Testing Gauge/Gravity Duality: The Eleven-Dimensional PP-wave

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Abstract

The gauge/gravity duality in the interaction between M theory objects has taught us a lot about quantum gravity. The eleven-dimensional PP-wave background provides a new arena for exploring this duality beyond flat and almost flat, i.e., weakly curved, backgrounds. In this thesis we discuss the gauge theories that describe the dynamics of interacting M theory objects, the supergravity calculations that capture these dynamics, the comparison of the two sides, and various objects (such as gravitons and membranes) in the eleven-dimensional PP-wave background. We only consider the one-loop gauge theory and linearized supergravity approximations.

Contents

Chapter 1

Introduction

The five superstring theories in ten dimensions, Type IIA, Type IIB, Type I, Heterotic $SO(32)$, Heterotic $E_8 \times E_8$ are known to be related to each other via a web of dualities [1]. Underlying these theories, there is believed to be a fundamental M theory whose low energy effective theory is the eleven-dimensional supergravity.

Matrix theory was proposed by [2, 3] as a candidate of M theory for flat target space. This is a $(0+1)$-dimensional quantum mechanical theory whose degrees of freedom reside in N by N matrices as a noncommutative generalization of the usual concept of target space coordinates, and it is argued that it should provide a nonperturbative description of quantum gravity in a manifestly unitary way.

The generalization of Matrix theory to a generic nonflat background was initiated in [4, 5]. In [6, 7], this generalization was carried out in detail to linear order in the background fields for a weakly curved background (i.e., a background whose metric is $g_{IJ} = \eta_{IJ} + h_{IJ}$, with η_{IJ} being the flat metric, and $|h_{IJ}| << 1$) which is independent of the light like direction x^-.

Not very long ago, another useful technique, different from the weakly curved background approximation, for understanding nonflat backgrounds was put forward in the context of string/M theory. This is the Penrose limit [8, 9, 10], a limit in which one zooms in, roughly speaking, to the close neighborhood of a null geodesic. In general, a space arising from Penrose limit may be highly curved, hence not restricted to the weak background approximation. More importantly, when taking the limit, the resulting background inherits the (super)symmetries of the original space. Hence one expects that upon taking the Penrose limit, the M theory in a nonflat background will become more tractable, provided that the original background has a large amount of (super)symmetries.

The Matrix theory in an eleven-dimensional Parallel-Plane-wave (henceforth abbreviated as PP-wave) background proposed in [10] is a good example of such an application of the Penrose limit. As a matter of fact, the main subject of study in this thesis will be the dynamics of this Matrix theory and other theories related to it. Here the original space is the product of four-dimensional Anti de Sitter space and seven-sphere $AdS_4 \times S^7$, or the product of seven-dimensional Anti de Sitter space and four-sphere $AdS_7 \times S^4$, and the eleven-dimensional PP-wave is the space resulting from their Penrose limits.

Although the Matrix theory (in a general background) has the form of a (0+1)-dimensional quantum mechanics, and can be interpreted as describing N D0-branes in the IIA string theory obtained from compactifying the eleven-dimensional M theory to ten dimensions, it does not only describe point particles. Philosophically, the reason is, since the Matrix theory is supposed to be a realization of the fundamental M theory, it must have the capability to describe the dynamics of all M theory objects (e.g., M2-brane, M5-brane from the perspective of eleven-dimensional supergravity, the fundamental string, NS5-brane, and higher-dimensional D-branes from the ten-dimensional IIA string perspective), after including all sectors with different values of N. Technically, the reasons are, to name a few, the Matrix theory can be obtained by "discretizing" the supermembrane theory [11, 12], the D0-brane theory can be T-dualized to give theories describing higher-dimensional D-branes, and by taking the "continuum limit" of the Matrix theory, one gets field theories living in higher dimensions. Depending on the problem at hand, e.g., the investigation of membrane scattering, one may choose the $(0 + 1)$-dimensional theory, or, higher-dimensional field theories as long as it is valid, whichever is more convenient, and the computed physical quantities should be the same. This is one of the points that the discussions in Sections 3.1 and 3.2 try to make.

The Matrix theory and related higher-dimensional field theories are nonabelian gauge theories, with which we can do perturbative loop computations in expansion of the gauge coupling. One interesting thing observed is the gauge/gravity duality in the dynamics of interacting M theory objects [13, 14, 15, 16, 17, 18, 19, 20, 7, 21, 22, 23, 24], where results of quantum loop computations on the gauge theories' side are shown to agree with results of solving classical equations of motion on the supergravity side. This duality is physically motivated recalling that eleven-dimensional supergravity is the low energy limit of M theory, although supersymmetry also seems to be a crucial ingredient to bridge the disparate regions of validity on the two sides. Of course, similar gauge/gravity duality also appears in many other settings, such as the well known AdS/CFT correspondence in IIB string theory (in that case, taking the Penrose limit actually gives us access to full string theory, not just the classical supergravity, on the AdS side [10]). As stated in the title, this thesis is devoted to the study of the gauge/gravity duality for interacting M theory objects in the eleven-dimensional PP-wave background. In this study we shall restrict our attention to one-loop on the gauge theory side, and to first order in κ_{11}^2 on the supergravity side, that is, linearized supergravity. As our contribution to the subject, this thesis discusses some generalities in the gauge/gravity comparison, the results on two graviton interaction in the absence of transfer of the momentum along the light-like x^- direction, the so-called M-momentum, and some initial steps taken towards understanding two membrane interaction with M-momentum transfer, based mainly on works in collaboration with Hok Kong Lee and Tristan McLoughlin.

The thesis is organized as follows:

In Chapter 2 we briefly review features of the eleven-dimensional PP-wave, giving the expressions for its 38 Killing vectors, 32 Killing spinors, and the (anti)commutators of the symmetry superalgebra [25, 26, 27], also describing how it arises as the Penrose limit of $AdS \times S$ space [8, 9, 28].

In Chapter 3 we discuss the gauge theories that describe M-theory objects in the eleven-dimensional PP-wave background.

Section 3.1 is devoted to the Matrix theory proposed by [10]. In this section we mainly review three different ways of deriving this matrix theory: the approach taken by [10] which starts with a superparticle and then nonabelianizes the resulting theory; the one taken by [29] where the Matrix theory is obtained by "discretizing" the supermembrane in 11-D PP-wave background; and the one taken by [29] which uses the framework developed in [6, 7] to derive the Matrix theory from the dynamics of multi D0-branes in the IIA supergravity background arising from a space-like compactification of 11-D PP-wave, in the weakly curved background approximation. We pay particular attention to the multi D0-brane approach because firstly it illustrates the connection between the dynamics in eleven dimensions and that in ten dimensions, as well as the appropriate limit [4] in which this connection is made, and secondly it illustrates how, given a large number of (super)symmetries, sometimes the weak background matrix theory actually needs no correction and is exact.

Section 3.2 is devoted to the three-dimensional theory describing multi spherical membranes in 11-D PP-wave. This theory was first obtained by [30] as the continuum limit of the matrix theory proposed by [10] expanded around its k-membrane vacuum. In Subsection 3.2.1 we present an alternative derivation which obtains this theory directly from the supermembrane theory, expanding around the single membrane vacuum to get an abelian theory and then carrying out a nonabelian generalization. In Subsection 3.2.2 we discuss the vacua and instantons of this theory in the two membrane case, presenting the instanton solution which interpolates between the flux one vacuum and the flux zero trivial vacuum, and briefly comment on the application of this instanton solution to the study of M-momentum transfer between two spherical membranes in an 11-D PP-wave background.

Section 3.3 is an application based on [21]. In short, [21] computes the two-graviton one-loop effective action for the Matrix theory in the eleven-dimensional PP-wave background, and compares it to the effective action on the supergravity side in the same background. Agreement is found for the effective action on both sides, to all orders of μ (i.e., beyond the weak background approximation), which provides evidence for the Matrix theory proposed by [10] being the correct description of M theory in the eleven-dimensional PP-wave background, and also points to the existence of a supersymmetric nonrenormalization theorem in this background. This section is mainly the computation on the gauge theory side given in [21].

In Chapter 4 we discuss the treatment of M theory objects' interaction on the supergravity side. On the supergravity side we use the method of source-probe analysis, which is valid when the source is much heavier than the probe.

Section 4.1 works out the light-cone Lagrangian, which is the supergravity quantity that should in the end be compared with the gauge theory one-loop effective Lagrangian, for a point particle and a membrane in an arbitrary (yet static, i.e., x^+-independent) background. In doing so, we give a careful treatment of the constrained Hamiltonian mechanics of the systems concerned.

Section 4.2 addresses the issue of finding the background fields that the probe feels, describing the diagonalization of the supergravity field equations for arbitrary static sources. The idea here is quite straightforward, and the results given are technical in nature, but since they are necessary for solving the field equations and have not been explicitly given elsewhere to the best of our knowledge, we present them in this section.

The above two sections provide the framework for the computation on the supergravity side. In Section 4.3 we apply the general formalism to the specific application of two graviton interaction in the absence of M-momentum transfer, completing our investigation of gauge/gravity duality in this particular physical problem. This section is based on the supergravity computation of [21].

In Chapter 5 we briefly review supersymmetric nonrenormalization theorems in the Matrix theory quantum mechanics in flat space, following [31, 32, 33, 34, 35, 36, 37], and comment on the generalization to a 11-D PP-wave background.

In Chapter 6, we make some concluding remarks, discussing possible future directions to pursue.

Chapter 2

Review of the Eleven-Dimensional PP-wave Geometry

The fields of eleven-dimensional supergravity are the metric $g_{\mu\nu}$, the three-form gauge potential $A_{\mu\nu\rho}$, and the gravitino ψ_μ.

In the eleven-dimensional PP-wave, the gravitino field ψ_μ vanishes, whereas the nonzero components of the metric $g_{\mu\nu}$ and the four-form field strength $F_{\mu\nu\rho\lambda}$ are given by

$$g_{+-} = 1,\ g_{++} = -\mu^2 \left[\frac{1}{9} \sum_{i=1}^{3} (x^i)^2 + \frac{1}{36} \sum_{a=4}^{9} (x^a)^2 \right],\ g_{AB} = \delta_{AB} \tag{2.1}$$

$$F_{123+} = \mu \tag{2.2}$$

Here μ is a parameter with the dimension of inverse length.

In our conventions, $\mu, \nu, \rho, \ldots$ take the values $+, -, 1, \ldots, 9$; $A, B, C, \ldots$ take the values $1, \ldots, 9$; $i, j, k, \ldots$ take the values $1, \ldots, 3$; and $a, b, c, \ldots$ take the values $4, \ldots, 9$. This solution to eleven-dimensional supergravity was first given by [25], and it also goes under the name KG space (where KG stands for Kowalski-Glikman).

2.1 Isometries and Supersymmetries of the 11-D PP-wave

The explicit expressions for the Killing vectors (as well as the algebra of the isometry group) and Killing spinors of the 11-D PP-wave were first given in [26]. Here we follow the exposition of the same subject given by [27]. A slight change of the notation of [27] is necessary to bring it in accordance with ours: exchange x^+ and x^-; change μ to $-\mu$; also in [27] the transverse index i goes from 1 to 9, thus corresponding to our transverse index A.

The isometry group of the 11-D PP-wave has 38 generators, coinciding with the dimension of the isometry algebras of $AdS_4 \times S^7$ and $AdS_7 \times S^4$. This is no accident, because the former solution can be obtained from the latter solutions via a Penrose limit [9], which we shall discuss shortly. The Killing vectors of 11-D PP-wave are

$$
\begin{aligned}
\xi_{e_-} &= -\partial_-, \quad \xi_{e_+} = -\partial_+ \\
\xi_{e_i} &= -\cos\left(\frac{\mu x^+}{3}\right)\partial_i - \sin\left(\frac{\mu x^+}{3}\right)\frac{\mu x^i}{3}\partial_- \\
\xi_{e_i^*} &= -\sin\left(\frac{\mu x^+}{3}\right)\frac{\mu}{3}\partial_i + \cos\left(\frac{\mu x^+}{3}\right)\frac{\mu^2 x^i}{9}\partial_- \\
\xi_{e_a} &= -\cos\left(\frac{\mu x^+}{6}\right)\partial_a - \sin\left(\frac{\mu x^+}{6}\right)\frac{\mu x^a}{6}\partial_- \\
\xi_{e_a^*} &= -\sin\left(\frac{\mu x^+}{6}\right)\frac{\mu}{6}\partial_a + \cos\left(\frac{\mu x^+}{6}\right)\frac{\mu^2 x^a}{36}\partial_- \\
\xi_{M_{ij}} &= x^i\partial_j - x^j\partial_i \\
\xi_{M_{ab}} &= x^a\partial_b - x^b\partial_a
\end{aligned}
\tag{2.3}
$$

A little explanation of the subscripts in the above expressions is in order: a generic so-called Cahen-Wallach space, of which the 11-D PP-wave is a special case, can be written as the coset space G/K, where G is the group whose algebra is spanned by $\{e_-, e_+, e_A, e_A^*\}$, and K is the subgroup whose algebra is spanned by e_A^*; M_{ij}, M_{ab} are the generators of $SO(3)$ and $SO(6)$, which are the symmetry transformations preserving the (++) component of the 11-D PP-wave metric.

Next let us look at the Killing spinors, which are determined by the condition that the supersymmetry variation of the gravitino field ψ_μ must vanish

$$\nabla_M \epsilon = \frac{1}{288} F_{PQRS} (\Gamma^{PQRS}{}_M + 8\Gamma^{PQR}\delta^S_M)\epsilon \tag{2.4}$$

where $\nabla_M \equiv \partial_M + \frac{1}{4}\omega_M^{AB}\Gamma_{AB}$ and the gamma matrices are those of $SO(10,1)$. There are 32 independent solutions to this equation, making the 11-D PP-wave a maximally supersymmetric solution of 11-D supergravity. These 32 Killing spinors depend only on x^+ and not the other coordinates, and are given by

$$\begin{aligned} \epsilon = \quad & \left[\cos\left(\frac{\mu x^+}{4}\right) + \sin\left(\frac{\mu x^+}{4}\right)\Gamma_{123}\right]\psi_- + \left[\cos\left(\frac{\mu x^+}{12}\right) + \sin\left(\frac{\mu x^+}{12}\right)\Gamma_{123}\right]\psi_+ \\ & + \frac{\mu}{6}\left(\sum_i x^i\Gamma_i - \frac{1}{2}\sum_a x^a\Gamma_a\right)\left[-\sin\left(\frac{\mu x^+}{12}\right) - \cos\left(\frac{\mu x^+}{12}\right)\Gamma_{123}\right]\Gamma_-\psi_+ \end{aligned} \tag{2.5}$$

where $\psi_\pm$ are arbitrary constant spinors satisfying $\Gamma_\pm\psi_\pm = 0$

Now let us look at the symmetry superalgebra. In the superalgebra, the commutator between two bosonic generators is realized as the Lie bracket of the two corresponding Killing vector fields. The commutator between a bosonic generator and a fermionic generator is realized as the spinorial Lie derivative $L_X\epsilon \equiv X^M\nabla_M\epsilon + \frac{1}{4}\nabla_{[M}X_{N]}\Gamma^{MN}\epsilon$, where X and ϵ are the corresponding Killing vector and Killing spinor, respectively. The anti-commutator between two fermionic generators is realized as the bilinear of the corresponding Killing spinors, namely, for two Killing spinors ϵ_1, ϵ_2, one simply takes $\bar{\epsilon}_1\Gamma^M\epsilon_2$ to get a Killing vector.

The commutators of the bosonic generators are

$$
\begin{aligned}
&[e_+, e_A] = e_A^*, \quad [e_+, e_i^*] = -\frac{\mu^2}{9} e_i, \quad [e_+, e_a^*] = -\frac{\mu^2}{36} e_a \\
&[e_i^*, e_j] = -\frac{\mu^2}{9} e_- \delta_{ij}, \quad [e_a^*, e_b] = -\frac{\mu^2}{36} e_- \delta_{ab} \\
&[M_{AB}, e_C] = -\delta_{AC} e_B + \delta_{BC} e_A \\
&[M_{AB}, e_C^*] = -\delta_{AC} e_B^* + \delta_{BC} e_A^*
\end{aligned}
\tag{2.6}
$$

The commutators of the bosonic generators and the fermionic generators, with $Q_\pm$ being the 32 fermionic generators generating shifts proportional to the constant spinors $\psi_\pm$ parametrizing the Killing spinors, are

$$
\begin{aligned}
&[e_-, Q_\pm] = 0, \quad [e_+, Q_-] = \frac{\mu}{4} \Gamma_{123} Q_-, \quad [e_+, Q_+] = \frac{\mu}{12} \Gamma_{123} Q_+ \\
&[e_i, Q_-] = \frac{\mu}{6} \Gamma_{123} \Gamma_i \Gamma_- Q_+, \quad [e_a, Q_-] = \frac{\mu}{12} \Gamma_{123} \Gamma_a \Gamma_- Q_+ \\
&[e_i^*, Q_-] = -\frac{\mu^2}{18} \Gamma_i \Gamma_- Q_+, \quad [e_a^*, Q_-] = -\frac{\mu^2}{72} \Gamma_a \Gamma_- Q_+ \\
&[M_{AB}, Q_\pm] = \frac{1}{2} \Gamma_{AB} Q_\pm
\end{aligned}
\tag{2.7}
$$

The anticommutators between fermionic generators are (suppressing spinor indices)

$$
\begin{aligned}
&\{Q_-, Q_-\} = -\Gamma_+ C^{-1} e_- \\
&\{Q_-, Q_+\} = -\sum_A \Gamma^A C^{-1} e_A - \frac{3}{\mu} \sum_i \Gamma_{123} \Gamma^i C^{-1} e_i^* - \frac{6}{\mu} \sum_a \Gamma_{123} \Gamma^a C^{-1} e_a^* \\
&\{Q_+, Q_+\} = -\Gamma_- C^{-1} e_+ - \frac{\mu}{6} \sum_{ij} \Gamma_- \Gamma_{123} \Gamma^{ij} C^{-1} M_{ij} + \frac{\mu}{12} \sum_{ab} \Gamma_- \Gamma_{123} \Gamma^{ab} C^{-1} M_{ab}
\end{aligned}
\tag{2.8}
$$

with C being the charge-conjugation matrix.

2.2 How the PP-wave Arises as a Penrose Limit of $AdS \times S$

We follow [9, 28] in our review of the Penrose limit in the context of 11-D supergravity (again, we exchange the x^+ and x^- in their notation).

The Penrose limit of a space-time can be thought as a blowup of the space-time along a null geodesic. The starting point for the derivation of the Penrose limit is the metric g and p-form potential ($p = 3$ in the 11-D supergravity context) A_p in a neighborhood of a conjugate-point-free [1] segment of a null geodesic γ in the original space-time given by

$$g = dV\left(dU + \alpha dV + \sum_A \beta_A dY^A\right) + \sum_{AB} C_{AB} dY^A dY^B \tag{2.9}$$

where α, β_A, C_{AB} are functions of all the coordinates, and C_{AB} is a symmetric positive-definite matrix. In this coordinate system, the null geodesic is parametrized by U, with $V = Y^A = 0$. Also

$$A_{UB_1B_2\ldots B_{p-1}} = A_{UVB_1B_2\ldots B_{p-2}} = 0 \tag{2.10}$$

where this form of A_p can be achieved by using its gauge freedom.

Then we rescale the coordinates to

$$U = u, \quad V = \Omega^2 v, \quad Y^A = \Omega y^A \tag{2.11}$$

with Ω being a positive real constant. Acting with this diffeomorphism on the tensor fields of the theory we obtain the Ω-dependent family of fields $g(\Omega), A_p(\Omega)$. Then the

[1] Recall the following definition of conjugate points [38]: Use γ to denote a geodesic. A solution η^a of the geodesic deviation equation $v^a\nabla_a(v^b\nabla_b\eta^c) = -R_{abd}{}^c\eta^b v^a v^d$ is called a Jacobi field on γ. A pair of points $p, q \in \gamma$ are said to be conjugate points, if there exists a Jacobi field η^a which is not identically zero but vanishes at both p and q. Roughly speaking, two points p and q are conjugate if an "infinitesimally nearby" geodesic intersects γ at both p and q.

Penrose limit of the original space-time is defined as

$$\bar{g} \equiv \lim_{\Omega \to 0} \Omega^{-2} g(\Omega), \quad \bar{A}_p \equiv \lim_{\Omega \to 0} \Omega^{-p} A_p(\Omega) \tag{2.12}$$

The limiting fields depend only on the coordinate u, which is the affine parameter along the null geodesic. The space-time resulting from this limit is expressed in Rosen coordinates. One can change to the more usual Brinkman (or harmonic) coordinates (we will not give the details of this coordinate transformation here) in which the metric takes the form

$$\bar{g} = 2dx^+ dx^- + \left(\sum_{BC} A_{BC}(x^+) x^B x^C \right) (dx^+)^2 + \sum_A dx^A dx^A \tag{2.13}$$

When A_{BC} is constant the above metric is a Lorentzian symmetric Cahen-Wallach space.

The nice thing about the Penrose limit is that, if the original space-time is a solution to the supergravity field equations, so is its Penrose limit; also, the number of (super)symmetries of the original space-time does not decrease in this limit. Hence, even if the M-theory of concern in the original space-time is beyond our analytical capability, we can still take a Penrose limit, usually making the problem more tractable and still getting physically meaningful data.

Now let us see how the Penrose limit works for the original space-times of interest, namely, $AdS_4 \times S^7$ and $AdS_7 \times S^4$, which are the near horizon geometries of M2 and M5 branes [2]. If we define $\rho \equiv \frac{R_{AdS}}{R_S}$ as the ratio of the radii of the AdS part and the S part, then $\rho = \frac{1}{2}$ for $AdS_4 \times S^7$ and $\rho = 2$ for $AdS_7 \times S^4$.

[2] A quick review of the near horizon geometries following [39]: The metric created by N M2 branes is $ds^2 = f^{-2/3}(-dt^2 + d\vec{x}^2) + f^{1/3}(dr^2 + r^2 d\Omega_7^2)$, $f = 1 + (R/r)^6$, where $R^6 = 32\pi^2 N l_p^6$ and $\vec{x}$ is a two-dimensional Euclidean vector. In the near horizon region $r << R$, so $f \approx (R/r)^6$. Plugging this into the metric and letting $r = R^{3/2}(u/2)^{1/2}$, one gets $ds^2 = (R/2)^2 \left[(du^2/u^2) + u^2(-dt^2 + d\vec{x}^2) \right] + R^2 d\Omega_7^2$, which is of the form $AdS_4 \times S^7$ with the AdS part in the Poincaré coordinates, and $R_{AdS} = R/2$, $R_S = R$. The derivation of the near horizon geometry of N M5 branes is similar.

The metric of the original $AdS \times S$ space-time is

$$R_S^{-2} g = \rho^2 \left[-d\tau^2 + (\sin\tau)^2 \left(\frac{dr^2}{1+r^2} + r^2 d\Omega_p^2 \right) \right] + d\psi^2 + (\sin\psi)^2 d\Omega_{8-p}^2 \qquad (2.14)$$

with $p = 2$ for the M2 case, and $p = 5$ for the M5 case. One can change the coordinates (ψ, τ) to (u, v)

$$u = \psi + \rho\tau, \quad v = \psi - \rho\tau \qquad (2.15)$$

in terms of which the metric becomes

$$R_S^{-2} g = dudv + \rho^2 \sin^2\left(\frac{u-v}{2\rho}\right) \left(\frac{dr^2}{1+r^2} + r^2 d\Omega_p^2 \right) + \sin^2\left(\frac{u+v}{2}\right) d\Omega_{8-p}^2 \quad (2.16)$$

One then takes the Penrose limit along the null geodesic parametrized by u. In practice this amounts to dropping the dependence on coordinates other than u, and replacing the spherical metric $d\Omega_{8-p}^2$ with the flat space metric $ds^2(E^{8-p})$ (because in the Penrose limit we are looking at a small region near the null geodesic on the sphere S^{8-p}, which is effectively flat space). The metric then becomes

$$R_S^{-2} \bar{g} = dudv + \rho^2 \sin^2\left(\frac{u}{2\rho}\right) ds^2(E^{p+1}) + \sin^2\left(\frac{u}{2}\right) ds^2(E^{8-p}) \qquad (2.17)$$

with $ds^2(E)$ denoting the metric of Euclidean flat space.

The above $\bar{g}$ is in Rosen coordinates, so let us change it to Brinkman coordinates. First introduce coordinates y^A, $A = 1, ..., 9$ so that

$$R_S^{-2} \bar{g} = dudv + \sum_A \frac{\sin^2(\lambda_A u)}{(2\lambda_A)^2} dy^A dy^A \qquad (2.18)$$

where

$$\lambda_A = \frac{1}{2\rho}, \quad A = 1, ..., p+1; \quad \lambda_A = \frac{1}{2}, \quad A = p+2, ..., 9 \qquad (2.19)$$

then change coordinates to (x^+, x^-, x^A) where

$$x^+ = \frac{u}{2}, \quad x^- = v - \frac{1}{4}\sum_A y^A y^A \frac{\sin(2\lambda_A u)}{2\lambda_A}, \quad x^A = y^A \frac{\sin(\lambda_A u)}{2\lambda_A} \tag{2.20}$$

The metric becomes

$$R_S^{-2}\bar{g} = 2dx^+dx^- - 4\left(\sum_A \lambda_A^2 x^A x^A\right)(dx^+)^2 + \sum_A dx^A dx^A \tag{2.21}$$

As is easily seen, the $\rho = \frac{1}{2}$ and $\rho = 2$ cases are isometric, with the explicit diffeomorphism given by $x^+ \to \frac{1}{2}x^+$, $x^- \to 2x^-$, and $(x^1, ..., x^6, x^7, ..., x^9) \to (x^4, ..., x^9, x^1, ..., x^3)$. Also, this metric is just the 11-D PP-wave, as is seen by rescaling all the coordinates by R_S, and then scaling x^+ and x^- oppositely by $\mu/3$.

Finally, let us give an alternative (and easier to remember) way of taking the Penrose limit of $AdS_{p+2} \times S^{9-p}$ ($p = 2$ or 5) to get the 11-D PP-wave, which is parallel to the treatment of $AdS_5 \times S^5$ in [10].

Let us use the global coordinates (τ, ρ, Ω_p) for the AdS part, and coordinates $(\beta, \theta, \Omega'_{7-p})$ similar to those in [10] for the S part. The metric is

$$ds^2 = \tilde{\rho}^2 R^2(-\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho d\Omega_p^2) + R^2(\cos^2\theta d\beta^2 + d\theta^2 + \sin^2\theta d\Omega'^{\,2}_{7-p}) \tag{2.22}$$

Notice that now we use $\tilde{\rho}$ to denote the ratio $\frac{R_{AdS}}{R_S}$, with ρ denoting a coordinate.

Consider the null geodesic γ given by $\rho = 0, \theta = 0, \beta = \tilde{\rho}\tau$. To zoom in to the neighborhood of γ, we introduce $\tilde{x}^{\pm} = \frac{\tau \pm (\beta/\tilde{\rho})}{2}$, and perform the rescaling

$$\tilde{x}^+ = x^+, \ \tilde{x}^- = \frac{x^-}{\tilde{\rho}^2 R^2}, \ \rho = \frac{r}{\tilde{\rho}R}, \ \theta = \frac{y}{R} \tag{2.23}$$

Now let us take the $R \to \infty$ limit. The infinite terms (i.e., terms proportional to R^2)

in the metric (2.22) cancel out, and we get

$$ds^2 = -4dx^+dx^- - (r^2 + \tilde{\rho}^2 y^2)(dx^+)^2 + (dr^2 + r^2 d\Omega_p^2) + (dy^2 + y^2 d\Omega'^2_{7-p}) \tag{2.24}$$

Notice that in the above metric, the ++ component arises from expanding $\cosh\rho$ and $\cos\theta$ to second order in ρ and θ, respectively. This metric is again just the 11-D PP-wave, as can be seen by rescaling x^+ by $\frac{\mu}{3}$ and x^- by $-\frac{3}{2\mu}$.

Chapter 3

The Gauge Theories Describing M-theory Objects in an Eleven-Dimensional PP-wave Geometry

First, a few remarks on the notations and conventions used in the rest of this thesis: other authors' results are cited at various places, however different authors use their own notations and conventions, differing by extra minus signs and numerical factors. We do not always bother to recast them in a uniform way, except when necessary, e.g. when comparing gauge theory results with supergravity results where these factors really do matter.

3.1 Matrix Theory

The action of Matrix theory in the 11-D PP-wave background was first proposed by Berenstein, Maldacena, and Nastase (BMN) in [10]. It is given by

$$\begin{aligned} S = \int dt \mathrm{Tr} \Bigg\{ \quad & \sum_{I=1}^{9} \frac{1}{2R} (D_0 X^I)^2 + i\psi^T D_0 \psi + \frac{(M^3 R)^2}{4R} \sum_{I,J=1}^{9} [X^I, X^J]^2 \\ & + (M^3 R) \sum_{J=1}^{9} \psi^T \gamma^J [\psi, X^J] + \frac{1}{2R} \left[-(\frac{\mu}{3})^2 \sum_{i=1}^{3} (X^i)^2 - (\frac{\mu}{6})^2 \sum_{a=4}^{9} (X^a)^2 \right] \\ & - i\frac{\mu}{4} \psi^T \gamma_{123} \psi - \frac{(M^3 R)\mu}{R} i \sum_{i,j,k=1}^{3} \epsilon_{ijk} (X^i X^j X^k) \Bigg\} \end{aligned} \tag{3.1}$$

where $D_0X^I = \partial_t X^I - i[X_0, X^I]$, and $D_0\psi = \partial_t\psi - i[X_0, \psi]$. Here X_0, X^I, and ψ are all N by N hermitian matrices. M is the 11-dimensional Planck mass, and R is the radius of light-like compactification in Discrete Light Cone Quantization (DLCQ) [3]. Eqn. (3.1) is a one-parameter generalization of the flat-space formula of [2].

3.1.1 Derivations

There are many derivations of (3.1). We will explain three of them. Different approaches are motivated by different (although usually related) physical pictures, and they give answers that agree. Being maximally supersymmetric (that is, having 32 supersymmetries, 16 of them linearly realized, and the other 16 nonlinearly realized) is a very restrictive condition, with very few theories satisfying it, so agreement is not surprising.

I. The BMN Approach

One approach is that taken by [10]. The starting point is the κ-symmetric action of a superparticle in the 11-D PP-wave

$$S = \int dt\; e^{-1} L_t^A L_t^A \tag{3.2}$$

where L_t^A is the pull-back of the 11-D PP-wave supervielbeins, which can be obtained by taking the Penrose limit of the supervielbeins of $AdS \times S$ spaces given in [40]. Upon gauge fixing the κ symmetry by choosing the fermionic light-cone gauge, and also fixing the bosonic light-cone gauge by setting $e = 1, x^+ = t$, S takes the form of free massive bosons X and fermions ψ with the masses $\sim \mu$. Then one generalizes this action to a nonabelian one by promoting the fields X and ψ to $N \times N$ hermitian matrices. One then adds the Myers term [41] $\sim \mu\epsilon_{ijk}\mathrm{Tr}(X^iX^jX^k)$ and the usual flat-space commutator terms. In this way one gets the matrix theory in 11-D PP-wave background.

II. Supermembrane Approach

Another approach, given in [29], is also based on the eleven-dimensional perspective. However, the starting point is not the superparticle, but the supermembrane.

The basic idea is the same as that in flat space [11].

First one writes out the action for the supermembrane in 11-D PP-wave, which consists of the kinetic term and the Wess-Zumino term

$$S = \int d^3\sigma \left[-\sqrt{-\det(\Pi_i^r \Pi_j^s \eta_{rs})} - \frac{1}{6}\epsilon^{ijk}\Pi_i^A \Pi_j^B \Pi_k^C B_{CBA} \right] \tag{3.3}$$

where $Z^M(\sigma) = (X^\mu(\sigma), \theta^\alpha(\sigma))$ are the target superspace embedding coordinates, $\Pi_i^A = \frac{\partial Z^M}{\partial \sigma^i} E_M^A$ is the pull-back of the supervielbein E_M^A to the membrane world-volume, and B_{MNP} is the three-form superfield whose bosonic part is the three-form $A_{\mu\nu\rho}$. The bosonic part of this action is worked out in Subsection 4.1.2 (see eqn. (4.32)), as a simple example of constructing the light cone Lagrangian for a membrane in an arbitrary eleven-dimensional supergravity background. As for the fermionic part of the action, one again uses the expressions for the supervielbeins in terms of the component world-volume and background fields given in [40], imposes fermionic light cone gauge to fix κ-symmetry, similar to the superparticle approach. The resulting supermembrane Hamiltonian is, in the notation of [29]

$$\begin{aligned} H = \int d^2\sigma \quad & \left\{ \frac{1}{p^+} \left[\frac{P_A^2}{2} + \frac{1}{4}\{X^A, X^B\}^2 + \frac{1}{2}\left(\left(\frac{\mu p^+}{3}\right)^2 (X^i)^2 + \left(\frac{\mu p^+}{6}\right)^2 (X^a)^2 \right) \right. \right. \\ & \left. \left. - \frac{\mu p^+}{6}\epsilon^{ijk}\{X^i, X^j\}X^k \right] - \frac{i}{2}\Psi^T\gamma^A\{X^A, \Psi\} - \frac{i}{4}\mu\Psi^T\gamma^{123}\Psi \right\} \end{aligned} \tag{3.4}$$

with the Poisson bracket defined as $\{X^A, X^B\} \equiv (\partial_1 X^A \partial_2 X^B - \partial_2 X^A \partial_1 X^B)$, not to be confused with an anti-commutator.

Now to pass to the Matrix theory action, one has to "regularize" the above supermembrane theory using the prescription of [11], which, roughly speaking, says one should expand functions on the membrane using a complete set of basis functions, and truncate the basis to a finite subset. The result is that one replaces functions on the membrane with hermitian $N \times N$ matrices, integrals with traces, and Poisson brackets with commutators

$$X^A(\sigma) \to \frac{1}{N} X^A, \quad \Psi(\sigma) \to \Psi, \quad p^+ \int d^2\sigma \to \frac{1}{R}\mathrm{Tr}, \quad \{,\} \to -i[,] \tag{3.5}$$

This gives the action (3.1).

III. Multiple D0-brane Approach

Yet another approach is from the ten-dimensional perspective, by considering the dynamics of multi D0-branes in IIA string theory, as given in [29]. That analysis is based on [6, 7], which gives the matrix theory in a general weakly curved background.

In [6] it was proposed that up to terms linear in the background metric perturbation h_{IJ} ($g_{IJ} = \eta_{IJ} + h_{IJ}$) and three-form A_{IJK}, with $I, J, K = 0, ..., 10$, the matrix theory action in a weakly curved (i.e., $|h_{IJ}| << 1$, $|A_{IJK}| << 1$, noting that A_{IJK} is also dimensionless) general 11-D background is given by, in the notation of that paper

$$S = S_{\text{flat}} + S_{\text{weak}} \tag{3.6}$$

with S_{flat} being the formula for flat space

$$S_{\text{flat}} = -\frac{1}{2R}\int d\tau Tr\left\{-D_\tau X_i D_\tau X_i + \frac{1}{2}[X_i, X_j][X_i, X_j] + \Theta_\alpha D_\tau \Theta_\alpha - \Theta_\alpha \gamma^i_{\alpha\beta}[X_i, \Theta_\beta]\right\} \tag{3.7}$$

with $i, j = 1, ..., 9$, and R being the radius of the light-like compactified x^- direction as in (3.1). The additional term

$$\begin{aligned} S_{\text{weak}} = \quad & \int d\tau \sum_{n=0}^{\infty} \sum_{i_1,...,i_n} \frac{1}{n}\bigg(\frac{1}{2}T^{IJ(i_1...i_n)}\partial_{i_1}...\partial_{i_n} h_{IJ}(0) + J^{IJK(i_1...i_n)}\partial_{i_1}....\partial_{i_n} A_{IJK}(0) \\ & + M^{IJKLMN(i_1...i_n)}\partial_{i_1}....\partial_{i_n} A^D_{IJKLMN}(0) + \text{fermionic terms}\bigg) \end{aligned} \tag{3.8}$$

with A^D being the six-form potential dual to the three-form A. Note that S_{weak} is in the form of the moments of the currents, the moments of the stress tensor

$T^{IJ(i_1 \dots i_n)}$, those of the membrane current $J^{IJK(i_1 \dots i_n)}$, and those of the five-brane current $M^{IJKLMN(i_1 \dots i_n)}$, coupling to the background fields. The currents are obtained by taking a trace which is symmetrized over all orderings of terms of the forms $\dot{X}^i$, $F_{ij} \equiv i[X^i, X^j]$, Θ, and $[X^i, \Theta]$ (we denote this symmetrized trace as STr). As examples,

$$\begin{aligned} T^{++} &= \frac{1}{R}\mathrm{STr}(1) \\ T^{+-} &= \frac{1}{R}\mathrm{STr}\left(\frac{1}{2}\dot{X}^i\dot{X}^i + \frac{1}{4}F_{ij}F_{ij} + \frac{1}{2}\Theta\gamma^i[X^i, \Theta]\right) \end{aligned} \tag{3.9}$$

There are two types of terms which contribute to higher moments of the currents. For example, the higher moments of the stress tensor are given by

$$T^{IJ(i_1 \dots i_n)} = \mathrm{Sym}\left(T^{IJ}; X^{i_1}, X^{i_2}, ..., X^{in}\right) + T^{IJ(i_1 \dots i_n)}_{\text{fermion}} \tag{3.10}$$

where the contributions $\mathrm{Sym}\left(\mathrm{STr}(Y); X^{i_1}, X^{i_2}, ..., X^{in}\right)$ are defined as the symmetrized average over all possible insertions of the matrices X^{i_k} into $\mathrm{STr}(Y)$ (where Y is the product of terms of the forms $\dot{X}^i$, F_{ij}, Θ, and $[X^i, \Theta]$). $T^{IJ(i_1 \dots i_n)}_{\text{fermion}}$ are additional terms containing fermions, with one simple example being

$$T^{+i(j)}_{\text{fermion}} = \frac{1}{8R}\mathrm{STr}\left(\Theta\gamma^{[ij]}\Theta\right) \tag{3.11}$$

The higher moments of the membrane current and five-brane current are given in a similar manner. For a more complete list of the moments of the currents, see Appendix A of [7].

Before relating the above matrix theory to multiple D0-brane dynamics in ten-dimensional IIA string theory and applying the formalism to the IIA background obtained from a space-like compactification of the 11-D PP-wave, we would like to make one remark: as pointed out in [29], using the matrix theory action given in eqn. (3.6), one can directly derive the matrix theory in 11-D PP-wave in the weakly curved approximation. The reason that we would like to present the somewhat indirect

10-D approach is that the 10-D viewpoint in term of D0-brane dynamics is quite informative, e.g., it clarifies the subtleties involved in light-like compactification as well as how dynamics in eleven dimensions is related to that in ten dimensions. Also, in [7] this procedure is actually used to predict the previously unknown multi D0-brane action in a general weakly curved type IIA supergravity background.

As pointed out by Seiberg [4] and Sen [5], light-like compactification and space-like compactification of M theory are related by a certain limit. More specifically, M theory with Planck scale M_P compactified on a light-like circle of radius R and momentum $P_- = \frac{N}{R}$ is the same as $\tilde{M}$ theory with Planck scale $\tilde{M}_P$ compactified on a spatial circle of radius R_s with N D0-branes in the limit

$$\begin{aligned} &R_s \to 0 \\ &\tilde{M}_P \to \infty \\ &R_s \tilde{M}_P^2 = R M_P^2 = \text{fixed} \\ &\tilde{M}_P \tilde{R}_i = M_p R_i = \text{fixed} \end{aligned} \tag{3.12}$$

where R_i is the characteristic length of the transverse metric. The IIA string theory resulting from $\tilde{M}$ theory has its string coupling and string scale given by $\tilde{g}_s = R_s^{3/4}(RM_P^2)^{3/4}$ and $\tilde{M}_s^2 = R_s^{-1/2}(RM_P^2)^{3/2}$, and is thus a weakly coupled string theory with large string tension in the Seiberg-Sen limit, which is a very simple theory and lies at the root of the simplification of the matrix theory. Also note that the condition $R_s\tilde{M}_P^2 = RM_P^2 = \text{fixed}$ is to ensure that the energies of the states we are interested in remain finite rather than going to zero in this limit. Now let us apply the above Seiberg-Sen limit to the weak 11-D background at hand.

Recall that our light-like compactified M theory has the background $g_{IJ} = \eta_{IJ} + h_{IJ}$, with the light-like (in the flat space limit) direction x^- being a circle of radius R. This theory can be obtained by an infinite boost of a space-like compactified theory as follows. Consider the $\tilde{M}$ theory with background metric $\tilde{g}_{IJ} = \eta_{IJ} + \tilde{h}_{IJ}$, where the space-like direction x^{10} is a circle of radius R_s. These two theories are related by a boost along the x^{10} direction with the boost parameter being

$$\gamma = \sqrt{\frac{R^2}{2R_s^2} + 1} \tag{3.13}$$

in the $R_s \to 0$ limit. The components $\tilde{h}_{IJ}$ and h_{IJ} are related in an obvious manner through this coordinate transformation.

The $\tilde{M}$ theory compactified on the space-like circle x^{10} is equivalent to type IIA string theory with the background metric, Ramond-Ramond (R-R) one-form, and dilaton given to leading order by

$$\begin{aligned} h^{IIA}_{\mu\nu} &= \tilde{h}_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}\tilde{h}_{10\ 10} \\ C_\mu &= \tilde{h}_{10\mu} \\ \phi &= \frac{3}{4}\tilde{h}_{10\ 10} \end{aligned} \tag{3.14}$$

where $\mu, \nu = 0, ..., 9$, with the string coupling and the string scale being $g_s = (R_s M_P)^{3/2}$, $M_s = R_s^{1/2} M_p^{3/2}$.

Combining the above two steps, namely, boosting and compactifying to ten dimensions, one obtains the following relations between the metric components

$$\begin{aligned} h^{IIA}_{00} &= \frac{3}{2}h_{+-} + \frac{R_s^2}{8R^2}h_{++} + \frac{R^2}{2R_s^2}h_{--} \\ h^{IIA}_{0i} &= \frac{R_s}{2R}h_{+i} + \frac{R}{R_s}h_{-i} \\ h^{IIA}_{ij} &= h_{ij} + \frac{1}{2}\delta_{ij}\left(-h_{+-} + \frac{R_s^2}{4R^2}h_{++} + \frac{R^2}{R_s^2}h_{--}\right) \\ \phi &= -\frac{3}{4}h_{+-} + \frac{3R_s^2}{16R^2}h_{++} + \frac{3R^2}{4R_s^2}h_{--} \\ C_0 &= \frac{R_s^2}{4R^2}h_{++} - \frac{R^2}{R_s^2}h_{--} \\ C_i &= \frac{R_s}{2R}h_{+i} - \frac{R}{R_s}h_{-i} \end{aligned} \tag{3.15}$$

where on the right-hand side we have kept only the leading term in R_s/R for each of the components of the metric h_{IJ}. In the above we have only considered the 10-D fields resulting from the 11-D metric. The relations for the other 10-D fields, i.e., the Neveu-Schwarz-Neveu-Schwarz (NS-NS) two-form and R-R three-form, can be worked out similarly.

Apparently the right-hand sides in the above expressions for the IIA string theory fields diverge when $R_s \to 0$. This is because we are not done with Seiberg-Sen limit yet. In the spirit of that limit, we should make the further rescaling in the IIA string theory action

$$R \to \left(\frac{R_s}{R}\right)^{1/2} R, \quad x^i \to \left(\frac{R_s}{R}\right)^{1/2} x^i, \quad h(\vec{x}) \to h(\vec{x}) \tag{3.16}$$

with the goal that the energies that we are interested in should remain finite when $R_s \to 0$. Note the above rescaling of the transverse direction x^i can be inferred from the rescaling of R_i in eqn. (3.12). Section 2.2.2 of [7] gives a nice illustration of this rescaling in the single D0-brane case.

The multi D0-brane action in weak IIA supergravity background is given by

$$S = S_{\text{flat}} + S_{\text{weak}} \tag{3.17}$$

with S_{flat} being the flat space expression, and

$$\begin{aligned} S_{\text{weak}} &= \int dt \sum_{n=0}^{\infty} \frac{1}{n!} \Big[\frac{1}{2} (\partial_{k_1} ... \partial_{kn} h^{IIA}_{\mu\nu}) I_h^{\mu\nu(k_1...k_n)} + (\partial_{k_1} ... \partial_{kn} \phi) I_\phi^{(k_1...k_n)} \\ &+ (\partial_{k_1} ... \partial_{kn} C_\mu) I_0^{\mu(k_1...k_n)} + (\partial_{k_1} ... \partial_{kn} \tilde{C}_{\mu\nu\lambda\rho\sigma\tau\zeta}) I_6^{\mu\nu\lambda\rho\sigma\tau\zeta(k_1...k_n)} \\ &+ (\partial_{k_1} ... \partial_{kn} B_{\mu\nu}) I_s^{\mu\nu(k_1...k_n)} + (\partial_{k_1} ... \partial_{kn} \tilde{B}_{\mu\nu\lambda\rho\sigma\tau}) I_5^{\mu\nu\lambda\rho\sigma\tau(k_1...k_n)} \\ &+ (\partial_{k_1} ... \partial_{kn} C^{(3)}_{\mu\nu\lambda}) I_2^{\mu\nu\lambda(k_1...k_n)} + (\partial_{k_1} ... \partial_{kn} \tilde{C}^{(3)}_{\mu\nu\lambda\rho\sigma}) I_4^{\mu\nu\lambda\rho\sigma(k_1...k_n)} \Big] \end{aligned} \tag{3.18}$$

Let us explain the meaning of the various terms in this action. Similar to the 11-D case, the additional term S_{weak} in the action due to the curved background takes the form of moments of currents coupling to 10-D background fields. I_h is the stress tensor, I_ϕ is the current coupling to the dilaton, I_0 and I_2 are D0 brane current and

D2 brane current coupling to the R-R one-form C and three-form $C^{(3)}$, respectively. I_6 and I_4 are D6 brane current and D4 brane current coupling to the dual seven form $\tilde{C}$ and dual five form $\tilde{C}^{(3)}$, respectively. And I_s and I_5 are the currents associated with fundamental string and NS5-brane which couple to NS-NS two form B and its dual $\tilde{B}$, respectively. It is understood that in the above action the $R_s \to 0$ limit is taken, with the rescaling of parameters as described earlier performed to ensure a finite result.

By requiring the action (3.17) to reproduce the action (3.6), the various currents I and their moments in (3.17) can be related to the 11-D stress tensor, membrane current, and five-brane current. The results are given in equations (17), (19), (21), (22) of [7], and we do not reproduce those equations here.

Finally it is time to apply the above formalism to the 11-D PP-wave [29]. The metric of 11-D PP-wave is given in light-cone coordinates, so we first (un)boost it with a boost parameter $\gamma = \sqrt{\frac{R^2}{2R_s^2} + 1}$. Under this boost,

$$x^+ \to \frac{R_s}{\sqrt{2}R} x^+, \quad x^- \to \frac{\sqrt{2}R}{R_s} x^- \tag{3.19}$$

As one can see, the 11-D PP-wave metric is invariant under this boost, except that its $(++)$ component, or equivalently μ^2, is rescaled by $\left(\frac{R_s}{\sqrt{2}R}\right)^2$. Combining this with the rescaling of dimensionful parameters R, x^A described earlier [7] to ensure finite energies, one sees that the overall effect of boost and rescaling is $\mu \to \mu/\sqrt{2}$.

Hence, in the unboosted frame, the 11-D metric becomes

$$ds^2 = -\left(1 + \frac{F^2}{4}\right) dt^2 + \left(1 - \frac{F^2}{4}\right) (dx^{10})^2 - \frac{F^2}{2} dt dx^{10} + \sum_{A=1}^{9} dx^A dx^A \tag{3.20}$$

with the positive quantity $F^2 \equiv -g_{++} = \mu^2 \left[\frac{1}{9} \sum_{i=1}^{3} (x^i)^2 + \frac{1}{36} \sum_{a=4}^{9} (x^a)^2\right]$. The four-form field strength in the unboosted frame is

$$F_{123(10)} = -F_{0123} = \frac{\mu}{2} \tag{3.21}$$

Next we perform a space-like compactification along the x^{10} direction to get a IIA supergravity background

$$ds^2_{IIA} = -e^{\frac{-2\phi}{3}} dt^2 + e^{\frac{2\phi}{3}} \delta_{AB} dx^A dx^B \tag{3.22}$$

with the dilaton, R-R one-form, R-R three-form, and NS-NS two-form given by

$$e^{\frac{4\phi}{3}} = 1 - \frac{F^2}{4}, \quad C_0 = -\frac{F^2}{4 - F^2}, \quad C_{0ij} = \frac{\mu}{6}\epsilon_{ijk}x^k, \quad B_{ij} = \frac{\mu}{6}\epsilon_{ijk}x^k \tag{3.23}$$

If we now use the approximation that the background is weakly curved, i.e., $F^2 << 1$ (recall that $-F^2 = g_{++} = h_{++}$), then to linear order in F^2, the dilaton, R-R one-form, and metric perturbation are

$$\phi \approx -\frac{3}{16}F^2, \quad C_0 \approx -\frac{F^2}{4}, \quad h^{IIA}_{00} \approx -\frac{F^2}{8}, \quad h^{IIA}_{AB} \approx -\frac{F^2}{8}\delta_{AB} \tag{3.24}$$

As a check, these expressions do agree with the general expressions given in eqn. (3.15), with h_{++} set to $-F^2$ and all other components of h set to zero (also after rescaling by $(R_s/R)^2$).

Now consider the multi D0-brane action (3.17) in the above weakly curved background. One finds that the terms arising from the eleven-dimensional metric are

$$\begin{aligned} S_2 &= \frac{1}{2}\partial_A\partial_B\phi I^{(AB)}_\phi + \frac{1}{4}\partial_A\partial_B h^{IIA}_{\mu\nu} I^{\mu\nu(AB)}_h + \frac{1}{2}\partial_A\partial_B C_0 I^{0(AB)}_0 \\ &= -\frac{1}{16}(\partial_A\partial_B F^2)\left[\frac{3}{2}I^{(AB)}_\phi + \frac{1}{2}I^{00(AB)}_h + \frac{1}{2}\delta_{CD}I^{CD(AB)}_h + 2I^{0(AB)}_0\right] \end{aligned} \tag{3.25}$$

Dropping higher-order terms that vanish in the Seiberg-Sen limit, eqn. (17) in [7] gives the following expressions for the currents I_ϕ, I_h, I_0

$$\begin{aligned} &I_\phi = T^{++} - \frac{1}{3}T^{+-} - \frac{1}{3}\delta_{CD}T^{CD} \\ &I^{00}_h = T^{++} + T^{+-}, \quad I^{CD}_h = T^{CD} \\ &I^0_0 = T^{++} \end{aligned} \tag{3.26}$$

and their (AB) moments are given by just adding superscript (AB) to both sides of the above expressions. Plugging these moments into eqn. (3.25), one finds

$$S_2 = -\frac{1}{4}(\partial_A \partial_B F^2) T^{++(AB)} \tag{3.27}$$

Using eqn. (3.10), one finds $T^{++(AB)} = \frac{1}{R}\mathrm{Tr}(X^A X^B)$. Plugging in the explicit expression for $F^2 = -g_{++}$ we see that S_2 gives the bosonic mass term in the matrix model (3.1). The terms arising from the eleven-dimensional three-form are

$$\begin{aligned} S_3 &= \partial_A B_{\mu\nu} I_s^{\mu\nu(A)} + \partial_A C_{\alpha\beta\gamma} I_2^{\alpha\beta\gamma(A)} \\ &= \partial_m \left(\frac{\mu}{6}\epsilon_{ijk} x^k\right) \left[I_s^{ij(m)} + 3 I_2^{0ij(m)}\right] \end{aligned} \tag{3.28}$$

Dropping higher-order terms that vanish in the Seiberg-Sen limit, eqn. (19) in [7] gives following expressions for the currents I_s, I_2

$$I_s^{ij} = 3J^{+ij}, \quad I_2^{0ij} = J^{+ij} \tag{3.29}$$

and their (m) moments are given by adding superscript (m) to both sides of the above expressions. Plugging these moments into eqn. (3.28), one finds

$$S_3 = \mu \epsilon_{ijk} J^{+ij(k)} \tag{3.30}$$

with

$$J^{+ij(k)} = -\frac{1}{6R}\mathrm{Tr}\left(i[X^i, X^j]X^k + \frac{i}{8}\Psi^T \gamma^{ijk}\Psi\right) \tag{3.31}$$

as given in Appendix A of [7]. Now it is easy to see that S_3 gives the fermionic mass term and the Myers term in (3.1).

We know that the Matrix model (3.1) is true to all orders in F^2, since the other two derivations of it described earlier did not require the background to be weakly curved. Hence it is interesting, from the perspective of this third derivation using the formula for multi D0-brane action in weakly curved backgrounds, that there are no corrections to the weak background result. Also, as can be computed, the scalar curvature of the IIA supergravity background obtained by compacting the 11-D PP-wave along x^{10} is given by

$$\mathcal{R} = -\frac{\mu^2}{8}(1 - F^2/4)^{-3/2}\left[a - b\mu^2\sum_i (x^i)^2 - c\mu^2\sum_a (x^a)^2\right] \tag{3.32}$$

with a, b, c being positive constants. This scalar curvature diverges as F^2 approaches 4, making the IIA background singular. This is due to the fact that $g_{10\ 10}$ changes sign when F^2 goes beyond 4, making our space-like compactification along x^{10} lose its validity. All in all, the reason that the Matrix model derived above turned out to be valid beyond the weak background approximation is likely to be due to the fact that the 11-D PP-wave we started with is maximally supersymmetric.

3.1.2 Supersymmetric Classical Vacua

The vacua of the Matrix model in the 11-D PP-wave are given by configurations which minimize the potential term. The nice thing about the potential term is that it can be written as a sum of squares,

$$V = \frac{R}{2}\mathrm{Tr}\left[\left(\frac{\mu}{3R}X^i + i\epsilon^{ijk}X^jX^k\right)^2 + \frac{1}{2}(i[X^a, X^b])^2 + (i[X^a, X^i])^2 + \left(\frac{\mu}{6R}\right)^2(X^a)^2\right] \tag{3.33}$$

in the notation of [29]. The last term in the above potential requires X^a to vanish at a minimum. Then one finds that the whole potential vanishes for

$$X^i = \frac{\mu}{3R}J^i \tag{3.34}$$

with J^i being the generators of an $SU(2)$ algebra in the N-dimensional representation

$$[J^i, J^j] = i\epsilon^{ijk}J^k \tag{3.35}$$

Hence the classical vacua are labelled by N-dimensional representations of $SU(2)$, which are specified by $(N_1, ..., N_k)$: partitions of N into sums of positive integers with the N_i's being the dimensions of the individual irreducible representations making up the N-dimensional representation. The interpretation of this result is that each vacuum corresponds to a set of membrane fuzzy spheres with radii $r_i = \frac{\mu}{6R}\sqrt{N_i^2 - 1} \approx \frac{\mu N_i}{6R}$ when the N_i's are large.

The above classical vacua preserve all of the sixteen linearly realized supersymmetries of the theory [10]. As it turns out these classical vacua are also exact quantum vacua [42].

One interesting observation about these vacua made in [30] is that they actually can also be interpreted as transverse five-branes in M theory. For example, the trivial vacuum $X^i = 0$, labelled by the partition $(1, 1, ..., 1)$, can be interpreted as a single five brane in the large N limit. The evidence for this is provided by comparing the spectrum of fluctuations about the single five-brane in 11-D supergravity with the set of protected excited states about the $X^i = 0$ vacuum on the Matrix theory side and finding precise agreement.

3.2 The Three-Dimensional Theory Describing Multiple Concentric Spherical Membranes

3.2.1 Derivation

The three-dimensional theory describing multiple concentric spherical membranes in the 11-D PP-wave background is a SYM-Higgs theory with 16 supersymmetries living on $R \times S^2$, first derived in [30]. The approach adopted in [30] is taking the Matrix quantum mechanics model in 11-D PP-wave proposed by [10], expanding it around the k-membrane vacuum where there are k copies of the N-dimensional irreducible representations of $SU(2)$, then by letting $N \to \infty$ going to the continuum limit to get a 3d theory with gauge group $U(k)$.

However, in a certain sense, the above approach is unnecessarily complicated, because the $(0 + 1)$d matrix model can be obtained by discretizing the supermembrane theory, and then what the above approach does is taking the continuum limit to get back to a 3d gauge theory. Here we take a different approach that goes directly from supermembrane theory to the 3d gauge theory, without the detour just described (see also Section 7 of [29]). We take the supermembrane action given in [29], which is already a 3d theory, expand it around the *single* spherical membrane vacuum, throw away higher-order terms, get a $U(1)$ 3d gauge theory, and then by adding adjoint indices to the fields obtain the nonabelian theory which describes multiple concentric membranes. The result of this simpler approach turns out to be the same as that of [30], which is not surprising because both can be regarded as deforming the usual 3d SYM-Higgs theory in flat space while preserving sixteen supersymmetries, and there are not so many such deformations! The following are the details of this approach.

I. The Abelian Theory

Ref. [29] gives the following supermembrane action in the 11-D PP-wave background

$$\begin{aligned}\mathcal{L}_0 \;=\; & \frac{1}{2}p^+\left(\dot{X}^A\right)^2 \\ & -\frac{1}{p^+}\left[\frac{1}{4}\{X^A,X^B\}^2+\frac{1}{2}\left(\left(\frac{\mu p^+}{3}\right)^2(X^i)^2+\left(\frac{\mu p^+}{6}\right)^2(X^a)^2\right)-\frac{\mu p^+}{6}\epsilon^{ijk}\{X^i,X^j\}X^k\right] \\ & +ib\Psi^T\dot{\Psi}+ic\Psi^T\gamma^A\{X^A,\Psi\}+\frac{i}{4}\mu\Psi^T\gamma^{123}\Psi \qquad (3.36)\end{aligned}$$

The indices: $A=1,...,9$, $i=1,2,3$, $a=4,...,9$. $\{f,g\}\equiv\epsilon^{rs}\partial_r f\partial_s g$ is the Poisson bracket on the membrane. Ψ is a 16 component real spinor. γ^A are the 16×16 gamma matrices of $SO(9)$, $\{\gamma^A,\gamma^B\}=2\delta^{AB}$; and we choose all the γ^A's to be real and symmetric. (One explicit representation of the γ^A's can be found in eqn. (5.B.1) of [43]; however in the following derivation we don't need to use those explicit expressions). $\gamma^{123}=\gamma^1\gamma^2\gamma^3$. b and c in the fermionic part of the lagrangian are real constants to be determined later on by requiring supersymmetry. (We don't take the values for b,c given in [29], since our conventions for spinors and gamma matrices shall differ from [29].)

We take the world volume of the membrane to be $R \times S^2$, with R parametrized by time t (which is the x^+ of the 11-D target space) and the unit sphere S^2 parametrized by (θ, ϕ) and the action is $S = \int dt d\theta d\phi \sin\theta \; \mathcal{L}$. Note that on a unit sphere the Poisson bracket of two functions $f(\theta, \phi)$ and $g(\theta, \phi)$ is given by

$$\{f, g\} \equiv \frac{1}{\sin\theta} \left(\partial_\theta f \partial_\phi g - \partial_\phi f \partial_\theta g \right) \tag{3.37}$$

A solution to the classical supermembrane equations of motion is one satisfying $\{X^i, X^j\} = \epsilon^{ijk} \frac{\mu p^+}{3} X^k$, which minimizes the potential. This is a sphere $X^i = x^i$, with

$$x^1 = \frac{\mu p^+}{3} \sin\theta \cos\phi, \;\; x^2 = \frac{\mu p^+}{3} \sin\theta \sin\phi, \;\; x^3 = \frac{\mu p^+}{3} \cos\theta \tag{3.38}$$

Fluctuations about this spherical membrane background give us a U(1) SYM theory. More specifically, let us expand about this background: $X^i = x^i + Y^i$. Then in terms of the fluctuation fields Y^i, X^a and Ψ, eqn. (3.36) becomes

$$\mathcal{L}_0 = \mathcal{L} + \mathcal{L}' \tag{3.39}$$

where $\mathcal{L}$ consists of terms quadratic in the fluctuation fields, and is given by

$$\begin{aligned} \mathcal{L}(Y^i, X^a, \Psi) \;\; = \;\; & \frac{1}{2} p^+ \left[\left(\dot{Y}^i \right)^2 + \left(\dot{X}^a \right)^2 \right] \\ & - \frac{1}{2p^+} \left[\left(F^k \right)^2 + \{x^i, X^a\}^2 + \left(\frac{\mu p^+}{6} \right)^2 (X^a)^2 \right] \\ & + ib \Psi^T \dot{\Psi} + ic \Psi^T \gamma^i \{x^i, \Psi\} + \frac{i}{4} \mu \Psi^T \gamma^{123} \Psi \end{aligned} \tag{3.40}$$

with $F^k \equiv \epsilon^{ijk} \{x^i, Y^j\} - \frac{\mu p^+}{3} Y^k$. $\mathcal{L}'$ consists of terms of cubic and quartic orders in the fluctuation fields. Using the shorthand notation x for x^i, Y for Y^i, X for X^a, and suppressing gamma matrices, we see that the cubic terms are of the forms $\{Y, Y\}\{x, Y\}$, $\{Y, Y\}Y$, $\{Y, X\}\{x, X\}$, $\Psi\{Y, \Psi\}$, and $\Psi\{X, \Psi\}$. The quartic terms are of the forms $\{Y, Y\}\{Y, Y\}$, $\{X, X\}\{X, X\}$, and $\{Y, X\}\{Y, X\}$.

By looking at the quadratic kinetic terms in $\mathcal{L}$, we determine that Y^i and X^a

both have mass dimension one half, and Ψ has mass dimension one. Noticing that the Poisson bracket $\{,\}$ contributes two spatial derivatives, and the appearance of x inside the Poisson bracket cancels one spatial derivative, we find that the cubic terms in $\mathcal{L}'$ are of mass dimensions $\frac{7}{2}$ and $\frac{9}{2}$, and the quartic terms in $\mathcal{L}'$ are of mass dimension 6. Hence all the terms in $\mathcal{L}'$ have mass dimensions greater than 3 and are thus non-renormalizable. So, in what follows we throw $\mathcal{L}'$ away, and only look at $\mathcal{L}$ as given in (3.40).

Consider the following $U(1)$ gauge transformation, with a time-independent transformation parameter $\Lambda(\theta, \phi)$ (the time-independence of Λ is a result of taking the gauge where the temporal component of the $U(1)$ gauge potential A_t vanishes):

$$\delta_{\text{gauge}} Y^i = \{\Lambda, x^i\}, \ \delta_{\text{gauge}} X^a = 0, \ \delta_{\text{gauge}} \Psi = 0 \tag{3.41}$$

As can be easily verified using the Jacobi identity of the Poisson bracket and the fact $\{x^i, x^j\} = \epsilon^{ijk} \frac{\mu p^+}{3} x^k$, the F^k we introduced above is gauge-invariant. Since Λ is time-independent, $\dot{Y}^i$ is also gauge-invariant. Hence this $U(1)$ gauge transformation leaves $\mathcal{L}$ unchanged.

Now let's show that our theory has 16 supersymmetries. Consider the following supersymmetry transformation with the parameter being a 16-component real spinor $\epsilon(t)$ (Recall that ϵ is time-dependent due to the supersymmetry algebra of the 11-D PP-wave target space) is

$$\begin{aligned} \delta Y^i &= i\epsilon^T \gamma^i \Psi + \lambda \{\Lambda_0, x^i\}, \ \delta X^a = i\epsilon^T \gamma^a \Psi \\ \delta \Psi &= e_1 \left(\dot{Y}^i \gamma^i + \dot{X}^a \gamma^a \right) \epsilon + e_2 F^k \gamma^k \gamma^{123} \epsilon \\ &\quad + e_3 \{x^i, X^a\} \gamma^i \gamma^a \epsilon + n\mu X^a \gamma^a \gamma^{123} \epsilon \end{aligned} \tag{3.42}$$

where $\dot{\Lambda}_0 = i\epsilon^T \Psi$. Also $\dot{\epsilon} = m\mu \gamma^{123} \epsilon$. We will determine the constants $\lambda, e_1, e_2, e_3, n$ in the supersymmetry transformation, the constant m in the expression for $\dot{\epsilon}$, and the constants b, c in the action (3.40) simultaneously by requiring supersymmetry.

The above supersymmetry transformation basically comes from truncating higher-order terms in the supersymmetry transformation of the original supermembrane theory when expanding around the vacuum background. The Λ_0 term in the supersymmetry transformation of Y^i is a compensating gauge transformation needed to remain in the $A_t = 0$ gauge. Recall that the supersymmetry transformation of A_t is $\delta A_t = \frac{i}{p^+}\epsilon^T\Psi$, and the gauge transformation is $\delta_{\text{gauge}}A_t = -\frac{1}{p^+}\partial_t\Lambda_0$. Requiring these two transformation to cancel so that we remain in the $A_t = 0$ gauge gives $\dot{\Lambda}_0 = i\epsilon^T\Psi$.

Demonstrating the supersymmetry is fairly easy and not so instructive, so we don't bother to write out the details and only explain the strategy here: Since the $SO(3)$ and $SO(6)$ parts don't mix, we'll look at their variations separately, namely, we consider $\delta\mathcal{L}$ that only depends on the Y^i's, and $\delta\mathcal{L}$ that only depends on the X^a's. When there are two time-derivatives on the fields (time-derivatives on ϵ can always be replaced by a term proportional to $\mu\gamma^{123}\epsilon$), we let one of them act on the boson and the other on the fermion. When there is only one time-derivative, we always move it to act on the bosons by integration by parts. We always let the Poisson bracket act on the bosons using integration by parts. Also, it helps to first look at terms containing two powers of μ explicitly (in comparison, there are many terms containing one power of μ, because the Poisson bracket of two x's gives a μ). Some useful gamma matrix identities are $\gamma^i\gamma^{123} = \frac{1}{2}\epsilon^{ijk}\gamma^j\gamma^k$, $\frac{1}{2}\epsilon^{jpq}\gamma_p\gamma_q\gamma_n = \delta^{jn}\gamma^{123} - \epsilon^{jnp}\gamma_p$, and $\gamma^k\gamma^{123}\gamma^i = \delta^{ki}\gamma^{123} - \epsilon^{kij}\gamma_j$. The fact $\{x^k, F^k\} = 0$ comes in handy, too.

We find the following values for the coefficients:

$$b = -1,\ c = -\frac{1}{p^+},\ \lambda = -\frac{1}{p^+},\ e_1 = \frac{p^+}{2},\ e_2 = -\frac{1}{2},\ e_3 = -\frac{1}{2},\ n = -\frac{p^+}{12},\ m = -\frac{1}{12} \tag{3.43}$$

plugging these values into the expressions for the Lagrangian and supersymmetry transformation, we get the action

$$\begin{aligned}
\mathcal{L}(Y^i, X^a, \Psi) &= \frac{1}{2}p^+\left[\left(\dot{Y}^i\right)^2 + \left(\dot{X}^a\right)^2\right] \\
&- \frac{1}{2p^+}\left[\left(F^k\right)^2 + \{x^i, X^a\}^2 + \left(\frac{\mu p^+}{6}\right)^2 (X^a)^2\right] \\
&- i\Psi^T\dot{\Psi} - i\,\frac{1}{p^+}\Psi^T\gamma^i\{x^i, \Psi\} + \frac{i}{4}\mu\Psi^T\gamma^{123}\Psi
\end{aligned} \tag{3.44}$$

and the supersymmetry transformation

$$\begin{aligned}
&\delta Y^i = i\epsilon^T\gamma^i\Psi - \frac{1}{p^+}\{\Lambda_0, x^i\},\ \delta X^a = i\epsilon^T\gamma^a\Psi \\
&\delta\Psi = \frac{p^+}{2}\left(\dot{Y}^i\gamma^i + \dot{X}^a\gamma^a\right)\epsilon - \frac{1}{2}F^k\gamma^k\gamma^{123}\epsilon \\
&\quad - \frac{1}{2}\{x^i, X^a\}\gamma^i\gamma^a\epsilon - \frac{p^+}{12}\mu X^a\gamma^a\gamma^{123}\epsilon
\end{aligned} \tag{3.45}$$

with $\dot{\epsilon} = -\frac{1}{12}\mu\gamma^{123}\epsilon$.

II. The Nonabelian Generalization

Generalizing the theory to the nonabelian case by adding adjoint indices to the fields is fairly straightforward. We use the latter half of the alphabet, e.g., m, n, p, as group adjoint indices. f_{mnp} is the real, totally antisymmetric structure constant of the gauge group. The gauge transformation of the fields is given by

$$\begin{aligned}
&\delta_{\text{gauge}}Y^i_m = \{\Lambda_m, x^i\} + gf^{mnp}\Lambda_n Y^i_p \\
&\delta_{\text{gauge}}X^a_m = gf^{mnp}\Lambda_n X^a_p \\
&\delta_{\text{gauge}}\Psi_m = gf^{mnp}\Lambda_n\Psi_p
\end{aligned} \tag{3.46}$$

where g is the gauge coupling, Λ_m is the time-independent gauge transformation parameter. Also, the adjoint indices are raised/lowered by the Kronecker delta.

Define the following gauge-covariant field strength and covariant-derivatives:

$$
\begin{aligned}
F^k_m &= \epsilon^{kij}\left(\{x^i, Y^j_m\} + \frac{g}{2} f^{mnp} Y^i_n Y^j_p\right) - \frac{\mu p^+}{3} Y^k_m \\
F^{ia}_m &= \{x^i, X^a_m\} + g f^{mnp} Y^i_n X^a_p \\
F^{ab}_m &= g f^{mnp} X^a_n X^b_p \\
\xi^i_m &= \{x^i, \Psi_m\} + g f^{mnp} Y^i_n \Psi_p
\end{aligned}
\tag{3.47}
$$

which all have the standard gauge transformation $\delta_{\mathrm{gauge}}(...)_m = g f^{mnp}\Lambda_n(...)_p$.

Now it's easy to see that, in terms of these gauge-covariant objects, the nonabelian generalization is

$$
\begin{aligned}
\mathcal{L}_{\mathrm{nonabelian}} = \quad & \frac{p^+}{2}\left[(\dot{Y}^i_m)^2 + (\dot{X}^a_m)^2\right] \\
& - \frac{1}{2p^+}\left[(F^k_m)^2 + (F^{ia}_m)^2 + \frac{1}{2}(F^{ab}_m)^2 + \left(\frac{\mu p^+}{6}\right)^2 (X^a_m)^2\right] \\
& -i\Psi^T_m \dot{\Psi}_m - i\frac{1}{p^+}\Psi^T_m \gamma^i \xi^i_m - i\frac{1}{p^+} g f^{mnp} \Psi^T_m \gamma^a X^a_n \Psi_p + i\frac{1}{4}\mu \Psi^T_m \gamma^{123} \Psi_m
\end{aligned}
\tag{3.48}
$$

The supersymmetry transformation is

$$
\begin{aligned}
\delta Y^i_m = \quad & i\epsilon^T \gamma^i \Psi_m - \frac{1}{p^+}\left(\{(\Lambda_0)_m, x^i\} + g f^{mnp} (\Lambda_0)_n Y^i_p\right) \\
\delta X^a_m = \quad & i\epsilon^T \gamma^a \Psi_m - \frac{1}{p^+} g f^{mnp} (\Lambda_0)_n X^a_p \\
\delta \Psi_m = \quad & \frac{p^+}{2}(\dot{Y}^i_m \gamma^i + \dot{X}^a_m \gamma^a)\epsilon - \frac{1}{2} F^k_m \gamma^k \gamma^{123} \epsilon \\
& - \frac{1}{2} F^{ia}_m \gamma^i \gamma^a \epsilon - \frac{1}{4} F^{ab}_m \gamma^a \gamma^b \epsilon - \frac{p^+}{12} \mu X^a_m \gamma^a \gamma^{123} \epsilon \\
& - \frac{1}{p^+} g f^{mnp} (\Lambda_0)_n \Psi_p
\end{aligned}
\tag{3.49}
$$

where $(\dot{\Lambda_0})_m = i\epsilon^T \Psi_m$ is the time-dependent compensating gauge transformation parameter, and we still have $\dot{\epsilon} = -\frac{1}{12}\mu\gamma^{123}\epsilon$ as in the abelian case.

The gauge-invariance of $\mathcal{L}_{\mathrm{nonabelian}}$ is easily seen since it is built from gauge-covariant objects. Demonstrating that it is invariant under the supersymmetry transformation given above is straightforward, although a bit tedious. Compared with flat space-time we now have many more terms due to two facts: time and space are now divided into $R \times S^2$; the nine transverse directions are now divided into $SO(3)$ and $SO(6)$. We relegate the detailed proof of the invariance of $\mathcal{L}_{\text{nonabelian}}$ in eqn. (3.48) under the sixteen supersymmetries to Appendix A.

III. Writing the Theory in a More Conventional Form

Let's first restore A_t. Now the gauge transformation parameter $\Lambda_m(t, \theta, \phi)$ can have time-dependence. The gauge transformation is

$$\begin{aligned}
\delta_{\text{gauge}} A_{mt} &= \dot{\Lambda}_m + g f^{mnp} \Lambda_n A_{pt} \\
\delta_{\text{gauge}} Y^i_m &= \{\Lambda_m, x^i\} + g f^{mnp} \Lambda_n Y^i_p \\
\delta_{\text{gauge}} X^a_m &= g f^{mnp} \Lambda_n X^a_p \\
\delta_{\text{gauge}} \Psi_m &= g f^{mnp} \Lambda_n \Psi_p
\end{aligned} \tag{3.50}$$

The action is

$$\begin{aligned}
\mathcal{L}_{\text{nonabelian}} = \quad & \frac{p^+}{2} \left[(D_t Y^i)^2_m + (D_t X^a)^2_m \right] \\
& - \frac{1}{2p^+} \left[(F^k_m)^2 + (F^{ia}_m)^2 + \frac{1}{2} (F^{ab}_m)^2 + \left(\frac{\mu p^+}{6} \right)^2 (X^a_m)^2 \right] \\
& - i \Psi^T_m (D_t \Psi)_m - i \frac{1}{p^+} \Psi^T_m \gamma^i \xi^i_m - i \frac{1}{p^+} g f^{mnp} \Psi^T_m \gamma^a X^a_n \Psi_p + i \frac{1}{4} \mu \Psi^T_m \gamma^{123} \Psi_m
\end{aligned} \tag{3.51}$$

where $(D_t Y^i)_m \equiv \dot{Y}^i_m - \{A_{mt}, x^i\} - g f^{mnp} A_{nt} Y^i_p$, $(D_t X^a)_m \equiv \dot{X}^a_m - g f^{mnp} A_{nt} X^a_p$,

$(D_t\Psi)_m \equiv \dot{\Psi}_m - gf^{mnp}A_{nt}\Psi_p$. The supersymmetry transformations are

$$\begin{aligned}
\delta Y^i_m = &\quad i\epsilon^T\gamma^i\Psi_m, \quad \delta X^a_m = i\epsilon^T\gamma^a\Psi_m, \quad \delta A_{mt} = \frac{i}{p^+}\epsilon^T\Psi_m \\
\delta\Psi_m = &\quad \frac{p^+}{2}((D_tY^i)_m\gamma^i + (D_tX^a)_m\gamma^a)\epsilon - \frac{1}{2}F^k_m\gamma^k\gamma^{123}\epsilon \\
&- \frac{1}{2}F^{ia}_m\gamma^i\gamma^a\epsilon - \frac{1}{4}F^{ab}_m\gamma^a\gamma^b\epsilon - \frac{p^+}{12}\mu X^a_m\gamma^a\gamma^{123}\epsilon
\end{aligned} \tag{3.52}$$

So far we have used the Poisson bracket extensively in our proof of gauge symmetry and supersymmetry, because this makes the derivation quite concise (e.g., the Jacobi identity of the Poisson bracket is used at many places of the proof). The fields transform as world volume scalars. This is OK for computation of any physical quantity, e.g., a scattering amplitude. However, let's also write the theory in the more familiar form where the fields transform as world volume scalars, vectors, and spinors [1].

First let's look at the bosonic part of (3.51). In the abelian case, the gauge transformation of the components of the Y^i's in the spherical coordinates is

$$\delta_{\text{gauge}}Y_r = 0, \;\; \delta_{\text{gauge}}Y_\theta = \frac{-1}{\sin\theta}\partial_\phi\left(\frac{\mu p^+}{3}\Lambda\right), \;\; \delta_{\text{gauge}}Y_\phi = \sin\theta\partial_\theta\left(\frac{\mu p^+}{3}\Lambda\right) \tag{3.53}$$

which suggests we should define new fields Φ, A_θ, A_ϕ as

$$\Phi = Y_r, \;\; A_\theta = \frac{1}{\sin\theta}Y_\phi, \;\; A_\phi = -\sin\theta Y_\theta \tag{3.54}$$

which has the simple gauge transformation:

$$\delta_{\text{gauge}}\Phi = 0, \;\; \delta_{\text{gauge}}A_\theta = \partial_\theta\left(\frac{\mu p^+}{3}\Lambda\right), \;\; \delta_{\text{gauge}}A_\phi = \partial_\phi\left(\frac{\mu p^+}{3}\Lambda\right) \tag{3.55}$$

In the nonabelian case, we do the same thing, i.e.,

$$\Phi_m \equiv Y_{mr}, \;\; A_{m\theta} \equiv \frac{1}{\sin\theta}Y_{m\phi}, \;\; A_{m\phi} \equiv -\sin\theta Y_{m\theta} \tag{3.56}$$

[1] There is a catch about the spinors, though, as we shall explain later.

Also, define the rescaled fields

$$\tilde{X^a}_m \equiv \frac{\mu\sqrt{p^+}}{3} X^a_m,\ \tilde{\Phi}_m \equiv \frac{\mu\sqrt{p^+}}{3}\Phi_m$$
$$\tilde{A}_{m\theta} \equiv \sqrt{p^+}A_{m\theta},\ \tilde{A}_{m\phi} \equiv \sqrt{p^+}A_{m\phi},\ \tilde{A}_{mt} \equiv \frac{\mu(p^+)^{3/2}}{3}A_{mt} \tag{3.57}$$

In terms of these rescaled fields, the bosonic part of the action (3.51) is

$$\begin{aligned}S_B = \int dt d\theta d\phi \left(\frac{3}{\mu}\right)^2 \sin\theta\big[&-\frac{1}{4}\tilde{F}_{m\mu\nu}\tilde{F}^{\mu\nu}_m - \frac{1}{2}(\tilde{D}_\mu\tilde{\Phi})_m(\tilde{D}^\mu\tilde{\Phi})_m - \frac{1}{2}\left(\frac{\mu}{3}\right)^2(\tilde{\Phi}_m)^2 \\ &-\frac{1}{2}(\tilde{D}_\mu\tilde{X}^a)_m(\tilde{D}^\mu\tilde{X}^a)_m - \frac{1}{2}\left(\frac{\mu}{6}\right)^2(\tilde{X}^a_m)^2 - \frac{1}{4}\tilde{g}^2 f^{mnp}f^{mrs}\tilde{X}^a_n\tilde{X}^b_p\tilde{X}^a_r\tilde{X}^b_s \\ &-\frac{1}{2}\tilde{g}^2 f^{mnp}f^{mrs}\tilde{\Phi}_n\tilde{X}^a_p\tilde{\Phi}_r\tilde{X}^a_s\big] + \frac{\mu}{3}\int V_m\wedge\tilde{F}_m\end{aligned} \tag{3.58}$$

where $\tilde{g} = -\left(\frac{3}{\mu}\right)(p^+)^{-3/2}g,\ \tilde{F}_{m\mu\nu} = \partial_\mu\tilde{A}_{m\nu} - \partial_\nu\tilde{A}_{m\mu} + \tilde{g}f^{mnp}\tilde{A}_{n\mu}\tilde{A}_{p\nu},\ (\tilde{D}_\mu\tilde{\Phi})_m = \partial_\mu\tilde{\Phi}_m + \tilde{g}f^{mnp}\tilde{A}_{n\mu}\tilde{\Phi}_p,\ (\tilde{D}_\mu\tilde{X}^a)_m = \partial_\mu\tilde{X}^a_m + \tilde{g}f^{mnp}\tilde{A}_{n\mu}\tilde{X}^a_p.$ and the one-form V_m has the following components: $V_{mt} = \tilde{\Phi}_m,\ V_{m\theta} = 0,\ V_{m\phi} = 0.$

The above action is of the form of a SYM-Higgs theory on a sphere of radius $\frac{3}{\mu}$. Besides the usual flat space terms, the presence of the three-form in the 11-D PP-wave background adds mass terms for the scalars $\tilde{\Phi}_m, \tilde{X^a}_m$, and also the term $\propto \int V_m\wedge\tilde{F}_m$. The last term, which in components is $\propto \int\tilde{\Phi}_m\tilde{F}_{m\theta\phi}$, comes from the corresponding three-form term in the supermembrane action. From the IIA $D2$ brane view point, it comes from the wedge product of the R-R one-form potential and the field strength of the world-volume gauge field. We see that our S_B has the same form of that of eqn. (7) in [30] (their μ is our $\mu/3$.)

Now let's look at the fermionic part of the action (3.51). Let's start with the abelian theory. In the abelian case, the $\Psi^T_m\gamma^i\xi^i_m$ term in the action (3.51) becomes

$$\begin{aligned}\Psi^T\gamma^i\{x^i,\Psi\} = \frac{\mu p^+}{3}\Psi^T\frac{1}{\sin\theta}[&(\cos\theta\cos\phi\gamma^1 + \cos\theta\sin\phi\gamma^2 - \sin\theta\gamma^3)\partial_\phi\Psi \\ &+(\sin\theta\sin\phi\gamma^1 - \sin\theta\cos\phi\gamma^2)\partial_\theta\Psi]\end{aligned} \tag{3.59}$$

The above expression contains unwanted explicit-ϕ-dependence which we will get rid of by applying an $SO(3)$ rotation on Ψ; after that, we will still have to apply another $SO(3)$ rotation on Ψ to bring the θ dependence into the standard form for a field on $R \times S^2$. Both rotations will have to depend on ϕ or θ.

The coefficients of the $\partial_\phi \Psi$ term form a vector $(\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta)$; and the coefficients of the $\partial_\theta\Psi$ term form another vector $(\sin\theta\sin\phi, -\sin\theta\cos\phi, 0)$. It is easily seen that, a rotation around the 3-axis by an angle $-\phi$ rotates the first vector into 1-3 plane and the second vector into 2-axis, thus eliminating the ϕ-dependence in both vectors. The explicit rotation matrix is given by $R_1 = \cos(\phi/2) - \sin(\phi/2)\gamma^1\gamma^2$ (recall that rotating the γ^i's is equivalent to rotating their coefficients). The second rotation is given by $R_2 = \cos(\theta/2) - \sin(\theta/2)\gamma^3\gamma^1$, which rotates the first vector into the 1-axis, keeping the second vector fixed. Define the rotated spinor ψ by $\Psi = R_1 R_2 \psi$. In terms of ψ, the fermionic part of the lagrangian (3.51) in the abelian case becomes

$$\mathcal{L}_F = -i\psi^T\dot{\psi} - i\frac{\mu}{3}\left[-\psi^T\gamma^2\partial_\theta\psi + \psi^T\frac{1}{\sin\theta}\gamma^1\partial_\phi\psi - \frac{1}{2}\psi^T\frac{\cos\theta}{\sin\theta}\gamma^2\psi\right] - i\frac{\mu}{12}\psi^T\gamma^{123}\psi \tag{3.60}$$

Now let's compare the above lagrangian with that of a Majorana (real) spinor in 3d η (which is a two-component spinor) with mass m on $R \times S^2$ with the radius of the sphere being $\frac{3}{\mu}$

$$\mathcal{L}_{3d} = i\eta^T\dot{\eta} + i\left(\frac{\mu}{3}\right)\left[-\eta^T\tau_1\partial_\theta\eta + \eta^T\tau_3\frac{1}{\sin\theta}\left(\partial_\phi - i\cos\theta\frac{\tau_2}{2}\right)\eta\right] + m\eta^T\tau_2\eta \tag{3.61}$$

(we have taken the 3d gamma matrices, denoted as $\tilde{\gamma}$'s to avoid confusion with the $SO(9)$ gamma matrices γ's, to be all real

$$\tilde{\gamma}^{\hat{t}} = i\tau_2,\ \tilde{\gamma}^{\hat{\theta}} = \tau_3,\ \tilde{\gamma}^{\hat{\phi}} = \tau_1 \tag{3.62}$$

with τ_1, τ_2, τ_3 being the Pauli matrices).

We see that $\mathcal{L}_F$ and $\mathcal{L}_{3d}$ have the same form, with the correspondence $\gamma^1 \rightarrow$

$\tau_3, \gamma^2 \to \tau_1, \frac{\mu}{12} \to m$, except that in the mass terms $i\gamma^{123}$ does not correspond to τ_2 (because $i\gamma^{123}$ commutes with γ^1, γ^2, while τ_2 anti-commutes with τ_3, τ_1). We would like to show that $\mathcal{L}_F$ can be written as the sum of 8 copies of $\mathcal{L}_{3d}$. However, this is going to be highly non-trivial. The analog of this in string theory is that, to relate the GS formalism to the RNS formalism, we have to perform a non-trivial bosonization and re-fermionization of the fermions. What we should do is probably to decompose Ψ into 8-component spinors Ψ_+ and Ψ_- with eigenvalue ± 1 under γ^9. $\Psi_\pm$ transform as 8_s's of $SO(8)$. Then we should exploit the triality of $SO(8)$ so that after some kind of three-dimensional analog of bosonization and re-fermionization, the $\Psi_\pm$ become $S_\pm$ which transform as 8_v's. Then we should combine $S_\pm$ to get eight 3d Majorana spinors $\eta^M, M = 1, ..., 8$ which transform as 8_v. This nontrivial nature of relating target space spinors to world-volume spinors is also the reason underlying the mismatch between $i\gamma^{123}$ and τ_2 noted above.

Carrying out the procedures outlined in the previous paragraph is an interesting exercise in group theory for its own sake. However we are not going to do it here. So we will just leave the fermionic part of the action as it is in eqn. (3.51), in terms of the world-volume scalar Ψ. (The fermions in [30] are also world-volume scalars and space-time spinors.)

3.2.2 Properties

In what follows we will mostly work in the static gauge where $A_{mt} = 0$, and also use the formalism where the fields are world-volume scalars, i.e., we shall use (3.48) as our Lagrangian, because in terms of the Y^i's, the $SO(3)$ symmetry in this formalism (let's call it the "Y-formalism") is manifest. Occasionally we go to the formalism given in (3.58) where the Y^i's are split into $\tilde{A}$'s and $\tilde{\phi}$ (let's call it the "A-Φ-formalism").

3.2.2.1 Supersymmetric Classical Vacua

Let us look at the potential term of (3.48)

$$V = \frac{1}{2p^+}\left[(F_m^k)^2 + (F_m^{ia})^2 + \frac{1}{2}(F_m^{ab})^2 + \left(\frac{\mu p^+}{6}\right)^2 (X_m^a)^2\right] \tag{3.63}$$

To minimize this potential, X_m^a has to vanish due to its mass term, hence F_m^{ia} and F_m^{ab} also vanish. Thus vacuum configurations must have Y_m^i that satisfy $F_m^k = 0$. And by looking at the supersymmetry transformation (3.49), one sees that the vacua preserve all the sixteen supersymmetries. In the A-Φ-formalism the form of these vacuum configurations is most easily seen [30]. In what follows we will focus on the case where the gauge group is $SU(2)$, which describes two membranes. Basically the classical vacua in this theory are just "abelian monopoles" in the $U(1) \subset SU(2)$. The following are the details.

Setting $\tilde{X}_m^a = 0$, $\tilde{A}_{1\mu} = \tilde{A}_{2\mu} = 0$, and $\tilde{\Phi}_1 = \tilde{\Phi}_2 = 0$, with the only nonzero fields being the $U(1)$ part $\tilde{A}_{3\mu}$, and $\tilde{\Phi}_3$ = constant on the sphere, the potential in (3.58) becomes

$$\left[\tilde{\Phi}_3 - \frac{\mu}{3}\frac{1}{\sin\theta}\tilde{F}_{3\theta\phi}\right]^2 \tag{3.64}$$

because the $\tilde{F}^2$ term, the mass term for $\tilde{\Phi}$, and the $V_m \wedge \tilde{F}_m$ term nicely form a complete square. Thus we need

$$\tilde{F}_{3\theta\phi} = \frac{3}{\mu}\sin\theta\ \tilde{\Phi}_3 \tag{3.65}$$

Then, by the Dirac quantization condition, one can work out that

$$\tilde{\Phi}_3 = \frac{\mu}{3}\frac{n}{\tilde{g}} \tag{3.66}$$

(which gives a flux $\int_{S^2}\tilde{F}_3 = \int d\theta d\phi \tilde{F}_{3\theta\phi} = \frac{2\pi n}{\tilde{g}}$), with the flux number n being integer. Recall that $\sqrt{(\tilde{\Phi}_m)^2}$ is the radial separation between the two membranes. In flat space the separation can be continuously varied, while here in the PP-wave we see that the radial separation is quantized in units of μ. In the appropriate limit where $\mu \to 0$, $n \to \infty$, the radial separation can then be regarded as varying continuously.

Going back to the Y-formalism, we find that in the classical vacuum labelled by flux number n

$$\begin{aligned} Y_1^k &= Y_2^k = 0 \\ Y_3^1 &= -n\frac{\mu p^+}{3g}\frac{\cos\phi(1\mp\cos\theta)}{\sin\theta} \\ Y_3^2 &= -n\frac{\mu p^+}{3g}\frac{\sin\phi(1\mp\cos\theta)}{\sin\theta} \\ Y_3^3 &= -n\frac{\mu p^+}{3g}(\pm 1) \end{aligned} \tag{3.67}$$

where the upper sign is for the northern hemisphere patch and the lower sign for the southern hemisphere patch. As can be easily verified, the above Y_m^i indeed yields a vanishing F_m^i.

3.2.2.2 Euclidean Instantons

Just as the theory itself can be regarded as arising from the continuum limit of the Matrix theory, the instantons in this theory can regarded as the continuum limit of the instantons in the Matrix theory (see [44] where instanton solutions in the matrix theory interpolating between an arbitrary vacuum and the trivial vacuum are found). Define the Euclidean time $\tau \equiv it$. Then the Euclidean action is given by, upon setting

$X_m^a = 0,\ \Psi_m = 0$

$$\begin{aligned} S_E &\equiv -iS = -\int d\tau d\theta d\phi \sin\theta \mathcal{L} \\ &= \int d\tau d\theta d\phi \sin\theta \left[\frac{p^+}{2}\left(\frac{\partial Y_m^k}{\partial \tau}\right)^2 + \frac{1}{2p^+}(F_m^k)^2\right] \\ &= \int d\tau d\theta d\phi \sin\theta \frac{p^+}{2}\left[\left(\pm\frac{\partial Y_m^k}{\partial \tau} + \frac{F_m^k}{p^+}\right)^2 - 2(\pm)\frac{\partial Y_m^k}{\partial \tau}\frac{F_m^k}{p^+}\right] \\ &= \int d\tau d\theta d\phi \sin\theta \frac{p^+}{2}\left(\pm\frac{\partial Y_m^k}{\partial \tau} + \frac{F_m^k}{p^+}\right)^2 \mp [K(\tau \to +\infty) - K(\tau \to -\infty)] \end{aligned} \tag{3.68}$$

where to get to the last line we have used that fact that the $\frac{\partial Y_m^k}{\partial \tau}\frac{F_m^k}{p^+}$ term can be written as a total τ-derivative of $K(\tau)$, where the quantity $K(\tau)$ is defined to be the integral over S^2

$$K(\tau) \equiv \frac{1}{2}\int d\theta d\phi \sin\theta \left[\epsilon^{kij} Y_m^k \{x^i, Y_m^j\} - \frac{\mu p^+}{3}(Y_m^k)^2 + \frac{g}{3}\epsilon^{kij}\epsilon^{mnp} Y_m^k Y_n^i Y_p^j\right] \tag{3.69}$$

Since we require Y_m^i to interpolate between two vacua, $K(\tau \to \pm\infty)$ can be evaluated using the expression (3.67) worked out earlier for Y_m^i in the final/initial vacuum.

Hence for given initial and final vacua, to minimize S_E the following BPS condition has to be satisfied

$$\pm\frac{\partial Y_m^k}{\partial \tau} + \frac{F_m^k}{p^+} = 0 \tag{3.70}$$

and then S_E is given just by the boundary term and is a function of the initial and final flux numbers (n_i, n_f). It is easy to verify that the BPS condition implies the equation of motion for Y_m^i.

Now let us look at the supersymmetry transformation (3.49), which becomes (using the BPS condition (3.70))

$$\begin{aligned} \delta\Psi_m &= \left(\frac{p^+}{2}\dot{Y}^i_m\gamma^i - \frac{1}{2}F^i_m\gamma^i\gamma^{123}\right)\epsilon \\ &= ip^+\frac{\partial Y^i_m}{\partial\tau}\gamma^i\left(\frac{1\mp i\gamma^{123}}{2}\right)\epsilon \end{aligned} \tag{3.71}$$

where the matrix $\frac{1\mp i\gamma^{123}}{2}$ has eight unity eigenvalues and eight zero eigenvalues, and is a projection operator. Thus we see that the instanton breaks eight of the sixteen supersymmetries.

As is usually the case, the BPS condition (3.70) is nonlinear in Y (recall that F^k_m is quadratic in Y) and finding its solution is not easy. Below we present the instanton solution interpolating between the $n = 1$ and $n = 0$ vacua.

In the $n = 0$ vacuum, $Y^k_m(n = 0) = 0$. In the $n = 1$ vacuum, $Y^k_m(n = 1) = \frac{\mu p^+}{3g}\delta^k_m$. As can be verified, this expression of $Y^k_m(n = 1)$ indeed gives a zero F^k_m, and its radial component $\Phi_m \equiv Y_{mr} = Y^k_m\frac{x^k}{\mu p^+/3} = \frac{x^m}{g}$ has a gauge invariant length $\sqrt{(\Phi_m)^2} = \frac{\mu p^+}{3g}$, as it should be for the $n = 1$ vacuum. This form of $Y^k_m(n = 1)$ is related to that given in (3.67) by a gauge transformation. We shall call this form of $Y^k_m(n = 1)$ the hedgehog gauge because of the form of Φ_m (we'll call that in (3.67) the sigma-3 gauge). As it turns out, the instanton has a nice form in this hedgehog gauge.

Take the trial solution

$$Y^k_m(\tau) = \omega(\tau)Y^k_m(n = 1) \tag{3.72}$$

and plug it into the BPS condition (3.70), one gets

$$\frac{d\omega}{d\tau} = \pm\frac{\mu}{3}(\omega - \omega^2) \tag{3.73}$$

whose solution is the kink function

$$\omega(\tau) = \frac{1}{1 + \exp\left[\mp\frac{\mu}{3}(\tau - \tau_0)\right]} \tag{3.74}$$

with τ_0 being the integration constant which gives the location of the kink. The solution with the upper sign corresponds to $n_i = 0, n_f = 1$, and the one with the lower sign corresponds to $n_i = 1, n_f = 0$. From now on let us take the one with the lower sign.

To summarize, the instanton solution that interpolates between $n_i = 1$ and $n_f = 0$ is given by

$$Y_m^k(\tau) = \frac{1}{1+\exp\left[\frac{\mu}{3}(\tau-\tau_0)\right]}\frac{\mu p^+}{3g}\delta_m^k \tag{3.75}$$

Note that this instanton solution, in the Y-formalism, is a constant on the sphere. The Euclidean action for this instanton is

$$S_E(n_i = 1, n_f = 0) = \frac{2\pi}{g^2}\left(\frac{\mu p^+}{3}\right)^3 \tag{3.76}$$

As a matter of fact, the function obtained from the Y_m^k in (3.75) by replacing δ_m^k with O^{km} also satisfies the BPS equation, where the matrix O^{km} is a constant (i.e., (τ, θ, ϕ)-independent) $SO(3)$ group element. However, this new solution is related to the original one in (3.75) by a gauge transformation, as one can verify.

We can expand the action about the above instanton solution, keeping up to terms quadratic in the fluctuations. There are bosonic zero modes and fermionic zero modes for the quadratic part of the action. As can be easily seen,

$$y_m^k(\tau) = \frac{dY_m^k}{d\tau_0} = \frac{d\omega}{d\tau_0}\frac{\mu p^+}{3g}\delta_m^k \tag{3.77}$$

is a bosonic zero mode, because it arises from shifting τ_0 which is a symmetry operation preserving the action. This bosonic zero mode is square integrable over $R \times S^2$. (There are also three bosonic zero modes arising from replacing δ_m^k with O^{km} as described above. However, since this is just a gauge transformation, and the resulting three zero modes are not square integrable, we usually discount them.)

The fermionic zero modes, which we denote as ψ_m, are given by the solutions to the Dirac equation in the instanton background

$$\frac{\partial \psi_m}{\partial \tau} - \frac{i}{p^+} g \epsilon^{mpn} Y_p^i \gamma^i \psi_n + i \frac{1}{4} \mu \gamma^{123} \psi_m = 0 \tag{3.78}$$

where we have assumed that ψ_m is (θ, ϕ)-independent. Take

$$\psi_m = \frac{d\omega}{d\tau} \gamma^m \epsilon \tag{3.79}$$

where $\left(\frac{1-i\gamma^{123}}{2}\right)\epsilon = 0$, i.e., $i\gamma^{123}\epsilon = \epsilon$. One can explicitly verify that this satisfies the Dirac equation. There are eight such fermionic zero modes given by the eight choices of ϵ. Comparing ψ_m with the supersymmetry transformation of Ψ_m given in (3.71) (where we now take the lower sign specific to our instanton), we see that they are just the eight supersymmetries broken by the instanton background, as expected. These eight fermionic zero modes are square integrable.

The physical meaning of this instanton: at $\tau = -\infty$, the two membranes are separated by one unit of distance (because $n_i = 1$) in the radial direction; at $\tau = +\infty$, these two membrane coincide (because $n_f = 0$). Since the radius of each membrane is proportional to its p_- momentum, this instanton describes the exchange of one unit of p_- momentum (the so called M-momentum) between the two membranes. Given this instanton, one could try to compute the transition amplitude when relative transverse velocity between the two membranes is turned on and compare the result with a supergravity computation, which we are currently investigating, in collaboration with Hok Kong Lee and Tristan McLoughlin. Also, note that for this instanton, the radial separation between the two concentric membranes $\sqrt{(\Phi_m)^2}$ goes from $\frac{\mu p^+}{3g}$ to zero, which is always much smaller than the size of the membranes $\sqrt{(x^i)^2} = \frac{\mu p^+}{3}$ when the gauge coupling g is large. Hence, if we also require that the separation between the two membranes in the x^4 through x^9 directions are much smaller than $\frac{\mu p^+}{3}$, the supergravity computation can be done in the near-membrane limit explained at the end of Section 4.2 and in Appendix F. For discussions on M-momentum transfer between membranes, gravitons, and other M theory objects, see [22, 23, 24].

We would like to make a few more comments on the application of instantons in this 3d gauge theory to the investigation of M-momentum transfer between two membranes. In flat space, because of the $SO(7)$ symmetry for the seven scalars corresponding to the dimensions transverse to a membrane (recall that x^- is non-dynamical in the light cone gauge), the separation vector between the two membranes for a given instanton can be rotated, thus producing a transverse velocity. Put in other words, suppose the initial separation is along the 4th direction and we want to turn on a velocity along the 5th direction, we can turn on a vev for the scalar of the 5th direction which is in the same gauge as that initial scalar of the 4th direction. In PP-wave, we no longer have this $SO(7)$ symmetry, instead the transverse directions are now divided into the radial direction Y_r of the Y^i's, and the six directions X^a. As we have seen, the instantons lie in the Y^i's. Hence given a instanton specified by Y, if we want to turn on a velocity along, say, the 4th direction, we have to find an X^4 that is in the appropriate gauge, namely, the gauge in accordance with Y. As it turns out, finding this gauge is not so straightforward technically.

Also, it is of interest to find instanton solutions with more general n_i, n_f rather than the simplest one presented above. Given the form of the BPS condition (3.70), one could try to expand the Y's in terms of spherical harmonics and consider the resulting nonlinear equations for the coefficients.

3.3 Two Graviton Interaction without M-momentum Transfer—Gauge Theory Computation

The content of this section is based on work in collaboration with Hok Kong Lee [21]. In short, [21] computes the two-graviton one-loop effective action for Matrix theory in the 11-D PP-wave background, and compares it to the effective action on the supergravity side in the same background. Agreement is found for the effective action on both sides, to all orders of μ. Besides providing further evidence for Matrix theory as a description of M-theory in the 11-D PP-wave background, this agreement also points to the existence of a supersymmetric nonrenormalization theorem in the 11-D PP-wave background. In this section, we mainly present the computation on the gauge theory side.

In their original paper [2], the authors computed graviton scattering in flat space using the Matrix theory and found exact agreement with eleven-dimensional supergravity. Since then, more detailed investigations have been performed in flat space [13, 14, 15, 16, 17, 18, 19, 20]. After the Matrix theory in a weak background was proposed by Taylor and Van Raamsdonk in [6], the case of a space weakly curved in the transverse directions was checked explicitly in [7].

Now that we have the Matrix theory in the 11-D PP-wave proposed by [10] at our disposal, which is exact in this curved background, we expect it to provide further test of the Matrix theory conjecture beyond the weak background approximation proposed by [6]. In addition, the 11-D PP-wave is different from the cases studied in [7], because the metric now has a nontrivial g_{++} component.

Finally a remark about terminology: we use the terms "effective potential", "effective action", and "effective Lagrangian" in an interchangeable sense, although strictly speaking we really mean the last one; this should not cause any confusion.

3.3.1 Brief Review of Known Results

In [14], the one-loop effective potential for two gravitons in flat spacetime background was computed in Matrix theory to be

$$V_{\text{eff}}^{1-\text{loop}} = \frac{15 N_p N_s v^4}{16 M^9 R^3 r^7} \tag{3.80}$$

where N_p and N_s are the numbers of D0-branes making up the probe graviton and source graviton, respectively, v and r are the transverse relative velocity and distance between them, M is the eleven-dimensional Planck mass, and R is the radius of compactification in DLCQ. This effective potential agrees precisely with the supergravity result [17].

In [7], the effective potential for a weakly curved background with nontrivial transverse metric components was computed. Again agreement was found. In fact, the only modification needed was the replacement of r by d, the geodesic distance between the two gravitons.

3.3.2 The Effective Potential

The main object for comparison on both sides of the proposed duality is the effective potential V_{eff}. The computation is carried out in the DLCQ formalism, which was proposed in Susskind's finite N conjecture [3], and further elucidated by [4, 5]. In this formalism x^- and $x^- + 2\pi R$ are identified. p_- is therefore quantized in units of $1/R$.

The implications of such a light-like compactification, however, are far from trivial [45]. One such complication arises from the longitudinal zero modes, which appear to cause perturbative amplitudes to diverge. In addition, there are concerns that the DLCQ of M-theory in the low energy limit is not necessarily the DLCQ of eleven-dimensional supergravity because some exotic degrees of freedom such as membranes wrapped around the lightlike direction may contribute.

Here we are going to take the viewpoint in [46]. Essentially, the presence of a source exerts a pressure that decompactifies the region surrounding it, rendering x^- effectively space-like by providing a nonzero g_{--} component in the metric. In the limit of large N, this bubble of eleven-dimensional space expands, and the approximation of supergravity as a low energy description is justified. This view is further elucidated in [47], and we do not expect new issues to arise in the PP-wave background.

One important fact is, the regions of validity for the gauge theory and supergravity are actually disjoint, as can be seen through the following argument by Hok Kong Lee (see also [47]): On the gauge theory side, it can be shown that the loop counting parameter is $NM^{-3}r^{-3}$, which must be small for loop expansion to make sense. This gives the gauge theory's region of validity $r > N^{1/3}M^{-1}$. On the supergravity side, the source graviton produces a metric component $h_{--} \sim NM^{-9}R^{-2}r^{-7}$, which gives an effective space-like compactification radius $R_s = \sqrt{h_{--}}R = N^{1/2}M^{-9/2}r^{-7/2}$. Recall that the string length is given by $l_s = M^{-3/2}R_s^{-1/2} = M^{3/4}r^{7/4}N^{-1/4}$. For supergravity to make sense, we must have $r > l_s$ (otherwise there is no such notion as spacetime). This gives supergravity's region of validity $r < N^{1/3}M^{-1}$.

Therefore, in general there is no reason why they should match, as each effective action is valid only within its own validity region. Thus a mismatch does not immediately invalidate the Matrix conjecture. An exact match, however, will point to the existence of a nonrenormalization theorem, which protects the terms evaluated from gaining higher-loop corrections. If such a nonrenormalization theorem does exist, then the agreement of both sides can be viewed as positive evidence for the Matrix conjecture. It is with these points in mind that the comparison of the effective action is made here.

On the Matrix theory side, the effective potential is computed up to 1-loop. As in flat space, it should correspond to terms of order κ_{11}^2 on the supergravity side. The relation $\kappa_{11}^2 = 16\pi^5/M^9$ [17] means only terms of order $1/M^9$ are relevant on the Matrix theory side for the purpose of such comparison.

A natural length scale that arises on the Matrix theory side is $1/(M^3R)^{1/2}$, which for convenience we will denote as $(\alpha)^{1/2}$. [2] In addition to the low velocity and large r approximation necessary to facilitate comparison in flat space, we will also assume that

$$\frac{\alpha^2\mu^2}{r^2} << 1 \tag{3.81}$$

where μ is the 123+ component of the four-form field strength.

This dimensionless number, as we will see in eqn. (3.82), is simply the relative strength of the new terms in the action arising from the PP-wave background to the quartic terms already present in flat space. In the opposite limit, $\frac{r^2}{\alpha^2\mu^2} << 1$, the effective potential on the Matrix theory side resums to give $1/\mu$ dependence[3], which does not appear possible to be reproduced on the supergravity side. In fact,

[2] This α should not be confused with the string scale α'.

[3] This can be seen in eqn. (3.94), a typical term in the effective potential.

this is nothing new. A similar issue arises already in flat space, where the effective potential only matches when we take the small v and large r limit, or more precisely, by expanding in the small parameter $v\alpha/r^2$. In other words, even with the existence of a nonrenormalization theorem, the results on both sides should only be compared at very large r, where supergravity is applicable.

3.3.3 Background Field Method

We will follow the background field method as reviewed in [48]. X is expanded into a background field B and a fluctuating field Y, i.e., $X = B + Y$. Only the part of the action that is quadratic in Y will be of interest below.

Recall the Matrix theory action in the DLCQ of M-theory in 11-D PP-wave background [10], given in eqn. (3.1)

$$S = \int dt \mathrm{Tr} \Bigg\{ \quad \sum_{I=1}^{9} \frac{1}{2R}(D_0 X^I)^2 + i\psi^T D_0 \psi + \frac{(M^3R)^2}{4R} \sum_{I,J=1}^{9} [X^I, X^J]^2$$
$$+ (M^3R) \sum_{J=1}^{9} \psi^T \gamma^J [\psi, X^J] + \frac{1}{2R} \left[-(\frac{\mu}{3})^2 \sum_{i=1}^{3} (X^i)^2 - (\frac{\mu}{6})^2 \sum_{a=4}^{9} (X^a)^2 \right]$$
$$- i\frac{\mu}{4} \psi^T \gamma_{123} \psi - \frac{(M^3R)\mu}{R} i \sum_{i,j,k=1}^{3} \epsilon_{ijk} (X^i X^j X^k) \Bigg\} \tag{3.82}$$

Taking the ratios of any of the μ-dependent terms to the μ-independent non-derivative terms gives the parameter in eqn. (3.81). In other words, the assumption stated in the previous section is identical to treating the new terms arising from the PP-wave background as a perturbation of flat space. Note that this is exactly the opposite of the approximation made in [29], where the μ-independent terms are treated as perturbations to the μ-dependent terms. While the computation of the 1-loop effective potential is possible in both limits on the Matrix theory side, an agreement with supergravity is possible only in the large r limit given in eqn. (3.81).

In what follows, unless stated otherwise, we will always assume the indices i goes from 1 to 3, a goes from 4 to 9, and I goes from 1 to 9. In addition to the action above, there are terms arising from the ghosts and gauge fixing, which we simply state below:

$$S_{\rm gf} = \int dt {\rm Tr}\Bigg[-\frac{1}{2R}(\partial_t X_0 + i[B_i, X_i])^2 \Bigg] \tag{3.83}$$

$$S_{\rm ghost} = \int dt {\rm Tr}\Bigg[\bar{c}\partial_t^2 c - \partial_t \bar{c}[X_0, c] + \bar{c}[B^i, [X^i, c]] \Bigg] \tag{3.84}$$

Thus, the complete Matrix theory action is

$$S_M = S + S_{\rm gf} + S_{\rm ghost} \tag{3.85}$$

To simplify the notation, we will put $M^3 R = 1/\alpha = 1$. This factor can be restored by dimensional analysis. It is also convenient to define $g^2 \equiv R$, which corresponds to a loop counting parameter in the Matrix theory.

3.3.3.1 Expansion about the Background

The fields X, ψ, and c are expanded in the following way, with a purely bosonic background

$$\begin{aligned}
X_\mu &= B_\mu + gY_\mu \quad &;& \quad \mu = 0, 1, 2, ..., 9 \\
B_I &= \begin{pmatrix} x_I & 0 \\ 0 & 0 \end{pmatrix} \quad &;& \quad Y_I = \begin{pmatrix} \zeta_I & z_I \\ \bar{z}_I & \widetilde{\zeta}_I \end{pmatrix} \\
B_0 &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad &;& \quad Y_0 = \begin{pmatrix} \zeta_0 & z_0 \\ \bar{z}_0 & \widetilde{\zeta}_0 \end{pmatrix} \\
\psi &= \begin{pmatrix} \eta & \theta \\ \bar{\theta} & \widetilde{\eta} \end{pmatrix} \quad &;& \quad c = \begin{pmatrix} \epsilon & c_1 \\ c_2 & \widetilde{\epsilon} \end{pmatrix}
\end{aligned} \tag{3.86}$$

Here we have set $N_p = N_s = 1$, i.e., we deal with 2×2 matrices. We will later restore N_p and N_s. The above background has the interpretation of one graviton (the source) sitting at the origin[4], while another graviton (the probe) approaches from the

[4]Another possible interpretation is a transverse five brane at the origin [30].

position given by x^I in the matrix B. We will use the shorthand $r^2 = \sum_{I=1}^{9}(x^I)^2$.

After a Wick rotation, where we define $S = iS^{(E)}$ and $\tau = it$, and at the same time rotating X_0 to $iX_0^{(E)}$, the quadratic part of the action is [5]

$$\mathcal{S}^{(E)}_{\rm boson} = \int d\tau \Bigg\{ -\frac{1}{2}\zeta_0\partial_\tau^2\zeta_0 - \frac{1}{2}\widetilde{\zeta}_0\partial_\tau^2\widetilde{\zeta}_0 + \frac{1}{2}\zeta_i(-\partial_\tau^2 + (\mu/3)^2)\zeta_i + \frac{1}{2}\zeta_a(-\partial_\tau^2 + (\mu/6)^2)\zeta_a$$
$$+ \frac{1}{2}\widetilde{\zeta}_i(-\partial_\tau^2 + (\mu/3)^2)\widetilde{\zeta}_i + \widetilde{\zeta}_a(-\partial_\tau^2 + (\mu/6)^2)\widetilde{\zeta}_a$$
$$+ \overline{z}_0(-\partial_\tau^2 + r^2)z_0 - 2i\partial_\tau x_I(\overline{z}_I z_0 - \overline{z}_0 z_I)$$
$$+ \overline{z}_i(-\partial_\tau^2 + r^2 + (\mu/3)^2)z_i + \overline{z}_a(-\partial_\tau^2 + r^2 + (\mu/6)^2)z_a - i\mu\epsilon_{ijk}x_i\overline{z}_j z_k \Bigg\} \quad (3.87)$$

$$\mathcal{S}^{(E)}_{\rm fermion} = \int d\tau \Bigg\{ \eta(i\partial_\tau - i\frac{\mu}{4}\gamma_{123})\eta + \widetilde{\eta}(i\partial_\tau - i\frac{\mu}{4}\gamma_{123})\widetilde{\eta} + 2\overline{\theta}(i\partial_\tau + x_I\gamma_I - i\frac{\mu}{4}\gamma_{123})\theta \Bigg\} \quad (3.88)$$

$$\mathcal{S}^{(E)}_{\rm ghost} = \int d\tau \Bigg\{ \overline{\epsilon}\partial_\tau^2\epsilon + \overline{\widetilde{\epsilon}}\partial_\tau^2\widetilde{\epsilon} + \overline{c}_1(\partial_\tau^2 - r^2)c_2 + \overline{c_2}(\partial_\tau^2 - r^2)c_1 \Bigg\} \quad (3.89)$$

3.3.3.2 The Sum over Mass

The partition function, $\mathcal{Z}$, of the above action can be computed as a product of functional determinants. The one-loop effective action Γ is then simply related to $\mathcal{Z}$ via

$$\exp(-\Gamma) = \mathcal{Z} \quad (3.90)$$

The one-loop effective potential is defined as

[5] For simplicity, all subsequent superscripts (E) on the Euclideanized fluctuation fields will be omitted.

$$\Gamma = -\int d\tau \; V_{\text{eff}} \tag{3.91}$$

To first approximation, however, it is not necessary to compute the functional determinants. As was suggested by Talfjord and Periwal [50], and [49], one could deduce the effective potential by simply evaluating the mass spectrum of the fluctuating fields. From the masses, the one-loop contribution to V_{eff} can be easily deduced using the formula

$$V_{\text{eff}}^{1-\text{loop}} = -\frac{1}{2}\left(\sum_{\text{real bosons}} m_b - \sum_{\text{real fermions}} m_f - \sum_{\text{real ghosts}} m_g\right) \tag{3.92}$$

The physical reason for this is that at large distances, i.e., the limit where supergravity is valid, all strings stretching between the D0-branes can be assumed to lie in their ground state. This result can also be verified using the complete expression for V_{eff} in terms of functional determinants. We provide an argument for this in Appendix B. In what follows, we will omit the superscript "1-loop", assuming this is understood. The contribution from tree level, which does not concern us here, is simply the Lagrangian with X replaced by B. Both contributions will be put back together at the end in eqn. (3.100).

One important point to note is that this method is valid only up to the lowest powers of v, as is already known in the flat space case. In flat space, the above formula reproduces every term predicted by a supergravity computation with the right coefficients, but the Matrix theory corrections to supergravity, i.e., terms with even higher powers of v and $1/r$ which would not be found in supergravity, will not come out with the correct coefficients. In fact, the parameter α can be treated as the counting parameter for this purpose. All terms of order α^3, which is basically κ_{11}^2 in the supergravity language, will be found on the supergravity side, but terms on the Matrix theory side with higher powers of α, which represent short distance effects, should be treated as corrections. To compute them correctly, one needs to make use of the complete expression in terms of functional determinants.

For our purpose, however, the above approach is sufficient. We are not interested in computing the correction to supergravity, rather we would like to check whether the terms already predicted by supergravity in the PP-wave background can be reproduced by a Matrix theory calculation.

3.3.4 A Simple Case

In the next section we will work out a more efficient method to compute V_{eff} without explicitly diagonalizing the mass matrix. Nevertheless, it is instructive to work out the simplest case in a direct approach to get the basic idea of the computation.

In this simple case, we put $x^8 = b$ and $x^9 = v\tau$, while all the other x^I are set to zero[6]. Here b is a constant, which can be interpreted as the impact parameter of the approaching probe graviton towards the source sitting at the origin. In this case, the mass matrix constructed from eqn.'s (3.87), (3.88) and (3.89) is easily diagonalized to give the mass spectrum listed in Table 3.1. It should be noted that the velocity in the table above is measured in Euclidean time τ, i.e., $v = \frac{\partial x}{\partial \tau}$. In a comparison with supergravity, a Wick rotation back into Minkowski time $t = -i\tau$ is required, which introduces extra minus signs in V_{eff}.

With the mass spectrum at hand, V_{eff} can be evaluated using eqn. (3.92)

$$\begin{aligned} V_{\text{eff}} = -\frac{1}{2}(2)(3\frac{\mu}{3} + 6\frac{\mu}{6} - 8\frac{\mu}{4}) - \frac{1}{2}\Big\{ & 6\sqrt{r^2 + \mu^2/3^2} + 10\sqrt{r^2 + \mu^2/6^2} + 2\sqrt{r^2 + \eta_+} \\ & + 2\sqrt{r^2 + \eta_-} - 8\sqrt{r^2 + \mu^2/4^2 + v} - 8\sqrt{r^2 + \mu^2/4^2 - v} - 4r \Big\} \end{aligned} \tag{3.93}$$

At this point it is useful to restore the factors of $M^3 R$, which we denote as $1/\alpha$. For instance, the first square root term in the about equation becomes

$$\sqrt{\frac{r^2}{\alpha^2} + \frac{\mu^2}{3^2}} \tag{3.94}$$

[6]Note that by putting all x^i to zero for $i = 1, 2, 3$, we ensure that in this case the Myers term will not contribute to the mass matrix.

$\mathbf{m^2}$	**Fields**
0	ζ^0
$\mu^2/3^2$	$\zeta^i \quad ; i = 1,2,3$
$\mu^2/6^2$	$\zeta^a \quad ; a = 4,...,9$
0	$\widetilde{\zeta}^0$
$\mu^2/3^2$	$\widetilde{\zeta}^i \quad ; i = 1,2,3$
$\mu^2/6^2$	$\widetilde{\zeta}^a \quad ; a = 4,...,9$
$r^2 + \mu^2/3^2$	$\overline{z}^i, z^i \quad ; i = 1,2,3$
$r^2 + \mu^2/6^2$	$\overline{z}^a, z^a \quad ; a = 4,...,8$
$r^2 + \eta_+$	$\overline{z}^0 + \overline{z}^9, z^0 + z^9$
$r^2 + \eta_-$	$\overline{z}^0 - \overline{z}^9, z^0 - z^9$
$\mu^2/4^2$	$\eta \quad (8)$
$\mu^2/4^2$	$\widetilde{\eta} \quad (8)$
$r^2 + \mu^2/4^2 + v$	$\theta \quad (8)$
$r^2 + \mu^2/4^2 - v$	$\overline{\theta} \quad (8)$
0	$\overline{\epsilon}, \epsilon$
0	$\overline{\widetilde{\epsilon}}, \widetilde{\epsilon}$
r^2	$\overline{c}_I, c_I \quad ; I = 1,2$

Table 3.1: **The Mass Spectrum for a Simple Case**. The numbers inside the round brackets indicate the number of physical degrees of freedom of the fermions with the given mass. $\eta_\pm$ is given by $\frac{1}{2}[\frac{\mu^2}{6^2} \pm \sqrt{(\frac{\mu^2}{6^2})^2 + 16v^2}]$.

This can in turn be written as

$$\frac{r}{\alpha}\sqrt{1+\frac{1}{3^2}(\frac{\alpha}{r^2})(\alpha\mu^2)} \tag{3.95}$$

The expression for V_{eff} given above, being a Matrix theory result, is only expected to match with supergravity in the large r limit (if it does at all!). Defining the large r limit by eqn. (3.81), we can then expand the one-loop effective potential in powers of $\alpha^2\mu^2/r^2$. Thus, expanding V_{eff} gives

$$V_{\text{eff}} = \alpha^3(\frac{15}{16}\frac{v^4}{r^7} + \frac{7}{96}\frac{\mu^2 v^2}{r^5} + \frac{1}{768}\frac{\mu^4}{r^3}) + O[\alpha^5] \tag{3.96}$$

Wick rotating v, and restoring N_p, N_s gives

$$V_{\text{eff}} = \frac{N_p N_s}{M^9 R^3}(\frac{15}{16}\frac{v^4}{r^7} - \frac{7}{96}\frac{\mu^2 v^2}{r^5} + \frac{1}{768}\frac{\mu^4}{r^3}) + O[\alpha^5] \tag{3.97}$$

The α^3 terms give the factor $1/M^9$, which translates into κ_{11}^2 in the supergravity language. This is the order we are interested in. We throw away the higher powers of α (which are always accompanied by powers of $1/r$) because they correspond to short distance corrections to supergravity, just as in flat space.

Here the first term is just the flat space result. The second and the third term are the interesting ones, with new $\mu^2 v^2$ and μ^4 dependence created by the PP-wave background. A comparison of their coefficients with supergravity will show exact agreement.

3.3.5 Mass Matrix Computation

In the more general cases, when the velocity and the impact parameter point in arbitrary directions, calculating the effective potential V_{eff} by finding the entire m^2 spectrum, then taking their square roots and expanding them in powers of μ and v becomes inefficient. In the most general case this involves finding the eigenvalues of mass matrices of very high dimension. Instead, it is possible to make use of the sum over mass formula in eqn. (3.92) without explicitly diagonalizing the mass matrix. Let us denote the square of the mass matrix as $W = M^2$. Since there is never any mixing between the bosons, the fermions and the ghosts, we can study their mass matrices separately.

In terms of W, the sum over mass formula becomes

$$V_{\rm eff}^{1-{\rm loop}} = -\frac{1}{2}{\rm tr}(\sqrt{W_b} - \sqrt{W_f} - \sqrt{W_g}) \tag{3.98}$$

The square root of W can be defined unambiguously by its expansion in powers of α/r^2 in the supergravity limit, as was discussed in Section 3.3.4. Note that M_b is defined to be the mass matrix for *real* bosons. If it is taken to be the mass matrix for the complex bosons, then there will be an extra factor of two in front of $\sqrt{W_b}$.

Simple Recipe for Mass Matrix

In this subsection we will give a simple recipe for writing out M^2 for both the bosons and the fermions. The mass for the ghosts is exactly the same as in the simple case of Section 3.3.4.

First of all, we should note that the mass of ζ^i and ζ^a are always $\mu/3$ and $\mu/6$, respectively, for $i = 1, 2, 3$ and $a = 4, ..., 9$. The mass of all eight physical degrees in η is always $\mu/4$. These are independent of the background B. Mixing occurs only among the z^I and among the θ and $\overline{\theta}$. Hence in what follows, we will denote the component arising from say $\overline{z}^I z^J$ in the bosonic Lagrangian simply as $(M^2)_{IJ}$ without mentioning z explicitly. Note also that M^2 is symmetric.

I. Rules for Bosons

1. $(M^2)_{00} = r^2$; $(M^2)_{ii} = r^2 + \mu^2/3^2$; $(M^2)_{aa} = r^2 + \mu^2/6^2$;
2. $\dot{x}^I = v_I$ mixes z^0 and z^I $\Rightarrow$ $(M^2)_{0I} = -2v_I$
3. $x^1 = b_1$ mixes z^2 and z^3... etc. $\Rightarrow$ $(M^2)_{jk} = i\mu\epsilon_{ijk}b_i$

Note that Rule 3 applies only to z^i but not z^a. Such mixing is the effect of the Myers term in the Matrix theory action.

II. Rules for Fermions

The mass matrix for the fermions can be written in a closed form

$$M^2 = r^2 + \mu^2/4^2 + \sum_{I=1}^{9} v_I \gamma_I + \sum_{i=1}^{3} \frac{i\mu x^i}{4}\{\gamma_i, \gamma_{123}\} \tag{3.99}$$

3.3.6 The General Case

Once the mass matrix squared $W = M^2$ is known, eqn. (3.98) can be used to compute the one-loop effective potential explicitly. In accordance with our earlier discussions, only terms up to order $\alpha^3 \sim 16\pi^5/M^9 = \kappa_{11}^2$ are kept. After restoring all factors of M^3R, N_p, and N_s, the sum of zero and one-loop effective potentials is given by

$$\begin{aligned} V_{\text{eff}}^{0,1-\text{loop}} &= \frac{N_p}{2R}(\sum_{I=1}^{9} v_I^2 + g_{++}) + \frac{N_p N_s}{M^9 R^3}\Bigg\{ \frac{15(\sum_{I=1}^{9} v_I^2)^2}{16r^7} - \frac{\mu^2 \sum_{i=1}^{3} v_i^2}{96r^5} - \frac{7\mu^2 \sum_{a=4}^{9} v_a^2}{96r^5} \\ &\quad + \frac{15\mu^2}{32r^7}\left[\sum_{i=1}^{3} x_i^2 \left(-\sum_{i=1}^{3} v_i^2 + \sum_{a=4}^{9} v_a^2\right) + 2(\sum_{i=1}^{3} x_i v_i)^2\right]\Bigg\} \\ &+ \frac{\mu^4 N_p N_s}{R^3 M^9}\frac{1}{768r^7}\left\{32\left[\sum_{i=1}^{3}(x^i)^2\right]^2 + \left[\sum_{a=4}^{9}(x^a)^2\right]^2 - 12\sum_{i=1}^{3}(x^i)^2 \cdot \sum_{a=4}^{9}(x^a)^2\right\} \end{aligned} \tag{3.100}$$

In (3.100), $\frac{N_p}{2R}(\sum_{I=1}^{9} v_I^2 + g_{++})$ is the zero-loop potential, and the rest is the one-loop potential. Note that, compared with (3.97), the one-loop potential given in (3.100) contains additional terms arising from the x^1, x^2, x^3 directions. This is the equation to be compared with the supergravity result. Notice the effective potential has manifest $SO(3) \times SO(6)$ symmetry, as should be expected from the symmetry of the original Matrix theory action. Just as in flat space [6], one should be able to recast this 1-loop effective potential in the form $T^{\mu\nu}G_{\mu\nu}$. A comparison with the supergravity side will indeed confirm this, as this is precisely the form of the effective potential on the supergravity side as derived in Appendix C.

Having computed the effective potential on the Matrix theory side, the next step will be to compare it with the result from a supergravity calculation. Before this can be done, the issue of gauge choice has to be addressed.

It is necessary to make a gauge choice when solving the Einstein equations. A gauge choice corresponds to a choice of the coordinate system one uses to describe the physics. On the Matrix theory side, such a choice of coordinates was made right from the very beginning: The action in eqn. (3.82) was written in coordinates that made the $SO(3) \times SO(6)$ symmetry manifest. Before a comparison is possible, a corresponding choice of coordinates, i.e., a choice of gauge has to be made on the supergravity side.

A comparison of the above equation with the general expression for V_{eff} in eqn. (4.113) will in the end determine the correct gauge choice for the supergravity computation. There will be a further discussion about gauge choice in the supergravity section.

Chapter 4

Interaction of M-theory Objects in Eleven-Dimensional Supergravity

On the supergravity side, we adopt the source-probe viewpoint, which is valid when the source is much heavier than the probe. Section 4.1 deals with the action of the probe via constrained Hamiltonian mechanics with the constraints arising from worldline/worldvolume diffeomorphism, for a point particle probe and a membrane probe. Section 4.2 deals with the determination of the background fields the probe feels, by diagonalizing the linearized supergravity field equations in the presence of the source. These first two sections provide the basis for investigating M-theory objects' interactions on the supergravity side, at linear κ_{11}^2 order. Section 4.3 is an application to the two graviton interaction without M-momentum transfer.

4.1 The Light Cone Lagrangian

The light cone Lagrangian is the quantity that will be computed on the supergravity side and then compared with the gauge theory result. We shall only consider bosonic degrees of freedom, because we are only concerned with the bosonic coordinates of the probe, and also the background fermionic fields are set to zero.

4.1.1 Point Particle Probe

Let us start with the case of a point particle (the graviton). By "light-cone Lagrangian" we mean a quantity $L_{lc}(x^-, p_-, x^A, \dot{x}^A)$ which will be defined below. Roughly speaking, it is the original Lagrangian Legendre transformed in the x^- degree of freedom. Appendix C gives a quick derivation of it. The dynamics of a point particle is a system with constraint due to time reparametrization symmetry, and has to be dealt with in the Hamiltonian formalism if one wants full rigor. The derivation in Appendix C is in the Lagrangian formalism and not so rigorous. Hence let us now give a careful derivation of the light-cone Lagrangian using the constrained Hamiltonian formalism. This is also a useful warm-up before we derive the same quantity for a membrane.

A very quick review about constraints: Constraints are relations between coordinates and momenta. Constraints that arise directly from the definition of momenta, i.e., without using the equations of motion, are called *primary constraints*. Constraints that arise when imposing the consistency requirement that the primary constraints are preserved in time evolution are called *secondary constraints*. Constraints that arise when imposing the consistency requirement that the secondary constraints are preserved in time evolution are called *tertiary constraints*, etc. Secondary, tertiary, etc., constraints are obtained by using the equations of motion. The classification of constraints into primary, secondary, tertiary, etc., constraints is of little importance in the final form of the Hamiltonian formalism. A more fundamental classification of constraints is to define *first-class constraints* and *second-class constraints* as follows: a constraint is called a first-class constraint if its Poisson bracket with every constraint vanishes weakly (i.e., vanishes on the submanifold defined by the constraints in phase space); a constraint that is not first-class is called a second-class constraint. This classification of constraints plays a central role in the Hamiltonian formalism, because first-class constraints are generators of gauge transformations. Equalities that hold only on the submanifold defined by the constraints in phase space are called weak equalities and usually denoted with the weak equality symbol "$\approx$", but in what follows we will simply use the equality symbol "$=$" for them, expecting no confusion.

For a comprehensive discussion on systems with constraints, see, for example, [51] and [52].

Consider a point particle of mass m in a general curved background $G_{\mu\nu}(x)$ (this background includes that generated by the source object, in the context of investigating M-theory objects' interaction on the supergravity side). The Lagrangian of the particle is

$$L(x^\mu, \dot{x}^\mu) = -m\sqrt{-G_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu} \tag{4.1}$$

where a dot means a derivative with respect to the world line parameter τ.

The momenta are then given by

$$p_\mu \equiv \frac{\partial L}{\partial \dot{x}^\mu} = \frac{m}{\sqrt{-G_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu}} G_{\mu\beta}(x)\dot{x}^\beta \tag{4.2}$$

As can be easily verified, $\gamma_1(x^\mu, p_\mu) \equiv G^{\mu\nu}(x)p_\mu p_\nu + m^2 = 0$, which is the primary constraint of our system. The Hamiltonian is given by $H \equiv p_\mu \dot{x}^\mu - L = 0$, which is expected because in general the Hamiltonian of a system with reparametrization symmetry vanishes.

The consistency condition $\dot{\gamma}_1 = 0$ trivially holds, hence we don't have any secondary constraint. So the only constraint of our system is $\gamma_1 = 0$, which is a first-class constraint and is the generator of the gauge symmetry — the reparametrization freedom of the world line.

The extended Hamiltonian is then given by $H_E = u^1\gamma_1$, where the arbitrary function u^1 is the Lagrange multiplier that embodies the gauge degree of freedom (i.e., reparametrization freedom). The evolution of any function $F(x^\mu, p_\mu)$ is given by $\dot{F} = [F, H_E]_{\mathrm{PB}}$, where $[F, G]_{\mathrm{PB}} \equiv \sum_{\mu=+,-,A} \left(\frac{\partial F}{\partial x^\mu} \frac{\partial G}{\partial p_\mu} - \frac{\partial F}{\partial p_\mu} \frac{\partial G}{\partial x^\mu} \right)$ is the Poisson bracket. Hence the equations of motion are

$$\begin{aligned}
\dot{x}^\mu &= 2u^1 G^{\mu\alpha}(x) p_\alpha \\
\dot{p}_\mu &= -u^1 \frac{\partial G^{\alpha\beta}(x)}{\partial x^\mu} p_\alpha p_\beta \\
\gamma_1 &= G^{\mu\nu}(x) p_\mu p_\nu + m^2 = 0
\end{aligned} \tag{4.3}$$

The next step is gauge fixing. Let us use the light-cone coordinates $x^\mu = \{x^+, x^-, x^A\}$, and impose the light-cone gauge condition $C_{\tilde{1}} \equiv x^+ - \tau = 0$. We see that $[\gamma_1, C_{\tilde{1}}]_{\text{PB}} = -2G^{+\nu}(x)p_\nu$ which can reasonably be assumed to be nonzero. Hence upon gauge fixing we get two second-class constraints $\chi_\alpha = (\gamma_1, C_{\tilde{1}})$.

Then, by the usual story of gauge fixing, now that we have two second-class constraints, we should replace Poisson brackets with the Dirac brackets (which we denote as $[,]_{\text{D}}$. Recall that the Dirac bracket for two phase space functions is defined as $[F, G]_{\text{D}} \equiv [F, G]_{\text{PB}} - [F, \chi_\alpha]_{\text{PB}} C^{\alpha\beta} [\chi_\beta, G]_{\text{PB}}$, where χ_α, χ_β are second-class constraints, and $C^{\alpha\beta}$ is the inverse matrix of $C_{\alpha\beta} \equiv [\chi_\alpha, \chi_\beta]_{\text{PB}}$), and in this case since the gauge fixing condition $C_{\tilde{1}}$ is time-dependent, we should also add a corresponding correction term (for a discussion on time-dependent gauge fixing, see, e.g., exercise 4.8 in [52]). One gets the follow law for time evolution

$$\dot{F} = [F, H_E]_{\text{D}} - C^{a\tilde{b}} \frac{\partial C_{\tilde{b}}}{\partial \tau} [F, \gamma_a]_{\text{PB}} = \frac{1}{2G^{+\nu}(x)p_\nu} [F, \gamma_1]_{\text{PB}} \tag{4.4}$$

where we have used the fact that $[F, H_E]_{\text{D}} = 0$. In particular, this gives

$$\begin{aligned}
\dot{x}^\mu &= \frac{G^{\mu\beta}(x) p_\beta}{G^{+\nu}(x) p_\nu} \\
\dot{p}_\mu &= -\frac{1}{2G^{+\nu}(x)p_\nu} \left(\frac{\partial G^{\alpha\beta}(x)}{\partial x^\mu} p_\alpha p_\beta \right) \\
\gamma_1 &= 0, \quad C_{\tilde{1}} = 0
\end{aligned} \tag{4.5}$$

It easy to verify that this indeed gives $\dot{x}^+ = 1$, which agrees with the gauge choice $x^+ = \tau$.

Using Dirac brackets is the longer (and more rigorous) way of fixing the gauge.

However, there is also a shortcut which avoids using Dirac brackets. Remember that the gauge freedom is embodied in u^1. Hence gauge fixing amounts to specifying u^1. So we just have to go to the equations of motion before gauge fixing, which contain u^1, and set $x^+ = \tau$. Then the x^+ equation gives us $u^1 = \frac{1}{2G^{+\nu}(x)p_\nu}$, which we can then substitute into other equations and get back the gauge fixed equations of motion obtained using Dirac brackets. This shortcut approach is what we shall use when investigating membranes.

Now let's restrict our attention to situations where the metric is static, i.e., independent of x^+, $G^{\mu\nu} = G^{\mu\nu}(x^-, x^A)$. Also, let's consider functions which are independent of x^+, p_+, i.e., $F = F(x^-, x^A, p_-, p_A)$. In this case, if we define the "light-cone Hamiltonian" as $H_{lc}(x^-, x^A, p_-, p_A) \equiv -p_+$, where $p_+(x^-, x^A, p_-, p_A)$ is obtained by solving $\gamma_1 = 0$ (in which x^+ is set to τ) for p_+ in terms of the other variables, then one can verify that

$$\dot{F} = \sum_{\mu=-,A} \left(\frac{\partial F}{\partial x^\mu} \frac{\partial H_{lc}}{\partial p_\mu} - \frac{\partial F}{\partial p_\mu} \frac{\partial H_{lc}}{\partial x^\mu} \right) \tag{4.6}$$

The above expression justifies the name of H_{lc}. It generates time evolution in the smaller phase space consisting of (x^-, x^A, p_-, p_A).

Using the gauge fixed equation of motion for $\dot{x}^\mu$ and also the expression of $p_+(x^-, x^A, p_-, p_A)$, we can express $\dot{x}^\mu$ as a function of (x^-, x^A, p_-, p_A). Now let's make the reasonable assumption that the relation $\dot{x}^B = \dot{x}^B(x^-, x^A, p_-, p_A)$ is invertible so that we can solve for $p_B(x^-, p_-, x^A, \dot{x}^A)$.

Now by making a Legendre transformation in the transverse degrees of freedom, we define the "light-cone Lagrangian" as

$$L_{lc}(x^-, p_-, x^A, \dot{x}^A) \equiv p_A \dot{x}^A - H_{lc} \tag{4.7}$$

Then it's easy to see that, in the L_{lc} formalism, the equations of motion are

$$p_A = \left(\frac{\partial L_{lc}}{\partial \dot{x}^A} \right)_{x^-, p_-, x^B}, \quad \dot{p}_A = \left(\frac{\partial L_{lc}}{\partial x^A} \right)_{x^-, p_-, \dot{x}^B} \tag{4.8}$$

The above two equations can be combined to give the expected Euler-Lagrange equation

$$\frac{d}{d\tau}\left(\frac{\partial L_{lc}}{\partial \dot{x}^A}\right) - \left(\frac{\partial L_{lc}}{\partial x^A}\right) = 0 \tag{4.9}$$

Also

$$\dot{x}^- = -\left(\frac{\partial L_{lc}}{\partial p_-}\right)_{x^-,x^A,\dot{x}^A}, \quad \dot{p}_- = \left(\frac{\partial L_{lc}}{\partial x^-}\right)_{p_-,x^A,\dot{x}^A} \tag{4.10}$$

One more look at L_{lc}: recall that $0 = H = p_\mu \dot{x}^\mu - L = p_+\dot{x}^+ + p_-\dot{x}^- + p_A\dot{x}^A - L$, which upon gauge-fixing $x^+ = \tau$ becomes $0 = p_+ + p_-\dot{x}^- + p_A\dot{x}^A - L = -H_{lc} + p_-\dot{x}^- + p_A\dot{x}^A - L = L_{lc} + p_-\dot{x}^- - L$, which gives $L_{lc} = L - p_-\dot{x}^-$, which is just the L' defined in eqn. (C.2) of Appendix C, as expected. (The P_p^+ in Appendix C is the p_- here.)

When investigating graviton interactions in a PP-wave, the background metric is $G_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$, with $g_{\mu\nu}$ being the unperturbed PP-wave, and $h_{\mu\nu}$ being the metric perturbation due to the source graviton. Keeping terms up to linear order in $h_{\mu\nu}$, one finds that the light cone Lagrangian for the probe graviton is

$$\begin{aligned} L_{lc} = p_-\Bigg\{ \frac{1}{2}\Bigg[& v^2 + g_{++} + h_{++} + g_{++}\left(\frac{1}{4}g_{++}h_{--} - h_{+-}\right) \\ & + \sum_A [2h_{+A} - h_{-A}(v^2 + g_{++})]v^A + \sum_{A,B} h_{AB}v^A v^B \Bigg] \\ & + \frac{1}{8}h_{--}v^4 - \frac{1}{2}v^2\left(h_{+-} - \frac{1}{2}g_{++}h_{--}\right)\Bigg\} \end{aligned} \tag{4.11}$$

which is the object one compares to the gauge theory effective Lagrangian (and gets agreement on).

Recall that the change of the longitudinal momentum p_- is governed by $\dot{p}_- = \left(\frac{\partial L_{lc}}{\partial x^-}\right)_{p_-,x^A,\dot{x}^A}$. Hence the x^--dependence of L_{lc} is what's responsible for the longitudinal momentum exchange between the probe graviton and source graviton. In the L_{lc} given in (4.11), x^--dependence comes in only through $h_{\mu\nu}$. Hence what one does is to Fourier transform $h_{\mu\nu}$ along the x^- direction, and solve the Einstein equations for the Fourier components of $h_{\mu\nu}$. When one is only considering transverse momentum transfer, one plugs the zeroth Fourier component of $h_{\mu\nu}$ into L_{lc}; when one is considering one unit of longitudinal momentum transfer, one plugs the first Fourier component of $h_{\mu\nu}$ in, etc.

Also, notice that in the L_{lc} given in (4.11), lower powers of velocity are accompanied by $h_{\mu\nu}$ with more lower $+$ indices, or equivalently fewer lower $-$ indices ("at higher level", in the language of Section 4.2). In Section 4.2, we shall see that the metric perturbations at lower level are easier to compute. Hence, in L_{lc} the v^4 term is the easiest to find, next is v^2, and then the v-independent term is the hardest to compute. In flat space, only the lowest level h_{--} is nonzero, and there is only the v^4 term. In the PP-wave, we have metric perturbations at all levels, and life is harder. In the membrane case the situation is similar.

4.1.2 Membrane Probe

Now we consider the membrane. A brief review of constrained Hamiltonian mechanics in field theory is given in Appendix D. The previous discussion on the point particle may be said to be merely a quest for rigor, but for the membrane the following discussion is a necessity. The reason is, although people have discussed the membrane in a fairly general background [12], the background there is taken to be the special case which is independent of x^-, which certainly is not true in the physical situation where there is p_- transfer (see discussion near the end of 4.1.1). Also, in [12], the background metric G_{--} and G_{-A} are set to zero using the target space diffeomorphism freedom. This is not the right gauge to use even in the investigation of only transverse momentum transfer (We know that in the two graviton interaction it is the background G_{--} that gives the v^4 term in the effective potential of the probe graviton). For the above two reasons, we would like to discuss the derivation of the light cone Lagrangian for a membrane in an arbitrary static background (note: by "static" we mean there is no x^+-dependence, yet x^--dependence is allowed).

Denote the background metric and three-form as $G_{\mu\nu}(x)$, $A_{\mu\nu\rho}(x)$, respectively, and the membrane embedding coordinates as $X^\mu(\sigma^i)$, with $\sigma^i, i = 0, 1, 2$ being the world-volume coordinates. Our membrane is considered to be a probe membrane, hence it does not have any back reaction on the background geometry. The background geometry can include the contribution from a source, though. The membrane Lagrangian density is given by

$$\mathcal{L}(X^\mu, \partial_i X^\mu) = T\left[\sqrt{-\det(g_{ij})} - \frac{1}{6}\epsilon^{ijk}A_{\mu\nu\rho}\partial_i X^\mu \partial_j X^\nu \partial_k X^\rho\right] \tag{4.12}$$

with T being the membrane tension, and $g_{ij} \equiv G_{\mu\nu}\partial_i X^\mu \partial_j X^\nu$ being the pullback metric. The momentum density is

$$\Pi_\lambda \equiv \frac{\partial \mathcal{L}}{\partial(\partial_0 X^\lambda)} = T\left[\sqrt{-\det(g_{ij})}\; g^{0k}(\partial_k X^\mu)G_{\lambda\mu} - A_{\lambda\nu\rho}\partial_1 X^\nu \partial_2 X^\rho\right] \tag{4.13}$$

Define

$$\tilde{\Pi}_\lambda \equiv \Pi_\lambda + T A_{\lambda\nu\rho}\partial_1 X^\nu \partial_2 X^\rho = T\sqrt{-\det(g_{ij})}\; g^{0k}(\partial_k X^\mu)G_{\lambda\mu} \tag{4.14}$$

Then it is easy to see we have the following primary constraint

$$\phi_0 \equiv G^{\lambda\xi}\tilde{\Pi}_\lambda\tilde{\Pi}_\xi + T^2\det(g_{rs}) = 0 \tag{4.15}$$

where $r, s = 1, 2$ label the spatial world-volume coordinates. It is also easily seen that, ϕ_0 only contains Π_μ, X^μ, and spatial derivatives of X^μ, as required for a constraint. One can also verify that there are two more primary constraints:

$$\phi_r \equiv \Pi_\lambda \partial_r X^\lambda, \;\; r = 1, 2 \tag{4.16}$$

Of course, we know the gauge freedom these constraints arise from: ϕ_0 comes from world-volume temporal reparametrization freedom and the ϕ_r's come from world-volume spatial reparametrization freedom. The Hamiltonian density is given by

$$\mathcal{H} \equiv \Pi_\lambda \partial_0 X^\lambda - \mathcal{L} = 0 \tag{4.17}$$

as it should be for a system with general covariance. The total Hamiltonian density is given by

$$\mathcal{H}_T = \mathcal{H} + c^i\phi_i = c^i\phi_i = c^0\phi_0 + c^r\phi_r \tag{4.18}$$

where the c^i's denote the Lagrange multiplier fields. $H_T = \int d\sigma^1 d\sigma^2\ \mathcal{H}_T$.

Computing the Poisson bracket with H_T (see Appendix D), one finds the equations of motion (using $\frac{\partial\phi_0}{\partial\Pi_\mu} = 2G^{\mu\xi}\tilde{\Pi}_\xi$, $\frac{\partial\phi_r}{\partial\Pi_\mu} = \partial_r X^\mu$)

$$\begin{aligned} \partial_0 X^\mu &= 2c^0 G^{\mu\xi}\tilde{\Pi}_\xi + c^r\partial_r X^\mu \\ \partial_0\Pi_\mu &= \partial_r\left(c^i\frac{\partial\phi_i}{\partial(\partial_r X^\mu)}\right) - c^i\frac{\partial\phi_i}{\partial X^\mu} \end{aligned} \tag{4.19}$$

where to facilitate the evaluation of $\partial_0\Pi_\mu$ we list below some useful expressions

$$\frac{\partial\phi_0}{\partial(\partial_r X^\mu)} = 2G^{\lambda\xi}\frac{\partial\tilde{\Pi}_\lambda}{\partial(\partial_r X^\mu)}\tilde{\Pi}_\xi + T^2\frac{\partial(\det(g_{rs}))}{\partial(\partial_r X^\mu)} \tag{4.20}$$

with

$$\begin{aligned} \frac{\partial\tilde{\Pi}_\lambda}{\partial(\partial_r X^\mu)} &= TA_{\lambda\mu\rho}(\delta_1^r\partial_2 X^\rho - \delta_2^r\partial_1 X^\rho) \\ \frac{\partial(\det(g_{rs}))}{\partial(\partial_r X^\mu)} &= G_{\mu\alpha}\left\{2\delta_1^r(\partial_1 X^\alpha)g_{22} + 2\delta_2^r(\partial_2 X^\alpha)g_{11} - 2g_{12}(\delta_1^r\partial_2 X^\alpha + \delta_2^r\partial_1 X^\alpha)\right\} \end{aligned} \tag{4.21}$$

$$\frac{\partial\phi_0}{\partial X^\mu} = \frac{\partial G^{\lambda\xi}}{\partial X^\mu}\tilde{\Pi}_\lambda\tilde{\Pi}_\xi + 2G^{\lambda\xi}\frac{\partial\tilde{\Pi}_\lambda}{\partial X^\mu}\tilde{\Pi}_\xi + T^2\frac{\partial(\det(g_{rs}))}{\partial X^\mu} \tag{4.22}$$

with

$$\begin{aligned} \frac{\partial\tilde{\Pi}_\lambda}{\partial X^\mu} &= T\frac{\partial A_{\lambda\nu\rho}}{\partial X^\mu}\partial_1 X^\nu\partial_2 X^\rho \\ \frac{\partial(\det(g_{rs}))}{\partial X^\mu} &= \frac{\partial G_{\alpha\beta}}{\partial X^\mu}(\partial_1 X^\alpha\partial_1 X^\beta g_{22} + g_{11}\partial_2 X^\alpha\partial_2 X^\beta - 2g_{12}\partial_1 X^\alpha\partial_2 X^\beta) \end{aligned} \tag{4.23}$$

also

$$\frac{\partial \phi_s}{\partial(\partial_r X^\mu)} = \delta_s^r \Pi_\mu, \quad \frac{\partial \phi_s}{\partial X^\mu} = 0 \quad r, s = 1, 2 \tag{4.24}$$

We need to check whether there are any secondary constraints. Using the above equations of motion, one shows after a somewhat tedious manipulation,

$$\partial_0 \phi_i = 0 \tag{4.25}$$

for arbitrary c^i, which means that there is no secondary constraint, and the ϕ_i's are all first class constraints with the Lagrange multipliers c^i's embodying the gauge degrees of freedom. This is an expected result, but it's reassuring to see that is the case for arbitrary backgrounds, not just the special backgrounds (e.g., flat background, x^--independent backgrounds) investigated previously.

Next we should gauge fix the membrane system. In the spirit explained in the point particle case, we will not do so by introducing Dirac brackets, but rather by specifying the c^i's, which is somewhat shorter.

Now use the light-cone coordinates $\{x^+, x^-, x^A\}$, and assume static background $G_{\mu\nu}(x^-, x^A)$ and $A_{\mu\nu\rho}(x^-, x^A)$. Use the light cone gauge $X^+ = \sigma^0$. Then the equation of motion for X^+ gives

$$c^0 = \frac{1}{2G^{+\xi}\tilde{\Pi}_\xi} \tag{4.26}$$

So now the constraint $\phi_0(X^-, X^A, \partial_r X^-, \partial_r X^A, \Pi_+, \Pi_-, \Pi_A) = 0$ (in which X^+ is set to σ^0) can be used to solve for $\Pi_+(X^-, X^A, \partial_r X^-, \partial_r X^A, \Pi_-, \Pi_A)$. Then we define the light cone Hamiltonian $\mathcal{H}_{lc}(X^-, X^A, \partial_r X^-, \partial_r X^A, \Pi_-, \Pi_A) \equiv -\Pi_+$

Before defining the light-cone Lagrangian, similar to the point particle case, we need to express Π_A as a function of $(X^-, X^A, \partial_r X^-, \partial_r X^A, \Pi_-, \partial_0 X^A)$. This is done by looking at the equations of motion for X^A:

$$\frac{\partial X^A}{\partial \sigma^0} = \frac{G^{A\xi}\tilde{\Pi}_\xi}{G^{+\nu}\tilde{\Pi}_\nu} + c^r \frac{\partial X^A}{\partial \sigma^r} \tag{4.27}$$

where we have used the expression for c^0 given above. We should fix the gauge by specifying the c^r's to be functions of $(X^-, X^A, \partial_r X^-, \partial_r X^A, \Pi_-, \Pi_A)$. Then using the expression $\Pi_+(X^-, X^A, \partial_r X^-, \partial_r X^A, \Pi_-, \Pi_A)$ obtained earlier, we can invert this equation of motion to find $\Pi_A(X^-, X^A, \partial_r X^-, \partial_r X^A, \Pi_-, \partial_0 X^A)$. Then we can define the light-cone Lagrangian

$$\mathcal{L}_{lc}(X^-, X^A, \partial_r X^-, \partial_r X^A, \Pi_-, \partial_0 X^A) \equiv \Pi_A \partial_0 X^A - \mathcal{H}_{lc} \tag{4.28}$$

• An Easy Example: A Probe Membrane in the Unperturbed PP-wave Background

Now, as an easy example, let us derive $\mathcal{L}_{lc}$ for a probe membrane in the unperturbed PP-wave background using the above prescription. We will see that we get the expected answer given in [29]

In the unperturbed PP-wave, the only nonzero component of $A_{\mu\nu\rho}$ is $A_{+ij} = \frac{\mu}{3}\epsilon_{ijk}x^k$, and we don't bother to write down the familiar PP-wave metric here again. Solving the constraint $\phi_0 = 0$, we find

$$\Pi_+ = \frac{-1}{2\Pi_-}\left\{-g_{++}\Pi_-^2 + \Pi_A\Pi_A + T^2\det(g_{rs})\right\} - T\frac{\mu}{3}\epsilon_{ijk}(\partial_1 X^i)(\partial_2 X^j)X^k \tag{4.29}$$

Also easily seen is $\det(g_{rs}) = \frac{1}{2}(\partial_1 X^A \partial_2 X^B - \partial_1 X^B \partial_2 X^A)^2$. Also the equations of motion for X^A are

$$\partial_0 X^A = \frac{\Pi_A}{\Pi_-} + c^r \partial_r X^A \tag{4.30}$$

To motivate a gauge choice, let us note that $\partial_0 \Pi_- = \partial_r(c^r \Pi_-)$. (This is just equation (2.21) in [12], in the particular case of PP-wave). Hence we choose $c^r = 0$, which means Π_- will be σ^0-independent. This gauge choice gives us

$$\Pi_A = \Pi_- \partial_0 X^A \tag{4.31}$$

Hence we find

$$\begin{aligned}
\mathcal{L}_{lc} \quad &= \Pi_A \partial_0 X^A - \mathcal{H}_{lc} \\
&= \Pi_- \partial_0 X^A \partial_0 X^A + \Pi_+ \\
&= \frac{1}{2}\Pi_- \partial_0 X^A \partial_0 X^A - \frac{\Pi_-}{2}\mu^2 \left[\frac{1}{9}(X^i)^2 + \frac{1}{36}(X^a)^2\right] \\
&- \frac{T^2}{4\Pi_-}(\partial_1 X^A \partial_2 X^B - \partial_1 X^B \partial_2 X^A)^2 - T\frac{\mu}{3}\epsilon_{ijk}(\partial_1 X^i)(\partial_2 X^j)X^k \qquad (4.32)
\end{aligned}$$

which agrees with [29] upon setting $T = -1$ (this minus sign is to conform to [12]'s convention eqn. (2.16) for the membrane action, which [29] follows.) and identifying our Π_- with their p^+.

• A Less Trivial Example: A Probe in the PP-wave Background Perturbed by a Source

Let us compute $\mathcal{L}_{lc}$ in the PP-wave background perturbed by some source. The background is now $G_{\mu\nu} = (G_{\mu\nu})_{pp} + h_{\mu\nu}$, and $A_{\mu\nu\rho} = (A_{\mu\nu\rho})_{pp} + a_{\mu\nu\rho}$, with the quantities with subscript pp being those of the unperturbed PP-wave background, and $h_{\mu\nu}, a_{\mu\nu\rho}$ being metric and three-form perturbations caused by the source. We only need the light-cone Lagrangian to linear order in the perturbation.

$$\mathcal{L}_{lc} = (\mathcal{L}_{lc})_{pp} + \delta\mathcal{L}_{lc} \qquad (4.33)$$

with $(\mathcal{L}_{lc})_{pp}$ being the expression given in the previous example of unperturbed PP-wave.

The computation is, as in the previous example, quite straightforward, although a bit tedious, because solving for Π_A in terms of $(X^-, X^A, \partial_r X^-, \partial_r X^A, \Pi_-, \partial_0 X^A)$ in the perturbed background requires some work. But the four-time-derivative term, i.e., the v^4 term in $\mathcal{L}_{lc}$ is not hard to find. Making the gauge choice $c^r = 0$, we find

$$\mathcal{L}_{lc}^{(4)} = \frac{\Pi_-}{8} h_{--}(\delta_{AB}\partial_0 X^A \partial_0 X^B)^2 \qquad (4.34)$$

(the superscript (4) means it's the four-time-derivative term) which is quite similar to the graviton case. To know this term, we only need h_{--} produced by the source. Of course, to get terms with lower powers of the velocity, we need other components of $h_{\mu\nu}$ and also the three-form perturbation $a_{\mu\nu\rho}$.

To summarize, to get the explicit expression for the light-cone Lagrangian, we only need to figure out the $h_{\mu\nu}$ and $a_{\mu\nu\rho}$ produced by the source, which is the subject of the next section.

4.2 Diagonalizing the Supergravity Field Equations for Arbitrary Static Sources

Now let us present the diagonalization of the linearized supergravity equations of motions for arbitrary sources. There is, of course, no highbrow knowledge involved here: we are just solving the linearized Einstein equations and Maxwell equations, which are coupled; and by "diagonalization" we basically just mean the prescription using which we get a decoupled Laplace equation for each component of the metric and three-form perturbations. The unperturbed background is the 11-D PP-wave, and we only consider static, i.e., x^+-independent, field configurations, thanks to the fact that the sources considered are taken to be static, i.e., with x^+-independent stress tensor and three-form current.

Since we leave the source arbitrary, what we'll present here are the left-hand side of the linearized equations. These are tensors whose computation is straightforward though a bit tedious: the reason we present them here is because they are necessary when solving the field equations, and to the best of our knowledge have not been explicitly given elsewhere.

A somewhat related problem is the diagonalization of the equations of motion when the source is absent. This requires field configurations with x^+-dependence. One good reference along this line is [53]. Roughly speaking, borrowing the language of electromagnetism, what's considered in [53] are electromagnetic waves in vacuum, while what we are considering here are electrostatics and magnetostatics for arbitrary sources.

Denote the metric perturbation $\delta g_{\mu\nu}$ as $h_{\mu\nu}$, and the gauge potential perturbation as $\delta A_{\mu\nu\rho} = a_{\mu\nu\rho}$. Once again, the nonzero components of the PP-wave background metric and the four-form field strength are given by

$$g_{+-} = 1,\ g_{++} = -\mu^2 \left[\frac{1}{9}\sum_{i=1}^{3}(x^i)^2 + \frac{1}{36}\sum_{a=4}^{9}(x^a)^2 \right],\ g_{AB} = \delta_{AB} \tag{4.35}$$

$$F_{123+} = \mu \tag{4.36}$$

In our conventions, $\mu, \nu, \rho, \ldots$ take the values $+, -, 1, \ldots, 9$; $A, B, C, \ldots$ take the values $1, \ldots, 9$; $i, j, k, \ldots$ take the values $1, \ldots, 3$; and $a, b, c, \ldots$ take the values $4, \ldots, 9$. Also, we follow [54] for the conventions of various tensors.

The nonzero components up to (anti)symmetry of the Christoffel symbol, Riemann tensor, etc., of the 11-D PP-wave are

$$\begin{aligned} \Gamma^A_{++} &= -\frac{1}{2}\partial_A g_{++}, \quad \Gamma^-_{+A} = \frac{1}{2}\partial_A g_{++} \\ R_{+A+B} &= -\frac{1}{2}\partial_A\partial_B g_{++}, \quad R_{++} = -\frac{1}{2}\partial_C\partial_C g_{++}, \quad R = 0 \end{aligned} \tag{4.37}$$

(We usually don't substitute the explicit expression of g_{++}, unless that brings significant simplification to the resulting formula)

Now let's add a source, thus perturbing the background. $h_{\mu\nu}, a_{\mu\nu\rho}$ are treated as rank-two and rank-three tensors, respectively, the covariant derivative ∇ acting on them is defined using the connection coefficient of the unperturbed PP-wave background, and indices are raised/lowered, traces are taken using the background metric $g_{\mu\nu}$. Let's deal with the Einstein equations first.

Define $\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}g_{\mu\nu}h$, where $h \equiv g^{\mu\nu}h_{\mu\nu}$. Without the source, the Einstein equation is

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - \kappa_{11}^2[T_{\mu\nu}]_A = 0 \tag{4.38}$$

Recall that the stress tensor of the gauge field is

$$[T_{\mu\nu}]_A = \frac{1}{12\kappa_{11}^2}\left(F_{\mu\lambda\xi\rho}F_\nu{}^{\lambda\xi\rho} - \frac{1}{8}g_{\mu\nu}F^{\rho\sigma\lambda\xi}F_{\rho\sigma\lambda\xi}\right) \tag{4.39}$$

The source perturbs the Einstein equation to

$$\delta\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right) - \kappa_{11}^2\delta[T_{\mu\nu}]_A = \kappa_{11}^2[T_{\mu\nu}]_S \tag{4.40}$$

with $[T_{\mu\nu}]_S$ standing for the stress tensor of the source.

As usual, it helps to proceed in an organized manner, grouping different terms in the above perturbed Einstein equations. One finds, $\delta\left(R_{\mu\rho} - \frac{1}{2}Rg_{\mu\rho}\right) = -\frac{1}{2}\nabla^\sigma\nabla_\sigma\bar{h}_{\mu\rho} + K_{\mu\rho} + Q_{\mu\rho}$, and $\kappa_{11}^2\delta[T_{\mu\nu}]_A = N_{\mu\nu} + L_{\mu\nu}$, where the explicit expressions of the symmetric tensors $\nabla^\sigma\nabla_\sigma\bar{h}_{\mu\nu}$, $K_{\mu\nu}$, $Q_{\mu\nu}$,$N_{\mu\nu}$, and $L_{\mu\nu}$ can be obtained after some work. Their definitions and components are given below [1]

- $\nabla^\sigma\nabla_\sigma\bar{h}_{\mu\nu}$

$$\begin{aligned}\nabla^\sigma\nabla_\sigma\bar{h}_{++} = \quad & g^{\mu\nu}\partial_\mu\partial_\nu\bar{h}_{++} + \left[-(\partial_A\partial_A g_{++})\bar{h}_{+-} + \frac{1}{2}(\partial_A g_{++}\partial_A g_{++})\bar{h}_{--}\right] \\ & +2\left[\partial_A g_{++}\partial_-\bar{h}_{+A} - \partial_A g_{++}\partial_A\bar{h}_{+-}\right]\end{aligned} \tag{4.41}$$

$$\nabla^\sigma\nabla_\sigma\bar{h}_{+-} = g^{\mu\nu}\partial_\mu\partial_\nu\bar{h}_{+-} - \frac{1}{2}(\partial_A\partial_A g_{++})\bar{h}_{--} + \partial_A g_{++}\partial_-\bar{h}_{-A} - \partial_A g_{++}\partial_A\bar{h}_{--} \tag{4.42}$$

$$\nabla^\sigma\nabla_\sigma\bar{h}_{+C} = g^{\mu\nu}\partial_\mu\partial_\nu\bar{h}_{+C} - \frac{1}{2}(\partial_A\partial_A g_{++})\bar{h}_{-C} + \partial_A g_{++}\partial_-\bar{h}_{AC} - \partial_C g_{++}\partial_-\bar{h}_{+-} - \partial_A g_{++}\partial_A\bar{h}_{-C} \tag{4.43}$$

$$\nabla^\sigma\nabla_\sigma\bar{h}_{--} = g^{\mu\nu}\partial_\mu\partial_\nu\bar{h}_{--} \tag{4.44}$$

[1]Notice that ∂_+ will never appear because we only consider the static case; also note $g^{\mu\nu}\partial_\mu\partial_\nu = -g_{++}\partial_-^2 + \partial_A\partial_A$ for static configurations.

$$\nabla^\sigma \nabla_\sigma \bar{h}_{-C} = g^{\mu\nu}\partial_\mu \partial_\nu \bar{h}_{-C} - \partial_C g_{++}\partial_- \bar{h}_{--} \tag{4.45}$$

$$\nabla^\sigma \nabla_\sigma \bar{h}_{CD} = g^{\mu\nu}\partial_\mu \partial_\nu \bar{h}_{CD} - \partial_C g_{++}\partial_- \bar{h}_{-D} - \partial_D g_{++}\partial_- \bar{h}_{-C} \tag{4.46}$$

- $K_{\mu\nu}$ Its definition is

$$K_{\mu\rho} \equiv \frac{1}{2}\left(R_\mu{}^\xi \bar{h}_{\xi\rho} + R_\rho{}^\xi \bar{h}_{\xi\mu}\right) + R^\sigma{}_{\mu\rho}{}^\xi \bar{h}_{\sigma\xi} + \frac{1}{2} g_{\mu\rho} R^{\xi\sigma}\bar{h}_{\xi\sigma} - \frac{1}{2} R\bar{h}_{\mu\rho} \tag{4.47}$$

Its components are given by

$$K_{++} = \left(-\frac{1}{2}\partial_A\partial_A g_{++}\right)\left(\bar{h}_{+-} + \frac{1}{2} g_{++}\bar{h}_{--}\right) + \frac{1}{2}\left(\partial_A\partial_B g_{++}\right)\bar{h}_{AB} \tag{4.48}$$

$$K_{+-} = \left(-\frac{1}{2}\partial_A\partial_A g_{++}\right)\bar{h}_{--} \tag{4.49}$$

$$K_{+A} = \left(-\frac{1}{4}\partial_C\partial_C g_{++}\right)\bar{h}_{-A} + \left(-\frac{1}{2}\partial_A\partial_B g_{++}\right)\bar{h}_{-B} \tag{4.50}$$

$$K_{--} = 0 \tag{4.51}$$

$$K_{-A} = 0 \tag{4.52}$$

$$K_{AB} = \frac{1}{2}\left[\partial_A\partial_B g_{++} - \frac{1}{2}\delta_{AB}\partial_C\partial_C g_{++}\right]\bar{h}_{--} \tag{4.53}$$

- $Q_{\mu\nu}$ Its definition is $Q_{\mu\rho} \equiv \frac{1}{2}(\nabla_\mu q_\rho + \nabla_\rho q_\mu) - \frac{1}{2} g_{\mu\rho}\nabla^\alpha q_\alpha$, where $q_\alpha \equiv \nabla^\beta \bar{h}_{\beta\alpha}$.

As one can recognize, $Q_{\mu\rho}$ contains the arbitrariness of making different gauge choices when solving the Einstein equation, where one makes a gauge choice by specifying the q_μ's. The components of $Q_{\mu\rho}$ are

$$
\begin{aligned}
&Q_{--} = \partial_- q_-, \quad Q_{-A} = \frac{1}{2}(\partial_- q_A + \partial_A q_-), \quad Q_{-+} = \frac{1}{2}(g_{++}\partial_- q_- - \partial_A q_A) \\
&Q_{AB} = \frac{1}{2}(\partial_A q_B + \partial_B q_A) - \frac{1}{2}\delta_{AB}(\partial_- q_+ - g_{++}\partial_- q_- + \partial_A q_A) \\
&Q_{+A} = \frac{1}{2}\left[\partial_A q_+ - (\partial_A g_{++}) q_-\right] \\
&Q_{++} = \frac{1}{2}(\partial_A g_{++}) q_A - \frac{1}{2} g_{++}(\partial_- q_+ - g_{++}\partial_- q_- + \partial_A q_A)
\end{aligned}
\tag{4.54}
$$

Let's make a few more remarks about gauge choice here. If one chooses the "Lorentz gauge" where all the q_μ's vanish, then $Q_{\mu\nu}$ all vanish. One can also choose, say, the "harmonic gauge" in which $\delta(g^{\rho\sigma}\Gamma^\mu{}_{\rho\sigma}) = 0$. (Note that in the unperturbed PP-wave background $g^{\rho\sigma}\Gamma^\mu{}_{\rho\sigma}$ vanishes.) These two gauges are in general different because $\delta(g^{\rho\sigma}\Gamma^\mu{}_{\rho\sigma}) = -h^{\rho\sigma}\Gamma^\mu{}_{\rho\sigma} + q^\mu$, or more explicitly, $\delta(g^{\rho\sigma}\Gamma^+{}_{\rho\sigma}) = q_-$, $\delta(g^{\rho\sigma}\Gamma^-{}_{\rho\sigma}) = -h_{-A}\partial_A g_{++} + q_+ - g_{++}q_-$, and $\delta(g^{\rho\sigma}\Gamma^A{}_{\rho\sigma}) = \frac{1}{2}h_{--}\partial_A g_{++} + q_A$. One may also choose gauges in between, of which our graviton computation in [21] is an example. We don't concern ourselves much with the issue of gauge choice here, because in any gauge, provided the q_μ's are set to some known functions, we shall be able to diagonalize the linearized equations. The gauge choice issue will resurface later when one compares the results of the supergravity calculation with that of the gauge theory calculation. There, for the results from both sides to match, one has to make a "most natural" gauge choice (usually motivated by the symmetry of the problem) on the supergravity side. We will discuss that in Section 4.3 in the specific example of two graviton interactions.

- $N_{\mu\nu}$ It is defined to be the part of $\kappa_{11}^2\delta[T_{\mu\nu}]_A$ that contains only the metric perturbation, but not the three-form gauge potential perturbation. Its components are given by

$$
\begin{aligned}
N_{++} &= \mu^2 \left(\frac{1}{3}\bar{h}_{+-} + \frac{1}{12} g_{++}\bar{h}_{--} - \frac{1}{3}\sum_{i=1}^{3}\bar{h}_{ii} + \frac{1}{6}\sum_{a=4}^{9}\bar{h}_{aa} \right) \\
N_{+-} &= \frac{\mu^2}{4}\bar{h}_{--}, \quad N_{+i} = \frac{\mu^2}{2}\bar{h}_{-i}, \quad N_{+b} = 0 \\
N_{--} &= 0, \quad N_{-i} = 0, \quad N_{-b} = 0 \\
N_{ij} &= -\frac{\mu^2}{4}\delta_{ij}\bar{h}_{--}, \quad N_{ib} = 0, \quad N_{ab} = \frac{\mu^2}{4}\delta_{ab}\bar{h}_{--}
\end{aligned} \tag{4.55}
$$

- $L_{\mu\nu}$ This is defined to be the part of $\kappa_{11}^2 \delta[T_{\mu\nu}]_A$ that contains only the three-form perturbation, but not the metric perturbation. Its components are given by

$$
\begin{aligned}
& L_{++} = \mu \left(\delta F_{123+} - \frac{1}{2} g_{++}\delta F_{123-} \right), \; L_{+-} = 0, \; L_{+i} = \frac{\mu}{4}\epsilon_{ijk}\delta F_{+jk-}, \; L_{+b} = \frac{\mu}{2}\delta F_{123b} \\
& L_{--} = 0, \quad L_{-i} = 0, \quad L_{-b} = 0 \\
& L_{ij} = \frac{\mu}{2}\delta_{ij}\delta F_{123-}, \; L_{ib} = \frac{\mu}{4}\epsilon_{ijk}\delta F_{bjk-}, \; L_{bd} = -\frac{\mu}{2}\delta_{bd}\delta F_{123-}
\end{aligned} \tag{4.56}
$$

Next let us deal with the Maxwell equation. In the absence of the source, it is

$$
\frac{1}{\sqrt{-g}}\partial_\lambda \left(\sqrt{-g}\; F^{\lambda\mu_1\mu_2\mu_3} \right) - \frac{\tilde{\eta}}{1152}\frac{\epsilon^{\mu_1\cdots\mu_{11}}}{\sqrt{-g}} F_{\mu_4\ldots\mu_7} F_{\mu_8\ldots\mu_{11}} = 0 \tag{4.57}
$$

where $\tilde{\eta}$ is either $+1$ or -1 depending on the convention, which we can always fix later by requiring the consistency of the conventions for the equations and the solutions that we consider. (As it turns out, in the two graviton interaction case [21] it does not matter because this $F \wedge F$ term has no effect on the final effective potential. Of course, for membrane interactions, that would no longer be the case.) When the source is present, we add its current $J^{\mu_1\mu_2\mu_3}$ to the left-hand side of the above equation, and get

$$
\delta \left[\frac{1}{\sqrt{-g}}\partial_\lambda \left(\sqrt{-g}\; F^{\lambda\mu_1\mu_2\mu_3} \right) - \frac{\tilde{\eta}}{1152}\frac{\epsilon^{\mu_1\cdots\mu_{11}}}{\sqrt{-g}} F_{\mu_4\ldots\mu_7} F_{\mu_8\ldots\mu_{11}} \right] = J^{\mu_1\mu_2\mu_3} \tag{4.58}
$$

We can write the left-hand side of the above equation as the sum of two totally antisymmetric tensors $Z^{\mu_1\mu_2\mu_3} + S^{\mu_1\mu_2\mu_3}$, where $Z^{\mu_1\mu_2\mu_3}$ is defined to be the part that contains the metric perturbation only, and $S^{\mu_1\mu_2\mu_3}$ is defined to be the part that contains the three-form perturbation only. One finds

$$
\begin{aligned}
&Z^{+-i} = \mu\epsilon_{ijk}\partial_j\bar{h}_{-k}, \quad Z^{+-b} = 0, \quad Z^{+ij} = \mu\epsilon_{ijk}(\partial_-\bar{h}_{-k} - \partial_k\bar{h}_{--}), \quad Z^{+ib} = 0, \quad Z^{+bc} = 0 \\
&Z^{-ij} = \mu\epsilon_{ijk}\left[\partial_k\left(\frac{1}{3}\bar{h} - \bar{h}_-^{\ -} - \sum_{i=1}^{3}\bar{h}_{ii}\right) - \partial_b\bar{h}_{kb}\right], \quad Z^{-ib} = \mu\epsilon_{ijk}\partial_j\bar{h}_{kb}, \quad Z^{-bc} = 0 \\
&Z^{ijk} = -\mu\epsilon_{ijk}\left[\partial_-\left(\frac{1}{3}\bar{h} - \bar{h}_-^{\ -} - \sum_{i=1}^{3}\bar{h}_{ii}\right) - \partial_b\bar{h}_{-b}\right], \quad Z^{ijb} = \mu\epsilon_{ijk}(\partial_-\bar{h}_{kb} - \partial_k\bar{h}_{-b}) \\
&Z^{ibc} = 0, \quad Z^{bce} = 0
\end{aligned} \tag{4.59}
$$

and

$$
S^{+-A} = g^{\mu\nu}\partial_\mu\partial_\nu a_{-+A} + \partial_B g_{++}\partial_- a_{BA-} - \partial_-(\nabla^\mu a_{\mu+A}) + \partial_A(\nabla^\mu a_{\mu+-}) \tag{4.60}
$$

$$
S^{+AB} = g^{\mu\nu}\partial_\mu\partial_\nu a_{-AB} - \partial_-(\nabla^\mu a_{\mu AB}) + \partial_A(\nabla^\mu a_{\mu-B}) - \partial_B(\nabla^\mu a_{\mu-A}) \tag{4.61}
$$

$$
\begin{aligned}
S^{-AB} = \quad & g^{\mu\nu}\partial_\mu\partial_\nu a_{+AB} - g_{++}S^{+AB} \\
& + \{[(\partial_A g_{++})(\partial_- a_{-+B}) + \partial_A(\nabla^\mu a_{\mu+B}) - \partial_A\left(a_{EB-}\partial_E g_{++}\right)] - [A \leftrightarrow B]\} \\
& -(\partial_D g_{++})\delta F_{D-AB} - \mu\frac{\tilde{\eta}}{24}\epsilon^{-AB\mu_4\ldots\mu_7 123+}\delta F_{\mu_4\ldots\mu_7}
\end{aligned} \tag{4.62}
$$

$$
\begin{aligned}
S^{ABE} = \quad & g^{\mu\nu}\partial_\mu\partial_\nu a_{ABE} - (\partial_A g_{++})(\partial_- a_{-BE}) - (\partial_B g_{++})(\partial_- a_{-EA}) - (\partial_E g_{++})(\partial_- a_{-AB}) \\
& -\partial_A(\nabla^\mu a_{\mu BE}) - \partial_B(\nabla^\mu a_{\mu EA}) - \partial_E(\nabla^\mu a_{\mu AB}) - \mu\frac{\tilde{\eta}}{24}\epsilon^{ABE\mu_4\ldots\mu_7 123+}\delta F_{\mu_4\ldots\mu_7}
\end{aligned} \tag{4.63}
$$

Notice that $S^{\mu_1\mu_2\mu_3}$ contains $\nabla^\mu a_{\mu\rho\lambda}$ and its derivatives. Those terms correspond to the gauge freedom for the three-form gauge potential. One could use the "Lorentz gauge" where $\nabla^\mu a_{\mu\rho\lambda} = 0$. But for the sake of generality, let's leave the gauge choice for the three-form arbitrary.

Now that we have collected the expressions for the various tensors, we are ready to diagonalize the field equations. Recall that the Einstein equation is

$$-\frac{1}{2}\nabla^\sigma\nabla_\sigma\bar{h}_{\mu\nu} + K_{\mu\nu} + Q_{\mu\nu} - N_{\mu\nu} - L_{\mu\nu} = \kappa_{11}^2[T_{\mu\nu}]_S \tag{4.64}$$

and the Maxwell equation is

$$Z^{\mu_1\mu_2\mu_3} + S^{\mu_1\mu_2\mu_3} = J^{\mu_1\mu_2\mu_3} \tag{4.65}$$

The right-hand sides of these equations are given by specifying the source that we consider (recall that the three-form current J is of order κ_{11}^2), hence we only need to concentrate on diagonalizing the left-hand sides.

As will be seen shortly, it is useful to define "level" for tensors: lower $+$/upper $-$ indices contribute $+1$ to the level; lower $-$/upper $+$ indices contribute -1 to level; and the upper A/lower A indices contribute zero to the level. We shall see that the field equations should be solved in ascending order of their levels. The following is the detailed prescription of the diagonalization procedure. Let us use the shorthand notation $(E.E.)_{\mu\nu}$ for the lower $(\mu\nu)$ component of the Einstein equation, and $(M.E.)^{\mu_1\mu_2\mu_3}$ for the upper $(\mu_1\mu_2\mu_3)$ component of the Maxwell equation.

- at level -2

The only field equation at this level is $(E.E.)_{--}$, which reads, upon using the expressions of the various tensors $\nabla^\sigma\nabla_\sigma\bar{h}_{\mu\nu}, K_{\mu\nu}, Q_{\mu\nu}$... etc., that we've given above

$$-\frac{1}{2}g^{\mu\nu}\partial_\mu\partial_\nu\bar{h}_{--} + Q_{--} = \kappa_{11}^2[T_{--}]_S \tag{4.66}$$

This equation can be immediately solved for $\bar{h}_{--}$ after specifying the source term and the gauge choice term Q_{--}.

- at level -1

We have $(E.E.)_{-A}$, which reads

$$-\frac{1}{2}\left[g^{\mu\nu}\partial_\mu\partial_\nu\bar{h}_{-A} - (\partial_A g_{++})(\partial_-\bar{h}_{--})\right] + Q_{-A} = \kappa_{11}^2[T_{-A}]_S \tag{4.67}$$

which can now be solved for $\bar{h}_{-A}$, using the $\bar{h}_{--}$ found previously. Also at this level is $(M.E.)^{+AB}$, which reads,

$$\begin{aligned}
&g^{\mu\nu}\partial_\mu\partial_\nu a_{-ij} - \partial_-(\nabla^\mu a_{\mu ij}) + \partial_i(\nabla^\mu a_{\mu-j}) - \partial_j(\nabla^\mu a_{\mu-i}) + \mu\epsilon_{ijk}(\partial_-\bar{h}_{-k} - \partial_k\bar{h}_{--}) = J^{+ij} \\
&g^{\mu\nu}\partial_\mu\partial_\nu a_{-ib} - \partial_-(\nabla^\mu a_{\mu ib}) + \partial_i(\nabla^\mu a_{\mu-b}) - \partial_b(\nabla^\mu a_{\mu-i}) = J^{+ib} \\
&g^{\mu\nu}\partial_\mu\partial_\nu a_{-bc} - \partial_-(\nabla^\mu a_{\mu bc}) + \partial_b(\nabla^\mu a_{\mu-c}) - \partial_c(\nabla^\mu a_{\mu-b}) = J^{+bc}
\end{aligned} \tag{4.68}$$

from which we can find a_{-AB}, upon specifying the gauge choice $\nabla^\mu a_{\mu\rho\lambda}$ for the three-form and using the $\bar{h}_{-A}$ and $\bar{h}_{--}$ found previously.

- at level 0

At this level we have $(E.E.)_{+-}$, $(M.E.)^{+-A}$, $(E.E.)_{AB}$, and $(M.E.)^{ABE}$.

$(E.E.)_{+-}$ is of the form

$$-\frac{1}{2}g^{\mu\nu}\partial_\mu\partial_\nu\bar{h}_{+-} = \text{known terms} \tag{4.69}$$

(From now on, we will not bother writing down the detailed equations; "known terms" refers to the gauge choice terms $Q_{\mu\nu}$, $\nabla^\mu a_{\mu\rho\lambda}$, source terms, and terms containing previously found $\bar{h}_{\mu\nu}$'s and $a_{\mu\nu\rho}$'s, one can write those down by looking up the expressions given earlier for the various tensors.) Hence solving it we get $\bar{h}_{+-}$. Solving $(M.E.)^{+-A}$ gives a_{-+A}.

$(E.E.)_{AB}$ and $(M.E.)^{ABE}$ are coupled, so a little more work is needed. The following are the details. First notice that the only unknown in $(M.E.)^{bce}$ is a_{bce}, hence solving this equation we find a_{bce} ($(M.E.)^{bce}$ contains the usual term $g^{\mu\nu}\partial_\mu\partial_\nu a_{bce}$ and also a term of the form $\partial_- a_{dfg}$ which comes from the $F\wedge F$ in the Maxwell equation, hence it is not quite a Laplace equation. But, that being said, one shouldn't have any difficulty solving it.)

$(M.E.)^{ibc}$ is of the form $g^{\mu\nu}\partial_\mu\partial_\nu a_{ibc}$ = known terms, solving which gives a_{ibc}.

$(M.E.)^{ijb}$ and $(E.E.)_{kb}$ are coupled in the following manner

$$
\begin{aligned}
g^{\mu\nu}\partial_\mu\partial_\nu a_{ijb} + \mu\epsilon_{ijk}\partial_-\bar{h}_{kb} &= \text{known terms} \\
-\frac{1}{2}g^{\mu\nu}\partial_\mu\partial_\nu\bar{h}_{kb} + \frac{1}{4}\mu\epsilon_{klm}\partial_- a_{lmb} &= \text{known terms}
\end{aligned} \tag{4.70}
$$

Decoupling these two equations are quite easy. Let us take a_{12b} and $\bar{h}_{3b}$ as the representative case. One sees that these two equations can be recombined to give

$$
\begin{aligned}
(g^{\mu\nu}\partial_\mu\partial_\nu + i\mu\partial_-)(\bar{h}_{3b} + ia_{12b}) &= \text{known terms} \\
(g^{\mu\nu}\partial_\mu\partial_\nu - i\mu\partial_-)(\bar{h}_{3b} - ia_{12b}) &= \text{known terms}
\end{aligned} \tag{4.71}
$$

Solving these equations gives $(\bar{h}_{3b} + ia_{12b})$ and $(\bar{h}_{3b} - ia_{12b})$, and in turn $\bar{h}_{3b}$ and a_{12b}.

$(M.E.)^{ijk}$ is coupled to $(E.E.)_{ij}$ and $(E.E.)_{bd}$ through the quantity $H \equiv \frac{2}{3}\sum_{i=1}^{3}\bar{h}_{ii} - \frac{1}{3}\sum_{a=4}^{9}\bar{h}_{aa}$ in the following manner

$$
\begin{aligned}
g^{\mu\nu}\partial_\mu\partial_\nu a_{123} + \mu\partial_- H &= \text{known terms} \\
-\frac{1}{2}g^{\mu\nu}\partial_\mu\partial_\nu\bar{h}_{ij} + \frac{1}{2}\mu\delta_{ij}\partial_- a_{123} &= \text{known terms} \\
-\frac{1}{2}g^{\mu\nu}\partial_\mu\partial_\nu\bar{h}_{bd} - \frac{1}{2}\mu\delta_{bd}\partial_- a_{123} &= \text{known terms}
\end{aligned} \tag{4.72}
$$

Combining the last two equations gives

$$
-g^{\mu\nu}\partial_\mu\partial_\nu H + 4\mu\partial_- a_{123} = \text{known terms} \tag{4.73}
$$

Recombining this with first equation, we get

$$
\begin{aligned}
(g^{\mu\nu}\partial_\mu\partial_\nu + 2i\mu\partial_-)(H + 2ia_{123}) &= \text{known terms} \\
(g^{\mu\nu}\partial_\mu\partial_\nu - 2i\mu\partial_-)(H - 2ia_{123}) &= \text{known terms}
\end{aligned} \tag{4.74}
$$

solving which individually gives H and a_{123}. Using the obtained expression for a_{123} one can then find $\bar{h}_{ij}$ and $\bar{h}_{bd}$. Thus we are done with $(E.E.)_{AB}$ and $(M.E.)^{ABE}$.

- at level 1

$(M.E.)^{-AB}$ is of the form $g^{\mu\nu}\partial_\mu\partial_\nu a_{+AB} =$ known terms, solving which gives a_{+AB}.

$(E.E.)_{+A}$ is of the form $-\frac{1}{2}g^{\mu\nu}\partial_\mu\partial_\nu\bar{h}_{+A} =$ known terms, solving which gives $\bar{h}_{+A}$.

- at level 2

$(E.E.)_{++}$ is of the form $-\frac{1}{2}g^{\mu\nu}\partial_\mu\partial_\nu\bar{h}_{++} =$ known terms, solving which gives $\bar{h}_{++}$.

Thus we have completely diagonalized the whole set of Einstein equations and Maxwell equations!

Let us use $\Box$ to denote $g^{\mu\nu}\partial_\mu\partial_\nu$ (which we call the Laplacian). In Appendix E we discuss the Green's function of $\tilde{\Box}$, the Fourier transform of $\Box$ along x^-. Since it does not have x^+ dependence, this Green's function is different from the scalar propagator discussed in [55], which has a closed form expression.

In Appendix F we give the expression for h_{--} (again Fourier transformed along the x^- direction) when the source is a membrane which is a sphere in the first three transverse directions, a point at the origin of the other six transverse directions, and moving along the trajectory $X^+ = \sigma^0$, $X^- = 0$, in the near-membrane limit (see Appendix F for what is meant by the "near-membrane limit"). In that case h_{--} has the form of a massive scalar Green's function in seven-dimensional Euclidean space, with its "mass" proportional to the k_- that the graviton carries. Plugging this h_{--} into (4.34) gives the v^4 term of the light cone Lagrangian of a probe membrane in the PP-wave background perturbed by this spherical source membrane, in the case when every point on the probe membrane is in the near-membrane limit with respect to the source membrane.

4.3 Two Graviton Interaction Without M-momentum Transfer—Supergravity Computation

Completing our investigation of the gauge/gravity duality appearing in two graviton interactions without M-momentum transfer in a PP-wave, the content of this section is basically taken from the computation on the supergravity side given in [21]. This is an application of the general formalism developed in Sections 4.1 and 4.2, although since [21] was done before the writing up of this thesis, the following differs from the general formalism in some nonessential details, e.g., it solves for $h_{\mu\nu}$ instead of $\bar{h}_{\mu\nu}$, and the term "level" is not mentioned explicitly. (The "solving the field equations in ascending order of the level" pattern can be very easily recognized though.) Because there is no M-momentum transfer in this application, many terms containing ∂_- which couple different equations simply disappear, making the diagonalization easier.

To find the two-body effective action, one only needs to solve for the metric perturbation caused by the source graviton at the linear order ($\sim \kappa_{11}^2$). The action is given by

$$S = S_G + S_A + S_P \tag{4.75}$$

S_G is the Einstein action for the metric, given by

$$S_G = \frac{1}{\kappa_{11}^2} \int d^{11}x \sqrt{|g|} R \tag{4.76}$$

S_A is the action for the three-form, given by

$$S_A = -\frac{2}{\kappa_{11}^2} \int d^{11}x \left\{ \frac{\sqrt{|g|}}{2 \cdot 2 \cdot 4!} F^{\mu\nu\lambda\xi} F_{\mu\nu\lambda\xi} + \frac{\tilde{\eta}}{12} \frac{1}{3!(4!)^2} \epsilon^{\mu_1 \ldots \mu_{11}} A_{\mu_1\mu_2\mu_3} F_{\mu_4 \ldots \mu_7} F_{\mu_8 \ldots \mu_{11}} \right\} \tag{4.77}$$

S_P is the action for the source graviton (the subscript P means "particle"), given by

$$S_P = C_P \frac{1}{2} \int_{-\infty}^{+\infty} d\xi \left(\frac{1}{\beta(\xi)} g_{\mu\nu}(y) \frac{dy^\mu}{d\xi} \frac{dy^\nu}{d\xi} - \beta(\xi) m^2 \right) \tag{4.78}$$

with C_P being some constant.

The above action gives the equations of motion for the metric, the three-form field, and the source graviton, listed below.

The Einstein equation is

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa_{11}^2 \left([T_{\mu\nu}]_A + [T_{\mu\nu}]_P \right) \tag{4.79}$$

The Maxwell equation is

$$\partial_\mu \left(\sqrt{|g|}F^{\mu\nu\lambda\xi}\right) - \frac{\tilde{\eta}}{1152}\epsilon^{\nu\lambda\xi\rho_1\ldots\rho_8}F_{\rho_1\ldots\rho_4}F_{\rho_5\ldots\rho_8} = 0 \tag{4.80}$$

The geodesic equation is

$$\frac{d^2y^\mu}{d\xi^2} + \Gamma^\mu_{\rho\nu}(y)\frac{dy^\rho}{d\xi}\frac{dy^\nu}{d\xi} = 0 \tag{4.81}$$

$[T_{\mu\nu}]_A$ and $[T_{\mu\nu}]_P$ are the stress tensors obtained by varying S_A and S_P with respect to the metric

$$[T_{\mu\nu}]_A = \frac{1}{12\kappa_{11}^2}\left(F_{\mu\lambda\xi\rho}F_\nu^{\lambda\xi\rho} - \frac{1}{8}g_{\mu\nu}F^{\rho\sigma\lambda\xi}F_{\rho\sigma\lambda\xi}\right) \tag{4.82}$$

$$[T_{\mu\nu}]_P(x) = \frac{C_P}{2}\frac{1}{\sqrt{|g(x)|}}g_{\mu\rho}(x)g_{\nu\lambda}(x)\int_{-\infty}^{+\infty}d\xi\frac{1}{\beta(\xi)}\frac{dy^\rho(\xi)}{d\xi}\frac{dy^\lambda(\xi)}{d\xi}\delta^{(11)}(x-y(\xi)) \tag{4.83}$$

Setting C_P to zero means the absence of the source graviton. In this case, a solution to the above equations of motion is the 11-D PP-wave background. Recall that the metric $g_{\mu\nu}$ and the four-form field strength of the unperturbed 11-D PP-wave background are given by

$$g_{+-} = 1,\ g_{++} = -\mu^2\left[\frac{1}{9}\sum_{i=1}^{3}(x^i)^2 + \frac{1}{36}\sum_{a=4}^{9}(x^a)^2\right],\ g_{AB} = \delta_{AB} \tag{4.84}$$

$$F_{123+} = \mu \tag{4.85}$$

As before, in our conventions, $\mu, \nu, \rho, \ldots$ take the values $+, -, 1, \ldots, 9$; $A, B, C, \ldots$ take the values $1, \ldots, 9$; $i, j, k, \ldots$ take the values $1, \ldots, 3$; and $a, b, c, \ldots$ take the values $4, \ldots, 9$

The introduction of a source graviton, i.e., a non-zero C_P, perturbs the above PP-wave solution to

$$g_{\mu\nu} \longrightarrow g_{\mu\nu} + h_{\mu\nu} \equiv G_{\mu\nu};\ F_{\mu\nu\rho\sigma} \longrightarrow F_{\mu\nu\rho\sigma} + f_{\mu\nu\rho\sigma} \tag{4.86}$$

It suffices to solve the geodesic equation at the zeroth order of C_P, which gives a solution

$$x^+ = \xi,\ x^- = 0,\ x^A = 0 \tag{4.87}$$

and the corresponding stress tensor of the source graviton is then

$$[T_{\mu\nu}]_P\,(x) = p^+ g_{\mu+} g_{\nu+} \delta(x^-) \prod_{A=1}^{9} \delta(x^A) \tag{4.88}$$

where $p^+ = \frac{C_P}{2\beta_0}$ is just the p_- of the source graviton (note that $\beta(\xi)$ is a constant β_0 for a geodesic) and in what follows we will use p^+ instead of C_P. Note that the order of κ_{11}^2 is the same as the order of p^+. Also note that the only non-vanishing component of $[T_{\mu\nu}]_P$ is $[T_{--}]_P = p^+ \delta(x^-) \prod_{A=1}^{9} \delta(x^A)$.

In what follows we will integrate everything over the x^- direction, thus getting rid of $\delta(x^-)$ and derivatives with respect to x^-. On the Matrix theory side, the effective potential was only computed up to 1-loop. In supergravity language, that means we are only looking at order κ_{11}^2. To find the effective potential on the supergravity side up to this order, we need only the linearized (i.e., to the linear order of p^+) Einstein equation and Maxwell equation.

We consider static solutions which have no x^+ dependence. Also, we restrict our attention to metric and gauge field perturbations that go to zero at infinity. The linearized Einstein equation in 11 dimension is

$$\delta R_{\mu\nu} = \kappa_{11}^2 \left[\delta T_{\mu\nu} + \frac{1}{9} g_{\mu\nu} \left(T^{\alpha\beta} h_{\alpha\beta} - g^{\alpha\beta} \delta T_{\alpha\beta} \right) \right] \equiv \mathcal{T}_{\mu\nu} \tag{4.89}$$

where the perturbation to the total stress tensor is given by

$$\delta T_{\alpha\beta} = [\delta T_{\alpha\beta}]_A + [T_{\alpha\beta}]_P \tag{4.90}$$

$[\delta T_{\alpha\beta}]_A$ is the perturbation to the stress tensor of the gauge field, which is to be expressed in terms of the perturbation to the field strength.

First look at the $(--)$ component of the Einstein equation, which is

$$\delta R_{--} = -\frac{1}{2}\sum_{A=1}^{9}\frac{\partial^2 h_{--}}{\partial x^A \partial x^A} \tag{4.91}$$

and

$$\mathcal{T}_{--} = \kappa_{11}^2 \delta T_{--} = \kappa_{11}^2 \left[T_{--}\right]_P = \kappa_{11}^2 p^+ \prod_{A=1}^{9} \delta(x^A) \tag{4.92}$$

where $[\delta T_{--}]_A = 0$ (as can be readily verified) has been used. This gives

$$h_{--} = \frac{\kappa_{11}^2 p^+}{\pi^4}\frac{15}{16}\frac{1}{|\vec{x}|^7} \tag{4.93}$$

where we use $\vec{x}$ to denote the nine-dimensional vector in the transverse directions.

The $(-A)$ component of the Einstein equation is

$$\delta R_{-A} = -\frac{1}{2}\sum_{B=1}^{9}\frac{\partial^2 h_{-A}}{\partial x^B \partial x^B} + \frac{1}{2}\sum_{B=1}^{9}\frac{\partial^2 h_{-B}}{\partial x^A \partial x^B} \tag{4.94}$$

and

$$\mathcal{T}_{-A} = 0 \tag{4.95}$$

which gives

$$h_{-A} = 0 \tag{4.96}$$

Now we look at the linearized Maxwell equation, in terms of the gauge potential perturbation $a_{\mu\nu\rho}$ (note $f_{\lambda\mu\nu\rho} = \partial_\lambda a_{\mu\nu\rho} - \partial_\mu a_{\nu\rho\lambda} + \partial_\nu a_{\rho\lambda\mu} - \partial_\rho a_{\lambda\mu\nu}$). We choose to work in the "Lorentz gauge" where $\sum_{D=1}^{9} \partial_D a_{\mu\nu D} = 0$. The upper $(AB+)$ component of the Maxwell equation gives

$$\sum_{D=1}^{9} \partial_D^2 a_{AB-} - \sum_{D=1}^{9} \partial_D \left[h_{--} F_{DAB+}\right] = 0 \tag{4.97}$$

Using the expression for h_{--} that we just found, we have

$$a_{ij-} = \frac{\mu\kappa_{11}^2 p^+}{\pi^4}\frac{15}{32}\sum_{k=1}^{3}\epsilon_{ijk}\frac{x^k}{|\vec{x}|^7} \tag{4.98}$$

while all other a_{AB-}'s vanish. Hence we find that the field strength is

$$\begin{aligned} f_{-ijk} &= \frac{\mu\kappa_{11}^2 p^+}{\pi^4}\frac{15}{32}\epsilon_{ijk}\left[7\frac{\sum_{i=1}^{3}(x^i)^2}{|\vec{x}|^9} - 3\frac{1}{|\vec{x}|^7}\right] \\ f_{-ijb} &= \frac{\mu\kappa_{11}^2 p^+}{\pi^4}\frac{15}{32}\sum_{k=1}^{3}\epsilon_{ijk}\left[7\frac{x^k x^b}{|\vec{x}|^9}\right] \end{aligned} \tag{4.99}$$

Next consider the upper (ABC) component of the Maxwell equation. Using the fact that $h_{-A} = 0$ and $a_{AB-} = 0$ except for a_{ij-}, we have

$$\sum_{D=1}^{9}\partial_D^2 a_{ABC} = 0 \tag{4.100}$$

hence, all $a_{ABC} = 0$. Now the $(A+-)$ component. Using $h_{-A} = 0$ we get

$$\sum_{D=1}^{9}\partial_D^2 a_{A-+} = 0 \tag{4.101}$$

thus $a_{A-+} = 0$. Now we go back to look at the $(+A)$ component of the Einstein equation. Using $h_{-A} = 0$, we get

$$\delta R_{+A} = -\frac{1}{2}\sum_{B=1}^{9}\frac{\partial^2 h_{+A}}{\partial x^B \partial x^B} + \frac{1}{2}\sum_{B=1}^{9}\frac{\partial^2 h_{+B}}{\partial x^A \partial x^B} \tag{4.102}$$

Using $a_{A-+} = 0$, $a_{ABC} = 0$, and $h_{-A} = 0$, we get

$$\mathcal{T}_{+A} = 0 \tag{4.103}$$

So we conclude that

$$h_{+A} = 0 \tag{4.104}$$

Now consider the $(+-)$ component of the Einstein equation

$$\delta R_{+-} = -\frac{1}{2}\sum_{A=1}^{9}\frac{\partial^2 h_{+-}}{\partial x^A \partial x^A} + \frac{1}{2}\sum_{A=1}^{9}\frac{\partial g_{++}}{\partial x^A}\frac{\partial h_{--}}{\partial x^A} \tag{4.105}$$

and

$$\mathcal{T}_{+-} = \frac{1}{6}\left(\mu^2 h_{--} - \mu f_{-123}\right) \tag{4.106}$$

In writing $\mathcal{T}_{+-}$, we made use of the following equations

$$\begin{aligned}
[\delta T_{+-}]_A &= \frac{\mu^2}{4\kappa_{11}^2} h_{--} \\
[\delta T_{ij}]_A &= \frac{1}{4\kappa_{11}^2}\delta_{ij}\left(-2\mu f_{-123} - \mu^2 h_{--}\right) \\
[\delta T_{bc}]_A &= \frac{1}{4\kappa_{11}^2}\delta_{bc}\left(2\mu f_{-123} + \mu^2 h_{--}\right) \\
[\delta T_{ib}]_A &= -\frac{\mu}{4\kappa_{11}^2}\sum_{j,k=1}^{3}\epsilon_{ijk} f_{-jkb}
\end{aligned} \tag{4.107}$$

Solving the $(+-)$ component of the Einstein equation, we get

$$h_{+-} = -\frac{\mu^2 \kappa_{11}^2 p^+}{\pi^4}\left[\frac{5}{64}\frac{\sum_{i=1}^{3}(x^i)^2}{|\vec{x}|^7} + \frac{1}{192}\frac{1}{|\vec{x}|^5}\right] \tag{4.108}$$

The (AB) component of the Einstein equation reads

$$\begin{aligned}
\delta R_{AB} &= -\frac{1}{2}\left[\sum_{C=1}^{9}\frac{\partial^2 h_{AB}}{\partial x^C \partial x^C} - \sum_{C=1}^{9}\frac{\partial^2 h_{AC}}{\partial x^B \partial x^C} - \sum_{C=1}^{9}\frac{\partial^2 h_{BC}}{\partial x^A \partial x^C} + \sum_{C=1}^{9}\frac{\partial^2 h_{CC}}{\partial x^A \partial x^B} + 2\frac{\partial^2 h_{+-}}{\partial x^A \partial x^B}\right] \\
&+ \frac{1}{4}\left[2h_{--}\frac{\partial^2 g_{++}}{\partial x^A \partial x^B} + 2g_{++}\frac{\partial^2 h_{--}}{\partial x^A \partial x^B} + \frac{\partial g_{++}}{\partial x^A}\frac{\partial h_{--}}{\partial x^B} + \frac{\partial g_{++}}{\partial x^B}\frac{\partial h_{--}}{\partial x^A}\right]
\end{aligned} \tag{4.109}$$

and

$$\begin{aligned}
\mathcal{T}_{ij} &= -\frac{1}{3}\delta_{ij}\left(2\mu f_{-123}+\mu^2 h_{--}\right) \\
\mathcal{T}_{bc} &= \frac{1}{6}\delta_{bc}\left(2\mu f_{-123}+\mu^2 h_{--}\right) \\
\mathcal{T}_{ib} &= -\frac{\mu}{4}\sum_{j,k=1}^{3}\epsilon_{ijk} f_{-jkb} \qquad (4.110)
\end{aligned}$$

So far the need to make a gauge choice for the metric has not arisen. Now to solve for h_{AB} we must make a gauge choice for the metric. Let $G^{\rho\sigma}$ and $\Gamma^{\mu}_{\rho\sigma}$ denote the complete inverse metric and Christoffel symbol, respectively (by "complete", we mean they include both the unperturbed and perturbed part). We shall fix the gauge by specifying $G^{\rho\sigma}\Gamma^{\mu}_{\rho\sigma}$.

As can be easily verified,

$$\begin{aligned}
G^{\rho\sigma}\Gamma^{+}_{\rho\sigma} &= \sum_{C=1}^{9}\partial_C h_{-C} = 0 \\
G^{\rho\sigma}\Gamma^{-}_{\rho\sigma} &= \sum_{C=1}^{9}\left(-h_{-C}\partial_C g_{++} + \partial_C h_{+C} - g_{++}\partial_C h_{-C}\right) = 0 \\
G^{\rho\sigma}\Gamma^{A}_{\rho\sigma} &= \sum_{C=1}^{9}\partial_C h_{AC} - \frac{1}{2}\partial_A\left(\sum_{C=1}^{9} h_{CC} + 2h_{+-} - g_{++}h_{--}\right) \qquad (4.111)
\end{aligned}$$

so we need to specify $G^{\rho\sigma}\Gamma^{A}_{\rho\sigma}$ to fix the gauge. Using the above expressions for $G^{\rho\sigma}\Gamma^{A}_{\rho\sigma}$, we can rewrite δR_{AB} as

$$\delta R_{AB} = -\frac{1}{2}\left[\sum_{C=1}^{9}\frac{\partial^2 h_{AB}}{\partial x^C\partial x^C} - \frac{\partial\left(G^{\rho\sigma}\Gamma^{A}_{\rho\sigma}\right)}{\partial x^B} - \frac{\partial\left(G^{\rho\sigma}\Gamma^{B}_{\rho\sigma}\right)}{\partial x^A} + \frac{1}{2}\left(\frac{\partial g_{++}}{\partial x^A}\frac{\partial h_{--}}{\partial x^B} + \frac{\partial g_{++}}{\partial x^B}\frac{\partial h_{--}}{\partial x^A}\right)\right] \qquad (4.112)$$

In general relativity one often uses the "harmonic gauge", where one sets $G^{\rho\sigma}\Gamma^{A}_{\rho\sigma} = 0$ (which is satisfied by the unperturbed PP-wave background). Here, however, we shall opt for a different gauge.

As derived in the Appendix C, the effective potential is given by

$$V_{\text{eff}} = \frac{N_p}{R}\left\{\frac{1}{2}\left[v^2 + g_{++} + h_{++} + g_{++}\left(\frac{1}{4}g_{++}h_{--} - h_{+-}\right)\right.\right.$$
$$\left. + \sum_A [2h_{+A} - h_{-A}(v^2 + g_{++})]v^A + \sum_{A,B} h_{AB} v^A v^B\right]$$
$$\left. + \frac{1}{8}h_{--}v^4 - \frac{1}{2}v^2\left(h_{+-} - \frac{1}{2}g_{++}h_{--}\right)\right\} \tag{4.113}$$

where N_p is the number of D0-branes forming the probe graviton, and $v^A \equiv \dot{x}^A$, $v^2 \equiv \sum_{A=1}^9 (v^A)^2$. As h_{+A}, h_{-A} all vanish, they simply drop out of the effective potential.

The computation on Matrix theory side in section 3.3.6 tells us that in the effective potential there are no terms of the form $v^a v^b$ for $a \neq b$, nor are there terms of the form $v^i v^a$. This suggests we choose the gauge such that $h_{ab} \propto \delta_{ab}$, and $h_{ia} = 0$. To make $h_{ab} \propto \delta_{ab}$, we set

$$G^{\rho\sigma}\Gamma^a_{\rho\sigma} = \frac{1}{2}h_{--}\partial_a g_{++} \tag{4.114}$$

then, to make $h_{ia} = 0$, we set

$$\partial_b\left(G^{\rho\sigma}\Gamma^i_{\rho\sigma}\right) = \frac{1}{2}\partial_i g_{++}\partial_b h_{--} - \frac{\mu}{2}\epsilon_{ijk} f_{-jkb} \tag{4.115}$$

which implies

$$G^{\rho\sigma}\Gamma^i_{\rho\sigma} = \frac{35}{96}\frac{\mu^2\kappa_{11}^2 p^+}{\pi^4}\frac{x^i}{|\vec{x}|^7} \tag{4.116}$$

Note that the above expression makes the gauge different from the "Lorentz gauge" where all the q_μ's vanish (see Section 4.2). Hence our gauge is something in between the harmonic gauge and the Lorentz gauge. In this gauge, the Einstein equation gives

$$h_{ab} = \delta_{ab}\frac{\mu^2\kappa_{11}^2 p^+}{\pi^4}\frac{1}{96}\left[\frac{15}{2}\frac{\sum_{k=1}^3 (x^k)^2}{|\vec{x}|^7} - \frac{1}{|\vec{x}|^5}\right] \tag{4.117}$$

$$h_{ij} = \delta_{ij}\frac{\mu^2\kappa_{11}^2 p^+}{\pi^4}\frac{1}{96}\left[-15\frac{\sum_{k=1}^{3}(x^k)^2}{|\vec{x}|^7} + \frac{1}{2}\frac{1}{|\vec{x}|^5}\right] + \frac{\mu^2\kappa_{11}^2 p^+}{\pi^4}\frac{15}{64}\frac{x^i x^j}{|\vec{x}|^7} \tag{4.118}$$

Now let us look at the upper $(AB-)$ component of the Maxwell equation. It gives the following equations

$$\sum_{D=1}^{9}\partial_D^2 a_{ij+} - g_{++}\sum_{D=1}^{9}\partial_D^2 a_{ij-} - \sum_{D=1}^{9}\partial_D g_{++}\left(\partial_D a_{ij-} + \partial_i a_{jD-} + \partial_j a_{Di-}\right)$$
$$+\mu\sum_{k=1}^{3}\epsilon_{ijk}\left\{-\sum_{D=1}^{9}\partial_D h_{Dk} + \sum_{m=1}^{3}(\partial_m h_{mk} - \partial_k h_{mm}) + \partial_k\left[\frac{1}{2}\left(g_{++}h_{--} + \sum_{D=1}^{9}h_{DD}\right)\right]\right\}$$
$$= 0$$

$$\sum_{D=1}^{9}\partial_D^2 a_{bc+} = 0$$
$$\sum_{D=1}^{9}\partial_D^2 a_{ib+} = 0 \tag{4.119}$$

Solving them gives

$$\begin{aligned} a_{ij+} &= \frac{\mu^3\kappa_{11}^2 p^+}{\pi^4}\left(\sum_{k=1}^{3}\epsilon_{ijk}x^k\right)\frac{1}{384\,|\vec{x}|^7}\left[-29\sum_{m=1}^{3}(x^m)^2 + \sum_{a=4}^{9}(x^a)^2\right] \\ a_{bc+} &= 0 \\ a_{ib+} &= 0 \end{aligned} \tag{4.120}$$

Hence the field strength is given by

$$\begin{aligned} f_{+ijk} &= \frac{\mu^3\kappa_{11}^2 p^+}{\pi^4}\epsilon_{ijk}\frac{1}{384\,|\vec{x}|^9}\left[-58\sum_{m=1}^{3}(x^m)^2 - 3\sum_{a=4}^{9}(x^a)^2 + 149\sum_{m=1}^{3}(x^m)^2\cdot\sum_{a=4}^{9}(x^a)^2\right] \\ f_{+ijb} &= \frac{\mu^3\kappa_{11}^2 p^+}{\pi^4}\left(\sum_{k=1}^{3}\epsilon_{ijk}x^k\right)\frac{5}{384}\frac{x^b}{|\vec{x}|^9}\left[-41\sum_{m=1}^{3}(x^m)^2 + \sum_{a=4}^{9}(x^a)^2\right] \end{aligned} \tag{4.121}$$

As can be easily checked, all the $a_{\mu\nu\rho}$ we have found indeed satisfy the Lorentz gauge.

Finally, we consider the $(++)$ component of the Einstein equation

$$\begin{aligned}\delta R_{++} \; &= \; -\frac{1}{2}\sum_{A=1}^{9}\partial_A^2 h_{++} + \frac{1}{2}\sum_{A,B=1}^{9}\partial_A g_{++}\partial_B h_{AB} - \frac{1}{4}\sum_{A,B=1}^{9}\partial_A g_{++}\partial_A h_{BB} \\ &+ \frac{1}{2}\sum_{A,B=1}^{9} h_{AB}\partial_A\partial_B g_{++} + \frac{1}{2}\sum_{A=1}^{9}\partial_A g_{++}\partial_A h_{+-} + \frac{1}{4}\sum_{A=1}^{9} g_{++}\partial_A g_{++}\partial_A h_{--} \\ &- \frac{1}{4}\sum_{A=1}^{9} h_{--}\left(\partial_A g_{++}\right)^2 \end{aligned} \tag{4.122}$$

and

$$\mathcal{T}_{++} = -\frac{\mu}{2}\left(2f_{+123} + \mu\sum_{i=1}^{3} h_{ii}\right) + \frac{\mu}{6} g_{++}\left(2f_{-123} + \mu h_{--}\right) \tag{4.123}$$

From this we find

$$h_{++} = \frac{\mu^4\kappa_{11}^2 p^+}{\pi^4}\,\frac{1}{6912\,|\vec{x}|^7}\left\{116\left[\sum_{i=1}^{3}(x^i)^2\right]^2 + 2\left[\sum_{a=4}^{9}(x^a)^2\right]^2 - 17\sum_{i=1}^{3}(x^i)^2\cdot\sum_{a=4}^{9}(x^a)^2\right\} \tag{4.124}$$

To summarize, the nonzero components of the metric perturbation are h_{--} [eqn. (4.93)], h_{+-} [eqn. (4.108)], h_{ab} [eqn. (4.117)], h_{ij} [eqn. (4.118)], and h_{++} [eqn. (4.124)]; and the nonzero components of the field strength perturbation are f_{-ijk}, f_{-ijb} [eqn. (4.99)], and f_{+ijk}, f_{+ijb} [eqn. (4.121)].

Substituting the expressions for the metric into our formula for $V_{\rm eff}$ in eqn. (4.113), averaging $h_{\mu\nu}$ over x^- (i.e., dividing by $2\pi R$), and noting that $\kappa_{11}^2 = \frac{16\pi^5}{M^9}$, $p^+ = \frac{N_s}{R}$, we find

$$
\begin{aligned}
V_{\text{eff}} &= \frac{N_p}{2R}(v^2+g_{++})+\frac{15}{16}\frac{N_pN_s}{M^9R^3}\frac{v^4}{|\vec{x}|^7} \\
&+ \frac{\mu^2 N_pN_s}{R^3M^9}\Bigg\{\left[-\frac{1}{96}\frac{1}{|\vec{x}|^5}-\frac{15}{32}\frac{(x^1)^2+(x^2)^2+(x^3)^2}{|\vec{x}|^7}\right]\sum_{i=1}^{3}(v^i)^2+\frac{15}{16}\frac{\sum_{i,j=1}^{3}x^ix^jv^iv^j}{|\vec{x}|^7} \\
&\qquad\qquad + \left[-\frac{7}{96}\frac{1}{|\vec{x}|^5}+\frac{15}{32}\frac{(x^1)^2+(x^2)^2+(x^3)^2}{|\vec{x}|^7}\right]\sum_{a=4}^{9}(v^a)^2\Bigg\} \\
&+ \frac{\mu^4 N_pN_s}{R^3M^9}\frac{1}{768\,|\vec{x}|^7}\left\{32\left[\sum_{i=1}^{3}(x^i)^2\right]^2+\left[\sum_{a=4}^{9}(x^a)^2\right]^2-12\sum_{i=1}^{3}(x^i)^2\cdot\sum_{a=4}^{9}(x^a)^2\right\}
\end{aligned}
\tag{4.125}
$$

Comparison of the above formula with eqn. (3.100) on the Matrix theory side shows exact agreement.

We would like to emphasize the approximation involved once again. We treated the source graviton as a perturbation to the exact PP-wave background, and the calculation was performed to first order in p^+. However, the solution that we found for these linearized equations is exact in μ.

So, we have finished our comparison of the effective potentials describing two graviton interactions on the gauge theory side and the supergravity side, and have found precise agreement at order κ_{11}^2, up to quantum corrections at short distances. Let us make a few comments on our result: our result at order μ^2 agrees with Taylor and Van Raamsdonk's proposal in [6] for Matrix theory in a weakly curved background up to linear terms. As mentioned in their discussion, their proposal is proven only in the case where the background is produced by well-defined Matrix theory configurations. This is not the case for the PP-wave background, so their proposal for the Matrix theory in this background, while convincing, is not a proven fact. Thus, the result at μ^2, i.e., terms linear in the background, can be treated as additional evidence for their proposal, similar to the explicit calculation in [7], this time with a nontrivial g_{++} metric component.

The result at order μ^4 is beyond linear order in the background, and hence is a new result. In fact, our calculation explicitly shows that there are no higher powers of μ in the effective action of the supergravity side at the order κ_{11}^2. On the Matrix theory side, higher powers of μ are also not expected for long distances. When they do appear, they are always accompanied by higher powers of α/r^2, which indicates that they are corrections to supergravity at short distances. However, as it stands, the corrections for velocity dependent terms are unreliable because they are computed using the sum over mass formula, which is exact only for terms independent of velocity or terms proportional to κ_{11}^2, as is shown in the Appendix B. Evaluating these corrections exactly requires going beyond the sum over mass formula, and an efficient way of handling the mass matrix will be of use.

As pointed out by [56, 57], Matrix theory in a generic curved background is not expected to agree with supergravity. In the PP-wave case, however, we do find precise agreement as has been shown above. This is likely to be a result of the large number of supersymmetries of the PP-wave background, or in other words, we can say that this agreement predicts the existence of a supersymmetric nonrenormalization theorem. This leads us naturally to the discussion in Chapter 5.

Chapter 5

Supersymmetric Nonrenormalization Theorems

The role of supersymmetric nonrenormalization theorems in Matrix theory has been crucial from the very beginning (see [2]). As a matter of fact, they are very likely to be the reason why the two theories, namely, gauge theory and supergravity, with disparate regions of validity (see Subsection 3.3.2), would give agreement in the first place.

The most convincing way to address this issue is brute force evaluation of higher loop contributions, as [14] did in showing that there is no correction to the v^4 term from two-loops in the context of graviton scattering in flat space. An alternative approach is to proceed without using detailed knowledge of the underlying theory. One wants to see how sixteen supersymmetries alone, together with the $SO(9)$ invariance of the transverse part of the flat space metric, and CPT invariance, would constrain the effective Lagrangian. Amazingly, to the v^4 order these global symmetries completely fix the effective Lagrangian, up to an overall constant. This approach was pioneered by [31], whose argument we will briefly review below. See also [32, 33, 34, 35, 36, 37].

In [31]'s notation, the bosonic part of the effective action can take the general form

$$S_{\text{boson}} = \int dt(f_1(r)v^2 + f_2(r)v^4 + \text{higher derivative terms}) \tag{5.1}$$

and one then uses supersymmetries to constrain the functions $f_1(r), f_2(r)$.

At order v^2, the supersymmetry transformations can be written in the form

$$\delta x^i = -i\epsilon\gamma^i\psi, \quad \delta\psi = \gamma^i v^i \epsilon + M\epsilon \tag{5.2}$$

where the matrix M contains v and two fermions. The closure of the above supersymmetry transformations then requires M to vanish, which in turn implies that f_1 must be a constant, which one normalizes to $\frac{1}{2}$. Hence adding the fermions the action at this order takes the form

$$S_1 = \int dt \left(\frac{1}{2}v^2 + i\psi\dot{\psi}\right) \tag{5.3}$$

At the v^4 order, the bosonic action can be written in the form

$$S_2 = \int dt \left(f_2^{(0)}(r)v^4 + ... + f_2^{(8)}(r)\psi^8\right) \tag{5.4}$$

where the ellipsis stands for $v^3\psi^2$, $v^2\psi^4$, and $v\psi^6$ terms.

The supersymmetry variation in general mixes terms with different numbers of fermions. However, the "top" term, i.e., the eight-fermion term, provides some simplifying clues. Its variation is

$$\delta(f_2^{(8)}(r)\psi^8) = \delta f_2^{(8)}(r)\psi^8 + f_2^{(8)}(r)\delta\psi^8 \tag{5.5}$$

where the first term is the only one that contains nine fermions, and thus does not mix with other terms. Hence this nine-fermion term must vanish by itself.

After the use of Fierz identities, the most general eight-fermion term of the effective Lagrangian can be written as

$$(\psi\gamma^{ij}\psi\psi\gamma^{jk}\psi\psi\gamma^{lm}\psi\psi\gamma^{mn}\psi)(g_1(r)\delta_{in}\delta_{kl} + g_2(r)\delta_{kl}x_i x_n + g_3(r)x_i x_k x_l x_n) \tag{5.6}$$

As argued earlier, the nine-fermion term in the supersymmetry variation of the above expression should vanish. Upon applying the operators $\gamma^q_{ac}\frac{d}{d\psi_c}\partial_q$ and $\gamma^q_{ac}\frac{d}{d\psi_c}x_q$ to the nine-fermion variation, one gets a set of coupled differential equation for the g's, whose solution is, unique up to a constant c

$$g_1(r) = \frac{2}{143}\frac{c}{r^{11}} \ , \quad g_2(r) = -\frac{4}{13}\frac{c}{r^{13}} \ , \quad g_3(r) = \frac{c}{r^{15}} \ . \tag{5.7}$$

This gives precisely the same eight-fermion Lagrangian, up to the overall constant, as obtained in [58] by an explicit loop computation. The other terms with different numbers of fermions in the order v^4 action S_2 can be further determined using the eight-fermion result obtained above; for details of that calculation see [34]. In the above proof(s) of supersymmetric nonrenormalization theorems, there are subtleties related to higher derivative terms and integration by parts. After taking those subtleties into account, one finds that the result is unaffected [37].

It is natural to ask whether we can give a similar proof of the supersymmetric nonrenormalization theorem in the 11-D PP-wave. The 11-D PP-wave has a non-vanishing g_{++} component which breaks the transverse $SO(9)$ symmetry into $SO(3)\times SO(6)$. As pointed out by [34] in their discussion section, when the $SO(9)$ is broken, the v^4 order effective action should take a form similar to that in flat space, with the coefficient function $f(\vec{x})$ for the v^4 term now being a harmonic function of the nine-vector $\vec{x}$, not just its length r, and the coefficient functions of the two, four, six, eight fermion terms being given by partial derivatives $\partial_i ... \partial_k f(\vec{x})$. Unlike in the $SO(9)$ invariant case, where supersymmetry constraints lead to ordinary differential equations with respect to r, now one has to solve partial differential equations with respect to $\vec{x}$. One would expect that this requires substantially more work. For example, one could introduce the dimensionless quantity ρ/z which is the ratio between the $SO(3)$ radius and the $SO(6)$ radius, and the functions' dependence on ρ/z would not be so easy to determine. Simplifying facts could appear after scrutinizing the system carefully enough, and it would be an interesting project to give a proof of the supersymmetric nonrenormalization theorem in 11-D PP-wave.

Chapter 6

Conclusion and Discussion

Gauge/gravity duality for M theory in flat and almost flat (i.e., weakly curved) backgrounds has been investigated extensively, and nice agreements have been found. In generic curved backgrounds, things are less clear, and we certainly want to understand them better. The two graviton results reported in this thesis can be regarded as a step in this direction.

We hope that our ongoing investigation of membrane interactions will shed some new light on M theory in generic curved backgrounds in the case of nonzero M-momentum transfer. There are also many other directions that are very natural to explore. One of them is going away from the Penrose limit towards M theory in the full $AdS \times S$, in the spirit of [59], which is in the IIB string context. To be more specific, we can add $1/R$ corrections ($R \to \infty$ in the Penrose limit) to the Matrix theory proposed by [10] and investigate the dynamics of that model. Another direction is M theory in backgrounds with fewer supersymmetries, e.g., those preserving sixteen supersymmetries considered in [27]. Since supersymmetric nonrenormalization theorems seem to be crucial for the gauge/gravity duality, those backgrounds that are not maximally supersymmetric should teach us something valuable. Finally, in this thesis we have restricted our attention to one loop in the gauge theory and linearized supergravity. In the future we will try to push our computation to higher loops and nonlinear supergravity.

Appendix A

Proof of Invariance of $\mathcal{L}_{\text{nonabelian}}$ under Sixteen Supersymmetries

In this appendix we prove that the action $\mathcal{L}_{\text{nonabelian}}$ given in eqn. (3.48) is invariant under sixteen supersymmetries. The supersymmetry transformation given in eqn. (3.49) can be written as

$$\delta = \delta_\epsilon + \delta_{\Lambda_0} \tag{A.1}$$

where δ_ϵ is the part containing ϵ, and δ_{Λ_0} is the compensating gauge transformation part containing Λ_0. So we have

$$\delta\mathcal{L}_{\text{nonabelian}} = \delta_\epsilon\mathcal{L}_{\text{nonabelian}} + \delta_{\Lambda_0}\mathcal{L}_{\text{nonabelian}} \tag{A.2}$$

As we will see in the following, it makes the proof neater to separate the supersymmetry variation $\delta\mathcal{L}_{\text{nonabelian}}$ into $\delta_\epsilon\mathcal{L}_{\text{nonabelian}}$ and $\delta_{\Lambda_0}\mathcal{L}_{\text{nonabelian}}$ at the very beginning.

Carrying out the variation explicitly, we find

$$\begin{aligned} \delta_{\Lambda_0}\mathcal{L}_{\text{nonabelian}} = \quad & -\dot{Y}^i_m(\{(\dot{\Lambda}_0)_m, x^i\} + gf^{mnp}(\dot{\Lambda}_0)_n Y^i_p) \\ & -\dot{X}^a_m(gf^{mnp}(\dot{\Lambda}_0)_n X^a_p) - \frac{ig}{p^+} f^{mnp}(\dot{\Lambda}_0)_n \Psi^T_p \Psi_m \end{aligned} \tag{A.3}$$

Deriving the result given in eqn. (A.3) involves some algebra. For example, to show that the g^3Y^4 term which arises from $\delta_{\Lambda_0}(F^k_m)^2$ vanishes, one has to deal with the product of three structure constants and use the Jacobi identity for the structure constants. We do not present the details of the derivation here, because the form of (A.3) is what one would expect, in two aspects. Firstly, it only depends on the time-derivative of Λ_0. The reason is, if Λ_0 were time-independent, δ_{Λ_0} would just be a usual time-independent gauge transformation of the form eqn. (3.46) with $\Lambda_m = -\frac{1}{p^+}(\Lambda_0)_m$, and we have already shown that $\mathcal{L}_{\text{nonabelian}}$ is invariant under time-independent gauge transformations. Secondly, it only contains g^0 and g^1 powers, but not g^2 and g^3 terms (recall that $\mathcal{L}_{\text{nonabelian}}$ has order g^2 terms, and δ_{Λ_0} has order g terms, so potentially $\delta_{\Lambda_0}\mathcal{L}_{\text{nonabelian}}$ could have up to g^3 order terms). The reason is, terms containing the time-derivative of Λ_0 come purely from $\delta_{\Lambda_0}(\dot{Y}^i_m)^2$, $\delta_{\Lambda_0}(\dot{X}^a_m)^2$, and $\delta_{\Lambda_0}(\Psi^T_m\dot{\Psi}_m)$, so are only of orders g^0 and g^1. Order g^2 and g^3 terms only contain Λ_0 not acted on by time-derivative, and must vanish, because $\mathcal{L}_{\text{nonabelian}}$ is invariant under time-independent gauge transformations order by order in g.

Now let us show that $\delta\mathcal{L}_{\text{nonabelian}}$ vanishes order by order in g (of course, what we really mean is it vanishes up to total derivatives; in what follows we drop the total derivative terms).

$$\delta\mathcal{L}_{\text{nonabelian}} = (\delta\mathcal{L}_{\text{nonabelian}})^{(0)} + (\delta\mathcal{L}_{\text{nonabelian}})^{(1)} + (\delta\mathcal{L}_{\text{nonabelian}})^{(2)} + (\delta\mathcal{L}_{\text{nonabelian}})^{(3)} \tag{A.4}$$

with the superscript denoting the power of g it contains. Also, as in the abelian case, we always move the time-derivative and Poisson bracket (spatial-derivative) to act on the boson through integration by parts. $(\delta\mathcal{L}_{nonabelian})^{(0)} = 0$ since this is just a sum of copies of the variation of the abelian Lagrangian.

Let us first consider $(\delta\mathcal{L}_{\text{nonabelian}})^{(3)}$. (We will return to the g^2 and g^1 terms afterwards.) Eqn. (A.3) already tells us that $(\delta_{\Lambda_0}\mathcal{L}_{\text{nonabelian}})^{(3)} = 0$. Also, easily seen $\delta_\epsilon\mathcal{L}_{\text{nonabelian}}$ does not contain any order g^3 terms. Hence $(\delta\mathcal{L}_{\text{nonabelian}})^{(3)} = 0$.

Next, let us consider $(\delta\mathcal{L}_{\text{nonabelian}})^{(2)}$. Here eqn. (A.3) again gives $(\delta_{\Lambda_0}\mathcal{L}_{\text{nonabelian}})^{(2)} = 0$, so the contribution comes purely from $(\delta_\epsilon\mathcal{L}_{\text{nonabelian}})^{(2)}$. There are four types of terms in it, which we write schematically as $X^3\epsilon\Psi$, $YX^2\epsilon\Psi$, $Y^2X\epsilon\Psi$, and $Y^3\epsilon\Psi$, and in what follows we'll write these structures as subscripts to denote the relevant types of variation of the Lagrangian.

First look at the $X^3\epsilon\Psi$ terms. These come from the variation of two terms in the Lagrangian, one being the Yukawa term $\Psi X\Psi$, the other being the $(F^{ab})^2$ term. It is given by

$$\begin{aligned}(\delta\mathcal{L}_{\text{nonabelian}})^{(2)}_{X^3\epsilon\Psi} = \quad & \frac{ig^2}{2p^+}f^{mnp}f^{mrs}X^a_rX^b_s(\gamma^a\gamma^b)_{\alpha\beta}\epsilon_\beta\gamma^c_{\alpha\lambda}X^c_n\Psi_{\lambda p} \\ & -\frac{ig^2}{2p^+}f^{mnp}X^a_nX^b_pf^{mrs}(\epsilon_\alpha\gamma^a_{\alpha\beta}\Psi_{\beta r})X^b_s \\ & -\frac{ig^2}{2p^+}f^{mnp}X^a_nX^b_pf^{mrs}X^a_r(\epsilon_\alpha\gamma^b_{\alpha\beta}\Psi_{\beta s}) \\ & =\frac{ig^2}{4p^+}f^{msp}f^{mnr}X^a_rX^b_sX^c_n\epsilon^T\gamma^a\gamma^b\gamma^c\Psi_p \end{aligned} \tag{A.5}$$

where to get to the last line we have renamed indices at various places and also used the Jacobi identity of the gauge group structure constants. This vanishes because

$$\begin{aligned} & f^{msp}f^{mnr}X^a_rX^b_sX^c_n\gamma^a\gamma^b\gamma^c \\ & \text{(by using the Jacobi identity for the structure constants)} \\ & = -(f^{mnp}f^{mrs}+f^{mrp}f^{msn})X^a_rX^b_sX^c_n\gamma^a\gamma^b\gamma^c \\ & \text{(renaming } n\leftrightarrow s, b\leftrightarrow c \text{ in the 1st term, and } r\leftrightarrow s, a\leftrightarrow b \text{ in the 2nd term)} \\ & = -f^{msp}f^{mrn}X^a_rX^c_nX^b_s\gamma^a\gamma^c\gamma^b - f^{msp}f^{mrn}X^b_sX^a_rX^c_n\gamma^b\gamma^a\gamma^c \\ & = -2f^{msp}f^{mrn}X^a_rX^b_nX^b_s\gamma^a + f^{msp}f^{mrn}X^a_rX^c_nX^b_s\gamma^a\gamma^b\gamma^c \\ & \quad -2f^{msp}f^{mrn}X^a_sX^a_rX^c_n\gamma^c + f^{msp}f^{mrn}X^b_sX^a_rX^c_n\gamma^a\gamma^b\gamma^c \\ & \text{(the two terms with only one gamma matrix cancel} \\ & \text{ upon renaming } n\leftrightarrow r \text{ in the 2nd of them)} \\ & = -2f^{msp}f^{mnr}X^a_rX^b_sX^c_n\gamma^a\gamma^b\gamma^c = 0 \end{aligned} \tag{A.6}$$

Thus we have shown $(\delta\mathcal{L}_{\text{nonabelian}})^{(2)}_{X^3\epsilon\Psi}=0$.

$(\delta\mathcal{L}_{\text{nonabelian}})^{(2)}_{Y^3\epsilon\Psi}=0$ by manipulations similar to the above.

Now look at $YX^2\epsilon\Psi$ terms. These come from the variations of three terms in the

Lagrangian, one being the $\Psi\xi$ term, one being the $(F^{ia})^2$ term, and the other being the Yukawa term $\Psi X \Psi$. We have

$$
\begin{aligned}
(\delta\mathcal{L}_{\text{nonabelian}})^{(2)}_{YX^2\epsilon\Psi} = \quad & -\frac{ig^2}{2p^+} f^{mnp} f^{mrs} Y^i_r X^a_n X^b_p \epsilon^T \gamma^a \gamma^b \gamma^i \Psi_s \\
& -\frac{ig^2}{p^+} f^{mnp} f^{mrs} Y^i_n X^a_p X^a_s \epsilon^T \gamma^i \Psi_r \\
& +\frac{ig^2}{p^+} f^{mnp} f^{mrs} Y^i_n X^a_p X^b_s \epsilon^T \gamma^i \gamma^a \gamma^b \Psi_r \\
& \text{(renaming the indices to bring the fields into the same form} \\
& \quad \text{and then using the Jacobi identity)} \\
& = 0 \qquad \text{(A.7)}
\end{aligned}
$$

$(\delta\mathcal{L}_{\text{nonabelian}})^{(2)}_{Y^2X\epsilon\Psi} = 0$ by similar steps.

So we have shown that $(\delta\mathcal{L}_{\text{nonabelian}})^{(2)} = 0$.

Next consider $(\delta\mathcal{L}_{\text{nonabelian}})^{(1)}$. At this order we have contributions from both $(\delta_{\Lambda_0}\mathcal{L}_{\text{nonabelian}})^{(1)}$ and $(\delta_\epsilon\mathcal{L}_{\text{nonabelian}})^{(1)}$. There are: terms containing μ, $\mu X^2\epsilon\Psi$, $\mu Y^2\epsilon\Psi$, $\mu YX\epsilon\Psi$; terms containing a Poisson bracket, $\{x, X\}X\epsilon\Psi$, $\{x, Y\}Y\epsilon\Psi$, $\{x, X\}Y\epsilon\Psi$, $\{x, Y\}X\epsilon\Psi$; terms containing a time-derivative, $\dot{X}X\epsilon\Psi$, $\dot{Y}Y\epsilon\Psi$, $\dot{X}Y\epsilon\Psi$, $\dot{Y}X\epsilon\Psi$; terms containing four spinors, $\epsilon\Psi^3$. Compared with g^2 order, g^1 order is more straightforward because structure constants only appear once in each term, hence it doesn't involve the Jacobi identity for the structure constants.

Let's first consider terms containing μ. The most complicated term is

$$
\begin{aligned}
(\delta\mathcal{L}_{\text{nonabelian}})^{(1)}_{\mu Y^2\epsilon\Psi} = ig\mu \quad [\quad & \frac{1}{3}Y^k_m\epsilon^{ijk}f^{mnp}(\epsilon^T\gamma^i\Psi_n)Y^j_p \\
& + \frac{1}{6}\epsilon^{kij}f^{mnp}Y^i_nY^j_p(\epsilon^T\gamma^k\Psi_m) \\
& + \frac{1}{24}\epsilon^{kij}f^{mnp}Y^i_nY^j_p(\gamma^k\gamma^{123})_{\alpha\beta}\gamma^{123}_{\beta\lambda}\epsilon_\lambda\Psi_{\alpha m} \\
& - \frac{1}{3}Y^k_m(\gamma^k\gamma^{123})_{\alpha\beta}\epsilon_\beta\gamma^i_{\alpha\lambda}f^{mnp}Y^i_n\Psi_{\lambda p} \\
& - \frac{1}{8}\epsilon^{kij}f^{mnp}Y^i_nY^j_p(\gamma^k\gamma^{123})_{\alpha\beta}\epsilon_\beta\gamma^{123}_{\alpha\lambda}\Psi_{\lambda m}] \\
& \text{(renaming indices to bring the fields into the same form} \\
& \text{and also using gamma matrix identities given before)} \\
= \quad & ig\mu\epsilon^{ijk}f^{mnp}Y^i_nY^j_p(\epsilon^T\gamma^k\Psi_m)\left(\frac{1}{3}+\frac{1}{6}-\frac{1}{24}-\frac{1}{3}-\frac{1}{8}\right) \\
= \quad & 0
\end{aligned}
\tag{A.8}
$$

The other terms containing μ all vanish by similar manipulations and we omit the details here. The terms containing the Poisson bracket and time-derivative also all vanish by similar steps and we don't bother to write the details down, either.

As always, it is worthwhile to write out the proof for the vanishing of the four spinor term in detail. So let us do this now. We have

$$
(\delta\mathcal{L}_{\text{nonabelian}})^{(1)}_{\epsilon\Psi^3} = \frac{g}{p^+}f^{mnp}\left[(\epsilon^T\gamma^A\Psi_n)(\Psi^T_m\gamma^A\Psi_p) - (\epsilon^T\Psi_n)(\Psi^T_m\Psi_p)\right] \tag{A.9}
$$

We just have to work out the Fierz transformation for $SO(9)$ spinors. Any 16×16 matrices can be expanded using the complete set of 256 matrices

$$
\{\gamma^M\} = \{1, \gamma^A, i\gamma^{AB}, i\gamma^{ABC}, \gamma^{ABCD}\} \tag{A.10}
$$

which satisfy $\text{tr}(\gamma^M\gamma^N) = 16\ \delta^{MN}$.

So we can expand

$$
\gamma^M\Psi_n\Psi^T_m\gamma^N = C_Q\gamma^Q \tag{A.11}
$$

with

$$C_Q = -\frac{1}{256}\mathrm{tr}(\gamma^N\gamma^Q\gamma^M\gamma^W)(\Psi_m^T\gamma^W\Psi_n) \tag{A.12}$$

Using the above result, we find

$$\begin{aligned}(\epsilon^T\gamma^A\Psi_n)(\Psi_m^T\gamma^A\Psi_p) = & -\frac{9}{16}(\Psi_m^T\Psi_n)(\epsilon^T\Psi_p) + \frac{7}{16}(\Psi_m^T\gamma^A\Psi_n)(\epsilon^T\gamma^A\Psi_p) \\ & + \frac{5}{16}(\Psi_m^T\gamma^{AB}\Psi_n)(\epsilon^T\gamma^{AB}\Psi_p) - \frac{3}{16}(\Psi_m^T\gamma^{ABC}\Psi_n)(\epsilon^T\gamma^{ABC}\Psi_p) \\ & - \frac{1}{16}(\Psi_m^T\gamma^{ABCD}\Psi_n)(\epsilon^T\gamma^{ABCD}\Psi_p) \end{aligned} \tag{A.13}$$

Upon contraction with f^{mnp} the γ^{AB} and γ^{ABC} terms vanish because these two types of matrices are antisymmetric. Hence we find

$$\begin{aligned} f^{mnp}(\epsilon^T\gamma^A\Psi_n)(\Psi_m^T\gamma^A\Psi_p) = & -\frac{9}{16}f^{mnp}(\Psi_m^T\Psi_n)(\epsilon^T\Psi_p) + \frac{7}{16}f^{mnp}(\Psi_m^T\gamma^A\Psi_n)(\epsilon^T\gamma^A\Psi_p) \\ & - \frac{1}{16}f^{mnp}(\Psi_m^T\gamma^{ABCD}\Psi_n)(\epsilon^T\gamma^{ABCD}\Psi_p) \end{aligned} \tag{A.14}$$

Similarly, we find

$$\begin{aligned} f^{mnp}(\epsilon^T\Psi_n)(\Psi_m^T\Psi_p) = & -\frac{1}{16}f^{mnp}(\Psi_m^T\Psi_n)(\epsilon^T\Psi_p) - \frac{1}{16}f^{mnp}(\Psi_m^T\gamma^A\Psi_n)(\epsilon^T\gamma^A\Psi_p) \\ & - \frac{1}{16}f^{mnp}(\Psi_m^T\gamma^{ABCD}\Psi_n)(\epsilon^T\gamma^{ABCD}\Psi_p) \end{aligned} \tag{A.15}$$

Hence we have

$$\begin{aligned} (\delta\mathcal{L}_{\mathrm{nonabelian}})^{(1)}_{\epsilon\Psi^3} = & \frac{g}{p^+}\left[f^{mnp}(\epsilon^T\gamma^A\Psi_n)(\Psi_m^T\gamma^A\Psi_p) - f^{mnp}(\epsilon^T\Psi_n)(\Psi_m^T\Psi_p)\right] \\ & \text{(using the Fierz identities derived above)} \\ = & \frac{g}{p^+}\left[-\frac{1}{2}f^{mnp}(\Psi_m^T\Psi_n)(\epsilon^T\Psi_p) + \frac{1}{2}f^{mnp}(\Psi_m^T\gamma^A\Psi_n)(\epsilon^T\gamma^A\Psi_p)\right] \\ & \text{(renaming indices } n \leftrightarrow p\text{)} \\ = & -\frac{1}{2}\frac{g}{p^+}\left[f^{mnp}(\epsilon^T\gamma^A\Psi_n)(\Psi_m^T\gamma^A\Psi_p) - f^{mnp}(\epsilon^T\Psi_n)(\Psi_m^T\Psi_p)\right] \\ = & -\frac{1}{2}(\delta\mathcal{L}_{\mathrm{nonabelian}})^{(1)}_{\epsilon\Psi^3} \end{aligned} \tag{A.16}$$

which gives

$$(\delta\mathcal{L}_{\text{nonabelian}})^{(1)}_{\epsilon\Psi^3} = 0 \tag{A.17}$$

So we have proved $(\delta\mathcal{L}_{\text{nonabelian}})^{(1)} = 0$. Thus $\delta\mathcal{L}_{\text{nonabelian}} = 0$.

Appendix B

The One-Loop Effective Potential

In this appendix we will prove the sum over mass formula in eqn. (3.92).

The one-loop effective action Γ of a theory expanded upon a background B in the background field method is given by

$$\Gamma = \frac{1}{2}[\mathrm{Tr}_{\mathrm{boson}}\ \ln(-\partial_\tau^2 + W_b(\tau)) - \mathrm{Tr}_{\mathrm{fermion}}\ \ln(-\partial_\tau^2 + W_f(\tau)) - \mathrm{Tr}_{\mathrm{ghost}}\ \ln(-\partial_\tau^2 + W_{gh}(\tau))] \tag{B.1}$$

Here $W = M^2$ is the mass matrix squared for the fluctuating fields, and the trace Tr is over both the functional space and the field component indices (which are, besides the $U(2)$ indices, the space-time indices $0, 1, .., 9$ for the bosons, and the 16 Dirac spinor indices for the fermions).

Take the trace of the boson, for example

$$\begin{aligned} \Gamma_{\mathrm{boson}} &= \frac{1}{2}\mathrm{Tr}_{\mathrm{boson}}\ \ln(-\partial_\tau^2 + W(\tau)) \\ &= -\frac{1}{2}\mathrm{Tr} \int_0^\infty \frac{ds}{s} \exp[-s(-\partial_\tau^2 + W(\tau))] \end{aligned} \tag{B.2}$$

The trace over functional space can be computed by sandwiching the operator between the "plane-wave" basis wave-functions $|\omega\rangle = \frac{1}{\sqrt{2\pi}} e^{-i\omega\tau}$ and the conjugate wave-functions $\langle\omega| = \frac{1}{\sqrt{2\pi}} e^{i\omega\tau}$

$$\Gamma_{\text{boson}} = -\frac{1}{2}\text{tr} \int d\tau \int \frac{d\omega}{2\pi} \int_0^\infty \frac{ds}{s} e^{i\omega\tau} \exp[-s(-\partial_\tau^2 + W(\tau))] e^{-i\omega\tau} \tag{B.3}$$

The trace tr is now over only the field component indices. If we define $V_{\text{eff}}(\tau)$ by

$$\Gamma = -\int d\tau V_{\text{eff}}(\tau) \tag{B.4}$$

Then, the bosonic part of V_{eff} becomes

$$V_{\text{eff}}(\text{boson}) = \frac{1}{2}\text{tr} \int \frac{d\omega}{2\pi} \int_0^\infty \frac{ds}{s} e^{i\omega\tau} \exp[-s(-\partial_\tau^2 + W(\tau))] e^{-i\omega\tau} \tag{B.5}$$

The operator in the middle can be rewritten as

$$\exp[-s(-\partial_\tau^2 + W(\tau))] = X e^{-sW(\tau)} e^{+s\partial_\tau^2} \tag{B.6}$$

Where X is defined as

$$\begin{aligned} X &\equiv \exp[-s(-\partial_\tau^2 + W(\tau))] e^{-s\partial_\tau^2} e^{+sW(\tau)} \\ &= 1 + \text{commutator terms} \end{aligned} \tag{B.7}$$

The commutator terms give corrections to supergravity, so for the purpose of this paper, which is to see whether the Matrix theory can reproduce supergravity results, we can ignore them. This claim will be proven shortly, after the result from approximating $X = 1$ is examined. In this approximation, we have

$$\begin{aligned} V_{\text{eff}}(\text{boson}) &= \frac{1}{2}\text{tr} \int \frac{d\omega}{2\pi} \int_0^\infty \frac{ds}{s} \exp[-s(\omega^2 + W(\tau))] \\ \Rightarrow \quad V_{\text{eff}} &= -\frac{1}{2}\text{tr } M \end{aligned} \tag{B.8}$$

Note that M, the square root of W, can be defined through its expansion in powers of $1/r$. Putting everything together, and minding the minus signs for the fermions and the ghosts, we get the sum over mass formula in eqn. (3.92). Now, we return to the claim made above, that the commutator terms in X will not contribute to terms in the supergravity limit. To show this, we first write X in a most general form

$$X = \sum_{n,m} K[{}^m_n](W(\tau)) s^n \partial_\tau^m \tag{B.9}$$

Here $K[{}^m_n](W(\tau))$ is a general function of $W(\tau)$ and its τ-derivatives, and is *defined* by eqn. (B.9). Looking back at the definition of X in eqn. (B.7), we see that n counts the number of terms involved in forming the commutator, and m is the number of derivatives not acting on W. For example, when $n = 0$, it implies $m = 0$, and $K[{}^0_0] = 1$, corresponding to the approximation we made above. All the other values of n correspond to commutator terms in X, and in particular, $K = 0$ when $n = 1$, because a commutator takes at least two terms.

Putting X in terms of K into V_{eff}, we will encounter the following factor inside the integrand:

$$e^{i\omega\tau} \partial^m e^{-i\omega\tau} = \sum_{l=0}^{m} \binom{m}{l} (-i\omega)^l \partial^{m-l} = \sum_{l=0}^{[m/2]} \binom{m}{2l} (-\omega^2)^l \partial^{m-2l} \tag{B.10}$$

$[m/2]$ is the biggest integer no larger than $m/2$. In the last line, we made use of the fact that ω will be integrated from $-\infty$ to $+\infty$ so that any odd functions in the integrand will give zero. As a result, only terms with even powers of ω are kept. Therefore, the effective potential becomes

$$\begin{aligned}
V_{\text{eff}}(\text{boson}) &= \frac{1}{4\pi}\sum_{n,m}\text{tr}\int d\omega\int_0^\infty \frac{ds}{s}K[{}^m_n]s^n e^{i\omega\tau}\partial^m e^{-i\omega\tau}e^{-sW}e^{-s\omega^2} \\
&= \frac{1}{4\pi}\sum_{n,m}\sum_{l=0}^{[m/2]}\text{tr}\int d\omega\int_0^\infty \frac{ds}{s}\tbinom{m}{2l}(-\omega^2)^l e^{-s\omega^2}K[{}^m_n]s^n\partial^{m-2l}e^{-sW} \\
&= \frac{1}{4\pi}\sum_{n,m}\sum_{l=0}^{[m/2]}\text{tr}\int_0^\infty \frac{ds}{s}\tbinom{m}{2l}(\frac{\partial^l}{\partial s^l}\sqrt{\frac{\pi}{s}})s^n K[{}^m_n]\partial^{m-2l}e^{-sW} \\
&= \text{tr}\sum_{n,m}\sum_{l=0}^{[m/2]}\frac{1}{4}\tbinom{m}{2l}\frac{1}{\Gamma(1/2-l)}K[{}^m_n]\partial^{m-2l}\int_0^\infty \frac{ds}{s}s^{n-l-1/2}e^{-sW} \\
&= \text{tr}\sum_{n,m}\sum_{l=0}^{[m/2]}\frac{1}{4}\tbinom{m}{2l}\frac{\Gamma(n-l-1/2)}{\Gamma(1/2-l)}K[{}^m_n]\partial^{m-2l}\frac{1}{W^{n-l-1/2}} \qquad \text{(B.11)}
\end{aligned}$$

Thus, the effective potential can be recast as

$$V_{\text{eff}} = \text{tr}\sum_{n,m}\sum_{l=0}^{[m/2]}\frac{1}{4}\tbinom{m}{2l}\frac{\Gamma(n-l-1/2)}{\Gamma(1/2-l)}\alpha^{2(n-l)-1}K[{}^m_n](W)\partial^{m-2l}\frac{1}{(\alpha^2 W)^{n-l-1/2}} \qquad \text{(B.12)}$$

As before, $\alpha = 1/(M^3R)$. The reason these factors of α are inserted will be clear shortly.

In a comparison of one-loop Matrix theory with supergravity, the relevant terms on the supergravity side are proportional to κ_{11}^2, which is of order α^3 on the Matrix theory side. This means that any higher powers of α are irrelevant for such a comparison as they represent Matrix theory corrections to supergravity, and finding them is not the purpose of this paper. In other words, to examine whether the Matrix theory can reproduce supergravity in the appropriate limit, only terms up to order α^3 need to be kept.

It makes sense, therefore, to examine each factor in V_{eff} and count the powers of α it contains. We begin with the mass matrix squared. By inspection of the explicit expressions given in Subsection 3.3.5, one sees that W can always be schematically written as

$$\begin{aligned} W &\sim \frac{1}{\alpha^2}r^2 + \frac{1}{\alpha}N \\ \Rightarrow \alpha^2 W &\sim r^2 + \alpha N \\ \Rightarrow (\alpha^2 W)^k &\sim 1 + \alpha N + (\alpha N)^2 + \cdots \end{aligned} \tag{B.13}$$

where powers of r and numerical coefficients have been suppressed in the last expression. For example, using eqn. (3.99), we have

$$\alpha N_f = \alpha \left(\sum_{I=1}^{9} v_I \gamma_I + \sum_{i=1}^{3} \frac{i\mu x^i}{4} \{\gamma_i, \gamma_{123}\} \right) + \alpha^2 \mu^2 / 4^2 \tag{B.14}$$

Similarly, αN_b can be constructed using the rules given in Subsection 3.3.5 (we omit its explicit expression here), while $\alpha N_{gh} = 0$. From the explicit expressions for N, it can be shown easily that tr $\alpha N = 0$, tr $[(\alpha N)^2] = 0 + O[\alpha^4]$, and tr $[(\alpha \partial N)^2] = 0 + O[\alpha^4]$. These facts are related to the large number of supersymmetries of our system and will be of use shortly. The last line in eqn. (B.13) is a symbolic statement that for any k, whether positive or negative, $(\alpha^2 W)^k$ will only give non-negative powers of α. Another important point to note is that every α arising from $(\alpha^2 W)^k$ is accompanied by a factor of N. Now look at $K[{}^m_n]$: Let $K[{}^m_n] = \sum_{p,q} K[{}^{m\ p}_{n\ q}]$ where p is the number τ-derivatives acting on W inside K, and q is the number of W inside K. By definition, we have

$$\frac{p+m}{2} + q = n \tag{B.15}$$

For $n = 0$ $\quad \Rightarrow K = 1$;

For $n = 1$ $\quad \Rightarrow K = 0$;

For $n \geq 2$ $\quad \Rightarrow K$ consists of commutators. In this case, we have

$$\begin{cases} q < n \\ \frac{p+m}{2} < n \end{cases} \tag{B.16}$$

For fixed q and $n \geq 2$, m and p have the following extremal values:

$m_{min} = 0 \quad \Rightarrow p_{max} = 2(n-q)$

$p_{min} = 1 \quad \Rightarrow m_{max} = 2(n-q) - 1$

The reason $p_{min} = 1$ is that ∂_τ^2 must act at least once on W to give non-vanishing commutators like $[W,[W,[W,\dot{W}]]]$ in K. Consider

$$\begin{aligned}
\alpha^{2(n-l)-1} K\left[{m \atop n}\right](W) \partial^{m-2l} \frac{1}{(\alpha^2 W)^{n-l-1/2}} &= \sum_{p,q} \alpha^{2(n-l)-1} K\left[{m \; p \atop n \; q}\right](W) \partial^{m-2l} \frac{1}{(\alpha^2 W)^{n-l-1/2}} \\
&= \sum_{p,q} \alpha^{2(n-q)-1-2l} K\left[{m \; p \atop n \; q}\right](\alpha^2 W) \partial^{m-2l} \frac{1}{(\alpha^2 W)^{n-l-1/2}} \\
&= \sum_{p,q} \alpha^{a} K\left[{m \; p \atop n \; q}\right](\alpha^2 W) \partial^{m-2l} \frac{1}{(\alpha^2 W)^{n-l-1/2}} \qquad \text{(B.17)}
\end{aligned}$$

where $a = 2(n-q) - 1 - 2l$. Noting $l \leq [m/2] \leq [m_{\max}/2] = [n-q-1/2] = n-q-1$, we must have

$$\begin{aligned}
l_{\max} &= n-q-1 \\
&\Rightarrow a \geq 1 \qquad \text{(B.18)}
\end{aligned}$$

From this derivation of the lower bound of a, we see that the equality holds only when $m = m_{\max} = 2(n-q) - 1$ and $l = l_{\max} = n-q-1$. Then, eqn. (B.15) gives

$$\begin{aligned}
m - 2l &= 1 \\
p = p_{\min} &= 1 \qquad \text{(B.19)}
\end{aligned}$$

A comparison of supergravity with one-loop Matrix theory means keeping terms only up to $\alpha^3 \sim \kappa_{11}^2$. Therefore, we need only consider the range of a to be

$$1 \leq a \leq 3 \qquad \text{(B.20)}$$

For $a = 3$: There can be no factors of αN from $\alpha^2 W$ because they increase the powers of α beyond 3, hence taking us beyond the limit of supergravity. Without any factors of αN, the effective potential is simply

$$\begin{aligned} V_{\text{eff}} &= f(r)\text{tr } 1 \\ \Rightarrow V_{\text{eff}} &\sim \text{tr } 1 = 0 \end{aligned} \tag{B.21}$$

Here, tr is again the trace over the boson minus the trace over the fermions and the ghosts. For $a = 2$: There can be at most one αN, either from $K(\alpha^2 W)$ or $\partial^{m-2l}(\alpha^2 W)^{-n+l+1/2}$ But for only one N, we have

$$\begin{aligned} V_{\text{eff}} &\sim \text{tr } \partial^k N \\ &= \partial^k \text{tr } N \\ &= 0 \end{aligned} \tag{B.22}$$

For $a = 1$: Now it is possible to have $(\alpha N)^2$ coming from one of the following three cases: (i) Both $(\alpha N)^2$ come from $\partial^{m-2l}(\alpha^2 W)^{-n+l+1/2}$

$$\begin{aligned} V_{\text{eff}} &\sim \alpha \text{tr } \partial^{m-2l} (\alpha N)^2 \\ &\sim \alpha \partial^{m-2l} \text{tr } (\alpha N)^2 \\ &\sim \alpha O[\alpha^4] \\ &\sim O[\alpha^5] \end{aligned} \tag{B.23}$$

(ii) Both $(\alpha N)^2$ come from $K(\alpha^2 W)$: Since we showed that $p = 1$ when $a = 1$, there is only one τ-derivative acting on W in K, we must have either: (a) $V_{\text{eff}} \sim \text{tr } N^2$ or (b) $V_{\text{eff}} \sim \text{tr } (N\partial_\tau N) \sim \frac{1}{2}\partial_\tau \text{tr } (N^2)$. In either case, $V_{\text{eff}} = 0 + O[\alpha^5]$.
(iii) One (αN) comes from $K(\alpha^2 W)$ and one from $\partial^{m-2l}(\alpha^2 W)^{-n+l+1/2}$: We already showed that $m - 2l = 1$ when $a = 1$, so we have

$$V_{\text{eff}} \sim K\partial(\alpha^2 W)^{-n+l+1/2} \tag{B.24}$$

This implies either: (a) $V_{\text{eff}} \sim \text{tr } (N\partial N)$ or (b) $V_{\text{eff}} \sim \text{tr } (\partial N \partial N)$

(a) is identical to case (iib) above. (b) is of order α^5 using the fact mentioned before. This exhausts all cases contributing to terms up to order $\alpha^3 \sim \kappa_{11}^2$ in V_{eff}. In particular, we have shown that none of the commutator terms in X, corresponding to $K[{}^{m}_{n}]$ with $n \geq 2$, contributes to terms relevant to supergravity. This completes the proof of the claim made following eqn. (B.7).

Appendix C

An Alternative Derivation of V_{eff} for the Point Particle Probe

In this appendix, we give an alternative derivation of the effective potential V_{eff} for a point particle probe (a probe graviton). The same quantity is derived in Subsection 4.1.1 under the name of light-cone Lagrangian, with a more rigorous treatment of constraints.

The Lagrangian of the probe graviton moving in the perturbed PP-wave with metric $G_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$, with $g_{\mu\nu}$ being the unperturbed PP-wave metric, and $h_{\mu\nu}$ being the metric perturbation due to the source graviton, is given by[1]

$$L = -m\sqrt{-G_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} = -m\sqrt{-(g_{\mu\nu} + h_{\mu\nu})\dot{x}^\mu\dot{x}^\nu} \tag{C.1}$$

We make a Legendre transformation

$$L' = L - P_p^+\dot{x}^- \tag{C.2}$$

where upon setting $x^+ = 1$

$$P_p^+ = \frac{\delta L}{\delta \dot{x}^-} = m\frac{1 + h_{-\nu}\dot{x}^\nu}{\sqrt{-G_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}} \tag{C.3}$$

[1]This approach is the one used in [17].

When we let $m \to 0$, this gives

$$G_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = (g_{\mu\nu} + h_{\mu\nu})\dot{x}^\mu \dot{x}^\nu = 2\dot{x}^- + g_{++} + v^2 + h_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = 0 \tag{C.4}$$

This is a quadratic equation for $\dot{x}^-$, which we will solve for $\dot{x}^-$, keeping only terms up to linear order in $h_{\mu\nu}$. We find

$$\begin{aligned}\dot{x}^- = -\Bigg\{ \frac{1}{2}\Bigg[& v^2 + g_{++} + h_{++} + g_{++}\left(\frac{1}{4}g_{++}h_{--} - h_{+-}\right) \\ & + \sum_A [2h_{+A} - h_{-A}(v^2 + g_{++})]v^A + \sum_{A,B} h_{AB}v^A v^B \Bigg] \\ & + \frac{1}{8}h_{--}v^4 - \frac{1}{2}v^2\left(h_{+-} - \frac{1}{2}g_{++}h_{--}\right) \Bigg\}\end{aligned} \tag{C.5}$$

Taking the limit $m \to 0$, the L' in eqn. (C.2) is simply $-P_p^+\dot{x}^-$, which is the effective potential V_{eff} that we need. It contains the interaction between the probe and the source up to terms linear in $h_{\mu\nu}$. An alternative way of writing such an interaction is

$$\frac{\delta L}{\delta G_{\mu\nu}} h_{\mu\nu} = T^{\mu\nu} h_{\mu\nu} \tag{C.6}$$

This structure was used by [6] to identify the effective potentials on both sides.

Appendix D

Constrained Hamiltonian Dynamics in Field Theory

For simplicity of notation let us consider field theory in $(1+1)$ dimensions. The generalization to field theory in higher dimensions is straightforward. Denote the field as $\eta(t,x)$, its momentum density $\pi(t,x) \equiv \frac{\partial \mathcal{L}(\eta, \partial_t \eta, \partial_x \eta)}{\partial(\partial_t \eta)}$, and the Hamiltonian density $\mathcal{H}(\eta, \partial_x \eta, \pi) \equiv \pi \partial_t \eta - \mathcal{L}$. The equations of motion are

$$\begin{aligned} \partial_t \eta &= \frac{\partial \mathcal{H}}{\partial \pi} \\ \partial_t \pi &= -\left(\frac{\partial \mathcal{H}}{\partial \eta} - \frac{\partial}{\partial x} \left(\frac{\partial \mathcal{H}}{\partial(\partial_x \eta)} \right) \right) \end{aligned} \tag{D.1}$$

The Poisson bracket in field theory is defined as follows. In field theory, we can have density $f(t,x) = f(\eta, \partial_x \eta, \pi, \partial_x \pi)$, or integrated density $F(t) = \int dx\ f(t,x) = \int dx\ f(\eta, \partial_x \eta, \pi, \partial_x \pi)$. Let us denote both densities and integrated densities collectively as ξ. For any two such objects ξ_1 and ξ_2, we define their Poisson bracket to be

$$[\xi_1, \xi_2]_{\mathrm{PB}} \equiv \int dy\ \left(\frac{\delta \xi_1}{\delta \eta(t,y)} \frac{\delta \xi_2}{\delta \pi(t,y)} - \frac{\delta \xi_2}{\delta \eta(t,y)} \frac{\delta \xi_1}{\delta \pi(t,y)} \right) \tag{D.2}$$

where $\frac{\delta(\ldots)}{\delta(\ldots)}$ is the functional derivative.

In terms of the Poisson bracket one can see that $\partial_t \eta(t,x) = \frac{\delta H}{\delta \pi(t,x)} = [\eta(t,x), H]_{\mathrm{PB}}$, and $\partial_t \pi(t,x) = -\frac{\delta H}{\delta \eta(t,x)} = [\pi(t,x), H]_{\mathrm{PB}}$ with $H \equiv \int dx\ \mathcal{H}(t,x)$ being the Hamiltonian. Using this, one also see that for any function $f(t,x) = f(\eta, \partial_x \eta, \pi, \partial_x \pi)$, we have

$$\partial_t f(t,x) = [f(t,x), H]_{\mathrm{PB}} \tag{D.3}$$

So far we have been talking about an unconstrained system. Now let us introduce constraints. Suppose the system has M primary constraints

$$\phi_m(\eta, \partial_x \eta, \pi) = 0, \quad m = 1, 2, ..., M \tag{D.4}$$

Define the total Hamiltonian density as $\mathcal{H}_T \equiv \mathcal{H} + u^m \phi_m$, and the total Hamiltonian as

$$H_T \equiv \int dx\ \mathcal{H}_T = H + \int dx\ u^m \phi_m \tag{D.5}$$

with $u^m(t,x)$ being the Lagrange multiplier fields. Then the time evolution is given by, for any function $f(t,x) = f(\eta, \partial_x \eta, \pi, \partial_x \pi)$

$$\partial_t f(t,x) = [f(t,x), H_T]_{\mathrm{PB}} \tag{D.6}$$

This completes our brief review of constrained Hamiltonian dynamics in field theory. We didn't introduce the extended Hamiltonian H_E which is different from the total Hamiltonian H_T only when there exist secondary constraints, since in the application to the membrane one can show that there is no secondary constraint (see Subsection 4.1.2).

Appendix E

The Green's Function for the x^- Fourier Transformed Laplacian $\tilde{\Box}$

In Section 4.2, upon diagonalizing the supergravity field equations, the components of the metric and three-form perturbations satisfy Laplace equations. (See, for example, eqn. (4.66).) So in this appendix we discuss the Green's function for the Laplace equation Fourier transformed along the x^- direction.

Let us use $\Box$ to denote the Laplacian $g^{\mu\nu}\partial_\mu\partial_\nu$, and its x^- Fourier transform $\tilde{\Box} \equiv g_{++}k_-^2 + \sum_{A=1}^9 \partial_A^2 = \sum_{A=1}^9 \tilde{\Box}_A$ where $\tilde{\Box}_i \equiv \frac{\partial^2}{(\partial x^i)^2} - \frac{1}{9}k_-^2\mu^2(x^i)^2$, $\tilde{\Box}_a \equiv \frac{\partial^2}{(\partial x^a)^2} - \frac{1}{36}k_-^2\mu^2(x^a)^2$,

Of course, $-\tilde{\Box}_A$ is just a harmonic oscillator's Hamiltonian along the x^A direction. Let us use ϕ_{n_A} and E_{n_A} to denote its normalized eigenfunctions and eigenvalues. Then $\Phi_{\{n_1,\ldots,n_9\}} \equiv \phi_{n_1}(x^1)\ldots\phi_{n_9}(x^9)$ and $E_{\{n_1,\ldots,n_9\}} \equiv \sum_{A=1}^9 E_{n_A}$ are the normalized eigenfunctions and eigenvalues of $-\tilde{\Box}$.

Use $\vec{x}$ to denote the nine vector $(x^1, \ldots, x^9)$. Define

$$K(\vec{x},\vec{y}) \equiv \sum_{\{n_1,\ldots,n_9\}} \frac{-1}{E_{\{n_1,\ldots,n_9\}}} \Phi_{\{n_1,\ldots,n_9\}}(\vec{x})\Phi^*_{\{n_1,\ldots,n_9\}}(\vec{y}) \tag{E.1}$$

which is the Green's function of $\tilde{\Box}$: $\tilde{\Box}_{\vec{x}}K(\vec{x},\vec{y}) = \delta(\vec{x}-\vec{y})$

The above Green's function doesn't have closed form. Thus although in principle $\bar{h}_{\mu\nu}$ and $a_{\mu\nu\rho}$ can all be obtained by integrating over the source using this Green's function, in practice one may have to take some limit to get an answer, e.g., the near-membrane limit in Appendix F.

Appendix F

h_{--} for a Spherical Source Membrane in the Near-Membrane Limit

In this appendix, we discuss the solution h_{--} to the field equation (4.66), in the near-membrane limit. Also see the end of Subsection 3.2.2.2 and the end of Section 4.2 for discussions on the near-membrane limit in the investigation of two membrane interactions.

Note $\bar{h}_{--} = h_{--}$. In the following we use the same symbol for the metric perturbation and its Fourier transform. Its meaning should be clear from context.

For a source membrane extending a sphere of radius r_0 in the (X^1, X^2, X^3) directions, with its center sitting at $X^- = 0, X^4 = ... = X^9 = 0$, and $X^+ = \sigma^0$, the $(--)$ component of its stress tensor is

$$[T_{--}]_S = T\delta(x^-)\delta(r - r_0)\delta(x^4)...\delta(x^9)\left(\frac{-1}{2}\right)\left(\frac{\mu r_0}{3}\right)^{-1} \tag{F.1}$$

where $r \equiv \sqrt{x^i x^i}$. Recall h_{--} satisfies eqn. (4.66)

$$-\frac{1}{2}\Box h_{--} + Q_{--} = \kappa_{11}^2 [T_{--}]_S \tag{F.2}$$

Let us make the gauge choice $q_- = 0$ (thus $Q_{--} = \partial_- q_- = 0$) and take its Fourier transform along the x^- direction

$$-\frac{1}{2}\tilde{\Box}h_{--} = \kappa_{11}^2 T\delta(r-r_0)\delta(x^4)...\delta(x^9)\left(\frac{-1}{2}\right)\left(\frac{\mu r_0}{3}\right)^{-1} \tag{F.3}$$

Integrating the Green's function K over the sphere is not such a great idea. Instead, we consider the near-membrane limit (so that the source membrane looks almost flat), where $w \equiv r - r_0 << r_0$, also $\sqrt{x^a x^a} << r_0$. In this limit (and assuming h_{--} only depends on $(w, x^4, ..., x^9)$), we get

$$\left[-\left(\frac{\mu r_0}{3}\right)^2 k_-^2 + \frac{\partial^2}{(\partial w)^2} + \frac{\partial^2}{(\partial x^4)^2} + ...\frac{\partial^2}{(\partial x^9)^2}\right] h_{--} = \left(\frac{\mu r_0}{3}\right)^{-1} \kappa_{11}^2 T\delta(w)\delta(x^4)...\delta(x^9) \tag{F.4}$$

which is just a massive scalar equation in seven-dimensional flat space. The solution to this is, defining $\xi \equiv \sqrt{w^2 + x^a x^a}$

$$h_{--} = -\frac{\kappa_{11}^2 T}{15\Omega_6\left(\frac{\mu r_0}{3}\right)}\frac{\exp\left(-\frac{\mu r_0}{3}k_-\xi\right)}{\xi^5}\left[3 + 3\left(\frac{\mu r_0}{3}k_-\xi\right) + \left(\frac{\mu r_0}{3}k_-\xi\right)^2\right] \tag{F.5}$$

where $\Omega_6 = \frac{16\pi^3}{15}$ is the area of unit six sphere.

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