \times 10^{-3}$	$7/3 - 1.4 \times 10^{-3}$	$(2, 2, -4)$
$12 + 5.7 \times 10^{-5}$	$28 + 3.8 \times 10^{-3}$	$7/3 - 1.4 \times 10^{-3}$	$(-2, -2, 4)$
$13 - 5.6 \times 10^{-5}$	$26 - 3.8 \times 10^{-3}$	$2 + 1.3 \times 10^{-3}$	$(1, 3, -4)$
$13 - 5.6 \times 10^{-5}$	$26 - 3.8 \times 10^{-3}$	$2 + 1.3 \times 10^{-3}$	$(-1, -3, 4)$

Table 3.5: Scaling limit of three-impurity $\mathfrak{su}(2)$ numerical spectrum at one loop in λ

theory results will be written in terms of $\lambda' = \lambda/J^2$ and J the angular momentum which is dual to the $\mathcal{R}$-charge. However, here we write our answers in terms of λ and $L = J + I$ (different sectors will have different relations). It is necessary to take this into account when comparing results and rewriting the string energies we will see string theory makes the following simple predictions for the large-L $\mathfrak{su}(2)$ expansion

coefficients $E^{(1,3)}_{\mathfrak{su}(2)}$ and $E^{(1,2)}_{\mathfrak{su}(2)}$:

$$\begin{aligned} E^{(1,2)}_{\mathfrak{su}(2)} &= (k_1^2 + k_2^2 + k_3^2)/2 \qquad & k_1 + k_2 + k_3 = 0 \\ E^{(1,3)}_{\mathfrak{su}(2)}/E^{(1,2)}_{\mathfrak{su}(2)} &= 2 \qquad & (k_1 \neq k_2 \neq k_3) \\ E^{(1,3)}_{\mathfrak{su}(2)}/E^{(1,2)}_{\mathfrak{su}(2)} &= \frac{7}{3} \qquad & (k_1 = k_2,\ k_3 = -2k_1)\ . \end{aligned} \tag{3.3.23}$$

Note that we must distinguish the case where all mode indices are unequal from the case where two indices are equal and different from the third. The last column of Table 3.5 displays the choice of indices $\{k_i\}$ that best fit each spectral series and the other columns display the deviation of the extrapolation coefficients from the string theory predictions of equation (3.3.23). As the lattice momenta increase, higher-order $1/L$ corrections to the spectrum become stronger and more data will be required to maintain a given level of precision of the polynomial fit. This effect can be seen directly in the extrapolated eigenvalues in Table 3.5. Nonetheless, it is clear from the table that the gauge theory match to the string theory prediction is extremely good.

We also note that the spectrum in Table 3.5 exhibits a degeneracy of eigenstates whose momentum labels are related by an overall sign flip (a symmetry that is implemented on the operator basis by a parity operator P which reverses the ordering of all fields within the trace). This degeneracy among "parity pairs" of gauge theory operators was observed in [57], where it was shown that it arises as a consequence of integrability (which can, in turn, be used to constrain the form of the Hamiltonian at higher loop order [59]). See [62] for further discussion on the implications of this degeneracy.

3.3.1.2 Two and three-loop order

A similar analysis can be performed on the two-loop $\mathfrak{su}(2)$ spin-chain Hamiltonian. As before, we use the Jordan-Wigner transformation restricted to an odd-impurity basis of operators to rewrite the two-loop Hamiltonian (3.3.5) in terms of position-space

fermionic oscillators, obtaining a result similar to equation (3.3.18):

$$\begin{aligned} H^{(4)}_{\mathfrak{su}(2)} &= \sum_{j=1}^{L}\Bigg\{-\frac{1}{2}\Big[b^\dagger_{j+2}b_j + b^\dagger_j b_{j+2} - 4\Big(b^\dagger_{j+1}b_j + b^\dagger_j b_{j+1}\Big)\Big] - 3\, b^\dagger_j b_j - 4\, b^\dagger_j b^\dagger_{j+1} b_j b_{j+1} \\ &\quad + b^\dagger_{j+1}b^\dagger_{j+2}b_j b_{j+1} + b^\dagger_j b^\dagger_{j+1} b_{j+1} b_{j+2} + b^\dagger_j b^\dagger_{j+2} b_j b_{j+2}\Bigg\}\,. \end{aligned} \tag{3.3.24}$$

Passing to momentum space, we obtain the two-loop analogue of equation (3.3.20):

$$\begin{aligned} H^{(4)}_{\mathfrak{su}(2)} &= -8\sum_{p=0}^{L-1}\sin^4\left(\frac{p\pi}{L}\right)\tilde{b}^\dagger_p\tilde{b}_p \\ &\quad + \frac{1}{L}\sum_{p,q,r,s=0}^{L-1}\left(e^{\frac{2\pi i(q+r)}{L}} + e^{\frac{-2\pi i(p+s)}{L}} + e^{\frac{4\pi i(q-s)}{L}} - 4\, e^{\frac{2\pi i(q-s)}{L}}\right)\tilde{b}^\dagger_p\tilde{b}^\dagger_q\tilde{b}_r\tilde{b}_s\,\delta_{p+q,r+s}\,. \end{aligned} \tag{3.3.25}$$

Although the two-loop Hamiltonian includes "long-range" interactions among non-neighboring lattice sites, the momentum-space Hamiltonian (3.3.25) conveniently terminates at two-body interaction terms. An equally important point is that, for fixed momenta $p, q, \ldots$, the one-body (two-body) operators scale as L^{-4} (L^{-5}) for large L (the corresponding scalings for the one-loop Hamiltonian were L^{-2} (L^{-3})). This special relation between density scaling and power of coupling constant is critical for matching to string theory.

We deal with the problem of finding the eigenvalues of the combined one- and two-loop Hamiltonian via Rayleigh-Schrödinger perturbation theory: at each value of the lattice length L we treat the one-loop operator $H^{(2)}_{\mathfrak{su}(2)}$ as a zeroth-order Hamiltonian and regard $H^{(4)}_{\mathfrak{su}(2)}$ as a first-order perturbation. The $O(\lambda^2)$ corrections to the spectrum of $H^{(2)}_{\mathfrak{su}(2)}$ are then found by taking expectation values of the perturbation $H^{(4)}_{\mathfrak{su}(2)}$ in the (numerically-determined) eigenvectors of $H^{(2)}_{\mathfrak{su}(2)}$. This is the recipe for non-degenerate first-order perturbation theory and we might worry that the previously-noted parity-pair degeneracy of the eigenvalues of $H^{(2)}_{\mathfrak{su}(2)}$ would force us to use the rules of degenerate perturbation theory. As discussed in [57, 63, 62], however, parity degeneracy can

be traced to the existence of a higher Abelian charge which is conserved to at least three-loop order. This charge can be used to show that the formulas of non-degenerate perturbation theory can be used without modification. The basic observation is that conservation of the Abelian charge guarantees that the matrix element of $H^{(4)}_{\mathfrak{su}(2)}$ between two degenerate eigenstates of $H^{(2)}_{\mathfrak{su}(2)}$ with different eigenvalues of the higher Abelian charge vanishes: this eliminates the vanishing energy-denominator singularities that would otherwise invalidate the non-degenerate first-order perturbation theory formulas (and similar arguments apply to the higher-order cases).

Using this method, we have evaluated the $O(\lambda^2)$ corrections to the spectrum of anomalous dimensions for lattice sizes from $L = 6$ to $L = 40$. As before, we fit the spectral data to a power series in $1/L$ to read off the leading scaling coefficients of the low-lying eigenvalues. As mentioned in the discussion of the two-loop Hamiltonian (3.3.25), we expect the two-loop eigenvalues to have the following scaling behavior in $1/L$:

$$E^{(2)}_L(\{k_i\}) = \frac{\lambda^2}{L^4} E^{(2,4)}(\{k_i\}) + \frac{\lambda^2}{L^5} E^{(2,5)}(\{k_i\}) + O(\lambda^2 L^{-6}) \ . \tag{3.3.26}$$

The numerical data confirm that the eigenvalues scale at least as fast as L^{-4}. The resulting numerical values for the leading scaling coefficients of low-lying eigenvalues, $E^{(2,4)}_{\mathfrak{su}(2)}$ and $E^{(2,5)}_{\mathfrak{su}(2)}$, are presented in Table 3.6. We will see later that string theory makes the following simple predictions for the two-loop large-L expansion coefficients:

$$\begin{aligned} E^{(2,4)}_{\mathfrak{su}(2)} &= -(k_1^2 + k_2^2 + k_3^2)^2/16 && k_1 + k_2 + k_3 = 0 \\ E^{(2,5)}_{\mathfrak{su}(2)}/E^{(2,3)}_{\mathfrak{su}(2)} &= 8 && (k_1 \neq k_2 \neq k_3) \\ E^{(2,5)}_{\mathfrak{su}(2)}/E^{(2,3)}_{\mathfrak{su}(2)} &= \frac{76}{9} && (k_1 = k_2,\ k_3 = -2k_1) \ . \end{aligned} \tag{3.3.27}$$

The low-lying levels in the table match the string theory predictions quite accurately and the decline in precision as one goes to higher energies is expected. As a consistency check we note that this time we have no freedom to choose the momenta (k_1, k_2, k_3)

associated with each state: they have been fixed in the one-loop matching exercise.

$E^{(2,4)}_{\mathfrak{su}(2)}$	$E^{(2,5)}_{\mathfrak{su}(2)}$	$E^{(2,5)}_{\mathfrak{su}(2)}/E^{(2,4)}_{\mathfrak{su}(2)}$	(k_1,k_2,k_3)
$-0.25-4.6\times10^{-9}$	$-2+8.0\times10^{-7}$	$8-3.4\times10^{-6}$	$(1,0,-1)$
$-2.25-1.4\times10^{-6}$	$-19+2.6\times10^{-4}$	$76/9+1.2\times10^{-4}$	$(1,1,-2)$
$-2.25-1.4\times10^{-6}$	$-19+2.6\times10^{-4}$	$76/9+1.2\times10^{-4}$	$(-1,-1,2)$
$-4+8.3\times10^{-7}$	$-32-1.1\times10^{-4}$	$8+3.0\times10^{-5}$	$(2,0,-2)$
$-12.25-9.9\times10^{-6}$	$-98+2.3\times10^{-3}$	$8-2.0\times10^{-4}$	$(1,2,-3)$
$-12.25-9.9\times10^{-6}$	$-98+2.3\times10^{-3}$	$8-2.0\times10^{-4}$	$(-1,-2,3)$
$-20.25+3.2\times10^{-3}$	-161.4	7.97	$(3,0,-3)$
$-36-2.8\times10^{-3}$	-304.6	8.46	$(2,2,-4)$
$-36-2.8\times10^{-3}$	-304.6	8.46	$(-2,-2,4)$
$-42.25+4.9\times10^{-3}$	-337.0	7.97	$(1,3,-4)$
$-42.25+4.9\times10^{-3}$	-337.0	7.97	$(-1,-3,4)$

Table 3.6: Scaling limit of three-impurity $\mathfrak{su}(2)$ numerical spectrum at two loops in λ

The three-loop $\mathfrak{su}(2)$ Hamiltonian (3.3.6) can be dealt with in a similar fashion. The position space operator version of this Hamiltonian is too long to record here, but its momentum space version is fairly compact:

$$
\begin{aligned}
H^{(6)}_{\mathfrak{su}(2)} \;=\;& 32\sum_{p=0}^{L-1}\sin^6\left(\frac{p\pi}{L}\right)\tilde{b}^\dagger_p\tilde{b}_p+\frac{1}{2L}\sum_{p,q,r,s=0}^{L-1}\Big\{-10\,e^{\frac{2\pi i(q+r)}{L}}+e^{\frac{2\pi i(2q+r)}{L}}+e^{\frac{2\pi i(q+2r)}{L}}+e^{\frac{2\pi i(q-3s)}{L}}\\
&+e^{\frac{2\pi i(2q-2r-3s)}{L}}+e^{\frac{2\pi i(3q-2r-3s)}{L}}+e^{\frac{2\pi i(q-r-3s)}{L}}+e^{\frac{2\pi i(2q-r-3s)}{L}}-e^{\frac{2\pi i(q-2s)}{L}}-10\,e^{\frac{2\pi i(q-r-2s)}{L}}\\
&-e^{\frac{2\pi i(2q-r-2s)}{L}}-e^{\frac{2\pi i(3q-r-2s)}{L}}-e^{\frac{2\pi i(q+r-2s)}{L}}+29\,e^{\frac{2\pi i(q-s)}{L}}-10\,e^{\frac{4\pi i(q-s)}{L}}+e^{\frac{6\pi i(q-s)}{L}}\\
&-e^{\frac{2\pi i(2q-s)}{L}}+e^{\frac{2\pi i(3q-s)}{L}}-e^{\frac{2\pi i(q+r-s)}{L}}+e^{\frac{2\pi i(2q+r-s)}{L}}+e^{\frac{2\pi i(q+2r-s)}{L}}\Big\}\tilde{b}^\dagger_p\tilde{b}^\dagger_q\tilde{b}_r\tilde{b}_s\,\delta_{p+q,r+s}\\
&+\frac{1}{L^2}\sum_{p,q,r,s,t,u=0}^{L-1}\Big\{e^{\frac{2\pi i(q+3r-2t-3u)}{L}}+e^{\frac{2\pi i(q+2r-s-2t-3u)}{L}}\\
&\qquad+e^{\frac{2\pi i(2q+3r-t-3u)}{L}}+e^{\frac{2\pi i(q+2r+s-u)}{L}}\Big\}\tilde{b}^\dagger_p\tilde{b}^\dagger_q\tilde{b}^\dagger_r\tilde{b}_s\tilde{b}_t\tilde{b}_u\,\delta_{p+q+r,s+t+u}\;. \qquad (3.3.28)
\end{aligned}
$$

It contains at most three-body operators and a careful examination of terms shows that, for fixed momenta, the one-body operators scale as L^{-6}, the two-body operators as L^{-7} and so on. We therefore expect the leading scaling coefficients in the $O(\lambda^3)$ eigenvalues to be $E^{(3,6)}_{\mathfrak{su}(2)}$ and $E^{(3,7)}_{\mathfrak{su}(2)}$, to use a by-now-familiar notation. To

find the eigenvalues to this order, we continue with the Rayleigh-Schrödinger perturbation theory strategy: the $O(\lambda^3)$ correction to any eigenvalue is the sum of the matrix element of $H^{(6)}_{\mathfrak{su}(2)}$ in the appropriate eigenvector of $H^{(2)}_{\mathfrak{su}(2)}$ plus the second-order sum-over-states contribution of $H^{(4)}_{\mathfrak{su}(2)}$. These two pieces can easily be computed numerically from the explicit Hamiltonian operators at a fixed L. Parity degeneracy and conservation of the higher Abelian charge mentioned above continue to hold, and we can again use non-degenerate perturbation theory formulas to compute the eigenvalue corrections. We have generated numerical eigenvalue data for lattices from $L = 6$ to $L = 40$ and the large-L scaling coefficients of the low-lying states extracted from those data are given in Table 3.7. As is by now well-known, the detailed match

$E^{(3,6)}_{\mathfrak{su}(2)}$	$E^{(3,7)}_{\mathfrak{su}(2)}$	$E^{(3,7)}_{\mathfrak{su}(2)}/E^{(3,6)}_{\mathfrak{su}(2)}$	(k_1, k_2, k_3)
0.1250	2.0003	16.003	$(1, 0, -1)$
4.125	58.03	14.07	$(1, 1, -2)$
4.125	58.03	14.07	$(-1, -1, 2)$
7.999	128.2	16.03	$(2, 0, -2)$
49.62	713.3	14.37	$(1, 2, -3)$
49.62	713.3	14.37	$(-1, -2, 3)$
91.15	$1,454$	15.96	$(3, 0, -3)$
263.8	$3,739$	14.17	$(2, 2, -4)$
263.8	$3,739$	14.17	$(-2, -2, 4)$

Table 3.7: Scaling limit of three-impurity $\mathfrak{su}(2)$ numerical spectrum at three loops in λ

to string theory breaks down at three-loop order, so there is no point in trying to match these results to string predictions.

3.3.2 A closed $\mathfrak{su}(1|1)$ subsector of $\mathfrak{su}(2|3)$

As we have discussed above, in [54] Beisert identified a closed $\mathfrak{su}(2|3)$ subsector of the full gauge theory which he further studied in [53] constructing its dilatation operator to three-loops. In the present setting the fields of $\mathfrak{su}(2|3)$ consist of three complex

scalars ϕ_a and two complex fermions ψ_α. In the closed $\mathfrak{su}(1|1)$ subspace we restrict to a single scalar denoted by Z and a single fermion labeled by ψ. Just as in the $\mathfrak{su}(2)$ sector, we use the fermionic position-space oscillators $b_j^\dagger,\ b_j$ to create or annihilate fermionic ψ insertions in a ground state composed of L scalars:

$$|L\rangle = \mathrm{tr}(Z^L) \qquad b_j^\dagger\,|L\rangle = \mathrm{tr}(Z_1\cdots Z_{j-1}\psi Z_{j+1}\cdots Z_L)\ . \tag{3.3.29}$$

In the notation of [54], the action of the Hamiltonian on basis states can be represented in terms of special permutation operators denoted by

$$\begin{Bmatrix} A_1 \dots A_N \\ B_1 \dots B_N \end{Bmatrix}\ ,$$

which replace all occurrences of the upper sequence of fields $A_1 \dots A_N$ in the trace by the lower sequence $B_1 \dots B_N$. Restricting Beisert's $\mathfrak{su}(2|3)$ Hamiltonian to the $\mathfrak{su}(1|1)$ subsector at one-loop order yields

$$H^{(2)}_{\mathfrak{su}(1|1)} = \begin{Bmatrix} Z\psi \\ Z\psi \end{Bmatrix} + \begin{Bmatrix} \psi Z \\ \psi Z \end{Bmatrix} - \begin{Bmatrix} Z\psi \\ \psi Z \end{Bmatrix} - \begin{Bmatrix} \psi Z \\ Z\psi \end{Bmatrix} + 2\begin{Bmatrix} \psi\psi \\ \psi\psi \end{Bmatrix}\ . \tag{3.3.30}$$

In terms of the position-space oscillators of equation (3.3.29), the $\mathfrak{su}(1|1)$ Hamiltonian can be assembled by inspection and takes the form

$$H^{(2)}_{\mathfrak{su}(1|1)} = \sum_{j=1}^{L} \left(b_j^\dagger b_j + b_{j+1}^\dagger b_{j+1} - b_{j+1}^\dagger b_j - b_j^\dagger b_{j+1} \right)\ . \tag{3.3.31}$$

There are no higher-body interaction terms at this order in λ. This fact can be checked by computing

$$\langle L| b_{i+1} b_i (H^{(2)}_{\mathfrak{su}(1|1)}) b_i^\dagger b_{i+1}^\dagger |L\rangle = 2\ , \tag{3.3.32}$$

which reproduces the two-body matrix element given by the last term in equa-

tion (3.3.30). In momentum space we obtain

$$H^{(2)}_{\mathfrak{su}(1|1)} = 4\sum_{p=0}^{L-1}\sin^2\left(\frac{p\pi}{L}\right)\tilde{b}^\dagger_p\tilde{b}_p\ . \tag{3.3.33}$$

In this notation it is clear that at one-loop the Hamiltonian is particularly simple being that of free fermions. The two-loop $\mathfrak{su}(1|1)$ momentum-space Hamiltonian can be extracted in the same manner (the position-space version is too long to print here):

$$\begin{aligned} H^{(4)}_{\mathfrak{su}(1|1)} &= -8\sum_{p=0}^{L-1}\sin^4\left(\frac{p\pi}{L}\right)\tilde{b}^\dagger_p\tilde{b}_p + \frac{1}{4L}\sum_{p,q,r,s=0}^{L-1}\Big\{e^{\frac{2\pi i(q-2r)}{L}} + e^{\frac{2\pi i(2q-r)}{L}} - 4\,e^{\frac{2\pi i(q-r)}{L}} \\ &\quad -2\,e^{\frac{2\pi i(q-2r-s)}{L}} - 2\,e^{\frac{2\pi i(q+s)}{L}} + e^{\frac{2\pi i(q-r+s)}{L}} + e^{\frac{2\pi i(2q-2r-s)}{L}}\Big\}\tilde{b}^\dagger_p\tilde{b}^\dagger_q\tilde{b}_r\tilde{b}_s\,\delta_{p+q,r+s}\ . \end{aligned} \tag{3.3.34}$$

Finally, the complete three-loop Hamiltonian for this subsector is

$$\begin{aligned} H^{(6)}_{\mathfrak{su}(1|1)} &= 32\sum_{p=0}^{L-1}\sin^6\left(\frac{p\pi}{L}\right)\tilde{b}^\dagger_p\tilde{b}_p - \frac{1}{16}\sum_{p,q,r,s=0}^{L-1} e^{\frac{60\pi i(q-r)}{L}}\Big\{2\,e^{-\frac{2\pi i(27q-29r)}{L}} + 2\,e^{-\frac{2\pi i(28q-29r)}{L}} \\ &\quad -4\,e^{-\frac{2\pi i(27q-28r)}{L}} + 37\,e^{-\frac{2\pi i(29q-28r)}{L}} - 6\,e^{-\frac{2\pi i(29q-27r)}{L}} + 8\,e^{-\frac{56\pi i(q-r)}{L}} - 72\,e^{-\frac{58\pi i(q-r)}{L}} \\ &\quad -6\,e^{-\frac{2\pi i(29q-29r-2s)}{L}} - 40\,e^{-\frac{2\pi i(29q-30r-s)}{L}} + 37\,e^{-\frac{2\pi i(29q-29r-s)}{L}} - 8\,e^{-\frac{2\pi i(29q-28r-s)}{L}} \\ &\quad +8\,e^{-\frac{2\pi i(27q-28r+s)}{L}} + 2\,e^{-\frac{2\pi i(28q-28r+s)}{L}} - 40\,e^{-\frac{2\pi i(29q-28r+s)}{L}} - 4\,e^{-\frac{2\pi i(27q-27r+s)}{L}} \\ &\quad +8\,e^{-\frac{2\pi i(29q-27r+s)}{L}} + 2\,e^{-\frac{2\pi i(27q-27r+2s)}{L}} + 8\,e^{-\frac{2\pi i(29q-30r-2s)}{L}}\Big\}\tilde{b}^\dagger_p\tilde{b}^\dagger_q\tilde{b}_r\tilde{b}_s\,\delta_{p+q,r+s} \\ &\quad +\frac{1}{16}\sum_{p,q,r,s,t,u=0}^{L-1}\Big\{2\,e^{\frac{2\pi i(q+2r-3s-2t)}{L}} - e^{\frac{2\pi i(q+3r-3s-2t)}{L}} - 4\,e^{\frac{2\pi i(q+2r-3s-t)}{L}} \\ &\quad -e^{\frac{2\pi i(2q+3r-3s-t)}{L}} + 8\,e^{\frac{2\pi i(q+2r-2s-t)}{L}} + 2\,e^{\frac{2\pi i(2q+3r-2s-t)}{L}} - 4\,e^{\frac{2\pi i(q+2r-3s-2t-u)}{L}} \\ &\quad +2\,e^{\frac{2\pi i(q+3r-3s-2t-u)}{L}} + 2\,e^{\frac{2\pi i(q+2r-2s+u)}{L}} \\ &\quad -4\,e^{\frac{2\pi i(q+2r-s+u)}{L}} - 4\,e^{\frac{2\pi i(q+2r-2s-t+u)}{L}}\Big\}\tilde{b}^\dagger_p\tilde{b}^\dagger_q\tilde{b}^\dagger_r\tilde{b}_s\tilde{b}_t\tilde{b}_u\,\delta_{p+q+r,s+t+u}\ . \end{aligned} \tag{3.3.35}$$

We note that $H^{(2)}_{\mathfrak{su}(1|1)}$, $H^{(4)}_{\mathfrak{su}(1|1)}$ and $H^{(6)}_{\mathfrak{su}(1|1)}$ terminate at one-body, two-body and three-body interactions, respectively. This will permit us to obtain the exact L-dependence of successive terms in the λ expansion of energy eigenvalues.

As in the $\mathfrak{su}(2)$ sector, we can use non-degenerate perturbation theory to extract

the L^{-1} scaling coefficients of the $\mathfrak{su}(1|1)$ eigenvalue spectrum up to three loops in λ. The scaling coefficients extrapolated from numerical diagonalization of lattices up to $L = 40$ are recorded for one-loop, two-loop and three-loop orders in Tables 3.8, 3.9, and 3.10, respectively. The same increase in the leading power of L^{-1} with corresponding order in λ that was noted in the $\mathfrak{su}(2)$ sector is found here too. It should also be noted that, because the impurities in this sector are fermions symmetrized on all group indices, the lattice momenta of all pseudoparticles must be different. The string theory prediction will amount to the following results for the one-loop and two-loop scaling coefficients:

$$E^{(1,2)}_{\mathfrak{su}(1|1)} = (k_1^2 + k_1k_2 + k_2^2) \qquad E^{(1,3)}_{\mathfrak{su}(1|1)} = 0$$
$$E^{(2,4)}_{\mathfrak{su}(1|1)} = -\frac{1}{4}(k_1^2 + k_1k_2 + k_2^2)^2 \qquad E^{(2,5)}_{\mathfrak{su}(1|1)} = -(k_1^2 + k_1k_2 + k_2^2)^2 \ . \quad (3.3.36)$$

The agreement of these predictions with the data in Tables 3.8 and 3.9 is excellent (with the usual caveat that data on larger and larger lattices is required to maintain a fixed precision as one goes to higher and higher energy levels).

$E^{(1,2)}_{\mathfrak{su}(1\|1)}$	$E^{(1,3)}_{\mathfrak{su}(1\|1)}$	$E^{(1,3)}_{\mathfrak{su}(1\|1)}/E^{(1,2)}_{\mathfrak{su}(1\|1)}$	(k_1, k_2, k_3)
$1 + 1.3 \times 10^{-10}$	-1.9×10^{-8}	-1.9×10^{-8}	$(1, 0, -1)$
$4 - 1.0 \times 10^{-7}$	1.8×10^{-5}	4.6×10^{-6}	$(2, 0, -2)$
$7 - 2.5 \times 10^{-7}$	4.4×10^{-5}	6.3×10^{-6}	$(1, 2, -3)$
$7 - 2.5 \times 10^{-7}$	4.4×10^{-5}	6.3×10^{-6}	$(-1, -2, 3)$
$9 - 3.9 \times 10^{-7}$	7.9×10^{-5}	8.7×10^{-6}	$(3, 0, -3)$
$13 - 4.0 \times 10^{-6}$	8.2×10^{-4}	6.3×10^{-5}	$(1, 3, -4)$
$13 - 4.0 \times 10^{-6}$	8.2×10^{-4}	6.3×10^{-5}	$(-1, -3, 4)$
$16 - 2.0 \times 10^{-5}$	4.1×10^{-3}	2.6×10^{-4}	$(4, 0, -4)$
$19 - 3.5 \times 10^{-5}$	7.3×10^{-3}	3.8×10^{-4}	$(2, 3, -5)$
$19 - 3.5 \times 10^{-5}$	7.3×10^{-3}	3.8×10^{-4}	$(-2, -3, 5)$

Table 3.8: Scaling limit of one-loop numerical spectrum of three-impurity $\mathfrak{su}(1|1)$ subsector

The scaling limit of the three-loop ratio $E^{(3,7)}_{\mathfrak{su}(1|1)}/E^{(3,6)}_{\mathfrak{su}(1|1)}$ is recorded for the first

$E^{(2,4)}_{\mathfrak{su}(1\|1)}$	$E^{(2,5)}_{\mathfrak{su}(1\|1)}$	$E^{(2,5)}_{\mathfrak{su}(1\|1)}/E^{(2,4)}_{\mathfrak{su}(1\|1)}$	(k_1, k_2, k_3)
-0.25	-0.99999	3.99995	$(1, 0, -1)$
-4.00006	-15.990	3.998	$(2, 0, -2)$
-12.251	-48.899	3.992	$(1, 2, -3)$
-12.251	-48.899	3.992	$(-1, -2, 3)$
-20.25	-80.89	3.995	$(3, 0, -3)$
-42.25	-168.2	3.98	$(1, 3, -4)$
-42.25	-168.2	3.98	$(-1, -3, 4)$
-64.00	-254.6	3.98	$(4, 0, -4)$
-90.26	-359.3	3.98	$(2, 3, -5)$
-90.26	-359.8	3.99	$(-2, -3, 5)$

Table 3.9: Scaling limit of two-loop numerical spectrum of three-impurity $\mathfrak{su}(1|1)$ subsector

few low-lying states in the spectrum in Table 3.10. These values are in disagreement with the corresponding three-loop predictions from the string theory as will be seen later. Given the well-established three-loop disagreement between the string and gauge theory in the $\mathfrak{su}(2)$ sector, however, this disagreement in the $\mathfrak{su}(1|1)$ subsector is not unexpected.

$E^{(3,7)}_{\mathfrak{su}(1\|1)}/E^{(3,6)}_{\mathfrak{su}(1\|1)}$	(k_1, k_2, k_3)
-86.41	$(1, 0, -1)$
-85.71	$(2, 0, -2)$
-83.74	$(1, 2, -3)$
-83.74	$(-1, -2, 3)$
-101.9	$(3, 0, -3)$
-96.01	$(1, 3, -4)$
-96.01	$(-1, -3, 4)$
-158.1	$(4, 0, -4)$

Table 3.10: Scaling limit of three-loop numerical spectrum of three-impurity $\mathfrak{su}(1|1)$ fermionic subsector

3.3.3 The $\mathfrak{sl}(2)$ sector

Finally we consider the closed $\mathfrak{sl}(2)$ sector the constituent fields of which are $SO(6)$ bosons Z carrying a single unit of $\mathcal{R}$-charge ($Z = \phi_5 + i\phi_6$), and each lattice site on the $\mathfrak{sl}(2)$ spin-chain is occupied by a single Z field acted on by any number of the spacetime covariant derivatives $\mathcal{D} \equiv \mathcal{D}_1 + i\mathcal{D}_2$. The total $\mathcal{R}$-charge of a particular operator is therefore equal to the lattice length L, and an I-impurity operator basis is spanned by single-trace operators carrying all possible distributions of I derivatives among the L lattice sites:

$$\mathrm{Tr}\left(\mathcal{D}^I Z\ Z^{L-1}\right)\ , \mathrm{Tr}\left(\mathcal{D}^{I-1} Z\ \mathcal{D}Z\ Z^{L-2}\right)\ ,\ \mathrm{Tr}\left(\mathcal{D}^{I-1} Z\ Z\mathcal{D}Z\ Z^{L-3}\right)\ , \ldots \quad (3.3.37)$$

The integer I counts the total number of derivatives in the operator and, since any number of impurities can occupy the same lattice site, one can think of n derivative insertions at the i^{th} lattice site as n bosonic oscillator excitations at the i^{th} lattice position:

$$(a_i^\dagger)^n \left|L\right\rangle \sim \mathrm{Tr}\left(Z^{i-1}\mathcal{D}^n Z Z^{L-i}\right)\ ,\ \ldots\ . \quad (3.3.38)$$

The ground state $|L\rangle$ is represented by a length L chain with no derivative insertions: $|L\rangle = \mathrm{Tr}\left(Z^L\right)$.

The one-loop $\mathfrak{sl}(2)$ spin-chain Hamiltonian (corresponding to the dilatation operator in this sector) was constructed in [53] and was defined by its action on basis states:

$$\begin{aligned} H^{(2)}_{\mathfrak{sl}(2)} &= \sum_{j=1}^{L} H^{\mathfrak{sl}(2)}_{j,j+1}\ , \\ H^{\mathfrak{sl}(2)}_{1,2}(a_1^\dagger)^j(a_2^\dagger)^{n-j}\left|L\right\rangle &= \sum_{j'=0}^{n}\left[\delta_{j=j'}\left(h(j)+h(n-j)\right) - \frac{\delta_{j\neq j'}}{|j-j'|}\right](a_1^\dagger)^{j'}(a_2^\dagger)^{n-j'}\left|L\right\rangle\ , \end{aligned} \quad (3.3.39)$$

(where $h(n) = 1 + \ldots + 1/n$ are the harmonic numbers). In other words, $H^{(2)}_{\mathfrak{sl}(2)}$ is a sum over the position-space Hamiltonian $H^{\mathfrak{sl}(2)}_{j,j+1}$ which acts on the j^{th} and $(j+1)^{th}$ (neighboring) lattice sites; the action of $H^{\mathfrak{sl}(2)}_{j,j+1}$ can be summarized by the explicit form given for $H^{\mathfrak{sl}(2)}_{1,2}$ above. Since it is only defined by its action on the state $(a_1^\dagger)^j (a_2^\dagger)^{n-j} |L\rangle$, it is difficult to immediately translate $H^{(2)}_{\mathfrak{sl}(2)}$ to momentum space. However, it is possible to expand it in powers of fields and use equation (3.3.39) to iteratively determine the expansion coefficients. The virial argument furthermore tells us that higher powers in the fields will determine higher powers of L^{-1} in the expansion of the energy. For our current purposes, it suffices to know the Hamiltonian expanded out to terms of fourth order in the fields and this truncation of the Hamiltonian can easily be constructed by inspection:

$$\begin{aligned} H^{(2)}_{\mathfrak{sl}(2)} &= -\sum_{j=1}^{L} \left[\left(a^\dagger_{j+1} - 2a^\dagger_j + a^\dagger_{j-1} \right) \left(a_j - \frac{1}{2} a^\dagger_j a^2_j \right) + \frac{1}{4} \left(a^{\dagger\,2}_{j+1} - 2a^{\dagger\,2}_j + a^{\dagger\,2}_{j-1} \right) a^2_j \right] \\ &\quad + \cdots \ . \end{aligned} \tag{3.3.40}$$

Transformation to momentum space gives

$$\begin{aligned} H^{(2)}_{\mathfrak{sl}(2)} = & \sum_{p=0}^{L-1} 4 \sin^2 \frac{p\pi}{L} \ \tilde{a}^\dagger_p \tilde{a}_p \\ & + \frac{1}{L} \sum_{p,q,r,s=0}^{L-1} \delta_{p+q,r+s} \left(-\sin^2 \frac{p\pi}{L} - \sin^2 \frac{q\pi}{L} + \sin^2 \frac{(p+q)\pi}{L} \right) \tilde{a}^\dagger_p \tilde{a}^\dagger_q \tilde{a}_r \tilde{a}_s \ + \cdots \ . \end{aligned} \tag{3.3.41}$$

This Hamiltonian acts on an I-impurity Fock space spanned by the generic states

$$\tilde{a}^\dagger_{k_1} \tilde{a}^\dagger_{k_2} \tilde{a}^\dagger_{k_3} \cdots |L\rangle \ , \tag{3.3.42}$$

with lattice momenta labeled by $k_i = 0, \ldots, L-1$, and subject to the constraint $\sum_i k_i = 0 \mod L$. Numerically diagonalizing this Hamiltonian on a range of lattice sizes, we obtain data from which we extract the numerical predictions for the one-

loop coefficients $E^{(1,2)}_{\mathfrak{sl}(2)}$ and $E^{(1,3)}_{\mathfrak{sl}(2)}$ presented in Table 3.11. String theory makes the following predictions for the scaling coefficients

$$E^{(1,2)}_{\mathfrak{sl}(2)} = (k_1^2 + k_1 k_2 + k_2^2) \qquad E^{(1,3)}_{\mathfrak{sl}(2)}/E^{(1,2)}_{\mathfrak{sl}(2)} = -2 \qquad k_1 \neq k_2 \neq k_3$$
$$E^{(1,2)}_{\mathfrak{sl}(2)} = 3n^2 \qquad E^{(1,3)}_{\mathfrak{sl}(2)}/E^{(1,2)}_{\mathfrak{sl}(2)} = -7/3 \qquad k_1 = k_2 = n, k_3 = -2n \ , \tag{3.3.43}$$

and we can easily verify that the agreement with Table 3.11 is excellent.

$E^{(1,2)}_{\mathfrak{sl}(2)}$	$E^{(1,3)}_{\mathfrak{sl}(2)}$	$E^{(1,3)}_{\mathfrak{sl}(2)}/E^{(1,2)}_{\mathfrak{sl}(2)}$	(k_1, k_2, k_3)
$1 + 1.2 \times 10^{-9}$	$-2 - 3.1 \times 10^{-7}$	$-2 - 3.1 \times 10^{-7}$	$(1, 0, -1)$
$3 - 7.6 \times 10^{-9}$	$-7 + 1.9 \times 10^{-6}$	$-7/3 + 6.3 \times 10^{-7}$	$(1, 1, -2)$
$3 - 7.6 \times 10^{-9}$	$-7 + 1.9 \times 10^{-6}$	$-7/3 + 6.3 \times 10^{-7}$	$(-1, -1, 2)$
$4 - 2.8 \times 10^{-7}$	$-8 + 6.9 \times 10^{-6}$	$-2 + 1.7 \times 10^{-6}$	$(2, 0, -2)$
$7 - 2.9 \times 10^{-7}$	$-14 + 7.1 \times 10^{-5}$	$-2 + 1.0 \times 10^{-5}$	$(1, 2, -3)$
$7 - 2.9 \times 10^{-7}$	$-14 + 7.1 \times 10^{-5}$	$-2 + 1.0 \times 10^{-5}$	$(-1, -2, 3)$
$9 - 4.1 \times 10^{-7}$	$-18 + 1.0 \times 10^{-4}$	$-2 + 1.0 \times 10^{-5}$	$(3, 0, -3)$
$12 + 8.4 \times 10^{-7}$	$-28 - 1.5 \times 10^{-4}$	$-7/3 - 1.2 \times 10^{-5}$	$(2, 2, -4)$
$12 + 8.4 \times 10^{-7}$	$-28 - 1.5 \times 10^{-4}$	$-7/3 - 1.2 \times 10^{-5}$	$(-2, -2, 4)$
$13 - 7.0 \times 10^{-6}$	$-26 + 1.7 \times 10^{-3}$	$-2 + 1.3 \times 10^{-4}$	$(1, 3, -4)$
$13 - 7.0 \times 10^{-6}$	$-26 + 1.7 \times 10^{-3}$	$-2 + 1.3 \times 10^{-4}$	$(-1, -3, 4)$
$16 - 1.4 \times 10^{-6}$	$-32 + 3.9 \times 10^{-4}$	$-2 + 2.4 \times 10^{-5}$	$(4, 0, -4)$
$19 - 7.5 \times 10^{-6}$	$-38 + 2.2 \times 10^{-3}$	$-2 + 1.1 \times 10^{-4}$	$(2, 3, -5)$
$19 - 7.5 \times 10^{-6}$	$-38 + 2.2 \times 10^{-3}$	$-2 + 1.1 \times 10^{-4}$	$(-2, -3, 5)$
$21 - 3.4 \times 10^{-6}$	$-42 + 8.8 \times 10^{-4}$	$-2 + 4.2 \times 10^{-5}$	$(1, 4, -5)$
$21 - 3.4 \times 10^{-6}$	$-42 + 8.8 \times 10^{-4}$	$-2 + 4.2 \times 10^{-5}$	$(-1, -4, 5)$

Table 3.11: Scaling limit of numerical spectrum of three-impurity $\mathfrak{sl}(2)$ sector at one-loop in λ

This concludes our virial calculation of higher impurity anomalous dimensions though we will return to spin-chains and the Bethe ansatz in a later chapter.

Chapter 4

Beyond the Penrose limit

4.1 Curvature corrections to the Penrose limit

In this section we expand the GS superstring action on $AdS_5 \times S^5$ in powers of $1/R^2$. We begin by constructing various necessary quantities including the Cartan 1-forms, the covariant derivative and the worldsheet Lagrangian. We then gauge fix the action by setting $x^+ = p_-\tau$ and calculate spacetime curvature corrections to the worldsheet metric by demanding consistency with the equations of motion. We similarly determine x^- by analyzing the x^- equation of motion and the covariant gauge constraints order-by-order. Finally we calculate the curvature corrections to the light-cone Hamiltonian.

For the moment, it is convenient to remove an overall factor of R^2 from the definition of the vielbeins $e^\mu{}_\nu$. In practice, this choice makes it easier to recognize terms that contribute to the Hamiltonian at the order of interest, and, in the end, allows us to avoid an additional rescaling operation on the fermions. We proceed by keeping terms to $\mathcal{O}(1/R^4)$, with the understanding that an extra factor of R^2 must be removed in the end. The covariant derivative

$$\mathcal{D}_a\theta^I = \partial_a\theta^I + \frac{1}{4}\partial_a x^\mu \omega_\mu^{\nu\rho}\Gamma_{\nu\rho}\theta^I - \frac{i}{2}\epsilon^{IJ}\Gamma_*\Gamma_\mu e^\mu{}_\nu \partial_a x^\nu \theta^J \tag{4.1.1}$$

may then be expanded to $\mathcal{O}(1/R^2)$:

$$\begin{aligned}\mathcal{D}_0\theta^I &= \left[\partial_0\theta^I - p_-\epsilon^{IJ}\Pi\theta^J\right] + \frac{1}{R}\left[\frac{p_-}{4}\left(z_j\Gamma^{-j} - y_{j'}\Gamma^{-j'}\right)\theta^I + \frac{1}{4}\epsilon^{IJ}\Gamma^-\Pi(\dot{x}^A\Gamma^A)\theta^J\right] \\ &\quad + \frac{1}{R^2}\left[\frac{1}{4}(\dot{z}_j z_k\Gamma^{jk} - \dot{y}_{j'}y_{k'}\Gamma^{j'k'})\theta^I + \frac{p_-}{4}\epsilon^{IJ}\Pi(y^2 - z^2)\theta^J - \frac{1}{2}\epsilon^{IJ}(\dot{x}^-)\Pi\theta^J\right] \\ &\quad + \mathcal{O}(R^{-3}) \qquad (4.1.2)\end{aligned}$$

$$\begin{aligned}\mathcal{D}_1\theta^I &= \partial_1\theta^I + \frac{1}{4R}\epsilon^{IJ}\Gamma^-\Pi(x'^A\Gamma^A)\theta^J \\ &\quad + \frac{1}{R^2}\left[\frac{1}{4}(z'_j z_k\Gamma^{jk} - y'_{j'}y_{k'}\Gamma^{j'k'})\theta^I - \frac{1}{2}\epsilon^{IJ}(x'^-)\Pi\theta^J\right] + \mathcal{O}(R^{-3})\ . \quad (4.1.3)\end{aligned}$$

Note that we have not rescaled the spinor field θ in the above expansion. This allows us to isolate the bosonic scaling contribution from the covariant derivative when combining various terms in the Lagrangian. Subsequently, the fermionic rescaling is performed based on the number of spinors appearing in each term. The worldsheet derivative notation is given by $\partial_\tau x = \partial_0 x = \dot{x}$ and $\partial_\sigma x = \partial_1 x = x'$. The various sectors of the worldsheet Lagrangian are assembled keeping x^- and its derivatives explicit; these will be removed by imposing the covariant gauge constraints. Keeping terms quartic in fermions, we have for the supervielbein and superconnection

$$\begin{aligned}L^\mu_{at} &= e^\mu{}_m\partial_a x^m - 4i\bar{\theta}^I\Gamma^\mu\left(\frac{\sinh^2(t\mathcal{M}/2)}{\mathcal{M}^2}\right)\mathcal{D}_a\theta^I \\ &\approx e^\mu{}_m\partial_a x^m - i\bar{\theta}^I\Gamma^\mu\left(t^2 + \frac{t^4\mathcal{M}^2}{12}\right)\mathcal{D}_a\theta^I \qquad (4.1.4)\\ L^I_{at} &= \frac{\sinh t\mathcal{M}}{\mathcal{M}}\mathcal{D}_a\theta^I \approx \left(t + \frac{t^3}{6}\mathcal{M}^2\right)\mathcal{D}_a\theta^I\ , \qquad (4.1.5)\end{aligned}$$

from which we construct the action

$$I = -\frac{1}{2\pi\alpha'}\int d^2\sigma\ \mathcal{L} \qquad (4.1.6)$$

with the Lagrangian density

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}h^{ab}e^{\mu}_{m}e^{\mu}_{n}\partial_a x^m \partial_b x^n - i\left(h^{ab}\delta^{IJ}+\epsilon^{ab}s^{IJ}\right)\left(\bar{\theta}^I\rho_a\Delta^{JK}\left(D_b\theta\right)^K\right. \\ &\quad \left. -i\left(\bar{\theta}^L\Gamma^{\mu}\left(D_a\theta\right)^L\right)\left(\bar{\theta}^I\Gamma^{\mu}\left(D_b\theta\right)^J\right)\right) \ , \end{aligned} \tag{4.1.7}$$

where

$$\begin{aligned}\rho_a &= e^{\mu}_{m}\partial_a x^m \Gamma^{\mu} \\ \Delta^{JK} &= \left(\delta^{JK}+\frac{\left(\mathcal{M}^2\right)^{JK}}{12}\right) \ . \end{aligned}$$

It will be useful to enforce the light-cone gauge condition $x^+ = p_-\tau$, with p_- a constant, at all orders in the theory. In the pp-wave limit, keeping the worldsheet metric flat in the light-cone gauge is consistent with the equations of motion however beyond leading order we are forced to consider curvature corrections to the worldsheet metric. In the purely bosonic case these corrections are kept implicit by defining gauge constraints in terms of canonical momenta. In the supersymmetric theory, we must explicitly calculate these corrections. The strategy is to expand the x^- equations of motion and the constraints in the rescaled coordinates (2.3.3) and solve for x^- and the worldsheet metric order-by-order. By varying x^- in the full Lagrangian we obtain

$$\partial_a\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_a x^-\right)}\right) = 0 \ , \tag{4.1.8}$$

which to lowest order in $1/R^2$ implies

$$\partial_0\left(2h^{00}p_-\right) = 0 \ , \tag{4.1.9}$$

so we can consistently choose the worldsheet metric flat at lowest order. From the conformal constraints

$$T_{ab} = L^{\mu}_{a}L^{\mu}_{b} - \frac{1}{2}h_{ab}h^{cd}L^{\mu}_{c}L^{\mu}_{d} \ , \tag{4.1.10}$$

we solve for x^-

$$\dot{x}^- = \sum_n \frac{a_n}{R^n} \qquad x'^- = \sum_n \frac{a'_n}{R^n} \ , \tag{4.1.11}$$

at lowest order

$$\begin{aligned} a_0 &= \frac{p_-}{2}(x^A)^2 - \frac{1}{2p_-}\left[(\dot{x}^A)^2 + (x'^A)^2\right] + i\bar{\theta}^I\Gamma^-\partial_0\theta^I - ip_-\epsilon^{IJ}\bar{\theta}^I\Gamma^-\Pi\theta^J \quad (4.1.12) \\ a'_0 &= -\frac{1}{p_-}\dot{x}^A x'^A + i\bar{\theta}^I\Gamma^-\partial_1\theta^I \ . \quad (4.1.13) \end{aligned}$$

By substituting these into (4.1.8) but now keeping the next order in $1/R^2$ we get

$$\begin{aligned} h^{00} = -1 + \frac{\tilde{h}^{00}}{R^2} + \mathcal{O}(R^{-4}) \qquad & h^{11} = 1 + \frac{\tilde{h}^{11}}{R^2} + \mathcal{O}(R^{-4}) \\ h^{01} &= \frac{\tilde{h}^{01}}{R^2} + \mathcal{O}(R^{-4}) \ , \end{aligned} \tag{4.1.14}$$

with

$$\begin{aligned} \tilde{h}^{00} &= \frac{1}{2}(z^2 - y^2) - \frac{1}{2p_-^2}\left[(\dot{x}^A)^2 + (x'^A)^2\right] + \frac{i}{2p_-}\bar{\theta}^I\Gamma^-(\partial_0\theta^I - s^{IJ}\partial_1\theta^J) \quad (4.1.15) \\ \tilde{h}^{01} &= \frac{1}{p_-^2}\dot{x}^A x'^A - \frac{i}{2p_-}\bar{\theta}^I\Gamma^-(\partial_1\theta^I - s^{IJ}\partial_1\theta^J) \ , \quad (4.1.16) \end{aligned}$$

which is consistent with the x^- equation of motion to $\mathcal{O}(1/R^2)$. Note that $\tilde{h}^{00} = -\tilde{h}_{00}$ and $\tilde{h}_{00} = \tilde{h}_{11}$.

Since the worldsheet metric is known to $\mathcal{O}(1/R^2)$, x^- can now be determined to this order from the constraints (4.1.10). So a_2, the correction to $\dot{x}^-$ is:

$$\begin{aligned} a_2 = \frac{-1}{2p_-}\Bigg(& a_0^2 - 2p_-y^2a_0 + a'^2_0 + \frac{1}{2}(\dot{z}^2z^2 - \dot{y}^2y^2) + \frac{p_-^2}{2}(y^4 - z^4) + \frac{1}{2}(z'^2z^2 - y'^2y^2) \\ & + (z^2 - y^2)(x'^A)^2 - \frac{1}{p_-^2}\left[(\dot{x}^A)^2 + (x'^A)^2\right](x'^A)^2 + \frac{i}{p_-}(x'^A)^2\bar{\theta}^I\Gamma^-\partial_0\theta^I \\ & - \frac{i}{p_-}(x'^A)^2 s^{IJ}\bar{\theta}^I\Gamma^-\partial_1\theta^J - ia_0\Delta_0^- - 2ip_-\Delta_2^- - 2ip_-(\Delta_0^-)_{\theta^4} + \frac{ip_-}{2}(y^2 - z^2)\Delta_0^- \\ & - 2i(\dot{x}^A\Delta_1^A + x'^A\Delta_1^A) - ia'_0\Delta'^-_0\Bigg) \ , \end{aligned} \tag{4.1.17}$$

where we have used the notation

$$\Delta_n^\mu \equiv \bar{\theta}^I \Gamma^\mu \mathcal{D}_0^n \theta^I \tag{4.1.18}$$

$$\Delta'^\mu_n \equiv \bar{\theta}^I \Gamma^\mu \mathcal{D}_1^n \theta^I \ , \tag{4.1.19}$$

$$\mathcal{D}_a = \mathcal{D}_a^0 + \frac{1}{R}\mathcal{D}_a^1 + \frac{1}{R^2}\mathcal{D}_a^2 + \mathcal{O}(R^{-3}) \ . \tag{4.1.20}$$

The component T_{01} is the current associated with translation symmetry on the closed-string worldsheet. Enforcing the constraint $T_{01} = 0$ is equivalent to imposing the level-matching condition on physical string states. This condition can be used to fix higher-order corrections to x'^-. However, since our goal is to examine curvature corrections to the pp-wave limit using first-order perturbation theory, we will only need to enforce the level-matching condition on string states that are eigenstates of the pp-wave theory. We therefore need only consider the equation $T_{01} = 0$ to leading order in the expansion, which yields (4.1.13) above. If we were interested in physical eigenstates of the geometry corrected to $\mathcal{O}(1/R^2)$ (i.e., solving the theory exactly to this order), we would be forced to solve $T_{01} = 0$ to $\mathcal{O}(1/R^2)$.

With $\dot{x}^-$ and an expansion of the worldsheet metric to the order of interest, we may proceed with expressing the Hamiltonian as the generator of light-cone time translation: $p_+ = \delta\mathcal{L}/\delta\dot{x}^+$.

The variation is completed prior to any gauge fixing (with the worldsheet metric held fixed). After computing the variation, the light-cone coordinates $x^\pm$ and the worldsheet metric corrections $\tilde{h}^{00}, \tilde{h}^{01}$ are to be replaced with their values found above. Hence, using a_0 and a_2 determined from the covariant gauge constraints (4.1.12,4.1.17), we remove x^- (x^+ has already been replaced with $p_-\tau$ in the above variations) and restore proper powers of R in the vielbeins (so that the desired corrections enter at $\mathcal{O}(1/R^2)$) and we rewrite the fermions in 16 component notation.

As expected, we find the pp-wave Hamiltonian at leading order

$$\mathcal{H}_{pp} = \frac{1}{2p_-}\left((\dot{x}^A)^2 + (x'^A)^2 + p_-^2(x^A)^2\right) - p_-\psi^\dagger\Pi\psi + \frac{i}{2}(\psi\psi' + \psi^\dagger\psi'^\dagger) \quad (4.1.21)$$

and a perturbation that is quite a bit more complicated. As it is merely an intermediate result we don't present it here.

4.2 Quantization of the Hamiltonian

We now want to calculate the energy corrections due to the rather complicated perturbation from the last section. However, canonical quantization requires that we express the Hamiltonian in terms of coordinates and conjugate momenta. At leading order in $1/R^2$, $\dot{x}^A$ is canonically conjugate to x^A and can be expanded in terms of creation and annihilation operators. Beyond leading order, however, the conjugate variable $p_A = \delta\mathcal{L}/\delta\dot{x}^A$ differs from $\dot{x}^A$ by terms of $\mathcal{O}(1/R^2)$. Substituting these $\mathcal{O}(1/R^2)$ corrected expressions for canonical momenta into the pp-wave Hamiltonian will give further $\mathcal{O}(1/R^2)$ corrections to the Hamiltonian. For example, bosonic momenta in the $SO(4)$ descending from the AdS_5 subspace

$$\begin{aligned} p_k &= \dot{z}_k + \frac{1}{R^2}\Big\{\frac{1}{2}y^2 p_k + \frac{1}{2p_-^2}\left[(p_A)^2 + (x'^A)^2\right]p_k - \frac{1}{p_-^2}(p_A x'^A)z'_k - \frac{i}{2p_-}p_k\bar{\theta}^I\Gamma^-\partial_0\theta^I \\ &\quad + \frac{i}{2p_-}p_k s^{IJ}\bar{\theta}^I\Gamma^-\partial_1\theta^J - \frac{ip_-}{4}\bar{\theta}^I\Gamma^- z_j\Gamma_k{}^j\theta^I - \frac{ip_-}{4}\bar{\theta}^I\Gamma^k\left(z_j\Gamma^{-j} - y_{j'}\Gamma^{-j'}\right)\theta^I \\ &\quad + \frac{i}{4}p_A\epsilon^{IJ}\bar{\theta}^I\Gamma^-\left(\Gamma_k\Pi\Gamma^A + \Gamma^A\Pi\Gamma_k\right)\theta^J + \frac{i}{2p_-}z'_k\bar{\theta}^I\Gamma^-\partial_1\theta^I - \frac{i}{2p_-}z'_k s^{IJ}\bar{\theta}^I\Gamma^-\partial_0\theta^J \\ &\quad + \frac{i}{4}x'^A s^{IJ}\epsilon^{JK}\bar{\theta}^I\Gamma^-\left(\Gamma_k\Pi\Gamma^A - \Gamma^A\Pi\Gamma_k\right)\theta^K\Big\} + \mathcal{O}(R^{-4})\ . \end{aligned} \quad (4.2.1)$$

The fermionic momenta, $\rho = \delta\mathcal{L}/\delta\dot{\psi}$, are given by

$$\begin{aligned} \rho &= ip_-\psi^\dagger + \frac{1}{R^2}\Big\{\frac{1}{4}y^2\rho + \frac{1}{8p_-^2}\left[(p_A^2) + (x'^A)^2\right]\rho + \frac{i}{4p_-}(p_A x'^A)\psi + \frac{i}{4p_-}(\rho\Pi\psi)\,\rho \\ &\quad - \frac{i}{8p_-}\left(\psi\rho' + \rho\psi'\right)\psi + \frac{i}{8p_-}\left(\psi\psi' - \frac{1}{p_-^2}\rho\rho'\right)\rho \\ &\quad + \frac{i}{48p_-}\left[\left(\psi\gamma^{jk}\rho\right)\left(\rho\gamma^{jk}\Pi\right) - (j,k,\rightleftharpoons j',k')\right]\Big\} + \mathcal{O}(R^{-4}) \end{aligned} \quad (4.2.2)$$

$$\begin{aligned}\rho^\dagger &= \frac{1}{R^2}\Big\{\frac{i}{4}p_- y^2\psi + \frac{i}{8p_-}\left[(p_A^2)+(x'^A)^2\right]\psi + \frac{1}{4p_-^2}\left(p_A x'^A\right)\rho - \frac{1}{4}\left(\rho\Pi\psi\right)\psi \\ &\quad -\frac{1}{8p_-^2}\left(\psi\rho' + \rho\psi'\right)\rho - \frac{1}{8}\left(\psi\psi' - \frac{1}{p_-^2}\rho\rho'\right)\psi\Big\} + \mathcal{O}(R^{-4})\ . \end{aligned} \tag{4.2.3}$$

Where we have used (4.1.12) and (4.1.13) to replace $\dot{x}^-$ and x'^- at leading order (in 16-component spinor notation), and (4.1.15) for the $\tilde{h}^{00}$ correction to the worldsheet metric. It is clear from the formula (4.2.2) for the conjugate momentum that our system is constrained. Primary constraints generally arise whenever we cannot express the time derivatives of canonical coordinates in terms of the momenta and the coordinates. Primary constraints can be categorized as either first or second class. Second-class constraints arise when canonical momenta do not have vanishing Poisson brackets with the primary constraints: $\{\rho_\psi, \chi_\psi\} \neq 0$, $\left\{\rho_{\psi^\dagger}, \chi_{\psi^\dagger}\right\} \neq 0$. (First-class constraints are characterized by the conditions $\left\{\rho_{\psi^\dagger}, \chi_{\psi^\dagger}\right\} = \{\rho_\psi, \chi_\psi\} = 0$, where $\chi_\psi = 0$ and $\chi_{\psi^\dagger} = 0$ are the constraint equations). Writing the equations for fermionic momenta as primary constraints, $\chi^1_\alpha = \rho_\alpha - ip_-\psi^\dagger_a \cdots = 0$, $\chi^2_\alpha = \rho^\dagger_\alpha \cdots = 0$, it is clear that they are second-class and so consistent quantization requires that the quantum anticommutator of two fermionic fields be identified with their Dirac bracket rather than with their classical Poisson bracket. The Dirac bracket is given in terms of Poisson brackets by (see, for example, [64] and [65])

$$\{A, B\}_{\rm D} = \{A, B\}_{\rm P} - \{A, \chi_N\}_{\rm P}\left(C^{-1}\right)^{NM}\{\chi_M, B\}_{\rm P}\ , \tag{4.2.4}$$

where

$$C_{NM} \equiv \{\chi_N, \chi_M\}_{\rm P}\ . \tag{4.2.5}$$

The indices N and M denote both the spinor index α and the constraint label $a = 1, 2$. For Grassmanian fields A and B, the Poisson bracket is defined by

$$\{A, B\}_{\rm P} = -\left(\frac{\partial A}{\partial \psi^\alpha}\frac{\partial B}{\partial \rho_\alpha} + \frac{\partial B}{\partial \psi^\alpha}\frac{\partial A}{\partial \rho_\alpha}\right) - \left(\frac{\partial A}{\partial \psi^{\dagger\alpha}}\frac{\partial B}{\partial \rho^\dagger_\alpha} + \frac{\partial B}{\partial \psi^{\dagger\alpha}}\frac{\partial A}{\partial \rho^\dagger_\alpha}\right) \ . \qquad (4.2.6)$$

As an example, the Dirac bracket $\{\rho_\alpha, \rho_\beta\}_{\rm D}$ is readily computed (to the order of interest): since $\{\rho_\alpha, \rho_\beta\}_{\rm D}$ contains

$$\{\rho_\alpha, \chi_{a\gamma}\} = \mathcal{O}(R^{-2}) \qquad \{\chi_{b\eta}, \rho_\beta\} = \mathcal{O}(R^{-2}) \ , \qquad (4.2.7)$$

an immediate consequence is that $\{\rho_\alpha, \rho_\beta\}_{\rm D}$ vanishes to $\mathcal{O}(1/R^4)$. To compute $\{\rho_\alpha, \psi_\beta\}_{\rm D}$, we note that

$$\begin{aligned} \{\rho_\alpha, \chi_{(2\gamma)}\}_{\rm P} &= -\delta_{\alpha\rho}\frac{\partial \chi_{(2\gamma)}}{\partial \psi_\rho} \\ &= \mathcal{O}(R^{-2}) \ , \end{aligned} \qquad (4.2.8)$$

and, to leading order,

$$(C^{-1})^{(2\gamma)(1\eta)} = -\frac{i}{p_-}\delta_{\gamma\eta} + \mathcal{O}(R^{-2}) \ , \qquad (4.2.9)$$

such that

$$\{\rho_\alpha, \psi_\beta\}_{\rm D} = -\delta_{\alpha\beta} - \frac{i}{p_-}\{\rho_\alpha, \chi_{(2\beta)}\}_{\rm P} \ . \qquad (4.2.10)$$

Similar manipulations are required for $\{\psi_\alpha, \psi_\beta\}_{\rm D}$, which does exhibit $\mathcal{O}(1/R^2)$ corrections. The second-class constraints on the fermionic sector of the system are removed by enforcing

$$\{\rho_\alpha(\sigma), \rho_\beta(\sigma')\}_{\rm D} = \mathcal{O}(R^{-4}) \ . \qquad (4.2.11)$$

$$\begin{aligned}\{\rho_\alpha(\sigma),\psi_\beta(\sigma')\}_{\rm D} &= -\delta_{\alpha\beta}\delta(\sigma-\sigma')+\frac{1}{4R^2}\delta(\sigma-\sigma')\Big\{\frac{-i}{p_-}(\rho\Pi)_\alpha\psi_\beta+\frac{i}{p_-}(\rho\Pi\psi)\delta_{\alpha\beta}\\ &+\frac{i}{2p_-}\left[\left(\psi\psi'\delta_{\alpha\beta}-\frac{1}{p_-}^2\rho\rho'\delta_{\alpha\beta}\right)+\psi'_\alpha\psi_\beta+\frac{1}{p_-^2}\rho'_\alpha\rho_\beta\right]\\ &+\frac{1}{2p_-^2}\left[(p_A)^2+(x'^A)^2\right]\delta_{\alpha\beta}+y^2\delta_{\alpha\beta}\Big\}\\ &-\frac{i}{8p_-R^2}\left(\psi_\alpha\psi_\beta+\frac{1}{p_-^2}\rho_\alpha\rho_\beta\right)\frac{\partial}{\partial\sigma'}\delta(\sigma-\sigma')+\mathcal{O}(R^{-4})\ , \quad (4.2.12)\end{aligned}$$

$$\begin{aligned}\{\psi_\alpha(\sigma),\psi_\beta(\sigma')\}_{\rm D} &= \frac{i}{4p_-R^2}\delta(\sigma-\sigma')\Big\{(\psi\Pi)_{(\alpha}\psi_{\beta)}-\frac{1}{p_-^2}(p_Ax'^A)\delta_{(\alpha\beta)}\\ &+\frac{1}{2p_-^2}\left[\psi'_{(\alpha}\rho_{\beta)}-\rho'_{(\alpha}\psi_{\beta)}+(\psi\rho'+\rho\psi')\delta_{(\alpha\beta)}\right]\Big\}\\ &+\frac{i}{8p_-^3R^2}\left(\rho_{(\alpha}\psi_{\beta)}-\psi_{(\alpha}\rho_{\beta)}\right)\frac{\partial}{\partial\sigma'}\delta(\sigma-\sigma')\\ &+\mathcal{O}(R^{-4}). \quad (4.2.13)\end{aligned}$$

Identifying these Dirac brackets with the quantum anticommutators of the fermionic fields in the theory naturally leads to additional $\mathcal{O}(1/R^2)$ corrections to the energy spectrum. We now make a field redefinition $\rho\to\tilde\rho$ and $\psi\to\tilde\psi$

$$\rho_\alpha\to\tilde\rho_\alpha = \rho_\alpha \quad (4.2.14)$$

$$\begin{aligned}\psi_\beta\to\tilde\psi_\beta &= \psi_\beta+\frac{i}{8p_-R^2}\Big\{(\psi'\psi)\psi_\beta-2(\rho\Pi\psi)\psi_\beta-\frac{1}{p_-^2}(\rho'\rho)\psi_\beta+\frac{2}{p_-^2}(p_Ax'^A)\rho_\beta\\ &+\frac{1}{p_-^2}\left[(\rho'\psi)\rho_\beta-(\rho\psi')\rho_\beta\right]+2ip_-\left[y^2\psi_\beta+\frac{1}{2p_-^2}\left((p_A)^2+(x'^A)^2\right)\psi_\beta\right]\Big\}, \quad (4.2.15)\end{aligned}$$

such that the Dirac brackets for the redefined fields take the usual form, $\{\tilde\rho_\alpha(\sigma),\tilde\psi_\beta(\sigma')\}=\delta_{\alpha\beta}\delta(\sigma-\sigma')$, and all the others are zero. This approach to enforcing the Dirac bracket structure amounts to adding $\mathcal{O}(1/R^2)$ correction terms to the Hamiltonian while keeping the standard commutation relations. After this redefinition we find for our Hamiltonian density (dropping the $\sim$'s from our equations) $\mathcal{H}=\mathcal{H}_{\rm pp}+\mathcal{H}_{\rm BB}+\mathcal{H}_{\rm BF}+\mathcal{H}_{\rm FF}$ where,

$$\mathcal{H}_{\rm pp} = \frac{p_-}{2}(x^A)^2 + \frac{1}{2p_-}\left[(p_A)^2 + (x'^A)^2\right] + i\rho\Pi\psi + \frac{i}{2}\psi\psi' - \frac{i}{2p_-^2}\rho\rho' \ , \quad (4.2.16)$$

$$\begin{aligned}\mathcal{H}_{\rm BB} = \frac{1}{R^2}\Bigg\{ & \frac{1}{4p_-}\left[-y^2\left(p_z^2 + z'^2 + 2y'^2\right) + z^2\left(p_y^2 + y'^2 + 2z'^2\right)\right] + \frac{p_-}{8}\left[(x^A)^2\right]^2 \\ & -\frac{1}{8p_-^3}\left\{\left[(p_A)^2\right]^2 + 2(p_A)^2(x'^A)^2 + \left[(x'^A)^2\right]^2\right\} + \frac{1}{2p_-^3}\left(x'^A p_A\right)^2\Bigg\}, \end{aligned} \quad (4.2.17)$$

$$\begin{aligned}\mathcal{H}_{\rm FF} = & -\frac{1}{4p_-^3R^2}\Bigg\{p_-^2\left[(\psi'\psi) + \frac{1}{p_-^2}(\rho\rho')\right](\rho\Pi\psi) - \frac{p_-^2}{2}(\psi'\psi)^2 - \frac{1}{2p_-^2}(\rho'\rho)^2 + (\psi'\psi)(\rho'\rho) \\ & +(\rho\psi')(\rho'\psi) - \frac{1}{2}\left[(\psi\rho')(\psi\rho') + (\psi'\rho)^2\right] + \frac{1}{12}(\psi\gamma^{jk}\rho)(\rho\gamma^{jk}\Pi\rho') \\ & -\frac{p_-^2}{48}\left(\psi\gamma^{jk}\psi - \frac{1}{p_-^2}\rho\gamma^{jk}\rho\right)\left(\rho'\gamma^{jk}\Pi\psi - \rho\gamma^{jk}\Pi\psi'\right) - (j,k \rightleftharpoons j',k')\Bigg\} \ , \end{aligned} \quad (4.2.18)$$

$$\begin{aligned}\mathcal{H}_{\rm BF} = \frac{1}{R^2}\Bigg\{ & -\frac{i}{4p_-^2}\left[(p_A)^2 + (x'^A)^2 + p_-^2(y^2 - z^2)\right]\left(\psi\psi' - \frac{1}{p_-^2}\rho\rho'\right) \\ & -\frac{1}{2p_-^3}(p_A x'^A)(\rho\psi' + \psi\rho') - \frac{i}{2p_-^2}\left(p_k^2 + y'^2 - p_-^2 z^2\right)\rho\Pi\psi \\ & +\frac{i}{4}(z'_j z_k)\left(\psi\gamma^{jk}\psi - \frac{1}{p_-^2}\rho\gamma^{jk}\rho\right) - \frac{i}{4}(y'_{j'}y_{k'})\left(\psi\gamma^{j'k'}\psi - \frac{1}{p_-^2}\rho\gamma^{j'k'}\rho\right) \\ & -\frac{i}{8}(z'_k y_{k'} + z_k y'_{k'})\left(\psi\gamma^{kk'}\psi - \frac{1}{p_-^2}\rho\gamma^{kk'}\rho\right) + \frac{1}{4p_-}(p_k y_{k'} + z_k p_{k'})\psi\gamma^{kk'}\rho \\ & +\frac{1}{4p_-}(p_j z'_k)\left(\psi\gamma^{jk}\Pi\psi + \frac{1}{p_-^2}\rho\gamma^{jk}\Pi\rho\right) - \frac{1}{4p_-}(p_{j'}y'_{k'})\left(\psi\gamma^{j'k'}\Pi\psi + \frac{1}{p_-^2}\rho\gamma^{j'k'}\Pi\rho\right) \\ & -\frac{1}{4p_-}(p_k y'_{k'} + z'_k p_{k'})\left(\psi\gamma^{kk'}\Pi\psi + \frac{1}{p_-^2}\rho\gamma^{kk'}\Pi\rho\right) - \frac{i}{2p_-^2}(p_k p_{k'} - z'_k y'_{k'})\psi\gamma^{kk'}\Pi\rho\Bigg\} \ . \end{aligned} \quad (4.2.19)$$

This system is quantized by imposing the standard (anti)commutator algebra for x^A, ψ and their conjugate variables p^A, ρ. We note that the Hamiltonian perturbation that is purely fermionic vanishes on the subspace of string zero-modes because all terms have at least one worldsheet spatial derivative. The bose-fermi mixing Hamiltonian still has terms which can lead to curvature corrections to the string zero-mode energies, but their net effect vanishes by virtue of non-trivial cancellations between

terms that split $SO(4) \times SO(4)$ indices and terms that span the entire $SO(8)$. How this comes about will be seen when we actually compute the matrix elements of this Hamiltonian.

4.3 Perturbative string spectrum

To compute the energy spectrum correct to first order in $\mathcal{O}(R^{-2})$, we use degenerate perturbation theory on the Fock space of eigenstates of the free Hamiltonian $\mathcal{H}_{\rm pp}$. The degenerate subspaces of the BMN theory are spanned by fixed numbers of creation operators with specified mode indices acting on the ground state $|J\rangle$, where $J = p_- R^2$ is the angular momentum (assumed large) of the string center of mass. For now, we restrict attention to "two-impurity states" generated by pairs of creation operators of equal and opposite mode number. For each positive mode number n, the 16 bosonic and fermionic creation operators can be combined in pairs to form the following 256 degenerate "two-impurity" states:

$$a_n^{A\dagger} a_{-n}^{B\dagger} |J\rangle \qquad b_n^{\alpha\dagger} b_{-n}^{\beta\dagger} |J\rangle \qquad a_n^{A\dagger} b_{-n}^{\alpha\dagger} |J\rangle \qquad a_{-n}^{A\dagger} b_n^{\alpha\dagger} |J\rangle \ . \tag{4.3.1}$$

The creation operators are classified under the $SO(4) \times SO(4)$ symmetry of the plane wave background. The bosonic creation operators $a_n^{A\dagger}$ decompose as $(\mathbf{4},\mathbf{1}) + (\mathbf{1},\mathbf{4})$, or, in the $SU(2)^2 \times SU(2)^2$ notation (see [48]), as $(\mathbf{2},\mathbf{2};\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{1};\mathbf{2},\mathbf{2})$. Similarly, the fermionic operators $b_n^{\alpha\dagger}$ decompose as $(\mathbf{2},\mathbf{1};\mathbf{2},\mathbf{1}) + (\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{2})$. It is useful to note that the two fermion irreps are eigenvectors, with opposite eigenvalue, of the Π operator. To find the perturbed energy spectrum, we must compute explicit matrix elements of $\mathcal{H}_{\rm int}$ in this basis and then diagonalize the resulting 256×256 matrix. We will compare the perturbed energy eigenvalues with general expectations from $PSU(2,2|4)$ as well as with the large $\mathcal{R}$-charge limit of the anomalous dimensions of gauge theory operators with two $\mathcal{R}$-charge defects.

The first step is to expand $\mathcal{H}_{\rm int}$ in creation and annihilation operators using (2.3.8,2.3.13) for x^A, ψ and the related expansions for p^A, ρ. As an example, we

quote the result for $H_{\rm BB}$ (keeping only terms with two creation and two annihilation operators):

$$\begin{aligned}\mathcal{H}_{\rm BB} &= -\frac{1}{32p_-R^2}\sum\frac{\delta(n+m+l+p)}{\xi}\times \\ &\Big\{2\Big[\xi^2-(p_-^4-k_lk_pk_nk_m)+\omega_n\omega_mk_lk_p+\omega_l\omega_pk_nk_m+2\omega_n\omega_lk_mk_p \\ &+2\omega_m\omega_pk_nk_l\Big]a^{\dagger A}_{-n}a^{\dagger A}_{-m}a^B_la^B_p+4\Big[\xi^2-(p_-^4-k_lk_pk_nk_m)-2\omega_n\omega_mk_lk_p+\omega_l\omega_mk_nk_p \\ &-\omega_n\omega_lk_mk_p-\omega_m\omega_pk_nk_l+\omega_n\omega_pk_mk_l\Big]a^{\dagger A}_{-n}a^{\dagger B}_{-l}a^A_ma^B_p+2\Big[8k_lk_pa^{\dagger i}_{-n}a^{\dagger j}_{-l}a^i_ma^j_p \\ &+2(k_lk_p+k_nk_m)a^{\dagger i}_{-n}a^{\dagger i}_{-m}a^j_la^j_p+(\omega_l\omega_p+k_lk_p-\omega_n\omega_m-k_nk_m)a^{\dagger i}_{-n}a^{\dagger i}_{-m}a^{j'}_la^{j'}_p \\ &-4(\omega_l\omega_p-k_lk_p)a^{\dagger i}_{-n}a^{\dagger j'}_{-l}a^i_ma^{j'}_p-(i,j\rightleftharpoons i',j')\Big]\Big\}\,, \end{aligned} \tag{4.3.2}$$

with $\xi\equiv\sqrt{\omega_n\omega_m\omega_l\omega_p}$. The expansion of the interaction terms involving fermi fields are too complicated to be worth writing down explicitly at this stage. Schematically, we organize the two-impurity matrix elements of the perturbing Hamiltonian as shown in Table 4.1.

$(\mathcal{H})_{int}$	$a^{A\dagger}_na^{B\dagger}_{-n}\left\|J\right\rangle$	$b^{\alpha\dagger}_nb^{\beta\dagger}_{-n}\left\|J\right\rangle$	$a^{A\dagger}_na^{\alpha\dagger}_{-n}\left\|J\right\rangle$	$a^{A\dagger}_{-n}b^{\alpha\dagger}_n\left\|J\right\rangle$
$\left\langle J\right\|a^A_na^B_{-n}$	$\mathcal{H}_{\rm BB}$	$\mathcal{H}_{\rm BF}$	0	0
$\left\langle J\right\|b^\alpha_nb^\beta_{-n}$	$\mathcal{H}_{\rm BF}$	$\mathcal{H}_{\rm FF}$	0	0
$\left\langle J\right\|a^A_nb^\alpha_{-n}$	0	0	$\mathcal{H}_{\rm BF}$	$\mathcal{H}_{\rm BF}$
$\left\langle J\right\|a^A_{-n}b^\alpha_n$	0	0	$\mathcal{H}_{\rm BF}$	$\mathcal{H}_{\rm BF}$

Table 4.1: Perturbation matrix in the space of two-impurity string states

4.3.1 Evaluating matrix elements

We now construct the matrix elements of the perturbing Hamiltonian in the space of degenerate two-impurity states. To convey a sense of what is involved, we display the

matrix elements of $\mathcal{H}_{\rm BB}$ (4.2.17) between the bosonic two-impurity Fock space states:

$$\begin{aligned}\left\langle J\left|a_n^A a_{-n}^B\left(\mathcal{H}_{\rm BB}\right)a_{-n}^{C\dagger}a_n^{D\dagger}\right|J\right\rangle &= \left(N_{\rm BB}(n^2\lambda')-2n^2\lambda'\right)\frac{\delta^{AD}\delta^{BC}}{J}\\ &+\frac{n^2\lambda'}{J(1+n^2\lambda')}\left[\delta^{ab}\delta^{cd}+\delta^{ad}\delta^{bc}-\delta^{ac}\delta^{bd}\right]\\ &-\frac{n^2\lambda'}{J(1+n^2\lambda')}\left[\delta^{a'b'}\delta^{c'd'}+\delta^{a'd'}\delta^{b'c'}-\delta^{a'c'}\delta^{b'd'}\right]\\ \approx (n_{\rm BB}-2)\ &\frac{n^2\lambda'}{J}\delta^{AD}\delta^{BC}+\frac{n^2\lambda'}{J}\left[\delta^{ab}\delta^{cd}+\delta^{ad}\delta^{bc}-\delta^{ac}\delta^{bd}\right]\\ &-\frac{n^2\lambda'}{J}\left[\delta^{a'b'}\delta^{c'd'}+\delta^{a'd'}\delta^{b'c'}-\delta^{a'c'}\delta^{b'd'}\right]+\mathcal{O}(\lambda'^2)\ ,\end{aligned} \tag{4.3.3}$$

where lower-case $SO(4)$ indices $a,b,c,d\in 1,\ldots,4$ indicate that A,B,C,D are chosen from the first $SO(4)$, and $a',b',c',d'\in 5,\ldots,8$ indicate the second $SO(4)$ ($A,B,C,D\in 5,\ldots,8$). We have also displayed the further expansion of these $\mathcal{O}(1/J)$ matrix elements in powers of λ' (using the BMN-limit energy eigenvalue condition $\omega_n/p_- = \sqrt{1+\lambda' n^2}$). This is to facilitate eventual contact with perturbative gauge theory via AdS/CFT duality. Note that $\mathcal{H}_{\rm BB}$ does not mix states built out of oscillators from different $SO(4)$ subgroups. There is a parallel no-mixing phenomenon in the gauge theory: two-impurity bosonic operators carrying spacetime vector indices do not mix with spacetime scalar bosonic operators carrying $\mathcal{R}$-charge vector indices.

Due to operator ordering ambiguities, we must introduce a contribution proportional to $\delta^{AD}\delta^{BC}$ to the matrix elements of $\mathcal{H}_{\rm BB}$ which depends on the particular ordering prescription chosen. $N_{\rm BB}(n^2\lambda')$ is an arbitrary function of $n^2\lambda'$, which is included to account for such ambiguities (we will shortly succeed in fixing it).

The calculation of the two-impurity matrix elements of the parts of $\mathcal{H}_{\rm int}$ that involve fermionic fields is rather involved and we found it necessary to employ symbolic manipulation programs to keep track of the many different terms. The final results are reasonably concise and they can be found in [66] however we will only record the

matrix elements of purely fermionic states here

$$\begin{aligned}
\left\langle J\middle| b_n^\alpha b_{-n}^\beta \left(\mathcal{H}_{\rm FF}\right) b_{-n}^{\gamma\dagger} b_n^{\delta\dagger}\middle| J\right\rangle &= \left(N_{\rm FF}(n^2\lambda') - 2n^2\lambda'\right)\frac{\delta^{\alpha\delta}\delta^{\beta\gamma}}{J} \\
&+\frac{n^2\lambda'}{24J(1+n^2\lambda')}\left[(\gamma^{ij})^{\alpha\delta}(\gamma^{ij})^{\beta\gamma} + (\gamma^{ij})^{\alpha\beta}(\gamma^{ij})^{\gamma\delta} - (\gamma^{ij})^{\alpha\gamma}(\gamma^{ij})^{\beta\delta}\right] \\
&-\frac{n^2\lambda'}{24J(1+n^2\lambda')}\left[(\gamma^{i'j'})^{\alpha\delta}(\gamma^{i'j'})^{\beta\gamma} + (\gamma^{i'j'})^{\alpha\beta}(\gamma^{i'j'})^{\gamma\delta} - (\gamma^{i'j'})^{\alpha\gamma}(\gamma^{i'j'})^{\beta\delta}\right] \\
&\approx (n_{\rm FF}-2)\,\frac{n^2\lambda'}{J}\delta^{\alpha\delta}\delta^{\beta\gamma} \\
&+\frac{n^2\lambda'}{24J}\left[(\gamma^{ij})^{\alpha\delta}(\gamma^{ij})^{\beta\gamma} + (\gamma^{ij})^{\alpha\beta}(\gamma^{ij})^{\gamma\delta} - (\gamma^{ij})^{\alpha\gamma}(\gamma^{ij})^{\beta\delta}\right] \\
&-\frac{n^2\lambda'}{24J}\left[(\gamma^{i'j'})^{\alpha\delta}(\gamma^{i'j'})^{\beta\gamma} + (\gamma^{i'j'})^{\alpha\beta}(\gamma^{i'j'})^{\gamma\delta} - (\gamma^{i'j'})^{\alpha\gamma}(\gamma^{i'j'})^{\beta\delta}\right] \\
&+\mathcal{O}(\lambda'^2)\ .
\end{aligned} \tag{4.3.4}$$

Because the interaction Hamiltonian is quartic in oscillators, normal-ordering ambiguities give rise to terms quadratic in oscillators, appearing as constant contributions to the diagonal matrix elements. There are normal-ordering contributions from each sector of the theory: $\mathcal{H}_{\rm BB}$ contributes a single term quadratic in bosonic oscillators; $\mathcal{H}_{\rm FF}$ yields a term quadratic in fermionic oscillators; $\mathcal{H}_{\rm BF}$ contributes one term quadratic in bosons and one quadratic in fermions. The bosonic contributions multiply terms of the form $a^\dagger a$, which are collected into the function $N_{\rm BB}(n^2\lambda')$ with one contribution from $\mathcal{H}_{\rm BB}$ and one contribution from $\mathcal{H}_{\rm BF}$. Similarly, $N_{\rm FF}(n^2\lambda')$, which collects terms multiplying $b^\dagger b$ receives one contribution from $\mathcal{H}_{\rm FF}$ and one contribution from $\mathcal{H}_{\rm BF}$. Normal-ordering contributions from both $a^\dagger a$ and $b^\dagger b$ terms are non-vanishing in the spacetime fermion subsector; all possible normal-ordering ambiguities appear in this subspace. The normal-ordering function $N_{\rm BF}(n^2\lambda')$ therefore must satisfy

$$N_{\rm BF}(n^2\lambda') = N_{\rm BB}(n^2\lambda') + N_{\rm FF}(n^2\lambda')\ . \tag{4.3.5}$$

The normal ordering functions are basically finite renormalizations which must be adjusted so that the spectrum reflects the $PSU(2,2|4)$ global supersymmetry of the classical worldsheet action (a symmetry we want to preserve at the quantum level).

We note that the eigenvalues in question are light-cone energies and thus dual to the gauge theory quantity $\Delta = D - J$, the difference between scaling dimension and $\mathcal{R}$-charge. Since conformal invariance is part of the full symmetry group, states are organized into conformal multiplets built on conformal primaries. A supermultiplet will contain several conformal primaries having the same value of Δ and transforming into each other under the supercharges. All 16 supercharges increment the dimension of an operator by 1/2, but only 8 of them (call them Q_α) also increment the $\mathcal{R}$-charge by 1/2, so as to leave Δ unchanged. These 8 supercharges act as 'raising operators' on the conformal primaries of a supermultiplet: starting from a super-primary of lowest $\mathcal{R}$-charge, the other conformal primaries are created by acting on it in all possible ways with the eight Q_α. Primaries obtained by acting with L_v factors of Q_α on the super-primary are said to be at level L_v in the supermultiplet (since the Q_α anticommute, the range is $L_v = 0$ to $L_v = 8$). The multiplicities of states at the various levels are then determined: for every $L_v = 0$ primary, there will in general be $C^8_{L_v}$ primaries at level L_v (where C^n_m is the binomial coefficient) and a total of $2^8 = 256$ conformal primaries summed over all L_v. If the $L_v = 0$ conformal primary is not a singlet, the total number of conformal primary states will be a multiple of 256. Since the number of two-impurity string states is exactly 256, we expect the super-primary level to be a singlet (in both spacetime and the residual $SO(4)$ $\mathcal{R}$-symmetry) and therefore necessarily a spacetime boson. This is the translation into string theory language of Beisert's careful analysis of supermultiplets of two-impurity BMN operators in $\mathcal{N} = 4$ super Yang-Mills theory [49].

These facts severely restrict the spectrum of eigenvalues. Although the two-impurity string states in question have the same J, they correspond to gauge theory states at different levels L_v in different supermultiplets. A string theory state corresponds to a gauge theory state of given L_v which is a member of a supermultiplet

built on a 'highest-weight' or super-primary state with $\mathcal{R} = J - L_v/2$. Since all the primaries in a supermultiplet have the same Δ, the string state should have an energy equal to $\Delta(\lambda, J - L_v/2)$ i.e., that of the super-primary state from which the gauge theory state descends.

A careful inspection of the way the normal ordering functions contribute to the energies of states in the two-impurity sector shows that states at levels $L_v = 0, 8$ are shifted by $N_{\rm BB}$ only. Similarly, levels $L_v = 2, 4, 6$ are shifted by $N_{\rm FF}$ or $N_{\rm BB}$ and one must have $N_{\rm BB} = N_{\rm FF}$ if those levels are to remain internally degenerate. Finally, levels $L_v = 1, 3, 5, 7$ are shifted by $N_{\rm BF}$ only. To agree with the spectrum outlined above, the level spacing must be uniform throughout the supermultiplet and this is only possible if we also set $N_{\rm BB} = N_{\rm BF}$. But then the constraint $N_{\rm BF} = N_{\rm BB} + N_{\rm FF}$ can only be met by setting $N_{\rm BB} = N_{\rm FF} = N_{\rm BF} = 0$, which then eliminates any normal-ordering ambiguity from the string theory. This is basically an exercise in using global symmetry conditions to fix otherwise undetermined finite renormalizations.

4.3.2 Diagonalizing the one-loop perturbation matrix

We are now ready to diagonalize the perturbing Hamiltonian and examine whether the resulting energy shifts have the right multiplet structure and whether the actual eigenvalues match gauge theory expectations. To simplify the problem, we will begin by diagonalizing the perturbation matrix expanded to first nontrivial order in both $1/J$ and λ'. Our results should, by duality, match one-loop gauge theory calculations and we will return to the problem of finding the string spectrum at higher orders in λ'. From the structure of the results just obtained for the perturbation matrices, we can see that the general structure of the energy eigenvalues of two-impurity states must be

$$E_{\rm int}(n) = 2 + n^2\lambda' \left(1 + \frac{\Lambda}{J} + \mathcal{O}(J^{-2})\right) + \mathcal{O}(\lambda'^2) \ , \tag{4.3.6}$$

where Λ is dimensionless and the dependence on $1/J$, λ' and mode number n is given by the matrix elements. As outlined above, the eigenvalues Λ must meet certain conditions if the requirements of duality and $PSU(2,2|4)$ symmetry are to be met.

The only way the expansion of (4.3.6) can be consistent with this is if $\Lambda = L_v + c$, where c is a pure numerical constant. Successive spacetime boson (or successive spacetime fermion) members of a supermultiplet must therefore have eigenvalues separated by exactly 2. We furthermore know that the multiplicity of the level-L_v eigenvalue must be $C^8_{L_v} = 1, 8, 28, ..., 1$ for $L_v = 0, 1, 2, ..., 8$. The representation content of the different levels under the $SO(3,1)$ spacetime and residual $SO(4)$ $\mathcal{R}$ symmetries can of course also be specified, if desired. Our program, then, is the following: we will first verify that the quantization procedure preserves the $PSU(2,2|4)$ supersymmetry by showing that the eigenvalues Λ satisfy the integer spacing and multiplicity rules just enumerated; in the process we will obtain specific values for Λ which we will then compare with what is known about one-loop gauge theory operator dimensions.

4.3.3 Details of the one-loop diagonalization procedure

We now confront the problem of explicitly diagonalizing the first-order perturbation matrix. The matrix block diagonalizes on the spacetime boson and spacetime fermion subspaces, as indicated in Table 4.1. Within these sub-blocks, there are further block diagonalizations arising from special properties of the one-loop form of the matrix elements of the perturbing Hamiltonian. For example, states of the form, $a^i_{-n} a^j_n |J\rangle$, mix only with themselves, thus providing a 16×16 dimensional diagonal sub-block. Within such sub-blocks, symmetry considerations are often sufficient to completely diagonalize the matrix or at least to reduce it to a low-dimensional diagonalization problem. In short, the problem reduces almost entirely to that of projecting the matrix elements of $\mathcal{H}_{\rm int}$ on subspaces of the two-impurity Fock space defined by various symmetry properties. Determining the $SO(4) \times SO(4)$ symmetry labels of each eigenstate in the diagonalization will furthermore enable us to precisely match string states with gauge theory operators. Although the projections onto the various

invariant subspaces are matters of simple algebra, the algebra is often too complicated to be done by hand and we have resorted to symbolic manipulation programs.

We begin with a discussion of the action of the purely bosonic perturbation $\mathcal{H}_{\rm BB}$ on the 64-dimensional Fock space created by pairs of bosonic creation operators. Part of this subspace mixes via $\mathcal{H}_{\rm BF}$ with the spacetime bosons created by pairs of fermionic creation operators, and we will deal with it later. There is, however, a subspace that only mixes with itself, through the purely bosonic perturbation $\mathcal{H}_{\rm BB}$. We will first deal with this purely bosonic block diagonalization, leading to eigenvalues we will denote by $\Lambda_{\rm BB}$. The 8 bosonic modes, $a^i, a^{i'}$, lie in the $SO(4) \times SO(4)$ representations $(\mathbf{2},\mathbf{2};\mathbf{1},\mathbf{1})$ and $(\mathbf{1},\mathbf{1};\mathbf{2},\mathbf{2})$ (we use $SU(2) \times SU(2)$ notation, rather than $SO(4)$, since it is unavoidable when we discuss fermions). A key fact is that the 16-dimensional spaces spanned by two $(\mathbf{2},\mathbf{2};\mathbf{1},\mathbf{1})$ oscillators or by two $(\mathbf{1},\mathbf{1};\mathbf{2},\mathbf{2})$ oscillators, i.e., $a^{i\dagger}a^{j\dagger}\,|J\rangle$ or $a^{i'\dagger}a^{j'\dagger}\,|J\rangle$, are closed under the action of the Hamiltonian. The $SO(4)$ representation content of the states created by such oscillator pairs is given by the formula $(\mathbf{2},\mathbf{2}) \times (\mathbf{2},\mathbf{2}) = (\mathbf{3},\mathbf{3}) + (\mathbf{3},\mathbf{1}) + (\mathbf{1},\mathbf{3}) + (\mathbf{1},\mathbf{1})$. By projecting the $\mathcal{O}(\lambda')$ part of (4.3.3) onto these subspaces, one can directly read off the eigenvalues $\Lambda_{\rm BB}$, with the results shown in Table 4.2. The projection onto invariant subspaces is a simple matter of symmetrization or antisymmetrization of oscillator indices. The most important point to note is that the eigenvalues are successive even integers, which is consistent with our expectations from extended supersymmetry. It is straightforward to match these states to gauge theory operators and compare energies with anomalous dimensions.

$SO(4)_{AdS} \times SO(4)_{S^5}$	$\Lambda_{\rm BB}$
$(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})$	-6
$(\mathbf{1},\mathbf{1};\mathbf{3},\mathbf{3})$	-2
$(\mathbf{1},\mathbf{1};\mathbf{3},\mathbf{1}) + (\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{3})$	-4

$SO(4)_{AdS} \times SO(4)_{S^5}$	$\Lambda_{\rm BB}$
$(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})$	2
$(\mathbf{3},\mathbf{3};\mathbf{1},\mathbf{1})$	-2
$(\mathbf{3},\mathbf{1};\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{3};\mathbf{1},\mathbf{1})$	0

Table 4.2: Energy shifts at $\mathcal{O}(1/J)$ for unmixed bosonic modes

The Fock space of spacetime bosons created by pairs of fermionic creation oper-

ators contains a similar pair of 16×16 diagonal sub-blocks. The construction and application of the relevant projection operators and the subsequent comparison with gauge theory operators is more complicated than on the bosonic side and we must develop some tools before we can obtain concrete results.

It is helpful to express the 8-component spinors of the string theory in a basis that allows us to define fermionic oscillators labeled by their $(\mathbf{2},\mathbf{1};\mathbf{2},\mathbf{1})$ and $(\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{2})$ representation content.

The original 32-component Majorana-Weyl spinors θ^I were reduced by the Weyl projection and a light-cone gauge condition to an 8-component spinor ψ^α (transforming in the 8_s of $SO(8)$). The remaining 8 components are further divided into spinors $\tilde{\psi}$ and $\hat{\psi}$ which are even or odd under the action of $\Pi \equiv \gamma^1\bar{\gamma}^2\gamma^3\bar{\gamma}^4$:

$$\begin{aligned} \Pi\tilde{\psi} &= -\tilde{\psi} \qquad & \Pi\tilde{b}^{\dagger\alpha} &= -\tilde{b}^{\dagger\alpha} \\ \Pi\hat{\psi} &= \hat{\psi} \qquad & \Pi\hat{b}^{\dagger\alpha} &= \hat{b}^{\dagger\alpha} \ . \end{aligned} \tag{4.3.7}$$

The spinors $\hat{\psi}$ transform in the $(\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{2})$ of $SO(4) \times SO(4)$, while $\tilde{\psi}$ transform in the $(\mathbf{2},\mathbf{1};\mathbf{2},\mathbf{1})$.

We denote the non-trivial $SU(2)$ generators of the $(\mathbf{2},\mathbf{1};\mathbf{2},\mathbf{1})$ irrep as Σ^+ and Ω^+, where the Σ act on the SO(4) descended from the AdS_5, and the Ω act on the $SO(4)$ coming from the S^5. The $(\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{2})$ generators are similarly labeled by Σ^- and Ω^-. Each set of spinors is annihilated by its counterpart set of $SU(2)$ generators:

$$\begin{aligned} \Sigma^+\hat{b}^{\dagger\alpha} = \Omega^+\hat{b}^{\dagger\alpha} &= 0 \\ \Sigma^-\tilde{b}^{\dagger\alpha} = \Omega^-\tilde{b}^{\dagger\alpha} &= 0 \ . \end{aligned} \tag{4.3.8}$$

We define the projection operators

$$\Pi_+ = \frac{1}{2}(1+\Pi) \qquad \Pi_- = \frac{1}{2}(1-\Pi) \ , \tag{4.3.9}$$

which project onto the $(\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{2})$ and $(\mathbf{2},\mathbf{1};\mathbf{2},\mathbf{1})$ irreps, respectively. The $\Pi_\pm$ pro-

jections commute with the $SO(4)$ generator matrices $\gamma^{ij}, \gamma^{i'j'}$, a fact which implies certain useful selection rules for the one-loop limit of (4.3.4). In brief, one finds that $\hat{b}^\dagger\hat{b}^\dagger\,|J\rangle$ states mix only with other $\hat{b}^\dagger\hat{b}^\dagger\,|J\rangle$ states and similarly for the $\tilde{b}^\dagger\tilde{b}^\dagger\,|J\rangle$ states. The generators of the four $SU(2)$ factors (4.3.8) of the manifest $SO(4)\times SO(4)$ symmetry can be expressed as 8×8 $SO(8)$ matrices as follows:

$$\begin{aligned}
\Sigma_1^\pm &= -\frac{1}{4i}(\gamma^2\gamma^3 \pm \gamma^1\gamma^4) & \Omega_1^\pm &= \frac{1}{4i}(-\gamma^6\gamma^7 \pm \gamma^5)\\
\Sigma_2^\pm &= -\frac{1}{4i}(\gamma^3\gamma^1 \pm \gamma^2\gamma^4) & \Omega_2^\pm &= \frac{1}{4i}(-\gamma^7\gamma^5 \pm \gamma^6)\\
\Sigma_3^\pm &= -\frac{1}{4i}(\gamma^1\gamma^2 \pm \gamma^3\gamma^4) & \Omega_3^\pm &= \frac{1}{4i}(-\gamma^5\gamma^6 \pm \gamma^7)\ .
\end{aligned} \tag{4.3.10}$$

We will use the representation for the γ^A given in the Appendix (A.16) when we need to make these generators explicit. For the $(\mathbf{2},\mathbf{1};\mathbf{2},\mathbf{1})$ spinors we define the components $\bar{w},\bar{x},\bar{y},\bar{z}$ according to

$$\tilde{b}^\dagger = \bar{w}\begin{pmatrix}1\\0\\0\\1\\0\\0\\0\\0\end{pmatrix} + \bar{x}\begin{pmatrix}0\\1\\-1\\0\\0\\0\\0\\0\end{pmatrix} + \bar{y}\begin{pmatrix}0\\0\\0\\0\\1\\0\\0\\-1\end{pmatrix} + \bar{z}\begin{pmatrix}0\\0\\0\\0\\0\\1\\1\\0\end{pmatrix}\ , \tag{4.3.11}$$

and rearrange them into two-component complex spinors:

$$\xi = \begin{pmatrix} \bar{z} + i\bar{x} \\ \bar{w} + i\bar{y} \end{pmatrix} \qquad \eta = \begin{pmatrix} \bar{w} - i\bar{y} \\ -\bar{z} + i\bar{x} \end{pmatrix} \quad \Leftarrow \quad \Sigma_i^+$$
$$\bar{\xi} = \begin{pmatrix} -\bar{z} + i\bar{x} \\ \bar{w} + i\bar{y} \end{pmatrix} \qquad \bar{\eta} = \begin{pmatrix} \bar{w} - i\bar{y} \\ \bar{z} + i\bar{x} \end{pmatrix} \quad \Leftarrow \quad \Omega_i^+. \tag{4.3.12}$$

The corresponding explicit $(\mathbf{2},\mathbf{1};\mathbf{2},\mathbf{1})$ generators are given by

$$\Sigma_1^+ = \begin{pmatrix} 0 & -1/2 \\ -1/2 & 0 \end{pmatrix} \qquad \Omega_1^+ = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$$
$$\Sigma_2^+ = \begin{pmatrix} 0 & i/2 \\ -i/2 & 0 \end{pmatrix} \qquad \Omega_2^+ = \begin{pmatrix} 0 & -i/2 \\ i/2 & 0 \end{pmatrix}$$
$$\Sigma_3^+ = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} \qquad \Omega_3^+ = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}. \tag{4.3.13}$$

One may similarly decompose the $(\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{2})$ spinors. These observations will make it possible to construct linear combinations of products of components of ψ^α transforming in chosen irreps of $SO(4) \times SO(4)$.

To project out the $(\mathbf{2},\mathbf{1};\mathbf{2},\mathbf{1})$ block of $\mathcal{H}_{\rm FF}$, we simply act on all indices of (4.3.4) with the Π_+ projection operator:

$$\begin{aligned}\left\langle J \middle| \tilde{b}_n^\alpha \tilde{b}_{-n}^\beta \left(\mathcal{H}_{\rm FF}\right) \tilde{b}_{-n}^{\gamma\dagger} \tilde{b}_n^{\delta\dagger} \middle| J \right\rangle \;=\; & -2\frac{n^2\lambda'}{J}\Pi_+^{\alpha\delta}\Pi_+^{\beta\gamma} + \frac{n^2\lambda'}{24J}\Big\{\Big[(\Pi_+\gamma^{ij}\Pi_+)^{\alpha\delta}(\Pi_+\gamma^{ij}\Pi_+)^{\beta\gamma} \\ & +(\Pi_+\gamma^{ij}\Pi_+)^{\alpha\beta}(\Pi_+\gamma^{ij}\Pi_+)^{\gamma\delta} - (\Pi_+\gamma^{ij}\Pi_+)^{\alpha\gamma}(\Pi_+\gamma^{ij}\Pi_+)^{\beta\delta}\Big] \\ & -\Big[(\Pi_+\gamma^{i'j'}\Pi_+)^{\alpha\delta}(\Pi_+\gamma^{i'j'}\Pi_+)^{\beta\gamma} + (\Pi_+\gamma^{i'j'}\Pi_+)^{\alpha\beta}(\Pi_+\gamma^{i'j'}\Pi_+)^{\gamma\delta} \\ & -(\Pi_+\gamma^{i'j'}\Pi_+)^{\alpha\gamma}(\Pi_+\gamma^{i'j'}\Pi_+)^{\beta\delta}\Big]\Big\}. \end{aligned} \tag{4.3.14}$$

The $SO(4)\times SO(4)$ representation content of this subspace is specified by $(\mathbf{2},\mathbf{1};\mathbf{2},\mathbf{1})\times(\mathbf{2},\mathbf{1};\mathbf{2},\mathbf{1}) = (\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})\oplus(\mathbf{1},\mathbf{1};\mathbf{3},\mathbf{1})\oplus(\mathbf{3},\mathbf{1};\mathbf{1},\mathbf{1})\oplus(\mathbf{3},\mathbf{1};\mathbf{3},\mathbf{1})$ and we must further project onto individual irreducible representations in order to identify the eigenvalues.

With the tools we have built up in the last few paragraphs, we are in a position to directly project out some of the desired irreducible representations. Bi-fermions such as $\tilde{b}^\dagger\tilde{b}^\dagger\,|J\rangle$ transforming as scalars under the first $SO(4)$ (i.e. under Σ_i^+) are constructed by making $SU(2)$ invariants out of the two-component spinors ξ and η. There are four such objects:

$$\begin{array}{ll} \xi_{-n}\tau_2\xi_n & \xi_{-n}\tau_2\eta_n \\ \eta_{-n}\tau_2\xi_n & \eta_{-n}\tau_2\eta_n\ , \end{array} \tag{4.3.15}$$

where τ_2 is the second Pauli matrix. At the same time, they must also comprise a $\mathbf{3}$ and a $\mathbf{1}$ under the second $SO(4)$ (i.e. under Ω_i^+). To identify the irreducible linear combinations, one has to re-express the objects in (4.3.15) in terms of the spinors $\bar{\xi}$ and $\bar{\eta}$ that transform simply under Ω_i^+. We find that

$$\begin{array}{ll} -\frac{1}{2}\left(\xi_{-n}\tau_2\eta_n - \eta_{-n}\tau_2\xi_n\right) & (\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1}) \quad \Lambda_{\rm FF} = -2 \\ \left.\begin{array}{l} \frac{1}{2}\left(\xi_{-n}\tau_2\eta_n + \eta_{-n}\tau_2\xi_n\right) \\ \frac{i}{2}\left(\xi_{-n}\tau_2\xi_n + \eta_{-n}\tau_2\eta_n\right) \\ -\frac{1}{2}\left(\xi_{-n}\tau_2\xi_n - \eta_{-n}\tau_2\eta_n\right) \end{array}\right\} & (\mathbf{1},\mathbf{1};\mathbf{3},\mathbf{1}) \quad \Lambda_{\rm FF} = 0\ . \end{array} \tag{4.3.16}$$

We simply have to re-express the ξ,η bilinears in terms of the original spinor creation operators $\tilde{b}$ in order to obtain an explicit projection of the matrix elements (4.3.14) onto irreducible subspaces and to obtain the eigenvalues Λ_{FF} associated with each irrep. A parallel analysis of states constructed by forming normalized $SU(2)$ invariants

from $\bar{\xi}$ and $\bar{\eta}$ gives another irrep and eigenvalue:

$$\left.\begin{array}{l} \frac{1}{2}\left(\bar{\xi}_{-n}\tau_2\bar{\eta}_n + \bar{\eta}_{-n}\tau_2\bar{\xi}_n\right) \\ \frac{i}{2}\left(\bar{\xi}_{-n}\tau_2\bar{\xi}_n + \bar{\eta}_{-n}\tau_2\bar{\eta}_n\right) \\ -\frac{1}{2}\left(\bar{\xi}_{-n}\tau_2\bar{\xi}_n - \bar{\eta}_{-n}\tau_2\bar{\eta}_n\right) \end{array}\right\} (\mathbf{3},\mathbf{1};\mathbf{1},\mathbf{1}) \qquad \Lambda_{\rm FF} = -4\ . \tag{4.3.17}$$

By similar arguments, whose details we will omit, one can construct the creation operator for the normalized $(\mathbf{3},\mathbf{1};\mathbf{3},\mathbf{1})$ and find the eigenvalue $\Lambda_{FF} = -2$.

An exactly parallel analysis of $\left\langle J\left|\hat{b}\hat{b}(\mathcal{H}_{\rm FF})\hat{b}^\dagger\hat{b}^\dagger\right|J\right\rangle$ on the 16-dimensional subspace spanned by $(\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{2})$ bi-fermions yields the same eigenvalue spectrum.

The overall results for this sector are displayed in Table 4.3.

$SO(4)_{AdS}\times SO(4)_{S^5}$	$\Lambda_{\rm FF}$
$(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})$	-2
$(\mathbf{1},\mathbf{1};\mathbf{3},\mathbf{1})$	0
$(\mathbf{3},\mathbf{1};\mathbf{1},\mathbf{1})$	-4
$(\mathbf{3},\mathbf{1};\mathbf{3},\mathbf{1})$	-2

$SO(4)_{AdS}\times SO(4)_{S^5}$	$\Lambda_{\rm FF}$
$(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})$	-2
$(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{3})$	0
$(\mathbf{1},\mathbf{3};\mathbf{1},\mathbf{1})$	-4
$(\mathbf{1},\mathbf{3};\mathbf{1},\mathbf{3})$	-2

Table 4.3: Energy shifts of states created by two fermions in (**2**,**1**;**2**,**1**) or (**1**,**2**;**1**,**2**)

To this point, we have been able to study specific projections of the $\mathcal{H}_{\rm BB}$ and $\mathcal{H}_{\rm FF}$ subsectors by choosing states that are not mixed by $\mathcal{H}_{\rm BF}$.

We now must deal with the subspace of spacetime boson two-impurity states that is not annihilated by $\mathcal{H}_{\rm BF}$. This 64-dimensional space is spanned by pairs of bosonic creation operators taken from different $SO(4)$ subgroups and pairs of fermionic creation operators of opposite Π-parity. The representation content of these creation-operator pairs is such that the states in this sector all belong to $(\mathbf{2},\mathbf{2};\mathbf{2},\mathbf{2})$ irreps. This space is of course also acted on by $\mathcal{H}_{\rm BB}$ and $\mathcal{H}_{\rm FF}$, so we will need the matrix elements of all three pieces of the Hamiltonian as they act on this subspace. By applying the appropriate projections to the general one-loop matrix elements, we obtain

the expressions

$$\left\langle J\left|a_n^A a_{-n}^B\left(\mathcal{H}_{\rm BB}\right)a_{-n}^{C\dagger}a_n^{D\dagger}\right|J\right\rangle \to -2\frac{n^2\lambda'}{J}\left(\delta^{ad'}\delta^{b'c}+\delta^{a'd}\delta^{bc'}+\delta^{ad}\delta^{b'c'}+\delta^{a'd'}\delta^{bc}\right)\ , \tag{4.3.18}$$

$$\left\langle J\left|b_n^\alpha b_{-n}^\beta\left(\mathcal{H}_{\rm BF}\right)a_{-n}^{A\dagger}a_n^{B\dagger}\right|J\right\rangle \ \to\ \frac{n^2\lambda'}{2J}\Big[\left(\Pi_+\gamma^{ab'}\Pi_-\right)^{\alpha\beta}-\left(\Pi_+\gamma^{a'b}\Pi_-\right)^{\alpha\beta}+\left(\Pi_-\gamma^{ab'}\Pi_+\right)^{\alpha\beta} -\left(\Pi_-\gamma^{a'b}\Pi_+\right)^{\alpha\beta}\Big]\ , \tag{4.3.19}$$

$$\begin{aligned}\left\langle J\left|b_n^\alpha b_{-n}^\beta\left(\mathcal{H}_{\rm FF}\right)b_{-n}^{\gamma\dagger}b_n^{\delta\dagger}\right|J\right\rangle \to\ & -2\frac{n^2\lambda'}{J}\left(\Pi_+^{\alpha\delta}\Pi_-^{\beta\gamma}+\Pi_-^{\alpha\delta}\Pi_+^{\beta\gamma}\right)\\ &+\frac{n^2\lambda'}{24J}\Big\{\Big[(\Pi_+\gamma^{ij}\Pi_+)^{\alpha\delta}(\Pi_-\gamma^{ij}\Pi_-)^{\beta\gamma}+(\Pi_+\gamma^{ij}\Pi_-)^{\alpha\beta}(\Pi_-\gamma^{ij}\Pi_+)^{\gamma\delta}\\ &-(\Pi_+\gamma^{ij}\Pi_-)^{\alpha\gamma}(\Pi_-\gamma^{ij}\Pi_+)^{\beta\delta}\Big]-\Big[(\Pi_+\gamma^{i'j'}\Pi_+)^{\alpha\delta}(\Pi_-\gamma^{i'j'}\Pi_-)^{\beta\gamma}\\ &+(\Pi_+\gamma^{i'j'}\Pi_-)^{\alpha\beta}(\Pi_-\gamma^{i'j'}\Pi_+)^{\gamma\delta}-(\Pi_+\gamma^{i'j'}\Pi_-)^{\alpha\gamma}(\Pi_-\gamma^{i'j'}\Pi_+)^{\beta\delta}\Big]\\ &+\Big[(\Pi_-\gamma^{ij}\Pi_-)^{\alpha\delta}(\Pi_+\gamma^{ij}\Pi_+)^{\beta\gamma}+(\Pi_-\gamma^{ij}\Pi_+)^{\alpha\beta}(\Pi_+\gamma^{ij}\Pi_-)^{\gamma\delta}\\ &-(\Pi_-\gamma^{ij}\Pi_+)^{\alpha\gamma}(\Pi_+\gamma^{ij}\Pi_-)^{\beta\delta}\Big]-\Big[(\Pi_-\gamma^{i'j'}\Pi_-)^{\alpha\delta}(\Pi_+\gamma^{i'j'}\Pi_+)^{\beta\gamma}\\ &+(\Pi_-\gamma^{i'j'}\Pi_+)^{\alpha\beta}(\Pi_+\gamma^{i'j'}\Pi_-)^{\gamma\delta}-(\Pi_-\gamma^{i'j'}\Pi_+)^{\alpha\gamma}(\Pi_+\gamma^{i'j'}\Pi_-)^{\beta\delta}\Big]\Big\}\ .\end{aligned} \tag{4.3.20}$$

Since the 64-dimensional space must contain four copies of the $(\mathbf{2},\mathbf{2};\mathbf{2},\mathbf{2})$ irrep, the diagonalization problem is really only 4×4 and quite easy to solve. The results for the eigenvalues appear in Table 4.4. Collecting the above results, we present the

$SO(4)_{AdS}\times SO(4)_{S^5}$	$\Lambda_{\rm BF}$
$(\mathbf{2},\mathbf{2};\mathbf{2},\mathbf{2})$	-4
$(\mathbf{2},\mathbf{2};\mathbf{2},\mathbf{2})\times 2$	-2
$(\mathbf{2},\mathbf{2};\mathbf{2},\mathbf{2})$	0

Table 4.4: String eigenstates in the subspace for which $\mathcal{H}_{\rm BF}$ has non-zero matrix elements

complete $SO(4)_{AdS}\times SO(4)_{S^5}$ decomposition of spacetime boson two-impurity states

in Table 4.5.

	$SO(4)_{AdS} \times SO(4)_{S^5}$	Λ
$\mathcal{H}_{\rm BB}$	$(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})$	-6
	$(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})$	2
	$(\mathbf{1},\mathbf{1};\mathbf{3},\mathbf{1}) + (\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{3})$	-4
	$(\mathbf{3},\mathbf{1};\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{3};\mathbf{1},\mathbf{1})$	0
	$(\mathbf{1},\mathbf{1};\mathbf{3},\mathbf{3})$	-2
	$(\mathbf{3},\mathbf{3};\mathbf{1},\mathbf{1})$	-2

	$SO(4)_{AdS} \times SO(4)_{S^5}$	Λ
$\mathcal{H}_{\rm FF}$	$(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})$	-2
	$(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})$	-2
	$(\mathbf{1},\mathbf{1};\mathbf{3},\mathbf{1}) + (\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{3})$	0
	$(\mathbf{3},\mathbf{1};\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{3};\mathbf{1},\mathbf{1})$	-4
	$(\mathbf{3},\mathbf{1};\mathbf{3},\mathbf{1}) + (\mathbf{1},\mathbf{3};\mathbf{1},\mathbf{3})$	-2
$\mathcal{H}_{\rm BF}$	$(\mathbf{2},\mathbf{2};\mathbf{2},\mathbf{2})$	0
	$(\mathbf{2},\mathbf{2};\mathbf{2},\mathbf{2}) \times 2$	-2
	$(\mathbf{2},\mathbf{2};\mathbf{2},\mathbf{2})$	-4

Table 4.5: Group decomposition of the 128 two-impurity spacetime bosons

By projecting out closed subspaces of the one-loop Hamiltonian we have successfully classified each of the energy levels in the bosonic Fock space with an $SO(4) \times SO(4)$ symmetry label. Similar arguments can be applied to the fermionic Fock space, where two-impurity string states have one bosonic and one fermionic oscillator. A summary of these results for all states, including spacetime fermions, is given in Table 4.6. The important fact to note is that the Λ eigenvalues and their multiplicities are exactly as required for consistency with the full $PSU(2,2|4)$ symmetry of the theory. This is a non-trivial result since the quantization procedure does not make the full symmetry manifest. It is also a very satisfying check of the overall correctness of the extremely complicated set of procedures we were forced to use. Finally it is straightforward to compare these string energies with the anomalous dimensions found in the previous chapter (see equation 3.2.24 and Table 3.4).

Level	0	2	4	6	8
Mult.	1	28	70	28	1
$\Lambda_{\rm Bose}$	-6	-4	-2	0	2

Level	1	3	5	7
Mult.	8	56	56	8
$\Lambda_{\rm Fermi}$	-5	-3	-1	1

Table 4.6: First-order energy shift summary: complete two-impurity string multiplet

4.4 Energy spectrum at all loops in λ'

To make comparisons with gauge theory dimensions at one loop in $\lambda = g_{YM}^2 N_c$, we expanded all string energies in powers of the modified 't Hooft coupling $\lambda' = g_{YM}^2 N_c/\mathcal{R}^2$. The string theory analysis is exact to all orders in λ' and it is possible to extract a formula for the $\mathcal{O}(1/J)$ string energy corrections which is exact in λ'. In practice, it is slightly more difficult to diagonalize the string Hamiltonian when the matrix elements are not expanded in small λ'. This is mainly because, beyond leading order, $\mathcal{H}_{\rm BF}$ acquires additional terms that mix bosonic indices in the same $SO(4)$ and also mix bi-fermionic indices in the same $(\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{2})$ or $(\mathbf{2},\mathbf{1};\mathbf{2},\mathbf{1})$ representation. Instead of a direct diagonalization of the entire 128-dimensional subspace of spacetime bosons, for example, we find it more convenient to exploit the 'dimension reduction' that can be achieved by projecting the full Hamiltonian onto individual irreps.

For example, the $(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})$ irrep appears four times in Table 4.5 and is present at levels $L_v = 0, 4, 8$ in the supermultiplet. To get the exact eigenvalues for this irrep, we will have to diagonalize a 4×4 matrix. The basis vectors of this bosonic sector comprise singlets of the two $SO(4)$ subgroups ($a^{\dagger a}a^{\dagger a}\,|J\rangle$ and $a^{\dagger a'}a^{\dagger a'}\,|J\rangle$) plus two bi-fermion singlets constructed from the $(\mathbf{2},\mathbf{1};\mathbf{2},\mathbf{1})$ and $(\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{2})$ creation operators ($\hat{b}^{\dagger\alpha}\hat{b}^{\dagger\alpha}\,|J\rangle$ and $\tilde{b}^{\dagger\alpha}\tilde{b}^{\dagger\alpha}\,|J\rangle$). It is a simple matter to project the general expressions for matrix elements of $\mathcal{H}_{\rm BB}$, etc., onto singlet states and so obtain the matrix as an explicit function of λ', n. The matrix can be exactly diagonalized and yields the following energies:

$$\begin{aligned}
E_0(n,J) &= 2\sqrt{1+\lambda' n^2} - \frac{n^2\lambda'}{J}\left[2 + \frac{4}{\sqrt{1+n^2\lambda'}}\right] + \mathcal{O}(1/J^2) \\
E_4(n,J) &= 2\sqrt{1+\lambda' n^2} - \frac{2n^2\lambda'}{J} + \mathcal{O}(1/J^2) \\
E_8(n,J) &= 2\sqrt{1+\lambda' n^2} - \frac{n^2\lambda'}{J}\left[2 - \frac{4}{\sqrt{1+n^2\lambda'}}\right] + \mathcal{O}(1/J^2)\ . \qquad (4.4.1)
\end{aligned}$$

The middle eigenvalue (L_4=4) is doubly degenerate, as it was in the one-loop limit.

There are two independent 2×2 matrices that mix states at levels $L = 2, 6$. According to Table 4.5, one can project out the antisymmetric bosonic and antisymmetric bi-fermionic states in the irrep $(\mathbf{1},\mathbf{1};\mathbf{3},\mathbf{1}) + (\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{3})$ or in the irrep $(\mathbf{3},\mathbf{1};\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{3};\mathbf{1},\mathbf{1})$. Our previous results can be used to carry out the needed projections and obtain explicit forms for the matrix elements of the perturbing Hamiltonian. The actual 2×2 diagonalization is trivial to do and both problems give the same result. The final result for the energy levels (using the same notation as before) is

$$\begin{aligned} E_2(n,J) &= 2\sqrt{1+\lambda' n^2} - \frac{n^2\lambda'}{J}\left[2 + \frac{2}{\sqrt{1+n^2\lambda'}}\right] + \mathcal{O}(1/J^2) \\ E_6(n,J) &= 2\sqrt{1+\lambda' n^2} - \frac{n^2\lambda'}{J}\left[2 - \frac{2}{\sqrt{1+n^2\lambda'}}\right] + \mathcal{O}(1/J^2)\ . \end{aligned} \tag{4.4.2}$$

We can carry out similar diagonalizations for the remaining irreps of Table 4.5, but no new eigenvalues are encountered: the energies already listed are the exact energies of the $L_v = 0, 2, 4, 6, 8$ levels. It is also easy to see that the degeneracy structure of the exact levels is the same as the one-loop degeneracy.

The odd levels of the supermultiplet are populated by the 128-dimensional spacetime fermions, and this sector of the theory can be diagonalized directly. Proceeding in a similar fashion as in the bosonic sector, we find exact energy eigenvalues for the $L_v = 1, 3, 5, 7$ levels (with unchanged multiplicities). The entire supermultiplet spectrum, bosonic and fermionic, can be written in terms of a single concise formula: to leading order in $1/J$ and all orders in λ', the energies of the two-impurity multiplet are given by

$$E_L(n,J) = 2\sqrt{1+\lambda' n^2} - \frac{n^2\lambda'}{J}\left[2 + \frac{(4-L_v)}{\sqrt{1+n^2\lambda'}}\right] + O(1/J^2)\ . \tag{4.4.3}$$

The degeneracies and irrep content are identical to what we found at one loop in λ'.

This expression can be rewritten, correct to order J^{-2}, as follows:

$$E_L(n,J) \simeq 2\sqrt{1+\frac{\lambda n^2}{(J-L_v/2)^2}} - \frac{n^2\lambda}{(J-L_v/2)^3}\left[2+\frac{4}{\sqrt{1+\lambda n^2/(J-L_v/2)^2}}\right]. \quad (4.4.4)$$

This shows that, within this expansion, the joint dependence on J and L_v is exactly what is required for extended supersymmetry multiplets. This is a rather nontrivial functional requirement, and a stringent check on the correctness of our quantization procedure. In order to make contact with gauge theory we expand (4.4.3) in λ', obtaining

$$\begin{aligned} E_L(n,J) \;&\approx\; \left[2+\lambda' n^2 - \frac{1}{4}(\lambda' n^2)^2 + \frac{1}{8}(\lambda' n^2)^3 + \dots\right] \\ &\quad + \frac{1}{J}\left[n^2\lambda'(L_v-6) + (n^2\lambda')^2\left(\frac{4-L_v}{2}\right) + (n^2\lambda')^3\left(\frac{3L_v-12}{8}\right) + \dots\right]. \quad (4.4.5) \end{aligned}$$

Finally we note that in our arguments using the multiplet structure we relied upon the gauge theory results of Beisert. We have seen that the string spectrum reflects the same group structure. However we would like to directly see how the supersymmetric Noether charges act on the string states and how they generate the multiplet structure that we found. As we are only interested the spectrum to leading order in perturbation theory we only need the states to zeroth order in $1/J$ i.e. we only need the charges in the Penrose limit and so can use the formulae given in [41] (rewritten slightly to agree with our notation). In [41] two sets of supersymmetric charges were found Q^+ and Q^-, it is apparent upon inspection that the set Q^+ change the number of oscillators and we are interested in the charges which leave the impurity number unchanged. The Q^- are given by

$$Q^- = \int d\sigma\; 2(p^I\bar{\gamma}^I\theta - x'^I\bar{\gamma}^I\bar{\theta} - ip_-x^I\bar{\gamma}^I\Pi\theta), \quad (4.4.6)$$

which we can expand in terms of creation and annihilation operators

$$\begin{aligned} Q^- &= \sum_n \frac{\bar{\gamma}^I}{\omega_n\sqrt{p_-}}\left[\left(\frac{(\omega_n-n)^2+p_-^2}{\sqrt{\omega_n-n}}\right)+\left(\frac{(\omega_n+n)^2+p_-^2}{\sqrt{\omega_n+n}}\right)\Pi\right]a_n^I b_n^\dagger \\ &\quad +\frac{\bar{\gamma}^I}{\omega_n\sqrt{p_-}}\left[\left(\frac{(\omega_n+n)^2+p_-^2}{\sqrt{\omega_n+n}}\right)+\left(\frac{(\omega_n-n)^2+p_-^2}{\sqrt{\omega_n-n}}\right)\Pi\right]a^{\dagger I}_{-n}b_{-n}, \quad (4.4.7) \end{aligned}$$

and we find a similar expression for $\bar{Q}^-$. We would naturally like to calculate the commutator of these operators with the interaction Hamiltonian and show that we can generate the energy spectrum from a primary state however this would be extremely tedious and we settle for a relatively minor example of how this would work. It is helpful to express p_- in terms of λ' and then expand to leading order as the expressions take a particularly simple form

$$Q^- = \sum_n \bar{\gamma}^I(a_n^I \hat{b_n}^\dagger + a_n^{\dagger I}\tilde{b_n}), \quad (4.4.8)$$

where we have removed an overall constant, and

$$\bar{Q}^- = \sum_n \bar{\gamma}^I(a_n^I \tilde{b_n}^\dagger + a_n^{\dagger I}\hat{b_n}) \ . \quad (4.4.9)$$

At leading order in λ' we know that the two-impurity primary state is $a_n^{\dagger\, i'}a^{\dagger i'}_{-n}\,|J\rangle$, we can act on this state with the above charges to find that the 8 states at the next level in the supermultiplet are

$$\frac{1}{\sqrt{8}}(\bar{\gamma}^{i'}b_n^\dagger a^{\dagger i'}_{-n} + \bar{\gamma}^{i'}b^\dagger_{-n}a_n^{\dagger\, i'})\,|J\rangle \ . \quad (4.4.10)$$

If our interpretation is correct these states should be at level one in the multiplet and have $\Lambda = -5$ and indeed it is straightforward to check that this is so. We can repeat this for the higher levels in the multiplet and we find consistent results.

Chapter 5

Higher Impurity States

5.1 Three-impurity string spectrum: leading order in λ'

The three-impurity Fock space block-diagonalizes into separate spacetime fermion and spacetime boson sectors. The bosonic sector contains states that are purely bosonic (composed of three bosonic string oscillators) and states with bi-fermionic components:

$$a_q^{A\dagger} a_r^{B\dagger} a_s^{C\dagger} \left|J\right\rangle \qquad a_q^{A\dagger} b_r^{\alpha\dagger} b_s^{\beta\dagger} \left|J\right\rangle \ . \tag{5.1.1}$$

Pure boson states are mixed by the bosonic part of the Hamiltonian $H_{\rm BB}$, while states with bi-fermionic excitations are mixed both by the purely fermionic Hamiltonian $H_{\rm FF}$ and the bose-fermi part $H_{\rm BF}$. The sector of spacetime fermion states is composed of purely fermionic excitations and mixed states containing two bosonic oscillators:

$$b_q^{\alpha\dagger} b_r^{\beta\dagger} b_s^{\gamma\dagger} \left|J\right\rangle \qquad a_q^{A\dagger} a_r^{B\dagger} b_s^{\alpha\dagger} \left|J\right\rangle \ . \tag{5.1.2}$$

Pure fermion states are mixed by $H_{\rm FF}$, and states with two bosonic excitations and a single fermionic excitation are mixed by $H_{\rm BB}$ and $H_{\rm BF}$. This block diagonalization of the perturbing Hamiltonian is displayed schematically in Table 5.1.

$(H)_{int}$	$a^{A\dagger}a^{B\dagger}a^{C\dagger}\left\|J\right\rangle$	$a^{A\dagger}b^{\alpha\dagger}b^{\beta\dagger}\left\|J\right\rangle$	$b^{\alpha\dagger}b^{\beta\dagger}b^{\gamma\dagger}\left\|J\right\rangle$	$a^{A\dagger}a^{B\dagger}b^{\alpha\dagger}\left\|J\right\rangle$
$\left\langle J\right\| a^A a^B a^C$	H_{BB}	H_{BF}	0	0
$\left\langle J\right\| a^A b^\alpha b^\beta$	H_{BF}	$H_{FF}+H_{BF}$	0	0
$\left\langle J\right\| b^\alpha b^\beta b^\gamma$	0	0	H_{FF}	H_{BF}
$\left\langle J\right\| a^A a^B b^\alpha$	0	0	H_{BF}	$H_{BB}+H_{BF}$

Table 5.1: Three-impurity string states

The three-impurity string states are subject to the usual level-matching condition on the mode indices: $q+r+s=0$. There are two generically different solutions of this constraint: all mode indices different ($q\neq r\neq s$) and two indices equal (eg. $q=r=n,\ s=-2n$). In the inequivalent index case, there are $16^3=4,096$ degenerate states arising from different choices of spacetime labels on the mode creation operators. In the case of two equivalent indices, the dimension of the degenerate subspace is half as large (there are fewer permutations on mode indices that generate linearly independent states). The two types of basis break up into irreducible representations of $PSU(2,2|4)$ in different ways and must be studied separately.

As in the two-impurity case, the problem of diagonalizing the perturbation simplifies enormously when the matrix elements are expanded to leading order in λ'. We will take this approach initially to obtain an overview of how degeneracies are lifted by the interaction. The generalization of the results to all loop orders in λ' (but still to first non-leading order in $1/J$) will be presented in the next section. It is once again the case that in the one-loop approximation, projection onto invariant subspaces under the manifest global $SO(4)\times SO(4)$ symmetry often diagonalizes the Hamiltonian directly (and at worst reduces it to a low-dimensional matrix). However we still require symbolic manipulation programs to organize the complicated algebra and perform explicit projections onto invariant subspaces. Once again we will spare the reader many of the details but they can be found in [63].

5.1.1 Matrix evaluation: inequivalent mode indices $(q \neq r \neq s)$

In the sector of spacetime bosons, the subspace of purely bosonic states $a_q^{A\dagger} a_r^{B\dagger} a_s^{C\dagger} |J\rangle$ is 512-dimensional. When each of the three mode indices (q, r, s) are different, states with bi-fermionic excitations $a_q^{A\dagger} b_r^{\alpha\dagger} b_s^{\beta\dagger} |J\rangle$ are inequivalent under permutation of the mode indices, and form a 1,536-dimensional subsector. The entire bosonic sector of the three-impurity state space therefore contains 2,048 linearly independent states. The fermionic sector decomposes in a similar manner. The subsector of purely fermionic states $b_q^{\alpha\dagger} b_r^{\beta\dagger} b_s^{\gamma\dagger} |J\rangle$ is 512-dimensional; fermionic states containing two bosonic excitations $a_q^{A\dagger} a_r^{B\dagger} b_s^{\alpha\dagger} |J\rangle$ are inequivalent under permutation of the mode indices, and comprise an additional 1,536-dimensional subsector.

Our first task is to evaluate the interaction Hamiltonian matrix. The matrix elements needed to fill out the spacetime boson sector are listed in Table 5.2.

$H_{\rm int}$	$a_s^{D\dagger} a_r^{E\dagger} a_q^{F\dagger} \|J\rangle$	$a_s^{D\dagger} b_r^{\gamma\dagger} b_q^{\delta\dagger} \|J\rangle$	$a_r^{D\dagger} b_q^{\gamma\dagger} b_s^{\delta\dagger} \|J\rangle$	$a_r^{D\dagger} b_s^{\gamma\dagger} b_q^{\delta\dagger} \|J\rangle$
$\langle J\| a_q^A a_r^B a_s^C$	$H_{\rm BB}$	$H_{\rm BF}$	$H_{\rm BF}$	$H_{\rm BF}$
$\langle J\| a_q^A b_r^\alpha b_s^\beta$	$H_{\rm BF}$	$H_{\rm FF} + H_{\rm BF}$	$H_{\rm BF}$	$H_{\rm BF}$
$\langle J\| a_s^A b_q^\alpha b_r^\beta$	$H_{\rm BF}$	$H_{\rm BF}$	$H_{\rm FF} + H_{\rm BF}$	$H_{\rm BF}$
$\langle J\| a_r^A b_s^\alpha b_q^\beta$	$H_{\rm BF}$	$H_{\rm BF}$	$H_{\rm BF}$	$H_{\rm FF} + H_{\rm BF}$

Table 5.2: Interaction Hamiltonian on spacetime boson three-impurity string states $(q \neq r \neq s)$

The purely bosonic, 512-dimensional block is

$$\begin{aligned}
\Big\langle J\Big| a_q^A a_r^B a_s^C (H_{\rm BB}) a_s^{D\dagger} a_r^{E\dagger} a_q^{F\dagger}\Big| J\Big\rangle \;&=\; \frac{\lambda'}{J}\delta^{AF}\delta^{BE}\delta^{CD}\left(rs + q(r+s) - q^2 - r^2 - s^2\right) \\
&+\frac{\lambda'}{2J}\Bigg\{\delta^{AF}\Big[(r^2+s^2)\big(\delta^{cd}\delta^{be} - \delta^{c'd'}\delta^{b'e'}\big) + (s^2-r^2)\big(\delta^{be}\delta^{c'd'} - \delta^{cd}\delta^{b'e'}\big) \\
&\qquad +2rs\big(\delta^{bd}\delta^{ce} - \delta^{bc}\delta^{de} - \delta^{b'd'}\delta^{c'e'} + \delta^{b'c'}\delta^{d'e'}\big)\Big] + \big(r \rightleftharpoons q,\ F \rightleftharpoons E,\ A \rightleftharpoons B\big) \\
&\qquad +\big(s \rightleftharpoons q,\ F \rightleftharpoons D,\ A \rightleftharpoons C\big)\Bigg\}\,. \qquad (5.1.3)
\end{aligned}$$

where the index a (a') etc. symbolizes the value of the vector index A, provided it is in the first (second) $SO(4)$. The off-diagonal entries that mix purely bosonic states $a_q^{A\dagger}a_r^{B\dagger}a_s^{C\dagger}\,|J\rangle$ with states containing bi-fermions $a_q^{A\dagger}b_r^{\alpha\dagger}b_s^{\beta\dagger}\,|J\rangle$ are given by a separate set of 512-dimensional matrices. For example

$$\left\langle J\middle|a_q^A a_r^B a_s^C (H_{\rm BF}) a_s^{D\dagger} b_r^{\alpha\dagger} b_q^{\beta\dagger}\middle|J\right\rangle \;=\; \frac{\lambda'}{2J}\delta^{CD} qr\left\{\left(\gamma^{ab'}\right)^{\alpha\beta} - \left(\gamma^{a'b}\right)^{\alpha\beta}\right\}\,, \quad (5.1.4)$$

and there are six similar blocks, each given by a simple permutation of the mode indices (q,r,s) in equation (5.1.4).

The pure-fermion part of the Hamiltonian, $H_{\rm FF}$, has non-vanishing matrix elements between states containing bi-fermionic excitations. For example,

$$\begin{aligned}\left\langle J\middle|b_q^\alpha b_r^\beta a_s^A (H_{\rm FF}) a_s^{B\dagger} b_r^{\gamma\dagger} b_q^{\delta\dagger}\middle|J\right\rangle \;=\; & -\frac{\lambda'}{2J}(q-r)^2\delta^{AB}\delta^{\alpha\delta}\delta^{\gamma\beta} \\ & +\frac{\lambda'}{24J}\delta^{AB} qr\Big\{\left(\gamma^{ij}\right)^{\alpha\gamma}\,\left(\gamma^{ij}\right)^{\beta\delta} - \left(\gamma^{ij}\right)^{\alpha\beta}\left(\gamma^{ij}\right)^{\gamma\delta} - \left(\gamma^{ij}\right)^{\alpha\delta}\left(\gamma^{ij}\right)^{\beta\gamma} \\ & -\left(\gamma^{i'j'}\right)^{\alpha\gamma}\left(\gamma^{i'j'}\right)^{\beta\delta} + \left(\gamma^{i'j'}\right)^{\alpha\beta}\left(\gamma^{i'j'}\right)^{\gamma\delta} + \left(\gamma^{i'j'}\right)^{\alpha\delta}\left(\gamma^{i'j'}\right)^{\beta\gamma}\Big\}\,, \quad (5.1.5)\end{aligned}$$

and there are other similar contributions related to this one by simple permutations of the mode indices (q,r,s).

The bose-fermi mixing Hamiltonian $H_{\rm BF}$ has diagonal matrix elements

$$\begin{aligned}\left\langle J\middle|b_q^\alpha b_r^\beta a_s^A (H_{\rm BF}) a_s^{B\dagger} b_r^{\gamma\dagger} b_q^{\delta\dagger}\middle|J\right\rangle \;=\; & \frac{\lambda'}{2J}\Big\{2s(q+r-s)\delta^{ab}\delta^{\alpha\delta}\delta^{\beta\gamma} - rs\Big[\left(\gamma^{ab}\right)^{\beta\gamma} - \left(\gamma^{a'b'}\right)^{\beta\gamma}\Big] \\ & -sq\Big[\left(\gamma^{ab}\right)^{\alpha\delta} - \left(\,\gamma^{a'b'}\right)^{\alpha\delta}\Big] - 2\Big[q^2+r^2+s^2-s(q+r)\Big]\delta^{a'b'}\delta^{\alpha\delta}\delta^{\beta\gamma}\Big\}, \quad (5.1.6)\end{aligned}$$

and off-diagonal elements

$$\left\langle J\middle|b_q^\alpha b_r^\beta a_s^A (H_{\rm BF}) a_r^{B\dagger} b_q^{\gamma\dagger} b_s^{\delta\dagger}\middle|J\right\rangle = -\frac{\lambda'}{2J}\delta^{\alpha\gamma} rs\Big\{\left(\delta^{ab}-\delta^{a'b'}\right)\delta^{\beta\delta} - \left(\gamma^{ab}\right)^{\beta\delta} + \left(\gamma^{a'b'}\right)^{\beta\delta}\Big\}. (5.1.7)$$

The remaining off-diagonal elements of $H_{\rm BF}$ are obtained by appropriate index permutations. We can find the matrix elements of the spacetime fermions in a similar fashion.

5.1.2 Matrix diagonalization: inequivalent mode indices ($q \neq r \neq s$)

We now turn to the task of diagonalizing the one-loop approximation to the perturbing Hamiltonian. To simplify the task, we again exploit the block diagonalizations that hold to leading order in λ' (but not to higher orders). As an example of the simplifications we have in mind, we infer from (5.1.4) that the matrix elements of $H_{\rm BF}$ between pure boson states $a_q^{A\dagger}a_r^{B\dagger}a_s^{C\dagger}\,|J\rangle$ and bi-fermionic spacetime bosons vanish to leading order in λ' if all three $SO(8)$ bosonic vector indices lie within the same $SO(4)$, descended either from AdS_5 or S^5. Restricting to such states brings the bosonic sector of the Hamiltonian into the block-diagonal form in Table 5.3. This leaves two

$H_{\rm int}$	$a^{a\dagger}a^{b\dagger}a^{c\dagger}\,\lvert J\rangle + a^{a'\dagger}a^{b'\dagger}a^{c'\dagger}\,\lvert J\rangle$	$a^{A\dagger}b^{\alpha\dagger}b^{\beta\dagger}\,\lvert J\rangle$
$\langle J\rvert\, a^a a^b a^c + \langle J\rvert\, a^{a'} a^{b'} a^{c'}$	$H_{\rm BB}$	0
$\langle J\rvert\, a^A b^\alpha b^\beta$	0	$H_{\rm FF} + H_{\rm BF}$

Table 5.3: Block-diagonal $SO(4)$ projection on bosonic three-impurity string states

64-dimensional subspaces of purely bosonic states on which the perturbation is block diagonal, as recorded in Table 5.4. As before since the interaction Hamiltonian has

$H_{\rm int}$	$a^{a\dagger}a^{b\dagger}a^{c\dagger}\,\lvert J\rangle$	$a^{a'\dagger}a^{b'\dagger}a^{c'\dagger}\,\lvert J\rangle$
$\langle J\rvert\, a^a a^b a^c$	$(H_{\rm BB})_{64\times 64}$	0
$\langle J\rvert\, a^{a'} a^{b'} a^{c'}$	0	$(H_{\rm BB})_{64\times 64}$

Table 5.4: SO(4) projection on purely bosonic states

manifest $SO(4)\times SO(4)$ symmetry, it is useful to project matrix elements onto irreps of that group before diagonalizing. A very important feature of the results which appear is that all the eigenvalues turn out to have a common simple dependence on mode indices. More precisely, the expansion of the eigenvalues for inequivalent mode

indices (q, r, s) at first order in λ' and $1/J$ can be written as

$$E_J(q, r, s) = 3 + \frac{\lambda'(q^2 + r^2 + s^2)}{2}\left(1 + \frac{\Lambda}{J} + O(J^{-2})\right) \ , \tag{5.1.8}$$

where Λ is a pure number that characterizes the lifting of the degeneracy in the various sectors. The notation $\Lambda_{\rm BB}$, $\Lambda_{\rm BF}$ and $\Lambda_{\rm FF}$ will be used to denote energy corrections arising entirely from the indicated sectors of the perturbing Hamiltonian.

In the $SO(4)$ projection in Table 5.4, we will find the set of 64 eigenvalues for both the $SO(4)_{AdS}$ and $SO(4)_{S^5}$ subsectors. We record this eigenvalue spectrum in Table 5.5, using an $SU(2)^2 \times SU(2)^2$ notation. For comparison, it is displayed alongside the bosonic 2-impurity spectrum found previously. In the three-impurity

$SO(4)_{AdS} \times SO(4)_{S^5}$	$\Lambda_{\rm BB}$
$(\mathbf{1},\mathbf{1};\mathbf{2},\mathbf{2})$	-8
$[\mathbf{1},\mathbf{1};(\mathbf{2}+\mathbf{4}),\mathbf{2}]+[\mathbf{1},\mathbf{1};\mathbf{2},(\mathbf{2}+\mathbf{4})]$	-6
$[\mathbf{1},\mathbf{1};(\mathbf{2}+\mathbf{4}),(\mathbf{2}+\mathbf{4})]$	-4
$[(\mathbf{2}+\mathbf{4}),(\mathbf{2}+\mathbf{4});\mathbf{1},\mathbf{1}]$	-2
$[(\mathbf{2}+\mathbf{4}),\mathbf{2};\mathbf{1},\mathbf{1}]+[\mathbf{2},(\mathbf{2}+\mathbf{4});\mathbf{1},\mathbf{1}]$	0
$(\mathbf{2},\mathbf{2};\mathbf{1},\mathbf{1})$	2

$SO(4)_{AdS} \times SO(4)_{S^5}$	$\Lambda_{\rm BB}$
$(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})$	-6
$(\mathbf{1},\mathbf{1};\mathbf{3},\mathbf{1})+(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{3})$	-4
$(\mathbf{1},\mathbf{1};\mathbf{3},\mathbf{3})$	-2
$(\mathbf{3},\mathbf{3};\mathbf{1},\mathbf{1})$	-2
$(\mathbf{3},\mathbf{1};\mathbf{1},\mathbf{1})+(\mathbf{1},\mathbf{3};\mathbf{1},\mathbf{1})$	0
$(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})$	2

Table 5.5: Three-impurity energy spectrum in the pure-boson $SO(4)$ projection (left panel) and two-impurity energy spectrum in the same projection (right panel)

case, the $(\mathbf{1},\mathbf{1};\mathbf{2},\mathbf{2})$ level in the $SO(4)_{S^5}$ subsector clearly descends from the two-impurity singlet $(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})$ in the same $SO(4)$ subgroup i.e., it arises from tensoring the singlet with an $SO(4)_{S^5}$ vector $(\mathbf{1},\mathbf{1};\mathbf{2},\mathbf{2})$. In the same manner, the three-impurity $[\mathbf{1},\mathbf{1};(\mathbf{2}+\mathbf{4}),\mathbf{2}]+[\mathbf{1},\mathbf{1};\mathbf{2},(\mathbf{2}+\mathbf{4})]$ level descends from the $SO(4)_{S^5}$ antisymmetric two-impurity state $(\mathbf{1},\mathbf{1};\mathbf{3},\mathbf{1})+(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{3})$, and the three-impurity $[\mathbf{1},\mathbf{1};(\mathbf{2}+\mathbf{4}),(\mathbf{2}+\mathbf{4})]$ level is from the two-impurity symmetric-traceless $(\mathbf{1},\mathbf{1};\mathbf{3},\mathbf{3})$ irrep. In the $SO(4)_{S^5}$ subsector, each of these levels receives a shift to the appropriate Λ of -2. The total multiplicity of each of these levels is also increased by a factor of four when the additional $(\mathbf{2},\mathbf{2})$ is tensored into the two-impurity state space. The $SO(4)_{AdS}$

subsector follows a similar pattern: the $(\mathbf{2},\mathbf{2};\mathbf{1},\mathbf{1})$, $[(\mathbf{2}+\mathbf{4}),\mathbf{2};\mathbf{1},\mathbf{1}]+[\mathbf{2},(\mathbf{2}+\mathbf{4});\mathbf{1},\mathbf{1}]$ and $[(\mathbf{2}+\mathbf{4}),(\mathbf{2}+\mathbf{4});\mathbf{1},\mathbf{1}]$ levels appear as three-impurity descendants of the two-impurity irrep spectrum $(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})+(\mathbf{3},\mathbf{1};\mathbf{1},\mathbf{1})+(\mathbf{1},\mathbf{3};\mathbf{1},\mathbf{1})+(\mathbf{3},\mathbf{3};\mathbf{1},\mathbf{1})$. In this subsector, however, the three-impurity Λ are identical to those in the two-impurity theory.

We can perform a projection on the purely fermionic subsector in Table 5.2 similar to that appearing in Table 5.4. In this case, instead of three bosonic impurities mixing with a single bosonic (plus a bi-fermionic) excitation, we are now interested in projecting out particular interactions between a purely fermionic state and a state with one fermionic and two bosonic excitations. We can do this because the off-diagonal elements between $\hat{b}^{\alpha\dagger}\hat{b}^{\beta\dagger}\hat{b}^{\gamma\dagger}\left|J\right\rangle$ and $a^{A\dagger}a^{B\dagger}b^{\alpha\dagger}\left|J\right\rangle$ vanish and the same is true for tilded states.

H_{int}	$\hat{b}^{\alpha\dagger}\hat{b}^{\beta\dagger}\hat{b}^{\gamma\dagger}\left\|J\right\rangle+\tilde{b}^{\alpha\dagger}\tilde{b}^{\beta\dagger}\tilde{b}^{\gamma\dagger}\left\|J\right\rangle$	$a^{A\dagger}a^{B\dagger}b^{\alpha\dagger}\left\|J\right\rangle$
$\left\langle J\right\| \hat{b}^{\alpha}\hat{b}^{\beta}\hat{b}^{\gamma}+\left\langle J\right\| \tilde{b}^{\alpha}\tilde{b}^{\beta}\tilde{b}^{\gamma}$	H_{FF}	0
$\left\langle J\right\| a^{A}a^{B}b^{\alpha}$	0	$H_{\text{BB}}+H_{\text{BF}}$

Table 5.6: Block-diagonal projection on fermionic three-impurity string states

Furthermore, the mixing matrix for pure fermions breaks into two 64-dimensional subblocks under this projection. By tensoring an additional $(\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{2})$ or $(\mathbf{2},\mathbf{1};\mathbf{2},\mathbf{1})$ impurity into the two-impurity states, we expect to see representations

$$\begin{aligned}
(\mathbf{1},\mathbf{2})\times(\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{2}) &= (\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{2})+[\mathbf{1},\mathbf{2};\mathbf{1},(\mathbf{2}+\mathbf{4})] \\
&\quad +[\mathbf{1},(\mathbf{2}+\mathbf{4});\mathbf{1},\mathbf{2}]+[\mathbf{1},(\mathbf{2}+\mathbf{4});\mathbf{1},(\mathbf{2}+\mathbf{4})]\ , \\
(\mathbf{2},\mathbf{1})\times(\mathbf{2},\mathbf{1};\mathbf{2},\mathbf{1}) &= (\mathbf{2},\mathbf{1};\mathbf{2},\mathbf{1})+[\mathbf{2},\mathbf{1};(\mathbf{2}+\mathbf{4}),\mathbf{1}] \\
&\quad +[(\mathbf{2}+\mathbf{4}),\mathbf{1};\mathbf{2},\mathbf{1}]+[(\mathbf{2}+\mathbf{4}),\mathbf{1};(\mathbf{2}+\mathbf{4}),\mathbf{1}]\ , \qquad (5.1.9)
\end{aligned}$$

for a total of 128 states. The projections onto the two 64-dimensional Π_+ and Π_- subspaces yield identical eigenvalues and multiplicities. The results for both subspaces

are presented in Table 5.7. The two-impurity bi-fermion states in Table 5.7 are

$SO(4)_{AdS} \times SO(4)_{S^5}$	$\Lambda_{\rm FF}$	$SO(4)_{AdS} \times SO(4)_{S^5}$	$\Lambda_{\rm FF}$
$(\mathbf{2},\mathbf{1};\mathbf{2},\mathbf{1}) + (\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{2})$	-3	$(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})$	-2
$[\mathbf{2},\mathbf{1};(\mathbf{2}+\mathbf{4}),\mathbf{1}] + [\mathbf{1},\mathbf{2};\mathbf{1},(\mathbf{2}+\mathbf{4})]$	-1	$(\mathbf{1},\mathbf{1};\mathbf{3},\mathbf{1}) + (\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{3})$	0
$[(\mathbf{2}+\mathbf{4}),\mathbf{1};\mathbf{2},\mathbf{1}] + [\mathbf{1},(\mathbf{2}+\mathbf{4});\mathbf{1},\mathbf{2}]$	-5	$(\mathbf{3},\mathbf{1};\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{3};\mathbf{1},\mathbf{1})$	-4
$[(\mathbf{2}+\mathbf{4}),\mathbf{1};(\mathbf{2}+\mathbf{4}),\mathbf{1}] + [\mathbf{1},(\mathbf{2}+\mathbf{4});\mathbf{1},(\mathbf{2}+\mathbf{4})]$	-3	$(\mathbf{3},\mathbf{1};\mathbf{3},\mathbf{1}) + (\mathbf{1},\mathbf{3};\mathbf{1},\mathbf{3})$	-2

Table 5.7: Spectrum of three-impurity states (top) and two-impurity states (bottom) created by Π_+-projected fermionic creation operators

spacetime bosons while the tri-fermion states are spacetime fermions. For comparison purposes, we have displayed both spectra. Note that the $O(1/J)$ energy corrections of the two types of state are simply displaced by -1 relative to each other.

This exhausts the subspaces that can be diagonalized by simple irrep projections. The remaining eigenvalues must be obtained by explicit diagonalization of finite dimensional submatrices obtained by projection onto representations with multiple occurrence. The upshot of these more complicated eigenvalue calculations is that the first-order λ' eigenvalues take on all integer values from $\Lambda = -8$ to $\Lambda = +2$, alternating between spacetime bosons and fermions.

5.1.3 Assembling eigenvalues into supermultiplets

We would now like to understand how the perturbed three-impurity spectrum reproduces the gauge theory structure of supersymmetry multiplets. This is relatively easy to infer from the multiplicities of the perturbed eigenvalues (and the multiplicities are a side result of the calculation of the eigenvalues themselves). The complete results for the eigenvalues Λ and their multiplicities are stated in Table 5.8 (we use the notation of (5.1.8), while the B and F subscripts are used to indicate bosonic and fermionic levels).

As explained previously for the two impurity case the eigenvalues Λ must satisfy certain conditions if the requirements of $PSU(2,2|4)$ symmetry and duality with are

Λ	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2
Multiplicity	4_B	40_F	180_B	480_F	840_B	1008_F	840_B	480_F	180_B	40_F	4_B

Table 5.8: Complete three-impurity energy spectrum (with multiplicities)

to be met.

The Λ eigenvalues in Table 5.8 are integer-spaced, which is consistent with supersymmetry requirements. However, because the range between top and bottom eigenvalues is 10, rather than 8, the $4,096$-dimensional space must be built on more than one type of extended supermultiplet, with more than one choice of c in the general formula $\Lambda = L_v + c$. This is to be contrasted with the two-impurity case, where the degenerate space was exactly 256-dimensional and was spanned by a single superconformal primary whose lowest member was a singlet under both Lorentz transformations and the residual $SO(4)$ R-symmetry. We can readily infer what superconformal primaries are needed to span the degenerate three-impurity state space by applying a little numerology to Table 5.8. The lowest eigenvalue is $\Lambda = -8$: it has multiplicity 4 and, according to Table 5.5, its $SO(4) \times SO(4)$ decomposition is $(\mathbf{1},\mathbf{1};\mathbf{2},\mathbf{2})$ (spacetime scalar, R-charge $SO(4)$ four-vector). According to the general arguments about how the full extended supermultiplet is built by acting on a 'bottom' state with the eight raising operators, it is the base of a supermultiplet of 4×256 states extending up to $\Lambda = 0$. By the same token, there is a highest eigenvalue $\Lambda = +2$: it has multiplicity 4 and, according to Table 5.5, its $SO(4) \times SO(4)$ decomposition is $(\mathbf{2},\mathbf{2};\mathbf{1},\mathbf{1})$ (spacetime vector, R-charge singlet). Using lowering operators instead of raising operators, we see that one derives from it a supermultiplet of 4×256 operators with eigenvalues extending from $\Lambda = -6$ to $\Lambda = +2$. By comparing with the *total* multiplicities of each allowed Λ (as listed in Table 5.8) we readily see that what remains are 8×256 states with eigenvalues running from $\Lambda = -7$ to $\Lambda = +1$ with the correct binomial coefficient pattern of multiplicities. The top and bottom states here are spacetime fermions and must lie in a spinor representation of the Lorentz group. It is not hard to see that they lie in the eight-dimensional

$SO(4) \times SO(4)$ irrep $(\mathbf{2},\mathbf{1};\mathbf{1},\mathbf{2}) + (\mathbf{1},\mathbf{2};\mathbf{2},\mathbf{1})$. This exhausts all the states and we conclude that the three-impurity state space is spanned by three distinct extended superconformal multiplets. The detailed spectrum is given in table below (where the last line records the total multiplicity at each level as given in Table 5.8 and the first line records the two-impurity spectrum for reference). Note the peculiar feature that certain energies are shared by all three multiplets: this is an accidental degeneracy that does not survive at higher loop order. It is worth making a few remarks at

Λ	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	
$\Delta_0 = 2$			1_B	8_F	28_B	56_F	70_B	56_F	28_B	8_F	1_B	scalar
$\Delta_0 = 3$	4	32	112	224	280	224	112	32	4			$SO(4)_{S^5}$ vector
			4	32	112	224	280	224	112	32	4	$SO(4)_{AdS_5}$ vector
		8	64	224	448	560	448	224	64	8		spinor
Total	4_B	40_F	180_B	480_F	840_B	1008_F	840_B	480_F	180_B	40_F	4_B	4,096

Table 5.9: Submultiplet breakup of the three-impurity spectrum

this point about the corresponding gauge calculation of anomalous dimension both for comparison with chapter three and with the final chapter on integrability. Minahan and Zarembo [52] found that the problem simplifies dramatically if we study the one-loop anomalous dimension of the special subset of single-trace operators of the form $\mathrm{tr}\,(\phi^{I_1}\cdots\phi^{I_L})$ as the one-loop anomalous dimension operator mixing matrix in this sector can be recast as the Hamiltonian of an integrable spin-chain solvable by the Bethe ansatz. Of particular interest to us they give a recipe for finding $1/J$ corrections to the anomalous dimensions of BMN operators of the type $\mathrm{tr}(Z^J W^I)$ where $Z = \phi^1 + i\phi^2$ and $W = \phi^3 + i\phi^4$ which correspond to string states of the type $a^{\dagger i'_1}\cdots a^{\dagger i'_I}\,|J\rangle$, which are symmetric and traceless in all indices. They found that

$$\gamma_{\mathfrak{su}(2)} = \frac{\lambda}{2L^3}\sum_n M_n k_n^2\,(L + M_n + 1) + O(L^{-4})\ . \tag{5.1.10}$$

The integer k_n represents pseudoparticle momenta on the spin-chain, and is dual to the string theory worldsheet mode indices; the quantity M_n labels the number of trace impurities with identical k_n. With I impurities, the spin-chain length is given

in terms of the R-charge by $L = J + I$, which leads to

$$\gamma_{\mathfrak{su}(6)} = \frac{\lambda}{2J^3} \sum_n M_n k_n^2 \left(J - 2I + M_n + 1\right) + O(J^{-4}) \ . \tag{5.1.11}$$

This expansion is similar in character to (5.1.8) and, for $I = 3$ (the three-impurity case), it matches that equation precisely with $\Lambda = -4$.

On the string theory side, three completely symmetrized $(\mathbf{1}, \mathbf{1}; \mathbf{2}, \mathbf{2})$ vectors form a tensor in the $(\mathbf{1}, \mathbf{1}; \mathbf{4}, \mathbf{4})$ irrep; such an irrep can be constructed from three $SO(4)_{S^5}$ vector (bosonic) creation operators. Table 5.5 shows that the corresponding string perturbation theory eigenvalue is (at one-loop order) $\Lambda = -4$ as well. We infer from Table 5.9 that this eigenvalue lies at level $L_v = 4$ of the $SO(4)_{S^5}$ vector superconformal multiplet (and this argument takes care of the gauge theory/string theory comparison for all other operators in that multiplet). The set of operators $\mathrm{tr}(Z^J W^I)$ is an $\mathfrak{su}(2)$ subsector of the more general $\mathrm{tr}\left(\phi^{I_1} \cdots \phi^{I_L}\right)$, which we refer to as a $\mathfrak{so}(6)$ sector as each lattice site is a $SO(6)$ vector. This $\mathfrak{su}(2)$ subspace is closed to all orders in perturbation theory (see the recent [67] for evidence that this may not be true non-perturbatively) and as mentioned in a previous chapter Beisert [53] has identified two other 'closed sectors' of operators in the gauge theory. In addition to the bosonic $\mathfrak{su}(2)$ sector, a bosonic $\mathfrak{sl}(2)$ sector and an $\mathfrak{su}(2|3)$ sector (of which the closed $\mathfrak{su}(2)$ sector is a subgroup) are also exactly closed.

In the string theory, the subsectors analogous to the gauge theory $\mathfrak{sl}(2)$ and a $\mathfrak{su}(1|1) \subset \mathfrak{su}(2|3)$ are constructed out of completely symmetrized $SO(4)_{AdS}$ bosons and completely symmetrized fermions of the same Π eigenvalue, respectively. They correspond to the central $L_v = 4$ levels of the remaining two supermultiplets in Table 5.9, and a calculation of their eigenvalues would complete the analysis of the match between three-impurity operators and string states at one-loop order. The Bethe ansatz for the general one-loop integrable spin-chain presented in [58] can easily be used to find similar results for these other sector. However for higher loops we use the virial results of chapter three and we can compare the results in this chapter

with those of Tables 3.5–3.11 taking care to use the correct relationship between L the lattice length and J the string angular momentum.

5.1.4 Two equivalent mode indices ($q = r = n,\ s = -2n$)

When two mode indices are allowed to be equal, the analysis becomes slightly more complicated as there are many more nonvanishing contributions to each matrix element. While the matrix elements are more complicated, the state space is only half as large when two mode indices are allowed to be equal. As a result, the fermionic and bosonic sectors of the Hamiltonian are each 1,024-dimensional.

To study this case, we make the mode index choice

$$q = r = n \qquad s = -2n\ . \tag{5.1.12}$$

The structure of matrix elements of the string Hamiltonian is quite complicated. For example the purely bosonic subsector is given by

$$\begin{aligned}
\langle J|a_n^A a_n^B a_{-2n}^C (H_{\rm BB}) a_{-2n}^{D\dagger} a_n^{E\dagger} a_n^{F\dagger}|J\rangle = \frac{n^2\lambda}{2J}\Big\{ & 5\,\delta^{BF}\,\delta^{cd}\delta^{ae} + 5\,\delta^{AF}\,\delta^{cd}\delta^{be} - 4\,\delta^{BF}\,\delta^{ad}\delta^{ce} \\
& +4\,\delta^{BF}\,\delta^{ac}\delta^{de} + 4\,\delta^{AF}\,\delta^{bc}\delta^{de} + 5\,\delta^{BE}\,\delta^{cd}\delta^{af} - 4\,\delta^{BE}\,\delta^{ad}\delta^{cf} + 4\,\delta^{BE}\,\delta^{ac}\delta^{df} + 4\,\delta^{AE}\,\delta^{bc}\delta^{df} \\
& -4\,\delta^{bd}\Big(\delta^{AF}\,\delta^{ce} + \delta^{AE}\,\delta^{cf}\Big) + 3\,\delta^{BF}\,\delta^{ae}\,\delta^{c'd'} + 3\,\delta^{AF}\,\delta^{be}\,\delta^{c'd'} + 3\,\delta^{BE}\,\delta^{af}\,\delta^{c'd'} \\
& -3\,\delta^{BF}\,\delta^{cd}\,\delta^{a'e'} - 3\,\delta^{AF}\,\delta^{cd}\,\delta^{b'e'} - 5\,\delta^{BF}\,\delta^{c'd'}\delta^{a'e'} - 5\,\delta^{AF}\,\delta^{c'd'}\delta^{b'e'} + 4\,\delta^{BF}\,\delta^{a'd'}\delta^{c'e'} \\
& +4\,\delta^{AF}\,\delta^{b'd'}\delta^{c'e'} - 4\,\delta^{BF}\,\delta^{a'c'}\delta^{d'e'} - 4\,\delta^{AF}\,\delta^{b'c'}\delta^{d'e'} - 3\,\delta^{BE}\,\delta^{cd}\,\delta^{a'f'} - 3\,\delta^{AE}\,\delta^{cd}\,\delta^{b'f'} \\
& -5\,\delta^{BE}\,\delta^{c'd'}\delta^{a'f'} - 5\,\delta^{AE}\,\delta^{c'd'}\delta^{b'f'} + 4\,\delta^{BE}\,\delta^{a'd'}\delta^{c'f'} + 4\,\delta^{AE}\,\delta^{b'd'}\delta^{c'f'} - 4\,\delta^{BE}\,\delta^{a'c'}\delta^{d'f'} \\
& -4\,\delta^{AE}\,\delta^{b'c'}\delta^{d'f'} + \delta^{AE}\,\delta^{bf}\Big(5\,\delta^{cd} + 3\,\delta^{c'd'}\Big) - 2\,\delta^{CD}\Big[9\Big(\delta^{BE}\delta^{AF} + \delta^{AE}\delta^{BF}\Big) - \delta^{be}\delta^{af} \\
& -\delta^{ae}\delta^{bf} + \delta^{ab}\delta^{ef} + \delta^{b'e'}\delta^{a'f'} + \delta^{a'e'}\delta^{b'f'} - \delta^{a'b'}\delta^{e'f'}\Big]\Big\}\ .
\end{aligned} \tag{5.1.13}$$

This matrix element exhibits the same antisymmetry in the $SO(4)_{AdS}$ and $SO(4)_{S^5}$ indices that is exhibited in equation (5.1.3).

We can perform a full symbolic diagonalization of the $1,024 \times 1,024$ bosonic and fermionic perturbation matrices to obtain the one-loop in λ', $O(1/J)$ energy correc-

tions. They can all be expressed in terms of dimensionless eigenvalues Λ according to the standard formula (5.1.8) modified by setting $q = r = n,\ s = -2n$:

$$E_J(n) = 3 + 3n^2\lambda' \left(1 + \frac{\Lambda}{J} + O(J^{-2})\right) \ . \tag{5.1.14}$$

The resulting spectrum is displayed in Table 5.10. The levels clearly organize

Λ_1 (S^5 vector)	$-23/3$	$-20/3$	$-17/3$	$-14/3$	$-11/3$	$-8/3$	$-5/3$	$-2/3$	$1/3$
Multiplicity	4_B	32_F	112_B	224_F	280_B	224_F	112_B	32_F	4_B
Λ_2 (AdS_5 vector)	$-19/3$	$-16/3$	$-13/3$	$-10/3$	$-7/3$	$-4/3$	$-1/3$	$2/3$	$5/3$
Multiplicity	4_B	32_F	112_B	224_F	280_B	224_F	112_B	32_F	4_B

Table 5.10: Spectrum of three-impurity string Hamiltonian with ($q = r = n,\ s = -2n$)

themselves into two superconformal multiplets built on vector primary states. Note that the spinor multiplet is absent and that the degeneracy between multiplets that was seen in the inequivalent mode index case has been lifted. The spinor multiplet is absent as it would contain a representation at level $L_v = 4$ arising from fermion creation operators completely symmetrized on $SO(4) \times SO(4)$ spinor indices; such a construct must vanish unless all the creation operator mode indices are different.

If we keep track of the $SO(4) \times SO(4)$ irrep structure, we find that the symmetric-traceless bosonic $SO(4)_{S^5}$ states arising from the closed $\mathfrak{su}(2)$ subsector fall into the $-11/3 \quad [280_B]$ level. This is the counterpart of the -4 $[280_B]$ level in Table 5.9. To compare with Minahan and Zarembo's Bethe ansatz calculation of the corresponding gauge theory operator dimension, we must evaluate equation (5.1.11) with the appropriate choice of parameters. In particular, $M_n = 2$ when two mode indices are allowed to coincide and, comparing with equation (5.1.14), we find perfect agreement with the string theory prediction $\Lambda = -11/3$. States at level $L_v = 4$ in the second multiplet in Table 5.10 correspond to operators in the $\mathfrak{sl}(2)$ closed sector of the gauge theory and the eigenvalue $\Lambda = -7/3$ $[280_B]$.

5.2 Three impurity string spectrum: all orders in λ'

In the previous section, we have studied the eigenvalue spectrum of the string theory perturbation Hamiltonian expanded to leading order in $1/J$ and to one-loop order in λ'. The expansion in λ' was for convenience only since our expressions for matrix elements are exact in this parameter. We should, in principle, be able to obtain results that are exact in λ' (but still of leading order in $1/J$). The simple one-loop calculations of the previous sections have given us an overview of how the perturbed string theory eigenvalues are organized into superconformal multiplets. This provides a very useful orientation for the more complex all-orders calculation, to which we now turn.

5.2.1 Inequivalent mode indices: $(q \neq r \neq s)$

Our first step is to collect the exact matrix elements of the perturbing Hamiltonian between three-impurity states of unequal mode indices. The block structure of the perturbation matrix in the spacetime boson sector is given in Table 5.2. The matrix elements between states consisting only of bosons are

$$\begin{aligned}\langle J|a_q^A a_r^B a_s^C (H_{\rm BB}) a_s^{D\dagger} a_r^{E\dagger} a_q^{F\dagger}|J\rangle = -\frac{1}{2\omega_q\omega_r\omega_s}\Big\{ & \delta^{BE}\omega_r\Big[\delta^{CD}\delta^{AF}(s^2+q^2(1+2s^2\lambda')) \\ & -(q^2+s^2)\delta^{cd}\delta^{af} - 2qs(\delta^{ad}\delta^{cf}-\delta^{ac}\delta^{df}) + (q^2-s^2)\delta^{af}\delta^{c'd'} - (q^2-s^2)\delta^{a'f'}\delta^{cd} \\ & +(q^2+s^2)\delta^{c'd'}\delta^{a'f'} + 2qs(\delta^{a'd'}\delta^{c'f'}-\delta^{a'c'}\delta^{d'f'})\Big] + \Big(C\rightleftharpoons B,\ D\rightleftharpoons E,\ s\rightleftharpoons r\Big) \\ & +\Big(A\rightleftharpoons B,\ F\rightleftharpoons E,\ q\rightleftharpoons r\Big)\Big\}\ , \end{aligned} \tag{5.2.1}$$

where we define for this section $\omega_q \equiv \sqrt{q^2+1/\lambda'}$ to simplify this and other similar expressions.

The other matrix elements are quite complicated and the explicit formulas, along with a collection of the Mathematica programs written to generate and work with them, are available on the web [68].

We were not able to symbolically diagonalize the complete perturbation matrix built from the exact (in λ') matrix elements: with the computing resources available to us, the routines for diagonalizing the full 2,048-dimensional matrices would not terminate in any reasonable time. As noted previously there are three protected $SO(4) \times SO(4)$ irreps that do not mix with any other irreps. It is a straightforward matter to project the perturbation matrix onto the dual of these irreps to obtain analytic expressions for the corresponding exact eigenvalues. In fact, the superconformal multiplet structure of the three-impurity problem is such that the energies/dimensions of all other irreps can be inferred from those of the three protected irreps. Hence, this method will give us exact expressions for all the energy levels of the three-impurity problem.

Consider first the $\mathfrak{sl}(2)$ closed sector. The dual sector is generated on the string theory side by bosonic creation operators completely symmetrized (and traceless) in $SO(4)_{AdS}$ vector indices. The simplest way to make this projection on equation (5.2.1) is to compute diagonal elements between the symmetrized states

$$a_q^{(a\dagger} a_r^{b\dagger} a_s^{c\dagger)} \left|J\right\rangle \ , \tag{5.2.2}$$

with $a \neq b \neq c$ (and, of course, $a, b, c \in 1, \ldots, 4$). The charges of the fermionic oscillators under this subgroup are $\pm 1/2$, so the three-boson state of this type cannot mix with one boson and two fermions (or any other state). Hence, the above projection of equation (5.2.1) yields the closed sector eigenvalue correction

$$\begin{aligned} \delta E_{AdS}(q,r,s,J) &= \frac{1}{J\omega_q\omega_r\omega_s}\Big\{ qs(1-qs\lambda')\omega_r + qr(1-qr\lambda')\omega_s + rs(1-rs\lambda')\omega_q \\ &\qquad + [qr + s(q+r)]\,\lambda'\omega_q\omega_r\omega_s \Big\} \\ &\approx \frac{1}{J}\Big\{ -2(q^2+qr+r^2)\lambda' - \frac{15}{8}\big(q^2r^2(q+r)^2\big)\lambda'^3 + \ldots \Big\} \ . \end{aligned} \tag{5.2.3}$$

To facilitate comparison with gauge theory results, we have performed a small-λ' expansion in the final line with the substitution $s \to -(q+r)$ (since the mode indices

satisfy the constraint $s+q+r=0$). The leading correction $-2(q^2+qr+r^2)\lambda'$ reproduces the one-loop eigenvalue $\Lambda_{\rm BB}=-2$ $[280_B]$ located at level $L_v=4$ in the $SO(4)_{AdS}$ multiplet in Table 5.9.

The closed $\mathfrak{su}(2)$ sector is generated by bosonic creation operators completely symmetrized and traceless in $SO(4)_{S^5}$ indices. Projection onto this irrep is achieved by choosing all mode operators in equation (5.2.1) to carry symmetrized, traceless $SO(4)_{S^5}$ labels (they can also be thought of as carrying charge $+1$ under some $SO(2)$ subgroup of $SO(4)_{S^5}$). Direct projection yields the $SO(4)_{S^5}$ eigenvalue

$$\begin{aligned}\delta E_{S^5}(q,r,s,J) &= -\frac{1}{J\omega_q\omega_r\omega_s}\Big\{\big[qr+r^2+q^2(1+r^2\lambda')\big]\,\omega_s+\big[qs+s^2+q^2(1+s^2\lambda')\big]\,\omega_r\\ &\qquad+\big[rs+s^2+r^2(1+s^2\lambda')\big]\,\omega_q-\big[rs+q(r+s)\big]\,\lambda'\omega_q\omega_r\omega_s\Big\}\\ &\approx\frac{1}{J}\Big\{-4(q^2+qr+r^2)\lambda'+(q^2+qr+r^2)^2\lambda'^2\\ &\qquad-\frac{3}{4}\big(q^6+3q^5r+8q^4r^2+11q^3r^3+8q^2r^4+3qr^5+r^6\big)\lambda'^3+\dots\Big\}\,. \qquad (5.2.4)\end{aligned}$$

This is the all-loop formula corresponding to gauge theory operator dimensions in the closed $\mathfrak{su}(2)$ subsector.

The eigenvalue of the symmetrized pure-fermion irrep can be obtained by evaluating the exact matrix element $H_{\rm FF}$ acting on three symmetrized fermionic creation operators with $SO(4)\times SO(4)$ indices chosen to lie in the same Π projection (with inequivalent mode indices). The exact energy shift for this irrep turns out to be

$$\begin{aligned}\delta E_{\rm Fermi}(q,r,s,J) &= -\frac{1}{4\,J\omega_q\omega_r\omega_s}\Big\{-4\big(rs+q(r+s)\big)\lambda'\omega_q\omega_r\omega_s\\ &\qquad+\Big[\omega_q\left(2s^2+4r^2s^2\lambda'+2r^2\right)+\big(s\to r,\ r\to q,\ q\to s\big)+\big(q\rightleftharpoons r\big)\Big]\Big\}\\ &\approx\frac{1}{J}\Big\{-3(q^2+qr+r^2)\lambda'+\frac{1}{2}(q^2+qr+r^2)^2\lambda'^2\\ &\qquad-\frac{3}{16}\big(2q^6+6q^5r+21q^4r^2+32q^3r^3+21q^2r^4+6qr^5+2r^6\big)\lambda'^3+\dots\Big\}\,. \qquad (5.2.5)\end{aligned}$$

The leading-order λ' correction $-3(q^2+qr+r^2)\lambda'$ reproduces the $\Lambda_{\rm FF}=-3$ $[580_F]$ eigenvalue at the $L_v=4$ level in the spinor multiplet in Table 5.9.

The argument we are making relies heavily on the claim that the perturbation matrix is block diagonal on the closed subsectors described above; we have evaluated the exact energy shift on these subsectors by simply taking the diagonal matrix element of the perturbing Hamiltonian in a particular state in each sector. We will now carry out a simple numerical test of the claimed block diagonalization of the full perturbing Hamiltonian. The basic idea is that, while it is impractical to algebraically diagonalize the full $2,048 \times 2,048$ perturbation matrices, it is quite easy to do a numerical diagonalization for a specific choice of λ' and mode indices q, r, s. One can then check that the numerical eigenvalues match the analytic predictions evaluated at the chosen coupling and mode indices. For definiteness, we choose

$$q = 1 \qquad r = 2 \qquad s = -3 \qquad \lambda' = 1 \ . \tag{5.2.6}$$

The predicted eigenvalue shifts of the three protected states, evaluated at the parameter choices of (5.2.6) are given in Table 5.11. These values come directly from eqns. (5.2.3,5.2.4,5.2.5) above (with J set to unity, for convenience). Since we want to

$\delta E: \ \lambda' = 1$		$q = 1, \ r = 2, \ s = -3$
$\delta E_{AdS}(1, 2, -3, J = 1)$	$=$	-16.255434067000426
$\delta E_{S^5}(1, 2, -3, J = 1)$	$=$	-20.137332508389193
$\delta E_{\text{Fermi}}(1, 2, -3, J = 1)$	$=$	-18.19638328769481

Table 5.11: Exact numerical eigenvalues of three-impurity protected sectors

compare these energies to a numerical diagonalization, we must maintain a high level of precision in the numerical computation. With the parameter choices of (5.2.6), the numerical diagonalization of the full $2,048 \times 2,048$ perturbation matrices yields the spectrum and multiplicities displayed in Table 5.12. The multiplicities are consistent with the superconformal multiplet structure we found in the one-loop analysis (given in Table 5.9). The predicted closed sector eigenvalues (listed in Table 5.11) match, to the precision of the calculation, entries in the list of numerical eigenvalues. These

energies also appear at the expected levels within the multiplets. $E_{AdS}(1,2,-3,J)$ and $E_{S^5}(1,2,-3,J)$ appear in bosonic levels with multiplicity 280_B, while energy $E_{\text{Fermi}}(1,2,-3,J)$ appears as a fermionic level with multiplicity 560_F; according to Table 5.9 these are uniquely identified as the central $L_v = 4$ levels of their respective multiplets, exactly where the protected energy levels must lie. All of this is clear evidence that the 'closed sector' states of the string theory do not mix with other states under the perturbing Hamiltonian, thus justifying our method of calculating their exact eigenenergies. At one loop, we found that the three superconformal mul-

$\delta E(1,2,-3,J=1)\ \lambda'=1$	Mult.
-30.821354623065	4_B
-26.9394561816763	4_B
-26.2093998737015	64_B
-25.4793435657269	112_B
-21.5974451243382	112_B
-20.8673888163637	448_B
-20.1373325083891	280_B
-16.2554340670003	280_B
-15.5253777590258	448_B
-14.7953214510512	112_B
-10.9134230096624	112_B
-10.1833667016878	64_B
-9.4533103937133	4_B
-5.57141195232456	4_B

$\delta E(1,2,-3,J=1)\ \lambda'=1$	Mult.
-28.8804054023706	8_F
-28.150349094396	32_F
-24.2684506530072	32_F
-23.5383943450326	224_F
-22.808338037058	224_F
-18.9264395956693	224_F
-18.1963832876947	560_F
-17.4663269797201	224_F
-13.5844285383314	224_F
-12.8543722303568	224_F
-12.1243159223822	32_F
-8.24241748099347	32_F
-7.51236117301893	8_F

(5.2.7)

Table 5.12: All loop numerical spectrum of three-impurity states ($q=1,\ r=2,\ s=-3,\ \lambda'=1,\ J=1$). Left panel: bosons; right panel: fermions

tiplets were displaced from each other by precisely the internal level spacing. This led to an accidental degeneracy which is lifted in the exact dimension formulas we have just derived. To explore this, it is useful to have formulas for the eigenvalues of all the levels in each multiplet. From the discussion previously, we know that each

level in the string energy spectrum can be connected by a simple integer shift in the angular momentum J. Since we are working at $O(1/J)$ in a large-J expansion, all contributions from this shift must come from the BMN limit of the theory. In other words, by sending $J \to J + 2 - L_v/2$ in the BMN formula for the energy

$$E = \sqrt{1 + \frac{n^2 g_{YM}^2 N_c}{(J + 2 - L_v/2)^2}} + \dots \ , \tag{5.2.8}$$

we can generate an expansion, to arbitrary order in λ', for each level L in the entire superconformal multiplet.

5.2.2 Two equal mode indices: $(q = r = n,\ s = -2n)$

An independent analysis is required when two mode indices are equal (specifically, we choose $q = r = n$, $s = -2n$). The all-loop matrix elements are complicated and we will refrain from giving explicit expressions for them (though the complete formulas can be found at [68]). As in the unequal mode index case, however, exact eigenvalues can easily be extracted by projection onto protected subsectors. In particular, the energy shift for states created by three bosonic mode creation operators with symmetric-traceless $SO(4)_{AdS}$ vector indices (the $\mathfrak{sl}(2)$ sector) turns out to be

$$\begin{aligned} \delta E_{AdS}(n,J) &= -\frac{n^2\lambda'}{J(1+n^2\lambda')\sqrt{4n^2+1/\lambda'}}\left\{\sqrt{4n^2+\frac{1}{\lambda'}}\left(3+4n^2\lambda'\right)+\omega_n\left(4+8n^2\lambda'\right)\right\} \\ &\approx \frac{1}{J}\left\{-7n^2\lambda' + n^4\lambda'^2 - \frac{17}{2}n^6\lambda'^3 + \dots\right\}. \end{aligned} \tag{5.2.9}$$

The leading order term in the small-λ' expansion is the $-7/3$ $[280_B]$ level $L = 4$ eigenvalue in the Λ_2 multiplet in Table 5.10. The energy shift of the $SO(4)_{S^5}$ partners of these states (belonging to the $\mathfrak{su}(2)$ closed sector) is

$$\begin{aligned} \delta E_{S^5}(n,J) &= -\frac{n^2\lambda'}{J(1+n^2\lambda')\sqrt{4n^2+1/\lambda'}}\left\{\sqrt{4n^2+\frac{1}{\lambda'}}\left(5+4n^2\lambda'\right)+\omega_n\left(6+8n^2\lambda'\right)\right\} \\ &\approx \frac{1}{J}\left\{-11n^2\lambda' + 8n^4\lambda'^2 - \frac{101}{4}n^6\lambda'^3 + \dots\right\}. \end{aligned} \tag{5.2.10}$$

The one-loop correction corresponds to the $-11/3$ $[280_B]$ level in the Λ_1 submultiplet of Table 5.10. As noted above, the protected symmetrized-fermion ($\mathfrak{su}(1|1)$) sector does not appear when two mode indices are equal. As in the previous section, we can do a numerical diagonalization of the full perturbation matrix to verify that the predicted eigenvalues are indeed exact and closed, but we will omit the details.

By invoking the angular momentum shift $J \to J + 2 - L_v/2$ in the BMN limit, we can use the energy shift of the $L_v = 4$ level to recover the exact energy shifts of all other levels in the superconformal multiplets of Table 5.10. The energy shifts of the vector multiplet containing the protected $SO(4)_{AdS}$ bosonic irrep at level $L_v = 4$ are given by the expression

$$\delta E_{AdS}(n, J, L_v) \approx \frac{1}{J}\left\{\frac{1}{2}(3L_v - 19)n^2\lambda' - \frac{1}{2}(9L_v - 38)n^4\lambda'^2 + \frac{1}{8}(99L_v - 464)n^6\lambda'^3 - \frac{1}{16}(645L_v - 3160)n^8\lambda'^4 + \dots\right\}. \quad (5.2.11)$$

The shifts of the multiplet containing the protected $SO(4)_{S^5}$ bosonic irrep are given by

$$\delta E_{S^5}(n, J, L_v) \approx \frac{1}{J}\left\{\frac{1}{2}(3L_v - 23)n^2\lambda' - \frac{1}{2}(9L_v - 52)n^4\lambda'^2 + \frac{1}{8}(99L_v - 598)n^6\lambda'^3 - \frac{1}{16}(645L_v - 3962)n^8\lambda'^4 + \dots\right\}. \quad (5.2.12)$$

Once again, we note that in order to get energies, rather than energy shifts, one must append the BMN energy of the original degenerate multiplet to these results. Unlike the unequal mode index case, there is no accidental degeneracy between superconformal multiplets spanning the three-impurity space, even at one loop in λ'. The level spacings within the two superconformal multiplets are the same, but the multiplets are offset from each other by an L_v-independent shift (but one that depends on λ' and mode indices).

5.3 N-impurity string energy spectra

We now further extend the calculation of string energies in the large-J expansion to N-impurity states.

5.3.1 The $SO(4)_{S^5}$ ($\mathfrak{su}(2)$) sector

We begin in the $\mathfrak{su}(2)$ sector spanned by symmetric-traceless pure-boson states excited in the S^5 subspace. Because we are restricting our attention to $SO(4)_{S^5}$ states symmetric in their vector indices, we form the following oscillators:

$$a_n = \frac{1}{\sqrt{2}}\left(a_n^5 + i a_n^6\right) \qquad \bar{a}_n = \frac{1}{\sqrt{2}}\left(a_n^5 - i a_n^6\right) \ , \tag{5.3.1}$$

and we are interested in matrix elements of the form

$$\langle J|\, a_{n_1} a_{n_2} \dots a_{n_{N_B}} (H_{\rm BB}) a^\dagger_{n_1} a^\dagger_{n_2} \dots a^\dagger_{n_{N_B}} |J\rangle \ . \tag{5.3.2}$$

Written in terms of these oscillators the relevant part of the Hamiltonian is

$$\begin{aligned} H_{\mathfrak{su}(2)} \ &= \ -\frac{1}{8p_- R^2} \sum_{n,m,l,p} \frac{\delta(n+m+l+p)}{\sqrt{\omega_n \omega_m \omega_l \omega_p}} \times \\ &\left\{ \omega_n\omega_m\omega_l\omega_p - (1-n\ m\ l\ p) - \omega_n\omega_l\ m\ p - \omega_m\omega_p\ n\ l + 4\ l\ p \right\} a^\dagger_{-n} a^\dagger_{-l} a_m a_p . \end{aligned} \tag{5.3.3}$$

The string states appearing in the matrix element of equation (5.3.2) have been written in the generic form $a^\dagger_{n_1} a^\dagger_{n_2} \dots a^\dagger_{n_{N_B}} |J\rangle$ and, as usual, they are subject to the level-matching condition $\sum_{j=1}^{N_B} n_j = 0$. The complete set of mode indices $\{n_1, n_2, \dots, n_{N_B}\}$ can contain one or more subsets of indices that are equal, while still satisfying the level-matching condition; this scenario complicates the calculation of energy eigenvalues to some extent. We will eventually compute the eigenvalues of interest for completely general string states, but for purposes of illustration we will start with the simplest case in which no two mode numbers are equal

$(n_1 \neq n_2 \neq \ldots \neq n_{N_B})$. Between states with completely distinct mode indices, the oscillator structure appearing in the Hamiltonian has the following matrix element:

$$\begin{aligned}\langle J|\, a_{n_1} a_{n_2} \ldots a_{N_B} (a^\dagger_{-n} a^\dagger_{-l} a_m a_p) a^\dagger_{n_1} a^\dagger_{n_2} \ldots a^\dagger_{N_B} |J\rangle \\ = \frac{1}{2} \sum_{\substack{j,k=1 \\ j\neq k}}^{N_B} \Big(\delta_{n_j+n}\, \delta_{n_k+l}\, \delta_{n_j-m}\, \delta_{n_k-p} + \delta_{n_j+n}\, \delta_{n_k+l}\, \delta_{n_k-m}\, \delta_{n_j-p} \\ + \delta_{n_j+l}\, \delta_{n_k+n}\, \delta_{n_j-m}\, \delta_{n_k-p} + \delta_{n_j+l}\, \delta_{n_k+n}\, \delta_{n_k-m}\, \delta_{n_j-p} \Big) \,. \end{aligned} \tag{5.3.4}$$

It is a straightforward exercise to compute the energy eigenvalue of the $SO(4)_{S^5}$ bosonic interaction Hamiltonian in the N_B-impurity symmetric-traceless irrep (with unequal mode indices): we simply attach the $H_{\rm BB}$ coefficient of the oscillator structure $a^\dagger_{-n} a^\dagger_{-l} a_m a_p$ to the right-hand side of equation (5.3.4) and carry out the summation over mode numbers. The result is remarkably compact:

$$\delta E_{S^5}(\{n_i\}, N_B, J) = -\frac{1}{J} \sum_{\substack{j,k=1 \\ j\neq k}}^{N_B} \frac{1}{2\,\omega_{n_j}\omega_{n_k}} \left[n_k^2 + n_j^2 \left(1 + n_k^2 \lambda'\right) + n_j n_k \left(1 - \omega_{n_j}\omega_{n_k}\lambda'\right) \right] \,, \tag{5.3.5}$$

This $\mathfrak{su}(2)$ formula can be checked against previously obtained string theory results with two and three impurities. The $\mathfrak{su}(2)$ eigenvalue for three impurities with unequal mode indices is

$$\begin{aligned}\delta E_{S^5}(n_1, n_2, n_3, J) \;=\; & -\frac{1}{J\omega_{n_1}\omega_{n_2}\omega_{n_3}} \Big\{ \left[n_1 n_2 + n_2^2 + n_1^2 (1 + n_2^2 \lambda') \right] \omega_{n_3} \\ & + \left[n_1 n_3 + n_3^2 + n_1^2 (1 + n_3^2 \lambda') \right] \omega_{n_2} + \left[n_2 n_3 + n_3^2 + n_2^2 (1 + n_3^2 \lambda') \right] \omega_{n_1} \\ & - \left[n_2 n_3 + n_1 (n_2 + n_3) \right] \lambda' \omega_{n_1}\omega_{n_2}\omega_{n_3} \Big\} \,. \end{aligned} \tag{5.3.6}$$

which is reproduced by equation (5.3.5). We now generalize the analysis completely by using eigenstates with M mode-index subsets, where all mode indices are equal

within these subsets:

$$\frac{\left(a_{q_1}^\dagger\right)^{N_{q_1}}}{\sqrt{N_{q_1}!}}\frac{\left(a_{q_2}^\dagger\right)^{N_{q_2}}}{\sqrt{N_{q_2}!}}\cdots\frac{\left(a_{q_M}^\dagger\right)^{N_{q_M}}}{\sqrt{N_{q_M}!}}\,|J\rangle\ .$$

The j^{th} subset contains N_{q_j} oscillators with equal mode index q_j, and the total impurity number is again N_B, such that

$$\sum_{i=1}^{M} N_{q_i} = N_B \qquad \sum_{i=1}^{M} N_{q_i} q_i = 0\ . \tag{5.3.7}$$

The matrix element of $a_{-n}^\dagger\, a_{-l}^\dagger\, a_m\, a_p$ between the above states is

$$\begin{aligned}
&\langle J|\,\frac{(a_{q_1})^{N_{q_1}}}{\sqrt{N_{q_1}!}}\cdots\frac{(a_{q_M})^{N_{q_M}}}{\sqrt{N_{q_M}!}}\left(a_{-n}^\dagger\, a_{-l}^\dagger\, a_m\, a_p\right)\frac{\left(a_{q_1}^\dagger\right)^{N_{q_1}}}{\sqrt{N_{q_1}!}}\cdots\frac{\left(a_{q_M}^\dagger\right)^{N_{q_M}}}{\sqrt{N_{q_M}!}}\,|J\rangle \\
&\quad=\sum_{j=1}^{M} N_{q_j}(N_{q_j}-1)\,\delta_{n+n_j}\,\delta_{l+n_j}\,\delta_{m-n_j}\,\delta_{p-n_j}+\frac{1}{2}\sum_{\substack{j,k=1\\ j\neq k}}^{M} N_{q_j}N_{q_k}\Big(\delta_{n+n_k}\,\delta_{l+n_j}\,\delta_{m-n_k}\,\delta_{p-n_j} \\
&+\delta_{n+n_j}\,\delta_{l+n_k}\,\delta_{m-n_k}\,\delta_{p-n_j}+\delta_{n+n_k}\,\delta_{l+n_j}\,\delta_{m-n_j}\,\delta_{p-n_k}+\delta_{n+n_j}\,\delta_{l+n_k}\,\delta_{m-n_j}\,\delta_{p-n_k}\Big)\ .
\end{aligned} \tag{5.3.8}$$

We thus obtain the completely general $\mathfrak{su}(2)$ energy shift for N_B-impurity string states containing M equal-mode-index subsets of oscillators:

$$\begin{aligned}
\delta E_{S^5}(\{q_i\},\{N_{q_i}\},M,J) &= -\frac{1}{2J}\Bigg\{\sum_{j=1}^{M} N_{q_j}(N_{q_j}-1)\left(1-\frac{1}{\omega_{q_j}^2\lambda'}\right) \\
&\quad+\sum_{\substack{j,k=1\\ j\neq k}}^{M}\frac{N_{q_j}N_{q_k}}{\omega_{q_j}\omega_{q_k}}\left[q_k^2+q_j^2\omega_{q_k}^2\lambda'+q_jq_k(1-\omega_{q_j}\omega_{q_k}\lambda')\right]\Bigg\}\ .
\end{aligned} \tag{5.3.9}$$

This master formula can be used to determine the $\mathfrak{su}(2)$ string energy spectrum to $O(J^{-1})$ for all possible physical string states in this sector.

We also note that equation (5.3.9) agrees perfectly with the corresponding near-pp-wave formula derived from the $\mathfrak{su}(2)$ string Bethe ansatz of [69] for completely general mode-number assignment. We will discuss this further at a later point.

5.3.2 The $SO(4)_{AdS}$ ($\mathfrak{sl}(2)$) sector

Following the derivation of equation (5.3.9) for the energy eigenvalues of arbitrary string states in the symmetric-traceless $SO(4)_{S^5}$ sector, it is straightforward to find the analogous expression for symmetric-traceless string states excited in the $SO(4)_{AdS}$ subspace, dual to operators in the $\mathfrak{sl}(2)$ sector of the corresponding gauge theory. We can define, for example,

$$a_n = \frac{1}{\sqrt{2}}\left(a_n^1 + ia_n^2\right) \qquad \bar{a}_n = \frac{1}{\sqrt{2}}\left(a_n^1 - ia_n^2\right) \ , \tag{5.3.10}$$

and carry out the above calculations by computing general matrix elements of $a^\dagger_{-n}a^\dagger_{-l}a_m a_p$ defined in terms of these oscillators. (Here we can project onto any (n,m)-plane in the AdS_5 subspace, as long as $n \neq m$.) General string energy eigenvalues in the $SO(4)_{AdS}$ symmetric-traceless irrep are thus found to be

$$\begin{aligned}\delta E_{AdS}(\{q_i\},\{N_{q_i}\},M,J) \ &= \ \frac{1}{2J}\Bigg\{\sum_{j=1}^{M} N_{q_j}(N_{q_j}-1)\left(1-\frac{1}{\omega_{q_j}^2\lambda'}\right) \\ &\quad + \sum_{\substack{j,k=1\\ j\neq k}}^{M} \frac{N_{q_j}N_{q_k}}{\omega_{q_j}\omega_{q_k}} q_j q_k \left[1 - q_j q_k \lambda' + \omega_{q_j}\omega_{q_k}\lambda'\right]\Bigg\}. \end{aligned} \tag{5.3.11}$$

5.3.3 The $\mathfrak{su}(1|1)$ sector

Based on the above results in the bosonic $SO(4)_{AdS}$ and $SO(4)_{S^5}$ symmetric-traceless sectors, we can easily formulate a conjecture for the N-impurity eigenvalue of symmetrized pure-fermion states in either the $(\mathbf{2},\mathbf{1};\mathbf{2},\mathbf{1})$ or $(\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{2})$ of $SO(4)\times SO(4)$, labeled by the $\mathfrak{su}(1|1)$ subalgebra. We first note that, since these states are composed of fermionic oscillators which are symmetrized in their spinor indices, no states in this sector can carry subsets of overlapping mode numbers (since they would automatically vanish). Furthermore, when restricting to states with completely unequal mode indices, we can see that the N-impurity eigenvalues obtained for the $\mathfrak{su}(2)$ and $\mathfrak{sl}(2)$

sectors (eqns. (5.3.5) and (5.3.11)) are obvious generalizations of the corresponding three-impurity formula. Namely, if the three-impurity eigenvalues take the generic form

$$\delta E(n_1, n_2, n_3, J) = \sum_{\substack{j,k=1 \\ j\neq k}}^{3} F(n_j, n_k) \ , \tag{5.3.12}$$

the N-impurity generalization is simply

$$\delta E(\{n_i\}, N, J) = \sum_{\substack{j,k=1 \\ j\neq k}}^{N} F(n_j, n_k) \ . \tag{5.3.13}$$

This factorization is a key signature of integrability [70] and we will have more to say about it in the next chapter. By carrying this over to the $\mathfrak{su}(1|1)$ sector, we find the N-impurity eigenvalue of $H_{\rm FF}$ between symmetrized $(\mathbf{2},\mathbf{1};\mathbf{2},\mathbf{1})$ or $(\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{2})$ fermions (the eigenvalues of both are necessarily degenerate):

$$\delta E_{\mathfrak{su}(1|1)}(\{n_i\}, N_F, J) = -\frac{1}{4J} \sum_{\substack{j,k=1 \\ j\neq k}}^{N_F} \frac{1}{\omega_{n_j}\omega_{n_k}} \left[n_j^2 + n_k^2 + 2n_j^2 n_k^2 \lambda' - 2\, n_j n_k \omega_{n_j}\omega_{n_k} \lambda' \right] \ . \tag{5.3.14}$$

One might carry out the direct N-impurity calculation in the $H_{\rm FF}$ sector analogous to the above calculations for $H_{\rm BB}$. This would be more complicated than in the bosonic sectors, and for the moment we leave equation (5.3.14) as it stands, withholding direct verification for a future study.

We know that the complete N-impurity spectrum should decompose into $PSU(2,2|4)$ multiplets. Given what we found in the two and three impurity case in addition to what we expect from duality with the gauge theory it is straightforward to work out what these multiplets will be. However actually deriving it from the Hamiltonian is presently beyond our abilities.

These N-impurity formula are of particular interest as they allow us to test and

motivate the quantum string Bethe ansätze of [69, 70] and to study integrability on both sides of the AdS/CFT correspondence. It is to this topic that we now turn our attention.

Chapter 6

Integrability and the Bethe Ansatz

There is considerable evidence that four-dimensional planar $\mathcal{N} = 4$ Yang-Mills theory is integrable. In particular, Minahan and Zarembo [52] derived the one-loop mixing matrix for the anomalous dimensions of operators consisting solely of scalar fields and identified this matrix with the Hamiltonian of an integrable $SO(6)$ closed spin-chain. Their result has been extended to the full set of $PSU(2,2|4)$ fields at one-loop [58] and to higher loops in the $SO(6)$ sector [57]. Furthermore, similar results have been found for certain $\mathcal{N} = 2$ Yang-Mills gauge theories which are dual to theories containing open strings. Specifically [71] studied an $Sp(N)$ superconformal theory with a hypermultiplet in the antisymmetric representation and four in the fundamental representation. This theory had previously been studied in the BMN limit by Berenstein et al [72] and semi-classical open spinning string solutions had been found in [73]. Similarly, DeWolfe and Mann [74] considered a defect conformal field theory [75] and showed that, at least for certain sectors, it is integrable. This theory is again dual to a string theory containing open strings and its BMN limit had been examined by Park and Lee [76]. Indeed many gauge theories show integrability if we restrict our attention to a subsector of fields, see for example references [77, 78] which are relevant to large-N QCD. It should also be noted that integrable $\mathfrak{sl}(2)$ spin-chains were discovered some time ago in phenomenologically-motivated studies of the scaling behavior of high-energy scattering amplitudes in QCD [79] (see also [80, 81, 82, 83] and see [84] for a review).

There have been parallel developments in the string theory where it was noticed, initially for the bosonic part [33] and later for the full action [34], that the classical coset sigma model possesses an infinite set of conserved charges. In [85] the authors constructed, via the Bäcklund transformation, a generating function for an infinite tower of mutually commuting charges and matched it to the gauge theory. In [86] the Riemann-Hilbert problem for the classical finite gap solutions of the classical sigma-model, restricted to an $S^3 \subset S^5$, was solved and it was shown that up to two-loops it is equivalent to the classical limit of the gauge theory Bethe equations. This result has been extended to include the full sigma model in a series of works [87, 88, 89, 90]. An approximate quantum version of the Bethe equations for the $S^2 \subset S^5$ sector of the string theory was conjectured by Arutyunov, Frolov and Staudacher [69] and this result was extended to several other sectors in [70]. Furthermore Berkovits [91] using the pure spinor formalism proved that the conserved charges persist in the quantum theory, see also [92] for a consideration of the conserved charges in the quantum theory near the BMN limit. Finally we note that in [93] the authors constructed a Lax representation of the classical bosonic string Hamiltonian in a specific gauge and this result was extended to the gauge fixed *physical* superstring in [94].

Given an integrable system in $1+1$ dimensions it can often be solved using the algebraic Bethe ansatz (see [95] for an introduction) which allows us to simultaneously diagonalize all the commuting conserved charges. This was first applied to the $SO(6)$ spin-chain Hamiltonian for purely scalar operators in [52] and was extended to the full set of $\mathfrak{psu}(2,2|4)$ operators by [58]. An asymptotic three-loop Bethe ansatz was constructed for the $\mathfrak{su}(2)$ sector in [96] and an all-loop conjecture was made in [55]. Similarly at one-loop there are ansätze for certain sectors of the $\mathcal{N} = 2$ theories mentioned above [71, 74] and as we have previously stated there have been proposals for the quantum string Bethe ansatz. Quantum integrability is tied to diffractionless, factorized scattering i.e., scattering processes that are only ever two body and in which the magnitudes of the momenta never change. Staudacher [70] focused on the scattering matrix as being potentially useful in studying these complicated systems

and by using the near-BMN energy spectra was able to conjecture quantum Bethe ansätze for a range of closed sectors.

In the next few sections we will consider some of the above statements in more detail, comparing the results with those of previous chapters and adding a few new ones.

6.1 The one-loop dilatation operator

To begin with we will consider the set of operators

$$Tr(\phi^N Z^{L-N}),\ Tr(\phi^{N-1} Z \phi Z^{L-N-1}), \dots \tag{6.1.1}$$

and the anomalous mixing matrix which in this sector can be written as

$$H^{(2)}_{\mathfrak{su}(2)} = \sum_{n=1}^{L} (1 - P_{n,n+1}) = \sum_{n=1}^{L} (1 - 2\vec{S}_n \cdot \vec{S}_{n+1}), \tag{6.1.2}$$

where the Z and ϕ's are modeled by up and down spins on a one dimensional lattice. Let us denote by $\Phi_{n_1,n_2,\dots,n_N}$ the wavefunction of the state where we have N down spins at the lattice sites $n_1, \dots, n_N$. Then Bethe's ansatz [97] for the eigenfunctions is

$$\Psi = \sum_{n_1<n_2<\cdots<n_N} c_{n_1,n_2\dots n_N} \Phi_{n_1,n_2\dots,n_N}\ , \tag{6.1.3}$$

with

$$c_{n_1,n_2\dots n_N} = \sum_{\sigma} \exp\left[i \left(\sum_{j=1}^{N} p_{\sigma_j} n_j + \frac{1}{2} \sum_{j<l} \phi_{\sigma_j,\sigma_l} \right) \right]\ , \tag{6.1.4}$$

where in this equation σ is the permutation operator and ϕ_{σ_j,σ_l} is proportional to the logarithm of the amplitude, $S(p_j, p_l)$, for the quasi-particles to exchange momenta when passing through each other. This ansatz gives eigenstates of $H^{(2)}_{\mathfrak{su}(2)}$ in general if

the scattering matrix satisfies

$$S(p_j, p_k) = \frac{\frac{1}{2}\cot\left(\frac{p_j}{2}\right) - \frac{1}{2}\cot\left(\frac{p_k}{2}\right) + i}{\frac{1}{2}\cot\left(\frac{p_j}{2}\right) - \frac{1}{2}\cot\left(\frac{p_k}{2}\right) - i} \ , \tag{6.1.5}$$

and the energy of this state will be given by

$$E_0 = \sum_{j=1}^{N} 4\sin^2\left(\frac{p_j}{2}\right) . \tag{6.1.6}$$

In fact there are an infinite number of conserved charges however their existence is not immediately evident in this construction. It is of course necessary to impose boundary conditions; by demanding periodicity we get the Bethe equations

$$e^{ip_j L} = \prod_{\substack{k=1 \\ k\neq j}}^{N} S(p_j, p_k) \qquad\qquad j = 1, \ldots, N \ , \tag{6.1.7}$$

and as we wish to only consider states with cyclic symmetry (because of the trace in the original gauge theory operators) the momenta must satisfy $\sum_j p_j = 0$. This method can be generalized and made more powerful by means of the algebraic Bethe ansatz [95] however we will not detail the construction merely taking those results we need from the literature.

It is possible to compare the above energy spectrum with the near-Penrose limit results of the string theory. We expand the the momenta in powers of $1/J$

$$p_{k,m_k} = \frac{2\pi n_k}{J} + \frac{p^{(1)}_{k,m_k}}{J^{3/2}} + \frac{p^{(2)}_{k,m_k}}{J^2} \cdots \ , \tag{6.1.8}$$

and we have allowed for the possibility of equal mode indices by including fractional powers. Substituting into the Bethe equations we find for the energy of an arbitrary number of impurities N consisting of M distinct mode numbers, n_j, of multiplicity

N_j,

$$\begin{aligned} E &= \frac{\lambda'}{2}\sum_{k=1}^{M} N_k n_k^2 - \frac{\lambda'}{2J}\sum_{k=1}^{M} N_k(N_k-1)n_k^2 \\ &\quad -\frac{\lambda'}{J}\sum_{\substack{k,j=1\\ j\neq k}}^{M} N_k N_j \frac{n_k(n_j^2+n_k^2)}{n_k-n_j}\ , \end{aligned} \tag{6.1.9}$$

which agrees with the general N-impurity string theory result at one-loop. We have expanded our result in terms of $1/J$, which is most natural in comparing with the string theory however in our virial expansion we expanded in $1/L$, L being the length of the lattice. Using $L = J + N$ it is straightforward to see that we get agreement with our virial calculations for the specific case of three impurities.

The proof of integrability and construction of the Bethe equations was extended to the entire $\mathfrak{psu}(2,2|4)$ by Beisert and Staudacher [58]. The Bethe ansatz for chains of spins in arbitrary representations of arbitrary simple Lie groups was developed some time ago [98] (see also [99] for an extension to supersymmetric spin-chains) but applied only recently to the specific case of the dilatation operator of $\mathcal{N} = 4$ SYM [52, 58]. In the notation of [58], the Bethe equations are expressed in terms of the Bethe roots (or rapidities) u_i associated with the various impurity insertions in the single-trace ground state $\mathrm{tr}\, Z^L$. The index i in the Bethe root u_i runs over the total number N of impurities. A second index $q_i = 1, \ldots, 7$ is used to associate each of the N Bethe roots with a particular simple root of the $\mathfrak{sl}(4|4)$ symmetry algebra associated with $\mathcal{N} = 4$ SYM. The Bethe ansatz then takes the form (see [58] and references therein for further details)

$$\left(\frac{u_i + \frac{i}{2}V_{q_i}}{u_i - \frac{i}{2}V_{q_i}}\right)^L = \prod_{j\neq i}^{N}\left(\frac{u_i - u_j + \frac{i}{2}M_{q_i,q_j}}{u_i - u_j - \frac{i}{2}M_{q_i,q_j}}\right)\ , \tag{6.1.10}$$

where V_{q_i} denotes the ${q_i}^{th}$ Dynkin coefficient of the spin representation and M is the Cartan matrix of the algebra. To be slightly more specific, if α_{q_i} are the root vectors associated with the nodes of the Dynkin diagram and μ is the highest weight

of the spin representation, then the Dynkin coefficient (for a bosonic algebra) is $V_{q_i} = 2\,\alpha^{(q_i)}\cdot\mu/(\alpha^{(q_i)})^2$ and the elements of the Cartan matrix are $M_{q_i,q_j} = 2\,\alpha^{(q_i)}\cdot\alpha^{(q_j)}/(\alpha^{(q_j)})^2$ (note that diagonal elements $M_{q_i,q_i} = 2$). Furthermore, since the spin-chain systems of interest to us are cyclic and carry no net momentum (analogous to the level-matching condition in the string theory), the Bethe roots u_i are subject to the additional constraint

$$1 = \prod_i^N \left(\frac{u_i + \frac{i}{2}V_{q_i}}{u_i - \frac{i}{2}V_{q_i}} \right) \ . \tag{6.1.11}$$

Finally, having found a set of Bethe roots u_i that solve the above equations, the corresponding energy eigenvalue (up to an overall additive constant) is given by

$$E = \sum_{j=1}^N \left(\frac{V_{q_j}}{u_j^2 + V_{q_j}^2/4} \right) \ . \tag{6.1.12}$$

It should be possible to solve these equations in the near-BMN limit and find the spectrum of arbitrary excitation states at one-loop. However because of the nesting of equations this is quite complicated and instead we shall focus on the two simple closed sectors examined in previous chapters, namely the $\mathfrak{sl}(2)$ and $\mathfrak{su}(1|1)$ sectors.

In the $\mathfrak{sl}(2)$ sector the highest weight is $-1/2$: the Dynkin diagram therefore has coefficient $V_{\mathfrak{sl}(2)} = -1$ and the Cartan matrix is $M_{\mathfrak{sl}(2)} = 2$. The Bethe equations (6.1.10,6.1.11) thus reduce to

$$\left(\frac{u_i - i/2}{u_i + i/2} \right)^L = \prod_{j\neq i}^N \left(\frac{u_i - u_j + i}{u_i - u_j - i} \right) \tag{6.1.13}$$

$$1 = \prod_i^N \left(\frac{u_i - i/2}{u_i + i/2} \right) \ . \tag{6.1.14}$$

Apart from a crucial minus sign, this is identical to the $\mathfrak{su}(2)$ Bethe equation (6.1.7).

The near-BMN $1/J$ correction for N- impurities is

$$\delta E = -\frac{\lambda'}{2J}\sum_{k=1}^{M} N_k(N_k-1)n_k^2 + \frac{\lambda'}{J}\sum_{\substack{k,j=1\\ j\neq k}}^{M} N_k N_j n_j n_k \tag{6.1.15}$$

and this formula agrees with the one-loop string answer and our virial expansion upon noting that in this sector $J = L$ (see table 3.11).

The one-loop Bethe ansatz [52,58] for the $\mathfrak{su}(1|1)$ sector is particularly simple as at this order the dilation operator is just the Hamiltonian of free fermions. To apply the general Bethe ansatz equation of equation (6.1.10), we note that the $\mathfrak{su}(1|1)$ Dynkin diagram is just a single fermionic node: the Cartan matrix is empty and the single Dynkin label is $V_{\mathfrak{su}(1|1)} = 1$. We therefore obtain the simple one-loop Bethe equation

$$\left(\frac{u_i + \frac{i}{2}}{u_i - \frac{i}{2}}\right)^L = 1 \tag{6.1.16}$$

which can be solved exactly for arbitrary impurity number. The general $\mathfrak{su}(1|1)$ Bethe roots are

$$u_i = \frac{1}{2}\cot\left(\frac{k_i\pi}{L}\right) \ , \tag{6.1.17}$$

and the energy eigenvalues computed from equation (6.1.12) are

$$E_{\mathfrak{su}(1|1)} = 4\sum_{i=1}^{N}\sin^2\left(\frac{\pi k_i}{L}\right) \ , \tag{6.1.18}$$

with the usual condition $\sum k_i = 0 \mod L$ from equation (6.1.11). This is just the sum of free lattice Laplacian energies and clearly matches the energies one would obtain from the one-loop $\mathfrak{su}(1|1)$ Hamiltonian of equation (3.3.33) (since the latter has no interaction terms). No expansion in $1/L$ was necessary in this argument, but it is straightforward to expand the energies in $1/L$ and verify the numerical results obtained in table 3.8 and equation (3.3.36).

6.2 Beyond one-loop

For the $\mathfrak{su}(2)$ sector the Bethe ansatz was extended to three loops in [96] and an all loop conjecture was proposed in [90]. This all-loop conjecture preserves the appropriate BMN scaling properties of the anomalous dimensions while maintaining integrability. In addition to the Bethe ansatz equations the authors of [90] further postulated the local part of the all-loop gauge transfer matrix and hence all-loop conserved charges. The validity of these higher loop Bethe ansätze is subject to the restriction that the length of the chain is assumed to be longer than the range of the interactions. The Bethe equations for this conjecture are

$$e^{ip_j L} = \prod_{\substack{k=1 \\ k\neq j}}^{N} S_{kj}(p_k, p_j) \ , \tag{6.2.1}$$

where the scattering matrix is now given by

$$S_{kj}(p_k, p_j) = \frac{\phi(p_k) - \phi(p_j) + i}{\phi(p_k) - \phi(p_j) - i} \ , \tag{6.2.2}$$

and the phase function $\phi(p)$ is

$$\phi(p) = \frac{1}{2}\cot\left(\frac{p}{2}\right)\sqrt{1 + 8g^2 \sin^2\left(\frac{p}{2}\right)}. \tag{6.2.3}$$

The functions $q_r(p)$ correspond to conserved charge densities and are given by

$$q_r(p) = \frac{1}{g^{r-1}} \frac{2\sin\left(\frac{1}{2}(r-1)p\right)}{r-1} \left(\frac{\sqrt{1 + 8g^2 \sin^2\left(\frac{p}{2}\right)} - 1}{2g \sin\left(\frac{p}{2}\right)} \right)^{r-1} \ , \tag{6.2.4}$$

with $q_1(p)$ being the momentum p and the second function $q_2(p)$ the energy of a single quasi-particle. The conserved charges are then just the sum of these densities over

all excitations. The dilation operator at all loops is given by

$$D(g) = L + g^2 \sum_{k=1}^{N} q_2(p_k) \ , \tag{6.2.5}$$

which we can compare with our string theory results by again expanding the momenta in powers of $1/J$. The general expression for the spectrum of the dimension operator is (as written in [69])

$$\begin{aligned} E_{Gauge} &= J + \sum_{k=1}^{M} N_k \omega_k - \frac{\lambda'}{J} \sum_{k=1}^{M} N_k (N_k - 1) \frac{n_k^2}{2\omega_k^2} \\ &\quad - \frac{\lambda'}{J} \sum_{k=1}^{M} N N_k \frac{n_k^2}{\omega_k} - \frac{\lambda'}{J} \sum_{\substack{k,j=1 \\ j \neq k}}^{M} 2 N_k N_j \frac{n_k^2 n_j}{n_k^2 - n_j^2} \left(n_j + n_k \frac{\omega_j}{\omega_k} \right) \ , \end{aligned} \tag{6.2.6}$$

where in this equation we have defined $\omega_k = \sqrt{1 + \lambda' n_k^2}$.

This agrees with the string theory to two loops but as expected disagrees at three-loops and beyond. It is possible to modify the scattering matrix in the manner of [69] to get agreement with the string theory to all orders in λ',

$$S_{kj} = \frac{\phi(p_k) - \phi(p_j) + i}{\phi(p_k) - \phi(p_j) - i} \exp \left[2i \sum_{n=0}^{\infty} \left(\frac{g^2}{2} \right)^{n+2} \left(q_{n+2}(p_k) q_{n+3}(p_j) - q_{n+3}(p_k) q_{n+2}(p_j) \right) \right] . \tag{6.2.7}$$

This scattering matrix was motivated by comparing the large L thermodynamic limit of the all loop spin-chain with the classical sigma model as follows. In the limit where we take L large and replacing $g^2/2 \to L^2 T$, so $T = \lambda / 16\pi^2 L^2$, we can scale the charge densities as

$$q_r(g) \to L^{-r} q_r(T) \ , \tag{6.2.8}$$

and the total charges as $Q_r(g) \to L^{-r+1} Q_r(T)$ where we assume that the number of

excitations is also $O(L)$. In particular the momentum and phase function scale as

$$p_k \to \frac{p_k}{L} \quad \text{and} \quad \phi(p_k) \to L\phi(p_k). \tag{6.2.9}$$

The rescaled phase is $\phi(p) = \frac{1}{p}\sqrt{1+4Tp^2}$ and we use this to eliminate p and write all the charges in terms of ϕ

$$q_r(\phi) = \frac{1}{\sqrt{\phi^2-4T}} \frac{1}{\left(\frac{1}{2}\phi + \frac{1}{2}\sqrt{\phi^2-4T}\right)^{r-1}}. \tag{6.2.10}$$

By introducing a distribution density we can convert the discrete sums into integrals

$$\rho_g(\phi) = \frac{1}{L}\sum_{k=1}^{N} \delta(\phi-\phi_k), \quad Q_r = \int_C d\phi\ \rho_g(\phi) q_r(\phi)\ , \tag{6.2.11}$$

where C is the discrete union of contours on which the distribution has support. The normalization of the density distribution is

$$\int d\phi\ \rho_g(\phi) = \alpha, \quad \alpha = \frac{N}{L}\ , \tag{6.2.12}$$

and in this limit the Bethe equations form a system of integral equations

$$\int_C d\phi' \frac{\rho_s(\phi')}{\phi-\phi'} = \frac{1}{2}\frac{1}{\sqrt{\phi^2-4T}} + n_k\pi\ , \tag{6.2.13}$$

which hold on each of the cuts C_k, $C = C_1 \cap \cdots \cap C_M$. In [86] Kazakov, Marshakov, Minahan and Zarembo studied the quasi-momentum of the integrable sigma model and showed that it satisfies a set of singular integral equations reminiscent of those above. In particular, they introduced a density $\sigma_s(x)$ satisfying

$$\int dx\ \sigma_s(x)\left(1-\frac{T}{x^2}\right) = \alpha\ ,$$

and the conserved charges can be written in terms of this density

$$Q_r = \int_C dx \, \frac{\sigma_s(x)}{x^r}. \tag{6.2.14}$$

Furthermore they obtained an integral equation for this density

$$⨍ dy \, \frac{\sigma(y)}{x-y} = \frac{\kappa x}{2(x^2-T)} + \pi n_k \quad \text{with} \quad x \in C_k \ , \tag{6.2.15}$$

with $\kappa = 1 + 2TQ_2$ the rescaled string energy. It is natural to call these equations classical string Bethe equations. Now, if we make the identification

$$d\phi = \left(1 - \frac{T}{x^2}\right) dx \quad \text{so that} \quad q_r(\phi) d\phi = \frac{dx}{x^r} \ , \tag{6.2.16}$$

we see that scaled gauge charge densities q_r and the string charge densities x^{-r} agree to all-loops. However as is apparent from their different normalizations, the string and gauge density distributions ρ_g and σ_s are not equal. If we define $\rho_s(\phi) := \sigma_s(x)$ we can write the string equations in terms of the gauge theory quantities. We can then ask what are the discrete equations of which these are the thermodynamic limit and indeed the answer is that of (6.2.7) [69].

It is possible to carry out an analogous derivation for the $\mathfrak{sl}(2)$ sector and this was done in [70] using the semi-classical string analysis for this non-compact sector of Kazakov and Zarembo [87].

The classical string BAE for this sector are

$$⨍ dx' \frac{\sigma_s(x')}{x-x'} = \pi n_k - \frac{x + m\left(\frac{\lambda}{4\pi J^2}\right)}{x^2 - \left(\frac{\lambda}{16\pi^2 J^2}\right)} \ , \tag{6.2.17}$$

where m is proportional to the total momentum of the string. We use the same expressions for the conserved charges as in the $\mathfrak{su}(2)$ case and we have the same relation between ϕ and x i.e. $\phi = x + \frac{T}{x}$. Assuming that $\rho_s(\phi) := \sigma_s(x)$ we can write

the string Bethe equations in terms of gauge theory variable as

$$\begin{aligned} 2\fint d\phi' \frac{\rho_s(\phi')}{\phi-\phi'} &= -2T\int \frac{d\phi'}{\sqrt{\phi'-4T}}\frac{\rho_s(\phi')}{\sqrt{\phi-4T}}\left(\frac{1}{x'}\sum_{r=0}^{\infty}\left(\frac{T}{xx'}\right)^r - \frac{1}{x}\sum_{r=0}^{\infty}\left(\frac{T}{xx'}\right)^r\right) \\ &\quad +2\pi n_k - p(\phi) \\ &= -2\int d\phi' \rho_s(\phi')\sum_r T^{r+1}\left(q_{r+2}(\phi')q_{r+1}(\phi) - q_{r+2}(\phi)q_{r+1}(\phi')\right) \\ &\quad +2\pi n_k - p(\phi)\ . \end{aligned} \tag{6.2.18}$$

This equation is the thermodynamic limit of

$$\begin{aligned} &\exp\left(-iLp_k - 2i\sum_{r=0}^{\infty}\left(\frac{g^2}{2}\right)^{r+1}\left(Q_{r+2}q_{r+1}(p_k) - Q_{r+1}q_{r+2}(p_k)\right)\right) \\ &\qquad = \prod_{j=1,j\neq k}^{M}\frac{\phi(p_k)-\phi(p_j)+i}{\phi(p_k)-\phi(p_j)-i}\ , \end{aligned} \tag{6.2.19}$$

which is very similar to the $su(2)$ case; the main differences being the range of summation and a minus sign. Checking this against the $\mathfrak{sl}(2)$ near-BMN N-impurity result we directly find agreement. As with the $\mathfrak{su}(2)$ case one can also study these equations in the strong coupling limit with $g \to \infty$ and $L \ll g^{1/2}$. In this case we find that the conformal dimensions scale like massive string modes at level n with masses $m^2 \propto \sqrt{n\lambda}$ and we note that the entire contribution to this dimension comes from the exponential parts of the scattering matrix. Staudacher [70] also proposed a quantum spin-chain for the $\mathfrak{su}(1|1)$ sector. In this case there was no semi-classical analysis and he guessed the form based mainly on the N-impurity near-BMN energy formula that we described earlier. Since then the authors of [90] have extended the semi-classical analysis to the full supersymmetric string and found that it agrees, in the thermodymanic, limit with Staudacher's prediction. Indeed given the full semi-classical Bethe equations it should be possible to guess the corresponding quantum ansatz for the full $\mathfrak{psu}(2,2|4)$ string.

6.3 Open string Bethe ansatz

As mentioned previously, the gauge duals of open string theories were studied in [71] and [74]. Chen, et al. considered an $\mathcal{N}=2$ superconformal $Sp(N_c)$ Yang-Mills theory with matter in four fundamental hypermultiplets which arises from the orientifold projection of $2N_c$ $D3$-branes in the presence of four $D7$-branes and an $O7$-plane. This theory contains a vector multiplet (V, W) in the adjoint representation with W describing the motion of the $D3$-branes transverse to the orientifold, a hypermultiplet (Z, Z') in the antisymmetric representation which describes the motion of the $D3$-branes along the $D7$-branes, and the four hypermultiplets $(\tilde{q}_A, q_A)$ which correspond to $D3-D7$ strings. There is a $SO(8)$ global symmetry which comes from the gauge symmetry of the $D7$-branes. Overall this theory has an R-symmetry $U(1) \times SU(2)_R$ and a global symmetry $SU(2)_L \times SO(8)$. We define $J = J^3_{SU(2)_R} + J^3_{SU(2)_L}$, which corresponds to the angular momentum along which we take the Penrose limit of the dual geometry, and with respect to this charge the bosonic fields have the following values

$$Z \;\; J=1; \quad Z' \;\; J=0; \quad W \;\; J=0; \quad q \;\; J=1/2; \quad \tilde{q} \;\; J=1/2. \tag{6.3.1}$$

Chen, et al. considered gauge invariant operators consisting of various chiral scalar fields and in particular states of the form

$$\mathcal{O}^{open}_{i_1 \cdots i_L} = \lambda_{pq} Q^p \Omega(\Phi_{i_1}\Omega) \ldots (\Phi_{i_L}\Omega) Q^q \ ,$$

with Q^p a linear combination of $(q, \tilde{q})$, Φ a linear combination of the chiral fields (Z, Z', W), λ^{pq} the Chan-Paton factors and Ω the invariant $Sp(N)$ tensor. The anomalous dimension mixing matrix restricted to these operators is

$$\Gamma_o = \frac{\lambda}{4\pi^2} \sum_{l=1}^{L-1} (1 - P_{l,l+1}) + \frac{\lambda}{4\pi^2} (\Sigma_1 + \Sigma_L). \tag{6.3.2}$$

The boundary terms are $\Sigma_1 = \Sigma(\otimes I_{3\times 3})^{L-1}, \Sigma_L = (I_{3\times 3}\otimes)^{L-1}\Sigma$, with Σ =diag$(0,0,1)$. This corresponds to the Hamiltonian of an integrable $SU(3)$ open spin-chain which can be solved exactly.

In [74] the authors considered a defect conformal theory originally constructed in [75] and showed that the dilatation operator restricted to a certain set of operators is integrable. This theory contains a $d = 3, \mathcal{N} = 4$ $SU(N_c)$ hypermultiplet in addition to the bulk $d = 4, \mathcal{N} = 4$, and describes a stack of N_c $D3$-branes and a $D5$-brane which is extended in three of the $D3$-brane directions, i.e. in the $D3$-brane world volume it has codimension one. The defect preserves an $SO(3,2)$ of the conformal group and eight of the supersymmetries. It also breaks the R-symmetry from $SU(4)$ to $SU(2)_H \times SU(2)_V$. We can decompose the bulk $d = 4, \mathcal{N} = 4$ vector multiplet into a $d = 3, \mathcal{N} = 4$ vector multiplet and a $d = 3, \mathcal{N} = 4$ adjoint hypermultiplet. The vector multiplet has bosonic fields $A_\mu, X^1, X^2, X^3, D_3 X^I$, with $\mu = 0, 1, 2$ and $I = 4, 5, 6$ and the hypermultiplet contains the component of the gauge field normal to the defect on addition to the scalars X^4, X^5, X^6 and $D_3 X^A$ with $A = 1, 2, 3$. The $d = 3, \mathcal{N} = 4$ $SU(N_c)$ hypermultiplet consists of the complex scalars q^m which couple canonically to the gauge fields. On the string side we take the Penrose limit by boosting along a geodesic in the $1-2$ plane and so we are particularly interested in the charges of these fields with respect to $SU(2)_H$. Taking J to be the $J^3_{SU(2)_H}$ we have

$$X^1, X^2, X^3 \ J = 1; \quad X^4, X^5, X^6 \ J = 0; \quad q^m, q^{\tilde{m}} \ J = 1/2 \ .$$

The authors of [74] furthermore found the anomalous mixing matrix, identified it with an integrable spin-chain Hamiltonian and proposed a set of Bethe ansatz equations (BAE) which solves this system. They took as their Bethe ground state

$$\bar{q_1} Z \dots Z q_2 \tag{6.3.3}$$

with $Z = X^1 + iX^2$ and considered impurities of the type $W = X^4 + iX^5$, corre-

sponding to a Dirichlet boundary condition in the dual string theory, and X^3, which is dual to an open string with Neumann boundary conditions (though as states of the type $\bar{q}_1 Z \dots X^3 \dots Z q_2$ are not closed under the action of the mixing matrix this is a little more complicated). We can summarize the BAE for operators dual to open string states with Dirichlet/Neumann boundary conditions as

$$e^{2ip_k L} = \prod_{j \neq k}^{N} S_{kj}(p_k, p_j) S_{kj}(p_k, -p_j),$$

where $S_{kj}(p_k, p_j)$ is the scattering matrix of the closed string. This formula has an obvious interpretation in terms of scattering in one dimension; as the impurity is taken the length of the string and back again it interacts with each of the other impurities twice and since the system is integrable each scattering process only involves two impurities at a time. We could also expect a phase due to scattering with the end points, however it was shown that this phase is zero for these particular types of impurities.

We will now compare these one-loop results and their obvious generalizations to all-loops to what we find from the string theory. The string theory calculation is a straightforward extension of the closed string case (one must change the mode expansion and correctly identify the relevant states) and here we will merely record the $1/J$ result for the spectrum. For strings moving on an $S^2 \subset S^5$ with either Neumann or Dirichlet boundary conditions the light-cone energy is given by:

$$\begin{aligned} E &= \sum_{k=1}^{M} N_k \sqrt{1 + \frac{\lambda' n_k^2}{4}} - \frac{\lambda'}{16J} \sum_{k}^{M} \frac{N_j(N_j - 1)}{\omega_j^2} \left(3n_j^2 + \frac{n_j^4}{2} \right) \\ &\quad - \frac{\lambda'}{16J} \sum_{\substack{j,k=1 \\ j \neq k}}^{M} \frac{N_j N_k}{\omega_j \omega_k} \left(2(n_j^2 + n_k^2) + \frac{\lambda' n_k^2 n_j^2}{2} \right) , \end{aligned} \tag{6.3.4}$$

with N total impurities, M the number of inequivalent mode indices, N_j the multiplicity of the mode number n_j and we have defined $\omega_k = \sqrt{1 + \frac{\lambda' n_k^2}{4}}$. We use the

one-loop scattering matrix

$$S_{kj}(p_k, p_j) = \frac{\phi(p_k) - \phi(p_j) + i}{\phi(p_k) - \phi(p_j) - i} \ , \tag{6.3.5}$$

where $\phi(p) = \frac{1}{2}\cot(\frac{p}{2})$, appropriate to the gauge theory dual of the closed string and we expand the momentum in powers of $1/J$. Assuming inequivalent mode numbers we have

$$p_k = \frac{\pi n_k}{J} + \frac{p_k^{(2)}}{J^2} + \dots \ , \tag{6.3.6}$$

and expanding the LHS of the Bethe equations

$$exp\left(2iL\left(\frac{\pi n_k}{J} + \frac{p_k^{(2)}}{J^2} + \cdots\right)\right) \simeq 1 + \frac{2ip_k^2}{J} + \frac{2i(N-1)\pi n_k}{J}\cdots \ , \tag{6.3.7}$$

using $L = J - 1 + N$. For the RHS we need the following

$$\begin{aligned} \frac{\phi(p_k) - \phi(p_j) + i}{\phi(p_k) - \phi(p_j) - i} &\simeq 1 + \frac{2\pi i}{J}\frac{n_k n_j}{n_j - n_k} \\ \frac{\phi(p_k) + \phi(p_j) + i}{\phi(p_k) + \phi(p_j) - i} &\simeq 1 + \frac{2\pi i}{J}\frac{n_k n_j}{n_j + n_k} \ , \end{aligned} \tag{6.3.8}$$

and putting both sides together we have

$$\begin{aligned} \frac{2ip_k^{(2)}}{J} + \frac{2i(N-1)\pi n_k}{J} &= \sum_{j\neq k}\frac{2\pi i}{J} n_k n_j \left(\frac{1}{n_j - n_k} + \frac{1}{n_j + n_k}\right) \\ \Rightarrow \frac{p_k^{(2)}}{\pi} &= \sum_{j\neq k}\left\{\frac{2n_k n_j^2}{(n_j^2 - n_k^2)} - n_k\right\} \ . \end{aligned} \tag{6.3.9}$$

The anomalous dimension is given by

$$\begin{aligned} E &= \sum_k \frac{\lambda}{2\pi}\sin^2\left(\frac{p_k}{2}\right) \\ &\simeq \sum \frac{\lambda n_k}{8J^2} + \frac{\lambda n_k p_k^{(2)}}{4\pi J^3}\cdots \ , \end{aligned} \tag{6.3.10}$$

and substituting in (6.3.9) we find that

$$E = \frac{\lambda'}{8}\sum_k n_k - \frac{\lambda'(N-1)}{4J}\sum_{k=1}^{N} n_k^2 \ , \tag{6.3.11}$$

where we have used $\sum_{j\neq k}\frac{n_j^2 n_k^2}{n_j^2-n_k^2}=0$ and $\sum_{j\neq k} n_k = (N-1)\sum_k n_k$. This agrees with the string theory result at one-loop for the case where we have inequivalent mode indices.

We can try to generalize this result by using the all-loop gauge theory ansatz of [55] described above and in this case we find that

$$\delta E = -\frac{\lambda'(N-1)}{4J}\sum_k \frac{n_k^2}{\omega_k} \ , \tag{6.3.12}$$

which obviously does not agree with the string theory prediction beyond two-loops. However we can make the same modifications appropriate to the quantum string, once again expand in powers $1/J$ and solve for the leading order correction to the quasi-momentum

$$\begin{aligned}\frac{p_k^{(2)}}{\pi} = \sum_{j\neq k}\Bigg\{ & \frac{2n_k n_j^2\omega_k}{n_j^2\omega_k^2 - n_k^2\omega_j^2} - n_k + \frac{1}{2}\frac{n_k^2(1-\omega_j)-n_j^2(1-\omega_k)}{n_k+n_j} \\ & + \frac{1}{2}\frac{n_k^2(1-\omega_j)-n_j^2(1-\omega_k)}{n_k-n_j}\Bigg\} \ .\end{aligned} \tag{6.3.13}$$

We now substitute this into our expression for the anomalous dimension and we find a $1/J$ correction

$$\begin{aligned}\delta E &= \frac{-\lambda'}{4J}\sum_{k=1}^{M}(M-1)\frac{n_k^2}{\omega_k} + \frac{\lambda'}{8J}\sum_{j\neq k}\left(\frac{2n_k^2}{\omega_k(n_k^2-n_j^2)}\right) \\ &= \frac{-\lambda'}{8J}\sum_{k\neq j}\frac{n_k^2+n_j^2+\frac{\lambda'}{4}n_k^2 n_j^2}{\omega_k\omega_j} \ ,\end{aligned} \tag{6.3.14}$$

which is exactly as found in the string theory calculation.

The case of confluent mode numbers is very similar to the closed string [69] where we find a generalized Stieltjes problem which can be solved as in [100]. Let N_k be the multiplicities of the degenerate mode numbers, p_{k,m_k} the momenta and M be the number of distinct mode numbers with $N = \sum_{k=1}^{M} N_k$ the total number of impurities.

$$\begin{aligned}\Delta E &= \sum_{k=1}^{M}\sum_{m_k=1}^{N_k} \frac{\lambda' n_k p^{(2)}_{k,m_k}}{4J\pi\omega_k} + \frac{\lambda' {p_{k,m_k}}^{(1)^2}}{8J\pi^2\omega_k^2} \\ &= -\frac{\lambda'}{16J}\sum_{k=1}^{M}\frac{N_k(N_k-1)}{\omega_k^2}\left(3n_k^2+\frac{\lambda' n_k^4}{2}\right) \\ &\quad +\frac{-\lambda'}{8J}\sum_{\substack{j,k=1\\k\neq j}}^{M} N_j N_k \frac{n_k^2+n_j^2+\frac{\lambda'}{4}n_k^2 n_j^2}{\omega_k\omega_j}\ , \end{aligned} \tag{6.3.15}$$

which again agrees with the string theory prediction.

6.4 Speculations

There is currently no explanation for the three-loop disagreement between the gauge theory and the string theory which has been found in the near-BMN limit and also in the case of the Frolov-Tseytlin spinning string [96]. Of course one possible and disappointing explanation for this could of be that the AdS/CFT duality is only approximate. However another, more optimistic, way of looking at these results is that we have been somewhat fortunate in finding the degree of agreement that we have. There is an order of limits issue on both sides of the duality, specifically, on the string side we first assume large J (or in the more general semi-classical analysis that L, the length, is large) and then we expand in terms of λ' whereas in the gauge theory we first expand in terms of small λ and then expand in $1/L$. The claim was made in [96] that the order of limits does matter and that *a priori* we should not expect to find agreement. That this might be true was first hinted at in [101] and in [96] it was shown that for the Inozemstev spin-chain (which is closely related to the all-loop conjecture used above) the order in which the limits are taken changes

the result. Indeed in [55] the authors discussed a concrete mechanism whereby this disagreement may arise. When we calculate the dilation operator in perturbation theory we consider Feynman diagrams that attach to a number of neighboring lattice sites on a finite length spin-chain. As the loop order increases, this region enlarges until it wraps completely around the trace and at this point the asymptotic methods used in the gauge theory cease to work. There are no wrapping interactions in the open string case however the fact that the generalization of the all-loop gauge answer fails to agree with the near-BMN limit suggest that there are other finite length contributions to the open spin-chain Hamiltonian. Of course we have not actually shown that this all-loop ansatz is the correct generalization though it does seem natural. We should also point out that the quantum string Bethe equations are only valid for large string tension $\sqrt{\lambda}$ and for $L \gg 1$ though we do not need $L \sim g$. That the quantum Bethe ansatz is not appropriate for small L may be seen in the strong coupling limit described above as the ansatz of [69] does not give the right prediction for the anomalous dimension of $L = 4$ and $M = 2$ Konishi descendant operator. One could hope to find a full non-asymptotic Bethe ansatz from the string theory by including higher $1/L^k$ corrections and which could then interpolate between gauge and string theories. One could fix some of these corrections by calculating the $1/J^2$ corrections to the near-BMN string energies however to date this has proven difficult.

Appendix A

Notation and conventions

We have attempted to use a consistent choice of indices and we list these here:

$$
\begin{aligned}
\mu, \nu, \rho = \text{ tangent space vector indices }, 0, \ldots, 9 \qquad & SO(9,1) \\
0, \ldots, 3 \qquad & SO(3,1) \\
m, n = \text{ spacetime indices }, 0, \ldots, 9 \qquad & \text{in ten dimensions} \\
0, \ldots, 3 \qquad & \text{in four dimensions} \\
\alpha, \beta, \gamma, \delta = \text{ spinor indices}, 1, \ldots, 16 \qquad & SO(9,1) \\
1, 2 \qquad & SL(2, \mathbf{C}) \\
M, N = \text{superspace indices} & \\
A, B = 1, \ldots, 8 \qquad & SO(8) \text{ vectors} \\
i, j, k = 1, \ldots, 4 \qquad & SO(4) \text{ vectors} \\
i', j', k' = 5, \ldots, 8 \qquad & SO(4)' \text{ vectors} \\
a, b = 0, 1 \qquad & \text{worldsheet coordinates } (\tau, \sigma) \\
I, J, K, L = 1, 2 \qquad & \text{two MW spinors of equal chirality.}
\end{aligned}
\tag{A.1}
$$

Though on occasion, in order not to resort to exotica, we have diverged from these

prescriptions. The following definitions are used throughout

$$\begin{aligned} R &\text{ Radius of } AdS_5 \text{ and } S^5 \\ N_c &\text{ Number of colors / Units of flux} \\ D &\text{ Dimension operator} \\ D_0 &\text{ Bare dimension operator} \\ \mathcal{R} &\text{ } \mathcal{R}\text{−charge} \\ \Delta_0 &= D_0 - \mathcal{R} \\ J &\text{ String angular momentum} \\ L_v &\text{ Level within a supermultiplet} \\ L &\text{ Spin chain length} \end{aligned} \tag{A.2}$$

The 32×32 Dirac gamma matrices are decomposed into a 16×16 representation according to

$$\begin{aligned} (\Gamma^\mu)_{32\times 32} &= \begin{pmatrix} 0 & \gamma^\mu \\ \bar{\gamma}^\mu & 0 \end{pmatrix} & \gamma^\mu\bar{\gamma}^\nu + \gamma^\nu\bar{\gamma}^\mu &= 2\eta^{\mu\nu} \\ \gamma^\mu &= (1, \gamma^A, \gamma^9) & \bar{\gamma}^\mu &= (-1, \gamma^A, \gamma^9) \\ \gamma^+ &= 1 + \gamma^9 & \bar{\gamma}^+ &= -1 + \gamma^9 \ . \end{aligned} \tag{A.3}$$

In particular, the notation $\bar{\gamma}^\mu$ lowers the $SO(9,1)$ spinor indices α, β:

$$\gamma^\mu = (\gamma^\mu)^{\alpha\beta} \qquad \bar{\gamma}^\mu = (\gamma^\mu)_{\alpha\beta} \ . \tag{A.4}$$

These conventions are chosen to match those of Metsaev in [41]. By invoking κ-

symmetry,

$$\bar{\gamma}^{+}\theta = 0 \implies \bar{\gamma}^{9}\theta = \theta \tag{A.5}$$

$$\bar{\gamma}^{-} = 1 + \bar{\gamma}^{9} \implies \bar{\gamma}^{-}\theta = 2\theta \ . \tag{A.6}$$

The antisymmetric product $\gamma^{\mu\nu}$ is given by

$$\begin{aligned} (\gamma^{\mu\nu})^{\alpha}{}_{\beta} &\equiv \frac{1}{2}(\gamma^{\mu}\bar{\gamma}^{\nu})^{\alpha}{}_{\beta} - (\mu \rightleftharpoons \nu) \\ (\bar{\gamma}^{\mu\nu})^{\alpha}{}_{\beta} &\equiv \frac{1}{2}(\bar{\gamma}^{\mu}\gamma^{\nu})_{\alpha}{}^{\beta} - (\mu \rightleftharpoons \nu) \ . \end{aligned} \tag{A.7}$$

We form the matrices Π and $\tilde{\Pi}$ according to:

$$\begin{aligned} \Pi &\equiv \gamma^{1}\bar{\gamma}^{2}\gamma^{3}\bar{\gamma}^{4} \\ \tilde{\Pi} &\equiv \gamma^{5}\bar{\gamma}^{6}\gamma^{7}\bar{\gamma}^{8} \ . \end{aligned} \tag{A.8}$$

These form the projection operators ($\Pi^2 = \tilde{\Pi}^2 = 1$)

$$\begin{aligned} \Pi_{+} &\equiv \frac{1}{2}(1+\Pi) \qquad & \Pi_{-} &\equiv \frac{1}{2}(1-\Pi) \\ \tilde{\Pi}_{+} &\equiv \frac{1}{2}(1+\tilde{\Pi}) \qquad & \tilde{\Pi}_{-} &\equiv \frac{1}{2}(1-\tilde{\Pi}) \ . \end{aligned} \tag{A.9}$$

The spinors θ^I represent two 32-component Majorana-Weyl spinors of $SO(9,1)$ with equal chirality. The 32-component Weyl condition is $\Gamma_{11}\theta = \theta$, with

$$\Gamma_{11} = \Gamma^{0}\ldots\Gamma^{9} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{32\times 32} \ . \tag{A.10}$$

The Weyl condition is used to select the top 16 components of θ to form the 16-component spinors

$$\theta^{I} = \begin{pmatrix} \theta^{\alpha} \\ 0 \end{pmatrix}^{I} \ . \tag{A.11}$$

It is useful to form a single complex 16-component spinor ψ from the real spinors θ^1 and θ^2:

$$\psi \;=\; \sqrt{2}(\theta^1 + i\theta^2) \; . \tag{A.12}$$

The 16-component Weyl condition $\gamma^9\theta = \theta$ selects the upper 8 components of θ, with

$$\gamma^9 = \gamma^1 \ldots \gamma^8 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{16\times 16} \; . \tag{A.13}$$

The 16-component Dirac matrices γ^μ can, in turn, be constructed from the familiar Spin(8) Clifford algebra, wherein (in terms of $SO(8)$ vector indices)

$$(\gamma^A)_{16\times 16} = \begin{pmatrix} 0 & \gamma^A \\ (\gamma^A)^T & 0 \end{pmatrix} \; , \tag{A.14}$$

and

$$\left\{\gamma^A, \gamma^B\right\}_{16\times 16} = 2\delta^{AB} \qquad \left(\gamma^A(\gamma^B)^T + \gamma^B(\gamma^A)^T = 2\delta^{AB}\right)_{8\times 8} \; . \tag{A.15}$$

The Spin(8) Clifford algebra may be constructed explicitly in terms of 8 real matrices

$$\begin{aligned} \gamma^1 &= \epsilon \times \epsilon \times \epsilon & \gamma^5 &= \tau_3 \times \epsilon \times 1 \\ \gamma^2 &= 1 \times \tau_1 \times \epsilon & \gamma^6 &= \epsilon \times 1 \times \tau_1 \\ \gamma^3 &= 1 \times \tau_3 \times \epsilon & \gamma^7 &= \epsilon \times 1 \times \tau_3 \\ \gamma^4 &= \tau_1 \times \epsilon \times 1 & \gamma^8 &= 1 \times 1 \times 1 \; , \end{aligned} \tag{A.16}$$

with

$$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ . \tag{A.17}$$

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