# INDEPENDENCE RESULTS FOR INDESCRIBABLE CARDINALS

Thesis by

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## Notation

Our set theoretical notation is quite standard: The letters $\alpha$, $\beta$, $\gamma$, ... denote ordinals, while $\kappa$, $\lambda$, $\mu$ are reserved for inaccessible cardinals unless it is explicitly mentioned that they are not. We let $[\alpha,\beta] = \{\zeta : \alpha \leq \zeta \leq \beta\}$ and $(\alpha,\beta) = \{\zeta : \alpha < \zeta < \beta\}$ and similarly for half-open intervals.

The notation $f: A \rightarrow B$ expresses that f is a function with domain A and range contained in B. $f[S]$ ($f^{-1}[S]$ resp.) denotes the pointwise image of S (preimage of S resp.).

As usual $\Sigma^m_n$ ($\Pi^m_n$ resp.) is the collection of all formulas in the $\in$ language of set theory with higher type variables and a unary predicate symbol whose prenex form has n alternating blocks of type m quantifiers starting with $\exists$ ($\forall$ resp.). A formula is $\Delta^m_n$ if it is equivalent with both a $\Sigma^m_n$ and a $\Pi^m_n$ formula and $\Sigma^m_0$ (or $\Pi^m_0$) if all quantifiers are of type $<$m. If we write down a formula $\Phi(X_1,...,X_n)$ then this is to mean that all free variables that occur in $\Phi$ are among $X_1, ..., X_n$.

$V_\alpha$ (the $\alpha$-th level of the von Neumann hierarchy) is often understood as the structure $\langle V_\alpha, \in \rangle$ (frequently with a finite sequence of possibly k-ary relations on $V_\alpha$ added on); however, we will simply write $V_\alpha \models \Phi(A)$ instead of $\langle V_\alpha, \in, A \rangle \models \Phi$ for example.

Unless remarked otherwise we will be working in ZFC. $ZF^-$ (depending on the context) and will always denote some suitable finite fragment of ZFC. Recall that we have a flat pairing function (i.e., one that does not raise the von Neumann rank) such that the coding and decoding is absolute for transitive models of a very weak fragment of ZF.

Recall also that for each m $\geq$ 1 and n $\geq$ 1 there is a formula $\chi_{\Sigma^m_n}(x,\mathfrak{X})$ in $\Sigma^m_m$

such that for any limit ordinal $\alpha$,

$$V_\alpha \models \Phi\ (\mathfrak{X}) \text{ iff } V_\alpha \models \chi_{\Sigma^m_n}\ (\text{`}\Phi\text{'},\mathfrak{X})$$

for all $\mathfrak{X} \in V_{\kappa+m}$ and $\Phi\ (\mathfrak{X})$ in $\Sigma^m_n$ where '$\Phi$' denotes a code of $\Phi$ in $V_\omega$. All the necessary details can be found in [Drake, 1974]. $\mathcal{M}$ is $\Sigma^m_o$ correct for $\kappa$ inside $\mathcal{N}$ iff $\mathcal{M}^{|V_{\kappa+m-2}|} \subseteq \mathcal{M}$ in $\mathcal{N}$; i.e., $\mathcal{M}$ is closed under sequences of length $|V_{\kappa+m-2}|$ in $\mathcal{N}$ where this means sequences of length $<\kappa$ in case $m = 1$. We call $\mathcal{M}$ $\Sigma^m_n$ $(n \geq 1)$ correct for $\kappa$ (in parameters from $V_{\kappa+m}$) inside $\mathcal{N}$ if in addition to this $\mathcal{M}$ correctly computes the $\Sigma^m_n$ facts that hold in $\mathcal{N}$ in parameters from $V_{\kappa+m} \cap \mathcal{M}$.

Given a model M of $ZF^-$ we will use the notation $(M)_\alpha$ to denote M's version of $V_\alpha$ for $\alpha \in M$. However we will also use the notation $M \models$ "$V_\alpha \models \Phi$" rather than $(M)_\alpha \models \Phi$. Regarding our forcing formalism we take V as a relative term for the ground model and construct generic extensions V[G] where partially ordered sets (posets) will be given preference over Boolean algebras. We contend that every element of V[G] has a name in V and let $V^P$ denote the class of all P names in V relative to a poset $P \in V$.

In general, $\tau$, $\sigma$ will denote names but we will also use the notation $\mathring{A}$ for a name for $A \in V[G]$. In particular, if Q is a poset $\in V[G]$ and $\tau \in V[G]$ a Q name then we denote by $\mathring{\tau} \in V^P$ a name for $\tau$. As usual $\check{x}$ is the canonical name for x that happens to be in the ground model. Very often (for instance if x is an ordinal) the $\check{x}$ notation will be supressed. By a nice name for a subset of $\sigma \in V^P$ we mean essentially a function that associates with each $\tau \in dom(\sigma)$ an antichain of P. Frequently the symbol $\Gamma$ will occur as a canonical name for the generic (for a given poset P). Thus we have for instance $p \Vdash p \in \Gamma$ for all $p \in P$. $Fn(I,2,\kappa)$ will always denote a $<\kappa$ support product of copies of the usual poset for adding a subset of $\kappa$ where the copies are indexed by the ordinals $\in$ I.

We say that a poset P is $<\kappa$ closed if for any decreasing sequence of length $<\kappa$ of conditions there is some condition extending all the conditions in the sequence. As a variant of this we will say that P is $<\kappa$ directed closed if for any directed $X \subseteq P$ (i.e., $\forall p,q \in X\ \exists r \in X\ r \leq p,q$) of size $<\kappa$ there is some $p \in P$ with $p \leq q$ for all $q \in X$. P is said to be $\kappa$ c.c. if every antichain has size $<\kappa$. P has the property $\kappa$ if every subset of P of size $\kappa$ has a subset of size $\kappa$ that consists of pairwise compatible conditions. Finally P is $\kappa$ centered if there is an equivalence relation on P with $\kappa$ equivalence classes such that any two equivalent conditions are compatible.

All the forcing terminology being used in this thesis and not being explained here can be found in [Baumgartner, 1983] and [Kunen, 1980].

## Introduction and Statement of Results

Indescribability is closely related to the reflection principles of Zermelo Fränkel set theory. In this axiomatic setting the universe of all sets stratifies into a natural cumulative hierarchy $(V_\alpha : \alpha \in On)$ such that any formula of the language for set theory that holds in the universe already holds in the restricted universe of all sets obtained by some stage.

The axioms of ZF prove the existence of many ordinals $\alpha$ such that this reflection scheme holds in the world $V_\alpha$ if one considers only first-order parameters over $V_\alpha$. Hanf and Scott [1961] noticed that one arrives at a large cardinal notion if one allows second order parameters, i.e., predicates over $V_\alpha$. For a given collection $\Omega$ of formulas in the $\in$ language of set theory with higher type variables and a unary predicate they define an ordinal $\alpha$ to be $\Omega$ indescribable if for all formulas $\Phi$ in $\Omega$ and $A \subseteq V_\alpha$,

$$(V_\alpha, \in, A) \models \Phi \ \Rightarrow \ \exists \beta < \alpha \ (V_\beta, \in, A \cap V_\beta) \models \Phi .$$

Since a sufficient coding apparatus is available this definition is (for the classes of formulas that we are going to consider) equivalent to the one that one obtains by allowing finite sequences of relations over $V_\alpha$ some of which are possibly k-ary. We will be interested mainly in certain standardized classes of formulas. Let $\Sigma^m_n$ ($\Pi^m_n$ resp.) denote the class of all formulas in the language introduced above whose prenex form has n alternating blocks of quantifiers of type m starting with $\exists$ ($\forall$ resp.). In [Hanf-Scott, 1961] it is shown that in ZFC, $\Pi^1_0$ indescribability is equivalent to inaccessibility and $\Pi^1_1$ indescribability coincides with weak compactness.

Thus the existence of $\Sigma^m_n$ (or $\Pi^m_n$) indescribable cardinals is unprovable in ZFC. However [Vaught, 1963] has shown that below any measurable cardinal there are many totally indescribable cardinals (where totally indescribable means $\Sigma^m_n$ indescribable for all m, n). Moreover, it follows from the results in [Jensen, 1967] that already below $\kappa(\omega)$ (i.e., the last cardinal with $\kappa \to (\omega)^{<\omega}_2$), there are stationary many totally indescribable cardinals. Therefore, indescribable cardinals are rather tame inhabitants of the large cardinal zoo; in fact $\Sigma^m_n$ and $\Pi^m_n$ indescribability relativizes down to L.

Since the $\Pi^1_0$ indescribability of $\kappa$ is a $\Pi^1_1$ property over $V_\kappa$, there are many $\Pi^1_0$ indescribables below the least $\Pi^1_1$ indescribable. Can this result be generalized to arbitrary m and n?

By computing the complexity of a truth definition for $\Pi^m_n$ and $\Sigma^m_n$ formulas [Levy, 1971] was able to prove that in ZFC

$$\pi^1_n = \sigma^1_{n+1} < \pi^1_{n+1} = \sigma^1_{n+2}$$

and

$$\pi^m_n, \sigma^m_n < \pi^m_{n+1}, \sigma^m_{n+1}$$

for m $\geq$ 2 and n $\geq$ 0, where $\pi^m_n$ ($\sigma^m_n$ resp.) is the least $\Pi^m_n$ ($\Sigma^m_n$ resp.) indescribable cardinal (if they exist). In fact he showed that the gaps are very large. Moreover he proved that

$$\sigma^m_n \neq \pi^m_n$$

for m $\geq$ 2 and n $\geq$ 1. The prominent question at this point is what can be said in ZFC about the relative size of $\sigma^m_n$ and $\pi^m_n$ for m $\geq$ 2 and n $\geq$ 1. [Moschovakis, 1976] provided an answer under the assumption that all sets are constructible; i.e., V = L. He showed that in L, $\sigma^m_n < \pi^m_n$.

The main results of this thesis are:

Theorem III.1.1. $(m \geq 2, n \geq 1)$

$$\mathrm{CON}(\mathrm{ZFC} + \exists\kappa,\kappa'(\kappa < \kappa', \kappa \text{ is } \Pi^m_n \text{ indescribable, and } \kappa' \text{ is } \Sigma^m_n \text{ indescribable}))$$

$$\Rightarrow \mathrm{CON}(\mathrm{ZFC} + \sigma^m_n > \pi^m_n + \mathrm{GCH}). \qquad \Box$$

Note that in ZFC + $\sigma^m_n > \pi^m_n$ one can prove that $L_{\sigma^m_n} \models (\exists\ \Sigma^m_n \text{ indescribable} \wedge \exists\ \Pi^m_n \text{ indescribable})$; thus by Gödel's second incompleteness theorem we cannot hope for a proof of CON(ZFC + $\exists\ \Sigma^m_n$ indescribable + $\exists\ \Pi^m_n$ indescribable) $\Rightarrow$ CON(ZFC + $\sigma^m_n > \pi^m_n$) that is formalizable within ZFC unless ZFC + $\sigma^m_n > \pi^m_n$ is inconsistent. Therefore III.1.1. is stated optimally.

There is also an Easton type result that says in effect that we have the ultimate freedom in arranging the relative sizes of $\sigma^m_n$ and $\pi^m_n$ (simultaneously for $m \geq 2$ and $n \geq 1$) as far as the axioms of ZFC are concerned.

Theorem III.7.1.Assuming the existence of $\Sigma^m_n$ indescribables for all $m \geq 2$, $n \geq 1$ and given a function $\mathcal{F}:\{(m,n): m \geq 2, n \geq 1\} \to \{0,1\}$ there is a poset $P_{\mathcal{F}} \in L[\mathcal{F}]$ such that GCH holds in $(L[\mathcal{F}])^{\mathcal{P}_{\mathcal{F}}}$ and

$$\Vdash^{L[\mathcal{F}]}_{P_{\mathcal{F}}} \begin{cases} \sigma^m_n < \pi^m_n & \text{if } \mathcal{F}(m,n) = 0 \\ \sigma^m_n > \pi^m_n & \text{if } \mathcal{F}(m,n) = 1. \end{cases}$$

$\Box$

The proof of III.1.1. is based on two important techniques for obtaining consistency result for large cardinals: The first technique is known as iterated forcing and originates in the [Solovay-Tennenbaum, 1971] consistency proof of Martin's axiom. Its underlying idea is that one defines the set of forcing condition by proceeding in stages and in our case at each stage we will kill off a potential candidate for a $\Sigma^m_n$ indescribable cardinal until we have taken care of all ordinals below a given $\Pi^m_n$ indescribable cardinal. The hard part in the proof of III.1.1. will be to guarantee that the forcing which we define preserves the $\Pi^m_n$ indescribability of the cardinal that we started out with. For this we will use master condition arguments. This way of reasoning was first employed by [Silver, 1971] and typically occurs in the following setting: We are given an elementary embedding $j: M \longrightarrow N$ where M and N are modeling a large enough fragment of ZF together with posets $P \in M$ and $Q = j(P) \in N$, and we want to find an M generic G and an N generic H such that j lifts to an embedding (that will again be called j) from M[G] into N[H]. A necessary and sufficient condition for this to work is to pick G and H with the property that

(∗) $\forall p \in G \quad j(p) \in H$.

If we are given a generic G for P, then a master condition for Q relative to G will be a condition $q \in Q$ such that for any N generic $H \subset Q$ with $q \in H$, (∗) holds for G and H. In order to make the proof of III.1.1. amenable to master condition arguments we must first find a reformulation of $\Pi$-indescribability in terms of elementary embeddings.

Theorem I.1. ($m \geq 1$, $n \geq 1$) $\kappa$ is $\Pi^m_n$ indescribable iff

$\forall M$ [M transitive, $M \models ZF^-$, $|M| = \kappa$, $\kappa \in M$, $M^{<\kappa} \subseteq M$ .⇒.

$\exists j$, N [N transitive, $|N| = |V_{\kappa+m-1}|$, N $\Sigma^m_{n-1}$ correct for $\kappa$,

$j: M \longrightarrow N$, cpt $j = \kappa$]]. □

where N is $\Sigma^m_{n-1}$ correct for $\kappa$ means $N^{|V_{\kappa+m-2}|} \subseteq N$ ($N^{<\kappa} \subseteq N$ for $m = 1$) and N correctly computes the $\Sigma^m_{n-1}$ facts that hold in parameters from $N \cap V_{\kappa+m}$.

It is an interesting observation that this reformulation suggests that for m $\geq$ 1, $\Pi^m_1$ indescribability can be construed as an analogue of the concept of hypermeasurability in [Mitchell, 1979] and we will also rephrase it in the context of extenders (cf. [Martin-Steel, 1988]).

As a warm up routine for the proof of III.1.1. we are going to apply this reformulation to evaluate the consistency strength of the failure of GCH at a $\Pi$ indescribable cardinal.

Theorem II.1. (n $\geq$ 1)

$$\text{CON (ZFC} + \exists\ \Pi^1_n \text{ indescribable cardinal)} \Longleftrightarrow$$

$$\text{CON (ZFC} + \exists\ \kappa\ \ (\kappa \text{ is } \Pi^1_n \text{ indescribable} \wedge 2^\kappa > \kappa^+)) \qquad \square$$

This theorem generalizes a result of Silver (cf. [Kunen, 1980]) about the consistency strength of the failure of GCH at a weakly compact cardinal. In the case of $\Pi^m_n$ (m $\geq$ 2, n $\geq$ 1) indescribability we obtain a result that is reminiscent of the failure of GCH at a measurable cardinal.

Theorem II.2. ($\ell \geq 1$, m $\geq$ 2, n $\geq$ 1)

$$\text{CON (ZFC} + \exists\ \Pi^{m+\ell-1}_n \text{ indescribable cardinal)} \Longleftrightarrow$$

$$\text{CON (ZFC} + \exists\ \kappa\ \ (\kappa \text{ is } \Pi^m_n \text{ indescribable} \wedge 2^\kappa = \kappa^{+\ell})) \qquad \square$$

# Table of Contents

> Ich bin dabei mit Seel' und Leib;
> Doch freilich würde mir behagen
> Ein wenig Freiheit und Zeitvertreib
> An schönen Sommerfeiertagen.
>
> J.W.v. Goethe
> Faust, part one of the
> tragedy, Studierzimmer

# CHAPTER I. A REFORMULATION OF $\Pi$ INDESCRIBABILITY IN TERMS OF ELEMENTARY EMBEDDINGS

We will introduce a reformulation of $\Pi$-indescribability which will enable us to show that various notions of forcing used in Chapter II and Chapter III preserve certain $\Pi$-indescribable cardinals.

Before we do this it is necessary to review a few elementary facts regarding the logical complexity of some frequently used set theoretical statements. Suppose we code a transitive set M of cardinality $|V_{\kappa+m-1}|$ (where $m \geq 1$) by a binary relation $\mathcal{E}$ on $V_{\kappa+m-1}$.

For any $\mathfrak{X} \in V_{\kappa+m}$ the statement "$\mathfrak{X} \in M$" is $\Sigma_1^m(\mathfrak{X},\mathcal{E})$ over $V_\kappa$ since it is equivalent to

$$\exists \mathfrak{F} \exists X \ \mathfrak{F} \text{ collapses } X \text{ to } \mathfrak{X}$$

where $\mathfrak{F}$ ranges over $V_{\kappa+m}$ and X ranges over $V_{\kappa+m-1}$ and "$\mathfrak{F}$ collapses X to $\mathfrak{X}$" is short for

$$\mathfrak{F} \text{ is a function} \wedge \text{dom } \mathfrak{F} \supseteq \{Y : Y\mathcal{E}X\} \wedge \forall Y \in \text{dom } \mathfrak{F} \ \forall Z\mathcal{E}Y \ \ Z \in \text{dom } \mathfrak{F}$$

$$\wedge \ \forall Y \in \text{dom } \mathfrak{F} \ \mathfrak{F}(Y) = \{\mathfrak{F}(Z) : Z\mathcal{E}Y\} \wedge \mathfrak{X} = \{\mathfrak{F}(Y) : Y\mathcal{E}X\}.$$

Hence this is $\Sigma_0^m(X,\mathfrak{X},\mathcal{E},\mathfrak{F})$.

Actually "$\mathfrak{X} \in M$" is $\Delta_1^m$ $(\mathfrak{X},\mathcal{E})$ over $V_\kappa$ since one can show that "$\mathfrak{X} \notin M$" can also be expressed in a $\Sigma_1^m$ $(X,\mathcal{E})$ way over $V_\kappa$ by using $|V_{\kappa+m} \cap M| \leq |V_{\kappa+m-1}|$.

But we will not need this in the sequel.

Next we examine "$M^{|V_{\kappa+m-2}|} \subseteq M$" (where for m = 1 we mean by this $M^{<\kappa} \subseteq M$). This is equivalent to

$$\forall X\, \exists Y\, \forall Z\, [Z \mathcal{E} Y \iff \exists x\, Z = X_x].$$

Here X, Y, Z range over $V_{\kappa+m-1}$ and x over $V_{\kappa+m-2}$ (for m = 1 x ranges over $V_\kappa$). Recall that any $X \in V_{\kappa+m-1}$ codes $|V_{\kappa+m-2}|$ many elements of $V_{\kappa+m-1}$ via $X_x = \{y \in V_{\kappa+m-2} : (x,y) \in X\}$, where x, y range over $V_{\kappa+m-2}$ (for m = 1 there is an analogous remark). So the whole formula is $\Sigma^m_0(\mathcal{E})$ over $V_\kappa$.

Finally let us investigate the complexity of the statement "M is $\Sigma^m_n$ correct for $\kappa$." For n = 0 recall that by definition this means $M^{|V_{\kappa+m-2}|} \subseteq M$ and we saw above that this is $\Sigma^m_0(\mathcal{E})$ over $V_\kappa$. Recall that for each m, n $\geq$ 1 there is a $\Sigma^m_n$ formula $\chi_{\Sigma^m_n}(.,.)$ that is universal for $\Sigma^m_n$ uniformly for all $V_\alpha$ (where $\alpha$ is a limit ordinal). So for n $\geq$ 1 we can express "M is $\Sigma^m_n$ correct for $\kappa$ in parameters from $V_{\kappa+m}$" by

$\forall X, Y \langle [\exists \mathcal{H}, Z[\mathcal{H}$ collapses Z to $\kappa \wedge \langle V_{\kappa+m-1}, \mathcal{E}\rangle \models$ "X codes a $\Sigma^m_n$ formula $\wedge$

$Y \in V_{Z+m} \wedge \chi^{V_Z}_{\Sigma^m_n}(X,Y)$"] $\Rightarrow$

$\exists \mathcal{F}, k\, \exists \mathcal{G}, \mathcal{Y}[\mathcal{F}$ collapses X to k $\wedge\ \mathcal{G}$ collapses Y to $\mathcal{Y} \wedge V_\kappa \models \chi_{\Sigma^m_n}(k,\mathcal{Y})]]$ .$\wedge$.

$\exists \mathcal{F}, k\, \exists \mathcal{G}, \mathcal{Y}[\mathcal{F}$ collapses X to k $\wedge\ \mathcal{G}$ collapses Y to $\mathcal{Y} \wedge V_\kappa \models \chi_{\Sigma^m_n}(k,\mathcal{Y})] \Rightarrow$

$\exists \mathcal{H}$, Z[$\mathcal{H}$ collapses Z to $\kappa \wedge \langle V_{\kappa+m-1}, \mathcal{E}\rangle \models \chi^{V_Z}_{\Sigma^m_n}(X,Y)]\rangle$

where $\mathcal{Y}$, $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ range over $V_{\kappa+m}$ and X, Y, Z over $V_{\kappa+m-1}$. Hence this formula is $\Delta^m_{n+1}(\mathcal{S},\kappa)$ over $V_\kappa$.

Theorem 1. $(m \geq 1, n \geq 1)$ $\kappa$ is $\Pi^m_n$ indescribable, iff

$\forall M[M \text{ trans} \wedge M \models ZF^- \wedge |M| = \kappa \wedge \kappa \in M \wedge M^{<\kappa} \subseteq M . \Rightarrow .$

$\exists j, N[N \text{ trans} \wedge |N| = |V_{\kappa+m-1}| \wedge N^{|V_{\kappa+m-2}|} \subseteq N \wedge$

$N$ is $\Sigma^m_{n-1}$ correct for $\kappa \wedge j: M \longrightarrow N \wedge \text{cpt}(j) = \kappa]]$.

Proof. Suppose $\kappa$ is $\Pi^m_n$ ind and assume towards a contradiction that for some M as above there is no j, N as above. We can code the structure $\langle M,\in\rangle$ by a binary relation E on $\kappa$ such that under the Mostowski collapsing function $\Pi$ for $\langle\kappa,E\rangle$ we have $\Pi(0) = \kappa$. We let $F \underset{\mathfrak{Df}}{=} \Pi^{-1} \upharpoonright \kappa$ and $T \underset{\mathfrak{Df}}{=} \{(n,\vec{\xi}): n < \omega, \vec{\xi} \in \kappa^{<\omega}$, n is a (code of a) first order formula that holds in $\langle\kappa,E\rangle$ under the assignment $\vec{\xi}\}$. We abbreviate by $\Phi(\kappa,E,F,T)$ the statement

$\langle\kappa,E\rangle$ is wellfounded and extensional $\wedge$

$F = \Pi^{-1} \upharpoonright \kappa \wedge \Pi(0) = \kappa \wedge$

T is the first order theory of $\langle\kappa,E\rangle$.

Note that E, F and T can be coded by subsets of $\kappa$ in the usual way. If $ZF^-$ denotes a sufficiently large finite fragment of ZFC, then "$\langle M,\in\rangle$ is bad" can be expressed as a $\Pi^m_n(\kappa,E,F,T)$ formula over $V_\kappa$ by quantifying over models of $ZF^-$:

$\forall \mathcal{M}[\mathcal{M}$ trans $\wedge\ \mathcal{M} \models ZF^- \wedge |\mathcal{M}| = |V_{\kappa+m-1}| \wedge$

$\mathcal{M}^{|V_{\kappa+m-2}|} \subseteq \mathcal{M} \wedge \mathcal{M}\ \Sigma^m_{n-1}$ correct for $\kappa \wedge \kappa$, E, F, T $\in \mathcal{M}$ . $\Rightarrow$ .

$\mathcal{M} \models$ "$\Phi(\kappa,E,F,T) \wedge \neg \exists j$, N[N trans $\wedge\ |N| = |V_{\kappa+m-1}| \wedge N^{|V_{\kappa+m-2}|} \subseteq N \wedge$

N is $\Sigma^m_{n-1}$ correct for $\kappa \wedge$ j: trans coll$\langle\kappa,E\rangle \longrightarrow N \wedge$ cpt j $= \kappa$]"].

By the $\Pi^m_n$ indescribability of $\kappa$ this formula must reflect to some inccessible $\lambda < \kappa$. Thus we have

$\langle\lambda,E \cap \lambda \times \lambda\rangle$ is wellfounded and extensional

and $F \cap (\lambda \times \lambda) =$ Mostowski collapsing function $^{-1}\restriction \lambda$, and 0 gets collapsed to $\lambda$

and $T \cap (\omega \times \lambda^{<\omega})$ is the first order theory of $\langle\lambda,E \cap (\lambda \times \lambda)\rangle$.

Let $\langle M^*,\in\rangle$ be the transitive collapse of $\langle\lambda,E \cap (\lambda \times \lambda)\rangle$. Since $\langle\lambda,E \cap (\lambda \times \lambda)\rangle \prec \langle\kappa,E\rangle$ and by the choice of E and F there is $j^*: M^* \longrightarrow M$ with cpt $j^* = \lambda$ and $j^*(\lambda) = \kappa$. In the usual way we can construct an elementary submodel $\langle X,\in\rangle < \langle M,\in\rangle$ with $|X| = |V_{\lambda+m-1}|$ and $X^{|V_{\lambda+m-2}|} \subseteq X$ and $j^*[M^*] \cup \{\lambda\} \subseteq X$. Then X is clearly $\Sigma^m_{n-1}$ correct for $\lambda$, since M is. Now collapse X to a transitive N. Denote by j the elementary embedding $j^*$ followed by this collapsing map. Obviously cpt j $= \lambda$ since $V_{\lambda+m-1} \cup \{\lambda\} \subseteq X$. We have just shown

$\exists j$, N[N trans $\wedge\ N = |V_{\lambda+m-1}| \wedge N^{|V_{\lambda+m-2}|} \subseteq N \wedge$

N is $\Sigma^m_{n-1}$ correct for $\lambda \wedge$ j: transcoll $\langle\lambda,E \cap \lambda \times \lambda\rangle \longrightarrow N \wedge$ cpt j $= \lambda$];

but this contradicts the fact that the above $\Pi^m_n$ statement holds at $V_\lambda$.

For the other direction in the theorem suppose that $\Phi(A)$ is $\Pi^m_n$ and $A \subseteq V_\kappa$ and $V_\kappa \models \Phi(A)$. Pick a transitive M with $|M| = |V_\kappa|$ and $M^{<\kappa} \subseteq M$ and $\kappa, A \in M$ and $M \models ZF^-$, where $ZF^-$ is a suitable finite fragment of ZFC. Then let N be transitive and $\Sigma^m_{n-1}$ correct for $\kappa$ and $j : M \longrightarrow N$ with cpt $j = \kappa$. Since $\Phi$ is $\Pi^m_n$ and $N \models ZF^-$ we get

$$N \models \text{“}\exists \alpha < j(\kappa)\, V_\alpha \models \Phi(j(A) \cap V_\alpha);\text{”}$$

hence, by elementarity of j

$$M \models \text{“}\exists \alpha < \kappa\, V_\alpha \models \Phi(A \cap V_\alpha)\text{”}$$

but $(M)_\alpha = V_\alpha$ for $\alpha \leq \kappa$ so that really

$$\exists \alpha < \kappa\, V_\alpha \models \Phi(A \cap V_\alpha);$$

i.e., $\Phi(A)$ reflects. □ end of Theorem 1.

We will conclude this chapter by making some observations which will not be used in the sequel but are nevertheless interesting in their own right.

The first observation involves the concept of hypermeasurability. Recall that a cardinal $\kappa$ is m-hypermeasurable ($m \geq 1$) if there is an elementary embedding of V into some transitive inner model N with critical point $\kappa$ such that $V_{\kappa+m} \subseteq N$. Theorem 1 suggests that $\Pi^m_1$ indescribability can be construed as an analogue of m-hypermeasurability for $m \geq 1$ in the same way that $\Pi^1_1$ indescribability relates to measurability.

[Mitchell, 1979] and Dodd-Jensen (cf. [Dodd, 1982]) have produced an analysis which shows that elementary embeddings $j:V \to M$ can be thought of as coming from an ultrapower by a system of measures. This leads to the notion of an extender. Given a transitive set X and a cardinal $\kappa$, we say that E is an extender with support X and critical point $\kappa$ if E is a function with dom $E = \langle X\rangle^{<\omega}$ (the set of all finite 1:1 sequences of elements of X) and for $s \in \langle X\rangle^{<\omega}$, $E(s) = E_s$ is a $\kappa$ complete measure on $\langle V_\kappa\rangle^{\operatorname{dom} s}$ (the set of all 1:1 sequences of elements of $V_\kappa$ of length dom s) such that for at least one $s \in \langle X\rangle^{<\omega}$ $E_s$ is nonprincipal and

(1) for $i_1, i_2 \in \operatorname{dom} s$ $\{b \in \langle V_\kappa\rangle^{\operatorname{dom} s} : b(i_1) \in b(i_2)\} \in E_s$ iff $s(i_1) \in s(i_2)$

(2) (Coherence) if $s \prec t$ (i.e., s is a subsequence of t) and $s = t \circ \pi$ for some $\pi:\operatorname{dom} s \xrightarrow{1:1} \operatorname{dom} t$ then for all $A \subseteq \langle V_\kappa\rangle^{\operatorname{dom} s}$

$$A \in E_s \text{ iff } A^\pi \in E_t$$

where $A^\pi = \{b \in \langle V_\kappa\rangle^{\operatorname{dom} t} : b \circ \pi \in A\}$

(3) (Normality) for any $s \in \langle X\rangle^{<\omega}$ and $f:\langle V_\kappa\rangle^{\operatorname{dom} s} \to V$ such that

$$\{b \in \langle V_k\rangle^{\operatorname{dom} s} : f(b) \in \cup \{b(i) : i \in \operatorname{dom} s\}\} \in E_s .$$

there exists some $t \in \langle X\rangle^{<\omega}$ with $s \prec t$ such that if $s = t \circ \pi$ with $\pi:\operatorname{dom} s \xrightarrow{1:1} \operatorname{dom} t$ then for some $i_o \in \operatorname{dom} t$

$$\{b \in \langle V_\kappa\rangle^{\operatorname{dom} t} : f(b \circ \pi) = b(i_o)\} \in E_t .$$

Given a transitive set M which models $ZF^-$ and is closed under sequences of length $<\kappa$ we define an M extender with support X and critical point $\kappa$ as above except that for

$s \in \langle X\rangle^{<\omega}$ $E_s$ is an ultrafilter in $P((V^{\kappa})^{\mathrm{dom}\, s}) \cap M$ and that in condition (3) we only consider $f \in M$.

Since each $E_s$ is countably complete and $M^{<\kappa} \subseteq M$ each ultrapower Ult $(M,E_s)$ is wellfounded and we can collapse it to a transitive $\langle M_s, \in\rangle$. Moreover the coherence condition (2) implies that we have a system of elementary embeddings $\langle i_{s,t} : M_s \longrightarrow M_t : s \prec t$ and $s, t \in \langle X\rangle^{<\omega}\rangle$ that commute; i.e., for $r, s, t \in X^{<\omega}$ with $r \prec s \prec t$, $i_{r,t} = i_{s,t} \circ i_{r,s}$. Therefore we can form the direct limit of the system $\langle M_s : s \in \langle X\rangle^{<\omega}\rangle$ which we denote by $\langle M^{\sim}, \in^{\sim}\rangle$. We say that E is a wellfounded extender if $\langle M^{\sim}, \in^{\sim}\rangle$ is wellfounded. Note that E will automatically be wellfounded if we work with countable 1:1 sequences rather than finite 1:1 sequences. In the context of extenders we can rephrase $\Pi^m_1$ $(m \geq 1)$ indescribability as follows:

> $\kappa$ is $\Pi^m_1$ indescribable iff for every transitive model M of $ZF^-$ with $|M| = \kappa$, $M^{<\kappa} \subseteq M$ and $\kappa \in M$ there is a wellfounded M extender with support $V_{\kappa+m-1}$ and critical point $\kappa$.

<u>Proof of this</u> Fix $m \geq 1$.

If $\kappa$ is $\Pi^m_1$ indescribable and M is a transitive model of $ZF^-$ of size $\kappa$ and closed under $<\kappa$ sequences by theorem 1 we can find a transitive N with $V_{\kappa+m-1} \subseteq N$ and an elementary embedding $j : M \longrightarrow N$ with $ctp(j) = \kappa$. Now for $s \in \langle V_{\kappa+m-1}\rangle^{<\omega}$ define $E_s$ by

$$A \in E_s \text{ iff } s \in j(A)$$

for $A \in P((V_\kappa)^{\mathrm{dom}\, s}) \cap M$. Then E is a wellfounded M extender since the directed system $\langle M_s : s \in \langle V_{\kappa+m-1}\rangle^{<\omega}\rangle$ can be represented inside N by using the fact that each $E_s$ is defined from j.

Conversely given any transitive M of size $\kappa$ with $M \models ZF^-$, $\kappa \in M$ and $M^{<\kappa} \subseteq M$ and a wellfounded M extender E with support $V_{\kappa+m-1}$ and critical point $\kappa$ we take N to be the transitive collapse of $\langle M^\sim, \in^\sim\rangle$ and let j denote the usual elementary embedding $j: M \longrightarrow N$ that we obtain from collapsing $\langle M^\sim, \in^\sim\rangle$. Clearly cpt(j) $= \kappa$ and by (1) together with the normality condition (3) $V_{\kappa+m-1} \subseteq N$. Now the proof of the easy direction of theorem 1 shows that $\kappa$ is $\Pi^m_1$ indescribable. $\square$ end of proof.

# CHAPTER II. THE CONSISTENCY STRENGTH OF THE FAILURE OF GCH AT A $\Pi$ INDESCRIBABLE CARDINAL

In this chapter we will evaluate the consistency strength of the failure of GCH at various $\Pi$ indescribable cardinals.

In the simple case of a $\Pi^1_0$ indescribable $\kappa$ one may force any number of new subsets of $\kappa$ and keep $\kappa$ inaccessible since the forcing is $<\kappa$ closed. However already in the case of $\Pi^1_1$ indescribable cardinal this approach fails: If V = L holds in the ground model and we add a new subset $X \subseteq \kappa$ via a $< \kappa$ Baire forcing then in the generic extension the $\Pi^1_1$ statement "$X \notin L$" does not reflect to any inaccessible $\lambda < \kappa$. For a given $\Pi$ indescribable cardinal $\kappa$ in the ground model we will define a forcing iteration of length $\kappa + 1$ and then use the characterization in I.1. to guarantee that $\kappa$ remains $\Pi$ indescribable in any generic extension.

First we look at $\Pi^1_n$ indescribable cardinals ($n \geq 1$). Theorem 1 generalizes a theorem of Silver (cf. [Kunen, 1980]) which says that the failure of GCH at a weakly compact cardinal is equiconsistent with the existence of a weakly compact cardinal.

**Theorem 1.** ($n \geq 1$) CON(ZFC + $\exists$ $\Pi^1_n$ indescribable cardinal) $\Longleftrightarrow$

CON(ZFC + $\exists\kappa$ ($\kappa$ is $\Pi^1_n$ indescribable $\wedge$ $2^\kappa > \kappa^+$))

**Proof.** Suppose $\kappa$ is $\Pi^1_n$ indescribable. We can assume V = L since $\kappa$ remains $\Pi^1_n$ indescribable in L. Fix $\ell \geq 2$, we will define a forcing iteration $P_{\kappa+1}$ such that

$$\Vdash_{P_{\kappa+1}} \text{"}\kappa \text{ is } \Pi^1_n \text{ indescribable} \wedge 2^\kappa = \kappa^{+\ell}\text{."}$$

For limit ordinals $\lambda \leq \kappa$ we let

$$P_\lambda = \lim_{\eta<\lambda} \text{dir } P_\eta \quad \text{if } \lambda \text{ is inaccessible}$$

$$P_\lambda = \lim_{\eta<\lambda} \text{inv } P_\eta \quad \text{otherwise}$$

and for $\alpha \leq \kappa$ let $\mathring{Q}_\alpha \in V^{P_\alpha}$ a canonical term with

$$\Vdash_{P_\alpha} \text{“}\mathring{Q}_\alpha \text{ is the trivial poset if } \alpha \text{ is not inaccessible and}$$

$$\mathring{Q}_\alpha = \text{Fn}(\alpha^{+\ell},2,\alpha) \text{ if } \alpha \text{ is inaccessible”.}$$

Note that for any inaccessible $\lambda \leq \kappa$

$$\forall \alpha < \lambda |P_\alpha| < \lambda$$

and hence for any Mahlo cardinal $\lambda \leq \kappa$

$$P_\lambda \text{ is } \lambda \text{ c.c.}$$

since $\{\alpha < \lambda : P_\alpha = \lim_{\eta<\alpha} \text{dir } P_\eta\}$ is stationary in $\lambda$. Moreover we have

$$\Vdash_{P_\lambda} \text{“}\lambda \text{ is inaccessible”}$$

since for any $\alpha < \lambda$

$$\Vdash_{P_\alpha} \text{“}P_{\alpha,\lambda} \text{ is } {<}\mu \text{ closed”}$$

where $\mu$ is the next inaccessible $\geq\alpha$ because $P_\alpha$ is clearly $\mu$ c.c. Since $|P_\lambda| = \lambda$ we also get

$$\Vdash_{P_\lambda} \text{GCH}^{\geq\lambda}$$

and a straightforward calculation shows

$$\Vdash_{P_{\kappa+1}} 2^{\kappa} = \kappa^{+\ell}.$$

To finish the proof we have to show

$$\Vdash_{P_{\kappa+1}} \text{“}\kappa \text{ is } \Pi^1_n \text{ indescribable.”}$$

Fix a condition $p^* \in P_{\kappa+1}$ and $\Phi$ in $\Pi^1_n$ and a $P_{\kappa+1}$ name $\mathring{A}$ for a subset of $V_\kappa$ and assume towards a contradiction

$$p^* \Vdash_{P_{\kappa+1}} \text{“}\Phi(\mathring{A}) \text{ describes } \kappa.\text{”}$$

By the reflection principle fix an ordinal $\delta >$ the least inaccessible above $\kappa$ with

$$V_\delta \models [ZF^- \wedge p^* \Vdash_{P_{\kappa+1}} \text{“}\Phi(\mathring{A}) \text{ describes } \kappa.\text{”}]$$

Note that in particular $P^{V_\delta}_{\kappa+1} = P_{\kappa+1}$.

Using standard arguments we can find a transitive M with $|M| = \kappa$ and $M^{<\kappa} \subseteq M$ and $\kappa \in M$ and some $p^M$, $\mathring{A}^M \in M$ and an elementary embedding $i : M \longrightarrow V_\delta$ with $\text{cpt } i = (\kappa^+)^M$ and $i(p^M) = p^*$ and $i(\mathring{A}^M) = \mathring{A}$. Since $\kappa$ is $\Pi^1_n$ indescribable we can find a transitive N with $|N| = |V_\kappa| = \kappa$ and $N^{<\kappa} \subseteq N$ and $\Sigma^1_{n-1}$ correct for $\kappa$ and an elementary embedding $j : M \longrightarrow N$ with $\text{cpt } j = \kappa$.

Note that $P_{\kappa+1}$ is $\kappa^+$ cc and $\text{cpt } i = (\kappa^+)^M$; hence we can use the pullback method to find an M generic $G^M$ for $P^M_{\kappa+1}$ and a V generic $G^V$ for $P_{\kappa+1}$ such that $p^* \in G^V$ and i lifts; i.e.,

$$\begin{array}{ccc} M[G^M] & \xrightarrow{i} & V_\delta[G^V] \\ P^M_{\kappa+1} & & P_{\kappa+1} \\ M & \xrightarrow{i} & V_\delta. \end{array}$$

Suppose for a moment that we could find an N generic $G^N$ for $P^N_{j(\kappa)+1}$ such that $N[G^N]$ is $\Sigma^1_{n-1}$ correct in $V[G^V]$ and j lifts; i.e.,

$$\begin{array}{ccc} M[G^M] & \xrightarrow{j} & N[G^N] \\ P^N_{\kappa+1} & & P^N_{j(\kappa)+1} \\ M & \xrightarrow{j} & N. \end{array}$$

Then we would arrive at a contradiction as follows. Since $\Phi$ is $\Pi^1_n$ we get

$$N[G^N] \models \exists\alpha < j(\kappa)\ V_\alpha \models \Phi(j(S) \cap V_\alpha)$$

where

$$S = \mathring{A}^{G^V} = i((\mathring{A}^M)^{G^M}) = (\mathring{A}^M)^{G^M} \in M[G^M]$$

and

$$S = j(S) \cap V_\kappa.$$

Hence by elementarity

$$M[G^M] \models \exists\alpha < \kappa\ V_\alpha \models \Phi(S \cap V_\alpha).$$

But by standard arguments $(M[G^M])_\alpha = (V[G^V])_\alpha$ for $\alpha \leq \kappa$ so that really in $V[G^V]$

$$\exists\alpha < \kappa\ V_\alpha \models \Phi(S \cap V_\alpha),$$

i.e., $\Phi(S)$ reflects, a contradiction. Hence in order to finish the proof we have only to construct $G^N$ with the properties above.

First note that $P^N_\kappa = P_\kappa$ since $N^{<\kappa} \subseteq N$. $P_\kappa \subseteq V_\kappa$ implies that we have $j(p) = p$ for all $p \in P_\kappa$. Let $G_\kappa = G^V \bigcap P_\kappa$. For any H that is $N[G_\kappa]$ generic for the tail $P^N_{\kappa,j(\kappa)}$ j will lift; i.e.,

$$\begin{array}{ccc} M[G_\kappa] & \longrightarrow & N[G_\kappa * H] \\ P_\kappa & & P^N_{j(\kappa)} \\ M & \longrightarrow & N. \end{array}$$

Now we construct an H that is $P^N_{\kappa,j(\kappa)}$ generic over $N[G_\kappa]$ such that $N[G_\kappa * H]$ is $\Sigma^1_{n-1}$ correct for $\kappa$ inside $V[G^V]$. First we need the following:

<u>Lemma</u> <u>1.1</u>. $N[G_\kappa]$ is $\Sigma^1_{n-1}$ correct for $\kappa$ inside $V[G_\kappa]$.

<u>Proof</u> <u>of</u> <u>1.1</u>. Since $\bar{P}$ is $\kappa$ c.c. and $N^{<\kappa} \subseteq N$ we get that $N[G_\kappa]^{<\kappa} \subseteq N[G_\kappa]$ in $V[G_\kappa]$ and hence $(N[G_\kappa])_\alpha = (V[G_\kappa])_\alpha$ for $\alpha \leq \kappa$ since $\kappa$ is inaccessible in $V[G]_\kappa$. Thus $N[G_\kappa]$ is $\Sigma^1_0$ correct for $\kappa$ inside $V[G_\kappa]$.

Before we continue with the proof we need the following:

<u>Fact</u> <u>1.2</u> ($\ell \geq 0$). Suppose $P \subseteq V_\kappa$ is $\kappa$ c.c. and $\tau \in V^P$. Then we can find $\tau^* \in V^P \cap V_{\kappa+\ell}$ with

$$\Vdash_P (\tau \in V_{\kappa+\ell} \Rightarrow \tau^* = \tau).$$

Proof of Fact 1.2. We use induction on $\ell$ and start with $\ell = 0$. Suppose we are given $\tau \in V^P$ and we already know that the claim holds for all terms of smaller rank than the rank of $\tau$.

Since P is $\kappa$ c.c. there is a cardinal $\lambda < \kappa$ with $\Vdash_P (\tau \in V_\kappa \Rightarrow |\tau| \leq \lambda)$. We pick a term $\mathring{f} \in V^P$ with $\Vdash_P (\tau \in V_\kappa \Rightarrow \mathring{f}: \lambda \xrightarrow{\text{onto}} \tau)$. For each $\alpha < \lambda$ we let

$$A_\alpha \text{ a max antichain } \{p \in P: \exists \sigma \in \operatorname{dom} \tau \; p \Vdash_P \mathring{f}(\alpha) = \sigma\}$$

and for each $p \in A_\alpha$ we pick $\sigma_{p,\alpha} \in \operatorname{dom} \tau$ with

$$p \Vdash_P \sigma_{p,\alpha} = \mathring{f}(\alpha).$$

Since for each $\sigma \in \operatorname{dom} \tau$ rank $(\sigma) <$ rank$(\tau)$ we can find terms $\sigma^*_{p,\alpha} \in V^P \cap V_\kappa$ such that

$$\Vdash_P (\sigma_{p,\alpha} \in V_\kappa \Rightarrow \sigma^*_{p,\alpha} = \sigma_{p,\alpha})$$

now define

$$\tau^* \underset{\mathfrak{Df}}{=} \{(\sigma^*_{p,\alpha}, p): p \in A_\alpha, \alpha < \lambda\}$$

then $\tau^* \subseteq V_\kappa$ and $|\tau^*| < \kappa$; hence $\tau^* \in V_\kappa$ and moreover

$$\Vdash_P (\tau \in V_\kappa \Rightarrow \tau^* = \tau).$$

Now suppose we have arrived at stage $\ell + 1$. Let

$$\tau^* \underset{\mathfrak{Df}}{=} \{(\sigma^*,p): (\sigma,p) \in \mathrm{dom}\ \tau\}$$

where for $\sigma \in \mathrm{dom}\ \tau \quad \sigma^* \in V^P \cap V_{\kappa+\ell}$ with

$$\Vdash_P (\sigma \in V_{\kappa+\ell} \Rightarrow \sigma = \sigma^*);$$

clearly $\tau^* \in V_{\kappa+\ell+1}$ (if we use flat pairing) and

$$\Vdash_P (\tau \in V_{\kappa+\ell+1} \Rightarrow \tau = \tau^*).$$

□ end of 1.2.

We can now continue with the proof of the lemma. We proceed by induction and show that for $k \leq n - 1$ $N[G_\kappa]$ is $\Sigma^1_k$ correct for $\kappa$ inside $V[G_\kappa]$. Let $k + 1 \leq n - 1$; it is enough to consider $\Phi(A)$ in $\Pi^1_{k+1}$ with $A \in (N[G_\kappa])_{\kappa+1}$ and $(V_\kappa \models \Phi(A))^{N[G_\kappa]}$ and to show that $(V_\kappa \models \Phi(A))^{V[G_\kappa]}$ since by induction hypothesis $N[G_\kappa]$ is $\Sigma^1_k$ correct in $V[G_\kappa]$ for $\kappa$. Pick $p \in G_\kappa$ and $\mathring{A} \in N^{P_\kappa} \cap V_{\kappa+1}$ (by applying fact 1.2. in N) such that $\mathring{A}^{G_\kappa} = A$ and $p \Vdash^N_{P_\kappa}$ "$V_k \models \Phi(\mathring{A})$."

Now in N (sine $\Phi$ is $\Pi^1_{\kappa+1}$):

$$\forall \mathcal{M}[\mathrm{trans}\ \mathcal{M} \wedge \mathcal{M} \models ZF^- \wedge |\mathcal{M}| = |V_\kappa| \wedge \mathcal{M}^{<\kappa} \subseteq \mathcal{M}$$

$$\wedge\ \mathcal{M}\ \Sigma^1_k \text{ correct for } \kappa \wedge P_\kappa, \mathring{A} \in \mathcal{M}\ .\Rightarrow .$$

$$\mathcal{M} \models [p \Vdash_{P_\kappa} \text{“}V_\kappa \models \Phi(\mathring{A})\text{”}]]$$

by using the induction hypothesis in N. But this formula is $\Pi^1_{k+1}(p,P_\kappa,\mathring{A},\kappa)$. Hence the same formula holds in V, but then by reflection principle in V

$$p \Vdash^N_{P_\kappa} \text{“}V_\kappa \models \Phi(\mathring{A}).\text{”}$$

So

$$(V_\kappa \models \Phi(A))^{V[G_\kappa]}. \qquad \square$$

end of 1.1.

Now we consider $N[G_\kappa]$'s version of the forcing at stage $\kappa$. Since $N[G_\kappa]^{<\kappa} \subseteq N[G_\kappa]$ in $V[G_\kappa]$ this is the initial segment of length $(\kappa^{+\ell})^{N[G_\kappa]}$ of the forcing $\mathrm{Fn}^{V[G_\kappa]}(\kappa^{+\ell},2,\kappa)$ that $V[G_\kappa]$ wants to do at stage $\kappa$. If G denotes the $V[G_\kappa]$ generic for $\mathrm{Fn}^{V[G_\kappa]}(\kappa^{+\ell},2,\kappa)$ coming from $G^V$, then $g \underset{\mathfrak{Df}}{=} G \cap \mathrm{Fn}^{N[G_\kappa]}(\kappa^{+\ell},2,\kappa)$ is certainly generic over $N[G_\kappa]$. Moreover the following lemma shows that $N[G_\kappa,g]$ is $\Sigma^1_{n-1}$ correct for $\kappa$ in $V[G^V]$.

Lemma 1.3. Suppose $\mathcal{N} \models ZF^-$ and $\mathcal{N}^{<\kappa} \subseteq \mathcal{N}$ and $\mathcal{N}$ is $\Sigma^1_n$ correct ($n \geq 0$) for $\kappa$. If G is $\mathrm{Fn}(\kappa^{+\ell},2,\kappa)$ generic over V and $g = G \cap \mathrm{Fn}((\kappa^{+\ell})^{\mathcal{N}},2,\kappa)$ then $\mathcal{N}[g]$ is $\Sigma^1_n$ correct in V[G] for $\kappa$.

Proof of 1.3. Note that $\mathcal{N}[g]^{<\kappa} \subseteq \mathcal{N}[g]$ in V[G] since all posets are $<\kappa$ closed (we can assume that $\mathcal{N}$ satisfied choice). We proceed by induction on n. To handle n + 1 it is enough to consider $\Phi(A)$ in $\Pi^1_{n+1}$ with $A \in (\mathcal{N}[g])_{\kappa+1}$ and $(V_\kappa \models \Phi(A))^{\mathcal{N}[g]}$ and to show that $(V_\kappa \models \Phi(A))^{V[G]}$ since by induction hypothesisN[g] is $\Sigma^1_n$ correct in V[G] for $\kappa$. Pick $p \in g$ and a nice $\mathrm{Fn}((\kappa^{+\ell})^{\mathcal{N}},2,\kappa)$ name $\mathring{A}$ for $A \subseteq V_\kappa$ with $p \Vdash^{\mathcal{N}}_{\mathrm{Fn}((\kappa^{+\ell})^{\mathcal{N}},2,\kappa)}$ "$V_k \models \Phi(\mathring{A})$".

Since $\mathrm{Fn}(\kappa^{+\ell},2,\kappa)$ has the $\kappa^+$ c.c. there is a complete suborder $Q \in \mathcal{N}$ of $\mathrm{Fn}((\kappa^{+\ell})^{\mathcal{N}},2,\kappa)$ with $|Q|^{\mathcal{N}} = \kappa$ and $p \in Q$ and $\mathring{A} \in \mathcal{N}^Q$. Moreover $\mathrm{Fn}((\kappa^{+\ell})^{\mathcal{N}},2,\kappa) \approx Q \times \mathrm{Fn}(\kappa^{+\ell})^{\mathcal{N}},2,\kappa)$ in $\mathcal{N}$ and $\mathrm{Fn}(\kappa^{+\ell},2,\kappa) \approx Q \times \mathrm{Fn}(\kappa^{+\ell},2,\kappa)$ in V. Hence if $\tilde{g} = g \cap Q$ then in $\mathcal{N}[\tilde{g}]$ $\Vdash_{\mathrm{Fn}((\kappa^{+\ell},2,\kappa)}$ "$V_\kappa \models \Phi(A)$." Note that $A \in (\mathcal{N}[\tilde{g}])_{\kappa+1}$. The following sublemma shows that $\mathcal{N}[\tilde{g}]$ is $\Sigma^1_{n+1}$ correct in $V[\tilde{g}]$ for $\kappa$.

<u>Sublemma</u> <u>1.3.1</u>. Suppose $\mathcal{N} \models ZF^-$ and $\mathcal{N}^{<\kappa} \subseteq \mathcal{N}$ and $\mathcal{N}$ is $\Sigma^1_m$ correct $(m \geq 0)$ for $\kappa$. If $Q \in \mathcal{N}$ is a $<\kappa$ Baire poset with $|Q| = \kappa$ in $\mathcal{N}$, then for any G which is V generic for Q, $\mathcal{N}[G]$ is $\Sigma^1_m$ correct for $\kappa$ in V[G].

<u>Proof</u> <u>of</u> <u>1.3.1</u>. Note that $\mathcal{N}[G]^{<\kappa} \subseteq \mathcal{N}[G]$ in V[G] since Q is $<\kappa$ Baire. We now proceed by induction on m.

To handle m + 1 it is enough to consider $\Phi(B)$ in $\Pi^1_{m+1}$ with $B \in (\mathcal{N}[G])_{\kappa+1}$ and $(V_\kappa \models \Phi(B))^{\mathcal{N}[G]}$ and to show that $(V_\kappa \models \Phi(B))^{V[G]}$ since by induction hypothesis $\mathcal{N}[G]$ is $\Sigma^1_m$ correct in V[G] for $\kappa$. Pick $q \in G$ and $\mathring{B} \in \mathcal{N}^Q$ a nice Q name for $B \subseteq V_\kappa$ with $q \Vdash^{\mathcal{N}}_Q V_\kappa \models \Phi(\mathring{B})$. Since $|Q| = \kappa$, we can assume that $Q \subseteq V_\kappa$; hence $\mathring{B} \in V_{\kappa+1}$. Now in $\mathcal{N}$ (since $\Phi$ is $\Pi^1_{m+1}$):

$$\forall \mathcal{M}[\text{trans } \mathcal{M} \wedge \mathcal{M} \models ZF^- \wedge |\mathcal{M}| = |V_\kappa| \wedge \mathcal{M}^{<\kappa} \subseteq \mathcal{M}$$

$$\wedge\ \mathcal{M}\ \Sigma^1_m \text{ correct for } \kappa \wedge Q, \mathring{B} \in \mathcal{M}\ .\Rightarrow .$$

$$\mathcal{M} \models q \Vdash_Q \text{“}V_\kappa \models \Phi(\mathring{B})\text{”}].$$

by using the induction hypothesis in $\mathcal{N}$. But this formula is $\Pi^1_{m+1}(q,Q,\mathring{B},\kappa)$. Hence the same formula holds in V, but then by reflection principle in V

$$q \Vdash_{Q} \text{“}V_\kappa \models \Phi(\mathring{B})\text{”}$$

so

$$(V_\kappa \models \Phi(B))^{V[G]}.$$

□
end of 1.3.1.

Since $\Phi$ is $\Pi^1_{n+1}$ the induction hypothesis applied within $\mathcal{N}[\tilde{g}]$ tells us that in $\mathcal{N}[\tilde{g}]$

$$\forall \mathcal{M}[\text{trans } \mathcal{M} \wedge \mathcal{M} \models ZF^- \wedge |\mathcal{M}| = |V_\kappa| \wedge \mathcal{M}^{<\kappa} \subseteq \mathcal{M} \wedge$$

$$\mathcal{M}\ \Sigma^1_n \text{ correct for } \kappa \wedge A \in \mathcal{M} . \Rightarrow .$$

$$\mathcal{M} \models \Vdash_{\overline{Fn(\kappa^{+\ell},2,\kappa)}} \text{“}V_\kappa \models \Phi(A)\text{”}].$$

But this formula is $\Pi^1_{n+1}(\kappa,A)$; hence by the sublemma it must hold in V[$\tilde{g}$], so in V[$\tilde{g}$]

$$\Vdash_{\overline{Fn(\kappa^{+\ell},2,\kappa)}} \text{“}V_\kappa \models \Phi(A).\text{”}$$

Therefore

$$(V_\kappa \models \Phi(A))^{V[G]}.$$

□
end of 1.3.

Next we have to construct h which is $N[G_\kappa,g]$ generic for the tail $P^{N[G_\kappa,g]}_{\kappa+1,j(\kappa)}$. h can be constructed in the usual way since $N[G_\kappa,g]^{<\kappa} \subseteq N[G_\kappa,g]$ in $V[G^V]$ and the tail is certainly $<\kappa$ closed and $|N[G_\kappa,g]| = \kappa$. Moreover $N[G_\kappa,g,h]$ will be $\Sigma^1_{n-1}$ correct in $V[G^V]$ since the tail is highly closed.

So we have arrived at

$$\begin{array}{ccc} M[G_\kappa] & \xrightarrow[j]{} & N[j(G_\kappa)] \\ P_\kappa & & P^N_{j(\kappa)} \\ M & \xrightarrow[j]{} & N \end{array}$$

where $j(G_\kappa) = G_\kappa * g * h$ and $N[j(G_\kappa)]$ is $\Sigma^1_{n-1}$ correct in $V[G^V]$ for $\kappa$.

In our final step we use a straightforward master condition argument to find a $N[j(G_\kappa)]$ generic K for $Fn^{N[j(G_\kappa)]}(j(\kappa)^{+\ell}, 2, j(\kappa))$ such that j lifts; i.e.,

$$\begin{array}{ccc} M[G^M] & \xrightarrow[j]{} & N[G^N] \\ M[G_\kappa] & \xrightarrow[j]{} & N[j(G_\kappa)] \end{array}$$

where $G^N = j(G_\kappa) * K$. K can be constructed in $V[G^V]$ by using that $Fn^{N[j(G_\kappa)]}(j(\kappa)^{+\ell}, 2, j(\kappa))$ is $<\kappa$ closed in $V[G^V]$ and $|N[j(G_\kappa)]| = \kappa$. $N[G^N]$ will still be $\Sigma^1_{n-1}$ correct for $\kappa$ in $V[G^V]$ since $Fn(j(\kappa)^{+\ell}, 2, j(\kappa))$ is $<j(\kappa)$ closed in $N[j(G_\kappa)]$.

□
end of theorem 1.

Actually the case m = 1 is very special when one tries to make GCH fail at a $\Pi^m_n$ indescribable cardinal. This is because of the following

FACT: if one can force $\kappa$ to be $\Pi^1_n$ ($n \geq 0$) indescribable by adding $\kappa^+$ many subsets of $\kappa$ then adding any number $\lambda > \kappa$ of subsets of $\kappa$ will force $\kappa$ to be $\Pi^1_n$ indescribable.

Thus in order to obtain the consistency of "$\kappa$ is $\Pi^1_n$ indescribable and $2^\kappa = \lambda$" (where $\lambda$ is some cardinal $>\kappa^+$) it is sufficient to define $P_{\kappa+1}$ by adding at any inaccessible stage $\mu \leq \kappa$ $\mu^+$ many subsets of $\mu$ and then to show that $\Vdash_{P_{\kappa+1}}$ "$\kappa$ is $\Pi^1_n$ indescribable" if $\kappa$ was already $\Pi^1_n$ indescribable in the ground model.

Next we evaluate the consistency strength of the failire of GCH at a $\Pi^m_n$ indescribable with $m \geq 2$, $n \geq 1$. It will turn out that this is consistencywise stronger than the existence of a $\Pi^m_n$ indescribable.

Theorem 2. ($m \geq 2$, $n \geq 1$, $\ell \geq 2$) $\mathrm{CON}(\mathrm{ZFC} + \exists\kappa(\Pi^m_n \text{ ind } \kappa \wedge 2^\kappa = \kappa^{+\ell})) \Longleftrightarrow.$ $\mathrm{CON}(\mathrm{ZFC} + \exists\kappa\ \Pi^{m+\ell-1}_n \text{ ind } \kappa)$.

Proof. "$\Leftarrow$"

Assume that $V = L$ and $\kappa$ is $\Pi^{m+\ell-1}_n$ indescribable. We define an iteration $P_{\kappa+1}$ by exactly the same clauses as in the proof of theorem 1 and we claim that

$$\Vdash_{P_{\kappa+1}} \text{"}\kappa \text{ is } \Pi^m_n \text{ indescribable} \wedge 2^\kappa = \kappa^{+\ell}\text{"}.$$

The proof closely follows the ideas in the proof of theorem 1 and we will keep the notation that we used as much as possible. The hard part of the proof is again to show that

$$\Vdash_{P_{\kappa+1}} \text{"}\kappa \text{ is } \Pi^m_n \text{ indescribable."}$$

Now we know that $\kappa$ is $\Pi^{m+\ell-1}_n$ indescribable in the ground model; hence there is a trans $N$, with $N \models ZF^-$, $|N| = |V_{\kappa+m+\ell-2}|$, ${}^{|V_{\kappa+m+\ell-3}|}N \subseteq N$ and $N$ $\Sigma^{m+\ell-1}_{n-1}$ correct for $\kappa$ and an elementary embedding $j: M \longrightarrow N$ with cpt $j = \kappa$. After constructing $G^M$ and $G^V$ as in the proof of 1 we have only to come up with a $P^N_{j(\kappa)+1}$ generic $G^N$ such that $N[G^N]$ is $\Sigma^m_{n-1}$ correct in $V[G^V]$ and j lifts; i.e.,

$$\begin{array}{ccc} M[G^M] & \longrightarrow & N[G^N] \\ P^M_{\kappa+1} & & P^N_{j(\kappa)+1} \\ M & \longrightarrow & N. \end{array}$$

As in the proof of theorem 1 $G_\kappa$ is $P^N_\kappa = P_\kappa$ generic but now $N[G_\kappa]^{\kappa^{+(m+\ell-3)}} \subseteq N[G_\kappa]$ in $V[G_\kappa]$ and $N[G_\kappa]$ is $\Sigma^{m+\ell-1}_{n-1}$ correct in $V[G_\kappa]$. Note that certainly $(\kappa^{+\ell})^{N[G_\kappa]} = (\kappa^{+\ell})^{V[G_\kappa]}$. So $N[G_\kappa]$ wants to do the same forcing at stage $\kappa$ of $P^N_{j(\kappa)+1}$ that $V[G_\kappa]$ wants to do at stage $\kappa$ of $P_{\kappa+1}$; namely, $Fn(\kappa^{+\ell},2,\kappa)$ which we denote by P from here on. Before we continue with the proof we need some technical lemmas.

Note that P is $\kappa^+$ c.c. and $<\kappa$ closed and has size $\kappa^{+\ell}$, so we can regard it as a subset of $V_{\kappa+\ell}$. Moreover $\Vdash_P V_{\check{\kappa}} = (V_\kappa)^\vee$.

Working in $N[G_\kappa]$ we define by induction on $r \geq 0$ names $\mathring{V}_{\kappa+r} \in (N[G_\kappa])^P$. Let $\mathring{V}_\kappa = (V_\kappa)^\vee$ and to define $\mathring{V}_{\kappa+r+1}$ choose for each nice P name $\tau \in N[G_\kappa]^P$ for a subset of $\mathring{V}_{\kappa+r}$ a maximal antichain $A_\tau \subseteq \{p \in P : p \Vdash^{N[G_\kappa]}_{P} \tau \subseteq \mathring{V}_{\kappa+r}\}$. Then let

$$\mathring{V}_{\kappa+r+1} = \bigcup_{\substack{\tau \text{ a nice P name} \\ \text{for a subset of } \mathring{V}_{\kappa+r}}} \{\tau\} \times A_\tau .$$

Lemma 2.1. (r $\geq$ 0)

$$\Vdash^{N[G_\kappa]}_{P} \overset{\circ}{V}_{\kappa+r} = V_{\check{\kappa}+r}$$

Proof. We use induction on r $\geq$ 0. By the $<\kappa$ closure of P $\Vdash^{N[G_\kappa]}_{P} \overset{\circ}{V}_\kappa = V_{\check{\kappa}}$.

Now assume G is P generic over $N[G_\kappa]$ and $\overset{\circ}{X} \in (N[G_\kappa])^P$ with $\overset{\circ}{X}^G \subseteq (N[G_\kappa,G])_{\kappa+r}$. By induction hypothesis pick $p_o \in G$ such that $p_o \Vdash^{N[G_\kappa]}_{P} \overset{\circ}{X} \subseteq \overset{\circ}{V}_{\kappa+r}$. Choose a nice name $\tau \in (N[G_{\kappa k}])^P$ for subset of $\overset{\circ}{V}_{\kappa+r}$ with $p_o \Vdash \overset{\circ}{X} = \tau$. Then $\{q \leq p_o : \exists p \geq q\ p \in A_\tau\}$ is dense below $p_o$; hence $A_\tau \cap G \neq \emptyset$. So $\overset{\circ}{X}^G = \tau^G \in \overset{\circ}{V}^G_{\kappa+r+1}$. Conversely if $\overset{\circ}{X}^G \in \overset{\circ}{V}^G_{\kappa+r+1}$ then, by induction hypothesis clearly $\overset{\circ}{X}^G \in (N[G_\kappa,G])_{\kappa+r+1}$.

□ end of 2.1.

Lemma 2.2. (r $\geq$ 1) every $\overset{\circ}{V}_{\kappa+r}$ and every nice P name for a subset of $\overset{\circ}{V}_{\kappa+r}$ can be coded in a highly absolute way by an element of $V_{\kappa+r+\ell}$.

Proof. Since $P \subseteq V_{\kappa+\ell}$ has the $\kappa^+$ c.c. every P antichain A has a code $\tilde{A} \in V_{\kappa+\ell}$. Now we proceed by induction on r.

r = 1: a nice P name $\overset{\circ}{X}$ for a subset of $V_\kappa$ looks like this:

$$\overset{\circ}{X} = \bigcup_{x \in V_\kappa} \{\check{x}\} \times A_x$$

where each $A_x$ is an antichain in P. Now we code $\overset{\circ}{X}$ by $\tilde{\overset{\circ}{X}} \underset{\mathfrak{Df}}{=}$ code of $\{(x,\tilde{A}_x) : x \in V_\kappa\}$ which is $\in V_{\kappa+\ell}$ since $|V_\kappa| = \kappa$.

Hence we can code $\overset{\circ}{V}_{\kappa+1}$ by

$$\tilde{\overset{\circ}{V}}_{\kappa+1} = \{(\tilde{\tau},\tilde{A}_{\tau}) : \tau \text{ a nice P name for a subset of } V_{\kappa}\}$$

clearly $\tilde{\overset{\circ}{V}}_{\kappa+1} \in V_{\kappa+1+\ell}$ .

A nice P name $\overset{\circ}{X}$ for a subset of $\overset{\circ}{V}_{\kappa+1}$ looks like this:

$$\overset{\circ}{X} = \bigcup_{\tau \in \mathrm{dom} \overset{\circ}{V}_{\kappa+1}} \{\tau\} \times B_{\tau}$$

where each $B_{\tau}$ is an antichain in P. By definition each $\tau \in \mathrm{dom}\ \overset{\circ}{V}_{\kappa+1}$ is a nice P name for a subset of $V_{\kappa}$; hence each $\tau \in \mathrm{dom}\ \overset{\circ}{V}_{\kappa+1}$ has a code $\tilde{\tau} \in V_{\kappa+\ell}$.

So we can code $\overset{\circ}{X}$ by

$$\tilde{\overset{\circ}{X}} = \{(\tilde{\tau},\tilde{B}_{\tau}) : \tau \in \mathrm{dom}\ \overset{\circ}{V}_{\kappa+1}\}.$$

Thus, clearly, $\tilde{\overset{\circ}{X}} \in V_{\kappa+1+\ell}$. The induction step is now straightforward using exactly the same argument.

Finally it is clear that there is a weak fragment $ZF^{-}$ of ZF such that for any trans M with $M \models ZF^{-}$ and $\kappa$, $P \in M$ M can correctly compute the coding and decoding if it has enough closure. □ end of 2.2.

Let us denote by G the P generic that we add at stage $\kappa$ of $P_{\kappa+1}$ (i.e., $G^{V} = G_{\kappa} * G$).

<u>Lemma</u> <u>2.3</u>. $N[G_{\kappa}, G]$ is $\Sigma^{m}_{n-1}$ correct for $\kappa$ in $V[G^{V}]$.

Proof. Note that in $V[G^V]$ $N[G_\kappa,G]^{\kappa^{+(m+\ell-3)}} \subseteq N[G_\kappa,G]$ because P is $\kappa^+$ c.c. An easy calculation shows that in $V[G^V]$ $|(V[G^V])_{\kappa+m-2}| = \kappa^{+(m+\ell-3)}$. Therefore we get that $(V[G^V])_{\kappa+m-1} = (N[G_\kappa,G])_{\kappa+m-1}$. If $n \geq 2$ we proceed by induction on $0 \leq k \leq n-1$. In order to handle $k+1 \leq n-1$ it is enough to consider $\Phi(A)$ in $\Pi^m_{k+1}$ with $A \in (N[G_\kappa,G])_{\kappa+m}$ and $N[G_\kappa,G] \models \Phi(A)$ and to show that $V[G^V] \models \Phi(A)$.

We pick a nice P name $\mathring{A} \in N[G_\kappa]^P$ for a subset of $\mathring{V}_{\kappa+m-1}$ with $\mathring{A}^G = A$ and $p \in G$ with

$$p \|\frac{N[G_\kappa]}{P} \text{“}V_\kappa \models \Phi(\mathring{A}).\text{”}$$

Note that by lemma 2.2 $\mathring{A}$ has a code $\tilde{\mathring{A}} \in V_{\kappa+m+\ell-1}$ such that decoding $\mathring{A}$ from $\tilde{\mathring{A}}$ is highly absolute. Now we have

$$\forall \mathcal{M}[\mathcal{M} \text{ trans} \wedge \mathcal{M} \models ZF^- \wedge \mathcal{M}^{|V_{\kappa+m+\ell-3}|} \subseteq \mathcal{M} \wedge |\mathcal{M}| = |V_{\kappa+m+\ell-2}|$$

$$\wedge \mathcal{M}\ \Sigma^{m+\ell-1}_k \text{ correct for } \kappa \wedge P, \tilde{\mathring{A}} \in \mathcal{M} \Rightarrow$$

$$\mathcal{M} \models \text{“}p \Vdash_P \text{“}V_\kappa \models \Phi(\text{the nice P name for a subset of } \mathring{V}_{\kappa+m-1} \text{ coded by } \tilde{\mathring{A}})\text{”}].$$

This holds because $\Phi$ is $\Pi^m_{k+1}$ and we can apply the induction hypothesis within $N[G_\kappa]$. Since this last formula is $\Pi^{m+\ell-1}_{k+1}(\kappa,P,\tilde{\mathring{A}})$ it must hold in $V[G_\kappa]$ because

$N[G_\kappa]$ is $\Sigma^{m+\ell-1}_{n-1}$ correct for $\kappa$ in $V[G_\kappa]$.

Hence we get in $V[G_\kappa]$

$$p\Vdash^{V[G_\kappa]}_{P} \text{“}V_\kappa \models \Phi(\mathring{A}).\text{”}$$

Therefore we conclude that in $V[G^V]$

$$V_\kappa \models \Phi(A).$$

□
end of 2.3.

Now we continue as in the proof of theorem 1 and construct H that is $N[G_\kappa,G]$ generic for the tail $P^N_{\kappa+1,j(k)}$. This is done in the usual way by observing that the tail is $<\mu$ closed in $N[G_\kappa,G]$ where $\mu$ is the least inaccessible $> \kappa + 1$ and by recalling that $|N[G_\kappa,G]| = \kappa$. Moreover the closure of the tail will yield that $N[G_\kappa,G,H]$ is $\Sigma^m_{n-1}$ correct in $V[G^V]$ and $N[G_\kappa,G,H]^{\kappa^{+(m+\ell-3)}} \subseteq N[G_\kappa,G,H]$ in $V[G^V]$. Since $j(p) = p$ for $p \in P_\kappa$ j lifts; i.e.,

$$\begin{array}{ccc} M[G^M] & \longrightarrow & N[j(G_\kappa)] \\ P_\kappa & & P^N_{j(\kappa)} \\ M & \longrightarrow & N \end{array}$$

where $j(G_\kappa) = G_\kappa * G * H$.

In our final step as in the proof of theorem 1 we use a straightforward master condition argument to pick K which is $N[j(G_\kappa)]$ generic for $Fn(j(\kappa)^{+\ell},2,j(\kappa))$ such that j lifts; i.e.,

$$\begin{array}{ccc} M[G^M] & \longrightarrow & N[G^N] \\ P^M_{\kappa+1} & & P^N_{j(\kappa)+1} \\ M & \longrightarrow & N \end{array}$$

where $G^N = G_\kappa * G * H * K$. $N[G^N]$ will be $\Sigma^m_{n-1}$ correct in $V[G^V]$ since $Fn(j(\kappa)^{+\ell}, 2, j(\kappa))$ is $<j(\kappa)$ closed in $N[j(G_\kappa)]$.

"$\Rightarrow$"

suppose $\kappa$ is $\Pi^m_n$ indescribable and $2^\kappa = \kappa^{+\ell}$; we claim that

$$(\kappa \text{ is } \Pi^{m+\ell-1}_n \text{ ind})^L.$$

Let $\Phi(A) \equiv \forall \mathfrak{X}\, \psi(\mathfrak{X}, A)$ where $\psi$ is $\Sigma^{m+\ell-1}_{n-1}$ and $\mathfrak{X}$ ranges over $V_{\kappa+m+\ell-1}$ and $A \in V_{\kappa+1} \cap L$ and assume

$$(V_\kappa \models \Phi(A))^L.$$

If $n \geq 2$ then we need the following lemma before we continue with the proof.

<u>Lemma</u> <u>2.4</u>. For $n \geq 2$, $\psi^L(\mathfrak{X}, A)$ is $\Sigma^m_{n-1}(\mathfrak{X}, A, \kappa)$ over $V_\kappa$.

<u>Proof</u>. Let T be a finite fragment of ZF + V = L such that for any trans model M of T we have $M = L_{On \cap M}$ and write $\psi(\mathfrak{X}, A)$ as

$$\exists \mathfrak{X}_1 \cdots Q\mathfrak{X}_{n-1}\, \theta(\mathfrak{X}, \vec{\mathfrak{X}}_i, A)$$

where $Q \in \{\exists, \forall\}$ and the $\mathfrak{X}_i$ range over $V_{\kappa+m+\ell-1}$ and $\theta$ is $\Sigma^{m+\ell-1}_0$.

Since $V_{\kappa+m+\ell-1} \cap L \subseteq L_{\kappa^{+(m+\ell-1)}}$ $(V_\kappa \models \psi(\mathfrak{B},A))^L$ is equivalent to

$$\exists\gamma_1 < \kappa^{+(m+\ell-1)}\ \exists\mathfrak{B}_1 \in L\gamma_1$$

$$\vdots$$

$$Q\gamma_{n-1} < \kappa^{+(m+\ell-1)}\ Q\mathfrak{B}_{n-1} \in L\gamma_{n-1}$$

$$\theta^L(\mathfrak{B},\vec{\mathfrak{B}}_i,A).$$

Now $\kappa^{+(m+\ell-2)} \le |V_{\kappa+m-1}|$ allows us to code each $L\gamma_i$ with $\gamma_i < \kappa^{+(m+\ell-1)}$ by a subset $\mathfrak{Y}_i \subseteq V_{\kappa+m-1}$; then the last formula is equivalent to

$$\exists\mathfrak{Y}_1 \models T \cdots Q\mathfrak{Y}_{n-1} \models T\ Q\,\mathcal{M}[\mathcal{M}\text{ trans} \wedge \mathcal{M} \models ZF^- \wedge$$

$$|\mathcal{M}| = |V_{\kappa+m-1}| \wedge \mathcal{M}^{|V_{\kappa+m-2}|} \subseteq \mathcal{M} \wedge \mathfrak{B}, \mathfrak{Y}_1, \ldots, \mathfrak{Y}_{n-1}, A \in \mathcal{M} \cdot \overset{\Rightarrow}{\wedge} \cdot$$

$$\mathcal{M} \models \exists\mathfrak{B}_1 \in \text{transcoll}\ \mathfrak{Y}_1 \cdots Q\mathfrak{B}_{n-1} \in \text{transcoll}\ \mathfrak{Y}_{n-1}\ \theta^L(\mathfrak{B},\vec{\mathfrak{B}}_i,A)].$$

Note that for any such $\mathcal{M}$ $\kappa^{+(m+\ell-2)} = (\kappa^{+(m+\ell-2)})^{\mathcal{M}}$ since $|V_{\kappa+m-2}| \ge \kappa^{+(m+\ell-3)}$; hence $V_{\kappa+m+\ell-2} \cap L \subseteq L^{\mathcal{M}}$ so that $(V_\kappa \models \theta)^L$ is absolute for $\mathcal{M}$.

$\square$
end of 2.4.

Now fix M with $|M| = \kappa$, M trans and $M \models ZF^-$ and $M^{<\kappa} \subseteq M$ and $\kappa \in M$. By the $\Pi^m_n$ indescribability of $\kappa$ there is a trans N with $N^{|V_{\kappa+m-2}|} \subseteq N$ and N $\Sigma^m_{n-1}$

correct and an elementary embedding $j : M \longrightarrow N$ with cpt $j = \kappa$. By the lemma we get for $n \geq 2$

$$(\forall \mathfrak{S} \in L \cap V_{\kappa+m+\ell-1}\ (V_\kappa \models \psi(\mathfrak{S},A))^L)^N.$$

This holds also if $n = 1$ since then $\psi(\mathfrak{S},A)$ is $\Sigma_0^{m+\ell-1}$ and we can use the argument from the last part of the proof of the lemma where we mentioned that $(V_\kappa \models \theta)^L$ is absolute for $\mathcal{M}$. Hence by the elementarity of j, for some $\alpha < \kappa$

$$(\forall \mathfrak{S} \in L \cap V_{\alpha+m+\ell-1}(V_\alpha \models \psi(\mathfrak{S},A \cap V_\alpha))^L)^M;$$

i.e.,

$$((V_\alpha \models \Phi(A \cap V_\alpha))^L)^M.$$

But then by $M^{<\kappa} \subseteq M$

$$(V_\alpha \models \Phi(A \cap V_\alpha))^L.$$

□
end of theorem 2.

# CHAPTER III. THE CONSISTENCY OF $\sigma_n^m > \pi_n^m (m \geq 2, n \geq 1)$.

## SECTION 1. General Remarks about the Construction.

The aim of this chapter is to prove the following.

Theorem 1.1. $(m \geq 2, n \geq 1)$

$$\text{CON}(\text{ZFC} + \exists\kappa, \kappa'(\kappa < \kappa', \kappa \text{ is } \Pi_n^m \text{ indescribable, and } \kappa' \text{ is } \Sigma_n^m \text{ indescribable})$$
$$\Rightarrow \text{CON}(\text{ZFC} + \sigma_n^m > \pi_n^m + \text{GCH})$$

In order to prove this theorem we will work in the theory ZF + V = L and assume that we have a $\Pi_n^m$ indescribable cardinal $\kappa$ and a $\Sigma_n^m$ indescribable cardinal $\kappa' > \kappa$. Then we will define a $\kappa + 1$ stage iteration $P_n^m$ such that

(1.2) $\Vdash_{P_n^m}$ "there are no $\Sigma_n^m$ indescribables $\leq\kappa$, $\kappa$ is $\Pi_n^m$ indescribable, $\kappa'$ is $\Sigma_n^m$ indescribable."

Hence we will obtain

$$\Vdash_{P_n^m} \sigma_n^m > \pi_n^m.$$

The definition of $P_n^m$ will be motivated by the fact that we want no $\Sigma_n^m$ indescribable cardinals in $V^{P_n^m}$ below $\kappa$. Thus $P_n^m$ will be essentially a $\kappa + 1$ stage forcing iteration where at stage $\lambda \leq \kappa$, where $\lambda$ is Mahlo we do some forcing that ensures that $\lambda$ will be

$\Sigma^m_n$ describable in $V^{P^m_n}$. (The fact that we also have to do something at stage $\kappa$ will play a role later in the proof that $P^m_n$ preserves the $\Pi^m_n$ indescribability of $\kappa$ — we already encountered this phenomenon in II.1. and II.2.)

Thus — as a first approximation — we can define the stages in $P^m_n$ as follows:

$$P^m_{n,0} = \{\emptyset\}$$

and for limit ordinals $\alpha \leq \kappa$:

$$P^m_{n,\alpha} = \lim_{\zeta<\alpha}\text{dir}\ P^m_{n,\zeta} \quad \text{if } \alpha \text{ is inaccessible}$$

and

$$P^m_{n,\alpha} = \lim_{\zeta<\alpha}\text{inv}\ P^m_{n,\zeta} \quad \text{otherwise.}$$

To define $P^m_{n,\lambda+1}$ for a Mahlo cardinal $\lambda \leq \kappa$ we pick a canonical term $Q^m_{n,\lambda} \in V^{P^m_{n,\lambda}}$ such that

$\Vdash_{P^m_{n,\lambda}}$ "$Q^m_{n,\lambda}$ is a certain finite step iteration of $\lambda$ is inaccessible and

GCH $\geq \lambda$ holds and $\lambda^{+\ell} = (\lambda^{+\ell})^L$ $(\ell \geq 1)$;

otherwise $Q^m_{n,\lambda}$ is the trivial poset."

To define the successor $P^m_{n,\beta+1}$ for an ordinal $\beta$ that is not a Mahlo cardinal we just pick a canonical name $Q^m_{n,\lambda} \in V^{P^m_{n,\lambda}}$ for the trivial poset; i.e., at stage $\beta$ where $\beta$ is not Mahlo we don't do anything. Now what exactly do we mean by "certain finite step iteration" ?

Suppose we are in $V^{P^m_{n,\lambda}}$ and $\lambda$ is inaccessible and $GCL^{\geq\lambda}$ holds and $\lambda^{+\ell} = (\lambda^{+\ell})^L$ for $\ell \geq 1$. In the first two stages (for n = 1 this will be done in one forcing) of $Q^m_{n,\lambda}$ we add an object that is modulo coding of type m over $V_\kappa$ (i.e., a subset of $V_{\kappa+m-1}$) and then force that a certain statement $\Phi^{\Sigma^m_n}$ holds about this object. The first poset will have size $\lambda^{+(m-1)}$ and will be $<\lambda^{+(m-1)}$ closed. The second poset will be $<\lambda^{+(m-2)}$ closed and $<\lambda^{+(m-1)}$ Baire and $\lambda^{+m}$ c.c. and of size $\lambda^{+m}$ (for n = 1 the first forcing will have size $\lambda^{+(m-1)}$ and $<\lambda^{+(m-1)}$ closed). Hence in particular after forcing with these posets, $\lambda$ is still inaccessible, $GCH^{\geq\lambda}$ holds, and $\lambda^{+\ell} = (\lambda^{+\ell})^L$ for $\ell \geq 1$. Recall that in the definition of $\Sigma^m_n$ describability of $\lambda$ we allow only parameters that are subsets of $V_\lambda$. Hence in the next (m − 1) steps we code down the object of type m on $V_\lambda$ that we first added successively until we end up with code $S_\lambda \subseteq \lambda$. This goes as follows: It will turn out that the object that we added at the first step of stage can be coded by a subset $A_\lambda \subseteq \lambda^{+(m-1)}$. Since $\lambda^{+(m-1)} = (\lambda^{+(m-1)})^L$ and $\lambda^{+(m-2)} = (\lambda^{+(m-2)})^L$ we can use almost disjoint forcing with the $\leq_L$ least almost disjoint family of size $\lambda^{+(m-1)}$ of constructable subsets of $\lambda^{+(m-2)}$. The forcing is $\lambda^{+(m-2)}$ centered and $<\lambda^{+(m-2)}$ closed. Thus after we do this forcing and obtain a code for $A_\lambda$ which is $\subseteq \lambda^{+(m-2)}$ we still have that $\lambda$ is inaccessible and $GCH^{\geq\lambda}$ holds and $\lambda^{+\ell} = (\lambda^{+\ell})^L$ for $\ell \geq 1$. Thus we can proceed in this fashion until we end up with a code $S_\lambda \subseteq \lambda$.

Now in the last step of our iteration $Q^m_{n,\lambda}$ we will force a club set $C_\lambda \subseteq \lambda$ such that

$$C_\lambda \cap \{\mu < \lambda : \mu \text{ inaccessible} \wedge V_\mu \models \Phi^{\Sigma^m_n} (\text{the object coded by } S_\lambda \cap V_\mu)\} = \emptyset.$$

Note that this forcing has size $\lambda$ and for each $\mu < \lambda$ there is a dense suborder that is $<\mu$ closed (it consists of all conditions whose top element $\geq \mu$).

Therefore we get the following:

$$\Vdash_{P^m_n} \quad Q^m_{n,\lambda} \text{ has for each } \mu < \lambda \text{ a dense suborder that is } < \mu \text{ closed.}$$

One can prove for each inaccessible $\lambda \leq \kappa$

(1.3) $\forall \alpha < \lambda \ |P^m_{n,\alpha}| < \lambda$

by induction on $\alpha$. For the successor step one has to use that for each $\alpha < \lambda$ $\Vdash_{P^m_n} |Q^m_{n,\alpha}| < \lambda$ since $\Vdash_{P^m_{n,\alpha}}$ "$\lambda$ is inac" because $|P^m_{n,\alpha}| < \lambda$ by induction hypothesis. From (1.3) we can deduce

$$\forall \lambda \leq \kappa \ (\lambda \text{ Mahlo} \Rightarrow P^m_{n,\lambda} \text{ is } \lambda \text{ c.c})$$

since $\{\alpha < \lambda \ P^m_{n,\alpha} = \operatorname{dirlim}_{\eta<\lambda} P^m_{n,\eta}\}$ is stationary in $\lambda$ by the Mahloness of $\lambda$. Hence we can conclude that for all Mahlo cardinals $\lambda \leq \kappa$

$$\Vdash_{P^m_{n,\lambda}} \text{“}\lambda \text{ is regular} \wedge GCH^{\geq\lambda} \text{ holds} \wedge \lambda^{+\ell} = (\lambda^{+\ell})^L \ (\ell \geq 1).\text{”}$$

In order to show that for Mahlo $\lambda \leq \kappa$

$$\Vdash_{P^m_{n,\lambda}} \text{"}\lambda \text{ is inaccessible"}$$

we need to observe that at any intermediate stage the iteration factors in a nice way.

Lemma 1.4. Let $0 < \alpha < \kappa$ and $P^m_{n,\alpha+1,\kappa+1}$ denote the tail of the iteration

in $V^{P^m_{n,\alpha+1}}$. If $\mu$ denotes the least inaccessible $\in (\alpha+1,\kappa+1)$ then

$$\Vdash_{P^m_{n,\alpha+1}} \text{"for each } \nu < \mu \ P^m_{n,\alpha+1,\kappa+1} \text{ has a dense suborder } {}^*P^m_{n,\alpha+1,\kappa+1} \text{ that is } <\nu \text{ closed."}$$

Proof. Let G be $P^m_{n,\alpha+1}$ generic and fix $\nu < \mu$. We claim that in V[G]:

(1.5) If $\beta \in (\alpha,\kappa+1]$ is a limit ordinal then $P^m_{n,\alpha+1,\beta}$ is either a direct or an inverse limit. Moreover if cf $\beta < \nu$ then $P^m_{n,\alpha+1,\beta}$ is an inverse limit.

In order to prove (1.5) note that $|P^m_{n,\alpha+1}| < \mu$ and that for $\beta \in (\alpha+1,\kappa+1)$ with $cf^{V[G]}(\beta) < \nu$ we have $P^m_{n,\beta} = \operatorname{liminv}_{\eta<\beta} P^m_{n,\eta}$(in V). Now we can use the same ideas as in [Baumgartner, 1983], Section 5.

For $\delta \in [\alpha+1,\kappa+1]$ we define

$${}^*P^m_{n,\alpha+1,\delta} \underset{\text{Df}}{=} \{f \in P^m_{n,\alpha+1,\delta} : \forall\zeta \in \text{dom } f \Vdash^{V[G]}_{P^m_{n,\alpha+1,\zeta}} \text{"f}(\zeta) \text{ is in the } <\nu \text{ closed dense suborder of } Q^m_{n,\zeta}\text{"}\}.$$

The first half of (1.5) implies that $*P^m_{n,\alpha+1,\delta}$ is dense. The full claim (1.5) shows that $*P^m_{n,\alpha+1,\delta}$ is $<\nu$ closed. □ end of 1.4.

Given this lemma it is easy to see that for Mahlo $\lambda \leq \kappa$

$$\Vdash_{P^m_{n,\alpha}} \text{"}\lambda \text{ is inaccessible."}$$

This also shows that at each Mahlo stage $\lambda \leq \kappa$ the requirements for doing a finite step iteration $Q^m_{n,\lambda}$ are satisfied. Then it will follow

$$\Vdash_{P^m_n} \text{"there are no } \Sigma^m_n \text{ indescribables } \leq\kappa\text{."}$$

Finally one can apply 1.4. to show by induction on $\alpha \leq \kappa$ that $\Vdash_{P^m_{n,\alpha}}$ GCH; hence we obtain

$$\Vdash_{P^m_n} \text{GCH.}$$

<u>Preservation</u> <u>of</u> <u>the</u> <u>$\Sigma^m_n$</u> <u>indescribability</u> <u>of</u> $\kappa'$.

This follows from the fact that the iteration $P^m_n$ can be defined inside any $V_\mu$ where $\mu > \kappa$ is inaccessible so that in particular $P^m_n \in V_\mu$. We prove the following general

<u>Lemma</u> <u>1.6</u>. Let $\kappa$ be $\Sigma^m_n$ indescribable and P be a poset with $P \in V_\kappa$. Then

$$\Vdash_P \text{"}\kappa \text{ is } \Sigma^m_n \text{ indescribable."}$$

<u>Proof</u>. Suppose G is P generic and $\Phi$ is $\Sigma_n^m$ and $A \in (V[G_\kappa])_{\kappa+1}$ and, in V[G]

$V_\kappa \models \Phi(A)$. Pick $\mathring{A} \in V^P$ with $(\mathring{A})^G = A$ and $p \in G$ with

(1.7) $p \Vdash_P$ "$V_\kappa \models \Phi(\mathring{A})$."

Let $\mu$ be the least inaccessible $<\kappa$ with $P \in V_\mu$. Now for any inaccessible $\nu \in (\mu,\kappa)$

pick $\mathring{A}_\nu \in V^P$ with

$$p \Vdash_P \mathring{A} \cap V_{\check{\nu}} = \mathring{A}_\nu .$$

For each such $\nu$ we can apply II.1.2. and find $\mathring{A}^*_\nu \in V^P \cap V_{\nu+1}$ with

$$p \Vdash_P \mathring{A}^*_\nu = \mathring{A}_\nu .$$

Then define

$$\mathring{A}^* \underset{\mathrm{Df}}{=} \bigcup_{\substack{\nu \in (\mu,\kappa) \\ \nu \text{ inaccessible}}} \mathring{A}^*_\nu .$$

Clearly

$$p \Vdash_P \mathring{A}^* = \mathring{A}$$

and we have $\mathring{A}^* \subseteq V_\kappa$ and for each inaccessible $\nu \in (\mu,\kappa)$

(1.8) $p \Vdash_P \mathring{A}^* \cap V_\nu = \mathring{A}^* \cap V_{\check{\nu}}$.

(Here consider $\mathring{A}^* \cap V_\nu$ as an element of $V^P$. Now (1.7) is $\Sigma_n^m(p,P,\check{\kappa},\mathring{A}^*)$ over $V_\kappa$

since it is equivalent to

$\exists \mathcal{M}[\mathcal{M}$ transitive, $\mathcal{M} \models ZF^-$ , $|\mathcal{M}| = |V_{\kappa+m-1}|$

$\mathcal{M}^{|V_{\kappa+m-2}|} \subseteq \mathcal{M}$, $\mathcal{M}$ $\Pi^m_{n-1}$ correct for $\kappa$, $\mathcal{M} \models [p \Vdash_P$ "$V_{\check{\kappa}} \models \Phi(\mathring{A}^*)$"]].

To see this note that for any such $\mathcal{M}$ the $\Pi^m_{n-1}$ correctness of $\mathcal{M}$ inside V for $\kappa$ will imply that with any P-generic H $\mathcal{M}[H]$ will still be $\Pi^m_{n-1}$ correct for $\kappa$ inside V[H] because $P \in V_\kappa$. By the $\Sigma^m_n$ indescribability of $\kappa$ in V this $\Sigma^m_n$ fact must reflect to some inaccessible $\lambda \in (\mu,\kappa)$. So we get

$$p \Vdash_P V_{\check{\lambda}} \models \Phi(\mathring{A}^* \cap V_\lambda)$$

where we consider $\mathring{A}^* \cap V_\lambda$ as an element of $V^P$. By applying (1.8) we obtain that in

$V[G]$, $(\mathring{A}^* \cap V_\lambda)^G = A \cap V_\lambda$. Therefore in V[G]

$$V_\lambda \models \Phi(A \cap V_\lambda);$$

i.e., $\Phi$ reflects. □ end of 1.6.

<u>Preservation of the $\Pi^m_n$ Indescribability of $\kappa$.</u>

This is the difficult part of the proof of (1.2) and we are going to use the reformulation in I.1.

Assume towards a contradiction that $\Phi$ is $\Pi^m_n$ and $\mathring{A} \in V^{P^m_n}$ and for some condition $p \in P^m_n$ we have

$$p \Vdash_{P^m_n} \text{"}\Phi(\mathring{A}) \text{ describes } \kappa.\text{"}$$

By the reflection principle we can find some ordinal $\delta >$ the least inaccessible above $\kappa$

such that $V_\delta \models [ZF^- \wedge p \Vdash_{P_n^m}$ "$\Phi(\mathring{A})$ describes $\kappa$"]. Note that $P_n^m \in V_\delta$.

Using standard arguments, one can construct a trans M of size $\kappa$ with $M^{<\kappa} \subseteq M$ and an elementary embedding $i: M \longrightarrow V_\delta$ with cpt $i > \kappa$ and some ${}^M p \in M$ and $M\mathring{A} \in M$ with $i({}^M p) = p$ and $i({}^M\mathring{A}) = \mathring{A}$. Using more or less straightforward master condition arguments we will construct a V generic ${}^V G$ for $P_n^m$ and an M generic ${}^M G$ for ${}^M(P_n^m)$ such that i lifts; i.e.,

$$\begin{array}{ccc} M[{}^M G] & \xrightarrow{i} & V_\delta[{}^V G] \\ & {}^M(P_n^m) \quad & P_n^m \\ M & \underset{i}{\longrightarrow} & V \end{array}$$

and also $p \in {}^V G$.

By applying I.1. we can also find a trans N with $|N| = |V_{\kappa+m-1}|$ and $N^{|V_{\kappa+m-2}|} \subseteq N$ and N is $\Sigma_{n-1}^m$ correct for $\kappa$ in V together with an elementary embedding $j: M \longrightarrow N$ with cpt $j = \kappa$.

In order to finish the proof we have to come up with an N generic ${}^N G$ for ${}^N(P_n^m)$ such that $N[{}^N G]$ is $\Sigma_{n-1}^m$ correct for $\kappa$ inside $V[{}^V G]$ and j lifts; i.e.,

$$\begin{array}{ccc} M[{}^M G] & \xrightarrow{j} & N[{}^N G] \\ & {}^M(P_n^m) \quad & {}^N(P_n^m) \\ M & \underset{j}{\longrightarrow} & N\,. \end{array}$$

Since $p \in {}^{V}G$ we get in V[G]

$$\Phi(\mathring{A}^{G}) \text{ describes } \kappa.$$

The $\Sigma^{m}_{n-1}$ correctness of $N[{}^{N}G]$ for $\kappa$ inside $V[{}^{V}G]$ implies

$$N[{}^{N}G] \models \text{“}V_\kappa \models \Phi(\mathring{A}^{G}).\text{”}$$

Then by elementarity of j and the fact that $\mathring{A}^{G} \in M$ (since cpt $j = \kappa$) we get

$$M[{}^{M}G] \models \exists\alpha < \kappa \;\; V_\alpha \models \Phi(\mathring{A}^{G} \cap V_\alpha).$$

It will be the case that $(M[{}^{M}G])^{<\kappa} \subseteq M[{}^{M}G]$ inside $V[{}^{V}G]$. Thus in $V[{}^{V}G]$

$$\exists\alpha < \kappa \;\; V_\alpha \models \Phi(\mathring{A}^{G} \cap V_\alpha).$$

Therefore we get a contradiction.

In order to find a generic ${}^{N}G$ with the properties above we use master condition arguments. At the last step of the last stage of the construction of ${}^{N}G$ where we add a certain club set $\subseteq j(\kappa)$ we will see that the iteration as we have defined it does not allow us to pick a master condition. We are going to avoid this problem by adding at each Mahlo stage $\lambda \leq \kappa$ of $P^{m}_{n}$ a $\lambda^{+}$ sequence (rather than just one) of objects of type $(m-1)$ over $V_\lambda$ and then we make a $\Sigma^{m}_{n}$ fact true for half of them and its negation for the other half. Then we code all these objects down until we end up with a $\lambda^{+}$ sequence of codes that are each $\subseteq \lambda$. Finally we add a $\lambda^{+}$ sequence (rather than just one) of closed sets $\subseteq \lambda$ each of which avoids a certain set of inaccessibles. We shall see that with this modification we will be able to find a master condition in the very last step of the construction of ${}^{N}G$.

The key point will be that in order to get a generic for the first step at stage $\kappa$ in ${}^{N}(P^{m}_{n})$ we do not merely take the generic for the first step at stage $\kappa$ in ${}^{V}(P^{m}_{n})$. Instead we choose a certain permutation of the sequence of generic objects at the first step of stage $\kappa$ of ${}^{V}(P^{m}_{n})$. This permutation will ensure that for certain $|M| = \kappa$ – many of the objects for which a $\Sigma^{m}_{n}$ fact holds in $V[{}^{V}G]$, a $\Pi^{m}_{n}$ fact (its negation) will hold in $N[{}^{N}G]$ and this fact will allow us to pick a master condition in the very last step of the construction of ${}^{N}G$.

The most difficult part of the proof will be to show that $N[{}^{N}G]$ will still be $\Sigma^{m}_{n-1}$ correct inside $V[{}^{V}G]$ for $\kappa$ in parameters from $V_{\kappa+m}$ after we do this permutation.

## SECTION 2. $\sigma^2_1/\pi^2_1$.

### 2.1. The Iteration $P^2_1$ .

In this section we will write $P_\alpha$ instead of $P^2_{1,\alpha}$. In Section 1 we have already outlined the general structure of this iteration and here we will restrict ourselves in exhibiting the 3 step iteration at stage $\lambda \leq \kappa$ (where $\lambda$ is Mahlo) that we use in order to make $\lambda$ $\Sigma^2_1$ describable in $V^{P_{\lambda+1}}$. So assume G is $P_\lambda$ generic over V = L. We have seen in Section 1 that in $V[G_\lambda]$ $\lambda$ is inaccessible, $(\lambda^{+\ell})^L = \lambda^{+\ell}$ for $\ell \geq 1$ and $GCH^{\geq\lambda}$ holds.

The forcing $Q_{(1)}$ that we use at the first step of stage $\lambda$ will add a sequence $(A\gamma : \lambda < \lambda^+)$ where for even $\gamma < \lambda^+$ $A\gamma$ is a pair consisting of a subset of $\lambda^+$ and a club set $\subseteq\lambda^+$ that is disjoint from it and for each odd $\gamma < \lambda^+$ $A\gamma$ is a subset of $\lambda^+$. $Q_{(1)}$ will be a $\lambda^+$ product with $<\lambda^+$ support of posets that are each $<\lambda^+$ closed and have size $\lambda^+$. So $Q_{(1)}$ is $<\lambda^+$ closed and has size $\lambda^+$. Hence if $(A\gamma : \gamma<\lambda^+)$ is generic for $Q_{(1)}$ then in $V[G_\lambda, \vec{A}\gamma]$ still $\lambda$ is inac, $(\lambda^+)^\ell = (\lambda^{+\ell})^L$ for $\ell \geq 1$ and $GCH^{\geq\lambda}$ holds. If we let $\tilde{A}_\gamma \underset{\mathfrak{Df}}{=} A_\gamma$ for odd $\gamma$ and $\tilde{A}_\gamma \underset{\mathfrak{Df}}{=}$ the first component in the pair $A_\gamma$ for even $\gamma$ then a $\Sigma^2_1$ fact is true about each $\tilde{A}_\gamma$ where $\gamma$ is even (namely, "$\tilde{A}_\gamma$ is not stationary") and a $\Pi^2_1$ fact is true about $\tilde{A}_\gamma$ where $\gamma$ is odd (namely "$\tilde{A}_\gamma$ is stationary"). Now in the next step $Q_{(2)}$ we will add for each $\tilde{A}_\gamma \subseteq \lambda^+$ a code $\subseteq \lambda$. Thus $Q_{(2)} = \prod_{\substack{\gamma<\lambda^+ \\ <\lambda \text{support}}} Q_{\tilde{A}_\gamma}$ where $Q_{\tilde{A}_\gamma}$ is the forcing for coding $\tilde{A}_\gamma \subseteq \lambda^+$ by a set $\subseteq \lambda$.

For this let $(T_\zeta : \zeta < \lambda^+)$ denote the $<_L$ least constructible sequence of size $\lambda^+$of almost disjoint subsets of $\lambda$ (note that $\lambda^+ = (\lambda^+)^L$ in $V[G_\lambda, \vec{A}_\gamma]$). The conditions in $Q_{\tilde{A}_\gamma}$ are pairs (s,b) where $s \subseteq \lambda$ and $b \subseteq \lambda^+$ and $|s|, |b| < \lambda$ and

$(s_1, b_1) \leq (s_2, b_2)$ iff $s_1 \supseteq s_2$, $b_1 \supseteq b_2$ and

$$\forall \alpha \in b_2 [\alpha \notin \tilde{A}_\gamma \Rightarrow T_\alpha \cap s_1 \subseteq s_2].$$

Clearly $Q_{\tilde{A}_\gamma}$ is $<\lambda$ closed and $\lambda$ centered (for this define $(s_1, b_2) \sim (s_2, b_2)$ iff $s_1 = s_2$). Moreover if $S_\gamma$ is $Q_{\tilde{A}_\gamma}$ generic then in $V[G_\lambda, \vec{A}_\gamma, S_\gamma]$ for all $\zeta \in \lambda^+$:

$$\zeta \in \tilde{A}_\gamma \Longleftrightarrow \tilde{S}_\gamma \cap T_\zeta \text{ is unbounded in } \lambda$$

where $\tilde{S}_\gamma \underset{\mathfrak{D}f}{=} \cup \{s : \exists b (s,b) \in S_\gamma\} \subseteq \lambda$.

If we pick a generic $(S_\gamma : \gamma < \lambda^+)$ for $Q_{(2)}$ then in $V[G_\lambda, \vec{A}_\gamma, \vec{S}_\gamma]$ for even $\gamma < \kappa^+$ we created a $\Sigma^2_1$ fact in the parameter $\tilde{S}_\gamma \subseteq \lambda$:

(2.1) The set coded by $\tilde{S}_\gamma$ is not stationary in $\lambda^+$.

Note that for odd $\gamma < \lambda^+$ $\tilde{A}_\gamma$ is still stationary in $V[G_\lambda, \vec{A}_\gamma, \vec{S}_\gamma]$ since by a standard $\Delta$ system argument $Q_{(2)}$ has the property $\lambda^+$. Moreover in $V[G_\gamma, \vec{A}_\gamma, \vec{S}_\gamma]$ we still have $\lambda$ is inaccessible, $\lambda^+ = (\lambda^+)^\ell$ for $\ell \geq 1$ and $GCH^{\geq \lambda}$ holds.

In the final step of the 3 step iteration at stage $\lambda$ we add a sequence $(C_\gamma : \gamma < \lambda^+)$ of club sets $\subseteq \lambda$ via a product forcing $Q_{(3)} = \prod_{\substack{\gamma < \lambda^+ \\ <\lambda \text{support}}} Q^\gamma_{(3)}$ such that for each $\gamma < \lambda^+$

$$C_\gamma \cap \{\mu < \lambda : \mu \text{ inaccessible} \wedge V_\mu \models \Phi^{\Sigma^2_1}(\tilde{S}_\gamma \cap V_\mu)\} = \emptyset.$$

Here $\Phi^{\Sigma^2_1}(\tilde{S}_\gamma)$ is the $\Sigma^2_1$ statement in (2.1). Obviously each $Q^\gamma_{(3)}$ has size $\lambda$. Thus by a standard $\Delta$ system argument $Q_{(3)}$ has the property $\lambda^+$. Furthermore for each $\alpha < \lambda$ $Q_{(3)}$ has a dense, $<\alpha$ closed suborder. Hence in particular $Q_{(3)}$ is $<\lambda$ Baire. So it is easy to see that for Mahlo $\lambda \leq \kappa$

$$\Vdash_{P_{\lambda+1}} \lambda \text{ is } \Sigma^2_1 \text{ describable.}$$

In fact we have for each Mahlo $\lambda \leq \kappa$

$$\Vdash_{P_{\kappa+1}} \lambda \text{ is } \Sigma^2_1 \text{ describable.}$$

since as we saw in Section 1, in $V^{P_{\lambda+1}}$ the tail $P_{\lambda+1,\kappa+1}$ is highly Baire.

## 2.2. Preservation of the $\Pi^2_1$ Indescribability of $\kappa$ in $V^{P_{\kappa+1}}$.

Assume towards a contradiction that for some condition $p^* \in P_{\kappa+1}$ and $\Phi$ in $\Pi^2_1$ and $\mathring{A} \in V^P$ we have

$$p^* \Vdash_{P_{\kappa+1}} \text{“}\Phi(\mathring{A}) \text{ describes } \kappa.\text{”}$$

Pick a $\delta>$ the least inaccessible greater than $\kappa$ such that

$$V_\delta \models [ZF^- \wedge p^* \Vdash_{P_{\kappa+1}} \text{“}\Phi(\mathring{A}) \text{ describes } \kappa\text{”}].$$

Note $P_{\kappa+1} \in V_\delta$.

By standard arguments we can find a transitive M with $|M| = \kappa$ and $M^{<\kappa} \subseteq M$ and an elementary embedding $i: M \longrightarrow V_\delta$ with cpt $i > \kappa$ such that for some ${}^M p$, ${}^M\mathring{A} \in M$ $i({}^M p) = p^*$ and $i({}^M\mathring{A}) = \mathring{A}$.

Now we have to find an M generic ${}^M G$ for ${}^M P_{\kappa+1}$ and a V generic ${}^V G$ for $P_{\kappa+1}$ such that $p^* \in {}^V G$ and i lifts:

$$\begin{array}{ccc} M[{}^M G] & \underset{i}{\longrightarrow} & V_\delta[{}^V G] \\ {}^M P_{\kappa+1} & & P_{\kappa+1} \\ M & \underset{i}{\longrightarrow} & V_\delta . \end{array}$$

<u>Construction</u> <u>of</u> $M_G$ <u>and</u> ${}^V G$.

Since cpt $i > \kappa$ and ${}^M P_\kappa = P_\kappa \subseteq V_\kappa$ we can take any V generic for $P_\kappa$, say $G_\kappa$ with $p^*|\kappa \in G_\kappa$ and i will lift; i.e.,

$$\begin{array}{ccc} M[G_\kappa] & \longrightarrow & V_\delta[G_\kappa] \\ P_\kappa & & P_\kappa \\ M & \longrightarrow & V_\delta . \end{array}$$

For the first step at stage $\kappa$ of $P^M_{\kappa+1}$ and $P_{\kappa+1}$ recall that $Q_{(1)}$ is $<\kappa^+$ closed. Since $|M[G_\kappa]| = \kappa$ any $M[G_\kappa]$ generic for ${}^M Q_{(1)}$ only has to meet $\kappa$ dense sets. So in $V[G_\kappa]$ we can pick a generic ${}^M\vec{A}_\gamma$ for ${}^M Q_{(1)}$ in the usual way.

$Q_{(1)}$ is also $<\kappa^+$ directed closed, hence there is a condition extending all i(q) for q in ${}^M\vec{A}_\gamma$. Any such condition in $Q_{(1)}$ is a master condition.

For the second and third step note that both $Q_{(2)}$ and $Q_{(3)}$ have the $\kappa^+$ c.c.; hence the pullback method works since cpt $i = (\kappa^+)^M > \kappa$.

Note that by the usual arguments $M[{}^MG]^{<\kappa} \subseteq M[{}^MG]$ inside $V[{}^VG]$.

By I.1. the $\Pi^2_1$ indescribability of $\kappa$ implies that in $V = L$ there is a trans N with $|N| = \kappa^+$, $N^\kappa \subseteq N$ (i.e., N is $\Sigma^2_0$ correct for $\kappa$) and an elementary embedding $j: M \longrightarrow N$ with cpt $j = \kappa$. Now we have to come up with an N generic ${}^NG$ for ${}^NP_{j(\kappa)+1}$ such that $N[{}^NG]^\kappa \subseteq N[{}^NG]$ inside $V[{}^VG]$ and such that j lifts; i.e.,

$$\begin{array}{ccc} M[{}^MG] & \xrightarrow[j]{} & N[{}^NG] \\ {}^MP_{\kappa+1} & & {}^NP_{j(\kappa)+1} \\ M & \xrightarrow[j]{} & N. \end{array}$$

<u>Construction</u> <u>of</u> $N[{}^NG]$.

First we find a ${}^NG_{j(\kappa)}$ that is ${}^NP_{j(\kappa)}$ generic over N. Note that $P_\kappa \subseteq V_\kappa$

and ${}^{N}P_\kappa = P_\kappa$. Since cpt $j = \kappa$ it will therefore suffice to find ${}^{N}G_{\kappa,j(\kappa)}$ that is $N[G_\kappa]$ generic for ${}^{N[G_\kappa]}P_{\kappa,j(\kappa)}$; then with ${}^{N}G_{j(\kappa)} \underset{\mathfrak{Df}}{=} G_\kappa * {}^{N}G_{\kappa,j(\kappa)}$ j will lift.

Since $P_\kappa$ is $\kappa$ c.c. we get $N[G_\kappa]^\kappa \subseteq N[G_\kappa]$ inside $V[G_\kappa]$. Hence the poset ${}^{N}Q_{(1)}$ that we use at the first step of stage $\kappa$ in ${}^{N}P_{j(\kappa)}$ is the same as $Q_{(1)}$ which we use at the first step of stage $\kappa$ in $P_{\kappa+1}$. Now consider the $<_L$ least permutation $\Pi$ in the ground model $V = L$ such that

$$\Pi\colon \text{Even}_{\kappa+} \longrightarrow \text{Even}_{\kappa+} - \text{Even}_{(\kappa+)^M}$$

$$\Pi\colon \text{Odd}_{\kappa+} \longrightarrow \text{Odd}_{\kappa+} \cup \text{Even}_{(\kappa+)^M}.$$

Recall that in $V = L$ $|(\kappa^+)^M| = \kappa$. Denote by $(A_\gamma : \gamma < \kappa^+)$ the generic sequence for $Q_{(1)}$ of $P_{\kappa+1}$. Then define for the noncritical $\gamma < \kappa^+$ (i.e., those $\gamma$ with $\gamma$ and $\Pi(\gamma)$ both even or both odd)

$${}^{N}A_\gamma \underset{\mathfrak{Df}}{=} A_{\Pi(\gamma)}.$$

And for the critical $\gamma < \kappa^+$ (i.e., $\gamma$ odd and $\Pi(\gamma) < (\kappa^+)^M$ even)

$${}^{N}A_\gamma \underset{\mathfrak{Df}}{=} \tilde{A}_{\Pi(\gamma)}.$$

Because $\Pi$ is in the ground model $({}^{N}A_\gamma : \gamma < \kappa^+)$ is certainly $N[G_\kappa]$ generic for $Q_{(1)}$. Hence for each critical $\gamma < \kappa^+$ ${}^{N}\tilde{A}_\gamma$ is stationary in $N[G_\kappa, {}^{N}\vec{A}_\gamma]$ but nonstationary in $V[G_\kappa, \vec{A}_\gamma]$.

By the $<\kappa^+$ closure of $Q_{(1)}$ $N[G_\kappa,{}^N\vec{A}_\gamma]^\kappa \subseteq N[G_\kappa,{}^N\vec{A}_\gamma]$ inside $V[G_\kappa,\vec{A}_\gamma]$. Hence $Q^N_{(2)}$ wants to add codes $\subseteq \kappa$ for each of the ${}^N\tilde{A}_\gamma \subseteq \kappa^+$ via almost disjoint forcing using the $<_L$ least constructible family $(T_\zeta : \zeta<\kappa^+)$ of almost disjoint subsets of $\kappa$. Hence with ${}^NS_\gamma \underset{\mathfrak{Df}}{=} S_{\Pi(\gamma)}$ the sequence $({}^NS_\gamma : \gamma<\kappa^+)$ is certainly generic for ${}^NQ_{(2)}$ over $N[G_\kappa,{}^N\vec{A}_\gamma]$. And by the $\kappa^+$ c.c. of $Q_{(2)}$ $N[G_\kappa,{}^N\vec{A}_\gamma,{}^N\vec{S}_\gamma]^\kappa \subseteq N[G_\kappa,{}^N\vec{A}_\gamma,{}^N\vec{S}_\gamma]$ inside $V[G_\kappa,\vec{A}_\gamma,\vec{S}_\gamma]$ (note again that $\Pi$ is the ground model). At the last step of stage $\kappa$ of ${}^NP_{j(\kappa)}$ we just take ${}^NC_\gamma \underset{\mathfrak{Df}}{=} C_{\Pi(\gamma)}$ and by the same argument $({}^NC_\gamma : C_\gamma<\kappa^+)$ is ${}^NQ_{(3)}$ generic over $N[G_\kappa,{}^N\vec{A}_\gamma,{}^N\vec{S}_\gamma]$ and we will have $N[G_\kappa,{}^N\vec{A}_\gamma,{}^N\vec{S}_\gamma,{}^N\vec{C}_\gamma]^\kappa \subseteq N[G_\kappa,{}^N\vec{A}_\gamma,{}^N\vec{S}_\gamma,{}^N\vec{C}_\gamma]$ inside $V[G^V]$. The tail ${}^NP_{\kappa+1,j(\kappa)}$ has from the viewpoint of $N[G_\kappa,{}^N\vec{A}_\gamma,{}^N\vec{S}_\gamma,{}^N\vec{C}_\gamma]$ a $<\kappa^+$ closed dense suborder. Since $|N| = \kappa^+$ in the usual way we can in $V[G^V]$ pick a generic ${}^NG_{\kappa+1,j(\kappa)}$ for ${}^NP_{\kappa+1,j(\kappa)}$ over $N[G_\kappa,{}^N\vec{A}_\gamma,{}^N\vec{S}_\gamma,{}^N\vec{C}_\gamma]$ and with ${}^NG_{j(\kappa)} \underset{\mathfrak{Df}}{=} G_\kappa * {}^N\vec{A}_\gamma * {}^N\vec{S}_\gamma * {}^N\vec{C}_\gamma * {}^NG_{\kappa+1,j(\kappa)}$ $N[{}^NG_{j(\kappa)}]^\kappa \subseteq N[{}^NG_{j(\kappa)}]$ inside $V[{}^VG]$.

Now we handle stage $j(\kappa)$ of ${}^NP_{j(\kappa)+1}$. At the first step we note that the forcing we encounter is $<j(\kappa)^+$ directed closed; hence a straightforward master condition argument works for the first step of stage $\kappa$ of ${}^MP_{\kappa+1}$.

In the next step we proceed analogously since by the $<j(\kappa)$ directed closure of the forcing at the second step of stage $j(\kappa)$ of ${}^{N}P_{j(\kappa)+1}$ we can always find a condition extending all $j(p)$ where p is a condition in the generic $({}^{M}S_{\gamma}:\gamma<(\kappa^{+})^{M})$ for the second step of stage $\kappa$ in ${}^{M}P_{\kappa+1}$.

Now we consider the very last step where we add a $j(\kappa)^{+}$ sequence of club sets. Before we finish the proof let us take a break and see what would happen if at stage $\kappa$ of $P_{\kappa+1}$ we had just added one subset of $\kappa^{+}$ rather than $\kappa^{+}$ many. *In this construction* denote the pair consisting of a subset of $\kappa^{+}$ together with a club set $\subseteq \kappa^{+}$ disjoint from it that we add in the first step of stage $\kappa$ of ${}^{M}P_{\kappa+1}$ by $A^{M}$ and the one in the first step of stage $\kappa$ of $P_{\kappa}$ by A. Let ${}^{M}\tilde{S} \subseteq \kappa$ be the code for ${}^{M}\tilde{A}$ and $\tilde{S} \subseteq \kappa$ be the code for $\tilde{A}$ and ${}^{M}C$ the club set that we add at the third step of stage $\kappa$ of ${}^{M}P_{\kappa+1}$. In order to lift the embedding j at the very last step of this construction we must find a club set $c^{*} \subseteq j(\kappa)$ of cardinality $<j(\kappa)$ in $N[G_{\kappa}, j({}^{M}A), j({}^{M}S)]$ such that $j(p) \subseteq c^{*}$ for all conditions p "in" ${}^{M}C$. But for all such p $p \subsetneq \kappa$ and $|p| < \kappa$; hence $j(p) = p$. Thus we must have $\kappa \in c^{*}$. The problem is now that in order for $c^{*}$ to be a condition in the forcing at the third step of stage $j(\kappa)$ of $P_{j(\kappa)+1}$ we must have $\kappa \notin c^{*}$. This is true since in $N[j(G_{\kappa}), j({}^{M}A), j({}^{M}S)]$ $\kappa < j(\kappa)$ is inaccessible and

(2.2) $V_\kappa \models \Phi^{\Sigma^2_1}(j({}^M\tilde{S}) \cap V_\kappa)$

where $\Phi^{\Sigma^2_1}(j({}^M\tilde{S}) \cap V_\kappa)$ says that the set $\subseteq \kappa^+$ which is coded by $j({}^M\tilde{S}) \cap V_\kappa$ is nonstationary in $\kappa^+$. To prove (2.2.) note that

$$j({}^M\tilde{S}) \cap V_\kappa = \tilde{S}^M$$

since $\tilde{S}^M \subseteq \kappa$ and that

$$\tilde{S}^M = i(\tilde{S}^M) = \tilde{S}$$

since $\tilde{S}^M \subseteq \kappa$.

But the set coded by $\tilde{S}$ (namely, $\tilde{A}$) is nonstationary in $N[G_\kappa]$ and hence in $N[G_k\, j({}^M A), j({}^M S)]$.

We shall now see how the permutation $\Pi$ that we chose avoids this problem. We define a condition $c^*$ for the forcing at step 3 of stage $j(\kappa)$ of ${}^N P_{j(\kappa)+1}$ by

$$\text{dom}\,(c^*) = \{j(\gamma): \gamma < (\kappa^+)^M]$$

and for $\gamma < (\kappa^+)^M$:

$$c^*_{j(\gamma)} = C^M_\gamma \cup \{\kappa\}$$

where $(C^M_\gamma : \gamma < \kappa^+)$ is the generic for the last step of stage $\kappa$ of ${}^M P_{\kappa+1}$. We have to show that $c^*$ is really a condition!

Notice that $|c^*| = \kappa < j(\kappa)$ and that for each $\gamma < (\kappa^+)^M$ $c^*_{j(\gamma)}$ is a closed subset of $j(\kappa)$ of cardinality $<j(\kappa)$. Now fix any $\gamma < (\kappa^+)^M$. We have to show that in $N[j(G_\kappa), j({}^M\vec{A}_\gamma), j({}^M\vec{S}_\gamma)]$

(2.3) $c^*_{j(\gamma)} \cap \{\mu < j(\kappa) : \mu \text{ inaccessible} \wedge V_\mu \models \Phi^{\Sigma^2_1}(j({}^M\tilde{S})_{j(\gamma)} \cap V_\mu)\} = \emptyset$

where $\Phi^{\Sigma^2_1}$ is the statement (2.1).

Clearly we don't have to worry about the $\mu < \kappa$ since $j({}^M\tilde{S})_{j(\gamma)} \cap V_\mu = j({}^M\tilde{S}_\gamma) \cap V_\mu = {}^M\tilde{S}_\gamma \cap V_\mu$ and since $V_{\mu+\omega}$ is the same whether computed in $M[G_\kappa, {}^M\vec{A}_\gamma, {}^M\vec{S}_\gamma]$or in $N[j(G_\kappa), j({}^M\vec{A}_\gamma), j({}^M\vec{S}_\gamma)]$. Now for $\mu = \kappa$ note that $j({}^M\tilde{S})_{j(\gamma)} \cap V_\kappa = {}^M\tilde{S}_\gamma = i({}^M\tilde{S}_\gamma) = (i^M\tilde{S})_{i(\gamma)} = \tilde{S}_\gamma$ (i.e., the $\gamma$-th code that we added at step 2 of stage $\kappa$ of $P_{\kappa+1}$). Recall that we defined ${}^N\tilde{S}_{\Pi^{-1}(\gamma)} = \tilde{S}_\gamma$ and that $\Pi^{-1}[(\kappa^+)^M] \subseteq \text{Odd}_{\kappa^+}$. Hence the set coded by $\tilde{S}_\gamma$, i.e., $\tilde{A}_\gamma = {}^N\tilde{A}_{\Pi^{-1}(\gamma)}$, is stationary in $N[G_\kappa, {}^N\vec{A}_\gamma]$. But then this set must also be stationary in $N[j(G_\kappa), j({}^M\vec{A}_\gamma), j({}^M\vec{S}_\gamma)]$ by some elementary chain condition and closure arguments. Thus we have shown (2.3).

Finally it is clear that $c^*$ extends $j(p)$ for all p in the generic ${}^M\vec{C}_\gamma$ and therefore $c^*$ is a master condition for the very last step of the construction. Routine arguments establish that $N[{}^NG]^\kappa \subseteq N[{}^NG]$ inside $V[{}^VG]$.

SECTION 3. $\sigma^m_1/\pi^m_1 (m \geq 3)$.

In this section we will always write $P_\alpha$ instead of $P^m_{1,\alpha}$. At stage $\lambda \leq \kappa$ (where $\lambda$ is Mahlo) we will do a (m+1)-step iteration to guarantee that $\lambda$ will be $\Sigma^m_1$ describable in $V^{P_{\lambda+1}}$.

Suppose that in $V[G_\lambda]$, where $G_\lambda$ is $P_\lambda$ generic we have: $\lambda$ is inac, $(\lambda^{+\ell})^L = \lambda^{+\ell}$ for $\ell \geq 1$ and $GCH^{\geq\lambda}$ holds. In the first step $Q_{(1)}$ of the (m−1) step iteration we force a sequence $(A_\gamma : \gamma < \lambda^+)$ where for each even $\gamma < \lambda^+$ $A_\gamma$ is a pair consisting of a subset of $\lambda^{+(m-1)}$ and a club set $\subseteq \lambda^{+(m-1)}$ disjoint from it. For odd $\gamma < \lambda^+$ $A_\gamma$ will simply be a "new" subset of $\gamma^{+(m-1)}$. Hence $Q_{(1)}$ is $\lambda^+$ product (with full support) of posets that are each $< \lambda^{+(m-1)}$ closed and have size $\lambda^{+(m-1)}$. Thus $Q_{(1)}$ is $<\lambda^{+(m-1)}$ closed and has size $\lambda^{+(m-1)}$. Note that $\tilde{A}_\gamma \subseteq \lambda^{+(m-1)}$ is stationary for odd $\gamma < \kappa^+$ and $\tilde{A}_\gamma \subseteq \lambda^{+(m-1)}$ is not stationary for even $\gamma < \kappa^+$ (where $\tilde{A}_\gamma$ is defined from $A_\gamma$ as in the $\pi^2_1/\sigma^2_1$ case). Moreover in $V[G_\gamma, \vec{A}_\gamma]$ $\lambda$ is still inaccessible and $(\lambda^{+\ell})^L = \lambda^{+\ell}$ for $\ell \geq 1$ and $GCH^{\geq\lambda}$ holds.

In the next (m−1) steps we code each $\tilde{A}_\gamma \subseteq \lambda^{+(m-1)}$ down until we end up with a code that is a subset of $\lambda$. At the k-th coding step $(1 \leq k \leq m-1)$ we will be given a sequence $(\tilde{S}^{m-k}_\gamma : \gamma < \lambda^+)$ where each $\tilde{S}^{m-k}_\gamma \subseteq \lambda^{+(m-k)}$ (for $k = 1$ $\tilde{S}^{m-1}_\gamma = \tilde{A}_\gamma$) and we want to code them by $(\tilde{S}^{m-(k+1)}_\gamma : \gamma < \lambda^+)$ where each $\tilde{S}^{m-(k+1)}_\gamma \subseteq \lambda^{+m-(k+1)}$ (define $\lambda^{+0} = \lambda$). The forcing $Q_{(k+1)}$ that does this is a product of length $\lambda^+$ (with full support if $k < m - 2$, with support $< \lambda^+$ if $k = m - 2$ and with support $< \lambda$ if $k = m - 1$) where each factor is $\lambda^{+(m-(k+1))}$ centered and $< \lambda^{+(m-(k+1))}$ closed and has size $\lambda^{+(m-k)}$. Thus $Q_{(k+1)}$ has the $\lambda^{+(m-k)}$ property and is $<\lambda^{+(m-(k+1))}$ closed and has size $\lambda^{+(m-k)}$. So throughout the coding: $\lambda$ is inaccessible, $(\lambda^{+\ell})^L = \lambda^{+\ell}$ for $\ell \geq 2$ and $GCH^{\geq\lambda}$ holds. Therefore to code $\tilde{S}_\gamma^{m-k} \subseteq \lambda^{+(m-k)}$ by some $\tilde{S}_\gamma^{m-(k+1)} \subseteq \lambda^{+(m-(k+1))}$ we can use almost disjoint forcing with the $<_L$ least constructible family of size $\lambda^{+(m-k)}$ of subsets of $\lambda^{+(m-(k+1))}$. In the (m+1) the step at stage $\lambda$ we add a sequence $(C_\gamma : \gamma < \lambda^+)$ of club subsets of $\lambda$ such that for each $\gamma < \kappa^+$

$$C_\gamma \cap \{\mu < \lambda : \mu \text{ inaccessible} \wedge V\mu \models \Phi^{\Sigma_1^m} (\tilde{S}_\gamma \cap V_\mu)\} = \emptyset$$

where $\Phi^{\Sigma_1^m}(\tilde{S} \cap V_\mu)$ says that the set $\subseteq \mu^{+(m-1)}$ coded by $\tilde{S} \cap V_\mu \subseteq \mu$ is nonstationary.

The forcing $Q_{(m+1)}$ that adds $(C_\gamma : \gamma < \lambda^+)$ is the analogue of the corresponding forcing in the $\sigma_1^2/\pi_1^2$ case. The proof that

$$\Vdash_{P_{\kappa+1}} \text{“}\kappa \text{ is } \Pi_1^m \text{ indescribable”}$$

is entirely analogous with the corresponding proof in the $\pi_1^2/\sigma_1^2$ case.

SECTION 4. $\sigma_3^2/\pi_3^2$ (The Generic Case).

4.1. The Iteration $P_3^2$.

We restrict ourselves to describing what we are going to do at the stage $\lambda \le \kappa$ where $\lambda$ is Mahlo in order to guarantee that

$$\Vdash_{P^2_{3,\lambda+1}} \text{“}\lambda \text{ is } \Sigma^2_3 \text{ describable.”}$$

In this section we are going to use $P_\lambda$ instead of $P^2_{3,\lambda}$. Choose a term $\overset{\circ}{Q}_\lambda \in V^{P_\lambda}$ such that

$\Vdash_{P_\lambda}$ $\overset{\circ}{Q}_\lambda$ is a certain 4 step iteration if $\lambda$ is inac,

$\lambda^{+\ell} = (\lambda^{+\ell})^L$ for $\ell \ge 1$ and GCH $^{\ge\lambda}$ holds;

otherwise $\overset{\circ}{Q}_\lambda$ is the trivial poset.

By “certain 4 step iteration” we mean the following: Suppose that $G_\lambda$ is $P_\lambda$ generic and in $V[G_\lambda]$ $\lambda$ is inac, $(\lambda^{+\ell})^L = \lambda^{+\ell}$ for $\ell \ge 1$ and $GCH^{\ge\lambda}$ holds. In the first step $Q_{(1)}$ we add a sequence $(F_\gamma : \gamma < \lambda^+)$ where each $F_\gamma : 2^{\lambda^+} \times 2^{\lambda^+} \longrightarrow 2^{\lambda^+}$ is a Lipshitz function; i.e., each $F_\gamma$ is really a function with domain $\bigcup_{\alpha<\lambda^+} 2^\alpha \times 2^\alpha$ and range contained in $2^{<\lambda^+}$ and for X, Y $\subseteq \lambda^+$ one defines $F_\gamma(X,Y) = \bigcup_{\alpha<\lambda^+} F_\gamma(X \cap \alpha, Y \cap \alpha)$. The poset $Q_{(1)}$ is a $\lambda^+$ product with $<\lambda^+$ support of copies of the forcing

$P_F$ where conditions in $P_F$ are functions f with

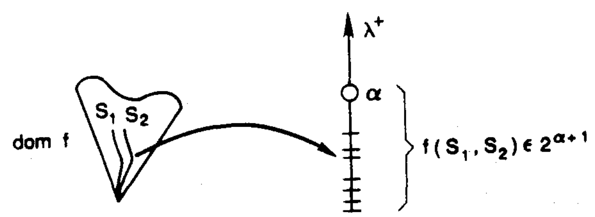
dom f as a subtree of $2^{<\lambda^+} \times 2^{<\lambda^+}$ of size $< \lambda^+ \wedge$

$\forall (s_1,s_2) \in$ dom f $[\exists \alpha < \lambda^+[\alpha \geq \text{dom } s_1 \wedge f(s_1,s_2) \in 2^{\alpha+1} \wedge f(s_1,s_2)(\alpha) = 0]$

$\wedge \forall \zeta[f(s_1,s_2)(\zeta) = 1 \Rightarrow \text{cf } \zeta = \lambda] \wedge$

$\forall (t_1,t_2) \in$ dom f $[(s_1,s_2)$ extends $(t_1,t_2) \Rightarrow f(s_1,s_2)$ extends $f(t_1,t_2)]]$

and for f, g $\in P_F$ $f \leq g$ iff $f \supseteq g$. The conditions for $P_F$ look like this




Clearly $|Q_{(1)}| = \lambda^+$ and $Q_{(1)}$ is $<\lambda^+$ closed. Hence in particular in $V[G_\lambda,(F_\gamma : \gamma < \lambda^+)]$ $\lambda$ is still inaccessible, $GCH^{\geq\lambda}$ holds, and $(\lambda^{+\ell})^L = \lambda^{+\ell}$ for $\ell \geq 1$.

In the next step, the forcing $Q_{(2)}$ which we will simply call Q from now on will make a $\Sigma^2_3$ statement true about the $F_\gamma$ for $\gamma$ even and a $\Pi^2_3$ statement true for the $F_\gamma$ with $\gamma$ odd. The $\Sigma^2_3$ statement about $F_\gamma$ will say

$\exists X \subseteq \lambda^+ \ \forall Y \subseteq \lambda^+ \ F_\gamma(X,Y)$ is nonstationary.

The $\Pi^2_3$ statement will be the negation

$\forall X \subseteq \lambda^+ \ \exists Y \subseteq \lambda^+ \ F_\gamma(X,Y)$ stationary.

For this note that $\lambda^+ = (\lambda^+)^L$; hence any $X \subseteq \lambda^+$ has a canonical code $\mathfrak{X} \in V_{\lambda+2}$ where $\mathfrak{X} = \{S: S \in P(\lambda) \cap L \wedge \exists\alpha(S$ is the $\underset{L}{<}$-$\alpha$-th subset of $\lambda \wedge \alpha \in X)\}$ and any $\mathfrak{X} \in V_{\lambda+2}$ codes $\{\alpha < \lambda^+$ : the $\underset{L}{<}$ $\alpha$-th constructible subset of $\lambda$ is $\in \mathfrak{X}\}$.

The poset Q will be a suborder of $Fn(\lambda^{++},2,\lambda)$. At many coordinates $\alpha < \lambda^{++}$ we will simply add a subset of $\lambda^+$; at many other $\alpha < \gamma^{++}$ we will add a club subset of $\lambda^+$ that is disjoint from $F_\gamma(S_1,S_2)$ where $(S_1,S_2)$ is some pair of subsets of $\lambda^+$ associated with $\alpha$ if certain conditions are satisfied.

First we partition $\lambda^{++}$ into cofinal pieces C, $A^\gamma(\gamma < \lambda^+)$ and $B^\gamma$ $(\gamma < \lambda^+)$ such that $\lambda^+ \subseteq C$. Then for each $\gamma < \lambda^+$ we pick an enumeration $(\tau^\gamma_\zeta : \zeta \in A^\gamma)$ of nice $Fn(\lambda^{++},2,\lambda^+)$ names for subsets of $\lambda^+$ and for each $\gamma < \lambda^+$ an enumeration $((\sigma^{1,\gamma}_\zeta,\sigma^{2,\gamma}_\zeta) : \zeta \in B^\gamma)$ of pairs of nice $Fn(\lambda^{++},2,\lambda^+)$ names for subsets of $\lambda^+$ such that for any nice $Fn(\lambda^{++},2,\lambda^+)$ name $\tau$ for a subset of $\lambda^+$ and any $\gamma < \lambda^+$ there are cofinally many $\zeta$ with $\tau^\gamma_\zeta = \tau$ and similarly for pairs of nice $Fn(\lambda^{++},2,\lambda^+)$ names for subsets of $\lambda^+$. In the sequel we will call such enumerations of tupels of nice names complete. Note that complete enumerations of length $\lambda^{++}$ always exist since $Fn(\lambda^{++},2,\lambda^+)$ has the $\lambda^{++}$ c.c. and size $\lambda^{++}$.

We now define the coordinates $<\alpha$ of Q by induction on $\alpha \leq \lambda^{++}$

for $\alpha = 0$

$$Q_0 \underset{\mathfrak{Df}}{=} \{\emptyset\}$$

if $\alpha \leq \lambda^{++}$ is a limit then

$$Q_\alpha \underset{\mathfrak{Df}}{\equiv} \{f \in Fn(\alpha,2,\lambda^+): \forall \eta < \alpha\ f|\eta \in Q_\eta\}$$

and if $\alpha \in C$ or $\alpha \in A^\gamma$ (some $\gamma < \lambda^+$) then

$$Q_{\alpha+1} \underset{\mathfrak{Df}}{\equiv} \{f \in Fn(\alpha+1,2,\lambda^+) : f|\alpha \in Q_\alpha\}$$

and if $\alpha \in B^\gamma$ for some $\gamma < \lambda^+$ then

$$Q_{\alpha+1} \underset{\mathfrak{Df}}{\equiv} \{f \in Fn(\alpha+ 1,2,\lambda^+): f|\alpha \in Q_\alpha \wedge$$

$$f|\alpha \Vdash_{Q_\alpha} \theta(\Gamma,\gamma,(\hat{\tau}^\gamma_\zeta : \zeta \in A^\gamma \cap \alpha), \hat{\sigma}^{1,\gamma}_\alpha, \hat{\sigma}^{2,\gamma}_\alpha, F_\gamma, f(\alpha)\}$$

where for $\zeta \in A^\gamma$

$$\hat{\tau}^\gamma_\zeta \underset{\mathfrak{Df}}{\equiv} \{(\eta,f): \exists g[(\eta,g) \in \tau^\gamma_\zeta \wedge f \leq g \wedge f \in Q_\zeta]\}$$

and similarly

$$\hat{\sigma}^{i,\gamma}_\alpha \underset{\mathfrak{Df}}{\equiv} \{\eta,f): \exists g[(\eta,g) \in \sigma^{i,\gamma}_\alpha \wedge f\leq g \wedge f \in Q_\alpha]\}$$

and when the formula $\theta$ is short for

$\gamma$ is odd $\wedge$ $[[\neg\exists\ \zeta \in A^\gamma \cap \alpha\ (\hat{\tau}^\gamma_\zeta = \hat{\sigma}^{1,\gamma}_\alpha \wedge \Gamma^\zeta = \hat{\sigma}^{2,\gamma}_\alpha) \wedge$

$(\hat{\sigma}^{1,\gamma}_\alpha, \hat{\sigma}^{2,\gamma}_\alpha) \in \text{dom } F_\gamma \wedge f(\alpha)$ kills $F_\gamma(\hat{\sigma}^{1,\gamma}_\alpha, \hat{\sigma}^{2,\gamma}_\alpha)]$ V $f(\alpha) = 0]$ .V.

$\gamma$ is even $\wedge$ $[[\Gamma^\gamma = \hat{\sigma}^{1,\gamma}_\alpha \wedge (\hat{\sigma}^{1,\gamma}_\alpha, \hat{\sigma}^{2,\gamma}_\alpha) \in \text{dom } F_\gamma \wedge f(\alpha)$ kills $F_\gamma(\hat{\sigma}^{1,\gamma}_\alpha, \hat{\sigma}^{2,\gamma}_\alpha)]$ V

$[\neg\exists\zeta \in A^\gamma \cap \alpha(\hat{\tau}^\gamma_\zeta = \hat{\sigma}^{1,\gamma}_\alpha \wedge \Gamma^\zeta = \hat{\sigma}^{2,\gamma}_\alpha) \wedge (\hat{\sigma}^{1,\gamma}_\alpha, \hat{\sigma}^{2,\gamma}_\alpha) \in \text{dom } F_\gamma \wedge$

$\wedge$ $f(\alpha)$ kills $F_\gamma(\hat{\sigma}^{1,\gamma}_\alpha, \hat{\sigma}^{2,\gamma}_\alpha)]$ V $f(\alpha) = 0]$.

Here $\Gamma^\zeta$ denotes the subset of $\lambda^+$ that we add at coordinate $\zeta$. Since compatibility in $Fn(\lambda^{++},2,\lambda^+)$ coincides with compatibility in the suborder Q, we get that Q has the $\lambda^{++}$ c.c. Moreover Q is $<\lambda$ closed because of the cofinality requirement that we put into the definition of the forcing for adding $(F_\gamma : \gamma < \lambda^+)$. However Q is not $<\lambda^+$ closed since for each $\gamma < \lambda^+ \Vdash^{V[G_\lambda]}_{Q_{(1)}}$ "$\forall X,Y \subseteq \lambda^+$ $F_\gamma(X,Y)$ is stationary" and, as we shall see, Q makes "many" of these sets nonstationary. Lemma 4.1.3. will show that Q is $<\lambda^+$ Baire. To prepare the proof of this we need some technical lemmas which will also be helpful later.

It is easy to modify the definition of the forcing $Q_{(1)}$ so that rather than add Lipshitz functions $F_\gamma : 2^{\lambda^+} \times 2^{\lambda^+} \longrightarrow 2^{\lambda^+}$ where $\gamma < \lambda^+$ we add a sequence $((F_\gamma, H_\gamma) : \gamma < \lambda^+)$ of pairs of Lipshitz functions with $F_\gamma, H_\gamma : 2^{\lambda^+} \times 2^{\lambda^+} \longrightarrow 2^{\lambda^+}$ such that for all X, Y $\subseteq \lambda^+$ $H_\gamma(X,Y)$ is a club subset of $\lambda^+$ which is disjoint from $F_\gamma(X,Y)$. We denote the modified poset by $Q'_{(1)}$ . It is a $\lambda^+$ product with $<\lambda^+$ support of a posets whose conditions are pairs (f,h) where

$f \in P_F \wedge$ h is a function $\wedge$ dom h = dom f

$\wedge\ \forall(s,t) \in$ dom h $[\exists\alpha < \lambda^+ [\alpha >$ dom s $\wedge\ h(s,t) \in 2^{\alpha+1} \wedge f(s,t) \in 2^{\alpha+1}$

$\wedge\ h(s,t)(\alpha) = 1] \wedge \{\zeta : h(s,t)(\zeta) = 1 = f(s,t)(\zeta)\} = \emptyset$

$\wedge\ \{\zeta : h(s,t)(\zeta) = 1\}$ is closed$] \wedge$

$\forall(s,t), (s',t') \in$ dom h[(s,t) extends $(s',t') \Rightarrow$ h(s,t) extends h(s',t')].

If $(f,h) \in Q'_{(1)}$ then $f \in Q_{(1)}$ and conversely for any $f \in Q_{(1)}$ there are h with $(f,h) \in Q'_{(1)}$. Moreover if D is dense in $Q_{(1)}$ then $\{(f,h) \in Q'_{(1)} : f \in D\}$ is dense in $Q'_{(1)}$. Thus if $((F_\gamma, H_\gamma) : \gamma < \lambda^+)$ is $Q'_{(1)}$ generic over $V[G_\lambda]$ then $(F_\gamma : \gamma < \lambda^+)$ is $Q_{(1)}$ generic over $V[G_\lambda]$ and $V[G_\lambda, \vec{F}_\gamma] \subseteq V[G_\lambda, (\vec{F}_\gamma, \vec{H}_\gamma)]$. In $V[G_\lambda, (\vec{F}_\gamma, \vec{H}_\gamma)]$ we now define the following poset $(\zeta \leq \lambda^{++})$

$$Q^*_\zeta \underset{\mathfrak{Df}}{=} \{q \in Q_\zeta : \forall \gamma < \lambda^+ \; \forall \beta \in B^\gamma [q(\beta) \neq 0 \Rightarrow$$

$$q|\beta \Vdash \frac{V[G_\lambda, (\vec{F}_\gamma, \vec{H}_\gamma)]}{Q_\beta} \text{ top } q(\beta) \in H_\gamma(\hat{\sigma}_\beta^{1,\gamma}, \hat{\sigma}_\beta^{1,\gamma})]\}.$$

Lemma 4.1.1. For each $\alpha \leq \lambda^{++}$ $Q^*_\alpha$ is $<\lambda^+$ closed.

Proof. Let $\alpha \leq \lambda^{++}$ and $(q_\eta : \eta < \lambda) \in V[G_\lambda, (\vec{F}_\gamma, \vec{H}_\gamma)]$ be a decreasing sequence of conditions in $Q^*_\alpha$. By induction on $\zeta \leq \alpha$ we will construct $q|\zeta$ such that

$q|\zeta \in Q^*_\zeta$

$\mathrm{dom}(q|\zeta) = \bigcup_{\eta<\lambda} \mathrm{dom}(q_\eta|\zeta)$

$q|\zeta \leq q_\eta|\zeta$ for all $\eta < \lambda$.

We check only the case of a successor as ordinal $\zeta + 1$. If $\zeta \in C$ or $\zeta \in A^{\bar{\lambda}}$ for some $\bar{\lambda} < \lambda^+$ there is no problem. Now assume that $\zeta \in B^\gamma$ for some $\gamma < \lambda^+$ and for some $\eta < \lambda$ $q_\eta(\zeta) \neq 0$.

Then take

$$q(\zeta) = \bigcup_{\substack{\eta<\lambda \\ \zeta \in \mathrm{dom}\, q_\eta}} q_\eta(\zeta) \cup \sup_{\eta<\lambda} \underset{\substack{\zeta \in \mathrm{dom}\, q_\eta \\ q_\eta(\zeta) \neq 0}}{\mathrm{top}} q_\eta(\zeta).$$

We get that $q|(\zeta + 1) \in Q_{\zeta+1}$ since (in any model) if for some X, Y $\subseteq \lambda^+$ $H_\gamma(X,Y)$ is defined, then it is disjoint from $F_\gamma(X,Y)$ and closed.

By the same argument $q|(\zeta + 1) \in Q^*_{\zeta+1}$. It is immediately clear that $q|(\zeta + 1) \leq q_\eta|(\zeta + 1)$ for all $\eta < \lambda$ and dom $q|(\zeta + 1) = \bigcup_{\eta<\lambda} \text{dom } q_\eta|(\zeta + 1)$.

□
end of 4.1.1

Lemma 4.1.2. For each $\alpha \leq \lambda^{++}$ $Q^*_\alpha$ is dense in $Q_\alpha$.

Proof. We use induction on $\alpha \leq \lambda^{++}$.

For $\alpha = 0$ this is true since $Q_0 = \{0\} = Q^*_0$ . For a successor ordinal $\alpha + 1$ we examine only the case where $\alpha \in B^\gamma$ for some $\gamma < \lambda^+$ and we are given $q \in Q_{\alpha+1}$ with $q(\alpha) \neq 0$. Since in any model $H_\gamma(X,Y)$ is always unbounded (if defined) we can pick an ordinal $\delta < \lambda^+$ with $\delta >$ top $q(\alpha)$ and a condition $q^* \leq q$ with $q^* \in Q^*_\alpha$ (by using that $Q^*_\alpha$ is dense in $Q_\alpha$) such that $q^* \Vdash^{V[G_\lambda,(\vec{F}_\gamma,\vec{H}_\gamma)]}_{Q_\alpha} \delta \in H_\gamma(\hat{\sigma}^{1,\gamma}_\alpha, \hat{\sigma}^{2,\gamma}_\alpha)$.

Hence $q^* \cup \{(\alpha, q(\alpha) \cup \{\delta\})\}$ is a condition in $Q^*_{\alpha+1}$ below q. If $\alpha$ is a limit ordinal and $q \in Q_\alpha$ there are two possibilities: If $\text{cf}(\alpha) \geq \lambda^+$ then dom q is bounded in $\alpha$ and we can by induction hypothesis find a condition $q^* \in Q^*_\alpha$ below q. If $\text{cf}(\alpha) \leq \lambda$ then we pick a normal sequence $(\lambda_\eta : \eta \leq \beta)$ with $\sup_{\eta<\beta} \lambda_\eta = \alpha$ and $\lambda_\beta = \alpha$ and $\beta = \text{cf}(\alpha) \leq \lambda$. Now using induction on $\eta \leq \beta$ we define a decreasing sequence $(q_\eta : \eta \leq \beta)$ with $q_\eta \in Q^*_{\lambda_\eta}$ and $q_\eta \leq q|\lambda_\eta$.

By 4.1.1. each $Q^*_{\lambda_\eta}$ is $<\lambda^+$ closed so the construction works at limit ordinals $\eta \le \beta$. Clearly $q_\beta \in Q^*_\alpha$ and extends q.

□
end of 4.1.2.

Lemma 4.1.3. For any $\alpha \le \lambda^{++}$ $Q_\alpha$ is $<\lambda^+$ Baire.

Proof. Suppose the lemma was false for some $\alpha \le \lambda^{++}$. Pick names $\vec{\mathring{A}}^\gamma$, $\vec{\mathring{B}}^\gamma$, $\mathring{C}$, $\vec{\mathring{\tau}}^\gamma_\zeta$, $\vec{\mathring{\sigma}}^{i,\gamma}_\zeta$ in $V[G_\lambda]^{Q_{(1)}}$ for the parameters that we need in the definition of $Q_\alpha$ and fix a condition $f \in Q_{(1)}$ with

(4.1.4) $$f \Vdash^{V[G_\lambda]}_{Q_{(1)}} \text{“} Q_\alpha(\text{defined from } \vec{\mathring{A}}^\gamma, \vec{\mathring{B}}^\gamma, \mathring{C}, \vec{\mathring{\tau}}^\gamma_\zeta, \vec{\mathring{\sigma}}^{i,\gamma}_\zeta) \text{ is not } <\lambda^+ \text{ Baire.”}$$

Pick some h such that $(f,h) \in Q'_{(1)}$. Now let $((F_\gamma,H_\gamma): \gamma < \lambda^+)$ be $Q'_{(1)}$ generic over $V[G_\lambda]$ and extending (f,h). In $V[G_\lambda,(\vec{F}_\gamma,\vec{H}_\gamma)]$ $Q_\alpha$ has a $<\lambda^+$ closed, dense suborder, namely $Q^*_\alpha$. So in particular in $V[G_\lambda,\vec{F}_\gamma]$ $Q_\alpha$ is $<\lambda^+$ Baire, contradicting (4.1.4).

□
end of 4.1.3.

As a corollary we get that for any $\alpha \le \lambda^{++}$

$$\Vdash^{V[G_\lambda,\vec{F}_\gamma]}_{Q_\alpha} \forall\gamma < \lambda^+ \text{ dom } F_\gamma = 2^{\lambda^+} \times 2^{\lambda^+} .$$

Therefore in the definition of $Q_{\alpha+1}$ we can omit the clause $(\hat{\sigma}^{1,\gamma}_\alpha,\hat{\sigma}^{2,\gamma}_\alpha) \in \text{dom } F_\gamma$ in the formula $\theta$.

Next we want to see that forcing with Q makes a $\Sigma^2_3$ or $\Pi^2_3$ statement about $F_\gamma$ true depending on whether $\gamma$ is even or odd. We must first prove a technical fact which will be used again later.

Lemma 4.1.5. For any condition $q \in Q$ and any ordinal $\delta < \lambda^+$ there is a stronger condition $q' \in Q$ with $\forall\gamma < \lambda^+\ \forall\beta \in B^\gamma[q(\beta) \neq 0 \Rightarrow \text{top } q'(\beta) > \delta]$.

Proof. Suppose this was false. Pick names $\mathring{C}$, $\vec{\mathring{B}}^\gamma$, $\vec{\mathring{A}}^\gamma$ and $\vec{\mathring{\tau}}^\gamma_\zeta$, $\vec{\mathring{\sigma}}^{1,\gamma}_\zeta$, $\vec{\mathring{\sigma}}^{2,\gamma}_\zeta$ in $V[G_\lambda]^{Q_{(1)}}$ and a condition $f \in Q_{(1)}$ with

(4.1.6.) $\quad f \Vdash^{V[G_\lambda]}_{Q_{(1)}}$ "there is no $q' \in Q\ (\mathring{C}, \vec{\mathring{A}}^\gamma, \vec{\mathring{B}}^\gamma, \vec{\mathring{\tau}}^\gamma_\zeta, \vec{\mathring{\sigma}}^{1,\gamma}_\zeta, \vec{\mathring{\sigma}}^{2,\gamma}_\zeta)$

with $q' \leq q \wedge \forall\gamma < \lambda^+\ \forall\beta \in \mathring{B}^\gamma(q(\beta) \neq 0$

$\Rightarrow \text{top } q'(\beta) > \delta)$;"

pick some h with (f,h) a condition in the modified poset $Q'_{(1)}$. Now let $((F_\gamma, H_\gamma) : \gamma < \lambda^+)$ be $Q'_{(1)}$ generic over $V[G_\lambda]$. Choose an increasing enumeration $(\alpha_\eta : \eta < \tilde{\lambda})$ of dom q where $|\tilde{\lambda}| = \lambda$.

In $V[G_\lambda, ((F_\gamma, H_\gamma) : \gamma < \lambda^+)$ we define a decreasing sequence $(q_\eta : \eta < \tilde{\lambda})$ where for $\eta < \tilde{\lambda}$

$$q_\eta \in Q^*_{\alpha_\eta}$$

$$q_\eta \leq q|\alpha_\eta$$

$$\forall\gamma < \lambda^+\ \ \forall\beta \in B^\gamma \cap \alpha_\eta(q(\beta) \neq 0 \Rightarrow \text{top } q_\eta(\beta) > \delta)$$

for $\eta = 0$ let $q_\eta = 0$.

If $\eta$ is a limit pick a condition $q_\eta \in Q^*_{\alpha_\eta}$ with $q_\eta \leq q_{\eta'}$ $\forall\eta' < \eta$. This is possible since $Q^*_{\alpha_\eta}$ is $<\lambda^+$ closed.

For successor $\eta + 1$ we examine only the case that $\alpha_\eta \in B^\gamma$ for some $\gamma < \lambda^+$ and $q(\alpha_\eta) \neq 0$. Similarly as in the successor case of the proof of 4.1.2., we can find some ordinal $\delta' > \delta$ and a condition $q^* \leq q_\eta$, $q|\alpha_{\eta+1}$ in $Q^*_{\alpha_{\eta+1}}$ such that top $q^*(\alpha_\eta) = \delta'$. Define this to be $q_{\eta+1}$.

Once the sequence $(q_\eta : \eta < \lambda)$ has been defined one can find a condition $q' \in Q^*_{\sup^+_{\eta<\tilde{\lambda}} \alpha_\eta}$ with the property that we want by either using the $<\lambda^+$ closure $Q^*_{\sup^+_{\eta<\tilde{\lambda}} \alpha_\eta}$ of or repeating the argument for the successor case in the inductive construction of the sequence above depending on whether dom q has a last element or not. In particular $q' \in Q$, so we get a contradiction with (4.1.6). $\square$ end of 4.1.5.

Next we want to see that after forcing with Q any $F_\gamma(X,Y)$ is stationary (for $X, Y \subseteq \lambda^+$) unless we kill it explicitly. To be more exact:

Lemma 4.1.7. Let G be a Q generic over $V[G_\lambda, \vec{F}_\gamma]$. In $V[G_\lambda, \vec{F}_\gamma, G]$ let $X, Y \subseteq \lambda^+$ and $\gamma_o < \lambda^+$ and assume

$$\forall \beta \in B^{\gamma_o}\ \forall q \in G[q(\beta) \neq 0 \Rightarrow ((\hat{\sigma}_\beta^{1,\gamma_o})^G, (\hat{\sigma}_\beta^{2,\gamma_o})^G) \neq (X,Y)].$$

Then $F_{\gamma_o}(X,Y)$ is stationary.

Proof. Pick names $\mathring{C}$, $\vec{\mathring{A}}^\gamma$, $\vec{\mathring{B}}^\gamma$, $\vec{\mathring{\tau}}^\gamma_\zeta$, $\vec{\mathring{\sigma}}^{i,\gamma}_\zeta$ in $V[G_\lambda]^{Q(1)}$ for the parameters in the definition of Q.

Let $\sigma^1, \sigma^2, \sigma \in V[G_\lambda]^{Q_{(1)}*Q}$ and $(\bar{f},\bar{q}) \in \vec{F}_\gamma * G$ with

$$(\bar{f},\bar{q}) \Vdash^{V[G_\lambda]}_{Q_{(1)}*Q} \text{“}\forall q \in G\ \forall \beta \in \mathring{B}^{\gamma_o}[q(\beta) \neq 0 \Rightarrow (\hat{\mathring{\sigma}}^{1,\gamma_o}_\beta, \hat{\mathring{\sigma}}^{2,\gamma_o}_\beta) \neq (\sigma^1,\sigma^2)]$$

$$\wedge\ \sigma \text{ is club in } \lambda^+.\text{”}$$

In order to finish the proof we have to find a condition $(f,q) \leq (\bar{f},\bar{q})$ and $s^1, s^2 \in 2^{<\lambda^+}$ and an ordinal $\alpha < \lambda^+$ such that

$$(s^1, s^2) \in \text{dom } f^{\gamma_o}$$

$$f^{\gamma_o}(s^1,s^2)(\alpha) = 1$$

$$(f,q) \Vdash^{V[G_\lambda]}_{Q_{(1)}*Q} \text{“}(\sigma^1,\sigma^2) \text{ extends } (s^1,s^2) \wedge \alpha \in \sigma.\text{”}$$

In order to come up with $(f,q)$ and $(s^1,s^2)$ and $\alpha$ we have to define a decreasing sequence $((f_\eta,q_\eta):\eta < \lambda)$ of conditions below $(\bar{f},\bar{q})$ and auxiliary sequences

$(\alpha_\eta:\eta < \lambda)$

$(\delta_\eta:\eta < \lambda)$

$(T_\eta:\eta < \lambda)$

$((b^\gamma_\eta:\gamma \in T_\eta):\ \eta < \lambda)$

$(((s^{1,\gamma}_{\beta,\eta}, s^{1,\gamma}_{\beta,\eta}):\beta \in b^\gamma_\eta):\gamma \in T_\eta):\eta < \lambda)$

$((s^1_\eta,s^2_\eta):\eta < \lambda)$

where $\delta_\eta^\gamma$, $\alpha_\eta < \lambda^+$ and $T_\eta \subseteq \lambda^+$ and $b_\eta^\gamma \subseteq \lambda^{++}$ and $s_{\beta,\eta}^{i,\gamma}$, $s_\eta^i \subseteq \lambda^+$ and at stage $\eta$ of the construction we have

$$\alpha_\eta, \delta_\eta \quad > \quad \sup_{\substack{\eta' < \eta \\ \gamma \in \bigcup_{\eta'<\eta} \operatorname{supp} f_{\eta'}}} \quad \operatorname{dom}_{(s,t) \in \operatorname{dom} f_{\eta'}^\gamma} \quad f_{\eta'}^\gamma(s,t) \;\cup\; \sup_{\eta'<\eta}(\alpha_{\eta'} \cup \delta_{\eta'})$$

$$f_\eta \Vdash_{Q_{(1)}}^{V[G_\lambda]} \begin{cases} T_\eta = \{\gamma < \lambda^+ : \mathring{B}^\gamma \cap \bigcup_{\eta'<\eta} \operatorname{dom} q_{\eta'} \neq \emptyset\} \cup \{\gamma_o\} \\ \forall \gamma \in T_\eta \quad b_\eta^\gamma = \mathring{B}^\gamma \cap \bigcup_{\eta'<\eta} \operatorname{dom} q_{\eta'} \end{cases}$$

$$(f_\eta, q_\eta) \Vdash_{Q_{(1)} * Q}^{V[G_\lambda]} \begin{cases} \hat{\mathring{\sigma}}_\beta^{i,\gamma} \cap \delta_\eta^\gamma = s_\beta^{i,\gamma} & (i = 1, 2, \gamma \in T_\eta, \beta \in b_\eta^\gamma) \\ \sigma^i \cap \delta_\eta^{\gamma_o} = s_\eta^i & (i = 1, 2) \\ \alpha_\eta \in \sigma \end{cases}$$

$$\forall \beta \in b_\eta^{\gamma_o}\ [\exists \eta' < \eta\ q_{\eta'}(\beta) \neq 0 \Rightarrow (s_{\beta,\eta}^{1,\gamma_o}, s_{\beta,\eta}^{2,\gamma_o}) \neq (s_\eta^1, s_\eta^2)]$$

$$\forall \gamma \in T_\eta\ \forall \beta \in b_\eta^\gamma [\exists \eta' < \eta\ \ q_{\eta'}(\beta) \neq 0 \Rightarrow \operatorname{top} q_\eta(\beta) > \delta_\eta].$$

We construct the sequences by induction on $\eta < \lambda$. If we have arrived at stage $\eta < \lambda$ and all requirements hold at the earlier stages of the construction we proceed as follows: Since $Q_{(1)} * Q$ is $<\lambda$ closed there is a condition $(f^*, q^*) \leq (\overline{f}, \overline{q})$ which extends all $(f_{\eta'}, q_{\eta'})$ for $\eta' < \eta$. Now pick $f^{**} \leq f^*$ and $T_\eta \subseteq \bigcup_{\eta'<\eta} \operatorname{dom} q_{\eta'}$ and $(b_\eta^\gamma : \gamma \in T_\eta)$ with

$$f^{**} \Vdash_{Q_{(1)}}^{V[G_\lambda]} \begin{cases} T_\eta = \{\gamma < \lambda^+ : \mathring{B}^\gamma \cap \bigcup_{\eta' < \eta} \operatorname{dom} q_{\eta'} \neq \emptyset\} \cup \{\gamma_o\} \\ \forall \gamma \in T_\eta \quad b_\eta^\gamma = \mathring{B}^\gamma \cap \bigcup_{\eta' < \eta} \operatorname{dom} q_{\eta'} . \end{cases}$$

Then pick ordinals $\alpha_\eta < \lambda^+$ and $\delta_\eta < \lambda^+$ and sets $s^{i,\gamma}_{\beta,\eta}$, $s^i_\eta \subseteq \lambda^+$ (for $\gamma \in T_\eta, \beta \in b^\gamma_\eta, i = 1,2$) and a condition $(f^{***}, q^{**}) \leq (f^{**}, q^*)$ such that

$$\alpha_\eta, \delta_\eta > \sup_{\substack{(s,t) \in \operatorname{dom} f^{**,\gamma} \\ \gamma \in \operatorname{supp} f^{**}}} \operatorname{dom} f^{**,\gamma}(s,t) \cup \sup_{\eta' < \eta} (\alpha_{\eta'} \cup \delta_{\eta'})$$

$$(f^{***}, q^{**}) \Vdash_{Q_{(1)} * Q}^{V[G_\lambda]} \begin{cases} \mathring{\sigma}^{i,\gamma}_\beta \cap \delta_\eta = s^{i,\gamma}_{\beta,\eta} \quad (i = 1, 2, \beta \in b^\gamma_\eta, \gamma \in T_\eta) \\ \sigma^i \cap \delta_\eta = s^i_\eta \qquad (i = 1,2) \\ \alpha_\eta \in \sigma \end{cases}$$

$$\forall \beta \in b^{\gamma_o}_\eta \, [\exists \eta' < \eta \; q_{\eta'}(\beta) \neq 0 \Rightarrow (s^{1,\gamma_o}_{\beta,\eta}, s^{2,\gamma_o}_{\beta,\eta}) \neq (s^1_\eta, s^2_\eta)].$$

Note that all this can be done since $Q_{(1)} * Q$ is $<\lambda^+$ Baire. Finally (using 4.1.5.) we pick $(f_\eta, q_\eta) \leq (f^{***}, q^{**})$ such that

$$\forall \gamma \in T_\eta \; \forall \beta \in b^\gamma_\eta [q^{**}(\beta) \neq 0 \Rightarrow \operatorname{top} q_\eta(\beta) > \delta_\eta].$$

This completes the definition of the sequences.

Now let $\alpha \underset{\mathfrak{Df}}{=} \sup_{\eta < \lambda} \alpha_\eta$ and pick a condition $f \in Q_{(1)}$ that extends all $f_\eta$ for $\eta < \lambda$ such that for all $\gamma \in \bigcup_{\eta < \lambda} T_\eta$ and all $\beta \in \bigcup_{\substack{\eta < \lambda \\ \gamma \in T_\eta}} b^\gamma_\eta$ $(s^{1,\gamma}_\beta, s^{2,\gamma}_\beta) \in \operatorname{dom} f^\gamma$ where

$s^{i,\gamma}_\beta \underset{\mathfrak{Df}}{=} \bigcup_{\eta<\lambda} s^{i,\gamma}_{\beta,\eta} (i = 1,2)$ and $(s^1,s^2) \in \text{dom } f^{\gamma o}$ where $s^i \underset{\mathfrak{Df}}{=} \bigcup_{\eta<\lambda} s^i_\eta (i = 1, 2)$ and

$f^\gamma(s^{1,\gamma}_\beta, s^{2,\gamma}_\beta)(\sup_{\substack{\eta<\lambda \\ \beta \in \text{dom } q_\eta \\ q_\eta(\beta)\neq 0}} \text{top } q_\eta(\beta)) = 0$ if $\exists\eta < \lambda\ q_\eta(\beta) \neq 0$ and $f^{\gamma o}(s^1,s^2)(\alpha) = 1.$

Note that if $\exists\eta < \lambda\ q_\eta(\beta) \neq 0$ then there is no conflict here.

We also define q by

$$\text{dom } q \underset{\mathfrak{Df}}{=} \bigcup_{\eta<\lambda} \text{dom } q_\eta$$

$$q_\eta(\beta) \underset{\mathfrak{Df}}{=} \begin{cases} \bigcup_{\eta<\lambda} q_\eta(\beta) \cup \{ \sup_{\substack{\eta<\lambda \\ q_\eta(\beta)\neq 0}} \text{top } q_\eta(\beta)\} & \text{if } \beta \in \bigcup_{\substack{\eta<\lambda \\ \gamma\in T_\eta}} b^\gamma_\eta \text{ and } q_\eta(\beta) \neq 0 \\ & \text{for some } \eta < \lambda \\ \bigcup_{\eta<\lambda} q_\eta(\beta) & \text{otherwise.} \end{cases}$$

Then (f,q) is a condition in $Q_{(1)} * Q$ below $(\overline{f},\overline{q})$ and $(s^1,s^2)$ and $\alpha$ have the properties that we want.

□
end of 4.1.7.

Corollary 4.1.8. For odd $\gamma < \lambda^+$

$$\Vdash^{V[G_\lambda,\vec{F}\gamma]}_{Q} \forall X \subseteq \lambda^+\ \exists Y \subseteq \lambda^+\ F_\gamma(X,Y) \text{ is stationary.}$$

Proof. Let G be Q generic over $V[G_\lambda,\vec{F}\gamma]$ and $X \subseteq \lambda^+$. Pick some $\alpha \in A^\gamma$ with $\hat{\tau}^G_\alpha = X$ (note that $(\tau^\gamma_\zeta : \zeta \in A^\gamma)$ is complete). We claim that $F_\gamma(X,G^\alpha)$ is stationary in $V[G_\lambda,\vec{F}_\gamma,G]$ where $G^\alpha$ is the set $\subseteq\lambda^+$ that G adds at coordinate $\alpha$. If $q \in G$, then for all $\beta \in B^\gamma \cap \alpha$ $(\hat{\sigma}_\beta^{2,\gamma})^G \neq G^\alpha$ since by the product lemma $G^\alpha$ is generic over $V[G_\lambda,\vec{F}_\gamma,G \cap Q_\alpha]$. For all $\beta \in B^\gamma$ with $\beta > \alpha$ we get that $q(\beta) \neq 0$ implies $(\hat{\sigma}_\beta^{1,\gamma},\hat{\sigma}_\beta^{1,\gamma}) \neq (\hat{\tau}_\alpha^\gamma\ ,G^\alpha)$ by the definition of $Q_{\beta+1}$. Hence by 4.1.7. $F_\gamma(X,G^\alpha)$ is stationary.

□ end of 4.1.8.

Lemma 4.1.9. For even $\gamma < \lambda^+$

$$\Vdash_{Q}^{V[G_\lambda,\vec{F}_\gamma]} \exists X \subseteq \lambda^+ \ \forall Y \subseteq \lambda^+ \ F_\gamma(X,Y) \text{ is nonstationary.}$$

Proof 4.1.9. Let G be a Q generic over $V[G_\lambda,\vec{F}_\gamma]$. We claim that in $V[G_\lambda,\vec{F}_\gamma,G]$ $\forall Y \subseteq \lambda^+$ $F_\gamma(G^\gamma,Y)$ is dead. Let $Y \subseteq \lambda^+$ and pick $\beta \in B^\gamma$ with $(\hat{\sigma}_\beta^{1,\gamma})^G = G^\gamma$ and $(\hat{\sigma}_\beta^{2,\gamma})^G = Y$. Since $\gamma$ is even, at coordinate $\beta$ we add a club subset of $\lambda^+$ disjoint from $F_\gamma(G^\gamma,Y)$.

□ end of 4.1.9.

Now let G be Q generic over $V[G_\lambda,\vec{F}_\gamma]$. The next step in our 4-step iteration will add a code $\tilde{S}_\gamma \subseteq \lambda$ for each $F_\gamma(\gamma < \lambda^+)$. Recall that $2^{<\lambda^+}$ is the same whether computed in $V[G_\lambda,\vec{F}_\gamma,G]$ or in $V[G_\lambda]$. But $V[G_\lambda]$ is just $L[G_\lambda]$ since we started out in V = L. Moreover $G_\lambda \subseteq P_\lambda \subseteq L_\lambda$; hence $2^{<\lambda^+} \subseteq L_{\lambda^+}[G_\lambda]$ and has order type $\lambda^+$ under the canonical wellordering $<_{L[G_\lambda]}$. Thus each $F_\gamma$ has a code $\tilde{F}_\gamma \subseteq \lambda^+$

with $\tilde{F}_\gamma \in V[G_\lambda, F_\gamma]$. Now let $Q_{(3)}$ be the $\lambda^+$ product with $<\lambda$ support of the posets $(Q_{\tilde{F}_\gamma} : \gamma < \lambda)$ where for each $\gamma < \lambda^+$ the poset $Q_{\tilde{F}_\gamma}$ adds a code $\tilde{S}_\gamma \subseteq \lambda$ for the set $\tilde{F}_\gamma \subseteq \lambda^+$ via the $\underset{L}{<}$ least almost disjoint family of size $\lambda^+$ of constructible subsets of $\lambda$. Note that $\lambda^+ = (\lambda^+)^L$ and $\lambda$ is inaccessible and $GCH^{\geq\lambda}$ holds in $V[G_\lambda, \vec{F}_\gamma, G]$. Thus as in the $\sigma_1^2|\pi_1^2$ case each $Q_{\tilde{F}_\gamma}$ is $\lambda$ centered and $<\lambda$ closed. Hence by a $\Delta$ system argument $Q_{(3)}$ has the property $\lambda^+$ and is $<\lambda$ closed. Therefore in particular $Q_{(3)} \times Q_{(3)}$ is $\lambda^+$ c.c. and does not add any new subsets of $\lambda^+$ all of whose initial segments are in $V[G_\lambda, \vec{F}_\gamma, G]$. Thus in $V[G_\lambda, \vec{F}_\gamma, G, \vec{S}_\gamma]$ we have:

(4.1.10.) $\exists X \subseteq \lambda^+ \ \forall Y \subseteq \lambda^+ \ \exists \mathcal{M} \ [\mathcal{M}$ trans, $\mathcal{M} \models ZF^-$, $|\mathcal{M}| = |V_{\lambda+1}|$, $\mathcal{M}^\lambda \subseteq \mathcal{M}$, $X, Y \in \mathcal{M}$, $\mathcal{M} \models$ "$\lambda^+ = (\lambda^+)^L \wedge$ if $\tilde{F} \subseteq \lambda^+$ is the set coded by $\tilde{S}_\gamma \subseteq \lambda$ via the $\underset{L}{<}$ least constructible family of size $\lambda^+$ of almost disjoint subsets of $\lambda$ and if F is the Lipshitz function with dom $F = \{(U,V) \in 2^{\lambda^+} \times 2^{\lambda^+} : \forall\alpha < \lambda^+$ $(U \cap \alpha, V \cap \alpha) \in L[G_\lambda]\}$ and rng $F \subseteq \{W \in 2^{\lambda^+} : \forall\alpha < \lambda^+ \ W \cap \alpha \in L[G_\lambda]\}$ that is coded by $\tilde{F}$ using $<_{L[G_\lambda]}$ on $2^{<\lambda^+}$ and if $\forall\alpha < \lambda^+ \ Y \cap \alpha \in L[G_\lambda]$ then $F(X,Y)$ is not stationary"$]$

for even $\gamma < \lambda^+$ and

(4.1.11.) $\forall X \subseteq \lambda^+ \; \exists Y \subseteq \lambda^+ \; \forall \mathcal{M} \, [\mathcal{M}$ trans, $\mathcal{M} \models ZF^-$, $|\mathcal{M}| = |V_{\lambda+1}|$, $\mathcal{M}^\lambda \subseteq \mathcal{M}$,

$X, Y \in \mathcal{M} \wedge \mathcal{M} \models$ "$\lambda^+ = (\lambda^+)^L \wedge \forall \alpha < \lambda^+ \;\; X \cap \alpha \in L[G_\lambda]$" $.\Rightarrow.$

$\mathcal{M} \models$ " if $\tilde{F} \subseteq \lambda^+$ is the set coded by $\tilde{S}_\gamma \subseteq \lambda$ via the $\underset{L}{<}$ least constructible family

of size $\lambda^+$ of almost disjoint subsets of $\lambda$ and if F is the Lipshitz function with

dom F $= \{(U,V) \in 2^{\lambda^+} \times 2^{\lambda^+} : \forall \alpha < \lambda^+ \; (U \cap \alpha, V \cap \alpha) \in L[G_\lambda]\}$

and rng F $\subseteq \{W \in 2^{\lambda^+} : \forall \alpha < \lambda^+ \;\; W \cap \alpha \in L[G_\lambda]\}$ that is coded by

$\tilde{F}$ using $<_{L[G_\lambda]}$ on $2^{<\lambda^+}$ then $(X,Y) \in$ dom F and F(X,Y) is stationary."]

for odd $\gamma < \lambda^+$.

Now the final step $Q_{(4)}$ in our 4-step iteration will be a $\lambda^+$ product with $<\lambda$ support of posets $(Q^\gamma_{(4)} : \gamma < \lambda^+)$ each of which adds a club set $C_\gamma \subseteq \lambda$ such that

$$C_\gamma \cap \{\mu < \lambda : \mu \text{ inaccessible } V_\mu \models \Phi^{\Sigma^2_3}(\tilde{S}_\gamma \cap V_\mu, G_\lambda \cap V_\mu, \lambda \cap \mu)\} = \emptyset$$

where $\Phi^{\Sigma^2_3}$ denotes the $\Sigma^2_3$ formula in (4.1.10). Note that for each $\gamma < \lambda^+$ $|Q^\gamma_{(4)}| = \lambda$ and for each $\delta < \lambda$, $Q^\gamma_{(4)}$ has a dense suborder that is $\delta$ closed. Therefore $Q_{(4)}$ has the property $\lambda^+$ and for each $\delta < \lambda$ has a dense suborder that is $\delta$ closed. So in particular $Q_{(4)} \times Q_{(4)}$ has the $\lambda^+$ c.c. and does not add any new subsets of $\lambda^+$ all of whose initial segments are in $V[G_\lambda, \vec{F}_\gamma, G, \vec{S}_\gamma]$. Hence in $V[G_\lambda, \vec{F}_\gamma, G, \vec{S}_\gamma, \vec{C}_\gamma]$ we still have (4.1.10.) and (4.1.11.). This completes the definition of the certain 4-step iteration that we want to do at stage $\lambda$. Clearly for even $\gamma < \lambda^+$

$$\Vdash_{P_\lambda * \overset{\circ}{Q}_\lambda}^{V} \text{“}\Phi^{\Sigma^2_3}(\tilde{S}_\gamma, G_\lambda, \lambda) \text{ describes } \lambda\text{”}$$

where $\Phi^{\Sigma^2_3}$ is as in (4.1.10). The rest of the clauses in the definition of the iteration $P_{\kappa+1}$ are as in Section 1. Note that as in Section 1:

$$\Vdash_{P_\lambda * \overset{\circ}{Q}_\lambda}^{V} \text{“the tail } P_{\lambda+1,\kappa+1} \text{ has a } <\alpha \text{ closed dense suborder for each } \alpha < \mu\text{”}$$

where $\mu$ is the least inaccessible $>\lambda$. Thus

$$\Vdash_{P_{\kappa+1}}^{V} \text{“there are no } \Sigma^2_3 \text{ indescribables } <\kappa.\text{”}$$

As a minor technical point one might wonder how we can uniformly (in $\lambda \leq \kappa$) pick the parameters that one needs to define the second step of the four-step iteration at stage $\lambda$. For this note that when we arrive at this step we are working in $L[G_\lambda, \vec{F}_\gamma]$ where $\vec{F}_\gamma$ is $V[G_\lambda]$ generic for the first step at stage $\lambda$. Thus we can simply pick the $<_{L[G_\lambda, \vec{F}_\gamma]}$ least family of parameters to define the poset at the second step.

## 4.2. Preservation of the $\Pi^2_3$ Indescribability of $\kappa$ in $V^{P_{\kappa+1}}$.

The aim of this section is to prove

$$\Vdash_{V}^{P_{\kappa+1}} \text{“}\kappa \text{ is } \Pi^2_3 \text{ indescribable.”}$$

Assume towards a contradiction $\Phi(\mathring{A})$ is $\Pi^2_3$ and $\mathring{A} \in V^{P_{\kappa+1}}$ is a name for a subset of $V_\kappa$ and $p \in P_{\kappa+1}$ with

$$p \Vdash_{P_{\kappa+1}} \text{"}\Phi(\mathring{A}) \text{ describes } \kappa.\text{"}$$

We pick an ordinal $\delta >$ the least inaccessible above $\kappa$ such that $V_\delta \models$ "$ZF^- \wedge p \Vdash_{P_{\kappa+1}}$ $\Phi(\mathring{A})$ describes $\kappa$."

Note in particular $P_{\kappa+1} = P^{V_\delta}_{\kappa+1} \in V_\delta$. By using standard arguments we can find a transitive M with $|M| = \kappa$ and $M^{<\kappa} \subseteq M$ and an elementary embedding $i: M \longrightarrow V_\delta$ with cpt (i) $> \kappa$ and ${}^M p$, ${}^M\mathring{A} \in M$ with $i({}^M p) = p$ and $i({}^M\mathring{A}) = \mathring{A}$. Since $\kappa$ is $\Pi^2_3$ indescribable in V we can find transitive N with $|N| = \kappa^+$ and $N^\kappa \subseteq N$ which is $\Sigma^2_2$ correct for $\kappa$ and an elementary embedding $j: M \longrightarrow N$ with cpt $j = \kappa$. In order to finish the proof we have to come up with a V generic ${}^V G$ for $P_{\kappa+1}$ with $p \in {}^V G$ an M generic ${}^M G$ for ${}^M P_{\kappa+1}$ and an N generic ${}^N G$ for ${}^N P_{j(\kappa)+1}$ such that i lifts and j lifts and $N[{}^N G]$ is $\Sigma^2_2$ correct for $\kappa$ inside $V[{}^V G]$. Then we get a contradiction as outlined in Section 1.

<u>Constructing</u> ${}^M G$ <u>and</u> ${}^V G$.

Note that $P^M_\kappa = P_\kappa$ since $M^{<\kappa} \subseteq M$ in V. So we pick a V generic $G_\kappa$ for $P_\kappa$ with $p|\kappa \in G_\kappa$ and i lifts, since clearly $i(p) = p$ for $p \in P_\kappa$.

$$\begin{array}{ccc} M[G_\kappa] & \xrightarrow[i]{} & V_\delta[G_\kappa] \\ P_\kappa & & P_\kappa \\ M & \xrightarrow[i]{} & V_\delta \ . \end{array}$$

In the next step we consider $Q_{(1)} * Q$. Let $(D_\alpha : \alpha < \kappa) \in V_\delta[G_\kappa]$ be an enumeration of all sets $\in M[G_\kappa]$ which are dense in ${}^M Q_{(1)} * {}^M Q$. Below any condition $({}^M\overline{f}, {}^M\overline{q})$ $\in {}^M Q_{(1)} * {}^M Q$ we can construct a decreasing sequence $(({}^M f_\eta, {}^M q_\eta) : \eta < \kappa)$ such that $({}^M f_\eta, {}^M q_\eta) \in D_\eta$(for $\eta < \kappa$) and there is a condition (f,q) extending all $(f_\eta, q_\eta)$ in $Q_{(1)} * Q$ where $f_\eta = i({}^M f_\eta)$ and $q_\eta = i({}^M q_\eta)$. The construction of this sequence takes place in $V[G_\kappa]$ but any initial piece of it is an element of $M[G_\kappa]$ since this is $<\kappa$ closed inside $V[G_\kappa]$ because $P_\kappa$ is $\kappa$ c.c. At stage $\eta$ of this construction we pick ${}^M T_\eta$ $\subseteq {}^M(\kappa^{++})$ and for each $\gamma \in T_\eta$ we pick ${}^M b^\gamma_\eta \subseteq {}^M(\kappa^{++})$ such that

$$ {}^M f_\eta \Vdash_{{}^M Q_{(1)}}^{M[G_\kappa]} \begin{cases} {}^M T_\eta = \{\gamma < \kappa^+ : {}^M \mathring{B}^\gamma \cap \bigcup_{\eta' < \eta} \operatorname{dom} {}^M q_{\eta'} \neq \emptyset\} \\ \\ \forall \gamma \in {}^M T_\eta \ \ b^\gamma_\eta = {}^M \mathring{B}^\gamma \cap \bigcup_{\eta' < \eta} \operatorname{dom} {}^M q_{\eta'} \end{cases} $$

and an ordinal ${}^{M}\delta_\eta < (\kappa^+)^M$ and sets ${}^{M}s^{1,\gamma}_{\beta,\eta}$ , ${}^{M}s^{2,\gamma}_{\beta,\eta} \subseteq (\kappa^+)^M$ for each $\gamma \in {}^{M}T_\eta$ and $\beta \in b^\gamma_\eta$ with

$${}^{M}\delta_\eta > \sup_{\substack{(s,t) \in \operatorname{dom} {}^{M}f^{\gamma}_{\eta'} \\ \gamma \in \operatorname{supp} {}^{M}f_{\eta'} \\ \eta' < \eta}} \operatorname{dom} {}^{M}f^{\gamma}_{\eta'}(s,t) \ \cup \ \sup_{\eta' < \eta} {}^{M}\delta_{\eta'}$$

$$({}^{M}f_\eta, {}^{M}q_\eta) \Vdash_{Q^M_{(1)} * Q^M}^{M[G_\kappa]} \ \forall \gamma \in {}^{M}T_\eta \ \forall \beta \in {}^{M}b^\gamma_\eta \ \ {}^{M}s^{i,\gamma}_{\beta,\eta} = {}^{M}\dot{\hat{\sigma}}^{i,\gamma}_{\beta} \cap {}^{M}\delta_\eta \ \text{ for i = 1, 2}$$

and

$$\forall \gamma \in {}^{M}T_\eta \ \forall \beta \in {}^{M}b^\gamma_\eta [\exists \eta' < \eta \ {}^{M}q_{\eta'}(\beta) \neq 0 \Rightarrow \operatorname{top} {}^{M}q_{\eta'}(\beta) > {}^{M}\delta_\eta].$$

Once the sequence $(({}^{M}f_\eta, {}^{M}q_\eta) : \eta < \kappa)$ has been defined we use the elementarity of $i: M[G_\kappa] \longrightarrow V_\delta[G_\kappa]$ to see that there is a condition (f,q) below all the $(f_\eta, q_\eta)$ for $\eta < \kappa$. Let ${}^{M}\vec{F}_\gamma * {}^{M}G$ denote the filter generated by $(({}^{M}f_\eta, {}^{M}q_\eta) : \eta < \kappa)$. Clearly ${}^{M}\vec{F}_\gamma * {}^{M}G$ is $M[G_\kappa]$ generic for ${}^{M}Q_{(1)} * {}^{M}Q$. Pick any $V[G_\kappa]$ generic $\vec{F}_\gamma * G$ for $Q_{(1)} * Q$ with $(f,q) \in \vec{F}_\gamma * G$. Then i lifts; i.e.,

$$M[G_\kappa, {}^{M}\vec{F}_\gamma, {}^{M}G] \ \underset{i}{\longrightarrow} \ V_\delta[G_\kappa, \vec{F}_\gamma, G]$$

$$M^{P_\kappa * {}^{M}Q_{(1)} * {}^{M}Q} \ \underset{i}{\longrightarrow} \ V_\delta^{P_\kappa * Q_{(1)} * Q} .$$

For the last two steps of the 4-step iteration at stage $\kappa$ where we code the Lipshitz functions by subsets of $\kappa$ and then add the club sets $\subseteq \kappa$ note that both posets are $\kappa^+$ c.c. Hence the pullback method works because $\mathrm{cpt}(i) = (\kappa^+)^M > \kappa$. So we have found an M generic ${}^M G$ for ${}^M P_{\kappa+1}$ and a V generic ${}^V G$ for $P_{\kappa+1}$ with

$$\begin{array}{ccc} M[{}^M G] & \xrightarrow[i]{} & V_\delta[{}^V G] \\ P^M_{\kappa+1} & & P_{\kappa+1} \\ M & \xrightarrow[i]{} & V_\delta \ . \end{array}$$

<u>Construction of</u> ${}^N G$.

Note that ${}^N P_\kappa = j(P_\kappa) \cap V_\kappa = P_\kappa$, since $P_\kappa \subseteq V_\kappa$ and $\mathrm{cpt}\ j = \kappa$. In particular $j(p) = p$ for all $p \in P_\kappa$. Thus if $\tilde{G}$ is any $N[G_\kappa]$ generic for the tail ${}^N P_{\kappa, j(\kappa)}$ then j will lift; i.e.

$$\begin{array}{ccc} M[G_\kappa] & \xrightarrow[j]{} & N[G_\kappa * \tilde{G}] \\ P_\kappa & & {}^N P_{j(\kappa)} \\ M & \xrightarrow[j]{} & N \ . \end{array}$$

We also want to choose $\tilde{G}$ in such a way that $N[G_\kappa * \tilde{G}]$ remains $\Sigma^2_2$ correct for $\kappa$ inside $V[{}^V G]$. First we note

Lemma 4.2.1. $N[G_\kappa]$ is $\Sigma^2_2$ correct for $\kappa$ in $V[G_\kappa]$.

Proof. Recall that $P_\kappa \subseteq V_\kappa$ and is $\kappa$ c.c. The proof is very similar to the proof of II.1.1. □ end of 4.2.1.

The $\kappa$ c.c. of $P_\kappa$ implies that $N[G_\kappa]$ is closed under $\kappa$ sequences inside $V[G_\kappa]$. Thus ${}^NQ_{(1)}$ (the poset at the first step of stage $\kappa$ of ${}^NP_{j(\kappa)}$) equals $Q_{(1)}$ (the poset at the first step of stage $\kappa$ of $P_{\kappa+1}$). In the ground model $V = L$ we pick the $\underset{L}{<}$ least permutation $\Pi$ of $\kappa^+$ such that

$$\Pi\colon\ \mathrm{Even}\,\kappa^+ \xrightarrow[\text{onto}]{1:1} \mathrm{Even}\,\kappa^+ - \mathrm{Even}\,(\kappa^+)^M$$

$$\mathrm{Odd}\ \kappa^+ \xrightarrow[\text{onto}]{1:1} \mathrm{Odd}\ \kappa^+ \cup \mathrm{Even}\,(\kappa^+)^M .$$

Now let ${}^NF_\gamma = F_{\Pi(\gamma)}$ for $\gamma < \kappa^+$. Clearly ${}^N\vec{F}_\gamma$ is $Q_{(1)}$ generic over $N[G_\kappa]$.

Lemma 4.2.2. $N[G_\kappa, {}^N\vec{F}_\gamma]$ is $\Sigma^2_2$ correct for $\kappa$ inside $V[G_\kappa, \vec{F}_\gamma]$.

Proof. Note that $N[G_\kappa, {}^N\vec{F}_\gamma] = N[G_\kappa, \vec{F}_\gamma]$ and that $|Q_{(1)}| = \kappa^+$. Thus $Q_{(1)}$ can be coded by a subset of $V_{\kappa+1}$. Moreover $Q_{(1)}$ is $<\kappa^+$ closed; thus $\Vdash_{Q_{(1)}} V_{\check{\kappa}+1} = (V_{\kappa+1})\check{}$. Hence every $A \in (N[G_\kappa, \vec{F}_\gamma])_{\kappa+2}$ has a nice $Q_{(1)}$ name $\mathring{A} \in (N[G_\kappa])_{\kappa+2}$. Now the proof of the lemma is a routine matter. □ end of 4.2.2.

The correctness argument for the next step involving the poset Q will require a lot more work. For this we need two more technical facts about Q. The first one will say that different choices of parameters all lead to isomorphic posets Q. We begin with a criterion which guarantees that we obtain a condition in Q, when we thin out certain coordinates of a given condition in Q.

Definition 4.2.3. Fix a family of parameters to define the poset Q. A set $S \subseteq \kappa^{++}$ is said to be *complete* (relative to these parameters) if for all $\gamma < \kappa^{+}$

$$\forall \zeta \in A^{\gamma} \cap S \;\; \forall (\eta,h) \in \tau^{\gamma}_{\zeta} \; (\text{dom } h \subseteq \zeta \Rightarrow \text{dom } h \subseteq S)$$

$$\forall \zeta \in B^{\gamma} \cap S \;\; \forall (\eta,h) \in \sigma^{i,\gamma}_{\zeta} \; (\text{dom } h \subseteq \zeta \Rightarrow \text{dom } h \subseteq S) \; (i = 1, 2)$$

and if $\kappa^{+} \subseteq S$.

Definition 4.2.4. Given a fixed family of parameters and $S \subseteq \kappa^{++}$ define

$$Q^{S} = \{q \in Q : \text{dom } q \subseteq S]$$

and for each $\zeta < \kappa^{++}$

$$Q^{S}_{\zeta} = \{q \in Q_{\zeta} : \text{dom } q \subseteq S]$$

and for $\gamma < \kappa^{+}$ and $\zeta \in A^{\gamma}$

$${}^{S}\tilde{\tau}^{\gamma}_{\zeta} = \{(\eta,h) : \exists f[(\eta,f) \in \tau^{\gamma}_{\zeta} \wedge h \leq f \wedge h \in Q^{S}_{\zeta}]\}$$

and similarly for $\zeta \in B^{\gamma}$ and $\sigma^{i,\gamma}_{\zeta}(i = 1, 2)$. If it is clear from the context we will drop the letter S and simply write, for instance, $\tilde{\tau}^{\gamma}_{\zeta}$.

Lemma 4.2.5. Suppose $S \subseteq \kappa^{++}$ is complete relative to a fixed family of parameters. Then for each $\zeta \leq \kappa^{++}$ and all $q \in Q_\zeta$

$$q|S \in Q_\zeta \text{ and}$$

$$Q_\zeta^S \subseteq_c Q_\zeta \quad .$$

Proof. We proceed by induction on $\zeta$. Note that the first claim implies the second claim of the lemma. The cases $\zeta = 0$ and $\zeta$ a limit ordinal are clear. To prove the first claim for a successor $\zeta + 1 < \kappa^{++}$ and a condition $q \in Q_{\zeta+1}$ we can assume $\zeta \in \operatorname{dom} q$ and $\zeta \in B^\gamma \cap S$ for some $\gamma < \kappa^+$. Suppose towards a contradiction that $q|S \notin Q_{\zeta+1}^S$. Then there is a condition $q' \in Q_\zeta^S$ with $q' \leq q|S$ and

$$q' \Vdash_{Q_\zeta^S}^{V[G_\kappa, \vec{F}_\gamma]} \neg\, \theta(\Gamma, \gamma, (\tilde{\tau}_\eta^\gamma : \eta \in A^\gamma \cap S \cap \zeta), \tilde{\sigma}_\zeta^{1,\gamma}, \tilde{\sigma}_\zeta^{2,\gamma}, F_\gamma, q(\zeta)).$$

This is true since the completeness of S implies that for any H which is $Q_\zeta$ generic over $V[G_\kappa, \vec{F}_\gamma]$ $(\tilde{\tau}_\eta^\gamma)^H = (\tilde{\tau}_\eta^\gamma)^{H \cap Q_\zeta^S}$ for all $\eta \in A^\gamma \cap S \cap \zeta$ and similarly for $\sigma_\zeta^{i,\gamma}$ (i = 1, 2). Moreover for $\eta \in A^\gamma \cap \zeta$ $((\hat{\sigma}_\zeta^{1,\gamma})^H, (\hat{\sigma}_\zeta^{2,\gamma})^H) = ((\hat{\tau}_\eta^\gamma)^H, H^\eta)$ implies $\eta \in S$.

If we define $q'' \in Q_\zeta$ by

$q'' \,|(S \cap \zeta) = q'$

$q'' \,|(\zeta - S) = q|(\zeta - S)$

then with the remarks above

$$q'' \Vdash_{Q_\zeta}^{V[G_\kappa,\overrightarrow{F}_\gamma]} \neg\, \theta(\Gamma,\gamma,(\hat{\tau}^\gamma_\eta : \eta \in A^\gamma \cap \zeta),\ \hat{\sigma}^{1,\gamma}_\zeta, \hat{\sigma}^{2,\gamma}_\zeta F_\gamma, q(\zeta))$$

which contradicts $q \in Q_{\zeta+1}$. □ end of 4.2.5.

Now let $(A^\gamma : \gamma < \kappa^+)$, $(B^\gamma : \gamma < \kappa^+)$, C, $(\tau^\gamma_\zeta : \zeta \in A^\gamma)$ $((\sigma^{1,\gamma}_\zeta, \sigma^{2,\gamma}_\zeta) : \zeta \in B^\gamma)(\gamma < \kappa^+)$ and $(\overline{A}^\gamma : \gamma < \kappa^+)$ $(\overline{B}^\gamma : \gamma < \kappa^+)$, $\overline{C}$, $(\overline{\tau}^\gamma_\zeta : \zeta \in \overline{A}^\gamma)$, $((\overline{\sigma}^{1,\gamma}_\zeta, \overline{\sigma}^{2,\gamma}_\zeta) : \zeta \in \overline{B}^\gamma)(\gamma < \kappa^+)$ be two families of parameters and Q and $\overline{Q}$ be the corresponding posets. We will define a sequence of functions $(e_\zeta : \zeta < \kappa^{++})$ such that

$\text{dom } e_\zeta$, $\text{rng } e_\zeta \subseteq \kappa^{++}$

$\zeta \subseteq \text{dom } e_\zeta \quad \zeta \subseteq \text{rng } e_\zeta$

$e_\zeta$ is 1:1

$|e_\zeta| < \kappa^{++}$

$\eta < \zeta \Rightarrow e_\eta \subseteq e_\zeta$

$\eta \in A^\gamma \cap \zeta (B^\gamma \cap \zeta$ resp., $C \cap \zeta$ resp.$) \iff e_\zeta(\eta) \in \overline{A}^\gamma (\overline{B}^\gamma$ resp., $\overline{C}$ resp.);

let $e_0 = \emptyset$ and for a limit $\zeta < \kappa^{++}$ let $e_\zeta = \bigcup_{\eta<\zeta} e_\eta$.

If we arrive at a successor $\zeta + 1$ and $\zeta \notin \text{dom } e_\zeta$ there are three cases: For $\zeta \in A^\gamma$ (for some $\gamma < \kappa^+$) fix the minimal $\eta \in \overline{A}^\gamma$ with $\eta \geq \sup^+ \text{rng } e_\zeta$ and

$\overline{\tau}^{\gamma}_{\eta} = (\hat{\tau}^{\gamma}_{\zeta})^{e_\zeta}$ (this is the image of $\hat{\tau}^{\gamma}_{\zeta} \subseteq \kappa^+ \times Q_\zeta$ under the shift map induced by $e_\zeta$; note that $\zeta \subseteq \mathrm{dom}\ e_\zeta$) and let $e_{\zeta+1}(\zeta) = \eta$. For $\zeta \in B^\gamma$ (for some $\gamma < \kappa^+$) fix the minimal $\eta \in \overline{B}^\gamma$ with $\eta \geq \sup^+ \mathrm{rng}\ e_\zeta$ and $\overline{\sigma}^{i,\gamma}_{\eta} = (\hat{\sigma}^{i,\gamma}_{\zeta})^{e_\zeta}$ (i = 1, 2) and let $e_{\zeta+1}(\zeta) = \eta$. For $\zeta \in C$ fix the minimal $\eta \in \overline{C}$ with $\eta \geq \sup^+ \mathrm{rng}\ e_\zeta$ and let $e_{\zeta+1}(\zeta) = \eta$.

If $\zeta \notin \mathrm{rng}\ e_\zeta \cup \{\eta\}$, then again there are $\zeta$ cases: For $\zeta \in \overline{A}^\gamma$ (for some $\gamma < \kappa^+$) fix the minimal $\xi \geq \sup^+ \mathrm{dom}\ e_\zeta\ .\cup.\ \zeta + 1$ with $\tau^{i,\gamma}_{\xi} = (\hat{\overline{\tau}}^{i,\gamma}_{\zeta})^{e_\zeta^{-1}}$. For $\zeta \in \overline{B}^\gamma$ (for some $\gamma < \kappa^+$) fix the minimal $\xi \geq \sup^+ \mathrm{dom}\ e_\zeta\ .\cup.\ \zeta + 1$ with $\sigma^{i,\gamma}_{\xi} = (\hat{\overline{\sigma}}^{i,\gamma}_{\zeta})^{e_\zeta^{-1}}$ (i = 1, 2). For $\zeta \in \overline{C}$ fix the minimal $\xi \in C$ with $\xi \geq \sup^+ \mathrm{dom}\ e_\zeta\ .\cup.\ \zeta + 1$. Then let $e_{\zeta+1}(\xi) = \zeta$.

Note that all this is possible since the sequences of terms and pairs of terms are complete.

Lemma 4.2.6. For each $\zeta$ with $\kappa^+ \leq \zeta < \kappa^{++}$ $\mathrm{dom}\ e_\zeta \subseteq \kappa^{++}$ is complete (rel. to the paramaters for Q) and $\mathrm{rng}\ e_\zeta \subseteq \kappa^{++}$ is complete (rel. to the parameters for $\overline{Q}$).

Proof. This is immediate from the definition of the sequence $(e_\zeta : \zeta < \kappa^{++})$.

□
end of 4.2.6.

Lemma 4.2.7. For any $\zeta < \kappa^{++}$

$$q^{e_\zeta} \in \overline{Q}^{\operatorname{rng} e_\zeta} \text{ for all } q \in Q^{\operatorname{dom} e_\zeta}$$

and

$$q^{e_\zeta^{-1}} \in Q^{\operatorname{dom} e_\zeta} \text{ for all } q \in \overline{Q}^{\operatorname{rng} e_\zeta}.$$

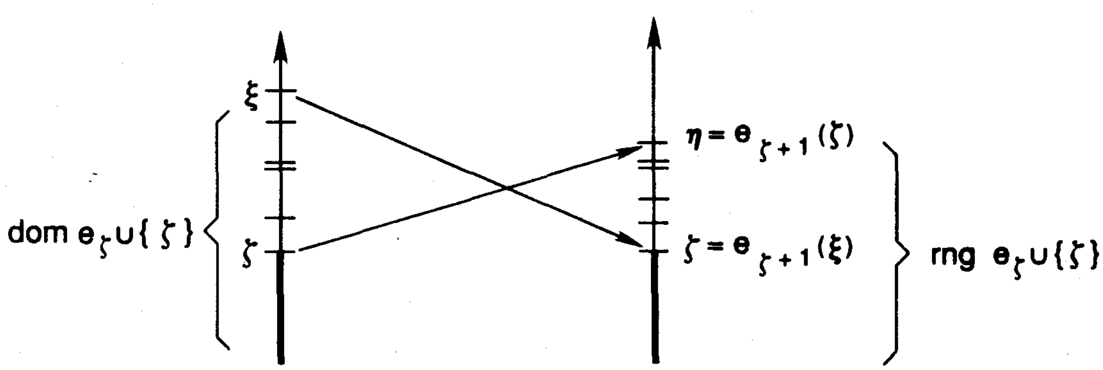
Thus $e_\zeta$ induces an isomorphism of $Q^{\operatorname{dom} e_\zeta}$ with $\overline{Q}^{\operatorname{rng} e_\zeta}$. Moreover, if H is $Q^{\operatorname{dom} e_\zeta}$ generic and $\overline{H} = \{q^{e_\zeta} : q \in H\}$

$$\forall \gamma < \kappa^+ \ \forall \nu \in A^\gamma \bigcap \operatorname{dom} e_\zeta \ (\tilde{\tau}_\nu^\gamma)^H = (\tilde{\tilde{\tau}}^\gamma_{e_\zeta(\nu)})^{\overline{H}}$$

$$\forall \gamma < \kappa^+ \ \forall \nu \in B^\gamma \bigcap \operatorname{dom} e_\zeta \ (\tilde{\sigma}_\nu^{i,\gamma})^H = (\tilde{\tilde{\sigma}}^{i,\gamma}_{e_\zeta(\nu)})^{\overline{H}}.$$

Proof. The case $\zeta = 0$ is clear.

Now suppose we are considering a successor $\zeta + 1 < \kappa^{++}$ and $q \in Q^{\operatorname{dom} e_{\zeta+1}}$. We can assume $\zeta \geq \kappa^+$. The worst case that can happen is that we have to add first $\zeta$ to the domain of $e_\zeta$ and then $\zeta$ to the range of this function in order to get $e_{\zeta+1}$ from $e_\zeta$:




clearly dom $e_\zeta \cup \{\xi\}$ is complete relative to the parameters of Q. Hence by 4.2.5. $q|(\operatorname{dom} e_\zeta \cup \{\xi\}) \in Q^{\operatorname{dom} e_\zeta \cup \{\xi\}}$.

Claim 1. $(q|(\text{dom } e_\zeta \cup \{\xi\}))^{e_{\zeta+1}} \in Q^{\text{rng } e_\zeta \cup \{\zeta\}}$.

Proof of Claim 1. It suffices to show

$$\overline{q} \underset{\mathfrak{Df}}{=} (q|(\text{dom } e_\zeta \cup \{\xi\}))^{e_{\zeta+1}}|(\zeta + 1) \in \overline{Q}_{\zeta+1}.$$

We can assume $\zeta \in \overline{B}^\gamma$ for some $\gamma < \kappa^+$ and $\zeta \in \text{dom } \overline{q}$. Let $\overline{H}_\zeta$ be $\overline{Q}_\zeta$ generic with $\overline{q}|\zeta \in \overline{H}_\zeta$. Pick a $\overline{Q}^{\text{rng } e_\zeta}$ generic $\overline{H}$ with $\overline{H} \cap \overline{Q}_\zeta = \overline{H}_\zeta$ and $(q|\text{dom } e_\zeta)^{e_\zeta} \in \overline{H}$. Then $H \underset{\mathfrak{Df}}{=} \{f \in Q^{\text{dom } e_\zeta} : f^{e_\zeta} \in \overline{H}\}$ is $Q^{\text{dom } e_\zeta}$ generic and $q|\text{dom } e_\zeta \in H$. Thus we get in $V[G_\kappa, \vec{F}_\gamma, H]$:

$$\theta(H, \gamma, ((\tilde{\tau}^\gamma_\nu)^H : \nu \in A^\gamma \cap \text{dom } e_\zeta), (\tilde{\sigma}^{1,\gamma}_\xi)^H (\tilde{\sigma}^{2,\gamma}_\xi)^H, F_\gamma, q(\xi))$$

because $q|(\text{dom } e_\zeta \cup \{\xi\}) \in Q^{\text{dom } e_\zeta \cup \{\xi\}}$. This yields in $V[G_\kappa, \vec{F}_\gamma, \overline{H}]$

$$\theta(\overline{H}, \gamma, ((\bar{\tilde{\tau}}^\gamma_\nu)^{\overline{H}} : \nu \in \overline{A}^\gamma \cap \text{rng } e_\zeta), (\bar{\tilde{\sigma}}^{1,\gamma}_\zeta)^{\overline{H}}, (\bar{\tilde{\sigma}}^{2,\gamma}_\zeta)^{\overline{H}}, F_\gamma, q^{e_{\zeta+1}}(\zeta)).$$

Since $\bar{\tilde{\sigma}}^{i,\gamma}_\zeta \subseteq \kappa^+ \times \overline{Q}_\zeta$ this is equivalent to

$$\theta(\overline{H}_\zeta, \gamma, ((\bar{\tilde{\tau}}^\gamma_\nu)^{\overline{H}_\zeta} : \nu \in \overline{A}^\gamma \cap \zeta), (\bar{\tilde{\sigma}}^{1,\gamma}_\zeta)^{\overline{H}_\zeta}, (\bar{\tilde{\sigma}}^{2,\gamma}_\zeta)^{\overline{H}_\zeta}, F_\gamma, \overline{q}(\zeta)).$$

□
end of proof of claim 1.

Claim 2. $q^{e_{\zeta+1}} \in \overline{Q}^{\text{rng } e_{\zeta+1}}$.

<u>Proof of the Claim 2</u>. We can assume that $\eta \in \overline{B}^{\gamma}$ for some $\gamma < \kappa^{+}$ and $q^{e_{\zeta+1}}(\eta) \neq 0$. We know already from claim 1 that $q^{e_{\zeta+1}}|\eta \in \overline{Q}_{\eta}$. Now let $\overline{H}$ be $\overline{Q}_{\eta}$ generic with $q^{e_{\zeta+1}}|\eta \in H$. By 4.2.6. $\overline{H} \cap \overline{Q}^{\mathrm{rng}\, e_{\zeta}}$ is $\overline{Q}^{\mathrm{rng}\, e_{\zeta}}$ generic. Then $H \underset{\mathfrak{Df}}{=} \{f \in Q^{\mathrm{dom}\, e_{\zeta}} : f^{e_{\zeta}} \in \overline{H} \cap \overline{Q}^{\mathrm{rng}\, e_{\zeta}}\}$ is $Q^{\mathrm{dom}\, e_{\zeta}}$ generic and $q|\mathrm{dom}\ e_{\zeta} \in H$. The completeness of dom $e_{\zeta} \cup \{\zeta\}$ relative to the parameters of Q yields $q|(\mathrm{dom}\ e_{\zeta} \cup \{\zeta\}) \in Q^{\mathrm{dom}\, e_{\zeta} \cup \{\zeta\}}$. Thus in $V[G_{\kappa}, \vec{F}_{\gamma}, H]$

$$\theta(H,\gamma,((\hat{\tau}_{\nu}^{\gamma})^{H} : \nu \in A^{\gamma} \cap \zeta), (\hat{\sigma}_{\zeta}^{1,\gamma})^{H}, (\hat{\sigma}_{\zeta}^{2,\gamma})^{H}, F_{\gamma}, q(\zeta)).$$

Since $\tilde{\sigma}_{\zeta}^{i,\gamma} \subseteq \kappa^{+} \times Q_{\zeta}$ this is equivalent to

$$\theta(H,\gamma,((\tilde{\tau}_{\nu}^{\gamma})^{H} : \nu \in A^{\gamma} \cap \mathrm{dom}\ e_{\zeta}), (\tilde{\sigma}_{\zeta}^{1,\gamma})^{H}, (\tilde{\sigma}_{\zeta}^{2,\gamma})^{H}, F_{\gamma}, q(\zeta)).$$

Therefore we obtain in $V[G_{\kappa}, \vec{F}_{\gamma}, \overline{H}]$

$$\theta(H,\gamma,((\bar{\tilde{\tau}}_{\nu}^{\gamma})^{\overline{H}} : \nu \in \overline{A}^{\gamma} \cap \mathrm{rng}\ e_{\zeta}), (\bar{\tilde{\sigma}}_{\eta}^{1,\gamma})^{\overline{H}}, (\bar{\tilde{\sigma}}_{\eta}^{2,\gamma})^{\overline{H}}, F_{\gamma}, \overline{q}(\eta)).$$

The completeness of rng $e_{\zeta} \cup \{\eta\}$ relative to the parameter for $\overline{Q}$ yields

$$\theta(\overline{H},\gamma,((\bar{\hat{\tau}}_{\nu}^{\gamma})^{\overline{H}} : \nu \in \overline{A}^{\gamma} \cap \eta), (\bar{\hat{\sigma}}_{\eta}^{1,\gamma})^{\overline{H}}, (\bar{\hat{\sigma}}_{\eta}^{2,\gamma})^{\overline{H}}, F_{\gamma}, q^{e_{\zeta+1}}(\eta)).$$

□
end of proof of claim 2.

A similar argument shows that for $q \in \overline{Q}^{\mathrm{rng}\, e_{\zeta+1}}$ $q^{e_{\zeta+1}^{-1}} \in Q^{\mathrm{dom}\, e_{\zeta+1}}$.

Finally it is immediate from the definition of $e_{\zeta+1}$ that the the part of the lemma about the terms holds and in fact we have already used this in the proofs of the claims above.

Now suppose $\zeta$ is a limit ordinal $<\kappa^{++}$ and let $q \in Q^{\mathrm{dom}\, e_\zeta}$. For each $\nu < \zeta$ $q|\mathrm{dom}\ e_\nu \in Q^{\mathrm{dom}\, e_\nu}$ by 4.2.6. Thus by induction hypothesis $(q|\mathrm{dom}\ e_\nu)^{e_\nu} \in \overline{Q}^{\mathrm{rng}\, e_\nu}$. Now one can use induction on $\nu \leq \kappa^{++}$ to show that $q^{e_\zeta}|\nu \in \overline{Q}_\nu$. At a successor $\nu + 1 < \kappa^{++}$ with $\nu \in \overline{B}^\gamma$ for some $\gamma < \kappa^+$ and $q^{e_\zeta}(\nu) \neq 0$ pick some $\xi < \zeta$ with $\nu \in \mathrm{dom}\ (q|\mathrm{dom}\ e_\xi)^{e_\xi}$. Then $(q|\mathrm{dom}\ e_\xi)^{e_\xi}|\nu \Vdash_{\overline{Q}_\nu} \theta$. Since $q^{e_\zeta}|\nu \leq (q|\mathrm{dom}\ e_\xi)^{e_\xi}|\nu$, we get $q^{e_\zeta}|\nu \Vdash_{\overline{Q}_\nu} \theta$. Hence $q^{e_\zeta}|\nu + 1 \in Q_{\nu+1}$. A similar argument shows that for $q \in \overline{Q}^{\mathrm{rng}\, e_\zeta}$ $q^{e_\zeta^{-1}} \in Q^{\mathrm{dom}\, e_\zeta}$. The rest of the lemma is immediate for a limit ordinal $\zeta$.

$\square$
end of 4.2.7.

Corollary 4.2.8. Up to isomorphism there is only one $\kappa^{++}$ iteration Q in $V[G_\kappa, \vec{F}_\gamma]$ that uses $(F_\gamma : \gamma < \kappa^+)$,

$\square$
end of 4.2.8.

The second technical result about Q illustrates that for any coordinate $\zeta < \kappa^{++}$, if we consider the tail $Q_{\zeta,\kappa^{++}}$ of the poset Q inside $V[G_\kappa, \vec{F}_\gamma]^{Q_\zeta}$, then this looks pretty much like the original poset modulo a minor modification.

Before we can prove this we need to set up some notation. Suppose that in $V[G_\kappa, \vec{F}_\gamma]$ we pick a family of parameters to define a $\kappa^{++}$ iteration Q. If $\delta < \kappa^{++}$ and $H_\delta$ is $Q_\delta$ generic over $V[G_\kappa, \vec{F}_\gamma]$, in $V[G_\kappa, \vec{F}_\gamma, H_\delta]$, let:

$$Q_{\delta,\zeta} \underset{\mathfrak{Df}}{=} \{s \in Fn([\delta,\zeta),2,\kappa^+): \exists q \in H_\delta\ q^\frown s \in Q_\zeta\}.$$

Pick a canonical name $\mathring{Q}_{\delta,\zeta} \in (V[G_\kappa, \vec{F}_\gamma]^{Q_\delta}$ for $Q_{\delta,\zeta}$. For each $q \in Q_\zeta (\delta \leq \zeta \leq \kappa^{++})$ pick a term $\mathring{q}_{\delta,\zeta} \in V[G_\kappa, \vec{F}_\gamma]^{Q_\delta}$ with

$$\Vdash_{Q_\delta} \mathring{q}_{\delta,\zeta} \in \mathring{Q}_{\delta,\zeta} \wedge [q|[\delta,\zeta) \in \mathring{Q}_{\delta,\zeta} \Rightarrow \mathring{q}_{\delta,\zeta} = q|[\delta,\zeta)].$$

<u>Lemma</u> <u>4.2.9</u>. For each $\zeta$ with $\delta \leq \zeta \leq \kappa^{++}$

$$\Phi_\delta: Q_\zeta \longrightarrow Q_\delta * \mathring{Q}_{\delta,\zeta}$$

$$q \longmapsto (q|\delta, \mathring{q}_{\delta,\zeta})$$

defines an isomorphism of $Q_\zeta$ with a dense suborder of $Q_\delta * \mathring{Q}_{\delta,\zeta}$.

□
end of 4.2.9.

Next we associate with each nice $Fn(\kappa^{++},2,\kappa^+)$ name $\tau$ for a subset of $\kappa^+$ a canonical name ${}_\delta\tau \in V[G_\kappa, \vec{F}_\gamma]^{Q_\delta}$ such that

(4.2.10.) $\Vdash_{Q_\delta} {}_\delta\tau = \{(\eta,h):\ \eta < \kappa^+, h \in Fn([\delta,\kappa^{++}),2,\kappa^+) \wedge$

$\exists f \in \Gamma\, \exists g \in Fn(\delta,2,\kappa^+)[(\eta,g^\frown h) \in \tau \wedge f \leq g]\}.$

Note that

$$\Vdash_{Q_\delta} \text{“}{}_\delta\tau \text{ is a nice } Fn([\delta,\kappa^{++}),2,\kappa^+) \text{ name for a subset of } \kappa^+ .\text{”}$$

Lemma 4.2.11. For any complete sequence $(\tau_\zeta : \zeta < \kappa^{++})$ of nice $Fn(\kappa^{++},2,\kappa^+)$ names for subsets of $\kappa^+$.

$$\Vdash_{Q_\delta} \text{“}({}_\delta\tau_\zeta : \zeta < \kappa^{++}) \text{ is a complete sequence of nice } Fn([\delta,\kappa^{++}),2,\kappa^+) \text{ names for subsets of } \kappa^+ .\text{”}$$

Proof. Let H be $Q_\delta$ generic over $V[G_\kappa,\vec{F}_\gamma]$ and in $V[G_\kappa,\vec{F}_\gamma,H]$ let $\sigma$ be a nice $Fn([\delta,\kappa^{++}),2,\kappa^+)$ name for a subset of $\kappa^+$. We pick $q \in H$ and $\mathring{\sigma} \in V[G_\kappa,\vec{F}_\gamma]^{Q_\delta}$ with $\mathring{\sigma}^H = \sigma$ and

$$q \Vdash_{Q_\delta} \text{“}\mathring{\sigma} \text{ is a nice } Fn([\delta,\kappa^{++}),2,\kappa^+) \text{ name for a subset of } \kappa^+ .\text{”}$$

Now define a nice $Fn(\kappa^{++},2,\kappa^+)$ name for a subset of $\kappa^+$ $\tau \in V[G_\kappa,\vec{F}_\gamma]$ via

$$\tau \underset{\mathfrak{Df}}{=} \{(\eta,h) : \eta < \kappa^+ \wedge h \in Fn(\kappa^{++},2,\kappa^+) \wedge h|\delta \in Q_\delta \wedge h|\delta \leq q \wedge$$
$$h|\delta \Vdash_{Q_\delta} (\eta, h|[\delta,\kappa^{++})) \in \mathring{\sigma}\}.$$

From (4.2.10) we obtain

$$q \Vdash_{Q_\delta} {}_\delta\tau = \mathring{\sigma}.$$

The completeness of $(\tau_\zeta : \zeta < \kappa^{++})$ in $V[G_\kappa,\vec{F}_\gamma]$ yields that there are arbitrarily large $\zeta < \kappa^{++}$ with $\tau_\zeta = \tau$. Obviously we have for any such $\zeta$

$$q \Vdash {}_\delta\tau_\zeta = \mathring{\sigma} .$$

□ end of 4.2.11.

In the sequel we will not always distinguish between ${}_{\delta}\tau$ (which is a name for a term) and its interpretation ${}_{\delta}\tau^{H}$ (which is an element of $(V[G_\kappa,\vec{F}_\gamma,H])^{Fn([\delta,\kappa^{++}),2,\kappa^{+})}$) unless there is reasonable danger of confusing these two objects.

We are now going to define what we mean by a *modified* $\delta$, $\kappa^{++}$ *iteration.* Suppose that in $V[G_\kappa,\vec{F}_\gamma]$ we have a partition of $\delta < \kappa^{++}$ into $A^{\gamma}(\gamma < \kappa^{+})$, $B^{\gamma}(\gamma < \kappa^{+})$ and C with $\kappa^{+} \subseteq C$ and for each $\zeta \in A^{\gamma}(\gamma < \kappa^{+})$ and $\eta \in B^{\gamma}(\gamma < \kappa^{+})$ we have nice $Fn(\kappa^{++},2,\kappa)$ names $\tau_{\zeta}^{\gamma}$, $\sigma_{\eta}^{1,\gamma}$, $\sigma_{\eta}^{2,\gamma}$ for subsets of $\kappa^{+}$.

Now let H be $V[G_\kappa,\vec{F}_\gamma]$ generic for the iteration $Q_\delta$ defined from the parameters. Suppose that in $V[G_\kappa,\vec{F}_\gamma,H]$ we partition $[\delta,\kappa^{++})$ into cofinal pieces $\overline{A}^{\gamma}(\gamma < \kappa^{+})$, $\overline{B}^{\gamma}$and $\overline{C}$ and we have sequences $(\overline{\tau}_{\zeta}^{\gamma}:\zeta \in \overline{A}^{\gamma})$ and $((\overline{\sigma}_{\zeta}^{1,\gamma},\overline{\sigma}_{\zeta}^{2,\gamma}):\zeta \in \overline{B}^{\gamma})$ that are, for each $\gamma < \kappa^{+}$, complete for $Fn([\delta,\kappa^{++}),2,\kappa^{+})$. Working in $V[G_\kappa,\vec{F}_\gamma,H]$ we define the modified $\delta$, $\kappa^{++}$ iteration $\tilde{Q}((\overline{A}^{\gamma}:\gamma < \kappa^{+})$, $(\overline{B}^{\gamma}:\gamma < \kappa^{+})$, $\overline{C}$, $(\overline{\tau}_{\zeta}^{\gamma}:\zeta \in \overline{A}^{\gamma},\gamma < \kappa^{+})$, $((\overline{\sigma}_{\zeta}^{1,\gamma},\overline{\sigma}_{\zeta}^{2,\gamma}))\ \zeta \in \overline{B}^{\gamma}))$ by induction on $\alpha \in [\delta,\kappa^{++}]$:

$$\tilde{Q}_{\delta,\delta} \underset{\mathfrak{Df}}{=} \{\emptyset\}$$

and for limit $\alpha$

$$\tilde{Q}_{\delta,\alpha} \underset{\mathfrak{Df}}{\equiv} \{f \in Fn([\delta,\alpha),2,\kappa^+) : \forall\beta \in [\delta,\alpha)\ f|(\delta,\beta) \in \tilde{Q}_{\delta,\beta}\}$$

for $\alpha \in \overline{A}^\gamma$ or $\overline{C}$ for some $\gamma < \kappa^+$:

$$\tilde{Q}_{\delta,\alpha+1} \underset{\mathfrak{Df}}{\equiv} \{f \in Fn([\delta,\alpha+1),2,\kappa^+) : f|[\delta,\alpha) \in \tilde{Q}_{\delta,\alpha}\}$$

and for $\alpha \in \overline{B}^\gamma$ for $\gamma < \kappa^+$

$$\tilde{Q}_{\delta,\alpha+1} \underset{\mathfrak{Df}}{\equiv} \{f \in Fn([\delta,\alpha+1),2,\kappa^+) : f|[\delta,\alpha) \in \tilde{Q}_{\delta,\alpha} \wedge f|[\delta,\alpha) \Vdash_{\tilde{Q}_{\delta,\alpha}} \tilde{\theta}\}$$

where the formula $\tilde{\theta}$ says:

$$\gamma \text{ is odd} \wedge [[\forall\zeta \in A^\gamma \cap \delta((\hat{\tau}^\gamma_\zeta)^H, H^\zeta) \neq (\hat{\bar{\sigma}}^{1,\gamma}_\alpha, \hat{\bar{\sigma}}^{2,\gamma}_\alpha) \wedge \forall\zeta \in \overline{A}^\gamma \cap \alpha\ (\hat{\bar{\tau}}^\gamma_\zeta, \Gamma^\zeta) \neq$$

$$(\hat{\bar{\sigma}}^{1,\gamma}_\alpha, \hat{\bar{\sigma}}^{2,\gamma}_\alpha) \wedge f(\alpha) \text{ is a condition for killing } F_\gamma(\hat{\bar{\sigma}}^{1,\gamma}_\alpha, \hat{\bar{\sigma}}^{2,\gamma}_\alpha)] \vee f(\alpha) = 0] \ .\vee.$$

$$\gamma \text{ is even} \wedge [[[H^\gamma = \hat{\bar{\sigma}}^{1,\gamma}_\alpha \vee [\forall\zeta \in A^\gamma \cap \delta((\hat{\tau}^\gamma_\zeta)^H, H^\zeta) \neq (\hat{\bar{\sigma}}^{1,\gamma}_\alpha, \hat{\bar{\sigma}}^{2,\gamma}_\alpha) \wedge$$

$$\forall\zeta \in \overline{A}^\gamma \cap \alpha\ ((\hat{\bar{\tau}}^\gamma_\zeta), \Gamma^\zeta) \neq (\hat{\bar{\sigma}}^{1,\gamma}_\alpha, \hat{\bar{\sigma}}^{2,\gamma}_\alpha)]] \wedge f(\alpha) \text{ kills } F_\gamma(\hat{\bar{\sigma}}^{1,\gamma}_\alpha, \hat{\bar{\sigma}}^{2,\gamma}_\alpha)] \vee f(\alpha) = 0]$$

where for $\eta \in \overline{A}^\gamma$

$$\hat{\bar{\tau}}^\gamma_\eta \underset{\mathfrak{Df}}{\equiv} \{(\eta,f) : \exists g[(\eta,g) \in \bar{\tau}^\gamma_\eta \wedge f \in \tilde{Q}_{\delta,\eta} \wedge f \leq g]\}$$

and similarly for $\bar{\sigma}^{i,\gamma}_\alpha$.

By an analogous proof as in 4.1.3. one can show that modified $\delta$, $\kappa^{++}$ iterations are $<\kappa^+$ Baire. Moreover the analogue of 4.2.7. shows that up to isomorphism there is only one $\delta$, $\kappa^{++}$ iteration in $V[G_\kappa, \vec{F}_\gamma H]$ (that is modified by referring to $(H,^\gamma : \gamma \in \text{Even}_{\kappa^+})$ and $(((\hat{\tau}^\gamma_\zeta)^H, H^\zeta) : \zeta \in A^\gamma \cap \delta, \gamma < \kappa^+)$ and that uses

$(F_\gamma : \gamma < \kappa^+))$. For any $\kappa^{++}$ iteration Q in $V[G_\kappa, \vec{F}_\gamma]$ defined from parameters $(A^\gamma : \gamma < \kappa^+)$, $(B^\gamma : \gamma < \kappa^+)$ and C and $(\tau^\gamma_\zeta : \zeta \in A^\gamma)$, $((\sigma^{1,\gamma}_\zeta, \sigma^{2,\gamma}_\zeta) : \zeta \in B^\gamma)$ and any H that is $Q_\delta$ generic over $V[G_\kappa, \vec{F}_\gamma]$ there is a canonical $\delta$, $\kappa^{++}$ iteration $\tilde{Q}$ in $V[G_\kappa, \vec{F}_\gamma, H]$ whose parameters arise from the parameters for Q. For this recall that for each $\gamma < \kappa^+$ $({}_\delta\tau^\gamma_\zeta : \zeta \in A^\gamma \bigcap [\delta, \kappa^{++}))$ and $(({}_\delta\sigma^{1,\gamma}_\zeta, {}_\delta\sigma^{2,\gamma}_\zeta) : \zeta \in B^\gamma \bigcap [\delta, \kappa^{++}))$ are complete in $V[G_\kappa, \vec{F}_\gamma, H]$ for $Fn([\delta, \kappa^{++}), 2, \kappa^+)$. The following lemma shows that at any intermediate stage $\delta < \kappa^{++}$ Q factors in a nice way.

<u>Lemma</u> <u>4.2.12</u>. In $V[G_\kappa, \vec{F}_\gamma, H]$ we have for each $\zeta \in [\delta, \kappa^{++}]$

$$\mathring{Q}^H_{\delta,\zeta} = \tilde{Q}_{\delta,\zeta}.$$

<u>Proof</u>. We proceed by induction on $\delta \le \zeta \le \kappa^{++}$. If $\zeta = \delta$ then note

$$\mathring{Q}^H_{\delta,\zeta} = \{\emptyset\} = \tilde{Q}_{\delta,\zeta}.$$

If $\zeta \le \kappa^{++}$ is a limit ordinal we distinguish two cases: If cf $(\zeta) \ge \kappa^+$ the claim follows from the induction hypothesis since all conditions have size $\le \kappa$, so we can assume cf $(\zeta) \le \kappa$. Clearly $\mathring{Q}^H_{\delta,\zeta} \subseteq \tilde{Q}_{\delta,\zeta}$. Now we suppose $q \in \tilde{Q}_{\delta,\zeta}$. Then we pick $S \subseteq [\delta, \zeta)$ cofinal in $\zeta$ with $|S| \le \kappa$. For each $\nu \in S$ there is $h_\nu \in H$ with $h_\nu \cup q|\nu \in Q_\nu$. Note that the sequence $(h_\nu : \nu \in S) \in V[G_\kappa, \vec{F}_\gamma]$ by the $<\kappa^+$ Baireness of $Q_\delta$. Thus we can pick $h \in H$ with $h \Vdash_{Q_\delta} \forall \nu \in S \;\; h_\nu \in \Gamma$.

Claim. $h^\frown q \in Q_\zeta$.

Proof of the Claim. Show by induction on $\nu$ that for $\delta \leq \nu \leq \zeta$ $h^\frown q|\nu \in Q_\nu$. If $\nu = \delta$ this is clear and also the case of $\nu$ being a limit ordinal is immediate. Now consider a successor $\nu + 1$. We can assume $\nu \in B^\gamma$ for some $\gamma < \kappa^+$ and $\nu \in$ dom q. Suppose towards a contradiction that we don't have $h^\frown q|\nu \Vdash \theta$. So there is a condition $f \in Q_\mu$ with $f \leq h^\frown q|\nu$ and $f \Vdash_{\overline{Q_\nu}} \neg\, \theta$. Let $\mu$ be the least ordinal $>\nu$ in S. Since $h \Vdash_{\overline{Q_\delta}} h_\mu \in \Gamma$ and $f|\delta \leq h$, we can find $g \in Q_\delta$ extending both $h_\mu$ and $f|\delta$. Now $g^\frown q|\mu \in Q_\mu$ and $g^\frown f|[\delta,\nu) \leq f$. Therefore $g^\frown q|\nu \Vdash_{\overline{Q_\nu}} \theta$ and $g^\frown f|[\delta,\nu) \Vdash_{\overline{Q_\nu}} \neg\, \theta$, clearly a contradiction since $f \leq h^\frown q|\nu$.

$\square$
end of proof of claim

The claim yields $q \in \mathring{Q}^H_{\delta,\zeta}$.

Finally consider the case of a successor $\zeta + 1 < \kappa^{++}$. Note that by induction hypothesis $\mathring{Q}^H_{\delta,\zeta} = \tilde{Q}_{\delta,\zeta}$. We will need the following:

Fact. If K is $\mathring{Q}^H_{\delta,\zeta}$ generic over $V[G_\kappa, \vec{F}_\gamma, H]$ then for each $\gamma < \kappa^+$ and $\nu \in A_\gamma \cap [\delta,\zeta]$

$$(({}_\delta\hat{\mathring{\tau}}^\gamma_\nu)^H)^K = (\mathring{\tau}^\gamma_\nu)^{\Phi_\delta^{-1}[H*K]}$$

where $\Phi_\delta$ is as in 4.2.9. and similarly for $\sigma^{i,\gamma}_\nu$ ($\nu \in B^\gamma \cap [\delta,\zeta]$).

Proof of the fact. If $\eta \in (({}_\delta\hat{\mathring{\tau}}^\gamma_\nu)^H)^K$ fix $h \in K \cap \mathring{Q}^H_{\delta,\nu}$ with $(\eta,h) \in ({}_\delta\hat{\mathring{\tau}}^\gamma_\nu)^H$ and let

$t \in Fn([\delta,\kappa^{++}),2,\kappa^{+})$ with $t \geq h$ and $(\eta,t) \in ({}_\delta\tau^\gamma_\nu)^H$. Then pick $f'' \in H$ and $g \in Fn(\delta,2,\kappa^{+})$ with $f'' \leq g$ and $(\eta,g^\frown t) \in \tau^\gamma_\nu$. Choose $f''' \in H$ such that $f'''^\frown h \in Q_\nu$ and let $f' \in H$ be a common extension of $f'''$ and $f''$. Now $(\eta,f'^\frown h) \in \hat\tau^\gamma_\nu$ since $f'^\frown h \in Q_\nu$ and $f'^\frown h \leq g^\frown t$. Clearly $\Phi_\delta(f'^\frown h) \in H * K$. Thus $\eta \in (\hat\tau^\gamma_\nu)^{\Phi_\delta^{-1}[H*K]}$.

Conversely assume $\eta \in (\hat\tau^\gamma_\nu)^{\Phi_\delta^{-1}[H*K]}$; then fix $f \in H$ and $h \in K$ with $f^\frown h \in Q_\nu$ and $(\eta,f^\frown h) \in \hat t^\gamma_\nu$. Pick $g \in Fn(\delta,2,\kappa^{+})$ and $t \in Fn([\delta,\kappa^{++}),2,\kappa^{+})$ with $f^\frown h \leq g^\frown t$ and $(\eta,g^\frown t) \in t^\gamma_\nu$. Now $(\eta,t) \in ({}_\delta\tau^\gamma_\nu)^H$ since $f \in H$ and $f \leq g$. But $h \leq t$ and $h \in \mathring{Q}^H_{\delta,\nu} = \tilde Q_{\delta,\nu}$; hence $(\eta,h) \in ({}_\delta\hat{\hat\tau}^\gamma_\nu)^H$ and $\eta \in (({}_\delta\hat{\hat\tau}^\gamma_\nu)^H)^K$.

□ end of the proof of the fact

Now we proceed with the successor case and consider $q \in \mathring{Q}^H_{\delta,\zeta+1}$. We pick $h \in H$ with $h^\frown q \in Q_{\zeta+1}$. We can assume $\zeta \in \mathrm{dom}\, q$ and $\zeta \in B^\gamma$ for some $\gamma < \kappa^{+}$. Then we obtain

$$h^\frown q|\zeta \Vdash^{V[G_\kappa,\vec F_\gamma]}_{Q_\zeta} \theta(\Gamma,\gamma,(\hat\tau^\gamma_\eta : \eta \in A^\gamma \cap \zeta), \hat\sigma^{1,\gamma}_\zeta, \hat\sigma^{2,\gamma}_\zeta F_\gamma, q(\zeta)).$$

The fact yields that

$$q|\zeta \Vdash^{V[G_\kappa,\vec F_\gamma,H]}_{\tilde Q_{\delta,\zeta}} \tilde\theta(H^\gamma,(((\hat\tau^\gamma_\eta)^H,H^\eta) : \eta \in A^\gamma \cap \delta),$$
$$\Gamma,\gamma,({}_\delta\hat{\hat\tau}^\gamma_\eta : \eta \in A^\gamma \cap [\delta,\zeta)), {}_\delta\hat{\hat\sigma}^{1,\gamma}_\zeta, {}_\delta\hat{\hat\sigma}^{2,\gamma}_\zeta, F_\gamma, q(\zeta)).$$

Hence $q \in \tilde Q_{\delta,\zeta+1}$.

Conversely if $q \in \tilde{Q}_{\delta,\zeta+1}$ we can assume $\zeta \in$ dom q and $\zeta \in B^\gamma$ for some $\gamma < \kappa^+$. Now we pick $h \in H$ with $h^\frown q|\zeta \in Q_\zeta$ and $h \Vdash_{Q_\delta} q \in \mathring{\tilde{Q}}_{\delta,\zeta+1}$, where $\mathring{\tilde{Q}}_{\delta,\zeta+1} \in V[G_\kappa, \vec{F}_\gamma]^{Q_\delta}$ is a canonical name for the poset $\tilde{Q}_{\delta,\zeta+1} \in V[G_\kappa, \vec{F}_\gamma, H]$ that we defined above. Now we have

$$h \Vdash^{V[G_\kappa,\vec{F}_\gamma]}_{Q_\delta} \text{“}q|\zeta \Vdash^{V[G_\kappa,\vec{F}_\gamma,\mathring{H}]}_{\mathring{Q}_{\delta,\zeta}} \tilde{\theta}(\mathring{H}^\gamma, (((\hat{\tau}^\gamma_\eta)^{\mathring{H}}, \mathring{H}^\eta) : \eta \in A^\gamma \cap \delta)$$

$$\mathring{K}, \gamma, ({}_\delta\hat{\hat{\tau}}^\gamma_\eta : \eta \in A^\gamma \cap [\delta,\zeta)), {}_\delta\hat{\hat{\sigma}}^{1,\gamma}_\zeta, {}_\delta\hat{\hat{\sigma}}^{2,\gamma}_\zeta, F_\gamma, q(\zeta))\text{”}$$

where $\mathring{H}$ is a canonical name $\in (V[G_\kappa, \vec{F}_\gamma])^{Q_\delta}$ for the $Q_\delta$ generic H and $\mathring{K}$ is a $Q_\delta$ name for the canonical $\mathring{Q}_{\delta,\zeta}$ name for the $\mathring{Q}_{\delta,\zeta}$ generic. By applying the fact we get

$$h^\frown q|\zeta \Vdash^{V[G_\kappa,\vec{F}_\gamma]}_{Q_\zeta} \theta\,(\Gamma, \gamma, (\hat{\tau}^\gamma_\eta : \eta \in A^\gamma \cap \zeta), \hat{\sigma}^{1,\gamma}_\zeta, \hat{\sigma}^{2,\gamma}_\zeta, F_\gamma, q(\zeta)).$$

Thus $h^\frown q \in Q_{\zeta+1}$ and therefore $q \in \mathring{Q}^H_{\delta,\zeta+1}$. $\square$ end of 4.2.12

We are also going to use the following specialization of this: Suppose that in $V[G_\kappa, \vec{F}_\gamma]$ we have defined a $\kappa^{++}$ iterationQ such that for some $\delta < \kappa^{++}$:

(4.2.13.) $\forall \gamma < \kappa^+ \;\; \forall \zeta \in A^\gamma \cap \delta \;\;$ supp $\tau^\gamma_\zeta \cap \text{Even}_{(\kappa^+)}M = \emptyset$

$\forall \gamma < \kappa^+ \;\; \forall \zeta \in B^\gamma \cap \delta \;\;$ supp $\sigma^{i,\gamma}_\zeta \cap \text{Even}_{(\kappa^+)}M = \emptyset$

where for $\tau \in V[G_\kappa, \vec{F}_\gamma]^{Fn(\kappa^{++},2,\kappa^+)}$ we define

$$\text{supp } \tau \underset{\mathfrak{Df}}{\equiv} \cup\{\text{dom } h : \exists\sigma\ (\sigma,h) \in \tau\}.$$

A similar argument as in the proof of 4.2.5. shows that for each $q \in Q_\delta$

$q|(\delta \sim \text{Even}_{(\kappa^+)^M}) \in Q_\delta^{\delta\sim\text{Even}_{(\kappa^+)^M}}$. The key point is that for any $Q_\delta$ generic H

$(\hat{\sigma}_\zeta^{1,\gamma})^H \neq H^\gamma$ for any $\gamma \in \text{Even}_{(\kappa^+)^M}$ and $\zeta \in B^\gamma \cap \delta$ since supp $\hat{\sigma}_\zeta^{1,\gamma} \subseteq \zeta \sim \text{Even}_{(\kappa^+)^M}$.

Therefore $Q_\delta^{\delta\sim\text{Even}_{(\kappa^+)^M}} \subseteq_c Q_\delta$. Now suppose H is $Q_\delta^{\delta\sim\text{Even}_{(\kappa^+)^M}}$ generic over

$V[G_\kappa, \vec{F}_\gamma]$. Then in $V[G_\kappa, \vec{F}_\gamma, H]$ let

$$^*Q_{\delta,\delta+{}^M(\kappa^+)+\zeta} \underset{\mathfrak{Df}}{\equiv} \{q \in Fn([\delta,\delta + {}^M(\kappa^+) + \zeta),2,\kappa^+) : \exists h \in H\ h \circ q \in Q_{\delta+\zeta}\}$$

where $\zeta \geq 0$ and $h \circ q \in Fn(\delta + \zeta,2,\kappa^+)$ is defined as follows:

$$h \circ q(\nu) \underset{\mathfrak{Df}}{\equiv} q(\delta + \xi) \text{ if } \nu \text{ is the } \xi\text{-th ordinal} \in \text{Even}_{(\kappa^+)^M}$$

$$h \circ q(\nu) \underset{\mathfrak{Df}}{\equiv} h(\nu) \text{ if } \nu \in \delta \sim \text{Even}_{(\kappa^+)^M}$$

$$h \circ q(\delta + \nu) \underset{\mathfrak{Df}}{\equiv} q(\delta + {}^M(\kappa^+) + \nu) \text{ for } 0 \leq \nu < \zeta.$$

We let $^*\mathring{Q}_{\delta,\delta+{}^M(\kappa^+)+\zeta} \in V[G_\kappa, \vec{F}_\gamma]^{Q_\delta^{\delta\sim\text{Even}_{(\kappa^+)^M}}}$ be a canonical name for

$^*Q_{\delta,\delta+(\kappa^+)^M+\zeta}$ and for each $q \in Q_{\delta+\zeta}$ we pick a canonical name $^*\mathring{q}_{\delta,\delta+(\kappa^+)^M+\zeta}$

$\in V[G_\kappa, \vec{F}_\gamma]^{Q_\delta^{\delta\sim\text{Even}_{(\kappa^+)^M}}}$ such that

$$\Vdash_{Q_\delta}^{\delta\text{-Even}_{(\kappa^+)^M}} [{}^*\mathring{q}_{\delta,\delta+{}^M(\kappa^+)+\zeta} \in {}^*\mathring{Q}_{\delta,\delta+{}^M(\kappa^+)+\zeta} \;.\wedge.$$

$$[q|\text{Even}_{(\kappa^+)^M}\widehat{\ }q|[\delta,\delta+\zeta) \in {}^*\mathring{Q}_{\delta,\delta+{}^M(\kappa^+)+\zeta} \Rightarrow$$

$$q|\text{Even}_{(\kappa^+)^M}\widehat{\ }q|[\delta,\delta+\zeta) = {}^*\mathring{q}_{\delta,\delta+{}^M(\kappa^+)+\zeta}]\,]$$

where for $g \in \text{Fn}(\text{Even}_{(\kappa^+)^M}, 2, \kappa^+)$ and $h \in \text{Fn}([\delta,\delta+\zeta), 2, \kappa^+)$ we define $g\widehat{\ }h \in \text{Fn}([\delta,\delta+(\kappa^+)^M+\zeta), 2, \kappa^+)$ by

$$g\widehat{\ }h\ (\delta+\nu) \underset{\mathfrak{Df}}{=} g(\xi) \text{ where } \xi \text{ is the } \nu\text{-th element of Even}_{(\kappa^+)^M}$$

and

$$g\widehat{\ }h\ (\delta+(\kappa^+)^M+\xi) \underset{\mathfrak{Df}}{=} h\ (\delta+\xi) \text{ for } \xi < \zeta.$$

As in 4.2.9 we have that

$$\begin{aligned} \Phi_\delta : Q_{\delta+\zeta} &\longrightarrow Q_\delta^{\delta\text{-Even}_{(\kappa^+)^M}} * {}^*\mathring{Q}_{\delta,\delta+{}^M(\kappa^+)+\zeta} \\ q &\longmapsto \left(q|(\delta\text{-Even}_{(\kappa^+)^M}),\ {}^*\mathring{q}_{\delta,\delta+{}^M(\kappa^+)+\zeta}\right) \end{aligned}$$

defines a dense embedding.

Working inside $V[G_\kappa, \vec{F}_\gamma, H]$ we are now going to define what we mean by a *special modified* $\delta$, $\kappa^{++}$ *iteration* ${}^*\tilde{Q}_{\delta,\kappa^{++}}$. This is very similar to a modified $\delta$, $\kappa^{++}$ iteration except that at the first ${}^M(\kappa^+)$ many coordinates $\geq\delta$ we have to add subsets of $\kappa^+$ such that: If $\gamma$ is the $\xi$-th even ordinal $< (\kappa^+)^M$ then the set $\subseteq \kappa^+$ that we add at coordinate $\delta+\xi$ will be a witness for the $\Sigma^2_3$ statement that we want to hold about $F_\gamma$. In analogy with (4.2.10.) we will also associate with each nice $Fn(\kappa^{++},2,\kappa^+)$ name $\tau \in V[G_\kappa,\vec{F}_\gamma]$ a canonical term ${}^*_\delta\tau \in V[G_\kappa,\vec{F}_\gamma]^{Q_\delta^{\delta\text{-}Even_{(\kappa^+)^M}}}$ such that

$$\Vdash_{Q_\delta^{\delta\text{-}Even_{(\kappa^+)^M}}} {}^*_\delta\tau = \{(\eta,q):\eta < \kappa^+,\ q \in Fn([\delta,\kappa^{++}),2,\kappa^+) \wedge$$
$$\exists h \in \Gamma\ \exists g \in Fn(\delta\text{-}Even_{(\kappa^+)^M},2,\kappa^+)[h \le g \wedge g \circ q \in \tau]\};$$

clearly

$$\Vdash_{Q_\delta^{\delta\text{-}Even_{(\kappa^+)^M}}} \text{“}{}^*_\delta\tau \text{ is a nice } Fn([\delta,\kappa^{++}),2,\kappa^+) \text{ name for a subset of } \kappa^+\text{”}$$

and just as in 4.2.11., for any complete sequence $(\tau_\zeta:\zeta < \kappa^{++})$ of nice $Fn(\kappa^{++},2,\kappa^+)$ names for subsets of $\kappa^+$

$$\Vdash_{Q_\delta^{\delta\text{-}Even_{(\kappa^+)^M}}} \text{“}({}^*_\delta\tau_\zeta:\zeta < \kappa^{++}) \text{ is a complete sequence of nice } Fn([\delta,\kappa^{++}),2,\kappa^+)$$
$$\text{names for subsets of } \kappa^+\text{”}$$

If we are in the special situation described in (4.2.13.) and H is $Q_\delta^{\delta\text{-}Even_{(\kappa^+)^M}}$ generic over $V[G_\kappa,\vec{F}_\gamma]$, then in $V[G_\kappa,\vec{F}_\gamma,H]$ there is a natural special modified $\delta$, $\kappa^{++}$ iteration ${}^*\tilde{Q}$ that refers to $(H^\gamma:\gamma \in Even_{\kappa^+} - Even_{(\kappa^+)^M})$ and $(((\tilde{\tau}^\gamma_\eta)^H,H^\eta):\eta \in A^\gamma \cap \delta,\ \gamma < \kappa^+)$ and uses $(F_\gamma:\gamma < \kappa^+)$ and whose parameters arise from the parameters of Q via the operation $\tau \to {}^{*}_{\delta}\tau$ above. The same ideas as in the case of modified iterations lead to the following factor lemma:

(4.2.14.) For all $\zeta \in [0,\kappa^{++}]$

$$*\mathring{Q}^{H}_{\delta,\delta+(\kappa^{+})^{M}+\zeta} = *\tilde{Q}_{\delta,\delta+(\kappa^{+})^{M}+\zeta}.$$

Once we are familiar with these facts it is rather easy to find a generic g for $Q^{N}$ (the poset at the second step of stage $\kappa$ in $P^{N}_{j(\kappa)+1}$) from the generic G for the poset Q that we use at the second step of stage $\kappa$ in $P_{\kappa+1}$.

In the ground model V = L let $\Pi^{*}$ denote the following injection defined on $\kappa^{++}$:

$$\Pi^{*}(\zeta) = \Pi(\zeta) \text{ if } \zeta \in \mathrm{Even}_{\kappa^{+}}.$$

$$\Pi^{*}(\zeta) = \zeta \text{ if } \zeta \in \kappa^{++} \sim \mathrm{Even}_{\kappa^{+}}.$$

Then $\Pi^{*}: \kappa^{++} \xrightarrow[\text{onto}]{1:1} \kappa^{++} \sim \mathrm{Even}_{(\kappa^{+})^{M}}$.

$\Pi^{*}$ induces a map $\Pi^{*}:\mathrm{Fn}(\kappa^{++},2,\kappa^{+}) \longrightarrow \mathrm{Fn}(\kappa^{++} \sim \mathrm{Even}_{(\kappa^{+})^{M}},2,\kappa^{+})$; using this map we can associate with each nice $\mathrm{Fn}(\kappa^{++},2,\kappa^{+})$ name $\tau$ for a subset of $\kappa^{+}$ a nice $\mathrm{Fn}(\kappa^{++} \sim \mathrm{Even}_{(\kappa^{+})^{M}},2,\kappa^{+})$ name $\tau^{\Pi^{*}}$ for a subset of $\kappa^{+}$ in the usual way. Recall that in $N[G_{\kappa},{}^{N}\vec{F}_{\gamma}]$ we have partitioned $(\kappa^{++})^{N}$ into cofinal pieces ${}^{N}C$, $({}^{N}A^{\gamma}:\gamma < \kappa^{+})$ and $({}^{N}B^{\gamma}:\gamma < \kappa^{+})$ and we have complete sequences $({}^{N}\tau^{\gamma}_{\eta}:\eta \in {}^{N}A^{\gamma},\gamma < \kappa^{+})$ and $(({}^{N}\sigma^{1,\gamma}_{\eta},{}^{N}\sigma^{2,\gamma}_{\eta}):\eta \in {}^{N}B^{\gamma},\gamma < \kappa^{+})$ of (pairs of) nice $\mathrm{Fn}((\kappa^{++})^{N},2,\kappa)$ names for subsets of $\kappa^{+}$ from which we define the iteration $({}^{N}Q_{\zeta}:\zeta$

$\leq \kappa^{++}$) using $({}^{N}F_\gamma : \gamma < \kappa^+)$.

Now in $V[G_\kappa, \vec{F}_\gamma]$ pick a partition of $\kappa^{++}$ into cofinal pieces ${}^*C$, $({}^*A^\gamma : \gamma < \kappa^+)$, $({}^*B^\gamma, \gamma < \kappa^+)$ such that ${}^*C \cap (\kappa^{++})^N = {}^NC$ and for $\gamma < \kappa^+$ ${}^NA^\gamma = {}^*A^{\Pi(\gamma)} \cap {}^N(\kappa^{++})$ and ${}^NB^\gamma = {}^*B^{\Pi(\gamma)} \cap {}^N(\kappa^{++})$ and pick for each $\gamma < \kappa^+$ complete sequences $({}^*\tau^\gamma_\eta : \eta \in {}^*A^\gamma)$ and $({}^*\sigma^{1,\gamma}_\eta, {}^*\sigma^{2,\gamma}_\eta) : \eta \in {}^*B^\gamma)$ of (pairs of) nice $Fn(\kappa^{++}, 2, \kappa^+)$ names for subsets of $\kappa^+$ such that for $\gamma < \kappa^+$ and $\eta \in {}^NA^\gamma$ ${}^*\tau^{\Pi(\gamma)}_\eta = ({}^N\hat{\tau}^\gamma_\eta)^{\Pi^*}$ and for $\eta \in {}^NB^\gamma$ ${}^*\sigma^{i,\Pi(\gamma)}_\eta = ({}^N\hat{\sigma}^{i,\gamma}_\eta)^{\Pi^*}$ (i = 1,2). Note that for these $\eta$:

$$\text{supp } {}^*\tau^{\Pi(\gamma)}_\eta \cap \text{Even}_{(\kappa^+)^M} = \emptyset \text{ and}$$

$$\text{supp } {}^*\sigma^{i,\Pi(\gamma)}_\eta \cap \text{Even}_{(\kappa^+)^M} = \emptyset.$$

Thus, if $(Q^*_\zeta : \zeta \leq \kappa^{++})$ denotes the $\kappa^{++}$ iteration that is defined in $V[G_\kappa, \vec{F}_\gamma]$ from these parameters and uses $\vec{F}_\gamma$ we have for each $\zeta$ with $0 \leq \zeta \leq {}^N(\kappa^{++})$

$$ {}^*Q_\zeta^{\zeta\text{-Even}_{(\kappa^+)^M}} \subseteq_c {}^*Q_\zeta$$

and

$$ {}^*Q_\zeta^{\zeta\text{-Even}_{(\kappa^+)^M}} = \{q^{\Pi^*} : q \in {}^NQ_\zeta\}.$$

By 4.2.8. Q and ${}^*Q$ are isomorphic. Let G (coming from $G^V$) be the Q generic and $G^*$ the pullback of G to ${}^*Q$ via an isomorphism of Q with ${}^*Q$. Let $g^* \underset{\mathfrak{Df}}{=} G^* \cap {}^*Q^{{}^N(\kappa^{++})\text{-Even}_{(\kappa^+)^M}}$

and g the pullback of $g^*$ to ${}^{N}Q$ via the map $\Pi^*$; i.e., $g = \{q \in {}^{N}Q : q^{\Pi^*} \in g^*\}$. Clearly g is ${}^{N}Q$ generic. Now we have to show

(4.2.15.) $N[G_\kappa, {}^{N}\vec{F}_\gamma, g]$ is $\Sigma^2_2$ correct for $\kappa$ inside $V[G_\kappa, \vec{F}_\gamma, G]$.

We begin with

Lemma 4.2.16. Suppose that $\mathcal{N} \models ZF^-$ is transitive and $\mathcal{N} \models ZF^-$ and $\mathcal{N}^\kappa \subseteq \mathcal{N}$ and $\mathcal{N}$ is $\Sigma^2_n$ $(n \geq 0)$ correct for $\kappa$ inside V where for some $S \subseteq \kappa$ $2^{<\kappa^+} \subseteq L[S]$ and $\kappa^+ = (\kappa^+)^L$. Let $(F_\gamma : \gamma < \kappa^+) \in \mathcal{N}$ be a sequence of Lipschitz functions $2^{\kappa^+} \times 2^{\kappa^+} \longrightarrow 2^{\kappa^+}$ and in $\mathcal{N}$ fix parameters $(A^\gamma : \gamma < \kappa^+)$, $(B^\gamma : \gamma < \kappa^+)$, C, $(\tau^\gamma_\zeta : \zeta \in A^\gamma)(\gamma < \kappa^+)$ and $((\sigma^{1,\gamma}_\zeta, \sigma^{2,\gamma}_\zeta) : \zeta < B^\gamma)(\gamma < \kappa^+)$. Then in V pick parameters $(\overline{A}^\gamma : \gamma < \kappa^+)$, $(\overline{B}^\gamma : \gamma < \kappa^+)$ $\overline{C}$, $(\overline{\tau}^\gamma_\zeta : \zeta \in \overline{A}^\gamma)(\gamma < \kappa^+)$ and $((\overline{\sigma}^{1,\gamma}_\zeta, \overline{\sigma}^{2,\gamma}_\zeta) : \zeta < B^\gamma)(\gamma < \kappa^+)$ such that $\overline{A}^\gamma \cap (\kappa^{++})^{\mathcal{N}} = A^\gamma$, $\overline{B}^\gamma \cap (\kappa^{++})^{\mathcal{N}} = B^\gamma$, $\overline{C} \cap (\kappa^{++})^{\mathcal{N}} = C$ and $\overline{\tau}^\gamma_\zeta = \tau^\gamma_\zeta$ for $\zeta \in A^\gamma$ and $\overline{\sigma}^{i,\gamma}_\zeta = \sigma^{i,\gamma}_\zeta (i = 1,2)$ for $\zeta \in B^\gamma$ for each $\gamma < \kappa^+$. If $Q \in \mathcal{N}$ and $\overline{Q} \in V$ are the corresponding iterations defined from these parameters both using $(F_\gamma : \gamma < \kappa^+)$ and if G is $\overline{Q}$ generic over V, then $\mathcal{N}[G \cap Q]$ is $\Sigma^2_n$ correct for $\kappa$ in V[G].

Proof. Under the hypotheses it is clear that Q is an initial segment of $\overline{Q}$ (i.e., $Q = \overline{Q}_{(\kappa^{++})^{\mathcal{N}}}$). Therefore $G \cap Q$ is $\mathcal{N}$ generic for Q. We proceed by induction on n.

The case n = 0 is clear: $\mathcal{N}[G \cap Q]^{\kappa} \subseteq \mathcal{N}[G \cap Q]$ inside V[G] because Q and $\overline{Q}$ are both $<\kappa^+$ Baire.

Now we handle the case n + 1. Because of the induction hypothesis it is enough to consider $\Phi(A)$ in $\Pi^2_{n+1}$ and $A \in (\mathcal{N}[G \cap Q])_{\kappa+2}$ with $\mathcal{N}[G \cap Q] \models$ "$V_\kappa \models \Phi(A)$" and to show that $V[G] \models$ "$V_\kappa \models \Phi(A)$." By the $\kappa^{++}$ c.c. of Q in $\mathcal{N}$, we fix $\delta < (\kappa^{++})^{\mathcal{N}}$ and a nice name $\mathring{A} \in Q_\delta$ for A and a condition $q \in Q_\delta \cap G$ with

$$q \Vdash^{\mathcal{N}}_{Q} \text{"}V_\kappa \models \Phi(\mathring{A})\text{."}$$

Then clearly

$$N[G \cap Q_\delta] \models \text{"}\Vdash_{\mathring{Q}^{G\cap Q_\delta}_{\delta,\kappa^{++}}} V_\kappa \models \Phi(A)\text{."}$$

Moreover, by the factor lemma 4.2.12. and by the analogue of 4.2.8. for modified $\delta$, $\kappa^{++}$ iterations

$\mathcal{N}[G \cap Q_\delta] \models$ "for all modified $\delta,\kappa^{++}$ iterations that refer to $(G^\gamma : \gamma \in \text{Even}_{\kappa^+})$ and $(((\hat{\tau}^\gamma_\zeta)^G, G^\zeta) : \zeta \in A^\gamma \cap \delta, \gamma < \kappa^+)$ and that use $(F_\gamma : \gamma < \kappa^+)$: $\Vdash V_\kappa \models \Phi(A)$."

The induction hypothesis applied within $\mathcal{N}[G \cap Q_\delta]$ yields in $\mathcal{N}[G \cap Q_\delta]$

$\forall \mathcal{M}[\mathcal{M}$ trans, $\mathcal{M} \models ZF^-$, $|\mathcal{M}| = |V_{\kappa+1}|$, $\mathcal{M}^{|V_\kappa|} \subseteq \mathcal{M}$, $\mathcal{M}$ $\Sigma^2_n$ correct for $\kappa$,

$(G^\gamma : \gamma \in \text{Even}_{\kappa^+}) \in \mathcal{M}$, $(((\hat{\tau}^\gamma_\eta)^G, G^\eta) : \eta \in A^\gamma \cap \delta, \gamma < \kappa^+) \in \mathcal{M}$,

$A \in \mathcal{M}$, $(F_\gamma : \gamma < \kappa^+) \in \mathcal{M}$, $\mathcal{M} \models \delta < \kappa^{++}$ $.\Rightarrow.$

$\mathcal{M} \models$ "for all modified $\delta,\kappa^{++}$ iterations that refer to $(G^{\gamma}:\gamma \in \mathrm{Even}_{\kappa^+})$ and $(((\hat{\tau}^{\gamma}_{\eta})^G, G^{\eta}):\eta \in A^{\gamma} \cap \delta, \gamma < \kappa^+)$ and use $(F_{\gamma}:\gamma < \kappa^+)$:

$\Vdash \ V_\kappa \models \Phi(A)$"].

Since $\kappa^+ = (\kappa^+)^L$, $(G^{\gamma}:\gamma < \kappa^+)$ and $((( \hat{\tau}^{\gamma}_{\eta})^G, G^{\eta}):\eta \in A^{\gamma} \cap \delta, \gamma < \kappa^+)$ which are all subsets of $\kappa^+$ can be coded by one subset of $V_{\kappa+1}$. We can also express $\mathcal{M} \models \delta < \kappa^{++}$ by picking, in $\mathcal{N}$ a wellorder of $V_{\kappa+1}$ of order type $\delta$ and then requiring that this wellorder of (i.e., a subset of $V_{\kappa+1}$) be in $\mathcal{M}$. This formula will be $\Sigma^2_1$ in this parameter. For each $\gamma < \kappa^+$ $F_\gamma \subseteq L[S]$ where $S \subseteq \kappa$. Hence we can use the canonical wellorder $<_{L[S]}$ on $2^{<\kappa^+}$ to code each $F_\gamma$ by a subset of $\kappa^+$. Then again these $\kappa^+$ subsets of $\kappa^+$ can be coded by one subset of $V_{\kappa+1}$. Therefore the last formula is $\Pi^2_{n+1}$ in a parameter $\in (\mathcal{N}[G \cap Q_\delta])_{\kappa+2}$. Since $\delta < \kappa^{++}$ we have $|Q_\delta| \leq \kappa^+ = |V_{\kappa+1}|$ and $\mathcal{N}[G \cap Q_\delta]$ is $\Sigma^2_{n+1}$ correct for $\kappa$ inside $V[G \cap Q_\delta]$. Hence the last formula which is $\Pi^2_{n+1}$ over $V_\kappa$ must hold in $V[G \cap Q_\delta]$. Together with another application of 4.2.12. this yields:

$V[G] \models$ "$V_\kappa \models \Phi(A)$." □ end of 4.2.16.

We now return to the proof of (4.2.15):

We assume that $\Phi(A)$ is $\Pi^2_2 \cup \Sigma^2_1$ and $A \in (N[G_\kappa, {}^N\vec{F}_\gamma, g])_{\kappa+2}$ with

$$N[G_\kappa, {}^N\vec{F}_\gamma, g] \models \text{“}V_\kappa \models \Phi(A)\text{”}$$

and we have to show

$$V[G_\kappa, {}^N\vec{F}_\gamma, G] \models \text{“}V_\kappa \models \Phi(A).\text{”}$$

Pick a $\delta < {}^N(\kappa^{++})$ and a nice name $\mathring{A} \in N[G_\kappa, {}^N\vec{F}_\gamma]^{{}^NQ_\delta}$ with $\mathring{A}^g = A$ and a condition $q \in Q_\delta \bigcap g$ such that

$$N[G_\kappa, {}^N\vec{F}_\gamma] \models \text{“}q \Vdash_{{}^NQ} V_\kappa \models \Phi(\mathring{A}).\text{”}$$

Lemma 4.2.16. together with the factor lemma 4.2.12. yields

(4.2.17.) $V[G_\kappa, {}^N\vec{F}_\gamma, g] \models$ for all modified $\delta$, $\kappa^{++}$ iterations that refer to

$$(g^\gamma : \gamma \in \text{Even}_{\kappa^+}) \text{ and } ((({}^N\hat{\tau}^\gamma_\eta)^g, g^\eta) : \eta \in {}^NA^\gamma \bigcap \delta, \gamma < \kappa^+)$$

$$\text{and use } ({}^NF_\gamma : \gamma < \kappa^+): \quad \Vdash \text{“}V_\kappa \models \Phi(A).\text{”}$$

Now recall that $V[G_\kappa, {}^N\vec{F}_\gamma, g] = V[G_\kappa, \vec{F}_\gamma, g^*]$, which we call $V^*$ from here on. Moreover by (4.2.14.) in $V^*$ the tail ${}^*Q_{\delta,\kappa^{++}}$ of ${}^*Q$ is equal to a special modified $\delta,\kappa^{++}$ iteration that is modified by referring to $((g^{*,\gamma} : \gamma \in \kappa^+ \sim \text{Even}_{(\kappa^+)^M})$ and $((({}^*\tilde{\tau}^\gamma_\eta)^{g^*}, g^{*,\eta}) : \eta \in {}^*A^\gamma \bigcap \delta, \gamma < \kappa^+))$ and uses $(F_\gamma : \gamma < \kappa^+)$. The correctness argument will be finished if we can show

(4.2.18.) $V^* \models$ for all special modified $\delta$, $\kappa^{++}$ iterations that refer to

$(g^{*,\gamma} : \gamma \in \text{Even}_{\kappa^+} \sim \text{Even}_{M(\kappa^+)})$ and

$((({}^*\tilde{\tau}^\gamma_\eta)^{g^*}, g^{*\eta}) : \eta \in {}^*A^\gamma \cap \delta, \gamma < \kappa^+))$ and that use $(F_\gamma : \gamma < \kappa^+)$

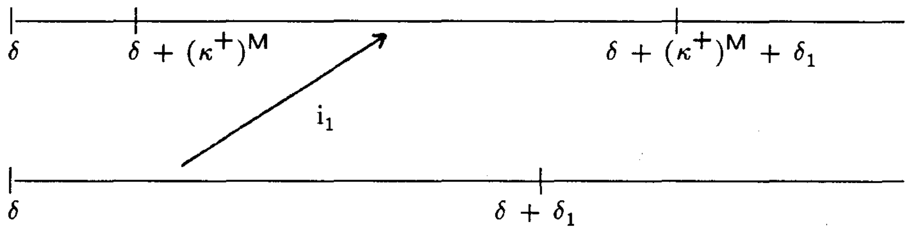
$\Vdash$ "$V_\kappa \models \Phi(A)$."

The rest of the correctness argument will therefore be concerned with establishing that (4.2.17.) implies (4.2.18.).

In the remainder of this section all modified $\delta$, $\alpha$ iterations (where $\alpha \in [\delta,\kappa^{++}]$) will be referring to $(g^\gamma : \gamma < \kappa^+)$ and $((({}^N\hat{\tau}^\gamma_\eta)^g, g^\eta) : \eta \in {}^NA^\gamma \cap \delta, \gamma < \kappa^+)$ and will use $({}^NF_\gamma : \gamma < \kappa^+)$ and all special modified $\delta$, $\alpha$ iterations will be referring to $(g^{*,\gamma} : \gamma \in \text{Even}_{\kappa^+} \sim \text{Even}_{M_{(\kappa^+)}})$ and $((({}^*\tilde{\tau}^\gamma_\eta)^{g^*}, g^{*,\eta}) : \eta \in {}^*A^\gamma \cap \delta, \gamma < \kappa^+)$ and will use $(F_\gamma : \gamma < \kappa^+)$.

The key point in proving that (4.2.17.) implies (4.2.18.) is the following back and forth property of modified and special modified $\delta$, $\kappa^{++}$ iterations in $V^*$: Suppose we are given a modified $\delta$, $\delta + \delta_1$ iteration $\overline{Q}_{\delta,\delta+\delta_1}$ $(\delta_1 < \kappa^{++})$ that is defined from a partition $\overline{C}$, $\overline{A}^\gamma(\gamma < \kappa^+)$ and $\overline{B}^\gamma(\gamma < \kappa^+)$ of $[\delta, \delta + \delta_1)$ and from sequences $(\overline{\tau}^\gamma_\zeta : \zeta \in A^\gamma)$ $(\gamma < \kappa^+)$ and $((\overline{\sigma}^{1,\gamma}_\zeta, \overline{\sigma}^{2,\gamma}_\zeta : \zeta \in \overline{B}^\gamma)$ $(\gamma < \kappa^+)$.

Let $i_1 : [\delta, \delta + \delta_1] \to [\delta + (\kappa^+)^M, \delta + (\kappa^+)^M + \delta_1)$ be defined by $i_1(\delta + \zeta) = \delta + (\kappa^+)^M + \zeta$ for $\zeta < \delta_1$




and let $\overline{\overline{C}} =_{\mathfrak{Df}} [\delta, \delta + (\kappa^+)^M) \cup i_1[C]$ and for $\gamma < \kappa^+$ $\overline{\overline{A}}^{\Pi(\gamma)} =_{\mathfrak{Df}} i_1[A^\gamma]$ and

$\overline{\overline{B}}^{\Pi(\gamma)} =_{\mathfrak{Df}} i_1[B^\gamma]$ and

$$\overline{\overline{\tau}}^{\Pi(\gamma)}_{i_1(\zeta)} = (\hat{\overline{\tau}}^{\gamma}_{\zeta})^{i_1} \text{ for } \zeta \in \overline{A}^\gamma$$

$$\overline{\overline{\sigma}}^{i,\Pi(\gamma)}_{i_1(\zeta)} = (\hat{\overline{\sigma}}^{i,\gamma}_{\zeta})^{i_1} \text{ for } \zeta \in \overline{B}^\gamma \ (i = 1, 2).$$

Let ${}^*\overline{Q}_{\delta,\delta+(\kappa^+)^M+\delta_1}$ denote the special modified $\delta, \delta + (\kappa^+)^M + \delta_1$ iteration that we define from these parameters. Since all these terms have support disjoint from $[\delta, \delta + (\kappa^+)^M)$ it is easy to see that

$${}^*\overline{Q}^{[\delta,\delta+(\kappa^+)^M+\delta_1) - [\delta,\delta+(\kappa^+)^M)}_{\delta,\delta+(\kappa^+)^M+\delta_1} \subseteq_c {}^*\overline{Q}_{\delta,\delta+(\kappa^+)^M+\delta_1}$$

and

$${}^*\overline{Q}^{[\delta,\delta+(\kappa^+)^M+\delta_1) - [\delta,\delta+(\kappa^+)^M)}_{\delta,\delta+(\kappa^+)^M+\delta_1} = \left\{q^{i_1} : q \in \overline{Q}_{\delta,\delta+\delta_1}\right\}.$$

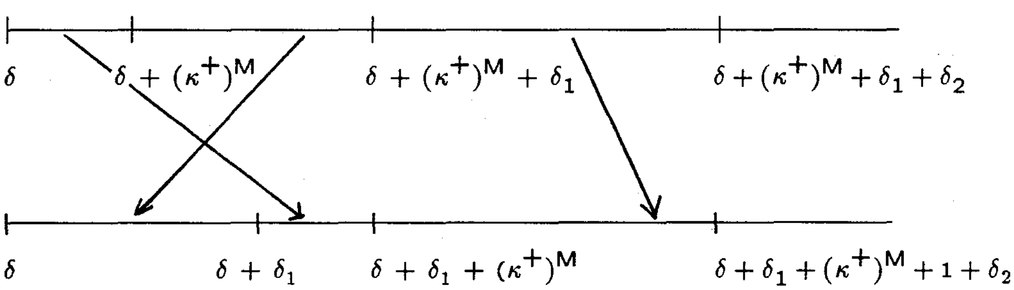
Now suppose we "enlarge" the family of parameters for ${}^*\overline{Q}_{\delta,\delta+(\kappa^+)^M+\delta_1}$; i.e., we have a partition of $[\delta, \delta + (\kappa^+)^M + \delta_1 + \delta_2)$ where $\delta_2 < \kappa^{++}$ into pieces that we denote again by $\overline{\overline{C}}$, $\overline{\overline{A}}^\gamma$, $\overline{\overline{B}}^\gamma$ $(\gamma < \kappa^+)$ and sequences $(\overline{\overline{\tau}}^\gamma_\zeta : \zeta \in \overline{\overline{A}}^\gamma)$ $(\gamma < \kappa^+)$ and $((\overline{\overline{\sigma}}^{1,\gamma}_\zeta, \overline{\overline{\sigma}}^{2,\gamma}_\zeta : \zeta \in \overline{\overline{B}}^\gamma)$ $(\gamma < \kappa^+)$. We denote the special modified $\delta$, $\delta + (\kappa^+)^M + \delta_1 + \delta_2$ iteration that we obtain from these parameters by ${}^*\overline{Q}_{\delta,\delta+(\kappa^+)^M+\delta_1+\delta_2}$.

Now define $i_2 : [\delta, \delta + (\kappa^+)^M + \delta_1 + \delta_2) \longrightarrow [\delta, \delta + \delta_1 + (\kappa^+)^M + 1 + \delta_2)$ by

$$i_2(\delta + \zeta) = \delta + \delta_1 + \zeta \text{ for } 0 \le \zeta < (\kappa^+)^M$$

$$i_2(\delta + {}^M(\kappa^+) + \zeta) = \delta + \zeta \text{ for } 0 \le \zeta < \delta_1$$

$$i_2(\delta + {}^M(\kappa^+) + \delta_1 + \zeta) = \delta + \delta_1 + (\kappa^+)^M + 1 + \zeta \text{ for } 0 \le \zeta < \delta_2.$$




Note that $\delta + \delta_1 + {}^M(\kappa^+) \notin \text{rng } i_2$. Now define $\overline{C} = i_2[\overline{\overline{C}}] \cup \{\delta + \delta_1 + (\kappa^+)^M\}$ and $\overline{A}^\gamma = i_2[\overline{\overline{A}}^{\Pi(\gamma)}]$ and $\overline{\overline{B}}^\gamma = i_2[\overline{\overline{B}}^{\Pi(\gamma)}]$ for $\gamma < \kappa^+$. Note that this does not conflict with the "old" partition of $[\delta, \delta + \delta_1)$. We will now "expand" the old sequences $(\overline{\tau}^{\gamma}_{\zeta} : \zeta \in \overline{A}^{\gamma} \cap \delta + \delta_1)(\gamma < \kappa^+)$ and $((\overline{\sigma}^{1,\gamma}_{\zeta}, \overline{\sigma}^{2,\gamma}_{\zeta} : \zeta \in \overline{B}^{\gamma} \cap \delta + \delta_1)(\gamma < \kappa^+)$ and define sequences $(\overline{\tau}^{\gamma}_{\zeta} : \zeta \in \overline{A}^{\gamma})(\gamma < \kappa^+)$ and $((\overline{\sigma}^{1,\gamma}_{\zeta}, \overline{\sigma}^{2,\gamma}_{\zeta}) : \zeta \in \overline{B}^{\gamma})(\gamma < \kappa^+)$ such that we have for the modified $\delta$, $\delta + \delta_1 + (\kappa^+)^M + 1 + \delta_2$ iteration $\overline{Q}_{\delta,\delta+\delta_1+(\kappa^+)^M+1+\delta_2}$ obtained from these parameters:

$$(4.2.19.)\quad \overline{Q}^{[\delta,\delta+\delta_1+(\kappa^+)^M+1+\zeta)-\{\delta+\delta_1+(\kappa^+)^M\}}_{\delta,\delta+\delta_1+(\kappa^+)^M+1+\zeta} \subseteq_c \overline{Q}_{\delta,\delta+\delta_1+(\kappa^+)^M+1+\zeta}$$

and

$$(4.2.20.)\quad \overline{Q}^{[\delta,\delta+\delta_1+(\kappa^+)^M+1+\zeta)-\{\delta+\delta_1+(\kappa^+)^M\}}_{\delta,\delta+\delta_1+(\kappa^+)^M+1+\zeta} =$$

$$\left\{q^{i_2} : q \in {}^*\overline{Q}_{\delta,\delta+(\kappa^+)^M+\delta_1+\zeta}\right\} \text{ for all } \zeta \in [0, \delta_2].$$

For $\delta + \delta_1 + {}^M(\kappa^+) + 1 + \zeta \in \overline{B}^{\gamma}$ for some $\gamma < \kappa^+$ simply let

$$\overline{\sigma}^{i,\gamma}_{\delta+\delta_1+(\kappa^+)^M+1+\zeta} \overset{=}{\mathfrak{Df}} (\hat{\hat{\overline{\overline{\sigma}}}}^{i,\Pi(\gamma)}_{\delta+(\kappa^+)^M+\delta_1+\zeta})^{i_2} \; (i = 1, 2).$$

For $\delta + \delta_1 + {}^M(\kappa^+) + 1 + \zeta \in \overline{A}^{\gamma}$ for some $\gamma < \kappa^+$ we have to look at two cases. If $\gamma$ is noncritical (i.e., either $\gamma$ and $\Pi(\gamma)$ are both even or both odd), then

$$\overline{\tau}^{\gamma}_{\delta+\delta_1+(\kappa^+)^M+1+\zeta} \overset{=}{\mathfrak{Df}} (\hat{\hat{\overline{\overline{\tau}}}}^{\Pi(\gamma)}_{\delta+(\kappa^+)^M+\delta_1+\zeta})^{i_2}$$

and if $\gamma$ is critical (i.e., $\gamma$ is odd and $\Pi(\gamma) < (\kappa^+)^M$ even), let

$$A_1 \text{ a maximal antichain } \subseteq \Big\{q \in {}^*\overline{Q}_{\delta,\delta+(\kappa^+)^M+\delta_1+\zeta} :$$

$$q \Vdash \hat{\overline{\tau}}^{\Pi(\gamma)}_{\delta+(\kappa^+)^M+\delta_1+\zeta} = \Gamma^{\delta+\xi}\Big\}.$$

where $\Pi(\gamma)$ is the $\xi$-th even ordinal $<(\kappa^+)^M$

and

$$A_2 \text{ a maximal antichain } \subseteq \Big\{q \in {}^*\overline{Q}_{\delta,\delta+(\kappa^+)^M+\delta_1+\zeta} :$$

$$q \Vdash \hat{\overline{\tau}}^{\Pi(\gamma)}_{\delta+(\kappa^+)^M+\delta_1+\zeta} \neq \Gamma^{\delta+\xi}\Big\}.$$

Assuming inductively that (4.2.19.) and (4.2.20.) hold for $\zeta$ we pick

$$\overline{\tau}^{\gamma}_{\delta+\delta_1+(\kappa^+)^M+1+\zeta} \in (V^*)^{\overline{Q}_{\delta,\delta+\delta_1+(\kappa^+)^M+1+\zeta}}$$

such that for $q \in A_1 \cup A_2$

$$q^{i_2} \Vdash_{\overline{Q}_{\delta,\delta+\delta_1+(\kappa^+)+1+\zeta}} \overline{\tau}^{\gamma}_{\delta+\delta_1+(\kappa^+)^M+1+\zeta} = \tau_q$$

where for $q \in A_1$

$\tau_q \mathrel{\overline{\overline{\mathfrak{Df}}}} \Gamma^{\delta+\delta_1+(\kappa^+)^M}$ (i.e., the canonical $\overline{Q}_{\delta,\delta+\delta_1+(\kappa^+)^M+1}$ name for the set that we add at coordinate $\delta + \delta_1 + (\kappa^+)^M$)

and for $q \in A_2$

$$^{\tau}q \;\overline{\overline{\mathfrak{D}f}}\; \left(\hat{\overline{\overline{\tau}}}^{\Pi(\gamma)}_{\delta+(\kappa^+)^M+\delta_1+\zeta}\right)^{i_2}.$$

It takes a routine but lengthy argument to show that (4.2.19.) and (4.2.20.) hold throughout this construction: Suppose we have arrived at some coordinate $\zeta_0 + 1$ and for all $\zeta \leq \zeta_0$ (4.2.19.) and (4.2.20.) for $\zeta_0$ hold. We can restrict ourselves to checking what happens if $\delta + (\kappa^+)^M + \delta_1 + \zeta_0 \in \overline{\overline{B}}^{\Pi(\gamma)}$ for a critical $\gamma < \kappa^+$. By induction hypothesis(4.2.19.) holds for $\zeta_0$. Thus $(V^*)^{\overline{Q}^{[\delta,\delta+\delta_1+(\kappa^+)^M+1+\zeta_0)\text{-}\{\delta+\delta_1+(\kappa^+)^M\}}_{\delta,\delta+\delta_1+(\kappa^+)^M+1+\zeta_0}}$ can be regarded as contained in $(V^*)^{\overline{Q}_{\delta,\delta+\delta_1+(\kappa^+)^M+1+\zeta_0}}$. Then (4.2.20.) for $\zeta_0$ together with the way in which we defined $(\overline{\tau}^{\gamma}_{\zeta} : \zeta \in \overline{A}^{\gamma} \cap (\delta+\delta_1+(\kappa^+)^M, \delta+\delta_1+(\kappa^+)^M+1+\delta_2))$ and the fact that no term at any coordinate in $\overline{Q}_{\delta,\delta+\delta_1+(\kappa^+)^M+1+\delta_2}$ can "see" the set that we added at coordinate $\delta + \delta_1 + (\kappa^+)^M$ in $\overline{Q}_{\delta,\delta+\delta_1+(\kappa^+)^M+1+\delta_2}$ yield:

It cannot happen that we want to kill at coordinate $\delta + (\kappa^+)^M + \delta_1 + \zeta_0$ in $^*\overline{Q}$ (at coordinate $\delta + \delta_1 + (\kappa^+)^M + 1 + \zeta_0$ in $\overline{Q}$ resp.) and save at cooordinate $\delta + \delta_1 + (\kappa^+)^M + 1 + \zeta_0$ in $\overline{Q}$ (at coordinate $\delta + (\kappa^+)^M + \delta_1 + \zeta_0$ in $\overline{Q}^*$ resp.).

This implies (4.2.20.) for $\zeta_0 + 1$. Then one proves (4.2.19.) for $\zeta_0 + 1$ as follows: Suppose $q \in \overline{Q}_{\delta,\delta+\delta_1+(\kappa^+)^M+1+\zeta_0+1}$ where we assume again that $\delta + \delta_1 + (\kappa^+)^M + 1 + \zeta_0 \in \overline{\overline{B}}^{\gamma}$ for some critical $\gamma < \kappa^+$ and $q(\delta+\delta_1+(\kappa^+)^M+1+\zeta) \neq 0$. Let $q'$ denote the unique element of $Fn([\delta,\delta + (\kappa^+)^M + \delta_1 + \zeta_0 + 1), 2, \kappa^+)$ so that

$$(q')^{i_2} = q|([\delta, \delta + \delta_1 + (\kappa^+)^M + 1 + \zeta_0 + 1) - \{\delta + \delta_1 + (\kappa^+)^M\}).$$

By the argument that we just used to establish (4.2.20.) it follows that $q' \in {}^*\overline{Q}_{\delta,\delta+(\kappa^+)^M+\delta_1+\zeta_0+1}$.

Hence by (4.2.20.) $(q')^{i_2} \in \overline{Q}^{[\delta,\delta+\delta_1+(\kappa^+)^M+1+\zeta_0+1)-\{\delta+\delta_1+(\kappa^+)^M\}}_{\delta,\delta+\delta_1+(\kappa^+)^M+1+\zeta_0+1}$.

Thus we get (4.2.19.) for $\zeta_0 + 1$.

We can also go through a similar procedure if we start with a special modified $\delta$, $\delta + (\kappa^+)^M + \delta_1$ iteration ${}^*\overline{Q}_{\delta,\delta+(\kappa^+)^M+\delta_1}$ ($\delta_1 < \kappa^{++}$). In order to find a modified $\delta$, $\delta + 1 + (\kappa^+)^M + \delta_1$ iteration $\overline{Q}_{\delta,\delta+1+(\kappa^+)^M+\delta_1}$ such that ${}^*\overline{Q}_{\delta,\delta+(\kappa^+)^M+\delta_1}$ is ismorphic via a coordinate induced embedding $i_1$ with the complete suborder $\overline{Q}^{[\delta,\delta+1+(\kappa^1)^M+\delta_1)-\{\delta\}}_{\delta,\delta+1+(\kappa^+)^M+\delta_1}$, we use the same ideas that established that $i_2$ from above had the right properties. Then for any extension $\overline{Q}_{\delta,\delta+1+(\kappa^+)^M+\delta_1+\delta_2}$ of $\overline{Q}_{\delta,\delta+1+(\kappa^+)^M+\delta_1}$ we can define an extension ${}^*\overline{Q}_{\delta,\delta+(\kappa^+)^M+\delta_1+1+\delta_2}$ of ${}^*\overline{Q}_{\delta,\delta+(\kappa^+)^M+\delta_1}$ such that $\overline{Q}_{\delta,\delta+1+(\kappa^+)^M+\delta_1+\delta_2}$ completely embeds into ${}^*\overline{Q}_{\delta,\delta+(\kappa^+)^M+\delta_1+1+\delta_2}$ via some $i_2$ such that $i_2 \circ i_1 = id_{{}^*\overline{Q}_{\delta,\delta+(\kappa^+)^M+\delta_1}}$.

This will work since the special modified iteration ${}^*\overline{Q}_{\delta,\delta+(\kappa^+)^M+\delta_1+1+\delta_2}$ does more killing than the modified iteration $\overline{Q}_{\delta,\delta+1+(\kappa^+)^M+\delta_1+\delta_2}$. Using this back and forth property of modified and special modified iterations we now show that roughly speaking generic extensions of $V^*$ via modified and special modified iterations satisfy the same $\Sigma^2_2$ statements (in parameters from $(V^*)_{\kappa+2}$).

Lemma 4.2.21. Suppose that in $V^*$ we are given a modified $\delta$, $\kappa^{++}$ iteration $\overline{Q}_{\delta,\kappa^{++}}$. Let q be a condition in $\overline{Q}_{\delta,\kappa^{++}}$ with

$$q \Vdash^{V^*}_{\overline{Q}_{\delta,\kappa^{++}}} \text{“}V_\kappa \models \Phi(A).\text{”}$$

Let $\delta_1 < \kappa^{++}$ such that $q \in \overline{Q}_{\delta,\delta+\delta_1}$ and there is a witness $\in (V^*)^{\overline{Q}_{\delta,\delta+\delta_1}}$ for the $\Sigma^2_2$ statement $\Phi$. We have seen how to define an initial piece $^*\overline{Q}_{\delta,\delta+(\kappa^+)^M+\delta_1}$ of a special modified $\delta$, $\kappa^{++}$ iteration which has a complete suborder isomorphic via some $i_1$ with $\overline{Q}_{\delta,\delta+\delta_1}$ and which has the property that for any extension $^*\overline{Q}_{\delta,\delta+(\kappa^+)^M+\delta_1+\delta_2}$ ($\delta_2 < \kappa^{++}$) of it we can define a complete embedding $i_2$ into an extension $\overline{\overline{Q}}_{\delta,\delta+\delta_1+(\kappa^+)^M+1+\delta_2}$ of $\overline{Q}_{\delta,\delta+\delta_1}$ with $i_2 \circ i_1 = id_{\overline{Q}_{\delta,\delta+\delta_1}}$. In this situation we have for *any* special modified $\delta$, $\kappa^{++}$ iteration $^*\overline{Q}_{\delta,\kappa^{++}}$ that extends $^*\overline{Q}_{\delta,\delta+(\kappa^+)^M+\delta_1}$:

$$q^{i_1} \Vdash^{V^*}_{^*\overline{Q}_{\delta,\kappa^{++}}} \text{“}V_\kappa \models \Phi(A).\text{”}$$

There is also an analogue for this lemma where one starts out with an initial segment of a special modified $\delta$, $\kappa^{++}$ iteration in $V^*$.

Proof. Suppose $\Phi(A) \equiv \exists X\ \forall Y\ \varphi(X,Y,A)$ where $\varphi$ is $\Sigma^2_0$. Then pick $\delta_1 < \kappa^{++}$ with $q \in \overline{Q}_{\delta,\delta+\delta_1}$ and such that there is a nice $\overline{Q}_{\delta,\delta+\delta_1}$ name for a subset of $V_{\kappa+1}$ say $\mathring{X}$ with

$$(4.2.22.)\quad q \Vdash^{V^*}_{\overline{Q}_{\delta,\kappa^{++}}} \text{“}V_\kappa \models \forall Y\ \varphi(\mathring{X},Y,A).\text{”}$$

Let $i_1 : \overline{Q}_{\delta,\delta+\delta_1} \longrightarrow {}^*\overline{Q}_{\delta,\delta+M(\kappa^+)+\delta_1}$ be a complete embedding such that we are in the situation described above. We claim:

$$q^{i_1} \Vdash^{V^*}_{{}^*\overline{Q}_{\delta,\kappa^{++}}} \text{“}V_\kappa \models \forall Y\ \varphi(\mathring{X}^{i_1},Y,A)\text{”}$$

for any extension ${}^*\overline{Q}_{\delta,\kappa^{++}}$ of ${}^*\overline{Q}_{\delta,\delta+M(\kappa^+)+\delta_1}$.

Suppose towards a contradiction that this fails for some extension ${}^*\overline{Q}_{\delta,\kappa^{++}}$. Then there is some $\delta_2 < \kappa^{++}$ and a condition $p \in {}^*\overline{Q}_{\delta,\delta+M(\kappa^+)+\delta_1+\delta_2}$ and a nice ${}^*\overline{Q}_{\delta,\delta+M(\kappa^+)+\delta_1+\delta_2}$ name for a subset of $V_{\kappa+1}$, say $\mathring{Y}$ such that $p \leq q^{i_1}$ and

$$p \Vdash^{V^*}_{{}^*\overline{Q}_{\delta,\kappa^{++}}} \text{“}V_\kappa \models \neg\,\varphi(\mathring{X}^{i_1},\mathring{Y},A).\text{”}$$

Now let $\overline{\overline{Q}}_{\delta,\delta+\delta_1+(\kappa^+)^M+1+\delta_2}$ be an extension of $\overline{Q}_{\delta,\delta+\delta_1}$ and

$i_2 : {}^*\overline{Q}_{\delta,\delta+M(\kappa^+)+\delta_1+\delta_2} \longrightarrow \overline{\overline{Q}}_{\delta,\delta+\delta_1+ M(\kappa^+)+1+\delta_2}$ be a complete embedding as in the situation described above; then

$$\text{(4.2.23.)}\quad p^{i_2} \Vdash_{\overline{\overline{Q}}_{\delta,\kappa^{++}}}^{V^*} \text{“}V_\kappa \models \neg\,\varphi(\mathring{X},\mathring{Y}^{i_2},A)\text{”}$$

for any modified $\delta$, $\kappa^{++}$ iteration $\overline{\overline{Q}}_{\delta,\kappa^{++}}$ that extends $\overline{\overline{Q}}_{\delta,\delta+\delta_1+(\kappa^+)^M+1+\delta_2}$. This is true because $\varphi$ is $\Sigma^2_0$ and $(\mathring{X}^{i_1})^{i_2} = \mathring{X}$.

On the other hand clearly $p^{i_2} \leq q$. Thus (4.2.23.) contradicts (4.2.22) since there is an isomorphism of $\overline{\overline{Q}}_{\delta,\kappa^{++}}$ with $\overline{Q}_{\delta,\kappa^{++}}$ that is the identity on $\overline{Q}_{\delta,\delta+\delta_1}$.

□
end of 4.2.21.

Given this lemma, the proof that (4.2.17.) implies (4.2.18.) is now very easy. First suppose that $\Phi(A)$ in (4.2.17.) is $\Sigma^2_2$. Then (4.2.21.) yields (4.2.18.) since up to isomorphism there is only one special modified $\delta$, $\kappa^{++}$ iteration in $V^*$. If $\Phi(A)$ in (4.2.17.) is $\Pi^2_2$, assume towards a contradiction that for some special modified $\delta$, $\kappa^{++}$ iteration ${}^*\overline{Q}$ there is a condition q with $q \Vdash_{{}^*\overline{Q}}$ “$V_\kappa \models \neg\,\Phi(A)$.” Then by the version of 4.2.21. that starts out with an initial piece of a special modified $\delta$, $\kappa^{++}$ iteration, we get that for some modified $\delta$, $\kappa^{++}$ iteration $\overline{Q}$ and some condition $p \in \overline{Q}$ $p\Vdash_{\overline{Q}}$ “$V_\kappa \models \neg\,\Phi(A)$” which contradicts (4.2.17.).

Next we consider the third step of stage $\kappa$ of ${}^N P_{j(\kappa)}$. Recall that in

$N[G_\kappa, {}^N\vec{F}_\gamma, g]$ we have for each $\gamma < \kappa^+$ a code ${}^N\tilde{F}_\gamma \subseteq \kappa^+$ for ${}^NF_\gamma$ by using that in $N[G_\kappa, {}^N\vec{F}_\gamma, g]$ $2^{<\kappa^+} \subseteq L[G_\kappa]$ which allows us to employ $<_{L[G_\kappa]}$ on $2^{<\kappa^+}$. But $2^{<\kappa^+} \cap L[G_\kappa]$ is the same whether computed in $V[G_\kappa, \vec{F}_\gamma, G]$ or $N[G_\kappa, {}^N\vec{F}_\gamma, g]$ since $N[G_\kappa, {}^N\vec{F}_\gamma, g]$ is closed under $\kappa$ sequences in $V[G_\kappa, \vec{F}_\gamma, G]$. Thus ${}^N\tilde{F}_\gamma = \tilde{F}_{\Pi(\gamma)}$ for $\gamma < \kappa^+$. Hence $Q_{{}^N\tilde{F}_\gamma}$ (of $N[G_\kappa, {}^N\vec{F}_\gamma, g]$) agrees with $Q_{\tilde{F}_{\Pi(\gamma)}}$ (of $V[G_\kappa, \vec{F}_\gamma, G]$). In short the forcing at step 3 of stage $\kappa$ in ${}^NP_{j(\kappa)}$ is just the forcing at step 3 of stage $\kappa$ of $P_{\kappa+1}$ with its factors permuted by $\Pi$. Thus with ${}^NS_\gamma \underset{\mathfrak{Df}}{=} S_{\Pi(\gamma)}$ (where $(S_\gamma : \gamma < \kappa^+)$ is the generic for the third step of stage $\kappa$ of $P_{\kappa+1}$), clearly $({}^NS_\gamma : \gamma < \kappa^+)$ is generic for the third step of stage $\kappa$ of ${}^NP_{j(\kappa)}$. Moreover the fact that the forcing at step 3 of stage $\kappa$ of ${}^NP_{j(\kappa)}$ has size $\kappa^+$ makes it easy to show that $N[G_\kappa, {}^N\vec{F}_\gamma, g, {}^N\vec{S}_\gamma]$ is $\Sigma^2_2$ correct for $\kappa$ in $V[G_\kappa, \vec{F}_\gamma, G, \vec{S}_\gamma]$. The $\kappa^+$ c.c. implies that $N[G_\kappa, {}^N\vec{F}_\gamma, g, {}^N\vec{S}_\gamma]$ is closed under $\kappa$ sequences in $V[G_\kappa, \vec{F}_\gamma, G, \vec{S}_\gamma]$.

A similar argument shows that if $(C_\gamma : \gamma < \kappa^+)$ denotes the generic for the 3rd step of stage $\kappa$ of $P_{\kappa+1}$ that comes from $G^V$, then with ${}^NC_\gamma = C_{\Pi(\gamma)}$ the sequence $({}^NC_\gamma : \gamma < \kappa^+)$ is generic for the forcing at the third step of stage $\kappa$ of ${}^NP_{j(\kappa)}$. Since this forcing has size $\kappa^+$, $N[G_\kappa, {}^N\vec{F}_\gamma, g, {}^N\vec{S}_\gamma, {}^N\vec{C}_\gamma]$ will be $\Sigma^2_2$ correct for $\kappa$ inside $V[G_\kappa, \vec{F}_\gamma, G, \vec{S}_\gamma, \vec{C}_\gamma]$ and by the $\kappa^+$ c.c. $N[G_\kappa, {}^N\vec{F}_\gamma, g, {}^N\vec{S}_\gamma, {}^N\vec{C}_\gamma]$ will be closed under $\kappa$ sequences in $V[G^V]$.

Recall that the tail ${}^NP_{\kappa+1, j(\kappa)}$ has a $<\mu$ closed dense suborder (where $\mu$ is the next inaccessible in $N > \kappa$). Since $|N[G_\kappa, {}^N\vec{F}_\gamma, g, {}^N\vec{S}_\gamma, {}^N\vec{C}_\gamma]| = \kappa^+$ we can in $V[G^V]$ construct an H which is generic over $N[G_\kappa, {}^N\vec{F}_\gamma, g, {}^N\vec{S}_\gamma, {}^N\vec{C}_\gamma]$ for the tail in the usual way. We know that j lifts with $j(G_\kappa) = G_\kappa * {}^N\vec{F}_\gamma * g * {}^N\vec{S}_\gamma * {}^N\vec{C}_\gamma * H$

$$\begin{array}{ccc} M[G_\kappa] & \underset{j}{\longrightarrow} & N[j(G_\kappa)] \\ P_\kappa & & {}^N P_{j(\kappa)} \\ M & \underset{j}{\longrightarrow} & N . \end{array}$$

Moreover $N[j(G_\kappa)]$ is closed under $\kappa$ sequences and $\Sigma^2_2$ correct for $\kappa$ inside $V[G^V]$ since the tail is highly Baire.

Next we consider stage $\kappa$ of ${}^M P_{\kappa+1}$ and stage $j(\kappa)$ of ${}^N P_{j(\kappa)+1}$. Denote by $({}^M F_\gamma : \gamma < (\kappa^+)^M)$, ${}^M g$, $({}^M S_\gamma : \gamma < (\kappa^+)^M)$ and $({}^M C_\gamma : \gamma < (\kappa^+)^M)$ the generics for the 4 steps of stage $\kappa$ of ${}^M P_{\kappa+1}$ that come from $G^M$. Since the first three steps of stage $j(\kappa)$ of $P^N_{j(\kappa)+1}$ are all $<j(\kappa)$ directed closed, it is easy to find master conditions in these cases. The fact that $|N[j(G_\kappa)]| = \kappa^+$ allows us to construct generics for the first 3 steps of stage $j(\kappa)$ of $P^N_{j(\kappa)+1}$ that contain these master conditions in the usual way, then j lifts.

$$\begin{array}{ccc} M[G_\kappa, {}^M\vec{F}_\gamma, {}^M g, {}^M\vec{S}_\gamma]] & \longrightarrow & N[j(G_\kappa), j({}^M\vec{F}_\gamma), j({}^M g), j({}^M\vec{S}_\gamma)] \\ & & \\ M & \longrightarrow & N \end{array}$$

To handle the last step of stage $j(\kappa)$ of $P^N_{j(\kappa)+1}$ where we add a sequence of club sets $\subseteq$ $j(\kappa)$ that avoid a certain set of inaccessibles, we proceeed just as in the $\sigma^2_1/\pi^2_1$ case. Define $c^*$ by

$$\mathrm{dom}(c^*) \underset{\mathfrak{Df}}{=} \{j(\gamma) : \gamma < (\kappa^+)^M\}$$

and for $\gamma < (\kappa^+)^M$ let

$$c^*_{j(\gamma)} \underset{\mathfrak{Df}}{=} C^M_\gamma \cup \{\kappa\}.$$

If we can verify that $c^*$ is a condition in the forcing at the last step of stage $\kappa$ of ${}^N P_{j(\kappa)+1}$, we are done. For this it is enough to show that in $N[j(G_\kappa),j({}^M\vec{F}_\gamma),j({}^M g),j({}^M\vec{S}_\gamma)]$ for each $\gamma < (\kappa^+)^M$

(4.2.24.) $c^*_{j(\gamma)} \bigcap \{\mu < j(\kappa) : \mu$ inaccessible $\wedge$

$$V_\mu \models \Phi^{\Sigma^2_3}(j({}^M\tilde{S})_{j(\gamma)} \bigcap V_\mu, j(G_\kappa) \bigcap V_\mu, j(\kappa) \bigcap \mu)\} = \emptyset$$

where $\Phi^{\Sigma^2_3}$ is the $\Sigma^2_3$ statement (4.1.10.). As in the $\sigma^2_1/\pi^2_1$ case we don't have to worry about the $\mu < \kappa$. For $\mu = \kappa$ recall that $j(\kappa) \bigcap \kappa = \kappa$, $j(G_\kappa) \bigcap V_\kappa = G_\kappa$ and $j({}^M\tilde{S})_{j(\gamma)} \bigcap V_\kappa = j({}^M\tilde{S}_\gamma) \bigcap V_\kappa = {}^M\tilde{S}_\gamma = i({}^M\tilde{S}_\gamma) = \tilde{S}_\gamma$ (which is the $\gamma$-th code that we add at the third step of stage $\kappa$ of $P_{\kappa+1}$). Recall also that $\tilde{S}_\gamma = {}^N\tilde{S}_{\Pi^{-1}(\gamma)}$ and $\Pi^{-1}[(\kappa^+)^M] \subseteq \mathrm{Odd}_{\kappa+}$. Thus in $N[G_\kappa, {}^N\vec{F}_\gamma, g]$ the $\Pi^2_3$ statement (4.1.11.) is true about $\tilde{S}_\gamma = {}^N\tilde{S}_{\Pi^{-1}(\gamma)}$. In the last two steps of stage $\kappa$ of ${}^N P_{j(\kappa)}$ we do not add any new subsets of $\kappa^+$ all of whose initial statements are in $N[G_\kappa, {}^N\vec{F}_\gamma, g]$, since both posets have the $\kappa^{+}$ property. Moreover the tail ${}^{N}P_{\kappa+1,j(\kappa)+1}$ is highly Baire in $N[G_\kappa, {}^{N}\vec{F}_\gamma, g, {}^{N}\vec{S}_\gamma, {}^{N}\vec{C}_\gamma]$. Thus we can conclude that the $\Pi^2_3$ fact (4.1.11.) holds about $\tilde{S}_\gamma = {}^{N}\tilde{S}_{\Pi^{-1}(\gamma)}$ in $N[j(G_\kappa), j({}^{M}\vec{F}_\gamma), j({}^{M}g), j({}^{M}\vec{S}_\gamma)]$ also; i.e., $V_\kappa \models \neg\, \Phi^{\Sigma^2_3}(j({}^{M}\tilde{S}_\gamma) \bigcap V_\kappa j(G_\kappa) \bigcap V_\kappa j(\kappa) \bigcap \kappa)$. Thus we have proved (4.2.24.)

## SECTION 5. $\sigma_n^2/\pi_n^2$ ($n \geq 2$).

### 5.1. Definition of the Iteration $P_n^2$.

We are going to define $\kappa + 1$ stage iterations $P_n^2$ by induction on n. These iterations are analogues of the one that we used in the "generic" case $\sigma_3^2/\pi_3^2$ and we will restrict ourselves to describing what must be changed. At stage $\lambda \leq \kappa$ of the iteration $P_n^2$, where $\lambda$ is a Mahlo cardinal, we will again have a four-step iteration: In the first step we will add a $\lambda^+$ sequence $(F_\gamma : \gamma < \lambda^+)$ where each $F_\gamma$ is a Lipshitz function $(2^{\lambda^+})^{n-1} \to 2^{\lambda^+}$.

In the second step we use a certain suborder of $Fn(\lambda^{++}, 2, \lambda^+)$ to make a $\Sigma_n^2$ statement true about the $F_\gamma$ with $\gamma$ even and $\Pi_n^2$ statement about the $F_\gamma$ with $\gamma$ odd.

Then in the third step we code each $F_\gamma$ by a subset $S_\gamma \subseteq \lambda$ exactly as in $\sigma_3^2/\pi_3^2$ case and finally in the 4th step we add a sequence $(C_\gamma : \gamma < \lambda^+)$ where each $C_\gamma \subseteq \lambda$ is club and avoids the set of all inaccessibles $\mu < \lambda$ such that a certain $\Sigma_n^2$ statement holds in $V_\mu$ just as in the $\sigma_3^2/\pi_3^2$ case.

### The $2^{nd}$ Step at Stage $\lambda$ in $P_n^2$.

In order to keep notation as simple as possible we will not give the full definition of a $\kappa^{++}$ iteration Q that makes a $\Sigma_n^2$ statement true about the $F_\gamma$ for even $\gamma$ and a $\Pi_n^2$ statement about $F_\gamma$ for odd $\gamma$. Instead we will work with one Lipshitz function

$F:(2^{\lambda^+})^{n-1} \rightarrow 2^{\lambda^+}$ and define an iteration $Q_{\Sigma^2_n}$ with

$$\Vdash_{Q_{\Sigma^2_n}} \exists X_1 \; \forall X_2 \cdots Q \; X_{n-1} \; \varphi(F(X_1,\ldots,X_{n-1}))$$

where the $X_i$ range over subsets of $\lambda^+$ and $Q \in \{\exists,\forall\}$ and $\varphi$ says that $F(X_1,\ldots,X_n)$ is stationary or is nonstationary depending on whether n is even or odd, respectively. We will also define an iteration $Q_{\Pi^2_n}$ such that

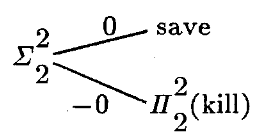
$$\Vdash_{Q_{\Pi^2_n}} \forall \; X_1 \; \exists X_2 \cdots Q \; X_{n-1} \; \varphi(F(X_1,\ldots,X_{n-1}))$$

with Q and $\varphi$ subject to the same conditions as above.

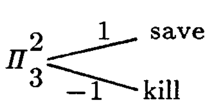
In order to define $Q_{\Sigma^2_n}$ and $Q_{\Pi^2_n}$ in both cases we first partition $\lambda^{++}$ into cofinal pieces C and $A^{(k)}$ $(1 \leq k \leq n-1)$ with $0 \in C$. Then for each k with $1 \leq k \leq n-1$ we choose complete sequences of k-tuples of nice $Fn(\lambda^{++},2,\lambda^+)$ names for subsets of $\lambda^+$ and enumerate them along the coordinates in $A^{(k)}$; i.e., $((\tau^{(1)}_\zeta,\ldots,\tau^{(k)}_\zeta):\zeta \in A^{(k)})$. Then we define our iterations to be suborders of $Fn(\lambda^{++},2,\lambda^+)$ where at each coordinate $\zeta \in \lambda^{++} \sim A^{(n-1)}$ we simply add a new subset of $\lambda^+$ and for $\zeta \in A^{(n-1)}$ we add a club set $\subseteq \lambda^+$ which is disjoint from $F(\hat{\tau}^{(1)}_\zeta,\ldots,\hat{\tau}^{(n-1)}_\zeta)$ if certain "*killing conditions*" are met. If these conditions are not satisfied we save $F(\hat{\tau}^{(1)}_\zeta,\ldots,\hat{\tau}^{(n-1)}_\zeta)$; i.e., we force with the trivial poset $\{\emptyset\}$.

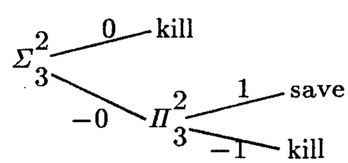
In order to describe these killing conditions we associate with each $n \geq 2$ a finite binary splitting graph for $\Sigma^2_n$ and a finite binary splitting graph for $\Pi^2_n$. The edges in these graphs will be labeled by integers. If some edge is labeled by say k $(0 < k \leq n - 2)$, this corresponds to the situation that in $V^{Q_\zeta}$ (where $Q_\zeta$ is the iteration restricted to coordinates $< \zeta$) $(\hat{\tau}^{(1)}_\zeta, ..., \hat{\tau}^{(k+1)}_\zeta) = (\hat{\tau}^{(1)}_\eta, ..., \hat{\tau}^{(k)}_\eta, G^\eta)$ for some $\eta \in A^{(k)} \bigcap \zeta$ where $G^\eta$ is the set that we add at coordinate $\eta$. The label $-k$ $(0 < k \leq n - 1)$ corresponds to the failure of this situation. The labels 0 and $-0$ express whether $\hat{\tau}^{(1)}_\zeta = G^o$ or not. Now in order to determine whether we kill or save at coordinate $\zeta$ (in $V^{Q_\zeta}$) we simply pick the path in the graph that corresponds to the various agreements and disagreements of $(\hat{\tau}^{(1)}_\zeta, ..., \hat{\tau}^{(n-1)}_\zeta)$ and check if it ends in a terminal mode that is labeled "kill" or "save." The graphs for $n = 2$ are very simple:

$\Pi^2_2$ (kill)

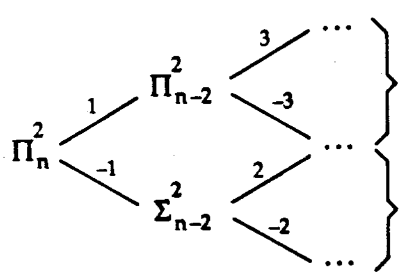



i.e., no edges. And for $n = 3$ we have



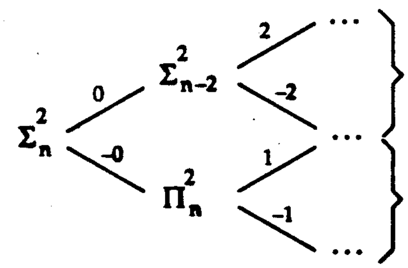



for n $\geq$ 3 we proceed by induction



graph for $\Pi^2_{n-2}$ where we add 2 to (subtract 2 from resp.) all positive labels including 0 (all negative labels including $-0$ resp.).

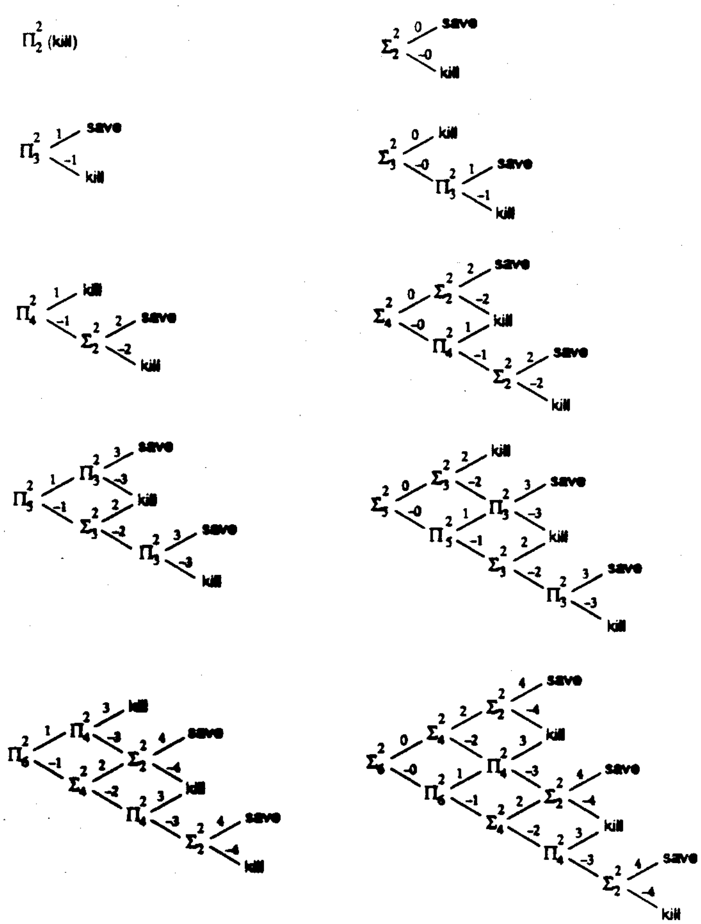
graph for $\Sigma^2_{n-2}$ with all labels changed as above

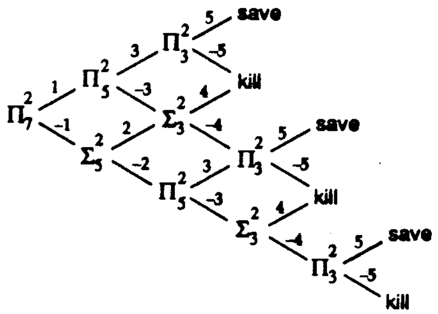


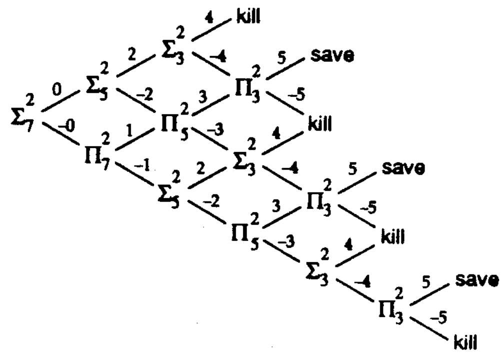
graph for $\Sigma^2_{n-2}$ with all labels changed as above

graph for $\Pi^2_n$

to illustrate what the graphs look like in practice we write them out up to n = 7.






As in the $\pi^2_3/\sigma^2_3$ case both $Q_{\Pi^2_n}$ and $Q_{\Sigma^2_n}$ are $<\lambda$ closed and $\lambda^{++}$ c.c. Moreover an argument analogous to the one in the $\pi^2_3/\sigma^2_3$ case shows that all these iterations are $<\lambda^+$ Baire.

Next we show that these iterations force indeed the $\Pi^2_n$ and $\Sigma^2_n$ statements about F that we want. Note that lemma 4.1.7. can be proved about $Q_{\Sigma^2_n}$ and $Q_{\Pi^2_n}$ by virtually the same idea.

Lemma 5.1.1. $Q_{\Pi^2_n}$ ($Q_{\Sigma^2_n}$ resp.) force the $\Pi^2_n$ ($\Sigma^2_n$ resp.) statements (over $V_\lambda$) that we want.

Proof. We have to inspect two cases.

Case 1. n is odd, say $n = 2\ell + 1$ ($\ell \geq 1$).

The $\Pi^2_{2\ell+1}$ statement says:

$$\forall X_1 \subseteq \lambda^+ \; \exists X_2 \subseteq \lambda^+ \cdots \exists X_{2\ell} \subseteq \lambda^+ \; F(X_1,\dots,X_{2\ell}) \text{ is stationary.}$$

From the inductive definition it follows that the top path in the graph for $\Pi^2_{2\ell+1}$ is labeled by the sequence 1, 2, ..., $2\ell - 1$ and ends in a save node.

Now assume that G is $Q_{\Pi^2_{2\ell+1}}$ generic over $V[G_\lambda, F]$ and work in $V[G_\lambda, F, G]$ if $X_1 \subseteq \lambda^+$ pick $\eta_1 \in A^{(1)}$ with $(\hat{\tau}^{(1)}_{\eta_1})^G = X_1$. Let $X_2 = G^\eta$ (the set that G adds at coordinate $\eta_1$). For $X_3 \subseteq \lambda^+$ now pick $\eta_3 \in A^{(3)}$ with $\left((\hat{\tau}^{(1)}_{\eta_3})^G, (\hat{\tau}^{(2)}_{\eta_3})^G, (\hat{\tau}^{(3)}_{\eta_3})^G\right) = (X_1, X_2, X_3)$ and let $X_4 = G^{\eta_3}$. Continue in this manner and define a tupel $(X_1, ..., X_{2\ell})$. Since the top path in the graph for $\Pi^2_{2\ell+1}$ is labeled 1, 3, ..., $2\ell - 1$ and ends in a save node, the analogue of lemma 4.1.7. for $Q_{\Pi^2_{2\ell+1}}$ yields that $F(X_1, ..., X_{2\ell})$ is stationary.

The $\Sigma^2_{2\ell+1}$ statement says:

$$\exists X_1 \subseteq \lambda^+ \ \forall X_2 \subseteq \lambda^+ \cdots \forall X_{2\ell} \subseteq \lambda^+ \ F(X_1, ..., X_{2\ell}) \text{ is not stationary.}$$

Again let G be $Q_{\Sigma^2_{2\ell+1}}$ generic over $V[G_\lambda, F]$ and work in $V[G_\lambda, F, G]$. From the inductive definitior it follows that the top path in the graph for $\Sigma^2_{2\ell+1}$ is labeled by the sequence 0, 2, ..., $2\ell - 2$ and ends in a killing node. Take $X_1 = G^0$. For $X_2 \subseteq \lambda^+$ pick $\eta_2 \in A^{(2)}$ $\left((\hat{\tau}^{(1)}_{\eta_2})^G, (\hat{\tau}^{(2)}_{\eta_2})^G\right) = (G^0, X_2)$ and let $X_3 = G^{\eta_2}$. Continue in this fashion and define a tupel $(X_1, ..., X_{2\ell-1})$. Now for $X_{2\ell} \subseteq \lambda^+$ find $\zeta \in A^{(2\ell)}$ with $\left((\hat{\tau}^{(1)}_\zeta)^G, ..., (\hat{\tau}^{(2\ell)}_\zeta)^G\right) = X_1, ..., X_{2\ell}$. Since the top path in the graph for $\Sigma^2_{2\ell+1}$ ends in a kill mode, we add the coordinate $\zeta$ a club set $\subseteq \lambda^+$ which is disjoint from $F(X_1, ..., X_{2\ell})$.

Case 2. n is even, say $n = 2\ell$ ($\ell \geq 1$).

Here a similar argument as in case 1 works. $\square$ end of 5.1.1.

Preservation of the $\Pi^2_n$ Indescribability of $\kappa$.

We assume towards a contradiction that there is a condition $p \in P^2_n$ and $\mathring{A} \in V^{P^2_n}$ and $\Phi$ in $\Pi^2_n$ with

$$p \Vdash_{P^2_n} \text{“}\Phi(\mathring{A}) \text{ describes } \kappa.\text{”}$$

Pick an ordinal $\delta >$ the least inaccessible above $\kappa$ such that

$$V_\delta \models ZF^- \wedge p \Vdash_{P^2_n} \text{“}\Phi(\mathring{A}) \text{ describes } \kappa.\text{”}$$

Pick an elementary embedding $i: M \longrightarrow V_\delta$ with cpt $i > \kappa$ and M trans, $|M| = \kappa$ and $M^{<\kappa} \subseteq M$. By the $\Pi^2_n$ indescribability of $\kappa$ in V there is trans N with $|N| = \kappa$, $N^\kappa \subseteq N$ and N is $\Sigma^2_{n-1}$ correct for $\kappa$ and an elementary embedding $j: M \longrightarrow N$ with cpt $j = \kappa$.

By the same argument as in the $\sigma^2_3/\pi^2_3$ case we can find a V generic ${}^V G$ for $P^2_n$ and an M generic ${}^M G$ for ${}^M P^2_n$ and that $p \in {}^V G$ and i lifts. Now we have to come up with an N generic ${}^N G$ for ${}^N P^2_n$ such that j lifts and such that $N[{}^N G]$ is still $\Sigma^2_{n-1}$ correct for $\kappa$ in $V[{}^V G]$.

The construction of ${}^N G$ is very similar to the construction in the $\sigma^2_3/\pi^2_3$ case. Only the part where we have to find a generic for stage $\kappa$ of ${}^N P^2_n$ from a generic for stage $\kappa$ of ${}^V P^2_n$ deserves a detailed exposition.

Clearly the forcing that $N[G_\kappa]$ wants to do in the first step of stage $\kappa$ of ${}^{N}P^2_n$ is the same as the one that $V[G_\kappa]$ wants to do at the first step of stage $\kappa$ of $P^2_n$, since $N[G_\kappa]$ is closed under $\kappa$ sequences inside $V[G_\kappa]$. Thus if $(F\gamma:\gamma < \kappa^+)$ are the $\kappa^+$ Lipshitz functions that we add at the first step of stage $\kappa$ in V, then with ${}^{N}F_\gamma \underset{\mathfrak{Df}}{=} F_{\Pi(\gamma)}$ $({}^{N}F_\gamma:\gamma < \kappa^+)$ is certainly generic over $N[G_\kappa]$ for the forcing that $N[G_\kappa]$ wants to do at the first step of stage $\kappa$ of $P^2_n$ and $N[G_\kappa, {}^{N}\vec{F}_\gamma]$ is $\Sigma^2_{n-1}$ correct for $\kappa$ and closed under $\kappa$ sequences in $V[G_\kappa, \vec{F}_\gamma]$.

Suppose that $(Q_\zeta:\zeta < \kappa^{++})$ denotes the $\kappa^{++}$ iteration that $V[G_\kappa, \vec{F}_\gamma]$ wants to use in order to make a $\Sigma^2_n$ ($\Pi^2_n$ resp.) statement true about $F_\gamma$ where $\gamma$ is even (odd resp.) and $({}^{N}Q:\zeta < \kappa^{++})$ is the $(\kappa^{++})^N$ iteration that $N[G_\kappa, {}^{N}\vec{F}_\gamma]$ wants to use for $({}^{N}F_\gamma:\gamma < \kappa^+)$. In the same way as in the $\sigma^2_3/\pi^2_3$ case we use the map $\Pi^*:\kappa^{++} \xrightarrow[\text{onto}]{1:1} \kappa^{++} \sim \text{Even}_{(\kappa^+)^M}$ to define a $\kappa^{++}$ iteration ${}^*Q$ in $V[G_\kappa, \vec{F}_\gamma]$ with the property that $\Pi^*$ induces an isomorphism of ${}^{N}Q$ with a complete suborder of ${}^*Q_{(\kappa^{++})^N}$. Recall that for critical $\gamma < \kappa^+$ ${}^*Q$ wants to make a $\Sigma^2_{n\text{-}1}$ statement true about $F_{\Pi(\gamma)} = {}^{N}F_\gamma$ and ${}^{N}Q$ wants to make $\Pi^2_{n-1}$ statement true about ${}^{N}F_\gamma$. However from the fact that $\text{rng}\ \Pi^* \cap \text{Even}_{(\kappa^+)^M} = \emptyset$, it follows that no term that appears in $Q^*$ at a coordinate $<(\kappa^{++})^N$ can possibly "see" the witness for the $\Sigma^2_{n-1}$ statement about $F_{\Pi(\gamma)}$ (for critical $\gamma$) that ${}^*Q$ adds at coordinate $\Pi(\gamma)$. Now the key observation is that the edge in the graph for $\Sigma^2_n$ which is labeled $-0$ leads into a subgraph which is identical with the graph for $\Pi^2_n$. Therefore, if G denotes the $V[G_\kappa, \vec{F}_\gamma]$ generic for Q and $G^*$, the generic for $^*Q$ that is obtained from G (by applying the fact that Q and $^*Q$ are isomorphic since the analogue of 4.2.7. clearly holds in the $\sigma^2_n/\pi^2_n$ case), and if $g^*$ is the restriction of $G^*$ to the complete suborder of $^*Q_{(\kappa^{++})^N}$ that is isomorphic to $^NQ$, then g (the pullback of $g^*$ via the isomorphism induced by $\Pi^*$) is clearly $N[G_\kappa, {}^N\vec{F}_\gamma]$ generic for $^NQ$. Now we claim

(5.2.1.) $N[G_\kappa, {}^N\vec{F}_\gamma, g]$ is $\Sigma^2_{n-1}$ correct for $\kappa$ in $V[G_\kappa, \vec{F}_\gamma, G]$.

Once we have proved this we can construct the rest of $^NG$ and carry out all the remaining correctness arguments exactly as in the $\sigma^2_3/\pi^2_3$ case.

<u>The Proof of (5.2.1.)</u>.

We start out exactly as in the $\sigma^2_3/\pi^2_3$ case and let $\Phi(A)$ be a formula in $\Sigma^2_{n-1} \cup \Pi^2_{n-1}$ and $A \in (N[G_\kappa, {}^N\vec{F}_\gamma, g])_{\kappa+2}$ and assume that

$$N[G_\kappa, {}^N\vec{F}_\gamma, g] \models \text{“}V_\kappa \models \Phi(A).\text{”}$$

Pick an ordinal $\delta < \kappa^{++}$ and a nice name $\mathring{A} \in (N[G_\kappa, \vec{F}_\gamma])^{Q_\delta}$ with $\mathring{A}^g = A$ and a condition $q \in g \cap Q_\delta$ such that

$$q \Vdash^{N[G_\kappa, {}^N\vec{F}_\gamma]}_{{}^NQ} \quad \text{“}V_\kappa \models \Phi(\mathring{A}).\text{”}$$

Clearly the analogues of the factor lemmas 4.2.12. and (4.2.14.) can also be established for the $\sigma^2_n/\Pi^2_n$ case by using the exact same arguments as in the $\sigma^2_3/\pi^2_3$ case and we have already remarked that the analogue of the isomorphism lemma 4.2.7. is also true in the $\sigma^2_n/\pi^2_n$ case. Hence we can apply the analogue of lemma 4.2.16. and obtain that in $V^* \underset{\mathfrak{D}f}{=} V[G_\kappa, \vec{F}_\gamma, g] = V[G_\kappa, {}^N\vec{F}_\gamma, g^*]$:

for all modified $\delta$, $\kappa^{++}$ iterations that use ${}^N\vec{F}_\gamma$

$\Vdash V_\kappa \models \Phi(A)$.

In order to finish the proof it certainly suffices to show:

for all special modified $\delta$, $\kappa^{++}$ iterations which use $\vec{F}_\gamma$

$\Vdash$ "$V_\kappa \models \Phi(A)$."

It goes without saying that modified $\delta$, $\kappa^{++}$ iterations and special modified $\delta$, $\kappa^{++}$ iterations in $V^*$ are defined totally analogous as in the $\sigma^2_3/\pi^2_3$ case. Moreover, modified iterations always use $({}^N F_\gamma : \gamma < \kappa^+)$ and special modified iterations use $(F_\gamma : \gamma < \kappa^+)$. Recall that for "many" $\gamma < \kappa^+$ modified and special modified iterations, both make a $\Sigma^2_n$ or a $\Pi^2_n$ statement true about ${}^N F_\gamma = F_{\Pi(\gamma)}$. However for the critical $\gamma$ (i.e., $\gamma$ odd and $\pi(\gamma) < (\kappa^+)^M$ even) the modified iteration makes a $\Pi^2_n$ statement true about ${}^N F_\gamma = F_{\Pi(\gamma)}$ and the special modified iteration makes a $\Sigma^2_n$ statement true about $F_{\Pi(\gamma)}$. Thus what is really going on here is the following:

We are working in a model (call it simply V) with a Lipshitz function $F:(2^{\gamma^+})^{n-1} \to 2^{\gamma^+}$. $\Phi(A)$ is a formula in $\Pi^2_{n-1} \cup \Sigma^2_{n-1}$ and $A \in V_{\kappa+2}$ such that for all $\Pi^2_n$ iterations $Q^\Pi$.

(5.2.2.) $\Vdash_{Q^\Pi}$ "$V_\kappa \models \Phi(A)$"

and we want to see that for all $\Sigma^2_n$ iterations $Q^\Sigma$

(5.2.3.) $\Vdash_{Q^{\Sigma}}$ "$V_\kappa \models \Phi(A)$."

In order to show this we use the (n − 1) *back and forth property of* $\Sigma^2_n/\Pi^2_n$. By this we mean the following:

For a given initial piece $Q^{\Pi}_{\delta_1}$ of a $\Pi^2_n$ iteration it is possible to define an initial piece $Q^{\Sigma}_{1+\delta_1}$ of a $\Sigma^2_n$ iteration and a complete embedding $i_1$ that is an isomorphism of $Q^{\Pi}_{\delta_1}$ with a complete suborder of $Q^{\Sigma}_{1+\delta_1}$. If we enlarge $Q^{\Sigma}_{1+\delta_1}$ to say $Q^{\Sigma}_{1+\delta_1+\delta_2}$ (where $\delta_2 < \kappa^{++}$), then we can define an extension $Q^{\Pi}_{\delta_1+2+\delta_2}$ of $Q^{\Pi}_{\delta_1}$ and a complete embedding $i_2$ that is an isomorphism of $Q^{\Sigma}_{1+\delta_1+\delta_2}$ with a complete suborder of $Q^{\Pi}_{\delta_1+2+\delta_2}$ with the property that $i_2 \circ i_1 = id_{Q^{\Pi}_{\delta_1}}$.

This process can be repeated until we have defined embeddings $i_1, i_2, ..., i_{n-1}$ and an initial segment $Q^{\Pi}$ of a $\Pi^2_n$ iteration and an initial $Q^{\Sigma}$ of a $\Sigma^2_n$ iteration such that $i_1, ..., i_{n-1}$ allow us to go back and forth n − 1 times between these two iterations in the way described above; i.e., $i_{k+1} \circ i_k = id$ for $1 \leq k \leq n-2$.

Moreover we can go through a similar procedure if we start with an initial segment of a $\Sigma^2_n$ iteration.

<u>Proof of the</u> (n − 1) <u>Back and Fourth Property for</u> $\Sigma^2_n/\Pi^2_n$ $(n \geq 1)$.

We first treat the case $\Sigma^2_{2n+1}/\Pi^2_{2n+1}$ $(n \geq 1)$. Fix $n \geq 1$. Suppose we start out with an initial segment ${}^{\Sigma}Q_{\delta_1}$ (where $\delta_1 < \kappa^{++}$) for a $\Sigma^2_{2n+1}$ iteration. Denote $[0,\delta_1)$

by $I_{1,\Sigma}$ and $[0, 1 + \delta_1)$ by $I_{1,\Pi}$ and define $i_1 : I_{1,\Sigma} \to I_{1,\Pi}$ by $i_1(\zeta) = 1 + \zeta$ for $\zeta < \delta_1$. Note that $0 \notin \text{rng } i_1$ and for reasons that become apparent below we call 0 *the new coordinate in* $I_{1,\Pi}$. Now define a $\Pi^2_{2n+1}$ iteration ${}^{\Pi}Q_{1+\delta_1}$ whose underlying partition of $I_{1,\Pi}$ is the partition induced by $i_1$ and where we do not associate a tupel of terms with 0. If no tupel of terms is assigned to coordinate $\zeta \in I_{1,\Sigma}$, then this is also true for $1 + \zeta \in I_{1,\Pi}$. If a k-tupel (with $k \neq 1$) of terms appears at coordinate $\zeta \in I_{1,\Sigma}$, then to coordinate $1 + \zeta \in I_{1,\Pi}$ we assign the tupel of terms each of which is the $i_1$-shift of the corresponding term in the tupel that appears at $\zeta \in I_{1,\Sigma}$. If a single term $\tau_\zeta$ is assigned to coordinate $\zeta \in I_{1,\Sigma}$, then we pick a canonical term $\tau^* \in V^{{}^{\Pi}Q_{1+\zeta}}$ such that, in $V^{{}^{\Pi}Q_{1+\zeta}}$

$$\tau^* = \begin{cases} \text{the set that we add at the new coordinate in } I_{1,\Pi} \text{ if certain} \\ \textit{altering coordinates} \text{ hold in } V^{{}^{\Sigma}Q_\zeta} \text{ about } \hat{\tau}_\zeta \\ (\hat{\tau}_\zeta)^{i_1} \text{ otherwise.} \end{cases}$$

We will explain in a moment what these *altering conditions* are and we are also going to show that in fact for each $\zeta \in I_{1,\Sigma}$

- $i_1$ induces an isomorphism of ${}^{\Sigma}Q_\zeta$ with ${}^{\Pi}Q^{1+\zeta\sim\{0\}}_{1+\zeta}$
  which is a complete suborder of ${}^{\Pi}Q_{1+\zeta}$ .

This shows in particular that $V^{{}^{\Sigma}Q_\zeta}$ can be regarded as being contained in $V^{{}^{\Pi}Q_{1+\zeta}}$ and the definition of $\tau^*$ makes sense.

In the second step of the construction for $\Sigma^2_{2n+1}/\Pi^2_{2n+1}$ assume we extend ${}^{\Pi}Q_{1+\delta_1}$ to say ${}^{\Pi}Q_{1+\delta_1+\delta_2}$ where $\delta_2 < \kappa^{++}$. Let us denote $[\,1 + \delta_1, 1 + \delta_1 + \delta_2)$ by $I_{2,\Pi}$ and $[\delta_1, \delta_1 + 2 + \delta_2)$ by $I_{2,\Sigma}$ and define $i_2 : I_{1,\Pi} \cup I_{2,\Pi} \to I_{1,\Sigma} \cup I_{2,\Sigma}$ by

$$i_2(0) = \delta_1$$

$$i_2(1 + \zeta) = \zeta \quad (0 \le \zeta < \delta_1)$$

$$i_2\,(1 + \delta_1 + \zeta) = \delta_1 + 2 + \zeta \quad (0 \le \zeta < \delta_2)$$

so that in particular $i_2 \circ i_1 = id_{I_{1,\Sigma}}$ and $\delta_1 + 1 \notin \text{rng}\, i_2$.

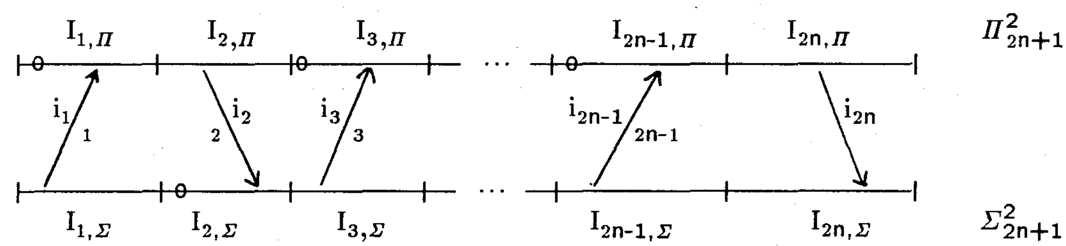
Now we define an extension ${}^{\Sigma}Q_{\delta_1+2+\delta_2}$ by choosing the partition of $I_{2,\Sigma}$ that is induced by $i_2$. We assign no tupel of terms to coordinate $\delta_1 + 1$ which we call the new coordinate in $I_{2,\Sigma}$. If a k tupel $(k \neq 2)$ is assigned to coordinate $\zeta \in I_{2,\Pi}$, then the k-tupel that consists of the $i_2$ shifts of the terms in this tupel will be assigned to coordinate $i_2(\zeta) \in I_{1,\Pi}$. If a pair $(\hat{\tau}^{(1)}_{\zeta}, \hat{\tau}^{(2)}_{\zeta})$ is associated with a coordinate $\zeta \in I_{2,\Pi}$, then we associate with $i_2(\zeta) \in I_{2,\Sigma}$ the pair $(\tau^*, (\hat{\tau}^{(2)}_{\zeta})^{i_2})$ where $\tau^* \in V^{{}^{\Sigma}Q_{i_2(\zeta)}}$ is a canonical term such that in $V^{{}^{\Sigma}Q_{i_2(\zeta)}}$

$$\tau^* = \begin{cases} \text{the set that we add at coordinate } \delta_1 + 1 \text{ if certain altering} \\ \text{conditions hold in } V^{{}^{\Pi}Q_{\zeta}} \text{ about } \hat{\tau}^{(1)}_{\zeta} \\ (\hat{\tau}^{(1)}_{\zeta})^{i_2} \quad \text{otherwise.} \end{cases}$$

Again we have to check that for $\zeta \in I_{2,\Pi}$, $i_2$ induces an isomorphism of ${}^{\Pi}Q_{\zeta}$ with ${}^{\Sigma}Q_{i_2(\zeta)}^{i_2(\zeta)-\{\delta_1+1\}}$ which is a complete suborder of ${}^{\Sigma}Q_{i_2(\zeta)}$.

We continue in this fashion until we have defined $I_{1,\Sigma}$, ..., $I_{2n-1,\Sigma}$ and $I_{1,\Pi}$, ..., $I_{2n-1,\Pi}$ and embeddings $i_1$, ..., $i_{2n-1}$. In the last step of the construction we will not introduce a new coordinate to define $I_{2n,\Sigma}$ from $I_{2n,\Pi}$ and we shift all terms at coordinates in $I_{2n,\Pi}$ to get the terms for the corresponding coordinates in $I_{2n,\Sigma}$.

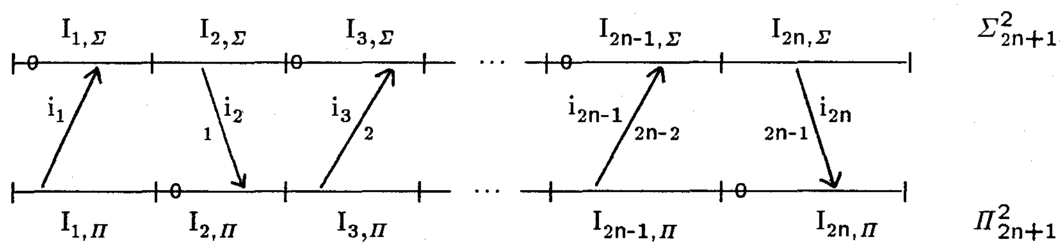
The schematic picture that one should have in mind when doing this construction looks like this:




The numbers below the arrows indicate the arity of the tupels whose first term gets changed at this stage of the construction. The symbol o indicates that we have a new coordinate in the interval where it occurs.

We are now going to explain what the altering conditions are. Suppose we are at stage $k \leq 2n-1$ of the construction above and k is odd (even resp.) and the k-tupel $(\tau_{\zeta}^{(1)}, ..., \tau_{\zeta}^{(k)})$ is assigned to coordinate $\zeta \in I_{k,\Sigma}$ ($\zeta \in I_{k,\Pi}$ resp.). Consider the graph for $\Sigma_{k+2}^2$ ($\Pi_{k+2}^2$ resp.). Among all its killing paths consider the *direct killing paths*, i.e., those killing paths whose last edge is labeled with a non-negative integer. Now the altering condition for $\hat{\tau}_\zeta^{(1)}$ is satisfied if one of the agreement scenarios prescribed by the direct killing paths hold about $(\hat{\tau}_\zeta^{(1)},...,\hat{\tau}_\zeta^{(k)})$ in $V^{{}^\Sigma Q_\zeta}$ (in $V^{{}^\Pi Q_\zeta}$ resp.).

Before we argue that the construction that we just described really works, let us explain what the constructions look like in the other cases: If we start with an initial segment ${}^\Pi Q_{\delta_1}$ of a $\Pi^2_{2n+1}$ iteration where $\delta_1 < \kappa^{++}$, we can define an initial segment ${}^\Sigma Q_{1+\delta_1}$ of a $\Sigma^2_{2n+1}$ iteration where all terms are obtained by shifting the terms from ${}^\Pi Q_{\zeta_1}$ so that they cannot see the $\Sigma$ witness that we add at coordinate 0 of ${}^\Sigma Q_{1+\delta_1}$. In steps 2, 3, ..., 2n we proceed analogously as in the steps 1, 2, ..., 2n−1 of the construction $\Pi^2_{2n+1}/\Sigma^2_{2n+1}$ where we start with an initial piece of a $\Sigma^2_{2n+1}$ iteration. Thus the schematic picture looks like this:




Again the numbers below the arrows indicate the arity of the tupels whose first term gets changed if the altering condition is met where now the altering condition for even (odd resp.) stage k with k ∈ {2, ..., 2n} is given by the direct killing paths in the graph for $\Sigma^2_{k+1}$ ($\Pi^2_{k+1}$ resp.). As before the symbols ∘ indicate that there is a new coordinate in the interval where they appear.

The constructions for the $\Pi^2_{2n}/\Sigma^2_{2n}$ case ($n \geq 1$) are totally analogous to the corresponding ones in the $\Pi^2_{2n+1}/\Sigma^2_{2n+1}$ case.

Now we have to check that these constructions really work. Let us examine the $\Sigma^2_{2n+1}/\Pi^2_{2n+1}$ case where we start with an initial piece of a $\Sigma^2_{2n+1}$ iteration. Suppose we are at step $k \leq 2n-1$ of the construction with k being odd and $\zeta \in I_{k,\Sigma}$. For the construction to work we have to show

(5.2.4.) $\quad {}^{\Pi}Q_{i_k(\zeta)} \restriction I_{1,\Pi} \cup \cdots \cup I_{k,\Pi} \sim \{\text{new coordinate} \in I_{k,\Pi}\} \subseteq_c {}^{\Pi}Q_{i_k(\zeta)}$

(5.2.5.) $\quad i_k$ induces an isomorphism of ${}^{\Sigma}Q_{\zeta}$ with

$$ {}^{\Pi}Q_{i_k(\zeta)} \restriction I_{1,\Pi} \cup \cdots \cup I_{k,\Pi} \sim \{\text{new coordinate} \in I_{k,\Pi}\}$$

where ${}^{\Pi}Q_{i_k(\zeta)}$ (${}^{\Sigma}Q_{\zeta}$ resp.) denotes the $\Pi^2_{2n+1}$ ($\Sigma^2_{2n+1}$ resp.) iteration up to coordinate $i_k(\zeta) \in I_{k,\Pi}$ ($\zeta \in I_{k,\Sigma}$ resp.).

We show this by induction on $\zeta \in I_{k,\Sigma}$. Clearly we can restrict ourselves to a successor ordinal $\zeta + 1 \in I_{k,\Sigma}$ where at coordinate $\zeta$ we possibly add a set that kills $F(\hat{\tau}^{(1)}_{\zeta},...,\hat{\tau}^{(2n)}_{\zeta})$. First we handle (5.2.5.). Note that $V^{{}^{\Sigma}Q_{\zeta}}$ can be thought of as being contained in $V^{{}^{\Pi}Q_{i_k(\zeta)}}$ in the sense that by induction hypothesis ${}^{\Sigma}Q$ is isomorphic to a complete suborder of ${}^{\Pi}Q_{i_{k(\zeta)}}$. In order to prove (5.2.5.) it will therefore suffice to show the following *two central facts.*

(5.2.6.) We cannot be on a killing path for $\Pi^2_{2n+1}$ in $V^{{}^{\Pi}Q_{i_k(\zeta)}}$ and on a saving path for $\Sigma^2_{2n+1}$ in $V^{{}^{\Sigma}Q_\zeta}$

and

(5.2.7.) We cannot be on a killing path for $\Sigma^2_{2n+1}$ in $V^{{}^{\Sigma}Q_\zeta}$ and on a saving path for $\Pi^2_{2n+1}$ in $V^{{}^{\Pi}Q_{i_k(\zeta)}}$.

Once these two central facts have been proven we consider the first claim (5.2.4.): For this it suffices to show that if $q \in {}^{\Pi}Q_{i_k(\zeta+1)}$, then $q'$ which is obtained from q by dropping the new coordinate in $I_{k,\Pi}$ is again a condition in ${}^{\Pi}Q_{i_k(\zeta+1)}$. So assume $q \in {}^{\Pi}Q_{i_k(\zeta+1)}$. There is a unique $q'' \in Fn(\zeta+1,2,\kappa^+)$ with $(q'')^{i_k} = q'$. If we can show that $q'' \in {}^{\Sigma}Q_{\zeta+1}$, then we are done since we have already shown (5.2.5.). But $q'' \in {}^{\Sigma}Q_{\zeta+1}$ follows easily from the first central fact.

If we are at step $k \leq 2n-2$ of this construction where k is even, we proceed in an analogous manner.

Finally at the last stage of this construction we do not have a new coordinate in $I_{2n,\Sigma}$ and all terms get shifted in $i_{2n}$ where we go from $I_{2n,\Pi}$ to $I_{2n,\Sigma}$ We have to show that

$i_{2n}$ induces a complete embedding of ${}^{\Pi}Q_\zeta$ into ${}^{\Sigma}Q_{i_{2n}(\zeta)}$ (for $\zeta \in I_{2n,\Pi}$).

In order to prove this we use the first of the two central facts and obtain $i_{2n}(q)$ is a condition in ${}^{\Sigma}Q_{i_{2n}(\zeta)}$ for all $q \in {}^{\Pi}Q_\zeta$. In order to show that each $q \in {}^{\Sigma}Q_{i_{2n}(\zeta)}$ has a

$i_{2n}$-reduction in ${}^{\Pi}Q_{\zeta}$, we proceed by the method used in the proof of 4.1.3.

In order to show that the construction for $\Pi^2_{2n+1}/\Sigma^2_{2n+1}$, starting with an initial piece of a $\Pi^2_{2n+1}$ iteration, and the constructions for $\Pi^2_{2n}/\Sigma^2_{2n}$ work we have only to prove the analogues of the two central facts (5.2.6.) and (5.2.7.).

<u>Proof of the Two Central Facts That Establish the</u> n − 1 <u>Back and Forth Property for</u> $\Sigma^2_n/\Pi^2_n$ ($n \geq 2$).

We prove by induction on $n \geq 2$ that the two central facts hold throughout the constructions for $\Sigma^2_n/\Pi^2_n$. The key ingredient here is the inductive definition of the graphs for $\Sigma^2_n$ and $\Pi^2_n$ which will be used all over the proof.

We begin the induction by examining the two basic cases $\Sigma^2_2/\Pi^2_2$ and $\Sigma^2_3/\Pi^2_3$. If we are given an initial piece of a $\Pi^2_2$ iteration ${}^{\Pi}Q_{\delta_1}$, then we can define an initial piece of $\Sigma^2_2$ iteration ${}^{\Sigma}Q_{1+\delta_1}$ by shifting all the terms in ${}^{\Pi}Q_{\delta_1}$ so that they cannot see the $\Sigma^2_2$ witness that we add a coordinate 0 in ${}^{\Sigma}Q_{1+\delta_1}$. Clearly the two central facts hold in this case. If we start with an initial piece of a $\Sigma^2_2$ iteration ${}^{\Sigma}Q_{\delta_1}$ and define an initial piece of a $\Pi^2_2$ iteration ${}^{\Pi}Q_{\delta_1}$ by using the same parameters, then clearly ${}^{\Sigma}Q_{\delta_1} \subseteq {}^{\Pi}Q_{\delta_1}$ since the graph for $\Pi^2_2$ has no saving paths at all.

Now we examine the case where we start with an initial segment of a $\Pi^2_3$

iteration ${}^{\Pi}Q_{\delta_1}$. In the first step we define an initial piece of a $\Sigma_3^2$ iteration ${}^{\Sigma}Q_{1+\delta_1}$ by shifting all the terms from ${}^{\Pi}Q_{\delta_1}$ so they cannot see the $\Sigma_3^2$ witness at coordinate 0 in ${}^{\Sigma}Q_{1+\delta_1}$. The two central facts clearly hold at this step. In the second step suppose we are at coordinate $\zeta \in I_{2,\Sigma}$ and on a killing path for $\Sigma_3^2$ in $V^{{}^{\Sigma}Q_\zeta}$ so that we have either 0, or $-0$, $-1$. If we have 0 in $V^{{}^{\Sigma}Q_\zeta}$ then there cannot be a 1 in $V^{{}^{\Pi}Q_{i_2(\zeta)}}$ with a term that appears at a coordinate $\in I_{1,\Pi}$ since none of these terms can see the $\Sigma_3^2$ witness that we add at the new coordinate $0 \in I_{1,\Sigma}$. Furthermore there cannot be a 1 in $V^{{}^{\Pi}Q_{i_2(\zeta)}}$ with a term that appears at a coordinate $\in I_{2,\Pi}$ since when going from $I_{2,\Sigma}$ to $I_{2,\Pi}$ we change all 1's in $I_{2,\Sigma}$ because we have 0 in $V^{{}^{\Sigma}Q_\zeta}$. Thus we have $-1$ in $V^{{}^{\Pi}Q_{i_2(\zeta)}}$; i.e., we kill in $V^{{}^{\Pi}Q_{i_2(\zeta)}}$. If we have $-0$, $-1$ in $V^{{}^{\Sigma}Q_\zeta}$ then we clearly must have $-1$ in $V^{{}^{\Pi}Q_{i_2(\zeta)}}$; i.e., again we kill in $V^{{}^{\Pi}Q_{i_2(\zeta)}}$.

Now suppose we start with an initial piece of a $\Sigma_3^2$ iteration ${}^{\Sigma}Q_{\delta_1}$. We can argue similarly as we did in the second step of the last case to see that if $\zeta \in I_{1,\Sigma}$ and we kill at $\zeta$ in $V^{{}^{\Sigma}Q_\zeta}$, then we also kill at $i_1(\zeta) \in I_{1,\Pi}$ in $V^{{}^{\Pi}Q_{i_1(\zeta)}}$. Conversely if we kill at $i_1(\zeta)$ in $V^{{}^{\Pi}Q_{i_1(\zeta)}}$, i.e., we have $-1$, then in $V^{{}^{\Sigma}Q_\zeta}$ $-$ 0 clearly implies $-1$; i.e., we kill at $\zeta$ in $V^{{}^{\Sigma}Q_\zeta}$. In the second step a similar argument shows that if we are on a killing path for $\Pi^2_3$ at a coordinate $\zeta \in I_{2,\Pi}$, then we are also on a killing path in $\Sigma_{Q_{i_2(\zeta)}}$.

Now suppose $n \geq 4$ and we have already proven the two central facts for all constructions for $\Sigma^2_k/\Pi^2_k$ with $k < n$. We will restrict ourselves to looking at an odd $2n + 1$ ($n \geq 2$) and consider the case of $\Sigma^2_{2n+1}/\Pi^2_{2n+1}$ where we start with an initial piece of a $\Pi^2_{2n+1}$ iteration. The arguments for the other cases are similar to the argument that we present here in detail.

First we want to argue that the two central facts are satisfied through the first $2n - 2$ stages of the construction. For this we make the following observation: The subgraph of the graph for $\Sigma^2_{2n+1}$ ($\Pi^2_{2n+1}$ resp.) which consists of all edges that are labeled by an integer of absolute value $\leq 2n - 3$ is identical with the graph for $\Sigma^2_{2n-1}$ ($\Pi^2_{2n-1}$ resp.) except that all nodes in the graph $\Sigma^2_{2n-1}$($\Pi^2_{2n-1}$ resp.) of the form $Q^2_k$ ($Q \in \{\forall,\exists\}$) have to be changed to $Q^2_{k+2}$ and we must replace each save node by the graph for $\Pi^2_3$ where the labels $1$, $-1$ get replaced by $2n-1$ and $-(2n-1)$ resp. and each kill node must be replaced by the graph for $\Sigma^2_3$ where the labels $0$, $-0$, $1$, $-1$ get replaced by $2n - 2$, $-(2n-2)$, $2n - 1$, $-(2n-1)$ resp. If we now apply the induction hypothesis about $\Sigma^2_{2n-1}/\Pi^2_{2n-1}$ together with this observation, then we see that up to the first $2n - 2$ stages of the construction for $\Sigma^2_{2n+1}/\Pi^2_{2n+1}$ we have the following:

Any time we are on a path in the subgraph of the graph for $\Pi^2_{2n+1}$ mentioned above that ends in "$\Sigma^2_3$" we cannot be on a path in the subgraph of the graph for $\Sigma^2_{2n+1}$ that ends in "$\Pi^2_3$" and similarly if we interchange $\Pi^2_{2n+1}$ and $\Sigma^2_{2n+1}$.

Then note that throughout the first 2n − 2 stages of the construction for $\Sigma^2_{2n+1}/\Pi^2_{2n+1}$ all the terms in 2n − 2 and 2n − 1 tupels merely get shifted. Moreover a $\Sigma^2_3$ iteration clearly does more killing than a $\Pi^2_3$ iteration. Hence the two central facts hold throughout the first 2n − 2 stages of the construction for $\Sigma^2_{2n+1}/\Pi^2_{2n+1}$.

Now we consider the last two stages of the construction for $\Sigma^2_{2n+1}/\Pi^2_{2n+1}$ where we start with an initial piece of a $\Pi^2_{2n+1}$ iteration. First we show that if we kill on the $\Pi^2_{2n+1}$ side, we cannot save on the $\Sigma^2_{2n+1}$ side. Inspection of the graph for $\Pi^2_{2n+1}$ tells us that there are two cases for killing paths:

*indirect killing*; i.e., the last edge in the path is labeled −(2n−1) and

*direct killing*; i.e., the last edge in the path is labeled 2n − 2.

On the other hand there are two ways of saving in the graph for $\Sigma^2_{2n+1}$:

*indirect saving;, i.e., the last two edges of the path are labeled* −(2n−2), *2n* − 1 *and*

*direct saving*; i.e., the last two edges of the path are labeled 2n − 3, 2n − 1.

It can never happen that we are on an indirect killing path for $\Pi^2_{2n+1}$ (i.e., −(2n−1)) and on a saving path for $\Sigma^2_{2n+1}$ (i.e., 2n − 1) since at stage 2n of the construction for $\Sigma^2_{2n+1}/\Pi^2_{2n+1}$ starting with an initial piece of a $\Pi^2_{2n+1}$ iteration, a 2n − 1 tupel gets altered only when we are on a direct killing path for $\Sigma^2_{2n+1}$.

It cannot happen that we are on a direct killing path for $\Pi^2_{2n+1}$ (i.e., ends in 2n − 2) and on an indirect saving path for $\Sigma^2_{2n+1}$ (i.e., the next-to-last edge is labeled −(2n−2)). This is so because at stage 2n − 1 of the construction for $\Sigma^2_{2n+1}/\Pi^2_{2n+1}$ (starting with an initial piece of a $\Pi^2_{2n+1}$ iteration) a 2n − 2 tupel gets altered only if we are on a direct killing paths in the graph of $\Pi^2_{2n}$. However the direct killing paths in the graph for $\Pi^2_{2n}$ can all be extended to save paths or indirect killing paths in the graph for $\Pi^2_{2n+1}$.

Finally we consider direct killing in $\Pi^2_{2n+1}$ and direct saving in $\Sigma^2_{2n+1}$. We observe that the direct killing paths in $\Pi^2_{2n+1}$ are just all the killing paths in $\Pi^2_{2n-1}$ extended by one edge which is labeled 2n − 2 and all the direct saving paths in $\Sigma^2_{2n+1}$ are just all saving paths in $\Sigma^2_{2n-1}$ extended by one edge labeled 2n − 1. Now we can apply our induction hypothesis about the $\Sigma^2_{2n-1}/\Pi^2_{2n-1}$ construction and hence this constellation can never arise.

Next we show that in the last two stages of the construction $\Sigma^2_{2n+1}/\Pi^2_{2n+1}$ (starting with an initial piece of $\Pi^2_{2n+1}$ iteration) we cannot kill on the $\Sigma^2_{2n+1}$ side and save on the $\Pi^2_{2n+1}$ side.

We have to check three cases here:

*Indirect killing path for* $\Sigma^2_{2n+1}$ (i.e., ending in $-(2n-1)$) versus *saving path in* $\Pi^2_{2n+1}$ (always ending in $2n - 1$) cannot occur since the first term in a $(2n - 1)$-tupel that gets altered at stage 2n of the construction for $\Sigma^2_{2n+1}/\Pi^2_{2n+1}$ (starting with an initial piece of a $\Pi^2_{2n+1}$ iteration) will then denote the set at the new coordinate $\in I_{2n,\Pi}$ . But no term at a coordinage $\in I_{1,\Pi} \cup \cdots \cup I_{2n,\Pi}$ can "see" this set.

Next we consider a *direct killing path in* $\Sigma^2_{2n+1}$ (i.e., ending in $2n - 2$) versus an *indirect saving path in* $\Pi^2_{2n+1}$ (i.e., ending in $-(2n-2)$, $2n - 1$). In this situation the $2n - 2$ agreement in $\Sigma^2_{2n+1}$ had to occur with a $(2n - 2)$-tupel whose first term denotes the set that we add at the new coordinate $I_{2n-1,\Sigma}$. Therefore the $2n - 1$ agreement in $\Pi^2_{2n+1}$ had to occur at a coordinate $\in I_{2n,\Pi}$. Then this must come from a $2n - 1$ agreement in $\Sigma^2_{2n+1}$ at a coordinage $\in I_{2n,\Sigma}$. However we assumed we were on a direct killing path in $\Sigma^2_{2n+1}$. Thus any such $2n - 1$ agreement would get destroyed at stage 2n of the construction for $\Sigma^2_{2n+1}/\Pi^2_{2n+1}$ when going from $I_{2n,\Sigma}$ to $I_{2n,\Pi}$ - a contradiction.

Finally we consider the case of a *direct killing path in* $\Sigma^2_{2n+1}$ (ending in $2n - 2$)

versus a *direct saving path in* $\Pi^2_{2n+1}$ (i.e., ending in 2n − 3, 2n − 1). We prove the following:

<u>Claim</u>: If we are on a direct killing path for $\Sigma^2_{2n+1}$ and on a direct saving path for $\Pi^2_{2n+1}$, then there cannot be a 2n − 1 agreement on the $\Pi^2_{2n+1}$ side with a 2n − 1 tupel that appears at a coordinate $\in I_{1,\Pi} \cup \cdots \cup I_{2n,\Pi}$.

The claim will give a contradiction to the fact that all direct saving paths for $\Pi^2_{2n+1}$ end in an edge that is labeled 2n − 1.

<u>Proof of the claim</u>: First we note that the (2n − 2) agreement on the $\Sigma^2_{2n+1}$ side cannot occur with a (2n − 2) tupel that appears at a coordinate $\in I_{2n,\Sigma}$ because in that case any (2n − 1) agreement on the $\Sigma^2_{2n+1}$ side had to occur at a coordinate $\in I_{2n,\Sigma}$. Since we are on a direct killing path for $\Sigma^2_{2n+1}$, the first term in any 2n − 1 tupel that gives a 2n − 1 agreement on the $\Sigma^2_{2n+1}$ side will be altered when going from $I_{2n,\Sigma}$ to $I_{2n,\Pi}$. This will result in −(2n − 1) on the $\Pi^2_{2n+1}$ side - a contradiction.

Now there are 2 possibilities for a direct killing path in $\Sigma^2_{2n+1}$. If its next-to-the-last edge is labeled −(2n−3), then the first term in any 2n − 3 agreement on the $\Pi^2_{2n+1}$ side has to agree with the set that we add at new coordinate $\in I_{2n-2,\Pi}$. From this it follows that any 2n − 2 agreement on the $\Sigma^2_{2n+1}$ side has to occur at a coordinate $\in I_{2n-1,\Sigma} \cup I_{2n,\Sigma}$ because none of the terms appearing at coordinates $\in I_{1,\Sigma} \cup \cdots \cup I_{2n-2,\Sigma}$ can see the set that we add at the new coordinate $I_{2n-2,\Pi}$. By the remark at the beginning of the proof of the claim, the 2n − 2 agreement on the

$\Sigma^2_{2n+1}$ side must therefore occur at a coordinate $\in I_{2n-1,\Sigma}$. Now we observe that the direct killing paths in $\Pi^2_{2n}$ are labeled exactly as the direct saving paths in $\Pi^2_{2n+1}$ if we delete their last edge. Therefore the first term in any $2n-2$ tupel that gives a $2n-2$ agreement on the $\Sigma^2_{2n+1}$ side has to agree with the set that we add at the new coordinate $\in I_{2n-1,\Sigma}$. This implies that any $2n-1$ agreement on the $\Sigma^2_{2n+1}$ side has to occur at a coordinate $\in I_{2n,\Sigma}$. But then we will end up with no $2n-1$ agreement on the $\Pi^2_{2n+1}$ since in the 2n-th step of the construction for $\Sigma^2_{2n+1}/\Pi^2_{2n+1}$ (starting with an initial piece of $\Pi^2_{2n+1}$ iteration) all these $2n-1$ agreements vanish - a contradiction.

So we have shown that the direct killing path in $\Sigma^2_{2n+1}$ cannot end with $-(2n-3)$, $2n-2$. Thus it must end in $2n-4$, $2n-2$. Inspection of the graph for $\Sigma^2_{2n+1}$ and $\Sigma^2_{2n}$ shows that we are on a direct saving path for $\Sigma^2_{2n}$ in this case.

Inspection of the graph for $\Pi^2_{2n+1}$ and $\Pi^2_{2n}$ shows that the direct saving paths in $\Pi^2_{2n+1}$ are obtained from the direct killing paths in $\Pi^2_{2n}$ by extending them with an edge labeled $2n-1$. We can assume by induction that:

> If we are on a direct killing path for $\Pi^2_{2n}$ and a direct saving path for $\Sigma^2_{2n}$, then there cannot be a $2n-2$ agreement on the $\Sigma^2_{2n}$ side with a $2n-2$ tupel that appears at a coordinate $\in I_{1,\Sigma} \cup \cdots \cup I_{2n-1,\Sigma}$ in the construction for $\Sigma^2_{2n}/\Pi^2_{2n}$ (starting with an initial piece of a $\Pi^2_{2n}$ iteration).

Thus any 2n − 2 agreement on the $\Sigma^2_{2n+1}$ side in the $\Sigma^2_{2n+1}/\Pi^2_{2n+1}$ construction has to occur at a coordinate $\in I_{2n,\Sigma}$. Again the remark at the beginning of the proof of the claim gives a contradiction.

□
end of proof of claim

To show the two central facts for the $\Sigma^2_{2n+1}/\Pi^2_{2n+1}$ construction (starting with an initial piece of $\Sigma^2_{2n+1}$ iteration) and the $\Sigma^2_{2n}/\Pi^2_{2n}$ construction, we use the same ideas with the obvious modifications.

The (n − 1) Back and Forth Property for $\Sigma^2_n/\Pi^2_n$ Implies (5.2.3.) (n ≥ 2).

The key point here is the following:

Lemma 5.2.8. Suppose we have a $\Sigma^2_n$ iteration $Q^\Sigma$ and formula $\Phi(\ )$ in $\Sigma^2_{n-1}$ together with a condition $q \in Q^\Sigma$ name $\mathring{A}$ for a subset of $V_{\kappa+1}$ such that

$$q \Vdash_{Q^\Sigma} \text{“}V_\kappa \models \Phi(\mathring{A})\text{.”}$$

If $\delta < \kappa^{++}$ is large enough so that $q \in Q^\Sigma_{\delta_1}$ and $\mathring{A} \in V^{Q^\Sigma_{\delta_1}}$ and there is a witness in $V^{Q^\Sigma_{\delta_1}}$ for the $\Sigma^2_n$ statement $\Phi$ and if $i_1 : Q^\Sigma_{\delta_1} \to Q^\Pi_{1+\delta_1}$ is a complete embedding as in the first stage of the construction that establishes the n − 1 back and forth property for $\Sigma^2_n/\Pi^2_n$ starting with an initial piece of a $\Sigma^2_n$ iteration, then we get for *any* $\Pi^2_n$ iteration

$\tilde{Q}^{\Pi}$ that extends $Q^{\Pi}_{1+\delta_1}$

$$i_1(q) \Vdash_{\tilde{Q}^{\Pi}} \text{“}V_\kappa \models \Phi(\mathring{A}^{i_1}).\text{”}$$

We also have an analogous version that starts out with an initial piece of a $\Pi^2_n$ iteration.

Proof. This will be proved by induction on $n \geq 2$ and we write down only the proof of the first version of the lemma (the proof of the second version is entirely analogous.)

We begin with the case $n = 2$.

Let $\Phi(\mathring{A}) \equiv \exists X\ \varphi(X,\mathring{A})$ where X ranges over $V_{\kappa+2}$ and $\varphi$ is $\Sigma^2_0$. Now pick $\delta_1 < \kappa^{++}$ large enough such that $q \in Q^{\Sigma}_{\delta_1}$ and $\mathring{A} \in V^{Q^{\Sigma}_{\delta_1}}$ and there is a nice name $\mathring{X} \in V^{Q^{\Sigma}_{\delta_1}}$ with

$$q \Vdash_{Q^{\Sigma}} \text{“}V_\kappa \models \varphi(\mathring{X},\mathring{A}).\text{”}$$

Now let $\tilde{Q}^{\Pi}$ be any $\Pi^2_n$ iteration that extends $Q^{\Pi}_{\delta_1}$; then

$$i_1(q) \Vdash_{\tilde{Q}^{\Pi}} \text{“}V_\kappa \models \varphi(\mathring{X}^{i_1},\mathring{A}^{i_1})\text{”}$$

since $\varphi$ is $\Sigma^2_0$ and $i_1 : Q^{\Sigma}_{\delta_1} \to Q^{\Pi}_{\delta_1}$ is a complete embedding.

The case $n = 3$ has already been demonstrated in the proof for $\sigma^2_3/\pi^2_3$ in 4.2.21. Now assume $n \geq 4$. Suppose $\Phi(\mathring{A}) \equiv \exists X\ \forall Y\ \varphi(X,Y,\mathring{A})$ where X, Y range over $V_{\kappa+2}$ and $\varphi$ is $\Sigma^2_{n-3}$. We pick $\delta_1 < \kappa^{++}$ large enough such that $q \in Q^{\Sigma}_{\delta_1}$ and $\mathring{A} \in Q^{\Sigma}_{\delta_1}$

and there is $\mathring{X} \in V^{Q^{\Sigma}_{\delta_1}}$ with

(5.2.9) $\quad q \Vdash_{Q^{\Sigma}} \forall Y\ \varphi(\mathring{X},Y,\mathring{A})$.

We assume towards a contradiction that there is some $\Pi^2_n$ iteration $\tilde{Q}^{\Pi}$ that extends $Q^{\Pi}_{1+\delta}$ and

(5.2.10.) $\neg\,(i_1(q) \Vdash_{\tilde{Q}^{\Pi}} \forall Y\ \varphi(\mathring{X}^{i_1},Y,\mathring{A}^{i_1}))$.

Then there is $\delta_2 < \kappa^{++}$ and a condition $q' \in \tilde{Q}^{\Pi}_{1+\delta_1+\delta_2}$ with $q' \leq i_1(q)$ and $\mathring{Y} \in V^{\tilde{Q}^{\Pi}_{1+\delta_1+\delta_2}}$ such that

(5.2.11.) $q' \Vdash_{\tilde{Q}^{\Pi}} \neg\,\varphi(\mathring{X}^{i_1},\mathring{Y},\mathring{A}^{i_1})$.

Recall that we can find an extension $\tilde{Q}^{\Sigma}_{\delta_1+2+\delta_2}$ of $Q^{\Sigma}_{\delta_1}$ and a complete embedding $i_2:\tilde{Q}^{\Pi}_{1+\delta_1+\delta_2} \longrightarrow \tilde{Q}^{\Sigma}_{\delta_1+2+\delta_2}$ as in the second stage of the construction that demonstrates the $(n-1)$ back and forth property for $\Sigma^2_n/\Pi^2_n$ (starting with an initial piece of a $\Sigma^2_n$ iteration).

Claim. $i_2(q') \Vdash_{\tilde{Q}^{\Sigma}} \neg\,\varphi(\mathring{X},\mathring{Y}^{i_2},\mathring{A})$ for any $\Sigma^2_n$ iteration $\tilde{Q}^{\Sigma}$ extending $\tilde{Q}^{\Sigma}_{\delta_1+2+\delta_2}$.

Proof of the claim: Assume towards a contradiction that we have some $\Sigma^2_n$ iteration

$\tilde{Q}^{\Sigma}$ and $\delta_3 < \kappa^{++}$ and a condition $q'' \in \tilde{Q}^{\Sigma}_{\delta_1+2+\delta_2+\delta_3}$ with $q'' \le i_2(q')$ such that

$$q'' \Vdash_{\tilde{Q}^{\Sigma}} \varphi(\mathring{X}, \mathring{Y}^{i_2}, \mathring{A})$$

and there is a witness in $V^{\tilde{Q}^{\Sigma}_{\delta_1+2+\delta_2+\delta_3}}$ for the $\Sigma^2_{n-3}$ statement $\varphi$.

Now proceed as in the 3rd stage of the construction that establishes the $(n-1)$ back and forth property for $\Sigma^2_n/\Pi^2_n$ (starting with an initial piece of a $\Sigma^2_n$ iteration) and find an extension $\tilde{\tilde{Q}}^{\Pi}_{1+\delta_1+\delta_2+2+\delta_3}$ of $\tilde{Q}^{\Pi}_{1+\delta_1+\delta_2}$. Since $\varphi$ is $\Sigma^2_{n-3}$ and we can go back and forth $n-3$ times between suitable extensions of $\tilde{Q}^{\Sigma}_{\delta_1+2+\delta_2+\delta_3}$ and $\tilde{\tilde{Q}}^{\Pi}_{1+\delta_1+\delta_2+2+\delta_3}$, we get by induction hypothesis:

For any $\Pi^2_n$ iteration $\tilde{\tilde{Q}}^{\Pi}$ extending $\tilde{\tilde{Q}}^{\Pi}_{1+\delta_1+\delta_2+2+\delta_3}$

(5.2.12.) $i_3(q'') \Vdash_{\tilde{\tilde{Q}}^{\Pi}} \varphi(\mathring{X}^{i_1}, \mathring{Y}, \mathring{A}^{i_1})$.

Note that clearly $\mathring{X}^{i_3} = \mathring{X}^{i_1}$ and $(\mathring{Y}^{i_2})^{i_3} = \mathring{Y}$ and $\mathring{A}^{i_3} = \mathring{A}^{i_1}$.

Now fix $\tilde{\tilde{Q}}^{\Pi}$ as above. Recall that there is an isomorphism of $\tilde{Q}^{\Pi}$ and $\tilde{\tilde{Q}}^{\Pi}$ that is the identity on $\tilde{Q}^{\Pi}_{1+\delta_1+\delta_2}$. Thus $i_3(q'') \le q'$ and (5.2.11.) and (5.2.12.) together are obviously a contradiction and the claim is proved.

□
end of proof of the claim.

Next fix any $\Sigma^2_n$ iteration $\tilde{Q}^{\Sigma}$ extending $\tilde{Q}^{\Sigma}_{\delta_1+2+\delta_2}$. From the claim we obtain

(5.2.13.) $i_2\ (q') \Vdash_{\tilde{Q}^{\Sigma}} \exists Y \neg\ \varphi\ (\mathring{X},Y,\mathring{A})$.

There is an isomorphism of $Q^{\Sigma}$ with $\tilde{Q}^{\Sigma}$ that is the identity on $Q^{\Sigma}_{\delta_1}$. Then $i_2(q') \leq q$ and (5.2.9.) and (5.2.13.) give a contradiction. Hence our assumption (5.2.10.) was false and we get

$$i_1\ (q) \Vdash_{\tilde{Q}^{\Pi}} \forall Y\ \ \varphi\ (\mathring{X}^{i_1},Y,\mathring{A}^{i_1})$$

and the first version of the lemma is proved. □ end of 5.2.8.

It is now easy to obtain (5.2.3.) from (5.2.2.) by using this lemma: Assume that (5.2.2) holds. If $\Phi(A)$ $\Sigma^2_{n-1}$ then by the second version of the lemma we obtain (5.2.3). If $\Phi(A)$ is $\Pi^2_{n-1}$ then we assume towards a contradiction that for some $\Sigma^2_n$ iteration $Q^{\Sigma}$ we have a condition $q \in Q^{\Sigma}$ with

$$q \Vdash_{Q^{\Sigma}} \text{“}V_\kappa \models \neg\ \Phi(\check{A})\text{.”}$$

Then by the first version of the lemma we obtain a $\Pi^2_n$ iteration $Q^{\Pi}$ and a condition $q' \in Q^{\Pi}$ such that

$$q' \Vdash_{Q^{\Pi}} \text{“}V_\kappa \models \neg\ \Phi(\check{A})\text{”}$$

which contradicts (5.2.2.).

SECTION 6. $\sigma_n^m/\pi_n^m (m\geq 3,\ n\geq 2)$.

The main ideas for establishing the consistency of $\sigma_n^m > \pi_n^m$ ($m \geq 3$, $n \geq 2$) have already been developed in the $\sigma_n^2/\pi_n^2$ case.

We will write $P_\alpha$ rather than $P^m_{n,\alpha}$ in this section. Let us describe the $(m + 2)$ step iteration that we use at stage $\lambda$ (where $\lambda$ is Mahlo) in order to make $\lambda$ $\Sigma_n^m$ describable in $V^{P_{\lambda+1}}$.

Suppose that $G_\lambda$ is $P_\lambda$ generic over $V = L$ and in $V[G_\lambda]$ $\lambda$ is inaccessible and $\lambda^{+\ell} = (\lambda^{+\ell})^L$ for $\ell \geq 1$ and $GCH^{\geq\lambda}$ holds. In the first step we add a sequence $(F_\gamma : \gamma < \lambda^+)$ where each $F_\gamma$ is a Lipshitz function $(2^{\lambda^{+(m-1)}})^{n-1} \to 2^{\lambda^{+(m-1)}}$. Thus the forcing $Q_{(1)}$ is a $\lambda^+$ product (with full support) of copies of the forcing notion $P_F$ where conditions in $P_F$ are functions f such that

dom f is a subtree of $(2^{<\lambda^{+(m-1)}})^{n-1}$ of size $<\lambda^{+(m-1)}$ and

$\forall (s_1,\ldots,s_{n-1}) \in \text{dom } f\ [\exists\alpha < \lambda^{+(m-1)}\ [\alpha \geq \text{dom } s_1 \wedge f(s_1,\ldots,s_n) \in 2^{\alpha+1}$

$\wedge\ f(s_1,\ldots,s_{n-1})(\alpha) = 0]$

$\wedge \forall\zeta[f(s_1,\ldots,s_{n-1})(\zeta) = 1 \Rightarrow cf(\zeta) = \lambda^{+(m-2)}]$

$\wedge \forall (t_1,\ldots,t_{n-1}) \in \text{dom } f\ [(t_1,\ldots,t_{n-1})$ extends $(s_1,\ldots,s_{n-1}) \Rightarrow f(t_1,\ldots,t_{n-1})$

extends $f(s_1,\ldots,s_{n-1})]]$

and for f, g $\in P_F$ we let $f \leq g$ iff $f \supseteq g$. Clearly $|Q_{(1)}| = \lambda^{+(m-1)}$ and $Q_{(1)}$ is

$<\lambda^{+(m-1)}$ closed. Hence if $(F_\gamma : \gamma < \lambda^+)$ is $Q_{(1)}$ generic then in $V[G_\lambda, \vec{F}_\gamma]$ we still have that $\lambda$ is inaccessible $\lambda^{+\ell} = (\lambda^{+\ell})^L$ for $\ell \geq 1$ and $GCH^{\geq\lambda}$ holds.

In the second step we will do an iteration $Q_{(2)}$ (that we call Q in the remainder of this section) which will make a $\Sigma^m_n$ fact true about $F_\gamma$ for $\gamma$ even and its negation about $F_\gamma$ for $\gamma$ odd. Q will be a certain suborder of $Fn(\lambda^{+m}, 2, \lambda^{+(m-1)})$. We partition $\lambda^{+m}$ into $\lambda^+$ many pieces say $(A_\gamma : \gamma < \lambda^+)$ and C with $\lambda^+ \subseteq C$ each of which has size $\lambda^{+m}$; then for each $\gamma < \lambda^+$ we partition $A_\gamma$ into pieces $(A^{(k)}_\gamma : 1 \leq k \leq n-1)$ each of which has size $\lambda^{+m}$.

Next for each $k \in \{1,\ldots,n-1\}$ we enumerate a complete sequence of k-tupels of nice $Fn(\lambda^{+m}, 2, \lambda^{+(m-1)})$ names for a subset of $\lambda^{+(m-1)}$ along the coordinates in $A^{(k)}_\gamma$. (Note that this is possible since $Fn(\lambda^{+m}, 2, \lambda^{+(m-1)})$ is $\lambda^{+m}$ c.c. and has size $\lambda^{+m}$.) The poset Q will add a new subset of $\lambda^{+(m-1)}$ at each coordinate in $\lambda^{+m} - \bigcup_{\gamma<\lambda^+} A^{(n-1)}_\gamma$. At a coordinate $\alpha \in A^{(n-1)}_\gamma$ (for some $\gamma < \lambda^+$) Q will add a club set $\subseteq \lambda^{+(m-1)}$ that is disjoint from $F_\gamma(\hat{\tau}^{(1)}_{\gamma,\alpha}, \ldots, \hat{\tau}^{(n-1)}_{\gamma,\alpha})$ (where $(\tau^{(1)}_{\gamma,\alpha}, \ldots, \tau^{(n-1)}_{\gamma,\alpha})$ is the tupel that appears at coordinate $\alpha \in A^{(n-1)}_\gamma$) if certain killing conditions are met. If these killing conditions are not satisfied we just force with the trivial poset $\{0\}$; i.e., we save $F_\gamma(\hat{\tau}^{(1)}_{\gamma,\alpha}, \ldots, \hat{\tau}^{(n-1)}_{\gamma,\alpha})$.

The killing conditions for $\alpha \in A^{(n-1)}_\gamma$ where $\gamma$ is even (i.e., the killing conditions in $\Sigma^m_n$) are again given by the graph for $\Sigma^m_n$. Similarly the graph for $\Pi^m_n$ tells us whether we kill at some $\alpha \in A_\gamma^{(n-1)}$ where $\gamma$ is odd. Now the graphs for $\Sigma_n^m$ ($\Pi_n^m$ resp.) look exactly like the graphs for $\Sigma_n^2$ ($\Pi_n^2$ resp.) except that the nodes are labeled $\Sigma_k^m$ and $\Pi_k^m$ instead of $\Sigma_k^2$ and $\Pi_k^2$. Clearly Q is $<\lambda^{+(m-2)}$ closed (because of the cofinality restriction in the definition of conditions in $Q_{(1)}$) and $\lambda^{+m}$ c.c. (since compatibility in Q agrees with compatibility in $Fn(\lambda^{+m},2,\lambda^{+(m-1)})$.

An analogous proof as in the $\sigma_3^2/\pi_3^2$ case shows that Q is $<\lambda^{+(m-1)}$ Baire. In particular this implies that for each $\gamma < \lambda^+$ $\Vdash^{V[G_\lambda,\vec{F}\gamma]}_{Q}$ dom $F_\gamma = (2^{\lambda^{+(m-1)}})^{n-1}$. Moreover the analogue of 4.1.7. can be proved for Q; hence after forcing with Q $F_\gamma(X_1,...,X_{n-1})$ (for $\gamma < \lambda^+$, $X_1,...,X_n \subseteq \lambda^{+(m-1)}$) will be stationary unless Q explicitly killed it. Therefore if G is Q generic over $V[G_\lambda,\vec{F}\gamma]$ we have in $V[G_\lambda,\vec{F}\gamma,G]$ for odd $\gamma < \kappa^+$:

$$\forall X_1 \subseteq \lambda^{+(m-1)}\ \exists X_2 \subseteq \lambda^{+(m-1)} \cdots QX_{n-1} \subseteq \lambda^{+(m-1)}\ \psi\ F_\gamma(X_1,...,X_{n-1})$$

where $Q = \exists$ ($Q = \forall$ resp.) and $\psi$ says $F_\gamma(X_1,...,X_{n-1})$ is stationary (nonstationary resp.) in $\lambda^{+(m-1)}$ for odd n (even n resp.). Clearly this is $\Pi_n^m(F_\gamma)$.

For even $\gamma < \kappa^+$ the negation of this statement will hold about $F_\gamma$ , i.e., a $\Sigma_n^m(F_\gamma)$ fact.

For each $\gamma < \lambda^+$ we can find a code $\tilde{F}_\gamma \subseteq \lambda^{+(m-1)}$ for $F_\gamma$ in $V[G_\lambda,\vec{F}\gamma,G]$. This uses the fact that $(2^{<\lambda^{+(m-1)}})^{V[G_\lambda,\vec{F}\gamma,G]} = (2^{<\lambda^{+(m-1)}})^{V[G_\lambda]}$ and $V[G_\lambda] = L[G_\lambda]$ where $G_\lambda \subseteq P_\lambda \subseteq L_\lambda$. Hence we can use the canonical wellordering $<_{L[G_\lambda]}$ on

$2^{<\lambda^{+(m-1)}}$ to do this coding.

The steps $Q_{(3)}, \ldots, Q_{(3+m-2)}$ will now code each $\tilde{F}_\gamma \subseteq \lambda^{+(m-1)}$ down to a subset $S_\gamma \subseteq \lambda$. This is done in exactly the same way as in the $\sigma_1^m/\pi_1^m$ proof. Finally in the last step we add a sequence $(C_\gamma : \gamma < \lambda^+)$ where each $C_\gamma \subseteq \lambda$ is club and

$$C_\gamma \cap \{\mu < \lambda : \mu \text{ inaccessible}$$

$$\wedge V_\mu \models \Phi^{\Sigma_n^m}(S_\gamma \cap V_\mu, G_\lambda \cap V_\mu, \lambda \cap \mu)\} = \emptyset.$$

Here $\Phi^{\Sigma_n^m}$ is the analogue of (4.1.10.) for $\Sigma_n^m$.

Now we can proceed as outlined in Section 1 and prove

$$\Vdash_{P_{\kappa+1}} \text{“there are no } \Sigma_n^m \text{ indescribables} \leq \kappa,$$

$$\kappa \text{ is } \Pi_n^m \text{ indescribable, } \kappa' \text{ is } \Sigma_n^m \text{ indescribable.”}$$

The hard part of the proof of $\Vdash_{P_{\kappa+1}}$ "$\kappa$ is $\Pi_n^m$ indescribable" is again to show (in the notation of Section 1) that

$$N[G_\kappa, {}^N\vec{F}_\gamma, g] \text{ is } \Sigma_{n-1}^m \text{ correct for } \kappa \text{ in } V[G_\kappa, \vec{F}_\gamma, G]$$

where $G_\kappa$ is the V generic for $P_\kappa$, $\vec{F}_\gamma$ is the generic for $Q_{(1)}$ of stage $\kappa$ of $P_{\kappa+1}$ and G is the Q generic of stage $\kappa$ of $P_{\kappa+1}$ and ${}^N\vec{F}_\gamma$ is the $Q_{(1)}$ generic for stage $\kappa$ of $P^N_{j(\kappa)}$ and g is the $Q^N$ generic for stage $\kappa$ of $P^N_{j(\kappa)}$.

The strategy for this is the same as in the $\sigma_n^2/\pi_n^2$ case; i.e., the key point is that $\Sigma_n^m/\Pi_n^m$ has the (n $-$ 1) back and forth property which is proved by the same arguments as in the $\sigma_n^2/\pi_n^2$ case.

SECTION 7. Oracles — The Final Word on Indescribability.

In order to state the final theorem we need to introduce the notion of an oracle. An oracle is simply a subset of $\omega$ that codes a function with domain $\{(m,n): m \geq 2,\ n \geq 1\}$ that takes values in $\{0,1\}$.

The final theorem is

Theorem 7.1. (ZFC) Assuming the existence of $\Sigma^m_n$ indescribables for all m and n and given any oracle $\mathfrak{F}$, there is a poset $P_{\mathfrak{F}} \in L[\mathfrak{F}]$ such that GCH holds in $(L[\mathfrak{F}])^{P_{\mathfrak{F}}}$ and

$$(7.2.)\qquad \Vdash^{L[\mathfrak{F}]}_{P_{\mathfrak{F}}} \begin{cases} \sigma^m_n < \pi^m_n & \text{if } \mathfrak{F}(m,n) = 0 \\ \sigma^m_n > \pi^m_n & \text{if } \mathfrak{F}(m,n) = 1. \end{cases}$$

□

Before defining $P_{\mathfrak{F}}$ (for a given oracle function $\mathfrak{F}$) we make some observations about small forcing and indescribability. In 1.6. we have already seen that a forcing of size $<\kappa$ cannot destroy the $\Sigma^m_n$ indescribability of $\kappa$. The same statement is true about a $\Pi^m_n$ indescribable cardinal $\kappa$. The proof for this is totally analogous to the $\Sigma^m_n$ case. However the reformulation in I.1. makes it possible to give an even easier proof.

To complete the picture we show (cf. Corollary 7.8.) that no poset can create new $\Sigma^m_n$ or $\Pi^m_n$ indescribable cardinals (for any m, n $\geq$ 1) that are larger than the cardinality of the forcing. First we prove:

Lemma 7.3. (ZFC) Suppose that $\kappa$ is inaccessible and P is a notion of forcing with $|P| < \kappa$. Let G be a P generic. Then, in V[G] for any $X \in V_{\kappa+1}$,

$$(7.4.)\qquad X \in V \iff X \subseteq V \wedge \forall s\,[s \in V \Rightarrow X \cap s \in V]$$

and for any $\mathfrak{X} \in V_{\kappa+m}$ $(m \geq 2)$

(7.5.) $\mathfrak{S} \in V \Longleftrightarrow \mathfrak{S} \subseteq V \wedge \forall \mathfrak{T}\, [\mathfrak{T} \in V \wedge |\mathfrak{T}| \leq \kappa .\Rightarrow. \mathfrak{S} \bigcap \mathfrak{T} \in V]$.

Here s ranges of $V_\kappa$ and $\mathfrak{T}$ over $V_{\kappa+m}$ (of V[G]).

Proof. To prove the nontrivial direction of (7.4.) assume towards a contradiction that for some condition $p^* \in G$ and some $\mathring{X} \in V^P$ we have

$$p^* \Vdash_P \text{“}\mathring{X} \subseteq (V)_\kappa \wedge \forall s\, [s \in V \Rightarrow \mathring{X} \bigcap s \in V] \wedge \mathring{X} \notin V.\text{”}$$

In V, pick a wellordering of $V_\kappa$ of order type $\kappa$ and let $seg_\alpha$ denote the segment of the first $\alpha$-many elements ($\alpha < \kappa$). We can (in V) for each $\alpha < \kappa$ pick $p_\alpha \leq p^*$ and $x_\alpha \in (V)_\kappa$ with

$$p_\alpha \Vdash \mathring{X} \bigcap seg_\alpha = x_\alpha .$$

$|P| < \kappa$ implies that there is some $p \in P$ with $p_\alpha = p$ for cofinally many $\alpha$.

Then let

$$\tilde{X} = \bigcup_{p_\alpha = p} x_\alpha$$

clearly

$$p \Vdash \tilde{X} = \mathring{X}$$

contradicting

$$p \Vdash \mathring{X} \notin V.$$

To prove the nontrivial direction in (7.5.) we assume (without loss of generality) $m = 2$ and suppose towards a contradiction that for some $p^* \in G$ and $\mathring{\mathfrak{S}}$ in $V^P$

$$p^* \Vdash \text{“}\mathring{\mathfrak{S}} \subseteq (V)_{\kappa+1} \wedge \forall \mathfrak{T}\, [\mathfrak{T} \in V \wedge |\mathfrak{T}| \leq \kappa .\Rightarrow. \mathring{\mathfrak{S}} \bigcap \mathfrak{T} \in V] \wedge \mathring{\mathfrak{S}} \notin V.\text{”}$$

If for some $p^{**} \leq p^*$ $p^{**} \Vdash |\mathring{\mathfrak{S}}| \leq \kappa$, then since $|P| < \kappa$ we can find $\mathfrak{T} \in (V)_{\kappa+2}$ with $|\mathfrak{T}| \leq \kappa$ and $p^{**} \Vdash \mathfrak{T} \supseteq \mathring{\mathfrak{S}}$. This clearly implies $p^{**} \Vdash \mathring{\mathfrak{S}} \in V$, a contradiction. Thus we can assume

$$p^* \Vdash |\mathring{\mathfrak{S}}| \geq \kappa.$$

Now we claim:

$$p^* \Vdash \exists \mathfrak{Y} \subseteq \mathring{\mathfrak{S}}[|\mathfrak{Y}| = \kappa \wedge \forall \mathfrak{Z}[\mathfrak{Y} \subseteq \mathfrak{Z} \subseteq \mathring{X} \Rightarrow \mathfrak{Z} \notin V]].$$

Proof of the Claim: Consider $B \underset{\mathfrak{Df}}{=}$ r.o. (P) and let H be B generic over V with $p^* \in H$. Since $|B| < \kappa$ we can find $\mathring{\mathfrak{Y}} \in V^B$ such that in V[H] $|\mathring{\mathfrak{Y}}^H| = \kappa$ and $\mathring{\mathfrak{Y}}^H \subseteq \mathring{\mathfrak{S}}^H$ and $\{\|\check{X} \in \mathring{\mathfrak{S}}\|^B : X \in \mathring{\mathfrak{S}}^H \cap (V)_{\kappa+1}\} = \{\|\check{X} \in \mathring{\mathfrak{Y}}\|^B : X \in \mathring{\mathfrak{Y}}^H \cap (V)_{\kappa+1}\}$. Then $\mathring{\mathfrak{S}}^H \notin V$ implies

$$\text{(7.6)} \qquad \neg \exists b \in H \ \forall X \in \mathring{\mathfrak{Y}}^H \cap (V)_{\kappa+1} \quad b \leq \|\check{X} \in \mathring{\mathfrak{S}}\|^B.$$

Now suppose $\mathring{\mathfrak{Z}} \in V^B$ with $\mathring{\mathfrak{Y}}^H \subseteq \mathring{\mathfrak{Z}}^H \subseteq \mathring{\mathfrak{S}}^H$; it follows that $\mathring{\mathfrak{Z}}^H \notin V$. Otherwise we can pick $\mathfrak{Z} \in V$ with $\|\mathring{\mathfrak{Z}} = \check{\mathfrak{Z}}\|^B \in H$ and then we get for all $X \in \mathring{\mathfrak{Y}}^H \cap (V)_{\kappa+1}$

$$\begin{aligned} \|\check{X} \in \mathring{\mathfrak{S}}\|^B &\geq \|\check{X} \in \mathring{\mathfrak{Z}}\|^B \cdot \|\mathring{\mathfrak{Z}} \subseteq \mathring{\mathfrak{S}}\|^B \\ &\geq \|\check{X} \in \check{\mathfrak{Z}}\|^B \cdot \|\check{\mathfrak{Z}} = \mathring{\mathfrak{Z}}\|^B \cdot \|\mathring{\mathfrak{Z}} \subseteq \mathring{\mathfrak{S}}\|^B \\ &= \|\check{\mathfrak{Z}} = \mathring{\mathfrak{Z}}\|^B \cdot \|\mathring{\mathfrak{Z}} \subseteq \mathring{\mathfrak{S}}\|^B \in H, \end{aligned}$$

since for $X \in \overset{\circ}{\mathfrak{Y}}^H \cap (V)_{\kappa+1}$ we clearly have $\|\check{X} \in \check{\mathfrak{Z}}\| = 1$ because $\overset{\circ}{\mathfrak{Y}}^H \subseteq \overset{\circ}{\mathfrak{Z}}^H = \mathfrak{Z}$.

But this contradicts (7.6). Hence $\overset{\circ}{\mathfrak{Y}}^H$ works and the claim is proved.

□
end of the proof of the claim

By the claim we can fix $\overset{\circ}{\mathfrak{Y}} \in V^P$ with

$$p^* \Vdash \text{“}\overset{\circ}{\mathfrak{Y}} \subseteq \overset{\circ}{\mathfrak{S}} \wedge |\overset{\circ}{\mathfrak{Y}}| = \kappa \wedge \forall \mathfrak{Z}\, [\overset{\circ}{\mathfrak{Y}} \subseteq \mathfrak{Z} \subseteq \overset{\circ}{\mathfrak{S}} \Rightarrow \mathfrak{Z} \notin V].\text{”}$$

Let $p^{**} \leq p^*$ and $\overset{\circ}{f} \in V^P$ such that

$$p^{**} \Vdash \overset{\circ}{f} : \kappa \xrightarrow[\text{onto}]{1:1} \overset{\circ}{\mathfrak{Y}}.$$

Then define (in V):

$$\mathfrak{Y} \underset{\mathfrak{Df}}{=} \{X \in (V)_{\kappa+1} : \exists p \leq p^{**}\ \exists \alpha < \kappa\ p \Vdash X = \overset{\circ}{f}(\alpha)\}.$$

Clearly $|\mathfrak{Y}| = \kappa$ and $p^{**} \Vdash \overset{\circ}{\mathfrak{Y}} \subseteq \check{\mathfrak{Y}}$. Now (in V) wellorder $\mathfrak{Y}$ in order type $\kappa$ and for $\alpha < \kappa$ denote by $seg_\alpha$ the segment of the first $\alpha$ element. Note that for $\alpha < \kappa$

$$p^{**} \Vdash \overset{\circ}{\mathfrak{S}} \cap seg_\alpha \in V.$$

Hence (in V) we can find for each $\alpha < \kappa$ $p_\alpha \leq p^{**}$ and $\mathfrak{Y}_\alpha$ with

$$p_\alpha \Vdash \overset{\circ}{\mathfrak{S}} \cap seg_\alpha = \mathfrak{Y}_\alpha.$$

Since $|P| < \kappa$ there must be some $p \leq p^{**}$ with $p = p_\alpha$ for cofinally many $\alpha$. Then

$$p \Vdash \text{“}\overset{\circ}{\mathfrak{S}} \cap \mathfrak{Y} = \bigcup_{p_\alpha = p} \mathfrak{Y}_\alpha \in V\text{”}$$

contradicting

$$p \Vdash \mathring{\mathfrak{X}} \cap \mathfrak{Y} \notin V.$$

□
end of 7.3.

We use this lemma to show

Lemma 7.7. (ZFC) If $\kappa$, P, G are as in 7.3. then in V[G] for any $\mathfrak{X} \in V_{\kappa+m}$ (where m $\geq 1$) the formula "$\mathfrak{X} \in V$" is $\Sigma_o^m$ $(\mathfrak{X},(V)_\kappa)$ over $V_\kappa$.

Proof. For m = 1 (7.4.) implies (s ranges over $V_\kappa$)

$$\mathfrak{X} \in V \text{ iff } \mathfrak{X} \subseteq (V)_\kappa \wedge \forall s\, [s \in (V)_\kappa \Rightarrow \mathfrak{X} \cap s \in (V)_\kappa].$$

Clearly this is $\Sigma_o^1$ $(\mathfrak{X},(V)_\kappa)$.

For m $\geq$ 2 we proceed by induction on m. Suppose $\mathfrak{X} \in V_{\kappa+m+1}$; then by (7.5.) (where $\mathfrak{T}$ ranges our $V_{\kappa+m+1}$)

$$\mathfrak{X} \in V \Longleftrightarrow \mathfrak{X} \subseteq (V)_{\kappa+m} \wedge \forall \mathfrak{T}[\mathfrak{T} \in V, |\mathfrak{T}| \leq \kappa \Rightarrow \mathfrak{X} \cap \mathfrak{T} \in V].$$

Now, by induction hypothesis this is $\Sigma_o^{m+1}$ because any $\mathfrak{T} \subseteq V_{\kappa+m}$ of cardinality $\leq\kappa$ can obviously be coded by some element of $V_{\kappa+m}$ so that the whole formula is $\Sigma_o^{m+1}$ $(\mathfrak{X},(V)_\kappa)$.

□
end of 7.7.

Remark. Of course lemma is not optimally phrased, but this version already allows us to prove:

Corollary 7.8. (ZFC) If $\kappa$ is inaccessible and P a poset of size $<\kappa$ and G is P generic, then for m, n $\geq 1$

$$(\kappa \text{ is } \Sigma_n^m \ (\Pi_n^m \text{ resp.) indescribable})^{V[G]}$$

implies

$$(\kappa \text{ is } \Sigma_n^m \ (\Pi_n^m \text{ resp.) indescribable})^{V}.$$

Proof. If $\Phi(A)$, where $A \subseteq (V)_\kappa$ is $\Sigma^m_n$ ($\Pi^m_n$ resp.) then by 7.7, in V[G] the formula $(\Phi(A))^V$ is $\Sigma^m_n$ $(A,(V)_\kappa)$ ($\Pi^m_n(A,(V)_\kappa)$ resp.). Then note that clearly $(V)_\kappa \subseteq V_\kappa$ (in V[G]). So we can use it as a parameter. □

end of 7.8.

We are now turning to the proof of 7.1. Suppose $\mathfrak{F}$ is an oracle and we have $\Sigma^m_n$ indescribables for all m, n. We know that in $L[\mathfrak{F}]$ the following picture holds for $m \geq 2$, $n \geq 1$:

$$(7.9) \qquad \cdots <^{L[\mathfrak{F}]} \sigma^m_n <^{L[\mathfrak{F}]} \pi^m_n <^{L[\mathfrak{F}]} \sigma^m_{n+1} <^{L[\mathfrak{F}]} \pi^m_{n+1} < \cdots .$$

For the sake of completeness we give a proof of this fact.

Proof of (7.9). Fix $m \geq 2$ and $n \geq 1$. We work in $L[\mathfrak{F}]$. Let $\kappa$ be the least $\Pi^m_n$ indescribable. The proof strategy is to find a $\Pi^m_n$ statement $\Phi(A,\kappa)$ with $A \subseteq V_\kappa$ such that $V_\kappa \models \Phi(A,\kappa)$ and any inaccessible $\lambda$ to which $\Phi$ reflects is $\Sigma^m_n$ indescribable. $\Phi(A)$ can be found as follows: We know that $\kappa$ being the least $\Pi^m_n$ indescribable is $\Sigma^m_n$ describable. We fix some $A \subseteq V_\kappa$ and a $\Sigma^m_n$ formula $\Psi(A)$ such that $V_\kappa \models \Psi(A)$ and $\Psi(A)$ does not reflect to any inaccessible $\lambda < \kappa$ and such that the witness in the $\Sigma^m_n$ formula $\Psi$ is least in the canonical wellordering $<_{L[\mathfrak{F}]}$ with property that it is a witness for a $\Sigma^m_n$ formula $\Psi'$ in a parameter $A'$ as above. We pick a sufficiently large finite fragment T of ZF + $V = L[\mathfrak{F}]$ such that for any transitive model M of T with $\mathfrak{F} \in M$ we have that $X \in M$ and $Y <_{L[\mathfrak{F}]} X$ imply $Y \in M$.

Then we take $\Phi(A,\kappa)$ to be the formula

$$\forall \mathcal{M}\ [\mathcal{M} \text{ transitive}, \mathcal{M} \models T, |\mathcal{M}| = |V_{\kappa+m-1}|, \mathcal{M}\ \Sigma^m_{n-1} \text{ correct for } \kappa,$$

$$\mathcal{M} \models \text{“}\kappa \text{ is not } \Sigma^m_n \text{ indescribable”} .\Rightarrow. \mathcal{M} \models \text{“}V_\kappa \models \Psi(A)\text{”}];$$

$\Phi$ is clearly $\Pi^m_n$ over $V_\kappa$ and by the choice of T we get $V_\kappa \models \Phi(A,\kappa)$. If $\lambda < \kappa$ is inaccessible and $V_\lambda \models \Phi(A \cap V_\lambda,\lambda)$, then $\lambda$ must be $\Sigma^m_n$ indescribable because we cannot have $V_\lambda \models \Psi(A \cap V_\lambda)$ by our choice of $\Psi(A)$. Thus $\Phi$ has the properties that we want.

□
end of the proof of (7.9).

Actually the proof that we just gave works for a large class of inner models. The key point is that the inner model under consideration (or at least its truncation up to the first measurable) must have a certain "good" wellorder.

We now resume the proof of 7.1. Working in $L[\mathfrak{F}]$ we define for $m \geq 2$ and $n \geq 1$ the poset $P^{m,n}_{\mathfrak{F}}$ to be the trivial poset if $\mathfrak{F}(m,n) = 0$. If $\mathfrak{F}(m,n) = 1$ then we use the exact same definition that we used for $P^m_n$ (with $\kappa = {}^{L[\mathfrak{F}]}\pi^m_n$ ) except that we replace L by $L[\mathfrak{F}]$ and we do something only at Mahlo stages $\geq {}^{L[\mathfrak{F}]}\sigma^m_n$ . Then we let

$$P_{\mathfrak{F}} \overset{=}{\text{Df}} \prod_{m\geq 2,n\geq 1} P^{m,n}_{\mathfrak{F}}.$$

We must show that (7.2.) holds. So fix $m' \geq 2$ and $n' \geq 1$. Note that $P_{\mathfrak{F}} \approx P_1 \times P_2 \times P_3$ where $P_1 \underset{\mathfrak{Df}}{=} \prod_{\substack{m<m' \text{ or} \\ m=m' \wedge n<n'}} P_{\mathfrak{F}}^{m,n}$ and $P_2 \underset{\mathfrak{Df}}{=} P_{\mathfrak{F}}^{m',n'}$ and $P_3 = \prod_{\substack{m>m' \text{ or} \\ m=m' \wedge n>n'}} P_{\mathfrak{F}}^{m,n}$.

First assume that $\mathfrak{F}(m',n') = 1$. We know from Section 1 that for each $\alpha < {}^{L[\mathfrak{F}]}\sigma^{m'}_{n'+1}$ $P_3$ has a $<\alpha$ closed, dense suborder. Hence $P_3$ is $< {}^{L[\mathfrak{F}]}\sigma^{m'}_{n'+1}$ Baire. Thus if $G_3$ is $P_3$ generic over $L[\mathfrak{F}]$, $(L[\mathfrak{F},G_3])_{{}^{L[\mathfrak{F}]}\sigma^{m'}_{n'+1}} = (L[\mathfrak{F}])_{{}^{L[\mathfrak{F}]}\sigma^{m'}_{n'+1}}$. This implies that in $L[\mathfrak{F},G_3]$ ${}^{L[\mathfrak{F}]}\pi^{m'}_{n'}$ is still $\Pi^{m'}_{n'}$ indescribable and that there are many $\Sigma^{m'}_{n'}$ indescribables above ${}^{L[\mathfrak{F}]}\pi^{m'}_{n'}$. It implies also that $L[\mathfrak{F},G_3]$'s version of $P_{\mathfrak{F}}^{m',n'}$ agrees with the $P_{\mathfrak{F}}^{m',n'}$ of $L[\mathfrak{F}]$. Thus if we denote by $\kappa'$ the least $\Sigma^{m'}_{n'}$ indescribable $>$ ${}^{L[\mathfrak{F}]}\pi^{m'}_{n'}$ in $L[\mathfrak{F},G_3]$ then for any $G_2$ that is $P_2$ generic over $L[\mathfrak{F},G_3]$: In $L[\mathfrak{F},G_3,G_2]$ ${}^{L[\mathfrak{F}]}\pi^{m'}_{n'}$ is $\Pi^{m'}_{n'}$ indescribable, $\kappa'$ is $\Sigma^{m'}_{n'}$ indescribable and there are no $\Sigma^{m'}_{n'}$ indescribables below ${}^{L[\mathfrak{F}]}\pi^{m'}_{n'}$.

Clearly $|P_1| < {}^{L[\mathfrak{F}]}\sigma^{m'}_{n'}$. Hence by 7.8. for any $G_1$ that is $P_1$ generic over $L[\mathfrak{F},G_3,G_2]$ we obtain that in $L[\mathfrak{F},G_3,G_2,G_1]$ there are no $\Sigma^{m'}_{n'}$ indescribables $\in[{}^{L[\mathfrak{F}]}\sigma^{m'}_{n'}, {}^{L[\mathfrak{F}]}\pi^{m'}_{n'}]$ and clearly we cannot have any $\Sigma^{m'}_{n'}$ indescribables below ${}^{L[\mathfrak{F}]}\sigma^{m'}_{n'}$. Also by $|P_1| < {}^{L[\mathfrak{F}]}\sigma^{m'}_{n'}$ we get that ${}^{L[\mathfrak{F}]}\pi^{m'}_{n'}$ is still $\Pi^{m'}_{n'}$ indescribable and $\kappa'$ is still $\Sigma^{m'}_{n'}$ indescribable in $L[\mathfrak{F},G_3,G_2,G_1]$. This shows that for $\mathfrak{F}(m',n') = 1$ we have

$$\Vdash^{L[\mathcal{F}]}_{P_3} \sigma^{m'}_{n'} > \pi^{m'}_{n'} .$$

Now we assume that $\mathcal{F}(m',n') = 0$. Then $P_{\mathcal{F}} \approx P_1 \times P_3$. The $<\sigma^{m'}_{n'+1}$ Baireness of $P_3$ implies

$$\Vdash^{L[\mathcal{F}]}_{P_3} \sigma^{m'}_{n'} = {}^{L[\mathcal{F}]}\sigma^{m'}_{n'} < {}^{L[\mathcal{F}]}\pi^{m'}_{n'} = \pi^{m'}_{n'} .$$

$|P_1| < {}^{L[\mathcal{F}]}\sigma^{m'}_{n'}$ together with the observation that no generic extension of $L[\mathcal{F}]$ can have any $\Pi^{m'}_{n'}$ indescribables $< {}^{L[\mathcal{F}]}\sigma^{m'}_{n'}$ yield that

$$\Vdash^{L[\mathcal{F}]}_{P_{\mathcal{F}}} \sigma^{m'}_{n'} < \pi^{m'}_{n'} .$$

Another factoring argument shows that for any cardinal $\mu$ $\Vdash^{L[\mathcal{F}]}_{P_{\mathcal{F}}} 2^{\mu} = \mu^{+}$. Hence we get

$$\Vdash^{L[\mathcal{F}]}_{P_{\mathcal{F}}} \text{GCH}.$$

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