# WAVES ON VORTEX FILAMENTS

Thesis by

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In Partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California

1973

(Submitted May 17, 1973)

## Acknowledgement

The author considers it an honor to be able to work under the supervision of Professor P. G. Saffman, who not only suggested the problems, but provided invaluable advice and constructive criticism without which this work could never have been completed. To him the author expresses his deepest gratitude.

The author would also like to thank Professor G. B. Whithan for his patient guidance during the first two years of study, and Professors H. B. Keller, D. S. Cohen and J. K. Knowles for their kind consent to serve on the examination committee.

The author's course of study was made possible by the Earle C. Anthony Fellowship, the Applied Mathematics Teaching and Research Assistantships, and the U. S. Air Force Office of Scientific Research Grant. To these sources of financial support the author is much obliged.

Finally, the author wishes to thank Miss Edith Williams for her speedy and intelligent typing, and his fiancé Molly for her understanding and encouragement.

Abstract

Various problems concerning waves on vortex filaments are considered. The local force balance method introduced by Moore and Saffman for the calculation of the induced velocity at a point of a vortex filament with arbitrary structure and shape is used to examine the effect of axial flow on the stability of trailing vortices and vortex rings. It is found that the effect is small in both cases. The method is extended to study the stability of vortex rings carrying electric charges, which are possible models for vortices in liquid helium. Two cases are considered—the conducting ring and the uniformly charged ring. In each of these cases it is found that the velocity of a charged ring is smaller than an uncharged one, and if the charges are strong enough, the ring may reverse its direction of motion. Furthermore, the charged ring becomes unstable when the charge effect is comparable to the vorticity effect. The motion and stability of a buoyant vortex ring are also considered. It is shown that a heavy ring travelling in the direction of gravity decelerates, thins and expands, while a light ring accelerates, fattens and contracts. The heavy ring is stable to disturbances of the centerline, but the light ring is unstable with a growth rate independent of wave number.

Intrinsic equations governing the curvature $\kappa$ and torsion $\tau$ of a vortex filament are obtained. They form a set of coupled nonlinear integro-partial differential equations. By retaining only the leading order term in the singularity of the Biot-Savart integral,

which corresponds to the localized induction hypothesis introduced by Arms and Hama, these can be reduced to a single nonlinear Schrödinger equation for the complex variable

$$\psi = \kappa\, e^{i \int_0^s \tau ds}$$

where s is the arclength. A complete set of steady state solutions for this equation is obtained. This includes the straight vortex, the helical vortex, the vortex ring, and a solitary wave form, all being limiting cases of a general periodic wave structure. A modified scheme is introduced to resolve an apparent nonuniformity of the solitary wave solution in the limit $\kappa \to 0$. Non-local effects (effects of the regular part of the Biot-Savart integral) are examined by means of an asymptotic expansion of the intrinsic equations in the small parameter $\epsilon = (\log \frac{1}{a})^{-1}$ where a is the core radius of the filament. It is shown that even in the tail ends of the solitary wave where the local effect fails to dominate, the solitary wave solution exists to $O(\epsilon^2)$.

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## I. INTRODUCTION

The subject of vortex motion was first brought to the attention of mathematicians and physicists more than a hundred years ago by Lord Kelvin when he presented his classical papers, "On Vortex Atoms," and "On Vortex Motion," to the Royal Society of Edinburgh in 1867. In these papers he explained and expanded the ideas of Helmholtz, and put forth the vortex atom theory, which states that atoms are actually vortices in ether. He proposed that a thorough investigation of the properties of vortices should be made which he hoped would shed light on the basic properties of matter through the vortex atom theory. This idea was shortlived. However, the proposal gave birth to a fascinating branch of applied mathematics; in the century that followed, the subject of vortex motion continued to attract the attention of mathematicians and physicists. Interesting results were obtained, but understanding of the problem as of today is still far from complete due to the complexity of the mathematics involved.

In this work, we consider the problem of waves on vortex filaments. By a vortex filament, we mean a thin tube of fluid whose surface is made up of vortex lines and into which most of the vorticity is concentrated. For an inviscid fluid with uniform density, it was shown by Helmholtz (1858) that such a tube moves with the fluid (i.e., the fluid elements constituting the tube are always the same ones) and has constant strength, the strength of a vortex filament being defined as:

$$\Gamma = \int \underset{\sim}{\omega} \cdot d\underset{\sim}{S} , \tag{1-1}$$

where $\underset{\sim}{\omega}$ is the vorticity and $d\underset{\sim}{S}$ is the surface element bounded by a closed curve on the tube which wraps around it once. The presence of a non-zero vorticity induces a velocity field given by

$$\underset{\sim}{u}_V(\underset{\sim}{x}) = \frac{1}{4\pi} \int \frac{(\underset{\sim}{x}' - \underset{\sim}{x}) \wedge \underset{\sim}{\omega}(\underset{\sim}{x}')}{|\underset{\sim}{x}' - \underset{\sim}{x}|^3} dV(\underset{\sim}{x}') , \tag{1-2}$$

where $\underset{\sim}{x}$ is the position vector of the point of interest and the integral is taken over the entire volume of non-zero vorticity. Thus, an isolated vortex filament in general moves under its own induction even in the absence of external forces.

Let us consider the problem of waves on an isolated vortex filament in an inviscid infinite fluid. There are several length scales involved. They are: the local core radius a, the local radius of curvature (of the center line) $\rho$, the characteristic amplitude of the waves D, and the characteristic wavelength of the waves $\lambda$. Almost all of the results obtained so far are for limiting cases. Kelvin (1880) treated the case $\frac{D}{a} \ll 1$ by solving the linearized equations of motion both inside and outside the filament and matching the solutions across the boundary. This is the so-called infinitesimal theory since the amplitudes of the disturbances (or waves) considered are infinitesimal quantities when compared to other length scales. This approach is exact, but it is limited to cases where the unperturbed state has extremely simple geometry in order for the equations to have solutions in closed form (Kelvin studied the perturbations on a straight vortex). Furthermore, the requirement that $\frac{D}{a} \ll 1$ becomes very restrictive when the filament is thin. For all practical purposes, the main use of the infinitesimal theory is to provide a check for other methods since they must agree as $\frac{D}{a} \to 0$. Another limiting case is that of small waves on thin filaments. By "small" waves we mean waves with $\frac{D}{\rho} \ll 1$ and $\frac{D}{\lambda} \ll 1$, and by a "thin" filament we mean $\frac{a}{\rho} \ll 1$ and $\frac{a}{\lambda} \ll 1$; that is, both the wave amplitudes and the core radius are small compared to the geometric length scales. This assumption allows us to make use of the mathematical idealization of a line vortex where the core radius is allowed to approach zero while the strength $\Gamma$ remains finite. The effect of vorticity due to such a line vortex can then be specified entirely by the strength $\Gamma$ and the position of the line to which the filament has contracted. This is given by the Biot-Savart law

$$\underset{\sim}{u}_V(\underset{\sim}{x}) = \frac{\Gamma}{4\pi} \oint \frac{(\underset{\sim}{x}' - \underset{\sim}{x}) \wedge d\underset{\sim}{x}'}{|\underset{\sim}{x} - \underset{\sim}{x}'|^3} , \qquad (1\text{-}3)$$

where $\oint$ denotes a line integral. The integral becomes divergent at $\underset{\sim}{x}' = \underset{\sim}{x}$ which is a consequence of the idealization that a is zero. The singularity can be removed by returning to a finite core near $\underset{\sim}{x}' = \underset{\sim}{x}$. The small wave assumption $(\frac{D}{\lambda} \ll 1, \frac{D}{\rho} \ll 1)$ allows us to expand around the unperturbed state. Note that now the wave amplitude D and the core radius a are independent. This approach appears to have been originated by J. J. Thomson (1883) when he considered small waves on vortex rings, and has been used extensively in various modified forms ever since. Quite recently Arms and Hama (Hama, 1962, 1963; Arms and Hama, 1965) introduced the localized induction hypothesis. The idea is to retain only the singular term in the Biot-Savart law and neglect logarithmic variations. This reduces (1-3) to a differential equation instead of an integral since the singular term depends only on local properties (the local curvature). Solutions in closed form with finite amplitudes can then be found within this order of approximation.

In essense, the three approaches mentioned above are just different degrees of trade-off between accuracy and generality. The exact infinitesimal analysis is applicable only to very simple configurations while the localized induction hypothesis gives crude results for problems with greater geometric complexity.

The three parts of the present work correspond to these three approaches. Part 1 deals with the infinitesimal theory. Chapter II examines the problem of infinitesimal waves on a straight vortex with and without axial velocity to illustrate the method and provide a basis of comparison for later results. Part 2 deals with small waves on a thin filament. Chapter III derives the equations of motion of a vortex filament with axial flow following the work of Moore and Saffman (1972). Chapter IV applies these equations to study the effect of axial velocity on the stability of aircraft trailing vortices. Chapter V is concerned with the stability of steady vortex rings. The cases considered are a ring with axial flow, an electrically charged conducting vortex ring, and a vortex ring with a uniform electric charge density. Chapter VI considers a bouyant ring, which is unsteady in the sense that its ring radius, core radius, and propagation speed are all functions of time. Part 3 examines the localized induction hypothesis and its extentions. Chapter VII derives the general intrinsic equations governing the curvature and torsion of a thin vortex filament. Chapter VIII solves the steady problem under the localized induction hypothesis and obtains among others a solitary wave solution. Chapter IX presents the modified hypothesis in which the logarithmic variation neglected by the localized induction hypothesis is introduced to eliminate an apparent non-uniformity and resolve an ambiguity in the original method. Chapter X develops an asymptotic expansion of the equations of motion as $\frac{a}{\rho} \rightarrow 0$, which has the localized induction hypothesis as the leading term, and applies the first order equations to examine the validity of the solitary wave solution.

# II. INFINITESIMAL WAVES ON A STRAIGHT VORTEX

§2. Thirteen years after the presentation of the vortex atom theory Kelvin published the first work concerning waves on vortex filaments titled, "Vibrations of a Columnar Vortex." In this paper he studied the infinitesimal disturbances on a straight vortex by solving the linearized equations of motion both inside and outside the filament and matching the solutions across the surface, requiring a continuity of pressure and the normal velocity. In 1967, Krishnamoorthy extended the analysis to include the effects of axial velocity. These works are partly reproduced in the present chapter.

## §3. The Governing Equations of the Infinitesimal Theory

The equations of motion of an incompressible, inviscid fluid with unit density are

$$\frac{\partial \underset{\frown}{u}}{\partial t} + (\underset{\frown}{u} \cdot \underset{\frown}{\nabla})\underset{\frown}{u} = - \underset{\frown}{\nabla} p, \tag{2-1}$$

$$\underset{\frown}{\nabla} \cdot \underset{\frown}{u} = 0, \tag{2-2}$$

where $\underset{\frown}{u}$ is the velocity and p is the pressure. Let us use cylindrical coordinates $(r, \theta, z)$ with corresponding velocity components $\underset{\frown}{u} =$ $(u, v, w)$. Then the unperturbed straight vortex is a circular cylinder about the z axis with equation

$$r = a_0, \tag{2-3}$$

and corresponding velocity and pressure

$$\underset{\sim}{u}_0 = (0,\ v_0(r),\ 0), \tag{2-4}$$

$$p_0 = \int^r \frac{v_0^2(r)}{r}\, dr. \tag{2-5}$$

The equation of the perturbed filament can be written as $r = a_0 + De^{i(nz + \omega t - m\theta)}$ and the disturbed velocity and pressure are given as

$$\underset{\sim}{u} = \underset{\sim}{u}_0 + \underset{\sim}{u}_1 = \underset{\sim}{u}_0 + \begin{pmatrix} u_1(r) \cos nz \sin(\omega t - m\theta) \\ v_1(r) \cos nz \cos(\omega t - m\theta) \\ w_1(r) \sin nz \sin(\omega t - m\theta) \end{pmatrix}, \tag{2-7}$$

$$p = p_0 + p_1(r) \cos nz \cos(\omega t - m\theta), \tag{2-8}$$

where D, $u_1, v_1, w_1$ and $p_1$ are infinitesimal quantities compared with $a_0$, $v_0$ and $p_0$, and their dependence on z, $\theta$ and t are chosen in view of the parity of the continuity equation (2-2). These perturbations describe sinusoidal disturbances along the vortex as well as undulations of the surface. The case $m = 0$ corresponds to periodic pulsation of the filament core ('sausaging'), and $|m| > 1$ corresponds to fluted distortions of the filament surface with no displacement of the centerline. The cases $m = \pm 1$ are interesting, they represent a sinusoidal disturbance of the centerline of the vortex with no change in the core radius, these are the modes that we are mainly interested in.

We now substitute the expressions in (2-7) and (2-8) into (2-1) and (2-2) and linearize about the unperturbed state. This leads to

$$\left(\omega - m\,\frac{v_0}{r}\right)u_1 - 2\,\frac{v_0}{r}\,v_1 = -\,\frac{dp_1}{dr}\,, \tag{2-9}$$

$$\left(\omega - m\,\frac{v_0}{r}\right)v_1 - \left(\frac{v_0}{r} + \frac{dv_0}{dr}\right)u_1 = m\,\frac{p_1}{r}\,, \tag{2-10}$$

$$\left(\omega - m\,\frac{v_0}{r}\right)w_1 = np_1, \tag{2-11}$$

$$\frac{du_1}{dr} + \frac{u_1}{r} + \frac{mv_1}{r} + nw_1 = 0\,. \tag{2-12}$$

If we include a constant axial velocity in the vortex, then the unperturbed velocity would be

$$\underset{\sim}{u_0} = \begin{pmatrix} 0 \\ v_0(r) \\ w_0 \end{pmatrix}, \tag{2-13}$$

where

$$w_0 = \begin{cases} W \text{ (constant)} & r < a_0 \\ 0 & r > a_0 \end{cases} \tag{2-14}$$

The equations for the perturbation quantities are then

$$\left(\omega - m\,\frac{v_0}{r} + nw_0\right)u_1 - 2\,\frac{v_0}{r}\,v_1 = -\frac{dp_1}{dr}, \tag{2-15}$$

$$\left(\omega - m\frac{v_0}{r} + nw_0\right)v_1 - \left(\frac{v_0}{r} + \frac{dv_0}{dr}\right)u_1 = m\frac{p_1}{r}, \qquad (2\text{-}16)$$

$$\left(\omega - m\frac{v_0}{r} + nw_0\right)w_1 = np_1 , \qquad (2\text{-}17)$$

with no change in the continuity equation. These reduce to (2-9) - (2-11) when W = 0.

For the a uniform distribution of vorticity inside the core, $v_0(r)$ is given by

$$v_0(r) = \begin{cases} \Omega r & r < a_0 \qquad (2\text{-}18) \\ \dfrac{\Omega a_0^2}{r} & r > a_0 \qquad (2\text{-}19) \end{cases} ,$$

and the appropriate boundary conditions are the continuity of $p_1$ and $u_1$ across the filament surface.

## §4. Solution of the Equations of Motion and the Dispersion Relation

For simplicity, let us first consider the case $w_0 = 0$. (2-9) and (2-10) can then be solved for $u_1$ and $v_1$ in terms of $w_1$. These are substituted into (2-12) to yield a second order equation of the Bessel type for $w_1$ which can be solved inside and outside of the filament to give $w_1$ as a function of r and parameters n, $\omega$ and m( $= + 1$). The two integration constants are determined by the boundary conditions. On the surface we can let $r = a_0$(the linearized boundary) to obtain a dispersion relation expressing $\omega$ in terms of n, m and $a_0$. Proceeding as such, we have, writing a instead of $a_0$ since the context is clear,

$$\frac{d^2w_1}{dr^2} + \frac{1}{r}\frac{w_1}{r} - \left(\frac{m^2}{r^2} - n^2\right)w_1 = 0 \quad r > a, \tag{2-20}$$

$$\frac{d^2w_1}{dr^2} + \frac{1}{r}\frac{w_1}{r} - \left(\frac{m^2}{r^2} - \nu^2\right)w_1 = 0 \quad r < a, \tag{2-21}$$

where

$$\nu^2 = n^2 \frac{[4\Omega^2 - (\omega - m\Omega)^2]}{(\omega - m\Omega)^2} \; . \tag{2-22}$$

The requirement that the solution should be bounded everywhere (at $r = \infty$ and $r = 0$) gives

$$w_1 = C_0 \; K_m(nr), \quad r > a \tag{2-23}$$

$$w_1 = C_1 \; J_m(\nu r), \quad r < a. \tag{2-24}$$

The boundary conditions on $p_1$[given by (2-11)] and $u_1$ allow us to eliminate $C_0$ and $C_1$ at the boundary $r = a$ so that we obtain the dispersion relation

$$\frac{J'_m(qa)}{(qa)J_m(qa)} + \frac{m}{q^2a^2k} = \frac{-K'_m(na)}{na\;K_m(na)}, \tag{2-25}$$

where

$$q^2 = \frac{n^2(1 - k^2)}{k^2}, \tag{2-26}$$

and

$$k = \frac{m\Omega - \omega}{2\Omega}. \tag{2-27}$$

We can treat (2-25) as a transcendental equation in k. It was pointed out by Kelvin (1880) that this equation has an infinite number of roots in the range $0 < k < 1$ and another infinite number of roots in $-1 < k < 0$, but none for $|k| > 1$. When axial velocity is included, a similar calculation leads to

$$\frac{J'_m(qa)}{qa\,J_m(qa)} + \frac{m}{q^2a^2k} = \frac{-4\Omega^2k^2}{(2\Omega k - nW)^2}\left(\frac{K'_m(na)}{na\,K_m(na)}\right), \qquad (2\text{-}28)$$

which reduces to (2-25) when $W = 0$.

Both (2-25) and (2-28) are too complicated to permit a complete discussion of all their roots, but it is possible to examine a limiting case. This is the case when $|na| \ll 1$ and $|qa| \ll 1$ so that both sides of the expression can be expanded in power series of na and qa. This was done by Kelvin for the case $W = 0$, and he obtained

$$\frac{\omega}{\Omega} = \frac{n^2a^2}{2}\,\tilde{L} + \cdots, \qquad (2\text{-}29)$$

where

$$\tilde{L} = \log\frac{2}{|na|} - C + \frac{1}{4}, \qquad (2\text{-}30)$$

and $C = 0.5772\cdots$ is Euler's constant. For the case $W \neq 0$, we have

$$\frac{1}{qa} - \frac{1}{4}\,qa - \frac{1}{96}\,q^3a^3 + \cdots = \frac{4\Omega^2k^2}{(2\Omega k - n\,W)^2}\left[\frac{1}{na} + na\left(\log\frac{2}{|na|} - C\right) + \cdots\right], \qquad (2\text{-}31)$$

which has been solved by Moore and Saffman (1972) assuming that W is $O(\Omega a)$ by treating it as a cubic in q, expressed in $\frac{\omega}{\Omega}$, it is

$$\frac{\omega}{\Omega} = \frac{n^2a^2}{2}\left[\tilde{L} - \frac{W^2}{\Omega^2a^2}\right] - \frac{n^2a^2}{2}\left[(\tilde{L} + \tfrac{1}{2}) - \frac{W^2}{\Omega^2a^2}\right]\frac{nW}{\Omega} + \cdots \qquad (2\text{-}32)$$

Note that it reduces to Kelvin's solution when $W = 0$.

The assumption that $|na| \ll 1$ simply means that we are considering long waves, since $n = \frac{2\pi}{\lambda}$, this assumption is equivalent to

$$\frac{a}{\lambda} \ll \frac{1}{2\pi}. \qquad (2\text{-}33)$$

The assumption that $qa \ll 1$ has no direct physical meaning but it turns out that this mode corresponds to low frequency oscillation (since $\frac{\omega}{\Omega} \ll 1$) while other modes have $\omega \sim \Omega$.

As we have pointed out earlier, the requirement $\frac{D}{a} \ll 1$ becomes severely restrictive when a gets small, thereby making these results of only limited direct application. However, it does provide a valuable check for more approximate methods like the cutoff method which we shall discuss in the next few chapters.

## III. EQUATIONS OF MOTION FOR A THIN VORTEX FILAMENT WITH AXIAL FLOW

§5 The infinitesimal theory is unsuitable for direct applications not only because of the assumption that $\frac{D}{a} \ll 1$, but to a greater extent due to the impossibility of solving the flow equations in closed form for almost any other case except the straight vortex. This difficulty is partially resolved if we consider a thin filament. The idea is that the velocity of a point on the vortex induced by the rest of the filament can be given by the Biot-Savart law (1-3) which is a formal solution of the flow equations if we assume the vortex behaves like a line vortex. This is indeed true when we consider contributions from points far away from the point in question. For the contribution of points close to the point of interest, the Biot-Savart integral becomes singular since the line vortex approximation which assumes a is small compared to all length scales cannot hold anymore as the distance $|\underset{\frown}{x}' - \underset{\frown}{x}|$ approaches zero. In fact, these neighboring points are the most influential, and their contribution must be determined by returning to the idea of a finite core.

This entire approach has been rigorized by Moore and Saffman (1972) when they derived the equations describing the motion of an arbitrary thin filament with axial flow by balancing the various forces acting on an element of the vortex. The Biot-Savart law is used for far field contributions and a matched asymptotic expansion is used for the influence of the neighboring points. This is the approach that we shall follow.

## §6. Local Force Balance for an Element of the Vortex

Let us recall the assumption of a thin filament, which is

$$\frac{a}{\rho} \ll 1, \tag{3-1}$$

where a is the core radius and $\rho$ is the local radius of curvature. In this limit is is meaningful to describe the filament by a position vector $\underset{\frown}{R} = \underset{\frown}{R}(\xi, t)$, where $\xi$ is a Lagrangian parameter along the filament and t is the time. Actually, $\underset{\frown}{R}$ describes the points on the line vortex to which the filament has contracted, it can also be thought of roughly as the centerline of the filament. It is also useful to introduce the arc length parameter $s = s(\xi, t)$ measured along the filament.

The velocity of a point on the filament is then given by

$$\underset{\frown}{u}(\xi, t) = \frac{\partial \underset{\frown}{R}}{\partial t}(\xi, t). \tag{3-2}$$

Let us consider an element $\Delta$ of the filament as shown in Fig. 3-1. We denote the curved surface by $\mathcal{C}$ and the plane ends by $E_1$(at $\xi$) and $E_2$(at $\xi + d\xi$). The length of the filament is taken as ds, then the force balance on $\Delta$ requires that

$$\underset{\frown}{F}_E + \underset{\frown}{F}_I = 0, \tag{3-3}$$

where $\underset{\frown}{F}_E$ is the force per unit length exerted by the surrounding fluid on the curved surface $\mathcal{C}$ and $\underset{\frown}{F}_I$ is the force per unit length exerted by the fluid inside the core on the surface including the

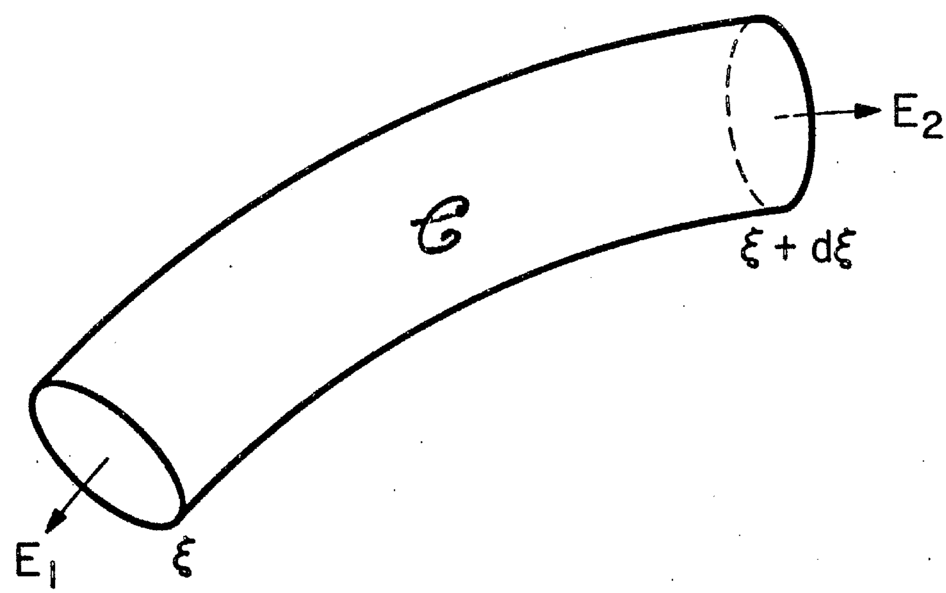



Fig. 3-1

effects of fluid inertia. In the absence of external body forces, $\underset{\sim}{F}_E$ and $\underset{\sim}{F}_I$ can be expressed as

$$\underset{\sim}{F}_E \, ds = \int_{\mathcal{C}} - p\underset{\sim}{\ell} \, dA, \tag{3-4}$$

$$\underset{\sim}{F}_I \, ds = - \frac{\partial}{\partial t} \int_{\Delta} \underset{\sim}{u} dV - \left[ \frac{\partial}{\partial s} \int_{A(s)} (p\underset{\sim}{s} + w\underset{\sim}{u}) dA \right] ds, \tag{3-5}$$

where $\underset{\sim}{\ell}$ denotes the outward normal on $\mathcal{C}$ , p is the pressure, w is the tangential component of the internal velocity relative to $\mathcal{C}$, $\underset{\sim}{s}$ is the tangent vector defined by

$$\underset{\sim}{s} = \frac{\partial \underset{\sim}{R}}{\partial s}, \tag{3-6}$$

$\underset{\sim}{u}$ is the velocity of the fluid, and A(s) is the cross sectional surface. The expressions (3-4) and (3-5) have been evaluated by Moore and Saffman (1972) using matched asymptotic expansions. We shall not present the details here but we shall summarize some of the intuitive arguments which they provided.

## §7. The Exterior Force $\underset{\sim}{F}_E$

The first contribution in the exterior force is the Kutta-Joukowski lift for flow past a cylinder with circulation, it is given by

$$\Gamma \underset{\sim}{Q} \wedge \underset{\sim}{s} , \tag{3-7}$$

where $\underset{\sim}{Q}$ is the relative velocity of the element to the surrounding fluid. To find $\underset{\sim}{Q}$ we argue as follows. The presence of the vortex filament induces a velocity everywhere in the fluid. At each point $\xi$ on the filament, there is associated such a velocity which can be thought of as the resulting velocity in the region $a \ll |\underset{\sim}{x} - \underset{\sim}{R}| \ll \rho$ when the swirl is subtracted off. We call this velocity $\underset{\sim}{V}(\xi, t)$ and it is defined as

$$\underset{\sim}{V}(\xi, t) = \underset{\sim}{V}_E(\xi, t) + \underset{\sim}{V}_I(\xi, t) \ , \qquad (3\text{-}8)$$

where $\underset{\sim}{V}_E$ is the velocity due to some external fields (eg. the presence of another vortex). $\underset{\sim}{V}_I(\xi, t)$ must be defined more carefully. A direct application of the Biot-Savart law leads to a divergent integral. To obtain a well defined quantity we subtract off the divergent part by considering the expression

$$\underset{\sim}{u}(\underset{\sim}{x};\xi, t) = \frac{\Gamma}{4\pi} \int \left\{ \frac{(\underset{\sim}{x}' - \underset{\sim}{x}) \wedge d\underset{\sim}{x}'}{|\underset{\sim}{x}' - \underset{\sim}{x}|^3} - \frac{(\underset{\sim}{x}^0 - \underset{\sim}{x}) \wedge d\underset{\sim}{x}^0}{|\underset{\sim}{x}^0 - \underset{\sim}{x}|^3} \right\}, \qquad (3\text{-}9)$$

where the first part of the integral is taken over the filament and the latter part taken over an osculating circle to the vortex filament at the point $\xi$. An osculating circle is a circle which lies on the plane formed by the tangent $\underset{\sim}{s}$ and the normal $\underset{\sim}{n}$ touching the point $\xi$ with radius equal to the local radius of curvature $\rho$. This expression is regular since both parts have the same singularity and therefore they cancel. We then define

$$\underset{\sim}{Q} = \underset{\sim}{V} - \frac{\partial \underset{\sim}{R}}{\partial t} \qquad (3\text{-}10)$$

where

$$\underset{\frown}{V}_I(\xi, t) = \underset{\frown}{u}(\underset{\frown}{R}(\xi, t);\xi, t).$$

The next important contribution to $\underset{\frown}{F}_E$ is in the form of a tension which is due to the curvature of the filament. This originates from the fact that when the filament is curved the velocity is higher on the concave side than the convex side and thus creates a pressure gradient directed towards the center of the filament. If we take $\underset{\frown}{n}$ as the principle normal (pointing inward) which is given by

$$\frac{\partial \underset{\frown}{s}}{\partial s} = \frac{\underset{\frown}{n}}{\rho}, \tag{3-11}$$

this force can be expressed in the form

$$\frac{T_0 \, \underset{\frown}{n}}{\rho},$$

where $T_0$ is found by detailed analysis of the flow near the point of interest to be

$$T_0 = \frac{\Gamma^2}{4\pi}\left(\log \frac{8\rho}{a} - \frac{1}{2}\right). \tag{3-12}$$

In the case where the core radius a varies with s, a tangential force is created, it is found to be

$$-\frac{\Gamma^2}{8\pi a^2}\frac{\partial a^2}{\partial s},$$

as we shall see later, this term will become important when there exist gravitational effects.

Other contributions are from the apparent mass effect and the pressure gradient effect in the surrounding fluid, but both of these are shown to be negligible by Moore and Saffman (1972).

Collecting the terms we have

$$\underset{\sim}{F}_E = \Gamma\left(\underset{\sim}{V}_I + \underset{\sim}{V}_E - \frac{\partial \underset{\sim}{R}}{\partial t}\right)\wedge \underset{\sim}{s} + \frac{\Gamma^2}{4\pi\rho}\left(\log\frac{8\rho}{a} - \frac{1}{2}\right)\underset{\sim}{n}$$
$$- \frac{\Gamma^2}{8\pi a^2}\frac{\partial a^2}{\partial s}\underset{\sim}{s}. \qquad (3\text{-}13)$$

## §8. The Interior Force $\underset{\sim}{F}_I$

The calculation of the interior force $\underset{\sim}{F}_I$ is as follows. Integration of the pressure term leads to a contribution of

$$\pi\frac{\partial}{\partial s}\left[\frac{1}{2}a^2\,\overline{v}^2\underset{\sim}{s}\right],$$

where $\overline{v}^2$ is the averaged swirl inside the core, and the condition that the pressure should be continuous on the surface is used. The Reynolds stress term contribution is found to be

$$\pi\frac{\partial}{\partial s}\left[-a^2\overline{w}^2\underset{\sim}{s} - a^2\overline{w}\frac{\partial \underset{\sim}{R}}{\partial t} - \frac{2}{\rho}\int_0^a r^2 v_0 w_0\,dr\,\underset{\sim}{b}\right], \qquad (3\text{-}14)$$

where $v_0$ and $w_0$ are the leading order swirl and axial velocities. If we define $\Lambda$ as

$$\Lambda = \frac{2\pi}{a^2\overline{w}\Gamma} \int_0^a r^2 v_0 w_0 \, dr \quad , \tag{3-15}$$

where $\overline{w}$ is the core averaged axial velocity, we have

$$\pi \frac{\partial}{\partial s}\left[- a^2\overline{w}^2 \underset{\frown}{s} - a^2\overline{w} \frac{\partial \underset{\frown}{R}}{\partial t} - \frac{\Lambda\Gamma\overline{w}a^2}{\pi\rho} \underset{\frown}{b}\right] \tag{3-16}$$

as the Reynolds stress term. Finally, the rate of change of momentum in the element $\Delta$ can be found as

$$- \pi a^2\overline{w} \frac{\partial \underset{\frown}{s}}{\partial t} - \pi \underset{\frown}{s} \frac{\partial}{\partial t}(a^2\overline{w}) - \pi a^2\overline{w}\underset{\frown}{s} \frac{\partial}{\partial t}\left(\log \frac{\partial s}{\partial \xi}\right). \tag{3-17}$$

Combining these, the interior force $\underset{\frown}{F}_I$ is given by

$$\underset{\frown}{F}_I = \pi \frac{\partial}{\partial s}\left[\frac{1}{2}a^2\overline{v}^2 \underset{\frown}{s}\right] - \pi \frac{\partial}{\partial s}\left[a^2\overline{w}^2 \underset{\frown}{s} + a^2\overline{w} \frac{\partial \underset{\frown}{R}}{\partial t} + \frac{\Lambda\Gamma\overline{w}a^2}{\rho\pi} \underset{\frown}{b}\right]$$

$$- \pi a^2\overline{w} \frac{\partial \underset{\frown}{s}}{\partial t} - \pi \frac{\partial}{\partial t}(a^2\overline{w})\underset{\frown}{s} - \pi a^2\overline{w} \frac{\partial}{\partial t}\left(\log \frac{\partial s}{\partial \xi}\right) \underset{\frown}{s}. \tag{3-18}$$

The force balance equation (3-3) has now become

$$\Gamma\left(\underset{\frown}{V}_E + \underset{\frown}{V}_I - \frac{\partial \underset{\frown}{R}}{\partial t}\right)\wedge \underset{\frown}{s} + \frac{\Gamma^2}{4\pi p}\left(\log \frac{8\rho}{a} - \frac{1}{2}\right)\underset{\frown}{n} - \frac{\Gamma^2}{8\pi a^2} \frac{\partial a^2}{\partial s} \underset{\frown}{s}$$

$$+ \pi \frac{\partial}{\partial s}\left[\frac{1}{2}a^2\overline{v}^2 \underset{\frown}{s}\right] - \pi \frac{\partial}{\partial s}\left[a^2\overline{w}^2\underset{\frown}{s} + a^2\overline{w} \frac{\partial \underset{\frown}{R}}{\partial t} + \frac{\Lambda\Gamma\overline{w}a^2}{\rho\pi} \underset{\frown}{b}\right]$$

$$- \pi a^2\overline{w} \frac{\partial \underset{\frown}{s}}{\partial t} - \pi \frac{\partial}{\partial t}(a^2\overline{w})\underset{\frown}{s} - \pi a^2\overline{w} \frac{\partial}{\partial t}\left(\log \frac{\partial s}{\partial \xi}\right) \underset{\frown}{s} = 0. \tag{3-19}$$

Before we invert this equation to obtain $\frac{\partial \underset{\sim}{R}}{\partial t}$, let us first check it against some known results. For an isolated vortex ring of radius R and core radius a, we have $\underset{\sim}{V}_E = 0$, $\underset{\sim}{V}_I = 0$, a = constant, $\rho = R =$ constant, $\frac{\partial}{\partial s}\left(\frac{\partial \underset{\sim}{R}}{\partial t}\right) = 0$, and so the velocity is given by

$$\frac{\partial \underset{\sim}{R}}{\partial t} = \frac{\Gamma}{4\pi R}\left(\log \frac{8R}{a} - \frac{1}{2} + \frac{2\pi^2 a^2 \overline{v}^2}{\Gamma^2} - \frac{4\pi^2 a^2 \overline{w}^2}{\Gamma^2}\right) \underset{\sim}{b}\,, \qquad (3\text{-}20)$$

where $\underset{\sim}{b}$ is the binormal, this agrees with the result found by an energy method (Saffman, 1970). Another check is to calculate the dispersion relation of the m = ± 1 modes for the centerline disturbance of a straight vortex by considering the perturbed vortex to be a helix of large pitch given by

$$\underset{\sim}{R} = D \cos\theta\, \underset{\sim}{e}_x + D \sin\theta\, \underset{\sim}{e}_y - \frac{(\theta + \omega t)}{n}\, \underset{\sim}{e}_z. \qquad (3\text{-}12)$$

By a straightforward calculation, using $\Lambda = \frac{1}{4}$, $\Gamma = 2\pi\Omega a^2$, $\overline{v}^2 = \frac{1}{2}\Omega^2 a^2$, and $\overline{w}^2 = W^2$ for a uniform core with top hat profile, we can recover the result (2-32) given by the infinitesimal theory. The important thing to note here is that the present derivation does not require $\frac{D}{a} \ll 1$ anymore, and thus extends the validity of (2-32) to waves that are small but not infinitesimal.

## §9. The "Cutoff" Method

At this point of the analysis we digress to describe an alternative approach known as the cutoff method. The method treats the singularity in the Biot-Savart integral by stopping the integration at a small distance $\ell$ from both sides of the singularity, therefore expressing the result as a function of $\ell$. The value of $\ell$ is then found by comparing to a known result (e.g., the result for the ring or the straight vortex). Let us do this for the perturbed straight vortex. The velocity is given by

$$\frac{\partial \underset{\sim}{R}}{\partial t} = \frac{\Gamma}{4\pi} \int_{[\ell]} \frac{d\underset{\sim}{R}' \wedge (\underset{\sim}{e}_x D - \underset{\sim}{R}')}{|\underset{\sim}{e}_x D - \underset{\sim}{R}'|^3}, \tag{3-22}$$

where the integral is a line integral over the helix given by (3-21) and $[\ell]$ means that a small distance $\ell$ is being cutoff from both sides of the point $\underset{\sim}{R}' = \underset{\sim}{e}_x D$. The frequency $\omega$ at the point $\theta = 0$ (and therefore every point since the origin is arbitrary) is

$$\omega = -\frac{1}{D}\left(\frac{\partial \underset{\sim}{R}}{\partial t}\right) \cdot \underset{\sim}{e}_y - n\left(\frac{\partial \underset{\sim}{R}}{\partial t} \cdot \underset{\sim}{e}_z\right). \tag{3-23}$$

Evaluating (3-22) and assuming $|na| \ll 1$. $\frac{D}{a} \ll 1$, we obtain

$$\frac{\omega}{\Omega} = \frac{n^2 a^2}{2}\left[\tilde{L} + \frac{1}{4} - \log\left(\frac{2\ell}{a}\right)\right]. \tag{3-24}$$

This would agree with (2-29) if

$$\ell = \frac{1}{2}\, e^{\frac{1}{4}}\, a. \tag{3-25}$$

The cutoff method assumes that this value of $\ell$ is the same for thin filaments of all shapes as long as the internal structure remains the same. This can be verified by applying the method to a vortex ring, the use of (3-25) leads to

$$\frac{\partial \underset{\sim}{R}}{\partial t} = \frac{\Gamma}{4\pi R}\left(\log\frac{8R}{a} - \frac{1}{4}\right)\underset{\sim}{b}\ , \tag{3-26}$$

which is the correct value for the speed of a vortex ring with uniform vorticity and no axial flow.

However, the cutoff method is only valid for $|na| \ll 1$ or $\frac{a}{\lambda} \ll \frac{1}{2\pi}$. In order to see this we calculate (3-23) and (2-29) both to the next order in $n^2a^2$, we find that the cutoff method gives a value of $\frac{\omega}{\Omega}$ which exceeds the exact value [found from the next order term in (2-29)] by

$$\frac{n^4a^4}{4}\left[\frac{26}{27}\left(\log\frac{2}{|na|} - C\right)^2 + \frac{55}{54}\left(\log\frac{2}{|na|} - C\right) + \frac{457}{864} - \frac{e^{\frac{1}{2}}}{8}\right]. \tag{3-27}$$

It can also be shown that in the absence of axial velocity the osculating circle method is equivalent to the cutoff method and therefore is also limited to long waves only.

## §10. The Internal Structure of the Core and the Velocity of the Filament

In the case of nonuniform vorticity and axial distributions the structure of the core may change in time and we must have rules to specify the time variation of a, v, w, and $\Lambda$. To leading order, it was found that

$$a = a(t), \tag{3-28}$$

$$v = v(r) = \frac{\Gamma}{2\pi r}\,\Phi\!\left(\frac{r}{a}\right), \tag{3-29}$$

and

$$w = W(t) + W_1(\xi, t) + \frac{\Gamma}{b_0} \Psi(\frac{r}{a}), \tag{3-30}$$

where $\Phi$, $\Psi$ and $b_0$ are determined by initial conditions and chosen so that

$$\Phi(1) = 1, \int_0^1 x \Psi(x)dx = 0. \tag{3-31}$$

It can be shown that these lead to an expression for $\Lambda$:

$$\Lambda = \int_0^1 \left[ x\Phi(x) + \frac{x\Gamma}{Wb_0} \Psi(x)\Phi(x) \right] dx. \tag{3-32}$$

Furthermore, if $\mathcal{L}$ denotes the total length of the filament (or length of a period if the filament is periodic), then the conservation of volume and circulation states that

$$a^2\mathcal{L} = \text{constant}, \tag{3-33}$$

$$W\mathcal{L} = \text{constant}. \tag{3-34}$$

$W_1(\xi, t)$ can be found in terms of $\mathcal{L}$ after a lengthy calculation to be

$$W_1(\xi, t) = - \int_0^s \underset{\sim}{s} \cdot \frac{\partial}{\partial s} (\underset{\sim}{V}_E + \underset{\sim}{V}_I) \, ds - \frac{s}{\mathcal{L}} \oint (\underset{\sim}{V}_E + \underset{\sim}{V}_I) \cdot \frac{\hat{n}}{\rho} ds. \tag{3-35}$$

We can now invert (3-19) by taking the vector product of both sides with $\underset{\sim}{s}$ to obtain

$$\frac{\partial \underset{\sim}{R}}{\partial t} = \underset{\sim}{V}_E + \underset{\sim}{V}_I + \frac{T}{\Gamma\rho}\,\underset{\sim}{b} - \left[\frac{2\pi a^2 W}{\Gamma}\,\underset{\sim}{s}\wedge\overset{\bullet}{\underset{\sim}{s}} + \Lambda W a^2\,\underset{\sim}{s}\wedge\frac{\partial}{\partial s}\left(\frac{\underset{\sim}{b}}{\rho}\right)\right] \qquad (3\text{-}36)$$

where

$$T = \frac{\Gamma^2}{4\pi}\left\{\log\frac{8\rho}{a} - \frac{1}{2} + \int_0^1 \frac{\Phi^2(x)dx}{x} - 4\pi^2\left[\frac{2a^2\int_0^1 x\,\Psi^2(x)dx}{b_0{}^2} + \frac{a^2W^2}{\Gamma^2}\right]\right\}, \qquad (3.37)$$

and

$$\overset{\circ}{\underset{\sim}{s}} = \frac{\partial}{\partial s}\left(\underset{\sim}{V}_E + \underset{\sim}{V}_I + \frac{T}{\Gamma\rho}\,\underset{\sim}{b}\right) - \left(\underset{\sim}{s}\cdot\frac{\partial}{\partial s}\left(\underset{\sim}{V}_E + \underset{\sim}{V}_I\right)\right)\underset{\sim}{s}$$

$$+\left[\frac{s}{\mathcal{L}}\frac{d\mathcal{L}}{dt} - \int^s \underset{\sim}{s}\cdot\frac{\partial}{\partial s}(\underset{\sim}{V}_E + \underset{\sim}{V}_I)ds\right]\frac{\underset{\sim}{n}}{\rho}. \qquad (3\text{-}38)$$

These equations represent a solution of the initial value problem. The actual computation is one of the "marching type" since $\overset{\bullet}{\underset{\sim}{s}}$ is known in terms of $\underset{\sim}{R}$. We have seen that the presence of axial velocity not only changes the cutoff length (by an amount proportional to the last term in (3-37) but also couples the internal dynamics of the core with the motion of the filament [through the terms in [ ] in (3.36)]. We note that this coupling effect cannot be obtained if we apply a straightforward cutoff method, which would give us (3-36) without the coupling terms in [ ].

In the next three chapters we shall apply these equations to study several examples of waves on thin filaments.

## IV. STABILITY OF TRAILING VORTICES WITH AXIAL FLOW

§11. Recent concern over air travel hazard caused by trailing velocities of giant aircraft has increased with the number of giant aircrafts in operation. Lack of reliable information about the breakup of these trailing vortices has led to the necessity of imposing long safety intervals between the successive landings of aircrafts and consequently valuable airport time is lost. The question of stability of trailing vortices is therefore one of considerable practical interest.

Trailing vortices are caused by the rolling up of the sheet vortex shed by the wings of an aircraft, and can be approximated by a pair of straight infinite vortices with strengths $\pm\,\Gamma$ and separation B. The values of $\Gamma$, B and core radius a depend on the characteristics of the wings.

As far back as 1938, observations of the presence of axial velocity in trailing vortices were reported (Hilton, 1938). Recent experiments described in Olsen, Goldberg, and Rogers (1971) confirmed this fact. Theoretical predictions were first made by Batchelor (1964) but were only partially correct. A recent theoretical explanation of the fact that the axial velocity can go in either direction was given by Moore and Saffman (1973). Further references on the topic can be found in a review by El-Ramly (1972).

## §12. The Governing Equations

Suppose the vortex pair is undisturbed. Then each filament experiences the velocity field induced by the other and as a result the pair drifts downwards at a constant speed of $\frac{\Gamma}{2\pi B}$. This effect is called mutual induction. This downward drift is removed in our stability analysis by attaching our coordinate system to the plane containing the unperturbed filaments, as shown in Fig. 4-1. In this coordinate system, the unperturbed filaments are given by

$$\underset{\sim}{R}_m^{\circ} = (x_m,\ (-1)^m \frac{B}{2},\ 0), \tag{4-1}$$

where m = 1 or 2 denoted the two filaments. We let the perturbed filaments take the form

$$\underset{\sim}{R}_m = (x_m,\ (-1)^m \frac{B}{2} + y_m\ e^{i\omega t + inx_m},\ z_m\ e^{i\omega t + inx_m}) \tag{4-2}$$

where $\omega$ is the frequency, n is the wave number per unit length, and $y_m$, $z_m$ are constants which are small compared to the separation B and the wavelength $\lambda = \frac{2\pi}{n}$. The core radius a is also assumed to be small compared to B and $\lambda$ but no restrictions exist between a and $y_m, z_m$.

It suffices to consider $\underset{\sim}{R}_1$ since the motion of $\underset{\sim}{R}_2$ requires only a change in subscripts and the replacement of $\Gamma$ by $-\Gamma$. The equations of motion of $\underset{\sim}{R}_1$ are given by (3-36) - (3-38) with $\underset{\sim}{V}_E$ being the mutual induction. Each of these terms will be evaluated in the next section.

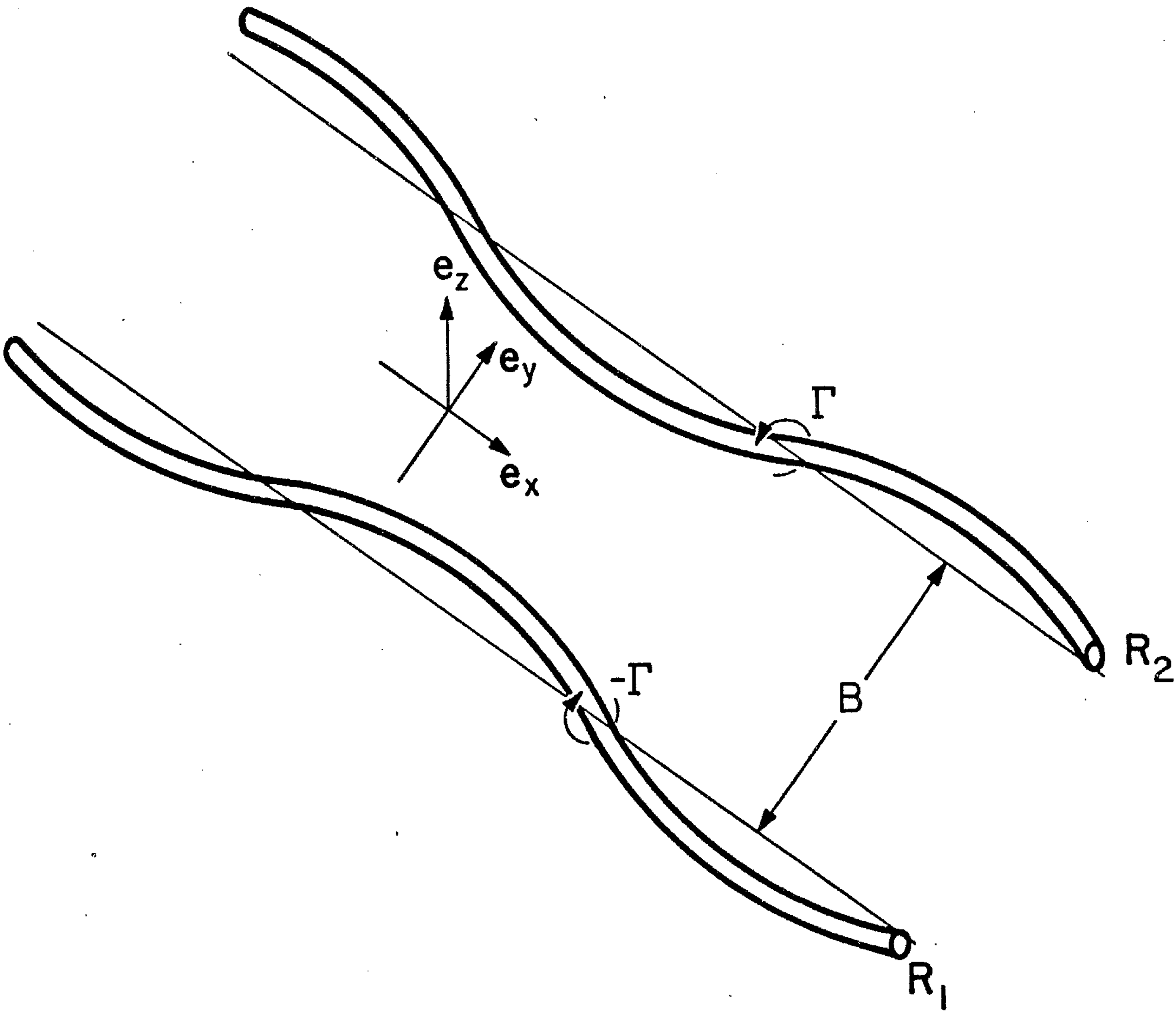



Fig. 4-1

The velocity of the perturbed filament is given by differentiating (4-2) with respect to t:

$$\frac{\partial \underset{\sim}{R}_1}{\partial t} = \omega\, y_1\, e^{i\omega t + inx_1}\, \underset{\sim}{e}_y + \omega\, z_1\, e^{i\omega t + inx_1}\, \underset{\sim}{e}_z . \qquad (4\text{-}3)$$

When this is equated to the expression for $\frac{\partial \underset{\sim}{R}_1}{\partial t}$ calculated from (3.36) - (3.38), a linear eigenvalue problem for $\omega$ is obtained. The existence of an imaginary part in $\omega$ denotes instability, and the stability boundary is given by Im $\omega = 0$.

## §13. The Components of the Velocity

The external velocity $\underset{\sim}{V}_E$ in this case is given by the mutual induction formula

$$\underset{\sim}{V}_E = \frac{\Gamma}{4\pi} \int \frac{(\underset{\sim}{R}_2 - \underset{\sim}{R}_1) \wedge d\underset{\sim}{R}_2}{|\underset{\sim}{R}_2 - \underset{\sim}{R}_1|^3} . \qquad (4\text{-}4)$$

Now

$$(\underset{\sim}{R}_2 - \underset{\sim}{R}_1) = (x_2 - x_1)\, \underset{\sim}{e}_x + \left[B + (y_2\, e^{i\omega t + inx_2} - y_1\, e^{i\omega t + inx_1})\right] \underset{\sim}{e}_y$$

$$+ \left(z_2\, e^{i\omega t + inx_2} - z_1\, e^{i\omega t + inx_1}\right) \underset{\sim}{e}_z, \qquad (4.5)$$

$$d\underset{\sim}{R}_2 = \left(\underset{\sim}{e}_x + iny_2\, e^{i\omega t + inx_2}\, \underset{\sim}{e}_y + inz_2\, e^{i\omega t + inx_2}\, \underset{\sim}{e}_z\right) dx_2, \qquad (4.6)$$

$$|\underset{\sim}{R}_2 - \underset{\sim}{R}_1|^3 = \left[(x_2 - x_1)^2 + B^2 + 2B(y_2\, e^{i\omega t + inx_2} - y_1\, e^{i\omega t + inx_1}) + O(y_1^2)\right]^{3/2}, \tag{4-7}$$

and if we denote $(x_2 - x_1)$ by $x$, we have upon linearization

$$\underset{\sim}{V}_E = -\frac{\Gamma}{4\pi}\int_{-\infty}^{\infty} \frac{B\,dx}{(x^2 + B^2)^{3/2}}\,\underset{\sim}{e}_z + \frac{\Gamma}{4\pi}\, e^{i\omega t + inx_1}\int_{-\infty}^{\infty}\left\{\left[\frac{-nBz_2 \sin nx\, dx}{(x^2 + B^2)^{3/2}}\right]\underset{\sim}{e}_x\right.$$

$$+ \left[\frac{z_2 \cos nx - z_1 + z_2\, nx \sin nx}{(x^2 + B^2)^{3/2}}\, dx\right]\underset{\sim}{e}_y$$

$$\left. + \left[\left(\frac{-y_2 \cos nx + y_1 - nx\, y_2 \sin nx}{(x^2 + B^2)^{3/2}} + \frac{3B^2(y_2 \cos nx - y_1)}{(x^2 + B^2)^{3/2}}\right)dx\right]\underset{\sim}{e}_z\right\}. \tag{4-8}$$

The first term is $O(1)$ and is just the downwash velocity $\frac{\Gamma}{2\pi B}$. The $\underset{\sim}{e}_x$ term represents longitudinal convection and does not enter into the analysis to this order of approximation. The last two terms in the $\underset{\sim}{e}_z$ component can be combined by an integration by parts, and the final expression for the first order terms upon integration becomes

$$\underset{\sim}{V}_E^{(1)} = \frac{\Gamma}{2\pi B^2}\, e^{i\omega t + inx_1}\{[-z_1 + (nBK_1(nB) + n^2B^2K_0(nB))z_2]\underset{\sim}{e}_y$$

$$+ [-y_1 + nBK_1(nB)y_2]\underset{\sim}{e}_z\}, \tag{4-9}$$

where $K_0$ and $K_1$ are modified Bessel functions of the second kind.

Now we turn to $\underset{\sim}{V}_I$, according to (3-7), it is

$$\underset{\sim}{V}_I = \frac{\Gamma}{4\pi}\left\{\int \frac{(\underset{\sim}{R}_1' - \underset{\sim}{R}_1)\wedge d\underset{\sim}{R}_1'}{|\underset{\sim}{R}_1' - \underset{\sim}{R}_1|^3} - \int \frac{(\underset{\sim}{R}^\circ - \underset{\sim}{R}_1)\wedge d\underset{\sim}{R}^\circ}{|\underset{\sim}{R}^\circ - \underset{\sim}{R}_1|^3}\right\}, \quad (4.10)$$

where the second integral is taken over the osculating circle. We can express the first integral as

$$\frac{\Gamma}{2\pi}\, e^{i\omega t + inx_1}\left[\lim_{\delta\to 0}\int_\delta^\infty \frac{(\cos nx + nx\sin nx - 1)}{x^3}\,dx\right](-z_1\underset{\sim}{e}_y + y_1\underset{\sim}{e}_z) \quad (4\text{-}11)$$

which can be evaluated in terms of $\epsilon = n\delta$ to be

$$\frac{\Gamma}{2\pi}\, e^{i\omega t + inx_1}\, n^2\left\{\frac{1}{2}\left[\frac{\cos\epsilon - 1}{\epsilon^2} + \frac{\sin\epsilon}{\epsilon} - \mathrm{Ci}(\epsilon)\right]\right\}(-z_1\underset{\sim}{e}_y + y_1\underset{\sim}{e}_z) \quad (4\text{-}12)$$

with Ci as the integral cosine. Expanding for small $\epsilon$ leads to

$$\frac{\Gamma}{4\pi}\, e^{i\omega t + inx_1}\, n^2\left\{-\log\epsilon - C + \frac{1}{2}\right\}(-z_1\underset{\sim}{e}_y + y_1\underset{\sim}{e}_z) \quad (4\text{-}13)$$

where we recall C as being Euler's constant. The second integral over the osculating circle can also be expressed in terms of $\delta$ as $\delta$ approaches zero as

$$\frac{\Gamma}{4\pi}\, e^{i\omega t + inx_1}\, n^2\left\{\log\frac{4\rho}{\delta}\right\}(-z_1\underset{\sim}{e}_y + y_1\underset{\sim}{e}_z)\,. \quad (4\text{-}14)$$

The radius of curvature $\rho$ in this case has been found to be

$$\rho = \frac{1}{n^2\sqrt{y_1^2 + z_1^2}}. \quad (4\text{-}15)$$

Subtracting (4-14) from (4-13) yields

$$\underset{\sim}{V}_I = \frac{\Gamma}{4\pi} e^{i\omega t + inx_1} n^2 \left[\frac{1}{2} - C - \log 4 + \log n + \log\sqrt{y_1^2 + z_1^2}\right] (-z_1 \underset{\sim}{e}_y + y_1 \underset{\sim}{e}_z) . \tag{4-16}$$

To calculate the "tension" term, we first write down the binormal $\underset{\sim}{b}$:

$$\underset{\sim}{b} = \frac{(-z_1 \underset{\sim}{e}_y + y_1 \underset{\sim}{e}_z) e^{i\omega t + inx_1}}{\sqrt{y_1^2 + z_1^2}}, \tag{4-17}$$

then we have

$$\begin{aligned}\left(\frac{T}{\Gamma\rho} \underset{\sim}{b}\right) &= \frac{\Gamma}{4\pi} e^{i\omega t + inx_1} n^2 \left(\log \frac{8\rho}{a} - \frac{1}{2} + A\right) (-z_1 \underset{\sim}{e}_y + y_1 \underset{\sim}{e}_z) \\ &= \frac{\Gamma}{4\pi} e^{i\omega t + inx_1} n^2 \Big(\log 8 - 2 \log n - \log \sqrt{y_1^2 + z_1^2} \\ &\qquad - \log a - \frac{1}{2} + A\Big) (-z_1 \underset{\sim}{e}_y + y_1 \underset{\sim}{e}_z) ,\end{aligned} \tag{4-18}$$

where

$$A = \int_0^1 \Phi \frac{(x)}{x} dx - 4\pi^2 \left(\frac{2a^2 \int_0^1 \Psi^2(x) x \, dx}{b_0^2} + \frac{a^2 W^2}{\Gamma^2}\right) \tag{4-19}$$

is a dimensionless parameter describing the internal structure of the vortex. Combining (4-18) and (4-16) we have

$$\underset{\sim}{V}_I + \frac{T}{\Gamma\rho}\underset{\sim}{b} = \frac{\Gamma n^2}{4\pi}\left(\log\frac{2}{na_e} - C + \frac{1}{4}\right)\left(-z_1\underset{\sim}{e}_y + y_1\underset{\sim}{e}_z\right), \qquad (4\text{-}20)$$

with

$$a_e = a\exp\left[\frac{1}{4} - \frac{2\pi^2 a^2}{\Gamma^2}(\overline{v}^2 - 2\overline{w}^2)\right]. \qquad (4\text{-}21)$$

At this point we note that for a uniform vortex with no axial flow the result in (4-20) is consistent with Kelvin's result (2-29) for an isolated straight filament.

The remaining two terms in (3-36) which explicitly involves W can be easily calculated by using the expressions found for $\underset{\sim}{V}_I$, $\underset{\sim}{V}_E$ and T. They are

$$e^{i\omega t + inx_1}\ \text{in}\ \frac{a^2W}{B^2}\left\{\left[y_1 - nBK_1(nB)y_2 - \frac{1}{2}n^2B^2\Theta y_1 + n^2B^2\Lambda y_1\right]\underset{\sim}{e}_y\right.$$
$$\left. + \left[z_1 - (nBK_1(nB) + n^2B^2K_0(nB))z_2 + \frac{1}{2}n^2B^2\Theta z_1 - n^2B^2\Lambda z_1\right]\underset{\sim}{e}_z\right\}, \qquad (4\text{-}22)$$

where we have defined $\Theta$ to be

$$\Theta = \log\frac{2}{|na_e|} - C + \frac{1}{4}. \qquad (4\text{-}23)$$

Collecting all the terms, the first order term of $\frac{\partial \underset{\sim}{R}_1}{\partial t}$, which we denote by

$$\frac{\partial \underset{\sim}{R}_1^{(1)}}{\partial t}$$

is given by

$$\frac{\partial \underset{\sim}{R}_1^{(1)}}{\partial t} = e^{i\omega t + inx_1}\left\{\frac{\Gamma}{2\pi B^2}\left[-z_1 + (nBK_1(nB) + n^2B^2K_0(nB))z_2 - n^2B^2\frac{\Theta}{2}z_1\right]\right.$$

$$\left. + in\frac{a^2W}{B^2}\left[y_1 - nBK_1(nB)y_2 - \frac{1}{2}n^2B^2(\Theta - 2\Lambda)y_1\right]\right\}\underset{\sim}{e}_y$$

$$+ e^{i\omega t + inx_1}\left\{\frac{\Gamma}{2\pi B^2}\left[-y_1 + nBK_1(nB)y_2 + n^2B^2\frac{\Theta}{2}y_1\right]\right.$$

$$\left. + in\frac{a^2}{B^2}W\left[z_1 - (nBK_1(nB) + n^2B^2K_0(nB))z_2 + \frac{1}{2}n^2B^2(\Theta - 2\Lambda)z_2\right]\right\}\underset{\sim}{e}_z. \tag{4-24}$$

A similar expression is found for $\frac{\partial \underset{\sim}{R}_2^{(1)}}{\partial t}$ except that the subscripts are interchanged and $\Gamma$ is replaced by $-\Gamma$.

## §14. The Eigenvalue Equations and Determinations of Stability Boundary

Equating the expression for $\frac{\partial \underset{\sim}{R}_m}{\partial t}$ from (4-3) to that from (4-24) and the corresponding result for $\frac{\partial R_2}{\partial t}$ we obtain

$$i\omega y_1 = \frac{\Gamma}{2\pi B^2}\left\{-z_1 + (nBK_1(nB) + n^2B^2K_0(nB))z_2 - n^2B^2\frac{\Theta}{2}z_1\right\}$$

$$+ in\frac{a^2W}{B^2}\left\{y_1 - nBK_1(nB)y_2 - \frac{1}{2}n^2B^2(\Theta - 2\Lambda)y_1\right\}, \tag{4-25}$$

$$i\omega z_1 = \frac{\Gamma}{2\pi B^2}\left\{-y_1 + nBK_1(nB)y_2 + n^2B^2\frac{\Theta}{2}y_1\right\}$$

$$+ in\frac{a^2W}{B^2}\left\{z_1 - (nBK_1(nB) + n^2B^2K_0(nB))z_2 + \frac{1}{2}n^2B^2(\Theta - 2\Lambda)z_1\right\}, \tag{4-26}$$

$$i\omega y_2 = \frac{\Gamma}{2\pi B^2}\left\{z_2 - (nBK_1(nB) + n^2B^2K_0(nB))z_1 + n^2B^2\frac{\Theta}{2}z_2\right\}$$

$$+ \, in\frac{a^2}{B^2}W\left\{y_2 - nBK_1(nB)y_1 - \frac{1}{2}n^2B^2(\Theta - 2\Lambda)y_2\right\}, \qquad (4\text{-}27)$$

$$i\omega z_2 = \frac{\Gamma}{2\pi B^2}\left\{y_2 - nBK_1(nB)y_1 - n^2B^2\frac{\Theta}{2}y_2\right\}$$

$$+ \, in\frac{a^2W}{B^2}\left\{z_2 - (nBK_1(nB) + n^2B^2K_0(nB))z_1 + \frac{1}{2}n^2B^2(\Theta - 2\Lambda)z_2\right\}. \qquad (4\text{-}28)$$

This fourth order system of algebraic equations for $\omega$ can be simplified by introducing the symmetric and antisymmetric modes defined as (see fig. 4-2)

$$y_S = y_2 - y_1, \qquad z_S = z_2 + z_1, \qquad (4\text{-}29)$$

$$y_A = y_2 + y_1, \qquad z_A = z_2 - z_1. \qquad (4\text{-}30)$$

In terms of $y_S$, $z_S$, $y_A$, $z_A$ the equations decouple to become

$$\left.\begin{aligned} i\hat{\omega}y_S &= \left(1 - (\beta K_1 + \beta^2K_0) + \beta^2\frac{\Theta}{2}\right)z_S + in\hat{W}\left\{1 - \beta K_1 - \tfrac{1}{2}\beta^2(\Theta - 2\Lambda)\right\}y_S \\ i\hat{\omega}z_S &= \left(1 + \beta K_1 - \beta^2\frac{\Theta}{2}\right)y_S + in\hat{W}\left\{1 - (\beta K_1 + \beta^2K_0) + \tfrac{1}{2}\beta^2(\Theta - 2\Lambda)\right\}z_S \end{aligned}\right\} (4\text{-}31)$$

$$\left.\begin{aligned} i\hat{\omega}y_A &= \left(1 + (\beta K_1 + \beta^2K_0) + \beta^2\frac{\Theta}{2}\right)z_A + in\hat{W}\left\{1 - \beta K_1 - \tfrac{1}{2}\beta^2(\Theta - 2\Lambda)\right\}y_A \\ i\hat{\omega}z_A &= \left(1 - \beta K_1 - \beta^2\frac{\Theta}{2}\right)y_A + in\hat{W}\left\{1 - (\beta K_1 + \beta^2K_0) + \tfrac{1}{2}\beta^2(\Theta - 2\Lambda)\right\}z_A \end{aligned}\right\}$$

$$(4.32)$$

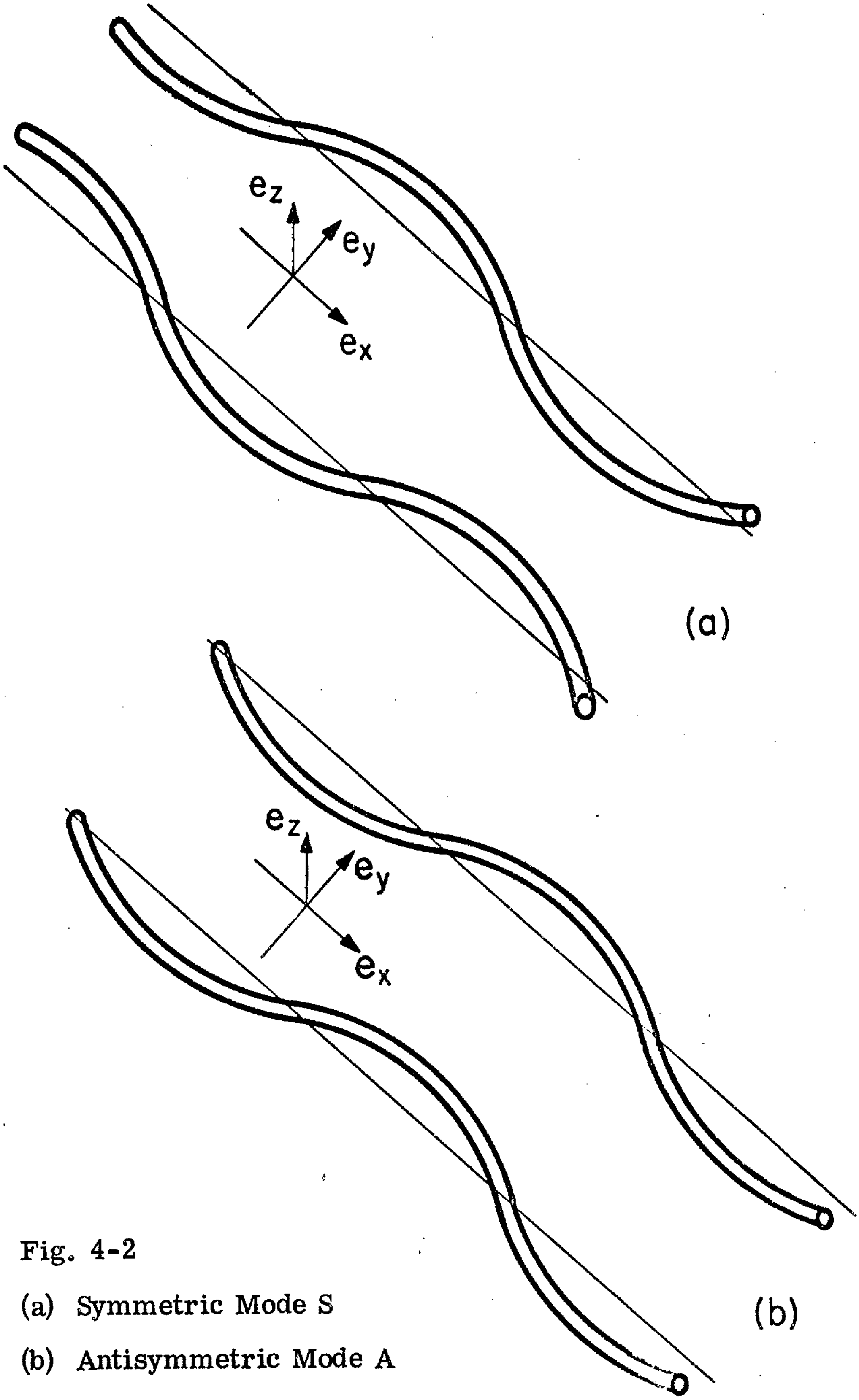



Fig. 4-2

(a) Symmetric Mode S

(b) Antisymmetric Mode A

where $\hat{\omega} = \omega \cdot \frac{2\pi B^2}{\Gamma}$, $\beta = nB$ and $\hat{W} = \frac{2\pi a^2}{\Gamma} W$. In order that $y_s$, $z_s$ are not identically zero, $\hat{\omega}$ must satisfy the determinant equation

$$\det \left[ \begin{array}{c|c} \hat{\omega} - n\hat{W}\left\{1 - \beta K_1 - \frac{1}{2}\beta^2(\Theta - 2\Lambda)\right\} & -i\left(1 - (\beta K_1 + \beta^2 K_0) + \beta^2 \frac{\Theta}{2}\right) \\ \hline -i\left(1 + \beta K_1 - \beta^2 \frac{\Theta}{2}\right) & \hat{\omega} - n\hat{W}\left\{1 - (\beta K_1 + \beta^2 K_0) + \frac{1}{2}\beta^2(\Theta - 2\Lambda)\right\} \end{array} \right] = 0, \tag{4-33}$$

for the symmetric mode, and a similar equation

$$\det \left[ \begin{array}{c|c} \hat{\omega} - n\hat{W}\left\{1 - \beta K_1 - \frac{1}{2}\beta^2(\Theta - 2\Lambda)\right\} & - i\left(1 - (\beta K_1 + \beta^2 K_0) + \beta^2 \frac{\Theta}{2}\right) \\ \hline - i\left(1 - \beta K_1 - \beta^2 \frac{\Theta}{2}\right) & \hat{\omega} - n\hat{W}\left\{1 - (\beta K_1 + \beta^2 K_0) + \frac{1}{2}\beta^2(\Theta - 2\Lambda)\right\} \end{array} \right] = 0 \tag{4-34}$$

for the antisymmetric mode. The stability boundaries for these modes are the values of $\beta$ for which Im $\hat{\omega} = 0$ in (4.33) and (4.34).

Let us first consider the case when W = 0. Then (4-33) and (4-34) reduce to

$$\hat{\omega}^2 + \left(1 + \beta K_1 - \beta^2 \frac{\Theta'}{2}\right) \left(1 - (\beta K_1 + \beta^2 K_0) + \beta^2 \frac{\Theta'}{2}\right) = 0 \tag{4-31}$$

and

$$\hat{\omega}^2 + \left(1 - \beta K_1 - \beta^2 \frac{\Theta'}{2}\right) \left(1 + (\beta K_1 + \beta^2 K_0) + \beta^2 \frac{\Theta'}{2}\right) = 0 \tag{4-32}$$

respectively, where

$$\Theta' = \log \frac{2}{|na|} - C + \frac{1}{4}. \qquad (4\text{-}33)$$

This agrees with the result obtained by Crow (1970). The stability boundaries for the symmetric and antisymmetric modes are given by

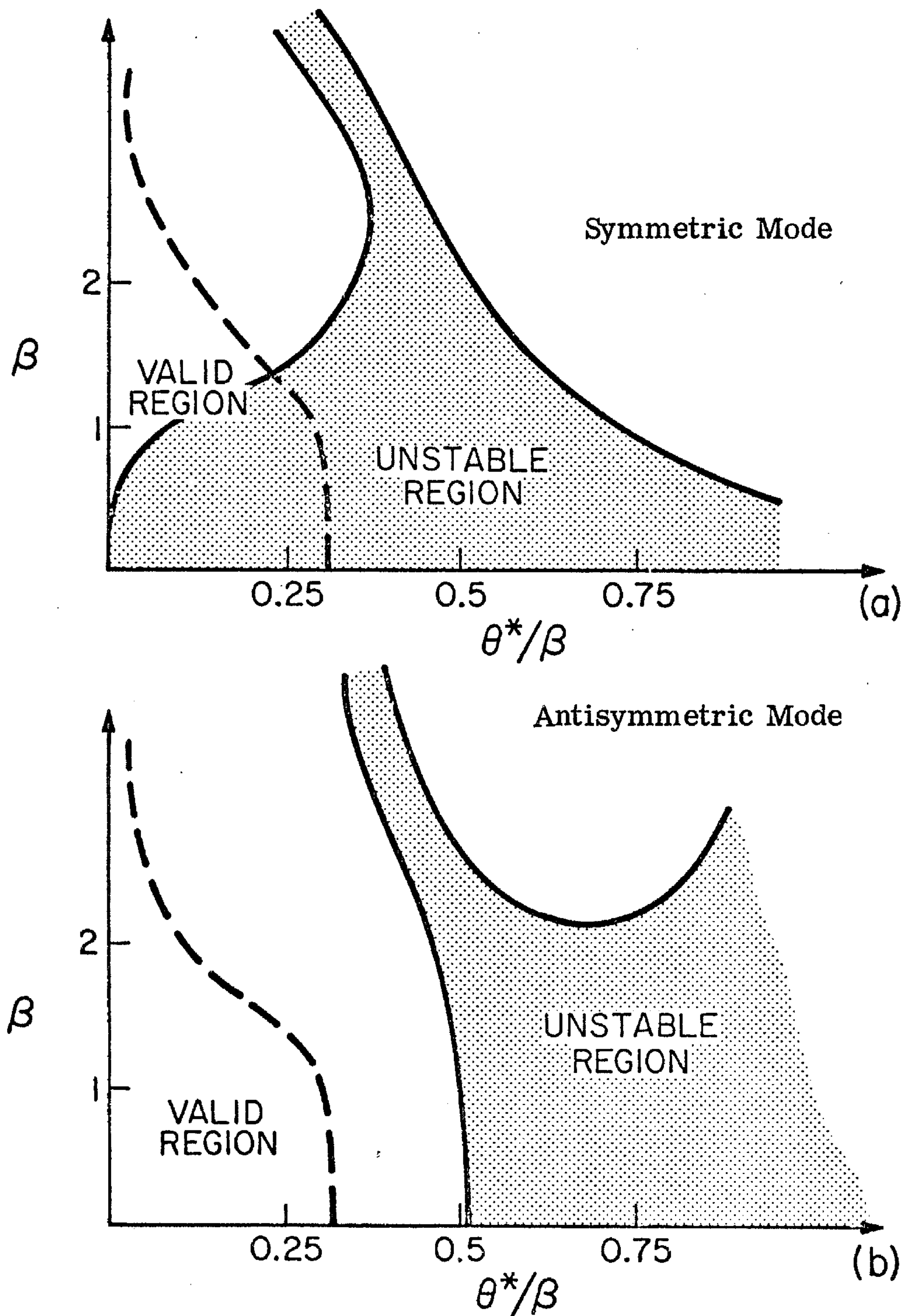
$$\left(1 + \beta K_1 - \beta^2 \frac{\Theta'}{2}\right) \cdot \left(1 - (\beta K_1 + \beta^2 K_0) + \beta^2 \frac{\Theta'}{2}\right) = 0, \qquad (4\text{-}34)$$

$$\left(1 - \beta K_1 - \beta^2 \frac{\Theta'}{2}\right) \cdot \left(1 + (\beta K_1 + \beta^2 K_0) + \beta^2 \frac{\Theta'}{2}\right) = 0. \qquad (4\text{-}35)$$

These are exhibited in Fig. 4-3 [after Crow (1970)] on a plot of $\beta$ vs $\frac{a}{B}$. The most unstable mode ($\beta$ for which Im $\hat{\omega}$ is a maximum) are represented by the dotted line. The value of $\frac{a}{B}$ depends on the characteristics of the wing. For elliptic loading, Spreiter and Sacks (1951) estimated that

$$\frac{a}{B} \doteq 0.197. \qquad (4\text{-}36)$$

This value was used by Crow to obtain the result that only the symmetric mode can be unstable and that the most unstable wavelength is 7.2 B (Crow has made an algebraic mistake which led him to the value 8.6B). However, this estimation of Spreiter and Sacks is suspicious. A recent theory of vortex roll-up suggests a time dependent core radius (Moore and Saffman, 1973), but this will not be treated here.




$\theta^* = na$

Fig. 4-3

Let us now examine the effect of axial velocity. The main consequence of a nonzero W is to replace a by $a_e$ (or $\Theta'$ by $\Theta$). Only to a smaller order [order $\frac{a^2W}{\Gamma}$, or O(a) since W is assumed to be $O(\Gamma/a)$] that the terms explicitly containing W alter the stability boundary. The most interesting effect is that a travelling wave is induced with speed given by

$$c = \frac{a^2}{B^2} W \left(2\beta K_1 + \beta^2 K_0 - \beta^2(\Theta - 2\Lambda)\right). \tag{4-37}$$

A similar result has been obtained by Widnall and Bliss (1971) but they have left out the term $2n^2a^2W\Lambda$ due to an incomplete estimation of the Reynolds stress term (3-14). Parks (1971) gave a travelling wave speed of O(W) which is in error, but it has been corrected in a private communication.

Thus we can conclude that axial velocity does not drastically alter the stability criteria for trailing vortices as far as the simple straight line vortex pair model is concerned. It will be interesting to repeat the analysis with a more sophisticated roll-up model like that given by Moore and Saffman (1973).

## V. The Stability of Steady Vortex Rings

§15. The vortex atom theory of Lord Kelvin stated that atoms are actually vortex rings in a perfect fluid (or ether). For this idea to be plausible, these rings must maintain their identities and therefore it is necessary for these rings to have a stable steady motion when no exterior forces are present. Kelvin (1867) stated without proof the first step towards this condition: that a thin isolated vortex ring in an infinite ideal fluid moves without change of shape with a steady velocity given by

$$\frac{\Gamma}{4\pi R}\left(\log\frac{8R}{a} - \frac{1}{4}\right)\underset{\sim}{b}, \tag{5-1}$$

where $\underset{\sim}{b}$ is the unit binormal and the ring is taken to have uniform vorticity. The next question is the stability of this steady motion. This was studied by J. J. Thomson in 1883 in his Adam Prize winning paper "A Treatise on the Motion of Vortex Rings," which showed that vortex rings are stable to small sinusoidal disturbances of the centerline. A question was raised on the validity of Thomson's work because his value for the speed of the ring, being

$$\frac{\Gamma}{4\pi R}\left(\log\frac{8R}{a} - 1\right)\underset{\sim}{b}, \tag{5-2}$$

did not agree with Kelvin's result. Subsequent investigations by Hicks (1884), Dyson (1893), Pocklington (1895), Gray (1912), Lamb (1932), Fraenkel (1970) and Saffman (1970) showed that Kelvin's value was correct, and that Thomson's error was due to an invalid approximation. It appears, however, that this error should not affect the stability criterion.

Quite recently, Widnall and Sullivan (1973) repeated Thomson's analysis using a rigorized cutoff method and obtained very similar results, but when a numerical evaluation was made on the stability conditions, some short wave instabilities were found. This result will be examined in detail later in the chapter.

Three cases of the motion and stability of steady vortex rings are considered in this chapter. They are: a ring with axial velocity, an electrically charged conducting vortex ring, and a uniformly charged vortex ring. The latter two cases are intended as models for vortices in liquid helium.

## §16. The Velocity of a Perturbed Vortex Ring with Axial Flow

For an isolated ring $\underset{\sim}{V}_E = 0$, and it was shown by Moore and Saffman (1972) that when equations (3-36) - (3-38) are applied to an unperturbed ring, we obtain its velocity as

$$\frac{\partial \underset{\sim}{R}}{\partial t} = \frac{\Gamma}{4\pi R}\left(\log\frac{8R}{a} - \tfrac{1}{2} + A\right)\underset{\sim}{b}, \qquad (5\text{-}3)$$

where A is defined in (4-19). In the case of uniform vorticity and no axial flow, A is $\frac{1}{4}$ and the result agrees with that of Kelvin.

Let $(\underset{\sim}{e}_x, \underset{\sim}{e}_y, \underset{\sim}{e}_z)$ be a coordinate system attached to the center of the moving unperturbed ring. On each point of the unperturbed ring, we define a local orthogonal system $(\underset{\sim}{e}_r, \underset{\sim}{e}_\theta, \underset{\sim}{e}_z)$ as shown in Fig. 5-1. This system is dependent on the location of the point and therefore is a function of $\theta$. In these coordinate systems, the perturbed ring can be expressed as

$$R = \left(R + r_1 e^{in\theta}\right)\cos\theta\, \underset{\sim}{e}_x + \left(R + r_1 e^{in\theta}\right)\sin\theta\, \underset{\sim}{e}_y + z_1 e^{in\theta}\, \underset{\sim}{e}_z$$

$$= \left(R + r_1 e^{in\theta}\right) \underset{\sim}{e}_r + z_1 e^{in\theta}\, \underset{\sim}{e}_z \;, \qquad (5\text{-}4)$$

where the perturbing quantities $r_1$ and $z_1$ are small compared to R.

We shall now calculate the velocity at a point of the perturbed ring, we recall that $\underset{\sim}{V}_I$ has been determined as

$$\underset{\sim}{V}_I = \frac{\Gamma}{4\pi}\left\{\int \frac{(\underset{\frown}{R}' - \underset{\frown}{R})\wedge d\underset{\frown}{R}'}{|\underset{\frown}{R}' - \underset{\frown}{R}|^3} - \int \frac{(\underset{\frown}{R}^\circ - \underset{\frown}{R})\wedge d\underset{\frown}{R}^\circ}{|\underset{\frown}{R}^\circ - \underset{\frown}{R}|^3}\right\} \qquad (5\text{-}5)$$

where the first integral is taken over the perturbed ring and the second over the osculating circle. If we let $\bar{\theta} = \theta' - \theta$, which is the angle between $\underset{\frown}{R}'$ and $\underset{\frown}{R}$, we have

$$\underset{\frown}{R}' - \underset{\frown}{R} = \left[R(\cos\bar{\theta} - 1) + r_1 e^{in\theta}\left(\cos\bar{\theta}\, e^{in\bar{\theta}} - 1\right)\right]\underset{\sim}{e}_r$$

$$+\left[R\sin\bar{\theta} + r_1 e^{in\theta}\sin\bar{\theta}\, e^{in\bar{\theta}}\right]\underset{\sim}{e}_\theta + z_1 e^{in\theta}\left(e^{in\bar{\theta}} - 1\right)\underset{\sim}{e}_z, \qquad (5\text{-}6)$$

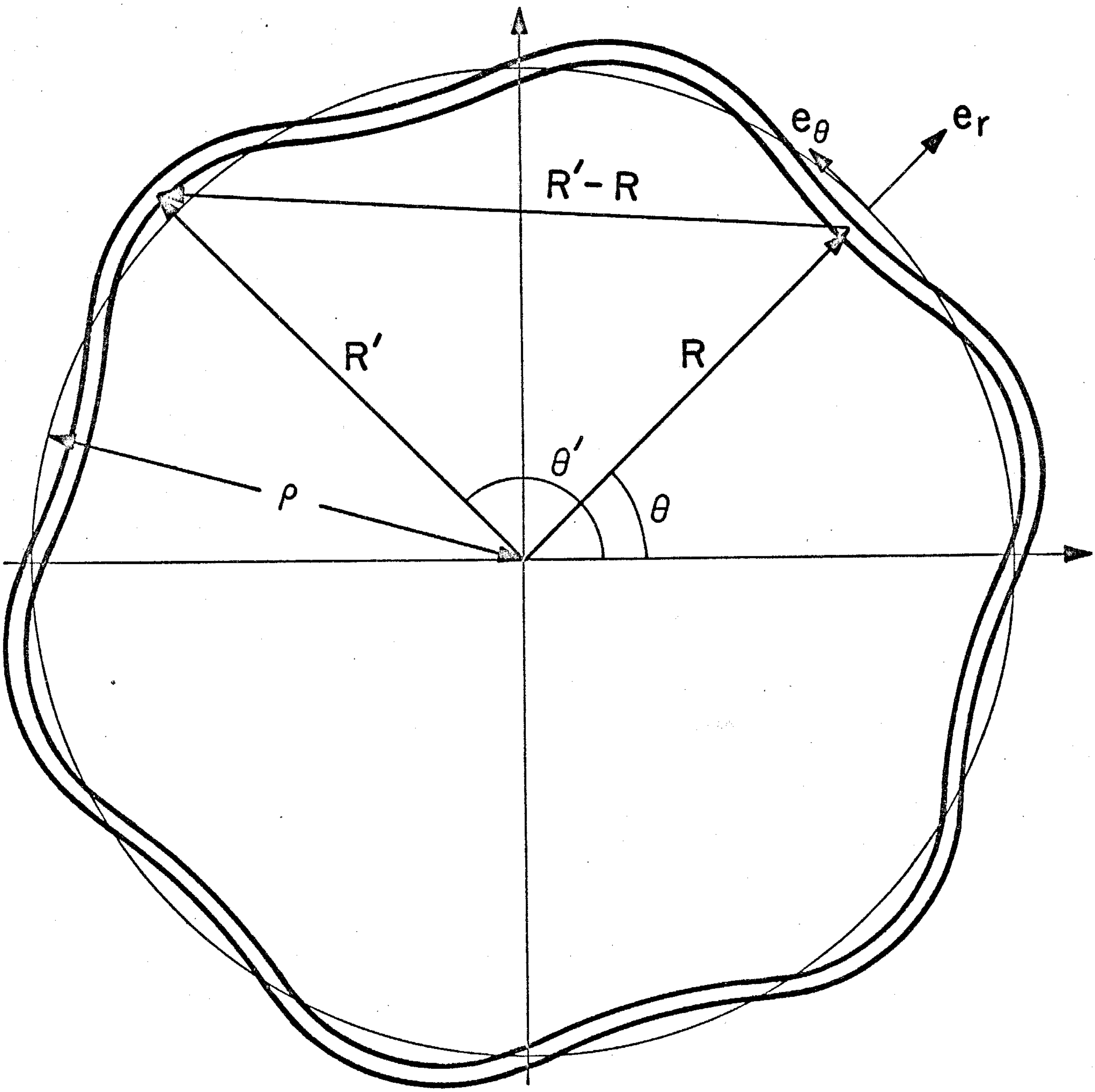



Fig. 5-1

$$d\underset{\sim}{R}' = \frac{d\underset{\sim}{R}'}{d\bar{\theta}}\, d\bar{\theta}$$

$$= \left\{\left[-R\sin\bar{\theta} + r_1 e^{in\theta}(-\sin\bar{\theta} + in\cos\bar{\theta})e^{in\bar{\theta}}\right]\underset{\sim}{e}_r\right.$$

$$\left. + \left[R\cos\bar{\theta} + r_1 e^{in\theta}(\cos\bar{\theta} + in\sin\bar{\theta})e^{in\bar{\theta}}\right]\underset{\sim}{e}_\theta + z_1 e^{in\theta}\left(ine^{in\bar{\theta}}\right)\underset{\sim}{e}_z\right\} d\bar{\theta}, \tag{5-7}$$

$$|\underset{\sim}{R}' - \underset{\sim}{R}|^{3} = R^{-3}\left[2(1-\cos\bar{\theta})\right]^{-\frac{3}{2}}\left\{1 - \frac{3}{2}\frac{r_1}{R}e^{in\theta}\left(1 + e^{in\bar{\theta}}\right) + O\left(\frac{r_1^2}{R^2}\right)\right\}, \tag{5-8}$$

so that the first integral in (5-4) becomes

$$\frac{\Gamma}{4\pi R}\left[\int \frac{(1-\cos\bar{\theta})\,d\bar{\theta}}{[2(1-\cos\bar{\theta})]^{3/2}}\right]\underset{\sim}{e}_z + \frac{\Gamma}{4\pi R}\frac{r_1}{R}e^{in\theta}\left[\int\left(\frac{e^{in\bar{\theta}}(1-\cos\bar{\theta}) + (e^{in\bar{\theta}} - \cos\bar{\theta})}{[2(1-\cos\bar{\theta})]^{3/2}}\right.\right.$$

$$\left.\left. - \frac{in\sin\bar{\theta}\, e^{in\bar{\theta}}}{[2(1-\cos\bar{\theta})]^{3/2}} - \frac{3}{2}\frac{(1+e^{in\bar{\theta}})(1-\cos\bar{\theta})}{[2(1-\cos\bar{\theta})]^{3/2}}\right)d\bar{\theta}\right]\underset{\sim}{e}_z$$

$$+ \frac{\Gamma}{4\pi R}\frac{z_1}{R}e^{in\theta}\left[\int \frac{\cos\bar{\theta}(1-e^{in\bar{\theta}}) + in\sin\bar{\theta}\, e^{in\bar{\theta}}}{[2(1-\cos\bar{\theta})]^{3/2}}\,d\bar{\theta}\right]\underset{\sim}{e}_r. \tag{5-9}$$

The osculating circle lies in the plane formed by $\underset{\sim}{n}$ and $\underset{\sim}{s}$, thus the components of the second integral are

$$\underset{\sim}{R}^{\circ} - \underset{\sim}{R} = \rho(\cos\bar{\theta} - 1)(-\underset{\sim}{n}) + \rho\sin\bar{\theta}\,\underset{\sim}{s}, \tag{5-10}$$

$$d\underset{\sim}{R}^{\circ} = (-\rho \sin\bar{\theta})(-\underset{\sim}{n}) + \rho\cos\bar{\theta}\,\underset{\sim}{s}, \tag{5-11}$$

$$\left|\underset{\sim}{R}^{\circ} - \underset{\sim}{R}\right|^{-3} = \rho^{-3}\,[2(1-\cos\bar{\theta})]^{-3/2}, \tag{5-12}$$

where $\underset{\sim}{s}$ and $\underset{\sim}{n}$ are the local tangent and principle normal at the point $\theta$, and $\rho$ is the local radius of curvature given by

$$\rho = R\left(1 - (n^2-1)\frac{r_1}{R}\,e^{in\theta}\right). \tag{5-13}$$

Therefore, the $\underset{\sim}{e}_r$ and $\underset{\sim}{e}_z$ components of the second integral in (5-5) are

$$\underset{\sim}{e}_r : \frac{\Gamma}{4\pi R}\int\frac{(1-\cos\bar{\theta})}{[2(1-\cos\bar{\theta})]^{3/2}}\,d\bar{\theta}\left(-n^2\frac{z_1}{R}\,e^{in\theta}\right), \tag{5-14}$$

$$\underset{\sim}{e}_z : \frac{\Gamma}{4\pi R}\int\frac{(1-\cos\bar{\theta})}{[2(1-\cos\bar{\theta})]^{3/2}}\,d\bar{\theta}\left(1 + (n^2-1)\frac{r_1}{R}\,e^{in\theta}\right). \tag{5-15}$$

The definition of $\underset{\sim}{V}_I$ requires it to be regular. This provides a check on our calculations. If we expand (5-9) near $\bar{\theta} = 0$ and retain only the singular terms, we would get

$$\frac{\Gamma}{4\pi R}\frac{z_1}{R}\,e^{in\theta}\left[-n^2\int\frac{d\bar{\theta}}{\bar{\theta}}\right]\underset{\sim}{e}_r + \frac{\Gamma}{4\pi R}\left[\left(1 + (n^2-1)\frac{r_1}{R}\,e^{in\theta}\right)\int\frac{d\bar{\theta}}{\bar{\theta}}\right]\underset{\sim}{e}_z. \tag{5-16}$$

It is clear that this possesses the same singularities as the expressions in (5-14) and (5-15) and so the difference will be nonsingular. Thus we have

$$\underset{\sim}{V}_I = z_1\, e^{in\theta}\left\{\frac{\Gamma}{4\pi R^2}\int \frac{\cos\bar\theta(1-e^{in\bar\theta}) + in\sin\bar\theta\, e^{in\bar\theta} + n^2(1-\cos\bar\theta)}{[2(1-\cos\bar\theta)]^{3/2}}\, d\bar\theta\right\}\underset{\sim}{e}_r$$

$$+ r_1\, e^{in\theta}\left\{\frac{\Gamma}{4\pi R^2}\int\left[e^{in\bar\theta}(1-\cos\bar\theta) + (e^{in\bar\theta} - \cos\bar\theta) - in\sin\bar\theta\, e^{in\bar\theta}\right.\right.$$

$$\left.\left. -\frac{3}{2}\left(1-e^{in\bar\theta}\right)(1-\cos\bar\theta) - (1-\cos\bar\theta)(n^2-1)\right]\frac{d\bar\theta}{[2(1-\cos\bar\theta)]^{3/2}}\right\}\underset{\sim}{e}_z, \qquad (5\text{-}17)$$

and by denoting the terms in { } as $U_z$ and $U_r$, we can write

$$\underset{\sim}{V}_I = z_1\, e^{in\theta}\; U_z\, \underset{\sim}{e}_r + r_1\, e^{in\theta}\; U_r\, \underset{\sim}{e}_z . \qquad (5\text{-}18)$$

With $\rho$ given by (5-13), the term $\frac{T}{\Gamma\rho}\, \underset{\sim}{b}$ is calculated to have $\underset{\sim}{e}_r$ and $\underset{\sim}{e}_z$ components given by

$$\underset{\sim}{e}_r :\ \frac{\Gamma}{4\pi R}\left\{\log\frac{8R}{a} - \tfrac{1}{2} + A\right\}\left(-\, n^2\, \frac{z_1}{R}\, e^{in\theta}\right) \equiv z_1\, e^{in\theta}\; T_z , \qquad (5\text{-}19)$$

$$\underset{\sim}{e}_z :\ \frac{\Gamma}{4\pi R}\left\{\log\frac{8R}{a} - \tfrac{1}{2} + A\right\} + \frac{r_1}{R}\, e^{in\theta}\, \frac{\Gamma}{4\pi R}\left\{(n^2-1)\left(\log\frac{8R}{a} - \tfrac{1}{2} + A\right)\right\}$$

$$\equiv V_0 + r_1\, e^{in\theta}\; T_r , \qquad (5\text{-}20)$$

where we recognize $V_0$ as the velocity of the unperturbed ring.

We are now left with the two terms involving W explicitly, a straightforward calculation gives them as

$$\left[-\frac{2\pi a^2}{\Gamma} W\, in\, r_1\, e^{in\theta} (U_r + T_r) - \Lambda W a^2 \left(in(n^2 - 1) \frac{r_1}{R^2} e^{in\theta}\right)\right] \underset{\sim}{e}_r$$

$$+ \left[\frac{2\pi a^2}{\Gamma} W\, in\, z_1\, e^{in\theta} (U_z + T_z) - \Lambda W a^2 \left(in^3 \frac{z_1}{R^2} e^{in\theta}\right)\right] \underset{\sim}{e}_z, \qquad (5\text{-}21)$$

where $\Lambda$, we recall, is given by (3-15). Combining these terms, we have the velocity of the perturbed ring

$$\frac{\partial \hat{R}}{\partial t} = V_0\, \underset{\sim}{e}_z + \left\{z_1 e^{in\theta}(U_z + T_z) - r_1 e^{in\theta}\left[\frac{in 2\pi a^2 W(U_r + T_r)}{\Gamma R} + \frac{in(n^2-1)\Lambda W a^2}{R^3}\right]\right\}\underset{\sim}{e}_r$$

$$+ \left\{r_1 e^{in\theta}(U_r + T_r) + z_1 e^{in\theta}\left[\frac{in 2\pi a^2 W(U_z + T_z)}{\Gamma R} - \frac{in^3 \Lambda W a^2}{R^3}\right]\right\}\underset{\sim}{e}_z. \qquad (5\text{-}22)$$

## §17. Stability of a Vortex Ring with Axial Flow

From equation (5-4), $\frac{\partial \hat{R}}{\partial t}$ in the moving frame is given by

$$\frac{\partial \hat{R}}{\partial t} = \frac{dr_1}{dt} e^{in\theta}\, \underset{\sim}{e}_r + \frac{dz_1}{dt} e^{in\theta}\, \underset{\sim}{e}_z, \qquad (5\text{-}22)$$

and by equating this to the first order terms in (5-22) we obtain

$$\frac{dr_1}{dt} = (U_z + T_z)\, z_1 - \left(\frac{in\, 2\pi a^2 W(U_r + T_r)}{R\Gamma} + \frac{in(n^2 - 1)\Lambda W a^2}{R^3}\right) r_1, \qquad (5\text{-}24)$$

$$\frac{dz_1}{dt} = (U_r + T_r)\, r_1 + \left(\frac{in\, 2\pi a^2 W(U_z + T_z)}{R\Gamma} - \frac{in^3 \Lambda W a^2}{R^3}\right) z_1. \qquad (5\text{-}25)$$

The solutions have the form $r_1 = r_0\, e^{i\omega t}$, $z_1 = z_0\, e^{i\omega t}$, which leads to an eigenvalue problem for $\omega$ with the characteristic equation

$$\omega^2 + \left(\frac{n\, 2\pi a^2 W(U_r + T_r - U_z - T_z)}{R\Gamma} + \frac{n(2n^2 - 1)\Lambda W a^2}{R^3}\right)\omega$$

$$+ (U_z + T_z)(U_r + T_r) + n^2 \left(\frac{2\pi a^2 W(U_r + T_r)}{R\Gamma} + \frac{(n^2 - 1)\Lambda W a^2}{R^3}\right) \cdot \qquad (5\text{-}26)$$

$$\left(\frac{2\pi a^2 W(U_z + T_z)}{R\Gamma} - \frac{n^2 \Lambda W a^2}{R^3}\right) = 0.$$

Again, the terms explicitly involving W are of $O(\frac{a^2 W}{\Gamma})$ [or O(a)] which are small compared to the other terms. Thus to leading order we have

$$\omega^2 + (U_z + T_z)(U_r + T_r) = 0, \qquad (5\text{-}27)$$

which has roots

$$\omega = \pm\, i\sqrt{(U_r + T_r)(U_z + T_z)}. \qquad (5\text{-}28)$$

This means that the ring is unstable only when

$$(U_r + T_r)(U_z + T_z) > 0, \qquad (5\text{-}29)$$

otherwise, the ring simply oscillates (or vibrates) with a frequency given by (5-28).

Similar to the case of trailing vortices, the main effect of the axial velocity is to replace the core radius a by an effective core radius $a_e$ and does not alter the stability criteria to leading order. Therefore we only need to analyze condition (5-29) provided we keep in mind that axial velocity effect is reflected in the definition of A.

In the limit as $\frac{a}{R} \to 0$, we get by retaining only the terms involving $\log \frac{R}{a}$ the following,

$$(U_r + T_r)(U_z + T_z) = \frac{\Gamma^2}{(4\pi R^2)^2}\left(\log \frac{R}{a}\right)^2 n^2(1 - n^2), \qquad (5\text{-}30)$$

which is less than or equal to zero for all values of n. Thus the ring in this limit is stable and we can check that this frequency agrees with the leading order frequency given by Thomson (1883). What Widnall and Sullivan (1973) did was to numerically evaluate $(U_r + T_r) \times (U_z + T_z)$ for various values of n (n being an integer). They have found that for almost any ratio of $\frac{a}{R}$, there exists an associated unstable mode $n^* = n^*(\frac{a}{R})$. This can be seen if we assume n is comparable to $\frac{R}{a}$, which gives the asymptotic values

$$(U_r + T_r) \sim (1 - n^2)\left(\log \frac{a}{2R} + C - 1 - A\right) - n^2(\log n + 1) + \frac{1}{4}\log n, \qquad (5\text{-}31)$$

$$(U_z + T_z) \sim n^2\left(\log \frac{a}{2R} + C - 1 - A\right) + (n^2 - 1)(\log n - 1) + \frac{1}{4}\log n. \qquad (5\text{-}32)$$

Now $(U_r + T_r)$ and $(U_z + T_z)$ no longer change sign at the same values of n but there exist bands of values of n for which the log n terms balances the log $\frac{a}{R}$ terms, thereby resulting in positive values of the product $(U_r + T_r)(U_z + T_z)$. These instability bands are exhibited in Fig. 5-2, where we have plotted

$$\hat{V} = V_0 \frac{4\pi R}{\Gamma} = \log \frac{8R}{a} - \frac{1}{2} + A \quad (5\text{-}33)$$

versus the imaginary part of

$$\hat{\omega} = \frac{\omega R}{V_0}. \quad (5\text{-}34)$$

These results are identical to that given by Widnall and Sullivan (1973).

The dependence of n* on $\frac{R}{a}$ can be estimated from (5-31) and (5-32). The instability bands must lie between the zeros of $(U_r + T_r)$ and $(U_z + T_z)$. This leads to the approximate relation

$$n^* \doteq \frac{2R}{a} e^{\frac{5}{4} - C} \quad (5\text{-}35)$$

for a uniform core with no axial flow, which means that the corresponding unstable wavelength is

$$\lambda^* = \frac{2\pi R}{n^*} \doteq \frac{a\pi}{e^{\frac{5}{4} - C}}, \quad (5\text{-}36)$$

and therefore,

$$\frac{a}{\lambda^*} \doteq \frac{2}{\pi}. \quad (5\text{-}37)$$

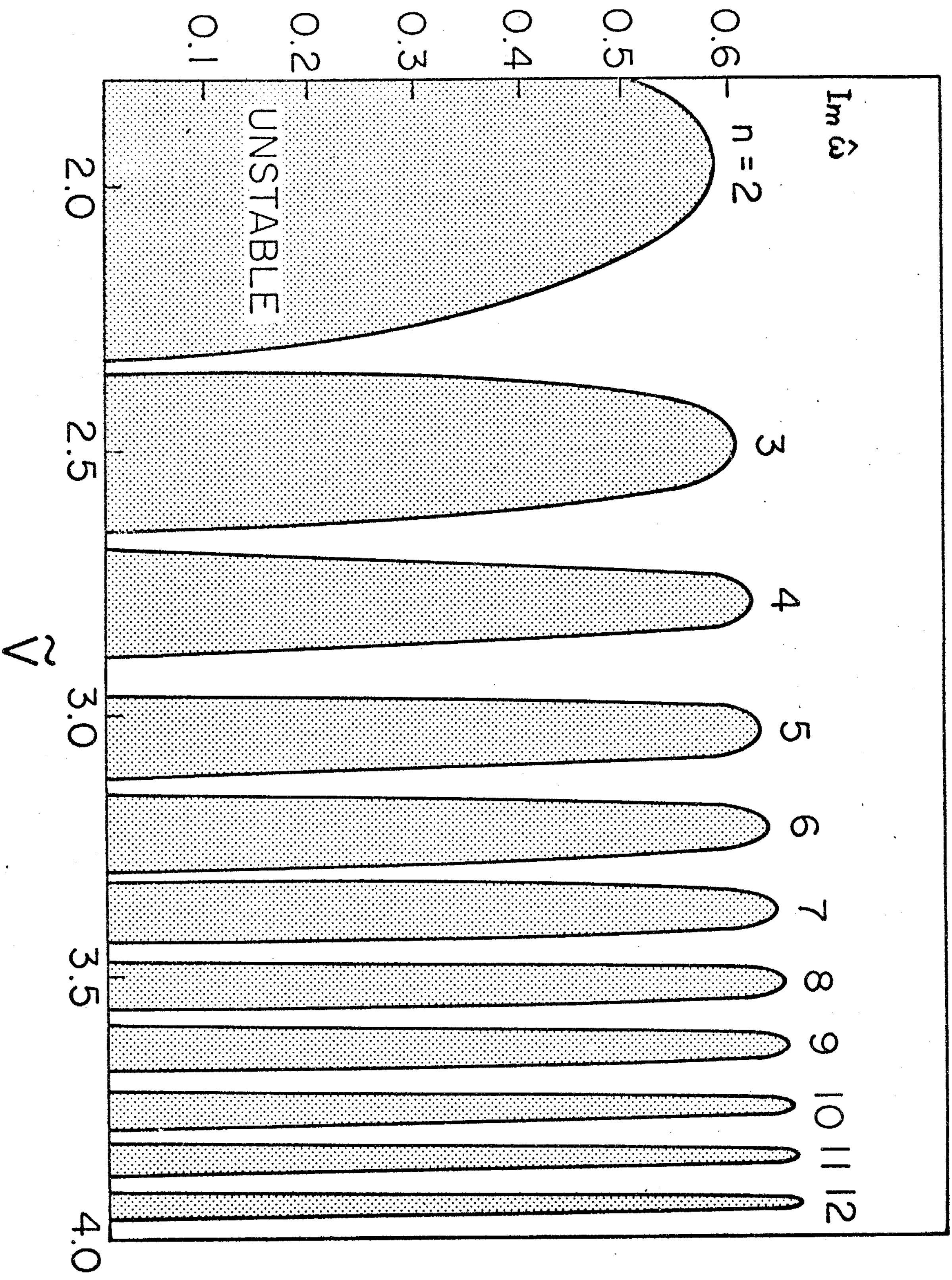

Im $\hat{\omega}$
0.6
0.5
0.4
0.3
0.2
0.1
n = 2
3
4
5
6
7
8
9
10
11
12
UNSTABLE
2.0
2.5
3.0
3.5
4.0
$\tilde{V}$

Fig. 5-2

Unfortunately, this value of core radius-to-wavelength ratio clearly violates the condition that

$$\frac{a}{\lambda} \ll \frac{1}{2\pi} , \tag{5-38}$$

which is necessary for the osculating circle method or the cutoff method to work, since the error of these methods are $O((\frac{2\pi a}{\lambda})^4)$ (see 3-29). Thus the short wave instability found by Widnall and Sullivan (1973) is not consistent with the method used and is not valid. These instabilities, which have been observed experimentally by Widnall et al., must be treated by the more exact infinitesimal analysis.

The only conclusion that can be drawn here is that vortex rings, with or without axial velocities, are stable to long sinusoidal perturbations of the centerline. Existence of shortwave instabilities is indicated both by experiments and extrapolation of long wave results to regions outside the region of validity but not proved.

## §18. The Cutoff Method as an Alternative

Before leaving this problem let us illustrate how the cutoff method used by Widnall and Sullivan (1973) can be applied to give the same results in the case of no axial velocity. Here the two terms $\frac{T}{\Gamma\rho}\underset{\wedge}{b}$ and $\underset{\wedge}{V}_I$ are combined into the singular Biot-Savart integral

$$\frac{\partial \underset{\wedge}{R}}{\partial t} = \frac{\Gamma}{4\pi} \oint \frac{(\underset{\wedge}{R}' - \underset{\wedge}{R}) \wedge d\underset{\wedge}{R}'}{|\underset{\wedge}{R}' - \underset{\wedge}{R}|^3} , \tag{5-39}$$

where $\oint$ means that a cutoff is used to remove the singularity at $\underset{\sim}{R}' = \underset{\sim}{R}$. If we define F(k) as

$$F(k) = \int_{-\pi}^{\pi} \frac{\cos k\bar{\theta}}{[2(1 - \cos\bar{\theta})]^{3/2}}\, d\bar{\theta}, \tag{5-40}$$

then by substituting (5-4) into (5-39) and expanding for small $\frac{z_1}{R}$, $\frac{y_1}{R}$, we can get

$$\frac{\partial \underset{\sim}{R}}{\partial t} = \frac{\Gamma}{4\pi R} \Big\{ (F(0) - F(1))\, \underset{\sim}{e}_z + \frac{z_1}{R}\, e^{in\theta} \Big[ F(1) - \tfrac{1}{2}(F(n+1) + F(n-1))$$

$$+ \frac{n}{2}(F(n+1) - F(n-1)) \Big] \underset{\sim}{e}_r + \frac{r_1}{R}\, e^{in\theta} \Big[ 2F(n)$$

$$- \tfrac{1}{2}(F(n+1) + F(n-1)) - F(1) - \frac{n}{2}(F(n+1) - F(n-1))$$

$$- \frac{3}{2}(F(0) + F(n) - F(1) - \frac{1}{2}(F(n+1) + F(n-1)) \Big] \underset{\sim}{e}_z \Big\}. \tag{5-41}$$

To evaluate F(k) we write

$$F(k) = 2 \lim_{\epsilon \to 0} \Big[ \int_{\epsilon}^{\pi} \frac{\cos k\bar{\theta}}{[2(1 - \cos\bar{\theta}]^{3/2}}\, d\bar{\theta} - \int_{\epsilon}^{\infty} \frac{\cos k\bar{\theta}\, d\bar{\theta}}{\bar{\theta}^3} - \frac{1}{8} \int_{\epsilon}^{\infty}$$

$$\frac{e^{-\bar{\theta}} \cos k\bar{\theta}}{\bar{\theta}}\, d\bar{\theta} \Big] + 2 \lim_{\epsilon \to 0} \Big[ \int_{\epsilon}^{\infty} \frac{\cos k\bar{\theta}\, d\bar{\theta}}{\bar{\theta}^3} + \int_{\epsilon}^{\infty} \frac{e^{-\bar{\theta}} \cos k\bar{\theta}\, d\bar{\theta}}{\bar{\theta}} \Big]. \tag{5-42}$$

In this form the first part is regular, and in fact can be considered as the cosine transform G(k) of a discontinuous function $g(\overline{\theta})$ defined by

$$g(\overline{\theta}) = \begin{cases} \dfrac{1}{[2(1 - \cos\overline{\theta})]^{3/2}} - \dfrac{1}{\overline{\theta}^3} - \dfrac{e^{-\overline{\theta}}}{8\overline{\theta}}, & \overline{\theta} \leq \pi \\ -\dfrac{1}{\overline{\theta}^3} - \dfrac{e^{-\overline{\theta}}}{8\overline{\theta}}, & \overline{\theta} > \pi\ . \end{cases} \tag{5-43}$$

G(k) can be evaluated numerically bya Fast Fourier Transform subroutine.

The singular part of (5-35) can be evaluated as the limit of $\epsilon \to 0$ of the expression

$$\lim_{\epsilon \to 0} \left[\frac{\cos k\epsilon}{\epsilon^2} - \frac{k \sin k\epsilon}{\epsilon} + k^2 \mathrm{Ci}(k\epsilon) - \frac{1}{4}\, \mathrm{Ci}\,(\epsilon\sqrt{1 + k^2})\right], \tag{5-44}$$

where Ci is the integral cosine whose expansion for small argument is

$$\mathrm{Ci}(x) = C + \log x, \tag{5-45}$$

with C being Euler's constant. Therefore, (5-44) becomes

$$\frac{1}{\epsilon^2} + k^2\left(-\frac{3}{2} + C + \log\epsilon + \log k\right) - \frac{1}{4}\left(C + \log k + \frac{1}{2}\log(1 + k^2)\right), \tag{5-46}$$

and from (5-42) we have

$$F(n) = G(n) + n^2 \left(-\frac{3}{2} + C + \log\epsilon + \log n\right) - \frac{1}{4}\left(C + \log n + \frac{1}{2}\log(1 + n^2)\right), \tag{5-47}$$

where we have dropped the term $\frac{1}{\epsilon^2}$ since examination of (5-41) shows that any terms not explicitly depending on n cancel out.

To evaluate $\epsilon$, which is the cutoff angle, we recall the assumption that the cutoff length $\ell$ is independent of the geometry and must, therefore, be the same as that found in the case of a perturbed straight filament (3-25), except now $\frac{1}{4}$ is replaced by the parameter A. This means

$$\log\frac{\ell}{R} = \log\frac{a}{2R} + \frac{1}{2} - A, \tag{5-48}$$

where A is defined in (4-19) with W = 0. The relation between $\ell$ and $\epsilon$ is a geometric one:

$$\begin{aligned}\epsilon &= \frac{\ell}{R + r_1 e^{in\theta}} + O\left(\frac{r_1^2}{R^2}\right) \\ &= \frac{\ell}{R}\left(1 - \frac{r_1}{R}\, e^{in\theta}\right) + O\left(\frac{r_1^2}{R^2}\right)\end{aligned} \tag{5-49}$$

and therefore

$$\log\epsilon = \log\frac{a}{2R} + \frac{1}{2} - A - \frac{r_1}{R}\, e^{in\theta}. \tag{5-50}$$

The additional $\frac{r_1}{R}$ term is only important in the leading order term

$(F(0) - F(1))$ since the other terms in (5-41) are already $O(\frac{r_1}{R})$.

By writing

$$H(k) = F(k) + \frac{r_1}{R} e^{in\theta} (k^2 - \frac{1}{4})$$

$$= k^2 \left(\log \frac{ak}{2R} + C - \frac{3}{2} + A\right) - \frac{1}{8} \log (1 + k^2) + G(k), \qquad (5\text{-}51)$$

and

$$\frac{\partial \hat{R}}{\partial t} = V_0 \, \underset{\sim}{e}_z + r_1 e^{in\theta} \, V_r \, \underset{\sim}{e}_z + z_1 e^{in\theta} \, V_z \, \underset{\sim}{e}_r, \qquad (5\text{-}25)$$

we have

$$V_0 = \frac{\Gamma}{4\pi R} (H(0) - H(1)), \qquad (5\text{-}53)$$

$$V_r = \frac{\Gamma}{4\pi R^2} \left[H(1) - \frac{1}{2}(H(n+1) + H(n-1)) + \frac{n}{2}(H(n+1) - H(n-1))\right], \qquad (5\text{-}54)$$

$$V_z = \frac{\Gamma}{4\pi R^2} \Big[2H(n) - \frac{1}{2}(H(n+1) + H(n-1)) - H(1) - \frac{n}{2} (H(n+1) - H(n-1))$$

$$- \frac{3}{2}(H(0) + H(n) - H(1) - \frac{1}{2}(H(n+1) + H(n-1)) + 1\Big] . \qquad (5\text{-}55)$$

We can check immediately that $V_0$ agrees with that found in the previous section, using the numerical values for G(k) (see Table 1). Some algebra confirms that $V_r$ and $V_z$ are identical with $(U_r + T_r)$ and $(U_z + T_z)$ when $W = 0$. The stability equations are now

Table I

| n | G(n) |
|---|---|
| 0 | 0.4535 |
| 1 | 0.0731 |
| 2 | 0.0253 |
| 3 | 0.0120 |
| 4 | 0.0080 |
| 5 | 0.0050 |
| 6 | 0.0045 |
| 7 | 0.0019 |
| 8 | 0.0032 |
| 9 | 0.0021 |
| 10 | 0.0026 |

$$\frac{dr_1}{dt} = V_z \, z_1, \tag{5-56}$$

$$\frac{dz_1}{dt} = V_r \, r_1, \tag{5-57}$$

and the stability boundary is

$$V_r V_z = 0. \tag{5-58}$$

The rest of the discussions in the previous section carried through entirely. The advantages of this form is that the form is very suitable for numerical calculations.

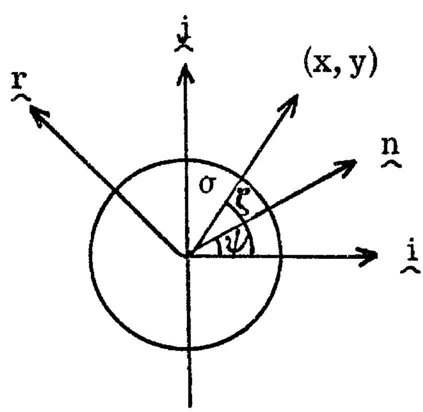
## §19. The Resultant Electric Force on a Charged Conducting Ring

In this section we shall calculate the resultant electric force on a charged conducting ring with small core radius as a first step towards the problem of stability of a charged conducting vortex ring. The method used will be the matched asymptotic expansion used by Moore and Saffman (1972) to calculate the velocity at a point of a vortex filament. The expansion parameter is $\frac{1}{\rho}$, where $\rho$ is the local radius of curvature ($\rho$ = R for a ring). In a strained coordinate system (with $\frac{1}{\rho}$ as the small parameter) the problem becomes two dimensional in the leading order and can therefore be solved. This near field solution is then matched to the inner limit of the far field solution obtained by considering the ring as a circular line charge. A mathematical discussion as well as some interesting examples concerning this technique are given by Fraenkel (1969).

We recall that a point on the filament is represented by the vector $\underset{\sim}{R}(\xi)$, where $\xi$ is a Lagrangian parameter along the filament. For any such point $\xi$ we define a local orthogonal coordinate system $(\underset{\sim}{i}, \underset{\sim}{j}, \underset{\sim}{s}(\xi))$ where $\underset{\sim}{s}$ is the local tangent of the filament. It is clear that $\underset{\sim}{i}$ and $\underset{\sim}{j}$ lie in the $\underset{\sim}{n} - \underset{\sim}{b}$ plane where $\underset{\sim}{n}$ is the principal normal and $\underset{\sim}{b}$ is the binormal of the filament. For a filament of arbitrary shape, $\underset{\sim}{i}$ and $\underset{\sim}{j}$ make an angle $\psi$ with $\underset{\sim}{n}$ and $\underset{\sim}{b}$, where $\psi$ is defined by

$$\frac{d\psi}{ds} = \tau, \tag{5-59}$$

with $\tau$ being the torsion and s the arc length. However, $\tau$ is zero for a circular ring, so that $\psi$ is a constant and can be taken to be zero without loss of generality. Since our discussion will be generalized to an arbitrary filament, we shall keep $\psi$ along in our calculations. (See Fig. 5-3).



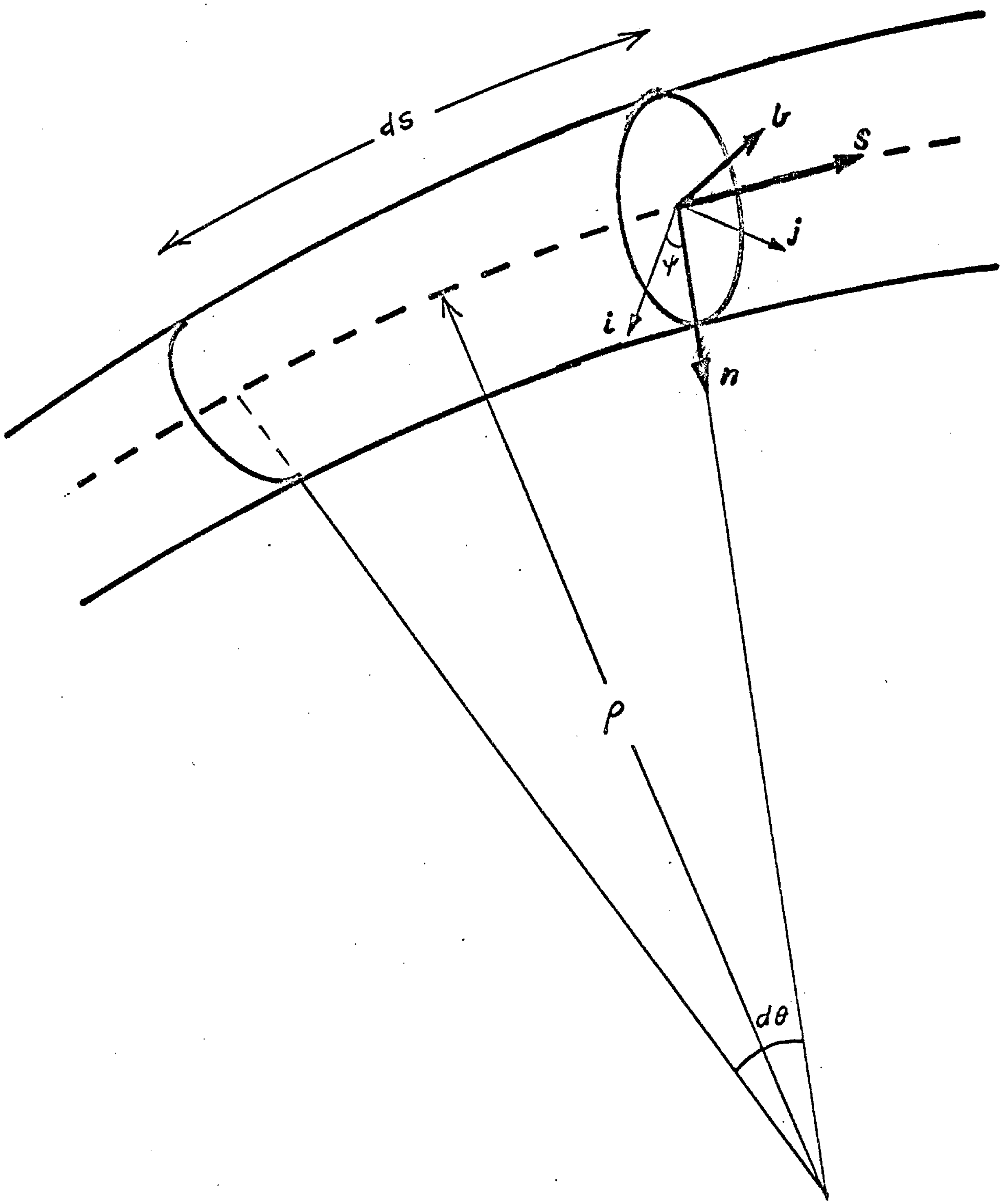



Fig. 5-3

With respect to this coordinate system, any point in space can be described by

$$\underset{\sim}{x} = \underset{\sim}{R}(\xi) + x\underset{\sim}{i} + y\underset{\sim}{j}. \tag{5-60}$$

We further introduce the local polar coordinates $(\sigma, \zeta)$ by

$$x = \sigma \cos \zeta, \quad y = \sigma \sin \zeta. \tag{5-61}$$

The cylindrical polar system $(\sigma, \zeta, s)$ possesses the metric

$$h_\sigma = 1, \; h_\zeta = \sigma, \; h_s = h = 1 - \left(\frac{\sigma}{\rho}\right) \cos(\zeta - \psi), \tag{5-62}$$

thus, the Laplacian operator in $(\sigma, \zeta, s)$ becomes

$$\nabla^2 = \frac{\partial}{\partial\sigma^2} + \frac{1}{\sigma}\frac{\partial}{\partial\sigma} + \frac{1}{\sigma^2}\frac{\partial^2}{\partial\zeta^2} + \frac{1}{h}\frac{\partial h}{\partial\sigma}\frac{\partial}{\partial\sigma} + \frac{1}{\sigma^2 h}\frac{\partial h}{\partial\zeta}\frac{\partial}{\partial\zeta} + \frac{1}{h}\frac{\partial}{\partial s}\left(\frac{1}{h}\frac{\partial}{\partial s}\right). \tag{5-63}$$

We shall now calculate the electric potential $\phi$ induced by the charges on the ring. The problem is

$$\left\{\begin{array}{ll} \nabla^2\phi = 0 \quad \text{outside the core}, & (5\text{-}64) \\ \\ \phi = \text{constant}, \; \dfrac{\partial\phi}{\partial\sigma} = \text{surface charge distribution on the core surface.} & (5\text{-}65) \end{array}\right.$$

The system of units that we shall use is the electrostatic units (e.s.u.).

For the near field solution $\phi$, where $\sigma \sim O(a) \ll \rho$, we can expand

$$\phi = \phi_0 + \phi_1 + \cdots, \tag{5-66}$$

where the subscripts denote orders of powers of $\frac{1}{\rho}$. We shall also assume that the core cross-section is circular to the leading order, so that

$$a = a_0 + a_1 + \cdots . \tag{5-67}$$

In this region, the variation along s will be $O(\frac{1}{\rho})$ compared to the variations in the $(\sigma, \zeta)$ plane, i.e., $\frac{\partial}{\partial s}$ is $O(\frac{1}{\rho})$ while $\frac{\partial}{\partial \sigma}$ and $\frac{1}{\sigma}\frac{\partial}{\partial \zeta}$ are O(1). Thus we have

$$\frac{\partial^2 \phi_0}{\partial \sigma^2} + \frac{1}{\sigma}\frac{\partial \phi_0}{\partial \sigma} + \frac{1}{\sigma^2}\frac{\partial^2 \phi_0}{\partial \zeta^2} = 0, \quad r \geqslant a_0, \tag{5-68}$$

as the leading order equation, which is the equation for the potential of an infinite circular cylinder. If we denote by $\lambda$ the charge per unit length ($\lambda = 2\pi a \Sigma$, where $\Sigma$ is charge per unit area); we have

$$\phi_0 = P_0 - 2\lambda \log \frac{\sigma}{a}, \tag{5-69}$$

where $P_0$ is some constant potential fixed by a particular choice of normalization. The value of $P_0$ does not affect the electric field $\hat{E}$ or the electric force.

Substitution of (5-69) into (5-64) yields an equation for $\phi_1$.

$$\frac{\partial^2\phi_1}{\partial\sigma^2} + \frac{1}{\sigma}\frac{\partial\phi_1}{\partial\sigma} + \frac{1}{\sigma^2}\frac{\partial^2\phi_1}{\partial\zeta^2} = \frac{-2\lambda}{\rho\sigma}\cos(\zeta - \psi), \tag{5-70}$$

which has solution of the form

$$\phi_1 = \left(B_1\sigma + \frac{B_2}{\sigma} - \frac{\lambda}{\rho}\,\sigma\log\frac{\sigma}{a}\right)\cos(\zeta - \psi). \tag{5-71}$$

The constants $B_1$ and $B_2$ will be determined by the boundary conditions that $\sigma = a$ is an equipotential surface and as $\sigma \to \infty$, this solution $\phi$ approaches $\Phi$ which is the solution for a circular line charge. The first condition requires that

$$\phi = \phi_0 + \phi_1 + \cdots = P_0 \quad \text{at } \sigma = a, \tag{5-72}$$

which means

$$\phi_1 = 0 \text{ at } \sigma = a, \tag{5-73}$$

or

$$\phi_1 = -2\lambda\frac{a_1}{a_0} \quad \text{at } \sigma = a_0, \tag{5-74}$$

where we have used (5-67). However, the form of $\phi_1$ (5.71) implies that

$$a_1 = a_1^*(\sigma)\cos(\zeta - \psi) \tag{5-75}$$

which is a rigid displacement of the centerline and does not enter into our analysis. Indeed, if we define the centerline as being the center of gravity of the cross-section, $a_1^*(\sigma)$ must be zero. Thus, we can write without loss of generality that

$$\phi_1 = 0 \quad \text{at } \sigma = a_0 . \tag{5-76}$$

Let us now turn to the far field solution $\Phi$, to which $\phi$ must match when $\sigma$ gets large. The region in which we shall compute $\Phi$ is

$$a \ll \sigma \ll \rho. \tag{5-77}$$

To find $\Phi$ we return to the cylindrical coordinate system located at the center of the ring, with radial component r, axial component z, and azimuthal angle $\theta$ (Fig. 5-4).

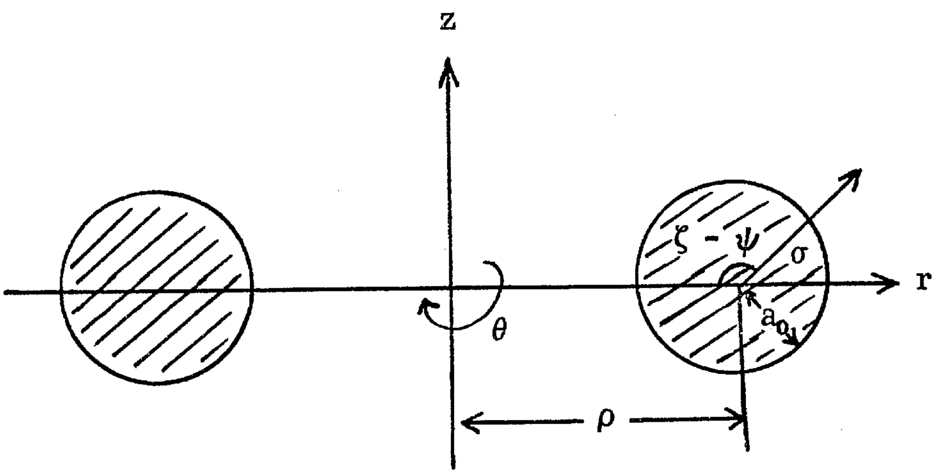



Fig. 5 - 4

Since $\sigma \gg a$, we can treat the ring as a circular line charge of radius $\rho$ with charge density $\lambda$ lying in the plane $z = 0$. The potential is given by

$$\Phi(r,0) = \lambda \int_V \frac{\delta(r' - \rho)\delta(z')r'dr'dz'd\theta'}{\left\{r^2 + r'^2 - 2rr' \cos\theta' + (z - z')^2\right\}^{\frac{1}{2}}} \tag{5-78}$$

where we have taken the point of interest to be $\theta = 0$, and V denotes the volume of the ring. This gives

$$\Phi = \lambda \int_0^{2\pi} \frac{d\theta'}{\left\{r^2 + r'^2 - 2rr'\cos\theta' + z^2\right\}^{\frac{1}{2}}} \; . \tag{5-79}$$

By letting $\theta' = 2\bar{\theta}$ and defining k to be

$$k^2 = \frac{4r\rho}{(r+\rho)^2 + z^2}, \tag{5-80}$$

the integral can be transformed into a complete elliptic integral of the first kind, which can then be integrated to give

$$\Phi = \frac{2\lambda\rho k}{\sqrt{r\rho}} K(k) \, , \tag{5-81}$$

where K is the complete elliptic integral of the first kind. Now if we write

$$\tau = \rho - X \, , \tag{5-82}$$

where X is increasing in the direction of the principal normal $\hat{n}$, then

$$k^2 = \frac{4\rho^2 - 4X\rho}{4\rho^2 - 4X\rho + X^2 + z^2} = \frac{4\rho^2 - 4X\rho}{4\rho^2 - 4X\rho + \sigma^2} \, . \tag{5-83}$$

Recalling that X, $\sigma \ll \rho$, we have

$$k^2 = 1 - \frac{\sigma^2}{4\rho^2 + 4X\rho} + O\left(\frac{1}{\rho^3}\right). \tag{5-84}$$

Let us define the complementary argument k′ by

$$k'^2 = 1 - k^2 = \frac{\sigma^2}{4\rho^2 + 4X\rho} = \frac{\sigma^2}{4\rho^2}\left(1 + \frac{X}{\rho} + O\left(\frac{1}{\rho^2}\right)\right) \tag{5-85}$$

and for later use, we note that

$$\frac{1}{k'} = \frac{2\rho}{\sigma}\left(1 - \frac{X}{2\rho}\right). \tag{5-86}$$

k′ is of $O(\frac{1}{\rho^2})$ and so K(k) can be expanded in k′ to give

$$\Phi = \frac{2\lambda\rho^{\frac{1}{2}}k}{\rho^{\frac{1}{2}}\left(1 - \frac{X}{\rho}\right)^{\frac{1}{2}}}\left[\log\frac{4}{k'} + O\left(\frac{1}{\rho^2}\right)\right]$$

$$= 2\lambda\left(1 + \frac{X}{2\rho}\right)\left(1 + O\left(\frac{1}{\rho^2}\right)\right)\left[\log\frac{8\rho}{\sigma} - \frac{X}{2\rho} + O\left(\frac{1}{\rho^2}\right)\right], \tag{5-87}$$

which leads to

$$\Phi = 2\lambda\log\frac{8\rho}{\sigma} + \frac{\lambda X}{\rho}\left[\log\frac{8\rho}{\sigma} - 1\right] + O\left(\frac{1}{\rho^2}\right). \tag{5-88}$$

We now match $\phi$ to $\Phi$ as $\sigma$ gets large. The leading order gives us a value for $P_0$

$$P_0 = 2\lambda \log \frac{8\rho}{a_0}\,, \tag{5-89}$$

which does not enter into the analysis. Matching of the first order terms leads to the equation

$$\left(B_1 - \frac{\lambda}{\rho} \log \frac{\sigma}{a_0}\right) = \frac{\lambda}{\rho} \left(\log \frac{8\rho}{a_0} - 1 - \log \frac{\sigma}{a_0}\right), \tag{5-90}$$

where we have made use of the relation $X = \sigma \cos(\zeta - \psi)$, this gives $B_1$ as

$$B_1 = \frac{\lambda}{\rho} (L - 1), \tag{5-91}$$

where L is given by

$$L = \log \frac{8\rho}{a_0}. \tag{5-92}$$

Therefore

$$\phi_1 = \left[(L - 1) + \frac{B_2\rho}{\sigma^2\lambda} + \log \frac{\sigma}{a_0}\right] \frac{\lambda}{\rho}\, \sigma \cos(\zeta - \psi), \tag{5-93}$$

and the condition $\phi_1 = 0$ at $\sigma = a_0$ gives

$$B_2 = - a_0^{\,2} (L - 1) \frac{\lambda}{\rho}. \tag{5-94}$$

Finally, we have

$$\phi_1 = \left[(L - 1) - \frac{a_0^{\,2}}{\sigma^2} (L - 1) - \log \frac{\sigma}{a_0}\right] \frac{\lambda}{\rho}\, \sigma \cos(\zeta - \psi). \tag{5-95}$$

Now, the surface charge density $\Sigma$ is given by

$$\Sigma = \frac{1}{4\pi} \frac{\partial \phi}{\partial \sigma}\Big|_{a_0} = -\frac{\lambda}{2\pi a_0} + \frac{\lambda}{4\pi\rho}\,[2L - 3]\,\cos(\zeta - \psi). \qquad (5\text{-}96)$$

Therefore, the outward force per unit area acting on the curved surface is

$$2\pi\,\Sigma^2\,\underset{\sim}{e}_r = \left[\frac{\lambda^2}{2\pi\,{a_0}^2} - \frac{\lambda^2}{2\pi\,a_0\,\rho}\,[2L - 3]\,\cos(\zeta - \psi) + O\left(\frac{1}{\rho^2}\right)\right]\underset{\sim}{e}_r. \qquad (5\text{-}97)$$

Since

$$\underset{\sim}{e}_r = \cos(\xi - \psi)\,\underset{\sim}{n} + \sin(\zeta - \psi)\,\underset{\sim}{b}, \qquad (5\text{-}98)$$

we have upon integration over the surface $\zeta$ the electric force per unit length as

$$\underset{\sim}{F} = -\frac{\lambda^2}{\rho}\left(\log\frac{8\rho}{a_0} - \frac{3}{2}\;\underset{\sim}{n}\right) + \frac{\lambda^2}{2a^2}\,\frac{\partial a^2}{\partial s}\,\underset{\sim}{s}. \qquad (5\text{-}99)$$

A contribution of $\frac{-\lambda^2}{2\rho}\,\underset{\sim}{n}$ which is due to the fact that the surface element of the tube is curved is balanced by an equal and opposite term caused by the pressure difference acting over the two ends of the element when we require the continuity of pressure across the surface.

## §20. The Velocity of a Charged Conducting Vortex Ring

The velocity of a charged conducting ring can be found by the local force balance method given in Chapter III. The exterior force now becomes

$$\underset{\sim}{F}_E = m_0 \Gamma \left(\underset{\sim}{s} \wedge \frac{\partial \underset{\sim}{R}}{\partial t}\right) + \frac{m_0 \Gamma^2}{4\pi\rho} \left\{\log \frac{8\rho}{a_0} - \frac{1}{2} - \mathcal{E} \left(\log \frac{8\rho}{a_0} - \frac{3}{2}\right)\right\} \underset{\sim}{n}$$

$$- (1 - \mathcal{E}) \frac{m_0 \Gamma^2}{8\pi a^2} \frac{\partial a^2}{\partial s} \underset{\sim}{s} \tag{5-100}$$

where $m_0$ is the density of the fluid, and $\mathcal{E}$ is defined to be

$$\mathcal{E} = \frac{4\pi\lambda^2}{m_0 \Gamma^2} = \frac{Q^2}{m_0 \pi \Gamma^2 \rho^2} \tag{5-101}$$

with Q as the total charge on the ring. $\mathcal{E}$ gives the ratio of the electric force to the tensile force due to swirl and curvature.

Since the ring is assumed to be a conductor, no electric field is present inside the core and therefore the internal force $\underset{\sim}{F}_I$ is unchanged. Actually, the core has a magnetic force induced by the swirling of the charged surface fluid elements but no magnetic force is present again because there is no charge inside. Thus $\underset{\sim}{F}_I$ is still given by

$$\underset{\sim}{F}_I = m_0 \frac{\partial}{\partial s} \left(\frac{1}{2} \pi a^2 \overline{\nu}^2 \underset{\sim}{s}\right), \tag{5-102}$$

and the component of the force balance equation perpendicular to $\underset{\sim}{s}$ is

$$\left(\underset{\sim}{s} \wedge \frac{\partial \underset{\sim}{R}}{\partial t}\right) + \frac{\Gamma}{4\pi\rho}\left\{\log\frac{8\rho}{a_0} - \frac{1}{2} + A - \mathcal{E}\left(\log\frac{8\rho}{a_0} - \frac{3}{2}\right)\right\}\underset{\sim}{b}, \qquad (5\text{-}103)$$

where A is the internal structure parameter defined earlier in (4-19). Inverting this, we get

$$\frac{\partial \underset{\sim}{R}}{\partial t} = \frac{\Gamma}{4\pi\rho}\left\{\log\frac{8\rho}{a_0} - \frac{1}{2} + A - \mathcal{E}\left(\log\frac{8\rho}{a_0} - \frac{3}{2}\right)\right\}\underset{\sim}{b}. \qquad (5\text{-}104)$$

This result agrees with that obtained by Pocklington (1895) using an energy method. It shows that the presence of the electric charges reduces the steady velocity of the ring. In fact, for strong enough charges the ring may reverse its direction (i.e., move in an opposite direction as the velocity induced by the swirl when no charge is present). For a suitable value of $\mathcal{E}$, given by

$$\mathcal{E} = \left(\log\frac{8\rho}{a_0} - \frac{1}{2} + A\right)\Big/\left(\log\frac{8\rho}{a_0} - \frac{3}{2}\right) \qquad (5\text{-}105)$$

the ring does not move at all.

## §21. The Resultant Electric Force on a Uniformly Charged Ring

Before we consider the stability of a charged vortex ring, let us first repeat our calculations for a uniformly charged ring. The difference between this case and the conducting ring is that there exists an electric field inside the core, the potential of which must be found by solving the Poisson equation. Let us denote the solution inside the core as $\varphi$, and we still have $\phi$ and $\Phi$ as the near and far field solution outside. At the boundary $\sigma = a$, we require

$$\varphi \big|_a = \phi \big|_a \quad , \tag{5-106}$$

$$\frac{\partial \varphi}{\partial \sigma}\Big|_a = \frac{\partial \phi}{\partial \sigma}\Big|_a , \tag{5-107}$$

and the equation for $\varphi$ inside the core is

$$\frac{\partial^2 \varphi}{\partial \sigma^2} + \frac{1}{\sigma}\frac{\partial \varphi}{\partial \sigma} + \frac{1}{\sigma^2}\frac{\partial^2 \varphi}{\partial \zeta^2} + \frac{1}{h}\frac{\partial h}{\partial \sigma}\frac{\partial \varphi}{\partial \sigma} + \frac{1}{r^2 h}\frac{\partial h}{\partial \zeta}\frac{\partial \varphi}{\partial \zeta} + \frac{\partial}{\partial s}\left(\frac{1}{h}\frac{\partial \varphi}{\partial s}\right) = -4\pi q \tag{5-108}$$

where q is the charge density inside, being $\frac{Q}{2\pi^2 a^2 R}$. The leading order in the strained coordinate system is

$$\frac{\partial^2 \varphi_0}{\partial \sigma^2} + \frac{1}{\sigma}\frac{\partial \varphi_0}{\partial \sigma} + \frac{1}{\sigma^2}\frac{\partial^2 \varphi_0}{\partial \zeta^2} = -\frac{4\lambda}{a^2} \tag{5-109}$$

where we have written $\lambda = \pi a^2 q$ as the charge per unit length, and expanded $\varphi$ as

$$\varphi = \varphi_0 + \varphi_1 + \cdots . \tag{5-110}$$

The solution for (5-109) is

$$\varphi_0 = P_0 - \frac{\lambda \sigma^2}{a^2}, \tag{5-111}$$

where again $P_0$ is an undetermined constant. The next order equation is

$$\frac{\partial^2\varphi_1}{\partial\sigma^2} + \frac{1}{\sigma}\frac{\partial\varphi_1}{\partial\sigma} + \frac{1}{\sigma^2}\frac{\partial^2\varphi_1}{\partial\zeta^2} = \frac{-2\lambda\sigma}{a^2\rho}\cos(\zeta - \psi), \tag{5-112}$$

which has solution

$$\varphi_1 = \left(\alpha\sigma - \frac{\lambda}{4a^2\rho}\sigma^3\right)\cos(\zeta - \psi), \tag{5-113}$$

where $\sigma$ is a constant to be determined by matching. At $\sigma = a$, we have

$$\varphi_1 = \left(\alpha a - \frac{\lambda a}{4\rho}\right)\cos(\zeta - \psi) = \left(\alpha^* - \frac{1}{4}\right)\frac{\lambda a}{\rho}\cos(\zeta - \psi), \tag{5-114}$$

$$\frac{\partial\varphi_1}{\partial\sigma} = \left(\alpha - \frac{3\lambda}{4\rho}\right)\cos(\zeta - \psi) = \left(\alpha^* - \frac{3}{4}\right)\frac{\lambda}{\rho}\cos(\zeta - \psi). \tag{5-115}$$

Again, $a = a_0 + a_1 + \cdots$, and $a_1$ represents a shift of the centerline and can be taken to be zero. Thus

$$a = a_0 + O\left(\frac{1}{\rho^2}\right), \tag{5-116}$$

and we can drop the subscript $_0$ to our order of approximation.

The far field solution $\Phi$ is unchanged. Therefore, the near field solution $\phi$ is given by

$$\phi = \phi_0 + \left[(L - 1)\sigma + \frac{B_2\rho}{\sigma\lambda} - \sigma L + \sigma\log\frac{8\rho}{\sigma}\right]\frac{\lambda}{\rho}\cos(\zeta - \psi), \tag{5-117}$$

and at $\sigma = a$

$$\phi_1 = \left[L - 1 + \frac{B_2\rho}{a^2\lambda}\right] \frac{\lambda a}{\rho} \cos(\zeta - \psi) = \left[L - 1 + B_2^*\right] \frac{\lambda a}{\rho} \cos(\zeta - \psi), \quad (5\text{-}118)$$

$$\frac{\partial \phi_1}{\partial \sigma} = [L - 1 - B_2^* - L + L - 1] \frac{\lambda}{\rho} \cos(\zeta - \psi) = (L - 2 - B_2^*) \frac{\lambda}{\rho} \cos(\zeta - \psi). \quad (5\text{-}119)$$

Equating (5-118) and (5-119) to (5-114) and (5-115) we obtain

$$\alpha^* = B_2^* + (L - 1) + \frac{1}{4}. \quad (5\text{-}120)$$

$$\alpha^* = L - 2 - B_2^* + \frac{3}{4}, \quad (5\text{-}121)$$

which leads to

$$B_2^* = -\frac{1}{4}, \ \alpha^* = L - 1. \quad (5\text{-}122)$$

Thus we have

$$\varphi = P_0 - \frac{\lambda\sigma^2}{a^2} + \left[(L - 1)\sigma - \frac{\sigma^3}{4a^2}\right] \frac{\lambda}{\rho} \cos(\zeta - \psi), \quad (5\text{-}123)$$

so that

$$\underset{\sim}{\nabla}\varphi = \left\{-\frac{2\lambda\sigma}{a^2} + \left[(L - 1) - \frac{3\sigma^2}{4a^2}\right] \frac{\lambda}{\rho} \cos(\zeta - \psi)\right\} \underset{\sim}{e}_r$$
$$- \left\{(L - 1) - \frac{\sigma^2}{4a^2}\right\} \frac{\lambda}{\rho} \sin(\zeta - \psi) \ \underset{\sim}{e}_\theta. \quad (5\text{-}124)$$

The resultant force per unit length is given by

$$\underset{\sim}{F} = -\frac{\lambda}{\pi a^2} \int \underset{\sim}{\nabla}\varphi \ dA. \quad (5\text{-}125)$$

Since

$$\underset{\sim}{e}_r = \cos(\zeta - \psi)\,\underset{\sim}{n} + \sin(\zeta - \psi)\,\underset{\sim}{b},$$

$$\underset{\sim}{e}_\theta = -\sin(\zeta - \psi)\,\underset{\sim}{n} + \cos(\zeta - \psi)\,\underset{\sim}{b}, \tag{5-126}$$

we have

$$\underset{\sim}{F} = -\frac{\lambda}{\pi a^2}\,\underset{\sim}{n}\int_0^{2\pi}\int_0^{a}\left\{\left[(L-1) - \frac{3\sigma^2}{4a^2}\right]\frac{\lambda}{\rho}\cos^2(\zeta - \psi) + \left[(L-1) - \frac{\sigma^2}{4a^2}\right]\right.$$

$$\left.\frac{\lambda}{\rho}\sin^2(\zeta - \psi)\right\}\sigma d\sigma d\zeta$$

$$= -\frac{\lambda^2}{a^2\rho}\left\{(L-1)\frac{\sigma^2}{2} - \frac{3\sigma^4}{16\sigma^4} + (L-1)\frac{\sigma^2}{2} - \frac{\sigma^4}{16a^2}\right\}\Bigg|_0^a\,\underset{\sim}{n}. \tag{5-127}$$

Again we have dropped the term due to curvature of the tube since it is balanced by the excess pressure acting over the ends. Therefore, the resultant force per unit length due to the charges is given by

$$\underset{\sim}{F} = \frac{-\lambda^2}{\rho}\left(\log\frac{8\rho}{a} - \frac{5}{4}\right)\underset{\sim}{n} + \frac{\lambda^2}{2a^2}\frac{\partial a^2}{\partial s}\,\underset{\sim}{s}. \tag{5-128}$$

## §22. The Velocity of a Uniformly Charged Vortex Ring

The exterior force $\underset{\sim}{F}_E$ in this case is given by

$$\underset{\sim}{F}_E = m_0\boldsymbol{\Gamma}\left(\underset{\sim}{s}\wedge\frac{\partial \underset{\sim}{R}}{\partial t}\right) + \frac{m_0\boldsymbol{\Gamma}^2}{4\pi\rho}\left\{\log\frac{8\rho}{a} - \frac{1}{2} - \mathcal{E}\left(\log\frac{8\rho}{a} - \frac{5}{4}\right)\right\}\underset{\sim}{n} - \frac{\boldsymbol{\Gamma}^2}{8\pi a^2}\frac{\partial a^2}{\partial s}(1-\mathcal{E})\underset{\sim}{s}.$$

(5-129)

The interior force $\underset{\sim}{F}_I$ requires more care. The electric field inside the core gives an additional term to the pressure. Furthermore, the charges moving with the fluid elements experiences a magnetic force due to its own induction. This would be quite complicated since it couples with the velocity distribution. Fortunately, self-induced magnetic fields are in general extremely weak compared to the electric field. In fact, the self-induced magnetic force is $O\left(\frac{v^2}{C_L^2}\right)$ when compared to the electric force, where $C_L$ is the velocity of light in vacuo, and we can definitely ignore it in our analyses.

Since the electric field is radial to leading order, we can expand the velocity and pressure inside by

$$\left.\begin{aligned} v &= v_0(\sigma) + v_1(\sigma, \zeta) + \cdots \\ u &= u_1(\sigma, \zeta) + \cdots \\ p &= p_0(\sigma) + p_1(\sigma, \zeta) + \cdots \end{aligned}\right\} \tag{5-130}$$

These are to be substituted into the two dimensional steady equations of motion [since the axial effect is of $O\left(\frac{1}{\rho^2}\right)$]

$$\left.\begin{aligned} u\frac{\partial u}{\partial \sigma} + \frac{v}{\sigma}\frac{\partial u}{\partial \zeta} - \frac{v^2}{\sigma} &= -\frac{1}{m_0}\frac{\partial p}{\partial \sigma} - \frac{\lambda}{\pi a^2}\frac{\partial \varphi}{\partial \sigma}, \\ u\frac{\partial v}{\partial \sigma} + \frac{v}{\sigma}\frac{\partial v}{\partial \zeta} + \frac{uv}{\sigma} &= \frac{-1}{m_0 \sigma}\frac{\partial p}{\partial \zeta} - \frac{\lambda}{\pi a^2}\frac{1}{\sigma}\frac{\partial \varphi}{\partial \zeta} \\ \frac{\partial u}{\partial \sigma} + \frac{u}{\sigma} + \frac{1}{\sigma}\frac{\partial v}{\partial \zeta} &= 0. \end{aligned}\right\} \tag{5-131}$$

The leading order equation is

$$\frac{1}{m_0}\frac{\partial p_0}{\partial \sigma} = \frac{v_0^{\,2}}{\sigma} + \frac{2\lambda^2\sigma}{a^2}\cdot\frac{1}{\pi a^2} \tag{5-132}$$

and the first order equations are

$$\frac{v_0}{\sigma}\frac{\partial u_1}{\partial \zeta} - \frac{2v_0 v_1}{\sigma} = -\frac{1}{m_0}\frac{\partial p_1}{\partial \sigma} - \left[(L-1) - \frac{3\sigma^2}{4a^2}\right]\frac{\lambda^2}{p\pi a^2}\cos(\zeta-\psi) \tag{5-133}$$

$$u_1\frac{\partial v_0}{\partial \sigma} + \frac{v_0}{\sigma}\frac{\partial v_1}{\partial \xi} + \frac{u_1 v_0}{\sigma} = -\frac{1}{m_0\sigma}\frac{\partial p_1}{\partial \zeta} + \left[(L-1) - \frac{\sigma^2}{a^2}\right]\frac{\lambda^2}{p\pi a^2}\sin(\zeta-\psi) \tag{5-134}$$

$$\frac{\partial u_1}{\partial \sigma} + \frac{u_1}{\sigma} + \frac{1}{\sigma}\frac{\partial v_1}{\partial \zeta} = 0. \tag{5-135}$$

Examination of the form of these first order equations allow us to conclude that the first order pressure $p_1$ must have the form

$$p_1 = p_1{}^*(\sigma)\cos(\zeta-\psi) \tag{5-136}$$

which vanishes when we integrate over the area. Thus, $\underset{\sim}{F}_I$ is given by

$$\underset{\sim}{F}_I = m_0\pi\frac{\partial}{\partial s}\left[\left(\frac{1}{2}a^2\bar{v}^2 + \frac{\lambda^2}{2\pi}\right)\underset{\sim}{s}\right] \tag{5-137}$$

correct to $O(\rho^{-2})$. Adding to $\underset{\sim}{F}_E$ and inverting, we obtain

$$\frac{\partial \underset{\sim}{R}}{\partial t} = \frac{\Gamma}{4\pi\rho}\left\{\log\frac{8\rho}{a} - \frac{1}{2} + A - \mathcal{E}\left(\log\frac{8\rho}{a} - \frac{7}{4}\right)\right\}\underset{\sim}{b} \tag{5-138}$$

as the velocity of a uniformly charged vortex ring.

## §23. The Motion of a Charged Vortex Filament of Arbitrary Shape

The discussion in this section will apply to both the conducting core and the uniformly charged core. It is simply an extension of the ideas of Chapter III. Let us consider a charged vortex filament of small core radius, then the interior force $\underset{\sim}{F}_I$ will be given by (5-102) or (5-137) depending on the core we are interested in. The exterior force $\underset{\sim}{F}_E$ will be different from that of the ring. It must be found in a way such that the near field solution matches to a far field solution which is consistent with the shape of the filament. To do this we define

$$\underset{\sim}{G} = \lambda^2 \int \left( \frac{(\underset{\sim}{R} - \underset{\sim}{R}')ds'}{|\underset{\sim}{R} - \underset{\sim}{R}'|^3} - \frac{(\underset{\sim}{R} - \underset{\sim}{R}^\circ)\, ds^\circ}{|\underset{\sim}{R} - \underset{\sim}{R}^\circ|^3} \right), \tag{5-139}$$

where the first integral is taken over the filament and the second is taken over the osculating circle at $\underset{\sim}{R}$. The definition of force $\underset{\sim}{G}$ is analogous to the definition of the velocity $\underset{\sim}{V}_I$. The total force due to the filament is then

$$\underset{\sim}{F}_E = \underset{\sim}{G} + \underset{\sim}{F}_E{}^\circ, \tag{5-140}$$

where $\underset{\sim}{F}_E^\circ$ is the exterior force due to the osculating circle at the point of interest. However, $\underset{\sim}{F}_E^\circ$ is just the force on a ring which coincides with the osculating circle and has been found in the previous sections [(5-101) and (5-125)], all we now need is to find $\underset{\sim}{G}$. From the definition of $\underset{\sim}{G}$; it is regular everywhere, and in most cases can be numerically evaluated. This means that the velocity at a point of a charged vortex filament of arbitrary shape is given by

$$\frac{\partial \underset{\sim}{R}}{\partial t} = \frac{1}{\Gamma}(\underset{\sim}{s} \wedge \underset{\sim}{G}) + \frac{\Gamma}{4\pi\rho}\left\{\log\frac{8\rho}{a} - \frac{1}{2} + A - \mathcal{E}F^{\circ}\right\}\underset{\sim}{b} + \underset{\sim}{V}_E + \underset{\sim}{V}_I \qquad (5\text{-}141)$$

where $\rho$ is the local radius of curvature, and $F^{\circ}$ is proportional to the electric force contribution of the osculating ring. For a conducting ring, $F^{\circ}$ is $\left(\log\frac{8\rho}{a} - \frac{3}{2}\right)$; for a uniformly charged ring, it is $\left(\log\frac{8\rho}{a} - \frac{7}{4}\right)$.

## §24. The Stability of Charged Vortex Rings

The stability of a charged vortex ring can be studied using the techniques of §17 and §18. Now W = 0, and the equation of motion is given by (5-141). This can be done as follows. We recall the definition of the perturbed ring as

$$\underset{\sim}{R} = \left(R + r_1 e^{in\theta}\right)\underset{\sim}{e}_r + z_1 e^{in\theta}\ \underset{\sim}{e}_z. \qquad (5\text{-}142)$$

We now calculate

$$\lambda^2 \int \frac{\underset{\sim}{s} \wedge (\underset{\sim}{R} - \underset{\sim}{R}')\, ds'}{|\underset{\sim}{R} - \underset{\sim}{R}'|^3},$$

substitution of (5-142) gives

$$\lambda^2 \int \frac{\underset{\sim}{s}\wedge(\underset{\sim}{R} - \underset{\sim}{R}')\, ds'}{|\underset{\sim}{R} - \underset{\sim}{R}'|^3} = \frac{\lambda^2}{R}\left[\int \frac{(\cos\bar\theta - 1)d\bar\theta}{[2(1-\cos\bar\theta)]^{3/2}}\right]\underset{\sim}{e}_z$$

$$- \frac{\lambda^2}{R}\frac{z_1}{R}e^{in\theta}\left[\int \frac{(\cos n\bar\theta - 1)\, d\bar\theta}{[2(1-\cos\bar\theta)]^{3/2}}\right]\underset{\sim}{e}_r$$

$$+ \frac{\lambda^2}{R}\frac{r_1}{R}e^{in\theta}\left[\int \frac{e^{in\bar\theta}(\cos\bar\theta - 1) + (\cos n\bar\theta\cos\bar\theta - 1) - \frac{3}{2}(e^{in\bar\theta}+1)(\cos\bar\theta - 1)}{[2(1-\cos\bar\theta)]^{3/2}}\, d\bar\theta\right]\underset{\sim}{e}_z, \tag{5.143}$$

where we have used

$$\underset{\sim}{s} = in\frac{r_1}{R}e^{in\theta}\underset{\sim}{e}_r + \underset{\sim}{e}_\theta + in\frac{z_1}{R}e^{in\theta}\underset{\sim}{e}_z;\, ds' = (R + r_1 e^{in\theta}e^{in\bar\theta})d\bar\theta. \tag{5-144}$$

The integration over the osculating circle is similar to that done in §16, we have

$$\underset{\sim}{R}^\circ - \underset{\sim}{R} = \rho(\cos\bar\theta - 1)(-\underset{\sim}{n}) + \rho\sin\bar\theta \cdot \underset{\sim}{s}, \tag{5-145}$$

so that

$$(\underset{\sim}{R}^\circ - \underset{\sim}{R})\wedge\underset{\sim}{s} = \rho(\cos\bar\theta - 1)\,\underset{\sim}{b}, \tag{5-146}$$

and therefore

$$\lambda^2 \int \frac{\underset{\sim}{s} \wedge (\underset{\sim}{R} - \underset{\sim}{R}^{\circ})\, ds^{\circ}}{|\underset{\sim}{R} - \underset{\sim}{R}^{o}|^3} = \frac{\lambda^2}{R} \left( \int \frac{(\cos\bar{\theta} - 1)}{[2(1 - \cos\bar{\theta})]^{3/2}} d\bar{\theta} \right) \left( - n^2 \frac{z_1}{R} e^{in\theta} \right)$$

$$+ \frac{\lambda^2}{R} \left( \int \frac{(\cos\bar{\theta} - 1)\, d\bar{\theta}}{[2(1 - \cos\bar{\theta})]^{3/2}} \right) \left( 1 + (n^2 - 1) \frac{r_1}{R} e^{in\theta} \right). \qquad (5\text{-}147)$$

It is straightforward to check that (5-147) and (5-143) have the same singularities at $\bar{\theta} = 0$ and thus $\underset{\sim}{s} \wedge \underset{\sim}{G}$ is regular.

The stability equations can now be obtained by substituting into (5-141) and equating the coefficients to that of (5-23). In the limit when $\frac{a}{R} \to 0$ where only terms with $\log \frac{R}{a}$ need to be retained, we get

$$\frac{dr_1}{dt} = \frac{\Gamma}{4\pi R^2} n^2 \log \frac{R}{a} [1 - \mathcal{E}] z_1, \qquad (5\text{-}148)$$

$$\frac{dz_1}{dt} = \frac{\Gamma}{4\pi R^2} (1 - n^2) \log \frac{R}{a} [1 - \mathcal{E}] r_1, \qquad (5\text{-}149)$$

which gives a stable oscillation with frequency $\omega$ given by

$$\omega = \pm \frac{i\Gamma}{4\pi R^2} \frac{R}{a} (1 - \mathcal{E}) \sqrt{n^2(1 - n^2)}. \qquad (5\text{-}150)$$

Note that the modes zero and one are neutrally stable.

However, this limiting result is only valid if $|1 - \mathcal{E}|$ is finite. For $|1 - \mathcal{E}|$ very close to zero, (5-148) and (5-149) no longer represent the leading terms. In fact, it is suspected that as $|1 - \mathcal{E}| \to 0$, the ring can become unstable since those terms neglected in (5-148) and (5-149) would dictate the sign of $\omega^2$ and may very well give rise to an imaginary $\omega$. This conjecture is confirmed by numerical evaluation, and the results are exhibited in Fig. 5-5. It is found that for a given ratio $\frac{a}{R}$, there exists a band of values of $\mathcal{E}$ within which the ring is unstable both to long and short waves. These instability bands decrease in width as $\frac{a}{R} \to 0$. Furthermore, if we let $\mathcal{E}$ increase from 0, it will be seen that the instability sets in before the value of $\mathcal{E}$ at which the ring reverses direction is reached.

Hence we conclude that the charged vortex rings (conducting or uniformly charged) are unstable to centerline disturbances when the charge and the swirl effects become comparable. The instabilities are not limited to short waves only, as opposed to the case when no electric charges were present.

CONDUCTING RING

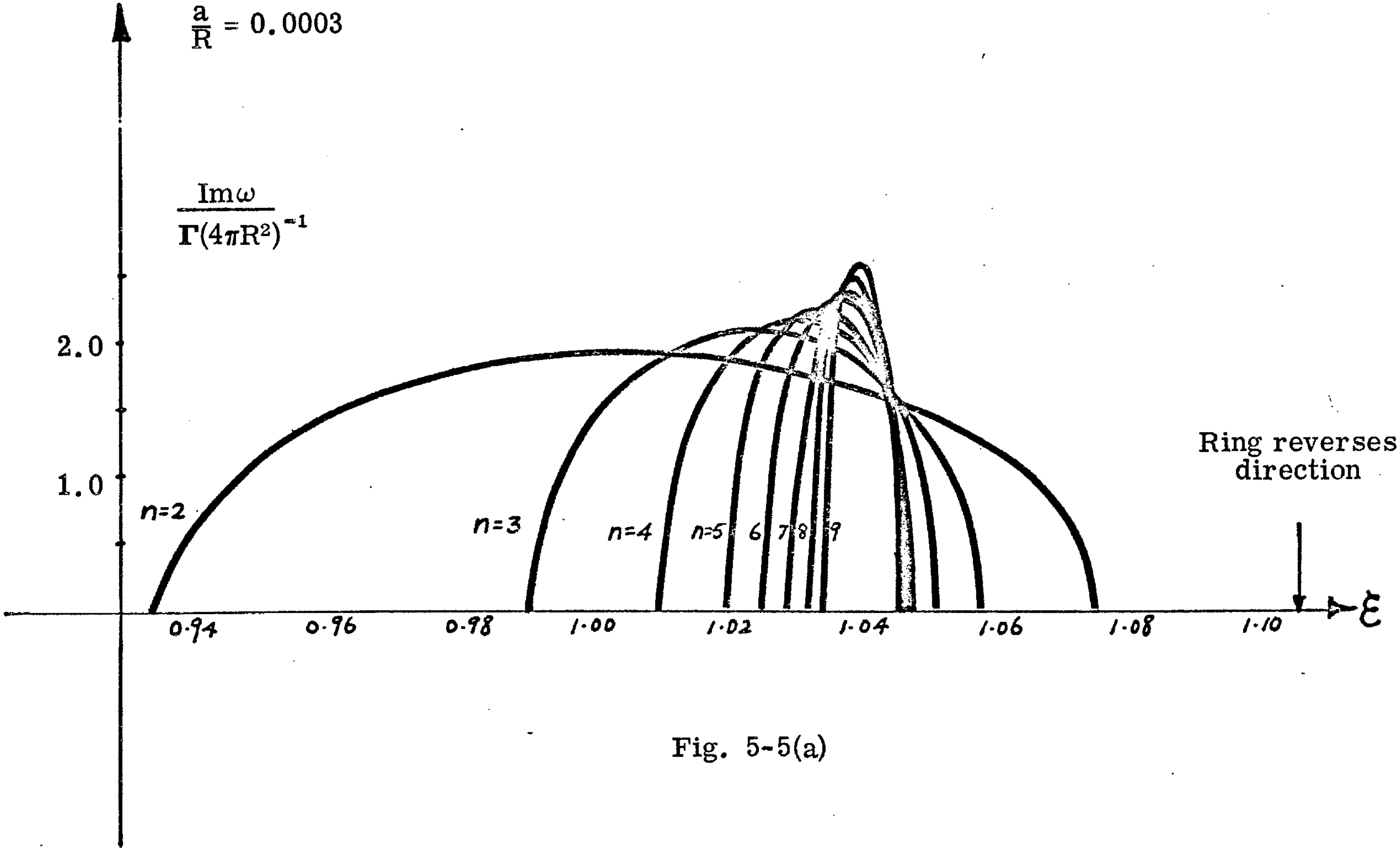
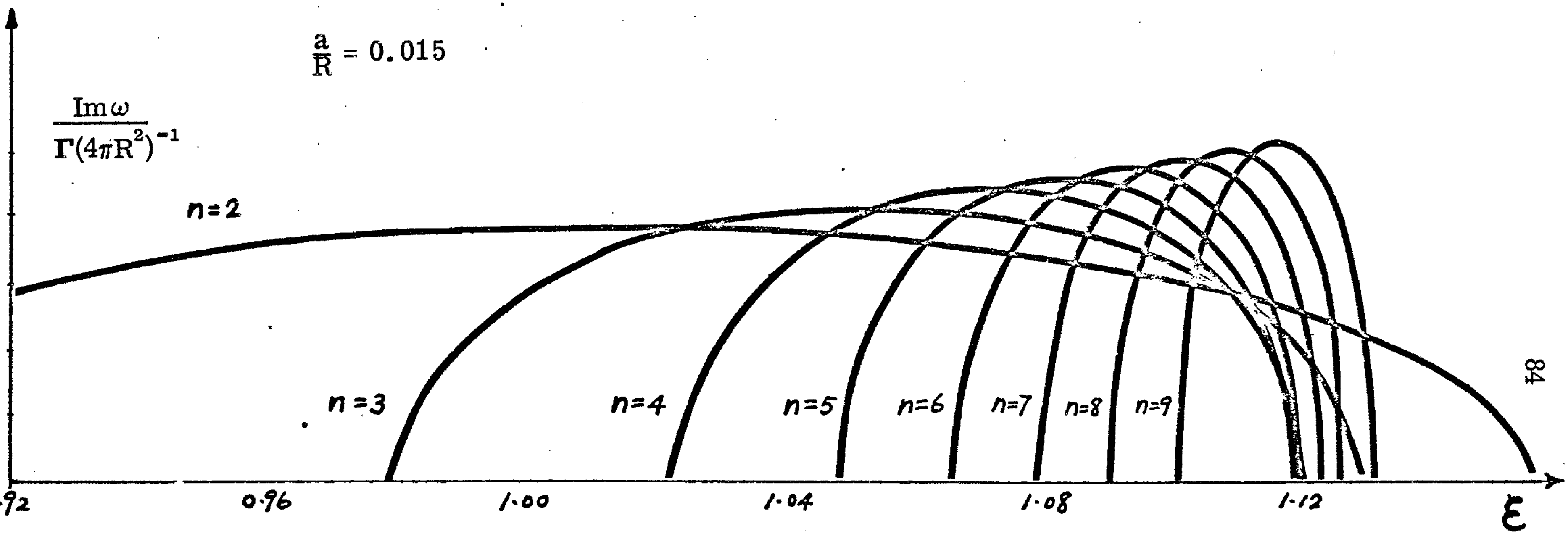



Fig. 5-5(a)




*Mode 2 starts to become unstable at* $\mathcal{E} = 0{\cdot}88$

*Ring reverses direction of propagation at* $\mathcal{E} = 1{\cdot}26$

*Modes 7, 8, 9 correspond to short waves* $\left(\frac{na}{R} > 0{\cdot}1\right)$

Fig. 5-5(b)

# VI. STABILITY OF AN UNSTEADY VORTEX RING--THE BUOYANT RING

§25. When light cream is dropped into cold water, a vortex ring is formed, but it rapidly disintegrates into a number of smaller rings. This process of disintegration is repeated several times for each of the daughter rings until they collide and mix turbulently. This phenomenon is vividly exhibited in the fluid mechanics film "Flow Instabilities" (Mollo-Christensen, 1969) and is known as vortex ring cascading. Results from the previous chapter show that vortex rings are stable to long wave disturbances; in this chapter, we shall include gravity and examine its effect on stability.

The behavior of vortex rings under gravity is of practical interest due to concern over the spread of radioactive clouds during a nuclear explosion where the heated gas eventually forms a vortex ring substantially lighter than the surrounding air. Of course, in such cases intense heat transfer is present and viscosity may also be important (Onufriev, 1967). However, we can consider the stability of a buoyant vortex ring in an inviscid fluid with no heat transfer as a first step towards a more complex theory, besides,it is probably a reasonable model for industrial chimney exhaust of waste gas which may be relevant to environmental engineers.

The existence of buoyancy effect renders the vortex ring unsteady. The velocity $V_0$, ring radius R and core radius a are all functions of time. If we call the direction in which gravitational force acts as "downwards," then we shall show that the heavy ring moving downwards or a light ring moving upwards will decelerate, expand, and become thinner; while a light ring moving downwards or a heavy ring moving upwards experience an opposite change, i.e., they accelerate, contract, and fatten. This may at first sound a little surprising, since one would expect a heavy ring moving in the direction of gravity to pick up momentum and therefore accelerate. However, it will be seen that while such a ring indeed picks up momentum, it does so not by accelerating but by increasing its radius R, and since the velocity is proportional to $R^{-1}$, the ring decelerates. The decreases in the core radius a is a consequence of the conservation of volume of the ring.

## §26. The Velocity of an Unperturbed Buoyant Vortex Ring

Let $m_1$ be the density of the fluid inside the ring and $m_0$ be the density of the surrounding fluid. The buoyancy force per unit length is given by

$$(m_1 - m_0)\, \underset{\sim}{g} \pi a^2. \tag{6-1}$$

If $\underset{\sim}{e}_z$ is the direction of motion of the vortex ring, then we shall consider

$$\underset{\sim}{g} = \pm\, g\, \underset{\sim}{e}_z\,, \tag{6-2}$$

which corresponds to the ring having its axis parallel to $\underset{\sim}{g}$. Two different cases are present, (i) $(m_1 - m_0)\,\pi a^2 \underset{\sim}{g} \cdot \underset{\sim}{e}_z > 0$ and (ii) $(m_1 - m_0)\pi a^2 \underset{\sim}{g} \cdot \underset{\sim}{e}_z < 0$. The first includes heavy rings travelling downwards or light rings travelling upwards, while the second includes light rings travelling downwards and heavy rings upwards. We refer to these cases as cases of negative and positive buoyancy respectively. The buoyancy force (6-1) must be included in the exterior force per unit length $\underset{\sim}{F}_E$.

We now consider the flow inside the core which is given by, to leading order;

$$\frac{1}{m_1}\frac{\partial p}{\partial \sigma} = \frac{v_\Omega^2}{\sigma} - g \sin\zeta\,, \tag{6-3}$$

$$\frac{1}{m_1 \sigma}\frac{\partial p}{\partial \zeta} = -\,g\cos\zeta. \tag{6-4}$$

These give

$$p = m_1 \int^{\sigma} \frac{v_\Omega^2}{\sigma}\, d\sigma - m_1 g \sigma \sin\zeta\,, \tag{6-5}$$

and when integrated over the cross section gives

$$\underset{\sim}{F}_I = m_1 \pi \frac{\partial}{\partial s}\left(\frac{1}{2}\, a^2 \overline{v^2}\, \underset{\sim}{s}\right), \tag{6-6}$$

since the term containing g integrates to zero. Thus $\underset{\sim}{F}_I$ is unchanged by the presence of gravity and (3-38) gives the velocity at a point of the unperturbed ring as

$$\frac{\partial \underset{\sim}{R}}{\partial t} = \frac{\Gamma}{4\pi R}\left(\log\frac{8R}{a} - \frac{1}{2} + \frac{Am_1}{m_0}\right)\underset{\sim}{e}_z + g\,\frac{(m_1 - m_0)\,\pi a^2}{\Gamma\, m_0}\,\underset{\sim}{e}_r, \tag{6-7}$$

where we have used the relationships

$$\underset{\sim}{s} = \underset{\sim}{e}_\theta, \qquad n = -\,\underset{\sim}{e}_r, \qquad \underset{\sim}{b} = \underset{\sim}{e}_z. \tag{6-8}$$

We note here that if $g = 0$ and $A = \frac{1}{4}$ then (6-7) reduces to the velocity of a uniform vortex ring with density difference but not under gravity, and agrees with the value given by Basset (1888). When $m_1 = 0$, we recover the hollow vortex (Hicks, 1884).

The radial component in the velocity which contains g causes a change in the radius of the ring R given by

$$\frac{dR}{dt} = \frac{g(m_1 - m_0)\pi a^2}{\Gamma\, m_0} = \frac{g(m_1 - m_0)}{\Gamma\, m_0}\;\frac{\mathcal{V}}{2\pi R} \tag{6-9}$$

where $\mathcal{V}$ is the volume of the ring, being

$$\mathcal{V} = 2\pi^2 a^2 R = \text{constant} \tag{6-10}$$

since the volume of the ring is conserved. Thus

$$R = R_0\,(1 + \beta t)^{\frac{1}{2}}\;, \tag{6-11}$$

where $R_0$ is the radius of the ring at $t = 0$ and $\beta$ is the buoyancy factor given by

$$\beta = \frac{g(m_1 - m_0)\,\mathcal{V}}{\pi \Gamma m_0 R_0^{\,2}} = \frac{g(m_1 - m_0)}{\Gamma\, m_0} \left(\frac{2\pi a_0^{\,2}}{R_0}\right) \tag{6-12}$$

with $a_0$ as the core radius at $t = 0$. The conservation of volume also gives

$$a = a_0\,(1 + \beta t)^{-1/4}. \tag{6-13}$$

Using (6-11) and (6-13), the propagation velocity of the ring is found to be

$$V_0 = \frac{\Gamma}{4\pi R_0\,(1 + \beta t)^{\frac{1}{2}}} \left\{\log \frac{8R_0}{a_0} - \frac{1}{2} + A\,\frac{m_1}{m_0} + \frac{3}{4}\log(1 + \beta t)\right\}. \tag{6-14}$$

From these we can see that the ring decelerates, expands, and gets thinner if $\beta > 0$, which corresponds to the negative buoyancy case. It must be pointed out that while the ring velocity approaches zero as $t \to \infty$, the ring never actually stops since $\int_0^t V_0 dt$ is unbounded as $t \to \infty$. For $\beta < 0$ (positive buoyancy), the ring accelerates, expands and fattens. In fact, the ring ceases to be a toroid at a finite time given by

$$t = \frac{1 - \left(\frac{a_0}{R_0}\right)^{4/3}}{-\beta} \tag{6-15}$$

which is the time when $a = R$. Of course, our equations of motion would no longer be valid if $\frac{a}{R} \not\ll 1$.

The tangential component of the force balance equation simply gives

$$\frac{\Gamma^2}{8\pi a^2}\,\frac{\partial a^2}{\partial s} = 0 \tag{6-16}$$

which implies that a is independent of t in the leading order.

## §27. Stability of a Buoyant Vortex Ring

The velocity of the perturbed buoyant ring is given by

$$\frac{\partial \underset{\sim}{R}}{\partial t} = \underset{\sim}{V}_I + \frac{\Gamma}{4\pi\rho}\left[\log\frac{8\rho}{a} - \frac{1}{2} + A\frac{m_1}{m_0}\right]\underset{\sim}{b} + \frac{g(m_1 - m_0)\pi a^2}{\Gamma m_0}\,\underset{\sim}{s}\wedge\underset{\sim}{e}_z . \tag{6-17}$$

We first examine the last term, with $\underset{\sim}{s}$ given by (5-144), we have

$$\frac{g(m_1 - m_0)\pi a^2}{\Gamma m_0}\left[\underset{\sim}{e}_r + in\frac{r_1}{R}\,e^{in\theta}\,\underset{\sim}{e}_\theta\right] . \tag{6-18}$$

The first two terms in the right hand side are very similar to those given in (5-54) and (5-55) for a perturbed vortex ring with axial flow except that now W = 0. Before we write them out, we first look at the tangential force balance equation, which is

$$-\frac{\Gamma^2}{8\pi a^2}\,\frac{\partial a^2}{\partial s} + \frac{g(m_1 - m_0)\pi a^2}{m_0}\,\underset{\sim}{e}_z\cdot\underset{\sim}{s} = 0. \tag{6-19}$$

By letting $\pi a^2 = \frac{\mathcal{V}}{2\pi R}$, we can integrate this equation with respect to $\theta$ to obtain

$$\log a = \log a_0 - \frac{1}{4} \log (1 + \beta t) + \frac{2\pi R_0{}^2}{\Gamma} \beta \frac{z_1}{R} e^{in\theta}. \tag{6-20}$$

This means that an $O\left(\frac{z_1}{R}\right)$ dependence on s is present in a, resulting in an additional $O\left(\frac{z_1}{R}\right)$ velocity in the $\underset{\sim}{e}_z$ direction of magnitude

$$-\frac{\beta}{2} \frac{R_0{}^2}{R^2} z_1 e^{in\theta} \underset{\sim}{e}_z \tag{6-21}$$

which must be included in the stability equations.

In the moving frame, we have

$$\frac{\partial \underset{\sim}{R}}{\partial t} = \frac{dR}{dt} \underset{\sim}{e}_r + \frac{dr_1}{dt} e^{in\theta} \underset{\sim}{e}_r + \frac{dz_1}{dt} e^{in\theta} \underset{\sim}{e}_z + V_0 \underset{\sim}{e}_z, \tag{6-22}$$

but $\frac{\partial \underset{\sim}{R}}{\partial t}$ given by (6-17) is

$$\frac{\partial \underset{\sim}{R}}{\partial t} = \frac{\beta R_0{}^2}{2R} \underset{\sim}{e}_r + \frac{\Gamma}{4\pi R} \left[\log \frac{8R}{a} - \frac{1}{2} + \frac{Am_1}{m_0}\right] \underset{\sim}{e}_z$$

$$+ V_z z_1 e^{in\theta} \underset{\sim}{e}_r + \left(V_r r_1 - \frac{\beta}{2} \frac{R_0{}^2}{R^2} z_1\right) e^{in\theta} \underset{\sim}{e}_z, \tag{6-23}$$

thus

$$V_0 = \frac{\Gamma}{4\pi R} \left[\log \frac{8R}{a} - \frac{1}{2} + \frac{Am_1}{m_0}\right], \tag{6-24}$$

$$\frac{dR}{dt} = \frac{\beta R_0{}^2}{2R}, \tag{6-25}$$

$$\frac{dr_1}{dt} = V_z z_1, \tag{6-26}$$

$$\frac{dz_1}{dt} = V_r\, r_1 - \frac{\beta}{2}\,\frac{R_0{}^2}{R^2}\, z_1, \tag{6-27}$$

and $V_z$ and $V_r$ are defined by

$$V_z = \frac{\Gamma}{4\pi R^2}\left[H(1) - \frac{1}{2}(H(n+1) + H(n-1)) + \frac{n}{2}(H(n+1) - H(n-1))\right], \tag{6-28}$$

$$V_r = \frac{\Gamma}{4\pi R^2}\left[2H(n) - \frac{1}{2}(H(n+1) + H(n-1)) - \frac{n}{2}(H(n+1) - H(n-1)) - H(1)\right.$$

$$\left. - \frac{3}{2}\left\{H(0) + H(n) - H(1) - \frac{1}{2}(H(n+1) + H(n-1))\right\} + 1\right], \tag{6-29}$$

where

$$H(n) = H(n;t) = n^2\left[\log\frac{a_0}{2R_0} + \log n + C - \frac{3}{2} + \frac{Am_1}{m_0} - \frac{3}{4}\log(1+\beta t)\right]$$

$$+ \frac{1}{8}\log(1+n^2) + G(n). \tag{6-30}$$

Notice that in (6-30) we have used (6-11) and (6-13) for the time dependence of a and R, this is consistent with (6-25).

If we write

$$V_z = \frac{\Gamma}{4\pi R_0^2}\, V_z{}^*, \tag{6-31}$$

$$V_r = \frac{\Gamma}{4\pi R_0^2}\, V_r{}^*, \tag{6-32}$$

and let

$$t^* = \frac{\Gamma}{4\pi R_0{}^2\beta}\,\log(1+\beta t), \tag{6-33}$$

we would get

$$\frac{dr_1}{dt^*} = V_z^* z_1, \tag{6-34}$$

$$\frac{dz_1}{dt^*} = V_r^* r_1 - \frac{2\pi\beta R_0^2}{\Gamma} z_1, \tag{6-35}$$

where now

$$H(n, t^*) = n^2 \left[\log \frac{a_0}{2R_0} + \log n + C - \frac{3}{2} + \frac{Am_1}{m_0} - \frac{3\pi R_0^2\beta}{\Gamma} t^*\right] + \frac{1}{8}\log(1+n^2)$$

$$+ G(n) . \tag{6-36}$$

In the limit of $\frac{a}{R} \to 0$, we have

$$V_z^* \sim -n^2 \left[\log \frac{R_0}{a_0} + \frac{3\pi R_0^2\beta}{\Gamma} t^*\right] \tag{6-37}$$

$$V_r^* \sim (n^2 - 1) \left[\log \frac{R_0}{a_0} + \frac{3\pi R_0^2\beta}{\Gamma} t^*\right] \tag{6-38}$$

and by letting

$$\alpha = \frac{\pi R_0^2\beta}{\Gamma} \left(\log \frac{R_0}{a_0}\right)^{-1} \tag{6-39}$$

we get

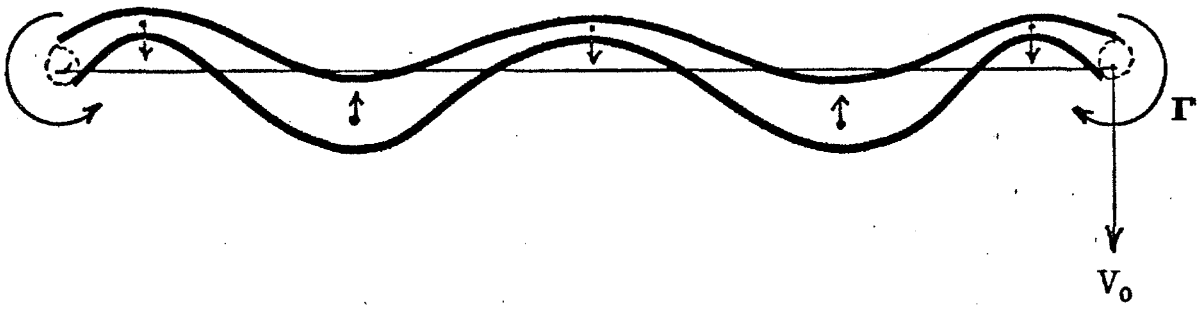
$$\left(\log \frac{R_0}{a_0}\right)^{-1} \frac{dr_1}{dt^*} = - n^2(1 + 3\alpha t^*) z_1, \tag{6-40}$$

$$\left(\log \frac{R_0}{a_0}\right)^{-1} \frac{dz_1}{dt^*} = (n^2 - 1)(1 + 3\alpha t^*) r_1 - 2\alpha z_1. \tag{6-41}$$

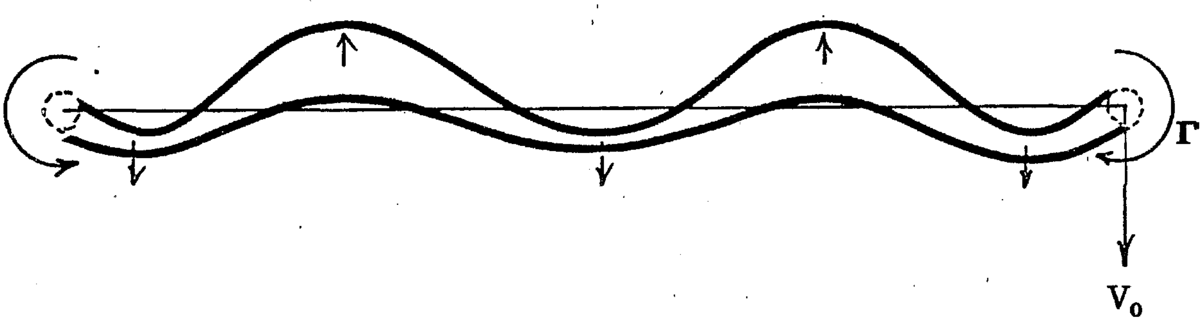
These equations represent a time dependent coupled oscillation with a growth or damping term, depending on the sign of $\alpha$. If $\alpha$ is positive, then the motion is damped and the system is stable; while for $\alpha$ negative, the motion is amplified and the system is unstable. From (6-38) we see that $\operatorname{sgn}(\alpha) = \operatorname{sgn}(\beta)$, therefore we conclude that negative buoyancy stabilizes the system while positive buoyancy renders the ring unstable. Numerical integration of (6-26) and (6-27) confirms this observation.

The mechanism of this stabilization or destabilization is as follows. The perturbation in the $\underset{\sim}{e}_z$ direction introduces a periodic dependence of the core radius a on the angle $\theta$. When $\beta > 0$, the dependence on a is such that it causes the portions moving ahead to fatten, those falling behind to thin. Since the dependence of the local velocity on a is like $\log \frac{1}{a}$, the fattened parts have velocities less than the mean velocity $V_0$ and the thinned parts have velocities greater than $V_0$. Thus a stabilizing mechanism is formed (see Fig. 6-1). For $\beta < 0$, the reverse situation happens and the ring becomes unstable because the thinned parts continue to move ahead faster than the mean velocity while the fattened parts slow down even further. Notice that the growth rate is independent of wave numbers and therefore all modes are equally unstable.

Some observations have been made on vortex rings formed by dropping light cream and heavy ink into water. It is found that the rings experience the cascading process in both cases. The number of daughter rings produced in each breakup ranges from 1 to 7 and with




(a) Stable configuration ($\beta > 0$)




(b) Unstable configuration ($\beta < 0$)

Fig. 6-1

no apparent pattern. Furthermore, the light rings break up much faster than the heavy ones. The eventual disintegration of the heavy rings shows that the foregoing theory does not completely explain the cascading phenomenon, and that the breakup of the heavy rings must be due to disturbances other than the long centerline perturbation. It is likely, however, that the light rings disintegrate due to the mechanism described. It must also be pointed out that these experiments are extremely crude and the rings formed are slow and fat and highly non-uniform. More precise experiments have to be performed in order that a valid comparison can be made.

## §28. Surface Tension Effect

In general, surface tension is present if the ring fluid is different from the surrounding fluid, and we shall examine its effects in this section. Let $\gamma$ be the surface tension, then the exterior force per unit length contains an additional term given by Moore and Saffman (1972) as

$$\frac{\partial}{\partial s}(2\pi a\gamma \underset{\sim}{s}) \, . \tag{6-42}$$

The interior pressure p now becomes

$$p = m_1 \int^{\sigma} \frac{v_\theta^2}{\sigma} dr - m_1 \, \sigma \, g \sin \zeta + \frac{\gamma}{a} \, , \tag{6-43}$$

and when integrated over the area gives

$$\underset{\sim}{F}_I = m_1 \frac{\partial}{\partial s} \left(\frac{\pi}{2} a^2 \overline{v}^2 \underset{\sim}{s}\right) - \frac{\partial}{\partial s}(\pi a\gamma \underset{\sim}{s}) . \tag{6-44}$$

Therefore, the presence of surface tension introduces a net force

$$\pi \frac{a\gamma}{\rho} \underset{\sim}{n} + \frac{\partial}{\partial s}(\pi a\gamma) \underset{\sim}{s}. \tag{6-45}$$

The term in the normal direction modifies the propagation velocity, leading to

$$V_0 = \frac{\Gamma}{4\pi R}\left\{\log \frac{8R}{a} - \frac{1}{2} + \frac{Am_1}{m_0} + \frac{4\pi^2\gamma a}{\Gamma^2 m_0}\right\}. \tag{6-46}$$

The term in the tangential direction must be included in the tangential force balance, so that now

$$\frac{\Gamma^2}{4\pi a}\frac{\partial a}{\partial s} - \frac{\partial}{\partial s}\left(\frac{\pi a\gamma}{m_0}\right) = \frac{g(m_1 - m_0)\pi a^2}{m_0} \underset{\sim}{e}_z \cdot \underset{\sim}{s}. \tag{6-47}$$

Let us define the dimensionless parameter $\Xi$ by

$$\Xi = \frac{4\pi^2 a_0 \gamma}{\Gamma^2 m_0}, \tag{6-48}$$

where $\Xi$ is the ratio of surface tension effect to swirl effect. Integration yields

$$\log a - \frac{a\Xi}{a_0} = \log a_0 - \frac{1}{4}\log(1+\beta t) - (1+\beta t)^{-\frac{1}{4}}\,\Xi + \frac{2\pi R_0{}^2\beta}{\Gamma R}\, z_1\, e^{in\theta}. \tag{6-49}$$

Since the combination

$$-\log a + \frac{a\Xi}{a_0}$$

appears in the leading order velocity, we can substitute (6-49) into (6-48) to obtain

$$\frac{\Gamma}{4\pi R_0 (1 + \beta t)^{1/2}} \left\{ \log \frac{8R_0}{a_0} - \frac{1}{2} + \frac{Am_1}{m_0} + \frac{3}{4} \log (1 + \beta t) + \Xi (1 + \beta t)^{-1/4} \right\} \underset{\sim}{e}_z$$

$$- \frac{\beta}{2(1 + \beta t)} z_1 e^{in\theta} \underset{\sim}{e}_z , \qquad (6\text{-}50)$$

which shows that the $O\left(\frac{z_1}{R}\right)$ contribution to $\underset{\sim}{e}_z$ is unaltered by the presence of surface tension, and the stability analysis in the previous section carries over completely. The only effect of $\gamma$ is to alter the mean propagation velocity of the ring.

## VII. THE INTRINSIC EQUATIONS OF MOTION

§29. Arms and Hama (Hama, 1962, 1963; Arms and Hama, 1965) introduced an approximate equation for the self-induced motion of a vortex filament which they used to study numerically the progressive deformation of vortex filaments with various initial profiles. This is known as the localized induction hypothesis in the sense that the motion of the filament is assumed to depend only on the local geometric structure, thus making the governing equation a differential, instead of an integral equation as in the Biot-Savart law. This hypothesis is valid for very thin filaments with geometric length scales much larger than the core radius a.

Betchov (1965) studied the analytical consequence of the localized induction equation by coupling it with the Frenet-Serret formulae of differential geometry to obtain a set of intrinsic equations, or equations which govern the curvature and torsion of the filament when it is considered as a space curve. They form a pair of coupled non-linear partial differential equations, from which he recovered the vortex ring and the helical vortex as particular solutions. He also found a stationary plane rotating loop solution which he claimed is not physical since it possesses a crossing point. Hasimoto (1972) rederived the intrinsic equations by a different method, the process of derivation showed that they are actually the real and imaginary of a nonlinear Schrödinger equation which occurs in several other contexts (Tsuzuki, 1971, Ono and Hasimoto, 1972). He also showed that the plane loop found by Betchov (1965) is the projection of a three-dimensional twist on the filament which he called a solitary wave.

This chapter derives a general set of intrinsic equations, independent of the localized induction hypothesis (which is contained as a particular case), the following chapters examine the consequences of these equations.

## §30. The Equation of Motion and its Relation to the Frenet-Serret Formulae

The general equation for the motion of a vortex filament can be put into the form

$$\left.\frac{\partial \underset{\frown}{R}}{\partial t}\right|_{\xi} = A\,\underset{\frown}{b} + B\,\underset{\frown}{n} + C\,\underset{\frown}{s} \qquad (7\text{-}1)$$

where the partial time derivative is taken at a fixed $\xi$($\xi$ as defined earlier, is a Lagrangian parameter along the filament), and A, B, C are functions of $\xi$ and t whose forms depend on the shape and structure of the filament. Of course, we recognize that in general the actual decomposition of the equation of motion (3-36) into the form of (7-1) may be rather cumbersome, but at the present moment we shall proceed on the assumption that this has been done.

We recall that the Frenet-Serret formulae of differential geometry give a set of relationships between the filament $\underset{\frown}{R}$ and its local tangent, normal and binormal, they are

$$\underset{\sim}{R}' = \underset{\sim}{s}, \tag{7-2}$$

$$\underset{\sim}{s}' = \kappa \underset{\sim}{n}, \tag{7-3}$$

$$\underset{\sim}{n}' = \tau \underset{\sim}{b} - \kappa \underset{\sim}{s}, \tag{7-4}$$

$$\underset{\sim}{b}' = -\tau \underset{\sim}{n}, \tag{7-5}$$

where $\kappa$ and $\tau$ are respectively the curvature and torsion of the filament, and $(\ )'$ denotes differentiation with respect to the arc-length parameter s. The idea is to get equations for $\kappa$ and $\tau$ by applying these to (7-1). However (7-1) as it stands now is not suitable for our purpose because the time differentiation is taken at fixed $\xi$ (instead of fixed s) which would not permit the interchange of order of differentiation when we differentiate (7-1) with respect to s. Therefore, we must calculate $\left.\frac{\partial \underset{\sim}{R}}{\partial t}\right|_s$.

The arclength s is a function of $\xi$ and t

$$s = s(\xi, t), \tag{7-6}$$

so that if we consider $\underset{\sim}{R} = \underset{\sim}{R}(s(\xi, t), t)$, we have

$$\left.\frac{\partial \underset{\sim}{R}}{\partial t}\right|_s = \left.\frac{\partial \underset{\sim}{R}}{\partial t}\right|_\xi - \left(\left.\frac{\partial \underset{\sim}{R}}{\partial s}\right|_t\right) \left(\left.\frac{\partial s}{\partial t}\right|_\xi\right). \tag{7-7}$$

Now, by (7-2) we know

$$\left.\frac{\partial \underset{\sim}{R}}{\partial s}\right|_t = \underset{\sim}{s}, \tag{7-8}$$

and $\left.\frac{\partial s}{\partial t}\right|_{\xi}$, which is the rate of change of arclength, can be expressed as

$$\left.\frac{\partial s}{\partial t}\right|_{\xi} = \frac{\partial}{\partial t} \int_{\xi_0}^{\xi} \left|\frac{\partial \underset{\sim}{R}}{\partial \xi}\right| d\xi = \int_{\xi_0}^{\xi} \frac{\partial}{\partial t} \left(\left|\frac{\partial \underset{\sim}{R}}{\partial \xi}\right|\right) d\xi. \tag{7-9}$$

The term in the integrand is calculated as follows:

$$\frac{\partial}{\partial t} \left(\left|\frac{\partial \underset{\sim}{R}}{\partial t}\right|\right) = \frac{\frac{\partial \underset{\sim}{R}}{\partial \xi}}{\left|\frac{\partial \underset{\sim}{R}}{\partial \xi}\right|} \cdot \frac{\partial}{\partial t}\left(\frac{\partial \underset{\sim}{R}}{\partial \xi}\right) = \underset{\sim}{s} \cdot \frac{\partial}{\partial t}\left(\frac{\partial \underset{\sim}{R}}{\partial \xi}\right),$$

since $\xi$ and t are independent of each other, we can interchange the order of differentiation and make use of (7-1) to get

$$\frac{\partial}{\partial t}\left(\frac{\partial \underset{\sim}{R}}{\partial \xi}\right) = \frac{\partial}{\partial \xi}\left[A\underset{\sim}{b} + B\underset{\sim}{n} + C\underset{\sim}{s}\right]$$

$$= \frac{\partial}{\partial s}\left[A\underset{\sim}{b} + B\underset{\sim}{n} + C\underset{\sim}{s}\right]\left(\left.\frac{\partial s}{\partial \xi}\right|_{t}\right)$$

$$= \left\{(A' + B\tau)\underset{\sim}{b} + (B' - A\tau + C\kappa)\underset{\sim}{n} + (C' - B\kappa)\underset{\sim}{s}\right\}\left(\left.\frac{\partial s}{\partial \xi}\right|_{t}\right), \tag{7-10}$$

where the last expression is obtained by applying the Frenet-Serret formulae. Therefore

$$\left.\frac{\partial s}{\partial t}\right|_{\xi} = \int_{\xi_0}^{\xi} (C' - B\kappa)\, \frac{\partial s}{\partial \xi}\, d\xi$$

$$= C - \int_0^s B\kappa ds + c(t). \qquad (7\text{-}11)$$

where we have chosen $s(\xi_0) = 0$ and $c(t)$ is a function of time, giving the tangential velocity of the point $s = 0$. Substitution into (7-9) and (7-7) gives

$$\left.\frac{\partial \hat{R}}{\partial t}\right|_{s} = A\hat{b} + B\hat{n} + \left[\int_0^s B\kappa ds + c(t)\right] \hat{s}. \qquad (7\text{-}12)$$

In fact, (7-12) can be obtained from (7-1) more directly if we note that the difference between

$$\left.\frac{\partial \hat{R}}{\partial t}\right|_{s} \text{ and } \left.\frac{\partial \hat{R}}{\partial t}\right|_{\xi}$$

must be in the tangential direction, so that we can write

$$\left.\frac{\partial \hat{R}}{\partial t}\right|_{s} = A\hat{b} + B\hat{n} + \hat{C}\,\hat{s} \qquad (7\text{-}13)$$

for some unknown $\hat{C}$. Now

$$\frac{\partial}{\partial s}\left(\left.\frac{\partial \hat{R}}{\partial t}\right|_{s}\right)$$

must be normal to $\hat{s}$ since it is merely $\frac{\partial}{\partial t}(\hat{s})$, and so by requiring

$$\frac{\partial}{\partial s}\left(\frac{\partial \underset{\frown}{R}}{\partial t}\Big|_s\right)$$

to have no tangential component we must have

$$\left(\frac{\partial \hat{C}}{\partial s}\right) - B\kappa \;=\; 0, \tag{7-14}$$

which immediately gives

$$\hat{C} = \int_0^s B\kappa ds + c(t), \tag{7-15}$$

and therefore (7-12). This simpler method of derivation was suggested by Professor P. G. Saffman.

## §31. The Complex Transformation and the Schrödinger Type Equations.

Let us introduce the following complex functions

$$\underset{\frown}{N} = (\underset{\frown}{n} + i\underset{\frown}{b})e^{i\int_0^s \tau ds}, \tag{7-16}$$

$$\psi = \kappa e^{i\int_0^s \tau ds}, \tag{7-17}$$

$$\phi = A e^{i\int_0^s \tau ds}, \tag{7-18}$$

$$\theta = B e^{i\int_0^s \tau ds}, \tag{7-19}$$

and note that

$$\underset{\sim}{N} \cdot \underset{\sim}{N} = 0, \;\; \underset{\sim}{N} \cdot \underset{\sim}{\overline{N}} = 2, \;\; \underset{\sim}{s} \cdot \underset{\sim}{N} = \underset{\sim}{s} \cdot \underset{\sim}{\overline{N}} = 0. \tag{7-20}$$

Now

$$\underset{\sim}{N}' = (\tau \underset{\sim}{b} - \tau \underset{\sim}{b} + i\tau \underset{\sim}{n} - i\tau \underset{\sim}{n} - \kappa \underset{\sim}{s}) e^{i \int_0^s \tau ds}$$

$$= - \psi \underset{\sim}{s}, \tag{7-21}$$

thus $(\underset{\sim}{N}')^{\cdot}$, where $(\;)^{\cdot}$ denotes $\frac{\partial}{\partial t}$, is given by

$$(\underset{\sim}{N}')^{\cdot} = - \dot{\psi} \underset{\sim}{s} - \psi \dot{\underset{\sim}{s}}. \tag{7-22}$$

$\dot{\underset{\sim}{s}}$ is calculated as follows

$$\dot{\underset{\sim}{s}} = (\underset{\sim}{R}')^{\cdot} = (\dot{\underset{\sim}{R}})'$$

$$= (A' + B\tau)\underset{\sim}{b} + (B' - A\tau)\underset{\sim}{n} + \kappa \left( \int_0^s B\kappa ds + c(t) \right) \underset{\sim}{n}, \tag{7-23}$$

which can be expressed in terms of $\underset{\sim}{N}$, $\psi$, $\phi$ and $\theta$ as

$$\dot{\underset{\sim}{s}} = \frac{i}{2}[\phi' \underset{\sim}{\overline{N}} - \overline{\Phi}' \underset{\sim}{N}] + \tfrac{1}{2}[\theta' \underset{\sim}{\overline{N}} + \overline{\theta}' \underset{\sim}{N}] + \tfrac{1}{2}[\psi \underset{\sim}{\overline{N}} + \overline{\psi} \underset{\sim}{N}] \left\{ \int_0^s \tfrac{1}{2}[\overline{\psi}\theta + \psi\overline{\theta}] ds + c(t) \right\}. \tag{7-24}$$

Substitution into (7-22) leads to

$$(\underset{\frown}{N}')^{\cdot} = -\dot{\psi}\underset{\frown}{s} - \frac{i\psi}{2}[\phi'\overline{\underset{\frown}{N}} - \overline{\phi}'\underset{\frown}{N}] - \frac{\psi}{2}[\theta'\overline{\underset{\frown}{N}} + \overline{\theta}'\underset{\frown}{N}] - \frac{\psi}{2}[\psi\overline{\underset{\frown}{N}} + \overline{\psi}\underset{\frown}{N}]\cdot$$

$$\left\{\int_0^s \tfrac{1}{2}[\overline{\psi}\theta + \psi\overline{\theta}]ds + c(t)\right\}. \tag{7-25}$$

We now calculate $(\dot{\underset{\frown}{N}})'$. We can let $\dot{\underset{\frown}{N}}$ to have the form

$$\dot{\underset{\frown}{N}} = f_1\underset{\frown}{N} + f_2\overline{\underset{\frown}{N}} + f_3\underset{\frown}{s}. \tag{7-26}$$

Since

$$0 = \frac{\partial}{\partial t}(\underset{\frown}{N}\cdot\overline{\underset{\frown}{N}}) = \dot{\underset{\frown}{N}}\cdot\overline{\underset{\frown}{N}} + \dot{\overline{\underset{\frown}{N}}}\cdot\underset{\frown}{N} = 2\,\overline{f}_1 + 2\,f_1, \tag{7-27}$$

we see that $f_1$ is imaginary and can be written as

$$f_1 = iZ, \tag{7-28}$$

where Z is some real function of s and t. Similarly,

$$0 = \frac{\partial}{\partial t}(\underset{\frown}{N}\cdot\underset{\frown}{N}) = 2\underset{\frown}{N}\cdot\dot{\underset{\frown}{N}} = 4f_2, \tag{7-29}$$

and

$$0 = \frac{\partial}{\partial t}(\underset{\frown}{N}\cdot\underset{\frown}{s}) = \dot{\underset{\frown}{N}}\cdot\underset{\frown}{s} + \underset{\frown}{N}\cdot\dot{\underset{\frown}{s}}$$

$$= f_3 + i\phi' + \theta' + \psi\left\{\int_0^s \tfrac{1}{2}[\overline{\psi}\theta + \overline{\theta}\psi]ds + c(t)\right\}, \tag{7-30}$$

so it follows that

$$f_2 = 0, \tag{7-31}$$

and

$$f_3 = - i\phi' - \theta' - \psi\left\{\int_0^s \tfrac{1}{2}[\overline{\psi}\theta + \psi\overline{\theta}]ds + c(t)\right\}. \qquad (7\text{-}31)$$

This gives us

$$(\mathring{\underset{\sim}{N}})' = i(Z'\underset{\sim}{N} + Z\underset{\sim}{N}' - \phi''\underset{\sim}{s} - \phi'\underset{\sim}{s}') - \left[\theta'' + \psi'\left\{\int_0^s \tfrac{1}{2}[\overline{\psi}\theta + \psi\overline{\theta}]ds + c(t)\right\}\underset{\sim}{s}\right.$$

$$\left. + \frac{\psi}{2}[\overline{\psi}\theta + \psi\overline{\theta}]\underset{\sim}{s} + \theta'\underset{\sim}{s}' + \psi\left\{\int_0^s \tfrac{1}{2}[\overline{\psi}\theta + \psi\overline{\theta}]ds + c(t)\right\}\underset{\sim}{s}'\right]. \qquad (7\text{-}32)$$

But

$$\underset{\sim}{s}' = \tfrac{1}{2}[\overline{\psi}\underset{\sim}{N} + \psi\underset{\sim}{\overline{N}}],$$

therefore

$$(\mathring{\underset{\sim}{N}})' = i\left[Z'\underset{\sim}{N} - Z\psi\underset{\sim}{s} - \phi''\underset{\sim}{s} - \frac{\phi'}{2}(\overline{\psi}\underset{\sim}{N} + \psi\underset{\sim}{\overline{N}})\right]$$

$$- \left[\theta'' + \psi'\left\{\int_0^s \tfrac{1}{2}[\overline{\psi}\theta + \psi\overline{\theta}]ds + c(t)\right\} + \frac{\psi}{2}(\overline{\psi}\theta + \psi\overline{\theta})\right]\underset{\sim}{s}$$

$$- \frac{\psi}{2}(\overline{\psi}\underset{\sim}{N} + \psi\underset{\sim}{\overline{N}})\left\{\int_0^s \tfrac{1}{2}[\overline{\psi}\theta + \psi\overline{\theta}]ds + c(t)\right\} - \frac{\theta'}{2}(\overline{\psi}\underset{\sim}{N} + \psi\underset{\sim}{\overline{N}}). \qquad (7\text{-}33)$$

We now equate the coefficients of $\underset{\sim}{s}$, $\underset{\sim}{N}$ and $\underset{\sim}{\overline{N}}$ in (7-25) and (7-33) to get

$$\dot{\psi} = i(Z\psi + \phi'') + \theta'' + \psi'\left\{\int_0^s \tfrac{1}{2}(\overline{\psi}\theta + \psi\overline{\theta})ds + c(t)\right\} + \frac{\psi}{2}(\overline{\psi}\theta + \psi\overline{\theta}), \qquad (7\text{-}34)$$

$$Z' = \tfrac{1}{2}(\overline{\psi}\phi' + \psi\overline{\phi}') + \tfrac{i}{2}(\overline{\psi}\theta' - \psi\overline{\theta}'), \tag{7-35}$$

the coefficients of $\underset{\frown}{N}$ in (7-25) and (7-33) are identical.

(7-34) and (7-35) form a system of complex non-linear integropartial differential equations of the Schrödinger type for the unknowns $\kappa$ and $\tau$, provided A and B are given (as functionals of $\kappa$ and $\tau$ or as functions of s and t or as both). Knowing $\kappa$ and $\tau$, we can resubstitute into the Frenet-Serret formulae to solve for $\underset{\frown}{R}$.

§32. Steady State Equations

Equations (7-34) and (7-35) are too complicated to be analysed in full. However, if we specialize to steady state and look for equations which describe a steadily propagating filament with no change of shape, we can reduce them to a set of ordinary integro-differential equations. To do this we assume that both $\kappa$ and $\tau$ are functions of s only, and similarly for A and B. This leads to

$$c\kappa' = \tau' A + 2\tau A' - B'' + \tau^2 B - \kappa^2 B - \kappa' \int_0^s B\kappa ds, \tag{7-36}$$

$$-c\tau\kappa = Z\kappa + A'' - \tau^2 A + \tau' B + 2\tau B' + \tau\kappa \int_0^s B\kappa ds, \tag{7-37}$$

$$Z' = (A' - \tau B)\kappa . \tag{7-38}$$

If we let $A = \kappa$ and $B = 0$, which we shall see would correspond to the localized induction hypothesis, then (7-36), (7-37) and (7-38) reduce to

$$c\kappa' = \tau'\kappa + 2\tau\kappa', \tag{7-39}$$

$$-c\tau\kappa = Z\kappa + \kappa'' - \tau^2\kappa. \tag{7-40}$$

$$Z' = \kappa'\kappa. \tag{7-41}$$

Equation (7-41) can be integrated to give

$$Z = \frac{\kappa^2}{2} + \Omega(t), \tag{7-42}$$

where $\Omega$ is an integration constant which in general is determined by the requirement that $\psi$ is real at $s = 0$; however, in a steady motion there is no initial conditions and $\Omega$ becomes an arbitrary constant. Therefore we have

$$(c - 2\tau)\kappa' = \tau'\kappa, \tag{7-43}$$

$$-(\Omega + c\tau)\kappa = \kappa'' - \tau^2\kappa + \frac{\kappa^3}{2}, \tag{7-44}$$

which are exactly the equations found by Hasimoto (1972), and have been shown to be equivalent to those of Betchov (1965).

# VIII. THE LOCALIZED INDUCTION HYPOTHESIS AND ITS COMPLETE STEADY STATE SOLUTIONS

§33. The localized induction concept assumes that the motion of the filament can be well approximated by retaining only the leading singular term in the Biot-Savart integral, which we recall is

$$\frac{\Gamma}{4\pi}\int\frac{(\hat{\underset{\frown}{R}}-\underset{\frown}{R})\wedge d\hat{\underset{\frown}{R}}}{|\underset{\frown}{R}-\hat{\underset{\frown}{R}}|^3} \quad . \tag{8-1}$$

To obtain this leading term,expand in Taylor series about the singular point $\hat{\underset{\frown}{R}}=\underset{\frown}{R}$. Let the point of interest be $\underset{\frown}{R}=\underset{\frown}{R}(\xi_0)$, then

$$\hat{\underset{\frown}{R}}=\underset{\frown}{R}(\xi_0+\xi)=\underset{\frown}{R}\;(\xi_0)+\xi\;\frac{\partial\underset{\frown}{R}}{\partial\xi}(\xi_0)+\frac{(\xi)^2}{2!}\;\frac{\partial^2\underset{\frown}{R}}{\partial\xi^2}\;(\xi_0)+\cdots \tag{8-2}$$

where $\xi$ is small. Now

$$\hat{\underset{\frown}{R}}-\underset{\frown}{R}=\xi\;\frac{\partial\underset{\frown}{R}}{\partial\xi}\;(\xi_0)+\frac{(\xi)^2}{2!}\;\frac{\partial^2\underset{\frown}{R}}{\partial\xi^2}\;(\xi_0)+\cdots, \tag{8-3}$$

and

$$d\hat{\underset{\frown}{R}}=\frac{d\hat{\underset{\frown}{R}}}{d\xi}\;d\xi=\frac{\partial\underset{\frown}{R}}{\partial\xi}(\xi_0)+\xi\;\frac{\partial^2\underset{\frown}{R}}{\partial\xi^2}\;(\xi_0)+\cdots, \tag{8-4}$$

so that the numerator of the integrand becomes

$$\left(\frac{\partial\underset{\frown}{R}}{\partial\xi}\;(\xi_0)\wedge\frac{\partial^2\underset{\frown}{R}}{\partial\xi^2}\;(\xi_0)\right)\;\frac{\xi^2}{2}\;, \tag{8-5}$$

and the denominator is just

$$|\xi|^{-3} \left|\frac{\partial \underset{\frown}{R}}{\partial \xi}\right|^{-3} (1 + O(\xi)). \tag{8-6}$$

Therefore, the integral (8-1) becomes

$$\frac{\Gamma}{4\pi} \int \frac{\dfrac{\partial \underset{\frown}{R}}{\partial \xi}(\xi_0) \wedge \dfrac{\partial^2 \underset{\frown}{R}}{\partial \xi^2}(\xi_0)}{2\left|\dfrac{\partial \underset{\frown}{R}}{\partial \xi}(\xi_0)\right|^3} \frac{d\xi}{|\xi|} . \tag{8-7}$$

If we integrate this over the range $\delta \leq |\xi| \leq d_0$, where $d_0$ is an arbitrary constant, we have

$$\frac{\Gamma}{4\pi} \left[\frac{\dfrac{\partial \underset{\frown}{R}}{\partial \xi}(\xi_0) \wedge \dfrac{\partial^2 \underset{\frown}{R}}{\partial \xi^2}(\xi_0)}{\left|\dfrac{\partial \underset{\frown}{R}}{\partial \xi}(\xi_0)\right|^3}\right] \log(\tfrac{1}{\delta}) + O(1) . \tag{8-8}$$

Here $\delta$ is assumed to be a small constant proportional to the core radius. We recognize that the term in the brackets is just $\kappa \underset{\frown}{b}$, where $\kappa$ is the local curvature and $\underset{\frown}{b}$ the local binormal. Thus we have the result that the velocity of the filament at a point is approximated by the equation

$$\frac{\partial \underset{\frown}{R}}{\partial t} \doteq \frac{\Gamma}{4\pi} \log\left(\frac{1}{\delta}\right) \kappa \underset{\frown}{b}. \tag{8-9}$$

By letting the normalized time t* to be

$$t^* = \frac{\Gamma}{4\pi} \log\left(\frac{1}{\delta}\right) t, \tag{8-10}$$

we obtain the localized induction equation

$$\frac{\partial \hat{R}}{\partial t^*} = \kappa \hat{b}. \tag{8-11}$$

We see that (8-11) is equivalent to the assumption that

$$B = 0, \; A = \frac{\Gamma}{4\pi} \log\left(\frac{1}{\delta}\right) \kappa, \tag{8-12}$$

or

$$\phi = \psi, \; \theta = 0. \tag{8-13}$$

This reduces (7-34), (7-35) to a single equation

$$\dot{\psi} = i\left[\psi'' + \psi\left(\frac{|\psi|^2}{2} + \Omega\right)\right] + c\psi', \tag{8-14}$$

with Ω being an arbitrary function of time. This is a nonlinear Schrödinger equation which is the same equation considered by Hasimoto (1972). Note that the real and imaginary parts of the steady equation are just (7-43) and (7-44), which must be true for consistency.

## §34. The Complete Solution of the Steady Localized Induction Equations

We shall now look for the solutions of (7-43) and (7-44). The first of these can be integrated to give

$$\tau = \tfrac{1}{2}\left(c - \frac{H}{\kappa^2}\right), \tag{8-15}$$

where H is a constant of integration which must be determined by an "initial condition" (i.e., a condition at a certain point on the filament). Substitution of this into (7-44) gives

$$\kappa'' = \nu^2\kappa - \frac{\kappa^3}{2} + \frac{H^2}{4\kappa^3}, \tag{8-16}$$

where

$$\nu^2 = -\Omega - \frac{c^2}{4}. \tag{8-17}$$

For $\nu$ real, (8-16) describes a non-linear oscillator with an inverse cubic restoring force; and for $\nu$ imaginary, no real solution exists. This can be seen by letting

$$\zeta = \kappa^2. \tag{8-18}$$

so that after integrating once, (8-16) becomes

$$(\zeta')^2 = 4\nu^2\zeta^2 - \zeta^3 - H^2 + J\zeta \equiv g(\zeta), \tag{8-19}$$

where J is another constant of integration. Equations (8-18) and (8-19) imply that for a real solution of $\kappa$ to exist, the polynomial

$g(\zeta)$ must possess two non-negative roots between which $g(\zeta)$ is positive. This is analgous to nonlinear oscillations in a potential well; if one of these roots is a double root, then an absorbing barrier exists and the period of oscillation becomes infinite.

Let us examine the roots of $g(\zeta)$. In order for two real roots to exist (which is equivalent to the requirement that all roots are real), the discriminant $\Delta(g)$ of $g(\zeta)$ must be non-negative, i.e.,

$$\Delta(g) = -2^8 \nu^6 H^2 - 2^4 \nu^4 J^2 + 2^3 3^2 J H^2 \nu^2 + 27H^4 + 2^2 J^3 \geq 0, \qquad (8\text{-}20)$$

which actually is a condition on J given H and $\nu$. The implications of the additional requirements that $g(\zeta)$ should be positive between two non-negative roots can be seen by writing $g(\zeta) = 0$ as

$$\zeta^3 = 4\nu^2\zeta^2 + J\zeta - H^2 \qquad (8\text{-}21)$$

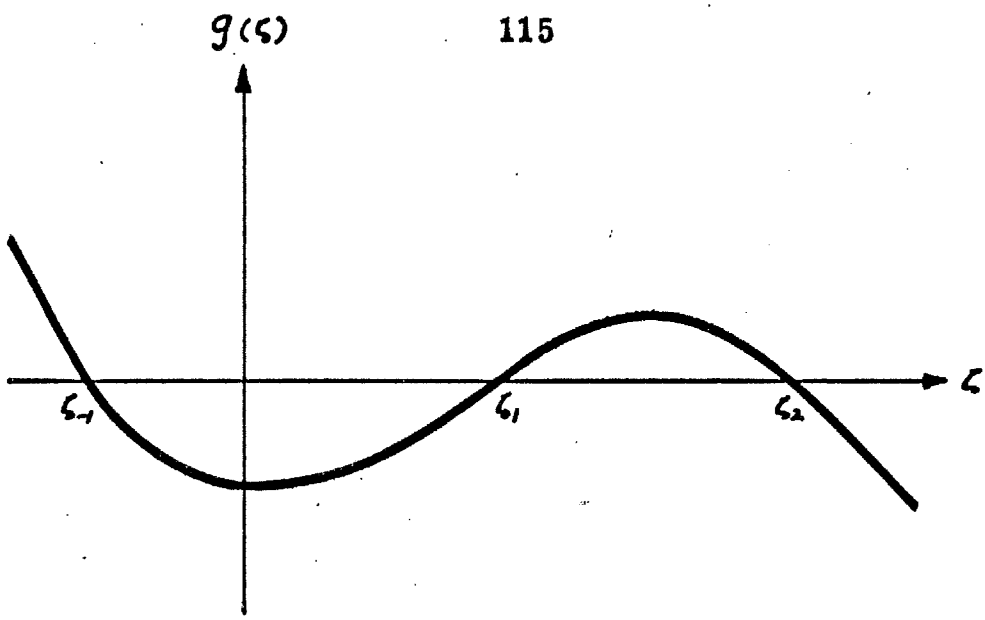
with some geometric arguments. First we note that we cannot have all three roots positive, since the sign of the constant term ($-H^2$) is the same as the sign of $\zeta^3$. Thus we only need to consider the following four cases:

(i) $\Delta \neq 0$, $H \neq 0$ (2 positive simple roots) [fig. 8 - 1(i)],

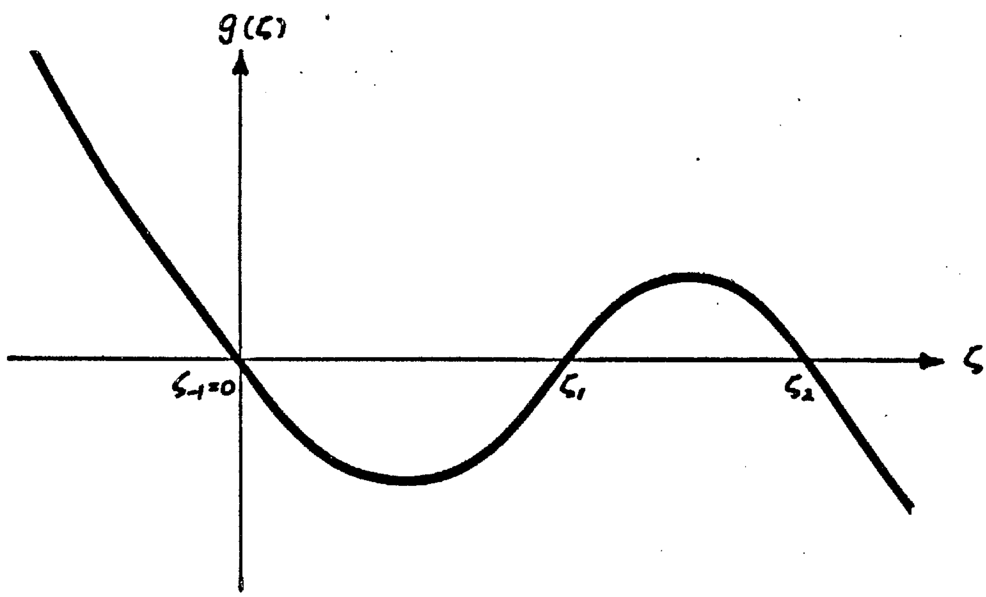
(ii) $\Delta \neq 0$, $H = 0$ (3 simple roots, one of them zero) [fig. 8 - 1(ii)],

(iii) $\Delta = 0$, $H = 0$ (a double root at zero) [fig. 8 - 1(iii)],

(iv) $\Delta = 0$, $H \neq 0$ (a positive double root) [fig. 8 - 1(iv)].

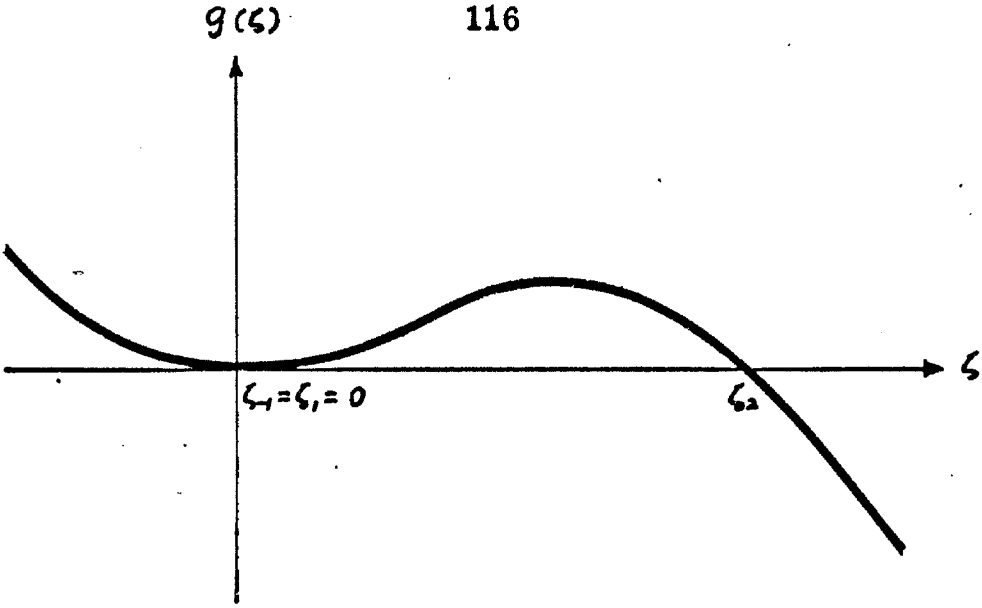



(i) $\Delta \neq 0$, $H \neq 0$, $\nu^2 > 0$ ($\nu^2 < 0$ not admissible)

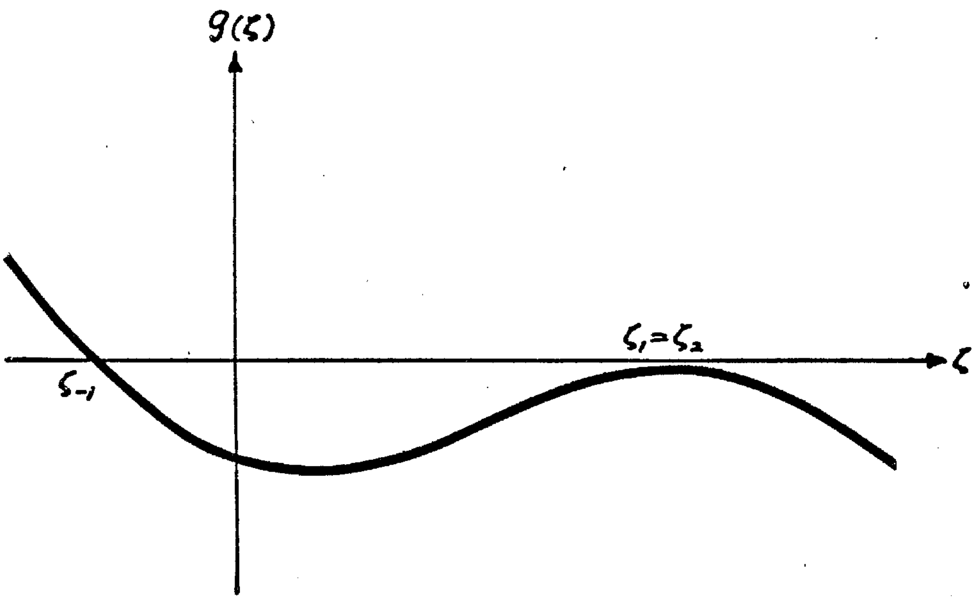



(ii) $\Delta \neq 0$, $H = 0$, $\nu^2 > 0$

Fig. 8-1




(iii) $\Delta = 0$ , $H = 0$ , $\nu^2 > 0$




(iv) $\Delta = 0$, $H \neq 0$ , $\nu^2 > 0$
(if $\nu^2 = 0$, triple root at $\zeta = 0$)

Fig. 8-1

Let us write the cubic as

$$g(\zeta) = (\zeta_2 - \zeta)(\zeta - \zeta_1)(\zeta + \zeta_{-1}) \tag{8-22}$$

where $\zeta_2$, $\zeta_1$, $\zeta_{-1}$ are all non-negative and $\zeta_2 \geqslant \zeta_1$. By letting $\eta = \zeta + \zeta_{-1}$, we have

$$g(\eta) = (\zeta_2 + \zeta_{-1} - \eta)(\eta - \zeta_1 - \zeta_{-1})\eta = (\eta_2 - \eta)(\eta - \eta_1)\eta \tag{8-23}$$

where $\eta_1 = \zeta_1 + \zeta_{-1}$, $\eta_2 = \zeta_2 + \zeta_{-1}$ are both non-negative. If we further let $t^2 = \eta$, the equation (8-19) transforms to

$$2tt' = \sqrt{g(t)} \tag{8-24}$$

$$\frac{s}{2} = \int_t^{\eta_2} \frac{dt}{\sqrt{(t^2 - \eta_1)(\eta_2 - t^2)}} \; ; \quad \eta_1 \leqslant t \leqslant \eta_2, \tag{8-25}$$

which is a standard elliptic integral, and has the solution

$$t = \sqrt{\eta_2}\ \mathrm{dn}\ \left(\frac{\sqrt{\eta_2}s}{2}\ \Big|\ \frac{\eta_2 - \eta_1}{\eta_2}\right) . \tag{8-26}$$

Since $\kappa = \zeta^{\frac{1}{2}} = (\eta - \zeta_{-1})^{\frac{1}{2}} = (t^2 - \zeta_{-1})^{\frac{1}{2}}$, we have

$$\kappa = \left\{(\zeta_2 + \zeta_{-1})\ \mathrm{dn}^2\ \left((\zeta_2 + \zeta_{-1})^{\frac{1}{2}}\ \frac{s}{2}\ \Big|\ \frac{\zeta_2 - \zeta_1}{\zeta_2 + \zeta_{-1}}\right) - \zeta_{-1}\right\}^{\frac{1}{2}} \tag{8-27}$$

where $\mathrm{dn}(\cdot\,|\,m)$ denotes the Jacobian elliptic function of the third kind with modulus m. Case (i) corresponds to $\zeta_2$, $\zeta_1$, $\zeta_{-1}$ all non-negative, which is a periodic wave with both $\tau$ and $\kappa$ varying. Case (iii) is when $\zeta_1 = 0$, and corresponds to the case when $\tau$ is constant but $\kappa$ is varying periodically. If $\tau = c = 0$, then it reduces to an elastica form found by Hasimoto (1971). Case (ii) corresponds to $\zeta_1 = \zeta_{-1} = 0$. Here the modulus m is 1 and dn degenerates to the hyperbolic secant sech. We can also check that $\zeta_2$ is just $4\nu^2$ in this case, and therefore

$$\kappa = 2\nu\, \mathrm{dn}^2\, (\nu s\,|1)$$

$$= 2\nu \,\mathrm{sech}\, \nu s, \tag{8-28}$$

$$\tau = \frac{c}{2}. \tag{8-29}$$

This is the solitary wave solution found by Hasimoto (1972). Case (iv) is when $\zeta_2 = \zeta_1$ and the solution for $\kappa$ degenerates to a constant, ($\kappa = \kappa_0$), and the expression

$$\Omega = -\nu^2 - \frac{c^2}{4}$$

simply gives the relation of the rotation rate to the curvature and torsion of the helix. If $c = 0$, then the solution represents a vortex ring with radius $\frac{1}{\kappa_0}$. Finally, if $c = 0$ and $\kappa_0 = 0$, we have a straight filament.

## §35. The Form of $\underset{\frown}{R}(\xi, t)$ Corresponding to the Solitary Wave Solution

From the Fundamental Theorem for space curves, we know that given $\kappa$ and $\tau$, there exists one and only one space curve, determined completely but for its position in space with the curvature $\kappa$ and torsion $\tau$. Thus we can recover the shape of the solitary wave solution from (8-28) and (8-29) via the Frenet-Serret formulae. Making use of the fact that $\tau$ is a constant, we have

$$\hat{b}'' = -\tau\hat{n}' = -\tau(-\kappa\hat{s} + \tau\hat{b}), \tag{8-30}$$

or

$$\hat{b}'' + \tau^2\hat{b} = \tau\kappa\hat{s}, \tag{8-31}$$

differentiating, we get

$$\left\{\frac{1}{\kappa}\left[\hat{b}'' + \tau^2\hat{b}\right]\right\}' = \tau\kappa\hat{n} = -\kappa\hat{b}', \tag{8-32}$$

which is just

$$\hat{b}''' - \frac{\kappa'}{\kappa}\hat{b}'' + (\tau^2 + \kappa^2)\hat{b}' - \frac{\kappa'}{\kappa}\tau^2\hat{b} = 0. \tag{8-33}$$

Substituting values of $\kappa$ and $\tau$ from (8-28) and (8-29) gives

$$\frac{d^3\hat{b}}{d\sigma^3} + \tanh\sigma\frac{d^2\hat{b}}{d\sigma^2} + \left(\frac{\tau^2}{\nu^2} + 4\,\mathrm{sech}^2\sigma\right)\frac{d\hat{b}}{d\sigma} + \frac{\tau^2}{\nu^2}\tanh\sigma\,\hat{b} = 0, \tag{8-34}$$

where $\sigma = \nu s$. We now define the vector $\hat{B}$ as

$$\hat{B} = \frac{d\hat{b}}{d\sigma} + \tanh\sigma\,\hat{b}. \tag{8-35}$$

Then we have

$$\frac{d^2\underset{\frown}{B}}{d\sigma^2} + \left(\frac{\tau^2}{\nu^2} + 2\,\mathrm{sech}^2\sigma\right)\underset{\frown}{B} = 0, \qquad (8\text{-}36)$$

which has solutions

$$0,\ \left(i\,\frac{\tau}{\nu} + \tanh\sigma\right)e^{i\frac{\tau}{\nu}\sigma},\ \left(-\frac{i\tau}{\nu} + \tanh\sigma\right)e^{-\frac{i\tau}{\nu}\sigma}. \qquad (8\text{-}37)$$

From these we can construct a solution for $\underset{\frown}{b}$:

$$\underset{\frown}{b} = \begin{pmatrix} 2\mu\,\frac{\tau}{\nu}\,\mathrm{sech}\,\sigma \\ \mu\left[-\left(1-\frac{\tau^2}{\nu^2}\right)\sin\frac{\tau}{\nu}\sigma + \frac{2\tau}{\nu}\tanh\sigma\cos\frac{\tau}{\nu}\sigma\right] \\ \mu\left[\left(1-\frac{\tau^2}{\nu^2}\right)\cos\frac{\tau}{\nu}\sigma + \frac{2\tau}{\nu}\tanh\sigma\sin\frac{\tau}{\nu}\sigma\right] \end{pmatrix}, \qquad (8\text{-}38)$$

where

$$\mu = \nu^2(\nu^2+\tau^2)^{-1}. \qquad (8\text{-}39)$$

The normalizing constants are determined by the requirement that $\underset{\frown}{b}$ is a unit vector, and some ± signs are dropped since they would not alter the shape of the curve in view of the fundamental theorem for space curves. Successive integration gives $\underset{\frown}{n}$ and $\underset{\frown}{s}$ as

$$\underset{\frown}{n} = \begin{pmatrix} 2\mu\,\mathrm{sech}\,\sigma\tanh\sigma \\ -\left[(1-2\mu\tanh^2\sigma)\cos\frac{\tau}{\nu}\sigma - 2\mu\frac{\tau}{\nu}\tanh\sigma\sin\frac{\tau}{\nu}\sigma\right] \\ -\left[(1-2\mu\tanh^2\sigma)\sin\frac{\tau}{\nu}\sigma + 2\mu\frac{\tau}{\nu}\tanh\sigma\cos\frac{\tau}{\nu}\sigma\right] \end{pmatrix}, \qquad (8\text{-}40)$$

$$\underset{\frown}{s} = \begin{pmatrix} 1 - 2\mu \ \mathrm{sech}^2\sigma \\ -2\mu \ \mathrm{sech}\ \sigma \left(\tanh\sigma \cos\frac{\tau}{\nu}\sigma + \frac{\tau}{\nu}\sin\frac{\tau}{\nu}\sigma\right) \\ -2\mu \ \mathrm{sech}\ \sigma \left(\tanh\sigma \sin\frac{\tau}{\nu}\sigma - \frac{\tau}{\nu}\cos\frac{\tau}{\nu}\sigma\right) \end{pmatrix} . \tag{8-41}$$

Integrating $\underset{\frown}{s}$ gives $\underset{\frown}{R}(s)$ as

$$\underset{\frown}{R}(s) = \begin{pmatrix} s - \frac{2\mu}{\nu}\tanh\nu s \\ \frac{2\mu}{\nu}\ \mathrm{sech}\ \nu s \cos\tau s \\ \frac{2\mu}{\nu}\ \mathrm{sech}\ \nu s \ \sin\tau s \end{pmatrix} . \tag{8-42}$$

The time dependence of $\underset{\frown}{R}$ is established by the relation

$$\frac{\partial \underset{\frown}{R}}{\partial t^*} = \kappa \underset{\frown}{b} + c\underset{\frown}{s} \tag{8-43}$$

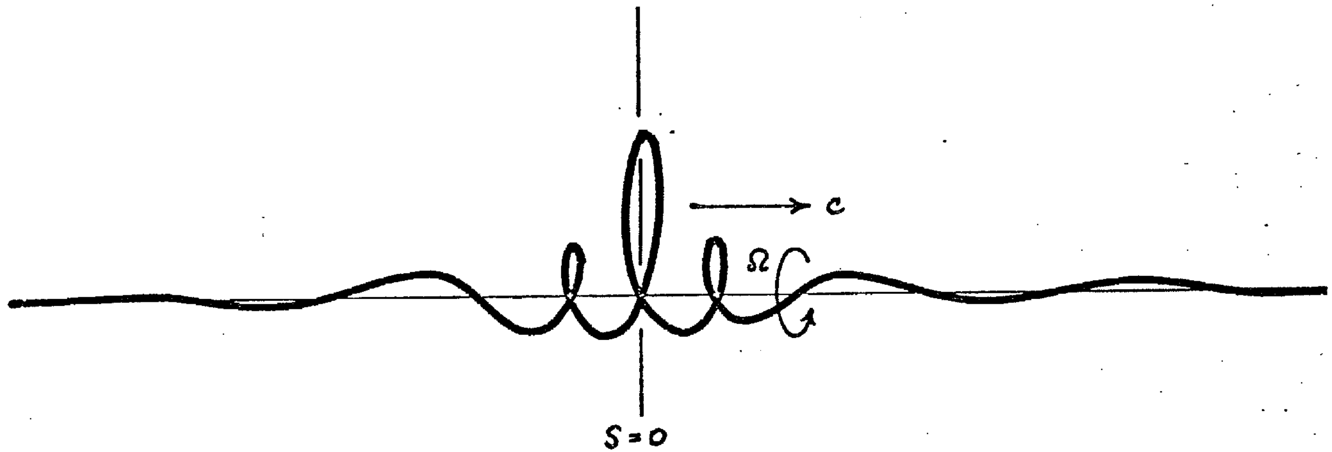
which yields

$$\underset{\frown}{R}(s, t^*) = \begin{pmatrix} s + ct^* - \frac{2\mu}{\nu}\tanh\nu s \\ \frac{2\mu}{\nu}\ \mathrm{sech}\ \nu s \cos(\tau s - \Omega t^*) \\ \frac{2\mu}{\nu}\ \mathrm{sech}\ \nu s \sin(\tau s - \Omega t^*) \end{pmatrix}, \tag{8-44}$$

and the velocity

$$\frac{\partial \underset{\frown}{R}}{\partial t^*}(s, t^*) = \begin{pmatrix} c \\ \Omega\frac{2\mu}{\nu}\ \mathrm{sech}\ \nu s \sin(\tau s - \Omega t^*) \\ -\Omega\frac{2\mu}{\nu}\ \mathrm{sech}\ \nu s \cos(\tau s - \Omega t^*) \end{pmatrix} . \tag{8-45}$$

At $s = 0$, we see that this represents a point propagating with velocity c and rotating at an angular velocity $\Omega$ about a circle of radius $\frac{2\mu}{\nu}$. A numerical plot of the filament shape has been given by Hasimoto (1972), a rough sketch is given in Fig. 8-2.




Solitary Wave on a Vortex Filament

Fig. 8-2

## IX. THE MODIFIED LOCALIZED INDUCTION HYPOTHESIS

§36. Let us now take a closer look at the assumptions of the localized induction hypothesis. We recall that the governing equation is (8-4), which is

$$\frac{\partial \underset{\sim}{R}}{\partial t} = \frac{\Gamma}{4\pi} \log\left(\frac{1}{\delta}\right) \kappa \underset{\sim}{b}. \tag{9-1}$$

The assumption made was that $\log\left(\frac{1}{\delta}\right)$ can be considered as a large constant so that the time scaling in (8-10) is independent of s. Detailed examination of the singular term shows that $\delta$ is in fact proportional to $\kappa a$ where a is the core radius and therefore is a function of $\kappa$. Thus the transformation (8-10) depends on s through $\kappa$, and $t^*$ and s cannot be justifiably considered as independent variables. Furthermore, as the filament becomes straight (as in the case for the tail ends of the solitary wave), $\kappa a \to 0$ and $\log\left(\frac{1}{\delta}\right) \to \infty$ which renders $t^*$ undetermined.

To eliminate these problems we considered a modified equation in which the logarithmic dependence is included

$$\frac{\partial \underset{\sim}{R}}{\partial t^*} = \kappa^*(1 - \epsilon \log \kappa^*)\, \underset{\sim}{b}, \tag{9-2}$$

where now

$$\kappa^* = \frac{\kappa}{K} \tag{9-3}$$

is the dimensionless curvature, normalized against some characteristic curvature k (say, $\kappa$ at s = 0), and

$$t^* = \frac{\Gamma}{4\pi} k \log\left(\frac{1}{ka}\right) t = \frac{\Gamma}{4\pi} k \epsilon^{-1} t, \qquad (9\text{-}4)$$

with

$$\epsilon = \left(\log\left(\frac{1}{ka}\right)\right)^{-1}. \qquad (9\text{-}5)$$

Notice that t* defined by (9-4) is independent of $\kappa$. The dimensional form of (9-2) is just

$$\frac{\partial \underset{\sim}{R}}{\partial t} = \frac{\Gamma}{4\pi} \kappa \log\left(\frac{1}{\kappa a}\right) \underset{\sim}{b}. \qquad (9\text{-}6)$$

## §37. Solutions of the Steady Modified Localized Induction Equations

Equation (9-2) corresponds to the case

$$A = \kappa^*(1 - \epsilon \log \kappa^*), \quad B = 0. \qquad (9\text{-}7)$$

Substituting these into (7-36) - (7-38) and dropping the superscript* (since the context is clear), we have

$$[c - 2\tau(1 - \epsilon(\log \kappa + 1))]\kappa' = \kappa(1 - \epsilon \log \kappa)\tau', \qquad (9\text{-}8)$$

$$-(\Omega + c\tau)\kappa = \kappa''(1 - \epsilon(\log \kappa + 1)) - \frac{\epsilon\kappa'^2}{\kappa} - \tau^2\kappa(1 - \epsilon \log \kappa)$$

$$+ \frac{\kappa^3}{2}\left(1 - \epsilon\left(\log \kappa + \tfrac{1}{2}\right)\right). \qquad (9\text{-}9)$$

First we check that for $\epsilon = 0$, this reduces to the equations of the localized induction hypothesis (we must keep in mind that $\epsilon = 0$ would make t* undefined and is therefore only a formal check).

The first of these equations can be integrated to give $\tau$ in terms of $\kappa$

$$\tau = \frac{1}{2(1 - \epsilon \log \kappa)} \left\{ c\left[1 - \epsilon\left(\log \kappa - \tfrac{1}{2}\right)\right] - \frac{H}{\kappa^2}\right\}. \qquad (9\text{-}10)$$

This reduces to (8-15) when $\epsilon = 0$. Furthermore, $\tau$ given by (9-10) approaches $\frac{c}{2}$ in both limits of large and small $\kappa$, which is an indication that the logarithmic term only has a weak effect on the solution form. We substitute (9-10) into (9-9) to get an equation for $\kappa$

$$\kappa''(1 - \epsilon(\log \kappa + 1)) - \frac{\epsilon \kappa'^2}{\kappa} + \left\{\Omega + \frac{c^2}{4(1 - \epsilon \log \kappa)} \left[1 - \frac{1}{(1 - \epsilon \log \kappa)^2} \cdot \left(\frac{H}{c\kappa^2} - \frac{\epsilon}{2}\right)^2\right]\right\} \kappa + \frac{\kappa^3}{2}\left(1 - \epsilon\left(\log \kappa + \tfrac{1}{2}\right)\right) = 0. \qquad (9\text{-}11)$$

Let us introduce $W(\kappa)$ as

$$W(\kappa) = \frac{\kappa'^2}{2}, \qquad (9\text{-}12)$$

and rewrite (9-11) as an equation for W in $\kappa$

$$\frac{dW(\kappa)}{d\kappa} = \epsilon \frac{2W}{\kappa(1-\epsilon(\log\kappa+1))} + \frac{\kappa}{(1-\epsilon(\log\kappa+1))}\left\{\Omega + \frac{c^2}{4(1-\epsilon\log\kappa)^2}\cdot\right.$$

$$\left.\left[1 - \frac{1}{(1-\epsilon\log\kappa)^2}\left(\frac{H}{c\kappa^2} - \frac{\epsilon}{2}\right)^2\right]\right\} + \frac{\kappa^3}{2}\,\frac{(1-\epsilon(\log\kappa+\frac{1}{2}))}{(1-\epsilon(\log\kappa+1))} = 0, \qquad (9\text{-}13)$$

which can be integrated to give

$$W(\kappa) = \frac{1}{[1-\epsilon(\log\kappa+1)]^2}\left\{-\Omega\left(1-\epsilon\left(\log\kappa+\tfrac{1}{2}\right)\right)\frac{\kappa^2}{2} - \frac{c^2\kappa^2}{8}\cdot\right.$$

$$\left[1 - \frac{\epsilon^2}{2\kappa^2}\int^{\kappa}\frac{x[1-\epsilon(\log x+1)dx}{[1-\epsilon\log x]^3}\right]$$

$$+\frac{HC\epsilon}{4}\left[\frac{1}{(1-\epsilon\log\kappa)} - \frac{\epsilon}{2(1-\epsilon\log\kappa)^2}\right]$$

$$-\frac{H^2}{8}\,\frac{1}{\kappa^2(1-\epsilon\log\kappa)^2} + \frac{J}{8}$$

$$\left.-\frac{\kappa^4}{8}\left[1-\epsilon(2\log\kappa+1)+\epsilon^2((\log\kappa)^2+\log\kappa+1)\right]\right\}, \qquad (9\text{-}14)$$

where J is the same integration constant as in (8-19). Again, by letting $\zeta = \kappa^2$, we have

$$(\zeta')^2 = g(\zeta) \equiv 8W(\zeta), \qquad (9\text{-}15)$$

therefore

$$g(\zeta) = \frac{1}{[1 - \epsilon(\frac{1}{2}\log\zeta + 1)]^2}\left\{- 4\Omega\left[1 - \tfrac{1}{2}\epsilon(\log\zeta + 1)\right]\zeta^2\right.$$

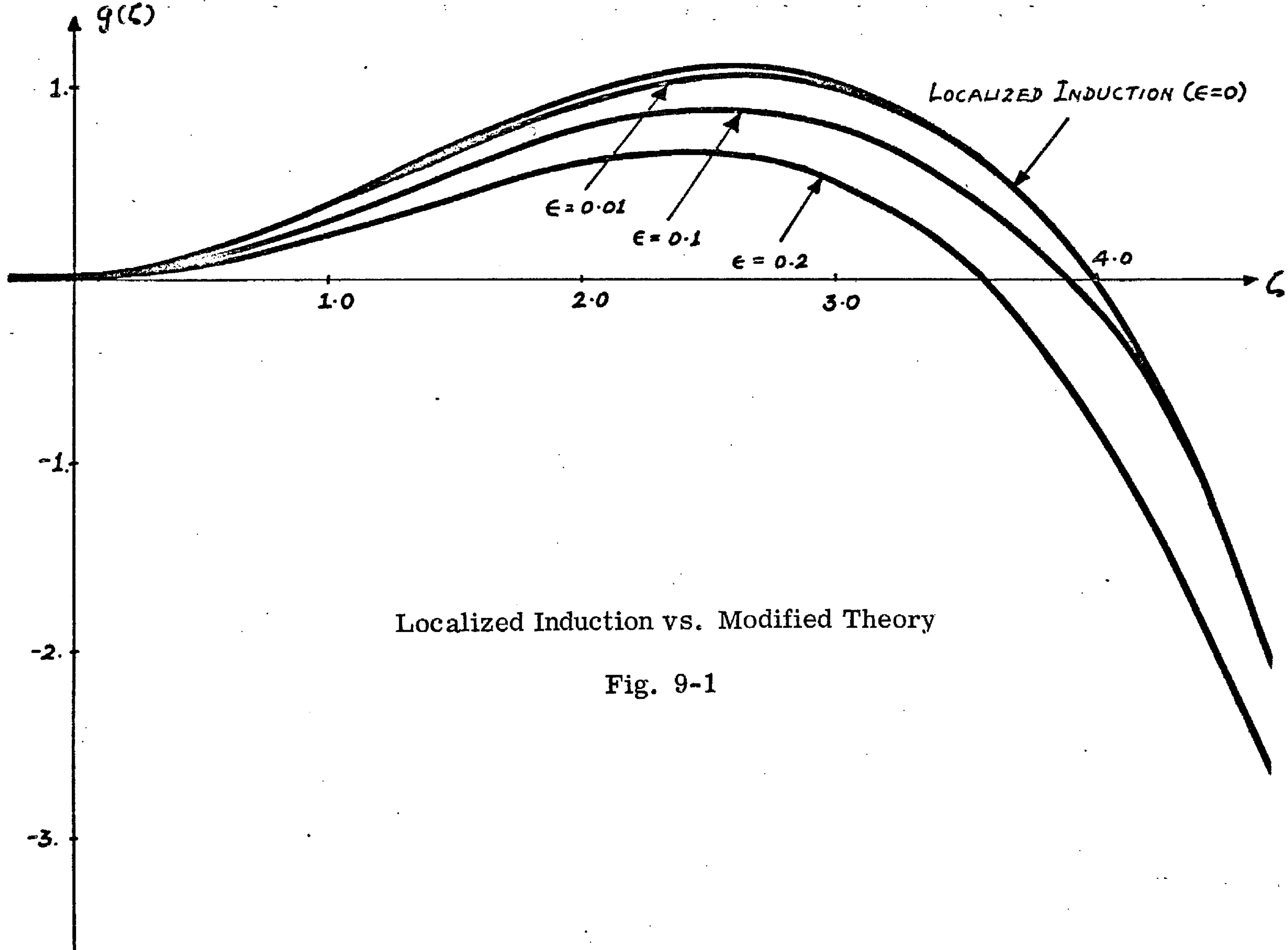
$$- c^2\left[1 - \frac{\epsilon^2}{2\zeta}\int^{\zeta}\frac{[1 - \epsilon(\frac{1}{2}\log y + 1)]}{[1 - \frac{1}{2}\epsilon\ \log y]^3}\,dy\right]\zeta^2$$

$$- 2\epsilon HC\left[\frac{1}{(1 - \frac{1}{2}\epsilon\log\zeta)} - \frac{\epsilon}{2(1 - \frac{1}{2}\epsilon\log\zeta)^2}\right]\zeta$$

$$- \frac{H^2}{(1 - \frac{1}{2}\epsilon\ \log\zeta)^2} + J\zeta$$

$$\left. + \left[1 - \epsilon(\log\zeta + 1) + \frac{\epsilon^2}{4}\ ((\log\zeta)^2 + 2\log\zeta + 4)\right]\zeta^3\right\}. \qquad (9\text{-}16)$$

Thus $g(\zeta)$ is a polynomial modified by small logarithmic terms, it is qualitatively similar to that found for the localized induction case ($\epsilon = 0$). For small $\epsilon$ and moderate $\kappa$, the effect is small. However, a solitary wave solution has values of $\kappa$ going to zero (at the tail ends), since $\log\kappa \to -\infty$ as $\kappa \to 0$, the logarithmic effect may be significant. A numerical plot of $g(\zeta)$ is made and compared to that for $\epsilon = 0$ (fig. 9-1), where it is found that qualitative agreement is very good even for $\epsilon$ as large as 0.2, only the maximum curvature at the center is lowered for large values of $\epsilon$ for given $\Omega$ and c. This implies that the dropping of the logarithmic term only creates a uniform error of $O(\epsilon)$ even when $\kappa \to 0$, and does not cause any qualitative




Localized Induction vs. Modified Theory

Fig. 9-1

difference. The localized induction hypothesis seems to be a good first approximation of the Biot-Savart law, and this is indeed what we shall show in the next chapter.

# X. ASYMPTOTIC EXPANSION OF THE BIOT-SAVART LAW OF INDUCTION

§38. Let us consider the solitary wave solution given by (8-28) and (8-29). We have just seen that the logarithmic term is not qualitatively important. However, another possible objection is raised against the localized induction solution as $\kappa \to 0$. For a point near the tail end, where $|s| \to \infty$ ($s = 0$ being the midpoint), the induced velocity given by the localized induction hypothesis (8-11) is

$$\frac{\partial \hat{R}}{\partial t^*} \propto e^{-\nu |s|}. \tag{10-1}$$

Now if we substitute into the exact Biot-Savart integral and expand for large s, it will be seen that

$$\frac{\partial \hat{R}}{\partial t^*} \propto \frac{\epsilon}{s^2} + O\left(\frac{1}{s^3}\right). \tag{10-2}$$

Regardless of how small $\epsilon$ is, the $O(\epsilon)$ term with its inverse square decay will eventually dominate the $O(1)$ term which decays exponentially. This cannot be eliminated even with the introduction of the logarithmic term. Thus it appears that the solitary wave solution does not approximate the Biot-Savart laws of induction uniformly. The reason for this is that near the tail end of a solitary wave, the filament is effectively straight and the singularity (which is proportional to the curvature) no longer dominates the integral.

The regular part of the integral, whose contribution to the induced velocity obeys the inverse square law, must be taken into account. In the normalized time scale $t^*$, this contribution is $O(\epsilon)$ and is comparable to the $\kappa^*\log\kappa^*$ term in the modified localized induction equation. It must be included in a uniform approximation of the Biot-Savart integral to $O(\epsilon^2)$.

## §39. Analysis of the Solitary Wave Solution to $O(\epsilon^2)$

Let the equation of motion be given by

$$\frac{\partial \hat{R}}{\partial t^*} = \kappa^*(1 - \epsilon \log \kappa^*)\, \hat{b} + \epsilon\, I_1\, \hat{b} + \epsilon\, I_2\, \hat{n} + \epsilon\, I_3\, \hat{s}, \tag{10-3}$$

where $I_1$, $I_2$, and $I_3$ are the binormal, normal and tangential components of the Biot-Savart integral with the singularity removed (they are the components of $\hat{V}_I$ except for a constant different in $\hat{b}$ component). For a steady solution $I_3$ does not enter into the analysis and we have

$$A = \kappa^* + \epsilon(I_1 - \kappa^*\log\kappa^*) \equiv \kappa^* + \epsilon\alpha, \tag{10-4}$$

$$B = \epsilon\ I_2 \equiv \epsilon\beta. \tag{10-5}$$

Note that $\alpha$ and $\beta$ have integral dependence on $\kappa^*$ and $\tau^*$, so (10-3) is actually an integro-differential equation. We now expand $\kappa^*$ and $\tau^*$ in $\epsilon$:

$$\kappa^* = \kappa_0 + \epsilon\kappa_1, \tag{10-6}$$

$$\tau^* = \tau_0 + \epsilon\tau_1, \tag{10-7}$$

and substitute into (7-36) - (7-38). Equating coefficients of $\epsilon$, we have

$$O(1): \begin{cases} (c - 2\tau_0)\kappa_0' = \tau_0', & (10\text{-}8) \\ -(\Omega + c\tau_0)\kappa_0 = Z_0\kappa_0 + \kappa_0'' - \tau_0^2\kappa_0, & (10\text{-}9) \\ Z_0 = \frac{\kappa_0^2}{2}, & (10\text{-}10) \end{cases}$$

$$O(\epsilon) \begin{cases} c\kappa_1' = \tau_1'\kappa_0 + 2\tau_1\kappa_0' + 2\tau_0(\kappa_1' + \alpha') - \beta'' + \tau_0^2\beta - \kappa_0^2\beta - \kappa_0'\int^s \kappa_0\beta ds & (10\text{-}11) \\ -(\Omega + c\tau_0)\kappa_1 - c\kappa_0\tau_1 = Z_0\kappa_1 + Z_1\kappa_0 + \alpha'' + \kappa_1'' - \tau_0^2(\alpha + \kappa_1) - 2\kappa_0\tau_0\tau_1 & \\ \qquad + 2\tau_0\beta' + \tau_0\kappa_0\int^s \kappa_0\beta\, ds, & (10\text{-}12) \\ Z_1 = \int^s [\kappa_0(\alpha' + \kappa_1') + \kappa_0'\kappa_1 - \tau_0\kappa_0\beta] ds. & (10\text{-}13) \end{cases}$$

The O(1) equations are just the localized induction equations and therefore

$$\kappa_0 = 2\nu \operatorname{sech} \nu s, \quad \tau_0 = \frac{c}{2}. \tag{10-14}$$

To examine the $O(\epsilon)$ equation, we have to compute $\alpha$ and $\beta$, which are integrals involving $\kappa^*$ and $\tau^*$. In fact, we only need $\alpha$ and $\beta$ to leading order. If we write

$$\alpha = \alpha_0 + \alpha_1 + \cdots, \tag{10-15}$$

$$\beta = \beta_0 + \beta_1 + \cdots, \tag{10-16}$$

all we need to know are $\alpha_0$ and $\beta_0$, which are

$$\alpha_0 = I_1(\kappa_0, \tau_0) - \kappa_0 \log \kappa_0, \tag{10-17}$$

$$\beta_0 = I_2(\kappa_0, \tau_0). \tag{10-18}$$

Now $I_1$ and $I_2$ are given by

$$I_1 = \underset{\sim}{b} \cdot \int_{[d_0]} \frac{(\hat{\underset{\sim}{R}} - \underset{\sim}{R}) \wedge d\hat{\underset{\sim}{R}}}{|\hat{\underset{\sim}{R}} - \underset{\sim}{R}|^3}, \tag{10-19}$$

$$I_2 = \underset{\sim}{n} \cdot \int_{[d_0]} \frac{(\hat{\underset{\sim}{R}} - \underset{\sim}{R}) \wedge d\hat{\underset{\sim}{R}}}{|\hat{\underset{\sim}{R}} - \underset{\sim}{R}|^3} \tag{10-20}$$

where $[d_0]$ means that the integration is stopped at $|\xi| = d_0$, $d_0$ being the same constant as that in §33. Explicit expressions for $\alpha_0$ and $\beta_0$ in terms of s are not available, however, we have seen in §38 that the $O(\epsilon)$ terms become important only when s gets large. Therefore, we are mainly concerned with the asymptotic behavior of (10-11) - (10-13) as $|s| \to \infty$, and we only need the asymptotic values of $\alpha_0$ and $\beta_0$ for large s.

## §40. Limits of the $O(\epsilon)$ Equations as $|s| \to \infty$.

Since the solitary wave solution is symmetric with respect to s, we can consider $s \to +\infty$ without loss of generality. In this limit

$$\kappa_0 \sim O(e^{-s}), \tag{10-21}$$

$$\tau_0 \sim O(1), \tag{10-22}$$

$$\alpha_0,\ \beta_0 \sim O(s^{-2}), \tag{10-23}$$

so that

$$Z_0 \sim O(e^{-2s}),\ Z_1 \sim O(e^{-s}) \tag{10-24}$$

and the leading terms of (10-11) and (10-12) are

$$\beta_0'' - \tau_0^2\beta_0 - 2\tau_0\alpha_0' = 0, \tag{10-25}$$

$$\kappa_1'' - \nu^2\kappa_1 = \tau_0^2\alpha_0 - \alpha_0'' - 2\tau_0\beta_0'. \tag{10-26}$$

Equation (10-26) gives $\kappa_1$ in terms of $\kappa_0$, $\tau_0$ and in fact has an exponentially decaying solution for $\kappa_1$. Equation (10-25) is very interesting. Since neither $\tau_1$ nor $\kappa_1$ appears in it, it appears to be an additional constraint relating $\kappa_0$ and $\tau_0$ through $\alpha_0$ and $\beta_0$ as $s \to \infty$. Let us examine its implications.

We use equation (8-1) in the t* coordinates, we have

$$\hat{\underset{\sim}{R}} = \begin{pmatrix} \zeta - \frac{2\mu}{\nu}\tanh \nu\zeta \\ \frac{2\mu}{\nu}\operatorname{sech} \nu\zeta \cos(\tau_0\zeta - \Omega t^*) \\ \frac{2\mu}{\nu}\operatorname{sech} \nu\zeta \sin(\tau_0\zeta - \Omega t^*) \end{pmatrix}, \tag{10-27}$$

$$\hat{\underset{\frown}{R}} - \underset{\frown}{R} = \begin{pmatrix} (\zeta - s) - \frac{2\mu}{\nu}(\tanh \nu\zeta - \tanh \nu s) \\ \frac{2\mu}{\nu}[\operatorname{sech} \nu\zeta \cos(\tau_0 \zeta - \Omega t^*) - \operatorname{sech} \nu s \cos(\tau_0 s - \Omega t^*)] \\ \frac{2\mu}{\nu}[\operatorname{sech} \nu\zeta \sin(\tau_0 \zeta - \Omega t^*) - \operatorname{sech} \nu s \sin(\tau_0 s - \Omega t^*)] \end{pmatrix}, \tag{10-28}$$

$$d\hat{\underset{\frown}{R}} = \begin{pmatrix} 1 - 2\mu \operatorname{sech}^2\zeta \\ -\frac{2\mu\tau_0}{\nu} \operatorname{sech} \nu\zeta \sin(\tau_0\zeta - \Omega t^*) - 2\mu \operatorname{sech} \nu\zeta \tanh \nu\zeta \cdot \cos(\tau_0\zeta - \Omega t^*) \\ \frac{2\mu\tau_0}{\nu} \operatorname{sech} \nu\zeta \cos(\tau_0\zeta - \Omega t^*) - 2\mu \operatorname{sech} \nu\zeta \tanh \nu\zeta \cdot \sin(\tau_0\zeta - \Omega t^*) \end{pmatrix}, \tag{10-29}$$

$$\left|\hat{\underset{\frown}{R}} - \underset{\frown}{R}\right|^{-3} = (\zeta - s)^{-3}\left(1 + O\left(\frac{1}{s}\right)\right). \tag{10-30}$$

So that (8-1) as $s \to \infty$ becomes

$$\frac{\partial \underset{\wedge}{R}}{\partial t^*} = \epsilon \begin{bmatrix} O\left(\dfrac{1}{s^3}\right) \\ \hdashline -\displaystyle\int_{[d_0]} \frac{2\mu\tau_0}{\nu} \cdot \left[\frac{\operatorname{sech}\nu\zeta \cos(\tau_0\zeta - \Omega t^*) - \dfrac{2\mu\nu}{\tau_0} \operatorname{sech}\nu\zeta \tanh\nu\zeta \sin(\tau_0\zeta - \Omega t^*)}{(s-\zeta)^2}\right] d\zeta \\ + O\left(\dfrac{1}{s^3}\right) \\ \hdashline -\displaystyle\int_{[d_0]} \frac{2\mu\tau_0}{\nu} \cdot \left[\frac{\operatorname{sech}\nu\zeta \sin(\tau_0\zeta - \Omega t^*) + \dfrac{2\mu\nu}{\tau_0} \operatorname{sech}\nu\zeta \tanh\nu\zeta \cos(\tau_0\zeta - \Omega t^*)}{(s-\zeta)^2}\right] d\zeta \\ + O\left(\dfrac{1}{s^3}\right) \end{bmatrix} + \underset{\wedge}{S} \qquad (10\text{-}31)$$

where $\underset{\wedge}{S}$ is the singular part of the integral which is $O(e^{-s})$. As $s \to \infty$, we have from (8-40) and (8-38) that

$$\underset{\wedge}{b} \sim \begin{pmatrix} O(e^{-s}) \\ -M\sin(\tau_0 s - \Omega t^*) + N\cos(\tau_0 s - \Omega t^*) + O(e^{-s}) \\ M\cos(\tau_0 s - \Omega t^*) + N\sin(\tau_0 s - \Omega t^*) + O(e^{-s}) \end{pmatrix}, \qquad (10\text{-}32)$$

$$\underset{\sim}{n} \sim \begin{pmatrix} O(e^{-s}) \\ M\cos(\tau_0 s - \Omega t^*) + N\sin(\tau_0 s - \Omega t^*) + O(e^{-s}) \\ M\sin(\tau_0 s - \Omega t^*) - N\cos(\tau_0 s - \Omega t^*) + O(e^{-s}) \end{pmatrix}, \qquad (10\text{-}34)$$

where we have written

$$M = \frac{\nu^2 - \tau_0{}^2}{\nu^2 + \tau_0{}^2} = \mu\left(1 - \frac{\tau_0{}^2}{\nu^2}\right) = -(1 - 2\mu), \qquad (10\text{-}34)$$

$$N = \frac{2\nu\tau_0}{\nu^2 + \tau_0{}^2} = \frac{2\mu\tau_0}{\nu}. \qquad (10\text{-}35)$$

We now rewrite (10-31) as

$$\frac{\partial \underset{\sim}{R}}{\partial t^*} = \epsilon \begin{bmatrix} O \\ P\cos\Omega t^* + Q\sin\Omega t^* \\ -P\sin\Omega t^* + Q\cos\Omega t^* \end{bmatrix} + O\left(\frac{1}{s^3}\right), \qquad (10\text{-}36)$$

where

$$P = -\int_{[d_0]} \frac{2\mu\tau_0}{\nu} \frac{\left[\operatorname{sech}\nu\zeta\cos\tau_0\zeta - \frac{2\mu\nu}{\tau_0}\operatorname{sech}\nu\zeta\tanh\nu\zeta\sin\tau_0\zeta\right]d\zeta}{(s-\zeta)^2}$$

(10-37)

$$Q = -\int_{[d_0]} \frac{2\mu\tau_0}{\nu} \frac{\left[\operatorname{sech}\nu\zeta\cos\tau_0\zeta + \frac{2\mu\nu}{\tau_0}\operatorname{sech}\nu\zeta\tanh\nu\zeta\sin\tau_0\zeta\right]d\zeta}{(s-\zeta)^2}$$

(10-38)

We can express $\alpha_0$ and $\beta_0$ in terms of M, N, P, Q by taking the $\hat{b}$ and $\hat{n}$ components of (10-36):

$$\alpha_0 = P(-M \sin \tau_0 s + N \cos \tau_0 s) + Q(M \cos \tau_0 s + N \sin \tau_0 s), \quad (10\text{-}39)$$

$$\beta_0 = P(M \cos \tau_0 s + N \sin \tau_0 s) + Q(M \sin \tau_0 s - N \cos \tau_0 s). \quad (10\text{-}40)$$

The forms of P and Q imply that

$$P, Q \sim O\left(\frac{1}{s^2}\right), \quad (10\text{-}41)$$

$$\frac{dP}{ds}, \frac{dQ}{ds} \sim O\left(\frac{1}{s^3}\right), \quad (10\text{-}42)$$

which means that $P'$ and $Q'$ can be ignored to our order of expansion. Thus,

$$\left.\begin{aligned} \beta_0'' - \tau_0^2\beta_0 &= -2\tau_0^2 P(M\cos\tau_0 s + N\sin\tau_0 s) - 2\tau_0^2 Q(M\sin\tau_0 s - \\ &\qquad N\cos\tau_0 s), \\ -2\tau_0\alpha_0' &= 2\tau_0 P(M\cos\tau_0 s + N\sin\tau_0 s) + 2\tau_0^2 Q(M\sin\tau_0 s - N\cos\tau_0 s), \end{aligned}\right\} \quad (10\text{-}43)$$

which shows that (10-25) is identically satisfied. Furthermore, (10-26) reduces to

$$\kappa_1'' - \nu^2\kappa_1 = 0, \quad (10\text{-}44)$$

giving, to leading order,

$$\kappa_1 \sim e^{-\nu|s|}. \quad (10\text{-}45)$$

These results show that the localized induction hypothesis is better than it appears in the sense that the corresponding solitary wave solution is self-consistent even in regions when the local effects fail to dominate. The $O(\epsilon)$ correction to the $O(1)$ solution is uniformly small for all values of $\kappa$, and thus the expansions (10-6) and (10-7) are justified. Even with an exponentially decaying curvature $\kappa$, agreement with the induced velocity given by the Biot-Savart integral appears to exist to $O(\epsilon^2)$ when the regular part of the integral is taken into account.

## Appendix. Conservation Forms of the Nonlinear Schrödinger Equation

For the Korteweg-de Vries equation

$$u_t + uu_x + \delta^2 u_{xxx} = 0, \tag{A-1}$$

interesting information can be deduced from the conversion laws and constants of motion (Miura,1968, Miura, Gardner and Kruskal, 1968, Su and Gardner,1969, Kruskal, Miura and Gardner,1970). The real and imaginary parts of the nonlinear Schrödinger equation

$$\dot{\psi} = i\left[\psi'' + \psi\left(\tfrac{1}{2}\,|\psi|^2 + \Omega\right)\right], \tag{A-2}$$

which is equivalent to (8-14) in the non-propagating frame, can also be put into conservation forms. If we define $\rho$ and u as

$$\rho = \kappa^2, \quad u = 2\tau, \tag{A-3}$$

where

$$\psi = \kappa\, e^{i\int^s \tau ds}, \tag{A-4}$$

we have

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial s}(\rho u) = 0, \tag{A-5}$$

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial}{\partial s}\left(\rho u^2 - \frac{\rho^2}{2} + \frac{1}{\rho}\left(\frac{\partial \rho}{\partial s}\right)^2 - \frac{\partial^2 \rho}{\partial s^2}\right) = 0. \tag{A-6}$$

Thus, for a solitary wave, when $\kappa \to 0$ exponentially as $|s| \to \infty$, we immediately have

$$I_\rho = \int_{-\infty}^{\infty} \rho \, ds, \tag{A-7}$$

$$I_{\rho u} = \int_{-\infty}^{\infty} (\rho u) \, ds, \tag{A-8}$$

as constants of motion. By expanding (B-6) and using (B-5) we have

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial s}\left(\frac{u^2}{2} - \rho + \frac{1}{2\rho^2}\left(\frac{\partial \rho}{\partial s}\right)^2 - \frac{1}{\rho}\frac{\partial^2 \rho}{\partial s^2}\right) = 0, \tag{A-9}$$

which leads to

$$I_u = \int_{-\infty}^{\infty} u ds. \tag{A-10}$$

In addition to these three integrals, the form of the pair (B-5) and (B-6) suggests a further integral explicitly dependent on x and t

$$J = \int_{-\infty}^{\infty} [s\rho - t(\rho u)] ds, \tag{A-11}$$

which corresponds to one found by Miura, Gardner and Kruskal (1968) for the Korteweg-de Vries equation.

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## Epilogue

For God's Sake

Let us sit upon the ground and tell sad stories
Of vortex filaments.
How some have been ill-posed, some singular,
Some poisoned by their self-induction, some core size killed,
Some haunted by the mathematics they have involved.

All murderous.

For within the swirling motion that rounds the mortal circulation
Of a vortex
Keeps futility his court,
And there the non-linearity sits
Scoffing at his state and grinning at his theories
Allowing him a breath, a little scene to linearize, compute
and fill with approximations
And then at last he comes and with a little inconsistency bores through
the costly hopes and

Farewell..........

Shakespeare
Richard II. Act 3 Scene 2