UNIVERSAL INFINITE PARTIAL ORDERS

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ABSTRACT

We consider infinite partial orders in which the order or comparability relations are transitive, non-reflexive, and non-symmetric. Our purpose is to construct for each infinite cardinal $\aleph$ a so-called "$\aleph$-universal" partial order in which every partial order of cardinality $\aleph$ can be isomorphically embedded. Using the Axiom of Choice we easily construct an $\aleph$-universal partial order of cardinality $2^{\aleph}$, while for those infinite cardinals $\aleph$ for which $\aleph^{\aleph_0} = 2^{\aleph}$, the General Continuum Hypothesis enables us to construct an $\aleph$-universal partial order of cardinality $\aleph$.

TABLE OF CONTENTS

UNIVERSAL INFINITE PARTIAL ORDERS

1. Classes-fundamental concepts.

We abstract from experience the concept of the collection of a definite group of objects into a whole; this new entity, this collection into a whole, we then call a <u>class</u>. The objects are called the <u>elements</u> or <u>members</u> of the class; the class is said <u>to contain</u> or <u>to be composed</u> of its elements, while the elements are spoken of as <u>belonging to</u> or <u>being contained in</u> the class. To exhibit a class which contains the elements a, b, c (and possibly others) we write $\{ a, b, c, \cdots \}$; we denote the class of objects a which satisfy some property P by $\{ a : a \text{ satisfies } P \}$. The assertion that an object a belongs to a class A is written $a \varepsilon A$; the assertion that a does not belong to A is written $a \not\varepsilon A$.

Many paradoxes have been stated in connection with the concept of class; some of them arise from the use of the word "all" and from our intuitive feeling that, given an object a and a class A we should be able to assert exactly one of the statements: $a \varepsilon A$, $a \not\varepsilon A$. Perhaps the simplest such paradox is furnished by the following self-contradictory "class": "the class C, which is composed of all those classes which are not members of themselves". If we assume that $C \varepsilon C$ we immediately conclude that $C \not\varepsilon C$, and conversely.

As a result of the existence of such paradoxes, and in order to establish a firm foundation for the construction of a consistent theory, it has been found necessary to single out certain "well-behaved" classes which are then called <u>sets</u>, and to lay down various axioms governing the behavior of both sets and classes. We shall not attempt to give here any complete exposition of an axiomatic theory of classes; instead we shall simply mention those concepts, rules, and results which are pertinent to our subject.

1.1. We have under consideration a universe of classes and a universe of sets, with the understanding that every set is a class. A class which is not a set is called a proper class. Certain classes, such as the empty-or void-class Λ , the finite classes, and the denumerable classes, are immediately defined to be sets. In addition, many methods are defined for the construction of new classes, together with rules stipulating when the newly formed classes are to be sets.

1.2. A class A is called a subclass (subset) of a class (set) B if every element of A is also an element of B, in symbols $A \subseteq B$. For every class A, $\Lambda \subseteq A$; if $A \subseteq B$, the class of those elements of B which are not also in A is denoted by $B - A$. If $A \subseteq B$ and $B \subseteq A$ then A and B are identical, which we denote by $A = B$. The collection of all subclasses of a class A is again a class, denoted by $\Gamma(A)$; if A is a set, $\Gamma(A)$ is also a set.

As an example, the set $\{ x, y, z \}$ has the eight subsets:

$T_1 = \Lambda$ $\qquad$ $T_5 = \{ y, z \}$

$T_2 = \{ x \}$ $\qquad$ $T_6 = \{ z, x \}$

$T_3 = \{ y \}$ $\qquad$ $T_7 = \{ x, y \}$

$T_4 = \{ z \}$ $\qquad$ $T_8 = \{ x, y, z \}$.

Finally, classes having no elements in common are said to be disjoint.

1.3. Given any two objects a and b we may form the set $\{ a, \{ a, b \} \}$, which consists essentially of the two objects but has the property that within it the object a is distinguished as, say, "first", and the object b as "second". The set $\{ a, \{ a, b \} \}$ may thus be called the <u>ordered pair</u> containing a and b (in that order), and is usually denoted by $< a, b >$.

1.4. Given two classes A and B, the collection of all ordered pairs $< a, b >$ with $a \varepsilon A$ and $b \varepsilon B$ is again a class, called the <u>cross product</u> of A and B and denoted by $A \times B$. If A and B are both sets, then $A \times B$ is a set.

A simple example shows the reason for using the notation $A \times B$: Let $A = \{ 0, 1 \}$ (two elements) and $B = \{ x, y, z \}$ (three elements). Then $A \times B$ has the six (= 2×3) elements:

$< 0, x >$, $< 1, x >$,

$< 0, y >$, $< 1, y >$,

$< 0, z >$, $< 1, z >$.

1.5. A class M, all of whose elements are ordered pairs, is called a (<u>single-valued</u>) <u>mapping</u> if M contains no two pairs $< a, b >$ and $< a, c >$ with b distinct from c. M is said to be <u>bi-unique</u> if in addition it contains no two pairs $< a, b >$ and $< c, b >$ with a distinct from c. The collection of first (second) elements of the pairs of M is a class called the <u>domain</u> (<u>range</u>) of M and denoted by D(M) [R(M)]; M is said to <u>map</u> D(M) <u>onto</u> R(M). For an element $\alpha \varepsilon D(M)$ there is a unique element $\beta \varepsilon R(M)$ for which $<\alpha, \beta> \varepsilon M$;

this element β is called the image of α by M and is denoted by $M ' \alpha$, so that we have $< \alpha, M ' \alpha > \varepsilon M$. If the domain and range of a mapping M are sets, then M itself is a set and is called a function. The converse, M^c, of a bi-unique mapping M is the class of pairs of M taken with the first and second elements interchanged; if M is a function then M^c is a function. If M and N are mappings with $D(N) = R(M)$, then the composition MN of the mappings M and N is the class of ordered pairs $< a, c >$ for which there exists an element b with $< a, b > \varepsilon M$ and $< b, c > \varepsilon N$; MN is a mapping and if both M and N are functions, then MN is a function.

1.6. For no proper class A and set X is there a bi-unique mapping of A onto X. This has the immediate consequence that a mapping is a function if and only if its domain is a set.

1.7. Given two classes A and B, a mapping M for which $D(M) = A$ and $R(M) \subseteq B$ is said to map A into B; to express this we write $M:A \to B$. The collection of all such mappings is a class which we shall denote by B^A; if both A and B are sets then B^A is again a set.

Again, a simple example will show the reason for using the notation B^A: Let $A = \{ x, y, z \}$ (three elements) and $B = \{ 0, 1 \}$ (two elements). Then B^A has the eight ($= 2^3$) elements:

$$M_1 = \begin{Bmatrix} < x, 0 > , \\ < y, 0 > , \\ < z, 0 > \end{Bmatrix} , \qquad M_5 = \begin{Bmatrix} < x, 0 > , \\ < y, 1 > , \\ < z, 1 > \end{Bmatrix} ,$$

$$M_2 = \left\{\begin{matrix} <x, 1>, \\ <y, 0>, \\ <z, 0> \end{matrix}\right\}, \quad M_6 = \left\{\begin{matrix} <x, 1>, \\ <y, 0>, \\ <z, 1> \end{matrix}\right\},$$

$$M_3 = \left\{\begin{matrix} <x, 0>, \\ <y, 1>, \\ <z, 0> \end{matrix}\right\}, \quad M_7 = \left\{\begin{matrix} <x, 1>, \\ <y, 1>, \\ <z, 0> \end{matrix}\right\},$$

$$M_4 = \left\{\begin{matrix} <x, 0>, \\ <y, 0>, \\ <z, 1> \end{matrix}\right\}, \quad M_8 = \left\{\begin{matrix} <x, 1>, \\ <y, 1>, \\ <z, 1> \end{matrix}\right\}.$$

Suppose now that we form the eight subsets $T_i = \{ w\colon w \in A$ and $M_i ‘ w = 1 \}$ of A, $i = 1, 2, \ldots, 8$. Then $T_1, \ldots, T_8$ are exactly the subsets of the example in 1.2., showing the essential similarity between the classes $\Gamma(A)$ and $\{0, 1\}^A$.

1.8. Let M be a mapping with domain A such that for each $\alpha \in A$, $M ‘ \alpha$ is a class. The collection of elements x such that there exists an $\alpha \in A$ for which $x \in M ‘ \alpha$ is a class called the <u>class union of the</u> $M ‘ \alpha$ <u>over the index class</u> A and is denoted by $\bigcup \{ M ‘ \alpha \colon \alpha \in A \}$. The collection of elements x which belong to <u>all</u> the $M ‘ \alpha$ is a class called the <u>class intersection of the</u> $M ‘ \alpha$ <u>over the index class</u> A and is denoted by $\bigcap \{ M ‘ \alpha \colon \alpha \in A \}$. The union and intersection of two classes S and T are denoted more simply by $S \cup T$ and $S \cap T$. Finally, all of the above statements hold if the words " mapping" and "class" are <u>simultaneously</u> replaced by "function" and "set".

1.9. Finally, all of the results to be presented in this paper require the following assumption. Axiom of Choice: *Let* F *be a function with domain* X (which by 1.6. is a set) *such that for each* $\alpha \varepsilon X$, $F`\alpha$ *is a non-void set. Then there exists a function* $f: X \rightarrow \bigcup \{ F`\alpha : \alpha \varepsilon X \}$ *which has the property that for each* $\alpha \varepsilon X$, $f`\alpha \varepsilon F`\alpha$. The collection of all such functions f is a set which we denote by $\prod \{ F`\alpha : \alpha \varepsilon X \}$. It is easily seen that this (product) set is a generalization of the cross product defined in 1.4. . The Axiom of Choice can be stated also in the following equivalent form: *Given any set* X *there exists a function* $F: \Gamma(X) \rightarrow X$ *such that for each non-void subset* T *of* X, $F`T \varepsilon T$, *and* $F`\Lambda = F`X$.

2. Cardinals

Perhaps the simplest property possessed by a set is the "number" of elements belonging to it. For a given finite set this number may be ascertained (uniquely) by some counting process; for infinite sets the concept of number requires a new definition. This definition may be gained by looking more closely at what is meant by the statement "the finite sets X and Y have the same number of elements". For example it is evident that the sets $\{a, b, c\}$ and $\{\alpha, \beta, \gamma\}$ both have three elements. However, in this case the peculiar nature of the elements themselves leads us to a further observation, namely that we may associate α with a, β with b, and γ with c. In effect, these associations, expressed in terms of ordered pairs, furnish us with a <u>bi-unique</u> function $F = \{ < a, \alpha > , < b, \beta > , < c, \gamma > \}$ which maps $\{a, b, c\}$ <u>onto</u> $\{\alpha, \beta, \gamma\}$. It is easily seen that there are exactly $n!$ $(0! = 1)$ such functions between any two sets of n elements each and no such functions between sets of differing numbers of elements. Thus the notion of "the same number of elements" for finite sets is equivalent to the existence of bi-unique functions; extending this idea gives us the concept of "number" for infinite sets.

2.1. Two sets S and T are said to be <u>equivalent</u>, written $S \sim T$, if there exists a bi-unique function F which maps S onto T. The identity function $I = \{ < a, a > : a \varepsilon S \}$ shows that $S \sim S$ for each set S; the relation of equivalence is <u>reflexive</u>. If $S \sim T$ by a function F, the function F^c leads to the statement that $T \sim S$;

the relation of equivalence is symmetric. If $S \sim T$ by a function F and $T \sim U$ by a function G, then $S \sim U$ by the function FG; the relation of equivalence is transitive. Thus the relation of equivalence splits the universe of sets into disjoint collections within each of which any two sets are equivalent. We consider each collection to be an entity which we call a cardinal; an arbitrary cardinal is denoted by the symbol $\mathfrak{m}$. Each set S belongs to exactly one cardinal $\mathfrak{m}$; we say that S has cardinality $\mathfrak{m}$ or is of cardinality $\mathfrak{m}$ and write $|S| = \mathfrak{m}$. The finite cardinals are simply the non-negative integers n; the infinite cardinals are usually denoted by the symbol $\aleph$; $\aleph_0$ denotes the cardinality of the set of (non-negative) integers.

We next consider various operations between cardinals. For this purpose we shall need representative sets having given cardinalities, and it can be shown that we can construct for each cardinal $\mathfrak{m}$ a standard representative set $S(\mathfrak{m})$ of cardinality $\mathfrak{m}$ with the property that for distinct cardinals $\mathfrak{m}$ and $\mathfrak{n}$, $S(\mathfrak{m})$ and $S(\mathfrak{n})$ are disjoint.

2.2. For two cardinals $\mathfrak{m}$ and $\mathfrak{n}$ we say that " $\mathfrak{m}$ is less than or equal to $\mathfrak{n}$", in symbols $\mathfrak{m} \leq \mathfrak{n}$, if there exists a subset of $S(\mathfrak{n})$ which is equivalent to $S(\mathfrak{m})$. This definition is easily seen to agree with the usual one for non-negative integers m and n. We easily show that this relation is transitive by forming the composition of appropriate functions. F. Bernstein has shown that if $\mathfrak{m} \leq \mathfrak{n}$ and $\mathfrak{n} \leq \mathfrak{m}$, then $\mathfrak{m} = \mathfrak{n}$. We say that " $\mathfrak{m}$ is strictly less than $\mathfrak{n}$", in symbols $\mathfrak{m} < \mathfrak{n}$, if $\mathfrak{m} \leq \mathfrak{n}$ and $\mathfrak{m} \neq \mathfrak{n}$. By the Axiom of Choice (1.9.) it will result in 3.10. that for any two cardinals $\mathfrak{m}$ and $\mathfrak{n}$ we must have exactly one of the three relations $\mathfrak{m} < \mathfrak{n}$, $\mathfrak{m} = \mathfrak{n}$, or $\mathfrak{n} < \mathfrak{m}$.

2.3. If F is a function with domain X such that for each $\alpha \in X$, $F`\alpha$ is a cardinal, then we define the <u>sum</u> $\sum\{F`\alpha : \alpha \in X\}$ <u>of the cardinals</u> $F`\alpha$ <u>over the index set</u> $X$ to be $\sum\{F`\alpha : \alpha \in X\} = |\bigcup\{S(F`\alpha) : \alpha \in X\}|$. (The disjointness of the $S(F`\alpha)$ is of prime importance here.) For finite sums of non-negative integers this definition is seen to agree with the one for ordinary addition. It is clear that for each $\beta \in X$, $F`\beta \leq \sum\{F`\alpha : \alpha \in X\}$. The sum of two cardinals $\mathfrak{m}$ and $\mathfrak{n}$ is simply denoted by $\mathfrak{m} + \mathfrak{n}$.

2.4. If F is a function with domain X such that for each $\alpha \in X$, $F`\alpha$ is a cardinal, then we define the <u>product</u> $\prod\{F`\alpha : \alpha \in X\}$ <u>of the cardinals</u> $F`\alpha$ <u>over the index set</u> $X$ to be $\prod\{F`\alpha : \alpha \in X\} = |\prod\{S(F`\alpha) : \alpha \in X\}|$. For finite products of non-negative integers this definition is seen to agree with the one for ordinary multiplication (see 1.4.). It is clear that for each $\beta \in X$, $F`\beta \leq \prod\{F`\alpha : \alpha \in X\}$. The product of two cardinals $\mathfrak{m}$ and $\mathfrak{n}$ is simply denoted by $\mathfrak{m} \times \mathfrak{n}$.

2.5. For two cardinals $\mathfrak{m}$ and $\mathfrak{n}$ we define $\mathfrak{m}$ <u>to the power</u> $\mathfrak{n}$, in symbols $\mathfrak{m}^{\mathfrak{n}}$, to be $\mathfrak{m}^{\mathfrak{n}} = |S(\mathfrak{m})^{S(\mathfrak{n})}|$. For any set S, $2^{|S|}$ is clearly the cardinality of $\Gamma(S)$; it is easily shown by assuming the contrary that for every cardinal $\mathfrak{m}$, $\mathfrak{m} < 2^{\mathfrak{m}}$. From this we see that there is no "largest" cardinal. We shall make frequent use of the General Continuum Hypothesis: <u>If</u> $\aleph$ <u>is an</u>

<u>infinite cardinal there exists no cardinal</u> $\mathfrak{m}$ <u>for which</u> $\aleph < \mathfrak{m} < 2^{\aleph}$. This hypothesis has been shown by W. Sierpinksi [1] to imply the Axiom of Choice.

2.6. Some easily proved results concerning cardinals are the following:

(1) $(\mathfrak{m}_1^{\mathfrak{m}_2})^{\mathfrak{m}_3} = \mathfrak{m}_1^{\mathfrak{m}_2 \times \mathfrak{m}_3}$;

(2) if $\mathfrak{m}_1 \leq \mathfrak{m}_2$ and $\mathfrak{m}$ is arbitrary, then $\mathfrak{m}_1^{\mathfrak{m}} \leq \mathfrak{m}_2^{\mathfrak{m}}$ and $\mathfrak{m}^{\mathfrak{m}_1} \leq \mathfrak{m}^{\mathfrak{m}_2}$.

If X is an index set and $\mathfrak{m}(\alpha)$ is a unique cardinal associated with the element α of X, then:

(3) $\Sigma\{\mathfrak{m} : \alpha \varepsilon X\} = |X| \times \mathfrak{m}$;

(4) $\Pi\{\mathfrak{m} : \alpha \varepsilon X\} = \mathfrak{m}^{|X|}$;

(5) $\Pi\{\mathfrak{m}^{\mathfrak{m}(\alpha)} : \alpha \varepsilon X\} = \mathfrak{m}^{\Sigma\{\mathfrak{m}(\alpha) : \alpha \varepsilon X\}}$;

(6) $\Pi\{[\mathfrak{m}(\alpha)]^{\mathfrak{m}} : \alpha \varepsilon X\} = [\Pi\{\mathfrak{m}(\alpha) : \alpha \varepsilon X\}]^{\mathfrak{m}}$;

(7) if $\mathfrak{n}(\alpha)$ is another unique cardinal associated with α, and if $\mathfrak{m}(\alpha) \leq \mathfrak{n}(\alpha)$ for each $\alpha \varepsilon X$, then

$$\Sigma\{\mathfrak{m}(\alpha) : \alpha \varepsilon X\} \leq \Sigma\{\mathfrak{n}(\alpha) : \alpha \varepsilon X\} \text{ and}$$

$$\Pi\{\mathfrak{m}(\alpha) : \alpha \varepsilon X\} \leq \Pi\{\mathfrak{n}(\alpha) : \alpha \varepsilon X\} .$$

Let F be a function with domain Y such that for each $\beta \varepsilon Y$, $F`\beta$ is a set and the $F`\beta$ are pair wise disjoint;

define $X = \bigcup \{ F`\beta : \beta \in Y \}$. If for each $\alpha \in X$ $\mathfrak{m}(\alpha)$ is a unique cardinal, then:

(8) $$\Sigma \{ \Sigma \{ \mathfrak{m}(\alpha) : \alpha \in F`\beta \} : \beta \in Y \} = \Sigma \{ \mathfrak{m}(\alpha) : \alpha \in X \} ;$$

(9) $$\Pi \{ \Pi \{ \mathfrak{m}(\alpha) : \alpha \in F`\beta \} : \beta \in Y \} = \Pi \{ \mathfrak{m}(\alpha) : \alpha \in X \} ;$$

(10) $$\Pi \{ \Sigma \{ \mathfrak{m}(\alpha) : \alpha \in F`\beta \} : \beta \in Y \} = \Sigma \{ \Pi \{ \mathfrak{m}(\alpha) : \alpha \in R(f) \} : f \in X^Y \text{ and } f`\beta \in F`\beta \text{ for } \beta \in Y \} .$$

We next introduce some structure into our classes and develop some powerful tools of investigation and construction.

3. Partial orderings of classes.

Many classes arising in our experience and in mathematics exhibit relations between their elements, for example: "a is south of b", "a is below b", "a is the father of b", "a is a subset of b". Some of these relations are transitive: if a is south of b and b is south of c, then a is south of c. On the other hand, "a is the father of b" is not transitive. We also note that two objects need not be comparable by a given (transitive) relation: an iceberg is neither above nor below the surface of the body of water in which it floats. We abstract from these and other examples the notion of a (transitive) partial ordering of a class.

3.1. For a class A, a subclass B of $A \times A$ is called a <u>partial ordering</u> of A (or A is <u>partially ordered by</u> B) if it satisfies the conditions:

(1*) for no $a \varepsilon A$ is $< a, a > \varepsilon B$ (B is non-reflexive);

(2*) for no $a, b \varepsilon A$ with $a \neq b$ are both $< a, b >$ and $< b, a > \varepsilon B$ (B is non-symmetric);

(3*) if $< a, b >$ and $< b, c > \varepsilon B$, then $< a, c > \varepsilon B$ (B is transitive).

Note that B may be void. We usually write "$a < b$" (to be read "a is less than b") in place of $< a, b > \varepsilon B$; with this convention we say that the class A is partially ordered by the relation "<" if:

(1) for no $a \varepsilon A$ do we have $a < a$;

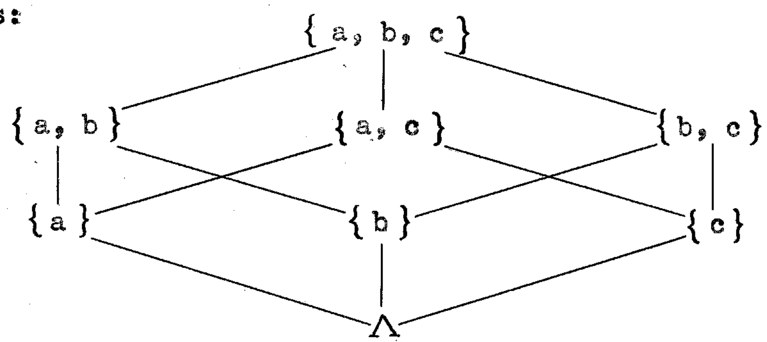
(2) For no $a, b \varepsilon A$ with $a \neq b$ do we have both $a < b$ and $b < a$;

(3) if $a < b$ and $b < c$ then $a < c$.

In place of $a < b$ we sometimes write $b > a$ ("b is greater than a"). Clearly, any subclass of a partially ordered class A is again partially ordered by the same relation; it is called a suborder of A. A set which is partially ordered is called (together with its relation "$<$") simply a *partial order*.

Elements a and b of A for which either $a < b$ or $b < a$ holds are said to be *comparable*, while if neither $a < b$ nor $b < a$ holds then they are said to be *non-comparable*; if $a < b$ then the pair $\langle a, b \rangle$ (ε B) is spoken of as an *order relation* or *comparability*, while if neither $a < b$ nor $b < a$ holds then both $\langle a, b \rangle$ and $\langle b, a \rangle$ ($\not\varepsilon$ B of course) are called *non-comparabilities*. An element m of A for which there is no $a \varepsilon A$ with $a > m$ is called a *maximal element* of A, and analogously for a *minimal element*. If $D \subseteq A$ and $b \varepsilon A$ such that for each $a \varepsilon D$ we have either $b > a$ or $b = a$, then b is called an *upper bound* (*in* A) for D, and analogously for a *lower bound*.

An example of a partial order is furnished by the eight subsets $T_1, \ldots, T_8$ of $\Gamma(\{a, b, c\})$ given in 1.2. when the ordering is by subset inclusion. This partial order can be exhibited graphically as follows:



3.2. A partially ordered class A is said to be <u>linearly ordered</u> by "<" (the class B of 3.1. is called a <u>linear ordering</u> of A) if in addition we have:

(4) for every two distinct elements a, b of A, either $a < b$ or $b < a$ holds.

Every subclass of a linearly ordered class is itself linearly ordered. A linearly ordered set is called simply a <u>linear order</u>. If a subset T of a partial order S is itself a linear order, then it is called a <u>chain</u> of S. If $a < b$ in a linearly ordered class then we say that "a precedes b" or that "b follows a". If F is a function with domain D such that for each $d \varepsilon D$, $F ` d$ is a non-void linear order, then $\prod \{ F ` d : d \varepsilon D \}$ can be made into a partial order which we shall denote by $P(\prod \{ F ` d : d \varepsilon D \} )$ by introducing the order relations: for $f_1, f_2 \varepsilon \prod \{ F ` d : d \varepsilon D \}$, we define $f_1 < f_2$ if and only if $f_1 ` d < f_2 ` d$ in $F ` d$ for every $d \varepsilon D$. If every $F ` d$ is a certain linear order L, then we denote the partial order $P(\prod \{ F ` d : d \varepsilon D \})$ more simply by $P(L^D)$.

3.3. A partially ordered class A is said to be <u>well-ordered</u> by "<" (the class B of 3.1. is called a <u>well-ordering</u> of A) if in addition we have:

(5) every non-void (partially ordered) subclass U of A has a minimal element.

Every subclass of a well-ordered class is itself well-ordered. A well-ordered class A is immediately seen to be linearly ordered by considering all subclasses having two elements; the minimal element of each non-void subclass U is thus unique and is called the "first element" of U. If ρ is an element of A, we define $\bar{\rho}$ to be the subclass $\bar{\rho} = \{ \pi : \pi \varepsilon A$ and $\pi < \rho$ in $A \}$ ($\bar{\rho}$ does <u>not</u> contain ρ). A <u>segment</u> of A is any subclass U of A which satisfies the condition: if $\sigma \varepsilon U$ and $\tau \varepsilon A$ such that $\tau < \sigma$ in A, then $\tau \varepsilon U$; if U is a segment not equal to A, then U is called a <u>proper segment</u> of A; every proper segment U of A is easily seen to equal $\bar{\rho}$ where ρ is the first element of $A - U$. A can have at most one maximal element; if it exists, this maximal element is called the <u>last element</u> of A. Every element ρ which is not a last element of A has a <u>successor</u> ρ_*, namely the first element of the subclass $\{ \pi : \pi > \rho$ in $A \}$. If ρ is an element for which $\bar{\rho}$ has a last element ρ^*, then ρ is the successor to ρ^* and ρ^* is called the <u>predecessor</u> of ρ . If an element ρ has no predecessor and is not the first element of A, then it is called a <u>limit element</u> of A; we write $\rho = \lim \{ \sigma : \sigma \varepsilon V \}$ if V is any subclass of $\bar{\rho}$ having the property that for each $\pi \varepsilon \bar{\rho}$ there exists a σ in V such that $\pi < \sigma$ ($\bar{\rho}$ itself is such a subclass). More generally, if V is any subclass of A, $\lim \{ \sigma : \sigma \varepsilon V \}$ is defined to be the first element of A not less than any element of V, if such an element exists.

We are now already in a position to prove the following strong form of transfinite induction.

3.4. Theorem: <u>Let A be a class and W a well-ordered class. Let there be mappings</u> $T : W \to \Gamma(W)$ <u>and</u> $R : \Gamma(A) \times W \to A$. <u>Then there exists a unique mapping</u> $F : W \to A$ <u>such that for each</u>

$\rho \in W$, $F`\rho = R` < F(\rho, T), \rho >$, <u>where</u> $F(\rho, T) = \{F`\mu : \mu \in \bar{\rho} \cap T`\rho\}$.

<u>Proof:</u>

1) Let C be the class of mappings $F(U):U \to A$ whose domains are segments U of W and which have the property that for each $\rho \in U$, $F(U)`\rho = R` < F(U)[\rho, T], \rho >$, where $F(U)[\rho, T] = \{F(U)`\mu : \mu \in \bar{\rho} \cap T`\rho\}$. Note that if U and V are segments of W, then either $U \subseteq V$ or $V \subseteq U$.

2) Let $F(U)$ and $F(V) \in C$ with $U \subseteq V$. Suppose there exists a $\rho \in U$ such that $F(U)`\rho \neq F(V)`\rho$. Let σ be the first such ρ. Then $F(U)`\mu = F(V)`\mu$ for all $\mu \in \bar{\sigma}$, so that $F(U)[\sigma, T] = F(V)[\sigma, T]$. But then

$$F(U)`\sigma = R` < F(U)[\sigma, T], \sigma > =$$

$$R` < F(V)[\sigma, T], \sigma > = F(V)`\sigma ,$$

a contradiction. Thus any two mappings in C agree on the smaller of their two domains.

3) Define $F = \bigcup\{F(U):F(U) \in C\}$. By 2), F is a mapping $F:D(F) \to A$. Since $D(F)$ is a union of segments of W, $D(F)$ is itself a segment of W. For $\rho \in D(F)$, there exists an $F(U) \in C$ such that $\rho \in U$ and $F`\mu = F(U)`\mu$ for all $\mu \in U$. Thus $F(\rho, T) = F(U)[\rho, T]$ and

$$F`\rho = F(U)`\rho = R` < F(U)[\rho, T], \rho > =$$

$$R` < F(\rho, T), \rho > .$$

Hence F has the desired property on the domain $D(F)$ and by 2), F is unique.

4) Suppose that $D(F) \neq W$. Let ρ be the first element of $W - D(F)$, so that $D(F) = \overline{\rho}$. We may then define a mapping $G: D(F) \cup \{\rho\} \to A$ as follows: for $\mu \varepsilon D(F)$, $G'\mu = F'\mu$, while $G'\rho = R' < F(\rho, T), \rho >$. But then we clearly would have $G \varepsilon C$, so that $D(G) \subseteq D(F)$ and $\rho \varepsilon D(F)$, a contradiction. Thus $D(F) = W$, which completes the proof of Theorem 3.4. •

Now, using the second form of the Axiom of Choice (1.9.), we obtain the following useful corollary to Theorem 3.4. •

3.5. Theorem: *Let* A *be a set and* W *a well-ordered class. Let there be mappings* $T: W \to \Gamma(W)$ *and* $R: \Gamma(A) \times W \to \Gamma(A)$. *For each* $\rho \varepsilon W$ *consider the following implication: if there is a mapping* $F(\rho): \overline{\rho} \to A$ *such that for each* $\mu \varepsilon \overline{\rho}$, $F(\rho)'\mu \ \varepsilon \ R' < F(\rho)[\mu, T], \mu >$, *then* $R' < F(\rho)[\rho, T], \rho >$ *is a non-void subset of* A, *where for* $\mu \varepsilon \overline{\rho} \cup \{\rho\}$, $F(\rho)[\mu, T] = \{ F(\rho)'\nu : \nu \varepsilon \overline{\mu} \cap T'\mu \}$. *If this implication holds for every* $\rho \varepsilon W$, *then there exists a (not necessarily unique) mapping* $F: W \to A$ *such that for each* $\rho \varepsilon W$, $F'\rho \ \varepsilon R' < F(\rho, T), \rho >$, *where* $F(\rho, T) = \{ F'\mu : \mu \varepsilon \overline{\rho} \cap T'\rho \}$.

Proof:

1) By the Axiom of Choice let there be a function $Z: \Gamma(A) \to A$ such that for non-void subsets B of A, $Z' B \varepsilon B$, while $Z'\Lambda = Z'A$. (The theorem is vacuously satisfied if A is void.) We define a mapping

$G:\Gamma(A) \times W \to A$ as follows: for $B \subseteq A$ and $\rho \in W$, $G ` < B, \rho >$ $= Z ` R ` < B, \rho >$. Then by Theorem 3.4. there exists a mapping $F:W \to A$ such that for each $\rho \in W$, $F`\rho = G ` < F(\rho, T), \rho >$, where $F(\rho, T) = \{ F`\mu : \mu \in \bar{\rho} \cap T`\rho \}$.

2) Suppose there exists a $\rho \in W$ such that $F`\rho \notin R ` < F(\rho, T), \rho >$; let σ be the first such ρ. Then for all $\mu \in \bar{\sigma}$, $F`\mu \in$ $R ` < F(\mu, T), \mu >$ so that the mapping $F \cap \bar{\sigma} \times A$ is a mapping $F(\sigma): \bar{\sigma} \to A$ such that for each $\mu \in \bar{\sigma}$, $F(\sigma)`\mu \in$ $R ` < F(\sigma) [\mu, T], \mu >$. By the hypothesis of the theorem then $R ` < F(\sigma) [\sigma, T], \sigma >$ is a non-void subset of A. But then

$$F`\sigma = G ` < F(\sigma, T), \sigma > = G ` < F(\sigma)[\sigma, T], \sigma >$$

$$= Z ` R ` < F(\sigma)[\sigma, T], \sigma > \in R ` < F(\sigma)[\sigma, T], \sigma >$$

$$= R ` < F(\sigma, T), \sigma >,$$

contrary to our assumption. Thus F satisfies the conclusion of Theorem 3.5. .

We note that Theorems 3.4. and 3.5. have several natural corollaries according as $D(R) = \Gamma(A)$ instead of $\Gamma(A) \times W$, or $T`\rho = \bar{\rho}$ for all $\rho \in W$, or both. In the remainder of this paper, when we refer to these theorems it is to be understood that we are using which ever corollary is applicable.

3.6. A bi-unique mapping M whose domain A and range B are partially ordered classes is said to be an <u>isomorphism between the partially ordered classes A and B</u> (in symbols $A \underset{M}{\cong} B$) if we have the condition: $x < y$ in A if and only if $M{}^{\backprime}x < M{}^{\backprime}y$ in B. Two partially ordered classes A and B are said to be <u>isomorphic</u> (<u>to one another</u>) if there exists an isomorphism between them; isomorphic linearly ordered classes are said to be <u>similar</u> (<u>to one another</u>). The relation of isomorphism is reflexive ($A \underset{I}{\simeq} A$ under the identity mapping $I{}^{\backprime}a = a$); symmetric (if $A \underset{M}{\simeq} B$ then $B \underset{M^c}{\simeq} A$), and transitive (if $A \underset{M}{\simeq} B$ and $B \underset{N}{\simeq} C$ then $A \underset{MN}{\simeq} C$). Thus the universe of partially ordered classes is split into disjoint collections of isomorphic partially ordered classes. As we did in 2.1. for collections of equivalent sets, we may think of each such collection as an entity which we call an <u>order type</u> and assume that for each order type we have a particular abstract representative.

3.7. The order types possessed by well-ordered <u>sets</u> are called <u>ordinals</u>; we shall denote ordinals by small Greek letters; a well-ordered set R which belongs to an ordinal ρ is said to have <u>ordinality</u> ρ. The collection of ordinals is taken to be a class which we denote by Ord. We introduce order relations into Ord as follows: Let ρ, σ be ordinals and R, S be the abstract representatives of ordinalities ρ, σ respectively. We say that " ρ is less than σ", in symbols $\rho < \sigma$ or $\sigma > \rho$, if R is similar to some <u>proper</u> segment of S. To show that these order relations constitute a partial ordering of Ord, it is sufficient to show that for any two well-ordered sets V and W there can exist at most one function $f: V \to W$ whose range is a segment of W and which is an isomorphism between V and $R(f)$. Suppose by way of contradiction that f and g are two distinct such isomorphisms and let π be the first element of V such that $f‘\pi \neq g‘\pi$; we may assume that $f‘\pi > g‘\pi$ in W. Since R(f) is a segment of W and $f‘\pi > g‘\pi$ in W, we may define $\nu = f^c‘g‘\pi$; then $\pi = f^c‘f‘\pi > f^c‘g‘\pi = \nu$ in V, since f is an isomorphism. But now $\pi > \nu$ in V implies that $g‘\pi > g‘\nu$ in W, so that $f‘\nu = g‘\pi > g‘\nu$ in W, contrary to the minimality of π; thus $f = g$ and we have a partial ordering of Ord.

3.7.1. Next we show that Ord is actually linearly ordered. Let S, T be the representative well-ordered sets for arbitrary ordinals σ, τ respectively. Define $Z:\Gamma(S) \to S$ to be the function such that for $U \subseteq S$, $U \neq S$, $Z‘U$ = first element of $S - U$, and $Z‘S = Z‘\Lambda$. Then by Theorem 3.4. there exists a unique function $F:T \to S$ such that for each $\rho \in T$, $F‘\rho = Z‘\{F‘\mu : \mu \in \bar{\rho}\}$. We easily verify the following three statements: If F is bi-unique and $R(F) \neq S$, then F is an isomorphism between T and the proper segment R(F) of S, so that $\tau < \sigma$. If F is bi-unique and $R(F) = S$, then F is an isomorphism between T and S, so that $\tau = \sigma$. If F is not bi-unique, let ρ be the first element of T for which $F‘\rho = F‘\mu$ for some $\mu < \rho$; then $F \cap \bar{\rho} \times S$ is bi-unique and $R(F \cap \bar{\rho} \times S) = S$; $F \cap \bar{\rho} \times S$ is then an isomorphism between S and the proper segment $\bar{\rho}$ of T, so that $\sigma < \tau$. Thus Ord is linearly ordered.

3.7.2. We see from 3.7.1. that for the representative well-ordered set R of ordinality ρ there is a bi-unique correspondence between the set of proper segments of R and the collection of ordinals $\mu < \rho$, namely $\bar{\rho}$. Since there is also a bi-unique correspondence between the set of proper segments of R and the set of elements of R (3.3.), we see that $\bar{\rho}$ is actually similar to R. Thus, $\bar{\rho}$ is a set (1.6.) and is well-ordered with ordinality ρ . We take $\bar{\rho}$ as the standard well-ordered set of ordinality ρ . At the same time we also see from the above that Ord is well-ordered.

3.7.3. Suppose that Ord were a set. Being well-ordered it would have some ordinality Ω , from which it would follow by 3.7.2. that Ord = $\bar{\Omega}$. But this leads to the contradiction $\Omega \varepsilon \bar{\Omega}$, since Ω is assumed to be in Ord. Thus, Ord is a well-ordered proper class.

3.7.4. We introduce addition and multiplication into Ord as follows. Let ρ , σ ε Ord. $\rho + \sigma$ is defined as the ordinality of the (clearly well-ordered) set $\bar{\rho} \cup \bar{\sigma}$, where the order relations of $\bar{\rho}$ and $\bar{\sigma}$ are retained and it is understood that every element of $\bar{\rho}$ precedes every element of $\bar{\sigma}$. To define $\rho \times \sigma$: for $< \alpha, \beta >$ and $< \mu , \nu > \varepsilon$ $\bar{\rho} \times \bar{\sigma}$, we define $< \alpha, \beta > < < \mu , \nu >$ if and only if $\beta < \nu$, or, $\beta = \nu$ and $\alpha < \mu$; these order relations are easily seen to constitute a well-ordering of $\bar{\rho} \times \bar{\sigma}$, and we define $\rho \times \sigma$ to be the ordinality of this set $\bar{\rho} \times \bar{\sigma}$.

3.8. The Axiom of Choice is equivalent to the Well Ordering Theorem: <u>For every set</u> S <u>there exists a well-ordering of</u> S. Clearly, we may pick out the first element of each non-void subset of a well-ordered set, so that the Well Ordering Theorem implies the (second form of the) Axiom of Choice (1.9.). To prove the converse: Let A be a set and $Z:\Gamma(A) \to A$ a function such that for each subset T of A, T

$\neq A$, $Z`T \varepsilon A - T$, and $Z`A = Z`\Lambda$ . By Theorem 3.4. there exists a unique mapping $F: Ord \to A$ such that for each $\rho \varepsilon Ord$, $F`\rho = Z`\{ F`\mu : \mu \varepsilon \bar{\rho} \}$. F cannot be bi-unique, by 1.6.. Let π be the first non-zero ordinal for which $F`\pi = F`0$; then it is easily shown that $F \cap \bar{\pi} \times A$ is a bi-unique function which maps $\bar{\pi}$ onto A and thus furnishes A with a well-ordering of ordinality π.

3.9. The Axiom of Choice is equivalent to the Maximal Principle: Every partial order P has a maximal chain C (meaning that no element of P - C is comparable with every element of C); thus if every chain of P has an upper bound in P then P has a maximal element. First, let A be a set and B the set of well-ordered subsets of A (a given subset of A may appear in B with many well-orderings). B is partially ordered by the relations: for C, D ε B, C < D in B, if and only if C is a proper segment of D. Application of the Maximal Principle to B yields a well-ordering of A. To prove the converse: Let P be a partial order with an imposed well-ordering $P = \{ p(\mu): \mu \varepsilon \bar{\rho}$ for some $\rho \varepsilon Ord \}$ (this well-ordering has no necessary connection with the partial ordering of P). Then, taking both A and W in Theorem 3.4. to be the well-ordered set P, the range of the function F is a maximal chain of P provided that we define the function $R: \Gamma(P) \times P \to P$ as follows:

for $B \subseteq P$ and for $p \varepsilon P$,

$$R`\langle B, p \rangle = \begin{cases} p & \text{if } B \cup \{p\} \text{ is a chain of } P, \\ & \text{and} \\ p(0) & \text{otherwise} \end{cases}$$

3.10. By the Well Ordering Theorem there is for every cardinal $\mathfrak{m}$ the non-void set of ordinals which are the ordinalities of well-ordered sets of cardinality $\mathfrak{m}$; if ρ is such an ordinal we have $|\bar{\rho}| = \mathfrak{m}$ and write simply $|\rho| = \mathfrak{m}$. The first ordinal so associated with $\mathfrak{m}$ is called the <u>initial</u> <u>ordinal</u> <u>of</u> $\mathfrak{m}$. The association of each infinite cardinal $\aleph$ with its initial ordinal ω easily yields the result that the collection of infinite cardinals is a proper class well-ordered by magnitude and actually similar to Ord. Using the isomorphism which provides this similarity, we may index the infinite cardinals as $\aleph_\alpha$ ($\alpha \varepsilon$ Ord) and denote by ω_α the initial ordinal of $\aleph_\alpha$. It is evident that each ω_α must be a limit ordinal (3.3.). If V is any set of cardinals $\mathfrak{m}$, then $\lim \{\mathfrak{m} : \mathfrak{m} \varepsilon V\}$ is defined to be the smallest cardinal $\geq \mathfrak{m}$ for all $\mathfrak{m} \varepsilon V$; if λ is a limit ordinal or zero, $\aleph_\lambda = \lim \{\mathfrak{m} : \mathfrak{m} < \aleph_\lambda\}$ and $\aleph_\lambda$ is called a <u>limit</u> <u>cardinal</u>. For any set S of cardinality $\aleph_\alpha$ ($\alpha \varepsilon$ Ord) we may assume that there is a well-ordering of S under which we may write $S = \{s(\mu) : \mu \varepsilon \overline{\omega_\alpha}\}$. Finally, the General Continuum Hypothesis now states that for $\alpha, \beta \varepsilon$ Ord with $\alpha > \beta$, $\aleph_\alpha \geq 2^{\aleph_\beta}$, equality holding if and only if $\alpha = \beta + 1$.

3.11. We have the following facts concerning cardinals and ordinals. If V is a set of cardinals, some of which are infinite, then $\sum\{\mathfrak{m} : \mathfrak{m} \varepsilon V\} = \lim \{\mathfrak{m} : \mathfrak{m} \varepsilon V\}$. Let $\alpha \varepsilon$ Ord; we consider functions f having some domain B such that for each $x \varepsilon B$, $f\text{‘}x$ is a cardinal $< \aleph_\alpha$, and such that $\aleph_\alpha = \sum\{f\text{‘}x : x \varepsilon B\}$. Among these functions f there will be one of minimal cardinality; we denote this minimal cardinal by $\mathfrak{m}(\alpha)$ and notice that $\aleph_0 \leq \mathfrak{m}(\alpha) \leq \aleph_\alpha$. If $\mathfrak{m}(\alpha) < \aleph_\alpha$ we call $\aleph_\alpha$ <u>singular</u>; if $\mathfrak{m}(\alpha) = \aleph_\alpha$ we call $\aleph_\alpha$ <u>regular</u>; every non-limit cardinal is regular; a regular limit cardinal is called <u>inaccessible</u>. For $\alpha \in \mathrm{Ord}$ and $\mathfrak{m} \leq \aleph_\alpha$, $\mathfrak{m} \times \aleph_\alpha = \aleph_\alpha$. By the General Continuum Hypothesis, we have:

$$\aleph_\alpha^{\mathfrak{m}} = \begin{cases} 1 & 0 = \mathfrak{m}, \\ \aleph_\alpha & 0 < \mathfrak{m} < \mathfrak{m}(\alpha), \\ \aleph_{\alpha+1} & \mathfrak{m}(\alpha) \leq \mathfrak{m} \leq \aleph_\alpha, \text{ and} \\ 2^{\mathfrak{m}} & \aleph_\alpha \leq \mathfrak{m}. \end{cases}$$

<u>If</u> λ <u>is a limit ordinal and</u> B <u>is a set of ordinals</u> $\beta < \lambda$ <u>such that</u> $\lambda = \lim \{\beta : \beta \in B\}$, <u>then</u> $\omega_\lambda = \lim \{\omega_\beta : \beta \in B\} = \lim \{\omega_{\beta+1} : \beta \in B\}$. Among such sets B there will be one of minimal cardinality; this minimal cardinal is in fact the $\mathfrak{m}(\lambda)$ of the preceding paragraph. <u>If</u> $\mu, \nu \in \mathrm{Ord}$ <u>and</u> $\mu > 0$, <u>then there exist unique ordinals</u> $\Theta(\mu,\nu)$ <u>and</u> $\Phi(\mu,\nu)$ <u>such that</u> $\Theta(\mu,\nu) \leq \nu$, $\Phi(\mu,\nu) < \mu$, <u>and</u> ν <u>has the representation</u> $\nu = \mu \times \Theta(\mu,\nu) + \Phi(\mu,\nu)$.

4. Linear orders and linear extensions of partial orders.

The investigations reported in this thesis are the results of an attempt to generalize the well known fact that every denumerable linear order is similar to some set of rational numbers. In particular, we generalize the following proof of that fact: Let L be a denumerable linear order with its elements numbered $a_o, a_1, \dots, a_n, \dots$. We map a_o onto the dyadic rational $r_o = 1 = 1/2^o$. Suppose we have mapped $a_o, \dots, a_n$ onto dyadic rationals $r_o, \dots, r_n$ in such a way that the linear order $\{ r_o, \dots, r_n \}$ is similar to the linear order $\{ a_o, \dots, a_n \}$. We then map a_{n+1} onto the smallest dyadic rational r_{n+1} of the form $k/2^{n+1}$, k odd and $0 < r_{n+1} < 2$, which yields $\{r_o, \dots, r_n, r_{n+1}\}$ similar to $\{a_o, \dots, a_n, a_{n+1}\}$. Continuing by induction gives us the desired set of (dyadic) rationals.

The above proof has essentially two parts, namely, the existence of appropriate minimal dyadic rationals and an induction to obtain an isomorphism. In the following discussion we construct for each infinite cardinal $\aleph_\alpha$ a set of "dyadic rationals", show the existence of the necessary minimal rationals, and present an inductive process in a form which is of use for further work.

4.1. Definition: Let $\alpha \varepsilon$ Ord. If the functions $f: \overline{\omega_\alpha} \to \overline{2}$ are ordered according to first differences, the set $\overline{2}^{\overline{\omega_\alpha}}$ becomes a linear order which we shall denote by $L(\overline{2}^{\overline{\omega_\alpha}})$. We shall call a function $f: \overline{\omega_\alpha} \to \overline{2}$ a rational if there exists an ordinal $\eta(f) \varepsilon \overline{\omega_\alpha}$, necessarily unique, such that $f ‘ \eta(f) = 1$ while $f ‘ \mu = 0$ for $\eta(f) < \mu < \omega_\alpha$. Throughout further discussions, η will be used exclusively to denote this "terminating ordinal" of the rationals. We shall denote by L_α the suborder of $L(\bar{2}^{\overline{\omega_\alpha}})$ consisting of the rationals.

Regarding these linear orders L_α we prove the following theorem concerning the existence of minimal elements.

4.2. Theorem: Let $\alpha \varepsilon$ Ord. Then L_α is a linear order of cardinality $\Sigma \{2^{|\eta|}: \eta \varepsilon \overline{\omega_\alpha}\}$, so that $\aleph_\alpha \leq |L_\alpha| \leq 2^{\aleph_\alpha}$. Let ρ be any ordinal ε $\overline{\omega_\alpha}$ and let A be any subset of L_α such that for each $f \varepsilon A$, $\eta(f) < \rho$. Then there exists a (unique) minimal element $h(A, \rho)$ in L_α such that $\eta[h(A, \rho)] = \rho$ and $h(A, \rho) > f$ in L_α for every $f \varepsilon A$. Define $B = \{g: g \varepsilon L_\alpha, \ \eta(g) < \rho$, and $g > f$ in L_α for every $f \varepsilon A\}$. Then for every $g \varepsilon B$, $g > h(A, \rho)$ in L_α. Assuming the General Continuum Hypothesis, L_α has cardinality $\aleph_\alpha$.

Proof:

1) $L_\alpha = \bigcup \{T(\eta): \eta \varepsilon \overline{\omega_\alpha}\}$ where $T(\eta) = \{f: f \varepsilon L_\alpha$ and $\eta(f) = \eta\}$, and the $T(\eta)$ are disjoint. But $|T(\eta)| = 2^{|\eta|} \leq 2^{\aleph_\alpha}$ so that $\aleph_\alpha \leq \Sigma\{|T(\eta)|: \eta \varepsilon \overline{\omega_\alpha}\} = |L_\alpha| \leq \Sigma\{2^{\aleph_\alpha}: \eta \varepsilon \overline{\omega_\alpha}\} = \aleph_\alpha \times 2^{\aleph_\alpha} = 2^{\aleph_\alpha}$. Assuming the General Continuum Hypothesis, $|T(\eta)| \leq \aleph_\alpha$, so that $\aleph_\alpha \leq |L_\alpha| = \Sigma\{|T(\eta)|: \eta \varepsilon \overline{\omega_\alpha}\}, \leq \Sigma\{\aleph_\alpha: \eta \varepsilon \overline{\omega_\alpha}\} = \aleph_\alpha^2 = \aleph_\alpha$.

2) If A is void, the theorem is clearly satisfied by the rational $h(A, \rho)$ defined as follows: $h(A, \rho)`\rho = 1$ while $h(A, \rho)`\mu = 0$ for $\mu \varepsilon \overline{\omega_\alpha}$, $\mu \neq \rho$. In the remainder of the proof, A is assumed to be non-void.

3) We define a function $R:\Gamma[\Gamma(A)] \times \bar{\rho} \to \Gamma(A)$ as follows:

for $\mu \varepsilon \bar{\rho}$ and for the void collection of subsets of A, that is for $\Lambda \varepsilon \Gamma[\Gamma(A)]$, $R ‘ < \Lambda, \mu > = \{ f : f \varepsilon A$ and $f ‘ 0 = \max \{ k ‘ 0 : k \varepsilon A \} \}$;

for $\mu \varepsilon \bar{\rho}$ and for $Y \varepsilon \Gamma[\Gamma(A)]$ such that $Y \neq \Lambda$, we put

$Y(*) = \bigcap \{ c : c \varepsilon Y \}$ and

$R ‘ < Y, \mu > = \{ f : f \varepsilon A$ and $f ‘ \mu = \max \{ k ‘ \mu : k \varepsilon Y(*) \} \}$;

if $Y(*)$ is void, then so is $R ‘ < Y, \mu >$.

Then by Theorem 3.4. there exists a unique function $F : \bar{\rho} \to \Gamma(A)$ such that for each $\mu \varepsilon \bar{\rho}$, $F ‘ \mu = R ‘ < F(\bar{\mu}), \mu >$, where $F(\bar{\mu}) = \{ F ‘ \nu : \nu \varepsilon \bar{\mu} \}$. This function F is seen to have the following properties:

a) For $\mu \leq \nu < \rho$, $F ‘ \mu \supseteq F ‘ \nu$, and the set $\{ \mu : \mu \varepsilon \bar{\rho}$ and $F ‘ \mu$ is non-void $\}$ is a segment of $\bar{\rho}$.

b) For $\nu \varepsilon \bar{\rho}$ with $f_1, f_2 \varepsilon F ‘ \nu$, we have $f_1, f_2 \varepsilon F ‘ \mu$ and $f_1 ‘ \mu = f_2 ‘ \mu$ for all $\mu \leq \nu$.

c) If $\nu \varepsilon \bar{\rho}$, $f_1 \varepsilon F ‘ \nu$, and $f_2 \varepsilon A$ such that $f_2 ‘ \mu = f_1 ‘ \mu$ for all $\mu \leq \nu$, then $f_2 \varepsilon F ‘ \mu$ for all $\mu \leq \nu$.

d) For any $\nu \varepsilon \bar{\rho}$, $F ‘ \nu$ is non-void if and only if $\nu = 0$, or, $\nu > 0$ and there exists some $f \varepsilon A$ such that $f \varepsilon F ‘ \mu$ for all $\mu < \nu$.

e) For $\sigma \varepsilon \bar{\rho}$ such that there is an $f_1 \varepsilon F ‘ \sigma$, and for every $\nu \leq \sigma$, there exists no $f_2 \varepsilon A$ such that $f_2 ‘ \mu = f_1 ‘ \mu$ for all $\mu < \nu$ while $f_2 ‘ \nu > f_1 ‘ \nu$.

4) We define the rational $h(A, \rho)$ of L_α as follows:

a) For $\mu \varepsilon \bar{\rho}$ such that $F ‘ \mu$ is void, $h(A, \rho) ‘ \mu = 0$.

b) For $\mu \varepsilon \bar{\rho}$ such that $F`\mu$ is non-void, $h(A,\rho)`\mu = f`\mu$ for any $f \varepsilon F`\mu$, by 3b).

c) $h(A,\rho)`\rho = 1$ and $h(A,\rho)`\mu = 0$ for $\rho < \mu < \omega_\alpha$.

By 3a), we have thus defined $h(A,\rho)$ over the whole domain $\overline{\omega_\alpha}$. Clearly $h(A,\rho) \varepsilon L_\alpha$ and $\eta[h(A,\rho)] = \rho$.

5) Suppose there is a $\nu \varepsilon \bar{\rho}$ and an $f \varepsilon A$ such that $f`\mu = h(A,\rho)`\mu$ for all $\mu < \nu$. Then we can show that $f \varepsilon F`\mu$ for all $\mu < \nu$. For suppose the contrary and let $\tau$ be the first ordinal $< \nu$ such that $f \notin F`\tau$. Then either $\tau = 0$ or $f \varepsilon F`\mu$ for all $\mu < \tau$, so that by 3d), there exists some $k \varepsilon F`\tau$. By 3b), $k`\mu = f`\mu = h(A,\rho)`\mu$ for all $\mu < \tau$, while by the hypothesis on $f$ and by 4), $f`\tau = h(A,\rho)`\tau = k`\tau$. From this, 3c) yields $f \varepsilon F`\tau$ a contradiction. Thus indeed $f \varepsilon F`\mu$ for all $\mu < \nu$.

6) Suppose that f is an element of A such that $f \varepsilon F`\mu$ for all $\mu \varepsilon \bar{\rho}$. We shall show that $h(A,\rho) > f$ in $L_\alpha$. By 4), $h(A,\rho)`\mu = f`\mu$ for all $\mu < \rho$, while $h(A,\rho)`\rho = 1 > 0 = f`\rho$, since $\eta(f) < \rho$. Thus $h(A,\rho) > f$ in L_α.

7) Suppose that $f \varepsilon A$ such that for some $\mu \varepsilon \bar{\rho}$, $f \notin F`\mu$. We shall show that in this case also, $h(A,\rho) > f$ in $L_\alpha$. Let $\sigma$ be the first ordinal $\varepsilon \bar{\rho}$ such that $f \notin F`\sigma$. Then either $\sigma = 0$ or $f \varepsilon F`\mu$ for all $\mu < \sigma$, so that by 3d), there exists some $k \varepsilon F`\sigma$. By 3b) and 4), $k \varepsilon F`\mu$ for all $\mu \leq \sigma$, $k`\mu = f`\mu = h(A,\rho)`\mu$ for all $\mu < \sigma$, and $k`\sigma = h(A,\rho)`\sigma$. If $f`\sigma = k`\sigma$ then by 3c) $f \varepsilon F`\sigma$, a contradiction. By 3e), $f`\sigma \not> k`\sigma$. Thus $f`\sigma < k`\sigma = h(A,\rho)`\sigma$ and we have $h(A,\rho) > f$ in L_α.

8) Let t be contained in L_α such that $\eta(t) = \rho$ and $t < h(A, \rho)$ in L_α. We shall show that there exists an element f of A such that $f > t$ in L_α. Since $\eta(t) = \eta[h(A, \rho)] = \rho$, t and $h(A, \rho)$ must differ at some first ordinal $\sigma < \rho$. This requires that $t‘\sigma = 0 < 1 = h(A, \rho)‘\sigma$, so that by 4), $F‘\sigma$ is non-void. Let $f \varepsilon F‘\sigma$. By 3b), $f \varepsilon F‘\mu$ for all $\mu < \sigma$ and by 4), $h(A, \rho)‘\mu = f‘\mu$ for all $\mu \leq \sigma$, while $t‘\mu = h(A, \rho)‘\mu = f‘\mu$ for all $\mu < \sigma$ and $t‘\sigma < h(A, \rho)‘\sigma = f‘\sigma$. Thus $f > t$ in L_α. Steps 6), 7), and 8) assure that $h(A, \rho)$ is the minimal element k of L_α having the properties that $\eta(k) = \rho$ and $k > f$ in L_α for all $f \varepsilon A$. We need only show that for each $g \varepsilon B$, $g > h(A, \rho)$ in L_α.

9) Suppose that there exists some $k \varepsilon A$ such that $k \varepsilon F‘\mu$ for all $\mu \varepsilon \bar{\rho}$; let $g \varepsilon B$. We shall show that in this case $g > h(A, \rho)$ in L_α. By 4), $h(A, \rho)‘\mu = k‘\mu$ for all $\mu \varepsilon \bar{\rho}$. Since $g > k$ in L_α and since $\eta(g)$, $\eta(k) < \rho$, g and k must differ at some first ordinal $\sigma < \rho$, and $g‘\sigma > k‘\sigma$. But then $g‘\mu = k‘\mu = h(A, \rho)‘\mu$ for all $\mu < \sigma$, while $g‘\sigma > k‘\sigma = h(A, \rho)‘\sigma$, so that $g > h(A, \rho)$ in L_α. In the remainder of the proof we assume that there exists no $k \varepsilon A$ such that $k \varepsilon F‘\mu$ for all $\mu \varepsilon \bar{\rho}$.

10) In accordance with the assumption we have made in 9), let π be the first ordinal $\leq \rho$ such that there exists no $f \varepsilon A$ for which $f \varepsilon F‘\mu$ for all $\mu < \pi$, $\pi > 0$. We shall show that π must be a limit ordinal. For suppose that π had a predecessor π^*. By the minimal property π, there exists a $k \varepsilon A$ such that $k \varepsilon F‘\mu$ for all $\mu < \pi^*$, or $\pi^* = 0$, so that by 3d), there exists some $f \varepsilon F‘\pi^*$.

Then by 3b), $f \varepsilon F`\mu$ for all $\mu < \pi$, contrary to the assumed property of π. Thus π must be a limit ordinal; the meaning attached to π in this step will be retained throughout the following steps.

11) There exists no ordinal $\tau < \pi$ such that $h(A, \rho)`\tau = 1$ while $h(A, \rho)`\mu = 0$ for $\tau < \mu < \pi$. For suppose the contrary. Since π is a limit ordinal, $\tau + 1 < \pi$ so that by the minimal property of π, there exists some $k \varepsilon A$ such that $k \varepsilon F`\mu$ for all $\mu < \tau + 1$. By 4), then, $h(A, \rho)`\mu = k`\mu$ for all $\mu \leq \tau$. If $k`\mu = 0$ for all μ such that $\tau < \mu < \pi$, then $k`\mu = h(A, \rho)`\mu$ for all $\mu \varepsilon \overline{\pi}$ so that by 5), $k \varepsilon F`\mu$ for all $\mu \varepsilon \overline{\pi}$, a contradiction. Thus suppose $\delta$ to be the first ordinal such that $\tau < \delta < \pi$ and $k`\delta = 1$. Then $k`\mu = h(A, \rho)`\mu$ for all $\mu < \delta$ while $k`\delta = 1 > 0 = h(A, \rho)`\delta$ so that $k > h(A, \rho)$ in L_α, contrary to 6) and 7).

12) Suppose that $\pi = \rho$; let $g \varepsilon B$. We shall show that $g > h(A, \rho)$ in L_α. Since $\eta(g) < \rho = \eta[h(A, \rho)]$, g and $h(A, \rho)$ must differ at some first ordinal $\delta \leq \rho$. If $\delta = \rho$, then $h(A, \rho)`\eta(g) = g`\eta(g) = 1$ while $h(A, \rho)`\mu = g`\mu = 0$ for $\eta(g) < \mu < \rho = \pi$, contrary to 11). Thus $\delta < \rho = \pi$. Since π is a limit ordinal, $\delta + 1 < \pi$ so that by the minimal property of π there exists some $k \varepsilon A$ such that $k \varepsilon F`\mu$ for all $\mu < \delta + 1$. By 4), then, $k`\mu = h(A, \rho)`\mu = g`\mu$ for all $\mu < \delta$ while $k`\delta = h(A, \rho)`\delta \neq g`\delta$. If $g`\delta < h(A, \rho)`\delta$ then $g < k$ in $L_\alpha$, contrary to the definition of B. Thus $g`\delta > h(A, \rho)`\delta$ and we have $g > h(A, \rho)$ in L_α.

13) Suppose finally that $\pi < \rho$; let $g \varepsilon B$. We shall show that in this last case also, $g > h(A, \rho)$ in L_α. By 10) and 3d), $F`\pi$ is void,

so that by 3a) and 4), $h(A,\rho)`\mu = 0$ for $\pi \leq \mu < \rho$. As in 12), $g$ and $h(A,\rho)$ must differ at some first ordinal $\delta \leq \rho$. If $\delta = \rho$, then we must have $\eta(g) < \pi$. But then $h(A,\rho)`\eta(g) = 1 = g`\eta(g)$ while $h(A,\rho)`\mu = g`\mu = 0$ for $\eta(g) < \mu < \pi$, contrary to 11). If $\pi \leq \delta < \rho$, then necessarily $g`\delta = 1 > 0 = h(A,\rho)`\delta$ while $g`\mu = h(A,\rho)`\mu$ for all $\mu < \delta$, so that $g > h(A,\rho)$ in $L_\alpha$. Finally, if $\delta < \pi$, $\delta + 1 < \pi$ since $\pi$ is a limit ordinal, so that by the minimal property of $\pi$, there exists some $k \in A$ such that $k \in F`\mu$ for all $\mu < \delta + 1$. Then by 4), $g`\mu = h(A,\rho)`\mu = k`\mu$ for all $\mu < \delta$ while $k`\delta = h(A,\rho)`\delta \neq g`\delta$. If $g`\delta < h(A,\rho)`\delta$ then $g < k$ in L_α, contrary to the definition of B. Thus $g`\delta >$ $h(A,\rho)`\delta$ and we have $g > h(A,\rho)$ in L_α. This completes the proof of Theorem 4.2. .

4.3. If P is a partial order and Q a linear order on the same set of elements, then Q is called a <u>linear</u> <u>extension</u> of P if every order relation of P is also an order relation of Q. If P is a partial order and L a linear order we say that a bi-unique function $F: P \to L$ generates a linear extension of P if $a < b$ in P implies that $F`a < F`b$ in L; the linear extension generated is the set of elements of P together with the order relations transmitted from L to P by means of the converse function F^c. If P is already a linear order, F constitutes an isomorphism between P and $R(F) \subseteq L$; we say that F <u>embeds</u> P <u>in</u> L, or that F <u>is</u> <u>an</u> <u>embedding</u> <u>of</u> P <u>in</u> L.

We now use Theorem 4.2. to prove a theorem concerning linear extensions of partial orders; the theorem is stated and proved in the following form for use in 5.10. .

4.4. Theorem: Let $\alpha \varepsilon$ Ord. Let P be any partial order of cardinality $\aleph_\alpha$ with an imposed well-ordering $P = \{ p(\mu) : \mu \varepsilon \bar{\omega}_\alpha \}$. For $\nu < \omega_\alpha$ denote by P_ν the set $\{ p(\mu) : \mu \varepsilon \bar{\nu} \}$. Consider any two ordinals ρ and σ such that $0 \leq \rho < \sigma \leq \omega_\alpha$; define $W = P_\sigma - P_\rho$ and for $\rho \leq \nu \leq \sigma$, $W_\nu = P_\nu - P_\rho$. We define a function $H : P \to \Gamma(P)$ as follows: for $p \varepsilon P$, $H‘p = \{ x \;\; x < p \text{ in } P \}$. If there is a function $G : P_\rho \to L_\alpha$ which generates a linear extension of P_ρ in such a manner that for each $p(\mu) \varepsilon P_\rho$, $\eta[G‘p(\mu)] = \mu$, then there exists a unique function extending G to the domain P_σ, $F(G,\sigma) : W \to L_\alpha$, which has the following two properties:

a) For each $p(\mu) \varepsilon W$, $\eta[F(G,\sigma)‘p(\mu)] = \mu$. Now if for each $p(\nu) \varepsilon W$ we let $A(G,\nu) = \{ G‘p(\mu) : p(\mu) \varepsilon P_\rho \cap H‘p(\nu) \}$ $\cup \{ F(G,\sigma)‘p(\mu) : p(\mu) \varepsilon W_\nu \cap H‘p(\nu) \}$, then $f \varepsilon A(G,\nu)$ implies that $\eta(f) < \nu$ so that the rational $h[A(G,\nu),\nu]$ of Theorem 4.2. exists. The second property of $F(G,\sigma)$ is that

b) For each $p(\nu) \varepsilon W$, $F(G,\sigma)‘p(\nu) = h[A(G,\nu),\nu]$.

These properties of $F(G,\sigma)$ assure that the combined function $G \cup F(G,\sigma)$ is a bi-unique function from P_σ into L_α which generates a linear extension of P_σ in such a manner that for each $p(\mu) \varepsilon P_\sigma$, $\eta[G \cup F(G,\sigma)‘p(\mu)] = \mu$.

Proof:

1) Define a function $T : W \to \Gamma(W)$ as follows: for $p \varepsilon W$, $T‘p = W \cap H‘p$. For $p \varepsilon W$ and $B \subseteq L_\alpha$ define $B\langle p\rangle = B \cup \{ G‘q : q \varepsilon P_\rho \cap H‘p \}$.

2) We define a function $R:\Gamma(L_\alpha) \times W \to L_\alpha$ as follows:

a) For $p(\nu) \varepsilon W$ and $B \subseteq L_\alpha$ such that there exists an $f \varepsilon B$ with $\eta(f) \geq \nu$, $R` < B, p(\nu) >$ = the unique rational $k \varepsilon L_\alpha$ which has $\eta(k) = 0$.

b) For $p(\nu) \varepsilon W$ and $B \subseteq L_\alpha$ such that $f \varepsilon B$ implies that $\eta(f) < \nu$, we have also that $f \varepsilon B < p(\nu) >$ implies that $\eta(f) < \nu$ so that the rational $h[B < p(\nu) >, \nu]$ exists; we put $R` < B, p(\nu) > = h[B < p(\nu) >, \nu]$.

By Theorem 3.4. there exists a unique function $F(G,\sigma):W \to L_\alpha$ such that for each $p(\nu) \varepsilon W$,

$$F(G,\sigma)`p(\nu) = R` < F(G,\sigma)[p(\nu), T], p(\nu) > ,$$

where

$$F(G,\sigma)[p(\nu), T] = \{ F(G,\sigma)`p(\mu):p(\mu) \varepsilon W_\nu \cap T`p(\nu) \} .$$

3) Suppose that there exists a $p(\mu) \varepsilon W$ with $\mu > 0$ such that $\eta[F(G,\sigma)`p(\mu)] = 0$. Let $p(\nu)$ be the first such $p(\mu)$. Then $f \varepsilon F(G,\sigma)[p(\nu), T]$ implies that $f = F(G,\sigma)`p(\mu)$ for some $p(\mu) \varepsilon W_\nu$, so that by the minimal property of $p(\nu)$, $\eta(f) = \mu < \nu$, by 2). But then $F(G,\sigma)[p(\nu),T]$ is a subset $B \subseteq L_\alpha$ for which $f \varepsilon B$ implies that $\eta(f) < \nu$, so that $F(G,\sigma)`p(\nu) = R` < B, p(\nu) > = h[B < p(\nu) >, \nu]$ and thus we have $\eta[F(G,\sigma)`p(\nu)] = \nu > 0$, contrary to our supposition. Thus for each $p(\mu) \varepsilon W$, $\eta[F(G,\sigma)`p(\mu)] = \mu$, by 2).

4) By 3), for each $p(\nu) \varepsilon W$, $f \varepsilon F(G,\sigma)[p(\nu), T]$ implies that $\eta(f) < \nu$. Then by the definition of $F(G,\sigma)$, we have that

$$F(G,\sigma)`p(\nu) = h\{ F(G,\sigma)[p(\nu), T] < p(\nu) >, \nu \}.$$

But

$$F(G,\sigma)[p(\nu),T] < p(\nu) > = \{ F(G,\sigma)`p(\mu) : p(\mu) \varepsilon W_\nu \cap T`p(\nu) \}$$
$$\cup \{ G`p(\mu) : p(\mu) \varepsilon P_\rho \cap H`p(\nu) \} = A(G,\nu),$$

since $W_\nu \cap T`p(\nu) = W_\nu \cap H`p(\nu)$. Thus, $F(G,\sigma)`p(\nu) = h[A(G,\nu),\nu]$.

5) Let $p(\mu)$ and $p(\nu)$ be comparable elements of P_σ with $\mu<\nu$. If $\nu < \rho$ then by the assumption on G, $G`p(\mu) < G`p(\nu)$ in L_α if and only if $p(\mu) < p(\nu)$ in P. If $\rho \leq \nu < \sigma$ then

$$G \cup F(G,\sigma)`p(\mu) \varepsilon F(G,\sigma)[p(\nu),T] < p(\nu) >$$

if and only if $p(\mu) < p(\nu)$ in P; that is, $G \cup F(G,\sigma)`p(\mu) < F(G,\sigma)`p(\nu)$ in L_α if and only if $p(\mu) < p(\nu)$ in P. This completes the proof of Theorem 4.4. .

4.5. Definition: Let $\alpha \varepsilon$ Ord. We call a linear order L $\aleph_\alpha$-universal if every linear order of cardinality $\aleph_\alpha$ can be embedded in L.

The following theorem, obtained in a somewhat different manner by N. Cuesta Dutari [2; 243, Theorem 15], can be obtained as a direct corollary to Theorem 4.4. .

4.6. Theorem: *Let* $\alpha \varepsilon$ Ord. *Let* L *be any linear order of cardinality* $\aleph_\alpha$ *with an imposed well-ordering* $L = \{p(\mu) : \mu \varepsilon \overline{\omega}_\alpha\}$ *Then there exists an embedding* $F: L \to L_\alpha$ *of* L *in* L_α *having the property that for each* $p(\mu) \varepsilon L$, $\eta[F`p(\mu)] = \mu$ *; that is,* L_α *is an* $\aleph_\alpha$*-universal linear order. Assuming the* General Continuum Hypothesis, L_α *has cardinality* $\aleph_\alpha$.

<u>Proof</u>:

In Theorem 4.4. let P be the linear order L of the present theorem. Let $0 = \rho$ and $\sigma = \omega_\alpha$ so that G is the null function and we have $F(\Lambda, \omega_\alpha): L \to L_\alpha$. The function $F = F(\Lambda, \omega_\alpha)$ clearly satisfies the conclusion of Theorem 4.6. .

Theorem 4.6. has also been obtained by W. Sierpinski [3; 62, Theorem 3] for non-limit cardinals $\aleph_\alpha$. The method of proof would work as well on all regular cardinals but not on singular cardinals. The proof is based on the following statement, which is due to F. Hausdorff [4; 181-182]. *Let* $\alpha \in$ Ord. *A sufficient condition that a linear order* L *be* $\aleph_\alpha$*-universal is that* L *have the two properties:*

a) *If* $E \subseteq L$ *such that* $|E| < \aleph_\alpha$, *then there exist elements* $a(E)$, $b(E)$ *in* L *such that for every* $c \in E$, $a(E) < c < b(E)$ *in* L.

b) *If* $E_1, E_2 \subseteq L$ *such that* $|E_1|, |E_2| < \aleph_\alpha$ *and such that for each* $a_1 \in E_1$ *and each* $a_2 \in E_2$ *we have* $a_1 < a_2$ *in* L, *then there exists an element* $b(E_1, E_2)$ *in* L *such that for each* $a_1 \in E_1$ *and each* $a_2 \in E_2$ *we have* $a_1 < b(E_1, E_2) < a_2$ *in* L.

That this condition is not a necessary one is easily seen by considering appropriate subsets of L_α for singular cardinals $\aleph_\alpha$.

As another corollary to Theorem 4.4. the theorem on linear extensions of a partial order, due to E. Szpilrajn [5], may be given in the following form.

4.7. Theorem: *Let* $\alpha \in$ Ord. *Consider any partial order* P *of cardinality* $\aleph_\alpha$ *with an imposed well-ordering*, $P = \{ p(\mu) : \mu \in \overline{\omega}_\alpha \}$. *Let* a *and* b *be distinct non-comparable elements of* P. *Then there* *exists a function* $F:P \to L_\alpha$ *which generates a linear extension of* P *in which* $a < b$ *and which also has the property that for each* $p(\mu) \in P$, $\eta[F{}^{\backprime}p(\mu)] = \mu$.

Proof:

1) Let us construct a new well-ordering of P, $P = \{q(\mu) : \mu \in \overline{\omega}_\alpha\}$, in such a way that $q(0) = b$ and $q(1) = a$. Then let $0 = \rho$, $\sigma = \omega_\alpha$, and let $F(\Lambda, \omega_\alpha):P \to L_\alpha$ be the function of Theorem 4.4. for this newwell-ordering of P. Since $b = q(0) \nless q(1) = a$ in P, we have that $F(\Lambda, \omega_\alpha){}^{\backprime}q(1) < F(\Lambda, \omega_\alpha){}^{\backprime}q(0)$ in L_α by Theorem 4.4. so that $F(\Lambda, \omega_\alpha)$ generates a linear extension L of P in which $a < b$.

2) Consider now the original well-ordering of P under which we may write $L = \{p(\mu) : \mu \in \overline{\omega}_\alpha\}$. Then the function $F:L \to L_\alpha$ of Theorem 4.6. clearly satisfies the requirements of Theorem 4.7.

5. Universal infinite partial orders

If P and Q are partial orders and a bi-unique function $F:P \rightarrow Q$ is an isomorphism between P and $R(F) \subseteq Q$, we say that F _embeds_ P _in_ Q or that F _is an embedding of_ P _in_ Q.

We consider now any partial order P and denote by S the set of all linear extensions L of P. By the theorem of E. Szpilrajn [5] (which applies to both finite and infinite partial orders), the set of order relations which are common to all the $L \in S$ is exactly the partial ordering of P. Let us define the "elemental" function $F:P \rightarrow P(\prod\{L:L \in S\})$ by: for $a \in P$ and $F'a = F(a) \in \prod\{L:L \in S\}$, $F(a)'L = a$ for each $L \in S$. Then by the above remark, F is clearly an embedding of P in $P(\prod\{L:L \in S\})$; we say that S _realizes_ P. Various subsets of S may analogously realize P; among such realizing sets of linear extensions of P there will be one of minimal cardinality (≥ 1, of course) because of the well-ordering of the class of cardinals. B. Dushnik and E.W. Miller [6] call this minimal cardinal the _dimension_ of P and prove the following statements: (1) _Every infinite partial order has a dimension less than or equal to its cardinality._ (2) _If a partial order_ P _has an embedding in a second partial order_ Q, _then dimension_ P $\leq$ _dimension_ Q. (3) _For_ $\alpha \in \text{Ord}$ _and_ $\mathfrak{m}$ _satisfying_ $0 < \mathfrak{m} \leq \aleph_\alpha$, _there exists a partial order of cardinality_ $\aleph_\alpha$ _and dimension_ $\mathfrak{m}$.

We make immediate use of the above observations in the next two theorems.

5.1. Definition: Let $\alpha \in \mathrm{Ord}$ and $\mathfrak{m}$ be any cardinal satisfying $0 < \mathfrak{m} \leq \aleph_\alpha$. We call a partial order P $\aleph_\alpha(\mathfrak{m})$-universal if every partial order Q of cardinality $\aleph_\alpha$ and dimension $\leq \mathfrak{m}$ has an embedding in P. If $\mathfrak{m} = \aleph_\alpha$, we call P simply $\aleph_\alpha$-universal by virtue of 5.(1).

5.2. Definition: Let $\alpha \in \mathrm{Ord}$ and $\mathfrak{m}$ be any cardinal satisfying $0 < \mathfrak{m} \leq \aleph_\alpha$; let $\mathfrak{m}$ have initial ordinal ζ ($\zeta \leq \omega_\alpha$). We consider the partial order $P(L_\alpha^{\bar{\zeta}})$. A function $F\colon \bar{\zeta} \to L_\alpha$ will be called a limited series if there exists an ordinal $\delta \in \overline{\omega_\alpha}$ such that for each $\mu \in \bar{\zeta}$, $\eta(F`\mu) = \delta$; this ordinal δ will be denoted by $\Delta(F)$. We denote by $P_\alpha(\mathfrak{m})$ the suborder of $P(L_\alpha^{\bar{\zeta}})$ consisting of the limited series.

We now show that each $P_\alpha(\mathfrak{m})$ is $\aleph_\alpha(\mathfrak{m})$-universal.

5.3. Theorem: Let $\alpha \in \mathrm{Ord}$ and $\mathfrak{m}$ be any cardinal satisfying $0 < \mathfrak{m} \leq \aleph_\alpha$. Let P be any partial order of cardinality $\aleph_\alpha$ with an imposed well-ordering, $P = \{p(\mu): \mu \in \overline{\omega_\alpha}\}$, which has a dimension $\mathfrak{n}$ where $0 < \mathfrak{n} \leq \mathfrak{m}$. Then there exists an embedding $F\colon P \to P_\alpha(\mathfrak{m})$ of P in $P_\alpha(\mathfrak{m})$ having the property that for each $p(\mu) \in P$ and $F`p(\mu) = F(\mu) \in P_\alpha(\mathfrak{m})$, $\Delta[F(\mu)] = \mu$. That is, $P_\alpha(\mathfrak{m})$ is an $\aleph_\alpha(\mathfrak{m})$-universal partial order; also $P_\alpha(\mathfrak{m})$ has dimension $\mathfrak{m}$. In particular, $P_\alpha(\aleph_\alpha)$ is an $\aleph_\alpha$-universal partial order of dimension $\aleph_\alpha$ and cardinality $2^{\aleph_\alpha}$. Assuming the General Continuum Hypothesis, $P_\alpha(\mathfrak{m})$ has cardinality $\aleph_\alpha$ whenever $\mathfrak{m} < \aleph_\alpha$.

Proof:

1) $P_\alpha(\mathfrak{m}) = \bigcup\{T(\delta): \delta \in \overline{\omega_\alpha}\}$ where $T(\delta) = \{F : F \in P_\alpha(\mathfrak{m})$ and

$\Delta(F) = \delta \}$, and the $T(\delta)$ are disjoint. Now $|T(\delta)| = 2^{m \times |\delta|}$. If $m = \aleph_\alpha$, then $2^{m \times |\delta|} = 2^{\aleph_\alpha}$ for $\delta > 0$, so that $|P_\alpha(\aleph_\alpha)| = \Sigma \{ |T(\delta)| : \delta \in \overline{\omega_\alpha} \} = 2^{\aleph_\alpha}$. However, if $m < \aleph_\alpha$, then by the General Continuum Hypothesis we have $|T(\delta)| \leq \aleph_\alpha$ so that $\aleph_\alpha \leq |P_\alpha(m)| = \Sigma \{ |T(\delta)| : \delta \in \overline{\omega_\alpha} \} \leq \Sigma \{ \aleph_\alpha ; \delta \in \overline{\omega_\alpha} \} = \aleph_\alpha^2 = \aleph_\alpha$.

2) Denote by ζ the initial ordinal of m and by τ the initial ordinal of n, $\tau \leq \zeta$; let K be a realizing set of cardinality n of linear extensions of P with an imposed well-ordering, $K = \{ L(\mu) : \mu \in \overline{\tau} \}$. By Theorem 4.4., taking $0 = \rho$ and $\sigma = \omega_\alpha$, let $G(\mu) : L(\mu) \to L_\alpha$ be an embedding of $L(\mu)$ in L_α having the property that for each $p(\nu) \in L(\mu)$, $\Delta[G(\mu) ` p(\nu)] = \nu$ for each $\mu \in \overline{\tau}$ . If $\tau < \zeta$ , define $G(\mu) = G(0)$ for $\tau \leq \mu < \zeta$ . For each $p(\nu) \in P$, define a function $F(\nu)$: $\zeta \to L_\alpha$ by putting $F(\nu) ` \mu = G(\mu) ` p(\nu)$ for each $\mu \in \overline{\zeta}$. Then each $F(\nu) \in P_\alpha(m)$ and we may define the desired function $F : P \to P_\alpha(m)$ simply by putting $F ` p(\nu) = F(\nu)$ for each $p(\nu) \in P$. By the definition of K, F is an embedding of P in $P_\alpha(m)$; further, F has the property that for each $p(\mu) \in P$ and $F ` p(\mu) = F(\mu) \in P$, $\Delta[F(\mu)] = \mu$.

3) By 5.(2,3) and step 2) above, we have dimension $P_\alpha(m) \geq m$. We shall show that dimension $P_\alpha(m) \leq m$ by defining m linear extensions of $P_\alpha(m)$ and showing that these extensions realize $P_\alpha(m)$; we shall then have dimension $P_\alpha(m) = m$.

4) For each $\mu \varepsilon \bar{\xi}$ we define a linear extension $L(\mu)$ of $P_\alpha(m)$ as follows. For distinct elements F_1, F_2 of $P_\alpha(m)$, we define $F_1 < F_2$ in $L(\mu)$ if and only if:

a) $F_1{}^{\backprime}\mu < F_2{}^{\backprime}\mu$ in L_α; or

b) $F_1{}^{\backprime}\mu = F_2{}^{\backprime}\mu$ and $F_1{}^{\backprime}\nu > F_2{}^{\backprime}\nu$ in L_α for the first ordinal $\nu \varepsilon \bar{\xi}$ for which $F_1{}^{\backprime}\nu \neq F_2{}^{\backprime}\nu$.

It is easily seen that each $L(\mu)$ is a linear extension of $P_\alpha(m)$. Thus in order to show that the set $\{ L(\mu): \mu \varepsilon \bar{\xi} \}$ realizes $P_\alpha(m)$, we need only consider non-comparable elements of $P_\alpha(m)$.

5) Let F_1 and F_2 be distinct non-comparable elements of $P_\alpha(m)$. Since these elements are distinct there exists a first ordinal $\rho \varepsilon \bar{\xi}$ such that $F_1{}^{\backprime}\rho \neq F_2{}^{\backprime}\rho$, and we may assume that $F_1{}^{\backprime}\rho < F_2{}^{\backprime}\rho$ in L_α. Thus we have $F_1 < F_2$ in $L(\rho)$. Since F_1 and F_2 are non-comparable in $P_\alpha(m)$, there must exist an ordinal $\sigma \varepsilon \bar{\xi}$ such that $\sigma \neq \rho$ and $F_1{}^{\backprime}\sigma \geq F_2{}^{\backprime}\sigma$ in L_α. If $F_1{}^{\backprime}\sigma > F_2{}^{\backprime}\sigma$ in L_α, then we have $F_1 > F_2$ in $L(\sigma)$. If $F_1{}^{\backprime}\sigma = F_2{}^{\backprime}\sigma$, then ρ is the first ordinal in $\bar{\xi}$ for which $F_1{}^{\backprime}\rho \neq F_2{}^{\backprime}\rho$, and the fact that $F_1{}^{\backprime}\rho < F_2{}^{\backprime}\rho$ in L_α implies that again $F_1 > F_2$ in $L(\sigma)$. Thus the set $\{ L(\mu): \mu \varepsilon \bar{\xi} \}$ realizes $P_\alpha(m)$, showing that dimension $P_\alpha(m) \leq m$ and completing the proof of Theorem 5.3. .

It is easily seen that Theorem 5.3. reduces to Theorem 4.6. when $m = 1$. Also it should be noted that steps 3), 4), and 5) of the proof of Theorem 5.3. are direct generalizations of the corresponding steps given by H. Komm [7; 510-511] in his proof of essentially the same theorem for the case $\aleph_\alpha = \aleph_0$.

5.4. We see from Theorem 5.3. that it should be possible to construct for each $\alpha \in$ Ord some suborder of $P_\alpha(\aleph_\alpha)$ which would be of cardinality $\aleph_\alpha$ and also would be $\aleph_\alpha(\mathfrak{m})$-universal for <u>all</u> $\mathfrak{m} < \aleph_\alpha$ <u>simultaneously</u>. To see what suborder of $P_\alpha(\aleph_\alpha)$ should be constructed, we consider a partial order P of cardinality $\aleph_\alpha$ and dimension $\mathfrak{m} < \aleph_\alpha$ with an imposed well-ordering $P = \{ p(\mu) : \mu \in \overline{\omega_\alpha} \}$. Let $F:P \to P_\alpha(\mathfrak{m})$ be the embedding of Theorem 5.3. . We must extend this embedding to some embedding $G:P \to P_\alpha(\aleph_\alpha)$; this problem is equivalent to extending each limited series $F \in P_\alpha(\mathfrak{m})$ to a limited series $H \in P_\alpha(\aleph_\alpha)$. The most obvious way is to define H by: $H'\mu = F'\mu$ for $\mu \in \overline{\zeta}$ where ζ is the initial ordinal of $\mathfrak{m}$ ($\zeta < \omega_\alpha$), and $H'\mu = F'0$ for $\zeta \leq \mu < \omega_\alpha$. This method can easily be shown to yield the desired result, but we shall find that the following method leads to a further application.

Suppose we have a limited series $F \in P_0(n)$, $n < \aleph_0$, and wish to extend F to a limited series $H \in P_0(\aleph_0)$. We may make a "periodic" extension of F, that is, we may define for each $\mu \in \overline{\omega_0}$, $H'\mu = F'r$, where $0 \leq r < n$ and r is congruent to μ modulo n. By analogy, if F is a limited series in $P_\alpha(\mathfrak{m})$, $\mathfrak{m}$ with initial ordinal $\zeta < \omega_\alpha$, we may extend F periodically to a function $H \in P_\alpha(\aleph_\alpha)$ by means of: for $\mu \in \overline{\omega_\alpha}$, $H'\mu = F'\Phi(\zeta,\mu)$, (3.11.) where $0 \leq \Phi(\zeta,\mu) < \zeta$ and $\Phi(\zeta,\mu)$ is essentially " congruent to μ modulo ζ ". We now make use of this idea of periodicity.

5.5. Definition: Let $\alpha \in \text{Ord}$. We call a partial order P $\aleph_\alpha$-sub-universal if every partial order of cardinality $\aleph_\alpha$ and dimension $< \aleph_\alpha$ has an embedding in P.

5.6. Definition: Let $\alpha \in \text{Ord}$. A function $F \in P_\alpha(\aleph_\alpha)$ will be called a periodic limited series if there exists a non-zero ordinal $\pi \in \overline{\omega_\alpha}$ such that $F`\mu = F`\Phi(\pi,\mu)$ for each $\mu \in \overline{\omega_\alpha}$. The minimal (non-zero) such ordinal π will be denoted by $\Pi(F)$. We denote by P_α the suborder of $P_\alpha(\aleph_\alpha)$ consisting of the periodic limited series.

5.7. Theorem: Let $\alpha \in \text{Ord}$ and let P be any partial order of cardinality $\aleph_\alpha$ with an imposed well-ordering, $P = \{ p(\mu) : \mu \in \overline{\omega_\alpha} \}$, which has a dimension $\mathfrak{m}$ where $0 < \mathfrak{m} < \aleph_\alpha$. Then there exists an embedding $H: P \to P_\alpha$ of P in P_α having the property that for each $p(\mu) \in P$ and $H`p(\mu) = H(\mu) \in P_\alpha$, $\Delta[H(\mu)] = \mu$ and $\Pi[H(\mu)] \leq \zeta$, the initial ordinal of $\mathfrak{m}$. That is, P_α is an $\aleph_\alpha$-sub-universal partial order; also, P_α has dimension $\mathfrak{m}(\alpha)$ where $\lim \{ \mathfrak{m} : \mathfrak{m} < \aleph_\alpha \} \leq \mathfrak{m}(\alpha) \leq \aleph_\alpha$, so that if $\aleph_\alpha$ is a limit cardinal, P_α has dimension $\aleph_\alpha$. Assuming the General Continuum Hypothesis, P_α has cardinality $\aleph_\alpha$.

Proof:

1) $P_\alpha = \bigcup \{ \bigcup \{ T(\delta, \pi) : 0 < \pi < \omega_\alpha \} : \delta < \omega_\alpha \}$ where $T(\delta, \pi) = \{ F : F \in P_\alpha, \Delta(F) = \delta, \text{and } \Pi(F) = \pi \}$, and the $T(\delta, \pi)$ are disjoint. Now $|T(\delta, \pi)| = 2^{|\delta| \times |\pi|}$, so that, assuming the General Continuum Hypothesis, $|T(\delta, \pi)| \leq \aleph_\alpha$ and we have

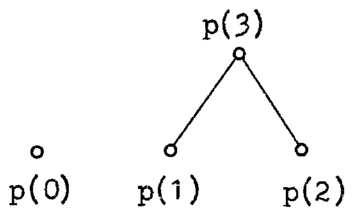
$\aleph_\alpha \leq \quad \Sigma\{\,\Sigma\{|T(\delta,\pi)|:0<\pi<\omega_\alpha\}:\delta<\omega_\alpha\} = |P_\alpha|$
$\leq \quad \Sigma\{\,\Sigma\{\aleph_\alpha:0<\pi<\omega_\alpha\}:\delta<\omega_\alpha\} = \aleph_\alpha^3 = \aleph_\alpha.$

2) Let $F:P \rightarrow P_\alpha(m)$ be the embedding of P in $P_\alpha(m)$ of Theorem 5.3., and consider each image $F\text{`}p(\mu) = F(\mu) \varepsilon P_\alpha(m)$. We define new functions $H(\mu) \varepsilon P_\alpha$ by: for $\rho \varepsilon \overline{\omega_\alpha}$, $H(\mu)\text{`}\rho = F(\mu)\text{`}\Phi(\zeta,\rho)$; clearly $\Delta[H(\mu)] \doteq \mu$ and $\Pi[H(\mu)] \leq \zeta$. We now define the desired function $H:P \rightarrow P_\alpha$ by putting each $H\text{`}p(\mu) = H(\mu)$; H clearly satisfies the conclusion of the theorem.

3) By 5.(2,3) and step 2) above, dimension $P_\alpha \geq m$ for all $m < \aleph_\alpha$; by Theorem 5.3. and 5.(2), since $P_\alpha \subseteq P_\alpha(\aleph_\alpha)$, dimension $P_\alpha \leq \aleph_\alpha$. Thus P_α has dimension $m(\alpha)$ satisfying $\lim\{m : m < \aleph_\alpha\}$ $\leq m(\alpha) \leq \aleph_\alpha$; this completes the proof of Theorem 5.7. .

5.8. In our final theorem we shall show that for certain ordinals α, P_α is actually $\aleph_\alpha$-universal. So far, in Theorems 5.3. and 5.7. we have embedded partial orders P in the partial orders $P_\alpha(m)$, $P_\alpha(\aleph_\alpha)$, or P_α by means of realizing sets of linear extensions; essentially we embedded all of P at once. We now consider means of embedding partial orders inductively. To make clear the ideas we shall show how to embed inductively in P_0 a partial order $P = \{p(\mu):\mu \varepsilon \overline{\omega_0}\}$ whose first four elements form the partial order

Q:



$p(3) > p(1)$,
$p(3) > p(2)$.

We wish to map each $p(\mu)$ of P onto some periodic limited series $F(\mu) \varepsilon P_o$ so that $\Delta[F(\mu)] = \mu$. To exhibit the embeddings graphically we shall consider the linear order L_o to be the set of dyadic rationals between 0 and 2; that is, we shall associate with each function $f \varepsilon L_o$ the dyadic rational

$$\sum_{n=o}^{\infty} \frac{f\text{`}n}{2^n} = \sum_{n=o}^{\eta(f)} \frac{f\text{`}n}{2^n} .$$

5.8.1. We see immediately that the partial order $\{p(0), p(1), p(2)\}$ (no comparabilites) has dimension 2 and is in fact realized by the two linear extensions

$$L_o: \begin{array}{l} p(2) \\ p(0) \\ p(1) \end{array} \qquad \text{and} \qquad L_1: \begin{array}{l} p(1) \\ p(0) \\ p(2) \end{array} .$$

Thus we can embed $\{ p(0), p(1), p(2) \}$ in P_o by mapping each $p(i)$, $i = 0, 1, 2$, onto $F(i) \varepsilon P_o$ where:

$$F(0)\text{`}\mu = \frac{1}{2^0} \quad \text{for all } \mu \ \varepsilon \ \overline{\omega_o},$$

$$F(1)\text{`}\mu = \begin{cases} \dfrac{1}{2^1} & \text{for } \mu \text{ even,} \\[2ex] \dfrac{3}{2^1} & \text{for } \mu \text{ odd, and} \end{cases}$$

$$F(2)\text{`}\mu = \begin{cases} \dfrac{5}{2^2} & \text{for } \mu \text{ even,} \\[2ex] \dfrac{1}{2^2} & \text{for } \mu \text{ odd.} \end{cases}$$

But this embedding cannot be extended to an embedding of Q in P_o for the following reason. Suppose we map p(3) onto some function F(3) in P_o. We must have both $F(3)'\mu > F(1)'\mu$ and $F(3)'\mu > F(2)'\mu$ for all $\mu \in \overline{\omega_o}$. This would require that $F(3)'\mu > F(0)'\mu$ for all $\mu \in \overline{\omega_o}$, and we could not obtain the necessary non-comparability < p(3), p(0) >. The difficulty, of course, lies in the fact that we have utilized only two of the six possible linear extensions of { p(0), p(1), p(2) } ; every realizing set of linear extensions of Q has a linear extension in which both p(0) > p(1) and p(0) > p(2). We next show how to overcome this difficulty.

5.8.2. We define a sequence of standard functions $F(\mu) \in P_o$, $\mu \in \overline{\omega_o}$, satisfying $\Delta[F(\mu)] = \mu$ as follows:

$$F(0)'\mu = \frac{1}{2^0} \quad \text{for all } \mu \in \overline{\omega_o},$$

$$\Delta[F(0)] = 0 \quad \text{and} \quad \Pi[F(0)] = 1;$$

$$F(1)'\mu = \begin{cases} \frac{1}{2^1}, & \mu \equiv 0 \text{ modulo } 2, \\ \frac{3}{2^1}, & \mu \equiv 1 \text{ modulo } 2, \end{cases}$$

$$\Delta[F(1)] = 1 \quad \text{and} \quad \Pi[F(1)] = 2;$$

$$F(2)'\mu = \begin{cases} \frac{1}{2^2}, & \mu \equiv 0, 1 \text{ modulo } 8, \\ \frac{3}{2^2}, & \mu \equiv 2, 3 \text{ modulo } 8, \\ \frac{5}{2^2}, & \mu \equiv 4, 5 \text{ modulo } 8, \\ \frac{7}{2^2}, & \mu \equiv 6, 7 \text{ modulo } 8, \end{cases}$$

$\Delta[F(2)] = 2$ and $\Pi[F(2)] = 8$;

$$F(3)`\mu = \begin{cases} \frac{1}{2^3}, & \mu \equiv 0, \ldots, 7 \quad \text{modulo } 64, \\ \frac{3}{2^3}, & \mu \equiv 8, \ldots, 15 \quad \text{modulo } 64, \\ \frac{5}{2^3}, & \mu \equiv 16, \ldots, 23 \quad \text{modulo } 64, \\ \frac{7}{2^3}, & \mu \equiv 24, \ldots, 31 \quad \text{modulo } 64, \\ \frac{9}{2^3}, & \mu \equiv 32, \ldots, 39 \quad \text{modulo } 64, \\ \frac{11}{2^3}, & \mu \equiv 40, \ldots, 47 \quad \text{modulo } 64, \\ \frac{13}{2^3}, & \mu \equiv 48, \ldots, 55 \quad \text{modulo } 64, \\ \frac{15}{2^3}, & \mu \equiv 56, \ldots, 63 \quad \text{modulo } 64, \end{cases}$$

$\Delta[F(3)] = 3$ and $\Pi[F(3)] = 64$;

etc.

The first four of these functions are shown in Fig. 1. It is clear from the method of formation that each $F(\mu)$ has $\Pi[F(\mu)] = \Pi\{2^\nu : \nu \leq \mu\} = 2^{\Sigma\{\nu : \nu \leq \mu\}} = 2^{\frac{1}{2}\mu(\mu+1)}$. It is also clear that for each $n \in \overline{\omega_0}$, every linear order formed from the set of elements $\{a(0), \ldots, a(n)\}$ is embedded in L_0 by the associations $a(i) \rightarrow F(i)`\mu$ for some $\mu < 2^{\frac{1}{2}n(n+1)}$.

5.8.3. We embed Q in P_0 as follows. Since there are no comparabilities between $p(0)$, $p(1)$, and $p(2)$, we map $p(i)$ onto $F(i)$, $i = 0, 1, 2$,. Now we cannot map $p(3)$ onto $F(3)$ directly since we

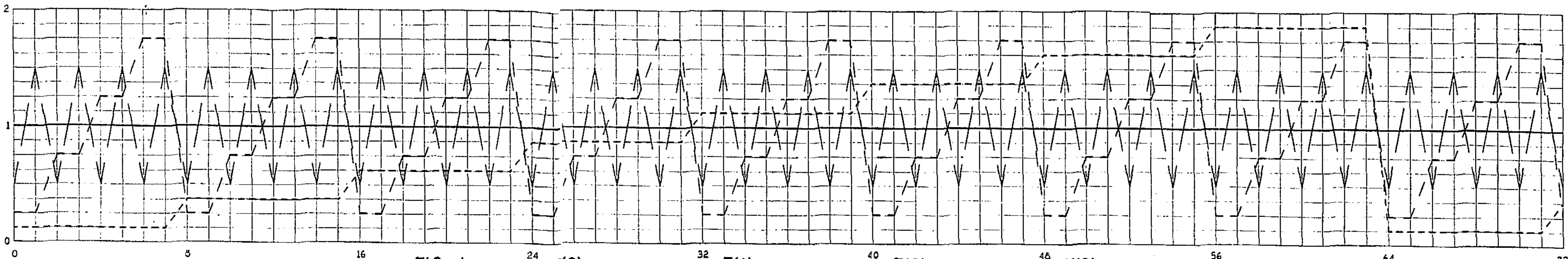


FIG. 1 F(0) ——— F(1) — — F(2) - - - - - F(3) --------

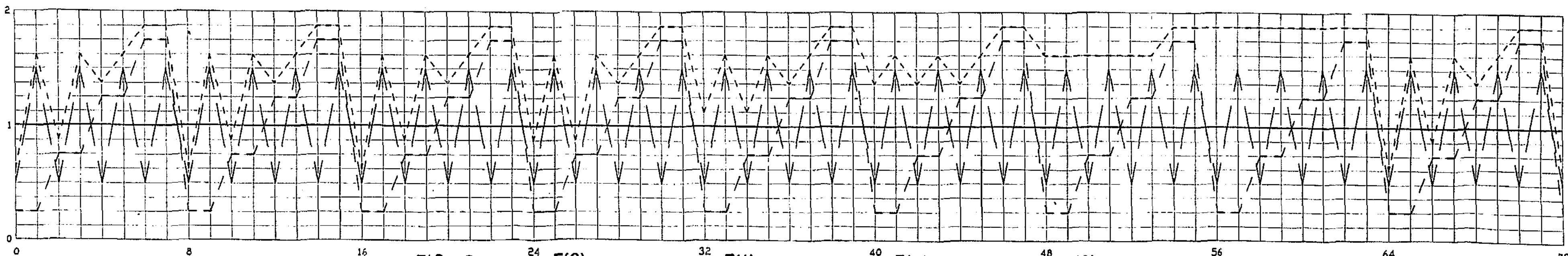


FIG. 2 F(0) ——— F(1) — — F(2) - - - - - G(3) --------

need $p(3) > p(1)$ and $p(3) > p(2)$. Instead we define a new function $G(3)$ as follows. For $\mu \in \overline{\omega_0}$ for which the associations $p(i) \to F(i)`\mu$, $i = 0, 1, 2, 3$, generate a linear extension of $Q$, $G(3)`\mu = F(3)`\mu$. For all other $\mu \in \overline{\omega_0}$, $G(3)`\mu$ is the minimal rational of the form $\frac{k}{2^3}$, $0 < \frac{k}{2^3} < 2$, k odd, for which the associations $p(i) \to F(i)`\mu$, $i = 0, 1, 2$, and $p(3) \to G(3)`\mu$ do generate a linear extension of Q. This embedding of Q is shown in Fig. 2. It is clear that this embedding of Q could be inductively extended to an embedding of all of P in P_0. Our last theorem generalizes these ideas.

5.9. Definition: Let $\lambda \in$ Ord. We say that $\aleph_\lambda$ is a <u>denumerable type</u> cardinal if $\aleph_\lambda^{\aleph_0} = 2^{\aleph_\lambda}$. $\aleph_0$ is a denumerable type cardinal. By the General Continuum Hypothesis (3.11) if $\lambda > 0$, $\aleph_\lambda$ is a denumerable type cardinal if and only if λ is a limit ordinal which is the limit of a denumerable increasing sequence of ordinals $< \lambda$. In the following theorem we assume the General Continuum Hypothesis for $\lambda > 0$.

5.10. Theorem: <u>Let $\aleph_\lambda$ be a denumerable type cardinal and let P be any partial order of cardinality $\aleph_\lambda$ with an imposed well-ordering, $P = \{p(\mu): \mu \in \overline{\omega_\lambda}\}$. Then there exists an embedding $F: P \to P_\lambda$ of P in P_λ having the property that for each $p(\mu) \in P$ and $F`p(\mu) = F(\mu) \in P_\lambda$, $\Delta[F(\mu)] = \mu$, and if $\lambda = 0$, $\Pi[F(\mu)] \leq 2^{\frac{1}{2}\mu(\mu+1)}$. That is, P_λ is an $\aleph_\lambda$-universal partial order; also, P_λ has cardinality and dimension $\aleph_\lambda$.</u>

Proof:

1) P_λ has cardinality and dimension $\aleph_\lambda$ by Theorem 5.7. .

2) We define subsets $P(n)$ and P_n of P for each integer $n \geq 0$ as follows. If $\lambda = 0$, $P(n) = \{ p(n) \}$. If $\lambda > 0$, let $\lambda = \lim \{ \lambda(n) : n \in \overline{\omega_o} \}$ with $\lambda(n) < \lambda(n+1)$ for each $n \in \overline{\omega_o}$, so that $\omega_\lambda = \lim \{ \omega_{\lambda(n)} : n \in \overline{\omega_o} \}$. Then $P(0) = \{ p(\mu) : \mu < \omega_{\lambda(0)} \}$ and for each integer $n > 0$, $P(n) = \{ p(\mu) : \omega_{\lambda(n-1)} \leq \mu < \omega_{\lambda(n)} \}$. For all values of λ we put $P_n = \bigcup \{ P(r) : r \leq n \}$.

3) For each integer $n \geq 0$ let $w(n) = \{ f : f \in L_\lambda^{P(n)}$ and for each $p(\mu) \in P(n)$, $\eta[f\,'\,p(\mu)] = \mu \}$. If $\lambda = 0$, $|w(n)| = 2^n$ and we may write $w(n) = \{ f(n,\mu) : \mu < 2^n \}$ where we may assume that $f(n,\mu) < f(n, \mu+1)$ in L_o. If $\lambda > 0$, $|w(n)| = \prod \{ 2^{|\mu|} : p(\mu) \in P(n) \}$ $= 2^{<n>}$, where $<n> = \Sigma \{ |\mu| : p(\mu) \in P(n) \} = \Sigma \{ |\mu| : \mu < \omega_{\lambda(n)} \}$ $= \aleph_{\lambda(n)}$, so that $|w(n)| = \aleph_{\lambda(n)+1}$; thus we may well-order $w(n)$ and write $w(n) = \{ f(n,\mu) : \mu < \omega_{\lambda(n)+1} \}$.

4) We define functions $g(n,\zeta) : P(n) \to L_\lambda$ for each integer $n \geq 0$ and for each $\zeta \in \overline{\omega_\lambda}$ as follows:

a) Suppose that $\lambda = 0$. We put $g(0,\zeta) = f(0,0)$ for all $\zeta \in \overline{\omega_o}$. For integers $n > 0$ and for each $\zeta \in \overline{\omega_o}$, let $\zeta(n) = \Phi(2^{\frac{1}{2}n(n+1)}, \zeta)$ and $\zeta(n,n) = \Theta[2^{\frac{1}{2}n(n-1)}, \zeta(n)]$, so that $\zeta(n) < 2^{\frac{1}{2}n(n+1)}$ and $\zeta(n) = 2^{\frac{1}{2}n(n-1)} \times \zeta(n,n) + r$ with $r < 2^{\frac{1}{2}n(n-1)}$ and $\zeta(n,n) < 2^n$; we put $g(n,\zeta) = f[n, \zeta(n,n)]$. These constructions correspond exactly to those of 5.8.2. .

b) Suppose that $\lambda > 0$. We put $g(0, \zeta) = f[0, \zeta(0)]$ for each $\zeta \in \overline{\omega_\lambda}$, where $\zeta(0) = \Phi(\omega_{\lambda(o)+1}, \zeta)$. For integers $n > 0$ and for each $\zeta \in \overline{\omega_\lambda}$, let $\zeta(n) = \Phi(\omega_{\lambda(n)+1}; \zeta)$ and $\zeta(n,n) = \Theta[\omega_{\lambda(n-1)+1}, \zeta(n)]$, so that $\zeta(n) < \omega_{\lambda(n)+1}$ and $\zeta(n) = \omega_{\lambda(n-1)+1} \times \zeta(n,n) + \nu$ with $\nu < \omega_{\lambda(n-1)+1}$ and $\zeta(n,n) < \omega_{\lambda(n)+1}$; we put $g(n, \zeta) = f[n, \zeta(n,n)]$. These constructions are generalizations of those in 2) above.

5) We show by induction on n that if $n \in \overline{\omega_o}$ and $G:P_n \to L_\lambda$ is any function such that for each $p(\mu) \in P_n$, $\eta[G\text{`}p(\mu)] = \mu$, then there exists an ordinal $\gamma(G) \in \overline{\omega_\lambda}$ such that $G = \bigcup\{g[r, \gamma(G)]: r \leq n\}$. If $\lambda = 0$, $\gamma(G)$ may be taken $< 2^{\frac{1}{2}n(n+1)}$; if $\lambda > 0$, $\gamma(G)$ may be taken $< \omega_{\lambda(n)+1}$.

This statement is clearly true for $n = 0$; let $n > 0$ and assume that the statement holds for $n - 1$. We define a function $H:P_{n-1} \to L_\lambda$ by: $H = G \cap P_{n-1} \times L_\lambda$. Then for each $p(\mu) \in P_{n-1}$, $\eta[H\text{`}p(\mu)] = \mu$, so that by our inductive assumption there exists an ordinal $\gamma(H)$; if $\lambda = 0$, $\gamma(H) < 2^{\frac{1}{2}n(n-1)}$; if $\lambda > 0$, $\gamma(H) < \omega_{\lambda(n-1)+1}$; that is, $H = \bigcup\{g[r, \gamma(H)]:r < n\}$. Now, $G - H$ is a function $G - H:P(n) \to L_\lambda$ such that for each $p(\mu) \in P(n)$, $\eta[G - H\text{`}p(\mu)] = \mu$, so that there exists an ordinal $\zeta \in \overline{\omega_\lambda}$ such that $G - H = f(n, \zeta)$; if $\lambda = 0$, $\zeta < 2^n$; if $\lambda > 0$, $\zeta < \omega_{\lambda(n)+1}$. We define an ordinal γ by means of: if $\lambda = 0$, $\gamma = 2^{\frac{1}{2}n(n-1)} \times \zeta + \gamma(H)$; if $\lambda > 0$, $\gamma = \omega_{\lambda(n-1)+1} \times \zeta + \gamma(H)$. Thus, for $\lambda = 0$, $\gamma < 2^{\frac{1}{2}n(n+1)}$, while for $\lambda > 0$, $\gamma < \omega_{\lambda(n)+1}$.

Clearly, for $r < n$, $g(r, \gamma) = g[r, \gamma(H)]$ while $g(n, \gamma) = f(n, \zeta)$, so that $G = \bigcup\{g[r, \gamma(H)]: r < n\} \cup f(n, \zeta) = \bigcup\{g(r, \gamma): r \leq n\}$, and we may take $\gamma(G) = \gamma$.

6) Now, depending on the order relations in P, we define functions $k(n, \zeta): P(n) \to L_\lambda$ for each integer $n \geq 0$ and for each $\zeta \varepsilon \omega_\lambda$ by induction on n. Let $n \varepsilon \overline{\omega_o}$ and $\zeta \varepsilon \overline{\omega_\lambda}$ and suppose that we have defined functions $k(r, \zeta): P(r) \to L_\lambda$ for every $r < n$ in such a manner that for each $r < n$ and each $p(\mu) \varepsilon P(r)$, $\eta[k(r,\zeta)`p(\mu)] = \mu$ and that $\bigcup\{k(r, \zeta): r < n\}$ generates a linear extension of P_{n-1} (with the convention that $P_{-1} = \Lambda$). If $\bigcup\{k(r, \zeta): r < n\} \cup g(n, \zeta)$ generates a linear extension of P_n, we define $k(n, \zeta) = g(n, \zeta)$; otherwise we define $k(n, \zeta) = F[\bigcup\{k(r, \zeta): r < n\}, \Omega(n)]$, the function of Theorem 4.4. for the present partial order $P = \{p(\mu): \mu \varepsilon \overline{\omega_\lambda}\}$, where $\Omega(n) = n + 1$ if $\lambda = 0$ and $\Omega(n) = \omega_{\lambda(n)}$ if $\lambda > 0$. By induction, if $\bigcup\{g(r, \zeta): r \leq n\}$ generates a linear extension of P_n, then $k(r, \zeta) = g(r, \zeta)$ for $r \leq n$. Also by induction, for each $\zeta \varepsilon \overline{\omega_\lambda}$, $\bigcup\{k(n, \zeta): n \varepsilon \overline{\omega_o}\}$ generates a linear extension of P.

7) We define functions $F(\mu): \overline{\omega_\lambda} \to L_\lambda$ for each $\mu \varepsilon \overline{\omega_\lambda}$ by putting $F(\mu)`\zeta = k(n, \zeta)`p(\mu)$ for each $\zeta \varepsilon \overline{\omega_\lambda}$, where n is the integer such that $p(\mu) \varepsilon P(n)$. Clearly, by 4) and 6), each $F(\mu) \varepsilon P_\lambda(\aleph_\lambda)$, and $\Delta[F(\mu)] = \mu$. Furthermore, by 4) and 6), each $F(\mu) \varepsilon P_\lambda$, and in fact, for $\lambda = 0$, $\Pi[F(\mu)] \leq 2^{\frac{1}{2}\mu(\mu+1)}$ and for $\lambda > 0$, $\Pi[F(\mu)] \leq \omega_{\lambda(n)+1}$. Now we can define the desired function $F: P \to P_\lambda$ simply by setting $F`p(\mu) = F(\mu)$ for each $p(\mu) \varepsilon P$. Because of the closing remark in 6), we need only check

that F preserves all the non-comparabilities in P.

8) Let $n \varepsilon \overline{\omega_0}$ and let $G: P_n \to L_\lambda$ be <u>any</u> function which generates a linear extension of P_n such that for each $p(\mu) \varepsilon P_n$, $\eta[G \text{`} p(\mu)] = \mu$. By the statement in 5), there exists an ordinal $\gamma(G) \varepsilon \overline{\omega_\lambda}$ such that $G = \bigcup \{ g[r, \gamma(G)] : r \leq n \}$. But then by the next to last sentence of 6), $k[r, \gamma(G)] = g[r, \gamma(G)]$ for $r \leq n$, so that $G = \bigcup \{ k[r, \gamma(G)] : r \leq n \}$. But now any non-comparability in P can be realized by the extensions $\bigcup \{ k[n, \gamma(G_1)] : n \varepsilon \overline{\omega_0} \}$ and $\bigcup \{ k[n, \gamma(G_2)] : n \varepsilon \overline{\omega_0} \}$ of two such functions G to the domain P, by Theorems 4.4. and 4.7. . Thus the function F is the desired embedding for Theorem 5.10. .

5.11. We could, of course, perform a process analogous to that in Theorem 5.10. for any limit cardinal $\aleph_\lambda$, but we could not make the statement of step 5) once n had passed ω_0. Step 5) was crucial in proving step 8), which assures us that the function F preserves the non-comparabilities of P and is thus an embedding of P in P_λ. For non-limit cardinals, Theorems 5.3. and 5.7. are the best results the author has been able to obtain. Thus, at this point, the existence of $\aleph_\alpha$-universal partial orders of cardinality $\aleph_\alpha$ remains an open question for all but the denumerable type limit cardinals.

It appears to the author that P_α is about the largest subset of $P_\alpha(\aleph_\alpha)$ which can be specified by a rule of formation and still be of cardinality $\aleph_\alpha$. It might be worth while to investigate further the properties of P_α; in particular, for non-limit cardinals $\aleph_{\alpha+1}$, it would be interesting to determine whether $P_{\alpha+1}$ <u>could</u> be $\aleph_{\alpha+1}$-universal by settling whether the dimension of $P_{\alpha+1}$ is $\aleph_\alpha$ or $\aleph_{\alpha+1}$.

REFERENCES

[1] W. Sierpinski, "L'hypothèse généralisée du continu et l'axiome du choix," Fundamenta Mathematicae, vol. 34 (1947), pp. 1-5.

[2] N. Cuesta Dutari, "Teoria decimal de los tipos de orden," Revista Matematica Hispano-Americana, vol. 3 (1943), pp. 186-205, 242-268.

[3] W. Sierpinski, "Sur une propriété des ensembles ordonnés," Fundamenta Mathematicae, vol. 36 (1949), pp. 56-67.

[4] F. Hausdorff, Grundzüge der Mengenlehre, Leipzig, 1914.

[5] E. Szpilrajn, "Sur l'extension de l'order partiel," Fundamenta Mathematicae, vol. 16 (1930), pp. 386-389.

[6] B. Dushnik and E.W. Miller, "Partially ordered sets," American Journal of Mathematics, vol. 63 (1941), pp. 600-610.

[7] H. Komm, "On the dimension of partially ordered sets," American Journal of Mathematics, vol. 70 (1948), pp. 507-520.