Large N Dualities in Topological String Theory

Thesis by

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Abstract

We investigate the phenomenon of large N duality in topological string theory from three different perspectives: worldsheets, matrix models, and melting crystals.

In the first part, we utilize the technique of mirror symmetry to generalize the worldsheet derivation of the duality, originally given by Ooguri and Vafa for the A-model on the conifold, to the A-model on more general geometries. We also explain how the Landau-Ginzburg models can be used to perform the worldsheet derivation of the B-model large N dualities.

In the second part, we consider a class of A-model large N dualities where the open string theory reduces through the Chern-Simons theory on a lens space to a matrix model. We compute and compare the matrix model spectral curve and the Calabi-Yau geometry mirror to the closed string geometry, confirming the predictions of the duality.

Finally in the third part, we propose a crystal model that describes the A-model on the resolved conifold. This is a generalization of the crystal for $\mathbf{C}^3$. We also consider a novel unitary matrix model for the Chern-Simons theory on the three-sphere and show how the crystal model for the resolved conifold is derived from the matrix model. Certain non-compact D-branes are naturally incorporated into the crystal and the matrix model.

Contents

Chapter 1

Introduction

Conventional quantum field theories such as QED or the Weinberg-Salem model are defined by their Lagrangians and the perturbation theories follow from them. A Feynman diagram consists of worldlines in spacetime of particles meeting at interaction points (Figure 1.1 (a)). The amplitude of each diagram is computed according to definite rules. In contrast to these field theories, the construction of any string theory begins by defining the S-matrix elements perturbatively without specifying the Lagrangian. A Feynman diagram of string theory is a surface swept by strings propagating in spacetime, i.e., the worldsheet of strings (Figure 1.1 (b)). Each diagram is assigned an amplitude from which the S-matrix element is constructed. Each S-matrix element has a quantum loop expansion in the coupling g_s, where the larger the genus (i.e., the number of handles of the surface) becomes, the higher the order of the quantum correction is. The amplitude of a diagram is the integral, over the moduli space of the surface, of a correlation function evaluated in a 2-dimensional conformal field theory (CFT). (For superstrings, we also need to sum over the spin structures.)

In bosonic string theory, we use the non-supersymmetric CFT based on maps from the

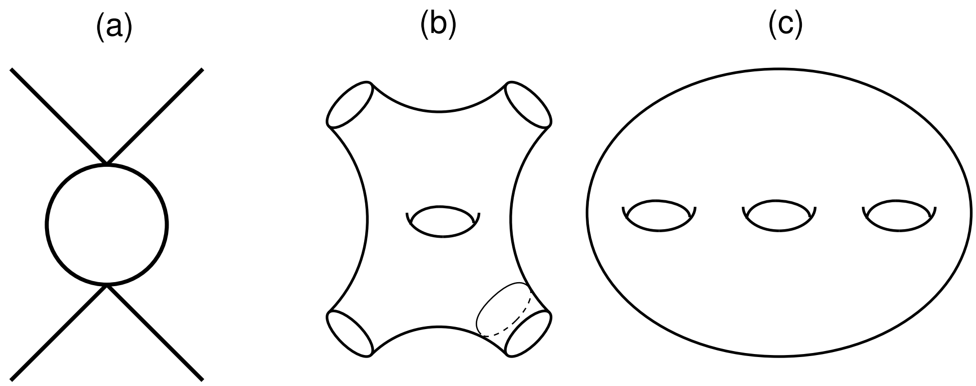


Figure 1.1: (a) In a usual quantum field theory, a Feynman diagram consists of worldlines of particles. (b) A string theory Feynman diagram is a worldsheet of strings moving around in spacetime. (c) In topological string theory, we consider vacuum amplitudes without external strings.

surface to spacetime. Quantum consistency requires that the dimension of spacetime is 26. The models obtained in this way have unstable vacua and are unrealistic.

In superstring theory, the CFT is also based on maps from the surface to spacetime, but we take the supersymmetric extension of the CFT. We find five types of consistent superstring theories with stable vacua in a flat metric background. These are type IIA, type IIB, type I, heterotic $SO(32)$, and heterotic $E_8 \times E_8$ string theories. All of them require 10 spacetime dimensions. In order to have a 4-dimensional physics out of string theory we need to compactify the extra 6 dimensions. This is typically done by choosing the 6 dimensions of spacetime to be a Calabi-Yau manifold. We say that the internal CFT is a non-linear sigma model with a Calabi-Yau target space. Models based on the heterotic $E_8 \times E_8$ theory comes rather close to the standard model of particle physics and has been studied extensively since the mid '80s. However, type IIA/B string theories have also been widely investigated because of their rich physics and their applications to 4-dimensional supersymmetric gauge theories. Over the last couple of years, phenomenological models based on the compactification of type IIA/B string theories with various p-form fluxes or/and intersecting branes have been enthusiastically discussed as a new way to connect string theory to the world in which we actually live.

Topological string theory is a yet another kind of string theory, which has a close relationship to type IIA/B string theories. As a CFT on the worldsheet, we choose a twisted version of the non-linear sigma model with a Calabi-Yau target space. After twisting the theory fermions on the worldsheet are no longer spinors, having acquired integer spins. There are two ways to twist, and the resulting theories are called A-model and B-model. The string theory based on a twisted non-linear sigma model is topological in the sense that there is no degree of freedom propagating in spacetime. The natural objects to compute are therefore vacuum amplitudes, i.e., diagrams without external strings attached (Figure 1.1 (c)). It turns out that these amplitudes of topological string theory conveniently package important terms in the effective action of type IIA/B theory compactified on the Calabi-Yau manifold.

The relevance to the 4-dimensional physics of type IIA/B string theories is the major physical motivation for studying topological string theory. There are at least two other good reasons to be interested in topological strings. Topological string theory provides simplified models of string theory, quantum gravity, and gauge theory. We will see that old ideas about quantum gravity, like quantum foam, are concretely realized in the context of topological string theory. Topological string theory is also an extremely nice framework to investigate the structure of topology and geometry of spaces such as Calabi-Yau manifolds. There are many issues, mirror symmetry and Gromov-Witten/Donaldson-Thomas correspondence to name a few, that have attracted vast attention from mathematicians. Developments of topological string theory owe much to interactions of physicists with mathematicians.

String theory can be regarded as a quantum field theory with an infinite number of fields arising from excitation modes of a string. There have been various efforts made to write down the Lagrangians for string theories, with varying degrees of success depending on the model. This approach is called string field theory. In general, string field theory is more successful for open strings than for closed strings.

String field theory is particularly tractable when applied to open topological string theory due to the absence of propagating degrees of freedom. Chern-Simons gauge theory is an example of string field theory for open topological strings, and will be discussed throughout this thesis.

String field theory applied to closed topological strings is a topological model of quantum gravity. In Chapter 4, we will discuss a recently proposed formulation of topological quantum gravity.

The large N duality is an old concept in theoretical physics, proposed by 't Hooft in 1974 [1]. Let us consider a $U(N)$ gauge theory whose fields are all matrices $(M^i{}_j)$. Assume that the Lagrangian is schematically of the form

$$L \sim \frac{1}{\lambda}\left(\mathrm{Tr}M^2 + \mathrm{Tr}M^3 + \mathrm{Tr}M^4 + ...\right), \tag{1.0.1}$$

where numerical factors and spacetime integrals are suppressed. Let us pay a special attention to the contraction of indices. This is done by representing a propagator by a double line (Figure 1.2 (a)), and an interaction vertex as in Figure 1.2 (b). A Feynman diagram

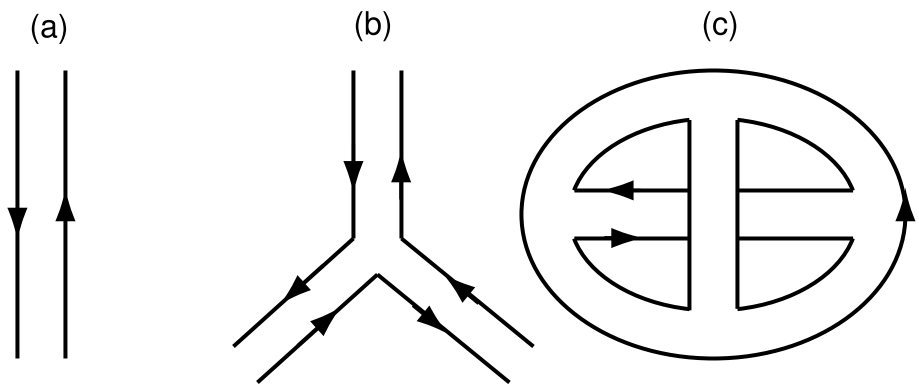


Figure 1.2: (a) A propagator is represented by a double line with orientation specified by arrows. (b) An interaction vertex is drawn in a way that the index contraction is manifest. (c) A typical Feynman diagram in the double line scheme.

looks like Figure 1.2 (c). Any such double line diagram can be drawn on a compact oriented surface of some topology. The genus of the surface is related to the number e of propagators, the number v of vertices, and the number h of closed paths (index loops) in the diagram by

$$2 - 2g = v - e + h. \tag{1.0.2}$$

Let us consider the dependence of a Feynman diagram on the coupling constant λ and N. From eq. (1.0.1), we see that a propagator contributes λ while each interaction vertex contributes λ^{-1}. The exponent in the power of N is the number h of index loops. Thus a

Feynman diagram is proportional to

$$\lambda^{e-v}N^h = \lambda^{2g+h-2}N^h = \lambda^{2g-2}t^h, \tag{1.0.3}$$

where $t = \lambda N$ is called the 't Hooft parameter. The vacuum amplitude is obtained by summing all such diagrams, and therefore has the structure

$$\begin{aligned} F_{\text{gauge}}(\lambda, N) &= \sum_{g=0}^{\infty}\sum_{h=1}^{\infty} F_{g,h}\lambda^{2g+h-2}N^h \\ &= \sum_{g=0}^{\infty}\lambda^{2g-2}\left(\sum_{h=1}^{\infty}F_{g,h}t^h\right) \\ &= \sum_{g=0}^{\infty}\lambda^{2g-2}F_g(t), \end{aligned} \tag{1.0.4}$$

where we have defined

$$F_g(t) := \sum_{h=1}^{\infty} F_{g,h}t^h. \tag{1.0.5}$$

Eq. (1.0.4) has the structure of an amplitude in a closed string theory. This argument was originally made by 't Hooft to suggest that for large N, QCD may be approximated by a closed string theory.

As a gauge theory, it is possible to take an open string field theory. Then the double line Feynman diagrams are precisely the open string worldsheets.

Our argument suggests that an open string theory may be equivalent to some closed string theory. The important examples of this *large N duality*, or *open/closed duality*, include the Matrix theory formulation of M-theory in the discrete lightcone quantization, AdS/CFT correspondence, and double scaled matrix models for 2-dimensional string theories.

In this thesis, we investigate the large N dualities in topological string theory from three different perspectives. In the following sections in this chapter we review the basics of topological string theory and the prototype example of a topological string large N duality proposed by Gopakumar and Vafa.

In chapter 2, we will study the dualities from the worldsheet point of view. The conformal field theories on the worldsheet relevant for the topological large N duality can be realized as the infrared limit of certain field theory models, the gauged linear sigma model, and the Landau-Ginzburg model. Ooguri and Vafa showed how the duality of Gopakumar and Vafa can be derived by a dynamical generation of boundaries on the closed string worldsheets. We will be able to extend the worldsheet derivation of the large N duality to other A-models in more general geometries by using mirror symmetry. We also extend the derivation to B-model large N dualities. Along the way, we uncover how the intricate boundary conditions like the Wilson line carried by a D-brane are manifested on the worldsheet theories of closed strings.

In chapter 3, we will describe a test of a class of large N dualities. In this class of large N dualities, the open string theory reduces to the Chern-Simons gauge theory on a lens space $S^3/\mathbf{Z}_p$. The Chern-Simons theory further reduces to an integral of $N \times N$ matrices, whose potential is Gaussian but whose measure is a non-compact modification of the unitary matrix measure. The dynamics of eigenvalues in a matrix model are in the large N limit captured by a 2-dimensional surface called the spectral curve. Large N duality predicts a precise relation between the spectral curve and the geometry in which closed strings propagate. We will compute both the spectral curve of the matrix model and the geometry for closed string propagation and confirm the prediction. This provides a quantitative test of this class of large N dualities.

In a recent development, topological string theory has been reformulated in terms of statistical models. In chapter 4, we will propose a new statistical model for the topological A-model on the resolved conifold. We will also present a new unitary matrix formulation of the Chern-Simons theory on S^3. It will then be shown that the crystal model is naturally derived from the unitary matrix model. We will see that non-compact D-branes associated with certain knot invariants can be incorporated into the crystal and the matrix models.

This thesis is based on the author's contribution to the subject, which appeared in papers [2],[3], and [4].

1.1 Basics of topological string theory

The most important construction of topological string theory is based on an $\mathcal{N} = (2,2)$ non-linear sigma model whose target space is a complex 3-dimensional Calabi-Yau manifold X. The non-linear sigma model has $\mathcal{N} = (2,2)$ superconformal symmetry with generators

$$(T, G^{\pm}, J) \quad \text{and} \quad (\bar{T}, \bar{G}^{\pm}, \bar{J}). \tag{1.1.1}$$

Here T is the energy momentum tensor, $G^{\pm}$ are the supercurrents, and J is the $U(1)$ R-current in the holomorphic sector. The corresponding generators in the anti-holomorphic sector are denoted with a bar.

We can make the non-linear sigma model into a topological field theory by twisting:

$$T_{\text{twisted}} := T + \frac{1}{2}\partial J, \qquad \bar{T}_{\text{twisted}} := \bar{T} \mp \frac{1}{2}\bar{\partial}\bar{J} \tag{1.1.2}$$

These twisted energy-momentum tensors satisfy the $c = 0$ Virasoro algebra. The two choices of sign leads to distinct topological field theories. The theory with the $-$ sign is called the A-model, while the $+$ sign leads to the so-called B-model.

Since we have the $c = 0$ Virasoro algebra, we can define topological string theory by analogy with bosonic string theory:

$$\text{spin } 1: \; G^{+}, G^{\mp} \quad \leftrightarrow \quad J_{\text{BRST}}, \bar{J}_{\text{BRST}} \tag{1.1.3}$$

$$\text{spin } 2: \; G^{-}, G^{\pm} \quad \leftrightarrow \quad b, \bar{b} \tag{1.1.4}$$

$$J, \mp\bar{J} \quad \leftrightarrow \quad J_{\text{ghost}} = - : bc :, \bar{J}_{\text{ghost}} = - : \bar{b}\bar{c} : \tag{1.1.5}$$

The identification of the R-current with the ghost current in the last line is possible because they are anomalous in the same way when the target space is a Calabi-Yau 3-fold.

The topological string amplitudes are defined as

$$F_g(t) := \int_{\mathcal{M}_g} \left\langle \prod_{k=1}^{3g-3} \int_{\Sigma} G^{-} \cdot \mu_k \int_{\Sigma} \bar{G}^{\pm} \cdot \bar{\mu}_k \right\rangle_{\text{twisted}} . \tag{1.1.6}$$

Here $\mu_k, k = 1, ..., 3g-3$ are the Beltrami differentials on the genus g Riemann surface, and $\mathcal{M}_g$ is the moduli space of genus g Riemann surfaces. (We restrict ourselves to $g \geq 2$.)

The basic properties of topological string amplitudes are their independence on half of the moduli of the Calabi-Yau 3-fold. The A-model amplitudes depend only on the Kähler moduli, while the B-model amplitudes depend only on the complex structure moduli.

So far we have been discussing closed topological string theory. Open topological string theory is defined by introducing D-branes. D-branes in our context are submanifolds of the Calabi-Yau on which open strings can end. The condition that the BRST symmetries above be preserved requires that the submanifolds are Lagrangian in the A-model, and holomorphic in the B-model, respectively. Open string amplitudes are defined in way similar to closed strings:

$$F_{g,h}(t) := \int_{\mathcal{M}_{g,h}} \left\langle \prod_{k=1}^{6g-3h-6} \int_{\Sigma} G^- \cdot \mu_k \right\rangle_{\text{twisted}}, \tag{1.1.7}$$

where $\mathcal{M}_{g,h}$ is the moduli space of genus g Riemann surfaces with h holes.

1.2 Gopakumar-Vafa duality

Conifold is a local approximation to the most typical singularity of a Calabi-Yau manifold. The conifold singularity can be smoothed in one of the two distinct ways. One is deformation and the other is resolution. (See figure 1.3.)

In 1998, Gopakumar and Vafa proposed that open topological string theory on the re-

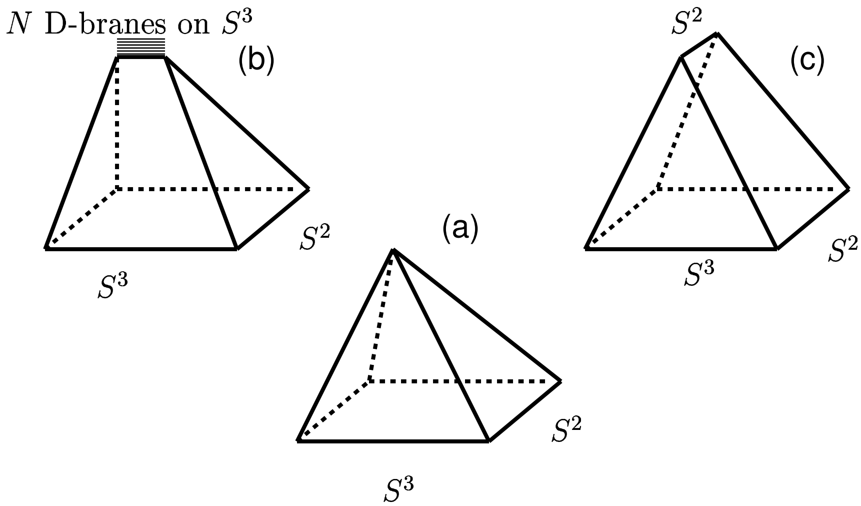


Figure 1.3: The singular conifold geometry (a) can be deformed to T^*S^3 (b). We wrap N D-branes on $S^3 \subset T^*S^3$, obtaining a $U(N)$ Chern-Simons theory on S^3. The conifold can also be resolved to $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbf{CP}^1$ (c). This is the geometry on which closed string theory is defined.

solved conifold is dual to closed topological string theory on the deformed conifold [5]. More precisely, one considers N coincident D-branes wrapping S^3 of the deformed conifold. The open string theory living on these D-branes was shown to reduce to $U(N)$ Chern-Simons gauge theory on S^3 [6]. The dictionary is such that the 't Hooft parameter $g_s N$ is identified with the complexified Kähler modulus t:

$$g_s N = t. \tag{1.2.1}$$

The partition function of the $U(N)$ Chern-Simons theory at level k is given by

$$Z_{\mathrm{CS}}(N, k, U(N)) = \frac{1}{(k+N)^{N/2}} \prod_{\alpha>0} 2 \sin \frac{\pi \alpha \cdot \rho}{k+N}. \tag{1.2.2}$$

Here $2\pi i/(k+N) = g_s$ is the string coupling constant, and the product is over the positive roots of $SU(N) \subset U(N)$, which are given by $\alpha_{ij} = e_i - e_j \in \mathbf{C}^N (\simeq$ Cartan subalgebra) for $i < j$. $\rho = (1/2)\sum_{\alpha>0}\alpha = \sum_{i=1}^{N}(\frac{N+1}{2} - i)e_i$ is the Weyl vector. Gopakumar and Vafa showed that this can be rewritten as

$$Z = M(q)e^{-\sum_{n=1}^{\infty}\frac{e^{-nt}}{n[n]^2}}. \tag{1.2.3}$$

Here $M(q) = \prod_{j=1}^{\infty}(1-q^j)^j$ is known as the McMahon function, and $[n] = q^{n/2} - q^{-n/2}, q = e^{-g_s}$. Eq. (1.2.3) is precisely the partition function of the resolved conifold.

There are generalizations of the topological string large N duality to A-models on other geometries and also to B-models.

Chapter 2

Worldsheet derivation

In the large N dualities in topological string theory, the string coupling constant g_s is the same for both open and closed string sides, and the 't Hooft couplings of the open string are identified with geometric moduli of the closed string side. Thus, it should be possible to give a microscopic explanation of the duality order-by-order in the string coupling expansion; namely, from the point of view of the string worldsheet. (This should also be the case for the AdS/CFT correspondence when the coupling constant does not run.) In [7], the worldsheet derivation was given for the case when the open string side is defined on the cotangent space of a three-sphere T^*S^3 with N D-branes wrapping the base S^3. The closed string side is the resolved conifold whose Kähler modulus t, which is the size of the blown up S^2, is identified with the 't Hooft coupling g_sN of the open string side. The strategy was to start with the closed string side and expand string amplitudes for small t. It has been noted in [8, 9] that in the linear sigma model description the closed string worldsheet at $t = 0$ develops a new non-geometric phase—the C phase—besides the geometric phase (Higgs phase), which flows to the non-linear sigma model for the resolved conifold in the IR limit. In [7], it was shown that the C phase can contribute to closed string amplitudes only in the following two cases:

(1) Each domain in the C phase has the topology of a disk.

(2) The entire worldsheet is in the C phase.

Moreover it was found that each disk in the C phase contributes a factor of t to the amplitudes. By interpreting C domains as holes on the worldsheet, the sum over disks in the C phase reproduces the open string Feynman diagram expansion with t being identified with the 't Hooft coupling. On the other hand, worldsheets that are entirely in the C phase do not correspond to any open string diagrams. It was shown that the sum of such worldsheets to all orders in the string coupling constant expansion correctly captures the gauge group volume vol $U(N)$ for the open string, which indeed does not come from Feynman diagrams but is needed to reproduce open string amplitudes.

Thus, this approach shows that the closed string theory on the resolved conifold has the open string expansion with the Kähler modulus t being equal to the 't Hooft coupling on the open string side. One may be more ambitious and try to reproduce the D-brane boundary condition. Some of the basic features of the boundary condition can been seen in this approach: at the interface of the C phase and the geometric phase, one finds that the worldsheet is pulled toward the apex of the conifold where the S^2 shrinks to zero size at $t = 0$ and the S^3 emerges after the conifold transition. However the precise boundary condition for the D-branes wrapping the S^3—for example, the Neumann boundary condition along the branes—has not been reproduced in this approach since the linear sigma model does not describe the geometry with S^3 of finite size. Although the size of S^3, being a complex structure modulus, is not relevant in the A-model and can be infinitesimal, it is desirable to understanding how D-brane boundary conditions are reproduced from the closed string point of view.

In this part, we will clarify the relation between D-branes in open string and C phases in closed string by studying cases in which there are several D-branes on the open string side. We will consider the open string on the $\mathbf{Z}_p$ quotient of T^*S^3, whose base is the lens space $S^3/\mathbf{Z}_p$ [10]. Since the fundamental group of $S^3/\mathbf{Z}_p$ is $\mathbf{Z}_p$, there are p different types of D-branes whose holonomies along the homotopy generator are given by $e^{2\pi i a/p}$ $(a = 0, 1, ..., p-1)$. It turns out that the closed string dual has p different C phases on the worldsheet, and we show how each C phase is identified with the corresponding D-brane by studying the behavior of linear sigma model variables at the interface of each C phase with the geometric phase. We find it is useful to use the Landau-Ginzburg B-model that is T-dual to the linear sigma model [11] since much of the analysis in the B-model can be carried out at the classical level. We will also discuss how this can be seen from the point of view of the mirror manifold.

It is straightforward to apply the worldsheet derivation in [7] and in this paper to other toric Calabi-Yau manifolds that are known to have open string duals [12], for example the one described by $Q_1 = (1, -1, 0, 1, -1)$ and $Q_2 = (0, 0, 1, -2, 1)$. It would be interesting to analyze closed string theories on more general toric manifolds and to discover new large N dualities.

This chapter is organized as follows: In section 2, we will review the worldsheet derivation of [7] in the conifold case. We will use the Landau-Ginzburg B-model to simplify some of the steps in the original derivation. We will then extend the analysis to the quotients of the conifold and explain the correspondence between D-branes and C phases from the Landau-Ginzburg description. In section 3, we will discuss the open/closed string duality as seen from the point of view of the mirror manifolds.

2.1 D-branes and phases

2.1.1 Review of the conifold case

In [5], it was conjectured that the A-type topological closed string theory on the resolved conifold with the Kähler modulus t is equivalent to the A-type topological open string theory on the cotangent space T^*S^3 (or equivalently on the deformed conifold) with N D-branes wrapping the base S^3. Here the Kähler modulus t of the closed string side is identified with the 't Hooft coupling g_sN of the open string side, with the string coupling constant g_s being the same on both sides. There have been several non-trivial checks of the conjecture [13, 14, 15, 16]. Finally, a worldsheet derivation of the open/closed string duality was given in [7]. In this subsection, we will review the derivation, using the mirror Landau-Ginzburg model [11] to simplify some of the steps.

The strategy is to start with the closed string side, expand string amplitudes around $t = 0$, and show that a sum over open string Feynman diagrams emerges in the t-expansion. Since $t = 0$ is a singular limit of the target space, it is useful to describe the worldsheet by the linear sigma model. The worldsheet theory consists of four chiral superfields $\Phi_1, \Phi_2, \Phi_3, \Phi_4$

with charges given by

$$Q = (1, 1, -1, -1), \tag{2.1.1}$$

and a gauge multiplet coupled to Q. Following [8], we rearrange the gauge multiplet into a twisted chiral multiplet Σ. When t is non-zero, there is a potential for the scalar field σ, which is the lowest component of Σ and the linear sigma model flows to the non-linear sigma model with the resolved conifold as the target space. In this description, the singularity at $t = 0$ is characterized by the fact that the potential for σ disappears in the limit and it can become indefinitely large without costing the worldsheet action—a new non-compact and non-geometric phase emerges on the worldsheet [8]. The idea of the derivation of the large N duality, advocated in [5] and quantified in [7], is to regard domains in this new phase as holes on the worldsheet, and phase boundaries as representing D-branes.

The derivation of [7] can be streamlined by using the mirror of the linear sigma model, which is found by performing the T-dual transformation on phase rotations of Φ_j, keeping Σ as a spectator [11]. The T-dual of Φ_j are twisted chiral superfields Y_j with periodicity $Y_j \sim Y_j + 2\pi i$. Combined with Σ, the mirror is a Landau-Ginzburg model with the superpotential W given by

$$W = \Sigma(Y_1 + Y_2 - Y_3 - Y_4 - t) - \sum_{j=1}^{4} e^{-Y_j}. \tag{2.1.2}$$

Note that, since the original model is A-twisted, the mirror Landau-Ginzburg model is B-twisted. It is easy to check that, when t is not equal to zero, there is no flat direction for the superpotential. It is believed that the Landau-Ginzburg model in this case flows to the mirror of the sigma model for the conifold. At $t = 0$, the superpotential remains flat for

$$\begin{aligned} y_1 = y_2 &= -\log\sigma + \pi i, \\ y_3 = y_4 &= -\log\sigma, \end{aligned} \tag{2.1.3}$$

with σ being arbitrary. Here y_i is the lowest component scalar field in Y_i. Following [7], we call this flat direction the C phase.[1]

In [7], it was argued that the C phase is described by a Landau-Ginzburg model for a single variable X with an effective superpotential

$$W_{eff} = -\frac{t}{X}. \tag{2.1.4}$$

It is straightforward to derive this from the B-model. When σ is large, the potential for Y_i becomes steep and we can integrate them out using the Gaussian approximation. Since the mirror Landau-Ginzburg model is B-twisted, it is sufficient to consider an integral over constant maps,

$$\int dY_1 dY_2 dY_3 dY_4 \exp(-W) \sim \frac{1}{\Sigma^2}\exp(t\Sigma). \tag{2.1.5}$$

The pre-factor $1/\Sigma^2$ gives a non-canonical measure for Σ. We can absorb it by changing the variable $\Sigma \to X = \Sigma^{-1}$. This gives the effective superpotential (2.1.4) for the canonically

[1] This is called the C phase in order to distinguish it from the Coulomb phase of the model, which is decoupled from the geometric phase in the IR limit. For a more detailed specification of the C phase, see [7].

normalized variable X.

We found that there are two phases in the worldsheet theory: the geometric phase, which flows in the IR to the non-linear sigma model on the conifold, and the C phase, which is non-geometric and is described by the effective Landau-Ginzburg model with the superpotential $W_{eff} = -t/X$. The functional integral then includes a sum over domains in the C phase on the worldsheet. Let us state the following two facts that were shown in [7]:

(1) A domain in the C phase contributes to a topological string amplitude only if (a) the domain has the topology of a disk or (b) the entire worldsheet is in C phase.

(2) Each C domain of the disk topology contributes to the amplitude by a factor of t. This follows from the integral

$$\oint dX e^{t/X} = 2\pi i t. \tag{2.1.6}$$

Thus, if we regard each C domain of the disk topology as a hole on the worldsheet, the closed topological string amplitude is expressed as a sum of open topological amplitudes with a boundary of each hole being weighted with a factor of t.

When the entire worldsheet is in the C phase, the closed topological string amplitude at genus g is given by $\chi(\mathcal{M}_g)/t^{2g-2}$, where $\chi(\mathcal{M}_g)$ is the Euler characteristic of the moduli space of genus g surfaces and is equal to $B_{2g}/2g(2g-2)$. The negative power in t (for $g \geq 2$) reflects the singularity at $t \to 0$. Summing this up over all genera, one obtains [7]:

$$\sum_g \chi(\mathcal{M}_g) \left(\frac{g_s}{t}\right)^{2g-2} = \sum_g \frac{B_{2g}}{2g(2g-2)N^{2g-2}} = \log \mathrm{vol}\, U(N), \tag{2.1.7}$$

where we used $t = g_s N$. Thus, the sum over the worldsheet in the pure C phase gives the gauge volume factor vol $U(N)$ for the gauge theory.

This establishes that the closed topological string theory on the resolved conifold is equivalent to some open topological string theory, with gauge group $U(N)$ and 't Hooft coupling t. We have not yet shown on which D-branes open strings are ending. According to the conjecture in [5], the D-branes should be wrapping the base S^3 of the deformed conifold. To see how the boundary condition emerges from the closed string dual, we first note that the transition from the resolved conifold to the deformed conifold is a local operation near the conifold singularity. Thus, away from the base S^3, we can approximate the deformed conifold by the geometric phase (Higgs phase) of the linear sigma model for the resolved conifold. Since A-model amplitudes are independent of the complex structure, we can make the size of S^3 as small as we like, making the approximation increasingly accurate. In the C phase, all the Φ^i fields in the sigma model become massive. Thus, $\Phi^i \to 0$ as we approach the "hole" on the closed string worldsheet, and it is roughly where the base S^3 is located. Reproducing the precise boundary condition for D-branes wrapping S^3—for example, deriving the Neumann boundary condition along S^3–is difficult in this approach since the linear sigma model does not describe the geometry of the deformed conifold with finite S^3.

Although reproducing a precise boundary condition for each D-brane may be difficult, one may ask if we can distinguish different types of D-branes in this approach. In the following subsections, we will demonstrate that it is possible.

2.1.2 Gauge theory on $S^3/\mathbf{Z}_2$

As the first example in which there is more than one type of D-brane, we consider the Chern-Simons gauge theory on the lens space $S^3/\mathbf{Z}_2$. Classical solutions are flat connections, which are labeled by holonomy matrices for the homotopy generator of the space. Since the fundamental group is $\mathbf{Z}_2$ in this case, the $U(N)$ gauge theory can have a holonomy matrix with N_1 eigenvalues being $(+1)$ and N_2 eigenvalues being (-1), where $N = N_1 + N_2$. This breaks the gauge group into $U(N_1) \times U(N_2)$. This can be realized by considering the topological string theory on $T^*S^3/\mathbf{Z}_2$, with N_1 D-branes wrapping the lens space with the trivial bundle and N_2 D-branes wrapping the same space with the bundle twisted by (-1).

According to the conjecture in [10], the target space of the closed string dual is the $\mathbf{Z}_2$ quotient of the resolved conifold. This space has two Kähler moduli, which are naturally identified with the two 't Hooft couplings g_sN_1 and g_sN_2 of the open string. This conjecture has also been tested in non-trivial ways [10, 17, 3].

The closed string worldsheet is described by a linear sigma model with five chiral multiplets $\Phi_i, i = 0, ..., 4$ with two sets of charges,

$$\begin{aligned} Q_1 &= (-2, 1, 1, 0, 0), \\ Q_2 &= (-2, 0, 0, 1, 1), \end{aligned} \tag{2.1.8}$$

and two gauge multiplets coupled to these charges. Since $Q_1 - Q_2 = (0, 1, 1, -1, -1)$, solving the D term constraint and dividing by the $U(1)$ gauge symmetry coupled to this combination of charges reduces $\phi_1, ..., \phi_4$ to the resolved conifold. In the cone $r_1 < 0, r_1 - r_2 < 0$, since $\phi_0 \neq 0$, we can use the remaining Q_1 gauge symmetry to fix the phase of ϕ_0. This leaves out a residual $\mathbf{Z}_2$ gauge symmetry acting as

$$(\phi_0, \phi_1, \phi_2, \phi_3, \phi_4) \to (\phi_0, -\phi_1, -\phi_2, \phi_3, \phi_4). \tag{2.1.9}$$

Thus we find the $\mathbf{Z}_2$ quotient of the resolved conifold as the target space.

As in the previous subsection, we rearrange the gauge multiplets into twisted chiral superfields, Σ_1 and Σ_2, and perform the T-dual transformation along phase rotations of the five chiral superfields to arrive at the B-twisted Landau-Ginzburg model with the superpotential

$$W = \Sigma_1(-2Y_0 + Y_1 + Y_2 - t_1) + \Sigma_2(-2Y_0 + Y_3 + Y_4 - t_2) - \sum_{i=0}^{5} e^{-Y_i}, \tag{2.1.10}$$

where the two Kähler moduli, t_1 and t_2, are linearly coupled to the gauge multiplets.

Let us examine when this superpotential has flat directions. Solving $\partial W/\partial y_i = 0$, we find

$$\begin{aligned} y_0 &= -\log(2\sigma_1 + 2\sigma_2), \\ y_1 = y_2 &= -\log(-\sigma_1), \\ y_3 = y_4 &= -\log(-\sigma_2). \end{aligned} \tag{2.1.11}$$

By substituting this into the remaining equations,

$$\frac{\partial W}{\partial \sigma_a} = -2y_0 + y_a + y_{a+1} - t_a = 0 \quad (a = 1, 2), \tag{2.1.12}$$

we find

$$\begin{aligned} e^{t_1} &= 4\left(1 + \frac{\sigma_2}{\sigma_1}\right)^2, \\ e^{t_2} &= 4\left(1 + \frac{\sigma_1}{\sigma_2}\right)^2. \end{aligned} \tag{2.1.13}$$

Since both t_1 and t_2 depend only on the ratio σ_1/σ_2, this is possible only if they satisfy the relation

$$\Delta = 16(e^{-t_1} - e^{-t_2})^2 - 8(e^{-t_1} + e^{-t_2}) + 1 = 0, \tag{2.1.14}$$

which is obtained by eliminating σ_1/σ_2 from the two equations in (2.1.13). The subspace of the Kähler moduli space where $\Delta = 0$ is known as the singular locus.[2] If the Kähler moduli satisfy $\Delta = 0$, the superpotential has a flat direction corresponding to the scaling of σ_1 and σ_2 while keeping their ratio fixed.

This model has two different C phases. For a generic point on the singular locus, only one of the two C phases emerges. But there is a particular point where both coexist. Let us

[2]The singular locus can also be derived from the linear sigma model point of view [18]. In this case, we have to take into account quantum corrections in the linear sigma model. This is in contrast to the mirror Landau-Ginzburg description, where the singular locus (2.1.14) is derived from the classical analysis of the superpotential (2.1.10).

consider the limit[3]

$$t_1, t_2 \to -\infty, \quad t_1 - t_2 \to 0. \tag{2.1.15}$$

In this limit, the condition (2.1.14) for the singular locus gives

$$t_1 - t_2 = \pm e^{t_1/2} + O(e^{t_1}). \tag{2.1.16}$$

For such t_1, t_2, we can solve (2.1.13) as

$$\sigma_2 = -\sigma_1 \pm e^{t_1/2}\sigma_1 + O(e^{t_1}). \tag{2.1.17}$$

Substituting this into (2.1.11), we find two flat directions:

$$\begin{aligned} C_+ \text{ phase}: \quad & \sigma_1 = -\sigma_2 = \sigma, \\ & y_0 = -\frac{t_1}{2} - \log\sigma, \\ & y_1 = y_2 = -\log\sigma, \\ & y_3 = y_4 = -\log\sigma + \pi i. \end{aligned} \tag{2.1.18}$$

$$\begin{aligned} C_- \text{ phase}: \quad & \sigma_1 = -\sigma_2 = \sigma, \\ & y_0 = -\frac{t_1}{2} - \log\sigma + \pi i, \\ & y_1 = y_2 = -\log\sigma, \\ & y_3 = y_4 = -\log\sigma + \pi i. \end{aligned} \tag{2.1.19}$$

[3]This limit is motivated by the fact that the two C phases coexist as we will show below. A geometric motivation for the limit will be made clear in section 3.

Both are complex 1-dimensional in the 7-dimensional space of (σ, y) and are parametrized by σ. Note that the two phases are distinguished by the value of y_0. When e^{t_1} and e^{t_2} are small but finite, either C_+ or C_- solves $dW = 0$ depending on the sign $(\pm)$ on the right-hand side of (2.1.16). In the limit (2.1.15), both phases coexist.

In this model, the flat coordinates $\hat{t}_+, \hat{t}_-$ are non-linear functions of the parameters t_1, t_2 in the superpotential (2.1.10). They can be computed either by the integrals $\int d\sigma dy e^{-W}$ with a different choice of contours, or by going to the mirror of the $\mathbf{Z}_2$ quotient of the resolved conifold and performing period integrals. From the latter point of view, $\hat{t}_1$ and $\hat{t}_2$ are periods of two 3-cycles in the mirror manifold. We will discuss the latter point of view in more detail in section 3. In terms of the flat coordinates, the condition $\Delta = 0$ is equivalent to $\hat{t}_+ = 0$ **or** $\hat{t}_- = 0$, where one of the two C phases emerges. The limit (2.1.15) corresponds to $\hat{t}_+ = 0$ **and** $\hat{t}_- = 0$ consistently with the fact that both C phases are realized in the limit.

Let us examine the limit more closely. The flat coordinates are expressed in the limit as

$$\begin{aligned} \hat{t}_+ &= e^{t_1/2} + t_1 - t_2 + O(e^{t_1}), \\ \hat{t}_- &= -e^{t_1/2} + t_1 - t_2 + O(e^{t_1}). \end{aligned} \tag{2.1.20}$$

Comparing this with (2.1.16), we find that the C_+ (C_-) phase emerges at $\hat{t}_+ = 0$ (at $\hat{t}_- = 0$). The two C phases coexist when both flat coordinates vanish. The two flat coordinates are exchanged as $(t_1, t_2) \to (t_1 + 2\pi i, t_2 + 2\pi i)$ and at the same time the two C phases are also exchanged.

Thus, at $\hat{t}_+ = \hat{t}_- = 0$, both C phases as well as the geometric phase coexist on the worldsheet. We claim that the two C phases correspond to the two types of D-branes with different holonomies around the homotopy generator γ of S^3/Z_2. This can be shown in the following three steps:

(1) We first note that C_+ and C_- are distinguished by the values of y as in (2.1.18) and (2.1.19), and that they are related to each other by a shift of y_0 by πi, one-half of the periodicity of y_0.

(2) Since a shift of y_i in the imaginary direction is T-dual to a phase rotation of ϕ_i for each $i = 0, ..., 5$, the C branches represent D-branes wrapping around the phase rotations of ϕ's. Since C_+ and C_- differ by the shift of one half of the period of y_0, the corresponding two types of D-branes in term of the dual ϕ variables are related to each other by a multiplication of (-1) to the holonomy of the gauge field around a 2π phase rotation of ϕ_0.

(3) What remains is to identify this (-1) as the relative holonomy around the homotopy generator γ on the D-brane worldvolume $S^3/\mathbf{Z}_2$. Since the fundamental groups of $S^3/\mathbf{Z}_2$ and $T^*S^3/\mathbf{Z}_2$ are isomorphic, and since the conifold transition is a local operation near the singularity, we can lift γ from the base and describe it in the linear sigma model variables. To see that γ is homotopic to the 2π phase rotation of ϕ_0, we just have to note that the latter is gauge equivalent via the Q_1 gauge transformation to the π rotation,

$$(\phi_0, e^{i\theta}\phi_1, e^{i\theta}\phi_2, \phi_3, \phi_4), \quad 0 \leq \theta \leq \pi, \tag{2.1.21}$$

and that this path is closed because of the $\mathbf{Z}_2$ quotient described in the third paragraph of this subsection.

We have established that the two C phases emerge in the limit $\hat{t}_+, \hat{t}_- \to 0$. The boundary conditions at the interface of the C phases and the geometric phase are related to each other by the shift of πi of the value of y_0. Via the T-duality, they are mapped to boundary conditions on the linear sigma model variables related to each other by a multiplication of (-1) to the holonomies around the homotopy generator of $S^3/\mathbf{Z}_p$, i.e., the two types of D-branes expected in the open string dual.[4]

For the same reason as in the case of the conifold discussed in [7] and reviewed in the last subsection, C domains contribute to topological string amplitudes only if they are of the disk topology or if they cover the entire worldsheet.

The large N duality conjecture states that the 't Hooft couplings for the two types of D-branes are given by the flat coordinate $\hat{t}_\pm$ on the closed string side. This can be shown as follows: Each of the C branches is complex 1-dimensional parametrized by σ in (2.1.18) and (2.1.19). An integral in the direction transverse to σ imposes one linear constraint on the five Y fields. For a large value of σ, the remaining four Y fields can be integrated out in the Gaussian approximation. As in the conifold case, this results in a superpotential linear in Σ with the non-canonical measure of $1/\Sigma^2$. Using the variable $X = 1/\Sigma$, one finds an effective superpotential $\sim 1/X$ with the canonical measure. To find the coefficient of the $1/X$ potential, we note the following two well-known facts:

[4]Since the open string is in the adjoint representation of the gauge group, only the relative holonomy of the two types of D-branes has an invariant meaning. This is T-dual to the fact that only the relative value of y_0 in the two C phases is relevant because of the translational invariance. In fact, gauge theory amplitudes are invariant under exchange of N_1 and N_2, the numbers of the two types of D-branes [10].

(1) The genus-g closed string amplitude for the Landau-Ginzburg model with the superpotential $W = t/X$ is proportional to t^{2-2g} [19].

(2) The singular part of the genus-g closed string amplitude for small $\hat{t}_\pm$ is proportional to $\hat{t}_\pm^{2-2g}$ [20].

Note that both statements follow from studies on the closed string side and do not assume the open/closed string duality. Comparing them, we find that the effective superpotential in the $C_\pm$ phase is given by $\hat{t}_\pm/X$, respectively. The disk amplitude is then computed exactly as in the conifold case, giving rise to the factor $\hat{t}_\pm$ for the $C_\pm$ phase. This is what we wanted to show.

2.1.3 Gauge theory on $S^3/\mathbf{Z}_p$

It is straightforward to generalize the result in the previous subsection to the case of the $\mathbf{Z}_p$ quotient of the conifold for $p \geq 2$. In this case, the conjectured gauge theory dual is on the lens space $S^3/\mathbf{Z}_p$ [10]. Since the fundamental group of the space is $\mathbf{Z}_p$, there are p different types of D-branes whose holonomies around the homotopy generator are given by $e^{2\pi i a/p}$ $(a = 0, 1, ..., p-1)$. We would like to see how they are identified with p different C phases in the closed string dual.

The worldsheet of the closed string on the $\mathbf{Z}_p$ quotient of the resolved conifold can be described by the linear sigma model with $(p+3)$ chiral fields $\Phi_0, \Phi_1, ..., \Phi_{p+2}$ coupled to p

gauge fields with the following charge vectors [3]:

$$
\begin{array}{lllllllllllll}
 & & \Phi_0, & \Phi_1, & \Phi_2, & \Phi_3, & \Phi_4, & \Phi_5, & \ldots, & \Phi_{4+j}, & \cdots, & \Phi_{p+2} & \\
Q_0 & =(& 0, & 1, & 1, & -1, & -1, & 0, & \ldots, & 0, & \ldots, & 0), & \\
Q_j & =(& -(j+1), & j, & 0, & 0, & 0, & 0, & \ldots, & 1, & \ldots, & 0), & \\
Q_{p-1} & =(& -p, & p-1, & 1, & 0, & 0, & 0, & \ldots, & 0, & \ldots, & 0), &
\end{array}
\tag{2.1.22}
$$

where $j = 1, \cdots, p-2$. Let us show that that this indeed describes the $\mathbf{Z}_p$ quotient of the resolved conifold in the cone,

$$
\begin{aligned}
& r_0 < 0, \\
& r_{p-1} < 0, \\
& -r_j + \frac{j+1}{p} r_{p-1} < 0 \quad (1 \le j \le p-2).
\end{aligned}
\tag{2.1.23}
$$

We write $U(1)_a$ for the $U(1)$ generated by Q_a, and the corresponding D term is $D_a := \sum_i Q_{ai}|\phi_i|^2 - r_a$ $(a = 1, ..., p-1)$. If $r_0 < 0$, $\{D_0 = 0\}/U(1)_0$ describes the resolved conifold. If $r_{p-1} < 0$, $D_{p-1} = 0$ does not allow ϕ_0 to vanish. Thus, we can use the $U(1)_{p-1}$ gauge symmetry to fix the phase of ϕ_0. Since ϕ_0 carries $(-p)$ units of Q_{p-1}, there is a $\mathbf{Z}_p$ residual gauge symmetry of $U(1)_{p-1}$ given by

$$
(\phi_0, \phi_1, \phi_2, \phi_{3 \le i \le p-1}) \to (\phi_0, e^{-\frac{2\pi i}{p}} \phi_1, e^{\frac{2\pi i}{p}} \phi_2, \phi_{3 \le i \le p-1}).
\tag{2.1.24}
$$

Other gauge groups $U(1)_{1\leq j\leq p-2}$ are completely fixed if $-r_j + \frac{j+1}{p}r_{p-1} < 0$ since $0 = D_j - \frac{j+1}{p}D_{p-1} = \left(\frac{j-p+1}{p}\right)|\phi_1|^2 - \frac{j+1}{p}|\phi_2|^2 + |\phi_{4+j}|^2 - r_j + \frac{j+1}{p}r_{p-1}$ requires ϕ_{4+j} to be non-zero. Thus, we obtain the $\mathbf{Z}_p$ quotient of the resolved conifold as a space of solutions to the D term constraints up to gauge transformations.

The mirror Landau-Ginzburg model has the superpotential

$$\begin{aligned} W &= \Sigma_0\big(Y_1 + Y_2 - Y_3 - Y_4 - t_0\big) + \sum_{j=1}^{p-2}\Sigma_j\big(-(j+1)Y_0 + jY_1 + Y_{4+j} - t_j\big) \\ &\quad + \Sigma_{p-1}\big(-pY_0 + (p-1)Y_1 + Y_2 - t_{p-1}\big) - \sum_{j=0}^{p-1} e^{-Y_j}. \end{aligned} \tag{2.1.25}$$

As in the previous subsection, we look for flat directions of the potential. Solving $\partial W/\partial y_i = 0$ $(i = 0, \ldots, p+2)$ gives

$$\begin{aligned} y_0 &= -\log\left(\sum_{j=1}^{p-1}(j+1)\sigma_j\right), \\ y_1 &= -\log\left(-\sigma_0 - \sum_{j=1}^{p-1} j\sigma_j\right), \\ y_2 &= -\log\left(-\sigma_0 - \sigma_{p-1}\right), \\ y_3 &= -\log\sigma_0, \\ y_4 &= -\log\sigma_0, \\ y_{4+i} &= -\log(-\sigma_i), \qquad (i = 1, \ldots, p-2). \end{aligned} \tag{2.1.26}$$

Substituting them into $\partial W/\partial\sigma_a = 0$ $(a = 0, \ldots, p-1)$ gives p relations of the form

$$
\begin{aligned}
e^{t_0} &= \frac{\sigma_0^2}{(\sigma_0+\sigma_{p-1})(\sigma_0+\sum_j j\sigma_j)}, \\
e^{t_k} &= \frac{(-\sum_j (j+1)\sigma_j)^{k+1}}{\sigma_k(\sigma_0+\sum_j j\sigma_j)^k}, \\
e^{t_{p-1}} &= \frac{(-\sum_j (j+1)\sigma_i)^p}{(\sigma_0+\sigma_{p-1})(\sigma_0+\sum_j j\sigma_j)^{p-1}},
\end{aligned} \tag{2.1.27}
$$

where $k = 1, \ldots, p-2$. Note that the right-hand sides are functions of the ratios of σs. Since there are p relations for $(p-1)$ variables, there is no solution for generic values of t and thus no flat direction for W. The singular locus, where a flat direction emerges, is determined by eliminating $\sigma_1/\sigma_0, \ldots, \sigma_{p-1}/\sigma_0$ from these p equations. The analysis of each of C phase can be done as in the $\mathbf{Z}_2$ case. For example, since each C phase is complex 1-dimensional parametrized by the scaling of σs, the functional integral over σ in the transverse direction imposes $(p-1)$ linear constraints on $(p+3)$ Y variables, leaving four linear combinations of Y free. The functional integrals of these four fields can be done in the Gaussian approximation, giving rise to the effective Landau-Ginzburg model with the $1/X$ superpotential.

To understand how p different C phases emerge, it is useful to make the following change of variables,

$$
\begin{aligned}
Y_0' &:= Y_0 + \frac{t_{p-1}}{p}, \qquad & Y_2' &:= -pY_0 + (p-1)Y_1 + Y_2 - t_{p-1}, \\
Y_4' &:= -Y_1 - Y_2 + Y_3 + Y_4 + t_0, \qquad & Y_{4+j}' &:= -(j+1)Y_0 + jY_1 + Y_{4+j} - t_j,
\end{aligned} \tag{2.1.28}
$$

so that the superpotential takes the form

$$W = -\Sigma_0 Y_4' + \sum_{j=1}^{p-2} \Sigma_j Y_{4+j}' + \Sigma_{p-1} Y_2' - \big(e^{\frac{t_{p-1}}{p}} e^{-Y_0'} + e^{-Y_1} + e^{-Y_2' - pY_0' + (p-1)Y_1}$$

$$+ e^{-Y_3} + e^{-Y_4' - 2Y_2' - pY_0 + (p-2)Y_1 + Y_3 + t_0} + \sum_{j=1}^{p-2} e^{-Y_{4+j}' + \frac{j+1}{p} t_{p-1} - (j+1)Y_0' + jY_1 - t_j}\big). \quad (2.1.29)$$

In the limit

$$\begin{aligned} & t_0 \to 0, \\ & t_{p-1} \to -\infty, \\ & \frac{j+1}{p} t_{p-1} - t_j \to -\infty, \end{aligned} \quad (2.1.30)$$

the superpotential becomes

$$\begin{aligned} W \;\; = \;\; & -\Sigma_0 Y_4' + \sum_{j=1}^{p-2} \Sigma_j Y_{4+j}' + \Sigma_{p-1} Y_2' - \big(e^{-Y_1} + e^{-Y_2' - pY_0' + (p-1)Y_1} \\ & + e^{-Y_3} + e^{-Y_4' - 2Y_2' - pY_0' + (p-2)Y_1 + Y_3}\big). \end{aligned} \quad (2.1.31)$$

Extremizing W leads to p different families of solutions

$$
\begin{aligned}
C_k \text{ phase}: \qquad & y_0' = -\log(-\sigma_0) + \frac{2\pi i}{p}k, \\
& y_1 = -\log(-\sigma_0), \quad y_2' = 0, \\
& y_3 = -\log(\sigma_0), \quad y_4' = 0, \\
& y_{4+j}' = 0, \quad \sigma_j = 0 \qquad (1 \leq j \leq p-2),
\end{aligned}
\tag{2.1.32}
$$

where $k = 0, 1, ..., p-2$. We found that p different C phases coexist in the limit (2.1.30).

The p different C phases are related to each other as $C_k \to C_{k+1}$ under the shift $y_0' \to y_0' + 2\pi i/p$. In terms of the original y variables, the shift is expressed as

$$
\begin{aligned}
&(y_0, y_1, y_2, y_3, y_4, y_{4+j}) \\
&\quad \to \left(y_0 + \frac{2\pi i}{p}, y_1, y_2, y_3, y_4, y_{4+j} + \frac{2\pi i}{p}(j+1) \right).
\end{aligned}
\tag{2.1.33}
$$

As explained in the second paragraph of this subsection, the homotopy generator of $S^3/\mathbf{Z}_p$ is the path

$$
(\phi_0, e^{-i\theta}\phi_1, e^{i\theta}\phi_2, \phi_3, ...), \quad 0 \leq \theta \leq \frac{2\pi}{p}. \tag{2.1.34}
$$

Note that a 2π rotation of ϕ_1, keeping other variables fixed, is contractible even if one is away from the apex of the conifold, since ϕ_1 can vanish while maintaining the D term constraints . Thus, the $(-2\pi/p)$ rotation of ϕ_1 in the above can be replaced by the $2\pi(p-1)/p$ rotation.

Under the $U(1)_{p-1}$ gauge transformation with respect to the charge vector Q_{p-1} in (2.1.22), this is gauge equivalent to a 2π phase rotation of ϕ_0. Since the map from C_k to C_{k+1} $(k = 0, ..., p-1)$ involves the $2\pi i/p$ shift of y_0, their holonomies under the 2π phase rotation of ϕ_0 differ by $e^{2\pi i/p}$. Namely, the relative holonomy of C_k and C_{k+1} around the homotopy generator of $S^3/\mathbf{Z}_p$ is equal to $e^{2\pi i/p}$, precisely reproducing the large N duality stated in the first paragraph of this subsection.

In principle, we can carry out the analysis further in this approach and find the flat coordinates explicitly in the Landau-Ginzburg description. However, it is more convenient and geometrically more intuitive to use the mirror manifold, which can be obtained by partially performing the functional integral of the Landau-Ginzburg model and making some change of variables [11]. We will to discuss this in the next section.

2.2 B-model large N dualities

The large N duality from the point of view of the B-model has led to the discovery of the relationship between the spectral density of matrix models and Calabi-Yau geometry. In this section, we will give a worldsheet derivation of this and related dualities.

2.2.1 Mirror of the $\mathbf{Z}_p$ quotient of the resolved conifold

In the last section, we considered the $\mathbf{Z}_p$ quotient of the resolved conifold. Here we will study the same problem, but from the point of view of its mirror manifold. The mirror Calabi-Yau manifold is given by the equation [3],

$$\begin{aligned} 0 &= G(x_1, x_2, u, v) \\ &\equiv x_1^2 + x_2^2 + (e^v - 1)(e^{pu+v} - 1) + e^{t_0} - 1 - e^{t_{p-1}/p+u+v} - \sum_{j=1}^{p-2} e^{\frac{j+1}{p} t_{p-1} - t_j + (j+1)u + v}. \end{aligned} \quad (2.2.1)$$

The non-linear sigma model on the above non-compact Calabi-Yau can be realized as the IR limit of the Landau-Ginzburg model with chiral superfields Λ, X_1, X_2, U, V and the superpotential

$$W = \Lambda G(X_1, X_2, U, V), \quad (2.2.2)$$

where the scalar components of U and V are defined modulo $2\pi i$. Indeed, as long as the geometry is smooth so that there is no solution to $G = dG = 0$, the only solutions to $dW = 0$ are $\Lambda = G = 0$. Excitations transverse to $\Lambda = G = 0$ are massive. Hence in the low-energy the theory flows to the non-linear sigma model on the geometry $G = 0$. Such a Landau-Ginzburg model was considered in [9] in the case of the deformed conifold. This Landau-Ginzburg model is related to the model used in subsection 2.1.3 by partially carrying out the functional integral and by making a change of variables [11]. We will show that the worldsheet phase structure found in subsection 2.1.3 can also be obtained from this Landau-Ginzburg model.

In the limit

$$t_0 \to 0,$$
$$t_{p-1} \to -\infty,$$
$$\frac{j+1}{p} t_{p-1} - t_j \to -\infty, \tag{2.2.3}$$

we have

$$G \sim x_1^2 + x_2^2 + (e^{pu+v} - 1)(e^v - 1) \tag{2.2.4}$$

and the geometry $G = 0$ develops p conifold singularities at

$$(x_1, x_2, u, v) = (0, 0, \frac{2\pi i}{p} a, 0),$$
$$a = 0, 1, ..., p-1. \tag{2.2.5}$$

The singularities of the geometry are reflected in the worldsheet theory as the appearance of new non-compact directions in the space of zero-energy configurations. Namely, the worldsheet theory develops p new flat directions where λ, the lowest component of Λ, is large while (x_1, x_2, u, v) are fixed to the locations of conifold singularities.

We see that Λ plays the same role as the Σ field, and C phases can be defined as flat directions where λ becomes large. Different C phases are distinguished by the values of u. In each C domain, since λ is large, we can integrate out x_1, x_2, u and v by Gaussian approximation, which produces the measure $d\lambda/\lambda^2$. Thus, the effective Landau-Ginzburg model of each of the C phases is again with the $1/X$ superpotential. The coefficient of the superpotential is given by the flat coordinate $\hat{t}_a$ $(a = 0, \ldots, p-1)$, since each C phase is associated with shrinking of one of p 3-cycles.

In the language of the present subsection it is easy to generalize the analysis of the singular locus and the periods. Let us introduce new parameters $d_0, d_1, ..., d_{p-1}$ as functions of $t_0, ..., t_{p-1}$ so that G is expressed as

$$G = x_1^2 + x_2^2 + (e^{pu+v} - 1)(e^v - 1) + d_0 + \sum_{j=1}^{p-1} d_j e^{ju+v}. \tag{2.2.6}$$

The limit (2.2.3) is equivalent to $d_j \to 0$ for $j = 0, 1, ..., p-1$. Let us evaluate Δ and the periods $\hat{t}_a$ in this limit. Suppose (u, v) are near $(2\pi i a/p, 0)$ and write $(u, v) = (2\pi i a/p + \delta u, \delta v)$. Assuming that δu and δv are of the order $\mathcal{O}(d)$, we can expand G to the quadratic order in the variation and find

$$G \sim x_1^2 + x_2^2 + (p\delta u + \delta v)\delta v + d_0 + \sum_{j=1}^{p-1} d_j (e^{\frac{2\pi i}{p} a})^j (1 + j\delta u + \delta v). \tag{2.2.7}$$

Completing the squares and evaluating the constant piece in the leading order puts $G = 0$ in the form

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 - \mu = 0, \tag{2.2.8}$$

with $\mu = \sum_{j=0}^{p-1} d_j e^{\frac{2\pi i}{p} aj}$. Thus, we can choose the flat coordinate $\hat{t}_a$ as

$$\hat{t}_a \sim \sum_{j=0}^{p-1} d_j e^{\frac{2\pi i}{p} aj}. \tag{2.2.9}$$

Repeating this for all $a = 0, 1, ..., p-1$, we find that the discriminant Δ to the leading order in d is given by

$$\Delta \sim \hat{t}_0 \hat{t}_1 \cdots \hat{t}_{p-1} \sim \prod_{a=0}^{p-1} \sum_{j=0}^{p-1} d_j e^{\frac{2\pi i}{p} aj} = \det_{i,j}(d_{i-j \bmod p}). \tag{2.2.10}$$

2.2.2 $x_1^2 + x_2^2 + x_3^2 + w'(x_4)^2 + f(x_4) = 0$

Essentially the same analysis applies to the geometry

$$x_1^2 + x_2^2 + x_3^2 + w'(x_4)^2 + f(x_4) = 0, \tag{2.2.11}$$

which has been extensively studied in the context related to gauge theory/matrix model correspondence [21]. Here $w(x) = \frac{1}{n+1} x^{n+1} + ...$ and f are polynomials of degrees $n+1$ and $n-1$, respectively. This means that $w'(x)^2 + f_{n-1}(x) = x^{2n} + ...$ is an arbitrary polynomial of degree $2n$ with a unit coefficient of x^{2n}. The non-linear sigma model on the geometry can be realized as the IR limit of the Landau-Ginzburg model with superpotential

$$W = \Lambda(X_1^2 + X_2^2 + X_3^2 + w'(X_4)^2 + f(X_4)). \tag{2.2.12}$$

When all the coefficients in f become small, n conifold singularities appear and n new branches develop. The ith one is characterized by large values of λ and $x_1 = x_2 = x_3 = 0, x_4 = a_i$ where $w'(x) = \prod_{a=1}^{n}(x - a_i)$. We define the ith Coulomb domain to be the place where the scalar field λ in the lowest component of Λ becomes large, and x_4 is frozen to a_i. In this case, the ith Coulomb domain is described by the Landau-Ginzburg model with $W = s_i/X$, where $s_i = \int_{A_i} \Omega = \int_{A_i} dx_1 dx_2 dx_3 dx_4/dG$ is the flat coordinate. A_i is the S^3 obtained by deforming the conifold singularity at $x_4 = a_i$. Applying the usual arguments, we get the 't Hooft expansion for the B-model open string on the blow-up of $x_1^2 + x_2^2 + x_3^2 + w'(x_4)^2 = 0$. Hence this B-model open string is large N dual to the B-model closed string on $G = x_1^2 + x_2^2 + x_3^2 + w'(x_4)^2 + f(x_4) = 0$. This proves the large N duality of the type used in [21].

We find it interesting to look at the deformed conifold case from this point of view. In this case, $w(x) = x^2/2$ and the Landau-Ginzburg superpotential is

$$W = \Lambda(X_1^2 + X_2^2 + X_3^2 + X_4^2 - \mu). \tag{2.2.13}$$

In the previous examples, the effective superpotential $W_{eff} \sim 1/X$ is given by applying the Gaussian approximation to the chiral multiplet fields, which is valid when $|\lambda| \gg 1$. In the case of the deformed conifold, this approximation is exact for any λ since W is already quadratic in $X_1, ..., X_4$. Thus, we obtain $W_{eff} = \mu/X$ without any approximation.[5] Therefore the sigma model on the deformed conifold is equivalent to the Landau-Ginzburg model with the $1/X$ superpotential [19, 22].

[5]The point arose from discussions with Donal O'Connell.

Since the $1/X$ Landau-Ginzburg model—effective theory in the C phase—is equivalent to the sigma model for the deformed conifold, full topological string amplitudes in this case are computable with worldsheets in pure C phase alone. This means that, from the point of view of the open string dual, perturbative open string Feynman diagrams should not contribute to topological string amplitudes. Indeed, this is consistent with the fact that the corresponding hermitian matrix model is Gaussian and there are no perturbative contributions [21]. This is a nontrivial check of our worldsheet analysis.

Chapter 3

Testing large N dualities with matrix models

In this chapter, we consider a class of large N dualities in topological string theory and test the dualities by using the matrix model formulation of open string theory.

In chapter 2 we discussed the large N dualities that involve simple conifold transitions as well as the $\mathbf{Z}_p$ quotient of the geometries. While an explicit form of the partition function for Chern-Simons theory on $S^3/\mathbf{Z}_p$ is known [23], this form includes the summation over all vacua in the theory. For the purpose of checking large N dualities, we want only the contribution from a single vacuum. For these reasons it has not been possible to exhibit the large N duality of $T^*(S^3/\mathbf{Z}_p)$ at the level of partition function. In [24, 10] it was shown that Chern-Simons theory on $S^3/\mathbf{Z}_p$ has a matrix model description. In [10] it was also shown that holomorphic Chern-Simons theory on $\mathbf{CP}^1$s inside the mirror (call it $\widetilde{X}$) of $T^*(S^3/\mathbf{Z}_p)$ has a matrix model description. Further, these matrix models are identical. For the case $p = 2$ the partition function of this matrix model was calculated perturbatively and was shown to agree with the holomorphic anomaly [25] predictions from the large N dual geometry, providing solid evidence for the proposed duality. For Chern-Simons theory on S^3 the matrix model was solved to all genera using orthogonal polynomials in [26] and the orientifold of the conifold was studied in [27]. The manifold $\widetilde{X}$ is given by the blowup of

$$xy = F_p(e^u, e^v), \tag{3.0.1}$$

where

$$F_p(e^u, e^v) = (e^v - 1)(e^{v+pu} - 1), \tag{3.0.2}$$

and by the general arguments by Dijkgraaf and Vafa [21, 28], the spectral curve of the corresponding matrix model should be a complex structure deformation of $F_p = 0$. In [17] Halmagyi and Yasnov found an expression for the spectral curve of the matrix model for Chern-Simons theory on $S^3/\mathbf{Z}_p$. This involved first showing that the matrix model has square root branch cuts, therefore its spectral curve has only two sheets. This led to an explicit expression for the resolvent, depending on $p-1$ parameters d_i, which in principle could be found perturbatively by performing the A-cycle integrals. The spectral curve can be read off from the resolvent and the d_i correspond to complex structure moduli. This will be reviewed in section 3.1.

In section 3.2 we use toric geometry to construct a resolution of the $\mathbf{Z}_p$-quotient of the resolved conifold. This is a particular A_{p-1} fibration over $\mathbf{CP}^1$. Then using the Hori-Vafa mirror map we can write down the mirror geometry and find that after a suitable coordinate redefinition, the nontrivial Riemann surface inside this 3-fold is precisely the spectral curve found in [17]. This explicitly identifies the large N dual of $T^*(S^3/\mathbf{Z}_p)$ for all $p > 1$. The matching of the geometries proves the equivalence of the leading order (in g_s) free energy between the matrix model and the A-model closed string on this particular fibration. This is the first check of this large N duality for $p > 2$.

3.1 The matrix model spectral curve

The partition function of Chern-Simons theory on the lens space $S^3/\mathbf{Z}_p$ [10] can be written as a matrix integral over p sets of eigenvalues, which we label by an index $I \in \{0, .., p-1\}$. The Ith set contains N_I eigenvalues. The measure is a product of two factors, a self interaction term (Δ_1) and a term containing the interaction between different sets of eigenvalues (Δ_2):

$$\Delta_1(u) = \prod_I \prod_{i \neq j} \left(2 \sinh \left(\frac{u_i^I - u_j^I}{2} \right) \right)^2, \tag{3.1.1}$$

$$\Delta_2(u) = \prod_{I<J} \prod_{i,j} \left(2 \sinh \left(\frac{u_i^I - u_j^J + d^{IJ}}{2} \right) \right)^2, \tag{3.1.2}$$

where $d^{IJ} = 2\pi i(I - J)/p$. The potential has an overall factor of p relative to the S^3 case:

$$V(u) = p \sum_{I,i} \frac{(u_i^I)^2}{2}. \tag{3.1.3}$$

In the above notations the Chern-Simons partition function becomes

$$Z \sim \int \prod_{I=0}^{p-1} \prod_{i=1}^{N_I} du_i^I \Delta_1(u) \Delta_2(u) \exp\left(-\frac{1}{g_s} V(u) \right). \tag{3.1.4}$$

We define individual resolvents for each set of the eigenvalues by

$$\omega_I(z) = g_s \sum_i \coth\left(\frac{z - u_i^I}{2}\right), \tag{3.1.5}$$

and the total resolvent, which we are most interested in, is

$$\omega(z) = \sum_I \omega_I\left(z - \frac{2\pi i I}{p}\right). \tag{3.1.6}$$

Anticipating taking the large N limit we also introduce 't Hooft parameters $S_I = g_s N_I$ and $S = \sum_I S_I$. The equation of motion for each eigenvalue is

$$p u_i^I = g_s \sum_{j\neq i} \coth\left(\frac{u_i^I - u_j^I}{2}\right) + g_s \sum_{J\neq I}\sum_j \coth\left(\frac{u_i^I - u_j^J + d^{IJ}}{2}\right). \tag{3.1.7}$$

From the large N limit of this equation we can derive

$$\frac{1}{2}\omega^2(z) - p\sum_I \left(z - \frac{2\pi i I}{p}\right)\omega_I\left(z - \frac{2\pi i I}{p}\right) = f(z), \tag{3.1.8}$$

where $f(z)$ is a regular function,

$$f(z) = p\, g_s \sum_I \sum_i (u_i^I + 2\pi i I/p - z)\coth\left(\frac{z - u_i^I - 2\pi i I/p}{2}\right) + \frac{S^2}{2}. \tag{3.1.9}$$

Given the large N limit of the equations of motion it is possible to find the total resolvent $\omega(z)$. This method, which we will now review, has been developed and checked in [17].

We assume that the eigenvalues spread only along the real line. This assumption leads to the correct result for our case. Note that we do not make any assumption on the type of cuts. In the total resolvent $\omega(z)$, the individual resolvents come with relative shifts of the argument by $2\pi i I/p$. Therefore the cuts in the total resolvent are now separated by $2\pi i I/p$. For example, if $\omega_0(z)$ jumps at the point z, all other individual resolvents $\omega_I(z - 2\pi i I/p)$ with $I \neq 0$ are regular at this point. This means that on the Ith cut the total resolvent jumps only due to the resolvent $\omega_I(z)$. From this it follows that

$$\frac{1}{2}\left(\omega_+\left(z+\frac{2\pi i I}{p}\right)+\omega_-\left(z+\frac{2\pi i I}{p}\right)\right) = pz, \quad (I\text{th cut}) \tag{3.1.10}$$

and so every cut is a square root.

We label the contour around the Ith cut as A_I. From (3.1.6) it is clear that

$$\lim_{z\to\infty} \omega(z) = S, \tag{3.1.11}$$

$$\lim_{z\to-\infty} \tilde{\omega}(z) = -S, \tag{3.1.12}$$

where $\tilde{\omega}(z)$ is the value of the resolvent on the second sheet. From (3.1.5) we also have that

$$\frac{1}{2}\oint_{A_I} \omega(z)dz = 2\pi i S_I. \tag{3.1.13}$$

Since the integral over the $A = \sum_I A_I$ cycle is fixed by (3.1.12), there are only $p-1$

independent periods. Now we construct a regular function,

$$g(Z) = e^{\omega/2} + Z^p e^{-\omega/2}, \tag{3.1.14}$$

which has the limiting behavior,

$$\lim_{Z\to\infty} g(Z) = e^{-S/2} Z^p, \tag{3.1.15}$$

$$\lim_{Z\to 0} g(Z) = e^{-S/2}, \tag{3.1.16}$$

and is thus of the form

$$g(Z) = e^{-S/2}(Z^p + d_{p-1}Z^{p-1} + ... + d_1 Z + 1). \tag{3.1.17}$$

The function $g(Z)$ depends on $p-1$ moduli d_n, which could be found by evaluating the period integrals (3.1.13). Since we have already fixed the integral over the cycle $A = \sum_I A_I$ by (3.1.16), there are only $p-1$ independent A-periods.

We can solve (3.1.14) for $\omega(Z)$ to get

$$\frac{\omega(Z)}{2} = \log\left(\frac{1}{2}\left(g(Z) - \sqrt{g^2(Z) - 4Z^p}\right)\right), \tag{3.1.18}$$

the function under the square root is a polynomial of degree $2p$, it has $2p$ distinct roots that depend on only $(p-1)$ parameters. Thus the spectral curve consists of two cylinders glued together along p cuts. Note that the center of the Ith cut is at the point $z = 2\pi i I/p$. The cycles around each cut are A cycles. There is also a set of non-compact cycles, the B cycles. Each B_I cycle starts at infinity on the physical sheet (the sheet where the resolvent is finite) and goes through the Ith cut to a point at infinity on the other sheet.

From (3.1.18) we see that the spectral curve is given by

$$(e^v - 1)(e^{pu+v} - 1) + e^S - 1 + e^v \sum_{n=1}^{p-1} d_n e^{nu} = 0, \tag{3.1.19}$$

which is a complex structure deformation of $F_p = 0$ (from (3.0.1)). Here $z \equiv u$ and $v = (S - \omega)/2$. It is worth mentioning that the functions $d_n(S_I)$ are available as a power series in 't Hooft parameters S_Is which is valid only in the region of small S_I. We will see that this region corresponds to negatively large values of Kähler parameters in the A-model.

3.2 The $\mathbf{Z}_p$-quotient of the resolved conifold

We will now construct the $\mathbf{Z}_p$-quotient of the resolved conifold $(\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbf{CP}^1)$ using standard toric methods.[1] We will then apply the Hori-Vafa mirror map, from which we can obtain the Riemann surface that is the nontrivial part of the mirror geometry.

The deformed conifold (T^*S^3) can be written as $\det A = \mu$, where

$$A = \begin{pmatrix} z_1 & z_3 \\ z_2 & z_4 \end{pmatrix}. \tag{3.2.1}$$

[1] A good reference for the basics of toric geometry is [29]

We can orbifold this geometry by a $\mathbf{Z}_p$ symmetry generated by

$$A \mapsto \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} A, \tag{3.2.2}$$

where $\alpha = e^{2\pi i/p}$. The S^3 is given by $|z_1|^2 + |z_3|^2 = \mu$ (assuming μ is real and positive) and the $\mathbf{Z}_p$ acts on this as $(z_1, z_3) \to (\alpha z_1, \alpha z_3)$, which gives the lens space $S^3/\mathbf{Z}_p = L(p,1)$, thus the entire 3-fold is now $T^*(S^3/\mathbf{Z}_p)$.

We now perform this orbifold action on the other side of the proposed large N duality. The resolved conifold can be expressed as

$$A \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = 0, \tag{3.2.3}$$

where $(\lambda_1 : \lambda_2)$ are the homogeneous coordinates on $\mathbf{CP}^1$. We simply extend the $\mathbf{Z}_p$ action given by (3.2.2) to the resolved conifold. Clearly this will act trivially on the base $\mathbf{CP}^1$ but nontrivially on the fiber coordinates since these are contained in the matrix A. Thus the resulting orbifold is a particular fibration of an A_{p-1}-singularity over $\mathbf{CP}^1$. We will show that the large N dual of $T^*(S^3/\mathbf{Z}_p)$ is the blowup of this space.

The procedure of blowing up can be conveniently described in the language of toric geometry. First we need the fan for the singular manifold. The standard two coordinate patches of $\mathbf{CP}^1$ can be used to trivialize the $\mathcal{O}(-1) + \mathcal{O}(-1) \to \mathbf{CP}^1$ bundle. On the first patch we have coordinates $(u, x_1, x_2) = (\lambda_1/\lambda_2, z_1, z_2)$ and on the second we have $(v, y_1, y_2) =$

$(\lambda_2/\lambda_1, -z_3, -z_4)$. On the overlap of the two patches these coordinates are related by $v = 1/u, y_1 = ux_1, y_2 = ux_2$. $\mathbf{Z}_p$ acts on the fiber coordinates. For the quotient of the first patch, therefore, we introduce invariant coordinates $(\xi_1, \xi_2, \xi_3) := (x_1^p, x_2^p, x_1x_2)$ satisfying $\xi_1\xi_2 = \xi_3^p$. Similarly we introduce (η_1, η_2, η_3) satisfying $\eta_1\eta_2 = \eta_3^p$ for the second patch. These two sets of coordinates are glued together by $v = 1/u, \eta_1 = u^p\xi_1, \eta_2 = u^p\xi_2, \eta_3 = u^2\xi_3$.

Toric geometry is designed to encode these transition functions (see the appendix for notations). Namely, we get the relations

$$\begin{aligned} &u_{12} + u_{13} = pu_{14}, \quad &&u_{22} + u_{23} = pu_{24}, \\ &u_{21} = -u_{11}, &&u_{22} = pu_{11} + u_{12}, \\ &u_{23} = pu_{11} + u_{13}, &&u_{24} = 2u_{11} + u_{14} \end{aligned} \tag{3.2.4}$$

for the lattice vectors in M. The first two relations imply that u_{14} and u_{24} are not vertices of the dual cones $\check{\sigma}_1$ and $\check{\sigma}_2$, respectively, where $\check{\sigma}_1$ is spanned by u_{11}, u_{12}, u_{13}, and $\check{\sigma}_2$ is spanned by u_{21}, u_{22}, u_{23}. We choose a basis such that $u_{11} = (1,0,0), u_{12} = (0,1,0), u_{14} = (0,0,1)$, this determines all remaining vectors u_{ij}.

From this we find that the cone σ_1 is generated by v_1, v_2, v_3, and the cone σ_2 is generated by v_4, v_2, v_3, where $v_1 = (1,0,0), v_2 = (0,p,1), v_3 = (0,0,1), v_4 = (-1,p,2)$. After a $GL(3,\mathbf{Z})$ transformation $(x', y', z') = (x+y, x, x+z)$, they become $v_i = (\nu_i, 1)$, where $\nu_1 = (1,1), \nu_2 = (p,0), \nu_3 = (0,0), \nu_4 = (p-1,-1)$. The toric fan of $A_{p-1} \to \mathbf{CP}^1$ consists of the cones σ_1, σ_2 and their faces.

To obtain a smooth manifold, we subdivide the cones σ_1, σ_2 by introducing $v_i = (\nu_i, 1)$

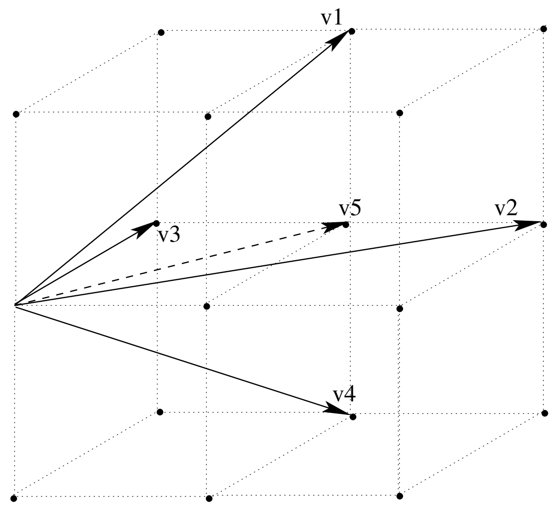


Figure 3.1: *The fan for the resolution of the* $\mathbf{Z}_2$ *orbifold of* $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbf{CP}^1$.

where $\nu_i = (i-4, 0)$ for $i = 5, \ldots, p+3$. The fan Δ of the smooth toric manifold is the union of $[v_1, v_3, v_5]$, $[v_1, v_{p+3}, v_2]$, $[v_1, v_5, v_6]$, $[v_1, v_6, v_7]$, $\ldots$ $[v_1, v_{p+2}, v_{p+3}]$ from σ_1 and $[v_4, v_3, v_5]$, $[v_4, v_{p+3}, v_2]$, $[v_4, v_5, v_6]$, $[v_4, v_6, v_7]$, $\ldots$ $[v_4, v_{p+2}, v_{p+3}]$ from σ_2 (see figure 3.1). Here $[u, v, w]$ denotes the cone spanned by the vectors u, v, w. It is easily seen that each cone now has volume equal to 1, where we normalize the volume of the cone generated by $[1,0,0], [0,1,0], [0,0,1]$ to 1. The toric web diagrams for these fans are drawn for the cases $p = 1, 2,$ and 3 in figure 3.2. This geometry and other A_{p-1} fibrations over $\mathbf{CP}^1$ were recently considered in [30, 31] for the purposes of geometric engineering [32]. Setting $m = 0$ in [31] gives the above geometry.

From the data of the fan Δ we can read off the charge vectors $Q_a, a = 0, 1, \cdots, p-1$ of the corresponding linear sigma model. They are the generators of the lattice $D = \{Q \in$

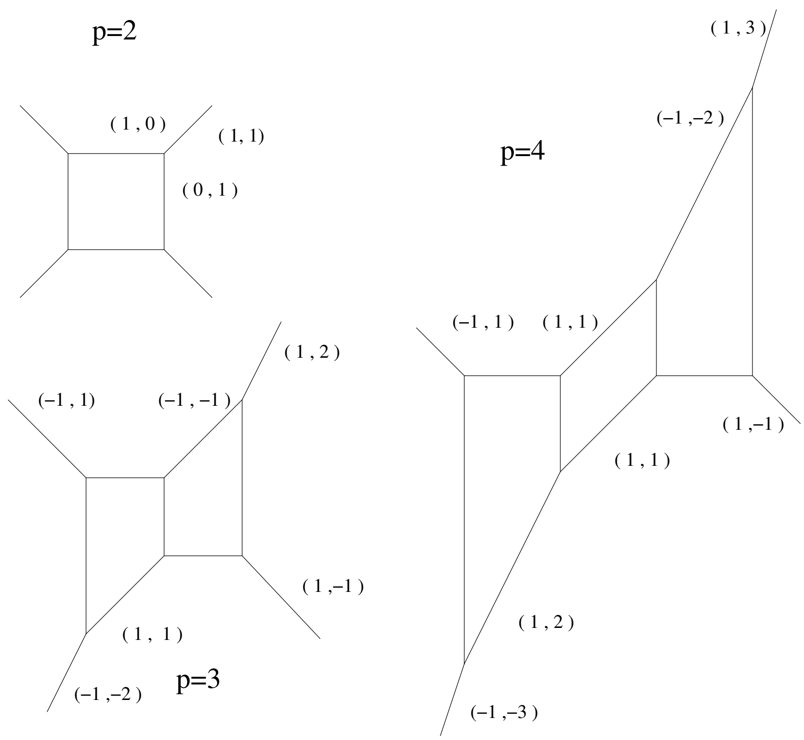


Figure 3.2: *The toric web diagrams of the large N dual of $T^*(S^3/\mathbf{Z}_p)$ for some values of p.*

$\mathbf{Z}^{p+3}|\sum_i Q_i v_i = 0\}$ [18]. After rearranging the columns, these are given by

$$\begin{array}{llccccccccccc} Q_0 & = (& 0, & 1, & 1, & -1, & -1, & 0, & \ldots, & 0, & 0, & 0, & \cdots\ 0), \\ Q_j & = (& -(j+1), & j, & 0, & 0, & 0, & 0, & \ldots, & 0, & \overset{(5+j)\text{th}}{1}, & 0, & \cdots\ 0), \\ Q_{p-1} & = (& -p, & p-1, & 1, & 0, & 0, & 0, & \ldots, & 0, & 0, & 0, & \cdots,\ 0), \end{array} \tag{3.2.5}$$

where $1 \leq j \leq p-2$. According to [11, 33], the mirror geometry is given by $zw = \sum_{i=0}^{p+2} e^{-y_i}$, where the fields are related by $\sum_i Q_{ai} y_i = t_a$ and one of the y_i variables is to be set to zero. Eliminating y_2, y_4, y_{4+j} $(1 \leq j \leq p-2)$, and setting $y_1 = 0$, we get the Riemann surface inside this 3-fold to be

$$0 = 1 + e^{-y_0} + e^{-y_3} + e^{-t_{p-1}} e^{-py_0} + e^{t_0 - t_{p-1}} e^{-py_0 + y_3} + \sum_{j=1}^{p-2} e^{-t_j} e^{-(j+1)y_0}. \tag{3.2.6}$$

Now after the coordinate transformation $u = -y_0 - t_{p-1}/p, v = y_3 + t_0 + \pi i$, we can write this as

$$(e^v - 1)(e^{pu+v} - 1) + e^{t_0} - 1 - e^v \left(e^{t_{p-1}/p} e^u + \sum_{j=1}^{p-2} e^{-t_j + (j+1)t_{p-1}/p} e^{(j+1)u} \right) = 0, \tag{3.2.7}$$

which is precisely (3.1.19).

Chapter 4

Calabi-Yau crystals

Topological string theory [6, 25] is currently undergoing a drastic paradigm change. Reshetikhin, Okounkov, and Vafa [34] realized that various amplitudes for the topological A-model on $\mathbf{C}^3$ can be expressed in terms of classical statistical models of a melting crystal. Iqbal, Nekrasov, and Vafa [35] proposed interpreting the crystals in terms of quantum foam or Kähler gravity, which is the target space theory of A-model closed string theory. Mathematically speaking, this means that Gromov-Witten invariants are related to the so-called Donaldson-Thomas invariants [36, 37].

Central to the dramatic paradigm shift in topological string theory is the interpretation of the Calabi-Yau crystal as describing the violent fluctuations of topology and geometry at microscopic scales. This is reminiscent of geometric transition, where open string theory and closed string are related via a local change of topology and geometry. Or rather, when crystal picture is combined with geometric transition, one naturally expects that the geometric change is part of the gravitational fluctuations or quantum foam. In this chapter, we realize this expectation and make it precise.

We propose a crystal melting model that describes the A-model closed strings on the resolved conifold $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbf{CP}^1$. Our model is a simple modification of the model for $\mathbf{C}^3$. The Kähler gravity interpretation leads one to view the gluing prescription of the topological vertices as computing partition functions of crystal models for general toric Calabi-Yau manifolds. Although the resolved conifold was discussed in that context in [35], what we propose is different from the prescription described there. Indeed it was emphasized in [38] that "the global rule of melting is absent for closed strings on toric Calabi-Yau manifolds with more than one fixed point of the toric action." One of the purposes of the present paper is to amend this situation.

Our model is obtained from the large N dual Chern-Simons theory on S^3. We show that the Chern-Simons theory can be formulated as a simple unitary matrix model that involves a theta function. This representation is then used to obtain a free field formula for the Chern-Simons theory, which is interpreted in terms of a statistical model.

It is also possible to introduce non-compact D-branes to the crystal, enlarging the arena of study to include open strings. It was shown in [38] for $\mathbf{C}^3$ that this corresponds to having defects in the crystal. In Chern-Simons theory, the observables are the Wilson loops that go around the circles in various knots and links in the 3-manifold. We show that the computation of a Wilson loop along an unknot can be nicely done in the unitary matrix model. This then translates to a natural crystal model with defects that represents some number of non-compact D-branes intersecting the $\mathbf{CP}^1$ in the resolved conifold. The fact that these D-branes fit neatly into the crystal shows that our model of the Calabi-Yau crystal is a natural one.

The crystal melting model is also a useful computational tool. For one crystal model, there are two ways to represent it in terms of free bosons and fermions. We use this freedom to explicitly compute certain amplitudes in Chern-Simons theory. We find an interesting phenomenon where the Kähler modulus of the resolved conifold is shifted by a multiple of g_s in the presence of non-compact D-branes. We also discuss the possible application of the crystal representation to prove more general examples of topological string large N dualities to all order in g_s.

The plan of the paper is as follows: In section 4.1 we propose a crystal melting model and demonstrate that it computes the partition function for the resolved conifold. In section 4.2 we explain how the crystal picture naturally arises from the dual open string theory. We also discuss the non-perturbative mismatch. In section 4.3 we derive from Chern-Simons theory the crystal models for non-compact D-branes, realizing them as defects in the crystal. Section 4.4 discusses the possible application of the crystal computation as a way to prove large N dualities to all order in g_s.

4.1 Crystal melting model for the resolved conifold

Let us recall the crystal model for $\mathbf{C}^3$ [34]. The zero-energy configuration is the positive octant $x, y, z \geq 0$ in $\mathbf{R}^3$ filled with atoms. Here an atom at (x_0, y_0, z_0) is a filled box $\{(x_0 + s_x, y_0 + s_y, z_0 + s_z) | 0 \leq s_x, s_y, s_z \leq 1\}$. We consider removing atoms from the corner. The allowed configurations are defined recursively as follows: The configuration where the whole octant is filled is allowed. If an allowed configuration has an atom at (x_0, y_0, z_0) such that there are no atoms in the region $\{(x,y,z)|x < x_0, y < y_0, z < z_0\}$, one can remove the atom at (x_0, y_0, z_0) to obtain another allowed configuration. The allowed configurations are also called 3D Young diagrams, in analogy with the familiar counterpart in two dimensions. The partition function is

$$Z = \sum_{\pi} q^{|\pi|}, \tag{4.1.1}$$

where the summation is over 3D Young diagrams π, and $q = e^{-g_s}$, $|\pi|$ is the number of atoms removed. This partition function agrees with the partition function of A-model closed strings on $\mathbf{C}^3$. This fact can be proved by the use of free field techniques familiar in string theory [34]. Below we generalize the technique to the situations of our interest.

The model we propose for the resolved conifold is the following: We add one more condition that further restricts the allowed configurations: Atoms in the region $x \geq N$ cannot be removed. Here N is related to the Kähler modulus t as $t = g_s N$. Note that this condition introduces a "wall" that together with the original three walls constitutes the toric diagram for the resolved conifold.

Now we demonstrate that this crystal melting program indeed reproduces the partition function for the resolved conifold. For this purpose, we express the partition function in

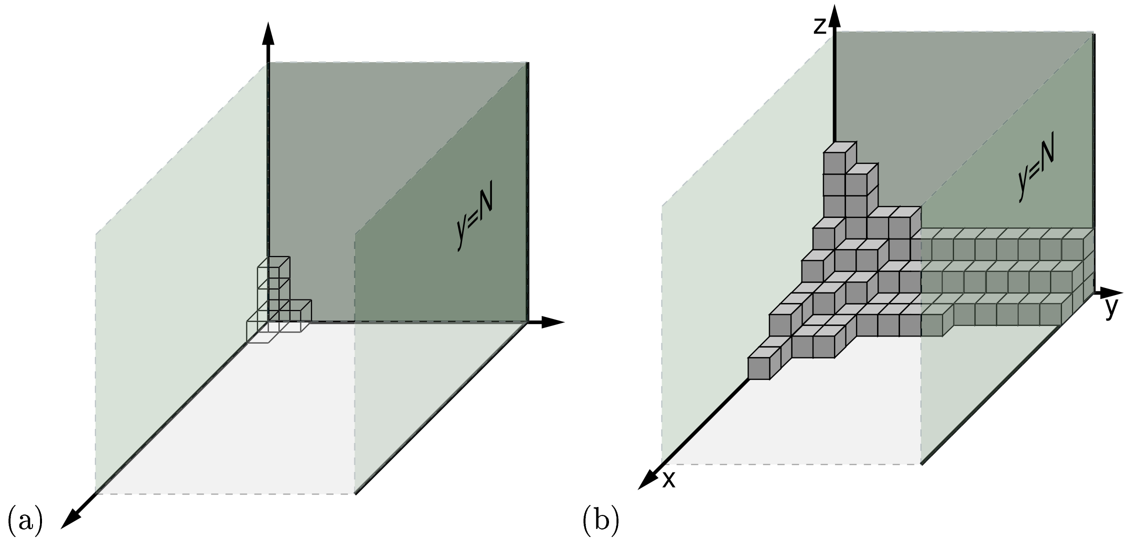


Figure 4.1: (a) The crystal melting model for the resolved conifold $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbf{CP}^1$. The edges, drawn as solid lines, of the positive octant bounded by the wall at $y = N$ form the toric diagram of the resolved conifold. (b) Many atoms have been removed from the crystal. Atoms cannot be removed from the region beyond the wall at $y = N$.

terms of free fermions and bosons:

$$\psi(z) = \sum_{r \in \mathbf{Z}+1/2} \frac{\psi_r}{z^{r+1/2}},\ \bar{\psi}(z) = \sum_r \frac{\bar{\psi}_r}{z^{r+1/2}}, \tag{4.1.2}$$

$$\{\psi_r, \bar{\psi}_s\} = \delta_{r+s,0}, \tag{4.1.3}$$

$$\phi(z) = x_0 - i\alpha_0 \log z + i \sum_{n \neq 0} \frac{\alpha_n}{n z^n}, \tag{4.1.4}$$

$$[\alpha_m, \alpha_n] = m\delta_{m+n,0}. \tag{4.1.5}$$

These are related via

$$i\partial\phi(z) =: \psi\bar{\psi}(z) :, \quad \psi(z) =: e^{i\phi(z)} :, \quad \bar{\psi}(z) =: e^{-i\phi(z)} : . \tag{4.1.6}$$

Now we define

$$\Gamma_{\pm}(z) = \exp \sum_{n>0} \frac{z^{\pm n}}{n} \alpha_{\pm n}. \tag{4.1.7}$$

It is well-known that neutral (zero momentum in the bosonic language) fermionic Fock states are labeled by (2D) Young diagrams μ, which we denote as $\mu = (\mu_1 \geq \mu_2 ... \geq \mu_d > 0)$. More explicitly, such Fock states are given by

$$\begin{aligned} |\mu\rangle &= \prod_{i=1}^{\infty} \psi_{i-\mu_i-1/2} |0\rangle\rangle \\ &= \prod_{i=1}^{d} \bar{\psi}_{-a_i} \psi_{-b_i} |0\rangle, \end{aligned} \tag{4.1.8}$$

where $|0\rangle\rangle$ is the state that is annihilated by all $\bar{\psi}_r, r \in \mathbf{Z} + 1/2$, and we have defined

$$a_i = \mu_i - i + 1/2, \quad b_i = \mu_i^t - i + 1/2. \tag{4.1.9}$$

μ^t is the transposed Young diagram. The Virasoro zero mode L_0 counts the number $|\mu|$ of boxes in the Young diagram μ:

$$L_0 |\mu\rangle = |\mu| |\mu\rangle. \tag{4.1.10}$$

Two Young diagrams λ and μ are said to interlace (and we write $\lambda \succ \mu$) if they satisfy

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq ... \tag{4.1.11}$$

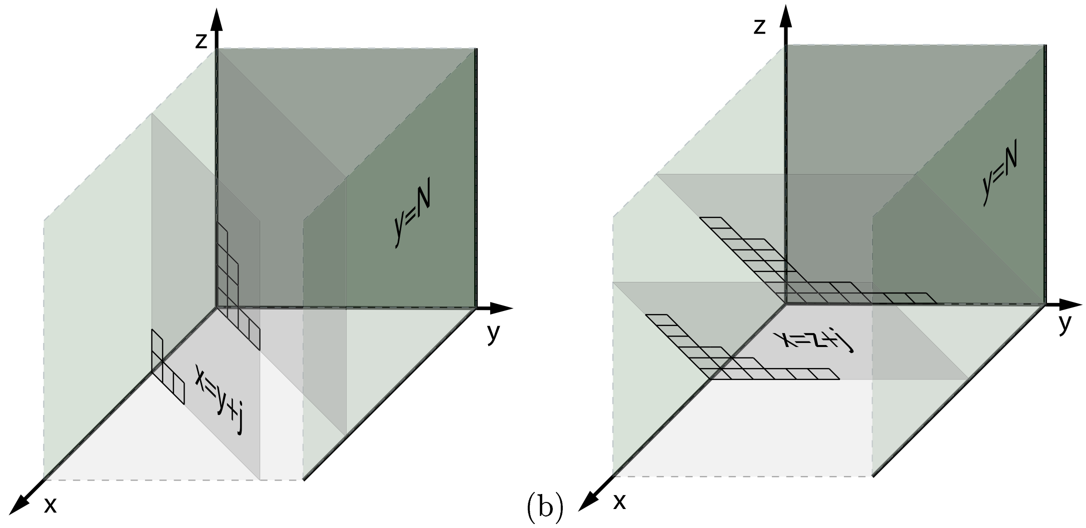
In other words, λ and μ interlace if and only if λ contains μ and μ contains λ with the first row removed. The interlacing condition is equivalent to the local condition for two Young diagrams one finds by slicing the allowed configuration of the crystal by the planes $x = y + j$ and $x = y + j + 1$ [34]. The operators $\Gamma_\pm(z)$ are useful because of the properties

$$\begin{aligned} \Gamma_+(1)|\lambda\rangle &= \sum_{\lambda \succ \mu} |\mu\rangle, \\ \Gamma_-(1)|\lambda\rangle &= \sum_{\mu \succ \lambda} |\mu\rangle. \end{aligned} \tag{4.1.12}$$

The partition function for the crystal can be written as

$$\begin{aligned} Z_{\text{crystal}}(q, t = g_s N) &= \langle 0| \left(\prod_{n=1}^{\infty} q^{L_0} \Gamma_+(1) \right) q^{L_0} \left(\prod_{m=1}^{N} \Gamma_-(1) q^{L_0} \right) |0\rangle \\ &= \langle 0| \prod_{n=1}^{\infty} \Gamma_+(q^{n-1/2}) \prod_{m=1}^{N} \Gamma_-(q^{-(m-1/2)}) |0\rangle. \end{aligned} \tag{4.1.13}$$

This can be understood as slicing the crystal by planes $x = y + j, j \in \mathbf{Z}$. Note that we have a finite product of vertex operators acting on $|0\rangle$. This restricts a 3D Young diagram to a trivial 2D Young diagram on the slice $x = y - N$. The interlacing conditions then imply that the 2D Young diagrams must have at most one row on the slice $x = y - N + 1$, two rows on $x = y - N + 2$, etc. Thus the free-field correlator represents a crystal model bounded by a



(a) (b)

Figure 4.2: (a) Closed string slicing: This slicing by planes $x = y + j$ allows one to compute the closed string amplitude as in eq. (4.1.14). (b) Open string slicing: Another slicing of the crystal by planes $x = z + j$ corresponds to the free-field representation eq. (4.2.1) obtained from Chern-Simons theory.

wall at $y = N$. See figure 4.2(a).

Now we can explicitly compute the partition function:

$$\begin{aligned} Z_{\text{crystal}}(q, t = g_s N) &= \langle 0| e^{-\sum_{n>0} \frac{\alpha_n}{n[n]}} e^{-\sum_{n>0} \frac{1-q^{Nn}}{n[n]} \alpha_{-n}} |0\rangle \\ &= e^{\sum_{n>0} \frac{1-q^{Nn}}{n[n]^2}} \\ &= M(q) e^{-\sum_{n>0} \frac{e^{-nt}}{n[n]^2}}. \end{aligned} \tag{4.1.14}$$

Here $[n] = q^{n/2} - q^{-n/2}$. Taking $N \to \infty$ pushes the wall at $y = N$ to infinity, and the partition function reduces to the result for $\mathbf{C}^3$.

This crystal model and the resulting amplitude are different from those discussed in [35]. While our crystal has a fixed finite size in the y direction, in [35] the distance between the two crystal corners are not fixed because two finite size 3D partitions are connected through a region of length $t = g_s N$. Consequently, instead of a single power of $M(q)$ in our model, the model in [35] gives the square of $M(q)$. More generally, a closed string partition function contains $M(q)^{\chi(X)/2}$, where $\chi(X)$ is the Euler characteristic of the target space X. If the target space X is non-compact, the definition of the Euler characteristic is ambiguous. In the context of the large N duality, it is known [5] that one should assign the value 2 to the Euler characteristic of the resolved conifold as we just did. This is natural in the sense that the target space admits one Kähler deformation but no complex structure deformation, and the general formula for a (compact) Calabi-Yau manifold is $\chi(X) = 2\,[(\#\text{Kähler deformations}) - (\#\text{Complex structure deformations})]$.

4.2 Large N dual open string theory

In this section, we study the crystal melting problem from the point of view of the large N duality.

The crystal model in the previous section can also be expressed as

$$\begin{aligned} & Z_{\text{crystal}}(q, t = g_s N) \\ = \; & \langle 0| \prod_{n=1}^{\infty} \Gamma_+(q^{n-1/2}) \mathbf{1}_{d^t \le N} \prod_{m=1}^{\infty} \Gamma_-(q^{-(m-1/2)}) |0\rangle. \end{aligned} \tag{4.2.1}$$

Here $\mathbf{1}_{d^t \le N}$ is the operator that projects onto the subspace spanned by $|\mu\rangle$ such that the Young diagram μ has at most N columns. This free-field expression corresponds to slicing the crystal by planes $z - x = j, j \in \mathbf{Z}$. See figure 4.2(b).

We call this the "open string slicing" because, as we will see below, this representation of the crystal naturally arises from Chern-Simons theory.

4.2.1 Unitary matrix model for Chern-Simons theory

The large N duality of Gopakumar and Vafa relates $U(N)$ Chern-Simons theory on S^3 to topological closed string on the resolved conifold $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbf{CP}^1$. The dictionary is that the Kähler modulus t of the closed string theory geometry is identified with the 't Hooft parameter $g_s N$. Certain amplitudes on the resolved conifold, including the closed string amplitudes, can be computed within the framework of Chern-Simons theory. We now develop a unitary matrix model formulation of Chern-Simons theory, which will be used to derive the crystal model for the resolved conifold later.

The partition function of the $U(N)$ Chern-Simons theory on S^3 is given by [23]:

$$Z_{\text{CS}}(N,k,U(N)) \;=\; \frac{1}{(k+N)^{N/2}} \prod_{\alpha>0} 2\sin\frac{\pi\alpha\cdot\rho}{k+N}. \tag{4.2.2}$$

Here $2\pi i/(k+N) = g_s$ is the string coupling constant, and the product is over the positive roots of $SU(N) \subset U(N)$, which are given by $\alpha_{ij} = e_i - e_j \in \mathbf{C}^N(\simeq$ Cartan subalgebra) for $i<j$. $\rho = (1/2)\sum_{\alpha>0}\alpha = \sum_{i=1}^{N}(\frac{N+1}{2} - i)e_i$ is the Weyl vector. Now note the following formula by Weyl for the denominator of the Lie algebra characters

$$\prod_{\alpha>0} 2\sinh(\alpha\cdot u) = \sum_{w\in W} \epsilon(w) e^{w(\rho)\cdot u}, \tag{4.2.3}$$

where W is the Weyl group isomorphic to the permutation group S_N and u is an arbitrary element of the Cartan subalgebra of $U(N)$. We use this identity to rewrite $Z_{\rm CS}(N,k,U(N)) =:$ $Z_{\rm CS}(g_s, N)$ as

$$\begin{aligned} Z_{\rm CS}(g_s, N) &= \frac{e^{-\frac{N(N-1)\pi i}{4}}}{(k+N)^{N/2}} \sum_{w\in W} \epsilon(w) e^{g_s w(\rho)\cdot\rho} \\ &= \left(\frac{g_s}{2\pi}\right)^{N/2} e^{-\frac{\pi i}{4}N^2} q^{-\frac{N(N-1)}{12}} \tilde{Z}_{\rm CS}(g_s, N). \end{aligned} \tag{4.2.4}$$

Here we have factored out the non-trivial part of the partition function:

$$\tilde{Z}_{\rm CS}(g_s, N) = \sum_{w\in W} \epsilon(w) q^{\frac{1}{2}(w(\rho)-\rho)^2}. \tag{4.2.5}$$

q is again e^{-g_s}. In what follows, we "analytically continue" in g_s and regard g_s as a complex parameter with a positive real part. We can introduce another sum over the Weyl group and an integral over the maximal torus as follows:[1]

$$\begin{aligned} & \tilde{Z}_{\rm CS}(g_s, N) \\ = & \frac{1}{|W|} \sum_{w,w'\in W} \epsilon(w)\epsilon(w') q^{\frac{1}{2}(w(\rho)-w'(\rho))^2} \\ = & \frac{1}{|W|} \int \prod_{i=1}^{N} \left(\frac{d\theta_i}{2\pi} \vartheta_{00}(e^{i\theta_i}; q) \right) \sum_{w,w'\in W} \epsilon(w)\epsilon(w') e^{i(w(\rho)-w'(\rho))\cdot\theta}. \end{aligned} \tag{4.2.6}$$

[1]Here we use the identity $q^{m^2/2} = \int_0^{2\pi} \frac{d\theta}{2\pi} \vartheta_{00}(e^{i\theta}; q) e^{im\theta}$. If we instead use $q^{m^2/2} = \int_{-\infty}^{\infty} \frac{du}{\sqrt{2\pi g_s}} e^{-\frac{u^2}{2g_s}} e^{mu}$, we get the matrix model with a non-compact integration region introduced in [24, 10]. The matrix model there can be transformed to our unitary matrix model via $u = i(\theta + 2\pi n)$, performing the sum over $n \in \mathbf{Z}$ and a modular transformation. The author thanks Hirosi Ooguri for pointing this out.

Here

$$\vartheta_{00}(e^{i\theta};q) := \sum_{m\in\mathbf{Z}} q^{\frac{m^2}{2}} e^{im\theta} \tag{4.2.7}$$

is one of Jacobi's theta functions. By making use of the Weyl denominator formula in eq. (4.2.3) again, we get

$$\tilde{Z}_{\rm CS} \;=\; \frac{1}{|W|}\int\left(\prod_{i=1}^{N}\frac{d\theta_i}{2\pi}\vartheta_{00}(e^{i\theta_i})\right)\left(\prod_{\alpha>0}2\sin\frac{\alpha\cdot\theta}{2}\right)^2. \tag{4.2.8}$$

The second factor now represents the Haar measure for $U(N)$ pushed down to the maximal torus. The partition function can be written in a very simple form,

$$\tilde{Z}_{\rm CS} = \int_{U(N)} dU \det\vartheta_{00}(U;q), \tag{4.2.9}$$

where the measure is normalized so that the volume of $U(N)$ is unity. This expression holds for any gauge group of the Chern-Simons theory on S^3 when the corresponding Haar measure is used.

4.2.2 Crystal from Chern-Simons theory

Now we use the product formula for the theta function

$$\begin{aligned}\vartheta_{00}(e^{i\theta};q) &= \prod_{j=1}^{\infty}(1-q^j)(1+e^{i\theta}q^{j-1/2})(1+e^{-i\theta}q^{j-1/2}) \\ &= \left(\prod_{j=1}^{\infty}(1-q^j)\right)\exp\left[\sum_{n>0}(-1)^n\frac{e^{in\theta}+e^{-in\theta}}{n[n]}\right]\end{aligned} \tag{4.2.10}$$

to write

$$\begin{aligned} \tilde{Z}_{\mathrm{CS}} &= \left(\prod_{j=1}^{\infty}(1-q^j)\right)^N \int dU \exp\left[\sum_{n>0}(-1)^n \frac{\mathrm{Tr}U^n + \mathrm{Tr}U^{-n}}{n[n]}\right] \\ &= \left(\prod_{j=1}^{\infty}(1-q^j)\right)^N \int dU \exp\left[\sum_{n>0} \frac{\mathrm{Tr}U^n + \mathrm{Tr}U^{-n}}{n[n]}\right]. \end{aligned} \tag{4.2.11}$$

To obtain the free-field expression for the partition function, we introduce the coherent states:

$$|U\rangle := \exp\left[\sum_{n>0} \frac{1}{n} \mathrm{Tr}\ U^n \alpha_{-n}\right] |0\rangle. \tag{4.2.12}$$

These states satisfy

$$\alpha_n|U\rangle = \mathrm{Tr}\ U^n|U\rangle, \tag{4.2.13}$$

$$\int dU|U\rangle\langle U| = \mathbf{1}_{d\leq N}, \tag{4.2.14}$$

where $\mathbf{1}_{d\leq N}$ is the projection to the subspace spanned by $|\mu\rangle$ such that the number of rows in μ is less than or equal to N. This formalism was extensively used in the context of 2D Yang-Mills theory, which has recently been attracting some attention. See [39] and the references therein.

By making use of $|U\rangle$, we can write

$$
\begin{aligned}
\tilde{Z}_{\rm CS} &= \left(\prod_{j=1}^{\infty}(1-q^j)\right)^N \langle 0|e^{\sum_{n>0}\frac{\alpha_n}{n[n]}}\int dU|U\rangle\langle U|e^{\sum_{n>0}\frac{\alpha_{-n}}{n[n]}}|0\rangle \\
&= \left(\prod_{j=1}^{\infty}(1-q^j)\right)^N \langle 0|e^{\sum_{n>0}\frac{\alpha_n}{n[n]}}\mathbf{1}_{d\le N}e^{\sum_{n>0}\frac{\alpha_{-n}}{n[n]}}|0\rangle. \qquad (4.2.15)
\end{aligned}
$$

Since $\alpha_n \to -\alpha_n$ is equivalent to $R \to R^t$ (c.f. eq. (4.1.8)), we finally obtain

$$
\begin{aligned}
\tilde{Z}_{\rm CS} &= \left(\prod_{j=1}^{\infty}(1-q^j)\right)^N \langle 0|e^{-\sum_{n>0}\frac{\alpha_n}{n[n]}}\mathbf{1}_{d^t\le N}e^{-\sum_{n>0}\frac{\alpha_{-n}}{n[n]}}|0\rangle \\
&= \xi(q)^{-N} Z_{\rm crystal}(q; t = g_s N). \qquad (4.2.16)
\end{aligned}
$$

We have demonstrated that the open string slicing in eq. (4.2.1) naturally arises from Chern-Simons theory. Note that there is a mismatch by the factor $\xi(q)^{-N}$ between $\tilde{Z}_{\rm CS}$ and $Z_{\rm crystal}$, where $\xi(q) = 1/\prod_{j=1}^{\infty}(1-q^j)$ is the "renormalization factor", which was found in [38] to be associated with a non-compact D-brane. As in [38], we use the modular property of $\eta(q) = q^{1/24}\xi(q)^{-1}$, namely $\eta(q) = \sqrt{2\pi/g_s}\eta(\tilde{q}), \tilde{q} = e^{-4\pi^2/g_s}$, to argue that it does not contribute to the perturbative amplitudes at genus no less than 2 when comparing the open and closed string sides. For lower genus amplitudes, the mismatch is absorbed into the usual ambiguities.

4.3 Adding D-branes

We can add non-compact D-branes to the system. In the language of Chern-Simons gauge theory, this corresponds to placing Wilson lines going through circles of links. In the case of an unknot, we will be able to see the connection to the description in [38].

On the open string side, we consider placing a stack of M non-compact D-branes in T^*S^3 intersecting the S^3 along an unknot S^1 [13]. Since the new D-branes are non-compact, we treat them as non-dynamical, acting as a source to the gauge fields on S^3 via an interaction. This interaction is obtained by integrating out the degrees freedom coming from the open strings stretching between the compact D-branes wrapping the S^3 and the non-compact D-branes. Let $U \in U(N)$ and $V \in U(M)$ be the holonomies along the unknot for the gauge fields on the compact and the non-compact D-branes, respectively. Then the interaction can be represented as

$$\int \mathcal{D}A e^{-S_{CS}[A]+\sum_{n=1}^{\infty}\frac{1}{n}\mathrm{Tr}U^n\mathrm{Tr}V^n} = Z_{CS}(S^3)\langle e^{\sum_{n=1}^{\infty}\frac{1}{n}\mathrm{Tr}U^n\mathrm{Tr}V^n}\rangle. \tag{4.3.1}$$

The expectation value can be expanded with the help of Frobenius' formula:

$$\langle e^{\sum_{n=1}^{\infty}\frac{1}{n}\mathrm{Tr}U^n\mathrm{Tr}V^n}\rangle = \sum_{\mu}\langle \mathrm{Tr}_\mu U\rangle \mathrm{Tr}_\mu V. \tag{4.3.2}$$

Here Tr_μ denotes the trace in the representation of $U(N)$ or $U(M)$ specified by the Young diagram μ.

It is natural to expect that $\langle \mathrm{Tr}_\mu U\rangle$ in eq. (4.3.2) is computed by the unitary matrix model in subsection 4.2.1 by inserting $\mathrm{Tr}_\mu U$. We now show that this is indeed correct; however, with the subtlety that the Wilson line and hence the non-compact D-branes have non-canonical framing.

The object we would like to compute is

$$\int dU \det \vartheta_{00}(U;q)\mathrm{Tr}_\mu U. \tag{4.3.3}$$

Going back to the eigenvalue integral, this is

$$\int \prod_{i=1}^{N} \frac{d\theta_i}{2\pi} \vartheta_{00}(e^{i\theta_i}) \det\left[(e^{i\theta_j})^{N-i}\right] \det\left[(e^{-i\theta_j})^{N-i}\right] \frac{\det\left[(e^{i\theta_j})^{\mu_i+N-i}\right]}{\det\left[(e^{i\theta_j})^{N-i}\right]}, \tag{4.3.4}$$

where we have used the Jacobi-Trudy formula $\mathrm{Tr}_\mu \mathrm{diag}(x_1, ..., x_N) \equiv s_\mu(x_1, ..., x_N)$ $= \det x_j^{\mu_i+N-i} / \det x_j^{N-i}$ for the Schur polynomial.[2] After canceling factors between the numerator and the denominator and performing the integrals, the matrix integral reduces to

$$\begin{aligned} & \frac{1}{N!} \sum_{\sigma,\sigma' \in S_N} \mathrm{sgn}\sigma \ \mathrm{sgn}\sigma' \prod_{j=1}^{N} q^{\frac{1}{2}\left(\mu_{\sigma(j)} - \sigma(j) + \sigma'(j)\right)^2} \\ = & \sum_{\sigma \in S_N} \mathrm{sgn}\sigma \prod_{j=1}^{N} q^{\frac{1}{2}(\mu_j - j + \sigma(j))^2} \\ = & \det\left[q^{\frac{1}{2}(\mu_i - i + j)^2}\right]. \end{aligned} \tag{4.3.5}$$

Up to μ-independent factors, this equals

$$q^{\frac{1}{2}\sum_{i=1}^{N} \mu_i(\mu_i - 2i + N + 1)} \det\left[q^{(j-\frac{N+1}{2})(\mu_i - i + N)}\right]. \tag{4.3.6}$$

The power of q can be written as $q^{(\kappa_\mu + N|\mu|)/2}$, where $\kappa_\mu = 2\sum_{(i,j)\in\mu}(i-j) = \sum_i \mu_i(\mu_i - 2i+1)$.

[2] A good reference on symmetric functions and the group theory relevant to us is [40].

This is the factor one obtains when the framing of the Wilson loop is shifted by one unit [23]. The determinant is of the form that appears in the numerator of the Jacobi-Trudy formula. Hence we have shown that

$$\frac{\int dU \det \vartheta_{00}(U;q)\mathrm{Tr}_\mu U}{\int dU \det \vartheta_{00}(U;q)} = q^{(\kappa_\mu+N|\mu|)/2}\mathrm{Tr}_\mu \mathrm{diag}(q^{-\frac{N-1}{2}}, q^{-\frac{N-3}{2}}, ..., q^{\frac{N-1}{2}}). \quad (4.3.7)$$

Relative to the result for the canonically framed unknot [13], we see that the matrix model computes amplitudes in the framing shifted by one unit.

This vacuum expectation value of the Wilson loop can be represented as a crystal melting model as follows:

$$\begin{aligned} \int dU \det \vartheta_{00}(U;q)\mathrm{Tr}_\mu U &= \xi(q)^{-N}\langle 0|e^{\sum_{n>0}\frac{\alpha_n}{n[n]}}\int dU \mathrm{Tr}_\mu U|U\rangle\langle U|e^{\sum_{n>0}\frac{\alpha_{-n}}{n[n]}}|0\rangle \\ &= \xi(q)^{-N}\langle 0|e^{\sum_{n>0}\frac{\alpha_n}{n[n]}}\int dU \sum_{\vec{k}}\frac{1}{z_{\vec{k}}}\chi_\mu(C(\vec{k}))\prod_{j=1}^{\infty}\alpha_j^{k_j}|U\rangle\langle U|e^{\sum_{n>0}\frac{\alpha_{-n}}{n[n]}}|0\rangle, \end{aligned} \quad (4.3.8)$$

where χ_μ is the $S_{|\mu|}$ character of the representation specified by μ, $\vec{k} = (k_1, k_2, ...)$ is an infinite vector with non-negative integer components, and $C(\vec{k})$ is the conjugacy class of $S_{|\mu|}$ specified by $\vec{k}$. Now the powers of α_j can be moved to the left to act on $\langle 0|$. This yields

$$\begin{aligned} \int dU \det \vartheta_{00}(U;q)\mathrm{Tr}_\mu U &= \xi(q)^{-N}\langle \mu|e^{\sum_{n>0}\frac{\alpha_n}{n[n]}}\int dU|U\rangle\langle U|e^{\sum_{n>0}\frac{\alpha_{-n}}{n[n]}}|0\rangle \\ &= \xi(q)^{-N}\langle \mu|e^{\sum_{n>0}\frac{\alpha_n}{n[n]}}\mathbf{1}_{d\le N}e^{\sum_{n>0}\frac{\alpha_{-n}}{n[n]}}|0\rangle \\ &= \xi(q)^{-N}\langle \mu^t|e^{-\sum_{n>0}\frac{\alpha_n}{n[n]}}\mathbf{1}_{d^t\le N}e^{-\sum_{n>0}\frac{\alpha_{-n}}{n[n]}}|0\rangle \\ &= \xi(q)^{-N}\langle \mu^t|\prod_{n=1}^{\infty}\Gamma_+(q^{n-1/2})\mathbf{1}_{d^t\le N}\prod_{m=1}^{\infty}\Gamma_-(q^{-(m-1/2)})|0\rangle \\ &= \xi(q)^{-N}q^{\sum_{i=1}^{\infty}(i-1/2)\mu_i^t}Z_{\mathrm{crystal}}^{\mathrm{D-branes}}, \end{aligned} \quad (4.3.9)$$

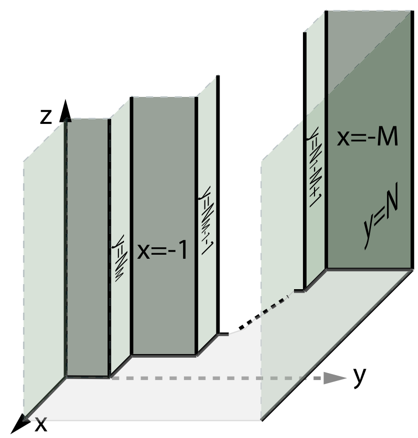


Figure 4.3: The initial configuration of the crystal with defects representing multiple non-compact D-branes intersecting $\mathbf{CP}^1$ in the resolved conifold. The defects introduce faces at $y = \mu_1^t = N_1 - M + 1, \mu_2^t = N_2 - M + 2, ..., \mu_{M-1}^t = N_{M-1} - 1, \mu_M^t = N_M$.

where we have defined

$$Z_{\text{crystal}}^{\text{D-branes}} := q^{-\sum_{i=1}^{\infty}(i-1)\mu_i^t} \langle \mu^t | \prod_{n=1}^{\infty} \Gamma_+(q^n) \mathbf{1}_{d^t \leq N} \prod_{m=1}^{\infty} \Gamma_-(q^{-(m-1)}) |0\rangle. \tag{4.3.10}$$

This free-field correlator together with the power of q represents, in the open string slicing, the partition function of the crystal melting model whose initial configuration is shown in figure 4.3. The power of q ensures that the initial configuration has zero energy.

It is possible to express the multi-D-brane crystal in the closed string slicing, which is a slight generalization of the free-field representation in [38].

$$\begin{aligned} Z_{\text{crystal}}^{\text{D-branes}} &= \langle 0 | \prod_{n=1}^{\infty} \Gamma_+(q^{n-1/2}) \prod_{m=1}^{N_M} \Gamma_-(q^{-(m-1/2)}) \Gamma_+(q^{-(N_M+1/2)}) \\ &\times \prod_{m=N_M+2}^{N_{M-1}} \Gamma_-(q^{-(m-1/2)}) \Gamma_+(q^{-(N_{M-1}+1/2)}) \prod_{m=N_{M-1}+2}^{N_{M-2}} \Gamma_-(q^{-(m-1/2)}) \Gamma_+(q^{-(N_{M-2}+1/2)}) ... \\ &\times \prod_{m=N_2+1}^{N_1} \Gamma_-(q^{-(m-1/2)}) \Gamma_+(q^{-(N_1+1/2)}) \prod_{m=N_1+2}^{N+M} \Gamma_-(q^{-(m-1/2)}) |0\rangle. \end{aligned} \tag{4.3.11}$$

Here $\mu_1^t = N_1 - M + 1, \mu_2^t = N_2 - M + 2, ..., \mu_{M-1}^t = N_{M-1} - 1, \mu_M^t = N_M$. In the closed string slicing, it is possible to explicitly evaluate the correlator to write it as a product. This also provides us with an interpretation of N_i as positions of D-branes and exhibits an interesting shift in the Kähler modulus:

$$\begin{aligned} Z_{\text{crystal}}^{\text{D-branes}} &= \left(\prod_{n=1}^{\infty} \prod_{1 \leq m \leq N+M, m \neq N_j+1} \frac{1}{1 - q^{n+m-1}} \right) \\ &\quad \times \prod_{i=1}^{M} \prod_{N_i+2 \leq m \leq N+M, m \neq N_j+1} \frac{1}{1 - q^{m-N_i-1}} \\ &= \xi(q)^M \prod_{i<j} (1 - e^{a_j - a_i}) M(q) \\ &\quad \times \prod_{i=1}^{M} e^{-\sum_{n=1}^{\infty} \frac{e^{-n\tilde{t}}}{n[n]^2}} \prod_{i=1}^{M} e^{\sum_{n=1}^{\infty} \frac{e^{-na_i} + e^{-n(\tilde{t}-a_i)}}{n[n]}}. \end{aligned} \tag{4.3.12}$$

Here we have defined $a_i := g_s(N_i + 1/2),\ i = 1, ..., M$ and $\tilde{t} := g_s(N + M) = t + g_s M$. Again, $\xi(q)$ can essentially be ignored in the perturbative computation due to the modular property of $\eta(q) = q^{1/24}\xi(q)^{-1}$. The factor $(1 - e^{a_j - a_i})$ is the contribution of an annulus diagram between the ith and jth D-branes.[3] This is the amplitude for M non-compact D-branes in the resolved conifold, which can be defined as the Kähler quotient

$$\{(X_I) \in \mathbf{C}^4 : |X_1|^2 + |X_2|^2 - |X_3|^2 - |X_4|^2 = \text{Re } \tilde{t}\}/U(1), \tag{4.3.13}$$

[3]The author thanks Nick Halmagyi for pointing this out.

with $U(1)$ action by charges $(1,1,-1,-1)$. The geometry of the D-branes is [41]

$$|X_1|^2 - \text{Re}(a_i) = |X_2|^2 - \text{Re}\ (\tilde{t} - a_i) = |X_3|^2 = |X_4|^2, \quad \sum_I \arg X_I = 0. \tag{4.3.14}$$

One thing that is interesting in our computation is that the Kähler parameter is shifted from $t = g_s N$ to $\tilde{t} = t + g_s M$. It has been known (see, for example, [42]) that the presence of D-branes can shift the effective size of the geometry by the string coupling times the number of D-branes. Here we have found another such phenomenon. The genus zero part of eq. (4.3.12) in the case of a single D-brane agrees with the results in [33].

The fact that the non-compact D-branes can be nicely incorporated to the crystal confirms that our crystal model of the resolved conifold is a natural one.

4.4 More general large N dualities, instanton counting, and geometric engineering

So far we have been discussing the Calabi-Yau crystal in the context of the Gopakumar-Vafa duality ($T^*S^3 \Leftrightarrow \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbf{CP}^1$), the simplest example of a large N duality in topological string theory. There is a family of generalizations of the large N duality which is worth considering in relation to Calabi-Yau crystal. The example of Gopakumar and Vafa is simple enough to prove the duality (at least at the level of free energies and some open string amplitudes) by direct calculations. However, our derivation of the resolved conifold crystal from Chern-Simons theory can be viewed as a complicated way of proving the duality. In this section we discuss the possible application of the ideas in the present paper to prove more general large N dualities.

Aganagic, Klemm, Marino, and Vafa made a conjecture in [10] that the duality of Gopakumar and Vafa still holds after taking a $\mathbf{Z}_n$ orbifold on the both sides of duality. On the closed string side, this produces A-type topological closed string theory living on the particular fibration of the A_{p-1} ALE space over $\mathbf{CP}^1$. The geometry has p Kähler moduli, the sizes of the base $\mathbf{CP}^1$ and p additional $\mathbf{CP}^1$ that blow up the A_{p-1} singularity. On the open string side, we again get Chern-Simons theory, this time living on the lens space $L(p,1) \simeq S^3/\mathbf{Z}_p$. Also after taking the orbifold, the relevant open string theory is a sector of Chern-Simons theory that contains one classical solution. A classical solution can be specified by a holonomy $\exp[2\pi i/N \mathrm{diag}(\overbrace{1,...,1}^{N_1}, \overbrace{2,...,2}^{N_2}, ..., \overbrace{p,...,p}^{N_p})]$ along the generator of the homotopy group. The Kähler parameters are then to be identified with linear combinations of the 't Hooft parameters $g_s N_i, i = 1, ..., p$.

The $p = 2$ duality was tested via perturbative computations by the original authors who proposed the duality [10]. For general p and a related duality, checks have been done by showing that the matrix models describing the sector of Chern-Simons theory leads to the spectral curves that are the nontrivial parts of the Calabi-Yau manifolds mirror to the A-model closed string geometries [17, 3, 43]. The worldsheet derivation of the Gopakumar-Vafa duality [7] has also been generalized for these large N dualities [4].

There are $p + 1$ choices ($m = 0, 1, ..., p$ in the notation of [31]) one can make when one fibers the A_{p-1} ALE space over $\mathbf{CP}^1$. The closed string geometry that is dual to the $S^3/\mathbf{Z}_p$

Chern-Simons theory is precisely the fibration $m = 0$ [3] that was shown to correspond to Nekrasov's instanton counting [44, 45] for the 5D $SU(p)$ gauge theory with a vanishing Chern-Simons term [30, 35]. (For the correspondence with a non-zero Chern-Simons term, see [46].) Nekrasov's correspondence between topological closed strings and 5D gauge theory has been discussed in [47, 48] by making use of the topological vertex [42]. As discussed in the introduction, the computation via the topological vertex is closely related to the Calabi-Yau crystal. In particular, the computation takes the form of an expansion in $q = e^{-g_s}$.

As we saw in the previous section, the Chern-Simons theory also naturally leads to an expansion in q. Hence, it is plausible that one will be able to prove the generalized large N dualities to all order in g_s by proving that the partition functions are the same on both sides as functions of q [49].

Let us also make an observation on the appearance of the unitary matrix model. In [50], the question of finding matrix models that compute the Seiberg-Witten solutions of $\mathcal{N} = 2$ gauge theories was addressed. The matrix models in [10] can be regarded as computing amplitudes in the 5D gauge theories with the same number of supercharges. By taking a double scaling limit, which is the familiar field theory limit of geometric engineering [32], one can compute amplitudes for 4D $\mathcal{N} = 2$ gauge theories from these matrix models. By using the technique in this paper, it is possible to rewrite the matrix models in [10] as unitary matrix models. These are similar to, and can be regarded as generalizations of, the unitary matrix model (Gross-Witten one-plaquette model [51]) that was considered in [50] for the $SU(2)$ gauge theory.

Chapter 5

Conclusion

In this thesis, we studied large N dualities in topological string theory from three different perspectives.

In Chapter 2 of this thesis, we reviewed and generalized the worldsheet derivation of the Gopakumar-Vafa large N duality in topological string theory. In the A-model case, we extended the proof of duality to geometries that involve replacing lens spaces $S^3/\mathbf{Z}_p$ by p $\mathbf{CP}^1$s. We saw from the behavior of the mirror Landau-Ginzburg model on the boundary of the C-phase that it knows about the rather intricate boundary condition, namely, the holonomy around the generator of $\pi_1(S^3/\mathbf{Z}_p)$. For the B-model, we considered the Calabi-Yau geometries of the form $x_1^2+x_2^2+x_3^2+w'(x)^2+f(x_4)=0$. The worldsheet derivation was implemented by using the Landau-Ginzburg model with superpotential $W=\Lambda(X_1^2+X_2^2+X_3^2+w'(X_4)^2+f(X_4))$. It would be extremely interesting to develop a similar worldsheet derivation of the AdS/CFT correspondence.

In chapter 3 we presented a quantitative tree-level test of large N dualities that involved trading D-branes wrapping a lens space $S^3/\mathbf{Z}_p$ with p $\mathbf{CP}^1$s of finite size as discussed in chapter 2. For this purpose, we took the matrix model of [10] and studied its spectral curve. We found that the spectral curve is precisely the Riemann surface that appears in the geometry mirror to the A-model closed string geometry. Since the genus-zero free energy is the prepotential of the Riemann surface, this provides a tree-level test of the large N conjecture.

In chapter 4, we proposed a crystal model that describes the A-model on the resolved conifold. The crystal is bounded by a wall at finite distance from the corner at which melting starts. The distance corresponds to the Kähler modulus of the geometry. We also formulated a unitary matrix model for Chern-Simons theory on S^3 and used it derive the Calabi-Yau crystals. It would be interesting to see if this kind of model can be generalized to other geometries, including those related to the Chern-Simons theory on $S^3\mathbf{Z}_p$.

Topological string theory is a subject that has been developing continuously since it was introduced. We now try to summarize the more recent developments in topological string theory.

In [52], Witten proposed that the topological B-model whose target space is a Calabi-Yau supermanifold $\mathbf{CP}^{3|4}$ is dual to the perturbative 4D $\mathcal{N} = 4$ super Yang-Mills theory. This is the first instance where the twistor formulation of the 4-dimensional field theory plays an essential role in string theory, and it uncovered a rather unexpected connection between string theory and perturbative gauge theory, triggering a surge of investigations. S-duality in topological string theory [53, 54] was proposed partially inspired by Witten's work.

A new connection was found between topological string theory and the physical string theory. According to the new proposal [55], a partition function of the BPS blackhole obtained by wrapping D-branes on cycles in a Calabi-Yau manifold is the absolute value squared of the topological string partition function. It is suggested that the blackhole formulation is the non-perturbative formulation of the topological string theory.

Another new proposal [56] states that Hitchin's functionals, which give rise to special holonomy metrics as extrema, can be regarded as the action of topological field theories. In particular, there are functionals that realize A-models and B-models, and another functional unifies them by defining a gravity theory on a 7-dimensional G_2 manifold. Pestun and Witten [57] examined this proposal in the case of B-model and found that at the one-loop level the naive quantization of the Hitchin functional fails. However, they found that there is an extension of the Hitchin functional that allows for a one-loop quantization.

Chern-Simons theory is also attracting new attention. Gukov, Schwarz, and Vafa in [58] studied the relation of the homological knot invariant of Khovanov and Rozansky to the BPS spectrum of states in topological open string theory. In [59], Beasley and Witten revisited the observation by Lawrence and Rozansky that the partition function of the Chern-Simons theory on a Seifert manifold localizes to the flat connections that are the critical points of the action. They were able to provide an explanation for this observation by using the technique of non-Abelian localization.

We saw that topological string theory is a rapidly developing field. The author believes that the progress reported in this thesis leads to deeper understandings of topological strings, superstrings, and the quantum physics of gravity and gauge theory.

Appendix A

Basics of toric geometry

A fan Δ in $N = \mathbf{Z}^n$ is a collection of strongly convex rational polyhedral cones in $N_{\mathbf{R}} = N \otimes_{\mathbf{Z}} \mathbf{R}$, such that (i) a face of any cone in Δ is also a cone in Δ and (ii) the intersection of two cones in Δ is a face of each cone.

The procedure to construct the charts and transition functions is the following [29]:

1. Let M be the dual lattice of N. For each cone $\sigma_i \in \Delta$ of maximal dimension, define the dual cone as $\check{\sigma}_i := \{u \in M_{\mathbf{R}} | (u, v) \geq 0, \forall v \in \sigma_i\}$.

2. For each dual cone $\check{\sigma}_i$, choose lattice points $u_{i,j}$ $(j = 1, 2, \cdots, r_i)$ such that $\check{\sigma}_i \cap M = \mathbf{Z}_+ u_{i,1} + \cdots + \mathbf{Z}_+ u_{i,r_i}$ $(\mathbf{Z}_+ = \{0, 1, 2, \cdots\})$.

3. For each dual cone $\check{\sigma}_i$, find a set of fundamental relations of $u_{i,1}, \cdots, u_{i,r_i}$ in the form $\sum_{j=1}^{r_i} p_{s,j} u_{i,j} = 0$, $s = 1, \cdots, R_i$.

4. For each cone σ_i, define the patch as $U_{\sigma_i} := \{(z_{i,1}, \cdots, z_{i,r_i}) \in \mathbf{C}^{r_i} | z_{i,1}^{p_{s,1}} \cdots z_{i,r_i}^{p_{s,r_i}} = 1, \forall s\}$.

5. For each pair of cones σ_i, σ_j, find a set of fundamental relations of $u_{i,1}, \ldots, u_{i,r_i}, u_{j,1},$

$\ldots, u_{j,r_j}$ in the form $\sum_{l=1}^{r_i} q_{i,j,l} u_{i,l} + \sum_{l=1}^{r_j} q'_{i,j,l} u_{j,l} = 0$.

6. Glue the two patches via $z_{i,1}^{q_{i,j,1}} \cdots z_{i,r_i}^{q_{i,j,r_i}} z_{j,1}^{q'_{i,j,1}} \cdots z_{j,r_j}^{q'_{i,j,r_j}} = 1$.

Bibliography

[1] G. 't Hooft, "A planar diagram theory for strong interactions," *Nucl. Phys.* **B72** (1974) 461.

[2] T. Okuda, "Derivation of Calabi-Yau crystals from Chern-Simons gauge theory," *JHEP* **03** (2005) 047, `hep-th/0409270`.

[3] N. Halmagyi, T. Okuda, and V. Yasnov, "Large N duality, lens spaces and the Chern-Simons matrix model," *JHEP* **04** (2004) 014, `hep-th/0312145`.

[4] T. Okuda and H. Ooguri, "D branes and phases on string worldsheet," *Nucl. Phys.* **B699** (2004) 135–150, `hep-th/0404101`.

[5] R. Gopakumar and C. Vafa, "On the gauge theory/geometry correspondence," *Adv. Theor. Math. Phys.* **3** (1999) 1415–1443, `hep-th/9811131`.

[6] E. Witten, "Chern-Simons gauge theory as a string theory," *Prog. Math.* **133** (1995) 637–678, `hep-th/9207094`.

[7] H. Ooguri and C. Vafa, "Worldsheet derivation of a large N duality," *Nucl. Phys.* **B641** (2002) 3–34, `hep-th/0205297`.

[8] E. Witten, “Phases of N = 2 theories in two dimensions,” *Nucl. Phys.* **B403** (1993) 159–222, hep-th/9301042.

[9] E. Witten, “Some comments on string dynamics,” hep-th/9507121.

[10] M. Aganagic, A. Klemm, M. Marino, and C. Vafa, “Matrix model as a mirror of Chern-Simons theory,” *JHEP* **02** (2004) 010, hep-th/0211098.

[11] K. Hori and C. Vafa, “Mirror symmetry,” hep-th/0002222.

[12] M. Aganagic and C. Vafa, “G(2) manifolds, mirror symmetry and geometric engineering,” hep-th/0110171.

[13] H. Ooguri and C. Vafa, “Knot invariants and topological strings,” *Nucl. Phys.* **B577** (2000) 419–438, hep-th/9912123.

[14] J. M. F. Labastida and M. Marino, “Polynomial invariants for torus knots and topological strings,” *Commun. Math. Phys.* **217** (2001) 423–449, hep-th/0004196.

[15] P. Ramadevi and T. Sarkar, “On link invariants and topological string amplitudes,” *Nucl. Phys.* **B600** (2001) 487–511, hep-th/0009188.

[16] J. M. F. Labastida, M. Marino, and C. Vafa, “Knots, links and branes at large N,” *JHEP* **11** (2000) 007, hep-th/0010102.

[17] N. Halmagyi and V. Yasnov, “The spectral curve of the lens space matrix model,” hep-th/0311117.

[18] D. R. Morrison and M. Ronen Plesser, "Summing the instantons: Quantum cohomology and mirror symmetry in toric varieties," *Nucl. Phys.* **B440** (1995) 279–354, `hep-th/9412236`.

[19] D. Ghoshal and C. Vafa, "C = 1 string as the topological theory of the conifold," *Nucl. Phys.* **B453** (1995) 121–128, `hep-th/9506122`.

[20] C. Vafa, "A stringy test of the fate of the conifold," *Nucl. Phys.* **B447** (1995) 252–260, `hep-th/9505023`.

[21] R. Dijkgraaf and C. Vafa, "Matrix models, topological strings, and supersymmetric gauge theories," *Nucl. Phys.* **B644** (2002) 3–20, `hep-th/0206255`.

[22] D. Ghoshal and S. Mukhi, "Landau-Ginzburg model for a critical topological string," in *Modern Quantum Field Theory, Proceedings, 2nd International Colloquium*, S. R. D. et. al., ed. Bombay, India, January 5–11, 1994. http://theory.tifr.res.in/group/colloq/articles.html.

[23] E. Witten, "Quantum field theory and the Jones polynomial," *Commun. Math. Phys.* **121** (1989) 351.

[24] M. Marino, "Chern-Simons theory, matrix integrals, and perturbative three-manifold invariants," `hep-th/0207096`.

[25] M. Bershadsky, S. Cecotti, H. Ooguri, and C. Vafa, "Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes," *Commun. Math. Phys.* **165** (1994) 311–428, hep-th/9309140.

[26] M. Tierz, "Soft matrix models and Chern-Simons partition functions," *Mod. Phys. Lett.* **A19** (2004) 1365–1378, hep-th/0212128.

[27] N. Halmagyi and V. Yasnov, "Chern-Simons matrix models and unoriented strings," *JHEP* **02** (2004) 002, hep-th/0305134.

[28] R. Dijkgraaf and C. Vafa, "A perturbative window into non-perturbative physics," hep-th/0208048.

[29] B. R. Greene, "String theory on Calabi-Yau manifolds," hep-th/9702155.

[30] A. Iqbal and A.-K. Kashani-Poor, "Instanton counting and Chern-Simons theory," *Adv. Theor. Math. Phys.* **7** (2004) 457–497, hep-th/0212279.

[31] A. Iqbal and A.-K. Kashani-Poor, "SU(N) geometries and topological string amplitudes," hep-th/0306032.

[32] S. Katz, A. Klemm, and C. Vafa, "Geometric engineering of quantum field theories," *Nucl. Phys.* **B497** (1997) 173–195, hep-th/9609239.

[33] M. Aganagic and C. Vafa, "Mirror symmetry, D-branes and counting holomorphic discs," hep-th/0012041.

[34] A. Okounkov, N. Reshetikhin, and C. Vafa, "Quantum Calabi-Yau and classical crystals," hep-th/0309208.

[35] A. Iqbal, N. Nekrasov, A. Okounkov, and C. Vafa, "Quantum foam and topological strings," hep-th/0312022.

[36] D. Maulik, N. Nekrasov, A. Okounkov, and R. Pandharipande, "Gromov-Witten theory and Donaldson-Thomas theory, I," math.AG/0312059.

[37] D. Maulik, N. Nekrasov, A. Okounkov, and R. Pandharipande, "Gromov-Witten theory and Donaldson-Thomas theory, II," math.AG/0406092.

[38] N. Saulina and C. Vafa, "D-branes as defects in the Calabi-Yau crystal," hep-th/0404246.

[39] S. Cordes, G. W. Moore, and S. Ramgoolam, "Lectures on 2-d Yang-Mills theory, equivariant cohomology and topological field theories," *Nucl. Phys. Proc. Suppl.* **41** (1995) 184–244, hep-th/9411210.

[40] J. Fulton and W. Harris, *Representation Theory. A First Course.* Graduate Texts in Mathematics Vol. 129. Springer-Verlag, 1999.

[41] M. Aganagic, A. Klemm, and C. Vafa, "Disk instantons, mirror symmetry and the duality web," *Z. Naturforsch.* **A57** (2002) 1–28, hep-th/0105045.

[42] M. Aganagic, A. Klemm, M. Marino, and C. Vafa, "The topological vertex," hep-th/0305132.

[43] V. Yasnov, "Chern-Simons matrix model for local conifold," `hep-th/0409136`.

[44] N. A. Nekrasov, "Seiberg-Witten prepotential from instanton counting," *Adv. Theor. Math. Phys.* **7** (2004) 831–864, `hep-th/0206161`.

[45] N. Nekrasov and A. Okounkov, "Seiberg-Witten theory and random partitions," `hep-th/0306238`.

[46] Y. Tachikawa, "Five-dimensional Chern-Simons terms and Nekrasov's instanton counting," *JHEP* **02** (2004) 050, `hep-th/0401184`.

[47] T. Eguchi and H. Kanno, "Topological strings and Nekrasov's formulas," *JHEP* **12** (2003) 006, `hep-th/0310235`.

[48] T. Eguchi and H. Kanno, "Geometric transitions, Chern-Simons gauge theory and Veneziano type amplitudes," *Phys. Lett.* **B585** (2004) 163–172, `hep-th/0312234`.

[49] T. Okuda. Work in progress.

[50] R. Dijkgraaf and C. Vafa, "On geometry and matrix models," *Nucl. Phys.* **B644** (2002) 21–39, `hep-th/0207106`.

[51] D. J. Gross and E. Witten, "Possible third order phase transition in the large N lattice gauge theory," *Phys. Rev.* **D21** (1980) 446–453.

[52] E. Witten, "Perturbative gauge theory as a string theory in twistor space," *Commun. Math. Phys.* **252** (2004) 189–258, `hep-th/0312171`.

[53] A. Neitzke and C. Vafa, "N = 2 strings and the twistorial Calabi-Yau," hep-th/0402128.

[54] N. Nekrasov, H. Ooguri, and C. Vafa, "S-duality and topological strings," *JHEP* **10** (2004) 009, hep-th/0403167.

[55] H. Ooguri, A. Strominger, and C. Vafa, "Black hole attractors and the topological string," *Phys. Rev.* **D70** (2004) 106007, hep-th/0405146.

[56] R. Dijkgraaf, S. Gukov, A. Neitzke, and C. Vafa, "Topological M-theory as unification of form theories of gravity," hep-th/0411073.

[57] V. Pestun and E. Witten, "The Hitchin functionals and the topological B-model at one loop," hep-th/0503083.

[58] S. Gukov, A. Schwarz, and C. Vafa, "Khovanov-Rozansky homology and topological strings," hep-th/0412243.

[59] C. Beasley and E. Witten, "Non-Abelian localization for Chern-Simons theory," hep-th/0503126.