On critical values of L-functions for holomorphic forms on $GSp(4) \times GL(2)$

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To my Parents

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Abstract

Let F be a genus two Siegel newform and g a classical newform, both of squarefree levels and of equal weight ℓ. We derive an explicit integral representation for the degree eight L-function $L(s, F \times g)$. As an application, we prove a reciprocity law — predicted by Deligne's conjecture — for the critical special values $L(m, F \times g)$ where $m \in \mathbb{Z}, 2 \leq m \leq \frac{\ell}{2} - 1$. The proof of our integral representation has two major components: the generalization of an earlier integral representation due to Furusawa and a "pullback formula" relating the complicated Eisenstein series of Furusawa with a simpler one on a higher rank group. The critical value result follows from our integral representation using rationality results of Garrett and Harris and the theory of nearly holomorphic forms due to Shimura.

Contents

Chapter 1

Introduction

1.1 Overview and main results

If $\mathcal{M}$ is an arithmetic or geometric object, then we can often associate a very interesting invariant to it. This invariant is a complex analytic function known as the L-function for $\mathcal{M}$ and is denoted $L(s, \mathcal{M})$.

The simplest example of an L-function is the Riemann Zeta function defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p (1 - p^{-s})^{-1}$$

for $\mathrm{Re}(s) > 1$ and by analytic continuation elsewhere. This function encodes rich number theoretic information about the distribution of primes. For instance, the fact that $\zeta(s)$ has no zeroes on the line $\mathrm{Re}(s) = 1$ is equivalent to the prime number theorem.

One can also associate L-functions to Dirichlet characters, number fields, elliptic curves and modular forms. The case of modular forms is particularly interesting because they, in a sense, include all the other examples mentioned so far. The classical modular forms are holomorphic functions that live on the upper-half plane and satisfy certain symmetries. These functions and their generalizations arise naturally from fundamental investigations in several mathematical areas. Indeed, there is a quote attributed to Martin Eichler saying that there are five basic operations in arithmetic: addition, subtraction, multiplication, division and modular forms.

The theory of automorphic representations provides a natural setting in which to study modular forms and their generalizations. The fascinating work of Langlands associates L-functions to automorphic representations and use these to express deep relationships between number theory, geometry and representation theory. One of the tools that has been successfully used to study L-functions and their special values is the method of integral representations; this is sometimes called the Rankin-Selberg method after Rankin and Selberg's fundamental work in this direction. Often, sharper and more explicit results are obtained when one restricts attention to holomorphic forms. The papers [GH93], [GK92], [HK91] treating the triple-product L-function are good examples, and in fact, provided an inspiration for this thesis.

A conjecture of Deligne [Del73] predicts that certain special values of such families of L-functions are algebraic numbers up to multiplication by certain period integrals. Indeed, let $L(s, \mathcal{M})$ be an arithmetically defined (or motivic) L-function associated to an arithmetic object $\mathcal{M}$. Then Deligne's conjectures assert that for certain critical points m,

(a) $L(m, \mathcal{M})$ is the product of a suitable transcendental number Ω and an algebraic number $A(m, \mathcal{M})$,

(b) If σ is an automorphism of $\mathbb{C}$, then $A(m, \mathcal{M})^\sigma = A(m, \mathcal{M}^\sigma)$.

To give a simple example, Deligne's conjecture for the Riemann zeta function is just the fact that $\zeta(m)/\pi^m$ is a rational number for all positive even integers m.

In this thesis, we prove a key special case of Deligne's conjecture when $\mathcal{M}$ corresponds to the product $F \times g$ where F is a Siegel modular form of genus two and g a classical modular form; see Theorem 3.6.1 below. As is often the case for such problems, the key ingredient in our proof is the interpretation of the transcendental factor as the period arising from a certain integral representation of Rankin Selberg type.

Fix odd, squarefree integers M, N. Let F be a genus two Siegel newform of level M and g an elliptic newform of level N; see § 2.8 for the precise definitions of these terms. One can associate a degree eight L-function $L(s, F \times g)$ to the pair (F, g). We assume that F and g have the same even integral weight ℓ and have trivial central characters. We also make the following assumption about F:

Suppose

$$F(Z) = \sum_{S>0} a(S) e(tr(SZ))$$

is the Fourier expansion; then we assume that

$$a(T) \neq 0 \text{ for some } T = \begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix} \qquad (1.1.1)$$

such that $-d = b^2 - 4ac$ *is the discriminant of the imaginary quadratic field* $L = \mathbb{Q}(\sqrt{-d})$*, and all primes dividing* MN *are inert in* L*.*

Given a Hecke character Λ of L, we define in §3.1 a Siegel Eisenstein series $E_{\Upsilon^\sharp}(g,s)$ on $GU(3,3;L)(\mathbb{A})$. Let R denote the subgroup of $GSp(4) \times GU(1,1;L)$ consisting of elements $h = (h_1, h_2)$ such that $h_1 \in GSp(4), h_2 \in GU(1,1;L)$ and h_1, h_2 have the same multiplier. We define in §3.2 an embedding $\iota : R \hookrightarrow GU(3,3;L)$. Let Φ, Ψ denote the adelizations of F, g respectively. We can extend the definition of Ψ to $GU(1,1;L)(\mathbb{A})$ by defining $\Psi(ag) = \Psi(g)$ for all $a \in L^\times(\mathbb{A}), g \in GL(2)(\mathbb{A})$. Our integral representation is as follows.

Theorem 3.6.1. *We have*

$$\int_{g \in Z(\mathbb{A})R(\mathbb{Q})\backslash R(\mathbb{A})} E_{\Upsilon^\sharp}(\iota(g_1,g_2),s)\overline{\Phi}(g_1)\Psi(g_2)\Lambda^{-1}(\det g_2)dg = A(s)L(3s+\frac{1}{2}, F \times g)$$

where $g = (g_1, g_2)$*,* Λ *is a suitable Hecke character of* L *and* $A(s)$ *is an explicit normalizing factor, defined in §3.6.*

The first step towards proving Theorem 3.6.1 is achieved in §2 where we extend an integral representation due to Furusawa. Let π, σ be the automorphic representations corresponding to F, g respectively. Furusawa [Fur93] defined a certain integral of Rankin Selberg type and proved that it factorizes as a product of local zeta integrals $Z_p(s)$. He computed the local zeta integrals only in the case when the local representations π_p, σ_p are both unramified and showed that they equal the local L-functions up to a normalizing factor. Thus, he was able to find an integral representation for $L(s, F \times g)$ in the full level case, i.e., when $M = 1, N = 1$. In §2 we compute, under certain conditions, the local integrals $Z_p(s)$ when the local representations are either Steinberg or unramified. This allows us to generalize the Furusawa integral representation to our case; see Theorem 2.8.7 for the precise statement.

The generalization is not straightforward. The explicit evaluation of the local zeta integral involves several steps. First of all, we need to perform certain technical volume and double-coset computations. These computations — easy in the unramified case — are tedious and challenging for the remaining cases and are carried out in §2.3. Secondly, it is necessary to suitably choose the sections of the Eisenstein series at the bad places to insure that the local zeta integrals do not vanish. Thirdly, and perhaps most crucially, the local computations require an explicit knowledge of the local Whittaker and Bessel functions. The formulae for the Whittaker model are well known in all cases; the same, however, is not true for the Bessel model. In fact, the only case where the local Bessel model for a finite place was computed before this work was when π_p is unramified [BFF97, Sug85]. However, that does not suffice for the cases when we have π_p Steinberg. As a preparation for the calculations in these cases, we find in §2.4, an explicit formula for the Bessel function for π_p when it is Steinberg. We believe this is of independent interest.

The second and final step towards proving Theorem 3.6.1 is achieved in §3. We prove a certain pullback formula (Theorem 3.2.1) that expresses our earlier Eisenstein series as the inner product of the cusp form and the pullback of the simpler higher-rank *Siegel* Eisenstein series $E_{\Upsilon^\sharp}$. Formulas in this spirit were first proved in a classical setting by Shimura [Shi97]. Unfortunately, Shimura only considers certain special types of Eisenstein series in his work which does not include ours (except in the full level case $M = 1, N = 1$). Furthermore his methods are *classical* and cannot be easily modified to deal with our case. The complicated sections at the ramified places and the need for precise factors make the *adelic* language the right choice for our purposes. We provide a complete proof of the pullback formula for our Eisenstein series which explicitly gives the precise factors at the ramified places needed by us.

Combining the results of §2 and §3, we deduce Theorem 3.6.1. It seems appropriate to mention here that the referee of our paper [Sah09] has indicated that it was well-known to some experts that one could use such a pullback formula to rewrite the Furusawa integral representation.

From Theorem 3.6.1, we easily conclude that $L(s, F \times g)$ is a meromorphic function whose only possible pole on the right of the critical line $\mathrm{Re}(s) = \frac{1}{2}$ is simple and at $s = 1$. Moreover, with the aid of basic techniques and results due to Garrett, Harris and Shimura, we prove the following Theorem.

Theorem 4.5.1. *Suppose that the Fourier coefficients of F and g are totally real and* *algebraic and that $\ell \geq 6$. For a positive integer k, $1 \leq k \leq \frac{\ell}{2} - 2$, define*

$$A(F, g; k) = \frac{L(\frac{\ell}{2} - k, F \times g)}{\pi^{5\ell-4k-4}\langle F, F\rangle\langle g, g\rangle}.$$

Then we have,

(a) $A(F, g; k)$ is algebraic

(b) For an automorphism σ of $\mathbb{C}$, $A(F, g; k)^\sigma = A(F^\sigma, g^\sigma; k)$.

We remark here that the completely unramified case $M = 1, N = 1$ of the above theorem was already known by the works of Heim [Hei99] and Böcherer–Heim [BH06], who used a very different integral representation from the one in this thesis. Also, just the algebraicity part of the above Theorem (i.e., part (a)) has been proved for the right-most critical value (corresponding to $k = 1$) in various settings earlier by Furusawa [Fur93], Pitale–Schmidt [PS09] and the author [Sah09].

To relate Theorem 4.5.1 to the conjecture of Deligne for motivic L-functions mentioned at the beginning of this introduction, we note that Yoshida [Yos01] has shown that the set of all critical points for $L(s, F \times g)$ is $\{m : 2 - \frac{\ell}{2} \leq m \leq \frac{\ell}{2} - 1, m \in \mathbb{Z}\}$. In particular, the critical points are always non-central (since the weight ℓ is even) and so the L-value is expected to be non-zero. Assuming the existence of a motive attached to F (this seems to be now known for our cases by the work of Weissauer [Wei05]) and the truth of Deligne's conjecture for the standard degree 5 L-function of F, Yoshida also computes the corresponding motivic periods. According to his calculations, the relevant period for the point m is precisely the quantity $\pi^{4m+3\ell-4}\langle F, F\rangle\langle g, g\rangle$ that appears in our theorem above (once we substitute $m = \frac{\ell}{2} - k$). We note here that Yoshida only deals with the full level case; however, as the periods remain the same (up to a rational number) for higher level, his results remain applicable to our case.

Thus, Theorem 4.5.1 is compatible with (and implied by) Deligne's conjecture, and furthermore, it covers all the critical values to the *right* of $\mathrm{Re}(s) = \frac{1}{2}$ *except* for the L-value at the point 1. The proof for the critical values to the left of $\mathrm{Re}(s) = \frac{1}{2}$ would follow from the expected functional equation. Extending our result to $L(1, F \times g)$ is intimately connected to proving the analyticity of the L-function at that point (see Corollary 3.6.2).

These questions, related to analyticity and the functional equation are also of interest for other applications and will be considered in a future paper.

We also note that the integral representation (Theorem 3.6.1) is of interest for several other applications. For instance, we hope that this integral representation will pave the way to certain new results involving stability, hybrid subconvexity, and non-vanishing results for the L-function under consideration following the methods of [MR]. We are also hopeful that we can prove results related to non-negativity of the central value $L(\frac{1}{2}, F)$. These results appear to be new for holomorphic Siegel modular forms. For example, the non-negativity result is known in the case of *generic* automorphic representations by Lapid and Rallis [LR02]; however, automorphic representations associated to Siegel modular forms are never generic. Another interesting application of the integral representation would be to the construction of p-adic L-functions. We intend to address these questions elsewhere.

We expect most of the results of this thesis to hold for arbitrary totally real base fields. It would be particularly interesting to work out the special value results when the Hilbert-Siegel modular forms have different weights for each Archimedean place. This case will be considered in a future work.

1.2 Notation

- The symbols $\mathbb{Z}$, $\mathbb{Z}_{\geq 0}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}_p$ and $\mathbb{Q}_p$ have the usual meanings. $\mathbb{A}$ denotes the ring of adeles of $\mathbb{Q}$, $\mathbb{A}_f$ the finite adeles. For a complex number z, $e(z)$ denotes $e^{2\pi i z}$.

- For a matrix M we denote its transpose by M^t. Denote by J_n the $2n$ by $2n$ matrix given by

$$J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

  We use J to denote J_2 and let s_1 denote the matrix $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$.

- For a positive integer n define the group $GSp(2n)$ by

$$GSp(2n, R) = \{g \in GL_{2n}(R) | g^t J_n g = \mu_n(g) J_n, \mu_n(g) \in R^\times\}$$

for any commutative ring R.

Define $Sp(2n)$ to be the subgroup of $GSp(2n)$ consisting of elements $g_1 \in GSp(2n)$ with $\mu_n(g_1) = 1$.

- For a quadratic extension L of $\mathbb{Q}$ define

$$GU(n, n) = GU(n, n; L)$$

by

$$GU(n, n)(\mathbb{Q}) = \{g \in GL_{2n}(L) | (\overline{g})^t J_n g = \mu_n(g) J_n, \mu_n(g) \in \mathbb{Q}^\times\}$$

where $\overline{g}$ denotes the conjugate of g.

- Let $\widetilde{H} = GU(3,3), \widetilde{H}_1 = U(3,3), H = GSp(6), H_1 = Sp(6), \widetilde{G} = GU(2,2)$, $\widetilde{G}_1 = U(2,2)$, $G = GSp(4), G_1 = Sp(4), \widetilde{F} = GU(1,1), \widetilde{F}_1 = U(1,1)$.

- Define

$$\widetilde{\mathbb{H}}_n = \{Z \in M_n(\mathbb{C}) | i(\overline{Z} - Z) \text{ is positive definite}\},$$
$$\mathbb{H}_n = \{Z \in M_n(\mathbb{C}) | Z = Z^t, i(\overline{Z} - Z) \text{ is positive definite}\}.$$

For $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GU(n, n)(\mathbb{R})$, $Z \in \widetilde{\mathbb{H}}_n$ define

$$J(g, Z) = CZ + D.$$

The same definition works for $g \in GSp(2n)(\mathbb{R}), Z \in \mathbb{H}_n$.

- For a commutative ring R we denote by $I(2n, R)$ the Borel subgroup of $GSp(2n, R)$ consisting of the set of matrices that look like $\begin{pmatrix} A & B \\ 0 & \lambda(A^t)^{-1} \end{pmatrix}$ where A is lower-triangular and $\lambda \in R^\times$.

- For a quadratic extension L of $\mathbb{Q}$ and v be a finite place of $\mathbb{Q}$, define $L_v = L \otimes_{\mathbb{Q}} \mathbb{Q}_v$.

  $\mathbb{Z}_L$ denotes the ring of integers of L and $\mathbb{Z}_{L,v}$ its v-closure in L_v.For a prime p, let $\mathbb{Z}_{L,p}^{\times}$ denote the group of units in $\mathbb{Z}_{L,p}$.

  If p is inert in L, the elements of $\mathbb{Z}_{L,p}^{\times}$ are of the form $a + b\sqrt{-d}$ with $a, b \in \mathbb{Z}_p$ and such that at least one of a and b is a unit. Let $\Gamma_{L,p}^{0}$ be the subgroup of $\mathbb{Z}_{L,p}^{\times}$ consisting of the elements with $p|b$.

- For a positive integer N the subgroups $\Gamma_0(N)$ and $\Gamma^0(N)$ of $SL_2(\mathbb{Z})$ are defined by

$$\Gamma_0(N) = \{A \in SL_2(\mathbb{Z}) \mid A \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{N}\},$$

$$\Gamma^0(N) = \{A \in SL_2(\mathbb{Z}) \mid A \equiv \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \pmod{N}\}.$$

  For p a finite place of $\mathbb{Q}$, their local analogues $\Gamma_{0,p}$ (resp. Γ_p^0) are defined by

$$\Gamma_{0,p} = \{A \in GL_2(\mathbb{Z}_p) \mid A \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{p}\},$$

$$\Gamma_p^0 = \{A \in GL_2(\mathbb{Z}_p) \mid A \equiv \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \pmod{p}\}.$$

- The local Iwahori subgroup I_p is defined to be the subgroup of $K_p = G(\mathbb{Z}_p)$ consisting of those elements of K_p that when reduced mod p lie in the Borel subgroup of $G(\mathbb{F}_p)$. Precisely,

$$I_p = \{A \in K_p \mid A \equiv \begin{pmatrix} * & 0 & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \pmod{p}\}$$

- Let $\widetilde{R}$ denote the subgroup of $\widetilde{G} \times \widetilde{F}$ consisting of elements $h = (h_1, h_2)$ such that $h_1 \in \widetilde{G}, h_2 \in \widetilde{F}$ and $\mu_2(h_1) = \mu_1(h_2)$. Let R denote the subgroup of $\widetilde{R}$ consisting of those (h_1, h_2) where $h_1 \in G$.

  For a fixed element $g \in \widetilde{G}(\mathbb{A})$, let $\widetilde{F}_1[g](\mathbb{A})$ denote the subset of $\widetilde{F}(\mathbb{A})$ consisting of all elements h_2 such that $\mu_2(g) = \mu_1(h_2)$.

Chapter 2

Extending the Furusawa Integral Representation

2.1 Preliminaries

Bessel models

We recall the definition of the Bessel model of Novodvorsky and Piatetski-Shapiro [NPS73] following the exposition of Furusawa [Fur93].

Let $S \in M_2(\mathbb{Q})$ be a symmetric matrix. We let $\mathrm{disc}(S) = -4\ \det(S)$ and put $d = -\mathrm{disc}(S)$. If $S = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$ then we define the element $\xi = \xi_S = \begin{pmatrix} b/2 & c \\ -a & -b/2 \end{pmatrix}$.

Let L denote the subfield $Q(\sqrt{-d})$ of $\mathbb{C}$.

We always identify $\mathbb{Q}(\xi)$ with L via

$$\mathbb{Q}(\xi) \ni x + y\xi \mapsto x + y\frac{\sqrt{(-d)}}{2} \in L, x, y \in \mathbb{Q}. \tag{2.1.1}$$

We define a subgroup $T = T_S$ of GL_2 by

$$T(\mathbb{Q}) = \{g \in GL_2(\mathbb{Q}) | g^t Sg = det(g)S\}. \tag{2.1.2}$$

The center of T is denoted by Z_T. It is not hard to verify that $T(\mathbb{Q}) = Q(\xi)^\times$ and $Z_T(\mathbb{Q}) = \mathbb{Q}^\times$. We identify $T(\mathbb{Q})$ with $L^\times$ via (2.1.1)).

We can consider T as a subgroup of G via

$$T \ni g \mapsto \begin{pmatrix} g & 0 \\ 0 & det(g).(g^{-1})^t \end{pmatrix} \in G. \tag{2.1.3}$$

Let us denote by U the subgroup of G defined by

$$U = \{u(X) = \begin{pmatrix} 1_2 & X \\ 0 & 1_2 \end{pmatrix} | X^t = X\}.$$

Let B be the subgroup of G defined by $R = TU$.

Let ψ be a non trivial character of $\mathbb{A}/\mathbb{Q}$. We define the character $\theta = \theta_S$ on $U(\mathbb{A})$ by $\theta(u(X)) = \psi(tr(S(X)))$. Let Λ be a character of $T(\mathbb{A})/T(\mathbb{Q})$ such that $\Lambda | Z_T(\mathbb{A}^\times) = 1$. Via (2.1.1) we can think of Λ as a character of $L^\times(\mathbb{A})/L^\times$ such that $\Lambda|\mathbb{A}^\times = 1$. Denote by $\Lambda \otimes \theta$ the character of $B(\mathbb{A})$ defined by $(\Lambda\otimes\theta)(tu) = \Lambda(t)\theta(u)$ for $t \in T(\mathbb{A})$ and $u \in U(\mathbb{A})$.

Let π be an automorphic cuspidal representation of $G(\mathbb{A})$ with trivial central character and V_π be its space of automorphic forms.

Then for $\Phi \in V_\pi$, we define a function B_Φ on $G(\mathbb{A})$ by

$$B_\Phi(h) = \int_{B(\mathbb{A})/B(\mathbb{Q})Z_G(\mathbb{A})} (\Lambda \otimes \theta)(r)^{-1}\Phi(rh)dr. \tag{2.1.4}$$

The $\mathbb{C}$ - vector space of function on $\widetilde{G}(\mathbb{A})$ spanned by $\{B_\Phi | \Phi \in V_\pi\}$ is called the global Bessel space of type (S, Λ, ψ) for π. We say that π has a global Bessel model of type (S, Λ, ψ), if the global Bessel space has positive dimension, that is if there exists $\Phi \in V_\pi$ such that $B_\Phi \neq 0$. In §2.1–§2.7 , we assume that:

There exists S, Λ, ψ such that π has a global Bessel model of type (S, Λ, ψ). (2.1.5)

Eisenstein series

We briefly recall the definition of the Eisenstein series used by Furusawa in [Fur93]. Let P be the maximal parabolic subgroup of $\widetilde{G}$ consisting of the elements in $\widetilde{G}$ that look like $\begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{pmatrix}$. We have the Levi decomposition $P = MN$ with $M = M^{(1)}M^{(2)}$ where the groups $M, N, M^{(1)}, M^{(2)}$ are as defined in [Fur93].

Precisely,

$$M^{(1)}(\mathbb{Q}) = \left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \overline{a}^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid a \in L^\times \right\} \simeq L^\times. \tag{2.1.6}$$

$$M^{(2)}(\mathbb{Q}) = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha & 0 & \beta \\ 0 & 0 & \lambda & 0 \\ 0 & \gamma & 0 & \delta \end{pmatrix} \mid \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GU(1,1)(\mathbb{Q}), \lambda = \mu_1 \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \right\} \tag{2.1.7}$$
$$\simeq GU(1,1)(\mathbb{Q}).$$

$$N(\mathbb{Q}) = \left\{ \begin{pmatrix} 1 & x & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\overline{x} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & a & y \\ 0 & 1 & \overline{y} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid a \in \mathbb{Q}, x, y \in L \right\}. \tag{2.1.8}$$

We also write

$$m^{(1)}(a) = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \overline{a}^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$m^{(2)} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha & 0 & \beta \\ 0 & 0 & \lambda & 0 \\ 0 & \gamma & 0 & \delta \end{pmatrix}.$$

Let σ be an irreducible automorphic cuspidal representation of $GL_2(\mathbb{A})$ with central character ω_σ. Let χ_0 be a character of $L^\times(\mathbb{A})/L^\times$ such that $\chi_0 \mid \mathbb{A}^\times = \omega_\sigma$.

Finally, let χ be a character of $L^\times(\mathbb{A})/L^\times = M_1(\mathbb{A})M_1(\mathbb{Q})$ defined by

$$\chi(a) = \Lambda(\overline{a})^{-1}\chi_0(\overline{a})^{-1}. \tag{2.1.9}$$

Then defining

$$\Pi(m_1m_2) = \chi(m_1)(\chi_0 \otimes \sigma)(m_2), m_1 \in M_1(\mathbb{A}), m_2 \in M_2(\mathbb{A}) \tag{2.1.10}$$

we extend σ to an automorphic representation Π of $M(\mathbb{A})$. We regard Π as a representation of $P(\mathbb{A})$ by extending it trivially on $N(\mathbb{A})$. Let δ_P denote the modulus character of P. If $p = m_1m_2n \in P(\mathbb{A})$ with $m_i \in M_i(\mathbb{A})(i = 1, 2)$ and $n \in N(\mathbb{A})$,

$$\delta_P(p) = |N_{L/\mathbb{Q}}(m_1)|^3 \cdot |\mu_1(m_2)|^{-3}, \tag{2.1.11}$$

where $||$ denoted the modulus function on $\mathbb{A}$.

Then for $s \in \mathbb{C}$, we form the family of induced automorphic representations of $\widetilde{G}(\mathbb{A})$

$$I(\Pi, s) = \mathrm{Ind}_{P(\mathbb{A})}^{\widetilde{G}(\mathbb{A})}(\Pi \otimes \delta_P^s) \tag{2.1.12}$$

where the induction is normalized. Let $f(g, s)$ be an entire section in $I(\Pi, s)$ viewed concretely as a complex-valued function on $\widetilde{G}(\mathbb{A})$ which is left $N(\mathbb{A})$-invariant and such that for each fixed $g \in \widetilde{G}(\mathbb{A})$, the function $m \mapsto f(mg, s)$ is a cusp form on $M(\mathbb{A})$ for the automorphic representation $\Pi \otimes \delta_P^s$. Finally we form the Eisenstein series $E(g, s) = E(g, s; f)$ by

$$E(g, s) = \sum_{\gamma \in P(\mathbb{Q})\backslash\widetilde{G}(\mathbb{Q})} f(\gamma g, s) \tag{2.1.13}$$

for $g \in \widetilde{G}(\mathbb{A})$.

This series converges absolutely (and uniformly in compact subsets) for $\mathrm{Re}(s) > 1/2$, has a meromorphic extension to the entire plane and satisfies a functional equation (see [Lan76, Fur93]).

2.2 The Rankin-Selberg integral

The global integral

The main object of study in this chapter is the following global integral of Rankin-Selberg type

$$Z(s) = Z(s, f, \Phi) = \int_{G(\mathbb{Q})Z_G(\mathbb{A})\backslash G(\mathbb{A})} E(g, s, f)\Phi(g)dg, \tag{2.2.1}$$

where $\Phi \in V_\pi$ and $f \in I(\Pi, s)$. $Z(s)$ converges absolutely away from the poles of the Eisenstein series.

Let $\Theta = \Theta_S$ be the following element of $\widetilde{G}(\mathbb{Q})$

$$\Theta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \alpha & 1 & 0 & 0 \\ 0 & 0 & 1 & -\overline{\alpha} \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ where } \alpha = \frac{b + \sqrt{-d}}{2c}$$

The 'basic identity' proved by Furusawa in [Fur93] is that

$$Z(s) = \int_{B(\mathbb{A})\backslash G(\mathbb{A})} W_f(\Theta h, s) B_\Phi(h) dh \tag{2.2.2}$$

where for $g \in \widetilde{G}(\mathbb{A})$ we have

$$W_f(g, s) = \int_{\mathbb{A}/\mathbb{Q}} f\left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & x \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} g, s\right) \psi(cx) dx, \tag{2.2.3}$$

and B_Φ is the Bessel model of type (S, Λ, ψ) defined in § 2.1.

The local integral

In this section v refers to any place of $\mathbb{Q}$. Let $\pi = \otimes_v \pi_v$ and $\sigma = \otimes_v \sigma_v$. Now suppose that Φ and f are factorizable functions with $\Phi = \otimes_v \Phi_v$ and $f(\cdot, s) = \otimes_v f_v(\cdot, s)$.

By the uniqueness of the Whittaker and the Bessel models, we have

$$W_f(g,s) = \prod_v W_{f,v}(g_v, s) \tag{2.2.4}$$

$$B_\Phi(h) = \prod_v B_{\Phi,v}(h_v, s) \tag{2.2.5}$$

for $g = (g_v) \in \widetilde{G}(\mathbb{A})$ and $h = (h_v) \in G(\mathbb{A})$ and *local* Whittaker and Bessel functions $W_{f,v}$, $B_{\Phi,v}$ respectively. Henceforth we write $W_v = W_{f,v}, B_v = B_{\Phi,v}$ when no confusion can arise.

Therefore our global integral breaks up as a product of local integrals

$$Z(s) = \prod_v Z_v(s) \tag{2.2.6}$$

where

$$Z_v(s) = Z_v(s, W_v, B_v) = \int_{B(\mathbb{Q}_v)\backslash G(\mathbb{Q}_v)} W_v(\Theta g, s) B_v(g) dg.$$

The unramified case

The local integral is evaluated in [Fur93] in the unramified case. We recall the result here.

Suppose that the characters $\omega_\pi, \omega_\sigma, \chi_0$ are trivial. Now let q be a finite prime of $\mathbb{Q}$ such that

(a) The local components π_q, σ_q and Λ_q are all unramified.

(b) The conductor of ψ_q is $\mathbb{Z}_q$.

(c) $S = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \in M_2(\mathbb{Z}_q)$ with $c \in \mathbb{Z}_q^\times$.

(d) $-d = b^2 - 4ac$ generates the discriminant of $L_q/\mathbb{Q}_q$.

Since σ_q is spherical, it is the spherical principal series representation induced from unramified characters α_q, β_q of $\mathbb{Q}_q^\times$.

Suppose M_0 is the maximal torus (the group of diagonal matrices) inside G and P_0 the Borel subgroup containing M_0 as Levi component. π_q is a spherical principal series representation, so there exists an unramified character γ_q of $M_0(\mathbb{Q}_q)$ such that $\pi_q = Ind_{P_0(\mathbb{Q}_q)}^{M_0(\mathbb{Q}_q)} \gamma_q$

, (where we extend γ_q to P_0 trivially). We define characters $\gamma_q^{(i)}(i = 1, 2, 3, 4)$ of $\mathbb{Q}_q^\times$ by

$$\gamma_q^{(1)}(x) = \gamma_q \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \gamma_q^{(2)}(x) = \gamma_q \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & x \end{pmatrix},$$

$$\gamma_q^{(3)}(x) = \gamma_q \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & x \end{pmatrix}, \qquad \gamma_q^{(4)}(x) = \gamma_q \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Now let $f_q(\ ,s)$ be the unique normalized $\widetilde{K_q}-$ spherical vector in $I_q(\Pi_q, s)$ and Φ_q be the unique normalized K_q- spherical vector in π_q. Let W_q, B_q be the coresponding vectors in the local Whittaker and Bessel spaces. The following result is proved in [Fur93]

Theorem 2.2.1 (Furusawa)**.** *Let $\rho(\Lambda_q)$ denote the Weil representation of $GL_2(\mathbb{Q}_q)$ corresponding to Λ_q. Then we have*

$$Z_q(s, W_q, B_q) = \frac{L(3s + \frac{1}{2}, \pi_q \times \sigma_q)}{L(6s + 1, \mathbf{1})L(3s + 1, \sigma_q \times \rho(\Lambda_q))}$$

where,

$$L(s, \pi_q \times \sigma_q) = \prod_{i=1}^{4} \Big((1 - \gamma_q^{(i)} \alpha_q(q) q^{-s})(1 - \beta_q^{(i)} \alpha_q(q) q^{-s}) \Big)^{-1},$$
$$L(s, \mathbf{1}) = (1 - q^{-s})^{-1},$$

$$L(s, \sigma_q \times \rho(\Lambda_q)) = \begin{cases} (1 - \alpha_q^2(q) q^{-2s})^{-1}(1 - \beta_q^2(q) q^{-2s})^{-1} & \text{if } q \text{ is inert in } L, \\ (1 - \alpha_q(q)\Lambda_q(q_1) q^{-s})^{-1}(1 - \beta_q(q)\Lambda_q(q_1) q^{-s})^{-1} & \text{if } q \text{ is ramified in } L, \\ (1 - \alpha_q(q)\Lambda_q(q_1) q^{-s})^{-1}(1 - \beta_q(q)\Lambda_q(q_1) q^{-s})^{-1} & \\ \cdot (1 - \alpha_q(q)\Lambda_q^{-1}(q_1) q^{-s})^{-1}(1 - \beta_q(q)\Lambda_q^{-1}(q_1) q^{-s})^{-1} & \text{if } q \text{ splits in } L, \end{cases}$$

where $q_1 \in \mathbb{Z}_q \otimes_{\mathbb{Q}} L$ *is any element with* $N_{L/\mathbb{Q}}(q_1) \in q\mathbb{Z}_q^\times$.

2.3 Strategy for computing the p-adic integral

Throughout this section we fix an odd prime p in $\mathbb{Q}$ such that p *is inert in* L. Moreover, we assume that $S \in M_2(\mathbb{Z}_p)$.

The fact that p is inert in L implies that if w, z are elements of $\mathbb{Z}_p$ then $w + z\xi \in (T(\mathbb{Q}_p) \cap K_p)$ if and only if at least one of w, z is an unit.

Moreover the additional assumption $S \in M_2(\mathbb{Z}_p)$ forces that a, c are units in $\mathbb{Z}_p$.

An explicit set of coset representatives

Recall the Iwahori subgroup I_p. It will be useful to describe a set of coset representatives of K_p/I_p.

But first some definitions.

Let Y be the set $\{0, 1, .., p-1\}$. Let $V = Y \cup \{\infty\}$ where ∞ is just a convenient formal symbol.

For $x = (n, q, r) \in \mathbb{Z}_p^3$, let $U_x \in U(\mathbb{Q}_p)$ be the matrix $\begin{pmatrix} 1 & 0 & n & q \\ 0 & 1 & q & r \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

For $y \in \mathbb{Z}_p$ define $Z_y = \begin{pmatrix} 1 & y & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -y & 1 \end{pmatrix} \in K_p$.

Also define $Z_\infty = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in K_p$.

In particular, the definitions U_x, Z_y make sense for $x \in Y^3, y \in V$. Now we define the following three classes of matrices. We call them matrices of class A, class B and class D, respectively.

(a) For $x = (n, q, r) \in Y^3, y \in V$, let $A_x^y = U_x J Z_y$.

(b) For $x = (n, q, r) \in Y^3$ with $q^2 - nr \equiv 0 \pmod p$ and $y \in V$, let $B_x^y = J U_x J Z_y$.

(c) For $\lambda, y \in V$, let
$$D_\lambda^y = \begin{cases} \begin{pmatrix} -\lambda & 0 & 0 & 1 \\ 1 & 0 & 0 & \lambda^{-1} \\ 0 & 1 & \lambda^{-1} & 0 \\ 0 & \lambda & -1 & 0 \end{pmatrix} Z_y & \text{if } \lambda \neq 0, \infty, \\ \begin{pmatrix} 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} Z_y & \text{if } \lambda = 0, \\ \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} Z_y & \text{if } \lambda = \infty. \end{cases}$$

Let S be the set obtained obtained by taking the union of the class A, class B and class D matrices, precisely $S = \{A_x^y\}_{\substack{y \in V \\ x \in Y^3}} \bigcup \{B_x^y\}_{\substack{y \in V, x=(n,q,r) \in Y^3 \\ q^2 - nr \equiv 0 \pmod p}} \bigcup \{D_\lambda^y\}_{\substack{\lambda \in V \\ y \in V}}$. Clearly S has cardinality $p^3(p+1) + p^2(p+1) + (p+1)^2 = (p+1)^2(p^2+1)$.

Lemma 2.3.1. *S is a complete set of coset representatives for K_p/I_p.*

Proof. Let us first verify that S has the *right* cardinality. Clearly the cardinality of K_p/I_p is the same as the cardinality of $G(\mathbb{F}_p)/I(4, \mathbb{F}_p)$. By [Kim98, Theorem 3.2], $|G(\mathbb{F}_p)| = p^4(p-1)^3(p+1)^2(p^2+1)$. On the other hand $I(4)$ has the Levi-decomposition

$$I(4) = \begin{pmatrix} g & 0 \\ 0 & v.(g^{-1})^t \end{pmatrix} \begin{pmatrix} 1_2 & X \\ 0 & 1_2 \end{pmatrix}$$

with g upper-triangular, X symmetric and $v \in GL(1)$. So $|I(4,\mathbb{F}_p)| = p^4(p-1)^3$. Thus $|G(\mathbb{F}_p)/I(4,\mathbb{F}_p)| = (p+1)^2(p^2+1)$ which is the same as the cardinality of S.

So it is enough to show that no two matrices in S lie in the same coset.

For a 2×2 matrix H with coefficients in $\mathbb{Z}_p$, we may reduce H mod p and consider the $\mathbb{F}_p$-rank of the resulting matrix; we denote this quantity by $r_p(H)$. It is easy to see that if the matrix $A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$ varies in a fixed coset of K_p/I_p, the pair $(r_p(A_1), r_p(A_3))$ remains constant.

Observe now that if A is of class A, then $r_p(A_3) = 2$; for A of class B, $r_p(A_3) < 2$ and $r_p(A_1) = 2$; while for A of class D we have $r_p(A_3) < 2$, $r_p(A_1) < 2$. This proves that elements of S of different classes cannot lie in the same coset.

Now we consider distinct elements of S of the same class, and show that they too must lie in different cosets.

For $x_1 = (n_1, q_1, r_1), x_2 = (n_2, q_2, r_2) \in Y^3$, $y_1, y_2 \in Y$, consider the elements $A_{x_1}^{y_1}, A_{x_2}^{y_2}$, $B_{x_1}^{y_1}, B_{x_2}^{y_2}$ of S. We have

$(A_{x_1}^{y_1})^{-1} A_{x_2}^{y_2} = (B_{x_1}^{y_1})^{-1} B_{x_2}^{y_2} =$

$$\begin{pmatrix} 1 & y_2 - y_1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -n_2 + n_1 & -n_2 y_2 - q_2 + n_1 y_2 + q_1 & 1 & 0 \\ y_1(n_1 - n_2) + q_1 - q_2 & y_1 y_2 (n_1 - n_2) + (y_1 + y_2)(q_1 - q_2) - r_2 + r_1 & y_1 - y_2 & 1 \end{pmatrix}.$$

So if the above matrix belongs to I_p, we must have $y_1 = y_2, n_1 = n_2$. That leads to $q_1 = q_2$, and finally by looking at the bottom row we conclude $r_1 = r_2$.

This covers the case of class A and class B matrices in S whose y-component is not equal to ∞.

Now $(A_{x_1}^{y_1})^{-1} A_{x_2}^{\infty} = (B_{x_1}^{y_1})^{-1} B_{x_2}^{\infty} =$

$$\begin{pmatrix} -y_1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ q_1 - q_2 & n_1 - n_2 & 0 & 1 \\ q_1 y_1 + r_1 - y_1 q_2 - r_2 & n_1 y_1 + q_1 - y_1 n_2 - q_2 & 1 & y_1 \end{pmatrix} \tag{2.3.1}$$

which cannot belong to I_p.

Also $(A^{\infty}_{x_1})^{-1}A^{\infty}_{x_2} = (B^{\infty}_{x_1})^{-1}B^{\infty}_{x_2} =$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -r_2+r_1 & -q_2+q_1 & 1 & 0 \\ -q_2+q_1 & -n_2+n_1 & 0 & 1 \end{pmatrix} \tag{2.3.2}$$

and if the above matrix lies in I_p we must have $x_1 = x_2$.

Thus we have completed the proof for class A and class B matrices.

To complete the proof of the lemma we need to show that no two class D matrices are in the same coset. The calculations for that case are similar to those above and are therefore omitted.

□

Reducing the integral to a sum

By [Fur93, p. 201]) we have the following disjoint union

$$G(\mathbb{Q}_p) = \coprod_{\substack{l\in\mathbb{Z} \\ 0\le m\in\mathbb{Z}}} B(\mathbb{Q}_p)\cdot h(l,m)\cdot K_p \tag{2.3.3}$$

where

$$h(l,m) = \begin{pmatrix} p^{2m+l} & 0 & 0 & 0 \\ 0 & p^{m+l} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & p^m \end{pmatrix}.$$

We wish to compute

$$Z_p(s) = \int_{B(\mathbb{Q}_p)\backslash G(\mathbb{Q}_p)} W_p(\Theta h, s) B_p(h) dh. \tag{2.3.4}$$

By (2.3.3) and (2.3.4) we have

$$Z_p(s) = \sum_{l\in\mathbb{Z}, m\ge 0} \int_{B(\mathbb{Q}_p)\backslash B(\mathbb{Q}_p)h(l,m)K_p} W_p(\Theta h, s) B_p(h) dh. \tag{2.3.5}$$

For $m \geq 0$ we define the subset T_m of S by

$$T_m = \{B^0_{(1,0,0)}, B^\infty_{(1,0,0)}, B^0_{(0,0,1)}, B^\infty_{(0,0,1)}, B^0_{(0,0,0)}, B^\infty_{(0,0,0)}, A^0_{(0,0,0)}, A^\infty_{(0,0,0)}\}$$

if $m > 0$,

$$T_0 = \{B^0_{(1,0,0)}, B^\infty_{(1,0,0)}, B^0_{(0,0,0)}, A^0_{(0,0,0)}\}.$$

Also, we use the notation $t_1 = B^0_{(1,0,0)}, t_2 = B^\infty_{(1,0,0)}, ..., t_8 = A^\infty_{(0,0,0)}$. Thus $T_m = \{t_i | 1 \leq i \leq 8\}$ if $m > 0$ and $T_0 = \{t_1, t_2, t_5, t_7\}$.

Proposition 2.3.2. *Let $l \in \mathbb{Z}, m \geq 0$. Then we have*

$$B(\mathbb{Q}_p)\backslash B(\mathbb{Q}_p)h(l,m)K_p = \coprod_{t \in T_m} B(\mathbb{Q}_p)\backslash B(\mathbb{Q}_p)h(l,m)tI_p.$$

Proof. Define two elements f and g in K_p to be (l,m)-equivalent if there exists $r \in B(\mathbb{Q}_p)$ and $k \in I_p$ such that $rh(l,m)fk = h(l,m)g$. Furthermore observe that if two elements of K_p are congruent mod p then they are in the same I_p-coset and therefore are trivially (l,m)-equivalent.

The proposition can be restated as saying that any $s \in S$ is (l,m)-equivalent to exactly one of the elements t with $t \in T_m$. This will follow from the following nine claims which we prove later below.

Claim 1. *Any class A matrix in S by left-multiplying by an appropriate element of $U(\mathbb{Z}_p)$ can be made congruent mod p to $A^y_{(0,0,0)}$ for some $y \in V$.*

Claim 2. *If $m > 0$ all the $A^y_{(0,0,0)}, y \in V \setminus \{0\}$ are (l,m)-equivalent. In the case $m = 0$ all the $A^y_{(0,0,0)}, y \in V$ are $(l,0)$-equivalent.*

Claim 3. *Any class B matrix in S by left-multiplying by an appropriate element of $U(\mathbb{Z}_p)$ can be made congruent mod p to one of the matrices*

$$B^{-\lambda}_{(1,\lambda,\lambda^2)}, B^\infty_{(1,\lambda,\lambda^2)} B^y_{(0,0,1)}, B^y_{(0,0,0)},$$

where $\lambda \in Y, y \in V$.

Claim 4. *The matrices* $B^y_{(1,\lambda,\lambda^2)}, \lambda \in Y, y \in \{-\lambda, \infty\}$ *are all* (l,m)*-equivalent to one of the matrices* $B^y_{(1,0,0)}, y \in \{0,\infty\}$.

Claim 5. *The matrices* $B^y_{(0,0,1)}, y \in V$ *by left-multiplying by an appropriate element of* $U(\mathbb{Z}_p)$ *can be made equal to one of the matrices* $B^y_{(0,0,1)}$ *with* $y \in \{0,\infty\}$.

Claim 6. *The matrices* $B^y_{(0,0,0)}, y \in V$ *are* (l,m)*-equivalent to one of the matrices* $B^y_{(0,0,0)}$ *with* $y \in \{0,\infty\}$. *In the case* $m = 0$ *these two matrices are also equivalent.*

Claim 7. *The matrices* $B^0_{(1,0,0)}, B^\infty_{(0,0,1)}$ *are* $(l,0)$*-equivalent and the matrices* $B^\infty_{(1,0,0)}, B^0_{(0,0,1)}$ *are also* $(l,0)$*-equivalent.*

Claim 8. *Any class D matrix* D^y_λ *by left-multiplying by an appropriate element of* $U(\mathbb{Z}_p)$ *can be made equal to a class B matrix.*

Claim 9. *No two elements of* T_m *are* (l,m)*-equivalent for any* $m \geq 0$.

Indeed claims 1, 2 imply that any class A matrix is (l,m)-equivalent to one of t_7, t_8 (and when $m = 0$, t_7 alone suffices). On the other hand claims 3, 4, 5, 6, 7 tell us that any class B matrix is (l,m)-equivalent to one of the $t_i, 1 \leq i \leq 6$ (and that just t_1, t_2, t_5 suffice if $m = 0$). Also claim 8 says that any class D matrix is also (l,m)-equivalent to one of the above. Since the class A, class B and class D matrix exhaust S, this shows that any element of S is (l,m)-equivalent to some element of T_m; in other words we do have the union stated in Proposition 2.3.2. Finally claim 9 completes the argument by implying that the union is indeed disjoint.

□

We now prove each of the above claims. The proofs are just computations, we simply multiply by suitable elements of R to get the results we desire.

Proof of claim 1. This follows from the fact that $U_{-x}A^y_x \equiv JZ_y \pmod p$ and $JZ_y = A^y_{(0,0,0)}$.

□

Proof of claim 2. We first deal with the case $m = 0$. For $y \in V, y \neq 0$ let $j = (-\frac{a}{y} + \frac{b}{2}) + \xi \in (T(\mathbb{Q}_p) \cap K_p)$ (here and elsewhere we interpret $1/\infty = 0$). Consider the element

$(A^0_{(0,0,0)})^{-1}h(l,0)^{-1}jh(l,0)A^y_{(0,0,0)}$. By direct calculation this equals

$$\begin{pmatrix} -\frac{a}{y} & 0 & 0 & 0 \\ -c & \frac{-cy^2+a-yb}{y} & 0 & 0 \\ 0 & 0 & -\frac{cy^2+a-yb}{y} & c \\ 0 & 0 & 0 & -\frac{a}{y} \end{pmatrix}$$

if $y \neq \infty$ and equals

$$\begin{pmatrix} a & 0 & 0 & 0 \\ b & -c & 0 & 0 \\ 0 & 0 & c & b \\ 0 & 0 & 0 & -a \end{pmatrix}$$

if $y = \infty$. Both of these matrices lie in I_p and this proves the claim for $m = 0$.

Now consider $m > 0$. For $y \in V, y \neq 0, \infty$, let $j = cy + p^m\xi \in (T(\mathbb{Q}_p) \cap K_p)$. Consider the element $(A^\infty_{(0,0,0)})^{-1}\ h(l,m)^{-1}j\ h(l,m)A^y_{(0,0,0)}$, which by direct calculation equals

$$\begin{pmatrix} -c & \frac{bp^m}{2} & 0 & 0 \\ cy - \frac{bp^m}{2} & cy^2 - \frac{yp^mb}{2} + p^{2m}a & 0 & 0 \\ 0 & 0 & -p^{2m}a - cy^2 + \frac{yp^mb}{2} & cy - \frac{bp^m}{2} \\ 0 & 0 & \frac{bp^m}{2} & c \end{pmatrix}$$

and this lies in I_p. Thus $A^y_{0,0,0}$ is (l,m)-equivalent to $A^\infty_{0,0,0}$ and this completes the proof of the claim. □

Proof of claim 3. Before proving this claim let us make a small remark. If $\lambda \in Y$ is such that λ^2 does not belong to Y one may ask what we mean by the notation $B^y_{(1,\lambda,\lambda^2)}$; in such a case, we understand λ^2 to refer to the unique element in Y that is congruent to $\lambda^2 \mod p$. *This convention will govern any such situation.*

We now begin proving the claim. Given a class B matrix $B^y_{(n,q,r)}$ with $n \neq 0$ we must have $q \equiv n\lambda, r \equiv n\lambda^2 \pmod p$ for some appropriate $\lambda \in Y$.

First assume that $y \neq -\lambda$. If $y \neq \infty$ put $s = \frac{\lambda - y + n(\lambda+y)}{n(y+\lambda)}, t = -\frac{1}{n(y+\lambda)}, u = 0$ and check that $(B^y_{n,q,r})^{-1}U_{s,t,u}B^{\infty}_{1,\lambda,\lambda^2}$ is congruent mod p to

$$\begin{pmatrix} -\lambda-y & 0 & -\frac{1}{n(y+\lambda)} & \frac{\lambda+n(y+\lambda)}{n(y+\lambda)} \\ \frac{\lambda+n(y+\lambda)}{n(y+\lambda)} & \frac{1}{n(y+\lambda)} & 0 & -\frac{1}{n(y+\lambda)} \\ 0 & 0 & -\frac{1}{y+\lambda} & \frac{\lambda+n(y+\lambda)}{y+\lambda} \\ 0 & 0 & 0 & n(y+\lambda) \end{pmatrix}.$$

If $y=\infty$ put $s=\frac{n-1}{n}, t=0, u=0$ and check that $(B^{\infty}_{n,q,r})^{-1}U_{s,t,u}B^{\infty}_{1,\lambda,\lambda^2}$ is congruent to

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{(n-1)\lambda}{n} & \frac{1}{n} & 0 & \frac{n-1}{n} \\ 0 & 0 & 1 & (n-1)\lambda \\ 0 & 0 & 0 & n \end{pmatrix}.$$

Both of these matrices belong to I_p.

Now suppose that $y=-\lambda$. Put $s=\frac{n-1}{n}, t=0, u=0$ and observe that $(B^{-\lambda}_{(n,q,r)})^{-1}U_{(s,t,u)}B^{-\lambda}_{(1,\lambda,\lambda^2)}$ is congruent mod p to

$$\begin{pmatrix} \frac{1}{n} & 0 & \frac{n-1}{n} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & n & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

which belongs to I_p also.

Finally assume that $n=0$. So $q=0$ as well. If $r=0$ there is nothing to prove. So suppose $r\neq 0$. If $y\neq\infty$ put $s=0, t=\frac{y(r-1)}{r}, u=\frac{r-1}{r}$ and observe that $(B^y_{(0,0,r)})^{-1}U_{(s,t,u)}B^y_{(0,0,1)}$ equals

$$\begin{pmatrix} 1 & 0 & -\frac{(r-1)y^2)}{r} & 0 \\ 0 & \frac{1}{r} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & r \end{pmatrix}$$

which belongs to I_p. If $y=\infty$ put $s=0, t=0, u=\frac{r-1}{r}$ and observe that $(B^y_{(0,0,r)})^{-1}U_{(s,t,u)}B^y_{(0,0,1)}$

equals

$$\begin{pmatrix} \frac{1}{r} & 0 & \frac{r-1}{r} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

which belongs to I_p.

Thus the claim is proved. □

Proof of claim 4. First suppose that $y = \infty$. If $m > 0$, put $j = c/\lambda + p^m\xi \in (T(\mathbb{Q}_p) \cap K_p)$. By direct calculation we verify that $(B^{\infty}_{(1,0,0)})^{-1}h(l,m)^{-1}jh(l,m)B^{\infty}_{(1,\lambda,\lambda^2)}$ is congruent mod p to

$$\begin{pmatrix} \frac{c}{\lambda} & 0 & 0 & 0 \\ c & \frac{c}{\lambda} & 0 & 0 \\ 0 & 0 & \frac{c}{\lambda} & -c \\ 0 & 0 & 0 & \frac{c}{\lambda} \end{pmatrix}$$

and this belongs to I_p.

If $m = 0$, put $j = \frac{(2c-b\lambda)}{2\lambda} + \xi \in (T(\mathbb{Q}_p) \cap K_p)$ and check that if we let $n \in Y \setminus 0$ be the element congruent $\mod p$ to $\frac{a\lambda^2 - b\lambda + c}{c}$ and $y \in Y \setminus 0$ be the element congruent $\mod p$ to $-\frac{c}{a\lambda}$ then $(B^{y}_{(n,0,0)})^{-1}h(l,0)^{-1}jh(l,0)B^{\infty}_{(1,\lambda,\lambda^2)}$ is congruent $\mod p$ to

$$\begin{pmatrix} \frac{c(a\lambda^2-b\lambda+c)}{a\lambda^2} & 0 & 0 & 0 \\ \frac{c-b\lambda}{\lambda} & -a & 0 & 0 \\ 0 & 0 & a & \frac{c-b\lambda}{\lambda} \\ 0 & 0 & 0 & -\frac{c(a\lambda^2-b\lambda+c)}{a\lambda^2} \end{pmatrix}$$

which lies in I_p. Hence $B^{\infty}_{(1,\lambda,\lambda^2)}$ is $(l,0)$-equivalent to $B^{y}_{(n,0,0)}$ and by the proof of Claim 3 it follows that it is $(l,0)$-equivalent to $B^{\infty}_{(1,0,0)}$.

Now assume that $y = -\lambda$. If $\lambda = 0$ there is nothing to prove so assume $\lambda \neq 0$. If $m > 0$, put $j = c + p^m\lambda\xi \in (T(\mathbb{Q}_p) \cap K_p)$. By a direct calculation we see that $(B^{0}_{(1,0,0)})^{-1}h(l,m)^{-1}jh(l,m)B^{-\lambda}_{(1,\lambda,\lambda^2)}$ is congruent to $cI_4 \pmod p$ and thus $B^{-\lambda}_{(1,\lambda,\lambda^2)}$ is (l,m)-equivalent to $B^{0}_{(1,0,0)}$. If $m = 0$, put $j = \frac{(2c-b\lambda)}{2\lambda} + \xi \in (T(\mathbb{Q}_p) \cap K_p)$ and check that if we let $n \in Y \setminus 0$ be the element that is congruent mod p to $\frac{a\lambda^2-b\lambda+c}{c}$ then,

$(B^0_{(n,0,0)})^{-1}h(l,0)^{-1}jh(l,0)B^{-\lambda}_{(1,\lambda,\lambda^2)}$ is congruent mod p to

$$\begin{pmatrix} \frac{c}{\lambda} & 0 & 0 & 0 \\ -a & \frac{a\lambda^2-b\lambda+c}{\lambda} & 0 & 0 \\ 0 & 0 & \frac{a\lambda^2-b\lambda+c}{\lambda} & a \\ 0 & 0 & 0 & \frac{c}{\lambda} \end{pmatrix}$$

which lies in I_p. Hence $B^{-\lambda}_{(1,\lambda,\lambda^2)}$ is $(l,0)$-equivalent to $B^0_{(n,0,0)}$ and by Claim 3 it follows that it is $(l,0)$-equivalent to $B^0_{(1,0,0)}$. □

Proof of claim 5. If $y=\infty$ there is nothing to prove. So assume $y \in Y$. Put $s=0, t=-y, u=0$ and observe that $(B^y_{(0,0,1)})^{-1}U_{(s,t,u)}B^0_{(0,0,1)}$ equals

$$\begin{pmatrix} 1 & 0 & y^2 & -y \\ 0 & 1 & -y & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

which is in I_p. □

Proof of claim 6. First consider the case $m>0$. Take $y \in V\backslash\{0,\infty\}$ and let $j = c/y + p^m\xi \in (T(\mathbb{Q}_p) \cap K_p)$. By direct calculation we verify that

$$(B^y_{(0,0,0)})^{-1}h(l,m)^{-1}jh(l,m)B^0_{(0,0,0)}$$

is congruent mod p to $\frac{c}{y}I_4$ and so $B^y_{(0,0,0)}$ is (l,m)-equivalent to $B^0_{(0,0,0)}$.

Now let $m=0$. Take $y \in V \setminus \{0,\infty\}$ and let $j = c/y + b/2 + \xi \in (T(\mathbb{Q}_p) \cap K_p)$. By direct calculation we verify that $(B^y_{(0,0,0)})^{-1}h(l,0)^{-1}jh(l,0)B^0_{(0,0,0)}$ equals

$$\begin{pmatrix} \frac{ay^2+by+c}{y} & 0 & 0 & 0 \\ -a & \frac{c}{y} & 0 & 0 \\ 0 & 0 & \frac{c}{y} & a \\ 0 & 0 & 0 & \frac{ay^2+by+c}{y} \end{pmatrix}$$

which lies in I_p.

Finally, if we take $j = b/2 + \xi \in (T(\mathbb{Q}_p) \cap K_p)$ we can verify that

$$(B^{\infty}_{(0,0,0)})^{-1} h(l,0)^{-1} j h(l,0) B^{0}_{(0,0,0)}$$

$$= \begin{pmatrix} -a & 0 & 0 & 0 \\ b & c & 0 & 0 \\ 0 & 0 & -c & b \\ 0 & 0 & 0 & a \end{pmatrix}$$

which lies in I_p. □

Proof of claim 7. Putting $j = \frac{b}{2} + \xi$ and $s = 1 - \frac{c}{a}, t = 0, u = 0$ we verify that

$$(B^{\infty}_{(0,0,1)})^{-1} h(l,0)^{-1} j U_{(s,t,u)} h(l,0) B^{0}_{(1,0,0)}$$

$$= \begin{pmatrix} -c & 0 & c-a & 0 \\ \frac{bc}{a} & c & \frac{b(a-c)}{a} & 0 \\ 0 & 0 & -a & b \\ 0 & 0 & 0 & a \end{pmatrix}$$

which lies in I_p.

Putting $j = -\frac{b}{2} + \xi$ and $u = 1 - \frac{a}{c}, t = 0, s = 0$ we verify that

$$(B^{\infty}_{(0,0,1)})^{-1} h(l,0)^{-1} j U_{(s,t,u)} h(l,0) B^{0}_{(1,0,0)}$$

$$= \begin{pmatrix} -a & -\frac{ba}{c} & 0 & \frac{b(a-c)}{c} \\ 0 & a & 0 & c-a \\ 0 & 0 & -c & 0 \\ 0 & 0 & -b & c \end{pmatrix}$$

which lies in I_p. □

Proof of claim 8. Suppose $y \neq \infty, \lambda \neq 0, \infty$. Put $s = 1 - 2\lambda y, t = y, u = 0$ and check that

$(D^y_\lambda)^{-1}U_{(s,t,u)}B^\infty_{(1,\lambda,\lambda^2)}$ is congruent mod p to

$$\begin{pmatrix} -1 & 0 & \frac{y}{\lambda} & \frac{1-2\lambda y}{2\lambda} \\ \lambda & 1 & -\frac{1}{2\lambda} & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2\lambda} \\ 0 & 0 & 0 & -\frac{1}{2} \end{pmatrix}.$$

Now suppose $y \neq \infty$ and put $s = 1, t = -y, u = 0$ and check that

$$(D^y_0)^{-1}U_{(s,t,u)}B^\infty_{(1,0,0)}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Next put $s = 0, t = -y, u = 1$ and check that $(D^y_\infty)^{-1}U_{(s,t,u)}B^0_{(0,0,1)}$ equals

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Now let $\lambda \neq 0, \infty$. Put $s = 1, t = 0, u = 0$ and check that $(D^\infty_\lambda)^{-1}U_{(s,t,u)}B^{-\lambda}_{(1,\lambda,\lambda^2)}$ is congruent mod p to

$$\begin{pmatrix} 1 & 0 & -1 & -\frac{1}{2\lambda} \\ 0 & -1 & \frac{1}{2\lambda} & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

Next put $s = 1, t = 0, u = 0$ and check that $(D^\infty_0)^{-1}U_{(s,t,u)}B^0_{(1,0,0)}$ is congruent mod p

to

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Finally put $s = 0, t = 0, u = 1$ and check that $(D^{\infty}_{\infty})^{-1}U_{(s,t,u)}B^{\infty}_{(0,0,1)}$ is congruent mod p to

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

□

Proof of claim 9. Suppose two elements f and g in T_m are (l,m)-equivalent. Then there exists $r \in B(\mathbb{Q}_p)$ and $k \in I_p$ such that $rh(l,m)fk = h(l,m)g$. Denote $r' = h(l,m)^{-1}rh(l,m)$ so that $g = r'fk$. Then r' is upper-triangular and belongs to K_p. Writing f, g in block form $f = \begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix}$, $g = \begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix}$ we conclude as in the proof of Lemma 2.3.1 that the $\mathbb{F}_p$-rank of f_3 equals the $\mathbb{F}_p$-rank of g_3. For class A matrices the $\mathbb{F}_p$-rank of the corresponding 2×2 block is 2 and for class B matrices it is less than 2. So it is not possible that one of f, g is class A and the other class B.

Thus we can assume that f and g are in the same class.

We first deal with the case $m > 0$.

Continuing with the generalities, let $r = tu$ with $T \in \mathbb{Q}_p, u \in U(\mathbb{Q}_p)$. Put

$$t' = h(l,m)^{-1}th(l,m), u' = h(l,m)^{-1}uh(l,m).$$

Thus $r' = t'u'$ and this forces $t' \in (T(\mathbb{Q}_p) \cap K_p), u' \in U(\mathbb{Z}_p)$. We must then have $t = x + zp^m\xi$ with $x \in \mathbb{Z}_p^{\times}, z \in \mathbb{Z}_p$. Let $u' = U_{(s,t,u)}$.

Let us first consider the class A case. We can check that $(A^0_{(0,0,0)})^{-1}t'u'A^{\infty}_{(0,0,0)}$ is congruent mod p to

$$\begin{pmatrix} 0 & x & 0 & 0 \\ x & -zc & 0 & 0 \\ -tx-zcu & -sx-zct & zc & x \\ -ux & -tx & x & 0 \end{pmatrix}$$

and so can never belong to I_p because x is a unit.

We now consider the class B case. Suppose for $(n_1, q_1, r_1), (n_2, q_2, r_2)$ we compute

$$(B^0_{(n_1,q_1,r_1)})^{-1} t' u' B^0_{(n_2,q_2,r_2)} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

and prove that C is never congruent to $0 \pmod p$. It will follow then that for any $y_1, y_2 \in V$, $B^{y_1}_{(n_1,q_1,r_1)}$ and $B^{y_2}_{(n_2,q_2,r_2)}$ are not (l,m)-equivalent because the introduction of the new terms Z_{y_1}, Z_{y_2} cannot affect C.

$(B^0_{(1,0,0)})^{-1} t' u' B^0_{(0,0,0)}$ is congruent mod p to

$$\begin{pmatrix} x & zc & sx+zct & tx+zcu \\ 0 & x & tx & ux \\ x & zc & sx+x+zct & tx+zcu \\ 0 & 0 & -zc & x \end{pmatrix}.$$

$(B^0_{(1,0,0)})^{-1} t' u' B^0_{(0,0,1)}$ is congruent mod p to

$$\begin{pmatrix} x & zc-tx-zcu & sx+zct & tx+zcu \\ 0 & x-ux & tx & ux \\ x & zc-tx-zcu & sx+x+zct & tx+zcu \\ 0 & -x & -zc & x \end{pmatrix}.$$

$(B^0_{(0,0,0)})^{-1} t' u' B^0_{(0,0,1)}$ is congruent mod p to

$$\begin{pmatrix} x & zc-tx-zcu & sx+zct & tx+zcu \\ 0 & x-ux & tx & ux \\ 0 & 0 & x & 0 \\ 0 & -x & -zc & x \end{pmatrix}.$$

Each of these three matrices have this property because x is a unit.

Now consider $(B^0_{(1,0,0)})^{-1}t'u'B^\infty_{(1,0,0)}$. This is congruent mod p to

$$\begin{pmatrix} zc & x-sx-zct & tx+zcu & sx+zct \\ x & -tx & ux & tx \\ zc & -sx-zct & tx+zcu & sx+x+zct \\ 0 & zc & x & -zc \end{pmatrix}$$

which cannot belong to I_p because x is a unit.

Next consider $(B^0_{(0,0,0)})^{-1}t'u'B^\infty_{(0,0,0)}$. This is congruent mod p to

$$\begin{pmatrix} zc & x & tx+zcu & sx+zct \\ x & 0 & ux & tx \\ 0 & 0 & 0 & x \\ 0 & 0 & x & -zc \end{pmatrix}$$

which cannot belong to I_p for the same reason.

Finally consider $(B^0_{(0,0,1)})^{-1}t'u'B^\infty_{(0,0,1)}$. This is congruent mod p to

$$\begin{pmatrix} -tx+zc-zcu & x & tx+zcu & sx+zct \\ x-ux & 0 & ux & tx \\ 0 & 0 & 0 & x \\ -ux & 0 & ux+x & tx-zc \end{pmatrix}$$

which can again not belong to I_p.

Thus we have completed the proof of the claim for $m > 0$.

For $m = 0$ we can only say that $t' = x + z\xi$ with atleast one of x, z an unit.

$(B^0_{(1,0,0)})^{-1}t'u'B^0_{(0,0,0)} =$

$$\begin{pmatrix} x+\frac{1}{2}zb & zc & sx+\frac{1}{2}zb+zct & tx+\frac{1}{2}zb+zcu \\ -za & x-\frac{1}{2}zb & tx-\frac{1}{2}zb-zas & ux-\frac{1}{2}zb-zat \\ x+\frac{1}{2}zb & zc & sx+x-\frac{1}{2}zb+\frac{1}{2}szb+zct & tx+\frac{1}{2}zb+zcu+za \\ 0 & 0 & -zc & x+\frac{1}{2}zb \end{pmatrix}$$

which if in I_p implies $p|z$ which in turn implies $p|x$, a contradiction.

$(B^{\infty}_{(1,0,0)})^{-1}t'u'B^{0}_{(0,0,0)} =$

$$\begin{pmatrix} -za & x-\frac{1}{2}zb & -zas+tx-\frac{1}{2}tzb & -zat+ux-\frac{1}{2}uzb \\ x+\frac{1}{2}zb & zc & sx+\frac{1}{2}szb+zct & tx+\frac{1}{2}tzb+zcu \\ 0 & 0 & -zc & x+\frac{1}{2}zb \\ x+\frac{1}{2}zb & zc & sx+x+zct+\frac{1}{2}szb-\frac{1}{2}zb & tx+\frac{1}{2}tzb+zcu+za \end{pmatrix}$$

which cannot belong to I_p for the same reason.

Put $G = z(ct+\frac{1}{2}sb-\frac{1}{2}b)$. $(B^{\infty}_{(1,0,0)})^{-1}t'u'B^{0}_{(1,0,0)} =$

$$\begin{pmatrix} za(s-1)-tx+\frac{1}{2}tzb & x-\frac{1}{2}zb & -zas+tx-\frac{1}{2}tzb & -zat+ux-\frac{1}{2}uzb \\ x(1-s)+G & zc & sx+\frac{1}{2}szb+zct & tx+\frac{1}{2}tzb+zcu \\ zc & 0 & -zc & x+\frac{1}{2}zb \\ \frac{1}{2}zb-G-sx & zc & x(s+1)+G & tx+\frac{1}{2}tzb+zcu+za \end{pmatrix}$$

which cannot belong to I_p for the same reason.

This completes the proof of the final claim. □

In which we calculate a certain volume

For any $t \in K_p$ we define the volume $I_t^{l,m}$ as follows.

$$I_t^{l,m} = \mathrm{vol}(B(\mathbb{Q}_p)\backslash B(\mathbb{Q}_p)h(l,m)tI_p). \tag{2.3.6}$$

In this subsection we shall explicitly compute the volume $I_t^{l,m}$. By Proposition 2.3.2, it is enough to do this for $t \in T_m$. The next two propositions state the results and the rest of the section is devoted to proving them.

Proposition 2.3.3. *Let $m > 0$. Let $M_{l,m}$ denote $\frac{p^{3l+4m}}{(p+1)(p^2+1)}$. Then the quantities $I_{t_i}^{l,m}$ for*

$1 \le i \le 8$ *are as follows.*

$$I_{t_1}^{l,m} = pM_{l,m} \qquad I_{t_5}^{l,m} = M_{l,m}$$
$$I_{t_2}^{l,m} = p^2 M_{l,m} \qquad I_{t_6}^{l,m} = pM_{l,m}$$
$$I_{t_3}^{l,m} = pM_{l,m} \qquad I_{t_7}^{l,m} = p^2 M_{l,m}$$
$$I_{t_4}^{l,m} = M_{l,m} \qquad I_{t_8}^{l,m} = p^3 M_{l,m}$$

Proposition 2.3.4. *For $m = 0$ the quantities $I_t^{l,m}$ are as follows.*

$$I_{t_1}^{l,m} = \frac{p^{3l+1}}{(p+1)(p^2+1)} \qquad I_{t_5}^{l,m} = \frac{p^{3l}}{(p+1)(p^2+1)}$$
$$I_{t_2}^{l,m} = \frac{p^{3l+2}}{(p+1)(p^2+1)} \qquad I_{t_7}^{l,m} = \frac{p^{3l+3}}{(p+1)(p^2+1)}$$

Remark. That the volume $I_t^{l,m}$ is finite can be viewed either as a *corollary* of the above propositions, or as a consequence of the fact that $\mathrm{vol}(B(\mathbb{Q}_p)\backslash B(\mathbb{Q}_p)h(l,m)K_p)$ is finite [Fur93, §3].

For each $t \in T_m$ define the subgroup G_t of K_p by

$$G_t = t^{-1}U(\mathbb{Z}_p)GL_2(\mathbb{Z}_p)t \cap I_p$$

where $U(\mathbb{Z}_p)$ is the subgroup of K_p consisting of matrices that look like $\begin{pmatrix} 1_2 & M \\ 0 & 1_2 \end{pmatrix}$ with $M = M^t \in M_2(\mathbb{Z}_p)$, and $GL_2(\mathbb{Z}_p)$ (more generally $GL_2(\mathbb{Q}_p)$) is embedded in $G(\mathbb{Q}_p)$ via $g \mapsto \begin{pmatrix} g & 0 \\ 0 & det(g)\cdot(g^{-1})^t \end{pmatrix}$.

Also let $G_t^1 = tG_tt^{-1}$ be the corresponding subgroup of $U(\mathbb{Z}_p)GL_2(\mathbb{Z}_p)$.

And finally, define

$$H_t = \{x \in GL_2(\mathbb{Z}_p) \mid \exists y \in U(\mathbb{Z}_p) \text{ such that } yx \in G_t^1\}. \tag{2.3.7}$$

It is easy to see that $H_t = U(\mathbb{Z}_p)G_t^1 \cap GL_2(\mathbb{Z}_p)$, thus H_t is a subgroup of $GL_2(\mathbb{Z}_p)$.

Lemma 2.3.5. *We have a disjoint union*

$$B(\mathbb{Q}_p)\backslash B(\mathbb{Q}_p)h(l,m)tI_p = \coprod_{y\in G_t\backslash I_p} B(\mathbb{Q}_p)\backslash B(\mathbb{Q}_p)h(l,m)G_t^1 ty.$$

Proof. Since $tI_p = \bigcup_{y\in G_t\backslash I_p} tG_t y = \bigcup_{y\in G_t\backslash I_p} G_t^1 ty$, the only thing to prove is that the union in the statement of the lemma is indeed disjoint.

So suppose that y_1, y_2 are two coset representatives of $G_t\backslash I_p$ and $rh(l,m)g_1ty_1 = h(l,m)g_2ty_2$ with $g_1, g_2 \in G_t^1, r \in B(\mathbb{Q}_p)$.

This means $ty_2y_1^{-1}t^{-1}$ is an element of K_p that is of the form $\begin{pmatrix} A & B \\ 0 & \det(A)\cdot(A^{-1})^t \end{pmatrix}$. Hence $ty_2y_1^{-1}t^{-1} \in U(\mathbb{Z}_p)GL_2(\mathbb{Z}_p)$. Thus $y_2y_1^{-1} \in t^{-1}U(\mathbb{Z}_p)GL_2(\mathbb{Z}_p)t \cap I_p = G_t$ which completes the proof.

□

By the above lemma it follows that

$$\begin{aligned} I_t^{l,m} &= \int_{G_t\backslash I_p} dg \cdot \int_{B(\mathbb{Q}_p)\backslash B(\mathbb{Q}_p)h(l,m)G_t^1} dt && (2.3.8)\\ &= p^{3(l+m)}[K_p : I_p]^{-1}[GL_2(\mathbb{Z}_p)U(\mathbb{Z}_p) : G_t^1] \int_{B(\mathbb{Q}_p)\backslash B(\mathbb{Q}_p)h(0,m)G_t^1} dt && (2.3.9) \end{aligned}$$

where we have normalized $\int_{U(\mathbb{Z}_p)GL_2(\mathbb{Z}_p)\backslash K_p} dx = 1$.

On the other hand,

$$\begin{aligned} B(\mathbb{Q}_p)\backslash B(\mathbb{Q}_p)h(0,m)G_t^1 &= B(\mathbb{Q}_p)\backslash B(\mathbb{Q}_p)h(0,m)U(\mathbb{Z}_p)G_t^1 \\ &= B(\mathbb{Q}_p)\backslash B(\mathbb{Q}_p)h(0,m)(U(\mathbb{Z}_p)G_t^1 \cap GL_2(\mathbb{Z}_p)) \\ &= T(\mathbb{Q}_p)\backslash T(\mathbb{Q}_p)h(m)H_t \end{aligned}$$

where $h(m) = \begin{pmatrix} p^m & 0 \\ 0 & 1 \end{pmatrix}$.

For each $t \in T_m$ let us define

$$A_t = [GL_2(\mathbb{Z}_p)U(\mathbb{Z}_p) : G_t^1]$$

and

$$V_{t,m} = \int_{T(\mathbb{Q}_p)\backslash T(\mathbb{Q}_p)h(m)H_t} dt.$$

We use the same normalization of Haar measures as in [Fur93], namely we have

$$\int_{T(\mathbb{Q}_p)\backslash T(\mathbb{Q}_p)h(m)GL_2(\mathbb{Z}_p)} dt = 1.$$

We summarize the computations above in the form of a lemma.

Lemma 2.3.6. *Let $m \geq 0$. For each $t \in T_m$ we have*

$$I_t^{l,m} = \frac{p^{3(l+m)}}{(p+1)^2(p^2+1)} \cdot A_t \cdot V_{t,m}.$$

Proof. This follows from equation (2.3.9). □

By exactly the same arguments as in [Fur93, p. 202-203], we see that

$$V_{t,m} = [GL_2(\mathbb{Z}_p) : H_t]^{-1}[T(\mathbb{Z}_p) : O_m^t] \tag{2.3.10}$$

where $O_m^t = T(\mathbb{Q}_p) \cap h(m)H_t h(m)^{-1}$.

Let Γ_p^0 (resp. $\Gamma_{0,p}$) be the subgroup of $GL_2(\mathbb{Z}_p)$ consisting of matrices that become lower-triangular (resp. upper-triangular) when reduced mod p.

Lemma 2.3.7. *(a) We have $H_{t_i} = \Gamma_p^0$ for $i = 1, 2, 5, 8$ and $H_{t_i} = \Gamma_{0,p}$ for $i = 3, 4, 6, 7$.*

(b) The quantities $A_{t_i} = [U(\mathbb{Z}_p)GL_2(\mathbb{Z}_p) : G_{t_i}^1]$ are as follows:

$$A_{t_1} = p(p+1) \qquad A_{t_5} = p+1$$

$$A_{t_2} = p^2(p+1) \qquad A_{t_6} = p+1$$

$$A_{t_3} = p^2(p+1) \qquad A_{t_7} = p^3(p+1)$$

$$A_{t_4} = p(p+1) \qquad A_{t_8} = p^3(p+1)$$

Proof. We will prove this directly using (2.3.7) and the definition of A_{t_i}.

First observe that the cardinality of $U(\mathbb{F}_p)GL_2(\mathbb{F}_p)$ is $p^3 \cdot (p^2-p)(p^2-1) = p^4(p-1)^2(p+1)$. Recall also that the images of Γ_p^0 and $\Gamma_{0,p}$ have cardinality $p(p-1)^2$ in $GL_2(\mathbb{F}_p)$.

Suppose

$$U = \begin{pmatrix} 1 & 0 & n & q \\ 0 & 1 & q & r \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, G = \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & d & -c \\ 0 & 0 & -b & a \end{pmatrix}.$$

We have

$$t_1^{-1}UGt_1 = \begin{pmatrix} a-nd+qb & b & nd-qb & -nc+qa \\ c-qd+rb & d & qd-rb & -qc+ra \\ a-nd+qb-d & b & nd-qb+d & -nc+qa-c \\ b & 0 & -b & a \end{pmatrix}.$$

By inspection, this belongs to I_p if and only if $b \equiv 0 \pmod p, n \equiv \frac{a}{d} - 1 \pmod p$. So $H_{t_1} = \Gamma_p^0$ and $A_{t_1} = \frac{p^4(p-1)^2(p+1)}{p(p-1)^2p^2} = p(p+1)$.

$$t_2^{-1}UGt_2 = \begin{pmatrix} d & c-qd+rb & ra-qc & qd-rb \\ b & a-nd+qb & qa-nc & nd-qb \\ 0 & b & a & -b \\ b & a-nd+qb-d & qa-c-nc & nd-qb+d \end{pmatrix}.$$

By inspection, this belongs to I_p if and only if $b \equiv 0 \pmod p, n \equiv \frac{a}{d} - 1 \pmod p, q \equiv \frac{c}{d}$. So $H_{t_2} = \Gamma_p^0$ and $A_{t_2} = \frac{p^4(p-1)^2(p+1)}{p(p-1)^2p} = p^2(p+1)$.

$$t_3^{-1}UGt_3 = \begin{pmatrix} a & b+nc-qa & nd-qb & -nc+qa \\ c & d+qc-ra & qd-rb & ra-qc \\ 0 & c & d & -c \\ c & d+qc-ra-a & qd-rb-b & ra-qc+a \end{pmatrix}.$$

By inspection, this belongs to I_p if and only if $c \equiv 0 \pmod p, r \equiv \frac{d}{a} - 1 \pmod p, q \equiv \frac{b}{a}$. So $H_{t_3} = \Gamma_{0,p}$ and $A_{t_3} = \frac{p^4(p-1)^2(p+1)}{p(p-1)^2p} = p^2(p+1)$.

$$t_4^{-1}UGt_4 = \begin{pmatrix} d+qc-ra & c & ra-qc & qd-rb \\ b+nc-qa & a & qa-nc & nd-qb \\ d+qc-ra-a & c & ra-qc+a & qd-rb-b \\ c & 0 & -c & d \end{pmatrix}.$$

By inspection, this belongs to I_p if and only if $c \equiv 0 \pmod{p}, r \equiv \frac{d}{a} - 1 \pmod{p}$. So $H_{t_4} = \Gamma_{0,p}$ and $A_{t_4} = \frac{p^4(p-1)^2(p+1)}{p(p-1)^2p^2} = p(p+1)$.

$$t_5^{-1}UGt_5 = \begin{pmatrix} a & b & nd-qb & -nc+qa \\ c & d & qd-rb & -qc+ra \\ 0 & 0 & d & -c \\ 0 & 0 & -b & a \end{pmatrix}.$$

By inspection, this belongs to I_p if and only if $b \equiv 0 \pmod{p}$. So $H_{t_5} = \Gamma_p^0$ and $A_{t_5} = \frac{p^4(p-1)^2(p+1)}{p(p-1)^2p^3} = (p+1)$.

$$t_6^{-1}UGt_6 = \begin{pmatrix} d & c & ra-qc & qd-rb \\ b & a & qa-nc & nd-qb \\ 0 & 0 & a & -b \\ 0 & 0 & -c & d \end{pmatrix}.$$

By inspection, this belongs to I_p if and only if $c \equiv 0 \pmod{p}$. So $H_{t_6} = \Gamma_{0,p}$ and $A_{t_6} = \frac{p^4(p-1)^2(p+1)}{p(p-1)^2p^3} = (p+1)$.

$$t_7^{-1}UGt_7 = \begin{pmatrix} d & -c & 0 & 0 \\ -b & a & 0 & 0 \\ qb-nd & nc-qa & a & b \\ rb-qd & qc-ra & c & d \end{pmatrix}.$$

By inspection, this belongs to I_p if and only if $c \equiv 0 \pmod{p}, n \equiv 0 \pmod{p}, q \equiv 0 \pmod{p}, r \equiv 0 \pmod{p}$. So $H_{t_7} = \Gamma_{0,p}$ and $A_{t_7} = \frac{p^4(p-1)^2(p+1)}{p(p-1)^2} = p^3(p+1)$.

$$t_8^{-1}UGt_8 = \begin{pmatrix} a & -b & 0 & 0 \\ -c & d & 0 & 0 \\ qc-ra & rb-qd & d & c \\ nc-qa & qb-nd & b & a \end{pmatrix}.$$

By inspection, this belongs to I_p if and only if $b \equiv 0 \pmod{p}, n \equiv 0 \pmod{p}, q \equiv 0 \pmod{p}, r \equiv 0 \pmod{p}$. So $H_{t_8} = \Gamma_p^0$ and $A_{t_8} = \frac{p^4(p-1)^2(p+1)}{p(p-1)^2} = p^3(p+1)$.

$\square$

Let t be such that $H_t = \Gamma_p^0$. Then by working through the definitions, we see that

$$O_m^t = x + p^{m+1} y \xi_0,\ \ x, y \in \mathbb{Z}_p. \tag{2.3.11}$$

On the other hand if t is such that $H_t = \Gamma_{0,p}$, then we see that

$$O_m^t = x + p^m y \xi_0,\ \ x, y \in \mathbb{Z}_p. \tag{2.3.12}$$

Lemma 2.3.8. *Let $m > 0$. Then we have $V_{t_i,m} = p^m$ for $i = 1, 2, 5, 8$ and $V_{t_i,m} = p^{m-1}$ for $i = 3, 4, 6, 7$.*

Proof. This follows from (2.3.10), (2.3.11), (2.3.12), Lemma 2.3.7 and [Fur93, Lemma 3.5.3] □

Proof of Proposition 2.3.3. The proof is a consequence of Lemma 2.3.6, Lemma 2.3.7 and Lemma 2.3.8. □

Let us now look at the case $m = 0$. In this case $T_0 = \{t_1, t_2, t_5, t_7\}$.

The groups H_{t_i} and the quantities $[GL_2(\mathbb{Z}_p) : H_{t_i}]^{-1}$ have already been calculated. On the other hand we now have

$$O_0^{t_i} = x + py\xi_0,\ \ x, y \in \mathbb{Z}_p. \tag{2.3.13}$$

for each $t_i \in T_0$.

Proof of Proposition 2.3.4. We have already calculated each A_{t_i}. Also by (2.3.10),(2.3.13) and Lemma 2.3.7 we conclude that each $V_{t_i,0} = 1$. Now the result follows as before, from Lemma 2.3.6.

□

Simplification of the local zeta integral

Recall the definition of the key local integral $Z_p(s)$ from §2.2. In (2.3.5) we reduced this integral to an useful sum. Now suppose that W_p and B_p are right I_p-invariant. Then

Proposition 2.3.2 allows us to further simplify that expression as follows.

$$Z_p(s) = \sum_{l\in\mathbb{Z},m\geq 0}\sum_{t\in T_m} W_p(\Theta h(l,m)t,s)\cdot B_p(h(l,m)t)\cdot I_t^{l,m} \tag{2.3.14}$$

Note that in the above formula we mildly abuse notation and use Θ to really mean its natural inclusion in $\widetilde{G}(\mathbb{Q}_p)$. We will continue to do this in the future for notational economy.

Remark. The importance of §2.3, where we calculated $I_t^{l,m}$ for each $t \in T_m$, is that we can now use the formula (2.3.14) to evaluate the local zeta integral whenever the local functions W_p and B_p can be explicitly determined.

2.4 The evaluation of the local Bessel functions in the Steinberg case

Background

Because automorphic representations of $GSp(4)$ are not necessarily generic, the Whittaker model is not always useful for studying L-functions. For many problems, the Bessel model is a good substitute. Explicit evaluation of local zeta integrals then often reduces to explicit evaluation of certain local Bessel functions. Formulas for the Bessel functions have been established in the following cases.

[Sug85] *unramified* representations of $GSp_4(\mathbb{Q}_p)$

[BFF97] *unramified* representations (the Casselman-Shalika like formula)

[Niw91] *class-one* representations on $Sp_4(\mathbb{R})$

[Miy00] large *discrete series* and P_J*-principal series* of $Sp_4(\mathbb{R})$

[Ish02] *principal series* of $Sp_4(\mathbb{R})$

In this section we give an explicit formula for the Bessel function for an unramified quadratic twist of the *Steinberg* representation of $GSp_4(\mathbb{Q}_p)$. By [Sch05] this is precisely the representation corresponding to a local newform for the Iwahori subgroup.

Throughout this section we let p be an odd prime that is inert in L. We suppose that the local component $(\omega_\pi)_p$ is trivial, the conductor of ψ_p is $\mathbb{Z}_p$ and $S = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \in M_2(\mathbb{Z}_p)$.

Because p is inert, L_p is a quadratic extension of $\mathbb{Q}_p$ and we may write elements of L_p

in the form $a+b\sqrt{-d}$ with $a,b\in\mathbb{Q}_p$; then $\mathbb{Z}_{L,p}=a+b\sqrt{-d}$ where $a,b\in\mathbb{Z}_p$. We identify L_p with $T(\mathbb{Q}_p)$ and ξ with $\sqrt{-d}/2$. Then $T(\mathbb{Z}_p)=\mathbb{Z}_{L,p}^\times$ consists of elements of the form $a+b\sqrt{-d}$ where a,b are elements of $\mathbb{Z}_p$ not both divisible by p.

We assume that Λ_p is trivial on the elements of $T(\mathbb{Z}_p)$ of the form $a+b\sqrt{-d}$ with $a,b\in\mathbb{Z}_p, p\mid b, p\nmid a$. Further, we assume that Λ_p is *not* trivial on the full group $T(\mathbb{Z}_p)$, that is, it is not unramified.

Finally, assume that the local representation π_p is an unramified twist of the Steinberg representation. This is representation IVa in [Sch05, Table 1]. The space of π_p contains a unique normalized vector that is fixed by the Iwahori subgroup I_p. We can think of this vector as the normalized local newform for this representation.

Bessel functions

Let $\mathfrak{B}$ be the space of locally constant functions φ on $G(\mathbb{Q}_p)$ satisfying

$$\varphi(tuh)=\Lambda_p(t)\theta_p(u)\varphi(h),\ \text{ for } t\in T(\mathbb{Q}_p), u\in U(\mathbb{Q}_p), h\in G(\mathbb{Q}_p).$$

Then by Novodvorsky and Piatetski-Shapiro [NPS73], there exists a unique subspace $\mathfrak{B}(\pi_p)$ of $\mathfrak{B}$ such that the right regular representation of $G(\mathbb{Q}_p)$ on $\mathfrak{B}(\pi_p)$ is isomorphic to π_p. Let B_p be the unique I_p-fixed vector in $\mathfrak{B}(\pi_p)$ such that $B_p(1_4)=1$. Therefore

$$B_p(tuhk)=\Lambda_p(t)\theta_p(u)\varphi(h), \tag{2.4.1}$$

where $t\in T(\mathbb{Q}_p), u\in U(\mathbb{Q}_p), h\in G(\mathbb{Q}_p), k\in K_p$.

Our goal is to explicitly compute B_p. By Proposition 2.3.2 and (2.4.1) it is enough to compute the values $B_p(h(l,m)t_i)$ for $l\in\mathbb{Z}, m\in\mathbb{Z}_{\geq 0}, t_i\in T_m$.

Let us fix some notation. Recall the matrices t_i which were defined in §2.3. Also we will frequently use other notation from §2.3. We now define

$$a_0^{l,m} = B_p(h(l,m)t_7), \qquad a_\infty^{l,m} = B_p(h(l,m)t_8),$$
$$b_0^{l,m} = B_p(h(l,m)t_2), \qquad {}^1b_0^{l,m} = B_p(h(l,m)t_1),$$
$$b_\infty^{l,m} = B_p(h(l,m)t_3), \qquad {}^1b_\infty^{l,m} = B_p(h(l,m)t_4),$$
$$c_0^{l,m} = B_p(h(l,m)t_5), \qquad c_\infty^{l,m} = B_p(h(l,m)t_6).$$

Lemma 2.4.1. *Let $m \geq 0, y \in \{0, \infty\}$. The following equations hold:*

(a) $a_y^{l,m} = 0 \quad$ *if* $l < -1$.

(b) ${}^1b_0^{l,m} = b_0^{l,m} = {}^1b_\infty^{l,0} = b_\infty^{l,0} = 0 \quad$ *if* $l < 0$.

(c) ${}^1b_\infty^{l,m} = b_\infty^{l,m} = 0 \quad$ *if* $l < -1$.

(d) $c_y^{l,m} = 0 \quad$ *if* $l < 0$.

Proof. First note that $U_{(0,0,p)}t_i \equiv t_i \pmod p$, hence they are in the same coset of K_p/I_p. Hence

$$\begin{aligned} B_p(h(l,m)t_i) &= B_p(h(l,m)U_{(0,0,p)}t_i) \\ &= B_p(U_{(0,0,p^{l+1})}h(l,m)t_i) \\ &= \psi_p(p^{l+1}c)B_p(h(l,m)t_i). \end{aligned}$$

Since the conductor of ψ_p is $\mathbb{Z}_p$ and c is a unit, it follows that $B_p(h(l,m)t_i) = 0$ for $l < -1$. This completes the proof of (a) and (c).

Next, observe that

$$\begin{aligned} c_y^{l,m} &= B_p(h(l,m)Z_y) \\ &= B_p(h(l,m)U_{(0,0,1)}Z_y) \\ &= B_p(U_{(0,0,p^l)}h(l,m)Z_y) \\ &= \psi_p(p^l c)B_p(h(l,m)Z_y). \end{aligned}$$

It follows that $B_p(h(l,m)Z_y) = 0$ for $l < 0$. This completes the proof of (d).

Next, we have

$$\begin{aligned} B_p(h(l,m)JU_{(1,0,0)}JZ_y) &= B_p(h(l,m)JU_{(1,0,0)}JU_{0,0,1}Z_y) \\ &= B_p(h(l,m)U_{0,0,1}JU_{(1,0,0)}JZ_y) \\ &= \psi_p(p^l c)B_p(h(l,m)JU_{(1,0,0)}JZ_y). \end{aligned}$$

It follows that ${}^1b_0^{l,m} = b_0^{l,m} = 0$ for $l < 0$.

Finally,

$$\begin{aligned} B_p(h(l,0)JU_{(0,0,1)}JZ_y) &= B_p(h(l,0)JU_{(0,0,1)}JU_{1,0,0}Z_y) \\ &= B_p(h(l,0)U_{1,0,0}JU_{(0,0,1)}JZ_y) \\ &= \psi_p(p^l a)B_p(h(l,0)JU_{(0,0,1)}JZ_y). \end{aligned}$$

It follows that ${}^1b_\infty^{l,0} = b_\infty^{l,0} = 0$ for $l < 0$. This completes the proof of (b).

□

By our normalization, we have $c_0^{0,0} = 1$. From Proposition 2.3.2, proof of Claim 6, it follows that $c_\infty^{0,0} = \Lambda_p(\frac{b+\sqrt{-d}}{2})$.

To get more information, we have to use the fact that the local Iwahori-Hecke algebra acts on B_p in a precise manner.

Hecke operators and the results

Henceforth we always assume that $l \geq -1, m \geq 0$. In particular, all equations that are stated without qualification will be understood to hold in the above range. We know that π_p is either $\mathrm{St}_{GSp(4)}$ or $\xi_0 \mathrm{St}_{GSp(4)}$ where ξ_0 is the non-trivial unramified quadratic character. Put $w_p = -1$ in the former case and $w_p = 1$ in the latter. Put

$$\eta_p = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & p & 0 & 0 \\ p & 0 & 0 & 0 \end{pmatrix}.$$

Also, for $y \in V$, define the matrices R_y as follows: If $y \in Y$,

$$R_y = (U_{(y,0,0)})^t,$$

and

$$R_\infty = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let $t \in G(\mathbb{Q}_p)$. By [Sch05], we know the following:

$$\sum_{y \in V} B_p(tZ_y) = 0, \tag{2.4.2}$$

$$B_p(t\eta_p) = w_p B_p(t), \tag{2.4.3}$$

$$\sum_{y \in V} B_p(tR_y) = 0. \tag{2.4.4}$$

(2.4.2) and Proposition 2.3.2 immediately imply

$$a_0^{l,m} + pa_\infty^{l,m} = 0, \qquad \text{for } m > 0 \tag{2.4.5}$$

$$pb_y^{l,m} + {}^1b_y^{l,m} = 0, \qquad \text{for } y \in \{0, \infty\} \tag{2.4.6}$$

$$pc_0^{l,m} + c_\infty^{l,m} = 0, \qquad \text{for } m > 0 \tag{2.4.7}$$

Next we act upon by η_p. Check that

$$(h(l+1,m)B^\infty_{(0,0,0)})^{-1}h(l,m)A^0_{(0,0,0)}\eta_p = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

So we have

$$\begin{aligned} a_0^{l,m} &= B_p(h(l,m)A^0_{(0,0,0)}) \\ &= w_p B_p(h(l,m)A^0_{(0,0,0)}\eta_p) \\ &= w_p B_p(h(l+1,m)B^\infty_{(0,0,0)}). \end{aligned}$$

Thus

$$a_0^{l,m} = w_p c_\infty^{l+1,m}. \tag{2.4.8}$$

We also have

$$(h(l+1,m)B^0_{(0,0,0)})^{-1}h(l,m)A^\infty_{(0,0,0)}\eta_p = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

So similarly, we conclude

$$a_\infty^{l,m} = w_p c_0^{l+1,m}. \tag{2.4.9}$$

Next, check that

$$(h(l,m)B^1_{(1,0,0)}\eta_p)^{-1}h(l-1,m+1)U_{(-1/p,0,0)}D^1_\infty = (Z^1)^t \in I_p.$$

Hence

$$B_p(h(l,m)B^1_{(1,0,0)}) = w_p B_p(h(l-1,m+1)D^1_\infty).$$

(Note that both sides are zero if $l=-1, m=0$).

By the proof of Proposition 2.3.2, $B_p(h(l,m)B^1_{(1,0,0)}) = b_0^{l,m}$ and $B_p(h(l-1,m+1)D^1_\infty) = \psi_p(p^{l-1}c)b_\infty^{l-1,m+1}$.

Thus we have proved

$$b_0^{l,m} = w_p\psi_p(p^{l-1}c)b_\infty^{l-1,m+1}. \tag{2.4.10}$$

At this point we pause and note that on account of (2.4.5)–(2.4.10) it is enough to compute the quantities $b_\infty^{l,m}, a_0^{l,m}, l \geq -1, m \geq 0, l+m \neq -1$. Of course, we already know that $a_0^{-1,0} = w_p\Lambda_p(\frac{b+\sqrt{-d}}{2})$.

Next, we use (2.4.4).

For each $x \in Y$, we can check that $A^0_{0,0,0}R_x = A^0_{-x,0,0}$. Furthermore, $A^0_{0,0,0}R_\infty = D^0_\infty$. Assuming $l+m \geq 0$ we have $B_p(h(l,m)A^0_{-x,0,0}) = a_0^{l,m}$ and $B_p(h(l,m)D^0_\infty = \psi_p(p^l c)b_\infty^{l,m}$. So using (2.4.4) we conclude

$$pa_0^{l,m} = -\psi_p(p^l c)b_\infty^{l,m}, \tag{2.4.11}$$

for $l+m \geq 0$.

However we can do more. Check that for $x \in Y$, $A^\infty_{(0,0,0)}R_x = A^\infty_{(0,0,-x)}$ and $A^\infty_{(0,0,0)}R_\infty \equiv D^0_0 \pmod p$. If $l \geq 0$ we have $B_p(h(l,m)A^\infty_{(0,0,-x)} = a_\infty^{l,m}$ and $B_p(h(l,m)D^0_0) = b_0^{l,m}$. So again using (2.4.4) we have

$$pa_\infty^{l,m} = -b_0^{l,m}, \tag{2.4.12}$$

for $l \geq 0$.

So (2.4.5), (2.4.10) and (2.4.12) imply that for $l \geq 0$, $m > 0$

$$b_\infty^{l,m} = -pb_0^{l,m} = -pw_p\psi_p(p^{l-1}c)b_\infty^{l-1,m+1}. \tag{2.4.13}$$

Now observe that $B^0_{0,0,0}R_\infty \equiv D^\infty_0 \pmod p$ and for $x \in Y$, $B^0_{0,0,0}R_x = B^0_{-x,0,0}$. Assuming $l+m \neq -1$ we have $B_p(h(l,m)D^\infty_0) = {}^1b_0^{l,m}$ and for $x \in y, x \neq 0$, $B_p(h(l,m)B^0_{-x,0,0}) = {}^1b_0^{l,m}$. Hence using (2.4.4)

$$c_0^{l,m} = -p\ {}^1b_0^{l,m} \tag{2.4.14}$$

So by equations (2.4.6) and (2.4.9) we have,

$$a_\infty^{l,m} = p^2\psi_p(p^l c)b_\infty^{l,m+1} \tag{2.4.15}$$

The above equation, along with our normalization tells us that

$$b_\infty^{-1,1} = \frac{1}{p^2}\psi_p(-\frac{c}{p})w_p. \tag{2.4.16}$$

Also, using (2.4.12),(2.4.13) and (2.4.15) we get

$$b_\infty^{l,m+1} = \frac{1}{p^4}b_\infty^{l,m} \tag{2.4.17}$$

for $l \geq 0, m > 0$.

(2.4.13), (2.4.17) and (2.4.16) imply :

$$b_\infty^{l,m} = -\frac{(-pw_p)^l}{p^{4l+4m+1}} \quad \text{if } l \geq 0, m \geq 1 \tag{2.4.18}$$

$$b_\infty^{-1,m} = \frac{1}{p^{4m-2}}\psi_p(-\frac{c}{p})w_p \quad \text{if } m \geq 1. \tag{2.4.19}$$

In the case $m = 0$, Proposition 2.3.2, proof of Claim 7, tells us that ${}^1b_\infty^{l,0} = \Lambda_p(\frac{b+\sqrt{-d}}{2})\ {}^1b_0^{l,0}$ which implies

$$b_\infty^{l,0} = \Lambda_p(\frac{b+\sqrt{-d}}{2})b_0^{l,0} = w_p\psi_p(p^{l-1}c)\Lambda_p(\frac{b+\sqrt{-d}}{2})b_\infty^{l-1,1} \tag{2.4.20}$$

EquationS (2.4.18)–(2.4.20), along with the earlier equations that specify the interdependence of various quantities, determine all the values $B_p(h(l,m)t_i)$. For convenience, we compactly state the facts proven above as two propositions. We only state it for $l \geq 0$ since that is the only case needed for our later applications. The values for $l = -1$ can be easily gleaned from these and the above equations.

Proposition 2.4.2. *Let $l \geq 0, m > 0$. Put $M = (-pw_p)^l p^{-4(l+m)}$. Then the following hold:*

(a) $B_p(h(l,m)t_1) = M \cdot \frac{-1}{p}$,

(b) $B_p(h(l,m)t_2) = M \cdot \frac{1}{p^2}$,

(c) $B_p(h(l,m)t_3) = M \cdot \frac{-1}{p}$,

(d) $B_p(h(l,m)t_4) = M$

(e) $B_p(h(l,m)t_5) = M$

(f) $B_p(h(l,m)t_6) = M \cdot (-p)$,

(g) $B_p(h(l,m)t_7) = M \cdot \frac{1}{p^2}$,

(h) $B_p(h(l,m)t_8) = M \cdot \frac{-1}{p^3}$.

Proposition 2.4.3. *Let $l \geq 0$. Put $M = (-pw_p)^l p^{-4l}$. Then the following hold:*

(a) $B_p(h(l,0)t_1) = M \cdot \frac{-1}{p}$,

(b) $B_p(h(l,0)t_2) = M \cdot \frac{1}{p^2}$,

(c) $B_p(h(l,0)t_5) = M$,

(d) $B_p(h(l,m)t_7) = M \cdot \frac{-\Lambda_p(\frac{b+\sqrt{-d}}{2})}{p^3}$.

2.5 The case unramified π_p, Steinberg σ_p

Assumptions

Suppose that the characters $\omega_\pi, \omega_\sigma, \chi_0$ are trivial. Let $p \neq 2$ be a finite prime of $\mathbb{Q}$ such that

(a) p is inert in $L = \mathbb{Q}(\sqrt{-d})$.

(b) The local components Λ_p and π_p are unramified.

(c) σ_p is the Steinberg representation (or its twist by the unramified quadratic character).

(d) The conductor of ψ_p is $\mathbb{Z}_p$

(e) $S = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \in M_2(\mathbb{Z}_p)$.

(f) $-d = b^2 - 4ac$ generates the discriminant of $L_p/\mathbb{Q}_p$.

Remark: σ_p is concretely realized as (possibly the unramified quadratic twist of) the special representation on the locally constant functions of $B_p \backslash GL_2(\mathbb{Q}_p)$ modulo the constant functions (where B_p is the standard Borel subgroup consisting of upper-triangular matrices). It corresponds to the local newform for the Iwahori subgroup $\Gamma_0(p)$ of $GL_2(\mathbb{Q}_p)$.

Description of B_p and W_p

For any choice of local Whittaker and Bessel functions W_p and B_p we can define the local zeta integral $Z_p(s)$ by (2.2.6). We now fix such a choice.

As in the unramified case from §2.1, we let B_p be the unique normalized K_p-vector in the local Bessel space. Sugano[Sug85] has computed the function B_p explicitly.

We now define W_p. Let $\widetilde{U_p}$ be the subgroup of $\widetilde{K_p}$ defined by

$$\widetilde{U_p} = \{z \in \widetilde{K_p} \mid z \equiv \begin{pmatrix} * & 0 & * & * \\ * & * & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \pmod{p}\}.$$

It is not hard to see that $I(\Pi_p, s)$ has $\widetilde{U_p}$-fixed vectors. Now let W_p be the unique $\widetilde{U_p}$-fixed vector in the local Whittaker space with the following properties:

- $W_p(e, s) = 1$,
- $W_p(g, s) = 0$ if g does not belong to $P(\mathbb{Q}_p)\widetilde{U_p}$

Concretely we have the following description of $W_p(\ s)$.

We know that $\sigma_p = \mathrm{Sp} \otimes \tau$ where Sp denotes the special (Steinberg) representation and τ is a (possibly trivial) unramified quadratic character. We put $a_p = \tau(p)$, thus $a_p = \pm 1$ is the eigenvalue of the local Hecke operator $T(p)$.

Let W'_p be the unique function on $GL_2(\mathbb{Q}_p)$ such that

$$W'_p(gk) = W'_p(g), \text{for } g \in GL_2(\mathbb{Q}_p), k \in \Gamma_{0,p}, \tag{2.5.1}$$

$$W'_p(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g) = \psi_p(-cx)W'_p(g), \text{for } g \in GL_2(\mathbb{Q}_p), x \in \mathbb{Q}_p, \tag{2.5.2}$$

$$W'_p \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} = \begin{cases} \tau(a)|a| & \text{if } |a|_p \leq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{2.5.3}$$

$$W'_p \left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) = \begin{cases} -p^{-1}\tau(a)|a| & \text{if } |a|_p \leq p, \\ 0 & \text{otherwise.} \end{cases} \tag{2.5.4}$$

We extend W'_p to a function on $GU(1,1)(\mathbb{Q}_p)$ by

$$W'_p(ag) = W'_p(g), \ \text{for } a \in L_p^\times, g \in GL_2(\mathbb{Q}_p).$$

Then, $W_p(\,s)$ is the unique function on $\widetilde{G}(\mathbb{Q}_p)$ such that

$$W_p(mnk,s) = W_p(m,s),\ \text{ for } m \in M(\mathbb{Q}_p), n \in N(\mathbb{Q}_P), k \in \widetilde{U_p}, \tag{2.5.5}$$

$$W_p(e) = 1 \text{ and } W_p(g,s) = 0 \text{ if } g \notin P(\mathbb{Q}_p)\widetilde{U_p}, \tag{2.5.6}$$

and

$$\begin{aligned} W_p\left(\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \overline{a}^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a_1 & 0 & b_1 \\ 0 & 0 & c_1 & 0 \\ 0 & d_1 & 0 & e_1 \end{pmatrix}, s\right) \\ = \left|N_{L/\mathbb{Q}}(a)\cdot c_1^{-1}\right|_p^{3(s+1/2)} \cdot W_p'\begin{pmatrix} a_1 & b_1 \\ d_1 & e_1 \end{pmatrix} \end{aligned} \tag{2.5.7}$$

for $a \in \mathbb{Q}_p^\times$, $\begin{pmatrix} a_1 & b_1 \\ d_1 & e_1 \end{pmatrix} \in GU(1,1)(\mathbb{Q}_p), c_1 = \mu_1\begin{pmatrix} a_1 & b_1 \\ d_1 & e_1 \end{pmatrix}$.

Let us use the following notation: For $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GU(1,1)$ we let

$$m^{(2)}(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & \beta & 0 \\ 0 & c & 0 & d \end{pmatrix}$$

where $\beta = \mu_1(\begin{pmatrix} a & b \\ c & d \end{pmatrix})$.

The results

For $i = 1, 2, 3, 4$, define the characters $\gamma_p^{(i)}$ of $\mathbb{Q}_p^\times$ as in §2.1. We now state and prove the main theorem of this section.

Theorem 2.5.1. *Let the functions B_p, W_p be as defined in §2.5. Then we have*

$$Z_p(s, W_p, B_p) = \frac{1}{p^2+1} \cdot \frac{L(3s+\frac{1}{2}, \pi_p \times \sigma_p)}{L(3s+1, \sigma_p \times \rho(\Lambda_p))}$$

where,

$$L(s, \pi_p \times \sigma_p) = \prod_{i=1}^{4} (1 - \gamma_p^{(i)}(p) a_p p^{-1/2} p^{-s})^{-1},$$

and

$$L(s, \sigma_p \times \rho(\Lambda_p)) = (1 - p^{-2s-1})^{-1}.$$

Before we begin the proof, we need a lemma.

Lemma 2.5.2. *We have the following formulae for $W_p(\Theta h(l,m)t_i, s)$ where $t_i \in T_m$.*

(a) If $m > 0$ then $W_p(\Theta h(l,m)t_i, s) = \begin{cases} p^{-6ms-3ls-3m-5l/2} a_p^l & \text{if } i \in \{1,5\} \\ p^{-6ms-3ls-3m-5l/2} a_p^l \cdot \frac{-1}{p} & \text{if } i \in \{3,7\} \\ 0 & \text{otherwise} \end{cases}$

(b) $W_p(\Theta h(l,0)t_i, s) = \begin{cases} p^{-3ls-5l/2} a_p^l & \text{if } i \in \{1,5\} \\ 0 & \text{if } i \in \{2,7\} \end{cases}$

Proof. We have

$$\Theta h(l,m) = h(l,m) \begin{pmatrix} 1 & 0 & 0 & 0 \\ p^m\alpha & 1 & 0 & 0 \\ 0 & 0 & 1 & -p^m\overline{\alpha} \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.5.8}$$

First consider the case $m > 0$.

We claim that $\Theta h(l,m)t_i \notin P(\mathbb{Q}_p)\widetilde{U_p}$ if $i \in \{2,4,6,8\}$.

Put $\Theta_m = \begin{pmatrix} 1 & 0 & 0 & 0 \\ p^m\alpha & 1 & 0 & 0 \\ 0 & 0 & 1 & -p^m\overline{\alpha} \\ 0 & 0 & 0 & 1 \end{pmatrix}$. Using (2.6.8), it suffices to prove that $\Theta_m t_i \notin$

$P(\mathbb{Z}_p)\widetilde{U_p}$. So take a typical element element

$$P = \begin{pmatrix} a & ax & at + ax\overline{y} & ay \\ 0 & m & m\overline{y} - \beta\overline{x} & \beta \\ 0 & 0 & \lambda/\overline{a} & 0 \\ 0 & \gamma & \gamma\overline{y} - \delta\overline{x} & \delta \end{pmatrix} \in P(\mathbb{Z}_p) \tag{2.5.9}$$

where all the variables lie in $\mathbb{Z}_L$ and $\begin{pmatrix} m & \beta \\ \gamma & \delta \end{pmatrix} \in GU(1,1)(\mathbb{Z}_p)$ with $\lambda = \mu_1 \begin{pmatrix} m & \beta \\ \gamma & \delta \end{pmatrix}$.

We have

$$P\Theta_m t_2 = \begin{pmatrix} ax & a + axp^m\alpha - at - ax\overline{y} & -(at + ax\overline{y})\overline{\alpha}p^m + ay & at + ax\overline{y} \\ m & mp^m\alpha - m\overline{y} + \beta\overline{x} & -(m\overline{y} - \beta\overline{x})\overline{(\alpha)}p^m + \beta & m\overline{y} - \beta\overline{x} \\ 0 & -\lambda(\overline{a})^{-1} & -\overline{\alpha}p^m\lambda(\overline{a})^{-1} & \lambda(\overline{a})^{-1} \\ \gamma & \gamma p^m\alpha - \gamma\overline{y} + \delta\overline{x} & -(\gamma\overline{y} - \delta\overline{x})\overline{\alpha}p^m + \delta & \gamma\overline{y} - \delta\overline{x} \end{pmatrix}$$

which, if it were an element of $\widetilde{U_p}$ would imply that $p \mid \lambda$, a contradiction.

We have

$$P\Theta_m t_4 = \begin{pmatrix} ax + (at + ax\overline{y})\overline{\alpha}p^m - ay & a + axp^m\alpha & -(at + ax\overline{y})\overline{\alpha}p^m + ay & at + ax\overline{y} \\ m + (m\overline{y} - \beta\overline{x})\overline{\alpha}p^m - \beta & mp^m\alpha & -(m\overline{y} - \beta\overline{x})\overline{\alpha}p^m + \beta & m\overline{y} - \beta\overline{x} \\ \overline{\alpha}p^m\lambda(\overline{a})^{-1} & 0 & -\overline{\alpha}p^m\lambda(\overline{a})^{-1} & \lambda(\overline{a})^{-1} \\ \gamma + (\gamma\overline{y} - \delta\overline{x})\overline{\alpha}p^m - \delta & \gamma p^m\alpha & -(\gamma\overline{y} - \delta\overline{x})\overline{\alpha}p^m + \delta & \gamma\overline{y} - \delta\overline{x} \end{pmatrix}$$

which, if it were an element of $\widetilde{U_p}$ would imply that $p \mid \lambda$, a contradiction.

We have

$$P\Theta_m t_6 = \begin{pmatrix} ax & a + axp^m\alpha & -(at + ax\overline{y})\overline{\alpha}p^m + ay & at + ax\overline{y} \\ m & mp^m\alpha & -(m\overline{y} - \beta\overline{x})\overline{\alpha}p^m + \beta & m\overline{y} - \beta\overline{x} \\ 0 & 0 & -\overline{\alpha}p^m\lambda(\overline{a})^{-1} & \lambda(\overline{a})^{-1} \\ \gamma & \gamma p^m\alpha & -(\gamma\overline{y} - \delta\overline{x})\overline{\alpha}p^m + \delta & \gamma\overline{y} - \delta\overline{x} \end{pmatrix}$$

which, if it were an element of $\widetilde{U_p}$ would imply that $p \mid \gamma, p \mid \delta$, a contradiction.

Finally we have

$$P\Theta_m t_8 = \begin{pmatrix} at + ax\overline{y})\overline{\alpha}p^m - ay & -at - ax\overline{y} & ax & a + axp^m\alpha \\ (m\overline{y} - \beta\overline{x})\overline{\alpha}p^m - \beta & -m\overline{y} + \beta\overline{x} & m & mp^m\alpha \\ \lambda\overline{\alpha}p^m(\overline{a})^{-1} & -\lambda(\overline{a})^{-1} & 0 & 0 \\ (\gamma\overline{y} - \delta\overline{x})\overline{\alpha}p^m - \delta & -\gamma\overline{y} + \delta\overline{x} & \gamma & \gamma p^m\alpha \end{pmatrix}$$

which, if it were an element of $\widetilde{U_p}$ would imply that $p \mid \gamma, p \mid \delta$, a contradiction. This completes the proof of the claim.

For the remaining t_i (i.e., $i \in \{1, 3, 5, 7\}$) we have the following decompositions:

$\Theta h(l, m)t_1$

$$= \begin{pmatrix} p^{2m+l} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & p^{-2m-l} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} m^{(2)}(\begin{pmatrix} p^{m+l} & 0 \\ 0 & p^m \end{pmatrix}) \begin{pmatrix} -1 & 0 & 0 & 0 \\ -p^m\alpha & -1 & 0 & 0 \\ 1 & 0 & -1 & p^m\overline{\alpha} \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$\Theta h(l, m)t_3 =$

$$\begin{pmatrix} p^{2m+l} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & p^{-2m-l} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} m^{(2)}(\begin{pmatrix} p^{m+l} & 0 \\ -p^m & p^m \end{pmatrix}) \begin{pmatrix} -1 & 0 & 0 & 0 \\ -p^m\alpha & -1 & 0 & 0 \\ 0 & -p^m\overline{\alpha} & -1 & p^m\overline{\alpha} \\ -p^m\alpha & 0 & 0 & -1 \end{pmatrix}$$

$\Theta h(l, m)t_5$

$$= \begin{pmatrix} p^{2m+l} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & p^{-2m-l} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} m^{(2)}(\begin{pmatrix} p^{m+l} & 0 \\ 0 & p^m \end{pmatrix}) \begin{pmatrix} 1 & 0 & 0 & 0 \\ p^m\alpha & 1 & 0 & 0 \\ 0 & 0 & 1 & p^m\overline{\alpha} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\Theta h(l,m)t_7$

$$= \begin{pmatrix} p^{2m+l} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & p^{-2m-l} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} m^{(2)}\left(\begin{pmatrix} 0 & -p^{m+l} \\ -p^m & 0 \end{pmatrix}\right) \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & p^m\overline{\alpha} & 0 & 0 \\ 0 & 0 & p^m\alpha & 1 \end{pmatrix}$$

Part (a) of the lemma now follows from the above decompositions and equations (2.5.1)-(2.5.7).

Let us now look at $m = 0$. Once again, let P be the matrix defined in (2.5.9). The same proof as above for t_2 shows that $P\Theta_m t_2 \notin \widetilde{U_p}$. As for t_7,

$$P\Theta_m t_7 = \begin{pmatrix} -at - ax\overline{y} & (at + ax\overline{y})\overline{\alpha} - ay & a + ax\alpha & ax \\ -m\overline{y} + \beta\overline{x} & (m\overline{y} - \beta\overline{x})\overline{\alpha} - \beta & m\alpha & m \\ -\lambda(\overline{a})^{-1} & \lambda\overline{\alpha}(\overline{a})^{-1} & 0 & 0 \\ -\gamma\overline{y} + \delta\overline{x} & (\gamma\overline{y} - \delta\overline{x})\overline{\alpha} - \delta & \gamma\alpha & \gamma \end{pmatrix}.$$

If the above matrix lies in $\widetilde{U_p}$ then we have $p \mid \gamma\alpha$ which implies $p \mid \gamma$. But that immediately implies, by looking at the bottom left entry, that $p \mid \delta\overline{x}$, hence (by looking at the second entry of the bottom row) $p \mid \delta$. Thus $p \mid \gamma, p \mid \delta$, a contradiction.

Thus $\Theta h(l,0)t_i \notin P(\mathbb{Q}_p)\widetilde{U_p}$ if $i \in \{2,7\}$. For t_1 and t_5 we have the above decompositions, from which part (b) follows via the equations (2.5.1)-(2.5.7).

□

Proof of Theorem 2.5.1. By (2.3.14) we have

$$Z_p(s, W_p, B_p) = \sum_{l\geq 0, m\geq 0} B_p(h(l,m)) \sum_{t_i\in T_m} W_p(\Theta h(l,m)t_i, s) \cdot I_{t_i}^{l,m} \tag{2.5.10}$$

We first look at the terms corresponding to $m > 0$. From Lemma 2.5.2 and Proposition 2.3.3 we have $\sum_{t_i\in T_m} W_p(\Theta h(l,m)t_i, s) \cdot I_{t_i}^{l,m} = 0$. So only terms corresponding to $m = 0$ contribute.

From Proposition 2.3.4 and Lemma 2.5.2 we have

$$\sum_{t_i\in T_0} W_p(\Theta h(l,0)t_i, s) \cdot I_{t_i}^{l,0} = \frac{1}{p^2+1} \cdot p^{-3ls+l/2} a_p^l.$$

Hence (2.6.9) reduces to

$$Z_p(s, W_p, B_p) = \frac{1}{p^2+1} \cdot \sum_{l \geq 0} B_p(h(l,0)) p^{-3ls+l/2} a_p^l.$$

Define $C(y) = \sum_{l \geq 0} B_p(h(l,0)) y^l$. We are interested in the quantity

$$Z_p(s, W_p, B_p) = \frac{1}{p^2+1} C(a_p p^{-3s+1/2}). \tag{2.5.11}$$

Sugano, in [Sug85, p. 544], has computed $C(y)$ explicitly. His results imply that

$$C(y) = \frac{H(y)}{Q(y)}$$

where $H(y) = 1 - \frac{y^2}{p^4}, Q(y) = \prod_{i=1}^4 (1 - \gamma_p^{(i)}(p) p^{-3/2} y)$.

Plugging in these values in (2.5.11) we get the desired result.

□

2.6 The case Steinberg π_p, Steinberg σ_p

Assumptions

Suppose that the characters $\omega_\pi, \omega_\sigma, \chi_0$ are trivial. Let $p \neq 2$ be a finite prime of $\mathbb{Q}$ such that

(a) p is inert in $L = \mathbb{Q}(\sqrt{-d})$.

(b) Λ_p is not trivial on $T(\mathbb{Z}_p)$; however it is trivial on $T(\mathbb{Z}_p) \cap \Gamma_p^0$.

(c) π_p is the Steinberg representation (or its twist by the unique non-trivial unramified quadratic character).

(d) σ_p is the Steinberg representation (or its twist by the unique non-trivial unramified quadratic character).

(e) The conductor of ψ_p is $\mathbb{Z}_p$.

(f) $S = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \in M_2(\mathbb{Z}_p)$.

(g) $-d = b^2 - 4ac$ generates the discriminant of $L_p/\mathbb{Q}_p$.

Remark. π_p corresponds to a local newform for the Iwahori subgroup I_p (see [Sch05]). Also, as in the previous section, σ_p corresponds to the local newform for the Iwahori subgroup $\Gamma_0(p)$ of $GL_2(\mathbb{Q}_p)$.

Description of B_p and W_p

Let Φ_p be the unique normalized local newform for the Iwahori subgroup I_p, as defined by Schmidt [Sch05]. Let w_p be the local Atkin-Lehner eigenvalue for π_p; this equals -1 when π_p is the Steinberg representation and equals 1 when π_p is the unramified quadratic twist of the Steinberg representation. We let B_p be the normalized vector that corresponds to Φ_p in the Bessel space. §2.4 was devoted to the computation of the values $B_p(h(l,m)t)$ for $l, m \in \mathbb{Z}, m \geq 0, t \in T_m$.

Because p is inert, L_p is a quadratic extension of $\mathbb{Q}_p$ and we may write elements of L_p in the form $a + b\sqrt{-d}$ with $a, b \in \mathbb{Q}_p$; then $\mathbb{Z}_{L,p} = a + b\sqrt{-d}$ where $a, b \in \mathbb{Z}_p$. We also identify L_p with $T(\mathbb{Q}_p)$ and ξ with $\sqrt{-d}/2$. We now define W_p. By Assumption (b) above, we have Λ_p is trivial on the elements of $T(\mathbb{Q}_p)$ of the form $a + b\sqrt{-d}$ with $a, b \in \mathbb{Z}_p, p \mid b, p \nmid a$. Take the canonical map $r : \widetilde{K}_p \to \widetilde{G}(\mathbb{F}_p)$ and define $I'_p = r^{-1}(I(4, \mathbb{F}_p))$.

Let s_1 denote the matrix $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$.

Let $W_p(\ ,s)$ be the unique vector in $I(\Pi_p, s)$ with the following properties:

- $W_p(1, s) = 1$,
- $W_p(s_1, s) = 1$,
- $W_p(gk, s) = W_p(g, s)$ is $k \in I'_p$,
- $W_p(g, s) = 0$ if g does not belong to $P(\mathbb{Q}_p)I'_p \sqcup P(\mathbb{Q}_p)s_1 I'_p$

Concretely we have the following description of $W_p(\ ,s)$:

We know that $\sigma_p = Sp \otimes \tau$ where Sp denotes the special (Steinberg) representation and τ is a (possibly trivial) unramified quadratic character. We put $a_p = \tau(p)$, thus $a_p = \pm 1$ is the eigenvalue of the local Hecke operator $T(p)$.

Let W'_p be the unique function on $GL_2(\mathbb{Q}_p)$ such that

$$W'_p(gk) = W'_p(g), \text{for } g \in GL_2(\mathbb{Q}_p), k \in \Gamma_{0,p}, \tag{2.6.1}$$

$$W'_p(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g) = \psi_p(-cx) W'_p(g), \text{for } g \in GL_2(\mathbb{Q}_p), x \in \mathbb{Q}_p, \tag{2.6.2}$$

$$W'_p \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} = \begin{cases} \tau(a)|a| & \text{if } |a|_p \leq 1, \\ 0 & \text{otherwise} \end{cases} \tag{2.6.3}$$

$$W'_p \left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) = \begin{cases} -p^{-1}\tau(a)|a| & \text{if } |a|_p \leq p, \\ 0 & \text{otherwise} \end{cases} \tag{2.6.4}$$

We extend W'_p to a function on $GU(1,1)(\mathbb{Q}_p)$ by

$$W'_p(ag) = W'_p(g), \text{ for } a \in L_p^\times, g \in GL_2(\mathbb{Q}_p).$$

Then, $W_p(\ s)$ is the unique function on $\widetilde{G}(\mathbb{Q}_p)$ such that

$$W_p(mnuk, s) = W_p(mu, s), \text{ for } m \in M(\mathbb{Q}_p), n \in N(\mathbb{Q}_P), u \in \{1, s_1\}, k \in I'_p, \tag{2.6.5}$$

$$W_p(t) = 0 \text{ if } t \notin P(\mathbb{Q}_p) I'_p \sqcup P(\mathbb{Q}_p) s_1 I'_p \tag{2.6.6}$$

$$\begin{aligned} &W_p \left(\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \overline{a}^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a_1 & 0 & b_1 \\ 0 & 0 & c_1 & 0 \\ 0 & d_1 & 0 & e_1 \end{pmatrix} u, s \right) \\ &= \left| N_{L/\mathbb{Q}}(a) \cdot c_1^{-1} \right|_p^{3(s+1/2)} \cdot \Lambda_p(a) W'_p \begin{pmatrix} a_1 & b_1 \\ d_1 & e_1 \end{pmatrix}, \end{aligned} \tag{2.6.7}$$

for $a \in \mathbb{Q}_p^\times, u \in \{1, s_1\}, \begin{pmatrix} a_1 & b_1 \\ d_1 & e_1 \end{pmatrix} \in GU(1,1)(\mathbb{Q}_p), c_1 = \mu_1 \begin{pmatrix} a_1 & b_1 \\ d_1 & e_1 \end{pmatrix}$.

The results

We now state and prove the main theorem of this section.

Theorem 2.6.1. *Let the functions B_p, W_p be as defined in subsection* 2.6. Then we have*

$$Z_p(s, W_p, B_p) = \frac{1-p}{p^2+1} \cdot \frac{p^{-6s-3}}{1 - a_p w_p p^{-3s-3/2}} \cdot L(3s + \frac{1}{2}, \pi_p \times \sigma_p)$$

where $L(s, \pi_p \times \sigma_p) = (1 + a_p w_p p^{-1} p^{-s})^{-1} (1 + a_p w_p p^{-2} p^{-s})^{-1}$.

Before we begin the proof, we need a lemma.

Lemma 2.6.2. *We have the following formulae for $W_p(\Theta h(l,m)t_i, s)$ where $t_i \in T_m$.*

$$W_p(\Theta h(l,m)t_i, s) = \begin{cases} p^{-6ms-3ls-3m-5l/2} a_p^l \cdot \frac{-1}{p} & \text{if } i = 3,4, \quad m > 0 \\ p^{-6ms-3ls-3m-5l/2} a_p^l & \text{if } i = 5,6, \quad m > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We have

$$\Theta h(l,m) = h(l,m) \begin{pmatrix} 1 & 0 & 0 & 0 \\ p^m\alpha & 1 & 0 & 0 \\ 0 & 0 & 1 & -p^m\overline{\alpha} \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.6.8}$$

Put $K'_p = r^{-1}(G(F_p))$. Thus $\Theta h(l,m)t_i \in P(\mathbb{Q}_p)K'_p$ when $m > 0$ and $\Theta h(l,m)t_i \in P(\mathbb{Q}_p)\Theta K'_p$ when $m = 0$. A direct computation shows that $P(\mathbb{Q}_p)K'_p$ and $P(\mathbb{Q}_p)\Theta K'_p$ are disjoint; the fact that $P(\mathbb{Q}_p)I'_p \subset P(\mathbb{Q}_p)K'_p$ then implies that $W_p(\Theta h(l,m)t_i, s) = 0$ for $m = 0$. From now on we assume $m > 0$.

We can check that $\Theta h(l,m)t_i \notin P(\mathbb{Q}_p)I'_p \sqcup P(\mathbb{Q}_p)s_1 I'_p$ if $i \in \{1,2,7,8\}$.

For the remaining t_i (i.e. $i \in \{3,4,5,6\}$) we have the decompositions:

$\Theta h(l,m)t_3 =$

$$\begin{pmatrix} p^{2m+l} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & p^{-2m-l} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} m^{(2)}(\begin{pmatrix} p^{m+l} & 0 \\ -p^m & p^m \end{pmatrix}) \begin{pmatrix} -1 & 0 & 0 & 0 \\ -p^m\alpha & -1 & 0 & 0 \\ 0 & -p^m\overline{\alpha} & -1 & p^m\overline{\alpha} \\ -p^m\alpha & 0 & 0 & -1 \end{pmatrix}$$

$\Theta h(l,m)t_4 =$

$$\begin{pmatrix} p^{2m+l} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & p^{-2m-l} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} m^{(2)}(\begin{pmatrix} p^{m+l} & 0 \\ -p^m & p^m \end{pmatrix}) s_1 \begin{pmatrix} 1 & p^m\alpha & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & p^m\alpha & 1 & 0 \\ p^m\overline{\alpha} & 0 & -p^m\overline{\alpha} & 1 \end{pmatrix}$$

$\Theta h(l,m)t_5$

$$= \begin{pmatrix} p^{2m+l} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & p^{-2m-l} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} m^{(2)}(\begin{pmatrix} p^{m+l} & 0 \\ 0 & p^m \end{pmatrix}) \begin{pmatrix} 1 & 0 & 0 & 0 \\ p^m\alpha & 1 & 0 & 0 \\ 0 & 0 & 1 & p^m\overline{\alpha} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\Theta h(l,m)t_6$

$$= \begin{pmatrix} p^{2m+l} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & p^{-2m-l} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} m^{(2)}(\begin{pmatrix} p^{m+l} & 0 \\ 0 & p^m \end{pmatrix}) s_1 \begin{pmatrix} 1 & p^m\alpha & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -p^m\overline{\alpha} & 1 \end{pmatrix}$$

The lemma now follows from the above decompositions and equations (2.6.1)-(2.6.7).

□

Proof of Theorem 2.6.1. By (2.3.14) we have

$$Z_p(s, W_p, B_p) = \sum_{l\geq 0, m\geq 0} \sum_{t_i \in T_m} B_p(h(l,m)t_i) W_p(\Theta h(l,m)t_i, s) \cdot I_{t_i}^{l,m} \tag{2.6.9}$$

From Proposition 2.3.3, Proposition 2.4.2 and Lemma 2.6.2 we have

$$\sum_{i\in\{3,4,5,6\}} B_p(h(l,m)t_i)W_p(\Theta h(l,m)t_i,s)\cdot I_{t_i}^{l,m} = \frac{(1-p)(-a_p w_p p^{-3s-5/2})^l(p^{-6s-3})^m}{p^2+1}.$$

Hence (2.6.9) implies

$$Z_p(s,W_p,B_p) = \frac{(1-p)p^{-6s-3}}{p^2+1}\cdot\frac{1}{1+a_p w_p p^{-2}p^{-3s-1/2}}\cdot\frac{1}{1-p^{-6s-3}}$$

This completes the proof.

□

2.7 The case Steinberg π_p, unramified σ_p

Assumptions

Suppose that the characters $\omega_\pi, \omega_\sigma, \chi_0$ are trivial. Let $p \neq 2$ be a finite prime of $\mathbb{Q}$ such that

(a) p is inert in $L = \mathbb{Q}(\sqrt{-d})$.

(b) Λ_p is not trivial on $T(\mathbb{Z}_p)$; however it is trivial on $T(\mathbb{Z}_p) \cap \Gamma_p^0$.

(c) π_p is the Steinberg representation (or its twist by the unique non-trivial unramified quadratic character) while σ_p is unramified.

(d) The conductor of ψ_p is $\mathbb{Z}_p$.

(e) $S = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \in M_2(\mathbb{Z}_p)$.

(f) $-d = b^2 - 4ac$ generates the discriminant of $L_p/\mathbb{Q}_p$.

Remark. π_p corresponds to a local newform for the Iwahori subgroup I_p (see [Sch05]).

Description of B_p and W_p

Let Φ_p be the unique normalized local newform for the Iwahori subgroup I_p, as defined by Schmidt [Sch05]. Let w_p be the local Atkin-Lehner eigenvalue for π_p; this equals -1 when

π_p is the Steinberg representation and equals 1 when π_p is the unramified quadratic twist of the Steinberg representation. We let B_p be the normalized vector that corresponds to Φ_p in the Bessel space. Section 2.4 was devoted to the computation of the values $B_p(h(l,m)t)$ for $l, m \in \mathbb{Z}, m \geq 0, t \in T_m$.

We now define W_p. Take the canonical map $r : \widetilde{K}_p \to \widetilde{G}(\mathbb{F}_p)$ and define $I'_p = r^{-1}(I(4, \mathbb{F}_p))$. Let $W_p(\ ,s)$ be the unique vector in $I(\Pi_p, s)$ with the following properties:

- $W_p(\Theta, s) = 1,$
- $W_p(1, s) = 1,$
- $W_p(gk, s) = W_p(g, s)$ is $k \in I'_p,$
- $W_p(g, s) = 0$ if g does not belong to $P(\mathbb{Q}_p)\Theta I'_p \sqcup P(\mathbb{Q}_p)I'_p$

Concretely we have the following description of $W_p(\ ,s)$.

Suppose σ_p is the principal series representation induced from the unramified characters α, β of $\mathbb{Q}_p^\times$. Let W'_p be the unique function on $GL_2(\mathbb{Q}_p)$ such that

$$W'_p(gk) = W'_p(g), \text{for } g \in GL_2(\mathbb{Q}_p), k \in GL_2(\mathbb{Z}_p), \tag{2.7.1}$$

$$W'_p(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g) = \psi_p(-cx)W'_p(g), \text{for } g \in GL_2(\mathbb{Q}_p), x \in \mathbb{Q}_p, \tag{2.7.2}$$

$$W'_p \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{cases} \left|\frac{a}{b}\right|_p^{\frac{1}{2}} \cdot \frac{\alpha(ap)\beta(b) - \alpha(b)\beta(ap)}{\alpha(p) - \beta(p)} & \text{if } \left|\frac{a}{b}\right|_p \leq 1, \\ 0 & \text{otherwise} \end{cases} \tag{2.7.3}$$

We extend W'_p to a function on $GU(1,1)(\mathbb{Q}_p)$ by

$$W'_p(ag) = W'_p(g), \text{ for } a \in L_p^\times, g \in GL_2(\mathbb{Q}_p).$$

Then, $W_p(\ s)$ is the unique function on $\widetilde{G}(\mathbb{Q}_p)$ such that

$$W_p(mnuk, s) = W_p(mu, s), \text{ for } m \in M(\mathbb{Q}_p), n \in N(\mathbb{Q}_P), u \in \{1, \Theta\}, k \in I'_p, \tag{2.7.4}$$

$$W_p(t) = 0 \text{ if } t \notin P(\mathbb{Q}_p)\Theta I'_p \sqcup P(\mathbb{Q}_p)I'_p \tag{2.7.5}$$

$$\begin{aligned} &W_p\left(\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \overline{a}^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a_1 & 0 & b_1 \\ 0 & 0 & c_1 & 0 \\ 0 & d_1 & 0 & e_1 \end{pmatrix} u, s\right) \\ &= \left|N_{L/\mathbb{Q}}(a) \cdot c_1^{-1}\right|_p^{3(s+1/2)} \cdot \Lambda_p(\overline{a}^{-1}) W'_p \begin{pmatrix} a_1 & b_1 \\ d_1 & e_1 \end{pmatrix}, \end{aligned} \tag{2.7.6}$$

for $a \in \mathbb{Q}_p^\times, u \in \{1, \Theta\}, \begin{pmatrix} a_1 & b_1 \\ d_1 & e_1 \end{pmatrix} \in GU(1,1)(\mathbb{Q}_p), c_1 = \mu_1 \begin{pmatrix} a_1 & b_1 \\ d_1 & e_1 \end{pmatrix}$.

The results

We now state and prove the main theorem of this section.

Theorem 2.7.1. *Let the functions B_p, W_p be as defined in subsection* 2.7. Then we have*

$$Z_p(s, W_p, B_p) = \frac{1}{(p+1)(p^2+1)} \cdot L(3s + \frac{1}{2}, \pi_p \times \sigma_p),$$

where $L(s, \pi_p \times \sigma_p) = (1 + w_p p^{-3/2}\alpha(p)p^{-s})^{-1}(1 + w_p p^{-3/2}\beta(p)p^{-s})^{-1}$.

Before we begin the proof, we need a lemma.

Lemma 2.7.2. *Let $t_i \in T_m, l \geq 0$. We have*

$$W_p(\Theta h(l,m)t_i, s) = \begin{cases} p^{-3ls-2l}\left(\frac{\alpha(p)^{l+1}-\beta(p)^{l+1}}{\alpha(p)-\beta(p)}\right) & \text{if } m = 0, i = 5 \\ p^{-6ms-3ls-3m-5l/2}\left(\frac{\alpha(p)^{l+1}-\beta(p)^{l+1}}{\alpha(p)-\beta(p)}\right) & \text{if } m > 0, i = 3,5 \\ 0 & \text{otherwise} \end{cases}$$

Proof. By the proof of Lemma 2.6.2 we have $\Theta h(l,m)t_i \notin P(\mathbb{Q}_p)\Theta I'_p$ if $m > 0$. As for the case $m = 0$, we can check that $\Theta h(l,0)t_i \notin P(\mathbb{Q}_p)\Theta I'_p$ if $i \in \{1,2,7\}$. On the other hand, again by the proof of Lemma 2.6.2, we have $\Theta h(l,m)t_i \in P(\mathbb{Q}_p)I'_p$ if and only if $m > 0$ and $i \in \{3,5\}$. The lemma now follows immediately from (2.7.1) - (2.7.6). $\square$

Proof of Theorem 2.7.1. We have

$$\begin{aligned} Z_p(s, W_p, B_p) = &\sum_{l \geq 0} W_p(\Theta h(l,0)t_5, s) B_p(h(l,0)t_5) \cdot I_{t_5}^{l,0} \\ &+ \sum_{l \geq 0, m > 0} \sum_{i \in \{3,5\}} W_p(\Theta h(l,m)t_i, s) B_p(h(l,m)t_i) \cdot I_{t_i}^{l,m} \end{aligned} \tag{2.7.7}$$

Using Proposition 2.4.3 , Proposition 2.3.4 and Lemma 2.7.2 we have

$$\sum_{i \in \{3,5\}} W_p(\Theta h(l,m)t_i, s) B_p(h(l,m)t_i) \cdot I_{t_i}^{l,m} = 0$$

and hence

$$\begin{aligned} Z_p(s, W_p, B_p) &= \sum_{l \geq 0} W_p(\Theta h(l,0)t_5, s) B_p(h(l,0)t_5) \cdot I_{t_5}^{l,0} \\ &= \frac{1}{(p+1)(p^2+1)} \sum_{l \geq 0} p^{-3ls-2l} \left(\frac{\alpha(p)^{l+1} - \beta(p)^{l+1}}{\alpha(p) - \beta(p)} \right) (-pw_p)^l p^{-l} \\ &= \frac{1}{(p+1)(p^2+1)} L(3s + \frac{1}{2}, \pi_p \times \sigma_p). \end{aligned}$$

This completes the proof of the theorem.

□

Remark. We might equally well have chosen W_p to be the simpler vector supported only on Θ (rather than on Θ and 1). The only reason we include 1 in the support of the section is because this definition will be necessary for §3.

2.8 The global integral and some results

Classical Siegel modular forms and newforms for the minimal congruence subgroup

For M a positive integer define the following global parahoric subgroups.

$$B(M) := Sp(4,\mathbb{Z}) \cap \begin{pmatrix} \mathbb{Z} & M\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ M\mathbb{Z} & M\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ M\mathbb{Z} & M\mathbb{Z} & M\mathbb{Z} & \mathbb{Z} \end{pmatrix},$$

$$U_1(M) := Sp(4,\mathbb{Z}) \cap \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ M\mathbb{Z} & M\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ M\mathbb{Z} & M\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{pmatrix},$$

$$U_2(M) := Sp(4,\mathbb{Z}) \cap \begin{pmatrix} \mathbb{Z} & M\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & M\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ M\mathbb{Z} & M\mathbb{Z} & M\mathbb{Z} & \mathbb{Z} \end{pmatrix},$$

$$U_0(M) := Sp(4,\mathbb{Q}) \cap \begin{pmatrix} \mathbb{Z} & M\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & M^{-1}\mathbb{Z} \\ M\mathbb{Z} & M\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ M\mathbb{Z} & M\mathbb{Z} & M\mathbb{Z} & \mathbb{Z} \end{pmatrix}.$$

When $M = 1$ each of the above groups is simply $Sp(4,\mathbb{Z})$. For $M > 1$, the groups are all distinct. If Γ' is equal to one of the above groups, or (more generally) is any congruence subgroup, we define $S_k(\Gamma')$ to be the space of Siegel cusp forms of degree 2 and weight k with respect to the group Γ'.

More precisely, let $\mathbb{H}_2 = \{Z \in M_2(\mathbb{C}) | Z = Z^t, i(\overline{Z} - Z) \text{ is positive definite}\}$. For any $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G$ let $J(g,Z) = CZ + D$. Then $f \in S_k(\Gamma')$ if it is a holomorphic function on $\mathbb{H}_2$, satisfies $f(\gamma Z) = \det(J(\gamma, Z))^k f(Z)$ for $\gamma \in \Gamma', Z \in \mathbb{H}_2$ and disappears at the cusps. We know that f has a Fourier expansion

$$f(Z) = \sum_{S>0} a(S,F) e(\mathrm{tr}(SZ)),$$

where $e(z) = \exp(2\pi iz)$ and S runs through all symmetric semi-integral positive-definite matrices of size two.

Now let M be a square-free positive integer. For any decomposition $M = M_1M_2$ into coprime integers we define, following Schmidt [Sch05], the subspace of oldforms $S_k(B(M))^{\mathrm{old}}$ to be the sum of the spaces

$$S_k(B(M_1) \cap U_0(M_2)) + S_k(B(M_1) \cap U_1(M_2)) + S_k(B(M_1) \cap U_2(M_2)).$$

For each prime p not dividing M there is the local Hecke algebra $\mathfrak{H}_p$ of operators on $S_k(B(M))$ and for each prime q dividing M we have the Atkin-Lehner involution η_q also acting on $S_k(B(M))$. For details, the reader may refer to [Sch05].

By a newform for the minimal congruence subgroup $B(M)$, we mean an element $f \in S_k(B(M))$ with the following properties

(a) f lies in the orthogonal complement of the space $S_k(B(M))^{\mathrm{old}}$.

(b) f is an eigenform for the local Hecke algebras $\mathfrak{H}_p$ for all primes p not dividing M.

(c) f is an eigenform for the Atkin-Lehner involutions η_q for all primes q dividing M.

Remark. By [Sch05], if we assume the hypothesis that a nice L-function theory for $GSp(4)$ exists, (b) and (c) above follow from (a) and the assumption that f is an eigenform for the local Hecke algebras at *almost* all primes.

Description of our newforms

Let M be an odd square-free positive integer and

$$F(Z) = \sum_{T>0} a(T)\mathrm{e}(\mathrm{tr}(TZ))$$

be a Siegel newform for $B(M)$ of even weight l.

Let N be an odd square-free positive integer and g be a normalized newform of weight l for $\Gamma_0(N)$. g has a Fourier expansion

$$g(z) = \sum_{n=1}^{\infty} b(n)e(nz)$$

with $b(1) = 1$. It is then well known that the $b(n)$ are all totally real algebraic numbers.

We make the following assumption:

$$a(T) \neq 0 \textit{ for some } T = \begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix} \tag{2.8.1}$$

such that $-d = b^2 - 4ac$ *is the discriminant of the imaginary quadratic field* $\mathbb{Q}(\sqrt{-d})$*, and all primes dividing* MN *are inert in* $\mathbb{Q}(\sqrt{-d})$.

We define a function $\Phi = \Phi_F$ on $G(\mathbb{A})$ by

$$\Phi(\gamma h_\infty k_0) = \mu_2(h_\infty)^l \det(J(h_\infty, iI_2))^{-l} F(h_\infty(i))$$

where $\gamma \in G(\mathbb{Q}), h_\infty \in G(\mathbb{R})^+$ and

$$k_0 \in (\prod_{p \nmid M} K_p) \cdot (\prod_{p | M} I_p).$$

Because we do not have strong multiplicity one for G we can only say that the representation of $G(\mathbb{A})$ generated by Φ is a *multiple* of an irreducible representation π. However that is enough for our purposes.

We know that $\pi = \otimes \pi_v$ where

$$\pi_v = \begin{cases} \text{holomorphic discrete series} & \text{if } v = \infty, \\ \text{unramified spherical principal series} & \text{if } v \text{ finite }, v \nmid M, \\ \xi_v \mathrm{St}_{GSp(4)} \text{where } \xi_v \text{ unramified, } \xi_v^2 = 1 & \text{if } v \mid M. \end{cases}$$

Next, we define a function Ψ on $GL_2(\mathbb{A})$ by

$$\Psi(\gamma_0 m k_0) = (\det\ m)^{\frac{l}{2}} (\gamma i + \delta)^{-l} g(m(i))$$

where $\gamma_0 \in GL_2(\mathbb{Q}),\ m = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GL_2^+(\mathbb{R})$, and

$$k_0 \in \prod_{p \nmid N} GL_2(\mathbb{Z}_p) \prod_{p | N} \Gamma_{0,p}$$

Let σ be the automorphic representation of $GL_2(\mathbb{A})$ generated by Ψ.

We know that $\sigma = \otimes\sigma_v$ where

$$\sigma_v = \begin{cases} \text{holomorphic discrete series} & \text{if } v = \infty, \\ \text{unramified spherical principal series} & \text{if } v \text{ finite }, v \nmid N, \\ \xi \mathrm{St}_{GL(2)} \text{where } \xi_v \text{ unramified}, \ \xi_v^2 = 1 & \text{if } v \mid N. \end{cases}$$

Description of our Bessel model

In order to use our results from the previous sections, we need to associate a Bessel model to π (or more accurately, we associate it to $\widetilde{\pi}$). This involves making a choice of (S, Λ, ψ). This subsection is devoted to doing that.

Let $\psi = \prod_v \psi_v$ be a character of $\mathbb{A}$ such that

- The conductor of ψ_p is $\mathbb{Z}_p$ for all (finite) primes p,
- $\psi_\infty(x) = e(-x)$, for $x \in \mathbb{R}$,
- $\psi|_{\mathbb{Q}} = 1$.

Put $L = \mathbb{Q}(\sqrt{-d})$. where d is the integer defined in (2.8.1).

First we deal with the case $M = 1$. In this case, our choice of S and Λ is identical to [Fur93]. To recall, put

$$T(\mathbb{A}) = \coprod_{j=1}^{h(-d)} t_j T(\mathbb{Q})T(\mathbb{R})(\Pi_{p<\infty} T(\mathbb{Z}_p)) \tag{2.8.2}$$

where $t_j \in \prod_{p<\infty} T(\mathbb{Q}_p)$ and $h(-d)$ is the class number of L.

Write $t_j = \gamma_j m_j \kappa_j$, where $\gamma_j \in GL_2(\mathbb{Q}), m_j \in GL_2^+(\mathbb{R})$, and $\kappa_j \in ((\Pi_{p<\infty} GL_2(\mathbb{Z}_p))$.

Choose

$$S = \begin{cases} \begin{pmatrix} d/4 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } d \equiv 0 \pmod 4 \\ \begin{pmatrix} (1+d)/4 & 1/2 \\ 1/2 & 1 \end{pmatrix} & \text{if } d \equiv 3 \pmod 4 \end{cases}$$

Let $S_j = \det(\gamma_j)^{-1}\gamma_j^t S\gamma_j$. Then, any primitive semi-integral two by two positive definite matrix with discriminant equal to $-d$ is $SL_2(\mathbb{Z})$-equivalent to some S_j. So, by our assumption, we can choose Λ a character of $T(\mathbb{A})/T(\mathbb{Q})T(\mathbb{R})((\Pi_{p<\infty}T(\mathbb{Z}_p))$ such that

$$\sum_{j=1}^{h(-d)} \Lambda(t_j)\overline{a(S_j)} \neq 0.$$

Thus, we have specified a choice of S and Λ for $M = 1$.

In the rest of this subsection, unless otherwise mentioned, assume $M > 1$.

Suppose p is a prime dividing M. We can identify L_p with elements $a + b\sqrt{-d}$ with $a, b \in \mathbb{Q}_p$. Let $\mathbb{Z}_{L,p}^\times$ denote the units in the ring of integers of L_p. The elements of $\mathbb{Z}_{L,p}^\times$ are of the form $a+b\sqrt{-d}$ with $a, b \in \mathbb{Z}_p$ and such that at least one of a and b is a unit. Let $\Gamma_{L,p}^0$ be the subgroup of $\mathbb{Z}_{L,p}^\times$ consisting of the elements with $p|b$. The group $\mathbb{Z}_{L,p}^\times/\Gamma_{L,p}^0$ is clearly cyclic of order $p+1$. Moreover, the elements $\{(-b+\sqrt{-d})/2\}$ where b is a positive integer satisfying $\{1 \leq b \leq 2p : b = d \pmod 2\}$ are distinct in $\mathbb{Z}_{L,p}^\times/\Gamma_{L,p}^0$. Note that $d = 0$ or $3 \pmod 4$ and hence $b = d \pmod 2$ implies that 4 divides $b^2 + d$. So we have the lemma:

Lemma 2.8.1. *There exists an integer b such that 4 divides $b^2 + d$ and $(-b + \sqrt{-d})/2$ is a generator of the group $\mathbb{Z}_{L,p}^\times/\Gamma_{L,p}^0$ for each $p|M$.*

Proof. By the comments above, we can choose, for each prime p_i dividing M, an integer b_i such that $b_i \equiv d \pmod 2$ and $(-b_i + \sqrt{-d})/2$ is a generator of the group $\mathbb{Z}_{L,p_i}^\times/\Gamma_{L,p_i}^0$. Now, using the Chinese Remainder theorem, choose b satisfying $b \equiv b_i \pmod{2p_i}$ for each i . □

Now we define

$$S = \begin{pmatrix} \frac{b^2+d}{4} & \frac{b}{2} \\ \frac{b}{2} & 1 \end{pmatrix}.$$

As in section 2.1 we define the matrix $\xi = \xi_S$ and the group $T = T_S$. We have $T(\mathbb{Q}) \simeq L^\times$. We write $T(\mathbb{Z}_p)$ for $T(\mathbb{Q}_p) \cap GL_2(\mathbb{Z}_p)$.

Let

$$T(\mathbb{A}) = \coprod_{j=1}^{h(-d)} t_j T(\mathbb{Q})T(\mathbb{R})(\Pi_{p<\infty}T(\mathbb{Z}_p) \tag{2.8.3}$$

where $t_j \in \prod_{p<\infty} T(\mathbb{Q}_P)$ and $h(-d)$ is the class number of L. For each $p|M$ put $\Gamma_{L,p}^0 = T(\mathbb{Z}_p) \cap \Gamma_p^0$. Note that under the isomorphism $T(\mathbb{Z}_p) \simeq \mathbb{Z}_{L,p}^\times$ sending $x + y\xi \mapsto x + y\frac{\sqrt{-d}}{2}$, our two definitions for $\Gamma_{L,p}^0$ agree, so there is no ambiguity.

Let $M = p_1 p_2 ... p_r$ be its decomposition into distinct primes. For each $1 \leq i \leq r$ we choose coset representatives $u_{k_i}^{(p_i)} \in T(\mathbb{Z}_{p_i})$ such that

$$T(\mathbb{Z}_{p_i}) = \coprod_{k_i=1}^{p_i+1} u_{k_i}^{(p_i)} \Gamma^0_{L,p_i}.$$

We write an r-tuple $(k_1, .., k_r)$ in short as $\widetilde{k}$. Let X denote the Cartesian product of the r sets $X_i = \{x : 1 \leq x \leq p_i\}$. For $\widetilde{k} \in X$, define

$$u_{\widetilde{k}} = \prod_{i=1}^{r} u_{k_i}^{(p_i)}.$$

Then it is easy to see that as $\widetilde{k}$ varies over X the elements $u_{\widetilde{k}}$ form a set of coset representatives of $\Pi_{p|M} T(\mathbb{Z}_p)/\Pi_{p|M}\Gamma^0_{L,p}$. Also note that $|X| = |SL_2(\mathbb{Z})/\Gamma^0(M)| = \Pi_{p_1|M}(p_i + 1)$. We denote the quantity $\Pi_{p_1|M}(p_i + 1)$ by $g(M)$.

Let $T(\mathbb{Z})$ denote the (finite) group of units in the ring of integers $\mathbb{Z}_L$ of L. Let $t(d)$ denote the cardinality of the group $T(\mathbb{Z})/\{\pm 1\}$. We know that,

$$t(d) = \begin{cases} 3 & \text{if } d = 3 \\ 2 & \text{if } d = 4 \\ 1 & \text{otherwise.} \end{cases}$$

Let $T_M^{\times}$ be the image of $T(\mathbb{Z})$ in $\Pi_{p|M} T(\mathbb{Z}_p)$. Then $T_M^{\times} \cap \Pi_{p|M}\Gamma^0_{L,p} = \{\pm 1\}$. Choose a set of elements $r_1, r_2, ..r_{t(d)}$ in $T(\mathbb{Z})$ such that they form distinct representatives in $T(\mathbb{Z})/\{\pm 1\}$. Let $\overline{r}_i$ denote the image of r_i in $T_M^{\times}$. We have

$$T_M^{\times} \Pi_{p|M} \Gamma^0_{L,p} = \coprod_{i=1}^{t(d)} \overline{r}_i (\Pi_{p|M}\Gamma^0_{L,p}). \tag{2.8.4}$$

Finally, choose $x_1, x_2, ..., x_{g(M)/t(d)}$ in $\Pi_{p|M} T(\mathbb{Z}_p)$ such that we have the disjoint coset decomposition:

$$\Pi_{p|M} T(\mathbb{Z}_p) = \coprod_{i=1}^{g(M)/t(d)} x_i T_M^{\times} \Pi_{p|M} \Gamma^0_{L,p} \tag{2.8.5}$$

This immediately gives us the fundamental coset decomposition:

$$T(\mathbb{A}) = \coprod_{\substack{1\le j\le h(-d)\\ 1\le k\le g(M)/t(d)}} t_j x_k T(\mathbb{Q})T(\mathbb{R})(\Pi_{p\nmid M}T(\mathbb{Z}_p))(\Pi_{p_i|M}\Gamma^0_{L,p_i}) \tag{2.8.6}$$

Also from (2.8.4) and (2.8.5) we immediately get another coset decomposition:

$$\Pi_{p|M}T(\mathbb{Z}_p) = \coprod_{\substack{1\le i\le g(M)/t(d)\\ 1\le j\le t(d)}} x_i\overline{r}_j\Pi_{p|M}\Gamma^0_{L,p} \tag{2.8.7}$$

But we know that an alternate set of coset representatives in the above equation is given by the elements $u_{\widetilde{k}}$. It follows that for any $1 \le i \le g(M)/t(d), 1 \le j \le t(d)$, there exists a unique $\widetilde{k} \in X$ such that $u_{\widetilde{k}}^{-1}x_i\overline{r}_j \in \Pi_{p|M}\Gamma^0_{L,p}$. This correspondence is bijective.

Write $t_jx_k = \gamma_{j,k}m_{j,k}\kappa_{j,k}$, where $\gamma_{j,k} \in GL_2(\mathbb{Q}), m_{j,k} \in GL_2^+(\mathbb{R})$, and $\kappa_{j,k} \in (\Pi_{p<\infty,p\nmid M}GL_2(\mathbb{Z}_p)\cdot \Pi_{p|M}\Gamma^0_p$. Also, by $(\gamma_{j,k})_f$ we denote the finite part of $\gamma_{j,k}$, that is, $(\gamma_{j,k})_f = \gamma_{j,k}m_{j,k}$.

Lemma 2.8.2. *For each j, the elements $\gamma_{j,1}^{-1}r_l\gamma_{j,k}$ form a system of representatives of $SL_2(\mathbb{Z})/\Gamma^0(M)$ as l, k vary over $1 \le l \le t(d)$, $1 \le k \le g(M)/t(d)$.*

Proof. Fix j. Let $1 \le l_2 \le t(d)$, $1 \le k_2 \le g(M)/t(d)$. We have

$$\gamma_{j,k_2}^{-1}r_{l_2}^{-1}r_l\gamma_{j,k} = m_{j,k_2}\kappa_{j,k_2}x_{k_2}^{-1}r_{l_2}^{-1}r_lx_k(m_{j,k}\kappa_{j,k})^{-1}.$$

Therefore $\gamma_{j,k_2}^{-1}r_{l_2}^{-1}r_l\gamma_{j,k} \in \left(GL_2^+(\mathbb{R})\Pi_{q<\infty}GL_2(\mathbb{Z}_q)\right) \cap GL_2(\mathbb{Q}) = SL_2(\mathbb{Z})$. Moreover, if it belongs to $\Gamma^0(M)$ then we must have $x_{k_2}^{-1}\overline{r}_{l_2}^{-1}\overline{r}_lx_k \in \Pi_{p|M}\Gamma^0_p$ and by (2.8.7) this can happen only if $l = l_2, k = k_2$. Now the lemma follows because the size of the set $\gamma_{j,1}^{-1}r_l\gamma_{j,k}$ equals the cardinality of $SL_2(Z)/\Gamma^0(M)$. □

Let $S_{j,k} = \det(\gamma_{j,k})^{-1}\gamma_{j,k}^tS\gamma_{j,k}$. So, looking at S and $S_{j,k}$ as elements of $GL_2(R)^+$ we have $S_{j,k} = \det(m_{j,k})\ (m_{j,k}^{-1})^tSm_{j,k}^{-1}$.

Lemma 2.8.3. *There exists j, k, $1 \le j \le h(-d)$, $1 \le k \le g(M)/t(d)$ such that $a(S_{j,k}) \ne 0$.*

Proof. By assumption (2.8.1), $a(T) \ne 0$ for some primitive semi-integral positive definite matrix T with discriminant equal to $-d$. By [Fur93, p.209] there exists j such that T is $SL_2(\mathbb{Z})$-equivalent to $S_{j,1}$. This means there is $R \in SL_2(\mathbb{Z})$ such that $T = R^tS_{j,1}R$. By

Lemma 2.8.2, we can find k, l such that $R = \gamma_{j,1}^{-1} r_l \gamma_{j,k} g$ where $g \in \Gamma^0(M)$. This gives us

$$\begin{aligned} T &= g^t \gamma_{j,k}^t r_l^t (\gamma_{j,1}^{-1})^t S_{j,1} \gamma_{j,1}^{-1} r_l \gamma_{j,k} g \\ &= \det(\gamma_{j,k})^{-1} g^t \gamma_{j,k}^t r_l^t S r_l \gamma_{j,k} g \\ &= \det(\gamma_{j,k})^{-1} g^t \gamma_{j,k}^t S \gamma_{j,k} g \\ &= g^t S_{j,k} g \end{aligned}$$

Hence $0 \neq a(T) = a(g^t S_{j,k} g) = a(S_{j,k})$, using the fact that the image of g^t in $Sp_4(\mathbb{Z})$ falls in $B(M)$ and F is a modular form for $B(M)$.

□

Proposition 2.8.4. *There exists a character* Λ *of* $T(\mathbb{A})/(T(\mathbb{Q})T(\mathbb{R})\Pi_{p<\infty, p\nmid M} T(\mathbb{Z}_p) \cdot \Pi_{p|M} \Gamma_{L,p}^0)$ *such that*

$$\sum_{\substack{1 \leq j \leq h(-d) \\ 1 \leq k \leq g(M)/t(d)}} \Lambda(t_j x_k)^{-1} \overline{a(S_{j,k})} \neq 0.$$

Moreover for any such Λ *we have* Λ_p *non-trivial on* $T(\mathbb{Z}_p)$ *for each prime* $p|M$*.*

Proof. By Lemma 2.8.3 we can find $S_{j,k}$ such that $a(S_{j,k}) \neq 0$. Hence using (2.8.6) we know that a character Λ satisfying the condition listed in the proposition exists.

Let Λ be such a character and p_i a fixed prime dividing M. We will show that Λ_{p_i} is not the trivial character on $T(\mathbb{Z}_{p_i})$.

For any $1 \leq j \leq h(-d)$ and $\widetilde{k} \in X$ we can write $t_j u_{\widetilde{k}} = \gamma_{j,\widetilde{k}} m_{j,\widetilde{k}} \kappa_{j,\widetilde{k}}$, where $\gamma_{j,\widetilde{k}} \in GL_2(\mathbb{Q}), m_{j,\widetilde{k}} \in GL_2^+(\mathbb{R})$ and $\kappa_{j,k} \in (\Pi_{p<\infty, p\nmid M} GL_2(\mathbb{Z}_p) \cdot \Pi_{p|M} \Gamma_p^0$.

We put $S_{j,\widetilde{k}} = \det(\gamma_{j,\widetilde{k}})^{-1} \gamma_{j,\widetilde{k}}^t S \gamma_{j,k}$

Suppose Λ_{p_i} is trivial on $T(\mathbb{Z}_{p_i})$. We claim that

$$\sum_{\substack{1 \leq j \leq h(-d) \\ \widetilde{k} \in X}} \Lambda(t_j u_{\widetilde{k}})^{-1} \overline{a(S_{j,\widetilde{k}})} = 0. \tag{2.8.8}$$

Suppose we fix $k_1, k_2, .., k_{i-1}, k_{i+1}, ..k_r$. For $1 \leq y \leq p_i + 1$, let $\widetilde{k}^y \in X$ be the r-tuple obtained by putting $k_i = y$. Then, by essentially the same argument as in Lemma 2.8.2 we see that $\gamma_{j,\widetilde{k}^1}^{-1} \gamma_{j,\widetilde{k}^y}$ form a set of representatives of $\Gamma^0(M/p_i)/\Gamma^0(M)$. In particular, this implies, by [Sch05, 3.3.3], that $\sum_y a(S_{j,\widetilde{k}^y}) = 0$, and therefore, because Λ_{p_i} is trivial on

$T(\mathbb{Z}_{p_i})$, we must have $\sum_y \Lambda(t_j u_{\widetilde{k}^y})^{-1} a(S_{j,\widetilde{k}^y}) = 0$. It follows, by breaking up

$$\sum_{\substack{1\le j\le h(-d)\\ \widetilde{k}\in X}} \Lambda(t_j u_{\widetilde{k}})^{-1}\overline{a(S_{j,\widetilde{k}})}$$

into quantities as above, (2.8.8) follows.

Given $1 \le k \le g(M)/t(d), 1 \le l \le t(d)$, let $\widetilde{k}(k,l)$ be the unique element in X such that

$$u^{-1}_{\widetilde{k}(k,l)} x_k \overline{r}_l \in \Pi_{p|M}\Gamma^0_{L,p}. \tag{2.8.9}$$

Such an element exists by our comment after (2.8.7). Suppose we write $r_l = \overline{r}_l r_{l,f} r_{l,\infty}$ where $r_{l,f} \in \Pi_{p\nmid M} T(\mathbb{Z}_p)$ and $r_{l,\infty} \in T(\mathbb{R})$

Then, using (2.8.9) we have

$$t_j u_{\widetilde{k}(k,l)} = r_l t_j x_k r^{-1}_{l,\infty} k$$

with $k \in (\Pi_{p<\infty, p\nmid M} GL_2(\mathbb{Z}_p) \cdot \Pi_{p|M}\Gamma^0_p$. In other words we can take $\gamma_{j,\widetilde{k}(k,l)} = r_l \gamma_{j,k}$.

But then $a(S_{j,\widetilde{k}(k,l)}) = a(S_{j,k})$. Also from (2.8.9) it is clear that $\Lambda^{-1}(t_j u_{\widetilde{k}(k,l)}) = \Lambda^{-1}(t_j x_k)$. On the other hand if we let k,l vary over all elements in the range $1 \le k \le g(M)/t(d), 1 \le l \le t(d)$, the corresponding $\widetilde{k}(k,l)$ vary over all $\widetilde{k} \in X$. As a result we conclude that

$$\sum_{\substack{1\le j\le h(-d)\\ \widetilde{k}\in X}} \Lambda(t_j u_{\widetilde{k}})^{-1}\overline{a(S_{j,\widetilde{k}})} = t(d) \sum_{\substack{1\le j\le h(-d)\\ 1\le k\le g(M)/t(d)}} \Lambda(t_j x_k)^{-1}\overline{a(S_{j,k})}. \tag{2.8.10}$$

But we have already shown that if Λ_{p_i} is trivial on $T(\mathbb{Z}_{p_i})$ then

$$\sum_{\substack{1\le j\le h(-d)\\ \widetilde{k}\in X}} \Lambda(t_j u_{\widetilde{k}})^{-1}\overline{a(S_{j,\widetilde{k}})} = 0.$$

The proof follows.

□

Consider now the global Bessel space of type (S, Λ, ψ) for $\widetilde{\pi}$. We shall prove that this space is non zero.

For that, we consider

$$B_{\overline{\Phi}}(h) = \int_{Z_G(\mathbb{A})B(\mathbb{Q})\backslash B(\mathbb{A})} (\Lambda \otimes \theta)(r)^{-1}\overline{\Phi}(rh)dr \tag{2.8.11}$$

where θ is defined as in Section 2.1 and $\overline{\Phi}(h) = \overline{\Phi(h)}$. We will show that this function is non-zero. In fact, we shall explicitly evaluate $B_{\overline{\Phi}}(g_\infty)$ for $g_\infty \in G(\mathbb{R})^+$.

Proposition 2.8.5. *Let $g_\infty \in G(\mathbb{R})^+$ and define $B_{\overline{\Phi}}(g_\infty)$ as in* (2.8.11). *The following hold:*

(a) If $M = 1$ we have

$$B_{\overline{\Phi}}(g_\infty) = \overline{\det(J(g_\infty, i))^{-l}}\mu_2(g_\infty)^l e(-tr(S \cdot \overline{g_\infty(i)}) \sum_{1\le j\le h(-d)} \Lambda(t_j)^{-1}\overline{a(S_j)}$$

(b) If $M > 1$ we have

$$B_{\overline{\Phi}}(g_\infty) = \frac{1}{g(M)}\overline{\det(J(g_\infty, i))^{-l}}\mu_2(g_\infty)^l e(-tr(S \cdot \overline{g_\infty(i)}) \sum_{\substack{1\le j\le h(-d)\\ 1\le k\le g(M)/t(d)}} \Lambda(t_j x_k)^{-1}\overline{a(S_{j,k})}$$

Remark. This is a mild generalization of [Sug85, (1-26)]. We present a proof below. But first, we need some preliminary results.

For any f a function on $\mathbb{H}_2$ and $g_\infty \in G(\mathbb{R})^+$ define

$$(f|g_\infty)(Z) = f(g_\infty(Z))\mu_2(g_\infty)^l \det(J(g_\infty, i))^{-l}.$$

Let M_2^{Sym} denote the space of symmetric two by two matrices. We shall think of M_2^{Sym} as a subgroup of G via $x \mapsto u(x)$.

Also, for any continuous function f on $G(\mathbb{Q})\backslash G(\mathbb{A})$ define

$$C_f(g) = \int_{M_2^{Sym}(\mathbb{Q})\backslash M_2^{Sym}(\mathbb{A})} f(u(X)g)\psi(\mathrm{tr}(SX))^{-1}dX.$$

The following lemma is the content of [Sug85, (1-19)]. However it is not proved there, so for convenience we include a proof here.

Lemma 2.8.6. *Let $g_\infty \in G(\mathbb{R})^+, g_f \in GL_2(\mathbb{A}_f)$. We consider g_f as an element of $G(\mathbb{A}_f)$*

via $g \mapsto \begin{pmatrix} g & 0 \\ 0 & det(g)\cdot(g^{-1})^t \end{pmatrix}$. *Then*

$$C_{\overline{\Phi}}(g_\infty g_f) = \overline{\det(J(g_\infty, i))^{-l}}\mu_2(g_\infty)^l a(g_f, S)e(-tr(S\cdot \overline{g_\infty}(i)))$$

where $a(g_f, T)$ *is the* (T)*'th Fourier coefficient of* $\overline{F|g_{\mathbb{R}}}$, *i.e.,*

$$\overline{F|g_{\mathbb{R}}(Z)} = \sum_T a(g_f, T)e(-tr\, T\overline{Z}),$$

and $g_{\mathbb{R}}$ *is defined by the equation* $g_f = g_{\mathbb{Q}}g_{\mathbb{R}}g_K$ *with* $g_{\mathbb{Q}} \in G(\mathbb{Q}), g_{\mathbb{R}} \in G(\mathbb{R})^+, g_K \in \prod_{p<\infty, p\nmid M} K_p \cdot \prod_{p|M} I_p$.

Proof. Put $U_p = \prod_{p<\infty,p\nmid M} K_p \cdot \prod_{p|M} I_p \subset G(\mathbb{A}_f)$. Define $\overline{\Phi}_f$ by $\overline{\Phi}_f(g) = \overline{\Phi}(gg_f)$. Then $\overline{\Phi}_f$ is left invariant by $G(\mathbb{Q})$ and right invariant by $g_{\mathbb{Q}}U_p g_{\mathbb{Q}}^{-1}$. From that it follows that $\overline{\Phi}_f(gg_\infty) = \overline{\Phi}_f(g_\infty)$ if $g \in g_{\mathbb{Q}}(\prod_{q<\infty} U(\mathbb{Z}_q))g_{\mathbb{Q}}^{-1}$. Also note that $C_{\overline{\Phi}}(g_\infty g_f) = C_{\overline{\Phi}_f}(g_\infty)$ and

$$\overline{\Phi}_f(g_\infty) = \mu_2(g_\infty)^l\overline{\det(J(g_\infty, i))^{-l}}(\overline{F|g_{\mathbb{R}})(g_\infty(i)}).$$

Finally, by approximation, we have

$$M_2^{Sym}(\mathbb{A}) = M_2^{Sym}(\mathbb{Q}) + \det(g_{\mathbb{Q}})^{-1}g_{\mathbb{Q}}\left(M_2^{Sym}(\mathbb{R})\prod_{q<\infty} M_2^{Sym}(\mathbb{Z}_q)\right)g_{\mathbb{Q}}^t.$$

Therefore

$$\begin{aligned}
C_{\overline{\Phi}_f}(g_\infty) &= \int_{M_2^{Sym}(\mathbb{Q})\backslash M_2^{Sym}(\mathbb{A})} \overline{\Phi}_f(u(X)g_\infty)\psi(\mathrm{tr}(SX))^{-1}dX \\
&= \int_{\det(g_{\mathbb{Q}})^{-1}g_{\mathbb{Q}}M_2^{Sym}(\mathbb{Z})g_{\mathbb{Q}}^t\backslash M_2^{Sym}(\mathbb{R})} \overline{\Phi}_f(u(X)g_\infty)e(\mathrm{tr}(SX))dX \\
&= \mu_2(g_\infty)^l\overline{\det(J(g_\infty, i))^{-l}}\sum_T a(g_f, T)e(\text{-}\mathrm{tr}(T\cdot\overline{g_\infty(i)})) \\
&\cdot\left(\int_{M_2^{Sym}(\mathbb{Z})\backslash M_2^{Sym}(\mathbb{R})} e(tr(T+S)\cdot X)dX\right) \\
&= \overline{\det(J(g_\infty, i))^{-l}}\mu_2(g_\infty)^l a(g_f, S)e(-\mathrm{tr}(S\cdot\overline{g_\infty(i)}))
\end{aligned}$$

□

Proof of Proposition 2.8.5. The case $M = 1$ is proved in [Sug85]. So we assume $M > 1$. Note that

$$B_{\overline{\Phi}}(g) = \int_{Z(\mathbb{A})T(\mathbb{Q})\backslash T(\mathbb{A})} C_{\overline{\Phi}}(tg)\Lambda^{-1}(t)dt.$$

Hence, using (2.8.6) and the fact that $C_{\overline{\Phi}}$ is right invariant by $\Pi_{p<\infty, p\nmid M} T(\mathbb{Z}_p) \cdot \Pi_{p|M}\Gamma^0_{L,p}$ we have

$$B_{\overline{\Phi}}(g_\infty) = [SL_2(\mathbb{Z}) : \Gamma^0(M)]^{-1} \sum_{j,k} \Lambda^{-1}(t_j x_k) \int_{Z_T(\mathbb{R})\backslash T(\mathbb{R})} C_{\overline{\Phi}}(t_j x_k t_\infty g_\infty) dt_\infty. \qquad (2.8.12)$$

Our Haar measure is normalized so that the compact set $Z_T(\mathbb{R})\backslash T(\mathbb{R})$ has volume 1. We henceforth write R^* instead of $Z_T(\mathbb{R})$ for simplicity. We have,

$$\begin{aligned}
&\int_{R^*\backslash T(\mathbb{R})} C_{\overline{\Phi}}(t_j x_k t_\infty g_\infty) dt_\infty \\
&= \int_{R^*\backslash T(\mathbb{R})} C_{\overline{\Phi}}(t_\infty g_\infty t_j x_k) dt_\infty \\
&= \int_{R^*\backslash T(\mathbb{R})} C_{\overline{\Phi}}(t_\infty g_\infty (\gamma_{j,k})_f) dt_\infty \\
&= \int_{R^*\backslash T(\mathbb{R})} \overline{\det(J(t_\infty g_\infty, i))^{-l}} \mu_2(t_\infty g_\infty)^l a((\gamma_{j,k})_f, S) e(-\mathrm{tr}(S \cdot t_\infty \overline{g_\infty}(i))) dt_\infty \\
&= \overline{\det(J(g_\infty, i))^{-l}} \mu_2(g_\infty)^l a((\gamma_{j,k})_f, S) (\int_{R^*\backslash T(\mathbb{R})} e(-\mathrm{tr}(S \cdot \overline{g_\infty}(i))) dt_\infty) \\
&= \overline{\det(J(g_\infty, i))^{-l}} \mu_2(g_\infty)^l a((\gamma_{j,k})_f, S) e(-\mathrm{tr}(S \cdot \overline{g_\infty(i)}))
\end{aligned} \qquad (2.8.13)$$

Let us compute $a((\gamma_{j,k})_f, S)$. We have

$$\begin{aligned}
\overline{F|m_{j,k}(Z)} &= \sum_{T>0} \overline{a(T)} e(-\mathrm{tr}\ T \cdot (m_{j,k}(\overline{Z}))) \\
&= \sum_{T>0} \overline{a(T)} e(-\mathrm{tr} \det(m_{j,k}^{-1}) \cdot ((m_{j,k})^t T m_{j,k}) \cdot \overline{Z}).
\end{aligned}$$

So, the S'th Fourier coefficient corresponds to $T = \det(m_{j,k})\ (m_{j,k}^{-1})^t S m_{j,k}^{-1} = S_{j,k}$. Thus

$$a((\gamma_{j,k})_f, S) = \overline{a(S_{j,k})}. \qquad (2.8.14)$$

Putting together (2.8.12),(2.8.13) and (2.8.14), we have the proof of the proposition.

□

Description of the Eisenstein series

This section describes the Eisenstein series on $\widetilde{G}(\mathbb{A})$. For each finite place v, recall that $\widetilde{K_v}$ is the maximal compact subgroup of $\widetilde{G}(\mathbb{Q}_v)$ and is defined by

$$\widetilde{K_v} = \widetilde{G}(\mathbb{Q}_v) \cap GL_4(\mathbb{Z}_{L,v}).$$

Let us now define

$$\widetilde{K_\infty} = \{g \in \widetilde{G}(\mathbb{R}) | \mu_2(g) = 1, g < iI_2 >= iI_2\}.$$

Equivalently

$$\widetilde{K_\infty} = U(2,2;\mathbb{R}) \cap U(4,\mathbb{R}).$$

We define

$$\rho_l(k_\infty) = \det(k_\infty)^{l/2} \det(J(k_\infty, i))^{-l}.$$

By [Ich07, p. 5], any matrix k_∞ in $\widetilde{K_\infty}$ can be written in the form $k_\infty = \lambda \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$ where $\lambda \in \mathbb{C}, |\lambda| = 1$, and $A+iB, A-iB \in U(2;\mathbb{R})$ with $\det(A+iB) = \overline{\det(A-iB)}$. Then,

$$\rho_l(k_\infty) = \det(A - iB)^{-l} \tag{2.8.15}$$

Note that if k_∞ has all real entries, i.e. $k_\infty \in \mathrm{Sp}(4, \mathbb{R}) \cap \mathrm{O}(4, \mathbb{R})$, then

$$\rho_l(k_\infty) = \det(J(k_\infty, i))^{-l}.$$

Extend Ψ to $GU(1,1;L)(\mathbb{A})$ by

$$\Psi(ag) = \Psi(g)$$

for $a \in L^\times(\mathbb{A}), g \in GL_2(\mathbb{A})$. We define subsets S_1, S_2, S_3 of the (finite) primes by

- S_1 is the set of primes that divide M but not N

- S_2 is the set of primes that divide $\gcd(M, N)$
- S_3 is the set of primes that divide N but not M

and put $S = S_1 \sqcup S_2 \sqcup S_3$. Let $L = \mathbb{Q}(\sqrt{-d}), \Lambda$ be as in §2.8. Note that all primes in S are odd and inert in L.

Now define the compact open subgroup $U^{\widetilde{G}}$ of $\widetilde{G}(\mathbb{A}_f)$ by

$$U^{\widetilde{G}} = \prod_{p \notin S} K_p^{\widetilde{G}} \prod_{p \in S_3} U_p^{\widetilde{G}} \prod_{p \in S_1 \cup S_2} I'_p \tag{2.8.16}$$

Define

$$f_\Lambda(g, s) = \delta_P^{s+\frac{1}{2}}(m_1 m_2)\Lambda(\overline{m_1})^{-1}\Psi(m_2)\rho_l(k_\infty) \quad \text{if } g = m_1 m_2 n \widetilde{k} k \in \widetilde{G}(\mathbb{A}) \tag{2.8.17}$$

where $m_i \in M^{(i)}(\mathbb{A}) \quad (i = 1, 2)$, $n \in N(\mathbb{A})$, $k = k_\infty k_0$ with $k_\infty \in K_\infty^{\widetilde{G}}$, $k_0 \in U^{\widetilde{G}}$ and $\widetilde{k} = \prod_p k_p \in \prod_p K_p^{\widetilde{G}}$ is such that $k_p = 1$ if $p \notin S_1 \sqcup S_2$, $k_p \in \{1, s_1\}$ for $p \in S_2$ and $k_p \in \{1, \Theta\}$ for $p \in S_1$. Put

$$f_\Lambda(g, s) = 0$$

if g is not of the form above.

We define the Eisenstein series $E_{\Psi,\Lambda}(g, s)$ on $\widetilde{G}(\mathbb{A})$ by

$$E_{\Psi,\Lambda}(g, s) = \sum_{\gamma \in P(\mathbb{Q}) \backslash \widetilde{G}(\mathbb{Q})} f_\Lambda(\gamma g, s). \tag{2.8.18}$$

The global integral

The global integral for our consideration is

$$Z(s) = \int_{Z_G(\mathbb{A})G(\mathbb{Q}) \backslash G(\mathbb{A})} E_{\Psi,\Lambda}(g, s)\overline{\Phi}(g)dg.$$

Then, by (2.2.6), Theorem 2.2.1, Theorem 2.5.1, Theorem 2.6.1 and Theorem 2.7.1 we have

$$Z(s) = \frac{Q_f Z_\infty(s)}{g(M/f)P_{MN}} \cdot \prod_{p|f} \frac{p^{-6s-3}}{1 - a_p w_p p^{-3s-3/2}} \cdot \frac{L(3s + \frac{1}{2}, \pi \times \sigma)}{\zeta_{MN}(6s + 1)L(3s + 1, \sigma \times \rho(\Lambda))} \tag{2.8.19}$$

where f denotes $\gcd(M, N)$ and

$$L(s, \pi \times \sigma) = \prod_{q<\infty} L(s, \pi_q \times \sigma_q)$$

$$L(s, \sigma \times \rho(\Lambda))) = \prod_{q<\infty, q\nmid M} L(s, \sigma_q \times \rho(\Lambda_q)),$$

$$\zeta_A(s) = \prod_{\substack{p\nmid A \\ p \text{ prime}}} (1 - p^{-s})^{-1},$$

$$P_A = \prod_{\substack{r|A \\ r \text{ prime}}} (r^2 + 1),$$

,

$$Q_A = \prod_{\substack{r|A \\ r \text{ prime}}} (1 - r),$$

and

$$Z_\infty(s) = \int_{B(\mathbb{R})\backslash G(\mathbb{R})} W_{f_\Lambda}(\Theta g, s) B_{\overline{\Phi}}(g) dg \tag{2.8.20}$$

As for the explicit computation of Z_∞, Furusawa's calculation in [Fur93], *mutatis mutandis*, works for us. The only real point of difference is the choice of S. Furusawa chooses

$$S = \begin{cases} \begin{pmatrix} \frac{d}{4} & 0 \\ 0 & 1 \end{pmatrix}, & \text{if } d \equiv 0 \pmod 4, \\ \begin{pmatrix} \frac{1+d}{4} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}, & \text{if } d \equiv 3 \pmod 4. \end{cases}$$

He computes $Z_\infty(s)$ for the case $d \equiv 0 \pmod 4$ and uses it to deduce the other case via a simple change of variables, using

$$\begin{pmatrix} \frac{1+d}{4} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{pmatrix}^t \begin{pmatrix} \frac{d}{4} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{pmatrix}.$$

In our case we have,

$$S = \begin{pmatrix} \frac{b^2+d}{4} & \frac{b}{2} \\ \frac{b}{2} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{b}{2} & 1 \end{pmatrix}^t \begin{pmatrix} \frac{d}{4} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{b}{2} & 1 \end{pmatrix}$$

and so a similar change of variables works.

Define $a(\Lambda) = a(F, \Lambda)$ by

$$a(\Lambda) = \begin{cases} \sum_{1 \le j \le h(-d)} \Lambda(t_j) a(S_j) & \text{if } M = 1 \\ \frac{1}{g(M)} \sum_{\substack{1 \le j \le h(-d) \\ 1 \le k \le g(M)/t(d)}} \Lambda(t_j x_k)^{-1} \overline{a(S_{j,k})} & \text{if } M > 1. \end{cases}$$

Then we have (cf. [Fur93, p. 214])

$$Z_\infty(s) = \pi \overline{a(\Lambda)} (4\pi)^{-3s - \frac{3}{2}l + \frac{3}{2}} d^{-3s - \frac{l}{2}} \cdot \frac{\Gamma(3s + \frac{3}{2}l - \frac{3}{2})}{6s + l - 1}.$$

Henceforth we simply write $L(s, F \times g)$ for $L(s, \pi \times \sigma)$. We can summarize our computations in the following theorem.

Theorem 2.8.7 (The integral representation)**.** *Let F and $E_{\Psi,\Lambda}$ be as defined previously. Then*

$$\int_{Z_G(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})} E_{\Psi,\Lambda}(g, s) \overline{\Phi}(g) dg = C(s) \cdot L(3s + \frac{1}{2}, F \times g)$$

where $C(s) =$

$$\frac{A(f) \pi \overline{a(\Lambda)} (4\pi)^{-3s - \frac{3}{2}l + \frac{3}{2}} d^{-3s - \frac{l}{2}} \Gamma(3s + \frac{3}{2}l - \frac{3}{2})}{g(M/f) P_{MN}(6s + l - 1) \zeta_{MN}(6s + 1) L(3s + 1, \sigma \times \rho(\Lambda))} \prod_{p|f} \frac{p^{-6s-3}}{1 - a_p w_p p^{-3s-3/2}}$$

with $f = \gcd(M, N)$.

Remark. Note that

$$C(\frac{l}{6} - \frac{1}{2}) = \frac{\pi^{4-2l} \quad \overline{a(F, \Lambda)}}{\zeta(l-2) L(\frac{l-1}{2}, \sigma \times \rho(\Lambda))} \times \text{ (an algebraic number)}.$$

Chapter 3

The Pullback Formula and the Second Integral Representation

3.1 Eisenstein series on $GU(3,3)$

Let $P_{\widetilde{H}} = M_{\widetilde{H}} N_{\widetilde{H}}$ be the Siegel parabolic of $\widetilde{H}$, with

$$M_{\widetilde{H}}(\mathbb{Q}) := \left\{ m(A,v) = \begin{pmatrix} A & 0 \\ 0 & v \cdot \overline{(A^{-1})}^t \end{pmatrix} | A \in GL_3(L), v \in \mathbb{Q}^\times \right\},$$

$$N_{\widetilde{H}}(\mathbb{Q}) := \left\{ n(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} | b \in M_3(L), \overline{b}^t = b \right\}.$$

For $s \in \mathbb{C}$, we form the induced representation

$$I(\Lambda, s) = \otimes_v I_v(\Lambda_v, s) = \mathrm{Ind}_{P_{\widetilde{H}}(\mathbb{A})}^{\widetilde{H}(\mathbb{A})}(\Lambda \| \cdot \|^{3s})$$

consisting of smooth functions Ξ on $\widetilde{H}(\mathbb{A})$ such that

$$\Xi(nm(A,v)g, s) = |v|^{-9(s+\frac{1}{2})} |N_{L/\mathbb{Q}}(\det A)|^{3(s+\frac{1}{2})} \Lambda(\det A) \Xi(g,s) \tag{3.1.1}$$

for $n \in N_{\widetilde{H}}(\mathbb{A})$, $m(A,v) \in M_{\widetilde{H}}(\mathbb{A})$, $g \in \widetilde{H}(\mathbb{A})$.

Finally, given such a section Ξ, we form the Eisenstein series $E_\Xi(h,s)$ by

$$E_\Xi(h,s) = \sum_{\gamma \in P_{\widetilde{H}}(\mathbb{Q}) \backslash \widetilde{H}(\mathbb{Q})} \Xi(\gamma h, s) \tag{3.1.2}$$

for Re(s) large, and defined elsewhere by meromorphic continuation.

Some compact subgroups

For each finite place p of $\mathbb{Q}$, define the maximal compact subgroups $K_p^{\widetilde{H}}, K_p^{\widetilde{F}}, \widetilde{K_p}$ of (respectively) $\widetilde{H}(\mathbb{Q}_p)$, $\widetilde{F}(\mathbb{Q}_p)$, $\widetilde{G}(\mathbb{Q}_p)$ by

$$K_p^{\widetilde{H}} = \widetilde{H}(\mathbb{Q}_p) \cap GL_6(\mathbb{Z}_{L,p}),$$

$$K_p^{\widetilde{F}} = \widetilde{F}(\mathbb{Q}_p) \cap GL_2(\mathbb{Z}_{L,p}),$$

$$\widetilde{K_p} = \widetilde{G}(\mathbb{Q}_p) \cap GL_4(\mathbb{Z}_{L,p}).$$

Let $U_p^{\widetilde{H}}$ be the subgroup of $K_p^{\widetilde{H}}$ defined by

$$U_p^{\widetilde{H}} = \left\{ z \in K_p^{\widetilde{H}} \mid z \equiv \begin{pmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * & * \end{pmatrix} \pmod p \right\}.$$

Let $r : K_p^{\widetilde{H}} \to \widetilde{H}(\mathbb{F}_p)$ be the canonical map and define the subgroup

$$I_p'^{\widetilde{H}} = r^{-1} I(6, \mathbb{F}_p).$$

Also, put

$$K_\infty^{\widetilde{H}} = \{g \in \widetilde{H}(\mathbb{R}) | \mu_3(g) = 1, g < iI_3 >= iI_3\},$$

$$\widetilde{K_\infty} = \{g \in \widetilde{G}(\mathbb{R}) | \mu_2(g) = 1, g < iI_2 >= iI_2\}$$

and

$$K_\infty^{\widetilde{F}} = \{g \in \widetilde{F}(\mathbb{R}) | \mu_1(g) = 1, g < i >= i\}.$$

By [Ich07, p.5], any matrix k_∞ in $K_\infty^{\widetilde{H}}$ (resp. $K_\infty^{\widetilde{F}}$) can be written in the form $k_\infty = \lambda \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$ where $\lambda \in \mathbb{C}, |\lambda| = 1$, and $A + iB, A - iB$ lie in $U(3;\mathbb{R})$ (resp. $U(1;\mathbb{R})$) with

$\det(A+iB) = \overline{\det(A-iB)}$.

For a positive even integer ℓ, define

$$\rho_\ell(k_\infty) = \det(A-iB)^{-\ell}. \tag{3.1.3}$$

Note that an alternate definition for $\rho_\ell(k_\infty)$ is simply

$$\rho_\ell(k_\infty) = \det(k_\infty)^{\ell/2}\det(J(k_\infty,i))^{-\ell}.$$

Also note that if k_∞ has all real entries, then

$$\rho_\ell(k_\infty) = \det(J(k_\infty,i))^{-\ell}.$$

A particular choice of section

Fix an element $Q \in H_1(\mathbb{Z})$ and an element $\Omega \in \widetilde{H}_1(\mathbb{Z})$. For any place v let Q_v (resp. Ω_v) denote the natural inclusion of Q (resp. Ω) into $\widetilde{H}(\mathbb{Q}_v)$.

We impose the following condition on Ω for all primes $p \in S_2$:

If $nm(A,v) \in P_{\widetilde{H}}(\mathbb{Q}_p) \cap \Omega_p I_p'^{\widetilde{H}} \Omega_p^{-1}$, *then* $\det(A) \in \Gamma^0_{L,p}$.

We next define, for each place v, a particular section $\Upsilon_v(s) \in I_v(\Lambda_v, s)$.

Recall that $I_v(\Lambda_v,s)$ consists of smooth functions Ξ on $\widetilde{H}(\mathbb{Q}_v)$ such that

$$\Xi(nm(A,t)g,s) = |t|_v^{-9(s+\frac{1}{2})}|N_{L/\mathbb{Q}}(\det A)|_v^{3(s+\frac{1}{2})}\Lambda_v(\det A)\Xi(g,s) \tag{3.1.4}$$

for $n \in N_{\widetilde{H}}(\mathbb{Q}_v)$, $m(A,t) \in M_{\widetilde{H}}(\mathbb{Q}_v)$, $g \in \widetilde{H}(\mathbb{Q}_v)$.

- Clearly $I_p(\Lambda_p, s)$ has a $K_p^{\widetilde{H}}$ fixed vector whenever Λ_p is unramified.

  For all finite places $p \notin S$, choose Υ_p to be the unique $K_p^{\widetilde{H}}$ fixed vector with

$$\Upsilon_p(1,s) = 1. \tag{3.1.5}$$

- For all finite places $p \in S_3$, choose Υ_p to be the unique $U_p^{\widetilde{H}}$ fixed vector with

$$\Upsilon_p(Q_p,s) = 1 \tag{3.1.6}$$

and

$$\Upsilon_p(t, s) = 0$$

if $t \notin P_{\widetilde{H}}(\mathbb{Q}_p)Q_pU_p^{\widetilde{H}}$.

- Suppose $p \in S_2$. Choose Υ_p to be the unique $I_p'^{\widetilde{H}}$ fixed vector with

$$\Upsilon_p(Q_p, s) = 1 \tag{3.1.7}$$

and

$$\Upsilon_p(t, s) = 0$$

if $t \notin P_{\widetilde{H}}(\mathbb{Q}_p)Q_pI_p'^{\widetilde{H}}$.

- Let $p \in S_1$.

Choose Υ_p to be the unique $I_p'^{\widetilde{H}}$ fixed vector with

$$\Upsilon_p(\Omega, s) = 1, \qquad \Upsilon_p(Q_p, s) = 1 \tag{3.1.8}$$

and

$$\Upsilon_p(t, s) = 0$$

if $t \notin P_{\widetilde{H}}(\mathbb{Q}_p)\Omega I_p'^{\widetilde{H}} \sqcup P_{\widetilde{H}}(\mathbb{Q}_p)Q_pI_p'^{\widetilde{H}}$. Note that such a vector exists by our assumption on Ω.

- Finally choose Υ_∞ to be the unique vector in $I_\infty(\Lambda_\infty, s)$ such that

$$\Upsilon_\infty(k_\infty, s) = \rho_\ell(k_\infty) \tag{3.1.9}$$

for $k_\infty \in K_\infty^{\widetilde{H}}$.

Let Υ be the factorizable section in $\mathrm{Ind}_{P_{\widetilde{H}}(\mathbb{A})}^{\widetilde{H}(\mathbb{A})}(\Lambda\|\cdot\|^{3s})$. defined by

$$\Upsilon(s) = (\otimes_v \Upsilon_v(s)).$$

As explained in (3.1.2), this gives rise to Eisenstein series $E_\Upsilon(g, s)$.

Also, for each place v, we define the local section $\Upsilon_v^\sharp$ by

$$\Upsilon_v^\sharp(g,s) = \Upsilon_v(gQ_v,s).$$

Define $\Upsilon^\sharp(s) = \otimes_v \Upsilon_v^\sharp(s)$.

Note that $\Upsilon^\sharp$ is right invariant by $\prod_{p\in S_1\sqcup S_2} Q I_p'^{\widetilde{H}} Q^{-1} \prod_{p\in S_3} Q U_p^{\widetilde{H}} Q^{-1} \prod_{\substack{p\notin S\\ p<\infty}} K_p^{\widetilde{H}}$.
So, by (3.1.2), we construct an Eisenstein series $E_{\Upsilon^\sharp}(g,s)$. Note that

$$E_{\Upsilon^\sharp}(g,s) = E_\Upsilon(gQ,s) \tag{3.1.10}$$

3.2 Statement of the pullback formula

We henceforth fix Q to equal the following matrix:

$$Q = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

An important embedding

We now define an embedding $\iota : \widetilde{R} \hookrightarrow \widetilde{H}$. Let $(g_1,g_2) \in \widetilde{R}(\mathbb{A})$, and put $h = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}$.
Then we define $\iota(g_1,g_2)$ to equal the element $p_0^{-1} h p_0 \in \widetilde{H}(\mathbb{A})$ where $p_0 \in GL_6(\mathbb{Q})$ is defined by

$$p_0 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}. \tag{3.2.1}$$

An essential feature of this embedding is the following. Suppose

$$g_1 = m_1(a)m_2(b)n \in P(\mathbb{A}),$$

$$g_2 = b$$

where

$$m_1(a) = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \overline{a}^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in M^{(1)}(\mathbb{A}),$$

$$n \in N(\mathbb{A}),\ b = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \widetilde{F}(\mathbb{A}),$$

and

$$m_2(b) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha & 0 & \beta \\ 0 & 0 & \lambda & 0 \\ 0 & \gamma & 0 & \delta \end{pmatrix} \in M^{(2)}(\mathbb{A}),$$

where $\lambda = \mu_1 \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. Then

$$\iota(g_1, g_2) \in P_{\widetilde{H}}(\mathbb{A}). \tag{3.2.2}$$

It is this key fact that enables us to pass from Klingen Eisenstein series on $\widetilde{G}(\mathbb{A})$ to Siegel Eisenstein series on $\widetilde{H}(\mathbb{A})$.

Henceforth, we fix

$$\Omega = \iota(\Theta, 1)Q.$$

The Pullback formula

For an element $g \in \widetilde{G}(\mathbb{A})$, let $\widetilde{F}_1[g](\mathbb{A})$ denote the subset of $\widetilde{F}(\mathbb{A})$ consisting of all elements h_2 such that $\mu_2(g) = \mu_1(h_2)$.

We will compute the integral

$$\mathcal{E}(g,s) = \int_{\widetilde{F_1}(\mathbb{Q})\backslash \widetilde{F}_1[g](\mathbb{A})} E_{\Upsilon^\sharp}(\iota(g,h),s)\Psi(h)\Lambda^{-1}(\det h)dh. \tag{3.2.3}$$

Define

$$\zeta^S(s) = \prod_{p\notin S}(1-p^{-s})^{-1},$$

$$L^S(s,\chi_{-D}) = \prod_{\substack{p\notin S\\ \gcd(p,D)=1}} (1-(\chi_{-D})_p(p)p^{-s})^{-1}$$

where χ_{-D} denotes the character of $\mathbb{A}^\times$ associated to L. Also define

$$P_S = \prod_{p\in S}(p^2+1).$$

Also, let $\rho(\Lambda)$ denote the representation of $GL_2(\mathbb{A})$ obtained from Λ by automorphic induction. Hence, for a prime $q\notin S$, we have:

$$L(s,\sigma_q\times\rho(\Lambda_q))$$
$$= \begin{cases} (1-\alpha^2(q)q^{-2s})^{-1}(1-\beta^2(q)q^{-2s})^{-1} & \text{if } q \text{ is inert in } L,\\ (1-\alpha(q)\Lambda_q(q_1)q^{-s})^{-1}(1-\beta(q)\Lambda_q(q_1)q^{-s})^{-1} & \text{if } q \text{ is ramified in } L,\\ (1-\alpha(q)\Lambda_q(q_1)q^{-s})^{-1}(1-\beta(q)\Lambda_q(q_1)q^{-s})^{-1} & \\ \cdot(1-\alpha(q)\Lambda_q^{-1}(q_1)q^{-s})^{-1}(1-\beta(q)\Lambda_q^{-1}(q_1)q^{-s})^{-1} & \text{if } q \text{ splits in } L,\end{cases}$$

where $q_1\in\mathbb{Z}_q\otimes_{\mathbb{Q}}L$ is any element with $N_{L/\mathbb{Q}}(q_1)\in q\mathbb{Z}_q^\times$.

Also for a prime $p\in S_3$, put

$$L(s,\sigma_p\times\rho(\Lambda_p)) = (1-p^{-2s-1})^{-1}.$$

Put

$$L(s,\sigma\times\rho(\Lambda)) = \prod_{q\nmid M} L(s,\sigma_q\times\rho(\Lambda_q)).$$

Now define

$$B(s) = \frac{B_\infty(s)L(3s+1,\sigma\times\rho(\Lambda))}{g(M)^2P_{S_3}L^S(6s+2,\chi_{-D})\zeta^S(6s+3)} \tag{3.2.4}$$

where

$$g(M) = \prod_{p|M}(p+1)$$

and

$$B_\infty(s) = \frac{(-1)^{\ell/2}2^{-6s-1}\pi}{6s+\ell-1}.$$

Then the pullback formula says:

Theorem 3.2.1 (Pullback formula)**.** *For $g \in \widetilde{G}(\mathbb{A})$ define $\mathcal{E}(g,s)$ as above and $E_{\Psi,\Lambda}(g,s)$ as in* (2.8.18). *Then we have*

$$\mathcal{E}(g,s) = B(s)E_{\Psi,\Lambda}(g,s).$$

We will prove the Pullback formula in §3.5 using the machinery developed in the next two sections.

3.3 The local integral and the unramified calculation

Definitions

We retain the notations and definitions of the previous section. Furthermore, for any prime p, we define the following compact subgroups of $\widetilde{F}(\mathbb{Q}_p)$:

- $\Gamma_{0,p}^{\widetilde{F}} = \{A \in K_p^{\widetilde{F}} \mid A \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{p}\}$
- Let $r_p : K_p^{\widetilde{F}} \to GU(1,1)(\mathbb{F}_p)$ be the canonical map and let $K_p'^{\widetilde{F}} = r_p^{-1}(GL_2(\mathbb{F}_p))$. Define
$$\Gamma_{0,p}'^{\widetilde{F}} = K_p'^{\widetilde{F}} \cap \Gamma_{0,p}^{\widetilde{F}}.$$

Some useful properties

First, we note some properties of the section $\Upsilon^\sharp$. Fix $(g_1, g_2) \in \widetilde{R}(\mathbb{A})$.

- Let p be a prime not dividing MN and $k_1 \in \widetilde{K_p}$, $k_2 \in K_p^{\widetilde{F}}$ with $\mu_2(k_1) = \mu_1(k_2)$. Then, note that
$$Q^{-1}\iota(k_1,k_2)Q \in K_p^{\widetilde{H}}.$$

Because $\Upsilon_p^\sharp$ is $K_p^{\widetilde{H}}$-fixed, it follows that

$$\Upsilon^\sharp(\iota(g_1k_1, g_2k_2), s) = \Upsilon^\sharp(\iota(g_1, g_2), s), \tag{3.3.1}$$

- Let $p|N, p \nmid M$. If $k_1 \in \widetilde{U_p}$, $k_2 \in \Gamma_{0,p}^{\widetilde{F}}$ with $\mu_2(k_1) = \mu_1(k_2)$ then check that

$$Q^{-1}\iota(k_1, k_2)Q \in U_p^{\widetilde{H}}. \tag{3.3.2}$$

Because $\Upsilon_p^\sharp$ is $U_p^{\widetilde{H}}$-fixed, it follows that

$$\Upsilon^\sharp(\iota(g_1k_1, g_2k_2), s) = \Upsilon^\sharp(\iota(g_1, g_2), s), \tag{3.3.3}$$

- Let p be a prime dividing M. If $k_1 \in I'_p$, $k_2 \in \Gamma'^{\widetilde{F}}_{0,p}$ with $\mu_2(k_1) = \mu_1(k_2)$ then check that

$$Q^{-1}\iota(k_1, k_2)Q \in I'^{\widetilde{H}}_p. \tag{3.3.4}$$

Because $\Upsilon_p^\sharp$ is $I'^{\widetilde{H}}_p$-fixed, it follows that

$$\Upsilon^\sharp(\iota(g_1k_1, g_2k_2), s) = \Upsilon^\sharp(\iota(g_1, g_2), s), \tag{3.3.5}$$

- Finally, let $k_1 \in \widetilde{K_\infty}$, $k_2 \in K_\infty^{\widetilde{F}}$ with $\mu_2(k_1) = \mu_1(k_2)$. Check that

$$Q^{-1}\iota(k_1, k_2)Q \in K_\infty^{\widetilde{H}}. \tag{3.3.6}$$

Hence we have

$$\Upsilon^\sharp(g_1k_1, g_2k_2, s) = \rho_\ell(k_1)\rho_\ell(k_2)^{-1}\Upsilon^\sharp(g_1, g_2, s). \tag{3.3.7}$$

The key local zeta integral

Let $\psi = \prod_v \psi_v$ be a character of $\mathbb{A}$ such that

- The conductor of ψ_p is $\mathbb{Z}_p$ for all (finite) primes p,
- $\psi_\infty(x) = e(x)$, for $x \in \mathbb{R}$,
- $\psi|_\mathbb{Q} = 1$.

Let W_Ψ be the Whittaker model for Ψ. It is a function on $\widetilde{F}(\mathbb{A})$ defined by

$$W_\Psi(g) = \int_{\mathbb{Q}\backslash\mathbb{A}} \Psi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) \psi(-x)dx.$$

We have the Fourier expansion

$$\Psi(g) = \sum_{\lambda\in\mathbb{Q}^\times} W_\Psi\left(\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} g\right) \tag{3.3.8}$$

By the uniqueness of Whittaker models, we have a factorization

$$W_\Psi = \otimes_v W_{\Psi,v}.$$

Now, for each place v, and elements $g_v \in \widetilde{F}(\mathbb{Q}_v), k_v \in \widetilde{K_v}$, define the local zeta integral

$$Z_v(g_v, k_v, s) = \int_{\widetilde{F}_1(\mathbb{Q}_v)} \Upsilon_v^\sharp(\iota(k_v, h_v), s) W_{\Psi,v}(g_v h_v)\Lambda_v^{-1}(\det h_v)dh_v, \tag{3.3.9}$$

The evaluation of this local integral at each place v lies at the heart of our proof of the pullback formula.

First of all, by (3.2.2) and the properties proved earlier, observe that it is enough to evaluate the integral for k_v lying in a fixed set of representatives of $(P(\mathbb{Q}_v) \cap \widetilde{K_v})\backslash\widetilde{K_v}/U_v$, where

$$U_v = \begin{cases} \widetilde{K_v} & \text{if } v \notin S \\ \widetilde{U_v} & \text{if } v \in S_3 \\ I'_v & \text{if } v \in S_1 \sqcup S_2 \end{cases}$$

For $1 \leq i \leq 5$, define the matrices $s_i \in G(\mathbb{Q})$ as follows:

$$s_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad s_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$s_4 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix}, \quad s_5 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Define the set $Y_\infty = \{1\}$ and for a (finite) prime p, define the set $Y_p \subset \widetilde{G}(\mathbb{Q}_p)$ as follows:

- $Y_p = \{1\}$ if $p \nmid MN$.
- $Y_p = \{1, s_1, s_2\}$ if $p|N, p \nmid M$.
- $Y_p = \{1, s_1, s_2, s_3, \Theta, \Theta s_2, \Theta s_4, \Theta s_5\}$ if $p|M$.

Remark. In the above definition, we consider the s_i and Θ as elements of $\widetilde{G}(\mathbb{Q}_p)$. This makes Y_v a subset of $\widetilde{G}(\mathbb{Z}_v)$ for all places v.

Lemma 3.3.1. *Y_v is a set of representatives for $(P(\mathbb{Q}_v) \cap \widetilde{K_v}) \backslash \widetilde{K_v} / U_v$ at all places v.*

Proof. For v infinite or v a prime not dividing MN, this is obvious. Now let p be a prime dividing N but not M. If W denotes the eight element Weyl group, then W is generated by s_1 and s_2. W is a set of representatives for $(P(\mathbb{Q}_p) \cap \widetilde{K_p}) \backslash \widetilde{K_p} / I_p^{\widetilde{G}}$ where $I_p^{\widetilde{G}}$ denotes the Iwahori subgroup of $\widetilde{G}(\mathbb{Q}_p)$. Since $\widetilde{U_p}$ is larger than $I_p^{\widetilde{G}}$, there is some collapsing, as expected. By explicit computation we find that $\{1, s_1, s_2\}$ do form a set of distinct representatives. The case when $p|M$ is also proved similarly by explicit computation. For brevity, we do not include the details here. $\square$

The rest of this section and the next will be devoted to evaluating at each place v the integral $Z_v(g_v, k_v, s)$ for every $k_v \in Y_v$, $g_v \in \widetilde{F}(\mathbb{Q}_v)$.

The local integral at unramified places

In the rest of this section, q will denote a prime that does not divide MN. Hence, both Λ_q and σ_q are unramified.

In particular, σ_q is a spherical principal series representation induced from unramified characters α, β of $\mathbb{Q}_q^\times$.

By abuse of notation we use q to also denote its inclusion in $\mathbb{Q}_q^\times$. Thus q is an uniformizer in our local field.

Let $\rho(\Lambda)$ denote the representation of $GL_2(\mathbb{A})$ obtained from Λ by automorphic induction. Hence we have:

$$L(s, \sigma_q \times \rho(\Lambda_q))$$
$$= \begin{cases} (1-\alpha^2(q)q^{-2s})^{-1}(1-\beta^2(q)q^{-2s})^{-1} & \text{if } q \text{ is inert in } L, \\ \\ (1-\alpha(q)\Lambda_q(q_1)q^{-s})^{-1}(1-\beta(q)\Lambda_q(q_1)q^{-s})^{-1} & \text{if } q \text{ is ramified in } L, \\ \\ (1-\alpha(q)\Lambda_q(q_1)q^{-s})^{-1}(1-\beta(q)\Lambda_q(q_1)q^{-s})^{-1} & \\ \cdot(1-\alpha(q)\Lambda_q^{-1}(q_1)q^{-s})^{-1}(1-\beta(q)\Lambda_q^{-1}(q_1)q^{-s})^{-1} & \text{if } q \text{ splits in } L, \end{cases}$$

where $q_1 \in \mathbb{Z}_q \otimes_{\mathbb{Q}} L$ is any element with $N_{L/\mathbb{Q}}(q_1) \in q\mathbb{Z}_q^\times$.

For a character χ of $\mathbb{Q}_q^\times$ define $L(s,\chi) = \begin{cases} (1-\chi(q)q^{-s}) & \text{if } \chi \text{ is unramified at } q, \\ 1 & \text{otherwise.} \end{cases}$

Proposition 3.3.2. *Let q be a prime such that $q \nmid MN$. Let $\mathbf{1}$ denote the trivial character and χ_{-D} denote the Hecke character associated to the quadratic extension $L/\mathbb{Q}$. Then, we have*

$$Z_q(g_q, 1, s) = W_{\Psi,q}(g_q) \cdot \frac{L(3s+1, \sigma_q \times \rho(\Lambda_q))}{L(6s+2, (\chi_{-D})_q)L(6s+3, \mathbf{1})}.$$

Proof. Let $K_q^{\widetilde{F}_1}$ denote the maximal compact subgroup of $\widetilde{F}_1(\mathbb{Q}_q)$ defined by

$$K_q^{\widetilde{F}_1} = \widetilde{F}_1(\mathbb{Q}_q) \cap GL_2(\mathbb{Z}_{L,q}).$$

Note that for $g \in \widetilde{F}_1(\mathbb{Q}_q), k_1, k_2 \in K_q^{\widetilde{F}_1}$, we have using (3.1.1), (3.3.1)

$$\begin{aligned} \Upsilon_q^\sharp(\iota(1, k_1 g k_2), s) &= \Upsilon_q^\sharp(\iota(m_2(k_1)m_2(k_1)^{-1}, k_1 g k_2), s) \\ &= \Upsilon_q^\sharp(\iota(m_2(k_1)^{-1}, g k_2), s) \\ &= \Upsilon_q^\sharp(\iota(1, g), s) \end{aligned}$$

In other words $\Upsilon_q^\sharp(\iota(1,g),s)$ only depends on the double coset $K_q^{\widetilde{F}_1} g K_q^{\widetilde{F}_1}$.

There are three distinct cases: q can be inert, split or ramified in L. We consider each of these cases separately.

Case 1. *q is inert in L.*

In this case, L_q is a quadratic extension of $\mathbb{Q}_q$. We may write elements of L_q in the form $a+b\sqrt{-d}$ with $a,b\in\mathbb{Q}_q$; then $\mathbb{Z}_{L,q}=a+b\sqrt{-d}$ where $a,b\in\mathbb{Z}_q$. Also note that Λ_q is trivial.

We know (Cartan decomposition) that

$$\widetilde{F}_1(\mathbb{Q}_q)=\bigsqcup_{n\geq 0}K_q^{\widetilde{F}_1}A_nK_q^{\widetilde{F}_1}$$

where $A_n=\begin{pmatrix} q^n & 0\\ 0 & q^{-n}\end{pmatrix}$. So (3.3.9) gives us

$$Z_q(g_q,1,s)=\sum_{n\geq 0}\Upsilon_q^{\sharp}(\iota(1,A_n),s)\int_{K_q^{\widetilde{F}_1}A_nK_q^{\widetilde{F}_1}}W_{\Psi,q}(g_qh_q)dh_q. \tag{3.3.10}$$

Given an element $k\in K_q^{\widetilde{F}_1}$ we can find $l\in\mathbb{Z}_{L,q}^{\times}$ such that $kl\in GL_2(\mathbb{Z}_q)$. It follows that if

$$GL_2(\mathbb{Z}_q)A_nGL_2(\mathbb{Z}_q)=\bigsqcup_i a_iGL_2(\mathbb{Z}_q),$$

where $a_i\in SL_2(\mathbb{Z}_q)$ then

$$K_q^{\widetilde{F}_1}A_nK_q^{\widetilde{F}_1}=\bigsqcup_i a_iK_q^{\widetilde{F}_1}.$$

The importance of this observation is that we can use the theory of Hecke operators for GL_2 to evaluate $\int_{K_q^{\widetilde{F}_1}A_nK_q^{\widetilde{F}_1}}W_{\Psi,q}(g_qh_q)dh_q$.

Recall that classically $T(q^k)$ denotes the Hecke operator corresponding to the set

$GL_2(\mathbb{Z}_q)S_kGL_2(\mathbb{Z}_q)$ where S_k comprises of the matrices of size 2 with entries in $\mathbb{Z}_q$ whose determinant generates the ideal (q^k). Also observe that

$$\begin{aligned}GL_2(\mathbb{Z}_q)S_{2n}GL_2(\mathbb{Z}_q)=&\begin{pmatrix} q^n & 0\\ 0 & q^n\end{pmatrix}GL_2(\mathbb{Z}_q)A_nGL_2(\mathbb{Z}_q)\\ &\bigsqcup\begin{pmatrix} q & 0\\ 0 & q\end{pmatrix}GL_2(\mathbb{Z}_q)S_{2n-2}GL_2(\mathbb{Z}_q).\end{aligned}$$

So we have

$$\int_{K_q^{\widetilde{F}_1} A_n K_q^{\widetilde{F}_1}} W_{\Psi,q}(g_q h_q) dh_q = \sum_i W_{\Psi,q}(g_q a_i) \tag{3.3.11}$$

$$= (\beta_{2n} - \beta_{2n-2}) W_{\Psi,q}(g_q) \tag{3.3.12}$$

where β_k is the eigenvalue corresponding to Ψ for the Hecke operator $T(q^k)$. We put $\beta_k = 0$ if $k < 0$.

Using [Bum97, Propostion 4.6.4] we have

$$\beta_k = \frac{q^{k/2}(\alpha(q)^{k+1} - \beta(q)^{k+1})}{\alpha(q) - \beta(q)} \tag{3.3.13}$$

for $k \geq 0$.

On the other hand, using (3.2.1) we see that $\iota(1, A_n)$ is the matrix

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & q^{-n} & 0 & 0 & 0 \\ 0 & 0 & q^{-n} - 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 - q^n & 0 & 0 & 0 & 0 & q^n \end{pmatrix}$$

We can write $C = PK$ where

$$P = \begin{pmatrix} q^n & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & q^{-n} & 0 & 0 \\ 0 & 0 & 0 & q^{-n} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in P_{\widetilde{H}}(\mathbb{Q}_q)$$

and

$$K = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1-q^n & q^n & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1-q^n & 0 & 0 & 0 & 0 & q^n \end{pmatrix}. \tag{3.3.14}$$

So, by (3.1.1) we have

$$\Upsilon_q^\sharp(\iota(1, A_n), s) = q^{-6n(s+1/2)} \Upsilon_q^\sharp(K, s) \tag{3.3.15}$$

Also $K \in K_q^{\widetilde{H}}$, hence $\Upsilon_q^\sharp(K, s) = 1$.

So, by (3.3.10),(3.3.12),(3.3.13),(3.3.15), we have

$$\begin{aligned} Z_q(g_q, 1, s) &= W_{\Psi,q}(g_q) \Bigg[\sum_{n \geq 0} q^{-6n(s+1/2)} \frac{q^n(\alpha(q)^{2n+1} - \beta(q)^{2n+1})}{\alpha(q) - \beta(q)} \\ &\qquad - \sum_{n \geq 1} q^{-6n(s+1/2)} \frac{q^{n-1}(\alpha(q)^{2n-1} - \beta(q)^{2n-1})}{\alpha(q) - \beta(q)} \Bigg] \\ &= W_{\Psi,q}(g_q) \frac{(1 - q^{-6s-3})(1 + q^{-6s-2})}{(1 - \alpha(q)^2 q^{-6s-2})(1 - \beta(q)^2 q^{-6s-2})} \\ &= W_{\Psi,q}(g_q) \cdot \frac{L(3s+1, \sigma_q \times \rho(\Lambda_q))}{L(6s+2, \chi_{-D}) L(6s+3, \mathbf{1})} \end{aligned}$$

Case 2. *q is split in L.*

We can identify L_q with $\mathbb{Q}_q \oplus \mathbb{Q}_q$ with $\mathbb{Q}_q$ embedded diagonally as $t \mapsto (t, t)$.

For $g \in GL_n(\mathbb{Q}_q)$ denote $g^* = J_n^{-1}(g^t)^{-1} J_n$. Note that for $n = 2$, $g^* = \frac{g}{\det g}$. Now there is a natural isomorphism of $GL_n(\mathbb{Q}_q)$ into $U(n,n)(\mathbb{Q}_q)$ given by $g \mapsto (g, g^*)$. Thus specializing to the $n = 2$ case, $g \mapsto (g, \frac{g}{\det g})$ takes $GL_2(\mathbb{Q}_q)$ isomorphically onto $\widetilde{F}_1(\mathbb{Q}_q)$.

Define $A_{m,k}$ to be the image of $\begin{pmatrix} q^{m+k} & 0 \\ 0 & q^m \end{pmatrix}$.

Thus $A_{m,k} = \begin{pmatrix} (q^{m+k}, q^{-m}) & 0 \\ 0 & (q^m, q^{-m-k}) \end{pmatrix}$.

The Cartan decomposition gives us

$$\widetilde{F}_1(\mathbb{Q}_q) = \bigsqcup_{\substack{k\geq 0\\ m\in\mathbb{Z}}} K_q^{\widetilde{F}_1} A_{m,k} K_q^{\widetilde{F}_1}.$$

Let q_1 denote the element $(q,1) \in L_q$. So $N_{L/\mathbb{Q}}(q_1) = q$. For brevity, let us denote $\Lambda_q(q_1)$ by λ. Note that for any integer m,

$$\Lambda_q(q^m, q^{-m}) = \lambda^{2m}.$$

Now, using (3.3.9), we have

$$Z_q(g_q, 1, s) = \sum_{\substack{k\geq 0\\ m\in\mathbb{Z}}} \Upsilon_q^{\sharp}(\iota(1, A_{m,k}), s)\lambda^{-4m-2k} \int_{K_q^{\widetilde{F}_1} A_{m,k} K_q^{\widetilde{F}_1}} W_{\Psi,q}(g_q h_q) dh_q. \tag{3.3.16}$$

Using the above conventions, and the notation of the inert case, we have

$$GL_2(\mathbb{Z}_q) S_k GL_2(\mathbb{Z}_q) = \begin{pmatrix} q^{-m} & 0 \\ 0 & q^{-m} \end{pmatrix} GL_2(\mathbb{Z}_q) A_{m,k} GL_2(\mathbb{Z}_q)$$
$$\bigsqcup \begin{pmatrix} q & 0 \\ 0 & q \end{pmatrix} GL_2(\mathbb{Z}_q) S_{k-2} GL_2(\mathbb{Z}_q).$$

So, we have

$$\int_{K_q^{\widetilde{F}_1} A_{m,k} K_q^{\widetilde{F}_1}} W_{\Psi,q}(g_q h_q) dh_q = (\beta_k - \beta_{k-2}) W_{\Psi,q}(g_q) \tag{3.3.17}$$

where we put $\beta_k = 0$ if $k < 0$. Now $\iota(1, A_{m,k})$ is the matrix C where

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & q^m & 0 & 0 & 0 \\ 0 & 0 & q^m - 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 - q^{m+k} & 0 & 0 & 0 & 0 & q^{m+k} \end{pmatrix}.$$

[Note that by C we actually mean the pair (C, C^*). This convention will be used throughout our treatment of the split case; thus the letters P, K etc. are really a shorthand for $(P, P^*), (K, K^*)$, etc.]

First we consider the case $m \geq 0$. We can write $C = PK$

where

$$P = \begin{pmatrix} q^{m+k} & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & q^m & -q^m & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$K = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & q^m & 1 & 0 & 0 \\ 0 & 0 & q^m - 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 - q^{m+k} & 0 & 0 & 0 & 0 & q^{m+k} \end{pmatrix}.$$

Since $P \in P_{\widetilde{H}}(\mathbb{Q}_q)$ we have, using (3.1.1)

$$\Upsilon_q^\sharp(\iota(1, A_{m,k}), s) = \lambda^{2m+k} q^{-3(2m+k)(s+1/2)} \Upsilon_q^\sharp(K, s) \tag{3.3.18}$$

Since $K \in K_q^{\widetilde{H}}$, $\Upsilon_q^\sharp(K, s) = 1$.

Thus when $m \geq 0$ we have

$$\Upsilon_q^\sharp(\iota(1, A_{m,k}), s) = \lambda^{2m+k} q^{-(6m+3k)(s+1/2)}. \tag{3.3.19}$$

Now suppose $0 \geq m \geq -k$. For convenience we temporarily put $n = -m$. So $0 \leq n \leq k$. We can write $C = PK$

where

$$P = \begin{pmatrix} q^{k-n} & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & q^{-n} & 0 & 0 \\ 0 & 0 & 0 & q^{-n} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$K = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1-q^{n} & q^{n} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1-q^{k-n} & 0 & 0 & 0 & 0 & q^{k-n} \end{pmatrix}.$$

Since $P \in P_{\widetilde{H}}(\mathbb{Q}_q)$ we have, using (3.1.1)

$$\begin{aligned} \Upsilon_q^{\sharp}(\iota(1, A_{m,k}), s) &= \lambda^{-2n+k} q^{-3k(s+1/2)} \Upsilon_q^{\sharp}(K, s) \\ &= \lambda^{2m+k} q^{-3k(s+1/2)} \Upsilon_q^{\sharp}(K, s) \end{aligned} \tag{3.3.20}$$

As before $\Upsilon_q^{\sharp}(K, s) = 1$.

So, when $-k \le m \le 0$ we have

$$\Upsilon_q^{\sharp}(\iota(1, A_{m,k}), s) = \lambda^{2m+k} q^{-3k(s+1/2)}. \tag{3.3.21}$$

Finally, consider the case $m \le -k$. For convenience we again put $n = -m$. So $0 \le k \le n$. We can write $C = PK$

where

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & q^{-n} & 0 & 0 \\ 0 & 0 & 0 & q^{-n} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & q^{k-n} \end{pmatrix}$$

and

$$K = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1-q^n & q^n & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ q^{n-k}-1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Since $P \in P_{\widetilde{H}}(\mathbb{Q}_q)$ we have, using (3.1.1)

$$\begin{aligned} \Upsilon_q^\sharp(\iota(1, A_{m,k}), s) &= \lambda^{-2n+k} q^{-3(2n-k)(s+1/2)} \Upsilon_q^\sharp(K, s) \\ &= \lambda^{2m+k} q^{-3(-2m-k)(s+1/2)} \Upsilon_q^\sharp(K, s) \end{aligned} \tag{3.3.22}$$

So, when $m \leq -k$ we have

$$\Upsilon_q^\sharp(\iota(1, A_{m,k}), s) = \lambda^{2m+k} q^{(6m+3k)(s+1/2)}. \tag{3.3.23}$$

Substituting (3.3.13),(3.3.17),(3.3.19),(3.3.21),(3.3.23) into (3.3.16) we obtain

$$\begin{aligned} &Z_q(g_q, 1, s) \\ &= W_{\Psi,q}(g_q) \sum_{k=0}^{\infty} (\beta_k - \beta_{k-2}) \Big[\sum_{m=1}^{\infty} \lambda^{-2m-k} q^{(-6m-3k)(s+1/2)} \\ &\qquad + \sum_{m=-k}^{0} \lambda^{-2m-k} q^{-3k(s+1/2)} + \sum_{m=-\infty}^{-k-1} \lambda^{-2m-k} q^{(6m+3k)(s+1/2)} \Big] \\ &= \frac{W_{\Psi,q}(g_q)(1-q^{-6s-3})(1-q^{-6s-2})}{(1-\alpha(q)\lambda q^{-3s-1})(1-\beta(q)\lambda q^{-3s-1})(1-\alpha(q)\lambda^{-1} q^{-3s-1})(1-\beta(q)\lambda^{-1} q^{-3s-1})} \\ &= W_{\Psi,q}(g_q) \cdot \frac{L(3s+1, \sigma_q \times \rho(\Lambda_q))}{L(6s+2, \chi_{-D}) L(6s+3, \mathbf{1})} \end{aligned}$$

Case 3. *q is ramified in L.*

We largely revert to the notation of the inert case. Write elements of L_q as $a + b\sqrt{-d}$ with $a, b \in \mathbb{Q}_q$; so $\mathbb{Z}_{L,q} = a + b\sqrt{-d}$ with $a, b \in \mathbb{Z}_q$. Also let $q_1 = \sqrt{-d}$; thus $N_{L/\mathbb{Q}}(q_1) \in q\mathbb{Z}_q^\times$. Put $\lambda = \Lambda_q(q_1)$. We have $\lambda^2 = 1$.

The Cartan decomposition takes the form

$$\widetilde{F}_1(\mathbb{Q}_q) = \bigsqcup_{n\geq 0} K_q^{\widetilde{F}_1} A_n K_q^{\widetilde{F}_1}$$

where $A_n = \begin{pmatrix} q_1^n & 0 \\ 0 & q_1^{-n} \end{pmatrix}$. So (3.3.9) gives us

$$Z_q(g_q, 1, s) = \sum_{n\geq 0} \Upsilon_q^\sharp(\iota(1, A_n), s) \int_{K_q^{\widetilde{F}_1} A_n K_q^{\widetilde{F}_1}} W_{\Psi,q}(g_q h_q) dh_q. \tag{3.3.24}$$

Now,

$$K_q^{\widetilde{F}_1} A_n K_q^{\widetilde{F}_1} = \begin{pmatrix} q_1^{-n} & 0 \\ 0 & q_1^{-n} \end{pmatrix} K_q^{\widetilde{F}_1} \begin{pmatrix} q^n & 0 \\ 0 & 1 \end{pmatrix} K_q^{\widetilde{F}_1}$$

.

So, by the same argument as in the inert case, we have,

$$\int_{K_q^{\widetilde{F}_1} A_n K_q^{\widetilde{F}_1}} W_{\Psi,q}(g_q h_q) dh_q = (\beta_n - \beta_{n-2}) W_{\Psi,q}(g_q) \tag{3.3.25}$$

where, of course, we put $\beta_n = 0$ for negative n.

Now $\iota(1, A_n)$ is the same matrix as in the inert case with q replaced by q_1. So the same choice of P and K work.

Thus, by (3.1.1) we have

$$\Upsilon_q^\sharp(\iota(1, A_n), s) = \lambda^n q^{-3n(s+1/2)} \tag{3.3.26}$$

Substituting (3.3.13),(3.3.25), (3.3.26) in (3.3.24) we have

$$\begin{aligned} Z_q(g_q, 1, s) &= W_{\Psi,q}(g_q) \Bigg[\sum_{n\geq 0} \lambda^n q^{-3n(s+1/2)} \frac{q^{n/2}(\alpha(q)^{n+1} - \beta(q)^{n+1})}{\alpha(q) - \beta(q)} \\ &\qquad - \sum_{n\geq 2} \lambda^n q^{-3n(s+1/2)} \frac{q^{n/2-1}(\alpha(q)^{n-1} - \beta(q)^{n-1})}{\alpha(q) - \beta(q)} \Bigg] \\ &= W_{\Psi,q}(g_q) \frac{(1 - q^{-6s-3})}{(1 - \alpha(q)\lambda q^{-3s-1})(1 - \beta(q)^\lambda q^{-3s-1})} \\ &= W_{\Psi,q}(g_q) \cdot \frac{L(3s+1, \sigma_q \times \rho(\Lambda_q))}{L(6s+2, \chi_{-D}) L(6s+3, \mathbf{1})} \end{aligned}$$

(Note that $L(s,\chi_{-D}) = 0$ in this case)

This completes the proof. $\square$

3.4 The local integral for the ramified and infinite places

The local integral for primes in S_3

Let r be a prime dividing N but not M. Note that r is inert by our assumptions. In this section we will prove the following proposition.

Proposition 3.4.1. *We have*

$$Z_r(g_r,k_r,s) = \begin{cases} \frac{1}{r^2+1} W_{\Psi,r}(g_r)\cdot L\big(3s+1,\sigma_r\times\rho(\Lambda_r)\big) & \text{if } k_r = 1 \\ 0 & \text{if } k_r = s_1 \text{ or } s_2. \end{cases}$$

where the local L-function $L(s,\sigma_r\times\rho(\Lambda_r))$ is defined by

$$L(s,\sigma_r\times\rho(\Lambda_r)) = (1-r^{-2s-1})^{-1}.$$

Proof. Recall that σ is the irreducible automorphic representation of $GL_2(\mathbb{A})$ generated by $\widetilde{\Psi}$. Let σ_r be the local component of σ at the place r. We know that $\sigma_r = \mathrm{Sp}\otimes\tau$ where Sp denotes the special (Steinberg) representation and τ is a (possibly trivial) unramified quadratic character. We put $a_r = \tau(r)$, thus $a_r = \pm 1$ is the eigenvalue of the local Hecke operator $T(r)$.

We first deal with the case $k_r = 1$. Let $\Gamma_{0,r}^{\widetilde{F}_1}$ denote the compact open subgroup of $\widetilde{F}_1(\mathbb{Q}_r)$ defined by

$$\Gamma_{0,r}^{\widetilde{F}_1} = \Gamma_{0,r}^{\widetilde{F}} \cap \widetilde{F}_1(\mathbb{Q}_r).$$

Note that for $g\in\widetilde{F}_1(\mathbb{Q}_r), k_1,k_2\in\Gamma_{0,r}^{\widetilde{F}_1}$, we have using (3.1.1), (3.3.3)

$$\begin{aligned} \Upsilon_r^\sharp(\iota(1,k_1gk_2),s) &= \Upsilon_r^\sharp(\iota(m_2(k_1)m_2(k_1)^{-1},k_1gk_2),s) \\ &= \Upsilon_r^\sharp(\iota(m_2(k_1)^{-1},gk_2),s) \\ &= \Upsilon_r^\sharp(\iota(1,g),s) \end{aligned} \tag{3.4.1}$$

In other words $\Upsilon_r^\sharp(\iota(1,g),s)$ only depends on the double coset $\Gamma_{0,r}^{\widetilde{F}_1} g \Gamma_{0,r}^{\widetilde{F}_1}$.

Because r is inert in L, L_r is a quadratic extension of $\mathbb{Q}_r$. We may write elements of L_r

in the form $a+b\sqrt{-d}$ with $a,b\in\mathbb{Q}_r$; then $\mathbb{Z}_{L,r}=a+b\sqrt{-d}$ where $a,b\in\mathbb{Z}_r$. Also note that Λ_r is trivial.

We know (Bruhat-Cartan decomposition) that

$$\begin{aligned}\widetilde{F}_1(\mathbb{Q}_r) = \Gamma_{0,r}^{\widetilde{F}_1} \quad &\cup \quad \Gamma_{0,r}^{\widetilde{F}_1} w \Gamma_{0,r}^{\widetilde{F}_1} \\ \cup \quad \bigsqcup_{n>0} \Gamma_{0,r}^{\widetilde{F}_1} A_n \Gamma_{0,r}^{\widetilde{F}_1} \quad &\cup \quad \bigsqcup_{n>0} \Gamma_{0,r}^{\widetilde{F}_1} A_n w \Gamma_{0,r}^{\widetilde{F}_1} \\ \cup \quad \bigsqcup_{n>0} \Gamma_{0,r}^{\widetilde{F}_1} w A_n \Gamma_{0,r}^{\widetilde{F}_1} \quad &\cup \quad \bigsqcup_{n>0} \Gamma_{0,r}^{\widetilde{F}_1} w A_n w \Gamma_{0,r}^{\widetilde{F}_1}.\end{aligned} \tag{3.4.2}$$

where $A_n = \begin{pmatrix} r^n & 0 \\ 0 & r^{-n} \end{pmatrix}$ and $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. So (3.3.9) gives us

$$\begin{aligned}Z_r(g_r,1,s) = {} & \Upsilon_r^\sharp(\iota(1,1),s) \int_{\Gamma_{0,r}^{\widetilde{F}_1}} W_{\Psi,r}(g_r h_r)\,dh_r \\ & + \Upsilon_r^\sharp(\iota(1,w),s) \int_{\Gamma_{0,r}^{\widetilde{F}_1} w \Gamma_{0,r}^{\widetilde{F}_1}} W_{\Psi,r}(g_r h_r)\,dh_r \\ & + \sum_{n>0} \Upsilon_r^\sharp(\iota(1,A_n),s) \int_{\Gamma_{0,r}^{\widetilde{F}_1} A_n \Gamma_{0,r}^{\widetilde{F}_1}} W_{\Psi,r}(g_r h_r)\,dh_r \\ & + \sum_{n>0} \Upsilon_r^\sharp(\iota(1,A_n w),s) \int_{\Gamma_{0,r}^{\widetilde{F}_1} A_n w \Gamma_{0,r}^{\widetilde{F}_1}} W_{\Psi,r}(g_r h_r)\,dh_r \\ & + \sum_{n>0} \Upsilon_r^\sharp(\iota(1,w A_n),s) \int_{\Gamma_{0,r}^{\widetilde{F}_1} w A_n \Gamma_{0,r}^{\widetilde{F}_1}} W_{\Psi,r}(g_r h_r)\,dh_r \\ & + \sum_{n>0} \Upsilon_r^\sharp(\iota(1,w A_n w),s) \int_{\Gamma_{0,r}^{\widetilde{F}_1} w A_n w \Gamma_{0,r}^{\widetilde{F}_1}} W_{\Psi,r}(g_r h_r)\,dh_r.\end{aligned} \tag{3.4.3}$$

Now $W_{\Psi,r}$ is an eigenvector for the Iwahori-Hecke algebra, hence each of the integrals in (3.4.3) evaluates to a constant multiple of $W_{\Psi,r}(g_r)$. Thus for some function $A(s)$ (not depending on g_r) we have

$$Z_r(g_r,1,s) = A(s) W_{\Psi,r}(g_r).$$

Since $W_{\Psi,r}(1)=1$ it follows that

$$Z_r(g_r,1,s) = Z_r(1,1,s) W_{\Psi,r}(g_r) \tag{3.4.4}$$

Given an element $k \in \Gamma_{0,r}^{\widetilde{F}_1}$ we can find $l \in \mathbb{Z}_{L,q}^{\times}$ such that $kl \in \Gamma_{0,r}$. It follows that if

$$\Gamma_{0,r} A_n \Gamma_{0,r} = \bigsqcup_i a_i \Gamma_{0,r},$$

where $a_i \in SL_2(\mathbb{Z}_q)$ then

$$\Gamma_{0,r}^{\widetilde{F}_1} A_n \Gamma_{0,r}^{\widetilde{F}_1} = \bigsqcup_i a_i \Gamma_{0,r}^{\widetilde{F}_1}.$$

From [Miy89, Lemma 4.5.6], we may choose $a_i = \begin{pmatrix} r^n & mr^{-n} \\ 0 & r^{-n} \end{pmatrix}$ where $0 \leq m < r^{2n}$. Using the formula in [GK92, Lemma 2.1], we have $W_{\Psi,r}(a_i) = r^{-2n}$ and hence

$$\sum_{a \in \Gamma_{0,r}^{\widetilde{F}_1} A_n \Gamma_{0,r}^{\widetilde{F}_1} / \Gamma_{0,r}^{\widetilde{F}_1}} W_{\Psi,r}(a) = 1 \tag{3.4.5}$$

Also, from [Miy89] we have

$$\Gamma_{0,r}^{\widetilde{F}_1} w A_n w \Gamma_{0,r}^{\widetilde{F}_1} = \bigsqcup_i b_i \Gamma_{0,r}^{\widetilde{F}_1}.$$

where $b_i = \begin{pmatrix} r^{-n} & 0 \\ -mr^{1-n} & r^n \end{pmatrix}$. Using the formula in [GK92, Lemma 2.1], and doing some simple manipulations, we have

$$\sum_{b \in \Gamma_{0,r}^{\widetilde{F}_1} w A_n w \Gamma_{0,r}^{\widetilde{F}_1} / \Gamma_{0,r}^{\widetilde{F}_1}} W_{\Psi,r}(b) = 1 \tag{3.4.6}$$

Next, we check that the quantities $\Upsilon_r^{\sharp}(\iota(1, A_n w), s)$, $\Upsilon_r^{\sharp}(\iota(1, wA_n), s)$, are both equal to 0. Indeed $\Upsilon_r^{\sharp}(\iota(1, A), s) = 0$ whenever $\iota(1, A)$ as an element of $\widetilde{H}(\mathbb{Q}_r)$ does not belong to $P_{\widetilde{H}}(\mathbb{Q}_r) Q U_r^{\widetilde{H}} Q^{-1}$. Let K be the matrix defined in (3.3.14) with q replaced by r. It suffices to prove that the quantities $K\iota(m(w), 1)Q, K\iota(1, w) \cdot Q$ do not belong to $(P_{\widetilde{H}}(\mathbb{Q}_r) \cap K_r^{\widetilde{H}}) Q U_r^{\widetilde{H}}$. We check this by taking a generic element P of $(P_{\widetilde{H}}(\mathbb{Q}_r) \cap K_r^{\widetilde{H}})$ and showing that $Q^{-1} P K_0 \notin U_r^{\widetilde{H}}$ where K_0 is one of the above quantities. That is a simple computation and is omitted.

On the other hand, putting

$$P = \begin{pmatrix} r^n & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & r^{-n} & 0 & 0 \\ 0 & 0 & 0 & r^{-n} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in P_{\widetilde{H}}(\mathbb{Q}_r)$$

we can check that

$$Q^{-1}P^{-1} \cdot (1, A_n)Q \in U_r^{\widetilde{H}},$$

hence

$$\Upsilon_r^\sharp(\iota(1, A_n), s) = r^{-6n(s+1/2)} \tag{3.4.7}$$

Also, putting

$$P = \begin{pmatrix} 0 & 0 & r^n & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & r^{-n} \\ 0 & 0 & 0 & 0 & 0 & r^{-n} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \in P_{\widetilde{H}}(\mathbb{Q}_r)$$

we can check that

$$Q^{-1}P^{-1} \cdot \iota(w, A_n w)Q \in U_r^{\widetilde{H}},$$

hence

$$\Upsilon_r^\sharp(\iota(1, wA_n w), s) = \Upsilon_r^\sharp(\iota(w, A_n w), s) = r^{-6n(s+1/2)} \tag{3.4.8}$$

So, using (3.4.5), (3.4.6), (3.4.7) and (3.4.14),

$$\begin{aligned} Z_r(1,1,s) &= \Upsilon_r^\sharp(\iota(1,1),s)\int_{\Gamma_{0,r}^{\widetilde{F}_1}} dh_r + \sum_{n>0}\Upsilon_r^\sharp(\iota(1,A_n),s)\Big[\int_{\Gamma_{0,r}^{\widetilde{F}_1}A_n\Gamma_{0,r}^{\widetilde{F}_1}} W_{\Psi,r}(h_r)dh_r \\ &\quad + \int_{\Gamma_{0,r}^{\widetilde{F}_1}A_n\Gamma_{0,r}^{\widetilde{F}_1}} W_{\Psi,r}(h_r)dh_r\Big] \\ &= [K_r^{\widetilde{F}} : \Gamma_{0,r}^{\widetilde{F}_1}]^{-1}\big(1+2\sum_{n>0}\Upsilon_r^\sharp(\iota(1,A_n),s)\big) \\ &= \frac{1}{r+1}(1+2\sum_{n>0} r^{-6n(s+1/2)}) \\ &= \frac{1}{r+1}\frac{1+r^{-6s-3}}{1-r^{-6s-3}} \end{aligned}$$

whence (3.4.4) implies

$$Z_r(g_r,1,s) = \frac{1}{r+1}W_{\Psi,r}(g_r)\cdot\frac{1+r^{-6s-3}}{1-r^{-6s-3}}. \tag{3.4.9}$$

Finally, we deal with the case when $k_r = s_1$ or s_2. The key observation is that if $k \in K_r^{\widetilde{F}_1}$ then for $i = 1, 2$,

$$s_i^{-1}m_2(k)s_i \in \widetilde{U_r}.$$

By the same argument as in (3.4.1), it follows that $\Upsilon_r^\sharp(s_i, g, s)$ only depends on the double coset $K_r^{\widetilde{F}_1} g \Gamma_{0,r}^{\widetilde{F}_1}$. So, if we can show that for all $h \in \widetilde{F}_1(\mathbb{Q}_r)$ we have $\sum_{a\in K_r^{\widetilde{F}_1}h\Gamma_{0,r}^{\widetilde{F}_1}/\Gamma_{0,r}^{\widetilde{F}_1}} W_{\Psi,r}(g_r a) = 0$, it would follow that $Z_r(g_r, s_i, s) = 0$.

If we define

$$W(g_r) = \sum_{a\in K_r^{\widetilde{F}_1}h\Gamma_{0,r}^{\widetilde{F}_1}/\Gamma_{0,r}^{\widetilde{F}_1}} W_{\Psi,r}(g_r a)$$

then $W(g_r k) = W(g_r)$ for all $k \in K_r^{\widetilde{F}_1}$; in other words W is a vector in the Whittaker space that is right $K_r^{\widetilde{F}_1}$ invariant. But the only such vector is the 0 vector and this completes the proof.

□

The local integral for primes in S_2

Proposition 3.4.2. *Let p be a prime dividing $\gcd(M,N)$ and $k_p \in Y_p$. We have*

$$Z_p(g_p, k_p, s) = \begin{cases} \frac{W_{\Psi,p}(g_p)}{(p+1)^2} & \text{if } k_p = 1 \text{ or } k_p = s_1 \\ 0 & \text{otherwise .} \end{cases}$$

Proof. Recall that σ is the irreducible automorphic representation of $GL_2(\mathbb{A})$ generated by Ψ. Let σ_p be the local component of σ at the place p. We know that $\sigma_p = \mathrm{Sp} \otimes \tau$ where Sp denotes the special (Steinberg) representation and τ is a (possibly trivial) unramified quadratic character. We put $a_p = \tau(p)$, thus $a_p = \pm 1$ is the eigenvalue of the local Hecke operator $T(p)$.

Let $\Gamma'^{\widetilde{F}_1}_{0,p}$ denote the compact open subgroup of $\widetilde{F}_1(\mathbb{Q}_p)$ defined by

$$\Gamma'^{\widetilde{F}_1}_{0,p} = \Gamma'^{\widetilde{F}}_{0,p} \cap \widetilde{F}_1(\mathbb{Q}_p).$$

We first consider the case $k_p = 1$. Note that for $g \in \widetilde{F}_1(\mathbb{Q}_p), k_1 \in \Gamma'^{\widetilde{F}_1}_{0,p}, k_2 \in \Gamma'^{\widetilde{F}_1}_{0,p}$, we have using (3.1.1), (3.3.5),

$$\begin{aligned} \Upsilon^{\sharp}_p(\iota(1, k_1 g k_2), s) &= \Upsilon^{\sharp}_p(\iota(m_2(k_1)m_2(k_1)^{-1}, k_1 g k_2), s) \\ &= \Upsilon^{\sharp}_p(\iota(m_2(k_1)^{-1}, g k_2), s) \\ &= \Upsilon^{\sharp}_p(\iota(1, g), s) \end{aligned} \tag{3.4.10}$$

In other words $\Upsilon^{\sharp}_p(\iota(1, g), s)$ only depends on the double coset $\Gamma'^{\widetilde{F}_1}_{0,p} g \Gamma'^{\widetilde{F}_1}_{0,p}$.

Because p is inert in L, L_p is a quadratic extension of $\mathbb{Q}_p$. We may write elements of L_p in the form $a + b\sqrt{-d}$ with $a, b \in \mathbb{Q}_p$; then $\mathbb{Z}_{L,p} = a + b\sqrt{-d}$ where $a, b \in \mathbb{Z}_p$. Also note that Λ_p is *not* trivial.

Fix a set U of representatives of $\mathbb{Z}^{\times}_{L,p}/\Gamma^0_{L,p}$. For definiteness we may take

$$U = \{1\} \cup \{b + \sqrt{-d} : b \in \mathbb{Z}, 0 \le b < p\}$$

For $l \in L^{\times}_p$ put $\widetilde{l} = \begin{pmatrix} l & 0 \\ 0 & \bar{l}^{-1} \end{pmatrix}$. We know that given $g \in \Gamma^{\widetilde{F}_1}_{0,p}$ there exists $l \in Z^{\times}_{L,p}$ such that $g\widetilde{l} \in \Gamma'^{\widetilde{F}_1}_{0,p}$. From this fact and the Bruhat-Cartan decomposition (3.4.2), it follows that

$$\begin{aligned}
\widetilde{F}_1(\mathbb{Q}_p) = & \bigsqcup_{l\in U} \Gamma'^{\widetilde{F}_1}_{0,p}\widetilde{l}\Gamma'^{\widetilde{F}_1}_{0,p} \quad \cup \quad \bigsqcup_{l\in U} \Gamma'^{\widetilde{F}_1}_{0,p} w\widetilde{l}\Gamma'^{\widetilde{F}_1}_{0,p} \\
& \cup \quad \bigsqcup_{\substack{n>0\\ l\in U}} \Gamma'^{\widetilde{F}_1}_{0,p} A_n\widetilde{l}\Gamma'^{\widetilde{F}_1}_{0,p} \quad \cup \quad \bigsqcup_{\substack{n>0\\ l\in U}} \Gamma'^{\widetilde{F}_1}_{0,p} A_n w\widetilde{l}\Gamma'^{\widetilde{F}_1}_{0,p} \\
& \cup \quad \bigsqcup_{\substack{n>0\\ l\in U}} \Gamma'^{\widetilde{F}_1}_{0,p} wA_n\widetilde{l}\Gamma'^{\widetilde{F}_1}_{0,p} \quad \cup \quad \bigsqcup_{\substack{n>0\\ l\in U}} \Gamma'^{\widetilde{F}_1}_{0,p} wA_n w\widetilde{l}\Gamma'^{\widetilde{F}_1}_{0,p}.
\end{aligned} \tag{3.4.11}$$

where as before $A_n = \begin{pmatrix} r^n & 0 \\ 0 & r^{-n} \end{pmatrix}$ and $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Now, in the proof of Proposition 3.4.1 we saw that the elements $\iota(1, A_n w), \iota(1, wA_n)$ of $\widetilde{H}(\mathbb{Q}_p)$ do not belong to $P_{\widetilde{H}}(\mathbb{Q}_p)QU_p^{\widetilde{H}}Q^{-1}$. In particular therefore, the elements $\iota(1, A_n w\widetilde{l})$, $\iota(1, wA_n\widetilde{l})$ of $\widetilde{H}(\mathbb{Q}_p)$ cannot belong to $P_{\widetilde{H}}(\mathbb{Q}_p)QI'^{\widetilde{H}}_pQ^{-1}$.

So (3.3.9) gives us

$$\begin{aligned}
Z_p(g_p, 1, s) = & \sum_{l\in U} \Lambda_p^{-2}(l)\Upsilon_p^\sharp(\iota(1,\widetilde{l}), s) \int_{\widetilde{l}\Gamma'^{\widetilde{F}_1}_{0,p}} W_{\Psi,p}(g_p h_p) dh_p \\
& + \sum_{n>0}\sum_{l\in U} \Lambda_p^{-2}(l)\Upsilon_p^\sharp(\iota(1,A_n\widetilde{l}), s) \int_{\Gamma'^{\widetilde{F}_1}_{0,p} A_n\widetilde{l}\Gamma'^{\widetilde{F}_1}_{0,p}} W_{\Psi,p}(g_p h_p) dh_p \\
& + \sum_{n>0}\sum_{l\in U} \Lambda_p^{-2}(l)\Upsilon_p^\sharp(\iota(1,wA_n w\widetilde{l}), s) \int_{\Gamma'^{\widetilde{F}_1}_{0,p} wA_n w\widetilde{l}\Gamma'^{\widetilde{F}_1}_{0,p}} W_{\Psi,p}(g_p h_p) dh_p.
\end{aligned} \tag{3.4.12}$$

If we choose a_i, b_i as in the proof of Proposition 3.4.1 then we have

$$\Gamma'^{\widetilde{F}_1}_{0,p} A_n\widetilde{l}\Gamma'^{\widetilde{F}_1}_{0,p} = \bigsqcup_i a_i \Gamma'^{\widetilde{F}_1}_{0,p},$$

$$\Gamma'^{\widetilde{F}_1}_{0,p} wA_n w\widetilde{l}\Gamma'^{\widetilde{F}_1}_{0,p} = \bigsqcup_i b_i \Gamma'^{\widetilde{F}_1}_{0,p}.$$

Hence, by the same argument as in the proof of that proposition, we have

$$\int_{\Gamma'^{\widetilde{F}_1}_{0,p} A_n\widetilde{l}\Gamma'^{\widetilde{F}_1}_{0,p}} W_{\Psi,p}(g_p h_p) dh_p = \int_{\Gamma'^{\widetilde{F}_1}_{0,p} wA_n w\widetilde{l}\Gamma'^{\widetilde{F}_1}_{0,p}} W_{\Psi,p}(g_p h_p) dh_p = [K_p^{\widetilde{F}} : \Gamma'^{\widetilde{F}_1}_{0,p}]^{-1}.$$

It is easy to check that the last quantity is equal to $\frac{1}{(p+1)^2}$.

So we have

$$Z_p(g_p,1,s)=\frac{W_{\Psi,p}(g_p)}{(p+1)^2}\Bigg(\sum_{l\in U}\Lambda_p^{-2}(l)\Upsilon_p^\sharp(\iota(1,\widetilde{l}),s) + \sum_{n>0}\sum_{l\in U}\Lambda_p^{-2}(l)\Upsilon_p^\sharp(\iota(1,A_n\widetilde{l}),s)+\sum_{n>0}\sum_{l\in U}\Lambda_p^{-2}(l)\Upsilon_p^\sharp(\iota(1,wA_nw\widetilde{l}),s)\Bigg). \tag{3.4.13}$$

We can check that for $n>0$, $\iota(1,A_n\widetilde{l})$ does not belong to $P_{\widetilde{H}}(\mathbb{Q}_p)QI_p'^{\widetilde{H}}Q^{-1}$, hence $\Upsilon_p^\sharp(\iota(1,A_n\widetilde{l}),s)=0$. We can also check that for $l\neq 1, l\in U$, $(1,\widetilde{l})$ does not belong to $P_{\widetilde{H}}(\mathbb{Q}_p)QI_p'^{\widetilde{H}}Q^{-1}$, hence $\Upsilon_p^\sharp(\iota(1,\widetilde{l}),s)=0$.

Also, putting

$$P=\begin{pmatrix} 0 & 0 & p^n\bar{l}^{-1} & 1 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0 & p^{-n}l\\ 0 & 0 & 0 & 0 & 0 & p^{-n}l\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\end{pmatrix}\in P_{\widetilde{H}}(\mathbb{Q}_p)$$

we can check that

$$Q^{-1}P^{-1}\iota(w,A_nw\widetilde{l})Q\in I_p'^{\widetilde{H}},$$

hence

$$\Upsilon_p^\sharp(\iota(1,wA_nw\widetilde{l}),s)=\Lambda_p(l)p^{-6n(s+1/2)} \tag{3.4.14}$$

Thus we have $\Lambda_p^{-2}(l)\Upsilon_p^\sharp(\iota(1,wA_nw\widetilde{l}),s)=\Lambda_p^{-1}(l)p^{-6n(s+1/2)}$ and hence for all $n>0$ we have

$$\sum_{l\in U}\Lambda_p^{-2}(l)\Upsilon_p^\sharp(\iota(1,wA_nw\widetilde{l}),s)=0.$$

So we conclude that

$$Z_p(g_p,1,s)=\frac{W_{\Psi,p}(g_p)}{(p+1)^2}.$$

Next, we deal with the case $k_p=s_1$.

If $k\in\Gamma_{0,p}'^{\widetilde{F}_1}$ then $s_1^{-1}m_2(k)s_1\in I_p'$. So, by the same argument as before, we know that $\Upsilon_p^\sharp(\iota(s_1,g),s)$ depends only on the double coset $\Gamma_{0,p}'^{\widetilde{F}_1}g\Gamma_{0,p}'^{\widetilde{F}_1}$.

Also, by explicit computation, we check that $\iota(s_1,A_nw\widetilde{l}),\iota(s_1,wA_n\widetilde{l})$ do not belong to $P_{\widetilde{H}}(\mathbb{Q}_p)QI_p'^{\widetilde{H}}Q^{-1}$ for any $n\geq 0$. Moreover, the quantity $\iota(s_1,A_n\widetilde{l})$ belongs to $P_{\widetilde{H}}(\mathbb{Q}_p)QI_p'^{\widetilde{H}}Q^{-1}$

if and only if $n = 0, l = 1$. On the other hand, for $n > 0$, the quantity $\iota(s_1, wA_nw\widetilde{l})$ does belong to $P_{\widetilde{H}}(\mathbb{Q}_p)QI_p'^{\widetilde{H}}Q^{-1}$. These last two facts are reflected in the following equations.

$$\iota(s_1, 1) = PQIQ^{-1}, \tag{3.4.15}$$

where

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in P_{\widetilde{H}}(\mathbb{Q}_p),$$

and

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in I_p'^{\widetilde{H}}.$$

$$\iota(s_1, w^{-1}A_nw\widetilde{l}) = PQIQ^{-1} \tag{3.4.16}$$

where

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -p^n\bar{l}^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -p^n l \end{pmatrix} \in P_{\widetilde{H}}(\mathbb{Q}_p),$$

and

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -p^{-n}l^{-1} & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -p^n\bar{l}^{-1} \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \in I_p'^{\widetilde{H}}.$$

So, we have

$$Z_p(g_p, s_1, s) = \frac{W_{\Psi,p}(g_p)}{(p+1)^2}\Big(1 + \sum_{n>0}\sum_{l\in U} \Lambda_p^{-2}(l)\Upsilon_p^\sharp(\iota(s_1, wA_nw\widetilde{l}), s)\Big). \tag{3.4.17}$$

But from (3.4.16) we see that $\Upsilon_p^\sharp(\iota(s_1, wA_nw\widetilde{l}), s) = \Lambda_p(l)p^{-6n(s+1/2)}$ and hence

$$\sum_{l\in U} \Lambda_p^{-2}(l)\Upsilon_p^\sharp(\iota(s_1, wA_nw\widetilde{l}), s) = 0.$$

This completes the proof that

$$Z_p(g_p, s_1, s) = \frac{W_{\Psi,p}(g_p)}{(p+1)^2}.$$

Next, we consider $k_p = s_2$. Let $\Gamma_p'^{0,\widetilde{F}_1} = J_1\Gamma_{0,p}'^{\widetilde{F}_1}J_1$ where $J_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

If $k \in \Gamma_p'^{0,\widetilde{F}_1}$ then $s_2^{-1}m_2(k)s_2 \in I_p'$. So, by the same argument as before, we know that $\Upsilon_p^\sharp(\iota(s_2, g), s)$ depends only on the double coset $\Gamma_p'^{0,\widetilde{F}_1}g\Gamma_{0,p}'^{\widetilde{F}_1}$.

Now, the Bruhat-Cartan decomposition (3.4.11) continues to hold when we replace the left $\Gamma_{0,p}'^{\widetilde{F}_1}$ in each term by $\Gamma_p'^{0,\widetilde{F}_1}$. So, to prove that $Z_p(g_p, s_2, s) = 0$ it is enough to prove that each of the elements $\iota(s_2, A_n\widetilde{l}), \iota(s_2, A_nw\widetilde{l}), \iota(s_2, wA_n\widetilde{l}), \iota(s_2, wA_nw\widetilde{l})$ cannot belong to $P_{\widetilde{H}}(\mathbb{Q}_p)QI_p'^{\widetilde{H}}Q^{-1}$ for any $n \geq 0$. This we do by an explicit computation. The details are omitted.

Next, take $k_p = s_3$. Once again, we check that if $k \in \Gamma_p'^{0,\widetilde{F}_1}$ then $s_3^{-1}m_2(k)s_3 \in I_p'$. On the other hand, an explicit computation again shows that the elements $\iota(s_3, A_n\widetilde{l}), \iota(s_3, A_nw\widetilde{l})$, $\iota(s_3, wA_n\widetilde{l})$, $\iota(s_3, wA_nw\widetilde{l})$ cannot belong to $P_{\widetilde{H}}(\mathbb{Q}_p)QI_p'^{\widetilde{H}}Q^{-1}$. So by exactly the same argument as the previous case, $Z_p(g_p, s_3, s) = 0$.

Next consider the case $k_p = \Theta$. Define

$$\Gamma_{1,p}^{\prime \widetilde{F}_1} = \{A \in \Gamma_{0,p}^{\prime \widetilde{F}_1} \mid A \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{p}\}.$$

We can check that if $k \in \Gamma_{1,p}^{\prime \widetilde{F}_1}$ then $\Theta^{-1} m_2(k) \Theta \in I'_p$. We know that given $g \in \Gamma_{0,p}^{\widetilde{F}_1}$, there exists $l \in \mathbb{Z}_{L,p}^{\times}$ such that $g\widetilde{l} \in \Gamma_{1,p}^{\prime \widetilde{F}_1}$. Thus, the Bruhat-Cartan decomposition (3.4.11) continues to hold when we replace the left $\Gamma_{0,p}^{\prime \widetilde{F}_1}$ in each term by $\Gamma_{1,p}^{\prime \widetilde{F}_1}$. An explicit computation again shows that the elements $\iota(\Theta, A_n\widetilde{l}), \iota(\Theta, A_n w\widetilde{l}), \iota(\Theta, wA_n\widetilde{l})$ never belong to $P_{\widetilde{H}}(\mathbb{Q}_p) Q I_p^{\prime \widetilde{H}} Q^{-1}$. On the other hand, if $n > 0$, then $\iota(\Theta, wA_n w\widetilde{l})$ does belong to $P_{\widetilde{H}}(\mathbb{Q}_p) Q I_p^{\prime \widetilde{H}} Q^{-1}$. Indeed,

$$\iota(\Theta, w^{-1} A_n w\widetilde{l}) = PQIQ^{-1} \tag{3.4.18}$$

where

$$P = \begin{pmatrix} 1 & \alpha & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -p^n\bar{l}^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\overline{\alpha} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -p^{-n}l \end{pmatrix} \in P_{\widetilde{H}}(\mathbb{Q}_p),$$

and

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -\alpha p^n l^{-1} & -1 - p^n l^{-1} & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -p^n \bar{l}^{-1}\overline{\alpha} \\ 0 & 0 & 0 & 0 & 1 & -1 + p^n \bar{l}^{-1} \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \in I_p^{\prime \widetilde{H}}.$$

So, we have

$$Z_p(g_p, \Theta, s) = \frac{W_{\Psi,p}(g_p)}{(p+1)^2} \Big(\sum_{n>0} \sum_{l \in U} \Lambda_p^{-2}(l) \Upsilon_p^{\sharp}(\iota(\Theta, wA_n w\widetilde{l}), s) \Big). \tag{3.4.19}$$

But from (3.4.18) we see that $\Upsilon_p^\sharp(\iota(\Theta, wA_nw\widetilde{l}), s) = \Lambda_p(l)p^{-6n(s+1/2)}$ and hence

$$\sum_{l \in U} \Lambda_p^{-2}(l)\Upsilon_p^\sharp(\iota(\Theta, wA_nw\widetilde{l}), s) = 0.$$

This completes the proof that

$$Z_p(g_p, \Theta, s) = 0.$$

Next consider the case $k_p = \Theta s_2$. Define the subgroup

$$\Gamma_p'^{\widetilde{F}_1} = \{A \in \Gamma_{0,p}'^{\widetilde{F}_1} \mid A \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{p}\}.$$

We can check that if $k \in \Gamma_p'^{\widetilde{F}_1}$ then $(\Theta s_2)^{-1} m_2(k) \Theta s_2 \in I_p'$. We know that given $g \in \Gamma_{1,p}'^{\widetilde{F}_1}$, there exists $x \in \mathbb{Z}/p\mathbb{Z}$ such that $gu(x) \in \Gamma_p'^{\widetilde{F}_1}$, where $u(x)$ is the matrix $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$. So, to prove that $Z_p(g_p, \Theta s_2, s) = 0$, it is enough to check that the elements $\iota(\Theta s_2, u(x)A_n\widetilde{l}), \iota(\Theta s_2, u(x)A_nw\widetilde{l}), \iota(\Theta s$ do not belong to $P_{\widetilde{H}}(\mathbb{Q}_p)QI_p'^{\widetilde{H}}Q^{-1}$. This can be done by an explicit computation (omitted for brevity).

Next, consider the case $k_p = \Theta s_4$. As before, we can check that if $k \in \Gamma_p'^{\widetilde{F}_1}$ then $(\Theta s_4)^{-1} m_2(k) \Theta s_4 \in I_p'$. To prove that $Z_p(g_p, \Theta s_4, s) = 0$, it is enough to check that the elements $\iota(\Theta s_4, u(x)A_n\widetilde{l}), \iota(\Theta s_4, u(x)A_nw\widetilde{l})$, $\iota(\Theta s_4, u(x)wA_n\widetilde{l})$, $\iota(\Theta s_4, u(x)wA_nw\widetilde{l})$ do not belong to $P_{\widetilde{H}}(\mathbb{Q}_p)QI_p'^{\widetilde{H}}Q^{-1}$. This is done by an explicit computation, which we omit.

Finally, we consider the case $k_p = \Theta s_5$. We can check that if $k \in \Gamma_{1,p}'^{\widetilde{F}_1}$ then $(\Theta s_5)^{-1} m_2(k) \Theta s_5 \in I_p'$. To prove that $Z_p(g_p, \Theta s_5, s) = 0$, it is enough to check that the elements $\iota(\Theta s_5, A_n\widetilde{l})$, $\iota(\Theta s_5, A_nw\widetilde{l})$, $\iota(\Theta s_5, wA_n\widetilde{l})$, $\iota(\Theta s_5, wA_nw\widetilde{l})$ do not belong to $P_{\widetilde{H}}(\mathbb{Q}_p)QI_p'^{\widetilde{H}}Q^{-1}$. This is done by an explicit computation, which we omit.

This completes the proof of the theorem.

□

The local integral for primes in S_1

In this subsection, we prove the following proposition.

Proposition 3.4.3. *Let p be a prime dividing M but not N and $k_p \in Y_p$. We have*

$$Z_p(g_p,k_p,s)=\begin{cases}\frac{W_{\Psi,p}(g_p)}{(p+1)^2} & \textit{if } k_p=1 \textit{ or } k_p=\Theta\\ 0 & \textit{otherwise .}\end{cases}$$

Proof. Recall that σ is the irreducible automorphic representation of $GL_2(\mathbb{A})$ generated by Ψ. Let σ_p be the local component of σ at the place p. We also let α,β be the unramified characters of $\mathbb{Q}_p^\times$ from which σ_p is induced.

Let $\Gamma'^{\widetilde{F}_1}_{0,p},\Gamma'^{\widetilde{F}_1}_{1,p}$ be as defined in the previous subsection.

We first consider the case $k_p=1$. As in the previous case, $\Upsilon^\sharp_p(\iota(1,g),s)$ only depends on the double coset $\Gamma'^{\widetilde{F}_1}_{0,p}g\Gamma'^{\widetilde{F}_1}_{0,p}$.

By explicit computation we check that, $\iota(1,A_n\widetilde{l})$, $\iota(1,A_nw\widetilde{l})$, $\iota(1,wA_n\widetilde{l})$, $\iota(1,wA_nw\widetilde{l})$ do not belong to $P_{\widetilde{H}}(\mathbb{Q}_p)\Omega I'^{\widetilde{H}}_pQ^{-1}$. So, by the results of the previous subsection, and by (3.4.11), we have $Z_p(g_p,1,s)=0$.

Next, consider the case $k_p=s_1$. Again, by explicit computation, we check that for $n>0$, $\iota(1,A_n\widetilde{l})$, $\iota(1,A_nw\widetilde{l})$, $\iota(1,wA_n\widetilde{l})$, $\iota(1,wA_nw\widetilde{l})$ do not belong to $P_{\widetilde{H}}(\mathbb{Q}_p)\Omega I'^{\widetilde{H}}_pQ^{-1}$. Furthermore $\iota(1,w\widetilde{l})$ does not belong to $P_{\widetilde{H}}(\mathbb{Q}_p)\Omega I'^{\widetilde{H}}_pQ^{-1}$ and $\iota(1,\widetilde{l})$ belongs only when $l\neq 1$. This last fact is reflected by the following equation: Let l satisfy the equation $l^2+b_1l+c_1=0$ and let $\alpha=(l-r)/2$ for some $r\in\mathbb{Z}$. Then we have

$$\iota(1,\widetilde{l})=P\Omega IQ^{-1} \tag{3.4.20}$$

where

$$P=\begin{pmatrix}0&-\frac{c}{2}&0&0&0&0\\1&\frac{c}{2l}&0&0&0&0\\0&0&\frac{l}{c}&0&0&0\\0&0&0&-\frac{b}{c}-\frac{1}{l}&-\frac{2}{c}&0\\0&0&0&1&0&0\\0&0&0&0&0&l\end{pmatrix}\in P_{\widetilde{H}}(\mathbb{Q}_p),$$

and

$$I = \begin{pmatrix} -\frac{2}{c} & 0 & 0 & 0 & 0 & 0 \\ \frac{b+r}{c} & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{c}{2} & -\frac{b+r}{2} & \frac{b+r}{2} \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in I_p'^{\widetilde{H}}.$$

Hence we have

$$Z_p(g_p, s_1, s) = \frac{W_{\Psi,p}(g_p)}{(p+1)^2} \left(\sum_{\substack{l \in U \\ l \neq 1}} \Lambda_p^{-2}(l) \Upsilon_p^{\sharp}(\iota(s_1, \widetilde{l}), s) + 1 \right). \tag{3.4.21}$$

where the 1 comes from the results of the previous subsection.

Noting that $\Upsilon_p^{\sharp}(\iota(s_1, \widetilde{l}), s) = \Lambda_p(l)$ and that $\sum_{\substack{l \in U \\ l \neq 1}} \Lambda_p^{-1}(l) = -1$,

we get

$$Z_p(g_p, s_1, s) = -\frac{W_{\Psi,p}(g_p)}{(p+1)^2} + \frac{W_{\Psi,p}(g_p)}{(p+1)^2} = 0.$$

Next, we consider $k_p = s_2$. Let $\Gamma_p'^{0,\widetilde{F}_1}$ be as in the previous subsection.

By the argument there, we know that $\Upsilon_p^{\sharp}(\iota(s_2, g), s)$ depends only on the double coset $\Gamma_p'^{0,\widetilde{F}_1} g \Gamma_{0,p}'^{\widetilde{F}_1}$.

To prove that $Z_p(g_p, s_2, s) = 0$ it is enough to prove that each of the elements $\iota(s_2, A_n\widetilde{l})$, $\iota(s_2, A_n w\widetilde{l})$, $\iota(s_2, wA_n\widetilde{l})$, $\iota(s_2, wA_n w\widetilde{l})$ cannot belong to $P_{\widetilde{H}}(\mathbb{Q}_p)\Omega I_p'^{\widetilde{H}} Q^{-1}$ for any $n \geq 0$. This we do by an explicit computation. The details are omitted.

Next, take $k_p = s_3$. Once again, an explicit computation shows that the elements $\iota(s_3, A_n\widetilde{l}), \iota(s_3, A_n w\widetilde{l}), \iota(s_3, wA_n\widetilde{l}), \iota(s_3, wA_n w\widetilde{l})$ cannot belong to $P_{\widetilde{H}}(\mathbb{Q}_p)\Omega I_p'^{\widetilde{H}} Q^{-1}$. So by exactly the same argument as the previous case, $Z_p(g_p, s_3, s) = 0$.

Next, consider the case $k_p = \Theta$. By explicit calculation, we check that for $n > 0$ the elements $\iota(\Theta, A_n w\widetilde{l})$, $\iota(\Theta, wA_n\widetilde{l})$, $\iota(\Theta, wA_n w\widetilde{l})$ do not belong to $P_{\widetilde{H}}(\mathbb{Q}_p)\Omega I_p'^{\widetilde{H}} Q^{-1}$.Also check that $\iota(\Theta, w\widetilde{l}) \notin P_{\widetilde{H}}(\mathbb{Q}_p)\Omega I_p'^{\widetilde{H}} Q^{-1}$. Also, provided $l \neq 1$, we have $\iota(\Theta, w\widetilde{l}) \notin P_{\widetilde{H}}(\mathbb{Q}_p)\Omega I_p'^{\widetilde{H}} Q^{-1}$. Thus, the only term that contributes is $\iota(\Theta, 1)$.

So by the same argument as before, we have

$$\begin{aligned} Z_p(g_p, \Theta, s) &= \Upsilon_p^\sharp(\iota(\Theta, 1), s) \int_{\Gamma'^{\widetilde{F}_1}_{0,p}} W_{\Psi,p}(g_p h_p) dh_p \\ &= \frac{W_{\Psi,p}(g_p)}{(p+1)^2} \end{aligned} \tag{3.4.22}$$

Next consider the case $k_p = \Theta s_2$. For $x \in \mathbb{Z}$, let $u(x)$ be as in the previous subsection. As before, to prove that $Z_p(g_p, \Theta s_2, s) = 0$, it is enough to check that the elements $\iota(\Theta s_2, u(x)A_n\widetilde{l}), \iota(\Theta s_2, u(x)A_n w\widetilde{l}), \iota(\Theta s_2, u(x)wA_n\widetilde{l}), \iota(\Theta, u(x)wA_n w\widetilde{l})$ do not belong to $P_{\widetilde{H}}(\mathbb{Q}_p)\Omega I'^{\widetilde{H}}_p Q^{-1}$. This can be done by an explicit computation (omitted for brevity).

Next, consider the case $k_p = \Theta s_4$. To prove that $Z_p(g_p, \Theta s_4, s) = 0$, it is enough to check that the elements $\iota(\Theta s_4, u(x)A_n\widetilde{l}), \iota(\Theta s_4, u(x)A_n w\widetilde{l})$, $\iota(\Theta s_4, u(x)wA_n\widetilde{l})$, $\iota(\Theta s_4, u(x)wA_n w\widetilde{l})$ do not belong to $P_{\widetilde{H}}(\mathbb{Q}_p)\Omega I'^{\widetilde{H}}_p Q^{-1}$. This is done by an explicit computation, which we omit.

Finally, we consider the case $k_p = \Theta s_5$. To prove that $Z_p(g_p, \Theta s_5, s) = 0$, it is enough to check that the elements $\iota(\Theta s_5, A_n\widetilde{l})$,$\iota(\Theta s_5, A_n w\widetilde{l})$, $\iota(\Theta s_5, wA_n\widetilde{l})$, $\iota(\Theta s_5, wA_n w\widetilde{l})$ do not belong to $P_{\widetilde{H}}(\mathbb{Q}_p)\Omega I'^{\widetilde{H}}_p Q^{-1}$. This is done by an explicit computation, which we omit.

□

The local integral at infinity

Proposition 3.4.4. *We have*

$$Z_\infty(g_\infty, 1, s) = B_\infty(s) W_{\Psi,\infty}(g_\infty),$$

where $B_\infty(s) = \frac{(-1)^{\ell/2} 2^{-6s-1}\pi}{6s+\ell-1}$.

Proof. Note that $K^{\widetilde{F}}_\infty$ is the maximal compact subgroup of $\widetilde{F}_1(\mathbb{R})$. Furthermore, note that any element h of $\widetilde{F}_1(\mathbb{R})$ can be written in the form

$$h = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix} k$$

where $x \in \mathbb{R}$, $b \in \mathbb{R}^+$, $k \in K^{\widetilde{F}}_\infty$. Let us henceforth denote $u(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, $t(b) = \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix}$.

We normalize our Haar measures such that $K^{\widetilde{F}}_\infty$ has volume 1. Also, note that Λ_∞ is trivial and for $k \in K_\infty^{\widetilde{F}}$, $g, h \in \widetilde{F}_1(\mathbb{R})$ we have

$$\Upsilon_\infty^\sharp(\iota(1, hk), s) W_{\Psi,\infty}(ghk) = \Upsilon_\infty^\sharp(\iota(1, h), s) W_{\Psi,\infty}(gh).$$

Hence we have

$$Z_\infty(g_\infty, 1, s) = \int_0^\infty \int_{-\infty}^\infty \Upsilon_\infty^\sharp(\iota(1, u(x)t(b)), s) W_{\Psi,\infty}(g_\infty u(x)t(b)) b^{-3} dx db \tag{3.4.23}$$

where dx, db are the usual Lebesgue measures.

Let $K_\infty^H = K_\infty^{\widetilde{H}} \cap H(\mathbb{R})$. To calculate $\Upsilon_\infty^\sharp(\iota(1, u(x)t(b)), s)$ we need to write the Iwasawa decomposition of $\iota(1, u(x)t(b))$. However, finding an explicit decomposition is not really necessary. Indeed, we know that there exists some decomposition

$$\iota(1, u(x)t(b)).Q = \begin{pmatrix} A & X \\ 1 & (A^t)^{-1} \end{pmatrix} K$$

with $K \in K_\infty^H$, $A \in GL_3(\mathbb{R})$ and that

$$\Upsilon_\infty^\sharp(\iota(1, u(x)t(b)), s) = |\det(A)|^{6(s+1/2)} \det(J(K, i))^{-\ell}. \tag{3.4.24}$$

Now, let $A_x^b = \iota(1, u(x)t(b)).Q$. By explicit computation, we see that

$$A_x^b = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{b} \\ 0 & 0 & 0 & 0 & 1 & -\frac{1}{b} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & b & 0 & 0 & -\frac{x}{b} \end{pmatrix}$$

By (3.4.24) we have

$$\det(J(A_x^b, i)) = \det(A)^{-1} \det(J(K, i)).$$

Since $\det(J(A_x^b, i)) = \frac{x-i(b^2+1)}{b}$ we have

$$\Upsilon_\infty^\sharp(\iota(1, u(x)t(b)), s) = |\det(A)|^{6(s+1/2)}\det(A)^{-\ell}b^\ell(x - i(b^2+1))^{-\ell}. \tag{3.4.25}$$

On the other hand, we have

$$(A_x^b)(i) = (AA^t i + XA^t).$$

By explicit computation, we see that

$$(A_x^b)(i) = \frac{1}{b^4+2b^2+x^2+1}\left[\begin{pmatrix} b^4+b^2+x^2 & 0 & -x \\ 0 & b^4+2b^2+x^2+1 & 0 \\ -x & 0 & b^2+1 \end{pmatrix} i \right.$$
$$\left. + \begin{pmatrix} -x & 0 & b^2+1 \\ 0 & 0 & 0 \\ b^2+1 & 0 & x \end{pmatrix}\right]$$

From this we get $\det(A) = \frac{b}{\sqrt{b^4+1+2b^2+x^2}}$.

Therefore, we have

$$\Upsilon_\infty^\sharp(\iota(1, u(x)t(b)), s) = b^{6s+3}(b^4+1+2b^2+x^2)^{-3(s+1/2)+\ell/2}(x - i(b^2+1))^{-\ell}. \tag{3.4.26}$$

On the other hand, we know that

$$W_{\Psi,\infty}(u(x)t(b)) = e^{2\pi i x}e^{-2\pi b^2}b^\ell$$

We will prove the proposition only for $g_\infty = 1$, the calculations in the general case are similar.

We need to evaluate the integral

$$\int_0^\infty \int_{-\infty}^\infty b^{6s+\ell}(x - i(b^2+1))^{-3(s+1/2)-\ell/2}(x + i(b^2+1))^{-3(s+1/2)+\ell/2}e^{-2\pi i x}e^{-2\pi b^2}dxdb \tag{3.4.27}$$

Putting $b^2 = y$, the above integral becomes

$$\frac{1}{2}\int_0^\infty \int_{-\infty}^\infty y^{3s+\frac{\ell-1}{2}}(x-i(y+1))^{-3(s+\frac{1}{2})-\frac{\ell}{2}}(x+i(y+1))^{-3(s+\frac{1}{2})+\frac{\ell}{2}}e^{2\pi ix}e^{-2\pi y}dxdy \quad (3.4.28)$$

Applying [GK92, (6.11)] to the inner integral, (3.4.28) becomes

$$\frac{(-1)^{\ell/2}(2\pi)^{6s+3}}{2\Gamma(3s+\frac{3}{2}+\frac{\ell}{2})\Gamma(3s+\frac{3}{2}-\frac{\ell}{2})}$$

times

$$\int_0^\infty e^{-2\pi(1+2t)}(t+1)^{3s+\frac{1}{2}+\frac{\ell}{2}}t^{3s+\frac{1}{2}-\frac{\ell}{2}}\left(\int_0^\infty y^{3s+\frac{\ell-1}{2}}e^{-4\pi y(1+t)}dy\right)dt. \quad (3.4.29)$$

Now, $\int_0^\infty y^{3s+\frac{\ell-1}{2}}e^{-4\pi y(1+t)}dy$ evaluates to

$$2^{-6s-\ell-1}\pi^{-3s-\frac{\ell}{2}-\frac{1}{2}}\Gamma(3s+\ell/2+\frac{1}{2}).$$

Using this, and the formula

$$\int_0^\infty e^{-2\pi(1+2t)}t^{3s+\frac{1}{2}-\frac{\ell}{2}} = 2^{-6s+\ell-3}e^{-2\pi}\pi^{-3s+\ell/2-\frac{3}{2}}\Gamma(3s+\frac{3}{2}-\frac{\ell}{2})$$

we see that (3.4.28) simplifies to

$$\frac{(-1)^{l/2}2^{-6s-1}\pi}{6s+\ell-1}W_{\Psi,\infty}(1).$$

□

3.5 Proof of the Pullback formula

In this section, we will prove Theorem 3.2.1.

Recall the definition of $\mathcal{E}(g,s)$ from §3.2. Our main step in computing $\mathcal{E}(g,s)$ will be the evaluation of the following integral:

$$\Upsilon_\Psi(g,s) = \int_{\widetilde{F}_1[g](\mathbb{A})} \Upsilon^\sharp(\iota(g,h),s)\Psi(h)\Lambda^{-1}(\det h)dh \quad (3.5.1)$$

By [Shi97], we know that the integral above converges absolutely and uniformly on compact sets for $\mathrm{Re}(s)$ large. We are going to evaluate the above integral for such s.

Note that $\widetilde{G}(\mathbb{A}) = P(\mathbb{A})\prod_v \widetilde{K_v}$. Moreover if $k \in \widetilde{K_v}$, we may write

$$k = m_2\left(\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}\right)k'$$

where $\lambda = \mu_2(k)$, so that $\mu_2(k') = 1$.

For any $p \in S_3$ we have, by the Bruhat decomposition,

$$\widetilde{K_p} = (P(\mathbb{Q}_p) \cap \widetilde{K_p})\widetilde{U_p} \sqcup (P(\mathbb{Q}_p) \cap \widetilde{K_p})s_1\widetilde{U_p} \sqcup (P(\mathbb{Q}_p) \cap \widetilde{K_p})s_2\widetilde{U_p}.$$

Also, for $p|M$, we have, by Lemma 3.3.1,

$$\widetilde{K_p} = \coprod_{s \in Y_p} (P(\mathbb{Q}_p) \cap \widetilde{K_p})sI'_p.$$

Recall that we defined the compact subgroup $U^{\widetilde{G}}$ of $\widetilde{G}(\mathbb{A}_f)$ in (2.8.16).

So write $g = m_1(a)m_2(b)nk$ where $k \in \prod_v \widetilde{K_v}$, $\mu_2(k) = 1$ and further write $k = k_\infty k_{\mathrm{ram}} k_{\mathrm{ur}}$ where

$$k_\infty \in \widetilde{K_\infty},\ k_{\mathrm{ur}} \in U^{\widetilde{G}}$$

and $k_{\mathrm{ram}} = \prod_v (k_{\mathrm{ram}})_v$, with

$$(k_{\mathrm{ram}})_v \in \begin{cases} \{1\} & \text{if } v \notin S \\ \{1, s_1, s_2\} & \text{if } v \in S_3 \\ \{1, s_1, s_2, s_3, \Theta, \Theta s_2, \Theta s_4, \Theta s_5\} & \text{if } v \in S_1 \sqcup S_2. \end{cases}$$

Therefore we have

$$\begin{aligned}\Upsilon_\Psi(g,s) &= \int_{\widetilde{F}_1[m_2(b)](\mathbb{A})} \Upsilon^\sharp(\iota(m_1(a)m_2(b)nk, b(b^{-1}h)), s)\Psi(h)\Lambda^{-1}(\det h)dh \\ &= \rho_\ell(k_\infty) \\ &\times \int_{\widetilde{F}_1[m_2(b)](\mathbb{A})} \Upsilon^\sharp(\iota(m_1(a)m_2(b)nk_{\mathrm{ram}}, b(b^{-1}h)), s)\Psi(h)\Lambda^{-1}(\det h)dh \\ &\qquad \text{(using properties from §3.3)} \\ &= \Lambda(a)\|N_{L/\mathbb{Q}}(a)\cdot\mu_2(b)^{-1}|^{3(s+1/2)}\rho_\ell(k_\infty) \\ &\times \int_{\widetilde{F}_1(\mathbb{A})} \Upsilon^\sharp(\iota(k_{\mathrm{ram}}, h), s)\Psi(bh)\Lambda^{-1}(\det h)dh \\ &\qquad \text{(using (3.1.1)).}\end{aligned}$$

We write

$$U_b(k_{\mathrm{ram}}, s) = \int_{\widetilde{F}_1(\mathbb{A})} \Upsilon^\sharp(\iota(k_{\mathrm{ram}}, h), s)\Psi(bh)\Lambda^{-1}(\det h)dh.$$

Thus we have

$$\Upsilon_\Psi(g,s) = \Lambda(a)|N_{L/\mathbb{Q}}(a)\cdot\mu_2(b)^{-1}|^{3(s+1/2)}\rho_l(k_\infty)\times U_b(k_{\mathrm{ram}}, s) \tag{3.5.2}$$

Recall the Whittaker expansion

$$\Psi(g) = \sum_{\lambda\in\mathbb{Q}^\times} W_\Psi\left(\begin{pmatrix}\lambda & 0\\ 0 & 1\end{pmatrix} g\right) \tag{3.5.3}$$

Therefore

$$U_b(k_{\mathrm{ram}}, s) = \sum_{\lambda\in\mathbb{Q}^\times} Z\left(\begin{pmatrix}\lambda & 0\\ 0 & 1\end{pmatrix} b, k_{\mathrm{ram}}, s\right) \tag{3.5.4}$$

where for $g \in \widetilde{F}(\mathbb{A})$, $k \in \prod_v \widetilde{K_v}$, $\mu_2(k) = 1$, we define

$$Z(g,k,s) = \int_{\widetilde{F}_1(\mathbb{A})} \Upsilon^\sharp(\iota(k,h), s)W_\Psi(gh)\Lambda^{-1}(\det h)dh.$$

Note that the uniqueness of the Whittaker function implies

$$Z(g,k,s) = \prod_v Z_v(g_v, k_v, s),$$

where the local zeta integral $Z_v(g_v, k_v, s)$ is defined as in (3.3.9).

So, by the results of the previous two sections, we have

$$Z(g, k_{\text{ram}}, s) = \begin{cases} B(s)W_\Psi(g) & \text{if } (k_{\text{ram}})_v \in Y_v' \text{ for all places } v \\ 0 & \text{otherwise} \end{cases} \tag{3.5.5}$$

where we define

$$Y_v' = \begin{cases} \{1\} & \text{if } v \notin S_1 \sqcup S_2 \\ \{1, s_1\} & \text{if } v \in S_2 \\ \{1, \Theta\} & \text{if } v \in S_1 \sqcup S_2. \end{cases}$$

From (3.5.2),(3.5.3),(3.5.4),(3.5.5) we conclude that

$$\Upsilon_\Psi(g, s) = B(s) f_\Lambda(g, s) \tag{3.5.6}$$

where $f_\Lambda(g, s)$ is defined as in §2.8.

We are now in a position to prove the Pullback formula.

Proof of Theorem 3.2.1. Recall the definition of $B(s)$ from (3.2.4). Also recall that we defined

$$\mathcal{E}(g, s) = \int_{\widetilde{F_1}(\mathbb{Q})\backslash \widetilde{F_1}[g](\mathbb{A})} E_{\Upsilon^\sharp}(\iota(g, h), s)\Psi(h)\Lambda^{-1}(\det h)dh. \tag{3.5.7}$$

The pullback formula states that

$$\mathcal{E}(g, s) = B(s)E_{\Psi,\Lambda}(g, s).$$

By abuse of notation, we use $\widetilde{R}(\mathbb{Q})$ to denote its image in $\widetilde{H}(\mathbb{Q})$. First, we recall from [Shi97] that $|P_{\widetilde{H}}(\mathbb{Q})\backslash\widetilde{H}(\mathbb{Q})/\widetilde{R}(\mathbb{Q})|$=2. We take the identity element as one of the double coset representatives, and denote the other one by τ. Thus

$$\widetilde{H}(\mathbb{Q}) = P_{\widetilde{H}}(\mathbb{Q})\widetilde{R}(\mathbb{Q}) \sqcup P_{\widetilde{H}}(\mathbb{Q})\tau\widetilde{R}(\mathbb{Q}).$$

Let us denote by R_1, R_2 the corresponding sets of coset representatives, i.e. $R_1 \subset \widetilde{R}(\mathbb{Q}), R_2 \subset \widetilde{R}(\mathbb{Q})$ and

$$P_{\widetilde{H}}(\mathbb{Q})\widetilde{R}(\mathbb{Q}) = \bigsqcup_{s \in R_1} P_{\widetilde{H}}(\mathbb{Q})s$$

and

$$P_{\widetilde{H}}(\mathbb{Q})\tau\widetilde{R}(\mathbb{Q}) = \bigsqcup_{s \in R_2} P_{\widetilde{H}}(\mathbb{Q})s.$$

Recall that we defined

$$E_{\Upsilon^\sharp}(h,s) = \sum_{\gamma \in P_{\widetilde{H}}(\mathbb{Q}) \backslash \widetilde{H}(\mathbb{Q})} \Upsilon^\sharp(\gamma h, s)$$

for Re(s) large. We can write $E_{\Upsilon^\sharp}(h,s) = E^1_{\Upsilon^\sharp}(h,s) + E^2_{\Upsilon^\sharp}(h,s)$ where

$$E^1_{\Upsilon^\sharp}(h,s) = \sum_{\gamma \in R_1} \Upsilon^\sharp(\gamma h, s)$$

and

$$E^2_{\Upsilon^\sharp}(h,s) = \sum_{\gamma \in R_2} \Upsilon^\sharp(\gamma h, s).$$

Now, by [Shi97, 22.9] the orbit of τ is 'negligible' for our integral, that is for all g,

$$\int_{\widetilde{F_1}(\mathbb{Q}) \backslash \widetilde{F}_1[g](\mathbb{A})} E^2_{\Upsilon^\sharp}(\iota(g,h),s)\Psi(h)\Lambda^{-1}(\det h)dh = 0.$$

It follows that

$$\mathcal{E}(g,s) = \int_{\widetilde{F_1}(\mathbb{Q}) \backslash \widetilde{F}_1[g](\mathbb{A})} E^1_{\Upsilon^\sharp}(g,h,s)\Psi(h)\Lambda^{-1}(\det h)dh. \tag{3.5.8}$$

On the other hand, by [Shi97, 2.7] we can take R_1 to be the following set:

$$R_1 = \{\iota(m_2(\xi)\beta, 1) : \xi \in \widetilde{F}_1(\mathbb{Q}), \beta \in P(\mathbb{Q}) \backslash \widetilde{G}(\mathbb{Q})\} \tag{3.5.9}$$

For Re(s) large, we therefore have

$$E^1_{\Upsilon^\sharp}(\iota(g,h),s) = \sum_{\substack{\xi \in \widetilde{F}_1(\mathbb{Q}) \\ \beta \in P(\mathbb{Q}) \backslash \widetilde{G}(\mathbb{Q})}} \Upsilon^\sharp(\iota((m_2(\xi)\beta g, h), s).$$

Substituting in (3.5.8) we have

$$\begin{aligned}
\mathcal{E}(g,s) &= \int_{\widetilde{F_1}(\mathbb{Q})\backslash \widetilde{F}_1[g](\mathbb{A})} \sum_{\substack{\xi\in\widetilde{F}_1(\mathbb{Q})\\ \beta\in P(\mathbb{Q})\backslash\widetilde{G}(\mathbb{Q})}} \Upsilon^\sharp(\iota(m_2(\xi)\beta g,h),s)\Psi(h)\Lambda^{-1}(\det h)dh \\
&= \int_{\widetilde{F_1}(\mathbb{Q})\backslash \widetilde{F}_1[g](\mathbb{A})} \sum_{\substack{\xi\in\widetilde{F}_1(\mathbb{Q})\\ \beta\in P(\mathbb{Q})\backslash\widetilde{G}(\mathbb{Q})}} \Upsilon^\sharp(\iota(\beta g,\xi^{-1}h),s)\Psi(\xi^{-1}h)\Lambda^{-1}(\det \xi^{-1}h)dh \\
&= \sum_{\beta\in P(\mathbb{Q})\backslash\widetilde{G}(\mathbb{Q})} \int_{\widetilde{F}_1[g](\mathbb{A})} \Upsilon^\sharp(\iota(\beta g,h),s)\Psi(h)\Lambda^{-1}(\det h)dh \\
&= \sum_{\beta\in P(\mathbb{Q})\backslash\widetilde{G}(\mathbb{Q})} \Upsilon_\Psi(\beta g,s) \\
&= B(s)\sum_{\beta\in P(\mathbb{Q})\backslash\widetilde{G}(\mathbb{Q})} f_\Lambda(\beta g,s) \\
&= B(s)E_{\Psi,\Lambda}(g,s)
\end{aligned}$$

Thus

$$\int_{\widetilde{F_1}(\mathbb{Q})\backslash \widetilde{F}_1[g](\mathbb{A})} E_{\Upsilon^\sharp}(g,h,s)\widetilde{\Psi}(h)\Lambda^{-1}(\det h)dh = B(s)E_{\Psi,\Lambda}(g,s) \tag{3.5.10}$$

for $\mathrm{Re}(s)$ large (so that all sums and integrals converge nicely and our manipulations are valid).

However, $E_{\Upsilon^\sharp}(g,h,s)$ is slowly increasing away from its poles, while $\Psi(h)$ is rapidly decreasing. Thus the left side above converges absolutely for $s\in\mathbb{C}$ away from the poles of the Eisenstein series. Hence (3.5.10) holds as an identity of meromorphic functions.

□

3.6 The integral representation

The following result was proved in the previous chapter:

$$\int_{Z_G(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})} E_{\Psi,\Lambda}(g,s)\overline{\Phi}(g)dg = C(s)\cdot L(3s+\frac{1}{2},F\times g)$$

where $C(s) =$

$$\frac{Q_f \pi \overline{a(\Lambda)} (4\pi)^{-3s-\frac{3}{2}\ell+\frac{3}{2}} d^{-3s-\frac{\ell}{2}} \Gamma(3s+\frac{3}{2}l-\frac{3}{2})}{g(M/f) P_{MN}(6s+\ell-1) \zeta^{MN}(6s+1) L(3s+1, \sigma \times \rho(\Lambda))} \prod_{p|f} \frac{p^{-6s-3}}{1-a_p w_p p^{-3s-3/2}}$$

where

$$f = \gcd(M, N),$$

$$Q_A = \prod_{\substack{r|A \\ r \text{ prime}}} (1-r),$$

and $g(A), P_A, \zeta^A$ are as defined earlier.

Recall the definition of $B(s)$ from (3.2.4) and let

$$A(s) = B(s)C(s).$$

Let R denote the subgroup of $\widetilde{R}$ consisting of elements $h = (h_1, h_2)$ such that $h_1 \in G, h_2 \in \widetilde{F}$ and $\mu_2(h_1) = \mu_1(h_2)$. The above Theorem, along with our pullback formula, implies the following result.

Theorem 3.6.1. *We have*

$$\int_{g \in Z(\mathbb{A})R(\mathbb{Q})\backslash R(\mathbb{A})} E_{\Upsilon^\sharp}(\iota(g_1, g_2), s) \overline{\Phi}(g_1) \Psi(g_2) \Lambda^{-1}(\det g_2) dg = A(s) L(3s + \frac{1}{2}, F \times g)$$

where $g = (g_1, g_2)$.

This new integral representation has a great advantage over the previous one: the Eisenstein series $E_{\Upsilon^\sharp}(g, s)$ is much simpler than $E_{\Psi,\Lambda}(g, s)$ (even though it lives on a higher rank group). This is because it is because it is induced from a *character* of the Siegel parabolic. Thus, it is more suitable for applications, especially with regard to special value results.

Corollary 3.6.2. *$L(s, F \times g)$ can be continued to a meromorphic function on the entire complex plane. It's only possible pole to the right of the critical line Re(s)= $\frac{1}{2}$ is at $s = 1$.*

Proof. The integral representation of Theorem 3.6.1 immediately proves the meromorphic continuation. Furthermore by [Ich04], we know that the only possible poles of the Eisenstein series $E_{\Upsilon^\sharp}(g, s)$ to the right of $s = 0$ are at $s = \frac{1}{6}$ and $s = \frac{1}{2}$. However, as we remark in the proof of Proposition 4.1.3, the pole at $s = \frac{1}{2}$ is impossible. So the only possible pole of the Eisenstein series in that half plane is at $s = \frac{1}{6}$ which corresponds to a pole of the L-functions at $s = 1$. $\square$

Chapter 4

Rationality of Eisenstein series and Deligne's conjecture on critical L-values

4.1 Eisenstein series on Hermitian domains

Let

$$\widetilde{G}^{+}(\mathbb{R}) = \{g \in \widetilde{G}(\mathbb{R}) : \mu_2(g) > 0\}.$$

Define the groups $G^{+}(\mathbb{R}), \widetilde{H}^{+}(\mathbb{R}), \widetilde{F}^{+}(\mathbb{R})$ similarly.

Also recall the definitions of the symmetric domains $\mathbb{H}_n$, $\widetilde{\mathbb{H}}_n$. We define the 'standard embedding' of $\widetilde{\mathbb{H}}_2 \times \widetilde{\mathbb{H}}_1$ into $\widetilde{\mathbb{H}}_3$ by

$$(Z_1, Z_2) \mapsto \begin{pmatrix} Z_1 & \\ & Z_2 \end{pmatrix}.$$

We use the same notation (Z_1, Z_2) to denote an element of $\widetilde{\mathbb{H}}_2 \times \widetilde{\mathbb{H}}_1$ and its image in $\widetilde{\mathbb{H}}_3$ under the above embedding. Note that this embedding restricts to an embedding of $\mathbb{H}_2 \times \mathbb{H}_1$ into $\mathbb{H}_3$.

We also define another embedding u of $\widetilde{\mathbb{H}}_2 \times \widetilde{\mathbb{H}}_1$ into $\widetilde{\mathbb{H}}_3$ by

$$u(Z_1, Z_2) = (Z_1, -\overline{Z_2}).$$

Clearly this embedding also restricts to an embedding of $\mathbb{H}_2 \times \mathbb{H}_1$ into $\mathbb{H}_3$.

Furthermore, the following is true, as can be verified by an easy calculation:

Let $g_1 \in \widetilde{G}_1(\mathbb{R}), g_2 \in \widetilde{F}_1(\mathbb{R})$, such that $g_1(i) = Z_1, g_2(i) = Z_2$. In the event that $(Z_1, Z_2) \in \mathbb{H}_2 \times \mathbb{H}_1$ we may even take $g_1 \in G_1(\mathbb{R}), g_2 \in SL_2(\mathbb{R})$.

Then

$$u(Z_1, Z_2) = (Q^{-1}\iota(g_1, g_2)Q)i.$$

Now, let us interpret the Eisenstein series on $GU(3,3)$ as a function on $\widetilde{\mathbb{H}}_3$. Recall the definitions of the sections $\Upsilon_v(s) \in \mathrm{Ind}_{P_{\widetilde{H}}(\mathbb{Q}_v)}^{\widetilde{H}(\mathbb{Q}_v)}(\Lambda_v \| \cdot \|_v^{3s})$. Also, for $Z \in \widetilde{\mathbb{H}}_n$, we set $\widehat{Z} = \frac{i}{2}(\overline{Z}^t - Z)$.

Lemma 4.1.1. *Let $g_\infty \in \widetilde{H}^+(\mathbb{R})$. Then*

$$\Upsilon_\infty(g_\infty, s) = \det(g_\infty)^{\ell/2} \det(J(g_\infty, i))^{-\ell} \det(\widehat{g_\infty(i)})^{3(s+1/2)-\ell/2}$$

Proof. Let us write $g_\infty = m(A, v)nk_\infty$ where $m(A, v) \in M(\mathbb{A})$, $n \in N(\mathbb{A})$ and $k \in K_\infty^{\widetilde{H}}$. Then, (3.1.1) and (3.1.9) tells us that

$$\Upsilon_\infty(g_\infty, s) = v^{-9(s+1/2)} |\det A|^{6s+3} \det(k_\infty)^{\ell/2} \det(J(k_\infty, i))^{-\ell}.$$

On the other hand, we can verify that

$$\widehat{g_\infty(i)} = v^{-1} A\overline{A}^t$$

and therefore

$$\det(\widehat{g_\infty(i)}) = v^{-3} |\det A|^2.$$

Also we see that

$$J(g_\infty, i) = v(\overline{A}^t)^{-1} J(k_\infty, i)$$

which implies

$$\det(J(g_\infty, i)) = v^3 \det(\overline{A})^{-1} \det(J(k_\infty, i)).$$

Finally

$$\det(g_\infty) = v^3 \det(k_\infty) \det(A) \det(\overline{A})^{-1}.$$

Putting the above equations together, we get the statement of the lemma. □

Corollary 4.1.2. *Let $s \in \mathbb{C}$, $u_f \in \widetilde{H}(\mathbb{A}_f)$ be fixed. Then the function Σ on $\widetilde{H}^+(\mathbb{R})$ defined* *by*

$$\Sigma(g_\infty) = \det(g_\infty)^{-\ell/2}\det(J(g_\infty,i))^{\ell}E_\Upsilon(u_f g_\infty, \frac{s}{3}+\frac{\ell}{6}-1/2)$$

depends only on $g_\infty(i)$.

Proof. We have

$$E_\Upsilon(u_f g_\infty, s) = \sum_{\gamma\in P_{\widetilde{H}(\mathbb{Q})}\backslash \widetilde{H}(\mathbb{Q})} \Upsilon_\infty(\gamma_\infty g_\infty, s)\Upsilon_f(\gamma_f u_f, s).$$

So, by the above lemma,

$$\Sigma(g_\infty) = \sum_{\gamma\in P_{\widetilde{H}(\mathbb{Q})}\backslash \widetilde{H}(\mathbb{Q})} \det(\gamma)^{\ell/2}\det(J(\gamma,Z))^{-\ell}\det(\widehat{\gamma(Z)})^{s} \tag{4.1.1}$$

where $Z = g_\infty(i)$.

□

Now,consider the coset decomposition

$$\widetilde{F}(\mathbb{A}) = \bigsqcup_{i=1}^{h}\widetilde{F}(\mathbb{Q})\widetilde{F}^{+}(\mathbb{R})\begin{pmatrix} t_i & \\ & t_i^* \end{pmatrix} U^{\widetilde{F}} \tag{4.1.2}$$

where $t_i \in \widetilde{F}(\mathbb{A}_f)$, $t_i^* = \overline{t_i}^{-1}$, and

$$U^{\widetilde{F}} = \prod_{p\notin S} K_p^{\widetilde{F}} \prod_{p\in S_3}\Gamma_{0,p}^{\widetilde{F}} \prod_{p\in S_1\sqcup S_2}\Gamma'^{\widetilde{F}}_{0,p}. \tag{4.1.3}$$

We note here that the constant h comes up because the class number of L may not be 1 and because the det map from $\Gamma'^{\widetilde{F}}_{0,p}$ to $\mathbb{Z}_{L,p}^{\times}$ is not surjective. In particular, note that if $M = 1$, we have $h = h(-d)$, the class number of L.

Also, we note that by the Cebotarev density theorem, we may choose t_i such that $(N_{L/\mathbb{Q}}t_i) = q_i^{-1}$ where q_i corresponds to an ideal of $\mathbb{Z}$ that splits in L. In particular $\gcd(q_i, MN) = 1$.

Now, let

$$\Gamma_i = SL_2(\mathbb{Z}) \cap \begin{pmatrix} t_i & \\ & t_i^* \end{pmatrix} K^{\widetilde{F}} \begin{pmatrix} t_i^{-1} & \\ & (t_i^*)^{-1} \end{pmatrix} \widetilde{F}(\mathbb{R})$$
$$= \Gamma_0(M) \cap \Gamma_0(Nq_i)$$

.

Also, we define the congruence subgroup $\Gamma_{M,N}$ of $Sp_4(\mathbb{Z})$ by

$$\Gamma_{M,N} = B(M) \cap U_2(N).$$

Recall the definition of $U^{\widetilde{G}}$ from (2.8.16). Let us define the compact open subgroup U^G of $G(\mathbb{A}_f)$ by

$$U^G = U^{\widetilde{G}} \cap G(\mathbb{A}). \tag{4.1.4}$$

Observe that

$$\Gamma_{M,N} = U^G Sp_4(\mathbb{R}) \cap Sp_4(\mathbb{Q}).$$

Next, put

$$s_i = \begin{pmatrix} t_i & \\ & t_i^* \end{pmatrix}$$

and

$$r_i = \iota(1, s_i) \in \widetilde{H}_1(\mathbb{A}_f).$$

For $Z \in \widetilde{\mathbb{H}}_3$, define the Eisenstein series $E^i_\Upsilon(Z; s)$ by

$$E^i_\Upsilon(Z; s) = \det(g_\infty)^{-\ell/2} \det(J(g_\infty, i))^\ell E_\Upsilon(Q^{-1} r_i Q g_\infty, s/3 + \ell/6 - 1/2), \tag{4.1.5}$$

where $g_\infty \in \widetilde{H}^+(\mathbb{R})$ is such that $g_\infty(i) = Z$. We note that $E^i_\Upsilon(Z, s)$ is well defined by Corollary 4.1.2.

Now, consider the function $E^i_\Upsilon(Z_1, Z_2; 0)$ for $Z_1 \in \mathbb{H}_2, \mathbb{Z}_2 \in \mathbb{H}_1$.

Proposition 4.1.3. *Assume $\ell \geq 6$. Then $E^i_\Upsilon(Z_1, Z_2; 0)$ is a modular form of weight ℓ for $\Gamma_{M,N} \times \Gamma_i$. Furthermore, for any s_0, the function $E^i_\Upsilon(Z_1, Z_2; s_0)$ (which is not holomorphic in Z_1, Z_2 unless $s_0 = 0$) transforms just like $E^i_\Upsilon(Z_1, Z_2; 0)$ under the action of $\Gamma_{M,N} \times \Gamma_i$.*

Proof. We know that $E_\Upsilon(g,s)$ converges absolutely and uniformly for $s > \frac{1}{2}$. So if $\ell > 6$, it follows that $E^i_\Upsilon(Z;0)$ is holomorphic. Furthermore, the case $\ell = 6$ corresponds to the point $s = \frac{1}{2}$ of $E_\Upsilon(g,s)$. From the general theory of Eisenstein series, we know that the residue of $E_\Upsilon(g,s)$ restricted to $K_\infty^{\widetilde{H}}$ at $s = \frac{1}{2}$ must be a constant function. However, because $E_\Upsilon(g,s)$ is an eigenfunction of $K_\infty^{\widetilde{H}}$ with non-trivial eigencharacter, this residue must be zero. Hence $E^i_\Upsilon(Z;0)$ is a holomorphic function of Z even for $\ell = 6$.

Let $A \in \Gamma_{M,N}, B \in \Gamma_i$. It suffices to show that

$$E^i_\Upsilon(AZ_1, BZ_2; s_0) = \det(J(A,Z_1))^\ell \det(J(B,Z_2))^\ell E^i_\Upsilon(Z_1,Z_2;s_0).$$

For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ denote $\widetilde{g} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$. Let $g_1 \in G_1(\mathbb{R}), g_2 \in SL_2(\mathbb{R})$ such that $g_1 i = Z_1, g_2 i = Z_2$. Put $s_1 = s_0/3 + \ell/6 - 1/2$. We have

$$\begin{aligned} E^i_\Upsilon(AZ_1, BZ_2; s_0) &= E^i_\Upsilon(u(AZ_1, \overline{-BZ_2}; s_0) \\ &= E^i_\Upsilon((Q^{-1}\iota(Ag_1, \widetilde{B}\widetilde{g_2})Q)i; s_0) \\ &= \det(J(Q^{-1}\iota(Ag_1, \widetilde{B}\widetilde{g_2})Q, i))^\ell E_\Upsilon(Q^{-1} r_i \iota(Ag_1, \widetilde{B}\widetilde{g_2})Q, s_1) \\ &= \det(J(Q^{-1}\iota(Ag_1, \widetilde{B}\widetilde{g_2})Q, i))^\ell E_{\Upsilon^\sharp}(\iota(Ag_1, s_i\widetilde{B}\widetilde{g_2}), s_1) \end{aligned}$$

Now, because $s_i^{-1}\widetilde{B}s_i \in U^{\widetilde{F}}$ we have

$$E_{\Upsilon^\sharp}(\iota(Ag_1, s_i\widetilde{B}\widetilde{g_2}), s_1) = E_{\Upsilon^\sharp}((g_1, s_i\widetilde{g_2}); s_1).$$

On the other hand, we can check that

$$\det(J(Q^{-1}(Ag_1, \widetilde{B}\widetilde{g_2})Q, i))^\ell = \det(J(A,Z_1))^\ell \det(J(B,Z_2))^\ell \det(J(g_1,i))^\ell \det(J(g_2,i))^l.$$

Putting everything together, we see that

$$E^i_\Upsilon(AZ_1, BZ_2; s_0) = \det(J(A,Z_1))^\ell \det(J(B,Z_2))^\ell E^i_\Upsilon(Z_1,Z_2;s_0)$$

as required. $\square$

4.2 The integral representation in classical terms

Henceforth, we assume $\ell \geq 6$. Recall the definitions of the compact open subgroups $U^G, U^{\widetilde{F}}$ from (4.1.4), (4.1.3) respectively. Let us define $U^R \subset R(\mathbb{A}_f)$ to be subgroup consisting of elements (g,h) with $g \in U^G$, $h \in U^{\widetilde{F}}$ and $\mu_2(g) = \mu_1(h)$. Also put $K^R_\infty = K_\infty \times K^{\widetilde{F}}_\infty$. Note that $K^R K^R_\infty$ is a compact subgroup of $R(\mathbb{A})$.

Also, define $V_{M,N} = [Sp_4(\mathbb{Z}) : \Gamma_{M,N}][K^{\widetilde{F}} : U^{\widetilde{F}}]$, where $K^{\widetilde{F}} = \prod_{p<\infty} K^{\widetilde{F}}_p$. We now rephrase Theorem 3.6.1 in classical terms.

Theorem 4.2.1. *For any k, we have*

$$\sum_i \Lambda^{-2}(t_i) \int_{\Gamma_i \backslash \mathbb{H}_1} \int_{\Gamma_{M,N} \backslash \mathbb{H}_2} E^i_\Upsilon(Z_1, -\overline{Z_2}; 1-k)\overline{F(Z_1)} g(q_i Z_2) \det(Y_1)^\ell \det(Y_2)^\ell dZ_1 dZ_2$$
$$= V_{M,N} A(\frac{\ell - 1 - 2k}{6}) L(\frac{\ell}{2} - k, F \times g)$$

where for $i = 1, 2$, we define the invariant measure dZ_i on $\mathbb{H}_{3-i}$ by

$$dZ_i = \frac{1}{2}(\det Y_i)^{i-4}) dX_i dY_i$$

where $Z_i = X_i + iY_i$.

Proof. By Theorem 3.6.1, it suffices to prove that for $g = (g_1, g_2)$,

$$V_{M,N} \int_{Z(\mathbb{A})R(\mathbb{Q})\backslash R(\mathbb{A})} E_{\Upsilon^\sharp}(\iota(g_1, g_2), \frac{\ell - 1 - 2k}{6})\overline{\Phi}(g_1)\Psi(g_2)\Lambda^{-1}(\det g_2) dg \qquad (4.2.1)$$
$$= \sum_i \Lambda^{-2}(t_i) \int_{\Gamma_i \backslash \mathbb{H}_1} \int_{\Gamma_{M,N} \backslash \mathbb{H}_2} E^i_\Upsilon(Z_1, -\overline{Z_2}; 1-k)\overline{F(Z_1)} g(q_i Z_2) \det(Y_1)^\ell \det(Y_2)^\ell dZ_1 dZ_2 \qquad (4.2.2)$$

Now, the quantity inside the integral in (4.2.1) is right invariant by $U^R K^R_\infty$. Also, we note that the volume of $U^R K^R_\infty$ is equal to $(V_{M,N})^{-1}$ (recall that we normalize the volume of the maximal compact subgroup to equal 1).

Hence we see that (4.2.1) equals

$$\int_{Z(\mathbb{A})R(\mathbb{Q})\backslash R(\mathbb{A})/U^R K_\infty^R} E_{\Upsilon^\sharp}(\iota(g_1,g_2),\frac{\ell-1-2k}{6})\overline{\Phi}(g_1)\Psi(g_2)\Lambda^{-1}(\det g_2)dg \tag{4.2.3}$$

Now, by strong approximation for $Sp_4(\mathbb{A})$ and (4.1.2) we know that

$$\begin{aligned}&Z(\mathbb{A})R(\mathbb{Q})\backslash R(\mathbb{A})/U^R K_\infty^R\\ &=\coprod_{i=1}^{h}(\Gamma_{M,N}\backslash Sp_4(\mathbb{R})/K_\infty)\times\begin{pmatrix}t_i & 0\\ 0 & t_i^*\end{pmatrix}(\Gamma_i\backslash SL_2(\mathbb{R})/SO(2)).\end{aligned}$$

Suppose $g\in Sp_4(\mathbb{R}), h\in SL_2(\mathbb{R})$. Also, put $s_i=\begin{pmatrix}t_i & \\ & t_i^*\end{pmatrix}$, $r_i=\iota(1,s_i)$, $g(i)=Z_1, h(i)=Z_2$.

We have

$$\begin{aligned}E_{\Upsilon^\sharp}(\iota(g,s_ih),\frac{\ell-1-2k}{6}) &= E_\Upsilon(Q^{-1}r_iQQ^{-1}\iota(g,h)Q,\frac{\ell-1-2k}{6})\\ &=\det(J(Q^{-1}\iota(g,h)Q,i))^{-\ell}E_\Upsilon^i(Z_1,-\overline{Z_2};1-k)\end{aligned}$$

On the other hand $\overline{\Phi}(g)=\overline{F(Z_1)\det(J(g,i))^{-\ell}}$ and $\Psi(s_ih)=g(q_iZ_2)\det(J(h,i))^{-\ell}$.

The result now follows from the observations

$$\det(J(Q^{-1}\iota(g,h)Q,i))=\det(J(g,i))\overline{\det(J(h,i))},$$

$$|\det(J(g,i))|^2=\det(Y_1),\ |\det(J(h,i))|^2=\det(Y_2).$$

and the fact that the Haar measure dg equals dZ_1dZ_2 under the above equivalence. □

Let us take a closer look at the quantity $A(\frac{\ell-1-2k}{6})$ that appears in the statement of the above theorem in the case when k is an integer, $1\le k\le\frac{\ell}{2}-2$. Write $a\sim b$ if a/b is rational. From the definition of $A(s)$, it is clear that

$$A(\frac{\ell-1-2k}{6})\sim\frac{\pi^{4+k-2\ell}\overline{a(\Lambda)}\sqrt{d}}{L(\ell+1-2k,\chi_{-d})\zeta(\ell-2k)\zeta(\ell+2-2k)}.$$

But it is well known that $\frac{L(\ell+1-2k,\chi_{-d})}{\pi^{\ell+1-2k}\sqrt{d}}$, $\frac{\zeta(\ell-2k)}{\pi^{\ell-2k}}$ and $\frac{\zeta(\ell+2-2k)}{\pi^{\ell+2-2k}}$ are all rational numbers. It follows that

$$A(\frac{\ell-1-2k}{6}) \sim \pi^{7k+1-5\ell}\overline{a(\Lambda)}. \tag{4.2.4}$$

Rationality of holomorphic Eisenstein series

Suppose f_1, f_2 are modular forms of weight ℓ for some congruence subgroup Γ of $Sp_{2n}(\mathbb{Z})$ containing $\{\pm 1\}$. We define the Petersson inner product

$$\langle f_1, f_2\rangle = \frac{1}{2}V(\Gamma)^{-1}\int_{\Gamma\backslash\mathbb{H}_n} f_1(Z)\overline{f_2(Z)}(\det Y)^{\ell-n-1}dX\,dY$$

where $V(\Gamma) = [Sp_{2n}(\mathbb{Z}) : \Gamma]$.

Note that these definitions are independent of our choice for Γ.

We henceforth use $E^i_{\Lambda,\ell}$ for E^i_{Υ} in order to show the dependence on Λ, ℓ and $a(F,\Lambda)$ for $a(\Lambda)$ to show the dependence on F.

By a result of M. Harris [Har97, Lemma 3.3.5.3], we know how $\mathrm{Aut}(\mathbb{C})$ acts on the Fourier coefficients of $E^i_{\Lambda,\ell}(Z;0)$. In particular he proves the following result.

Proposition 4.2.2 (Harris)**.** *The Fourier coefficients of $E^i_{\Lambda,\ell}(Z;0)$ lie in $\mathbb{Q}^{ab}$. Furthermore, if $\sigma \in Gal(\mathbb{Q}^{ab}/\mathbb{Q})$, then*

$$E^i_{\Lambda,\ell}(Z;0)^\sigma = E^i_{\Lambda^\sigma,\ell}(Z;0)$$

where $E^i_{\Lambda,\ell}(Z;0)^\sigma$ is obtained by letting σ act on the Fourier coefficients of $E^i_{\Lambda,\ell}(Z;0)$.

4.3 Nearly holomorphic Eisenstein series

Recall the definition of the Eisenstein series $E^i_{\Upsilon}(Z;s)$ from (4.1.5). The section Υ defining this Eisenstein series depends on the Hecke character Λ as well as on the integer ℓ. Henceforth, to make this dependence explicit, we use $E^i_{\Lambda,\ell}(Z;s)$ to denote $E^i_{\Upsilon}(Z;s)$. Moreover, for any other positive even integer k, we use $E^i_{\Lambda,k}(Z;s)$ to denote the Eisenstein series that is defined similarly except that the integer ℓ has been replaced by k everywhere. In particular, we know that $E^i_{\Lambda,k}(Z;0)$ is a holomorphic Eisenstein series (of weight k), whenever $k \geq 6$.

We can write any $Z \in \widetilde{\mathbb{H}}_n$ uniquely as $Z = X + iY$ where X, Y are Hermitian and Y is positive definite. We can also write any $Z \in \mathbb{H}_n$ uniquely as $Z = X + iY$ where X, Y are symmetric and Y is positive definite. These decompositions are compatible with each other in the obvious sense under the inclusion $\mathbb{H}_n \subset \widetilde{\mathbb{H}}_n$.

We briefly recall Shimura's theory of differential operators and nearly holomorphic functions. A thorough exposition of this material can be found in his book [Shi00].

Let $\mathbb{H}$ temporarily stand for $\mathbb{H}_n$ or $\widetilde{\mathbb{H}}_n$. For a non negative integer q, we let $\mathcal{N}^q(\mathbb{H})$ denote the space of all polynomials of degree $\leq q$ in the entries of Y^{-1} with holomorphic functions on $\mathbb{H}$ as coefficients.

Suppose Γ is a congruence subgroup of Sp_{2n} (if $\mathbb{H} = \mathbb{H}_n$) or $U(n,n)$ (if $\mathbb{H} = \widetilde{\mathbb{H}}_n$). For a positive integer k, we let $\mathcal{N}_k^q(\mathbb{H}, \Gamma)$ stand for the space of functions $f \in \mathcal{N}^q(\mathbb{H})$ satisfying

$$f(\gamma Z) = \det(J(\gamma, Z))^k f(Z)$$

for all $\gamma \in \Gamma, Z \in \mathbb{H}$, with the standard additional (holomorphy at cusps) condition on the Fourier expansion if $\mathbb{H} = \mathbb{H}_1 = \widetilde{\mathbb{H}}_1$. It is well-known that $\mathcal{N}_k^q(\mathbb{H}, \Gamma)$ is finite dimensional. In particular, if $q = 0$, then $\mathcal{N}_k^q(\mathbb{H}, \Gamma)$ is simply the corresponding space of weight k modular forms.

We let $N = n^2$ if $\mathbb{H} = \widetilde{\mathbb{H}}_n$ and $N = (n^2 + n)/2$ if $\mathbb{H} = \mathbb{H}_n$.

Whenever convergent, the Petersson inner product for nearly holomorphic forms is defined exactly as before.

Any $f \in \mathcal{N}_q^t(\mathbb{H}, \Gamma)$ has a Fourier expansion [Shi00, p. 117] as follows:

$$f(Z) = \sum_{T \in \mathcal{L}} Q_T((2\pi Y)^{-1}) e^{2\pi i TrTZ}$$

where $\mathcal{L}$ is a suitable lattice and for each T, Q_T is a polynomial in N variables and of degree $\leq t$. For an automorphism σ of $\mathbb{C}$ we define

$$f^\sigma(Z) = \sum_{T \in \mathcal{L}} Q_T^\sigma((2\pi Y^{[\sigma]})^{-1}) e^{2\pi i TrTZ}$$

where Q_T^σ is obtained by letting σ act on the coefficients of $\mathbb{Q}_T$ and

$$Y^{[\sigma]} = \begin{cases} Y^t & \text{if } \mathbb{H} = \widetilde{\mathbb{H}}_n \text{ and } \sqrt{-d}^\sigma = -\sqrt{-d} \\ Y & \text{otherwise} \end{cases}$$

We say that $f \in \mathcal{N}_q^t(\mathbb{H}, \Gamma; \overline{\mathbb{Q}})$ if $f \in \mathcal{N}_q^t(\mathbb{H}, \Gamma)$ and $f^\sigma = f$ for all $\sigma \in \mathrm{Aut}(\mathbb{C}/\overline{\mathbb{Q}})$. We will occasionally omit the weight q and the congruence subgroup Γ when we do not wish to specify those. In particular, we write $\mathcal{N}_q^t(\mathbb{H};\overline{\mathbb{Q}})$ to denote $\bigcup_\Gamma \mathcal{N}_q^t(\mathbb{H},\Gamma;\overline{\mathbb{Q}})$ where the union is taken over all congruence subgroups Γ.

Now, from (4.1.1), it is easy to see that for a positive integer k (assume $k \le \frac{\ell}{2} - 2$ to ensure convergence) we have $E^i_{\Lambda,\ell}(Z;1-k) \in \mathcal{N}^{3(k-1)}(\widetilde{\mathbb{H}}_3)$. Then, exactly the same proof as Proposition 4.1.3 tells us that the restriction of this function to $\mathbb{H}_2 \times \mathbb{H}_1$ is a nearly holomorphic modular form with respect to the appropriate subgroups. More precisely, we have

$$E^i_{\Lambda,\ell}(Z_1,Z_2;1-k) \in \mathcal{N}_\ell^{2(k-1)}(\mathbb{H}_2,\Gamma_{M,N}) \otimes \mathcal{N}_\ell^{(k-1)}(\mathbb{H}_1,\Gamma_i). \tag{4.3.1}$$

We remark here that for a general $f \in \mathcal{N}^{3(k-1)}(\widetilde{\mathbb{H}}_3)$ we can only say that $f(Z_1,Z_2) \in \sum \mathcal{N}^{\lambda_1}(\mathbb{H}_2) \otimes \mathcal{N}^{\lambda_2}(\mathbb{H}_1)$ where the sum should be extended over all (λ_1,λ_2) with $\lambda_1+\lambda_2 = 3(k-1)$. However, in this case, we know by (4.1.1) the exact nature of the polynomial of degree $3(k-1)$; thus we can conclude that $\lambda_1 = 2(k-1), \lambda_2 = k-1$.

To prove the desired algebraicity result for critical L-values, we will need to know arithmeticity properties for the nearly holomorphic modular forms in (4.3.1). That is the substance of the next proposition.

Proposition 4.3.1. *Let $\ell \ge 6$ and let k be an integer satisfying $1 \le k \le \frac{\ell}{2} - 2$. Then the function $E^i_{\Lambda,\ell}(Z_1,Z_2;1-k)$ on $\mathbb{H}_2 \times \mathbb{H}_1$ belongs to*

$$\pi^{3(k-1)}\left(\mathcal{N}_\ell^{2(k-1)}(\mathbb{H}_2,\Gamma_{M,N};\overline{\mathbb{Q}}) \otimes \mathcal{N}_\ell^{(k-1)}(\mathbb{H}_1,\Gamma_i;\overline{\mathbb{Q}})\right).$$

Furthermore, for an automorphism σ of $\mathbb{C}$, we have

$$(\pi^{-3(k-1)}E^i_{\Lambda,\ell}(Z_1,Z_2;1-k))^\sigma = \pi^{-3(k-1)}E^i_{\Lambda^\sigma,\ell}(Z_1,Z_2;1-k).$$

Proof. Since we already know (4.3.1) and since the Fourier coefficients of $E^i_{\Lambda,\ell}(Z_1,Z_2;1-k)$ are just sums of those of $E^i_{\Lambda,\ell}(Z;1-k)$, it is enough to prove that

$$(\pi^{-3(k-1)}E^i_{\Lambda,\ell}(Z;1-k))^\sigma = \pi^{-3(k-1)}E^i_{\Lambda^\sigma,\ell}(Z;1-k). \tag{4.3.2}$$

For positive integers p,q, we have the (modified) Maass-Shimura differential operator Δ_q^p that acts on the space of nearly holomorphic forms of weight q on $\widetilde{\mathbb{H}}_3$. This operator is defined in [Shi00, p.146]. By [Shi00, Theorem 14.12], we know that

$$\Delta_q^p \mathcal{N}_q^t(\widetilde{\mathbb{H}}_3; \overline{\mathbb{Q}}) \subset \pi^{3p} \mathcal{N}_{q+2p}^{t+3p}(\widetilde{\mathbb{H}}_3; \overline{\mathbb{Q}}).$$

However, more is true; in fact

$$((\pi i)^{-3p} \Delta_q^p f)^\sigma = (\pi i)^{-3p} \Delta_q^p (f^\sigma) \tag{4.3.3}$$

whenever $f \in \mathcal{N}_q^t(\widetilde{\mathbb{H}}_3)$. This easily follows from [Shi00, p. 118] since the Maass-Shimura operators are special cases of the operators considered there and the projection map is $\mathrm{Aut}(\mathbb{C})$-equivariant. An alternative way to directly see (4.3.3) is to observe that the action of the Maass-Shimura operator on the Fourier coefficients of a nearly holomorphic form can be explicitly computed and observed to satisfy the desired property. The details in the symplectic case was worked out by Panchishkin [Pan05, Theorem 3.7]; the calculations in the unitary case are very similar.

We know that $E^i_{\Lambda,\ell+2-2k}(Z;0) \in \mathcal{N}^0_{\ell+2-2k}(\widetilde{\mathbb{H}}_3; \overline{\mathbb{Q}})$. So, we can apply (4.3.3) when $t = 0, p = k-1, q = \ell+2-2k, f = E^i_{\Lambda,\ell+2-2k}(Z;0)$.

Moreover, by the result of Harris stated above,

$$E^i_{\Lambda,\ell+2-2k}(Z;0)^\sigma = E^i_{\Lambda^\sigma,\ell+2-2k}(Z;0).$$

So, (4.3.2) will follow if we know that

$$\Delta^{k-1}_{\ell-2(k-1)} E^i_{\Lambda,\ell+2-2k}(Z;0) = c \cdot i^{3(k-1)} \cdot E^i_{\Lambda,\ell}(Z;1-k) \tag{4.3.4}$$

for some rational number c (The superscript i should not be confused with the quantity $i = \sqrt{-1}$ that appears above!).

But (4.3.4) is precisely the content of Shimura's calculations in [Shi00, (17.27)]. We remark here that the Eisenstein series Shimura considers has different sections than ours at the finite places dividing MN; however that does not make a difference because the differential operator only depends on the archimedean section. In particular, we apply [Shi00, Theorem 12.13] to each term of the definition of our Eisenstein series using (4.1.1) and observe that (4.3.4) follows with $c = 2^{-3(k-1)} c^{k-1}_{\ell-2(k-1)}(\frac{\ell}{2} - k + 1)$ where $c^p_q(s)$ is defined as in [Shi00, (17.20)].

$\square$

4.4 Holomorphic projection

Shimura observed [Shi00, p. 123] that for $q > n + t$, there exists a holomorphic projection operator $\mathfrak{A}$ on $\mathcal{N}_q^t(\mathbb{H}_n)$. For a nearly holomorphic form $f \in \mathcal{N}_q^t(\mathbb{H}_n)$, $\mathfrak{A}f$ is a modular form of weight q (i.e. an element of $\mathcal{N}_q^0(\mathbb{H}_n)$). For any cusp form g of weight q on $\mathbb{H}_n$,

$$< f, g >=< \mathfrak{A}f, g > .$$

More precisely, by the proof of [Shi00, Theorem 15.3], we can write

$$f = \mathfrak{A}f + L_q f'$$

where L_q is a rational polynomial of certain differential operators and f' is a certain nearly holomorphic form. The differential operators which are used to define L_q are Aut($\mathbb{C}$)-equivariant by [Shi00, Theorem 14.12]. Thus, for an automorphism σ of $\mathbb{C}$, we have

$$f^\sigma = (\mathfrak{A}f)^\sigma + L_q(f'^\sigma).$$

So we can conclude that

$$\mathfrak{A}(f^\sigma) = (\mathfrak{A}f)^\sigma.$$

Furthermore because the space of modular forms is a direct sum of the space of Eisenstein series and the space of cusp forms, there exists an orthogonal projection from the space of modular forms on $\mathbb{H}_n$ to the space of cusp forms on $\mathbb{H}_n$. Because the space of Eisenstein series is preserved under automorphisms of $\mathbb{C}$, this cuspidal projection is also Aut($\mathbb{C}$)-equivariant.

From the above comments we conclude the existence of a projection map $\mathfrak{A}_{\text{cusp}}$ from $\mathcal{N}_q^{t_1}(\mathbb{H}_2, \Gamma_2) \otimes \mathcal{N}_q^{t_2}(\mathbb{H}_1, \Gamma_1)$ to $S_q(\mathbb{H}_2, \Gamma_2) \otimes S_q(\mathbb{H}_1, \Gamma_1)$ for $q > 2 + t_i$ and congruence subgroups $\Gamma_2 \subset Sp_4, \Gamma_1 \subset SL_2$. This projection map satisfies, for any $\mathfrak{E}(Z_1, Z_2) \in \mathcal{N}_q^{t_1}(\mathbb{H}_2, \Gamma_2) \otimes \mathcal{N}_q^{t_2}(\mathbb{H}_1, \Gamma_1)$, $F^{(1)} \in S_q(\mathbb{H}_2, \Gamma_2)$, $g^{(1)} \in S_q(\mathbb{H}_1, \Gamma_1)$, the following properties:

(a) $\langle\langle \mathfrak{A}_{\text{cusp}} \mathfrak{E}(Z_1, Z_2), F^{(1)}(Z_1)\rangle, g^{(1)}(Z_2)\rangle = \langle\langle(\mathfrak{E}(Z_1, Z_2), F^{(1)}(Z_1)\rangle, g^{(1)}(Z_2)\rangle$,

(b) $(\mathfrak{A}_{\text{cusp}}\mathfrak{E})^{\sigma} = \mathfrak{A}_{\text{cusp}}(\mathfrak{E}^{\sigma})$.

In particular, everything above can be applied to the case when $\mathfrak{E}(Z_1, Z_2) = \pi^{-3(k-1)}E^i_{\Lambda,\ell}(Z_1, Z_2; 1-k)$.

We use $g_i(z)$ to denote the cusp form $g(q_i z)$ on $\Gamma_0(Nq_i)$. We can rewrite Theorem 4.2.1 as follows.

$$\sum_i \Lambda^{-2}(t_i)\langle\langle E^i_{\Lambda,\ell}(Z_1, Z_2; 1-k), F(Z_1)\rangle, g_i(Z_2)\rangle$$
$$= \frac{V_{M,N}}{V(\Gamma_i)V(\Gamma_{M,N})}A(\frac{\ell-1-2k}{6})L(\frac{\ell}{2}-k, F\times g)$$

Note that we have used the fact that g_i has real Fourier coefficients. Together with (4.2.4) the above equation implies that

$$\sum_i \Lambda^{-2}(t_i)\langle\langle E^i_{\Lambda,\ell}(Z_1, Z_2; 1-k), F(Z_1)\rangle, g_i(Z_2)\rangle \sim \pi^{7k+1-5\ell}\overline{a(F,\Lambda)}L(\frac{\ell}{2}-k, F\times g). \quad (4.4.1)$$

4.5 Deligne's conjecture

Motives and periods

Let $L(s, \mathcal{M})$ be the L-function associated to a motive $\mathcal{M}$ over $\mathbb{Q}$. Suppose $\mathcal{M}$ has coefficients in an algebraic number field E; then $L(s, \mathcal{M})$ takes values in $E \otimes_{\mathbb{Q}} \mathbb{C}$.

Note that E sits naturally inside $E\otimes_{\mathbb{Q}}\mathbb{C}$. Let d be the rank of $\mathcal{M}$ and $d^{\pm}$ the dimensions of the $\pm$ eigenspace of the Betti realization of $\mathcal{M}$. Deligne defined the motivic periods $c^{\pm}(\mathcal{M})$ and conjectured that for all "critical points" m,

$$\frac{L(m, \mathcal{M})}{(2\pi i)^{md^{\epsilon}}c^{\epsilon}(\mathcal{M})} \in E$$

where $\epsilon = (-1)^m$.

Now, let F, g have algebraic Fourier coefficients. Assuming the existence of motives M_F, M_g attached to F, g respectively, Yoshida computed the critical points for $M_F \otimes M_g$. He also computed the motivic periods $c^{\pm}(M_F \otimes M_g)$ under the assumption that Deligne's conjecture holds for the degree 5 L-function for F. We note here that Yoshida only deals with the full level case; however as the periods remain the same (up to a rational number) for higher level, his results remain applicable to our case.

Yoshida's computations [Yos01, Theorem 13] show that Deligne's conjecture implies the following reciprocity law:

$$\left(\frac{L(m, F \times g)}{\pi^{4m+3\ell-4}\langle F, F\rangle\langle g, g\rangle}\right)^{\alpha} = \frac{L(m, F^{\alpha} \times g^{\alpha})}{\pi^{4m+3\ell-4}\langle F^{\alpha}, F^{\alpha}\rangle\langle g^{\alpha}, g^{\alpha}\rangle} \tag{4.5.1}$$

for all $2 - \frac{\ell}{2} \leq m \leq \frac{\ell}{2} - 1$, $\alpha \in \mathrm{Aut}(\mathbb{C})$.

In the next subsection we prove the above statement for all the critical points m to the *right* of $\mathrm{Re}(s) = \frac{1}{2}$ *except* for the point 1. The proof for the critical values to the left of $\mathrm{Re}(s) = \frac{1}{2}$ would follow from the expected functional equation. The proof that $L(1, F \times g)$ behaves nicely under the action of $\mathrm{Aut}(\mathbb{C})$ would probably require further work because we do not know that this quantity is even finite (see Corollary 3.6.2). Thus, the problem of extending our result to the remaining critical values is closely related to questions of analyticity and the functional equation for the L-function. These questions are also of interest for other applications, such as transfer to $GL(4)$ and will be considered in a future paper.

We also note that the integral representation (Theorem 3.6.1) is of interest for several other applications. Indeed, we hope that this integral representation will pave the way to stability, hybrid subconvexity, non-vanishing, non-negativity and p-adic results for the L-function under consideration. We intend to deal with these questions elsewhere.

The main result

Theorem 4.5.1. *Let $\ell \geq 6$. Further, assume that F has totally real algebraic Fourier coefficients and define*

$$A(F, g; k) = \frac{L(\frac{\ell}{2} - k, F \times g)}{\pi^{5\ell-4k-4}\langle F, F\rangle\langle g, g\rangle}.$$

Then, we have:

(a) $A(F, g; k) \in \overline{\mathbb{Q}}$,

(b) For all $\alpha \in Aut(\mathbb{C})$, $A(F, g; k)^{\alpha} = A(F^{\alpha}, g^{\alpha}; k)$.

Proof. Let U be the least common multiple of M, N and all the q_i. Let Γ_1 be the principal congruence subgroup of $Sp_4(\mathbb{Z})$ of level U and Γ_2 the principal congruence subgroup of

$SL_2(\mathbb{Z})$ of level U. For each i, we can write

$$\mathfrak{A}_{\text{cusp}}(\pi^{-3(k-1)}E^i_{\Lambda,\ell}(Z_1, Z_2; 1-k)) = \sum_r F_1^r(Z_1) f_1^r(Z_2) \tag{4.5.2}$$

where F_1^r (resp. f_1^r) is a cusp form for Γ_1 (resp. Γ_2); all of weight ℓ. Then

$$\sum_r \langle f_1^r, g_i\rangle\langle F_1^r, F\rangle = \pi^{-3(k-1)}\langle\langle E^i_{\Lambda,\ell}(Z_1, Z_2; 1-k), F(Z_1)\rangle, g_i(Z_2)\rangle. \tag{4.5.3}$$

We also have

$$\sum_r \langle (f_1^r)^\alpha, g_i^\alpha\rangle\langle (F_1^r)^\alpha, F^\alpha\rangle = \pi^{-3(k-1)}\langle\langle E^i_{\Lambda^\alpha,\ell}(Z_1, Z_2; 1-k), F^\alpha(Z_1), g_i^\alpha(Z_2)\rangle \tag{4.5.4}$$

using Proposition 4.3.1 and the properties of holomorphic projection stated above.

By (4.4.1) we know that

$$A(F, g; k) = W \cdot (\overline{a(F, \Lambda)})^{-1} \cdot \sum_i \Lambda^{-2}(t_i) \frac{\sum_r \langle f_1^r, g_i\rangle\langle F_1^r, F\rangle}{\langle F, F\rangle\langle g, g\rangle} \tag{4.5.5}$$

for some rational number W.

Making α act on both sides of the above equation we get

$$A(F, g; k)^\alpha = W \cdot (\overline{a(F^\alpha, \Lambda^\alpha)})^{-1} \cdot \sum_i (\Lambda^\alpha)^{-2}(t_i) \left(\frac{\sum_r \langle f_1^r, g_i\rangle\langle F_1^r, F\rangle}{\langle F, F\rangle\langle g, g\rangle}\right)^\alpha. \tag{4.5.6}$$

We also note that $\langle g, g\rangle = \langle g_i, g_i\rangle$.

Now by a result of Garrett [Gar92, p. 460], we know that for each r,

$$\left(\frac{\langle f_1^r, g_i\rangle\langle F_1^r, F\rangle}{\langle F, F\rangle\langle g, g\rangle}\right)^\alpha = \left(\frac{\langle (f_1^r)^\alpha, g_i^\alpha\rangle\langle (F_1^r)^\alpha, F^\alpha\rangle}{\langle F^\alpha, F^\alpha\rangle\langle g^\alpha, g^\alpha\rangle}\right).$$

so we have

$$A(F, g; k)^\alpha = W \cdot (\overline{a(F^\alpha, \Lambda^\alpha; k)})^{-1} \cdot \sum_i (\Lambda^\alpha)^{-2}(t_i) \left(\frac{\sum_r \langle (f_1^r)^\alpha, g_i^\alpha\rangle\langle (F_1^r)^\alpha, F^\alpha\rangle}{\langle F^\alpha, F^\alpha\rangle\langle g^\alpha, g^\alpha\rangle}\right). \tag{4.5.7}$$

Using (4.5.5) for $F^\alpha, g^\alpha, \Lambda^\alpha$, we conclude that

$$A(F, g; k)^\alpha = A(F^\alpha, g^\alpha; k).$$

□

Remark. The above result was already known in the completely unramified case ($M = 1, N = 1$) by the work of Böcherer and Heim [BH06] who used a different method.

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