PHOTODISINTEGRATION OF THE DEUTERON

Thesis by

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ABSTRACT

The Chew meson theory has been applied in an attempt to explain the observed resonance occurring in the photodisintegration of the deuteron at an energy of about 250 Mev. The resonance is interpreted as being due to the rescattering of a virtual photo-produced meson in the $J = 3/2$, $T = 3/2$ state from one of the two nucleons, the meson being finally absorbed. The impulse approximation is used to describe the scattering. Results are obtained for the total cross section as a function of energy in qualitative agreement with experiment, but though an asymmetry in the angular distribution is found, it does not give a different cross section at 0° and 180° as seems to be required by the data.

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I. INTRODUCTION.

For the past several years a considerable amount of experimental information has been accumulating concerning the photodisintegration of the deuteron at energies running up to almost 500 Mev.[1] These results indicate a resonance of sorts in the total cross section at a photon energy of about 225 Mev. Theoretical calculations have been carried up to about 150 Mev ignoring any explicit meson effects,[2] and somewhat further by means of including a meson magnetic moment.[3] These do not seem to predict any resonance behavior. In view of the observed resonances in both the meson nucleon scattering and meson photoproduction from hydrogen in the $J = 3/2$, $T = 3/2$ states one is tempted to suggest that the resonance in the deuteron photodisintegration is due to the scattering in a virtual state of a meson which has been produced by the photon, the meson being finally absorbed by one of the two outgoing nucleons. In order to attempt a quantitative description of the effect of such processes it is necessary to fix on a particular meson theory to describe the meson nucleon interactions. Since relativistic forms of meson theory have been notoriously unsuccessful in predicting anything, and since there is no reasonably valid approximation method which can be applied to them, it seems advisable to use Chew's form of meson theory.[4] This theory has several advantages. First, it is not a complete theory (and is rather more of a phenomenological approach), in the sense that no attempt is made to describe a large group of meson phenomena, such as S-wave interactions, relativistic effects, or heavy mesons. The S-wave interactions are just ignored, the rest is assumed to be describable by a cutoff on the momentum of any virtual meson, thus restricting one to low energies and providing

an extra parameter. The theory is constructed to agree with the important qualitative features observed in the meson nucleon interaction; namely strong P-wave couplings, pseudo scalar mesons and conservation of isotopic spin. Second, the coupling constant is small, so that there exist fairly reasonable approximation methods. Third, the theory is relatively easy to use, at least compared with the existing relativistic theories. Last, but not least, the theory agrees fairly well with all low energy meson effects, (except, of course, the S-waves).

The above being considered good and sufficient reasons, the Chew theory will be used in an attempt to describe the deuteron photodis-integration up to energies of several hundred Mev. Since even a relatively simple theory such as this becomes unpleasantly complicated when applied to two nucleon problems, some rather drastic approximations will of course be necessary. It is therefore not to be expected that more than mere qualitative features of the cross sections will be reproduced.

II. FORMAL EQUATIONS.

The Hamiltonian for a two-nucleon system interacting with both electromagnetic and meson fields may be written

$$H = H_o + H^+_{NM}(a) + H^-_{NM}(a) + H^+_{NM}(b) + H^-_{NM}(b) + V_2 \qquad (1)$$

where H_o denotes the free nucleons, and the free meson and photon field. We assume any forces between the nucleons to be correctly given by meson theory, so there is no separate term to describe the n-p interaction. V_2 represents all electromagnetic interactions, and for a $\underline{\sigma} \cdot \underline{\nabla}$ meson theory will have the form

$$V_2 = H_{\text{nucleon current}} + H_{\text{meson current}} + H_{\text{three-field}} \quad . \qquad (2)$$

$H^{\pm}_{NM}(a,b)$ denotes the interaction creating or destroying a meson on nucleon a or b. Denote the sum of these four H_{NM} terms by V_1.

In order to calculate a cross section for the deuteron photodisintegration we will be interested in evaluating the R-matrix for the above interaction $V_1 + V_2$ between a state of 2 nucleons, 1 photon, 0 mesons and a state of 2 nucleons, 0 photons and 0 mesons. Since electromagnetic interactions are weak, it will be sufficient to calculate R only to first order in V_2. Thus[5]

$$R = R_1 + V_2 + R_1 \frac{1}{a} V_2 + V_2 \frac{1}{a} R_1 + R_1 \frac{1}{a} V_2 \frac{1}{a} R_1 \qquad (3)$$

where R_1 is the transition matrix associated with the interaction V_1 above; that is:

$$R_1 = V_1 + V_1 \frac{1}{a} R_1 \quad . \qquad (4)$$

Now, observing that V_2 has only matrix elements connecting states of n mesons with states of $n \pm 0,1,2$ mesons, and remembering that R is to be calculated between 2 states of 0 mesons, we can write

$$(0|R|0) = (0|V_2|0) + \sum_{012} (0|R_1|0,1,2)\, \frac{1}{a}\, (0,1,2|V_2|0)$$

$$+ \sum_{012} (0|V_2|0,1,2)\, \frac{1}{a}\, (0,1,2|R_1|0)$$

$$+ \sum_{n,n'} (0|R_1|n)\, \frac{1}{a}\, (n|V_2|n')\, \frac{1}{a}\, (n'|R_1|0) \qquad (5)$$

where $n' = n \pm 0,1,2$. The term R_1 gives no contributions, since the disintegration also goes from a state of 1 photon to a state of 0 photons, and R_1 contains no e-m interactions.

The matrix element $(n|R_1|0)$ includes all possible Feynman diagrams leading from a state of 2 free nucleons (the initial state) to some intermediate state of n mesons. It may thus be split into two parts, as

$$(n|R_1|0) = (n|R_1'|0)\, [1 + \frac{1}{a}\, (0|R_1''|0)] \qquad (6)$$

where $(n|R_1'|0)$ has at <u>no</u> intermediate point in the diagram a state of 0 mesons. Thus $(n|R_1'|0)$ contains all diagrams going from 0 to n with no gaps, and $(0|R_1''|0)$ contains all diagrams going from 0 to 0.

In an analogous fashion, $(0|R_1|n)$ can also be broken down into the form

$$(0|R_1|n) = [1 + (0|R_1''|0)\, \frac{1}{a}]\, (0|R_1'|n) \quad . \qquad (7)$$

(Note if $n = 0$, then we have directly $(0|R_1|n) = (0|R_1''|0)$.) Hence the entire matrix element may be written

$$(0|R|0) = [1 + (0|R_1''|0)\,\tfrac{1}{a}]\left\{ (0|V_2|0) + \sum_{1,2} (0|R_1'|1\ 2)\,\tfrac{1}{a}\,(1\ 2|V_2|0) \right.$$

$$+ \sum_{1,2} (0|V_2|1\ 2)\,\tfrac{1}{a}\,(1\ 2|R_1'|0)$$

$$\left. + \sum_{n,n'\neq 0} (0|R_1'|n)\,\tfrac{1}{a}\,(n|V_2|n')\,\tfrac{1}{a}(n'|R_1'|0) \right\}$$

$$\cdot\,[1 + \tfrac{1}{a}\,(0|R_1''|0)] \quad . \tag{9}$$

Now observe that

$$(0|R_1''|0) = V + V\,\tfrac{1}{a}\,V + V\,\tfrac{1}{a}\,V\,\tfrac{1}{a}\,V + \cdots$$

where V is the exact nuclear force predicted by the meson theory. Hence

$$(\Psi| = (0| + (0|R_1''\ \tfrac{1}{a}$$

$$= (0| + (0|\ (V + V\,\tfrac{1}{a}\,V + \cdots\ )\,\tfrac{1}{a}$$

$$= (0| + (\Psi|V\,\tfrac{1}{a} \quad . \tag{10}$$

This is the equation given by Lippman and Schwinger[6] for the exact solution corresponding to a potential V. It is thus possible to replace $[1 + (0|R''|0)\,\tfrac{1}{a}](0|$ by $(\Psi_f|$, where Ψ_f represents the final state wave function of the scattering neutron and proton with the appropriate energy. Similarly, $[\tfrac{1}{a}\,(0|R''|0) + 1]$ may be replaced by the deuteron wave function $|\Psi_i)$.[7] We can thus finally write

$$(0|R|0) = (\Psi_f|V_2 + R_1' \frac{1}{a} V_2 + V_2 \frac{1}{a} R_1' + R_1' \frac{1}{a} V_2 \frac{1}{a} R_1'|\Psi_i) \qquad (11)$$

where Ψ_f and Ψ_i represent the desired initial and final states of the two-nucleon system. Thus Ψ_i may be chosen as the deuteron wave function, and Ψ_f as the wave function for n-p scattering of the appropriate energy.

We have thus in a sense changed our point of view, in that once it has been assumed that the n-p system can be exactly described by meson theory, we do not attempt to actually so describe it, but use phenomenological wave functions for Ψ_f and Ψ_i. Thus all explicit meson effects are removed from the problem except for those directly connected (i.e.with no gaps) to the electromagnetic interaction. All other meson effects are taken to be equivalent to putting in as initial and final states ordinary wave functions for two nucleons, provided it is assumed that meson theory is in principle capable of predicting two nucleon behavior properly.

It should be emphasized that as yet no approximation has been made to the meson theory; later on we shall drop the point of view of using an exact meson theory, and use for all interactions the Chew type of theory, which is not thought of as being in any sense a limit of a full scale relativistic theory, but rather interpreted merely as being the type of coupling indicated at low energies by experiment.

The expression (11) can be broken down still further as follows. Consider for example the term $V_2 \frac{1}{a} R_1'$. This is to be taken between two states of zero mesons. It may thus be written in the form

$$(\Psi_f|V_2 \frac{1}{a} R_1'|\Psi_i)$$

$$= (\Psi_f|V_2|1) \frac{1}{a}(1|1 + R_s|1) \frac{1}{a}(1|H_{NM}^{+}|\Psi_i)$$

$$+ (\Psi_f|V_2|2) \frac{1}{a}(2|H_{NM}^{-}|1) \frac{1}{a}(1|1 + R_s|1) \frac{1}{a}(1|H_{NM}^{+}|\Psi_i)$$

$$+ (\Psi_f|V_2|2) \frac{1}{a}(2|H_{NM}^{+}|3) \frac{1}{a}(3|R_1'|1) \frac{1}{a}(1|H_{NM}^{+}|\Psi_i) \quad , \qquad (12)$$

since V_2 can only connect 0 to 1 or 2 mesons. Here R_s represents the scattering amplitude for mesons on deuterons.

Now, if we assume a Chew type of theory, the coupling constant (after renormalization) is rather small ($f^2 \sim .1$) so that a weak coupling expansion may be expected to be adequate except for cases in which resonant (i.e.vanishing) energy denominators occur in intermediate states.*

In the above expression for $V_2 \frac{1}{a} R_1'$, all energy denominators for intermediate states lie before the photon interaction, are therefore of the form $E_o - E_{int.} = k - (k + \text{meson energies}) = -(\text{meson energies})$, and thus cannot vanish. These terms can therefore presumably be treated in a weak coupling expansion.

$$(\Psi_f|V_2 \frac{1}{a} R_1'|\Psi_i) \approx (\Psi_f|V_2 \frac{1}{a} H_{NM}^{+} + V_2 \frac{1}{a} H_{NM}^{+} \frac{1}{a} H_{NM}^{+}|\Psi_i) \quad . \qquad (13)$$

For the term $R_1' \frac{1}{a} V_2$, however, all meson effects occur after the photon has been absorbed, and thus it cannot be immediately argued that the energy denominators will not vanish. We may write

$$(\Psi_f|R_1' \frac{1}{a} V_2|\Psi_i) =$$

$$= (\Psi_f|H_{NM}^{-} \frac{1}{a}(1 + R_s) \frac{1}{a} V_2 + H_{NM}^{-} \frac{1}{a}(1 + R_s) \frac{1}{a} H_{NM}^{-} \frac{1}{a} V_2|\Psi_i)$$

$$+ (\Psi_f|H_{NM}^{-}|1) \frac{1}{a}(1|R_1'|3) \frac{1}{a}(3|H_{NM}^{-}|2) \frac{1}{a}(2|V_2|\Psi_i) \quad . \qquad (14)$$

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* For a discussion of the features of the Chew theory, see Appendix B.

The second term here contains at least an intermediate state of three mesons. The energy denominator here cannot therefore vanish unless the photon energy is at least three meson rest masses, (ie.) 420 Mev. This is beyond the region of interest here, so the second term will be dropped.

The same kinds of arguments may be applied to the term $R_1' \frac{1}{a} V_2 \frac{1}{a} R_1'$, resulting in

$$(\Psi_f | R_1' \frac{1}{a} V_2 \frac{1}{a} R_1' | \Psi_i) =$$

$$= (\Psi_f | H_{NM}^- \frac{1}{a} (1 + R_s) \frac{1}{a} V_2 \frac{1}{a} H_{NM}^+ | \Psi_i)$$

$$+ (\Psi_f | H_{NM}^- \frac{1}{a} (1 + R_s) \frac{1}{a} H_{NM}^- \frac{1}{a} V_2 \frac{1}{a} H_{NM}^+ \frac{1}{a} H_{NM}^+ | \Psi_i) . \qquad (15)$$

The second of these two terms contains one more f^2 than the first; also the energy denominator in it in some places requires at least a 2μ photon to vanish which the ones in the first do not. The second term should therefore be smaller than the first, and will be neglected. Thus we finally obtain the form to be used in explicit calculations. It can be broken up into two parts, to be treated separately. The first consists of all terms of order ef^2 or lower; this will be called the perturbation part. The second contains all terms with R_s in them, and will be called the resonant part. Written in detail, and with the appropriate part of V_2 explicitly put in, these are:

(i) Perturbation Part.

$$(\Psi_f | H_{\text{nucleon current}} + H_{NM}^- \frac{1}{a} H_3^+ + H_3^- \frac{1}{a} H_{NM}^+ + H_{NM}^- \frac{1}{a} H_{NM}^- \frac{1}{a} H_{\text{meson current}}$$

$$+ H_{NM}^- \frac{1}{a} H_{\text{meson current}} \frac{1}{a} H_{NM}^+ + H_{\text{meson current}} \frac{1}{a} H_{NM}^+ \frac{1}{a} H_{NM}^+$$

$$+ H_{NM}^- \frac{1}{a} H_{\text{nucleon current}} \frac{1}{a} H_{NM}^+ | \Psi_i) . \qquad (16)$$

(ii) Resonant Part.

$$(\Psi_f | H^-_{NM} \frac{1}{a} R_s \frac{1}{a} H_\pi + H^-_{NM} \frac{1}{a} R_s \frac{1}{a} H^-_{NM} \frac{1}{a} H_{\text{meson current}}$$

$$+ H^-_{NM} \frac{1}{a} R_s H_{\text{meson current}} \frac{1}{a} H^+_{NM}$$

$$+ H^-_{NM} \frac{1}{a} R_s \frac{1}{a} H_{\text{nucleon current}} \frac{1}{a} H^+_{NM} | \Psi_i) \, . \qquad (17)$$

The notation in the above two expressions is $H^- = H^-_a + H^-_b$; thus effects of coupling to either nucleon are implied. In Fig. (1) are shown diagrams for each of these terms, in the same order as above.

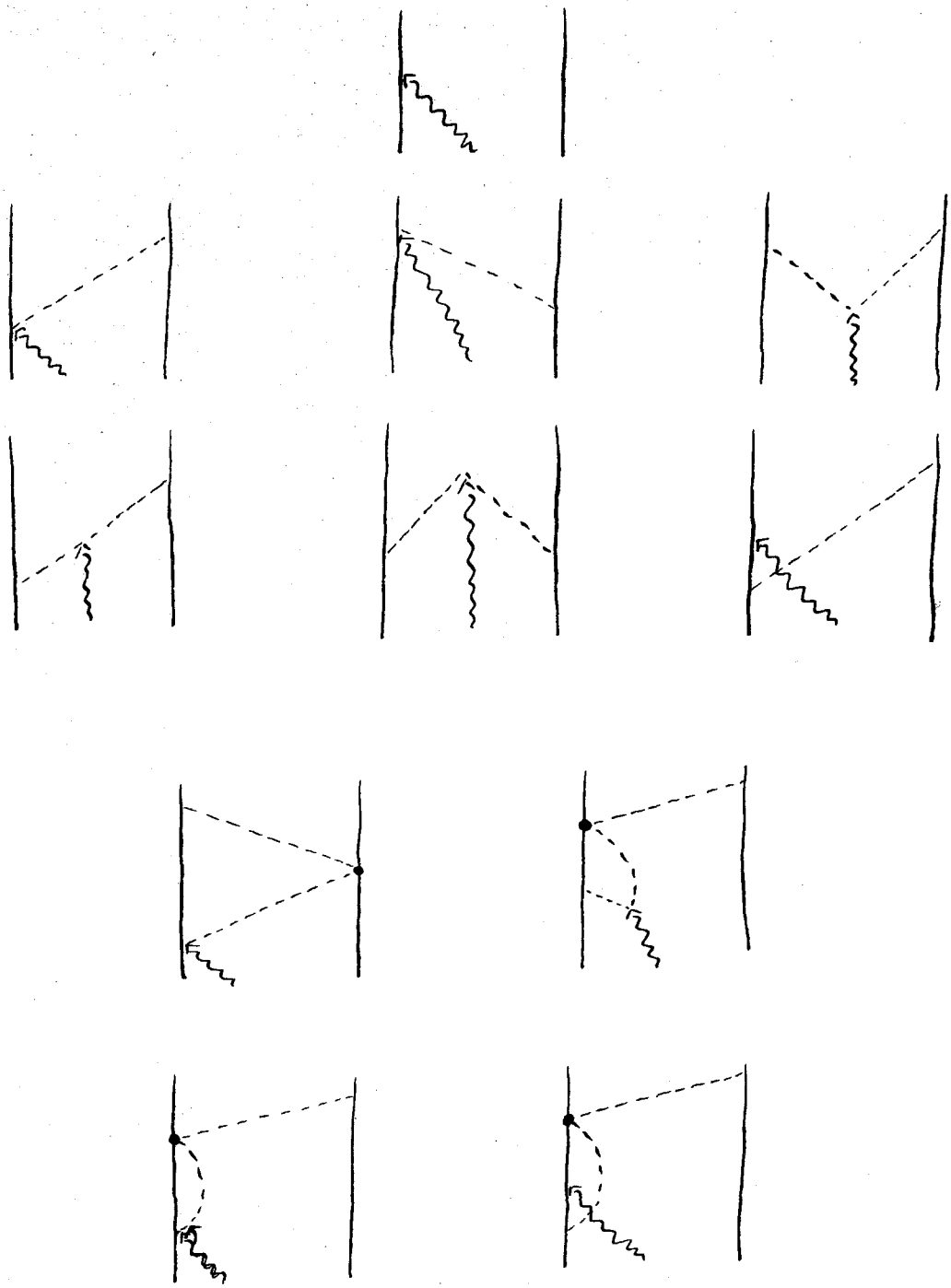

Fig. 1
(•) denotes re-scattering

III. PERTURBATION TERMS.

These do not contain any virtual scattering processes of meson or nucleons, so they should not be expected to produce any resonant effects. It may be expected therefore, that these terms should include the earlier calculation of Schiff and Marshall and Guth,[2] and in addition provide a kind of background for any resonant effects which may arise from other sources.

The explicit form of the perturbation interaction is given in equation (16). For the various interactions the following expressions will be used.*

$$H_{\text{nucleon current}} = \sum_{a,b} \left\{ -\frac{ie}{2M} \frac{1+\tau_3^a}{2} \underline{A} \cdot \underline{\nabla}_a - \frac{(\mu_p+\mu_n)+(\mu_p-\mu_n)\tau_3^a}{2} \underline{\sigma}_a \cdot \underline{H} \right\}$$

$$H_{NM}^{(-)} = ig \frac{\underline{\sigma}_a \cdot \underline{K}}{\sqrt{2\omega}} \underline{\tau}_a \cdot \underline{a}\, e^{i\underline{K}\cdot\underline{r}_a} + ig \frac{\underline{\sigma}_b \cdot \underline{K}}{\sqrt{2\omega}} \underline{\tau}_b \cdot \underline{a}\, e^{i\underline{K}\cdot\underline{r}_b}$$

$$H_{NM}^{(+)} = (H_{NM}^{(-)})^+$$

$$H_3^{(-)} = igeT \frac{\underline{\sigma}_a \cdot \hat{\underline{e}}}{\sqrt{2\omega}\sqrt{2k}} \underline{\tau}_a \cdot \underline{a}\, e^{i\underline{K}\cdot\underline{r}_a} + igeT \frac{\underline{\sigma}_b \cdot \hat{\underline{e}}}{\sqrt{2\omega}\sqrt{2k}} \underline{\tau}_b \cdot \underline{a}\, e^{i\underline{K}\cdot\underline{r}_b}$$

$$H_{\text{meson current}} = -e\,\underline{A} \cdot (\phi\, \underline{\nabla}\, \phi^* - \phi^*\, \underline{\nabla}\, \phi)$$

$H_{\text{nucleon current}}$ can be rewritten in what will turn out to be a more convenient form by using Siegert's Theorem.[8] This states

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* We use the notation $g = \sqrt{4\pi}\, f/\mu$, where f is Chew's coupling constant.

$$-\frac{ie}{2M}\underline{A}\cdot\underline{\nabla}+ie\,\underline{A}\cdot\underline{r}\,V_{exchange}=\underline{E}\cdot\underline{D}=-\frac{ie}{2}\sqrt{\frac{k}{2}}\,\hat{\underline{e}}\cdot\underline{r}$$

for the electric dipole part of the coupling. Here $V_{exchange}$ denotes the charge exchange part of the nucleon force. This exchange force term may be interpreted as the low photon energy contribution of the charged meson exchange effects; thus this should already be contained in the remaining meson terms. We shall therefore instead take

$$H'_{\text{nucleon current}}=-\frac{ie}{2}\sqrt{\frac{k}{2}}\,\hat{\underline{e}}\cdot\underline{r}$$

$$-\sum_{a,b}\frac{(\mu_p+\mu_n)+(\mu_p-\mu_n)\tau_3^a}{2}\,\underline{\sigma}_a\cdot\underline{H}$$

and include the additional term $-ie\,\underline{A}\cdot\underline{r}\,V_{exchange}$ in with the meson effects. The reasons for making this replacement of $\hat{\underline{e}}\cdot\underline{\nabla}$ by $\hat{\underline{e}}\cdot\underline{r}$ are the following:

(i) It allows a more direct comparison with the Schiff and Marshall and Guth calculations, since their coupling is taken to be $\hat{\underline{e}}\cdot\underline{r}$.

(ii) It emphasizes the deuteron wave function at larger distances, where it is better known.

The discussion will now be split into two parts; first a calculation of the photodisintegration amplitude produced by $H'_{\text{nucleon current}}$, and second that produced by explicit measure effects. In calculating the meson contribution, it must be remembered that the term $ie\,\underline{A}\cdot\underline{r}\,V_{exchange}$ must be subtracted, so it will be necessary to calculate it explicitly. In view of the interpretation of this term as the effect of charged meson exchange interactions at low energies, it may be expected that this subtraction will simply remove the charged meson exchange effect at low energies. This will be verified by direct comparison of such effects and the $V_{exchange}$ term.

In discussing the contribution of $H_{\text{nucleon current}}$, we first observe that previous calculations[2] have indicated that the $\underline{\sigma} \cdot \underline{H}$ term gives very small effects in the energy region of interest. This term can therefore be neglected, and we can take

$$H'_{\text{nucleon current}} = - \frac{ie}{2} \sqrt{\frac{k}{2}} \, \hat{\underline{e}} \cdot \underline{r} \ .$$

We therefore wish to calculate just the matrix element

$$\int \Psi_f^* (- \frac{ie}{2} \sqrt{\frac{k}{2}} \, \hat{\underline{e}} \cdot \underline{r}) \, \Psi_i \, d^3 \underline{r} \ .$$

The initial wave function here is to represent a deuteron, which we take as a triplet S-state, so that we may write

$$\Psi_i = \frac{u_i}{r} Y_{oo}(\Omega_r) \, X_1^m$$

where the radial function u_i is not at present specified in more detail. $Y_{oo}(\Omega r) = 1/\sqrt{4\pi}$ is the angular dependence of an S-state, and X_1^m denotes a triplet spin function of projection m.

The final state we shall take to be a plane wave, of momentum P_f which is given by overall energy conservation. In general its spin dependence may be either triplet or singlet; in this case the coupling is spin independent, so only the triplet state will contribute. The momentum of the nucleons in the final state is high enough so that the assumption of a plane wave state is probably fairly reasonable; in fact, it is exact for P-wave outgoing particles (as one has in this term) for a 50 per cent exchange force. Therefore

$$\Psi_f = e^{i \underline{P}_f \cdot \underline{r}} \, X_1^{m'} \ ,$$

and since only the P-wave part of this will contribute here, equivalently

$$\Psi_f = 4\pi i \sum_{m=-1}^{1} j_1(P_f r)\, Y^*_{1m}(\Omega_p)\, Y_{1m}(\Omega_r) \,.$$

The coupling $\hat{e} \cdot \underline{r}$ may be written as

$$\hat{e} \cdot \underline{r} = \sqrt{\frac{4\pi}{3}}\, r \sum_{m=-1}^{1} e^*_m\, Y_{1m}(\Omega_r)$$

where

$$e_1 = -\frac{e_x + ie_y}{\sqrt{2}}\,, \qquad e_{-1} = \frac{e_x - ie_y}{\sqrt{2}}\,, \qquad e_o = e_z \,.$$

It is then a straightforward calculation to evaluate the matrix element, and the result is

$$-e\sqrt{\frac{\pi k}{2}}\ \frac{\hat{e} \cdot \underline{P}_f}{P_f}\ \left| \int_o^\infty u^*_f\, r\, u_i\, dr \right| \cdot \delta_{mm'}$$

where

$$u_f = r\, j_1(P_f r) \,.$$

Defining

$$I(k) = \left| \int_o^\infty u^*_f\, r\, u_i\, dr \right| \,,$$

the final result is

$$-e\sqrt{\frac{\pi k}{2}}\ \frac{\hat{e} \cdot \underline{P}_f}{P_f}\ I(k)\ \delta_{mm'} \qquad (18)$$

and the final state can only be a triplet. The integral $I(k)$ has been evaluated for various choices of u_i to represent the deuteron by Marshall and Guth.[2]

Let us now continue on to meson effects. These included the terms

$$(\Psi_f| \; H^-_{NM} \; \frac{1}{a} \; H_3 + H_3 \; \frac{1}{a} \; H^+_{NM} + H^-_{NM} \; \frac{1}{a} \; H^-_{NM} \; \frac{1}{a} \; H_{mes} + H^-_{NM} \; \frac{1}{a} \; H_{mes} \; \frac{1}{a} \; H^+_{NM}$$

$$+ H_{mes} \; \frac{1}{a} \; H^+_{NM} \; \frac{1}{a} \; H^+_{NM} + H^-_{NM} \; \frac{1}{a} \; H_{\text{nucleon current}} \; \frac{1}{a} \; H^+_{NM} \; |\Psi_i) \; , \qquad (19)$$

with the provision that we must subtract the term $ie \; \underline{A} \cdot \underline{r} \; V_{exchange}$, which presumably is the three field and meson current contribution at low energies.

To verify this, and to see in detail how to subtract the $V_{exchange}$ contribution, it will be convenient to begin with a discussion of the form of the meson current and three field interaction diagrams. The nucleon current parts (that is, terms like $H^-_{NM} \; \frac{1}{a} \; H_{\text{nucleon current}} \; \frac{1}{a} \; H^+_{NM}$) will not be affected by the subtraction of $ie \; \underline{A} \cdot \underline{r} \; V_{exchange}$, and their discussion will consequently be deferred.

Observing that H^a_3 can only produce an S-wave meson relative to nucleon a, while H_{NM} only can couple P-wave mesons, and arguing that terms like $H^-_{NM}(a) \; \frac{1}{a} \; H^+_{NM}(a)$ are essentially included in the anomalous magnetic moment terms (which are known to be small), one can easily convince oneself that the only diagrams which need to be taken into account are those listed in Figure (2). The matrix element representing the sum of all these diagrams we shall denote by $(\underline{P}_f|M|\underline{P}_i)$ between plane wave states of the nucleons of relative momentum $\underline{P}_f$ and $\underline{P}_i$. Consider first the contribution from the H_3 couplings. The relevant diagrams here are shown in Figure(2a).

Fig. 2a

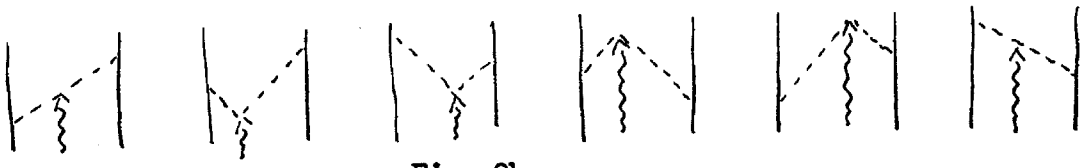

Fig. 2b

The plane wave matrix element, from a state of relative nucleon momentum $\underline{P}_i$ to a state of relative nucleon momentum $\underline{P}_f$ representing the first diagram may be directly written down. It is

$$\frac{1}{\sqrt{2k}} \, ig \; \underline{\sigma}_b \cdot \underline{K} \, \frac{1}{K^2 - \mu^2} \, ige \; \underline{\sigma}_a \cdot \hat{\underline{e}} \; (\tau_+^b \tau_-^a - \tau_-^b \tau_+^a)$$

since the interaction

$$H_3^{(emit)} = ige \; T_z \, \frac{\underline{\sigma} \cdot \hat{\underline{e}}}{\sqrt{2k}} \, ,$$

where T_z is the charge of the emitted meson. The operators τ_+ and τ_- change $n \to p$ and $p \to n$ respectively.

Conservation of momentum determines $\underline{K} = \underline{P}_b^f - \underline{P}_b^i$; in the c-m system, this gives $\underline{K} = - \underline{P}_f + \underline{P}_i + \underline{k}/2$. We first evalute the isotopic spin parts. The deuteron is an isotopic spin singlet state, so we want to evaluate

$$(\tau_+^b \tau_-^a - \tau_-^b \tau_+^a) \, X_o^o$$

where X_o^o is the isotopic spin singlet wave function. This is easily done; the result is

$$2X_1^o$$

where X_1^o is a triplet isotopic spin state of projection 0. The matrix

element thus becomes

$$-\frac{2eg^2}{\sqrt{2k}}\ \vec{\sigma}_b\cdot\underline{K}\ \vec{\sigma}_a\cdot\hat{\underline{e}}\ \frac{1}{K^2-\mu^2}\ .$$

The second diagram may be calculated in an analogous fashion. The result for both is:

$$-\frac{2eg^2}{\sqrt{2k}}\left\{\frac{\vec{\sigma}_b\cdot\underline{K}\ \vec{\sigma}_a\cdot\hat{\underline{e}}}{K^2-\mu^2}-\frac{\vec{\sigma}_a\cdot\underline{K}'\ \vec{\sigma}_b\cdot\hat{\underline{e}}}{K'^2-\mu^2}\right\} \tag{20}$$

where

$$\underline{K}' = \underline{P}_f - \underline{P}_i + \underline{k}/2, \qquad \underline{K} = -\underline{P}_f + \underline{P}_i + \underline{k}/2\ .$$

Look next at the terms in $H_{\text{meson current}}$. The diagrams included here are shown in Figure (2a), where it is understood that all 6 possible time orderings are to be included. The result here is:

$$-\frac{4eg^2}{\sqrt{2k}}\ \vec{\sigma}_b\cdot\underline{K}\ \frac{1}{K^2-\mu^2}\ \hat{\underline{e}}\cdot(\underline{P}_f-\underline{P}_i)\ \frac{1}{K'^2-\mu^2}\ \vec{\sigma}_a\cdot\underline{K}' \tag{21}$$

Adding (20) and (21) we find the entire plane wave matrix element from the H_3 and meson current couplings to be

$$(\underline{P}_f|M|\underline{P}_i) = -\frac{2eg^2}{\sqrt{2k}}\left\{\frac{\vec{\sigma}_b\cdot\underline{K}\ \vec{\sigma}_a\cdot\hat{\underline{e}}}{K^2-\mu^2}-\frac{\vec{\sigma}_a\cdot\underline{K}'\ \vec{\sigma}_b\cdot\hat{\underline{e}}}{K'^2-\mu^2}\right.$$

$$\left.+\,2\,\frac{\vec{\sigma}_b\cdot\underline{K}\ \hat{\underline{e}}\cdot(\underline{P}_f-\underline{P}_i)\ \vec{\sigma}_a\cdot\underline{K}'}{(K^2-\mu^2)(K'^2-\mu^2)}\right\} \tag{22}$$

We actually want the matrix element of these interactions not between plane wave states, but between the wave functions $\Psi_f + \Psi_i$ describing the outgoing n-p system and the deuteron respectively. Thus it is necessary to evaluate

$$(\Psi_f|M|\Psi_i) = \int \frac{d^3 P_f}{(2\pi)^3} \int \frac{d^3 P_i}{(2\pi)^3} (\Psi_f|P_f)(P_f|M|P_i)(P_i|\Psi_i)$$

$$= \int \frac{d^3 P_f}{(2\pi)^3} \int \frac{d^3 P_i}{(2\pi)^3} \int d^3 r_f \int d^3 r_i \, \Psi_f^*(r_f)$$

$$\cdot e^{iP_f \cdot r_f} (P_f|M|P_i) \, e^{-iP_i \cdot r_i} \, \Psi_i(r_i). \qquad (23)$$

If the exponential is written as

$$\exp -i \left[\frac{P_i + P_f}{2} \cdot (r_i - r_f) + \frac{P_i - P_f}{2} \cdot (r_i + r_f) \right] ,$$

then since $(P_f|M|P_i)$ is a function only of $P_f - P_i$, we can evaluate $\int d^3(P_i + P_f)$, and get $\delta(r_i - r_f)$ as the result. Thus

$$(\Psi_f|M|\Psi_i) = \int \frac{d^3 P}{(2\pi)^3} \int d^3 r \, \Psi_f^*(r) \, (P_f|M|P_i) \, e^{-iP \cdot r} \, \Psi_i(r)$$

where

$$P = P_f - P_i \, .$$

We want to calculate these meson terms for small photon energies, so we can verify that these terms actually do equal the coupling

$$(\Psi_f| \frac{ie}{\sqrt{2k}} \hat{e} \cdot r \, V_{exchange}|\Psi_i)$$

predicted by Siegert's Theorem. Therefore we shall first get the low k limit of $(\Psi_f|M|\Psi_i)$. This is easily found to be

$$- \frac{2eg^2}{\sqrt{2k}} \int \frac{d^3 P}{(2\pi)^3} \int d^3 r \quad \Psi_f^*(r) \left\{ \frac{\sigma_b \cdot P \; \sigma_a \cdot \hat{e} + \sigma_a \cdot P \; \sigma_b \cdot \hat{e}}{P^2 + \mu^2} \right.$$

$$\left. + 2 \, \frac{\sigma_b \cdot P \; \hat{e} \cdot P \; \sigma_a \cdot P}{(P^2 + \mu^2)^2} \right\} e^{-iP \cdot r} \, \Psi_i(r) \, . \qquad (24)$$

We must now calculate $(\Psi_f | \frac{ie}{\sqrt{2k}} \hat{e} \cdot \underline{r} \, V_{exchange} | \Psi_i)$ and compare it to the above.

To lowest order in g

$$V_{exchange} = \int ig \, \underline{\sigma}_a \cdot \underline{K} \quad e^{i\underline{K}\cdot\underline{r}_a} \frac{1}{K_\mu^2 - \mu^2} (-ig) \, \underline{\sigma}_b \cdot \underline{K} \, e^{-i\underline{K}\cdot\underline{r}_b}$$

$$(\tau^a_+ \tau^b_- + \tau^a_- \tau^b_+) \frac{d^3 \underline{K}}{(2\pi)^3} \ ,$$

where $K_\mu^2 - \mu^2 = -(\underline{K}^2 + \mu^2)$. Evaluation of the isotopic spin operators on X^o_o gives $-2X^o_o$. Hence (writing $-\underline{P}$ for $\underline{K}$, and $\underline{r}$ for $\underline{r}_a - \underline{r}_b$)

$$V_{exchange} = -2g^2 \int \frac{d^3 \underline{P}}{(2\pi)^3} \frac{\underline{\sigma}_b \cdot \underline{P} \ \underline{\sigma}_a \cdot \underline{P}}{(P^2 + \mu^2)} e^{-i\underline{P}\cdot\underline{r}} \ .$$

Since this is integrated on all $d^3 \underline{P}$, $\underline{P}$ may be replaced by $\underline{P} + \hat{\underline{e}}$ for any constant vector $\hat{\underline{e}}$, without changing anything. Thus

$$V_{exchange} = -2g^2 \int \frac{d^3 \underline{P}}{(2\pi)^3} \frac{\underline{\sigma}_b \cdot (\underline{P} + \hat{\underline{e}}) \ \underline{\sigma}_a \cdot (\underline{P} + \hat{\underline{e}})}{(\underline{P} + \hat{\underline{e}})^2 + \mu^2} e^{-i(\underline{P}+\underline{e})\cdot\underline{r}}$$

Since $\hat{\underline{e}}$ is arbitrary, the first order term in $\hat{\underline{e}}$ must by itself be zero.

$$0 = -2g^2 \int \frac{d^3 \underline{P}}{(2\pi)^3} \left[\left\{ \frac{\underline{\sigma}_b \cdot \underline{P} \ \underline{\sigma}_a \cdot \hat{\underline{e}} + \underline{\sigma}_a \cdot \underline{P} \ \underline{\sigma}_b \cdot \hat{\underline{e}}}{P^2 + \mu^2} \right.\right.$$

$$\left.\left. - 2 \frac{\underline{\sigma}_b \cdot \underline{P} \ \hat{\underline{e}} \cdot \underline{P} \ \underline{\sigma}_a \cdot \underline{P}}{(P^2 + \mu^2)^2} \right\} e^{-i\underline{P}\cdot\underline{r}} - i\hat{\underline{e}}\cdot\underline{r} \frac{\underline{\sigma}_b \cdot \underline{P} \ \underline{\sigma}_a \cdot \underline{P}}{P^2 + \mu^2} e^{-i\underline{P}\cdot\underline{r}} \right]$$

i.e., we have

$$i\hat{\underline{e}}\cdot\underline{r} \, V_{exchange} = -2g^2 \int \frac{d^3 \underline{P}}{(2\pi)^3} \left\{ \frac{\underline{\sigma}_b \cdot \underline{P} \ \underline{\sigma}_a \cdot \hat{\underline{e}} + \underline{\sigma}_b \cdot \hat{\underline{e}} \ \underline{\sigma}_a \cdot \underline{P}}{P^2 + \mu^2} \right.$$

$$\left. - 2 \frac{\underline{\sigma}_b \cdot \underline{P} \ \hat{\underline{e}} \cdot \underline{P} \ \underline{\sigma}_a \cdot \underline{P}}{(P^2 + \mu^2)^2} \right\} e^{-i\underline{P}\cdot\underline{r}} \qquad (25)$$

and taking the matrix element of this between initial and final states, we have, comparing to (24),

$$(\Psi_f| \frac{ie}{\sqrt{2k}} \hat{\underline{e}} \cdot \underline{r}\, V_{exchange}|\Psi_i) \equiv (\Psi_f|M|\Psi_i)_{k=o}$$

exactly as desired.

Thus the above calculation verifies that we have already included in the $\hat{\underline{e}} \cdot \underline{r}$ coupling $(\Psi_f|M|\Psi_i)_{k=o}$; the corrections to this coming from these meson parts of the perturbation term must therefore be given by

$$(\Psi_f|M|\Psi_i) - (\Psi_f|M|\Psi_i)_{k=o} \quad .$$

Let us now return to the problem of calculating $(\Psi_f|M|\Psi_i)$ explictly. This requires a particular choice of Ψ_f and Ψ_i. As before, we shall take Ψ_f to be a plane wave state. Thus all we really need is

$$\int \frac{d^3 \underline{P}_i}{(2\pi)^3} (\underline{P}_f|M|\underline{P}_i)(\underline{P}_i|\Psi_i) \tag{26}$$

where $(\underline{P}_f|M|\underline{P}_i)$ is the plane wave matrix element calculated above.

Using the mean value theorem, we can write

$$\int \frac{d^3\underline{P}}{(2\pi)^3} (\underline{P}_f|M|\underline{P}_i)(\underline{P}_i|\Psi_d)$$

$$= [\int d\Omega_p (\underline{P}_f|M|\underline{P}_i)]_{|\underline{P}_i|=P_{eff}} \int \frac{P_i^2\, dP_i}{(2\pi)^3} (\underline{P}_i|\Psi_d)$$

$$= [\int \frac{d\Omega_p}{4\pi} (\underline{P}_f|M|P_i)]_{P_{eff}} \Psi_d(0) \tag{27}$$

provided $(\underline{P}_i|\Psi_d)$ is everywhere ≥ 0 as a function of $|P_i|$. (This is actually the case for most of the commonly used deuteron wave functions.)

P_{eff} in the above expression will of course in general depend on $\underline{P}_f$, $\underline{k}$, etc., so we have not really gained anything by writing it this way. However, one might hope that P_{eff} will be much less than P_f, so that we could neglect it and obtain the result

$$\int \frac{d^3 \underline{P}}{(2\pi)^3} (\underline{P}_f|M|\underline{P}_i)(\underline{P}_i|\Psi_d) \simeq (\underline{P}_f|M|0)\, \Psi_d(0). \tag{28}$$

P_{eff} will probably be of the order of the average momentum in the deuteron, i.e., about 50 Mev. The smallest value of P_f which will interest us will be about 300 Mev. Hence such an approximation might not seem too unreasonable.

Since it is, however, possible to compute exactly (for some particular choices of the deuteron wave function) part of the above expression (that arising from the H_3 coupling), we shall look at the exact answers for these terms to see what error this approximation makes, and to see how the approximation may be improved. Furthermore, the above form implies a great sensitivity to the deuteron wave function at the origin, and it will be of interest to see if this apparent sensitivity really exists.

We first consider the diagram  , for which we found the plane wave matrix element to be

$$(\underline{P}_f|M'|\underline{P}_i) = \frac{2eg^2}{\sqrt{2k}} \quad \frac{\underline{\sigma}_b \cdot (\underline{P}_f - \underline{P}_i - \underline{k}/2)\ \underline{\sigma}_a \cdot \hat{\underline{e}}}{(k/2)^2 - (\underline{P}_f - \underline{P}_i - \underline{k}/2)^2 - \mu^2}\ . \tag{29}$$

We shall neglect the $(k/2)^2$ part of the energy denominator. This introduces no great error since $k^2/4 \ll P_f^2$. We thus wish to compute

$$\int d^3\underline{r} \int \frac{d^3\underline{P}_i}{(2\pi)^3} (\underline{P}_f|M'|\underline{P}_i)\, e^{-i\underline{P}_i\cdot\underline{r}}\, \Psi_d(\underline{r}) \quad .$$

The evaluation of this expression is straightforward and yields

$$-\frac{2eg^2}{\sqrt{2k}}\, \underline{\sigma}_b \cdot \underline{P}\ \ \underline{\sigma}_a \cdot \hat{\underline{e}}\ \mathcal{Y}(P)$$

where

$$\mathcal{Y}(P) = \frac{1}{\sqrt{4\pi}}\ \frac{1}{P} \int_o^\infty j_1(Pr)(\mu + \tfrac{1}{r})\, e^{-\mu r}\, u_1(r)\, dr \ ,$$

$$= \frac{1}{P} \int_o^\infty j_1(Pr)(\mu r + 1)\, e^{-\mu r}\, \Psi_d(r)\, dr,$$

and

$$\underline{P} = \underline{P}_f - \underline{k}/2.$$

It is first interesting to observe that the approximation (28) discussed above is equivalent to an approximate evaluation of $\mathcal{Y}(P)$ by assuming $\Psi_d(r)$ is a constant over the range of r in which the integrand in $\mathcal{Y}$ is appreciable. Thus

$$\mathcal{Y}(P) \tilde{\sim} \Psi_d(0)/P \int_o^\infty j_1(Pr)(\mu r + 1)\, e^{-\mu r}\, dr$$

$$= \frac{\Psi_d(0)}{P^2 + \mu^2} \ ,$$

where $\Psi_d(0)$ here represents the constant value of the deuteron wave function. The notation $\Psi_d(0)$ is therefore somewhat misleading in that it implies a strong sensitivity of the result to the deuteron wave function near the origin. Actually, as looking at the approximation from this point of view demonstrates, $\Psi_d(0)$ represents a sort of average value of the deuteron wave function, weighted by $j_1(Pr)(\mu r + 1)\, e^{-\mu r}$.

In order to discuss more carefully the dependence of $\mathcal{Y}(P)$ on a particular choice of deuteron wave functions, it will be convenient to set $\mu = 0$. This does not appreciably change the region in which the integrand is large; as can be seen from the original expression (29), it is equivalent to neglecting μ^2 compared to P^2, and thus introduces an error of ≤ 20 per cent.

The problem thus resolves itself into determining the sensitivity of the expression

$$\mathcal{Y}(P) = \frac{1}{P} \int_0^\infty j_1(pr)\Psi_d(r)\, dr$$

to a choice of $\Psi_d(r)$, and discovering to what extent the approximation

$$\mathcal{Y}(P) \approx \frac{\Psi_d(0)}{P^2}$$

is a valid one.

For very small momenta, i.e., as $P \to 0$, $j_1(Pr) \to Pr$. Hence as $P \to 0$, $\mathcal{Y}(P) \to \int_0^\infty r\, \Psi_d(r)\, dr$. This weights the deuteron wave function at large distances, and is therefore not sensitive to the choice of Ψ_d, since the asymptotic form of Ψ_d is fairly well determined. The approximation $\Psi_d(0)/P^2$ is clearly not valid, as can also be seen from the original point of view as expressed in equation (28).

For very large P, $j_1(Pr) \to -\frac{1}{Pr} \cos Pr$, so that

$$\mathcal{Y}(P) \to -\frac{1}{P^2} \int_0^\infty \frac{\cos Pr}{r}\, \Psi_d(r)\, dr.$$

This emphasizes small distances, so the approximation $\Psi_d(0)/P^2$ becomes valid, but the sensitivity to $\Psi_d(r)$ in unknown regions also becomes great.

One might hope, therefore, that in an intermediate region it will be possible to both use the approximation (28) and still not have $\mathcal{Y}(P)$ very sensitive to a choice of $\Psi_d(r)$. The momenta P of interest here are, roughly, $2\mu < P < 4\mu$, where μ is the meson mass. Since the first zero of $j_1(Pr)$ occurs at $Pr = 4.5$, it comes at a position comparable to this deuteron radius for the range of P. Hence one should not expect the dominant effect to be due to the inner part of the deuteron only, but the whole volume of the deuteron should contribute.

In Table (1) are shown the results for $P\mathcal{Y}(P)$ for several choices of $\Psi_d(r)$, and also the approximate value $\Psi_d(0)/P$. The exact values for the square well are obtained by numerical integration. As is clearly to be expected the approximation (28) is not at all good for wave functions which are not relatively flat over the region of the first loop of $j_1(Pr)$, such as the Hulthen wave function and the repulsive core wave function. For a square well, however, the approximation (28) is excellent, since the square well wave function is nearly flat over a large region.

As can be seen from the table, the variation of $\mathcal{Y}(P)$ with different deuteron wave functions is not excessive, only amounting to about 25 per cent at $P = 400$ Mev, and of course becoming greater as P increases. We can thus conclude that the choice of wave functions does not much matter, and we therefore choose a square well. For the square well, approximation (28) is excellent, so the final result will be to take

$$\mathcal{Y}(P) = \frac{\Psi_d(0)}{P^2 + \mu^2}$$

where $\Psi_d(0)$ is a square well wave function evaluated at the origin.

TABLE 1.

$$\int_0^\infty j_1(pr)\, \Psi_d(r)\, dr = p\, \psi(P).$$

p=	μ	2μ	3μ	4μ	5μ
(Hulthén.)	.123	.105	.102	.087	.078
(Approx.H)	.501	.250	.167	.125	.100
(Square)	.120	.103	.074	.055	.044
(Approx.Sq.)	.225	.112	.075	.056	.045

To summarize, then, in order to include the fact that the initial state is not a plane wave state, but is really a deuteron, all that is necessary is to drop the initial momentum $\underline{P}_i$ in the plane wave matrix element and multiply by $\Psi_d(0)$, where $\Psi_d(r)$ represents the square well wave function of the deuteron. This will be good, for the square well wave function, to an accuracy ranging from 3 per cent at P_f = 280 Mev to 2 per cent at P_f = 700 Mev. Furthermore, this result is not extremely sensitive to the choice of the deuteron wave function as a square well, in spite of the misleading notation $\Psi_d(0)$. The variation with wave functions depends on the photon energy, but is generally of the order of 25 per cent.

Thus, the matrix element of the perturbation meson effects, (including now both the H_3 coupling and the meson current coupling) may be written

$$-\frac{2eg^2}{\sqrt{2k}}\left\{\frac{\underline{\sigma}_b\cdot(\underline{P}_f-\underline{k}/2)\,\underline{\sigma}_a\cdot\hat{\underline{e}}}{P_f^2-\underline{P}_f\cdot\underline{k}+\mu^2}-\frac{\underline{\sigma}_b\cdot\underline{P}_f\,\underline{\sigma}_a\cdot\hat{\underline{e}}}{P_f^2+\mu^2}\right.$$

$$+\frac{\underline{\sigma}_a\cdot(\underline{P}_f+\underline{k}/2)\,\underline{\sigma}_b\cdot\hat{\underline{e}}}{P_f^2+\underline{P}_f\cdot\underline{k}+\mu^2}-\frac{\underline{\sigma}_a\cdot\underline{P}_f\,\underline{\sigma}_b\cdot\hat{\underline{e}}}{P_f^2+\mu^2}$$

$$-2\,\frac{\underline{\sigma}_b\cdot(\underline{P}_f-\underline{k}/2)\,\hat{\underline{e}}\cdot\underline{P}_f\;\underline{\sigma}_a\cdot(\underline{P}_f+\underline{k}/2)}{(P_f^2-\underline{P}_f\cdot\underline{k}+\mu^2)(P_f^2+\underline{P}_f\cdot\underline{k}+\mu^2)}$$

$$\left.+2\,\frac{\underline{\sigma}_b\cdot\underline{P}_f\;\hat{\underline{e}}\cdot\underline{P}_f\;\underline{\sigma}_a\cdot\underline{P}_f}{(P_f^2+\mu^2)(P_f^2+\mu^2)}\right\}\cdot\Psi_d(0), \qquad (30)$$

where this now includes the fact that the initial state is a deuteron, and not a plane wave state. $\Psi_d(0)$ represents, as argued above, the square well wave function at the origin. Its value is chosen as

$$\Psi_d(0) = 308\ (\text{Mev})^{3/2}\ .\ ^*$$

In addition to the above terms, there is a contribution coming from the nucleon current interaction occurring while a meson is present, which up to now has not been discussed in detail. This comes from the term

$$H_{NM}^{-}\ \frac{1}{a}\ H_{\text{nucleon current}}\ \frac{1}{a}\ H_{NM}^{+}$$

appearing in equation 19 (p. 15) the relevant diagrams here are

---

* This results from taking a range b = 2.07 x 10^{-13} cm[9] and adjusting depth V_o = 22.9 Mev to fit the triplet scattering length a_t = 5.39 x 10^{-13} cm as given by Blatt and Weisskopf, "Theoretical Nuclear Physics".

where processes in which the meson is reabsorbed by the same nucleon are ignored as these are essentially a vertex modification and included by the use of a renormalized coupling constant.[4] Nucleon recoil will here, as in the resonant terms to be discussed later, be neglected when a virtual meson is present. The nucleon current coupling here is therefore limited to that coming from the anomalous moments.

There is no need of a subtraction here, since no part of these diagrams is included in the $\hat{\underline{e}} \cdot \underline{r}$ coupling. The matrix element may therefore be calculated in a straightforward way, giving approximately:

$$\frac{1}{2} \frac{eg^2}{\sqrt{2k}} \left(\frac{\mu_p - \mu_n}{e} \right)^2 \left\{ 2i(\underline{\sigma}_a + \underline{\sigma}_b) \cdot \underline{P}_f \; \underline{P}_f \cdot \underline{k} \times \hat{\underline{e}} \right.$$

$$+ (\underline{\sigma}_b \cdot \underline{P}_f \; \underline{\sigma}_a \cdot \underline{k} + \underline{\sigma}_a \cdot \underline{P}_f \; \underline{\sigma}_b \cdot \underline{k}) \hat{\underline{e}} \cdot \underline{P}_f$$

$$\left. - (\underline{\sigma}_b \cdot \underline{P}_f \; \underline{\sigma}_a \cdot \hat{\underline{e}} + \underline{\sigma}_a \cdot \underline{P}_f \; \underline{\sigma}_b \cdot \hat{\underline{e}}) \underline{k} \cdot \underline{P}_f \right\}$$

$$\cdot \frac{\Psi_d(0)}{\omega_f^2(\omega_f - k/2)}$$

where

$$\omega_f^2 \simeq P_f^2 + k^2/4 \; . \tag{31}$$

The entire perturbation matrix element is then given by the sum of expressions (18), (30) and (31).

The cross section resulting from the perturbation terms may be easily computed, using the well-known Golden Rule,

$$d\sigma = 2\pi \frac{1}{6} \sum |(f|M|i)|^2 \, \varrho(E_f)$$

where the sum is over the initial three spin states, two polarizations and final four spin states. $\rho(E_f)$ denotes the density of final states, and is given by

$$\rho(E_f) = \frac{P_f M}{2 \cdot (2\pi)^3} d\Omega .$$

It is most convenient to split the matrix element M into parts resulting in a final triplet and singlet spin state. Since these do not interfere, the cross section breaks up into two parts,

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma_{\text{singlet}}}{d\Omega} + \frac{d\sigma_{\text{triplet}}}{d\Omega} .$$

The expressions for $d\sigma_s/d\Omega$ and $d\sigma_t/d\Omega$ are given below. In obtaining these results, the angular dependence in the denominators of $(f|M|i)$ has been expanded out to order $\underline{P}_f \cdot \underline{k} / P_f^2 + \mu^2$. Neglecting all terms of order $k^2/P_f^2 + \mu^2$ and higher, there is no contribution at all to a triplet final state from meson current and H_3 parts, and no contribution to a singlet final state from the nucleon anomalous moment terms.

$$\frac{d\sigma_s}{d\Omega} = \frac{1}{4\pi} \left(\frac{e^2}{4\pi}\right) (P_f Mk)(g^2 \Psi_d(0))^2 \left\{ \frac{4}{3} \frac{1}{(P_f^2 + \mu^2)^2} \left[1 - 4 \frac{P_f^2}{P_f^2 + \mu^2} + 4 \frac{P_f^4}{(P_f^2 + \mu^2)^2} \right] + \frac{8}{3} \frac{P_f^2}{(P_f^2 + \mu^2)^3} \left[1 - 2 \frac{P_f^2}{P_f^2 + \mu^2} \right] \sin^2 \theta + \frac{8}{3} \frac{P_f^4}{(P_f^2 + \ ^2)^4} \sin^4 \theta \right\} \qquad (32)$$

$$\frac{d\sigma_t}{d\Omega} = \frac{1}{4\pi}\left(\frac{e^2}{4\pi}\right)(P_f\, Mk)\left\{\frac{\pi}{2}|I(k)|^2 \sin^2\theta + \frac{4}{3}\left(\frac{\mu_p - \mu_n}{e}\right)^2 g^4|\Psi_d(0)|^2 \frac{P_f^4}{\omega_f^4(\omega_f - k/2)^2} \cdot (4 + \sin^2\theta + 5\sin^4\theta)\right\} \quad (33)$$

The total cross section may be obtained upon integration over $d\Omega$, giving

$$\sigma_s = \frac{\pi}{3}(P_f\, Mk)\left(\frac{e^2}{4\pi}\right) g^4|\Psi_d(0)|^2 \frac{4}{\pi}\frac{1}{(P_f^2+\mu^2)^2}\left(1 - \frac{8}{3}\frac{P_f^2}{P_f^2+\mu^2} + \frac{12}{5}\frac{P_f^4}{(P_f^2+\mu^2)^2}\right) \quad (34)$$

$$\sigma_t = \frac{\pi}{3}(P_f\, Mk)\left(\frac{e^2}{4\pi}\right)\left\{|I(k)|^2 + \frac{11}{3\pi}\left(\frac{\mu_p - \mu_n}{e}\right)^2 g^4|\Psi_d(0)|^2 \frac{P_f^4}{\omega_f^4(\omega_f - k/2)^2}\right\} \quad (35)$$

The numerical evaluation of these cross sections requires the calculation of $I(k)$ using the square well wave function previously written down. This is most easily done by numerical integration. Using the resulting values for $I(k)$, and chosing $f^2 = .1$, $\omega_{max} = M$, which fits the meson nucleon scattering data as determined by Gammel,[10] we find the results given in Table 2.

k	σ_s	σ_t	$\sigma_s + \sigma_t$
100 MeV	1.45 μb	42.3 μb	43.7 μb
150	1.61	21.3	22.9
200	1.57	12.5	14.1
250	1.55	9.24	10.8
300	1.49	7.43	8.9
350	1.43	6.60	8.0
400	1.39	6.40	7.8

TABLE 2.

No resonance is shown here, as is, of course, to be expected. The perturbation terms provide a fairly constant background of about 5 μb which diminishes quite slowly with photon energy.

The angular distribution resulting from these terms is seen to be symmetrical around 90°. This does not seem to agree with the experimental results, which indicate that an asymmetry forms well before the appearance of the resonance, and even as low as 20 Mev. However, at energies far below or far above the resonant area, the angular distribution tends to become symmetric.

This completes the discussion of the lowest order meson effects. We shall now proceed to an analysis of the resonant parts.

IV. RESONANT TERMS.

These were defined to consist of the cases in which one or more mesons was produced by H_3, $H_{\text{meson current}}$ or $H_{\text{nucleon current}}$, and then allowed to rescatter virtually in the deuteron before being absorbed. If it were not for the presence of the second nucleon, these processes could be described as photoproduction off of one nucleon, with the meson then being absorbed on the other nucleon. (The meson cannot be re-absorbed on the same nucleon if it is to be produced in a 3/2 3/2 state relative to it.) The fact that there are two nucleons, however, destroys this simple viewpoint. First, it allows the possibility of multiple scattering. Second, mesons produced in S-waves from nucelon (a), which in simple photoproduction cannot be enhanced by rescattering, can now rescatter from nucleon (b).

We shall discuss these difficulties in order, beginning with a discussion of scattering of mesons in deuterium, i.e., a discussion of the form of the scattering matrix R_s.

R_s was supposed to represent the amplitude to go from a state of one meson to another state of one meson under the interaction $H^{\pm}_{NM}(a) + H^{\pm}_{NM}(b)$. Thus R_s satisfies the equation

$$(A) \quad R_s = H^{\pm}_{NM}(a + b) + H^{\pm}_{NM}(a + b)\, \frac{1}{a}\, R_s \,. \tag{36}$$

In lowest order, for example, the diagrams contributing to R_s are given in Figure (3), remembering that we had chosen to divide things in such a way that a state of zero mesons never appeared.

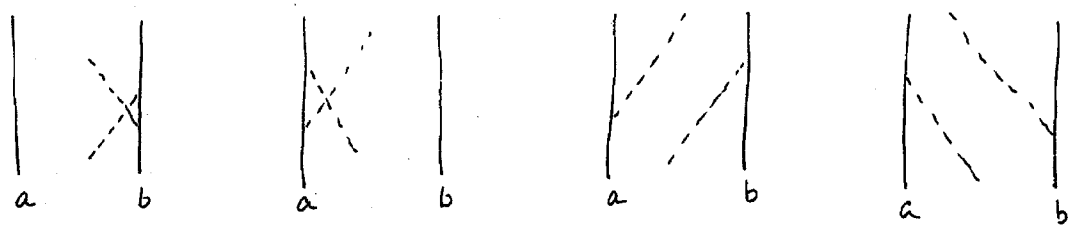

Fig. (3)

In a different Lorentz frame, diagrams (3c) and (3d) might look like Figure (4), in which there is a state of zero mesons. If we were to

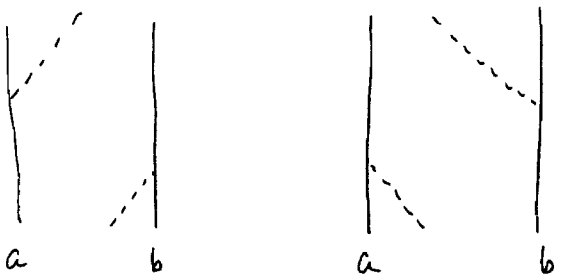

Fig. (4)

neglect terms such as in Figs. (3c) and (3d), we would be essentially dropping the field theory approach, and rather consider the meson to be simply a particle which can be scattered by either nucleon a or nucleon b with the potential

$$V_a + V_b = H^-_{NM}(a)\,\frac{1}{a}\,H^+_{NM}(a) + H^-_{NM}(b)\,\frac{1}{a}\,H^+_{NM}(b)\ .$$

In this point of view, R_s would satisfy

$$R_s = (V_a + V_b) + (V_a + V_b)\,\frac{1}{a}\,R_s \qquad (37)$$

and the lowest order diagrams here would be just those of Figs. (3a) and (3b). We shall adopt this view, and replace equation (36) as determining R_s by equation (37). This preserves the features of resonant scattering in the 3/2 3/2 state from either nucleon as given by field theory, and only neglects diagrams in which the resonant scattering is coupled to higher orders in f resulting from other processes. Examples of the types of diagrams included in (36) and (37) are given in Fig. (5).

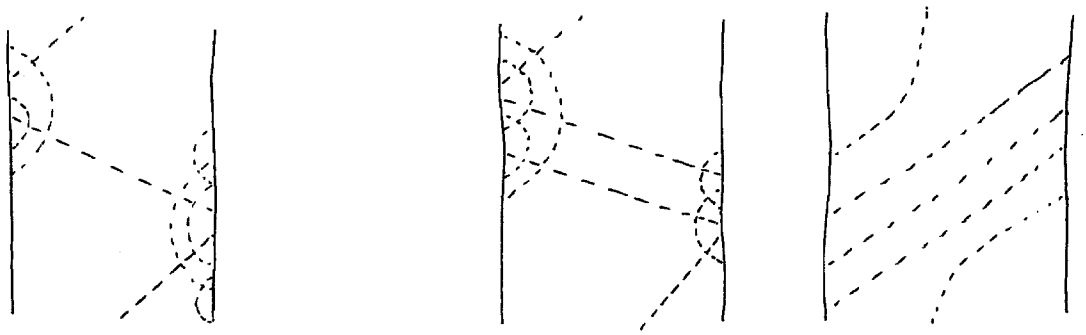

Fig. (5)

Determining R_s then resolves itself into the problem of finding the scattering from two sources when that from either source by itself is known.

Equation (37) may be rewritten as

$$R_s = V_a + V_b + V_a \frac{1}{a} V_a + V_b \frac{1}{a} V_b + V_a \frac{1}{a} V_b + V_b \frac{1}{a} V_a + \cdots$$

$$= R_a + R_b + R_a \frac{1}{a} R_b + R_b \frac{1}{a} R_a + \cdots \qquad (38)$$

where

$$R_{a,b} = V_{a,b} + V_{a,b} \frac{1}{a} R_{a,b} \,. \tag{39}$$

Thus R_a, R_b represent the exact scattering from nucleons a, b alone, and R_s is the sum of the amplitude to scatter from a or b, or first on a and then on b, etc.

One approximation which has often[11] been applied to problems of the interaction of mesons and deuterium is the impulse approximation, which consists of taking $R_s = R_a + R_b$. This one would expect to be fairly good whenever the two scatterers are fairly well separated compared to the range of interaction with the scattered particle. That is, even though the interaction of the scatterer and projectile may be quite strong, the scatterers are far enough apart so that the probability of the projectile bouncing from one to the other is quite small. There are difficulties with this approximation;[12] nevertheless we shall adopt it here.

It therefore is necessary to have an expression for the scattering of a π-meson from a single nucleon, located at some position $\underline{r}_a$. We shall take the scattering as appreciable only in the $J = 3/2$, $T = 3/2$ state, so that we can write

$$(\underline{k}'T'|R_a|\underline{k}T) = (\underline{k}'|P^a_{3/2}|\underline{k})(T'|Q^a_{3/2}|T)(k'|R|k)\, e^{-i(\underline{k}' - \underline{k})\cdot\underline{r}_a} \,, \tag{40}$$

where $P^a_{3/2}$ denotes the angular momentum 3/2 projection on nucleon a, and $Q^a_{3/2}$ the isotopic spin 3/2 projection on nucleon a. These are (13)

$$(\underline{k}'|P^a_{3/2}|\underline{k}) = \frac{1}{4\pi\, k'k} \left(3\,\underline{k}' \cdot \underline{k} - \underline{\sigma}_a \cdot \underline{k}'\ \underline{\sigma}_a \cdot \underline{k}\right)$$

$$(T'|Q^a_{3/2}|T) = \frac{1}{3} \left(3\underline{a}'^{+} \cdot \underline{a} - \underline{\tau}_a \cdot \underline{a}'^{+}\ \underline{\tau}_a \cdot \underline{a}\right) \,.$$

The factor $\exp -i(\underline{\kappa}' - \underline{\kappa}) \cdot \underline{r}_a$ comes in because the nucleon is not located at the origin. For $(\kappa'|R|\kappa)$ the approximate expression given by Gammel[10] will be used.

We thus finally end up with

$$R_s = R_a + R_b$$

where R_a, R_b are described above.

With this approximation for R_s, the resonant matrix elements may be written down in detail. The indices (a) and (b) denote the two nucleons.

$$\begin{aligned} & H^-_{NM}(a+b)\,\frac{1}{a}\,(R_a + R_b)\,\frac{1}{a}\,H_3(a+b) \\ & + H^-_{NM}(a+b)\,\frac{1}{a}\,(R_a + R_b)\,\frac{1}{a}\,H^-_{NM}(a+b)\,\frac{1}{a}\,H_{\text{meson current}} \\ & + H^-_{NM}(a+b)\,\frac{1}{a}\,(R_a + R_b)\,\frac{1}{a}\,H_{\text{meson current}}\,\frac{1}{a}\,H^+_{NM}(a+b) \\ & + H^-_{NM}(a+b)\,\frac{1}{a}\,(R_a + R_b)\,\frac{1}{a}\,H_{\text{nucleon current}}\,\frac{1}{a}\,H^+_{NM}(a+b)\,. \end{aligned} \tag{41}$$

As an example, the diagrams corresponding to the first line here are written in Fig. (6).

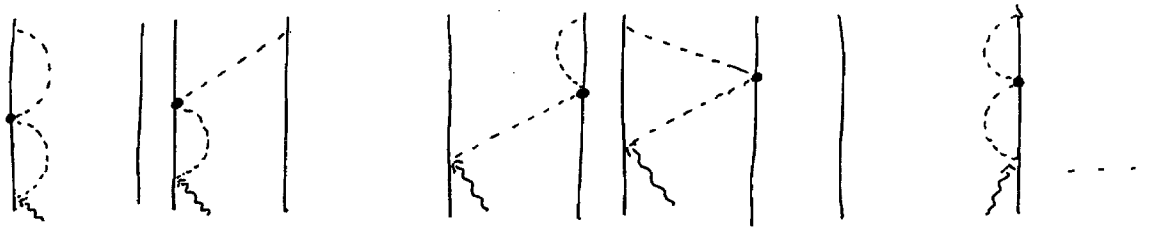

Fig. (6)

All but one of these terms (in Fig. (6)) may immediately be thrown out, once we assume the no-recoil approximation except when mesons are exchanged. The first and third diagrams cannot have the scattering occurring in the 3/2 3/2 state. The first and second diagrams have an S-wave meson to scatter. Thus only the fourth diagram remains.

Similar arguments may be applied to the other terms. The remaining terms are listed below:

$$
\begin{aligned}
&H^{-}_{NM}(a)\,\frac{1}{a}\,R_b\,\frac{1}{a}\,H_3(a) + H^{-}_{NM}(b)\,\frac{1}{a}\,R_a\,\frac{1}{a}\,H_3(b)\\
&+ H^{-}_{NM}(a)\,\frac{1}{a}\,R_b\,\frac{1}{a}\,(H^{-}_{NM}(a+b)\,\frac{1}{a}\,H_{meson} + H_{meson}\,\frac{1}{a}\,H^{+}_{NM}(a+b))\\
&+ H^{-}_{NM}(b)\,\frac{1}{a}\,R_a\,\frac{1}{a}\,(H^{-}_{NM}(a+b)\,\frac{1}{a}\,H_{meson} + H_{meson}\,\frac{1}{a}\,H^{+}_{NM}(a+b))\\
&+ H^{-}_{NM}(a)\,\frac{1}{a}\,R_b\,\frac{1}{a}\,(H_{nucleon}(a+b)\,\frac{1}{a}\,H^{+}_{NM}(a+b))\\
&+ H^{-}_{NM}(b)\,\frac{1}{a}\,R_a\,\frac{1}{a}\,(H_{nucleon}(a+b)\,\frac{1}{a}\,H^{+}_{NM}(a+b))\,. \qquad (42)
\end{aligned}
$$

The spirit of the impulse approximation, which we used above to calculate R_s, is that the amplitude to exchange mesons between the two nucleons is fairly small. Thus one expects that if a process of re-scattering can occur with, say, one meson exchange, the contribution from it will be larger than one with two meson exchanges, which is otherwise the same. To sketch the reason for this, consider the two terms

$$H^{-}_{NM}(a)\,\frac{1}{a}\,R_b\,\frac{1}{a}\,H_{nucleon}(b)\,\frac{1}{a}\,H^{+}_{NM}(b)$$

and

$$H^{-}_{NM}(a)\,\frac{1}{a}\,R_b\,\frac{1}{a}\,H_{nucleon}(a)\,\frac{1}{a}\,H^{+}_{NM}(a).$$

The first term may be rewritten as (symbolically)

$$\frac{H_{\text{nucleon}}(b)}{H^{-}_{NM}(b)} \left(H^{-}_{NM}(a) \frac{1}{a} R_b \frac{1}{a} H^{-}_{NM}(b) \frac{1}{a} H^{+}_{NM}(b)\right)$$

$$= \frac{H_{\text{nucleon}}(b)}{H^{-}_{NM}(b)} H^{-}_{NM}(a) \frac{1}{a} R_b \ ,$$

so that the ratio of the second to this is (again symbolically)

$$\frac{\frac{1}{a} H_{\text{nucleon}}(a) \frac{1}{a} H^{+}_{NM}(a)}{\frac{H_{\text{nucleon}}(b)}{H^{-}_{NM}(b)}} = \frac{(H_{NM})^2}{a^2} \ .$$

Since the coupling constant is small, and since there is no resonance here (due to not having any resonant scattering matrix present), this ratio is small.

Therefore, using essentially the impulse approximation, only the following terms remain.

$$\begin{aligned}
&H^{-}_{NM}(a) \frac{1}{a} R_b \frac{1}{a} H_3(a) + H^{-}_{NM}(b) \frac{1}{a} R_a \frac{1}{a} H_3(b) \\
&+ H^{-}_{NM}(a) \frac{1}{a} R_b \frac{1}{a} \left(H^{-}_{NM}(b) \frac{1}{a} H_{\text{meson}} + H_{\text{meson}} \frac{1}{a} H^{+}_{NM}(b)\right) \\
&+ H^{-}_{NM}(b) \frac{1}{a} R_a \frac{1}{a} \left(\text{"} \quad a \longleftrightarrow b \right) \\
&+ H^{-}_{NM}(a) \frac{1}{a} R_b \frac{1}{a} \left(H_{\text{nucleon}}(b) \frac{1}{a} H^{+}_{NM}(b)\right) \\
&+ H^{-}_{NM}(b) \frac{1}{a} R_a \frac{1}{a} \left(\text{"} \quad b \longleftrightarrow a \right) \ . \qquad (43)
\end{aligned}$$

Now, the last four terms here represent just the magnetic dipole and electric quadrupole photoproduction terms from one nucleon, with absorbtion on the other. In the photoproduction calculations, it is known that by far the greater contribution comes from the anomalous moment coupling of the nucleon and very little from the meson current terms.[14] The meson current terms may therefore be ignored, and the final result is

$$H^{-}_{NM}(a)\,\frac{1}{a}\,R_b\,\frac{1}{a}\,H_3(a) + H^{-}_{NM}(b)\,\frac{1}{a}\,R_a\,\frac{1}{a}\,H_3(b)$$
$$+ H^{-}_{NM}(a)\,\frac{1}{a}\,R_b\,\frac{1}{a}\,H_{nucleon}(b)\,\frac{1}{a}\,H^{+}_{NM}(b)$$
$$+ H^{-}_{NM}(b)\,\frac{1}{a}\,R_a\,\frac{1}{a}\,H_{nucleon}(a)\,\frac{1}{a}\,H^{+}_{NM}(a)\ . \qquad (44)$$

The corresponding diagrams are shown in Fig. (7), in the same order.

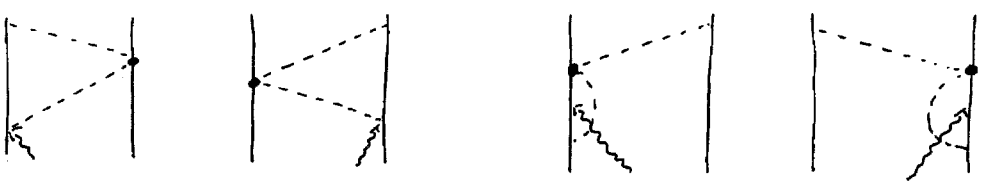

Fig. (7)

We shall now proceed to calculate these, starting with the H_3 terms.

A. H_3 terms.

The matrix element of interest is:

$$(\Psi_f | H^-_{NM}(a) \frac{1}{a} R_b \frac{1}{a} H^+_3(a) + H^-_{NM}(b) \frac{1}{a} R_a \frac{1}{a} H^+_3(b) | \Psi_i) \ . \qquad (45)$$

The energy denominators $\frac{1}{a}$ are $\frac{1}{E - H_o + i\varepsilon}$; these should really be replaced by renormalized propagators,[4]

$$S_r(E - H_o + i\varepsilon) = \frac{1}{E - H_o + i\varepsilon} \ \frac{1}{(1 + (f/\mu)^2 \Delta(E - H_o + i\varepsilon))}$$

where

$$\Delta(E - H_o) = \frac{3(E - H_o)}{\pi} \int_{\mu}^{\omega_{max}} \frac{\kappa^3 \, d\omega}{\omega^2(\omega - E - H_o)} \ .$$

Chew states that the function $\Delta(E)$ is appreciable only in the range $\mu < E < \omega_{max}$; for this case E is not in this range, so we shall ignore the renormalization effects.

Consider $H^-_{NM}(a) \frac{1}{a} R_b \frac{1}{a} H^+_3(a)$ first. The plane wave matrix element of this quantity is

$$\sum_{\underline{a},\underline{a}'} \int \frac{d^3\underline{\kappa}}{(2\pi)^3} ig \frac{\underline{\sigma}_a \cdot \underline{\kappa}'}{\sqrt{2\omega'}} \ \underline{\tau}_a \cdot \underline{a}' \ \frac{1}{E - E_2 + i\varepsilon} \ (\underline{\kappa}' \ \underline{a}' | R_b | \underline{\kappa} \ \underline{a})$$

$$\frac{1}{E - E_1 + i\varepsilon} \ igeT \ \frac{\underline{\sigma}_a \cdot \hat{\underline{e}}}{\sqrt{2k} \ \sqrt{2\omega}} \ \underline{\tau}_a \cdot \underline{a}^+ \qquad (46)$$

where

$$\underline{\kappa}' = \underline{P}_f - \underline{P}_i - \underline{k}/2 + \underline{\kappa}$$

$$E - E_1 = \left(k + \frac{P_i^2}{M} + \frac{k^2}{4M} \right) - \left(\omega + \frac{(P_i + k/2)^2}{2M} + \frac{(P_i + k/2 - \underline{\kappa})^2}{2M} \right)$$

$$E - E_2 = \left(k + \frac{P_i^2}{M} + \frac{k^2}{4M} \right) - \left(\omega' + \frac{P_f^2}{2M} + \frac{(P_i + k/2 - \underline{\kappa})^2}{2M} \right),$$

taken between states of relative momentum $\underline{P}_f$ and $\underline{P}_i$ for the nucleons. Note that recoil is included here during the exchange of mesons between the two nucleons, but not during the virtual scattering process from a single nucleon.

Note also that the $\exp -i(\underline{K}' - \underline{K}) \cdot \underline{r}_b$ factor no longer appears in $(\underline{K}'\ \underline{a}'|R_b|\underline{K}\ \underline{a})$, since it has been taken into account by conserving momentum. Thus in the above,

$$(\underline{K}'\ \underline{a}'|R_b|\underline{K}\ \underline{a}) = (\underline{K}'|P^b_{3/2}|\underline{K})(\underline{a}'|Q^b_{3/2}|\underline{a})(K'|R|K) .$$

The matrix element may be rewritten as a product of an isotopic spin part, an angular integral, and a radial (in K) integral, all of which can be separately computed. The isotopic spin part is

$$\frac{1}{3} \sum_{\underline{a},\underline{a}'} \underline{\tau}_a \cdot \underline{a}'(3\underline{a}'^{+} \cdot \underline{a} - \underline{\tau}_b \cdot \underline{a}'^{+}\ \underline{\tau}_b \cdot \underline{a})T_z\ \underline{\tau}_a \cdot \underline{a}^{+}$$

and this is to act on an isotopic spin singlet wave function describing the deuteron. The calculation is straightforward. Applying the above to

$$X^o_o = \frac{1}{\sqrt{2}} (X^a_{1/2}\ X^b_{-1/2} - X^a_{-1/2}\ X^b_{1/2})$$

we get the result $\frac{4}{3} X^o_1$, where

$$X_1 = \frac{1}{\sqrt{2}}(X^a_{1/2}\ X^b_{-1/2} + X^a_{-1/2}\ X^b_{1/2}) .$$

Taking the matrix element with a final triplet state X^o_1, we get as a result 4/3.

The angular part is

$$\frac{1}{4\pi K' K} \int d\Omega_{\underline{K}} \; \underline{\sigma}_a \cdot (\underline{g} + \underline{K})[3(\underline{g} + \underline{K}) \cdot \underline{K} - \underline{\sigma}_b \cdot (\underline{g} + \underline{K}) \, \underline{\sigma}_b \cdot \underline{K}]$$

where $\underline{g} = \underline{P}_f - \underline{P}_i - \underline{k}/2$. This integrates to

$$\frac{K^2}{KK'} \left\{ 3 \underline{\sigma}_a \cdot \underline{g} - \frac{1}{3} \underline{\sigma}_b \cdot \underline{g} \; \underline{\sigma}_a \cdot \underline{\sigma}_b \right\}$$

We have here assumed that there is no angular dependence in the energy denominators. In other words, we now take

$$E - E_1 = k - \omega - \frac{K^2}{2M}$$

$$E - E_2 = k/2 - \omega' - \frac{K^2}{2M}$$

where we have used the conservation law $k + \frac{P_i^2}{M} + \frac{k^2}{4M} = \frac{P_f^2}{M}$. The resulting matrix element may then be written

$$-\frac{4}{3} \frac{eg^2}{\sqrt{2k}} (3 \underline{\sigma}_a \cdot \underline{g} - \frac{1}{3} \underline{\sigma}_b \cdot \underline{g} \; \underline{\sigma}_a \cdot \underline{\sigma}_b) \; \underline{\sigma}_a \cdot \hat{\underline{e}} \cdot I'(k) \qquad (47)$$

where

$$I'(k) = \int \frac{K^4 dK}{(2\pi)^3} \frac{1}{\sqrt{2\omega'}} \frac{1}{E - E_2 + i\varepsilon} \frac{(K'|R|K)}{K'K} \frac{1}{E - E_1 + i\varepsilon} \frac{1}{\sqrt{2\omega}}$$

The other terms, $H^-_{NM}(b) \frac{1}{a} R_a \frac{1}{a} H^+_3(b)$, may be calculated immediately by just interchanging a and b in the above. This means $\underline{g} = \underline{P}_f - \underline{P}_i - k/2$ is replaced by $\underline{g} = -\underline{P}_f + \underline{P}_i - k/2$, since in the center of mass system, $\underline{P}^f_a = \underline{P}_f = -\underline{P}^f_b$, and $\underline{P}^i_a = \underline{P}_i - \underline{k}/2$, $\underline{P}^i_b = -\underline{P}_i - \underline{k}/2$. Since also $X^o_o \to -X^o_o$ upon interchange of a and b, the entire effect is therefore to change the sign of $\underline{k}$ in $\underline{g}$. This term therefore gives

$$-\frac{4}{3}\frac{eg^2}{\sqrt{2k}}\left(3\ \underline{\sigma}_b\cdot\underline{g}'-\frac{1}{3}\ \underline{\sigma}_a\cdot\underline{g}'\ \underline{\sigma}_b\cdot\underline{\sigma}_a\right)\ \underline{\sigma}_b\cdot\hat{\underline{e}}\cdot I'(k) \qquad (48)$$

where

$$\underline{g}' = \underline{P}_f - \underline{P}_i + \underline{k}/2 \ .$$

The entire resonant matrix element is the sum of expressions (47) and (48).

As discussed before, if we use a square well wave function for the deuteron, it is a good approximation to include the effect of the initial being a deuteron and not a plane wave by dropping $\underline{P}_i$ everywhere and multiplying by $\Psi_d(0)$. We again use a plane wave as the final state. The matrix element may be rewritten in two pieces, one given a triplet and one a singlet final state. We find: (dropping the term in $\frac{1}{3}\ \underline{\sigma}_b\cdot\underline{\sigma}_a$ from the matrix element)

(i) Triplet Final State:

$$-8\frac{eg^2}{\sqrt{2k}}\ I(k)\ \Psi_d(0)\ \left[\hat{\underline{e}}\cdot\underline{P}_f+\frac{i}{2}\left(\underline{\sigma}_a+\underline{\sigma}_b\right)\cdot\underline{P}_f\times\hat{\underline{e}}\right] \qquad (49)$$

(ii) Singlet Final State:

$$2i\frac{eg^2}{\sqrt{2k}}\ I(k)\ \Psi_d(0)\ \left(\underline{\sigma}_a-\underline{\sigma}_b\right)\cdot\underline{k}\times\hat{\underline{e}} \qquad (50)$$

The singlet contributions are clearly much smaller than the triplet, being in the ratio $\frac{1}{12}\frac{k}{P_f}$, and can therefore be neglected.

B. $H_{\text{nucleon current}}$ Terms.

Since the calculation of the H_3 terms has been sketched in some detail, it will suffice here to merely observe one simplifying relation.

The nucleon current coupling (for a non-recoiling nucleon, which we assume to be the case except when mesons are exchanged) is

$$H_{\text{nucleon current}} = -i\mu \frac{\underline{\sigma} \cdot \underline{k} \times \hat{\underline{e}}}{\sqrt{2k}}$$

where we can write, approximately,

$$\mu = \frac{\mu_p - \mu_n}{2} \tau_3 = \frac{\mu_p - \mu_n}{2} \underline{\tau} \cdot \underline{a}_3$$

This is seen to be the same as H_{NM}^{-} for a neutral meson of momentum $\underline{k} \times \hat{\underline{e}}$, aside from some numerical factors. This fact allows us to replace

$$R_a \frac{1}{a} H_{\text{nucleon current}}(a) \frac{1}{a} H_{NM}^{+}(a)$$

by the quantity

$$- \frac{(\mu_p - \mu_n)}{2g} (R_a - V_a)$$

where R_a represents the scattering of an incident neutral meson of momentum $\underline{k} \times \hat{\underline{e}}$, and V_a represent the $J = 3/2$, $T = 3/2$ contribution of the meson nucleon potential, as calculated to second order in the coupling constant. The resulting matrix element may be written as follows:

(i) To a triplet final state.

$$g^2(\mu_p - \mu_n)\, S(k)\, \Psi_d(0)\, [2i(\underline{\sigma}_a + \underline{\sigma}_b) \cdot \underline{P}_f\ \underline{P}_f \cdot \underline{k} \times \hat{\underline{e}}$$

$$+ (\underline{\sigma}_b \cdot \underline{P}_f\ \underline{\sigma}_a \cdot \underline{k} + \underline{\sigma}_a \cdot \underline{P}_f\ \underline{\sigma}_b \cdot \underline{k})\, \hat{\underline{e}} \cdot \underline{P}_f$$

$$- (\underline{\sigma}_b \cdot \underline{P}_f\ \underline{\sigma}_a \cdot \hat{\underline{e}} + \underline{\sigma}_a \cdot \underline{P}_f\ \underline{\sigma}_b \cdot \hat{\underline{e}})\, \underline{k} \cdot \underline{P}_f] \qquad (51)$$

where

$$g^2\, S(k) = \frac{1}{\sqrt{2\omega'}}\ \frac{1}{k/2 - \omega'}\ \frac{(K'|R_{3/2\ 3/2}|k) - (K'|V_{3/2\ 3/2}|k)}{4\pi K'k} \qquad (52)$$

and

$$\omega'^2 \approx P_f^2 + k^2/4 + \mu^2 .$$

(ii) As before, the singlet states will contribute much less, being of order $\frac{1}{6}\ \frac{k}{P_f}$ of the triplet.

The cross sections may be calculated in a straightforward way from the above matrix elements. The results, including interference from the perturbation terms, are:

$$\frac{d\sigma_{\text{triplet}}}{d\Omega} = \frac{1}{12\pi}\left(\frac{e^2}{4\pi}\right)(P_f\, Mk) \qquad (53)$$

$$\cdot \Big\{\ 32g^4|\Psi_d(0)|^2\ \left(\frac{P_f}{k}\right)^2 |I'(k)|^2\ (4 + \sin^2\theta)$$

$$+ 24\ \sqrt{\pi}\, g^2\, \Psi_d(0)\, \left(\frac{P_f}{k}\right) I(k)\ \mathrm{Re}\ I'(k)\ \sin^2\theta$$

$$+ 128\ g^4|\Psi_d(0)|^2\ \left(\frac{\mu_p - \mu_n}{e}\right)\sqrt{\frac{k}{2}}\ \frac{P_f^3}{k}\ \mathrm{Re}(I'(k)\, S^*(k))$$

$$\cdot \sin^2\theta \cos\theta$$

$$+ 4g^4 |\Psi_d(0)|^2 \left(\frac{\mu_p - \mu_n}{e}\right)^2 k P_f^4 |S(k)|^2 \cdot (4 + \sin^2\theta + 5\sin^4\theta) \Big\}$$

$$\sigma_{triplet} = \frac{\pi}{3}\left(\frac{e^2}{4\pi}\right)(P_f\, Mk) \Big\{ \frac{448}{3\pi} g^4 \Psi_d^2 \left(\frac{P_f}{k}\right)^2 |I'|^2 + \frac{16}{\sqrt{\pi}}\left(\frac{P_f}{k}\right) g^2 \Psi_d(0) \frac{P_f}{k} I(k)\, \mathrm{Re}\, I'(k) + \frac{88}{3\pi} g^4 \Psi_d^2 \left(\frac{\mu_p - \mu_n}{e}\right)^2 k P_f^4 |S(k)|^2 \Big\} \tag{54}$$

In order to arrive at numbers for these cross sections, it is necessary to calculate $I'(k)$. This had been defined by

$$I'(k) = \int \frac{K^4\, dK}{(2\pi)^3} \frac{1}{\sqrt{2\omega'}} \frac{1}{k/2 - \omega' - K^2/2M + i\varepsilon} \cdot \frac{(K'|R|K)}{K'K} \frac{1}{k - \omega - K^2/2M + i\varepsilon} \frac{1}{\sqrt{2\omega}} \tag{55}$$

For $(K'|R|K)$, we choose the form given by Gammel[10] which reproduced the exact results very accurately.

$$(K'|R|K) = (K'|V|K) \cdot \Big\{ 1 - \int \frac{K''^2\, dK''}{(2\pi)^3} \frac{(K''|V|K'')}{k - \omega'' + i\varepsilon} \Big\}^{-1} \cdot \Big\{ 1 + \frac{1}{(K'|V|K)} \int \frac{K''^2\, dK''}{(2\pi)^3} \frac{(K'|V|K'')(K''|V|K)}{k - \omega'' + i\varepsilon} - \int \frac{K''^2\, dK''}{(2\pi)^3} \frac{(K''|V|K'')}{k - \omega'' + i\varepsilon} \Big\}, \tag{56}$$

and $(\kappa'|V|\kappa) = 4\pi g^2/3 \cdot \kappa'\kappa / \sqrt{2\omega\ 2\omega'} \cdot 4/k - \omega - \omega'$. Using the relation

$$\lim_{\varepsilon \to 0} \frac{1}{k - \omega'' + i\varepsilon} = P\left(\frac{1}{k - \omega''}\right) - i\pi\,\delta(k - \omega''), \tag{57}$$

where P denotes the principal value at the pole, this may be written as

$$(\kappa'|R|\kappa) = (\kappa'|V|\kappa) \left\{ 1 + \frac{\Delta_I - i\Delta_I'}{1 - \Delta_F + i\Delta_F'} \right\} \tag{58}$$

where we define

$$\Delta_F = \frac{G^2}{3M^2} \int_{\mu}^{\omega_{max}} \frac{\kappa''^3\, d\omega''}{(k - \omega'')(k - 2\omega'')}$$

$$\Delta_I = \frac{G^2}{3M^2} (k - \omega - \omega') \int_{\mu}^{\omega_{max}} \frac{\kappa''^3\, d\omega''}{(k - \omega' - \omega'')(k - \omega'')(k - \omega - \omega'')} \tag{59}$$

$$\Delta_I' = \pi \frac{G^2}{3M^2} \frac{\kappa^3(k)}{\omega\omega'} (k - \omega - \omega')$$

$$\Delta_F' = -\pi \frac{G^2}{3M^2} \frac{\kappa^3(k)}{k}$$

Here G^2 represents $(8\pi g^2 M^2)/(Q\pi)^3$. One should observe that in all the energy denominators internal to the rescattering process, nucleon recoil has been neglected. It has not, however, been neglected in the processes of meson exchange between the two nucleons.

Using this form for $(\kappa'|R|\kappa)$, I(k) may be written as (*)

$$I(k) = \int \frac{\kappa^4 d\kappa}{(2\pi)^3} \frac{1}{\sqrt{2\omega'}} \frac{1}{k/2 - \kappa^2/2M - \omega'} \frac{(\kappa'|R|\kappa)}{\kappa'\kappa} \frac{1}{k - \kappa^2/2M - \omega} \frac{1}{\sqrt{2\omega}}$$

$$- i\pi \left(\frac{\kappa^3 \omega}{(2\pi)^3} \frac{1}{\sqrt{2\omega'}} \frac{1}{k/2 - \kappa^2/2M - \omega'} \frac{(\kappa'|R|\kappa)}{\kappa'\kappa} \frac{1}{\sqrt{2\omega}} \right) \Bigg|_{k - \kappa^2/2M - \omega = 0}$$

$$- i\pi \left(\frac{\kappa^3 \omega}{(2\pi)^3} \frac{1}{\sqrt{2\omega'}} \frac{(\kappa'|R|\kappa)}{\kappa'\kappa} \frac{1}{k - \kappa^2/2M - \omega} \frac{1}{\sqrt{2\omega}} \right) \Bigg|_{k/2 - \kappa^2/2M - \omega' = 0},$$

(60)

and the integral (that is, the first of the above three terms) becomes

$$I'(k) = \frac{G^2}{3M^2} (I_1(k) + I_2(k)) \tag{61}$$

where

$$I_1(k) = \int_{\mu}^{\omega_{max}} \frac{\kappa^3 d\omega}{\omega'} \frac{1}{k/2 - \kappa^2/2M - \omega'} \cdot \frac{1}{k - \omega - \omega'} \cdot \frac{1}{k - \kappa^2/2M - \omega}$$

$$I_2(k) = \frac{1}{1 - \Delta_F + i\Delta_F'} \int_{\mu}^{\omega_{max}} \frac{\kappa^3 d\omega}{\omega'} \frac{1}{k/2 - \kappa^2/2M - \omega'}$$

$$\left(\frac{\Delta_I - i\Delta_I'}{k - \omega - \omega'} \right) \frac{1}{k - \kappa^2/2M - \omega} \tag{62}$$

The integral $I_2(k)$ shall be further split up:

$$I_2(k) = \frac{G^2}{3M^2} \cdot \frac{1}{1 - \Delta_F + i\Delta_F'} \cdot \Pi(k) \tag{63}$$

---

* If the equations $k - \kappa^2/2M - \omega = 0$ and $k/2 - \kappa^2/2M - \omega' = 0$ have no solutions, then the indicated terms do not appear.

where

$$\Pi(k) = \int_{\mu}^{\omega_{max}} \frac{\kappa^3 \, d\omega'}{\omega'} \cdot \frac{1}{k/2 - \kappa^2/2M - \omega'} \; \frac{1}{k - \kappa^2/2M - \omega}$$

$$\cdot \left(G(k, \omega, \omega') - i\pi \frac{\kappa^3(k)}{\omega\omega'} \right) \tag{64}$$

and

$$\frac{\Delta_I(k, \omega, \omega')}{k - \omega - \omega'} = \frac{G^2}{3M^2} \int_{\mu}^{\omega_{max}} \frac{\kappa''^3 \, d\omega''}{(k - \omega' - \omega'')(k - \omega'')(k - \omega - \omega'')}$$

$$\equiv \frac{G^2}{3M^2} \; G(k, \omega, \omega') \; . \tag{65}$$

Finally, we write $\Pi(k)$ as

$$\Pi(k) = \Pi_1(k) + i\pi \kappa^3(k) \; \Pi_2(k) \tag{66}$$

These integrals can all be evaluated numerically. The resulting numbers are given in Table (3).

k (Mev)	$I_1(k)$	$\Pi_1(k)$	$\Pi_2(k)$	Re $I'(k)$	Im $I'(k)$
100	-.390	-.099	1.40×10^{3}	-1.00×10^{-6}	0×10^{-6}
150	-.407	-.122	1.40	-1.34	~ 0
200	-.420	-.126	1.30	-1.51	- .625
250	-.358	-.129	1.14	- .606	- 1.153
300	-.348	-.134	.973	- .060	- .739
350	-.326	-.142	.791	.018	- .461
400	-.303	-.164	.660	.013	- .382

TABLE 3.

Combining this with the contributions from the $i\varepsilon$ parts at the poles, we finally get Table (4).

k (Mev)	Re I(k)	Im I(k)	$\|I(k)\|^2$
100	-1.00×10^{-6}	0×10^{-6}	1.00×10^{-12}
150	-1.34	- .015	1.80
200	-1.38	- .905	2.72
250	- .016	-1.411	1.99
300	.539	- .728	.820
350	.521	- .366	.405
400	.461	- .295	.299

TABLE (4).

Similarly, to calculate S(k), Gammel's approximate formcan again be used. Here no integral is involved, since S(k) was defined simply by

$$g^2 S(k) = \frac{1}{\sqrt{2\omega'}} \frac{1}{k/2-\omega'} \frac{(\kappa'|R_{3/2\ 3/2}|k) - (\kappa'|V_{3/2\ 3/2}|k)}{4\pi \kappa' k}$$

$$= g^2 \frac{\sqrt{2}}{3} \frac{1}{\omega'^2(\omega' - k/2)} \frac{1}{\sqrt{k}} \left(\frac{\Delta_I(\omega',k,k) - i\Delta_F'}{1 - \Delta_F + i\Delta_F'} \right) \quad (67)$$

and Δ_I, Δ_F and Δ_F' are defined as before.

Using these results for S(k) and I(k), the cross sections may be computed directly to give the results shown in Table (5).

k (MeV)	$\sigma_{resonant}$ (*)	$\sigma_{perturbation}$	σ_{total}
100	-12.2	43.7	31.5
150	17.2	22.9	40.1
200	86.2	14.1	100.3
250	99.3	10.8	110.1
300	63.4	8.9	72.3
350	34.9	8.0	42.9
400	23.0	7.8	30.8

TABLE (5)

The theoretical cross section obtained above is plotted in Fig. (8). It is seen to be considerably too high for large photon energies, running from 20 per cent to 50 per cent above the experimental points. In view of the numerous approximations, it is questionable whether any better agreement than this could be expected. The theoretical results do, at least, predict accurately the qualitative features of the resonance, such as its position and general shape.

It may be useful to mention, however, several points in the approximation used which tend to overestimate the cross section. First, as was observed earlier, the choice of wave functions for the deuteron could alter the results by up to 40 per cent in the cross section. A repulsive core wave function would tend to reduce the results, a Hulthan well to increase them. Second, the fact that multiple scattering in the deuteron has been neglected also tends to overestimate the results.

* $\sigma_{resonant}$ here denotes the entire contribution to the cross section produced by the resonant terms, including interference with the perturbation terms.

Brueckner[12] has estimated that inclusion of multiple scattering effects may decrease the predictions of the impulse approximation in π-D scattering by up to 40 or 50 per cent. Finally, the coupling constant is not known completely accurately, and the resonant features of the cross section depend quite sensitively on it. The choice of $f^2 = .1$ made here is, as a matter of fact somewhat higher than values generally considered now. In order to indicate the sensitivity to the coupling constant, the cross section has been computed using $f^2 = .08$ and making the reasonable assumption that the R matrix for scattering is not altered much by this change. The resulting cross section is also shown in Fig. (8), and clearly agrees quite well with experiments, except for the dip before the resonance.

Regarding the angular distribution: an asymmetry is obtained from the resonant terms of the form $\sin^2\theta \cos\theta$. This asymmetry becomes very small at low and high energies, and also gives the same cross section at 0° and 180°, which is apparently inconsistent with the experiments. It may be that a $\cos\theta$ term can only be found by investigating the D state in the deuteron.

In conclusion, then, it may be stated that the qualitative features of the cross section are correctly predicted by this model; more than this cannot really be stated in view of the approximation required in the calculation.

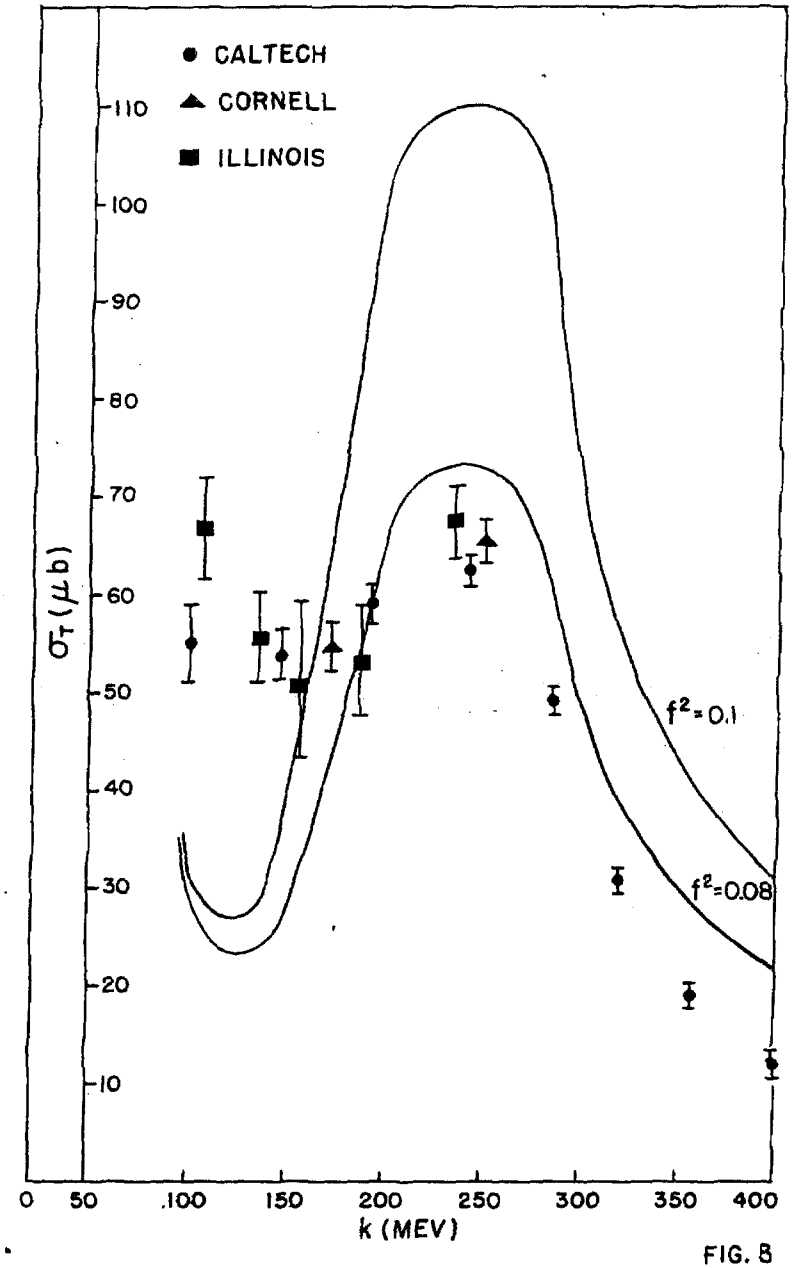
CALTECH
CORNELL
ILLINOIS
σ_T (μb)
110
100
90
80
70
60
50
40
30
20
10
0
50
100
150
200
250
300
350
400
k (MEV)
f^2=0.1
f^2=0.08

FIG. 8

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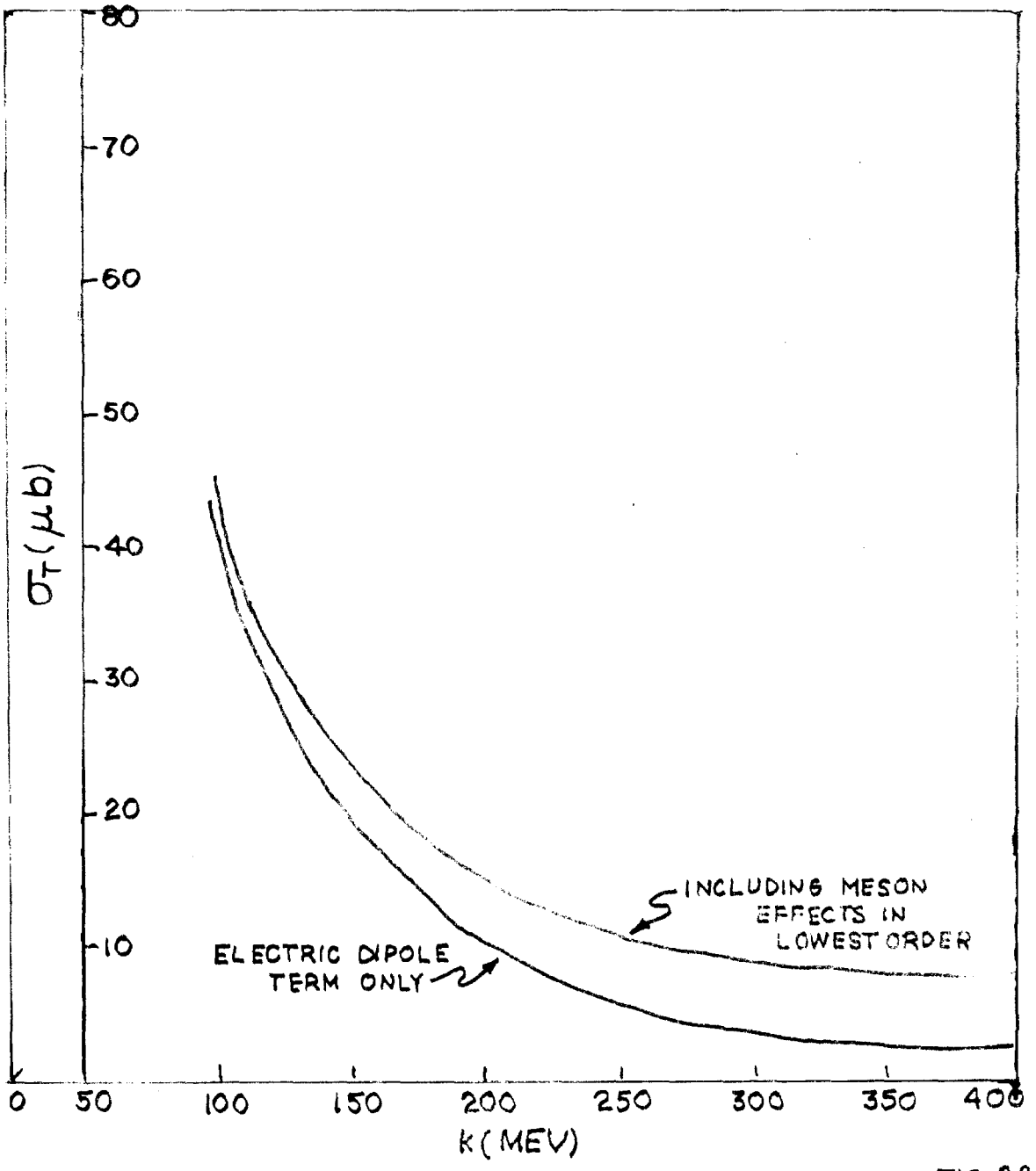

80
70
60
50
40
30
20
10
σ_T (μb)
INCLUDING MESON EFFECTS IN LOWEST ORDER
ELECTRIC DIPOLE TERM ONLY
0
50
100
150
200
250
300
350
400
k (MEV)

FIG. 8A

Appendix A.

FORMAL SCATTERING THEORY.[5,6]

If one is interested in determining the scattering cross section produced by an interaction V in the Schrodinger equation

$$(H_o + V)\Psi = E_o\Psi \tag{1}$$

the perturbation theory approach provides the answer in the following form. The cross section is given by 2π times the density of final states times the square of some matrix element, this matrix element being the sum of the perturbation series, to whatever order is desired. If the matrix element is denoted M_{fi}, the perturbation series give

$$M_{fi} = V_{fi} + \sum_n \frac{V_{fn}\,V_{ni}}{E_o - E_n} + \sum_{n,n'} \frac{V_{fn}\,V_{nn'}\,V_{n'i}}{(E_o - E_n)(E_o - E_{n'})} + \cdots . \tag{2}$$

One might try to sum this series exactly by rewriting it as

$$M_{fi} = (f|M|i) = (f|V|i) + \left(f|V \frac{1}{E_o - H_o} V|i \right) + \cdots \tag{3}$$

or in operator notation

$$\begin{aligned} M &= V + V\frac{1}{E_o - H_o}V + V\frac{1}{E_o - H_o}V\frac{1}{E_o - H_o}V + \cdots \\ &= V + V\frac{1}{E_o - H_o}M. \end{aligned} \tag{4}$$

We have thus obtained an integral equation for the operator M, and upon taking the matrix element of the operator between the initial and final states have also obtained an exact solution for the scattering amplitude.

The above procedure is, unfortunately, an oversimplification. The principal difficulty with it is that it makes no mention of how to treat the poles appearing in the energy denominators. It will therefore

be convenient to start from a time dependent point of view, and try to determine how these difficulties are to be overcome. It will turn out that the above equation for M is valid, but that the method of treating the pole will determine which of several quite distinct quantities will be obtained. We begin with the time dependent Schrodinger equation,

$$H\Psi = (H_o + V)\Psi = i \frac{\partial \Psi}{\partial t} \tag{5}$$

and are interested in a scattering process going from some initial eigenstate of H_o, ϕ_i, to another state ϕ_f. Usually the procedure is to say the initial state is fed in suddenly at time t_o, so that the exact state at time t is

$$\Psi_i(t) = e^{-iH(t-t_o)} \phi_i e^{-iE_i t_o} \tag{6}$$

Actually, the physical situation is always to feed in a beam of incident waves over a period of time T. Thus it might be better to write the wave function at time t as

$$\Psi_i(t) = \frac{1}{T} \int_{-T}^{o} e^{-iH(t-t_o)} \phi_i e^{-iE_i t_o} dt, \tag{7}$$

where this form implies producing the incident state continuously from time $-T$ to time 0. Instead of starting the incident beam suddenly, however, it will be convenient to start feeding it in gradually. This can be described by writing

$$\Psi_i(t) = \lim_{T\to\infty} \frac{1}{T} \int_{-T}^{o} e^{-iH(t-t_o)} \phi_i \, e^{-iE_i t_o} \, e^{t_o/T} \, dt$$

$$= \lim_{\varepsilon\to 0} \varepsilon \int_{-1/\varepsilon}^{o} e^{-iHt} \, e^{-i(E_i - H + i\varepsilon)t_o} \, \phi_i \, dt, \qquad (8)$$

where $\varepsilon = 1/T$. With this method of introducing the incident wave, the exact wave function at time t is given by $\Psi_i^{(+)}(t)$ (the (+) is put in to denote propagation from negative to positive time) and the probability to be in a final state ϕ_f at time t is

$$\omega_{fi}(t) = |f_{fi}(t)|^2 \cdot N^{-1} \qquad (9)$$

where

$$f_{fi}(t) = (\phi_f \, e^{-iE_f t}, \Psi_i^{(+)}(t)) \qquad (10)$$

and

$$N = (\Psi_i^{(+)}(t), \Psi_i^{(+)}(t)). \qquad (11)$$

N is time independent because H is Hermitian. Now

$$\Psi_i^{(+)}(t) = e^{-iHt} \, \varepsilon \int_{-\infty}^{o} e^{-i(E_i - H + i\varepsilon)t_o} \, \phi_i \, dt_i$$

$$= e^{-iHt} \, \frac{i\varepsilon}{E_i - H + i\varepsilon} \, \phi_i \qquad (12)$$

and since $H\phi_i = (V + E_i)\phi_i$, this results in

$$\Psi_i^{(+)}(0) = \frac{1}{E_i - H + i\varepsilon} \, (E_i - H + i\varepsilon) + \frac{1}{E_i - H + i\varepsilon} \, (H - E_i) \, \phi_i$$

$$= \phi_i + \frac{1}{E_i - H + i\varepsilon} \, V\phi_i$$

$$= \phi_i + \frac{1}{E_i - H_o + i\varepsilon} \, V\Psi_i^{(+)}(0) \, . \qquad (13)$$

Thus the transition amplitude is

$$f_{fi}(0) = \delta_{fi} + \frac{1}{E_i - E_f + i\varepsilon} \quad (f|R|i) \tag{14}$$

where we define

$$(f|R|i) = (\phi_f, V\Psi_i^{(+)}(0)). \tag{15}$$

The cross section is related to the transition rate, as it is necessary to calculate $d/dt\ f_{fi}(t)|_{t=0}$. This is

$$\begin{aligned} \frac{d}{dt}\ f_{fi}(t)|_{t=o} &= i(\phi_f, (E_f - H)\Psi_i^{(+)}(0)) \\ &= -i(\phi_f, V\ \Psi_i^{(+)}(0)) \\ &= -i(f|R|i). \end{aligned} \tag{16}$$

Therefore the transition rate is, at $t = 0$,

$$\frac{d}{dt}\ \omega_{fi}(t) = \left\{ 2\delta_{if}\ \mathrm{Im}(i|R|i) + \frac{2\varepsilon}{(E_f-E_i)^2+\varepsilon^2}\ |(f|R|i)|^2 \right\} \cdot N^{-1} \tag{17}$$

N is easily calculated

$$N = \sum_i\ |f_{fi}(t)|^2 = 1 + \frac{2}{\varepsilon}\ \mathrm{Im}(i|R|i) + \sum_i \frac{1}{(E_f-E_i)^2+\varepsilon^2}\ |(f|R|i)|^2, \tag{18}$$

and since N is constant in time,

$$0 = \frac{d}{dt}\sum_i\ |f_{fi}(t)|^2 = 2\ \mathrm{Im}(i|R|i) + \sum_i\ \frac{1\varepsilon}{(E_f-E_i)^2+\varepsilon^2}\ |(f|R|i)|^2 \tag{19}$$

Then

$$N = 1 + \frac{1}{\varepsilon}\ \mathrm{Im}(i|R|i). \tag{20}$$

In the limit as $\varepsilon \to 0$ and the volume of the box in which everything is quantized $\to \infty$, $N \to 1$. The reason for this is that the length of the incident wave train, which is $Tv = v/\varepsilon$, must be less than L, the side of the box. Hence as $\varepsilon \to 0$ and $L \to \infty$, εL stays finite so $\varepsilon L^3 \to \infty$, and therefore $\frac{1}{\varepsilon}(f|R|f) \sim (\varepsilon L^3)^{-1} \to 0$. The cross section is related to the rate of change of transition probability by dividing by the incident flux.

$$\sigma_{fi} = \frac{L^3}{v} \frac{d}{dt} \omega_{fi}(t)\Big|_{t=0} \tag{21}$$

Since $(i|R|i) \to 0$ as $L \to \infty$, we have left

$$\sigma_{fi} = \frac{1}{v} \lim_{\substack{\varepsilon \to 0 \\ L \to \infty}} \frac{2\varepsilon}{(E_f - E_i)^2 + \varepsilon^2} |(f|R|i)|^2 L^3 \tag{22}$$

as the cross section for the transition to a particular state f. Now

$$\lim_{\varepsilon \to 0} \frac{2\varepsilon L^3}{(E_f - E_i)^2 + \varepsilon^2} \longrightarrow 2\pi \delta(E_f - E_i) L^3 \tag{23}$$

which gives the density of final states (together with the statement of energy conservation) when integrated over the final states. Thus

$$\sigma = \frac{1}{v} 2\pi |(f|R|i)|^2 \varrho(E_f) \tag{24}$$

where ϱ is the density of states. The matrix element $(f|R|i)$ may be determined directly from an integral equation.

$$(f|R|i) = (\phi_f, V\Psi_i^{(+)}(0))$$

$$= (\phi_f, V[\phi_i + \frac{1}{E_i - H_o + i\varepsilon} V\Psi_i^{(+)}(0)])$$

$$= (\phi_f, V\phi_i) + \sum_I (\phi_f, V\phi_I) \frac{1}{E_i - E_I + i\varepsilon} (I|R|i) \qquad (25)$$

This, rewritten in operator form, is simply

$$R = V + V \frac{1}{E_i - H_o + i\varepsilon} R . \qquad (26)$$

As an example, consider the scattering of a particle of spin zero from a spherically symmetrical potential V. Let $\underline{\kappa}_f$ and $\underline{\kappa}_i$ denote the initial and final momenta of the particle. The cross section is then given by

$$\frac{d\sigma}{d\Omega} = \frac{1}{v} 2\pi |(\underline{\kappa}_f|R|\underline{\kappa}_i)|^2 \cdot \frac{\kappa M}{(2\pi)^3} \qquad (27)$$

Since V is spherically symmetric the scattering must conserve the angular momentum l, and its z component m. It must also be independent of m. Thus if we transform R to a representation in terms of the energy (or equivalently the magnitude of the momentum) and angular momentum, R will be diagonal.

$$(\underline{\kappa}_f|R|\underline{\kappa}_i) = \sum_{lm} (\underline{\kappa}_f|lm)(lm|\underline{\kappa}_i) (\kappa_f|R_l|\kappa_i), \qquad (28)$$

and conservation of energy gives $\kappa_f = \kappa_i$. Now

$$(\underline{\kappa}_f|lm) = \int \sum_{lm} i^L Y_{LM}(\Omega_{\underline{\kappa}_f}) Y^*_{LM}(\Omega_r) Y_{lm}(\Omega_r) \frac{d\Omega_r}{(2\pi)^3}$$

$$= i^l Y_{lm}(\Omega_{\underline{\kappa}_f}) \qquad (29)$$

and similarly

$$(lm|\underline{\kappa}_i) = (i^l Y_{lm}(\Omega_{\kappa_i}))^* . \tag{30}$$

We thus obtain

$$\frac{d\sigma}{d\Omega} = \frac{2\pi}{v} \left| \sum_{lm} Y^*_{lm}(\Omega_{\kappa_i}) Y_{lm}(\Omega_{\kappa_f}) (\kappa_f|R_l|\kappa_i) \right| \frac{\kappa M}{(2\pi)^3} \tag{31}$$

If the z-axis is chosen along the direction of the incident beam, then $Y_{lm}(\Omega_{\kappa_i}) \to Y_{lm}(0,0) = (\frac{2l+1}{4\pi})^{1/2} \delta_{m,o}$. Thus

$$\frac{d\sigma}{d\Omega} = \frac{2\pi}{v} \left| (4\pi)^{-1/2} \sum_l \sqrt{2l+1}\, Y_{lo}(\Omega_{\kappa_i}) (\kappa_f|R_l|\kappa_i) \right|^2 \frac{\kappa M}{(2\pi)^3}$$

$$= \frac{2\pi}{v} \left| \frac{1}{4\pi} \sum_l (2l+1) (\kappa_f|R_l|\kappa_i) P_l(\cos\theta_f) \right|^2 \frac{\kappa M}{(2\pi)^3} \tag{32}$$

Now if $(\kappa|R_l|\kappa)$ were defined as

$$(\kappa|R_l|\kappa) = -\frac{(2\pi)^3}{\kappa M} \frac{1}{\pi} \sin\delta_l e^{i\delta_l}, \tag{33}$$

the cross section would have the form

$$\frac{d\sigma}{d\Omega} = \frac{1}{\kappa^2} \left| \sum_l (2l+1) \sin\delta_l e^{i\delta_l} P_l(\cos\theta_f) \right|^2 \tag{34}$$

This is precisely the usual form obtained from a partial wave analysis.

Identifying the diagonal matrix element of R with this combination of the phase shifts can be done in general. If A denotes any representation in which R is diagonal, then

$$(f|R|i) = \sum_A (f|A)(A|i) R_A \tag{35}$$

If the quantities $(f|A)$ are normalized by

$$\sum_I \delta(E_i - E_I)(A|I)(I|A') = C(E_i)\ \delta_{AA'}\ , \tag{36}$$

then we can define

$$R_A = -\frac{\pi}{C(E_i)}\ \sin\delta_A\, e^{i\delta_A}\ . \tag{37}$$

$C(E_i)$ represents the density of states per unit energy range. The cross section is therefore

$$\frac{d\sigma}{d\Omega} = \frac{2\pi}{v}\left|\sum_A (f|A)(a|i)\,\frac{1}{\pi C}\sin\delta_A\, e^{i\delta_A}\right|^2 C$$

$$= \frac{2}{\pi C(E) v}\left|\sum_A (f|A)(a|i)\sin\delta_A\, e^{i\delta_A}\right|^2\ . \tag{38}$$

The expansion into the states A may be interpreted in terms of projection operators. Defining $P_A = |A)(A|$,

$$(f|R|i) = \sum_A (f|P_A|i)\, R_A \tag{39}$$

and since $P_A P_{A'} = P_A\, \delta_{AA'}$, we have

$$R_A = V_A + V_A\ \frac{1}{E - H_o + i\varepsilon}\ R_A\ . \tag{40}$$

Explicitly written out for the example treated above, this last equation is

$$(\kappa'|R_1|\kappa) = (\kappa'|V_1|\kappa) + \int \frac{\kappa''^2\, d\kappa''}{(2\pi)^3}\,(\kappa'|V_1|\kappa'')\,\frac{1}{E - E'' + i\varepsilon}\,(\kappa''|R_1|\kappa). \tag{41}$$

The cross section is then equally well written as

$$\frac{d\sigma}{d\Omega} = 2\pi\left|\sum_A (f|P_A|i)\, R_A\right|^2\ \varrho(E_f)\ . \tag{42}$$

It is possible to define another operator, called the reaction matrix, as follows. Write

$$(f|T|i) = \sum_A (f|P_A|i)\ T_A \tag{43}$$

and

$$T_A = \frac{R_A}{1 + i\pi C\ R_A}\ , \qquad R_A = \frac{T_A}{1 - i\pi C\ T_A} \tag{44}$$

Thus

$$-\pi C\ T_A = \tan\ \delta_A\ . \tag{45}$$

Again referring to the explicit example previously discussed, an integral equation may be derived for T_A. We had

$$(\kappa|R_1|\kappa) = (\kappa|V_1|\kappa) + \int \frac{\kappa''^2\, d\kappa''}{(2\pi)^3}\ (\kappa|V_1|\kappa'')\frac{1}{E - E''}\ (\kappa''|R_1|\kappa)$$

$$+ i\pi\ \frac{\kappa M}{(2\pi)^3}\ (\kappa|V_1|\kappa)\ (\kappa|R_1|\kappa)\ , \tag{46}$$

since

$$\frac{1}{E - E'' + i\varepsilon} = \frac{1}{E - E''} + i\pi\ \delta(E - E'').$$

Now, by definition,

$$(\kappa'|R_1|\kappa) = \frac{(\kappa'|T_1|\kappa)}{1 - i\pi\ \frac{\kappa M}{(2\pi)^3}\ (\kappa|T_1|\kappa)}\ . \tag{47}$$

Thus

$$(\kappa|T_1|\kappa) = (\kappa|V_1|\kappa) - i\pi \frac{\kappa M}{(2\pi)^3} (\kappa|V_1|\kappa)\ (\kappa|T_1|\kappa)$$

$$+ \int \frac{\kappa''^2\, d\kappa''}{(2\pi)^3} (\kappa|V_1|\kappa'') \frac{1}{E - E''} (\kappa''|T_1|\kappa)$$

$$+ i\pi \frac{\kappa M}{(2\pi)^3} (\kappa|V_1|\kappa)\ (\kappa|T_1|\kappa)$$

$$= (\kappa|V_1|\kappa) + \int \frac{\kappa''^2\, d\kappa''}{(2\pi)^3} (\kappa|V_1|\kappa'') \frac{1}{E - E''} (\kappa''|T_1|\kappa)\ . \qquad (48)$$

Thus, putting back in the projection operators, and writing the equation in operator form,

$$T = V + V \frac{1}{E - H_o} T. \qquad (49)$$

This is just the same equation as R satisfies, except that the principal value is taken at the pole.

In a problem with two potentials, $V = V_1 + V_2$, where one of them, say V_2, can be treated in lowest order perturbation theory, we have

$$R = (V_1 + V_2) + (V_1 + V_2) \frac{1}{E - H_o + i\varepsilon} R \qquad (50)$$

which, to terms linear in V_2, is

$$R = R_1 + V_2 + R_1 \frac{1}{E - H_o + i\varepsilon}\ V_2 + V_2\ \frac{1}{E - H_o + i\varepsilon}\ R_1$$

$$+ R_1\ \frac{1}{E - H_o + i\varepsilon}\ V_2\ \frac{1}{E - H_o + i\varepsilon}\ R_1 \qquad (51)$$

where

$$R_1 = V_1 + V_1\ \frac{1}{E - H_o + i\varepsilon}\ R_1\ . \qquad (52).$$

A similar equation holds for T.

Appendix B.

THE CHEW MESON THEORY.[4]

Low energy experimental work on the interaction of π mesons and nucleons has established fairly conclusively the dominance of the P-wave interaction. It is also a well known experimental fact that π mesons are pseudoscalar particles. A simple guess as to the form of the interaction at low energies is therefore

$$H_{NM} = \int \bar{\Psi} \sqrt{4\pi} \frac{f_o}{\mu} \underline{\sigma} \cdot \underline{\nabla} \quad \underline{\tau} \cdot \underline{\phi} \quad \Psi \, d^3\underline{x}, \tag{1}$$

where Ψ and ϕ are the nucleon and meson wave functions, respectively. This coupling is also part of the low energy limit of both the pseudoscalar direct and the pseudoscalar gradient relativistic theories, and as such may be thought to have some theoretical justification. The point of view adopted is not this, however. No attempt is made to discuss a relativistic theory; it is merely assumed that whatever the complete theory is, it has approximately this form at low energies.

Since this theory is to apply only to low energy phenomena, the nucleon will be treated as a static source, $\bar{\Psi}(x)\,\Psi(x) = \rho(\underline{x})$ being the density function of the nucleon. The coupling thus may be written

$$H_{NM} = \int i \sqrt{4\pi} \frac{f_o}{\mu} \frac{\underline{\sigma} \cdot \underline{K}}{\sqrt{2\omega_K}} \left\{ \underline{\tau} \cdot \underline{a}_{\underline{K}} \int e^{i\underline{K} \cdot \underline{x}} \rho(\underline{x}) \, d^3\underline{x} \right.$$

$$\left. - \underline{\tau} \cdot \underline{a}^{+}_{\underline{K}} \int e^{-i\underline{K} \cdot \underline{x}} \rho(\underline{x}) \, d^3\underline{x} \right\} \frac{d^3\underline{K}}{(2\pi)^3} \tag{2}$$

where the meson field has been expanded into creation and destruction

operators $\underline{a}^{+}_{\underline{K}}$ and $\underline{a}_{\underline{K}}$. Writing

$$v(\underline{K}) = \int e^{i\underline{K} \cdot \underline{x}} \rho(\underline{x})\, d^3 \underline{x} \tag{3}$$

we have

$$H_{NM} = \int \left\{ i\sqrt{4\pi}\, \frac{f_o}{\mu}\, \frac{\underline{\sigma} \cdot \underline{K}}{\sqrt{2\omega_K}}\, \underline{\tau} \cdot \underline{a}_K v(\underline{K}) - i\sqrt{4\pi}\, \frac{f_o}{\mu}\, \frac{\underline{\sigma} \cdot \underline{K}}{\sqrt{2\omega_K}}\, \underline{\tau} \cdot \underline{a}^{+}_K v(\underline{K}) \right\} \frac{d^3 \underline{K}}{(2\pi)^3} \tag{4}$$

If the nucleon were taken as a point source, $v(\underline{K}) = 1$, and the interaction would diverge at high meson momenta. The assumption is therefore made that for some unknown reasons (e.g., the appearance of heavy mesons, and recoil effects) this coupling is effectively cut off at high energies. In other words, the function $v(\underline{K})$ is assumed to produce a cutoff in momentum. This is generally taken as a square cutoff, so that

$$v(\underline{K}) = \begin{cases} 1 & K < K_{max} \\ 0 & K > K_{max} \end{cases}$$

All integrals over virtual mesons are then cut off at K_{max}, which is assumed to be somewhere around the nucleon mass, but other than that is treated as a new free parameter to be adjusted.

One further important modification is made. That is that the theory is renormalized, and the coupling constant f_o replaced by a renormalized constant f. It should be emphasized that this replacement is not necessary to make the theory finite as it is in the relativistic ones. The theory here is already finite. It is merely a computational method, which sums part of the perturbation series explicitly first, and treats the rest in the usual weak coupling way.

A short sketch of the renormalization procedure is the following. Consider, for example, meson nucleon scattering. Any term in the perturbation expansion will have the form

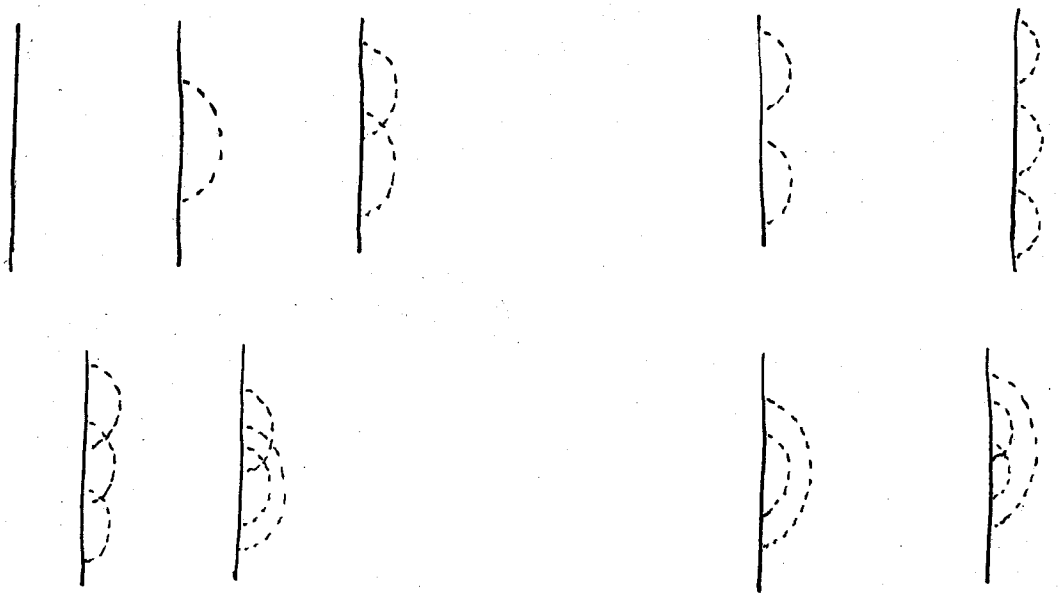
$$f_o^n{}_{ijk} \cdots V \frac{1}{E_o - E_i} V \frac{1}{E_o - E_j} V \frac{1}{E_o - E_k} V \cdots \qquad (5)$$

where $f_o V$ denotes the interaction (in this case H_{NM}), and E_i, etc. are the energies associated with the i^{th} intermediate state. E_o is the initial energy. The energy denominator $(E_o - E_i)^{-1}$ can be interpreted as a nucleon propagation function $S(E_o - E_i)$.

In the above form, the coupling constant is the "theoretical one"; that is, the constant appearing in the original equation. The energies are measured relative to zero energy for a free "theoretical" nucleon; that is, one not interacting with anything. The renormalization consists in re-expressing the calculation in terms of physical coupling constants and energies, i.e., those associated with a physical, or interacting, nucleon.

The free propagation of a physical nucleon is characterized by the diagrams of Figure 1; the interaction of a physical nucleon by those of Figure 2. These, in both cases, may be split into two classes, primitive and otherwise, according as they can be divided into sums of simpler diagrams or not. Thus, the diagrams of Figures 1a and 2a are primitive, the others are not.

Let $S'(E)$, $\Gamma(E_1, E_2)$ represent the propagation and interaction of a physical nucleon. These then include all the diagrams of Figures 1 and 2 respectively.



a b

Fig. 1

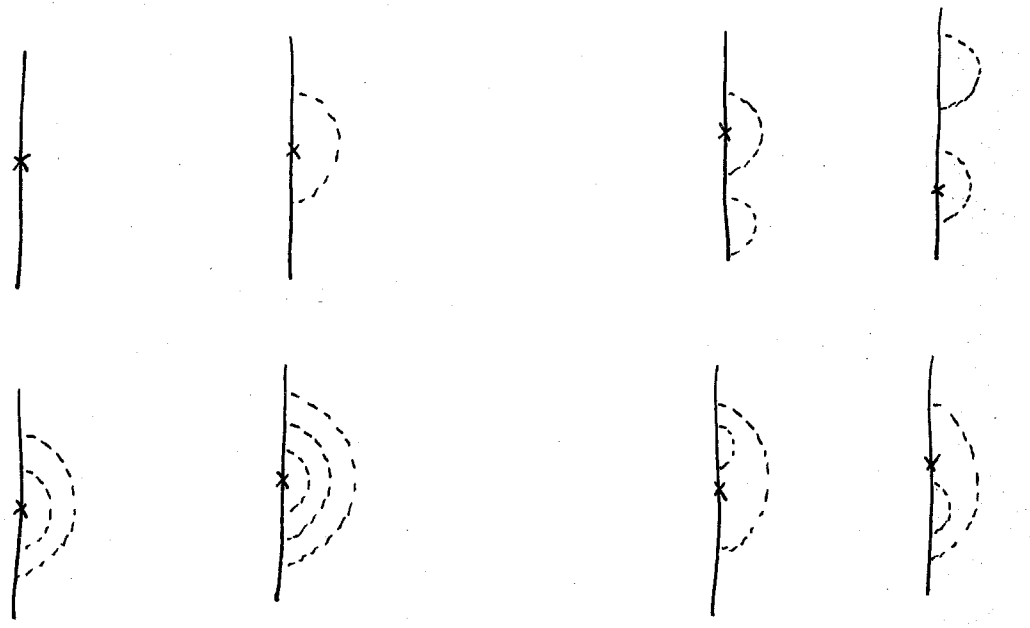


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Fig. 2

Now suppose $I(f_o, S(E), V)$ represents the exact matrix element for any process. Then this can also be constructed as follows. Let $I_o(f_o, S, V)$ be the sum of all irreducible (that is, with no self energy or vertex modifications) diagrams. Then the exact matrix element is obtained by everywhere replacing S by S', V by Γ in I_o. Thus

$$I(f_o, S(E), V) = I_o(f_o, S'(E), \Gamma).$$

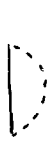
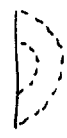
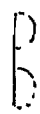
The renormalization procedure then consists of showing that it is possible to replace f_o, S', E and Γ by new quantities f, S_1', E', Γ_1, calculated in a different way, without altering I_o. In the relativistic case, S' and Γ are infinite, while S_1' and Γ_1 are finite. Here this replacement is simply a matter of convenience, since f turns out to be smaller than f_o. We thus want to define new functions so that

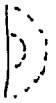
$$I_o(f_o, S'(E), \Gamma) = I_o(f, S_1'(E'), \Gamma_1).$$

Consider first the replacement of E by E', which corresponds to mass renormalization. If we define $\sum(E)$ to be the sum of all proper irreducible (that is, containing no self energy diagrams, and not being a combination of simpler self-energy diagrams) self energy diagrams, then

$$S'(E) = S(E) + S(E) \sum (E)\, S'(E).$$

Here $\sum (E)$ is to be computed using S' instead of S, so this is an integral equation for S'. $\sum$ includes the diagrams

 and  , but not  or  ; the diagram  is included in S' by iterating the equation once,

and  is included since Σ is calculated with S' instead of S.

Since $S(E) = 1/E$, we have

$$S'(E) = \frac{1}{E - \Sigma(E)} \ .$$

Now consider the function defined by

$$S''(E') = \frac{1}{E' - \Sigma''(E') - \Sigma''(0)}$$

where Σ'' is computed in the same way as Σ, only using S'' instead of S'. If we choose the zero of energy of E' such that $E' - \Sigma''(0) = E$, we have

$$S''(E') = \frac{1}{E - \Sigma''(E')} = S(E) + S(E)\ \Sigma''(E')\ S''(E').$$

Hence, since $\Sigma''(E')$ is the same function of $S''(E')$ that $\Sigma(E)$ is of $S'(E)$, we have that $S''(E')$ and $S'(E)$ satisfy the same integral equation, and hence

$$S''(E') = S'(E).$$

Thus we have

$$S'(E') = \frac{1}{E' - \Sigma(E') - \Sigma(0)} = S(E') + S(E')(\Sigma(E') - \Sigma(0))\ S'(E')$$

$$= S(E') + S(E')\ \Sigma_1(E')\ S'(E')$$

where we write $S'(E')$ for $S'(E)$, and $\Sigma(E') - \Sigma(0) = \Sigma_1(E')$. This completes the mass renormalization. We clearly have

$$I_o(f_o, S'(E), \Gamma) = I_o(f_o, S'(E'), \Gamma).$$

The charge renormalization procedure consists of the following.

Suppose we define

$$\begin{aligned} S'_r(E') &= Z_2^{-1}\ S'(E') \\ \Gamma_r(E'_1 E'_2) &= Z_1\ \Gamma(E'_1 E'_2) \\ f_r &= Z_2\ Z_1^{-1}\ f\ . \end{aligned} \tag{11}$$

Again E' denotes the energy measure relative to zero for a physical nucleon, whereas E denotes the energy relative to zero for a theoretical nucleon. Similarly f_r can be thought of as the physical coupling, and f_o is the theoretical one.

Consider any vertex in I_c. This has associated two S''s, one Γ and one f_o. Also each S' is shared between two vertices. Therefore, if the replacement is made as indicated in equations (11), we have

$$f_o S' \Gamma S' = Z_2^{-1} Z_1 f_r Z_2^{1/2} S'_r Z_1^{-1} \Gamma_r Z_2^{1/2} S'_r = f_r S'_r \Gamma_r S'_r ,$$

and hence

$$I(f_o S'(E') \Gamma) = I(f_r S'_r(E'), \Gamma_r) .$$

The final thing which must be done is to choose Z_1 and Z_2 and determine the equations satisfied by S'_r and Γ_r. We had

$$S'(E') = S(E') + S(E') \Sigma_1(E) S'(E). \tag{12}$$

Making the substitutions (11), we have

$$Z_2 S'_r(E') = S(E') + S(E') \Sigma_{1r}(E') S'_r(E') \tag{13}$$

since $\Sigma_{1r}(E')$, (which is Σ_1 with $f_o S'\Gamma$ replaced by $f S'_r \Gamma_r$) is related to $\Sigma_1(E')$ by

$$\Sigma_{1r}(E') = Z_2 \Sigma_1(E') .$$

(This is easily seen since there is one unbalanced S' in Σ_1.) Now expanding $\Sigma_{1r}(E')$ in powers of E', we obtain, remembering that Σ_1 starts linearly in E',

$$Z_2\, S'_r(E') = S(E') + S(E') \left(E' \left.\frac{\partial \Sigma}{\partial E'}\right|_{E'=0} + \Sigma_{2r}(E') \right) S'_r(E'). \tag{14}$$

In $S(E') = 1/E'$, so equation (14) becomes

$$Z_2\, S'_r(E') = S(E') + \left.\frac{\partial \Sigma}{\partial E'}\right|_{E'=0} S'_r(E') + S(E')\, \Sigma_{2r}(E')\, S'_r(E') . \tag{15}$$

Therefore, choosing $Z_2 = 1 + \left.\frac{\partial \Sigma}{\partial E'}\right|_{E'=0}$ results in

$$\begin{aligned} S'_r(E') &= S(E') + S(E')\, \Sigma_{2r}(E')\, S'_r(E') \\ &= \frac{1}{E' - \Sigma_{2r}(E')} \end{aligned} \tag{16}$$

where

$$\Sigma_{2r}(E') = \Sigma(E') - \Sigma(0) - E' \left.\frac{\partial \Sigma}{\partial E'}\right|_{E'=0} \tag{17}$$

where everything is calculated with f_r instead of f_o. In the same way, choosing $Z_1 = 1 - \Lambda(0,0)$ gives

$$\Gamma_r(E'_1\, E'_2) = V(E'_1\, E'_2)\, (1 + \Lambda_{2r}(E'_1\, E'_2)) \tag{18}$$

where

$$\Lambda_{2r}(E'_1\, E'_2) = \Lambda(E'_1\, E'_2) - \Lambda(0,0), \tag{19}$$

and again everything is computed using f_r instead of f_o. This last corresponds to charge renormalization in electrodynamics.

As an example of this, we may compute the f_r^2 corrections to the propagation function. To order f_r^2,

$$\Sigma(E') = 3\sqrt{4\pi}\,\frac{f_r}{\mu}\int \frac{\underline{\sigma}\cdot\underline{k}}{\sqrt{2\omega}}\,\frac{1}{E'-\omega}\,\frac{\underline{\sigma}\cdot\underline{k}}{\sqrt{2\omega}}\,\sqrt{4\pi}\,\frac{f_r}{\mu}\,\frac{d^3\underline{k}}{(2\pi)^3}$$

$$= 3\,\frac{4\pi}{8\pi^3}\left(\frac{f_r}{\mu}\right)^2\frac{1}{2}\int \frac{\underline{\sigma}\cdot\underline{k}\;\underline{\sigma}\cdot\underline{k}}{(E'-\omega)}\,d^3\underline{k}$$

$$= \frac{3}{2}\,\frac{16\pi^2}{8\pi^3}\left(\frac{f_r}{\mu}\right)^2\int_{\mu}^{\omega_{max}}\frac{k^3\,d\omega}{E'-\omega}$$

$$= \frac{3}{\pi}\left(\frac{f_r}{\mu}\right)^2\int_{\mu}^{\omega_{max}}\frac{k^3\,d\omega}{E'-\omega} \tag{20}$$

$$\Sigma(0) = -\frac{3}{\pi}\left(\frac{f_r}{\mu}\right)^2\int\frac{k^3 d\omega}{\omega} \tag{21}$$

$$E'\left.\frac{\partial\Sigma}{\partial E'}\right|_{E'=0} = -E'\,\frac{3}{\pi}\left(\frac{f_r}{\mu}\right)^2\int k^3\,d\omega\;\frac{1}{\omega^2} \tag{22}$$

Hence

$$\Sigma_{2r}(E') = \frac{3}{\pi}\left(\frac{f_r}{\mu}\right)^2\int k^3\,d\omega\left(\frac{1}{E'-\omega}+\frac{1}{\omega}+\frac{E'}{\omega^2}\right)$$

$$= \frac{3}{\pi}\left(\frac{f_r}{\mu}\right)^2\int k^3\,d\omega\;\frac{E'^2}{\omega^2(E'-\omega)} \tag{23}$$

and we have, finally,

$$S'_r(E') = \frac{1}{E'\left(1-\frac{3}{\pi}\left(\frac{f_r}{\mu}\right)^2 E'\int\frac{k^3\,d\omega}{\omega^2(E'-\omega)}\right)} \tag{24}$$

The lowest order corrections to the vertex operator are very small, and can generally be neglected.

To give an example of the use of the Chew theory, it will now be applied to set up the equations for meson nucleon scattering.

The method of calculation is the following. First, ignore all diagrams which contribute only to renormalization effects. These are already included by using f_r instead of f_o. Second, use lowest order

perturbation theory on the remaining diagrams, except for those which have energy denominators which can resonate. It is then easy to see that only the diagrams of Figure 3 will be included.

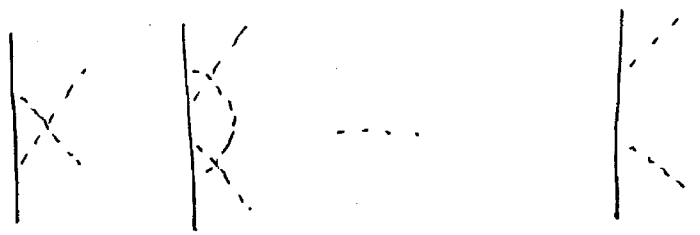


Fig. 3

We should everywhere use S_r' and Γ_r as the propagators and vertex parts; in practice these differ from S and V so little for most energies that we shall use S_r' only in Figure 2b, and never use Γ_r.

The reaction matrix for the diagram of Figure 3b is readily written down.

$$(\underline{\kappa}'\underline{a}'|T(3b)|\underline{\kappa}\ \underline{a}) = 4\pi \left(\frac{f_r}{\mu}\right)^2 \frac{\underline{\sigma}\cdot\underline{\kappa}'\ \underline{\Phi}\cdot\underline{a}'^{+}}{\sqrt{2\omega'}}\ S_r'(\omega_o)\ \frac{\underline{\sigma}\cdot\underline{\kappa}_o\ \underline{\tau}\cdot\underline{a}}{\sqrt{2\omega_o}} \tag{25}$$

An integral equation is easily set up for the reaction matrix of Figure 3a, which is identical with the second order Tamm-Dancoff or Brueckner-Watson approximations.

$$(\underline{\kappa}'\underline{a}'|T(3a)|\underline{\kappa}_o\ \underline{a}_o) = (\underline{\kappa}'\underline{a}'|V|\underline{\kappa}_o\ \underline{a}_o) + \sum_{\underline{a}''} \int \frac{d^3\underline{\kappa}''}{(2\pi)^3} (\underline{\kappa}'\ \underline{a}'|V|\underline{\kappa}''\ \underline{a}'')\ \frac{1}{\omega_o - \omega''}\ (\underline{\kappa}''\ \underline{a}''|T(3a)|\underline{\kappa}_o\ \underline{a}_o), \tag{26}$$

where

$$(\underline{\kappa}'\ \underline{a}'|V|\underline{\kappa}\ \underline{a}) = 4\pi\left(\frac{f_r}{\mu}\right)^2 \frac{\underline{\sigma}\cdot\underline{\kappa}\ \underline{\tau}\cdot\underline{a}}{\sqrt{2\omega}}\ \frac{1}{\omega_o-\omega'-\omega}\ \frac{\underline{\sigma}\cdot\underline{\kappa}'\ \underline{\tau}\cdot\underline{a}'^{+}}{\sqrt{2\omega'}} \tag{27}$$

and this represents just the first diagram of Figure 3a. This may be split into angular momentum and isotopic spin states. We have

$$\begin{aligned}(\underline{\kappa}'\ \underline{a}'|V|\underline{\kappa}\ \underline{a}) = \frac{(4\pi)^2}{3}\left(\frac{f_r}{\mu}\right)^2 \frac{1}{\sqrt{2\omega'}\ \sqrt{2\omega}}\ \frac{1}{\omega_o-\omega'-\omega}\\ - (\kappa'|P_{1/2}|\kappa)\ (\underline{a}'|Q_{1/2}|\underline{a})\\ - 2[(\underline{\kappa}'|P_{3/2}|\underline{\kappa})(\underline{a}'|Q_{1/2}|\underline{a}) + (\underline{\kappa}'|P_{1/2}|\underline{\kappa})(\underline{a}'|Q_{3/2}|\underline{a})]\\ + 4(\underline{\kappa}'|P_{3/2}|\underline{\kappa})(\underline{a}'|Q_{3/2}|\underline{a})\\ = \sum_{JT}(\kappa'|V_{JT}|\kappa)\ (\underline{\kappa}'|P_J|\underline{\kappa})\ (\underline{a}'|Q_T|\underline{a})\end{aligned} \tag{28}$$

where

$$\begin{aligned}(\underline{\kappa}'|P_{1/2}|\underline{\kappa}) &= \frac{1}{4\pi\kappa'\kappa}\ \underline{\sigma}\cdot\underline{\kappa}'\ \underline{\sigma}\cdot\underline{\kappa}\\ (\underline{\kappa}'|P_{3/2}|\underline{\kappa}) &= \frac{1}{4\pi\kappa'\kappa}\ (3\underline{\kappa}'\cdot\underline{\kappa} - \underline{\sigma}\cdot\underline{\kappa}'\ \underline{\sigma}\cdot\underline{\kappa})\\ (\underline{a}'|Q_{1/2}|\underline{a}) &= \frac{1}{3}\ \underline{\tau}\cdot\underline{a}'^{+}\ \underline{\tau}\cdot\underline{a}\\ (\underline{a}'|Q_{3/2}|\underline{a}) &= \frac{1}{3}\ (3\underline{a}'^{+}\cdot\underline{a} - \underline{\tau}\cdot\underline{a}'^{+}\ \underline{\tau}\cdot\underline{a})\end{aligned} \tag{29}$$

are angular momentum and isotopic spin projection operators for the states $J = 1/2,\ 3/2$ and $T = 1/2, 3/2$. The integral equation for the reaction matrix can then be broken up as

$$(\underline{\kappa}'\ \underline{a}'|T(3a)|\underline{\kappa}\ \underline{a}) = \sum_{JT} (\kappa'|T_{JT}|\kappa)\ (\underline{\kappa}'|P_J|\underline{\kappa})\ (\underline{a}'|Q_T|\ \underline{a}) \qquad (30)$$

and

$$(\kappa'|T_{JT}|\kappa_o) = (\kappa'|V_{JT}|\kappa_o) + \int \frac{\kappa''^2\, d\kappa''}{(2\pi)^3} (\kappa'|V_{JT}|\kappa'')\ \frac{1}{\omega_o - \omega''}\ (\kappa''|T_{JT}|\kappa_o) \qquad (31)$$

Solving this integral equation then gives the phase shifts in the various states through the relation

$$-\pi \tan \delta_{JT}(\kappa) = \frac{\kappa\omega}{(2\pi)^3}\ (\kappa|T_{JT}|\kappa). \qquad (32)$$

To summarize, the Chew theory is a phenomenological description of the principal experimental facts concerning low energy meson-nucleon processes. It is a very incomplete description in that no attempt is made to discuss S-wave interactions, nucleon pairs, heavy mesons or recoil effects. It is merely an attempt to see if all existing low energy observations are consistent with nothing more than strong P-wave couplings, pseudoscalar mesons and conservation of angular momentum and isotopic spin.

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