Regularity of the Anosov splitting

and

A new description of the Margulis measure

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Titel und Orden halten manchen Puff ab im Gedränge

Goethe [Go]

Acknowledgment

"...the best thing you can do is to emigrate and improve your mind." [Wd]

At the completion of a dissertation, the end and high point of a student career, thoughts wander back through the years of formal education.

Twenty years ago my family moved to Ethiopia for six years. I am grateful for this and all the other experiences, for the environment they have nurtured me in. I am indebted to my parents, the theologians, for encouraging my curiosity in science. My achievements are also theirs.

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Five years ago I came to the California Institute of Technology.
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This work is the beginning of a life as a mathematician. I look forward to sharing it with Kathy.

Abstract

The Anosov splitting into stable and unstable manifolds of hyperbolic dynamical systems has been known to be Hölder continuous always and differentiable under bunching or dimensionality conditions. It has been known, by virtue of a single example, that it is not always differentiable. High smoothness implies some rigidity in several settings.

In this work we show that the right bunching conditions can guarantee regularity of the Anosov splitting up to being differentiable with derivative of Hölder exponent arbitrarily close to one. On the other hand we show that the bunching condition used is optimal. Instead of providing isolated examples we prove genericity of the low-regularity situation in the absence of bunching. This is the first time a local construction of low-regularity examples is provided.

Based on this technique we indicate how horospheric foliations of nonconstantly curved symmetric spaces can be made to be nondifferentiable by a smoothly small perturbation.

In the last chapter the Hamenstädt-description of the Margulis measure is rendered for Anosov flows and with a simplified argument. The Margulis measure arises as a Hausdorff measure for a natural distance on (un)stable leaves that is adapted to the dynamics.

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0: Summary

One aim of this thesis is to understand the regularity of the stable and unstable distributions for Anosov systems. After a comprehensive introduction it begins with a chapter containing a theorem that guarantees a certain amount of regularity based on a bunching assumption on the rates of contraction and expansion of the Anosov system. The regularity obtained is between Hölder continuous and differentiable with almost Lipschitz derivative, depending on the amount of bunching.

In the following chapters it is shown that at least in the category of symplectic Anosov systems this regularity theorem is optimal. It is shown that generically symplectic Anosov systems fail to have Anosov splitting exceeding the regularity asserted by the regularity theorem. The proof extends over three chapters in which appropriate coordinates are obtained, a necessary condition for excessive regularity is derived, and perturbations causing this condition to be violated are constructed.

Finally, applications to geodesic flows in negative curvature are outlined.

Another independent aim of this thesis is to give a new description of the Bowen-Margulis measure of maximal entropy for Anosov flows. This is done in the last chapter.

The last section of the introduction contains a summary of results.

Anosov flows and diffeomorphisms

The dynamical systems we are going to consider are Anosov systems and geodesic flows on manifolds of negative curvature . We will usually assume that they are C^∞, but this is rarely needed.

Definition 1.1: Let M be a compact Riemannian manifold and $\varphi^t: M \to M$ a C^∞flow (*i.e.*, a one-parameter group of diffeomorphisms). φ^t is called an **Anosov flow** if $\dot\varphi \neq 0$ and the tangent bundle TM of M decomposes into a direct sum $TM = E^{su} \oplus E^{ss} \oplus E^{\varphi}$, where $E^{\varphi} = <\dot\varphi>$ is generated by the flow direction $\dot\varphi$ and

$$\left(\exists\, a, b, C \in \mathbb{R}^+\right)\left(\forall\, p \in M, t > 0, u \in E^{su}(p), v \in E^{ss}(p)\right)$$

$$\frac{1}{C} e^{-at} \|v\| \leq \|D\varphi^t(v)\| \leq C e^{-bt} \|v\|$$

$$\frac{1}{C} e^{-at} \|u\| \leq \|D\varphi^{-t}(u)\| \leq C e^{-bt} \|u\|$$

Remark 1.2: The classical reference is the book [A1] by **Anosov.**

In the Russian literature the names C-, U- or Y-system are also used (because the Russian word for "condition" begins with the cyrillic letter "Y", which is transliterated as "U" and sometimes replaced by "C" for "condition").

It actually turns out that only estimates from above are needed.

These estimates shall be called the Anosov estimates.

Theorem/Definition 1.3: i) The subbundles E^{su} and E^{ss} are tangent to foliations $W^{su}$and W^{ss} respectively, whose leaves are called the strong unstable and strong stable manifolds.

ii) The leaves of the foliations W^u and W^s obtained similarly from

$E^u := E^{su} \oplus E^{\varphi}$ and $E^s := E^{ss} \oplus E^{\varphi}$ are referred to as the (weak) unstable and (weak) stable manifolds respectively. The pair of subbundles (E^u, E^s) is referred to as the Anosov splitting.

iii) The leaves of the foliations W^{su}, W^{ss}, W^u and W^s are C^∞ injectively immersed copies of Euclidean space.

iv) The foliations W^{su}, W^{ss}, W^u and W^s are always Hölder continuous in the C^1-topology, in the sense that the distributions E^{su}, E^{ss}, $E^u := E^{su} \oplus E^{\varphi}$ and $E^s := E^{ss} \oplus E^{\varphi}$ are Hölder continuous.

v) The foliations W^{su} and W^u as well as W^{ss} and W^s form transversal pairs. In other words, at every intersection point of leaves of W^{su} and W^u (W^{ss} and W^s) the leaves intersect transversally. (This is by definition.)

vi) If (E^u, E^s) is the Anosov splitting for φ^t then (E^s, E^u) is the Anosov splitting for $\psi^t := \varphi^{-t}$. (Thus often it suffices to consider E^u in proofs.)

In order to make sense of the regularity claims here we need the

Definition 1.4: A distribution (*i.e.*, a subbundle of the tangent bundle of the manifold) on a differentiable manifold is said to be Hölder continuous with Hölder exponent β if in local coordinates it can be spanned by vector fields which have coefficients that are Hölder with exponent β.

Analogously regularity of higher order (Lipschitz, C^1, C^1 with Hölder derivative, C^k etc.) is defined via representations in local coordinates.

Alternative viewpoints for defining regularity are the following:

Definition 1.5: A foliation on a compact smooth manifold M is a decomposition $M = \bigcup_{\iota \in I} M_\iota$ (disjoint) such that $M_\iota \subset M$ are smooth submanifolds and for all $p \in M$ there exists an open neighborhood U of p, $k \in \mathbb{N}$ and a homeomorphism $h: U \to V \subset \mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$ such that h sends connected components of $U \cap M_\iota$ to sets of the form $V \cap (\mathbb{R}^k \times \{x\})$ for some $x \in \mathbb{R}^{n-k}$.

The foliation is said to be β-Hölder if h can be chosen to be β-Hölder. Similarly for Lipschitz, C^1, C^1 with Hölder derivative, C^k etc.

Or one can take as a measure of regularity of the stable and unstable foliations the regularity of the holonomy maps as defined in the last chapter.

Definition 1.6: Let M be a compact Riemannian manifold and $\phi: M \to M$ a diffeomorphism. ϕ is called an **Anosov diffeomorphism** if the tangent bundle TM of M decomposes into a direct sum $TM = E^u \oplus E^s$, such that

$$\left(\exists\, a, b, C \in \mathbb{R}^+\right)\left(\forall\, p \in M, n \in \mathbb{N}, u \in E^u(p), v \in E^s(p)\right)$$

$$\frac{1}{C} e^{-an} \|v\| \le \|D\phi^n(v)\| \le C e^{-bn} \|v\|$$

$$\frac{1}{C} e^{-an} \|u\| \le \|D\phi^{-n}(u)\| \le C e^{-bn} \|u\|$$

As in the case of flows, one obtains foliations W^u and W^s. In analogous situations their regularity and the smoothness of their leaves always correspond to that of the foliations W^u and W^s for flows. In other words a comprehensive discussion of flows and diffeomorphisms should subsume W^u and W^s for flows and diffeomorphisms under one label and treat the foliations W^{su} and W^{ss} for flows as peculiar to flows. Specifically, time changes for flows, (which have no counterpart for diffeomorphisms) can affect the regularity of W^{su} and W^{ss} while leaving the regularity of the weak foliations W^u and W^s unchanged [P].

Definition 1.7: Let $\varphi^t: M \to M$ be a C^∞ flow. A flow ψ^t is said to have been obtained from φ^t by a time change if there exists $f \in C^\infty(M, \mathbb{R}^+)$ such that $\dot\psi = f\dot\varphi$.

Plante [P] has shown, in essence, that Anosov flows in dimension three generically do not have strong stable and strong unstable foliations of class C^1. He explicitly indicates how a time change can lead to breakdown of regularity of the strong stable and unstable foliations.

We, in contrast, will prove here regularity results for the weak stable and unstable foliations. Therefore we want to point out that the time changes here do not play a role. On one hand, time changes leave the weak stable and unstable foliations invariant. On the other hand, the data controlling regularity appear in the form of a bunching condition, which is by definition invariant under time changes:

Definition: Let M be a compact Riemannian manifold and $\varphi^t: M \to M$ an Anosov flow. For $\alpha \in (0,2]$ φ^t is called α-bunched if

$$\Big(\exists\ \mu_1, \mu_2, \nu_2, \nu_1: M \times \mathbb{R}^+ \to \mathbb{R}\Big)\ \Big(\forall\ p \in M\Big)\ \Big(\forall\ v \in E^{ss}(p)\ ,\ u \in E^{su}(p)\Big)\ \Big(\forall\ t > 0\Big)$$

$$\frac{1}{C}\mu_1(p,t)\,\|v\| \le \|D\varphi^t(v)\| \le C\mu_2(p,t)\,\|v\|$$

$$\frac{1}{C}\nu_1(p,t)\,\|u\| \le \|D\varphi^{-t}(u)\| \le C\nu_2(p,t)\,\|u\|$$

and

$$\mu_2 \cdot \nu_2 \le \Big(min(\mu_1, \nu_1)\Big)^\alpha.$$

Remark 1.8: In the case of symplectic flows, (which we define below) $\mu_i = \nu_i$ and α-bunching is equivalent to $\mu_2^2 \le \mu_1^\alpha \le \mu_2^\alpha$ or $\nu_2^2 \le \nu_1^\alpha \le \nu_2^\alpha$.

The bunching condition can thus be viewed as a condition restricting the spread of the spectrum of Lyapunov exponents.

One can take μ_i, ν_i so that there are $\bar{\mu}_i$, $\bar{\nu}_i\colon M \to \mathbb{R}$ for which $\mu_i(p,t) = (\bar{\mu}_i(p))^t$ and $\nu_i(p,t) = (\bar{\nu}_i(p))^{-t}$.

Thus in this work we will use the following bunching condition:

Definition 1.9: Let φ^t be an Anosov flow on a Riemannian manifold M. For $\alpha \in (0,2]$ we call $p \in M$ α-bunched if one can take $0 < \mu_1 < \mu_2 < 1 < \nu_2 < \nu_1 < \infty$, $C > 0$ such that

$$\mu_2 \nu_2^{-1} \le \left(min\ (\mu_1, \nu_1^{-1})\right)^\alpha$$

and

$$\forall\ v \in E^{ss}(p), u \in E^{su}(p), t > 0$$

$$\frac{1}{C} \mu_1^t \|v\| \le \|D\varphi^t(v)\| \le C \mu_2^t \|v\|$$

$$\frac{1}{C} \nu_1^{-t} \|u\| \le \|D\varphi^{-t}(u)\| \le C \nu_2^{-t} \|u\|.$$

Otherwise call p α-spread.

We call p u-α-bunched if $\mu_2 \nu_2^{-1} \le \mu_1^\alpha$

and s-α-bunched if $\mu_2 \nu_2^{-1} \le \nu_1^{-\alpha}$.

φ^t is called α-bunched if every $p \in M$ is α-bunched with uniform C.

φ^t is called strongly α-bunched if there exists an $\epsilon > 0$ such that all $p \in M$ are $(\alpha+\epsilon)$-bunched.

Define (strong) u-α-bunching and (strong) s-α-bunching for φ^t in the obvious way.

Example 1.10: Although $\mathbb{R}^4$ is not compact, it is instructive to consider the linear diffeomorphism of $\mathbb{R}^4$ given by

$$\phi: (x, y, z, w) \mapsto (2\cdot x, 4\cdot y, z/2, w/4).$$

Here $$W^u(x_o, y_o, z_o, w_o) = \{(x, y, z, w) | \, z = z_o, w = w_o\}$$

and $$W^s(x_o, y_o, z_o, w_o) = \{(x, y, z, w) | \, x = x_o, y = y_o\}.$$

The leaves of W^u are characterized by the fact that $d(\phi^n(p), \phi^n(q)) \to 0$ for $q \in W^u(p)$, $n < 0$, the leaves of W^s by $d(\phi^n(p), \phi^n(q)) \to 0$ for $q \in W^s(p)$, $t > 0$.

As the Anosov estimates would suggest, this convergence is actually exponential. Here it is furthermore possible to discern different exponential rates of convergence within such a leaf. In the example we have fast stable leaves

$$W^{fu}(x_o, y_o, z_o, w_o) = \{(x, y, z, w) | \, x = x_o, z = z_o, w = w_o\}$$

and $$W^{fs}(x_o, y_o, z_o, w_o) = \{(x, y, z, w) | \, x = x_o, y = y_o, z = z_o\}.$$

These leaves are characterized by the fact that $e^{-3\cdot n}\cdot d(\phi^n(p), \phi^n(q)) \to 0$ for $q \in W^{fu}(p)$, $n < 0$ and $e^{3\cdot n}\cdot d(\phi^n(p), \phi^n(q)) \to 0$ for $q \in W^s(p)$, $n > 0$.

Example 1.11: To give a similar example on a compact manifold, consider the following map of the 4-torus: Project the linear map of $\mathbb{R}^4$ given by

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 13 & 8 \\ 0 & 0 & 8 & 5 \end{bmatrix}$$

to the torus $\mathbb{T}^4 = \mathbb{R}^4/\mathbb{Z}^4$ under the natural projection. (The map projects since it has unit determinant and thus the inverse also has integer entries and consequently the integer lattice $\mathbb{Z}^4$ is mapped onto itself.) The map has four distinct positive eigenvalues and, being symmetric, thus an orthonormal basis of eigenvectors. As above, one obtains two-dimensional contracting and expanding subspaces and these have one-dimensional fast contracting and expanding subspaces spanned by the eigenvectors associated to the smallest and largest eigenvalue respectively.

To be more explicit we compute the eigenvalues and eigenvectors here. Observe that $A=\begin{bmatrix} M & 0 \\ 0 & M^3 \end{bmatrix}$ with $M=\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, so the eigenvalues are those of M and their cubes. But
$det\, M-\lambda I=\begin{vmatrix} 2-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix}=\lambda^2-3\lambda+1$ has zeros $\frac{3\pm\sqrt{5}}{2}$, *i.e.*, $\lambda_1=\lambda:=\frac{3+\sqrt{5}}{2}$, $\lambda_2=\lambda^{-1}$.
Thus we have eigenvalues λ, λ^3, λ^{-1}, λ^{-3}. The associated eigenvectors are obtained from solving $\begin{bmatrix} 2\text{-}\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix} v=0$ and are (without normalization)

$\lambda \leftrightarrow v_1=(1,\lambda-2,0,0)$, $\lambda^3 \leftrightarrow v_2=(0,0,1,\lambda-2)$

$\lambda^{-1} \leftrightarrow w_1=(1,\lambda^{-1}-2,0,0)$, $\lambda^{-3} \leftrightarrow w_2=(0,0,1,\lambda^{-1}-2)$.

The unstable direction is spanned by v_1 and v_2, the fast unstable direction by v_2; the stable direction is spanned by w_1 and w_2, the fast stable direction by w_2.

In general, at a periodic point an Anosov flow (or diffeomorphism) will have a substructure of the stable and unstable manifold. It is called the stable and unstable filtration and consists of nested submanifolds in the stable manifold and nested submanifolds in the unstable manifold. The stable filtration will consist of a family of submanifolds W_i, $i=1,\dots,k$ of the stable manifold with the property that W_{i+1} is a submanifold of W_i and W_1 is the stable manifold of the periodic point p. Furthermore there are numbers λ_i such that $W_i=\{x\in M: \lambda_i^t d(\varphi^t x, \varphi^t p)\to 0\}$. In other words these manifolds correspond to "increasingly faster stable manifolds." The proof of this fact works along the same lines as the proof of existence of stable manifolds by Hadamard and thus these fast stable manifolds are also smooth and vary continously under perturbations of the Anosov system. Note that their tangent spaces at p correspond to sums of root spaces for the return map associated to those eigenvalues inside disks in $\mathbb{C}$ of different size. That is, if we take a closed disk in $\mathbb{C}$ of radius $r \leq 1$ and consider the eigenvalues of the return map contained in this disk, then the sum of the root spaces for these eigenvalues is the tangent space to one of the submanifolds mentioned above, in the case of $r = 1$ for example, the entire stable direction. In the future we will usually refer to the lowest-dimensional element W_k of the filtration as "the" fast stable leaf of p.

In a slightly different guise this issue arises for a return map directly, *i.e.*, we consider a periodic orbit p for an Anosov flow and take a hypersurface transversal to the flow through p. Thus, taking the return map to this transversal (locally) we are looking at a fixed point p of an Anosov diffeomorphism ϕ. Then here, too we have the stable and unstable filtration. As above, the stable filtration will consist of a family of submanifolds W_i, $i = 1, \ldots, k$ of the stable manifold with the property that W_{i+1} is a submanifold of W_i and W_1 is the stable manifold of the periodic point p. Furthermore there are numbers λ_i such that $W_i = \{x \in M: \lambda_i^n d(\phi^n x, \phi^n p) \to 0\}$.

Again, when $D\phi|_p$ has an eigenvalue μ such that $|\mu|$ is minimal, then we have a fast stable leaf with contraction at a rate $|\mu|$, which is a proper submanifold of the unstable leaf of p.

Geodesic Flows

To define geodesic flows consider a Riemannian manifold N with tangent bundle TN. For $t \in \mathbb{R}$ and a vector $v \in TN$ following the geodesic γ_v defined by $\dot{\gamma}_v(0)=v$ for time t gives a new vector $\varphi^t(v) := \dot{\gamma}_v(t)$.

Definition 1.12: Let N be a Riemannian manifold with tangent bundle TN. Define the geodesic flow $\varphi^t: TN \to TN$ by $\varphi^t(v) := \dot{\gamma}_v(t)$, where γ_v is the geodesic with $\dot{\gamma}_v(0)=v \in TN$.

Geodesic flows have a special structure. They can be described as Hamiltonian flows on TN. Let M be a $2n$-dimensional differentiable manifold. A symplectic form on M is a nondegenerate differential 2-form ω on M. ($\omega^n = \omega \wedge \omega \wedge \cdots \wedge \omega$ then defines a volume form.) In the setting of Riemannian geometry the cotangent bundle T^*N of N has a natural symplectic structure given in coordinates (u,v) on T^*N by $\sum_{i,j} du^i \wedge dv^j$. Via the isomorphism between TN and T^*N this gives a symplectic structure on TN. The geodesic flow then arises from the energy functional $H: TN \to \mathbb{R}$, $X \mapsto \frac{1}{2} g(X,X)$ as the flow generated by the Hamiltonian vector field ξ_H defined by $\omega(\xi_H, \cdot) = DH$. Being a Hamiltonian flow, the geodesic flow leaves H and ω invariant. Since H measures speed, geodesics have constant speed, *i.e.*, $\|v\| = \|\varphi^t(v)\|$. Thus one can restrict the geodesic flow to the unit tangent bundle $SN := \{v \in TN \mid \|v\| = 1\}$. This restriction corresponds simply to restricting a Hamiltonian flow to an energy surface. But while in general, Hamiltonian flows will have rather distinct restrictions to different energy surfaces, the restrictions of a geodesic flow to different sphere bundles $S_aN := \{v \in TN \mid \|v\| = a\}$ are isomorphic in that they differ only by a reparametrization [K]. (Thus no information is lost in this process.) Symplectic structures exist only in even-dimensional manifolds, but $\varphi^t|_{SN}$ admits a description as a "transversally" symplectic flow, *i.e.*, a flow preserving symplectic structures on transversals. In fact, the restriction of the symplectic form ω on TN to SN is an antisymmetric 2-form of rank $2n-2$, which is nondegenerate on transversals to the geodesic flow. We will often refer also to such a form as symplectic.

The symplectic form ω on TN is the exterior derivative of a 1-form θ on TN, which descends to SN. Thus the geodesic flow on SN is an example of a contact flow.

Definition 1.13: Let M be a differentiable manifold of dimension $2n-1$. A contact structure on M is a differential one-form θ such that $\theta \wedge (d\theta)^{n-1}$ is a volume on M. A contact flow is a flow preserving a contact structure.

In particular, geodesic flows (on SN) are contact flows.

A relation between geodesic flows and Anosov flows is given by

Theorem 1.14 [**A1,K**]: If N is compact and negatively curved then the geodesic flow is an Anosov flow.

(Negative curvature is not necessary for geodesic flows to be Anosov, but there can only be "very little" positive curvature. This is discussed briefly on page 10 of [A1] and in [Ho].) Let us note here that in the case of geodesic flows "the manifold" N is called the configuration space and its tangent bundle or unit tangent bundle is called the phase space M, *i.e.*, the manifold on which the geodesic flow acts.

Let us point out here that the presence of an invariant contact form makes the regularity of strong and weak distributions coincide, essentially by providing a canonical choice of time and thus preventing time changes that could reduce regularity of strong stable and unstable distributions. To be more specific $E^{su} = E^u \cap ker\theta$ and $E^{ss} = E^s \cap ker\theta$, so since θ is smooth, E^{su} and E^{ss} are as smooth as E^u and E^s respectively.

In the geometric setting the stable and unstable foliations correspond to the horospheric foliations: On the universal cover of a negatively curved Riemannian manifold horospheres are defined as level sets $H_v(t)$ of Busemann functions $h_v(x) := \lim_{t\to\infty}\Big((d(x,\gamma_v(t)) - t\Big)$ for $v \in SM$. We call $H_v(t) = \{x \in M \mid h_v(x) = t\}$ the horospheres at $\gamma_v(\infty)$. In the Poincaré disk model of the two dimensional real hyperbolic space they are circles touching the boundary of the disk with v as an inward normal. They can also be thought of as the boundaries of horoballs $B_v(0) := \bigcup_{t>0} B_t(\gamma_v(t))$, where B_t denotes the ball of radius t. If w is another inward normal to H_v then γ_v and γ_w are positively asymptotic $\big($*i.e.*, $d(\gamma_v(t), \gamma_w)$ is bounded for $t > 0\big)$. The inward normal bundle $E^{ss}(v)$ of H_v is thus the strong stable manifold of v. In other words, in the geometric setting strong stable manifolds project under the footpoint projection $TM \to M$ to horospheres, which are natural geometric objects. Thus the stable foliation in TM (or SM) is referred to as the horoshperic foliation. Note that reversing time along geodesics or by applying the flip map $J: TM \to TM$, $v \mapsto -v$ we get the strong unstable manifold: $E^{su}(v) = JE^{ss}(Jv)$. It is the outward normal bundle of the horosphere centered at $\gamma_v(-\infty)$. This reproves the previous remark that the regularity of the stable and unstable foliations coincides for geodesic flows.

Symplectic Flows

Our discussion here will also consider symplectic Anosov flows. The setting is that of an odd-dimensional manifold with a 2-form that is nondegenerate on transversals to the flow and invariant under the flow. We will usually refer to this form as an invariant symplectic form for the flow. (This terminology is consistent with that of [An], but very different from [AM], where this is called a "contact form" and our contact form an "exact contact form.") We provide a few more pertinent facts about symplectic structures.

Associated with a symplectic form is the concept of Lagrangian subspaces. A Lagrangian subspace of T_pM is a maximal linear subspace V so that $\omega_p|_V \equiv 0$. Note that the stable and unstable distributions consist of Lagrangian subspaces since invariance of ω combined with the Anosov estimates forces ω to vanish on stable and unstable subspaces. Take two vectors x, y in an unstable subspace. Then $|\omega(x,y)| = |\omega(\varphi^t x, \varphi^t y)| \leq \|\omega\| \cdot \|\varphi^t x\| \cdot \|\varphi^t y\| \leq \nu_2^{-2t} \|\omega\| \cdot \|x\| \cdot \|y\| \to 0$.

A frequently useful theorem about local representations of a symplectic form ω is

Theorem 1.15: (Darboux) At any point $p \in M$ there exist local coordinates in which ω is represented as $J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$, where I is the $n \times n$ identity matrix.

(This contrasts sharply with the situation for a Riemannian metric where a similar statement is true only for flat metrics.) This theorem simplifies thinking about a symplectic form in local terms. We will give a proof (of a stronger result) in a later chapter.

History

Let us now give an indication of some results that motivated this study. As mentioned above, the Anosov splitting associated with an Anosov flow (in particular a geodesic flow for a negatively curved manifold) is always Hölder continuous.

Theorem 1.16 [**A2**]: The Anosov splitting is Hölder continuous.

(We will reprove this result here.) It seems striking that the regularity of the Anosov splitting might be as low as C^{ϵ}. Thus its regularity has been investigated for some time. **Anosov** [A1] showed by constructing an example that the Anosov splitting may not be C^1.

Theorem 1.17 (**[A1], §24**): The Anosov splitting is not always C^1.

(Anosov also mentions, but does not carry out "simple but lengthy considerations in the neighborhood of periodic points." In a way that will be our undertaking.) His example is of a global nature in that it is constructed from a product of toral automorphisms, thus in a very specific and computable situation. Also the spectrum of that system is spread much further than necessary to destroy smoothness. It does, however, exhibit exactly the Hölder exponent that one would guess from the spectrum (in terms of the relation between spectral bunching and regularity to be established here). Indeed, in the terminology of our example 1.11, he modifies the automorphism $M \times A$ of $\mathbb{T}^6 = \mathbb{T}^2 \times \mathbb{T}^4$, which has eigenvalues λ and λ^3 and thus is $\frac{2}{3}$-bunched. The system he obtains is also $\frac{2}{3}$-bunched and he computes that the Hölder exponent of the stable and unstable distributions is $\frac{2}{3}$, which corresponds to the relations we shall discuss below. It is also more effective than the examples given in this work inasmuch as in his example the distributions are nondifferentiable almost everywhere. Although this is not entirely unlikely to be the case in the construction given here, we here only prove nondifferentiability at suitable periodic points.

Considering geodesic flows on negatively curved surfaces **Hopf** proved the following theorem in 1940:

Theorem 1.18 [Ho]: For the geodesic flow on a negatively curved surface the horospheric splitting is C^1.

In 1974 **Hirsch** and **Pugh** proved this theorem for manifolds of higher dimension under a pinching assumption on the sectional curvature K.

Definition 1.19: A manifold is said to be absolutely $a-$pinched if

$$a < \frac{inf\ K_{p,min}}{sup\ K_{p,max}},$$

(relatively) $a-$pinched if

$$a < inf\, \frac{K_{p,min}}{K_{p,max}},$$

where $a \in (0,1)$, sup and inf are taken over $p \in M$ and

$$K_{p,max} = sup\{|K(p,\Pi)| : \Pi \text{ a 2-plane in } T_pM\}$$

and

$$K_{p,min} = inf\{|K(p,\Pi)| : \Pi \text{ a 2-plane in } T_pM\}.$$

Hirsch and Pugh proved

Theorem 1.20 [HP]: For the geodesic flow on an absolutely 1/4-pinched manifold of negative curvature the Anosov splitting is C^1.

Their proof is very dynamical in nature and uses essentially that this pinching assumption guarantees strong 1-bunching. (We improve the theorem here as a corollary of our work, using that, by continuity of the sectional curvature, relative a-pinching implies strong $2\cdot\sqrt{a}$-bunching [K].) In fact Hirsch and Pugh proved that in the case of Anosov systems bunching implies some regularity:

__**Theorem**__ __**1.21**__ **[HP]**: For $\alpha\in(0,1]$ call a diffeomorphism ϕ HP-α-bunched if $\left(\exists C>1,\mu_1<\mu_2<1<\nu_2<\nu_1\right)\left(\forall p\in M\right)\left(\forall\, v\in E^s(p), u\in E^u(p), n\in\mathbb{N}\right)$

$$\frac{1}{C}\mu_1^n\,\|v\|\le\|D\phi^n(v)\|\le C\mu_2^n\,\|v\|$$
$$\frac{1}{C}\nu_1^{-n}\,\|u\|\le\|D\phi^{-n}(u)\|\le C\nu_2^{-n}\,\|u\|$$

with $$\mu_2\cdot\nu_2^{-1}<\left(min\ (\mu_1,\nu_1^{-1})\right)^{\alpha}.$$
Then the Anosov splitting of a HP-α-bunched Anosov flow is C^{α}.

(Note that this bunching condition is much more stringent than our strong bunching condition since some quantifiers have been exchanged in comparison to our definition 1.9.)

They prove the theorem for hyperbolic sets of diffeomorphisms.

The theory for surfaces has recently been completed by **Hurder** and **Katok.** Their methods are more local and more refined than those used by Hirsch and Pugh. Their results apply to volume preserving Anosov flows on three-dimensional manifolds. By distributions of smoothness $C^{1+\omega}$ we mean distributions that in local C^∞-coordinates are spanned by vector fields of class C^1 whose derivatives have modulus of continuity ω. Here we take $\omega(x):=-x\cdot log\,x$.

Theorem 1.22 [**HK**]: Let φ^t be a volume preserving C^∞ Anosov flow on a three-dimensional Riemannian manifold M. Then the Anosov splitting is $C^{1+\omega}$, where $\omega(x) := -x \cdot \log x$. If it is $C^{1+\overline{\omega}}$ with $\overline{\omega} = o(-x \cdot \log x)$ then it is C^∞.

For geodesic flows the last conclusion implies that the curvature is constant [HK, G].

While the theory for surfaces is thus complete in this respect there is no full understanding of the situation in higher dimensions. On one hand there have been studies very recently that explored the implications of high smoothness of the Anosov splitting for geodesic flows, which have yielded fascinating rigidity results. These have been initiated by work of **Kanai** [Kn], which has been developed further by **Katok, Feres** and **Flaminio.** Without trying to explain their methods we outline two results by Feres and Katok. The first result is a step towards the following

Conjecture 1.23: Let M be a compact C^∞ Riemannian manifold with negative curvature and C^∞ Anosov splitting for the geodesic flow. Then the geodesic flow on M is C^∞ isomorphic to that of a locally symmetric Riemannian space.

Indeed, one conjectures more boldly that the manifold has to be isometric to the locally symmetric Riemannian space.

This first result is contained in

__Theorem__ __1.24__ **(Feres & Katok [FK1]):** Let M be a compact C^∞ Riemannian manifold with 1/4-pinched negative curvature and C^∞ Anosov splitting. Then the geodesic flow on M is C^∞ isomorphic to that of a Riemannian space of constant negative curvature.

together with

__Theorem__ __1.25__ **(Feres, [F]):** If M is odd-dimensional the hypothesis of 1/4-pinching can be omitted in theorem 1.24.

The second result is:

__Theorem__ __1.26__: **(Feres & Katok, [FK2])** Let M be a five-dimensional manifold with an Anosov flow preserving a transverse symplectic form and a smooth ergodic probability measure. If the Anosov splitting is C^∞ then either
i) the manifold has a flow-invariant structure of an affine locally symmetric space
or
ii) its Oseledetz decomposition extends to a splitting of TM and the Lyapunov exponents are -2χ, $-\chi$, 0, χ, 2χ where χ is a flow-invariant measurable function.

This result is interesting here because it shows that those methods can be used for Anosov flows, but also because it complements the situation considered in the work at hand.

It is, by the way, believed that these rigidity results should be true with C^∞ replaced by C^2. We will be able to provide some evidence to support (at least the reasonableness of) this conjecture: Symplectic Anosov systems generically do not have C^2 Anosov splitting.

On the other hand, while this research is going on and the idea is being reinforced that having very smooth Anosov splitting is a rare occurrence, there has been no full understanding of the regularity issue in the absence of pinching or bunching. For example, so far there has been no complete answer to the question: Is the Anosov splitting C^1 in the absence of strong 1-bunching or 1/4-pinching? While there is the example by Anosov where this is not the case, the more delicate question remains: Does the Anosov splitting **typically** fail to be C^1 in the absence of (strong) 1-bunching? Put the other way around: Are theorems like those of Hirsch and Pugh optimal in some sense? It has been believed by most people in the field that this is indeed the case. Now finally we can shed some light on this issue. We show that already low regularity exceeding the degree provided by our regularity theorem is rather special.

The work by Katok and Hurder has not only been mentioned for "historical" reasons, but also because the local methods used there have inspired this study and proved quite useful in it. One major tool in their approach can be traced to an idea of **Anosov.** In [A1] he proves that the Anosov splitting need not be C^2 by observing that for a symplectic map having C^2 Anosov splitting forces a relation on the third jet of the return map at periodic points. He then shows that this relation can be violated. Katok and Hurder observed that this "relation" actually describes the vanishing of a globally defined cocycle. Exceeding the critical smoothness forces this cocycle to vanish and this permits them to carry out a bootstrapping process that gives arbitrarily high smoothness.

Let us briefly discuss the connection to this work. The condition we find to be necessary for C^α-regularity is of a different nature than that found for C^2-

regularity. Instead of a construct involving a jet of the return map we obtain a geometric horizontality condition on unstable directions of points in the fast stable leaf of a periodic point. If any global meaning of this condition is to be found one should expect it to have the nature of an invariant distribution or some invariance property of unstable directions, rather than being a cocycle.

It should be pointed out that while Anosov obtained his condition necessary for C^2 by considering a periodic point, Katok and Hurder obtain an expression at arbitrary points and observe that having C^2 Anosov splitting forces it to vanish at periodic points. (They then verify that they have a cocycle at hand.) Our coordinates here appear to be too specifically adapted to a periodic point to immediately be able to obtain global information. One particularly obvious weakness is the fact that we use the concept of a fast stable leaf here, which may not be defined everywhere. (This aspect motivates the desire to study perturbations of symmetric spaces of nonconstant negative sectional curvature: Here, as well as for small perturbations, fast stable leaves are well-defined everywhere by work of Pesin [Ps][1].) This then is an issue awaiting clarification: What intrinsic meaning does the condition developed in this work have and is it the manifestation of some global object or structure?

To return to the specific usefulness of Katok and Hurder's methods we remark that it is there that the semilocal analysis along orbits, both to obtain as well as to limit smoothness, was developed. It is when our bunching condition is violated that we can expect the Anosov splitting to "typically" fail to be highly regular.

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[1]Although this paper contains some errors, the conclusion we need is valid.

Hirsch and Pugh show in their papers that for $\alpha \leq 1$ HP-α-bunching implies C^α Anosov splitting. Using the more local methods of Katok and Hurder yields the following result:

__Theorem__ __1.27__: Let M be a Riemannian manifold and φ^t an α-bunched Anosov flow, $\alpha \in (0,1)$. Then the Anosov splitting of φ^t is $C^{\alpha-\epsilon}$ for all $\epsilon > 0$. If φ^t is strongly α-bunched then the Anosov splitting of φ^t is $C^{\alpha+\epsilon}$ for some $\epsilon > 0$.

In order to unify the treatment we used here:

__Definition__ __1.28__: A function $f: \mathbb{R} \to \mathbb{R}$ is said to be β-Hölder if f is $[\beta]$ times differentiable and the $[\beta]^{th}$ derivative is Hölder continuous in the usual sense with Hölder exponent $\beta - [\beta]$.
A distribution is said to be β-Hölder or Hölder with Hölder exponent β if in local coordinates it can be generated by vector fields represented by coordinate functions that are β-Hölder.

__Remark__ __1.29__: Note that 1/4-pinching implies strong 1-bunching. We thus proved that in the geodesic-flow-theorem of Hirsch and Pugh one can conclude that the Anosov splitting is indeed $C^{1+\epsilon}$ for some $\epsilon > 0$.

Recall that in the example given by Anosov in [A1] the diffeomorphism has contraction and expansion constants given by $0 < \nu^{-3} < \nu^{-1} < 1 < \nu < \nu^3 < \infty$. Thus his system is $\frac{2}{3}$-bunched. And indeed, by explicit computation, he shows that the Anosov splitting has Hölder exponent exactly $\frac{2}{3}$.

We expect that in the absence of α-bunching the Anosov splitting should typically fail to be C^α. The theory we develop can be summarized as follows.

Definition 1.30: Let M be a Riemannian manifold, $l \in \mathbb{N}$. Define

$$\mathfrak{S} := \left\{ \textit{transversally symplectic Anosov flows } \varphi^t : M \to M \textit{ on } M \textit{ with the } C^\infty\textit{-topology} \right\}.$$

Here the C^∞-topology is defined as follows:

We say that $\varphi_n \to \varphi \in \mathfrak{S}$ if $\forall \epsilon > 0,\ k \in \mathbb{N}\ \exists N \in \mathbb{N}\ \forall n \geq N$

$$\|\varphi_n - \varphi\|_{C^k} < \epsilon \text{ and } \|\omega_n - \omega\|_{C^l} < \epsilon,$$

where ω_n and ω denote invariant transversal symplectic forms for φ_n and φ respectively.

Remark 1.31: Since the invariant symplectic forms are not unique (one can, *e.g.*, rescale them) the closeness condition for them is intended to say that ω_n and ω can be chosen so as to be close.

Definition 1.32: Define $\mathfrak{S}_2 := \mathfrak{S}$

and $\mathfrak{S}_\alpha := \{\varphi^t \in \mathfrak{S} : \varphi^t \text{ has an } \alpha\text{-spread periodic orbit}\}.$

Remark 1.33: $\mathfrak{S}_\alpha$ is open in $\mathfrak{S}$. We shall use the induced topology.

Theorem 1.34: For $\alpha \in (0, 2]$ flows in $\mathfrak{S}_\alpha$ generically do not have Anosov splitting of class C^α.

Remark 1.35: We show that at the α-spread periodic orbit the regularity of both stable and unstable distributions can be made to be less than C^α by small perturbations causing a necessary condition for such regularity to be violated. We then remark that the set of Anosov flows violating this condition is open.

From our discussion it will be evident that, mutatis mutandis, we also have

Theorem 1.36: Let $\alpha \in (0,2]$. Denote by $\mathfrak{D}_\alpha$ the set of symplectic Anosov diffeomorphisms that possess an α-spread periodic orbit p. Then diffeomorphisms in $\mathfrak{D}_\alpha$ generically do not have Anosov splitting of class C^α.

Remark 1.37: Note that we are making statements about absence of regularity between C^2 and C^ϵ for small ϵ. Specifically we obtain genericity of failure of the Anosov splitting to be C^2, breakdown of differentiability in the absence of 1-bunching and the fact that the Hölder exponent of the Anosov splitting may be arbitrarily small.

Contemplating theorems 1.27 and 1.34 immediately raises an obvious question of what happens "in between." Ideally one should hope that the data on periodic points determine the bunching behavior to the following extent: "α-bunching along all periodic orbits forces α-bunching." In that case one could assert the following

Conjecture 1.38: If all periodic orbits are α-bunched then the flow is α-bunched and thus the Anosov splitting is $C^{\alpha-\epsilon}$ for all $\epsilon > 0$.
If there is a periodic orbit that is not α-bunched then the Anosov splitting is generically not C^α.
(This latter part is known for symplectic systems.)

Our hope is that this question can be resolved reasonably soon. The statement "α-bunching along all periodic orbits forces α-bunching" has a flavor of Lifschitz theory [L] and similar methods might yield a positive answer.

Statement of Results

In this section, intended for use as a reference, we briefly summarize the results obtained here.

Proving Regularity:

Definition 1.39: Let φ^t be an Anosov flow on a Riemannian manifold M. For $\alpha\in(0,2]$ we call $p\in M$ α-bunched if one can take $0<\mu_1<\mu_2<1<\nu_2<\nu_1<\infty$, $C>0$ such that

$$\mu_2\nu_2^{-1}\le\left(min\ (\mu_1,\nu_1^{-1})\right)^\alpha$$

and

$$\forall\ v\in E^s(p),u\in E^u(p),t>0\Big)$$

$$\frac{1}{C}\mu_1^t\,\|v\|\le\|D\varphi^t(v)\|\le C\mu_2^t\,\|v\|$$

$$\frac{1}{C}\nu_1^{-t}\,\|u\|\le\|D\varphi^{-t}(u)\|\le C\nu_2^{-t}\,\|u\|.$$

Otherwise call p α-spread.

We call p u-α-bunched if $\mu_2\nu_2^{-1}\le\mu_1^\alpha$

and s-α-bunched if $\mu_2\nu_2^{-1}\le\nu_1^{-\alpha}$.

φ^t is called α-bunched if every $p\in M$ is α-bunched with uniform C.

φ^t is called strongly α-bunched if there exists an $\epsilon>0$ such that all $p\in M$ are $(\alpha+\epsilon)$-bunched.

Define (strong) u-α-bunching and (strong) s-α-bunching for φ^t in the obvious way. Similarly for diffeomorphisms. Write

$$\mathfrak{A}:=\Big\{\phi\colon\ \phi\ \textit{is an Anosov system}\Big\}$$

$$\mathfrak{A}_\alpha:=\Big\{\phi\colon\ \phi\ \textit{is an }\alpha\textit{-bunched Anosov system}\Big\}$$

$$\mathfrak{A}_\alpha^S:=\Big\{\phi\colon\ \phi\ \textit{is a strongly }\alpha\textit{-bunched Anosov system}\Big\}$$

$$\mathfrak{A}_{i\text{-}\alpha}:=\Big\{\phi\colon\ \phi\ \textit{is an }i\text{-}\alpha\textit{-bunched Anosov system}\Big\},\ i=u,s$$

$$\mathfrak{A}_{i\text{-}\alpha}^S:=\Big\{\phi\colon\ \phi\ \textit{is a strongly }i\text{-}\alpha\textit{-bunched Anosov system}\Big\},\ i=u,s.$$

A manifold is said to be (relatively) a – pinched ($a \in (0,1)$) if

$$a < \inf_{p \in M} \frac{K_{p,min}}{K_{p,max}},$$

where

$$K_{p,max} = \sup\{|K(p,\Pi)| : \Pi \text{ a 2-plane in } T_pM\}$$

and
$$K_{p,min} = \inf\{|K(p,\Pi)| : \Pi \text{ a 2-plane in } T_pM\}.$$

Write
$$\mathfrak{G}_a := \left\{ a\text{-pinched geodesic flows} \right\}.$$

Remark 1.40: By [K] and continuity of curvature a-pinching implies strong $2 \cdot \sqrt{a}$-bunching, *i.e.*, $\mathfrak{G}_a \subset \mathfrak{A}^s_{2\sqrt{a}}$.

Theorem I: Denote by E^u, E^s the unstable and stable distributions respectively and let $i = u$, s.

The Anosov splitting is Hölder continuous. (Since $\mathfrak{A} = \bigcup_{(0,2]} \mathfrak{A}_\alpha$.)

If $\alpha \in (0,2]$, $\phi \in \mathfrak{A}_\alpha$, $\epsilon > 0$ then the Anosov splitting of ϕ is $C^{\alpha - \epsilon}$.

If $\alpha \in (0,2)$, $\phi \in \mathfrak{A}^s_\alpha$, then $\exists \epsilon > 0$ such that the Anosov splitting of ϕ is $C^{\alpha + \epsilon}$.

If $\alpha \in (0,2]$, $\phi \in \mathfrak{A}_{i\text{-}\alpha}$, $\epsilon > 0$ then E^i is $C^{\alpha - \epsilon}$.

If $\alpha \in (0,2)$, $\phi \in \mathfrak{A}^s_{i\text{-}\alpha}$, then $\exists \epsilon > 0$ such that E^i is $C^{\alpha + \epsilon}$.

If E^i for $\phi \in \mathfrak{A}$ has codimension one, then $\exists \epsilon > 0$ such that it is $C^{1+\epsilon}$.

If $a \in [0,1)$, $\phi \in \mathfrak{G}_a$ then $\exists \epsilon > 0$ such that the horospheric foliations of ϕ are $C^{2\sqrt{a}+\epsilon}$.

Low Regularity:

Definition 1.41: Define

$$\mathfrak{S} := \left\{ \text{transversally symplectic Anosov systems } \phi \text{ with the } C^\infty\text{-topology} \right\}.$$

For $\alpha \in (0,2]$ define
$$\mathfrak{S}_2 := \mathfrak{S}$$

and
$$\mathfrak{S}_\alpha := \{\phi \in \mathfrak{S} : \phi \text{ has an } \alpha\text{-spread periodic orbit}\}.$$

Remark 1.42: $\bigcup_{(0,2)} \mathfrak{S}_\alpha$ is open and dense (*i.e.*, generic) in $\mathfrak{S}$.

Theorem II:

If $\alpha \in (0,2]$ then $\phi \in \mathfrak{S}_\alpha$ generically does not have C^α Anosov splitting.

$\phi \in \mathfrak{S}$ generically does not have $C^{1+x|log\, x|}$ Anosov splitting.

Remark 1.43: The breakdown of regularity occurs at the α-spread periodic orbits. It can be achieved simultaneously for stable and unstable distributions.

Although the proof given here is missing some technical detail we also state

Theorem III: Compact quotients of symmetric spaces of nonconstant negative curvature admit a finite cover that can be perturbed so as not to have C^1 horospheric foliations. Compact quotients of spaces of constant negative curvature admit a finite cover that can be perturbed so as not to have C^2 horospheric foliations.

Margulis measure:

Finally we prove:

Theorem IV: The Margulis measure of a transitive Anosov flow arises from a Hausdorff measure for a natural distance on (un)stable leaves.

2: The regularity theorem

In the introduction we stated that by using the methods of Katok and Hurder one can obtain the following result:

Theorem 2.1: Let M be a Riemannian manifold and φ^t an α-bunched Anosov flow. Then the Anosov splitting of φ^t is $C^{\alpha-\epsilon}$ for all $\epsilon>0$.

Corollary 2.2: Let M be a Riemannian manifold and φ^t a strongly α-bunched Anosov flow. Then the Anosov splitting of φ^t is $C^{\alpha+\epsilon}$ for some $\epsilon>0$.

In order to unify the treatment we used here:

Definition 2.3: A function $f:\mathbb{R}\to\mathbb{R}$ is said to be β-Hölder if f is $[\beta]$ times differentiable and the $[\beta]^{th}$ derivative is Hölder continuous in the usual sense with Hölder exponent $\beta-[\beta]$.
A distribution is said to be β-Hölder or Hölder with Hölder exponent β if in local coordinates it can be generated by vector fields represented by coordinate functions that are β-Hölder.

In this chapter we will outline a proof of this fact. Since the methods are adapted from [HK] we will not go into as much detail here as in the later chapters.

Remark 2.4: Recall that it suffices to investigate the regularity of the unstable distribution.

Hölder continuity

In this paragraph we outline the proof of one part of this theorem, namely

Proposition 2.5: Let $\alpha\in(0,1]$, M a Riemannian manifold and φ^t a u-α-bunched Anosov flow. Then for any $\epsilon>0$ the unstable distribution is Hölder continuous with exponent $\alpha-\epsilon$.

Proof: We first choose appropriate coordinates. For all $p\in M$ take a hypersurface $\mathcal{T}_p$ transversal to $\dot\varphi$ of (small but) uniform size, so that the $\mathcal{T}_p$ depend C^∞ on p. For fixed p we abuse notation and write W^u for $W^u\cap\mathcal{T}_p$ and W^s for $W^s\cap\mathcal{T}_p$. Denote the tangent distributions to these W^u and W^s by E^u and E^s. These then are (by [A2]) Hölder continuous distributions on $\mathcal{T}_p$. We want to show that they have Hölder exponent α.

We choose **coordinates** $\Xi: M\times[-\epsilon,\epsilon]\to M$ such that

$\Xi_p:[-\epsilon,\epsilon]\to\mathcal{T}_p$

W^u and W^s are coordinate planes ($W^u\sim\mathbb{R}^k\times\{0\}$ and $W^s\sim\{0\}\times\mathbb{R}^l$)

the induced map $\phi^t:\mathcal{T}_p\to\mathcal{T}_{\varphi^t p}$ has differential at zero of the form $D\phi^t=\begin{bmatrix}A_t & o\\ o & C_t\end{bmatrix}$ with

$$\|A_t^{-1}\|<L\nu_2^{-t}\text{ and }\|C_t\|<L\mu_2^t$$

the coordinates $\Xi_p:[-\epsilon,\epsilon]\to\mathcal{T}_p$ are C^∞.

(Although we do not need this, we remark that they can be taken to depend Hölder continuously on p in the C^1-topology.)

We shall write the coordinates as (x,y) such that

$$\Xi_p(x,0)\in W^u(p)\text{ and }\Xi_p(0,y)\in W^s(p).$$

We define a **cone field**

$$V(\delta) := \left\{ \begin{array}{l} k+1\text{-dimensional distributions } v \text{ on } M \text{ such} \\ \text{that } v(p) \text{ contains } \dot\varphi(p) \text{ and is } \delta\text{-close to } E^u \end{array} \right\}$$

We will represent elements $v \in V(\delta)$ in the coordinates Ξ_p by identifying $v(p)$ with $v(p) \cap T\mathcal{T}_p$. Note that by the same token we can represent $v(q)$ in coordinates Ξ_p for $q \in \mathcal{T}_p$.

Thus δ-closeness is determined in the coordinates Ξ_p by representing $v(p)$ as the graph of a linear map $D: \mathbb{R}^k \to \mathbb{R}^l$ and using the norm topology.

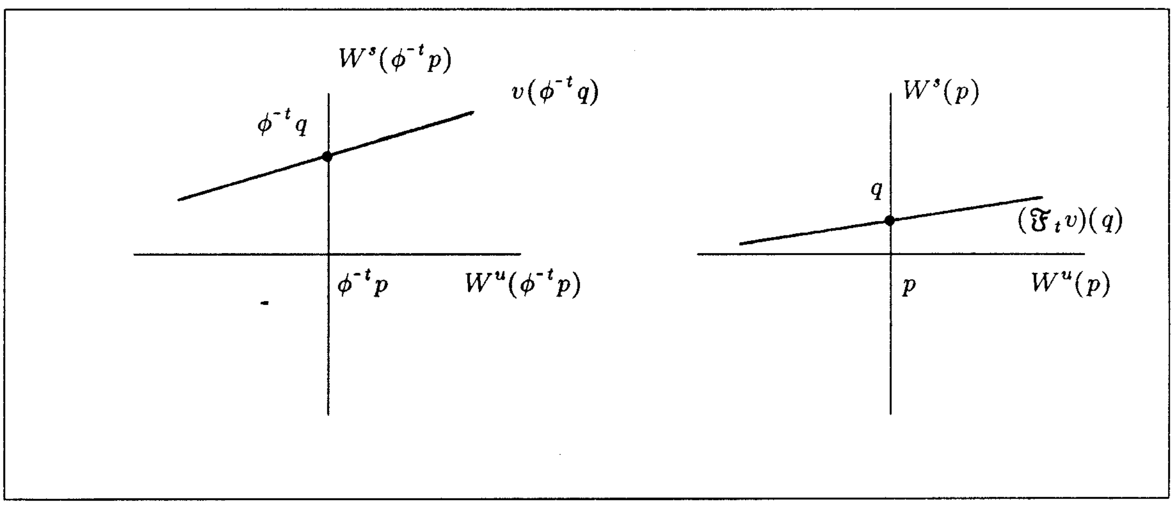
The flow φ^t acts on $V(\delta)$ by transformations $\mathfrak{F}_t$ defined by

$$\mathfrak{F}_t : v \mapsto \mathfrak{F}_t v : p \mapsto D\varphi^t(v(\varphi^{-t}p)),$$

i.e.,

$$(\mathfrak{F}_t v)(p) = D\varphi^t(v(\varphi^{-t}p)).$$

For t sufficiently large $\mathfrak{F}_t(V(\delta)) \subset V(\delta)$ and for $v \in V(\delta)$ we have $\mathfrak{F}_t v \to E^u$ as $t \to \infty$.



Now let $\beta = \alpha - \epsilon$ for some $\epsilon > 0$ and recall that for all $p \in M$ we have

$$\nu_2^{-1}\mu_2\mu_1^{-\beta} < 1.$$

In order to prove the proposition it suffices to show

Lemma 2.6: There are $K, \epsilon_o > 0$ such that if $v \in V(\delta(\epsilon_o))$ and $\|y\| < \epsilon_o$ then

$\left(\exists T \in \mathbb{R}\right)\left(\forall t \in [T, 2T]\right)$ $\quad \mathfrak{F}_t(V(\delta(\epsilon_o))) \subset V(\delta(\epsilon_o))$

and if $\quad \|v(0,y)\| < K\|y\|^\beta$

then $\quad \|(\mathfrak{F}_t v)(0,z)\| < K\|z\|^\beta,$

where $(0,z) = \phi^t(0,y)$.

This lemma implies the proposition since by applying it inductively we obtain the

Corollary 2.7: There are $K, \epsilon_o > 0$ such that if $v \in V(\delta(\epsilon_o))$ and $\|y\| < \epsilon_o$ then $\left(\exists T \in \mathbb{R}\right)\left(\forall t > T\right)$ $\quad \mathfrak{F}_t(V(\delta(\epsilon_o))) \subset V(\delta(\epsilon_o))$

and if $\quad \|v(0,y)\| < K\|y\|^\beta$

then $\quad \|(\mathfrak{F}_t v)(0,z)\| < K\|z\|^\beta,$

where $(0,z) = \phi^t(0,y)$.

Thus we showed that for all $t > T$ $\mathfrak{F}_t$ preserves the collection of distributions that are β-Hölder with given constant K in the stable direction. But since $\mathfrak{F}_t v \to E^u$ as $t \to \infty$ and this condition is closed (by equicontinuity) this forces E^u to be Hölder continuous in the stable direction with exponent $\beta = \alpha - \epsilon$. Since E^u is smooth in the directions of $\dot{\varphi}$ and E^u we conclude that E^u is $(\alpha - \epsilon)$-Hölder.

Proof of lemma: In this proof we consider only points on the stable leaf and thus write $v(y)$, etc. instead of $v(0,y)$, etc.

Suppose $\quad \|v(y)\| < K\|y\|^\beta.$

The differential of the map ϕ^t at y is

$$D\phi^t(y) = \begin{bmatrix} A_t & 0 \\ B_t & C_t \end{bmatrix}$$

and there exists $L > 1$ such that

$$\|A_1(y) - A_1(0)\| \le L\,\|y\|$$

$$\|A_1^{-1}(y) - A_1^{-1}(0)\| \le L\,\|y\|$$

$$\|C_1(y) - C_1(0)\| \le L\,\|y\|$$

$$\|B_t(y)\| \le L\,\|y\| \text{ for } t < 1.$$

Using that

$$\|A_t^{-1}(0)\| \le L\,\nu_2^{-t}$$

$$\|C_t(0)\| \le L\,\mu_2^{t}$$

$$B_t(0) = 0$$

one can show that

$$\left(\exists \delta > 0,\ L > 1\right)\left(\forall \|y\| < \delta\right)$$

$$\|A_t^{-1}(y)\| \le L\,\nu_2^{-t}$$

$$\|B_t(y)\| \le L(t)\|y\|$$

$$\|C_t(y)\| \le L\,\mu_2^{t}$$

$$\|y\| \le L\,\mu_1^{-t}\|\phi^t(y)\|.$$

The Lipschitz constant $L(t)$ for $B_t(y)$ depends on t and we do not attempt to control its growth. We may assume that it is nondecreasing in t.

The last statement follows from an analogous one about $\|C_t^{-1}(y)\|$ and the mean value theorem. The first one is shown in the same way as the third. Thus we prove the second and third statement here. We begin with the second one:

Since

$$D\phi^t(y) = \begin{bmatrix} A_t(y) & 0 \\ B_t(y) & C_t(y) \end{bmatrix}$$

and

$$D\phi^{t+s}(y) = D\phi^s(\phi^t y)\ D\phi^t(y),$$

i.e.,

$$\begin{bmatrix} A_{t+s}(y) & 0 \\ B_{t+s}(y) & C_{t+s}(y) \end{bmatrix} = \begin{bmatrix} A_s(\phi^t y) & 0 \\ B_s(\phi^t y) & C_s(\phi^t y) \end{bmatrix} \begin{bmatrix} A_t(y) & 0 \\ B_t(y) & C_t(y) \end{bmatrix}$$

we find

$$B_n(y) = B_1(\phi^{n-1}(y))\,A_{n-1}(y) + C_1(\phi^{n-1}(y))\,B_{n-1}(y).$$

Since $\|A_{n-1}(y)\|$ and $\|C_1(\phi^{n-1}(y))\|$ are bounded by constants depending only on n and inductively $B_1(\phi^{n-1}(y))$ and $B_{n-1}(y)$ are Lipschitz continuous we conclude that

$$\|B_n(y)\| \leq L(n)\|y\|.$$

The claim follows since $\|B_t(y)\| \leq L\|y\|$ for $t<1$.

For the third claim, $\|C_t(y)\| \leq L\mu_2^t$, it clearly suffices to show:

Claim: $(\exists K>1)$: $\|C_n(y) - C_n(0)\| \leq K\mu_2^n\|y\|$

Proof: We use induction. Note first that we can find $L>1>\lambda$ such that

$$\|C_1(y) - C_1(0)\| \leq L\|y\|$$

$$\|C_t(0)\| \leq L\mu_2^t$$

$$\|\phi^n y\| \leq L\lambda^n\|y\|$$

and assume $\|C_j(y) - C_j(0)\| \leq K\mu_2^j\|y\|$ for $i \leq n$, $\|y\|$, $\frac{1}{K} < \frac{\mu_2(1-\lambda)}{2L^4}$.

Then $\|C_i(\phi^m y)\| \leq$

$$\leq \|C_i(0)\| + \|C_i(\phi^m y) - C_i(0)\| \leq$$

$$\leq L\mu_2^i + K\mu_2^i\|\phi^m y\| \leq$$

$$\leq L\mu_2^i + KL\lambda^m\mu_2^i\|y\| =$$

$$= L\mu_2^i(1 + K\lambda^m\|y\|)$$

and $\|C_{n+1}(y) - C_{n+1}(0)\| \leq$

$$\leq \|C_n(\phi y)\| \cdot \|C_1(y) - C_1(0)\| +$$

$$+\sum_{j=1}^{n-1}\|C_{n-j}(\phi^{j+1}y)\| \cdot \|C_1(\phi^j y) - C_1(0)\| \cdot \|C_j(0)\| +$$

$$+\|C_1(\phi^n y) - C_1(0)\| \cdot \|C_n(0)\| \le$$

$$\le L\mu_2^n(1 + K\lambda\|y\|) \cdot L\|y\| +$$

$$+\sum_{j=1}^{n-1} L\mu_2^{n-j}(1 + K\lambda^{j+1}\|y\|) \cdot L^2\lambda^j\|y\| \cdot L\mu_2^j +$$

$$+ L^2\lambda^n\|y\| \cdot L\mu_2^n \le$$

$$\le \sum_{j=0}^{n} L\mu_2^{n-j}(1 + K\lambda^{j+1}\|y\|) \cdot L^2\lambda^j\|y\| \cdot L\mu_2^j =$$

$$= L^4\mu_2^n\|y\| \sum_{j=0}^{n}\lambda^j(1 + K\lambda^{j+1}\|y\|) \le$$

$$\le L^4\mu_2^n\|y\| \frac{1}{1-\lambda}(1 + K\frac{\lambda}{1+\lambda}\|y\|) <$$

$$< K\mu_2^{n+1}\|y\| \cdot \left[\frac{L^4}{\mu_2(1-\lambda)}\right] \cdot \left[\frac{1}{K} + \|y\|\right] <$$

$$< K\mu_2^{n+1}\|y\|. \qquad \text{since } \|y\|, \frac{1}{K} < \frac{\mu_2(1-\lambda)}{2L^4}. \quad \square$$

Now we use these estimates to investigate the action of $\mathfrak{F}_t$.

If we represent $v(p)$ by a linear map D as indicated above we can as well consider the graph of D as the image of the map represented in matrix form as $\begin{bmatrix} I \\ D \end{bmatrix}$ where I is the (k,k)-identity matrix.

$$\text{Then } (D\phi^t|_y)(v(p)) = \begin{bmatrix} A_t & 0 \\ B_t & C_t \end{bmatrix}\begin{bmatrix} I \\ D \end{bmatrix} = \begin{bmatrix} A_t \\ B_t + C_t D \end{bmatrix} \sim \begin{bmatrix} I \\ (B_t + C_t D)A_t^{-1} \end{bmatrix},$$

where $\sim$ indicates that the two maps have the same graph. In other words if we let $z = \phi^t y$ then $D(z) = (B_t(y) + C_t(y)D(y))A_t^{-1}(y)$ and thus

$\|D(z)\| =$

$$= \|(B_t(y) + C_t(y)D(y))A_t^{-1}(y)\|$$

$$\leq \|B_t(y)\|\,\|A_t^{-1}(y)\| + \|A_t^{-1}(y)\|\,\|C_t(y)\|\,\|D(y)\|$$

$$\leq L(t)\|y\| \cdot L\nu_2^{-t} + L\nu_2^{-t} \cdot L\mu_2^{t} \cdot K\|y\|^{\beta}$$

$$\leq L(t)\, L\mu_1^{-t}\|z\| \cdot L\nu_2^{-t} +$$

$$+ L^{2+\beta} \cdot \left[\nu_2^{-1}\mu_2\mu_1^{-\beta}\right]^{t} \mu_1^{\epsilon t} \cdot K\|z\|^{\beta}$$

$$< K\|z\|^{\beta}$$

for $t \in [T, 2T]$, where T is such that

$$L^{2+\beta} \cdot \mu_1^{\epsilon T} < \frac{1}{2}$$

and K such that $\quad K > 2\, L(t)\, L\mu_1^{-t} \cdot L\nu_2^{-t} \quad$ for $t \in [T, 2T]$.

We also take T large enough so that $\mathfrak{F}_t(V(\delta)) \subset V(\delta)$ for $t > T$. □

Differentiability

We now want to prove

Proposition 2.8: Let M be a Riemannian manifold and φ^t a strongly u-1-bunched Anosov flow. Then the unstable distribution is differentiable.

Remark 2.9: The previous section already yields that the unstable distribution is Lipschitz-continuous.

Again we need only show transversal differentiability ([HK], [LMM]).

Proof: Let us first explain how differentiability can be proved using estimates only and without knowledge of the derivative. For this introduction consider first function $f:\mathbb{R}\to\mathbb{R}$. Note that if f is differentiable at $x\in\mathbb{R}$ then

$$\frac{f(x+h)-f(x)}{h}\to f'(x)$$

and

$$\frac{f(x)-f(x-k)}{k}\to f'(x)$$

whence

$$\left|\frac{f(x+h)-f(x)}{h}-\frac{f(x)-f(x-k)}{k}\right|\xrightarrow[(h,k)\to 0]{}0$$

or equivalently

$$\frac{1}{hk}\cdot\left|kf(x+h)+hf(x-k)-(h+k)f(x)\right|\xrightarrow[(h,k)\to 0]{}0.$$

Conversely, the last statement implies differentiability at x.

If we now consider $f:\mathbb{R}^n\to\mathbb{R}$ then in order to prove differentiability it suffices to show that for any triple $(v_1,v_2,v_3)\in\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n$ such that $v_1+v_2+v_3=0$

$$\frac{1}{h_1h_2h_3}\cdot\left|h_2h_3f(x+v_1h_1)+h_1h_3f(x+v_2h_2)+h_1h_2f(x+v_3h_3)-(h_2h_3+h_1h_3+h_1h_2)f(x)\right|\to 0$$

as $(h_1,h_2,h_3)\to 0$ and convergence is uniform in $(v_1,v_2,v_3)\in\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n$.

The reason is that taking $v_3=0$ gives a proof of existence of directional derivatives analogous to the case of the real line (with convergence uniform in direction) and then this formula proves that directional derivatives depend linearly on the direction. For a function on a manifold we have to replace the arguments $(x+v_i h_i)$ by $c_i(h_i)$, where c_i is a curve with $\dot{c}_i(0)=v_i$. The rest of the proof is the same.

In order to prove differentiability of the unstable distribution we recall first that it again suffices to prove transversal differentiability (*i.e.*, in the stable direction).

Choose coordinates as in the previous paragraph. We shall consider

$$\mathfrak{B}:=\left\{\begin{array}{c}\text{triples } (v_1,v_2,v_3) \text{ of vector fields on } M \text{ such that:}\\ v_i(p)\in E^s(p)\cap T\mathcal{T}_p\\ v_1+v_2+v_3=0 \text{ and } \|v_1\|+\|v_2\|+\|v_3\|=1.\end{array}\right\}$$

$\dot{\varphi}$ acts on $\mathfrak{B}$ by: $$\mathfrak{E}_t(v_1,v_2,v_3):=(\tilde{v}_1,\tilde{v}_2,\tilde{v}_3)$$

where $$\tilde{v}_i(p)=\frac{D\phi^t v_i(\varphi^t p)}{\xi_t(v_1(\varphi^t p),v_2(\varphi^t p),v_3(\varphi^t p))}.$$

ξ_t is the scale factor needed to have $(\tilde{v}_1,\tilde{v}_2,\tilde{v}_3)\in\mathfrak{B}$.

Note that $$\xi_t< L\mu_2^t \text{ and } \xi_t^{-1}< L\mu_1^{-t}.$$

In light of the previous paragraph and the introduction to this one, it should now be clear that in order to show differentiability it suffices to prove

<u>**Lemma**</u> <u>**2.10**</u>: There exist $\epsilon>0, \rho<1, K\in\mathbb{R}$ such that **if**

$$\Big(\forall p\in M\Big)\Big(\forall (v_1,v_2,v_3)\in\mathfrak{B},\ 0<h_1,h_2,h_3<\epsilon\Big)$$

$$\left\|h_2h_3D(\gamma_{v_1(p)}(h_1))+h_1h_3D(\gamma_{v_2(p)}(h_2))+h_1h_2D(\gamma_{v_3(p)}(h_3))\right\|<Kh_1h_2h_3$$

where $\gamma_{v_i(p)}$ are geodesics in $\mathcal{T}_p$ with $\dot{\gamma}_{v_i(p)}(0)=v_i(p)$, **then**

$$\Big(\exists T\in\mathbb{R}\Big)\Big(\forall p\in M, t\in[T,2T], (\tilde{v}_1,\tilde{v}_2,\tilde{v}_3)\in\mathfrak{E}_t\mathfrak{B}, 0<h_1,h_2,h_3<\epsilon\Big)$$

$$\left\|\tilde{h}_2\tilde{h}_3 D(\gamma_{\tilde{v}_1(p)}(\tilde{h}_1)) + \tilde{h}_1\tilde{h}_3 D(\gamma_{\tilde{v}_2(p)}(\tilde{h}_2)) + \tilde{h}_1\tilde{h}_2 D(\gamma_{\tilde{v}_3(p)}(\tilde{h}_3))\right\| < \rho\, K\tilde{h}_1\tilde{h}_2\tilde{h}_3$$

where $\gamma_{\tilde{v}_i(p)}$ are geodesics in $\mathcal{T}_p$ with $\dot{\gamma}_{\tilde{v}_i(p)}(0) = \tilde{v}_i(p)$ and $\tilde{h}_i = \xi_t \cdot h_i$.

Proof: Let us first evaluate the above expression with arguments $\left(\phi^t(\gamma_{v_i(\varphi^t p)}(h_i))\right)$ instead of $\left(\gamma_{\tilde{v}_i(p)}(\xi_t h_i)\right)$. We use

$$D(\phi^t(\gamma_{v_i(\varphi^t p)}(h_i))) =$$

$$= B_t(\gamma_{v_i(\varphi^t p)}(h_i)) \cdot A_t^{-1}(\gamma_{v_i(\varphi^t p)}(h_i)) +$$

$$+ C_t(\gamma_{v_i(\varphi^t p)}(h_i)) \cdot D(\gamma_{v_i(\varphi^t p)}(h_i)) \cdot A_t^{-1}(\gamma_{v_i(\varphi^t p)}(h_i)).$$

Thus we get the following estimates:

$$\left\|\tilde{h}_2\tilde{h}_3 D(\phi^t(\gamma_{v_1(\varphi^t p)}(h_1))) + \tilde{h}_1\tilde{h}_3 D(\phi^t(\gamma_{v_2(\varphi^t p)}(h_2))) + \tilde{h}_1\tilde{h}_2 D(\phi^t(\gamma_{v_3(\varphi^t p)}(h_3)))\right\| =$$

$$= \xi_t^2 \cdot \left\|h_2 h_3 D(\phi^t(\gamma_{v_1(\varphi^t p)}(h_1))) + h_1 h_3 D(\phi^t(\gamma_{v_2(\varphi^t p)}(h_2))) + h_1 h_2 D(\phi^t(\gamma_{v_3(\varphi^t p)}(h_3)))\right\| \le$$

$$\le \xi_t^2 \cdot \left\|h_2 h_3 B_t(\gamma_{v_1(\varphi^t p)}(h_1)) A_t^{-1}(\Gamma_1) + h_1 h_3 B_t(\Gamma_2) A_t^{-1}(\Gamma_2) + h_1 h_2 B_t(\Gamma_3) A_t^{-1}(\Gamma_3)\right\| +$$

$$+ \xi_t^2 \cdot \left\|h_2 h_3 C_t(\Gamma_1) D(\Gamma_1) A_t^{-1}(\Gamma_1) + h_1 h_3 C_t(\Gamma_2) D(\Gamma_2) A_t^{-1}(\Gamma_2) + h_1 h_2 C_t(\Gamma_3) D(\Gamma_3) A_t^{-1}(\Gamma_3)\right\|$$

Here we used the shorthand Γ_i for the argument $\gamma_{v_i(\varphi^t p)}(h_i)$.

Note that the first of these two terms contains only differentiable expressions, hence is bounded by some (uniform) multiple of $h_1 h_2 h_3$. We estimate the second term by using contractiveness of C_t and A_t^{-1} to get the bound

$$L^2 \mu_2^t \nu_2^{-t} \xi_t^2 K h_1 h_2 h_3 = L^2 \mu_2^t \nu_2^{-t} \xi_t^{-1} K \tilde{h}_1 \tilde{h}_2 \tilde{h}_3 \le L^3 \mu_2^t \nu_2^{-t} \mu_1^{-t} K \tilde{h}_1 \tilde{h}_2 \tilde{h}_3 < \tfrac{1}{3} K \tilde{h}_1 \tilde{h}_2 \tilde{h}_3 \text{ for large } t.$$

For $t \in [T, 2T]$ we then choose K large enough so that the first term is also dominated by $\frac{1}{3}K\tilde{h}_1\tilde{h}_2\tilde{h}_3$. If we now pass to the argument $\left(\gamma_{\tilde{v}_i(p)}(\xi_t h_i)\right)$ instead of $\left(\phi^t(\gamma_{v_i(\varphi^t p)}(h_i))\right)$ we simply observe that the curves $\left(\gamma_{\tilde{v}_i(p)}(\xi_t s)\right)$ and $\left(\phi^t(\gamma_{v_i(\varphi^t p)}(s))\right)$ considered in these two instances are tangent at zero, so that the error term engendered is of order $o(\tilde{h}_1\tilde{h}_2\tilde{h}_3)$ and thus our estimates imply the lemma. □

Hölder continuity of the derivative

We now want to prove

Proposition 2.11: Let $\alpha > 1$, M a Riemannian manifold and φ^t a u-α-bunched Anosov flow. Then for $\epsilon > 0$ the unstable distribution is $C^{\alpha - \epsilon}$, *i.e.*, differentiable with derivative $(\alpha - \epsilon - 1)$-Hölder continuous.

Remark 2.12: Note that we can conclude that strong 1-bunching implies that the Anosov splitting is $C^{1+\epsilon}$ for some $\epsilon > 0$ and that if the unstable foliation is of codimension one then, since $\mu_1 = \mu_2$, *i.e.*, $\mu_2 \nu_2^{-1} < \mu_1^{\alpha} = \mu_1$ for $\alpha = 1$, this implies that the unstable distribution is of class $C^{1+\epsilon}$.

We already know that the unstable distribution is differentiable.

Proof: We shall be rather brief here, since the explanations relating our estimates to the result of the proposition are along the same lines as in the previous proofs. Let

$$\beta = \alpha - \epsilon - 1.$$

Then our bunching assumption is $\mu_2 \nu_2^{-1} \mu_1^{-\alpha} \leq 1$,

i.e.,

$$\mu_2 \nu_2^{-1} \mu_1^{-\beta-1-\epsilon} \leq 1.$$

We need to show that if

$$\| D_y D(y) - D_y D(0) \| \leq K \|y\|^{\beta}$$

then

$$\| D_z D(z) - D_z D(0) \| \leq K \|z\|^{\beta},$$

where z denotes $\phi^t(y)$ and D is the linear map representing E^u in our coordinates. The reader will note the uniformity in the estimates below.

We first distinguish linear and nonlinear parts by writing

$$D(y) = {}^L D(y) + {}^N D(y)$$

with ${}^{L}D(y)$ linear in y and ${}^{N}D(y)$ differentiable in y with $D_y\, {}^{N}D(0)=0$.

We use similar notation for the matrices A_t^{-1}, B_t and C_t, *i.e.*,

$$A_t^{-1}(y) = A_t^{-1}(0) + {}^{L}A_t^{-1}(y) + {}^{N}A_t^{-1}(y)$$

$$B_t(y) = {}^{L}B_t(y) + {}^{N}B_t(y)$$

$$C_t(y) = C_t(0) + {}^{L}C_t(y) + {}^{N}C_t(y).$$

Note that then our claim reduces to showing that if

$$\left\| D_y\, {}^{N}D(y) \right\| \le K \|y\|^{\beta}$$

then

$$\left\| D_z\, {}^{N}D(z) \right\| \le K \|z\|^{\beta},$$

where z denotes $\phi^t(y)$.

This is now a relatively easy computation:

$${}^{L}D(\phi^t y) + {}^{N}D(\phi^t y) =$$

$$= D(\phi^t y) =$$

$$= B_t(y)\, A_t^{-1}(y) + C_t(y)\, D(y)\, A_t^{-1}(y) =$$

$$= \left[{}^{L}B_t(y) + {}^{N}B_t(y) \right] \left[A_t^{-1}(0) + {}^{L}A_t^{-1}(y) + {}^{N}A_t^{-1}(y) \right] +$$

$$+ \left[C_t(0) + {}^{L}C_t(y) + {}^{N}C_t(y) \right] \left[{}^{L}D(y) + {}^{N}D(y) \right] \left[A_t^{-1}(0) + {}^{L}A_t^{-1}(y) + {}^{N}A_t^{-1}(y) \right] =$$

by separating linear parts

$$= \left[{}^{L}B_t(y) A_t^{-1}(0) \right] + \left[B_t(y) A_t^{-1}(y) - {}^{L}B_t(y) A_t^{-1}(0) \right] +$$

$$+ \left[C_t(0)\, {}^{L}D(y) A_t^{-1}(0) \right] + \left[C_t(y) D(y) A_t^{-1}(y) - C_t(0)\, {}^{L}D(y) A_t^{-1}(0) \right] =$$

$$= \left[B_t(y) A_t^{-1}(y) - {}^{L}B_t(y) A_t^{-1}(0) \right] + \left[C_t(y) D(y) A_t^{-1}(y) - C_t(0)\, {}^{L}D(y) A_t^{-1}(0) \right] +$$

$$+ \left[Linear\ part \right].$$

Therefore

$$^N D(\phi^t y) =$$

$$= \left[B_t(y) A_t^{-1}(y) - {}^L B_t(y) A_t^{-1}(0) \right] + \left[C_t(y) D(y) A_t^{-1}(y) - C_t(0)\, {}^L D(y) A_t^{-1}(0) \right] =$$

$$= \left[B_t(y) A_t^{-1}(y) - {}^L B_t(y) A_t^{-1}(0) \right] + \left[C_t(y)\, {}^L D(y) A_t^{-1}(y) - C_t(0)\, {}^L D(y) A_t^{-1}(0) \right] +$$

$$+ \left[C_t(y)\, {}^N D(y) A_t^{-1}(y) \right]$$

$$=: RHS$$

applying D_y to the left hand side now gives by the chain rule

$$D_z\, {}^N D(\phi^t(y))\, D_y \phi_t(y) = D_z\, {}^N D(\phi^t(y))\, C_t(y).$$

Thus
$$\left\| D_z\, {}^N D(\phi^t(y)) \right\| \leq \left\| D_y [RHS] \right\| \cdot \left\| C_t^{-1}(y) \right\|.$$

Note however that in $\left\| D_y [RHS] \right\|$ the first two terms of RHS only contribute $O(\|y\|)$. Thus (by the usual technique of enlarging constants in the end) we only consider the term $\left\| D_y \left[C_t(y)\, {}^N D(y) A_t^{-1}(y) \right] \right\|$.

Again, since ${}^N D(y)$ is Lipschitz in y we have

$$\left\| D_y \left[C_t(y)\, {}^N D(y) A_t^{-1}(y) \right] \right\| \leq \left\| C_t(y) \left[{}^N D_y D(y) \right] A_t^{-1}(y) \right\| + O(\|y\|)$$

and it suffices to control $\left\| C_t(y) \left[{}^N D_y D(y) \right] A_t^{-1}(y) \right\|$. But

$$\left\| C_t(y) \left[{}^N D_y D(y) \right] A_t^{-1}(y) \right\| \cdot \left\| C_t^{-1}(y) \right\| \leq$$

$$\leq \left\|C_t(y)\right\| \left\| {}^N D_y D(y)\right\| \left\|A_t^{-1}(y)\right\| \left\|C_t^{-1}(y)\right\| \leq$$

$$\leq \left\|C_t(y)\right\| \left\|A_t^{-1}(y)\right\| \left\|C_t^{-1}(y)\right\| K\|y\|^{\beta} \leq$$

$$\leq L\mu_2^t L\nu_2^{-t} L\mu_1^{-t} KL\mu_1^{-\beta t}\|\phi^t y\|^{\beta} \leq$$

$$\leq L^4\left[\mu_2\nu_2^{-1}\mu_1^{-1-\beta}\right]^t K\|z\|^{\beta} \leq$$

$$\leq L^4\left[\mu_2\nu_2^{-1}\mu_1^{-\alpha}\right]^t \mu_1^{\epsilon t} K\|z\|^{\beta}.$$

Thus for T large enough to make

$$L^4\mu_1^{\epsilon T} < \tfrac{1}{2}$$

and K large enough to overcome the $O(\|y\|)$-terms for $t \in [T, 2T]$, we obtain the desired result. □

This chapter is the first of those devoted to studying the absence of high regularity in insufficiently bunched symplectic systems. In this chapter we set the stage for some local calculations in a neighborhood of a periodic point. The first statement we want to prove is rather natural: At a periodic point p we can choose coordinates adapted to the symplectic structure and the stable and unstable filtrations of p. Since this is a local theorem we assume without loss of generality that we are considering an Anosov diffeomorphism of $\mathbb{R}^{2n}$ preserving a symplectic structure.

Proposition 3.1: Let $U \subset \mathbb{R}^{2n}$ be open, $0 \in U$ and $F: U \to \mathbb{R}^{2n}$ an Anosov diffeomorphism onto its image with 0 as a fixed point and preserving a symplectic form ω. Let $W_1^s \subset W_2^s \subset \ldots \subset W_n^s = W^s(0)$ be submanifolds of the stable leaf $W^s(0)$ of 0 with $0 \in W_i^s$ and $dim\, W_i^s = i$ and let $W_1^u \subset W_2^u \subset \ldots \subset W_n^u = W^u(0)$ be submanifolds of the unstable leaf $W^u(0)$ of 0 with $0 \in W_i^u$ and $dim\, W_i^u = i$.
Then there exist coordinates $(p_1, \ldots, p_n, q_1, \ldots, q_n)$ on U such that in these coordinates

i) $\omega = \sum dp_i \wedge dq_i$ (or $\omega = J := \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$, where I is the $n\times n$ identity matrix),

ii) $W_i^s = \{(p_1, \ldots, p_n, q_1, \ldots, q_n) \in U: p_1 = \ldots = p_n = q_1 = \ldots = q_{n-i} = 0\}$

iii) $W_i^u = \{(p_1, \ldots, p_n, q_1, \ldots, q_n) \in U: q_1 = \ldots = q_n = p_1 = \ldots = p_{n-i} = 0\}$

Definition 3.2: If there exists $I \subset \{1, \ldots, n\}$ such that $\{W_i^s : i \in I\}, \{W_i^u : i \in I\}$ are the stable and unstable filtrations of 0 these coordinates are called adapted coordinates.

Proof: We adapt the proof of Darboux' theorem given in Arnol'd's book [An] on Classical Mechanics.

Denote by M_i a $2i$-dimensional submanifold of U such that

$$W_i^s \subset M_i\,,\; W_i^u \subset M_i\,.$$

Let $p_1\colon U \to \mathbb{R}$ be such that $p_1|_{W^s} = 0$, $p_1|_{M_{n-1}} = 0$ and the Hamiltonian vector field P_1 (defined by $dp_1 = \omega(P_1, \cdot)$) is transversal to a hypersurface N_n containing W_n^u and M_{n-1}. If P_1^t denotes the Hamiltonian flow generated by P_1 then

$$\left(\forall z \in U\right)\left(\exists!\; y \in N_n,\; q_1(z) \in \mathbb{R}\right)\colon\; z = P_1^{q_1(z)}(y)\,.$$

Thus we have $q_1\colon U \to \mathbb{R}$ with $q_1|_{N_n} = 0$ and since dp_1 and dq_1 are independent, because the Poisson bracket $\{q_1, p_1\} = dq_1(P_1) = 1$, we find that

$$M_{n-1} = \{x \in U\colon p_1(x) = q_1(x) = 0\}\;.$$

Since $\{q_1, p_1\} = 1$ this completes the proof in case $n = 1$.

We now proceed inductively:

Denote by Q_1 the Hamiltonian vector field associated with q_1 and by Q_1^t the associated Hamiltonian flow.

<u>Claim</u>: $\omega|_{M_{n-1}}$ is nondegenerate.

This claim implies that $(M_{n-1}, \omega|_{M_{n-1}})$ is symplectic and hence, by induction hypothesis, has adapted coordinates $\{p_i, q_i\}_{i=2}^n$. We thus get coordinates on U:

$$\left(\forall z \in U\right)\left(\exists!\; y \in M_{n-1},\; s,\; t \in \mathbb{R}\right)\colon\; z = P_1^s(Q_1^t(y))\,.$$

Define $p_i(z) := p_i(y)\,,\; q_i(z) := q_i(y)\,,\; i = 2, \ldots, n\,.$

<u>Proof of claim</u>: For $x \in M_{n-1}$, $v \in T_x M_{n-1}$ we have $\omega(P_1, v) = dp_1(v) = 0$ and $\omega(Q_1, v) = dq_1(v) = 0$ since p_1 and q_1 vanish on M_{n-1}.

If $w \in T_x M_{n-1}$ is such that $(\forall v \in T_x M_{n-1})\; \omega(v, w) = 0$ then,

since $$\omega(P_1, v) = \omega(Q_1, v) = 0,$$

we have $(\forall v \in T_x U)\; \omega(v, w) = 0$, whence $w = 0$ since ω is nondegenerate. ✓

We now have to show that the coordinates $\{p_i,\ q_i\}_{i=1}^n$ are adapted.

ii) and iii) are satisfied by construction, so we conclude the proof by showing that the coordinates are symplectic. It suffices to show that $\{p_i, p_j\} = \{q_i, q_j\} = 0$ and $\{q_i, p_j\} = \delta_{ij}$.

Case 1: $i = j = 1$: Already known.

Case 2: $i = 1$ or $j = 1$: Follows from invariance of q_k and p_k under Q_1^t and P_1^t for $k \geq 2$.

Case 3: $i, j \geq 2$: q_i and p_j are invariant under Q_1^t and P_1^t, hence the corresponding Hamiltonian flows commute and thus Q_i and P_j are invariant under Q_1^t and P_1^t. Since Q_1^t and P_1^t are symplectic the Poisson brackets $\{q_i, p_j\} = \omega(Q_i, P_j)$ are invariant under Q_1^t and P_1^t and it suffices to evaluate them at points of M_{n-1}. Since q_1 and p_1 are integrals of the flows Q_i^t and P_j^t, Q_i and P_j are tangent to $M_{n-1} = \{x \in U : p_1(x) = q_1(x) = 0\}$. Thus $Q_i|_{M_{n-1}}$ and $P_j|_{M_{n-1}}$ are ω-Hamiltonian vector fields for $q_i|_{M_{n-1}}$ and $p_j|_{M_{n-1}}$ and the desired Poisson brackets are those obtained in the coordinates of M_{n-1}, which are, by induction hypothesis, as desired. □

We will from now on assume that we are using adapted coordinates.

Now we wish to pass from these adapted coordinates to special adapted coordinates that satisfy one further requirement. While the use of adapted coordinates needs little motivation, it will not be clear at this point why we need special adapted coordinates. This will only become clear when they are actually put to use. The idea, however, is, essentially, to isolate "slow" directions in some sense.

Recall the setting of the previous proposition. We define integers k, l and

m by taking $n-k$ to be the maximal dimension of elements of the stable filtration, m the minimal such dimension and $k+l+m=n$. We may of course have $l=0$. Considering points in the fast stable leaf

$$\{(p_1,\dots,p_n,q_1,\dots,q_n)\in U: p_1=\dots=p_n=q_1=\dots=q_{n-m}=0\}$$

(m being the lowest dimension of any element of the stable filtration) we can write the differential DF of the diffeomorphism F as

$$DF|_{(0,\dots,0,q_{n-m+1},\dots,q_n)}=\begin{bmatrix} A & O \\ B & C \end{bmatrix}$$

with C lower block triangular. This is because the stable filtration at p is an invariant subfiltration of

$$W_i^s=\{(p_1,\dots,p_n,q_1,\dots,q_n)\in U: p_1=\dots=p_n=q_1=\dots=q_{n-i}=0\},$$

whence the tangent spaces are subspaces of the form $\{(0,\dots,0,x_{n-j+1},\dots,x_n)\}$. Invariance of these forces the above form for $DF|_{(0,\dots,0,q_{n-m+1},\dots,q_n)}$. The block sizes correspond to the increments in dimension when passing to higher-dimensional elements of the filtration. Here we shall only use that the blocks can be taken of sizes k, l and m, *i.e.*, we think of C as

$$C=\begin{bmatrix} C_{slow} & 0 & 0 \\ * & C_{mid} & 0 \\ * & * & C_{fast} \end{bmatrix}\begin{matrix} k \\ l \\ m \end{matrix}$$
$$\qquad k \quad\; l \quad\; m$$

Furthermore $A=C^{t^{-1}}$ since F is symplectic, *i.e.*,

$$J=\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}=\begin{bmatrix} A^t & B^t \\ 0 & C^t \end{bmatrix}\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}\begin{bmatrix} A & O \\ B & C \end{bmatrix}=\begin{bmatrix} A^tB-B^tA & A^tC \\ -C^tA & 0 \end{bmatrix}.$$

This evidently implies that A is upper block triangular and the diagonal blocks are obtained from those of C by taking transposes of inverses.

We want to change our coordinate system in such a way as to have the upper left hand k-block of the matrix B identically zero at points $(0,\dots,q_{n-m+1},\dots,q_n)$. Recall that k here denotes the codimension in $W^s(0)$ of the highest-dimensional element of the filtration at 0. Simultaneously we want to maintain of course the conditions obtained in the previous proposition. We first have to do a little linear algebra in

Lemma 3.3: For $S\in gl_s(k,\mathbb{R}):=\{S\in gl(k,\mathbb{R}): S^t=S\}$ and $A\in GL(k,\mathbb{R})$ with $\|A^{-1}\|<1$ the equation $E-A^tEA=S$ has a unique solution $E\in gl_s(k,\mathbb{R})$.

Proof: $gl_s(k,\mathbb{R})\to gl_s(k,\mathbb{R})$, $E\mapsto E-A^tEA$ is linear. We need to show injectivity.

Suppose $E-A^tEA=0$, *i.e.*, $E=A^tEA$. It suffices to show $(v,Ev)=0$ for all $v\in S^{k-1}:=\{v\in\mathbb{R}^k:\|v\|=1\}$.

Let $v_0\in S^{k-1}$ be such that $|(v_0,Ev_0)|=K:=\max_{v\in S^{k-1}}|(v,Ev)|$. Since $\|A^{-1}\|<1$ there exist $\sigma>1$, $v_1\in S^{k-1}$ such that $\sigma v_0=Av_1$ whence

$$\sigma^2K=|(\sigma v_0,E\sigma v_0)|=|(Av_1,EAv_1)|=|(v_1,A^tEAv_1)|=|(v_1,Ev_1)|\le K.$$

But if $\sigma^2K\le K$ while $\sigma^2>1$ and $K\ge 0$ then $K=0$. □

Definition 3.4: Let $U\subset\mathbb{R}^{2n}$ and $F: U\to\mathbb{R}^{2n}$ be an Anosov diffeomorphism onto its image, which fixes a point $p\in U$ and preserves a symplectic form $\langle\cdot,\omega\cdot\rangle$. Then coordinates $(p_1,\dots,p_n,q_1,\dots,q_n)$ on U are called **special adapted coordinates** if in these coordinates

i) the stable filtration at p is a subfiltration of

$\{(p_1,\dots,p_n,q_1,\dots,q_n)\in U: p_1=\dots=p_n=q_1=\dots=q_{n-i}=0\}_{i=1}^n$,

ii) the unstable filtration at p is a subfiltration of

$\{(p_1,\dots,p_n,q_1,\dots,q_n)\in U: q_1=\dots=q_n=p_1=\dots=p_{n-i}=0\}_{i=1}^n$,

iii) $\omega=\sum dp_i\wedge dq_i$ (or $\omega=J:=\begin{bmatrix}0 & I\\ -I & 0\end{bmatrix}$, where I is the $n\times n$ identity matrix),

iv) The upper left hand $k\times k$ block of the matrix B defined by $DF|_{(0,\ldots,0,q_{n-m+1},\ldots,q_n)} = \begin{bmatrix} A & O \\ B & C \end{bmatrix}$ is zero for all $q_{n-m+1},\ldots,q_n$. Here $n-k$ is the dimension of the highest-dimensional element of the stable filtration at p properly contained in the stable manifold and m is the lowest dimension of elements of the stable filtration.

Proposition 3.5: Let $U\subset\mathbb{R}^{2n}$ and $F\colon U\to\mathbb{R}^{2n}$ be an Anosov diffeomorphism onto its image, which fixes a point $p\in U$ and preserves a symplectic form $<\cdot,\omega\cdot>$. Then there are special adapted coordinates for F.

Proof: Choose adapted coordinates and denote them by (s,t,u,v,w,z) with $s=(p_1,\ldots,p_k)\in\mathbb{R}^k$, $t=(p_{k+1},\ldots,p_{n-m})\in\mathbb{R}^l$, $u=(p_{n-m+1},\ldots,p_n)\in\mathbb{R}^m$, $v=(q_1,\ldots,q_k)\in\mathbb{R}^k$, $w=(q_{k+1},\ldots,q_{n-m})\in\mathbb{R}^l$, $z=(q_{n-m+1},\ldots,q_n)\in\mathbb{R}^m$.
Then, suppressing the argument $(0,\ldots,0,z)$, $DF|_{(0,\ldots,0,z)} = \begin{bmatrix} A & O \\ B & C \end{bmatrix}$ with C lower block triangular, as remarked above. The block sizes are k, l and m, where the block of size l may consist of smaller blocks or be absent if $l=0$. Note that $A = C^{t^{-1}}$ and A^tB is symmetric since F is symplectic, *i.e.*,

$$J=\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}=\begin{bmatrix} A^t & B^t \\ 0 & C^t \end{bmatrix}\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}\begin{bmatrix} A & O \\ B & C \end{bmatrix}=\begin{bmatrix} A^tB\text{-}B^tA & A^tC \\ -C^tA & 0 \end{bmatrix}.$$

Denote the upper left k-blocks of A, B, and C by a, b, and c respectively. Then a^tb is symmetric and thus $a^tb - e + a^tea = 0$ has a unique symmetric solution $e(z)\in gl(k,\mathbb{R})$ by the above lemma. Note that $e(0)=0$ since $b(0)=0$.

Let $$G(s,t,u,v,w,z) := (s,t,u-\tfrac{1}{2}D_z(s\,e(z)\,s), v+e(z)\,s, w, z),$$
where D_z denotes the differential with respect to z, or equivalently the matrix (or vector) of partial derivatives with respect to z. Then

$$DG=\begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ -D_z e(z)s & 0 & I & 0 & 0 & -\frac{1}{2}D_z D_z s e(z) s \\ e(z) & 0 & 0 & I & 0 & D_z e(z)s \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{bmatrix}$$

$$=:\begin{bmatrix} D & F \\ E & D^{t^{-1}} \end{bmatrix}.$$

Thus G is symplectic:

$$DG^t J DG = \begin{bmatrix} D^t E - E^t D & I - E^t F \\ F^t E - I & F^t D^{t^{-1}} - D^{-1} F \end{bmatrix} = J$$

since $F^t E = E^t F = 0$, and $D^{-1} F = F$ and $D^t E = E$ are symmetric. This proves iii).

G also preserves the stable filtration since if $s=0$ we have $DG=\begin{bmatrix} I & 0 \\ E & I \end{bmatrix}$.

Likewise the unstable filtration is preserved since for $z=0$ $DG=\begin{bmatrix} D & F \\ 0 & D^{t^{-1}} \end{bmatrix}$ and D is lower triangular. Thus i) and ii) are proved.

Finally, to prove iv), we consider points $(0,\dots,0,z)$ and observe that the representation of DF in the new coordinates is given by

$$DG\cdot DF\cdot DG^{-1} = \begin{bmatrix} A & 0 \\ EA+B-CE & C \end{bmatrix}.$$

Since $EA+B-CE = B - A^{t^{-1}}E + EA$ and A is upper block triangular (with blocks of size k, l and m) the upper left k-block of this expression is the product of the upper left k-blocks: $b - a^{t^{-1}}e + ea$.

But since A^t is nonsingular and $a^t b - e + a^t e a = 0$ we conclude that

$$b - a^{t^{-1}}e + ea = 0$$ as required. □

4: A necessary condition for high regularity

This chapter contains the central idea of this project. It is shown here that excessive regularity of the Anosov splitting forces a degeneracy that cannot be typical. The approach amounts to isolating the worst case that could occur in the estimates used in the proof of regularity from the bunching condition.

We consider a (transversally) symplectic Anosov flow φ^t on a compact Riemannian manifold.

Definition 4.1: A flow φ^t on a compact Riemannian manifold is said to be transversally symplectic if there is an antisymmetric 2-form ω such that ω is nondegenerate on hypersurfaces transversal to the flow.

For a periodic point $p \in M$ we can choose a small hypersurface $\mathcal{T}_p$ containing p and transversal to the flow.

Definition 4.2: For a periodic point $p \in M$ of φ^t denote by $\mathcal{T}_p$ a small hypersurface transversal to the flow. We define the induced map or return map Φ_p as follows: Φ_p is the restriction of the Poincaré return map of $\mathcal{T}$ to an open convex subset of $\mathcal{T}$ containing p such that the return time is smooth. We now denote this set by $\mathcal{T}$ or $\mathcal{T}_p$.

For purposes of illustration we can think of $\mathcal{T}_p$ as coordinatized by $\Xi: [-\epsilon, \epsilon]^{2n} \to \mathcal{T}_p$. Clearly $\Phi = \Phi_p$ is well-defined and smooth on a neighborhood of p in $\mathcal{T}_p$. Therefore we can consider its differential $D\Phi$ near p. Φ is symplectic with respect to the symplectic form ω on $\mathcal{T}_p$. Counting with multiplicities $D\Phi|_p$ has $2n$ eigenvalues, all off the unit circle since φ^t is an Anosov flow, and the set of eigenvalues is closed under the operation of taking reciprocals.

Definition 4.3: A periodic orbit p is said to be α-spread if the differential $D\Phi_p$ at p of the induced map Φ for the transversal $\mathcal{T}_p$ has eigenvalues λ_1 and λ_2 of modulus greater than one such that $1<|\lambda_2|^2<|\lambda_1|^\alpha$. We will assume $|\lambda_1|$ and $|\lambda_2|$ are the minimal and maximal moduli for eigenvalues outside the unit circle.

Proposition 4.4: Let $\alpha\in(0,2]$. Assume φ^t has an α-spread periodic orbit with $|\lambda_1|\neq|\lambda_2|^2$ and the unstable distribution is Hölder continuous with Hölder exponent α.
Then the unstable directions satisfy a geometric horizontality condition of positive codimension.

Remark 4.5: Here the case $\alpha>1$ implies a $C^{1+\epsilon}$-assumption. The expansion rates at the periodic point are $\nu_1=|\lambda_1|$ and $\nu_2=|\lambda_2|$.

Proof: We choose special adapted coordinates on $\mathcal{T}_p$ as obtained previously. From now on our discussion will take place in these coordinates.

By taking intersections of weak unstable leaves of the flow with $\mathcal{T}_p$ and passing to tangent directions again, we obtain a distribution on $\mathcal{T}_p$, which we also refer to as the unstable distribution. Near 0 (*i.e.*, near p) these directions are almost horizontal and thus we can represent these two-dimensional affine subspaces as the graphs of affine maps. Dropping the constant part, we retain the linear part $D\in M_{n,n}$. The unstable direction is thus represented as the graph of the linear map D or, equivalently, as the image of the linear map

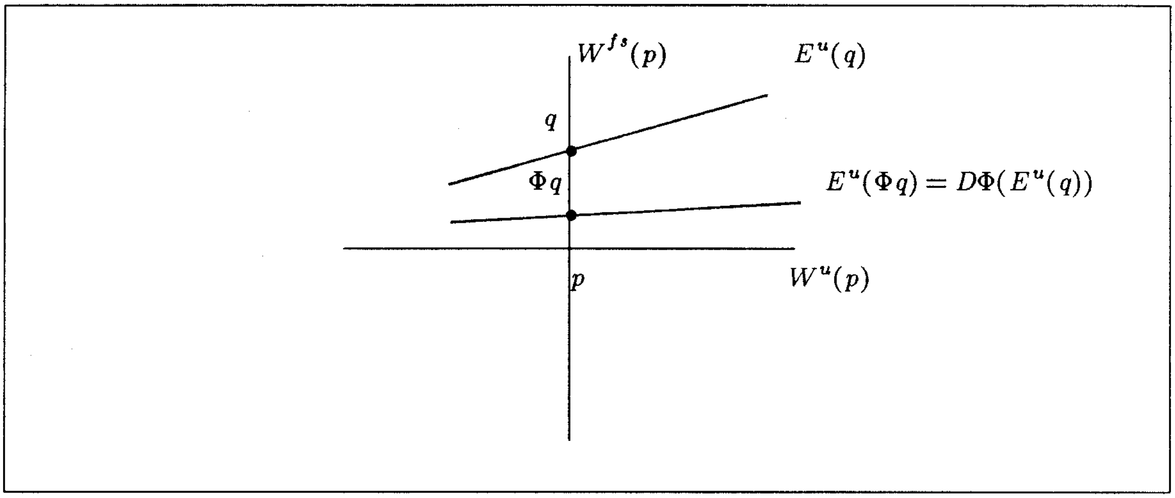
$$\begin{bmatrix} I \\ D \end{bmatrix}:\mathbb{R}^n\simeq\mathbb{R}^n\times\{(0,0)\}\to\mathbb{R}^{2n}.$$

We will be particularly interested in points $q=(0,\dots,0,z)$ on the fast stable leaf through 0 (*i.e.*, p). At these points the differential of Φ takes a special form.

Indeed when we obtained special adapted coordinates we saw that

$$D\Phi|_{(0,\ldots,0,z)} = \begin{bmatrix} A & 0 \\ B & C \end{bmatrix}$$

with C lower block triangular and $A^{-1} = C^{t}$.



The action of $D\Phi$ on $\begin{bmatrix} I \\ D \end{bmatrix}$ gives

$$\begin{bmatrix} A & 0 \\ B & C \end{bmatrix}\begin{bmatrix} I \\ D \end{bmatrix} = \begin{bmatrix} A \\ B + CD \end{bmatrix}.$$

To bring this into the form $\begin{bmatrix} I \\ D' \end{bmatrix}$ again we multiply from the right with $A^{-1} = C^{t}$ and obtain

$$\begin{bmatrix} I \\ D' \end{bmatrix} = \Phi^{*}\begin{bmatrix} I \\ D \end{bmatrix} = \begin{bmatrix} A \\ B + CD \end{bmatrix} A^{-1} = \begin{bmatrix} I \\ (B + CD)\,A^{-1} \end{bmatrix} = \begin{bmatrix} I \\ (B + CD)\,C^{t} \end{bmatrix}.$$

Now recall that the unstable distribution is flow invariant. Therefore $D' = D \circ \Phi$ and we just showed that

$$D(\Phi(0,\ldots,0,z)) = D' = (B + CD)\,C^{t} = (B + CD(0,\ldots,0,z))\,C^{t}.$$

Denote the upper left k-blocks of A, B, and C by a, b, and c respectively. Now, recalling that $b(0,\ldots,0,z) = 0$ and that C is lower block triangular, we compute the upper left k-block of this matrix equation. We get

$$d' = c \cdot d \cdot c^{t}$$

or
$$d(\Phi(0,\dots,0,z)) = c(0,\dots,0,z) \cdot d(0,\dots,0,z) \cdot c^{t}(0,\dots,0,z).$$

If we let $z_n := \Phi^n(0,0,0,z)$ we see that

$$d(z_n) = \left[\prod_{i=0}^{n-1} c(z_{n-i-1})\right] \cdot d(z_0) \cdot \left[\prod_{i=0}^{n-1} c^{t}(z_i)\right].$$

Observe that what we have accomplished here is to isolate the slowest contraction by virtue of our special adapted coordinates.

<u>Case 1</u>: Suppose that $\alpha \in (0,1]$ and that $d(\cdot)$ is Hölder continuous at 0 with Hölder exponent α. For $\delta > 0$ such that $(\nu_2+\delta)^2 < \nu_1^{\alpha}$ we choose a norm on $\mathbb{R}^k$ with respect to which $\|c^{-1}(0)\| \le \nu_2 + \delta$. Since c is Lipschitz continuous

$$\|c^{-1}(z)\| \le (\nu_2+\delta) \cdot (1 + C_1\|z\|).$$

Note also that $\|z_n\| \le C_2 \nu_1^{-n}$ and $\|d(z_n)\| \le C_3 \|z_n\|^{\alpha}$. But then

$$\begin{aligned} \|d(z_0)\| &\le \left[\prod_{i=0}^{n-1} \|c^{-1}(z_i)\|\right] \|d(z_n)\| \left[\prod_{i=0}^{n-1} \|c^{t^{-1}}(z_{n-i-1})\|\right] \\ &\le C_4 (\nu_2+\delta)^{2n} \nu_1^{-\alpha n} \prod_{i=0}^{n-1}(1 + C_5 \nu_1^{-i}) \\ &\le C_6 (\nu_2+\delta)^{2n} \nu_1^{-\alpha n} \\ &\to 0. \end{aligned}$$

Here we used that

$$\begin{aligned} \prod_{i=0}^{n-1}(1 + C_5\nu_1^{-i}) &= exp \sum_{i=0}^{n-1} log(1 + C_5 \nu_1^{-i}) \\ &\le exp \sum_{i=0}^{n-1} C_5 \nu_1^{-i} \\ &\le exp \sum_{i=0}^{\infty} C_5 \nu_1^{-i} \le \infty \end{aligned}$$

and
$$(\nu_2+\delta)^{2n} \nu_1^{-\alpha n} = \left[(\nu_2+\delta)^2 \nu_1^{-\alpha}\right]^n < 1.$$

Thus
$$\underline{\underline{d(z) \equiv 0}}.$$

<u>Case</u> <u>2</u>: $\alpha > 1$ and $\nu_1 < \nu_2^2$. Differentiating the recursion relation

$$d(\Phi(0,\ldots,0,z)) = c(0,\ldots,0,z)\cdot d(0,\ldots,0,z)\cdot c^{\mathrm{t}}(0,\ldots,0,z)$$

at 0 with respect to z gives $D_z d|_0\, D_z\Phi|_0 = c(0)\, D_z d|_0\, c^{\mathrm{t}}(0)$,

since
$$d(0,\ldots,0) = 0.$$

Thus
$$\|D_z d|_0\| \le \|D_z d|_0\|\, \|c(0)\| \|c^{\mathrm{t}}(0)\| \|D_z\Phi|_0^{-1}\| \le \|D_z d|_0\| \frac{\nu_1+\delta}{(\nu_2-\delta)^2}$$

by choosing a norm on $\mathbb{R}^k$ similarly to the choice made in case 1.

But since $\frac{\nu_1}{\nu_2^2} < 1$ we have for sufficiently small δ that $\frac{\nu_1+\delta}{(\nu_2-\delta)^2} < 1$ and thus

$$\|D_z d|_0\| = 0\,.$$

This, however, implies that if Dd has Hölder exponent $\alpha - 1$ then by the mean value theorem $\|d(z_n)\| \le C\|z_n\|^{\alpha}$. By the estimates in case 1 this again forces $d(z) \equiv 0$. □

<u>Remark</u> <u>4.6</u>: We know for dynamical reasons that the unstable directions are Lagrangian subspaces: Take two vectors x, y in an unstable subspace. Then $|\omega(x,y)| = |\omega(\varphi^t x, \varphi^t y)| \le \|\omega\|\cdot\|\varphi^t x\|\cdot\|\varphi^t y\| \le \nu_2^{-2t}\|\omega\|\cdot\|x\|\cdot\|y\| \to 0$. On the other hand this property has nothing to do with the condition obtained here. Indeed, since the pullback of ω to the subspace parametrized by $\begin{bmatrix} I \\ D \end{bmatrix}$ is given by $\begin{bmatrix} I \\ D \end{bmatrix}^{\mathrm{t}} \cdot \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \cdot \begin{bmatrix} I \\ D \end{bmatrix} = D - D^{\mathrm{t}}$, D represents a Lagrangian subspace iff D is symmetric. In other words the condition obtained here is not an obvious identity arising from the symplectic structure. Of course in the next chapter we will show that indeed this condition can be violated by (symplectic) perturbations.

The condition
$$\underline{\underline{d(z) \equiv 0}} \qquad \text{(H)}$$
can be thought of as a kind of horizontality condition on unstable directions. It says that along the fast stable leaf of p some part of D (having to do with slow contraction) is horizontal in our coordinates.

5: Genericity of low regularity

In this chapter we first prove a dynamical lemma that will help in designing the "right" perturbations for our purposes. We can then complete the proof of the main theorem.

Definition 5.1: Let φ^t be a flow on a topological space M. A point $x \in M$ is called negatively nonrecurrent if there exists an open neighborhood U of x in M such that $t_o := max\{t<0 \colon \varphi^t x \notin U\} > -\infty$ and $U \cap \bigcup_{t<t_o} \varphi^t x = \emptyset$.

Proposition 5.2: Let M be a Riemannian manifold and φ^t an Anosov flow with a periodic point $p \in M$. Then the fast stable leaf of p contains a negatively nonrecurrent point.

Remark 5.3: The fast stable leaf of p is the submanifold of the stable manifold of p that consists of points whose orbits approach p with maximal speed. The proof only uses that it is a submanifold of the stable leaf of p.

Proof: There exist C, $\chi > 0$ such that $\|D\varphi^t|_{E^s}\| \le C \cdot e^{-\chi t}$ for t$>$0. Denote by $\mathfrak{T}$ a hypersurface through p transversal to the flow direction and by $W^{fs}(p)$ the fast stable leaf of p for the return map Φ.

Let $q \in W^{fs}(p)$ and $B_1 = B_{\sqrt{\epsilon}}(q)$ where $\epsilon > 0$ is such that

$$T := inf\,\{t > t_o \colon \varphi^{-t} B_{\sqrt{\epsilon}}(q) \cap B_{\sqrt{\epsilon}}(q) \neq \emptyset\} > \frac{1}{\chi} \cdot log 5C,$$

where $$t_o := inf\,\{t > 0 \colon \varphi^{-t} B_{\sqrt{\epsilon}}(q) \cap B_{\sqrt{\epsilon}}(q) = \emptyset\};$$

i.e., $Ce^{-\chi T} < \frac{1}{5}$. Here $B_\epsilon(q)$ denotes the ϵ-ball around q in M.

From now on denote by $W^{fs}(p)$ the connected component containing p of the intersection of B_1 and the fast stable leaf of p and by $W^s(x)$ the connected component containing x of the intersection of B_1 and the stable leaf of x for

$x \in B_1$. For ϵ sufficiently small $B_2 := B_1 \cap \mathcal{T}$ has local product structure. (That is, local stable and unstable leaves intersect in exactly one point, so that we can introduce coordinates as follows: Coordinatize $W^s(p)$ and $W^u(p)$ and then for $x \in B_2$ define the coordinates of x as

$$(\textit{coordinates of } W^u(x) \cap W^s(p), \textit{ coordinates of } W^s(x) \cap W^u(p)).)$$

Let D be an ϵ-ball in the intersection of $\mathcal{T}$ and the weak unstable leaf of q and let

$$U_1 := \bigcup_{x \in D} B^s_\epsilon(x)$$

$$V_1 := \bigcup_{x \in D} \overline{B^s_{2\epsilon}(x)}$$

$$W_1 := \bigcup_{x \in D} B^s_{3\epsilon}(x)$$

where $\overline{}$ denotes closure and $B^s_\epsilon(x)$ denotes the ϵ-ball in $W^s(x)$, $x \in B_1$. For $x \in B_1$ let

$$U_x := U_1 \cap W^s(x),$$

$$V_x := V_1 \cap W^s(x),$$

$$W_x := W_1 \cap W^s(x),$$

$$S_x := V_x \setminus U_x.$$

Note that the S_x are spherical shells in $W^s(x)$.

It suffices to find a point in $U := U_q \cap W^{fs}(p)$ that does not return to U_1 in negative time.

Define

$$t\colon B_2 \to \mathbb{R}^+ \cup \{\infty\},\ x \mapsto t(x) := \inf\{t > 0 : \varphi^{-t}x \in \overline{U_1}\}$$

and take $x_0 \in U_2 := \overline{U}$ such that

$$t(x_0) = \min_{x_0 \in U_2} t(x) > T.$$

There is a smooth function $\tau \colon W_2 := W_{\varphi^{-t(x_0)}x_0} \to \mathbb{R}^+$ such that

$$\psi_1(x) := \varphi^{\tau(x)}x \in \mathfrak{T} \text{ for } x \in W_2$$

and
$$\tau(\varphi^{-t(x_0)}x_0) = t(x_0).$$

Thus $\psi_1 \colon W_2 \to \mathfrak{T}$ is a diffeomorphism onto its image and

$$\psi(W_2) \subset W^s(p) = W^s(x_0).$$

The intersection $U \cap S_1$ of the spherical shell

$$S_1 = \psi_1(S_{\varphi^{-t(x_0)}x_0}) \subset W^s(p)$$

with U consists of points **not** returning to U_1 for time t with

$$-T_1 := -\max_{x \in V_2} t(x) \leq t < 0,$$

where $V_2 := V_{\varphi^{-t(x_0)}x_0}$.

Claim: $U \cap S_1$ has a connected component U_3' such that $(\psi_1(W_2) \backslash S_1) \cup U_3'$ is connected. (That is, $U_q \cap W^{fs}(p)$ traverses the spherical shell S_1 from the inside to the outside.)

Proof: For ϵ small enough U is connected and has diameter at least ϵ. By the choice of T the diameter of S_1 is at most $\frac{4}{5} \cdot \epsilon$. Thus U contains points on the outside of the shell.

Since $x_0 \in U_2$ is on the inside of the shell there are points of U on the inside of the shell and the claim follows. (We showed that U contains points of both components of $\psi(W_2) \backslash S_1$.) ✓

Take $U_3 \subset U_3'$ closed and connected such that $(\psi_1(W_2) \backslash S_1) \cup U_3$ is connected. Note that $\psi_1^{-1}(U_3)$ connects the complement of $S_{\varphi^{-t(x_0)}x_0}$ in W_2 and is itself connected. In particular $diam(\psi_1^{-1}(U_3)) \geq \epsilon$.

Define
$$t \colon W_2 \to \mathbb{R}^+ \cup \{\infty\}, \; x \mapsto t(x) := inf\{t > 0 : \varphi^{-t}x \in \overline{U_1}\}$$

and take
$$x_1 \in \psi_1^{-1}(U_3)$$
such that
$$t(x_1) = \min_{x \in \psi_1^{-1}(U_3)} t(x) > T.$$

There is a smooth function $\tau: W_3 := W_{\varphi^{-t(x_1)}x_1} \to \mathbb{R}^+$ such that
$$\psi_2(x) := \varphi^{\tau(x)} x \in \mathcal{T} \text{ for } x \in W_3$$
and
$$\tau(\varphi^{-t(x_1)}x_1) = t(x_1).$$
Thus $\psi_2: W_3 \to \mathcal{T}$ is a diffeomorphism onto its image and $\psi(W_3) \subset W^s(x_1)$.

The spherical shell
$$S_2 := \psi_2(S_{\varphi^{-t(x_1)}x_1}) \subset W^s(x_1)$$
consists of points **not** returning to U_1 for time t with
$$-T_2 := -\max_{x \in V_3} t(x) \leq t < 0,$$
where
$$V_3 := V_{\varphi^{-t(x_1)}x_1}.$$

__**Claim:**__ $\psi_1^{-1}(U_3) \cap S_2$ has a connected component U_4' such that $(\psi_2(W_3) \setminus S_2) \cup U_4'$ is connected.

__**Proof:**__ As before. We use that $\psi_1^{-1}(U_3)$ is connected and $diam(\psi_1^{-1}(U_3)) \geq \epsilon$.✓

Note that $U_4 := \psi_1(U_4') \subset U_3$ consists of points not returning to U_1 for time t with
$$-T_1 - T_2 \leq t < 0$$
and that
$$T_i > T.$$
Iterating this argument gives a decreasing sequence $\{U_i\}_{i=3}^{\infty}$ of compact sets where U_i has return times beyond $(i-2) \cdot T$. The intersection is then nonempty and contained in U and consists of negatively nonrecurrent points. □

After these preliminaries we proceed to the proof of the main theorem.

Theorem 5.4: Let $\alpha \in (0,2)$. Denote by $\mathfrak{S}_\alpha$ the set of (transversally) symplectic Anosov flows that possess an α-spread periodic orbit p. (With the C^∞-topology as on page 22.)
Then flows in $\mathfrak{S}_\alpha$ generically do not have unstable distributions with Hölder exponent α.

Remark 5.5: By the next Lemma we first reduce the proof of the main theorem to showing that the condition (**H**) obtained in the previous chapter is generically violated.

Definition 5.6: A periodic orbit is said to have a 2-1-resonance if the maximal modulus of an eigenvalue of the return map to a transversal is the square of the minimal modulus of an eigenvalue of absolute value larger than one.

Remark 5.7: This is not the common usage of the term resonance.

Lemma 5.8: Flows in $\mathfrak{S}_\alpha$ generically have an α-spread periodic orbit without a 2-1-resonance.

Proof: It clearly suffices to exhibit perturbations destroying the resonance.

Consider a periodic point p with a 2-1-resonance and introduce adapted coordinates on a small transversal hypersurface as in the previous chapter. Let B_ϵ be an ϵ-ball about 0 in these coordinates and coordinatize the flow box $B := \bigcup_{\tau=0}^{\epsilon} \varphi^\tau B_\epsilon$ as $[0,\epsilon] \times B_\epsilon$ so that the flow φ^τ is represented by $(s,x) \mapsto (s+\tau, x)$, *i.e.*, the identity flow.

Sublemma: For $0 \le \tau \le \epsilon^{k+1}$ there exist symplectic maps $G_\tau : B_\epsilon \to B_\epsilon$ such that

$G_\tau \in C^\infty$,

G_τ is C^1-close to $id: B_\epsilon \to B_\epsilon$,

$G_\tau|_{B_\epsilon \setminus B_{\epsilon(1-\epsilon)}} = id$ and

$G_\tau(s,t,u,v,w,z) = (e^{\epsilon^{k+1}\tau}s, t, e^{-\epsilon^{k+1}\tau}u, e^{-\epsilon^{k+1}\tau}v, w, e^{\epsilon^{k+1}\tau}z)$ on B_{ϵ^3}.

We first show how this completes the proof of the Lemma.

Let $\eta \in C^\infty(\mathbb{R}, \mathbb{R}^+)$ be such that

$$\eta(x) = 0 \qquad (x<0)$$
$$\eta(x) = \epsilon^{k+1} \qquad (x>0)$$
$$\eta \text{ is } C^k\text{-small}$$

and on $B = [0,\epsilon]\times B_\epsilon$ redefine the flow $\varphi^\tau \simeq (s,x) \mapsto (s+\tau, x)$ so that

$$\overline{\varphi}^\tau(0,x) = (\tau, G_{\eta(\tau)}(x)).$$

This flow $\overline{\varphi}^\tau$ is a symplectic C^k-small perturbation of the original flow. It is easy to see that it has no 2-1-resonance: Near the origin we have

$$DG_\tau = \begin{bmatrix} e^{\epsilon^{k+1}\tau I} & & & & & \\ & I & & & \mathbf{O} & \\ & & e^{-\epsilon^{k+1}\tau I} & & & \\ & & & e^{-\epsilon^{k+1}\tau I} & & \\ & \mathbf{O} & & & I & \\ & & & & & e^{\epsilon^{k+1}\tau I} \end{bmatrix}$$

Thus the differential of the return map for the flow $\overline{\varphi}^\tau$ does not have the resonance, since its maximal and minimal eigenvalues outside the unit circle are multiplied by $e^{\epsilon^{2k+2}}$ and $e^{-\epsilon^{2k+2}}$, respectively, so their ratio is changed by a factor of $e^{2\epsilon^{2k+2}}$.

Proof of sublemma: We use a method devised by Moser that is exhibited in the proof of Darboux' theorem in [AM].
(This method is also the hat from which we pulled special adapted coordinates.)

Define $\overline{\gamma}(s,t,u,v,w,z) = \langle s,v\rangle - \langle u,z\rangle$ and $\gamma = \rho\overline{\gamma}$ where $\rho \in C^\infty(B_\epsilon, \mathbb{R})$ is such that

$$\rho \text{ is } C^k\text{-small},$$
$$\rho = \epsilon^{k+1} \quad \text{on } B_{\epsilon^2} \text{ and}$$
$$\rho = 0 \quad \text{on } B_\epsilon \setminus B_{\epsilon(1-\epsilon)},$$

and $\langle\cdot,\cdot\rangle$ denotes the standard inner products in $\mathbb{R}^k$ and $\mathbb{R}^m$.

Define a vector field X by $\omega(X,\cdot) = d\gamma$.
Since X is smooth with compact support it generates a complete flow G_τ.
Claim: The flow G_τ is symplectic.
Proof of claim: $G_0^*\omega = \omega$ and $\frac{d}{d\tau}(G_\tau^*\omega) = G_\tau^*(\pounds_X\omega) = G_\tau^* d(\omega(X,\cdot)) = G_\tau^* d^2\gamma = 0$, where $\pounds$ denotes the Lie derivative.

Finally, in order to study the maps G_τ near the origin we integrate the vector field $\overline{X}$ given by $\omega(\overline{X},\cdot) = d\overline{\gamma}$, which coincides with $\epsilon^{k+1}X$ near 0. We find that $\overline{X}(s,t,u,v,w,z) \doteq (s,0,-u,-v,0,z)^t$ and thus evidently G_τ is as desired. □

Proof of Theorem 5.4: It suffices to prove that the horizontality condition (**H**) of the previous chapter is generically violated.

Density:

We first prove density, *i.e.*, we want to show that if the condition (**H**) of the previous chapter is satisfied then there are arbitrarily small perturbations of the flow for which this condition is violated.

We pick a negatively nonrecurrent point q in the fast stable leaf of the periodic point p. We can in fact choose it such that it lies inside the neighborhood of p on which we have special adapted coordinates. Translate the coordinates so that q is the origin. We now construct a perturbation in a way similar to that used in the proof of Lemma 5.8.

Define $\overline{\gamma}(s,t,u,v,w,z)=\frac{1}{2}<s,s>=\frac{1}{2}\|s\|^2$ and $\gamma=\rho\overline{\gamma}$ where $\rho\in C^\infty(B_\epsilon,\mathbb{R})$ is such that

$$\begin{aligned} &\rho \text{ is } C^k\text{-small,} \\ &\rho=\epsilon^{k+1} \qquad \text{on } B_{\epsilon^2} \text{ and} \\ &\rho=0 \qquad \text{on } B_\epsilon\setminus B_{\epsilon(1-\epsilon)}, \end{aligned}$$

and $<\cdot,\cdot>$ denotes the standard inner product in $\mathbb{R}^k$.

Define a vector field X by $\omega(X,\cdot)=d\gamma$.

Since X is smooth with compact support it generates a complete flow G_τ.

__Claim__: The flow G_τ is symplectic.

__Proof of claim__: $G_0^*\omega=\omega$ and $\frac{d}{d\tau}(G_\tau^*\omega)=G_\tau^*(\pounds_X\omega)=G_\tau^*d(\omega(X,\cdot))=G_\tau^*d^2\gamma=0$, where $\pounds$ denotes the Lie derivative.

Finally, in order to study the maps G_τ near the origin we integrate the vector field $\overline{X}$ given by $\omega(\overline{X},\cdot)=d\overline{\gamma}$, which coincides with $\epsilon^{k+1}X$ near 0. We find that $\overline{X}(s,t,u,v,w,z)=(0,0,0,s,0,0)^{\mathrm{t}}$ and thus the induced flow $\overline{G}_\tau$ is given by $\overline{G}_\tau(s,t,u,v,w,z)=(s,t,u,v+\tau s,w,z)$. Consequently

$$D\overline{G}_\tau=\begin{bmatrix} \multicolumn{3}{c}{\mathbf{I}} & \mathbf{O} \\ I & 0 & 0 & \\ 0 & 0 & 0 & \mathbf{I} \\ 0 & 0 & 0 & \end{bmatrix}$$

$$=: \begin{bmatrix} I & 0 \\ E & I \end{bmatrix}$$

Let $\eta \in C^\infty(\mathbb{R}, \mathbb{R}^+)$ be such that

$$\eta(x) = 0 \qquad (x<0)$$

$$\eta(x) = \epsilon^{k+1} \qquad (x>0)$$

$$\eta \text{ is } C^k\text{-small}$$

and on $B = [0,\epsilon] \times B_\epsilon$ redefine the flow $\varphi^\tau \simeq (s,x) \mapsto (s+\tau, x)$ so that

$$\overline{\varphi}^\tau(0,x) = \big(\tau, G_{\eta(\tau)}(x)\big).$$

This flow $\overline{\varphi}^\tau$ is a symplectic C^k-small perturbation of the original flow. It is easy to see that it causes the condition (**H**) to be violated: Applying $D\overline{G}_\tau$ as above to $\begin{bmatrix} I \\ D \end{bmatrix}$ gives

$$\begin{bmatrix} I & 0 \\ E & I \end{bmatrix} \begin{bmatrix} I \\ D \end{bmatrix} = \begin{bmatrix} I \\ E+D \end{bmatrix}$$

and this means that the condition (**H**) is now violated because if D satisfies it, $E+D$ does not. Note, however, that we still have to show the

<u>**Claim:**</u> $E+D$ is the unstable direction at q for the flow $\overline{\varphi}^\tau$.

<u>**Proof:**</u> Here the choice of q as a nonrecurrent point is important.

The unstable direction at $\varphi^{-\epsilon} q$ can be computed as follows:

Take any distribution $\mathcal{E}$ along $\{\varphi^t(q)\}_{t<0}$, which is close to E^u. Then

$$\lim_{t \to \infty} D\varphi^t(\mathcal{E}(\varphi^{-t-\epsilon}(q)) = E^u(\varphi^{-\epsilon} q).$$

Since q is nonrecurrent this limit is not affected by the perturbation. Thus $E^u(\varphi^{-\epsilon} q)$ and $\overline{E}^u(\varphi^{-\epsilon} q)$ coincide. By invariance of $\overline{E}^u$ under $\overline{\varphi}^t$ the claim follows.

Openness:

We complete the proof by showing that the condition (**H**) is violated on an open set of flows in $\mathfrak{S}_\alpha$. We need to prove that if the condition (**H**) is violated for a flow φ^t then it is violated for all sufficiently close flows.

Let $\varphi^t \in \mathfrak{S}_\alpha$ be such that the condition (**H**) is violated. Consider a small perturbation $\overline{\varphi}^t \in \mathfrak{S}_\alpha$. Recall first of all that on page 22 the C^∞-topology on $\mathfrak{S}_\alpha$ was defined as follows:

We say that $\varphi_n \to \varphi \in \mathfrak{S}_\alpha$ if $\left(\forall\, \epsilon > 0,\ k \in \mathbf{N}\right)\left(\exists\, N \in \mathbf{N}\right)\left(\forall n \geq N\right)$

$$\|\varphi_n - \varphi\|_{C^k} < \epsilon \text{ and } \|\omega_n - \omega\|_{C^1} < \epsilon,$$

where ω_n and ω denote invariant transversal symplectic forms for φ_n and φ respectively.

Since the invariant symplectic forms are not unique (one can, *e.g.*, rescale them) the closeness condition for them is intended to say that ω_n and ω can be chosen so as to be close.

In order to show openness we observe that the condition (**H**) is an open condition in a fixed system of special adapted coordinates. Thus we are done once we show:

Lemma 5.9: If φ_n is C^k-close to φ then special adapted coordinates at p' are C^1-close to those at p.

Remark 5.10: Here p' denotes the periodic orbit of φ_n near p.

Proof: It suffices to show that adapted coordinates are close since the coordinate change to special adapted coordinates depends smoothly on the representation of the return map in adapted coordinates.

In order to show closeness of adapted coordinates it suffices to show that the filtrations at p' are C^1-close to those at p, since the construction of adapted coordinates depends continuously on the filtration and symplectic form.

But this in turn follows from standard contraction arguments, *i.e.*, the arguments used to construct stable leaves of periodic points. In this case the statement can be extracted from the arguments in [A1], Lemmata 9.2 and 9.3. The argument is as follows:

In a local calculation the local unstable filtrations are obtained by a Hadamard-argument using invariant cones. The data at our disposal here are, via local coordinates, ϕ and ϕ_n, two C^k-close diffeomorphisms of $\mathbb{R}^{2n}$ with 0 as a hyperbolic fixed point and ϕ_n C^k-close to ϕ. Since the root spaces of ϕ_n are close to those of ϕ (after possibly consolidating several into one), the corresponding elements of the filtration for ϕ and ϕ_n are tangent to each other. But since the Hadamard-argument gives uniform estimates on the derivatives this forces corresponding elements to be C^1-close. □

This chapter describes work that is complete except for explicitly performing some geometric calculations that are not points of mathematical difficulty but technical work.

Geodesic flows are an obvious target for application of this theory. In fact they were among the motivations for this study. The main difference is that in this case we want to make statements about perturbations of metrics. As far as the phase space picture is concerned, local perturbations of a Riemannian metric on N are in the phase space $M = SN$ much less localized than the perturbations considered here. An equivalent way of seeing the difficulty is the following: When constructing the perturbation in proving density of flows with Anosov splitting of low regularity we needed to produce some nonrecurrence. In the setting of a geodesic flow the appropriate concept of nonrecurrence is that of a geodesic that does not return to a certain neighborhood in the configuration space N. In general it is not clear whether such a geodesic can be found.

Note however that in our proof of existence of a negatively nonrecurrent point the only ingredient that is not available in the configuration space setting is the fact that one can make return times to a neighborhood large by shrinking the neighborhood. Recall that the proof there began as follows:

There exist C, $\chi > 0$ such that $\|D\varphi^t|_{E^s}\| \leq C \cdot e^{-\chi t}$ for t > 0.
Let $q \in W^{fs}(p)$ and $B_1 = B_{\sqrt{\epsilon}}(q)$ where $\epsilon > 0$ is such that

$$T := inf\,\{t > t_o : \varphi^{-t} B_{\sqrt{\epsilon}}(q) \cap B_{\sqrt{\epsilon}}(q) \neq \emptyset\} > \frac{1}{\chi} \cdot log 5C,$$

where $t_o := inf\,\{t > 0 : \varphi^{-t} B_{\sqrt{\epsilon}}(q) \cap B_{\sqrt{\epsilon}}(q) = \emptyset\}$;
i.e., $Ce^{-\chi T} < \frac{1}{5}$. Here $B_\epsilon(q)$ denotes the ϵ-ball around q in M.

In the case of a geodesic flow where we want to avoid returns to a neighborhood in the configuration space, this is not obviously so. There may be geodesics that are closed piecewise smooth loops, *i.e.*, geodesics returning to the initial point but with a different direction. In other words, sufficiently long return times to the patch where the Riemannian metric is perturbed cannot be expected. Before describing one setting where this problem can be avoided and that yields some interesting insights, let us point out that the argument given here for existence of perturbations is not the only one possible. Another method, (which yields perturbation results in a less fine topology than the C^∞–topology) only requires the existence of an orbit that has long return times in a somewhat average sense. (It involves controlling errors accumulating in encounters with the perturbed neighborhood.) It may be possible to find a geodesic with such properties in the configuration space. (Using that same argument one might also, by the way, attempt to design the perturbation of the metric in such a way that the errors introduced when the geodesic does return are only "large" if the geodesic crosses the perturbed patch in a direction almost parallel to the geodesic corresponding to $\bigcup_{t=-\epsilon}^{0} \varphi^t q$. This could conceivably make the issue of return time obsolete.)

Nice targets for application to geodesic flows are symmetric spaces of nonconstant negative curvature. (For example, the complex hyperbolic space.) These are symmetric spaces of negative curvature with minimal curvature -4 and maximal curvature -1. At every periodic geodesic the return map has expansion rates e and e^2. So while the system has smooth Anosov splitting due to its algebraic nature, its dynamical parameters are at the threshold for breakdown of differentiability of the Anosov splitting: they are 1-bunched, but no more. The primary reason this case is tractable with our methods is the following: A paper by **Borel** [B] has the following consequence:

Proposition 6.1 [B]: A compact locally symmetric space S of negative sectional curvature has finite covers with arbitrarily large injectivity radius.

(I am much indebted to Masahiko Kanai for pointing out this fact to me.)

Thus we can consider a compact quotient of S and pass to a finite cover with injectivity radius larger than the return times required in the nonrecurrence argument. In other words, we take the fairly crude approach of using the injectivity radius as a bound for return times. By transitivity of the isometry group every periodic orbit has expansion rates ϵ and ϵ^2. We can thus perturb the metric so as to obtain a 2-spread periodic orbit. (The proof of the corresponding statement in the category of Anosov flows is given above, when we "destroy a 2-1-resonance.") Due to the large return times guaranteed by the large injectivity radius, we can find a negatively nonrecurrent geodesic as we did before in the flow category. A local perturbation of the metric that produces the "twist" causing the condition (**H**) to be violated then completes the proof of

Theorem 6.2: Any compact locally symmetric space S of nonconstant negative sectional curvature has a finite cover admitting perturbations of the metric whose geodesic flows have horospheric foliations with modulus of continuity $1-\epsilon$ for some ϵ.

Perturbations here are in the C^∞-topology. Note that the proof given here is complete except that the local perturbations of the metric have not been given explicitly.

There are more reasons why it is worthwhile to study symmetric spaces. Aside from being very intriguing examples in which to observe breakdown of smoothness, the fact that fast stable directions are defined everywhere (and also for small perturbations) might yield some insights into the currently poorly understood nature of the geometric condition (**H**) obtained earlier. The rather homogeneous structure of symmetric spaces and the universal presence of fast stable leaves (even for perturbations) may be useful in trying to develop a global picture of this condition.

There is the possibility that there are no nontrivial perturbations of symmetric spaces of nonconstant curvature, (*e.g.*, the complex hyperbolic space) which still have C^1 Anosov splitting. Thus we have here some motivation to ask:

Question 6.3: Are symmetric spaces of nonconstant negative sectional curvature rigid in the category of manifolds with C^1 Anosov splitting?

Note however that this would imply by the regularity theory developed in earlier chapters, that no perturbation of the symmetric spaces considered is strongly 1-bunched. This is a priori not a very natural supposition.

The techniques developed here can also be applied to manifolds of constant negative curvature. In this case one perturbs to get a $(2-\epsilon)$-spread periodic geodesic and applies the machinery to show

Theorem 6.4: Any compact manifold S of constant negative sectional curvature has a finite cover admitting perturbations of the metric whose geodesic flows have horospheric foliations of regularity below $C^{2-\epsilon}$ for some $\epsilon > 0$.

7: A new description of the Margulis measure

In this chapter we show that the Margulis measure for Anosov flows arises from a Hausdorff measure for a natural distance on unstable leaves. This chapter generalizes and simplifies work of **Hamenstädt** [Hs] that was done in the setting of geodesic flows and used for her rigidity theory.

The Margulis measure is a measure maximizing entropy. Entropy is a word for two invariants of dynamical systems measuring their complexity or disorder. For a thorough introduction we recommend the book by **Walters** [W].

Measure theoretic or metric entropy is a notion in measurable dynamics and information theory developed by **Kolmogorov**. (One explanation for the term "metric" could be that it is shorter than and contained in measure theoretic.) It measures how rapidly sets become independent under the dynamics or growth of information under iteration.

Topological entropy, invented by **Adler, Konheim** and **McAndrew**, is an imitation of metric entropy in topological terms, using open sets rather than measurable ones. It can be thought of as measuring how many distinguishable orbits there are when one performs observations of limited accuracy over a finite stretch of time.

There is an interesting connection between these two concepts of entropy: topological entropy is never smaller than measure theoretic entropy (for any invariant probability measure). In fact, topological entropy is the supremum of all measure theoretic entropies taken over all invariant probability measures. In the case of Anosov systems, topological entropy is the maximum of measure theoretic entropies and there is a unique measure of maximal entropy. Let us emphasize this statement again:

Variational Principle: Topological entropy is the supremum of measure theoretic entropies. For Anosov systems, there is a unique measure of maximal entropy.

Definition: The unique measure of maximal entropy is called the Bowen-Margulis measure.

This measure was obtained independently in two very different ways by **Bowen** and **Margulis**. While Bowen [Bw1] constructed this measure as a measure for which periodic orbits are equidistributed and showed that it maximizes measure theoretic entropy, Margulis [M] constructed a measure with the property that its conditionals on unstable and stable leaves dilate uniformly in time. To be more precise, he obtained measures on unstable and stable manifolds and built from them a global measure in a local weighted product construction. Let us paraphrase the procedure to give a clearer idea.

Let $\varphi^t: M \to M$ be an Anosov flow, which has an invariant probability measure with a smooth density with respect to Lebesgue measure. Using the Riemannian structure one can define conditional measures on all unstable and stable manifolds. For the sake of this argument let us assume that we start with (Riemannian) Lebesgue measures on the leaves, *i.e.*, measures given by the volume form of the Riemannian metric on each leaf.

Denote this measure on $W^{su}(z)$ by λ_z. Pick $z_o \in M$ and $S_o \in W^{su}(z_o)$ open with compact closure. For $z \in M$, $S \subset W^{su}(z)$ measurable define

$$\mu_z(S) := \lim_{\tau\to\infty} \frac{\lambda_{\varphi^\tau(z)}(\varphi^\tau(S))}{\lambda_{\varphi^\tau(z_o)}(\varphi^\tau(S_o))}.$$

An important part of Margulis' paper is to show that this limit exists.

If we now define $\quad f(t) := \mu_{\varphi^t(z_o)}(\varphi^t(S_o))$

then

$$\mu_{\varphi^t(z)}(\varphi^t(S)) = \lim_{\tau\to\infty} \frac{\lambda_{\varphi^{t+\tau}(z)}(\varphi^{t+\tau}(S))}{\lambda_{\varphi^\tau(z_o)}(\varphi^\tau(S_o))} =$$

$$= \lim_{\tau\to\infty} \frac{\lambda_{\varphi^{t+\tau}(z_o)}(\varphi^{t+\tau}(S_o))}{\lambda_{\varphi^\tau(z_o)}(\varphi^\tau(S_o))} \cdot \frac{\lambda_{\varphi^{t+\tau}(z)}(\varphi^{t+\tau}(S))}{\lambda_{\varphi^{t+\tau}(z_o)}(\varphi^{t+\tau}(S_o))} =$$

$$= \lim_{\tau\to\infty} \frac{\lambda_{\varphi^{t+\tau}(z_o)}(\varphi^{t+\tau}(S_o))}{\lambda_{\varphi^\tau(z_o)}(\varphi^\tau(S_o))} \cdot \lim_{\tau\to\infty} \frac{\lambda_{\varphi^{t+\tau}(z)}(\varphi^{t+\tau}(S))}{\lambda_{\varphi^{t+\tau}(z_o)}(\varphi^{t+\tau}(S_o))} =$$

$$= \lim_{\tau\to\infty} \frac{\lambda_{\varphi^{t+\tau}(z_o)}(\varphi^{t+\tau}(S_o))}{\lambda_{\varphi^\tau(z_o)}(\varphi^\tau(S_o))} \cdot \lim_{\tau\to\infty} \frac{\lambda_{\varphi^{\tau}(z)}(\varphi^{\tau}(S))}{\lambda_{\varphi^{\tau}(z_o)}(\varphi^{\tau}(S_o))} =$$

$$= \mu_{\varphi^t(z_o)}(\varphi^t(S_o)) \cdot \mu_z(S) =$$

$$= f(t) \cdot \mu_z(S).$$

In particular for $z = z_o$, $S = S_o$ this implies

$$f(t+s) = \mu_{\varphi^{t+s}(z_o)}(\varphi^{t+s}(S_o)) =$$

$$= \mu_{\varphi^t(\varphi^s(z_o))}(\varphi^t(\varphi^s(S_o))) =$$

$$= f(t) \cdot \mu_{\varphi^s(z_o)}(\varphi^s(S_o)) =$$

$$= f(t) \cdot f(s).$$

Thus we jump to the conclusion that $f(t) = e^{dt}$. Sinai [S] showed that $d = h =$ topological entropy. Therefore

$$\mu \circ \varphi^t = e^{ht}\mu.$$

This essentially characterizes the Margulis measure, as we shall refer to it, since we only use and redescribe the construction of Margulis.

Let M be a compact Riemannian manifold and $\varphi^t: M \to M$ a C^2 Anosov flow. On $W^{su}(z)$ we define a distance η_z and a spherical measure $\sigma_z = \sigma$, which expand uniformly under the flow. σ is equivalent to the conditional **Margulis measure** [Hs, M, S] and for a Lyapunov metric σ equals the conditional Margulis measure on every leaf $W^{su}(z)$.

In order to have consistent notation for this chapter we recall the

<u>Definition</u> <u>7.1</u> [A]: A flow $\varphi^t: M \to M$ is called an **Anosov flow** if $\dot{\varphi} \neq 0$ and the tangent bundle is a Whitney sum $TM = E^{su} \oplus E^{ss} \oplus E^{\varphi}$, where $E^{\varphi} = < \dot{\varphi} >$ is generated by $\dot{\varphi}$ and there are $a > 0$, $b \geq 1$ so that

$$\left.\begin{array}{l} \|D\varphi^t u\| \leq b \cdot \|u\| \cdot e^{at} \text{ for } t \leq 0,\ u \in E^{su} \\ \|D\varphi^t v\| \leq b \cdot \|v\| \cdot e^{-at} \text{ for } t \geq 0,\ v \in E^{ss} \end{array}\right\} \qquad (\mathbf{A})$$

<u>Remark</u> <u>7.2</u>: Recall also that the distributions E^{su}, E^{ss}, $E^u := E^{su} \oplus E^{\varphi}$ and $E^s := E^{ss} \oplus E^{\varphi}$ are tangent to the foliations W^{su}, W^{ss}, W^u and W^s respectively, which are continuous in the C^1-topology. We will use here that being a smooth injectively immersed submanifold, every unstable leaf $W^{su}(z)$ has a distance d_z (and thus notions of openness and compactness) induced by Riemannian lengths of curves in $W^{su}(z)$.

There is a natural correspondence between points on two nearby unstable leaves:

<u>Definition</u> <u>7.3</u> [M]: $S \subset W^{su}(z)$ and $S' \subset W^{su}(z')$ are called ϵ**-equivalent** if there is a continuous $\phi: S\times[0,1] \to M$ so that:

$\phi(\,\cdot\,,0) = id$,

$\psi := \phi(\,\cdot\,,1): S \to S'$ is a homeomorphism,

$\phi(x,[0,1]) \subset W^{s}(x)$ and is of length less than ϵ for all $x \in S$.

<u>Remark</u> <u>7.4</u>: This is related to the local product structure briefly described in an earlier chapter (page 56). ψ is sometimes called the holonomy map. It is as regular as the foliation W^{s}. In the introduction we pointed out that the regularity of ψ can be used to define the regularity of W^{s}.

<u>Lemma</u> <u>7.5</u> [A]: After possibly changing a there exists a Riemannian metric on M, equivalent to the given metric and called a **Lyapunov adapted metric**, such that (**A**) holds with $b = 1$.

<u>Definition</u> <u>7.6</u> [Hs]: Fix $R \in \mathbb{R}$. For x, y $\in W^{su}(z)$ let

$$\eta(x,y) := \eta_{z,R}(x,y) := e^{-sup\ \{t \in \mathbb{R}:\ d_{\varphi^t(z)}(\varphi^t(x),\varphi^t(y)) \leq R\}}.$$

<u>Remark</u> <u>7.7</u>: Several properties of η are evident:

$\eta \circ \varphi^t = e^t \cdot \eta$ since we change the time parameter,

$\eta_{z',R} = \eta_{z,R}$ for $z' \in W^{su}(z)$ since the same is true for d_z, and finally

$\eta \geq 0$, $\eta(x,y) = \eta(y,x)$ and $\eta(x,y) = 0$ *iff* $x = y$.

Thus it looks as if η is a distance.

<u>Lemma</u> <u>7.8</u>: For x_1, x_2, $y \in W^{su}(z)$ and a, b as in (**A**) we have

$$\tfrac{1}{b}\big(\eta(x_1,x_2)\big)^a \leq \big(\eta(x_1,y)\big)^a + \big(\eta(x_2,y)\big)^a.$$

<u>Remark</u> <u>7.9</u>: Thus η^a is a distance if M is given a Lyapunov metric.

Proof: Let $t = -\log\eta(x_1,x_2)$ and $r_i = d_{\varphi^t(z)}(\varphi^t x_i, \varphi^t y)$ for $i = 1, 2$.
If any $r_i > Rb$ then $\eta^a(x_i, y) > \eta^a(x_1, x_2) > \frac{1}{b}\eta^a(x_1, x_2)$ and we are done.
Thus assume that both $r_i < Rb$ for $i = 1, 2$.

Claim: Then $\eta^a(x_i,y) \geq e^{-at} r_i / Rb$.

Proof of claim: It suffices to show that

$$d_{\varphi^{t+\tau}(z)}(\varphi^{t+\tau}x_i, \varphi^{t+\tau}y) > R \text{ for } \tau > \frac{1}{a}\cdot\log\frac{Rb}{r_i} > 0$$

since in that case $\qquad \eta^a(x_i,y) \geq (e^{-t-\tau})^a = e^{-at} r_i / Rb.$

But if $\gamma \subset W^{su}(\varphi^{t+\tau}z)$ is a curve joining $\varphi^{t+\tau}x_i$ and $\varphi^{t+\tau}y$ and

$$l(\gamma) = d_{\varphi^{t+\tau}(z)}(\varphi^{t+\tau}x_i, \varphi^{t+\tau}y),$$

then by (**A**) $r_i \leq l(\varphi^{-\tau}\circ\gamma) \leq b\cdot e^{-a\tau}\cdot l(\gamma) = b\cdot e^{-a\tau}\cdot d_{\varphi^{t+\tau}(z)}(\varphi^{t+\tau}x_i, \varphi^{t+\tau}y)$

and thus $\qquad d_{\varphi^{t+\tau}(z)}(\varphi^{t+\tau}x_i, \varphi^{t+\tau}y) \geq \frac{r_i}{b}e^{a\tau} > \frac{r_i}{b}\frac{Rb}{r_i} = R \qquad$ as claimed. ✓

But using the claim now we obtain

$$\begin{aligned} \eta^a(x_1,x_2)/b &= e^{-at}/b && \text{since } \eta^a(x_1,x_2) = e^{-at} \\ &\leq e^{-at}(r_1 + r_2)/Rb && \text{since } r_1 + r_2 \geq d_{\varphi^t(z)}(\varphi^t x_1, \varphi^t x_2) = R \\ &\leq \eta^a(x_1,y) + \eta^a(x_2,y). && \text{since } \eta^a(x_i,y) \geq e^{-at}r_i/Rb. \end{aligned}$$ □

Lemma 7.10: Omitting z in the subscript, we have

$$\eta_R \leq \eta_r \leq \left[\frac{Rb}{r}\right]^{\frac{1}{a}}\cdot\eta_R \text{ for } 0 < r \leq R.$$

Proof: Clearly $\eta_R \leq \eta_r$.

On the other hand $d_{\varphi^t(z)}(\varphi^t x, \varphi^t y) = r$ for $t = -\log\eta_r(x,y)$.

Thus for $\tau > \frac{1}{a}\cdot\log\frac{Rb}{r} > 0$ we get $d_{\varphi^{t+\tau}(z)}(\varphi^{t+\tau}x, \varphi^{t+\tau}y) > R$ as above and hence

$$\eta_R(x,y) \geq e^{-t-\tau} = \eta_r(x,y)\,e^{-\tau} > \eta_r(x,y)\left[\frac{r}{Rb}\right]^{\frac{1}{a}}.$$ □

Definition 7.11 [Hs]: For $S \subset W^{su}(z)$ let

$$\sigma_\epsilon(S) := inf\Big\{ \sum_{j=1}^{\infty} \epsilon_j^h \colon S \subset \bigcup_{j=1}^{\infty} B_{\eta_z}(x_j, \epsilon_j) \text{ with } x_j \in W^{su}(z) \text{ and } \epsilon_j \leq \epsilon \Big\}$$

and

$$\sigma(S) := \sigma_z(S) := \sup_{\epsilon > 0} \sigma_\epsilon(S).$$

Here h is topological entropy and $B_\eta(x_j, \epsilon_j)$ are ϵ_j–balls with respect to η around x_j.

Remark 7.12: σ is the h/a-dimensional **spherical measure** [Fd] on $W^{su}(z)$ arising from η^a. $\sigma_{\varphi^t(z)} \circ \varphi^t = e^{ht} \cdot \sigma_z$ and σ is Borel regular, *i.e.*,

$$\sigma(S) = sup\ \{\sigma(C) \colon C \subset S \text{ compact}\} \qquad \text{(see [Fd]).}$$

Lemma 7.13: If σ is constructed from a Lyapunov metric then for $\delta > 0$ there is an $\epsilon > 0$ so that if $S \subset W^{su}(z)$ and $S' \subset W^{su}(z')$ are ϵ–equivalent then

$$(1 - \delta) \cdot \sigma(S) < \sigma(S') < (1 + \delta) \cdot \sigma(S).$$

Proof: We will use that if $\{x\}$ and $\{x'\}$ are ϵ–equivalent and $\{x''\} := W^u(x') \cap W^{ss}(x)$ then $x'' = \varphi^\tau x'$ for some $\tau \in \mathbb{R}$, so there is a $C \in \mathbb{R}$ such that $\varphi^t x$ and $\varphi^t x'$ are $C \cdot \epsilon$–equivalent for $t > 0$.

For $\delta_j < \delta < 1$ there is a cover $B_{\eta_z}(x_j, \delta_j)$ of $S \subset \bigcup_{j=1}^{\infty} B_{\eta_z}(x_j, \delta_j)$ so that $\sum_{j=1}^{\infty} \delta_j^h \leq \sigma(\mathrm{S}) + \delta$.

Claim: $S' \subset \bigcup_{j=1}^{\infty} B_{\eta_{\psi(z)}}(\psi(x_j), \iota(\epsilon) \cdot \delta_j)$ where ι is such that $\lim_{\epsilon \to 0} \iota(\epsilon) = 1$.

Proof: If $\mathrm{y} \in \mathrm{S} \cap \mathrm{B}_{\eta_z}(\mathrm{x}, \delta)$ then $\mathrm{d}_{\varphi^{-\log\delta}(z)}(\varphi^{-\log\delta}\mathrm{x}, \varphi^{-\log\delta}\mathrm{y}) < R$.

By uniform continuity of E^{su} this implies

$$d_{\psi(\varphi^{-\log\delta}(z))}(\psi(\varphi^{-\log\delta}x), \psi(\varphi^{-\log\delta}y)) < R + \theta(C\epsilon)$$

with $\lim_{\epsilon \to 0} \theta(\epsilon) = 0$. But then we conclude by Lemma 7.10 that

$$\eta_{\psi(z)}(\psi(\mathrm{x}), \psi(\mathrm{y})) < \iota(\epsilon) \cdot \delta$$

where

$$\iota(\epsilon) := \left[\frac{R + \theta(C\epsilon)}{R}\right]^{\frac{1}{a}}.$$

Therefore $\psi(S \cap B_{\eta_z}(x_j, \delta_j)) \subset S' \cap B_{\eta_{\psi(z)}}(\psi(x_j), \iota(\epsilon)\delta_j)$ and the claim follows. ✓

Thus $\sigma(S') \leq \iota(\epsilon)^h \cdot \sigma(S)$. The other inequality follows similarly. □

Lemma 7.14 [M, S]: For a transitive C^2 Anosov flow φ^t we can construct the **Margulis measure** μ.

Its restriction μ^u_z to $W^{su}(z)$ has the following properties:

I) μ^u_z is positive on open and finite on compact sets.

II) $\mu^u_{\varphi^t(z)} \circ \varphi^t = e^{ht} \cdot \mu^u_z$

III) for $\delta > 0$ there exists $\epsilon > 0$ such that if $S \subset W^{su}(z)$ and $S' \subset W^{su}(z')$ are ϵ-equivalent then

$$(1-\delta) \cdot \mu^{su}(S) < \mu^{su}(S') < (1+\delta) \cdot \mu^{su}(S)$$

IV) μ^{su} is Borel regular

V) the μ^{su} are defined up to a global constant (not just up to a constant on each leaf).

Remark 7.15: III) is related to what is called "holonomy invariance" - which characterizes the Margulis measure by [BM]. We do not use this fact.

Lemma 7.16: There exist $0 < \alpha_1 < \alpha_2 < \infty$ such that

$$\alpha_1 \cdot \epsilon^h < \mu^{su}(B_\eta(x,\epsilon)) < \alpha_2 \cdot \epsilon^h \qquad \text{for all } x \in M.$$

Proof: Suppose $\mu^{su}(B_\eta(x_i,1)) \to 0$ for some $\{x_i\}_{i=1}^{\infty} \subset M$. By compactness of M we may assume that $x_i \to x$. For i large $S = \overline{B}_\eta(x,\frac{1}{2})$ is ϵ-equivalent to some $S' \subset B_\eta(x_i,1)$ and

$$\mu^{su}(B_\eta(x_i,1)) \geq \mu^{su}(S') \geq \tfrac{1}{2} \cdot \mu^{su}(S) > 0$$

by Lemma 7.14, a contradiction. So $0 < \alpha_1 < \mu^{su}(B_\eta(x,1))$.

$\mu^{su}(B_\eta(x,1)) < \alpha_2 < \infty$ is shown similarly.

The claim now follows, since

$$\mu^{su}(B_\eta(x,\epsilon)) = \mu^{su}(\varphi^{\log \epsilon}(B_\eta(\varphi^{-\log \epsilon}x,1))) = \epsilon^h \cdot \mu^{su}(B_\eta(\varphi^{-\log \epsilon}x,1)). \qquad \square$$

Lemma 7.17: $\alpha_2^{-1} \cdot \mu^{su} \le \sigma \le (2b)^{\frac{h}{a}} \cdot \alpha_1^{-1} \cdot \mu^{su}$.

Proof: 1) Let $S \subset W^{su}(z)$. By definition of σ_ϵ there is a covering by balls $B_\eta(x_j,\epsilon_j)$ with $\epsilon_j \le \epsilon$, $x_j \in W^{su}(z)$ and

$$S \subset \bigcup_{j=1}^{\infty} B_\eta(x_j,\epsilon_j),$$

such that
$$\sigma_\epsilon(S) + \delta \ge \sum_{j=1}^{\infty} \epsilon_j^h.$$

But then
$$\sigma_\epsilon(S) + \delta \ge \sum_{j=1}^{\infty} \epsilon_j^h \ge \alpha_2^{-1} \cdot \sum_{j=1}^{\infty} \mu^{su}(B_\eta(x_j,\epsilon_j)) \ge \alpha_2^{-1} \cdot \mu^{su}(S).$$

The left inequality follows when we let $\delta \to 0$ and $\epsilon \to 0$.

2) By Borel-regularity of σ and μ^{su} we can assume that $S \subset W^{su}(z)$ is compact.

Let $\epsilon > 0$ and $S_\epsilon = \{x \in W^{su}(z)\colon \exists y \in S\colon \eta(x,y) < \epsilon/(2b)^{1/a}\}$.

Suppose $\{x_j\}_{j=1}^m \subset S$ is a maximal subset so that the $B_\eta(x_j,\epsilon/(2b)^{1/a})$ are pairwise disjoint. Since $S \subset \bigcup_{j=1}^{m} B_\eta(x_j,\epsilon)$ by Lemma 7.8, we have

$$\sigma_\epsilon(S) \le \sum_{j=1}^{m} \epsilon^h = (2b)^{h/a} \sum_{j=1}^{m} \left[\frac{\epsilon}{a(2b)^{1/a}}\right]^h$$

$$\le (2b)^{h/a} \cdot \alpha_1^{-1} \cdot \sum_{j=1}^{m} \mu^{su}(B_\eta(x_j,\epsilon/(2b)^{1/a})) \qquad \text{by Lemma 7.16}$$

$$\le (2b)^{h/a} \cdot \alpha_1^{-1} \cdot \mu^{su}(S_\epsilon) \text{by disjointness of } B_\eta(x_j,\epsilon/(2b)^{1/a}) \subset S_\epsilon.$$

Letting $\epsilon \to 0$ gives $\sigma(S) \le (2b)^{h/a} \cdot \alpha_1^{-1} \cdot \mu^{su}(S)$ which is the second inequality. □

Theorem 7.18: Let M be a compact C^2-manifold and φ^t a transitive Anosov flow. Equip M with a Lyapunov metric. Then after normalization the measures σ of Definition 7.11 agree with the conditionals of the Margulis measure on every leaf.

Proof: σ has measurable densities $f_z: W^{su}(z) \to \mathbb{R}$ with respect to μ^{su}.

$f: M \to \mathbb{R}$, $z \mapsto f_z(z)$ is measurable by Lemmata 7.13 and 7.14.

f is φ^t-invariant since $\mu^{su} \circ \varphi^t = e^{ht} \cdot \mu^{su}$ and $\sigma \circ \varphi^t = e^{ht} \cdot \sigma$.

Since μ is ergodic $f \equiv const.$ $\mu - a.e.$

By Lemmata 7.13 and 7.14, we can normalize f so that $f_z \equiv 1$ $\mu^{su} - a.e.$ on each leaf $W^{su}(z)$. □

Remark 7.19: 1) If M carries an arbitrary Riemannian metric then after normalization the measure σ agrees with the conditionals of the Margulis measure on μ-almost every leaf. The reason is that the above proof still goes through because f is measurable since $(x,y,z,R) \mapsto \eta_{z,R}(x,y)$ is lower semicontinuous.

2) The above results are also true for the h-dimensional Hausdorff measure [Fd].

3) These results apply directly to geodesic flows since by [HIH] the standard metric on SM is a Lyapunov metric.

4) It is interesting to compare this construction with the one in [Bw2].

8: References

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