ELASTOSTATIC AND AEROELASTIC PROBLEMS RELATING TO THIN WINGS OF HIGH SPEED AIRPLANES

Thesis by

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In Partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California

1948

Acknowledgment

The writer wishes to express appreciation to Professor E. E. Sechler for his constant advice and encouragement throughout the work, to Professor P. Epstein for his suggestions in the theory of plates, and to Professor A. D. Michal, for his guidance in the mathematical work of Part II of this thesis.

He wishes also to express his gratitude to several friends and coworkers of the GALCIT swept wing research group for many interesting discussions and for help in numerical computations, to Miss H. S. Yu for help in evaluating many integrals, and to Mrs. E. Fox for typing the thesis.

This study has been greatly aided by the Wright Field Air Materiel Command which is sponsoring a swept wing research program at the California Institute, and this aid is gratefully acknowledged.

SUMMARY

This report is concerned with the statics and dynamics of very thin wings of high speed airplanes. With the modern tendency towards sweepback, which is necessary for supersonic airplanes, the wing constructions tend more and more to an ideal structure, hence for the static problems of this report, the wing is idealized to a thin cantilever elastic plate.

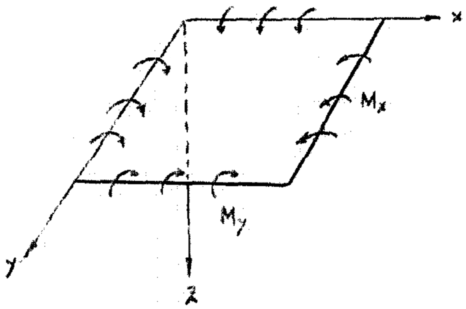
Part I gives a general formulation of the fundamental equations of deformation of thin elastic plates and the direct methods of solution. For small deflection of plates, the equations and boundary conditions are derived from the three-dimensional equations of elasticity developed in power series of the thickness of the plate. It is shown that the classical Poisson-Kirchhoff theory is coincident with the first approximation in this development. These equations are then transformed into oblique coordinates for treating problems concerning swept plates. Since the problem of the cantilever plate is very difficult to solve from the standpoint of biharmonic analysis, emphasis is laid on the direct methods of solution which lead to useful approximate solutions with desired accuracy. Section 1.21 discusses the relation between plate problems and equivalent variational problems. Section 1.22 contains a systematic review of the Rayleigh-Ritz method and all methods allied to it. Section 1.23 formulates the method of relaxation of boundary conditions, including the Trefftz method as one instance.

Part II discusses the general aeroelastic problems of high speed airplanes. For airplanes accelerating or decelerating through

the transonic region, the coefficients in the aeroelasticity equations are of transient nature. Such transient perturbations are new phenomena in aeronautics but are sufficiently important to warrant detailed investigation. A general mathematical treatment is given, though due to lack of aerodynamic data at present, no specific example is included. A general solution is obtained and this solution is expanded into a generalized power series which proves to be particularly useful when the transient perturbation is small. The present result includes the ordinary small perturbation theory for finite degrees of freedom as a particular case. Several results regarding small perturbations are given in section 2.6.

The next two parts give a detailed computation on the deflection of and stresses in cantilever plates. The deflection of rectangular cantilever plates is solved both by the Rayleigh-Ritz method and the method of relaxation of boundary conditions. For swept plates the Rayleigh-Ritz method is used. A theory of stress approximation without using the intermediate deflection function is developed in Part IV, and is applied to rectangular plates.

NOTATIONS



x, y, z Rectangular coordinates

$D = \frac{Et^3}{12(1-\nu^2)}$ Flexural rigidity of plates.
t = thickness

M_x, M_y, H_{xy} Resultant bending and twisting moments per unit length in plates. The bending moments are considered positive when they put the upper side of an element of the plate in compression. The sense of the twisting moment agrees with the couple due to the shearing stresses τ_{xy}.

Q_x, Q_y Resultant shearing forces per unit length in a plate, normal to middle surface. Their sign is the same as the corresponding sign of the shearing stresses.

$u, v, w,$ Elastic displacements in x, y, z directions respectively.

θ Sweep-back angle

λ, μ Elastic constants.

ν, E Poisson's ratio and Young's modulus respectively.

$$\kappa = \frac{\lambda}{\mu} = \frac{2\nu}{1-2\nu} \; ; \qquad A = \frac{2(\kappa+1)}{\kappa+2} = \frac{1}{1-\nu} .$$

q Intensity of distributed load per unit area of the middle surface of the plate.

n, t Used in subscripts denoting outer normal and tangential directions respectively.

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I. GENERAL FORMULATIONS OF THE FUNDAMENTAL EQUATIONS OF DEFORMATION OF THIN ELASTIC PLATES AND DIRECT METHODS OF SOLUTION

I.1 Fundamental Equations of Deformation

§ 1.11 Introduction

An exact theory of plates should be deduced from the general equations of elasticity for a three-dimensional continuum. The classical Poisson-Love theory (Refs. 1, 2, 3) which is based upon intuitive assumptions is unsatisfactory in this respect. If the deflection is fairly large, the finite displacement must be also taken into account; since although the six components of the strain tensor are small, some of the derivatives of the components of displacement might be large. Then it is necessary to differentiate between stresses referred to the surface elements of the undeformed and the deformed system.

The following formulation is concerned with small deflections. Within this limitation the fundamental equations are deduced by starting from the conditions in a thick plate, regarded as a three-dimensional continuum, and then passing to the limit of thin plate by expanding the stresses into power series in the thickness of the plate. The validity of the three-dimensional field equations and of the boundary conditions at the faces of the plate are retained during the limiting process. This concept was suggested by P. Epstein (Ref. 6) in 1942. In its application to elastic plates, higher order terms can readily be deduced, from which comparison to the classical theory can be made.

In explicit terms the basic assumptions in the following development are:

(1) That the thickness of the plate is small compared to other dimensions of the plate, and

(2) That the deformation of the plate is small compared to the thickness of the plate.

These assumptions are of course tacitly implied in the classical theory of thin plates, which in addition postulates that any line normal to the middle surface of the plate remains straight and normal to it in any strained condition, and that the stress component normal to the middle surface is small in comparison to other stress components and may be neglected in the stress-strain relations.

§ 1.12 <u>Fundamental Equations in Orthogonal Cartesian Coordinates</u>

Let (x,y,z) be a system of Cartesian coordinates, and let the plane $z = 0$ coincide with the middle surface of the plate in the unstrained state. The plate is assumed to be thin, that is, the thickness $2\epsilon h$ of the plate is small compared to other dimensions of the plate. The thickness may be variable. It is designated by $2\epsilon h(x,y)$ where 2ϵ is a small quantity of the dimension of the thickness and $h(x,y)$ a function of x and y describing the thickness distribution of the plate.

Using the usual notation σ_x, σ_y, τ_{xy} etc. for the components of the stress tensor, the general field equations are

$$\frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} = 0 \tag{1}$$

etc., by cyclic change of subscripts.

These stress components are connected to the components (u,v,w) of the displacement vector by relations like

$$\begin{aligned} \sigma_z &= 2\mu \frac{\partial w}{\partial z} + \lambda \left(\frac{\partial w}{\partial z} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right), \\ \tau_{zx} &= \mu \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right). \end{aligned} \tag{2}$$

Expressing the field equations (1) in terms of the components of displacements, they become*

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*In all the following formulas, there is a symmetry between u,v, and x,y. The third equation is usually omitted for simplicity in writing. It is always obtained by interchanging u into v, and x into y, and vice versa.

$$\frac{\partial^2 w}{\partial z^2} + \nabla^2 w + (1+\nu)\frac{\partial}{\partial z}\left(\frac{\partial w}{\partial z} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

$$\frac{\partial^2 u}{\partial z^2} + \nabla^2 u + (1+\nu)\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial z} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \qquad (3)$$

$$\frac{\partial^2 v}{\partial z^2} + \nabla^2 v + (1+\nu)\frac{\partial}{\partial y}\left(\frac{\partial w}{\partial z} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

where $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

The boundary conditions for a plate under distributed traction $p(x,y)$ over its faces applied in the direction normal to the middle plane of the plate may be written as

$$\sigma_z(\pm \epsilon h) = \pm p,$$
$$\tau_{zx}(\pm \epsilon h) = \tau_{zy}(\pm \epsilon h) = 0 \qquad (4)$$

where "$(\pm \epsilon h)$" means that the quantities attached are evaluated at $z = \pm \epsilon h$ respectively. By this method, a total traction of $2p$ per unit area of the middle surface of the plate is distributed half on top face and half on bottom face.* The conditions (4) are natural for plates of uniform thickness. But for plates of variable thickness the specifications of conditions (4) are rather artificial, as it would imply a system of loading consisting not only of normal pressure to the faces but also of shearing stress along the faces of the plate. However, the equations of deformation are developed according to the above specification first, and we will return to the more natural

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*Since obviously the middle surface of a plate under the boundary conditions $\sigma_z(\pm \epsilon h) = p$ will not deflect, i.e. $w(o) = 0$, it is evident by the principle of superposition that the differential equations involving $w(o)$ derived from the boundary condition (4) is the same as that derived from the conditions

$$\sigma_z(+\epsilon h) = 2p, \qquad \sigma_z(-\epsilon h) = 0 .$$

boundary conditions for plates of variable thickness at the end of this section.

Assuming that the stress components are analytic throughout the thickness, one obtains, by adding and subtracting the expanded form of (4) in power series, the equations

$$\begin{aligned}
&\sigma_z(0) + \frac{1}{2!}\epsilon^2 h^2 \sigma_z''(0) + \frac{1}{4!}\epsilon^4 h^4 \sigma_z^{(iv)}(0) + \cdots\cdots = 0 \,, \\
&\sigma_z'(0) + \frac{1}{3!}\epsilon^2 h^2 \sigma_z'''(0) + \frac{1}{5!}\epsilon^4 h^4 \sigma_z^{(v)}(0) + \cdots\cdots = \frac{p}{\epsilon h} \,, \\
&\tau_{zx}(0) + \frac{1}{2!}\epsilon^2 h^2 \tau_{zx}''(0) + \frac{1}{4!}\epsilon^4 h^4 \tau_{zx}^{(iv)}(0) + \cdots\cdots = 0 \,, \\
&\tau_{zx}'(0) + \frac{1}{3!}\epsilon^2 h^2 \tau_{zx}'''(0) + \frac{1}{5!}\epsilon^4 h^4 \tau_{zx}^{(v)}(0) + \cdots\cdots = 0 \,.
\end{aligned} \tag{5}$$

and two more equations obtained from the last two by changing the subscript x into y. The accents denote differentiation with respect to z.

Now the stress component

$$\tau_{xy} = \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \tag{2$_{bis}$}$$

is defined exclusively by partials with respect to x and y. But σ_x and σ_y contain partial derivative with respect to z. Let

$$A = \frac{2(\kappa+1)}{\kappa+2} = \frac{1}{1-\nu} \,, \qquad \kappa = \frac{\lambda}{\mu} \,, \tag{6}$$

and define the two-dimensional stress components

$$\begin{aligned}
q_{xx} &= 2\mu \left[A\frac{\partial u}{\partial x} + (A-1)\frac{\partial v}{\partial y} \right] , \\
q_{yy} &= 2\mu \left[(A-1)\frac{\partial u}{\partial x} + A\frac{\partial v}{\partial y} \right] ,
\end{aligned} \tag{7}$$

so that

$$\begin{aligned}
\sigma_x &= (A-1)\,\sigma_z + q_{xx} \,, \\
\sigma_y &= (A-1)\,\sigma_z + q_{yy} \,.
\end{aligned} \tag{8}$$

Substituting (5) and (8) into the field equations (1) which are certainly valid for the particular value of $z = 0$, one obtains, up to fourth order in ϵ ,

$$\frac{p}{\epsilon h} = \frac{\epsilon^2}{2!}\left\{\frac{h^2}{3}\sigma_z''' + \frac{\partial}{\partial x}(h^2\tau_{zx}'') + \frac{\partial}{\partial y}(h^2\tau_{zy}'')\right\}$$
$$+ \frac{\epsilon^4}{4!}\left\{\frac{h^4}{5}\sigma_z^{(v)} + \frac{\partial}{\partial x}(h^4\tau_{zx}^{(iv)}) + \frac{\partial}{\partial y}(h^4\tau_{zy}^{(iv)})\right\},$$

$$\frac{\partial}{\partial x}q_{xx} + \frac{\partial}{\partial y}\tau_{xy} - \frac{\epsilon^2}{2!}\left\{(A-1)\frac{\partial}{\partial x}(h^2\sigma_z'') + \frac{1}{3}h^2\tau_{zx}'''\right\}$$
$$- \frac{\epsilon^4}{4!}\left\{(A-1)\frac{\partial}{\partial x}(h^4\sigma_z^{(iv)}) + \frac{1}{5}h^4\tau_{zx}^{(v)}\right\} = 0, \tag{9}$$

and a third equation by interchanging x and y in (9), where all the partial derivatives are evaluated at $z = 0$.

If all the partial derivatives of the stress components in (9) are expressed in terms of partials with respect to x and y of the displacement components, we will obtain the fundamental differential equations defining the deformation of the elastic plate.

In the following, the fundamental differential equation is developed up to fourth order in ϵ. This requires that σ_z'', τ_{zx}'', τ_{zy}'', σ_z''', τ_{zx}''', τ_{zy}''' be accurate up to second order in ϵ, and $\sigma_z^{(iv)}$, $\sigma_z^{(v)}$, etc. up to zero order. Now the definitions (2) and conditions (5) give, when $z = 0$,

$$\frac{\partial w}{\partial z} = -(A-1)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - \frac{2-A}{4\mu}\epsilon^2 h^2 \sigma_z'',$$
$$\frac{\partial u}{\partial z} = -\frac{\partial w}{\partial x} - \frac{1}{2!}\frac{\epsilon^2 h^2}{\mu}\tau_{zx}'', \qquad \text{etc.} \tag{10}$$

From the field equations (3) and using (10), one obtains

$$\frac{\partial^2 w}{\partial z^2} = (A-1)\nabla^2 w + \frac{A\epsilon^2}{4\mu}\left\{\frac{\partial}{\partial x}(h^2\tau_{zx}'') + \frac{\partial}{\partial y}(h^2\tau_{zy}'')\right\},$$
$$\frac{\partial^2 u}{\partial z^2} = -\nabla^2 u - A\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{A}{4\mu}\epsilon^2\frac{\partial}{\partial x}(h^2\sigma_z''). \tag{11}$$

But from (1) and (8), the field equations can be written as

$$\sigma_z' + \frac{\partial}{\partial x}\tau_{zx} + \frac{\partial}{\partial y}\tau_{zy} = 0$$

$$\tau_{xz}' + \frac{\partial}{\partial x}q_{xx} + (A-1)\frac{\partial}{\partial x}\sigma_z + \frac{\partial}{\partial y}\tau_{xy} = 0, \qquad \text{etc.} \tag{12}$$

The partials σ_z'', σ_z''' etc. can be found by successive differentiation of this equation with respect to z. Note that σ_z'' etc. need only be correct to zero order in (10) and (11). Now from (2), (7), and (10),

$$\tau_{xy}' = -2\mu\frac{\partial^2 w}{\partial x\partial y}, \qquad q_{xx}' = -2\mu\left\{A\nabla^2 w - \frac{\partial^2 w}{\partial y^2}\right\}. \tag{13}$$

Hence by differentiating (12) with respect to , and using (13), (2), and (7), one obtains

$$\frac{1}{2\mu}\sigma_z'' = A\nabla^2\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right), \qquad \frac{1}{2\mu}\tau_{zx}'' = A\nabla^2\frac{\partial w}{\partial x}, \tag{14}$$

from which (10) and (11) can be written in the following form:

$$\frac{\partial w}{\partial z} = -(A-1)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - \frac{A(2-A)}{2}\epsilon^2 h^2\nabla^2\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right), \tag{15}$$

$$\frac{\partial u}{\partial z} = -\frac{\partial w}{\partial x} - A\epsilon^2 h^2\nabla^2\frac{\partial w}{\partial x},$$

$$\frac{\partial^2 w}{\partial z^2} = (A-1)\nabla^2 w + \frac{A^2\epsilon^2}{2}\left\{\frac{\partial}{\partial x}\left(h^2\nabla^2\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(h^2\nabla^2\frac{\partial w}{\partial y}\right)\right\}, \tag{16}$$

$$\frac{\partial^2 u}{\partial z^2} = -\nabla^2 u - A\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{A^2\epsilon^2}{2}\frac{\partial}{\partial x}\left\{h^2\nabla^2\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right\}.$$

Substituting (15) into the results of differentiation of the definitions (2) and (7), one obtains equations similar to (13) but now correct to second order in ϵ:

$$\frac{1}{2\mu}\tau_{xy}' = -\frac{\partial^2 w}{\partial x\partial y} - \frac{A\epsilon^2}{2}\left\{\frac{\partial}{\partial y}\left(h^2\nabla^2\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial x}\left(h^2\nabla^2\frac{\partial w}{\partial y}\right)\right\}$$

$$\frac{1}{2\mu}q_{xx}' = -A\nabla^2 w + \frac{\partial^2 w}{\partial y^2} - A\epsilon^2\left\{A\frac{\partial}{\partial x}\left(h^2\nabla^2\frac{\partial w}{\partial x}\right) + (A-1)\frac{\partial}{\partial y}\left(h^2\nabla^2\frac{\partial w}{\partial y}\right)\right\}. \tag{17}$$

Similarly, a second differentiation of (2) and (7) with respect to z gives

$$\frac{1}{2\mu}\tau''_{xy} = -\frac{1}{2}\nabla^2\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) - A\frac{\partial^2}{\partial x\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) + \frac{A^2\epsilon^2}{2}\frac{\partial^2}{\partial x\partial y}\left[h^2\nabla^2\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right],$$

$$\frac{1}{2\mu}q''_{xx} = -A(1+A)\nabla^2\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) + \nabla^2\frac{\partial v}{\partial y} + A\frac{\partial^2}{\partial y^2}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) \qquad (18)$$

$$+\frac{A^2\epsilon^2}{2}\left(A\nabla^2-\frac{\partial^2}{\partial y^2}\right)\left[h^2\nabla^2\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right].$$

By successive partial differentiation of the field equations (3), (2), and (7) with respect to z, and using (15) and (16), one obtains, correct to zero order in ϵ,

$$\frac{\partial^3 w}{\partial z^3} = (2A-1)\nabla^2\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right), \qquad \frac{\partial^3 u}{\partial z^3} = (1+A)\nabla^2\frac{\partial w}{\partial x},$$

$$\frac{\partial^4 w}{\partial z^4} = (1-2A)\nabla^4 w, \qquad \frac{\partial^4 u}{\partial z^4} = \nabla^4 u + 2A\nabla^2\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right), \text{ etc.} \qquad (19)$$

and

$$\frac{1}{2\mu}\tau'''_{xy} = (1+A)\nabla^2\frac{\partial^2 w}{\partial x\partial y},$$

$$\frac{1}{2\mu}q'''_{xx} = A(1+A)\nabla^4 w - (1+A)\nabla^2\frac{\partial^2 w}{\partial y^2},$$

$$\frac{1}{2\mu}\tau^{(IV)}_{xy} = \frac{1}{2}\nabla^4\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) + 2A\nabla^2\frac{\partial^2}{\partial x\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) \qquad (20)$$

$$\frac{1}{2\mu}q^{(IV)}_{xx} = A(2A+1)\nabla^4\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) - \nabla^4\frac{\partial v}{\partial y} - 2A\nabla^2\frac{\partial^2}{\partial y^2}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right).$$

The derivatives $\sigma''_z(0)$, $\sigma'''_z(0)$ etc. can now be computed.

From (12) and using (2), (7), and (15), one obtains, correct to second order in ϵ,

$$\frac{1}{2\mu}\sigma'_z = \frac{A\epsilon^2}{2}\left\{\frac{\partial}{\partial x}\left(h^2\nabla^2\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(h^2\nabla^2\frac{\partial w}{\partial y}\right)\right\},$$

$$\frac{1}{2\mu}\tau'_{zx} = -A\frac{\partial^2 u}{\partial x^2} - \frac{1}{2}\frac{\partial^2 u}{\partial y^2} + \left(\frac{1}{2}-A\right)\frac{\partial^2 v}{\partial x\partial y} \qquad (21)$$

$$+\frac{A(A-1)}{2}\epsilon^2\frac{\partial}{\partial x}\left[h^2\nabla^2\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right].$$

Similarly, by differentiating (12) with respect to z, and using (17) and (21),

$$\frac{1}{2\mu}\sigma_z'' = A\nabla^2\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) - \frac{A(A-1)}{2}\epsilon^2\nabla^2\left[h^2\nabla^2\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right],$$

$$\frac{1}{2\mu}\tau_{zx}'' = A\nabla^2\frac{\partial w}{\partial x} + \frac{A\epsilon^2}{2}\left\{\left(A\frac{\partial^2}{\partial x^2}+\nabla^2\right)\left(h^2\nabla^2\frac{\partial w}{\partial x}\right)+A\frac{\partial^2}{\partial x\partial y}\left(h^2\nabla^2\frac{\partial w}{\partial y}\right)\right\}. \tag{22}$$

Further differentiation of (12) leads to

$$\frac{1}{2\mu}\sigma_z''' = -A\nabla^4 w - \frac{A\epsilon^2}{2}\left\{\frac{\partial}{\partial x}\left[\left(A\frac{\partial^2}{\partial x^2}+\nabla^2\right)\left(h^2\nabla^2\frac{\partial w}{\partial x}\right)+A\frac{\partial^2}{\partial x\partial y}\left(h^2\nabla^2\frac{\partial w}{\partial y}\right)\right]\right. \tag{23}$$

$$\left.+\frac{\partial}{\partial y}\left[\left(A\frac{\partial^2}{\partial y^2}+\nabla^2\right)\left(h^2\nabla^2\frac{\partial w}{\partial y}\right)+A\frac{\partial^2}{\partial x\partial y}\left(h^2\nabla^2\frac{\partial w}{\partial x}\right)\right]\right\}.$$

and the following expressions in zero order:

$$\frac{1}{2\mu}\tau_{zx}^{(iv)} = -2A\nabla^4\frac{\partial w}{\partial x},$$

$$\frac{1}{2\mu}\sigma_z^{(v)} = 2A\nabla^6 w. \tag{24}$$

All the quantities in equation (9) have now been obtained in required form. A single step of substituting them into equation (9) gives the final result:

$$\frac{p}{\epsilon h} = \mu A\epsilon^2\left\{-\frac{h^2}{3}\nabla^4 w + \frac{\partial}{\partial x}\left(h^2\nabla^2\frac{\partial w}{\partial x}\right)+\frac{\partial}{\partial y}\left(h^2\nabla^2\frac{\partial w}{\partial y}\right)\right\}$$

$$+\frac{2\mu\epsilon^4}{4!}\left\{\left(6\frac{\partial h^2}{\partial x}+4h^2\frac{\partial}{\partial x}\right)\left[\left(A\frac{\partial^2}{\partial x^2}+\nabla^2\right)\left(h^2\nabla^2\frac{\partial w}{\partial x}\right)+A\frac{\partial^2}{\partial x\partial y}\left(h^2\nabla^2\frac{\partial w}{\partial y}\right)\right]\right.$$

$$+\left(6\frac{\partial h^2}{\partial y}+4h^2\frac{\partial}{\partial y}\right)\left[\left(A\frac{\partial^2}{\partial y^2}+\nabla^2\right)\left(h^2\nabla^2\frac{\partial w}{\partial y}\right)+A\frac{\partial^2}{\partial x\partial y}\left(h^2\nabla^2\frac{\partial w}{\partial x}\right)\right]$$

$$\left.+\frac{2h^4}{5}\nabla^6 w - 2A\frac{\partial}{\partial x}\left(h^4\nabla^4\frac{\partial w}{\partial x}\right) - 2A\frac{\partial}{\partial y}\left(h^4\nabla^4\frac{\partial w}{\partial y}\right)\right\}. \tag{25}$$

For a flat plate with constant thickness t, $h = 1$, $2\epsilon = t$, and equation (25) becomes

$$\frac{q}{D} = \nabla^4 w + \frac{t^2}{80}\left[5 + 11(1-\nu)\right]\nabla^6 w. \tag{26}$$

where $q = 2p$ = loading per unit area of the middle surface of the plate,

and
$$D = \frac{Et^3}{12(1-\nu^2)} \tag{27}$$

It is to be noted that the differential equation defining w and those defining u, v are independent of each other.

When the fourth order terms in are neglected, (26) reduces to the familiar form

$$\frac{q}{D} = \nabla^4 w \tag{28}$$

Boundary Conditions

The boundary conditions at the edges of a plate should be specified in consistence with the order of approximation adopted in the differential equations of equilibrium. In a thick plate subjected to given forces, either the stresses or the displacements have prescribed values at every point of the edge. For example, if one edge of a thick plate coincides with the surface x = constant, the boundary conditions at that edge are

1. $u = v = w = o$ for clamped edges,

2. $\sigma_x = \tau_{xy} = \tau_{xz} = o$ for free edges, and

3. Some mixed conditions for supported edges.

In passing the field equation to the limit of a thin plate, these boundary conditions pass to their respective integrated forms. In the first approximation, when the differential equation is of fourth order, two conditions are required at each edge. These may be specified by

1. $w = \frac{\partial w}{\partial n} = 0$ for clamped edges,

2. w and bending moment = o for simply supported edges, and

3. Bending moment = o. Resultant reaction = o for free edges, where the bending moments and resultant reaction are to be calculated only up to zero order in the thickness of the plate.

In the second approximation, the differential equation is of sixth order, so three conditions are required at each edge. These may be specified, for example, by

1. $w = 0$, $\frac{\partial u}{\partial z} = -\frac{\partial w}{\partial x} - A\epsilon^2 h^2 \nabla^2 \frac{\partial w}{\partial x} = 0,$ $\frac{\partial^3 u}{\partial z^3} = (1+A)\nabla^2 \frac{\partial w}{\partial x} = 0,$ for a clamped edge, x = constant.

2. w = o, bending moment = o, twisting moment = o, for simply supported edges,

3. Bending moment = twisting moment = vertical shear = o, for free edges;

while the quantities named in (3) are now to be calculated up to second order in ϵ.

For higher order approximations, the differential equations are of still higher order and more stringent specifications of boundary conditions are necessary.

Explicit expressions of forces and moments can be easily obtained. Consider the edge x = constant. According to our fundamental assumption, the stresses are expandible into power series in z throughout the thickness of the plate. Now

$$
\begin{aligned}
X &= \int_{-\epsilon h}^{\epsilon h} \sigma_x \, dz , & M_x &= \int_{-\epsilon h}^{\epsilon h} \sigma_x z \, dz , \\
Q_x &= \int_{-\epsilon h}^{\epsilon h} \tau_{xz} \, dz , & H_{xy} &= \int_{-\epsilon h}^{\epsilon h} \tau_{xy} z \, dz .
\end{aligned}
\tag{29}
$$

Hence

$$\bar{X} = 2\epsilon h\, \sigma_x(0) + \frac{2(\epsilon h)^3}{3!}\sigma_x'' + \cdots\cdots$$

$$M_x = 2\frac{(\epsilon h)^3}{3}\sigma_x'(0) + 2\frac{(\epsilon h)^5}{5\times 3!}\sigma_x'''(0) + \cdots\cdots$$

$$Q_x = 2\epsilon h\, \tau_{xz}(0) + \frac{2(\epsilon h)^3}{3!}\tau_{xz}''(0) + \cdots\cdots \tag{30}$$

$$H_{xy} = 2\frac{(\epsilon h)^3}{3}\tau_{xy}'(0) + 2\frac{(\epsilon h)^5}{5\times 3!}\tau_{xy}'''(0) + \cdots\cdots$$

The quantities $\sigma_x(0)$, $\sigma_x'(0)$, $\sigma_x''(0)$ etc. should be consistently evaluated according to the order of approximation used. For second approximations (up to fourth order in ϵ) they are given by equations (2), (8), (17), (18), (20), (21), (22), and (23). They reduce to the familiar forms in the first approximation as follows:

$$M_x = \frac{2(\epsilon h)^3}{3}\sigma_x'(0) = -D\left[\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right],$$

$$Q_x = 2\epsilon h\,\tau_{xz}(0) + \frac{2(\epsilon h)^3}{3}\tau_{xz}'' = -D\nabla^2\frac{\partial w}{\partial x}, \tag{31}$$

$$H_{xy} = \frac{2}{3}(\epsilon h)^3\tau_{xy}'(0) = -D(1-\nu)\frac{\partial^2 w}{\partial x\,\partial y};$$

where $D = \dfrac{Et^3}{12(1-\nu^2)}$, $t = 2\epsilon h$ = the thickness of the plate

The resultant reaction referred to above in the conditions for free edges leads to the well-known Kirchhoff's condition for free edges. (Refs. 9 or 10 or 11). In short, the statically equipollent resultant shearing force on the edge x = const. is a distributed load of

$$Q_x + \frac{\partial H_{xy}}{\partial y} = -D\left\{\frac{\partial^3 w}{\partial x^3} + (2-\nu)\frac{\partial^3 w}{\partial x\,\partial y^2}\right\}. \tag{32}$$

and a concentrated force of magnitude Hxy at the corners.

Similar formulas for forces and moments on other edges are easily obtained by coordinate transformations.

Calculation of Stresses

The stresses in the plate due to bending can be computed by series expansions similar to equation (30).

For the first approximation, the formulas for the maximum normal stresses are:

$$(\sigma_x)_{max} = \frac{6 M_x}{t^2} , \qquad (\sigma_y)_{max} = \frac{6 M_y}{t^2} . \tag{33}$$

The shearing stresses parallel to x- and y-axes are obtained from

$$(\tau_{xy})_{max} = - G t \frac{\partial^2 w}{\partial x \partial y} . \tag{34}$$

and the shearing stresses parallel to the z -axis are obtained from the fact that the shearing forces Q_x and Q_y are distributed along the thickness of the plate following the parabolic law, (cf. equations (18) and (30), and remembering that we are concerned only with the first approximation). As in the case of beams of rectangular cross section,

$$(\tau_{xz})_{max} = \frac{3}{2h} Q_{x_{max}} , \qquad (\tau_{yz})_{max} = \frac{3}{2h} Q_{y_{max}} . \tag{35}$$

Plates of Variable Thickness

Returning now to the plates of variable thickness, when the external forces are applied to the plate in the nature of air pressure, i.e. when the tractions $\pm p(x,y)$ are applied on the faces of the plate in the direction normal to the faces, the boundary conditions become

$$\begin{aligned} &\sigma_n (\pm \epsilon h) = \pm p(x,y) , \\ &\tau_{nt_1}(\pm \epsilon h) = 0 , \qquad \tau_{nt_2}(\pm \epsilon h) = 0 . \end{aligned} \tag{36}$$

where n is the outer normal to the face of the plate and t_1 the tangent to the face of the plate in the n-x plane, and t_2 that in the n-y plane. Since $\epsilon h(x,y)$ is assumed to be small, the following table of direction cosines is valid: (see Fig I.12:1)

	x	y	z
n	$-\epsilon h_x$	$-\epsilon h_y$	$\pm\left[1-\frac{\epsilon^2}{2}(h_x^2+h_y^2)\right]$
t_1	$1-\frac{\epsilon^2}{2}h_x^2$	0	$\pm[\epsilon h_x]$
t_2	0	$1-\frac{\epsilon^2}{2}h_y^2$	$\pm[\epsilon h_y]$

where the subscripts x and y denote partial differentiations. The "$\pm$" sign in the last column applies to faces on the $\pm$ side of the z-axis respectively.

By the general tensor transformation of stress-components,

$$\sigma_z(\pm\epsilon h) = \left[1-\frac{\epsilon^2}{2}(h_x^2+h_y^2)\right]^2\sigma_n(\pm\epsilon h) + \epsilon^2\Big[h_x^2\,\sigma_{t_1t_2}(\pm\epsilon h)$$
$$+ h_y^2\,\sigma_{t_1t_2}(\pm\epsilon h) + 2h_xh_y\,\tau_{t_1t_2}(\pm\epsilon h)\Big],$$

$$\tau_{zx}(\pm\epsilon h) = \mp\left[1-\frac{\epsilon^2}{2}(h_x^2+h_y^2)\right]\epsilon h_x\,\sigma_n(\pm\epsilon h) \pm \epsilon h_x\left(1-\frac{\epsilon^2}{2}h_x^2\right)\sigma_{t_1t_2}(\pm\epsilon h) \tag{37}$$
$$\pm\left[\left(1-\frac{\epsilon^2}{2}h_x^2\right)\epsilon h_y + \left(1-\frac{\epsilon^2}{2}h_y^2\right)\epsilon h_x\right]\tau_{t_1t_2}(\pm\epsilon h).$$

Now, up to first order in ϵ,

$$\sigma_{t_1t_1} = \sigma_x\,, \qquad \sigma_{t_2t_2} = \sigma_y \qquad \tau_{t_1t_2} = \tau_{xy} \tag{38}$$

So, correct to third order in ϵ, the stress components $\sigma_{t_1t_1}$ etc. on the right-hand side of (37) can be replaced by σ_x etc.

Expand $\sigma_z(\pm\epsilon h)$ etc. into a power series in $\pm\epsilon h$, and adding and subtracting the expanded form of eqs. (37), to obtain boundary conditions on the faces of the plate similar to eqs. (5), one obtains

$$\sigma_z(0) + \frac{\epsilon^2}{2} h^2 \sigma_z''(0) = \epsilon^2 \left[h_x^2 q_{xx}(0) + h_y^2 q_{yy}(0) + 2 h_x h_y \tau_{xy}(0) \right] + \cdots\cdots$$

$$\sigma_z'(0) = \frac{p}{\epsilon h} + \frac{\epsilon (h_x^2 + h_y^2)}{h} (A-2) p + \epsilon^2 \left[-\frac{h^2}{6} \sigma_z'''(0) + h_x^2 q_{xx}'(0) \right.$$
$$\left. + h_y^2 q_{yy}'(0) + 2 h_x h_y \tau_{xy}'(0) \right] + \cdots\cdots$$

$$\tau_{xz}(0) = -\epsilon h_x p + \epsilon^2 h \left[-\frac{h}{2} \tau_{xz}''(0) + h_x \sigma_x'(0) + (h_x + h_y) \tau_{xy}'(0) \right] + \cdots\cdots$$

$$\tau_{xz}'(0) = \frac{1}{h} \left[h_x \sigma_x(0) + (h_x + h_y) \tau_{xy}(0) \right] + \frac{\epsilon^2}{2h} \left[-\frac{h^3}{3} \tau_{zx}'''(0) \right.$$
$$\left. + h^2 h_x \sigma_x''(0) + h^2 (h_x + h_y) \tau_{xy}''(0) - h_x^3 \sigma_x(0) - (h_x^2 h_y + h_x h_y^2) \tau_{xy}(0) \right] + \cdots\cdots \tag{39}$$

and two similar expressions for $\tau_{zy}(0)$ and $\tau_{zy}'(0)$. All the stress components in these equations are evaluated at $z = 0$, and primes denote differentiations with respect to z. The definitions of q_{xx}, q_{yy} are given by eqs. (7), (8).

Substituting (39) into (1), and recalling (8), one obtains the equation of equilibrium:

$$\frac{p}{\epsilon^3 h} - \frac{h^2}{6} \sigma_z''' + h_x^2 q_{xx}' + h_y^2 q_{yy}' + 2 h_x h_y \tau_{xy}' + \frac{\partial}{\partial x} \left\{ -\frac{h^2}{2} \tau_{zx}'' + h h_x q_{xx}' \right.$$
$$\left. + h (h_x + h_y) \tau_{xy}' \right\} + \frac{\partial}{\partial y} \left\{ -\frac{h^2}{2} \tau_{zy}'' + h h_y q_{yy}' + h (h_x + h_y) \tau_{xy}' \right\} = 0. \tag{40}$$

where all the stress components are taken at $z = 0$. The terms σ_z''' etc. involved in this equation must be expressed in terms of the derivatives of (u, v, w) with respect to x and y. Up to the first order in ϵ, equations (13) through (24) are true in this case. Hence one obtains the differential equation:

$$
\begin{aligned}
q = {} & D \nabla^4 w + \nabla^2 D \nabla^2 w + 2 \frac{\partial D}{\partial x} \nabla^2 \frac{\partial w}{\partial x} + 2 \frac{\partial D}{\partial y} \nabla^2 \frac{\partial w}{\partial y} \\
& - (1-\nu) \left\{ \frac{\partial^2 D}{\partial x^2} \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 D}{\partial y^2} \frac{\partial^2 w}{\partial x^2} - 2 \frac{\partial^2 D}{\partial x \partial y} \frac{\partial^2 w}{\partial x \partial y} \right\} \qquad (41) \\
& + (1-\nu) \left\{ \left[\nabla^2 D - \frac{1-\nu}{3D} \left(\left(\frac{\partial D}{\partial x}\right)^2 + \left(\frac{\partial D}{\partial y}\right)^2 \right) \right] \frac{\partial^2 w}{\partial x \partial y} + \frac{\partial D}{\partial y} \frac{\partial^3 w}{\partial x \partial y^2} + \frac{\partial D}{\partial x} \frac{\partial^3 w}{\partial x^2 \partial y} \right\}.
\end{aligned}
$$

The boundary conditions at the edges of the plate are the same as before.

§ I.13 Differential Equations and Boundary Conditions in Oblique Coordinates

Oblique coordinates may be used to advantage in treating problems of swept plates. In oblique coordinates the coordinates of a point P are given by distances measured parallel to the ξ and η axes, respectively, from the η and ξ axes. (Fig. 1.13:1).

Let x-y be a rectangular coordinate system and let the η -axis be coincident with the y axis, and the ξ -axis be at an angle θ from the x-axis. θ is said to be positive when the ξ -axis lies in the first quadrant.

It is clear from the figure that the coordinate transformation is given by the following equations

$$x = \xi \cos\theta \quad , \qquad y = \eta + \xi \sin\theta \quad . \tag{1}$$

from which

$$ds^2 = dx^2 + dy^2 = d\xi^2 + d\eta^2 + 2\sin\theta \, d\xi \, d\eta \, . \tag{2}$$

Because of the computational complication induced by an oblique angle, a general survey of the swept plate problem is carried out only in the first approximation.

For flat plates of constant thickness t , equation (1.12:28) becomes, in oblique coordinates,

$$\frac{\partial^4 w}{\partial \xi^4} - 4\sin\theta \frac{\partial^4 w}{\partial \xi^3 \partial \eta} + (2 + 4\sin^2\theta)\frac{\partial^4 w}{\partial \xi^2 \partial \eta^2} - 4\sin\theta \frac{\partial^4 w}{\partial \xi \partial \eta^3} + \frac{\partial^4 w}{\partial \eta^4} = K, \tag{3}$$

where

$$K = \frac{q \cos^4\theta}{D} \; ,$$

q = the load per unit area of the plate , (4)

$$D = \frac{E t^3}{12(1-\nu^2)} \; .$$

This can also be written as

$$\left(\frac{\partial^2}{\partial\xi^2}-2\sin\theta\frac{\partial^2}{\partial\xi\partial\eta}+\frac{\partial^2}{\partial\eta^2}\right)\left(\frac{\partial^2 w}{\partial\xi^2}-2\sin\theta\frac{\partial^2 w}{\partial\xi\partial\eta}+\frac{\partial^2 w}{\partial\eta^2}\right)=k\,. \qquad (3)'$$

Expressions for forces and moments in terms of the partials of w with respect to the ξ, η coordinates can be obtained from equations (1.12:31) and (1.12:32). General expressions are very cumbersome in oblique coordinates. Forces and moments on sections $\xi = \text{const.}$, η = const. are expressed as follows:

M_ξ = bending moment per unit length acting on sections ξ = constant.

$$= -\frac{D}{\cos^2\theta}\left\{\frac{\partial^2 w}{\partial\xi^2}-2\sin\theta\frac{\partial^2 w}{\partial\xi\partial\eta}+(\sin^2\theta+\nu\cos^2\theta)\frac{\partial^2 w}{\partial\eta^2}\right\}. \qquad (5)$$

M_η = bending moment per unit length acting on sections η = const.

$= -D\left(\dfrac{\partial^2 w}{\partial n^2}+\nu\dfrac{\partial^2 w}{\partial\xi^2}\right)$ where n is the normal to the section η = const. (6)

$$= -\frac{D}{\cos^2\theta}\left\{(\sin^2\theta+\nu\cos^2\theta)\frac{\partial^2 w}{\partial\xi^2}-2\sin\theta\frac{\partial^2 w}{\partial\xi\partial\eta}+\frac{\partial^2 w}{\partial\eta^2}\right\}.$$

$H_{\xi t}$ = twisting moment per unit length acting on a section ξ = const. in the direction of the normal to the section.

$$= -\frac{D(1-\nu)}{\cos\theta}\left(\frac{\partial^2 w}{\partial\xi\partial\eta}-\sin\theta\frac{\partial^2 w}{\partial\eta^2}\right) \qquad (7)$$

$H_{\eta t}$ = twisting moment per unit length acting on a section η = const. in the direction of the normal to the section

$$= -\frac{D(1-\nu)}{\cos\theta}\left(-\sin\theta\frac{\partial^2 w}{\partial\xi^2}+\frac{\partial^2 w}{\partial\xi\partial\eta}\right) \qquad (8)$$

Q_ξ = Vertical (in z direction) shearing force per unit length acting in a section ξ = const.

$$= -\frac{D}{\cos^3\theta}\left\{\frac{\partial^3 w}{\partial\xi^3}-3\sin\theta\frac{\partial^3 w}{\partial\xi^2\partial\eta}+(1+2\sin^2\theta)\frac{\partial^3 w}{\partial\xi\partial\eta^2}-\sin\theta\frac{\partial^3 w}{\partial\eta^3}\right\}. \qquad (9)$$

Q_η = Vertical (in z -direction) shearing force per unit length acting in a section η = const.

$$= -\frac{D}{\cos^3\theta}\left\{-\sin\theta\frac{\partial^3 w}{\partial\xi^3}+(1+2\sin^2\theta)\frac{\partial^3 w}{\partial\xi^2\partial\eta}-3\sin\theta\frac{\partial^3 w}{\partial\xi\partial\eta^2}+\frac{\partial^3 w}{\partial\eta^3}\right\}. \qquad (10)$$

The determination of the deflection surface of a plate now consists of integrating equation (3) with appropriate boundary conditions. These conditions are summarized as follows:

(1) Built-in edge: w and $\frac{\partial w}{\partial n}$ must be zero. If the edge $\xi = 0$ is built-in, then $(w)_{\xi=0} = 0, \quad \left(\frac{\partial w}{\partial\xi}-\sin\theta\frac{\partial w}{\partial\eta}\right)_{\xi=0}=0$ (11)

(2) Simply-supported edge: w and the bending moment acting on the edge must be zero.

(3) Free edge: The bending moment M_n and the reaction $Q_n+\frac{\partial H_{nt}}{\partial t}$ along the edge must vanish. The twisting moment H_{nt} must also vanish at the sharp corner if that corner is unsupported. Proper expressions for M_n are to be chosen from (5) or (6) on edges ξ = const. or η = const. respectively. For the second condition on the reaction, the results are:

On the edge corresponding to $\xi=a$,

$$\left\{\frac{\partial^3 w}{\partial\xi^3}-3\sin\theta\frac{\partial^3 w}{\partial\xi^2\partial\eta}+(2+\sin\theta-\nu\cos^2\theta)\frac{\partial^3 w}{\partial\xi\partial\eta^2}\right.$$
$$\left.-\sin\theta\left[1+(1-\nu)\cos^2\theta\right]\frac{\partial^3 w}{\partial\eta^3}\right\}_{\xi=a}=0 \qquad (12)$$

and on the edge corresponding to $\eta=b$,

$$\left\{-\sin\theta\left[1+(1-\nu)\cos^2\theta\right]\frac{\partial^3 w}{\partial\xi^3}+(2+\sin^2\theta-\nu\cos^2\theta)\frac{\partial^3 w}{\partial\xi^2\partial\eta}\right.$$
$$\left.-3\sin\theta\frac{\partial^3 w}{\partial\xi\partial\eta^2}+\frac{\partial^3 w}{\partial\eta^3}\right\}_{\eta=b}=0. \qquad (13)$$

I.2 DIRECT METHODS OF SOLUTION

§ I.21 Equivalence Between a Differential System and a Variational Problem

A plate problem can usually be replaced by an equivalent variational problem. This equivalence is of great importance because direct methods of solving the variational problems sometimes offer powerful means to obtain useful approximate solutions. Conversely, a plate problem can be formulated according to the minimum energy theorems, which can be derived from the variation of strain energy of an elastic body either through variation in displacements or through variation in the stress tensor. From the former procedure there results the theorem of minimum potential energy (Ref. 8, § V, p. 66, or Ref. 19, §64) that "of all displacements satisfying given boundary conditions, those that satisfy the equilibrium conditions minimize the potential energy

$$V = U - \iiint_\tau F_i u_i \, dv - \iint_S \overset{n}{T}_i u_i \, ds \ , \tag{1}$$

where the surface integral is taken over that portion of the boundary on which the surface forces $\overset{n}{T}_i$ are prescribed." In this expression U is the strain energy stored in the plate

$$U = \frac{1}{2} \iiint e_{ij} \, p_{ij} \, dv \tag{2}$$

and F_i is the body force vector and $\overset{n}{T}_i$ the vector of surface force acting on an element of surface with an outer normal n_i. u_i is, as before, the displacement vector, and e_{ij}, p_{ij} the strain and stress tensor respectively. It should be emphasized that the displacements must satisfy the boundary conditions on that portion of the surface where the <u>displacements are prescribed</u>, and need to satisfy the same only. For example, the "natural" boundary conditions on free edges are satisfied as a consequence of this theorem, and need not be fulfilled in the choice of u_i .

In Kirchhoff's theory (See Ref. 2), he starts by assuming an expression for total strain energy of the bent plate which contains only terms depending on the action of bending and twisting moments, as follows:

$$U(w)=\frac{1}{2}D\iint_R \left\{ \left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)^2-2(1-\nu)\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2\right]\right\}dx\,dy\,, \qquad (3)$$

where the integration is extended over the entire region R of the mid-surface of the plate. The potential energy of the external forces and moments are given by the work function

$$W(w)=\iint_R q\,w\,dx\,dy-\int_L M_n\frac{\partial w}{\partial n}ds+\int_L\left(Q_n-\frac{\partial H_{nt}}{\partial s}\right)w\,ds \qquad (4)$$

where the line integral is integrated in the positive sense (so that n is to s as the x is to the y axis) along the boundary curve L of the region R. "<u>n</u>" is the outer normal to the boundary curve L of R, and s the length of the boundary curve along the boundary L.

The problem is then to find a unique function w, which minimizes the function

$$V(w)=U(w)-W(w) \qquad (5)$$

with the auxiliary condition of continuity on w and in conformity with the conditions of support at the edges L of the plate.

A necessary condition that V(w) reaches a minimum is that the first variation $\delta V(w)$ vanishes. This leads to the field equation

$$\nabla^4 w = \frac{q}{D} \tag{6}$$

and the boundary conditions

Either $\delta w = 0$ or

$$D\left[(1-\nu)\frac{\partial}{\partial s}\left\{\left(\frac{\partial^2 w}{\partial x^2}-\frac{\partial^2 w}{\partial y^2}\right)\frac{\sin 2\theta}{2} - \cos 2\theta \frac{\partial^2 w}{\partial x \partial y}\right\} - \cos\theta\left(\frac{\partial^3 w}{\partial x^3}+\frac{\partial^3 w}{\partial x \partial y^2}\right) - \sin\theta\left(\frac{\partial^3 w}{\partial y^3}+\frac{\partial^3 w}{\partial x^2 \partial y}\right)\right] - \left(Q_n - \frac{\partial H_{nt}}{\partial s}\right) = 0 \tag{7}$$

and $\delta \frac{\partial w}{\partial n} = 0$ or

$$D\left[(1-\nu)\left(\frac{\partial^2 w}{\partial x^2}\cos^2\theta + \sin 2\theta \frac{\partial^2 w}{\partial x \partial y} + \sin^2\theta \frac{\partial^2 w}{\partial y^2}\right. + \nu \nabla^2 w\right] + M_n = 0$$

where θ is the angle between the outer normal to the boundary and the x-axis. n and s are as usual the outer normal and length of the boundary curve. These results agree with those of Poisson-Love theory. Therefore, the form of the strain energy expression (3) is compatible with the first approximation of the procedure of Section 1.1.

In such a formulation of the fundamental field equation and boundary conditions, however, one is confronted with the problem of forming a proper expression for the potential energy to begin with. The expression (3) is found satisfactory as a first approximation, but what are the proper expressions of U in higher order approximations? For a satisfactory theory one must be able to derive systematically the proper expressions for strain energy of a bent plate to any order of approximation without imposing arbitrary assumptions regarding to the deformation of the plate. In all events it seems that this problem is as complicated and is of the same nature as the problem of the last chapter.

From another point of view, suppose that a differential system for the deformation of a plate is obtained by a certain process of reasoning. But instead of proceeding as usual from the general solution to the particular solution, which ofttimes is either impractical or impossible, one wishes to state the problem in a variational form so that useful approximate solutions can be obtained by direct methods. For example a differential system (6) and (7) can be equivalent to the problem of finding a solution of the equation $\delta V = 0$, where V is given by (5), (3), (4).

In general, a differential system that consists of a differential equation $D(w)$ 0 over a region R and boundary conditions $B_i(w)$ 0 on the boundary curve L, $(i = 1, 2, \cdots)$, can be represented as a problem to find a function w, so that

$$\iint_R D(w)\,\delta w\,dx\,dy + \sum_i \int_L B_i(w)\,\delta_i w\,ds = 0 \tag{8}$$

for arbitrary $\delta_i w$.

If an integral $\iint_R E(w)\,dx\,dy$ can be found of which the vanishing of the first variation leads to the above equation (8), then (8) can be written as $\delta \iint E(w)\,dx\,dy = 0$. Such an integral, in general, has a simple physical meaning, as, for example, an energy integral. But when one starts from a given differential system, it is not at all evident that such an integral can be found easily. The formulation of the equivalent variational equation as (8) therefore avoids this difficulty.

Based on convergence considerations, there is another reason for not attaching too much importance on finding the energy integral. In direct methods one approaches the true solution w by forming a sequence of approximating functions W_N $(N = 1, 2, 3 \ldots)$ and by passing N to infinity to get the required solution. But sometimes when an approximating sequence W_N is found so that the integral $\iint_R E(W_N)\,dx\,dy$

readily converges to the true minimum value, the function w_N itself, or its derivative, may not be convergent. In general, the convergence of w_N or its derivatives is better if the order of the derivatives of w occuring in the integrand E(w) is higher. For example, Courant remarked (Ref. 14) that the spectacular success of the direct procedure of W. Ritz was due to a fortunate choice of the problem of the vibration of plates for his illustration rather than the seemingly easier problems of the membrane. The convergence can be improved by adding to the original minimizing integral expressions which consist of higher order derivatives of w and which vanish for the true solution. For example, the plate problem can be formulated so as to find a function w which minimizes the functional

$$V'(w) = U - W + k \iint_R \left(\nabla^4 w - \frac{q}{D} \right)^2 dx\, dy \tag{9}$$

where U and W are given by (3) and (4) respectively and "k" is any positive constant. Such additional terms make $V'(w)$ more sensitive to variations of w without changing the true solution. In other words, the minimizing sequence attached to such a "sensitized" functional will be forced to be better as regards convergence. However, this "sensitization" certainly spoils the physical meaning of the original integral.

Equation (8) is not the only way to represent a differential system $D(w) = 0$ in a region R and $B_i(w) = 0,\ (i = 1, 2, \cdots)$ on the boundary L of R. It can be as well represented as

$$\iint_R D^2(w)\, dx\, dy + \sum_i \int_L B_i^2(w)\, ds = 0 \tag{10}$$

If the functions $D(w_N)$, $B_i(w_N)$, $(i = 1, 2, \cdots)$ where is an approximating functional sequence of w, be interpreted as error functions, then the problem is equivalent to finding a sequence $\{w_N\}$ so that the square error

$$\iint_R D^2(w_N)\,dx\,dy + \sum_i \int_L B_i^2(w_N)\,ds \tag{11}$$

be minimized.

The classical method of solving a variational problem by forming its Euler's equation and then solving it with appropriate boundary conditions is an indirect method. The alternative idea of looking for a single sequence of functions which solves the given minimal problem is called the direct method. First conceived by J. Bernoulli and later by Riemann (Ref. 15), it was developed by the fundamental work of Hilbert (Ref. 16). Its utilization to numerical calculations of the solutions was independently envisaged by Lord Rayleigh (Ref. 5) and Walther Ritz (Ref. 17, 18). Since then numerous applications and improvements have been made by many authors. (See Refs. 10, 11, and 19).

Stated in general terms, the problem is the following: Given an integral expression $V[\varphi]$ over a given domain R of the independent variables, whose boundary satisfies all the desired continuity assumptions; to find a function $\varphi = w$ for which $V[w] = d$, the lower limit of $V[\varphi]$ for the totality of all φs which satisfy the continuity conditions and boundary conditions.

It is assumed that such a lower limit exists, and that the integrand satisfies all the continuity and regularity conditions.

The assumption on the definability implies the existence of a so-called minimal sequence φ_1, φ_2, φ_3, ... of allowable functions, for which

$$\lim_{n \to \infty} V[\varphi_N] = d \tag{12}$$

The fundamental conception of all direct methods is such that each φ_N be obtained from a definite elementary minimal problem, and through the passing to the limit $N \to \infty$ to get the required solution w of the given minimal problem. These two principal steps, the construction of the minimal sequence on the basis of common minimal problems and the passing to the limit, characterize the direct methods.

§ 1.23 RAYLEIGH-RITZ METHOD AND ITS ALLIED METHODS

The Rayleigh-Ritz method consists in narrowing down the choice of φ (See the last paragraph of § 1.21) into a smaller class of functions. It is assumed that φ can be written as functions that satisfy the boundary conditions and involve a number of parameters. These parameters are then determined so as to make $V[\varphi]$ a minimum. The approximating functions must satisfy the boundary conditions, but in general do not the differential equations.

The deflection w of an elastic plate minimizes the potential energy integral (1.21:5). In order to obtain an approximate solution, one assumes that w can be represented with sufficient accuracy by a series of the form

$$W_N = \sum_{i=1}^{N} c_i \varphi_i(x, y) \tag{1}$$

where $\{\varphi_i\}$ is so chosen that w_N satisfies the same continuity conditions of w and boundary conditions on the edges where displacements are prescribed. If (1) is inserted for w into the integral V, equation (1.21:5), the latter becomes a quadratic function of the parameters c_i $(i = 1, 2, \cdots N)$. The minimizing conditions

$$\frac{\partial V_N}{\partial c_i} = 0 \qquad (i = 1, 2, \cdots N) \tag{2}$$

provide, therefore, N linear equations to determine the N unknown constants c_i. The approximate deflection surface w_N is thus determined. This is the Rayleigh-Ritz procedure (Refs. 5, 17, 18).

Galerkin's method (Ref. 20) is more closely related to equation (1.21:8). If w_N satisfies all the boundary conditions $B_i(w)$, $i = 1, 2, \cdots$, then the last member of (1.21:8) vanishes automatically when w_N is substituted into it, which now becomes

$$\iint_R D(w_N)\, \delta w \, dx\, dy = 0 \tag{3}$$

This cannot be satisfied by arbitrary variations δw, otherwise w_N would be an exact solution. The constants c_i in (1) can, however, be so chosen that (3) is satisfied for a set of N values of δw, $k_i \varphi_i(x,y)$, where $\{\varphi_i(x,y)\}_{,}\ i=1,2\cdots N$ are the coordinate functions of w_N, as defined by equation (1), and $\{k_i\}$ are positive constants. If now, N becomes infinite and the functions $\{\varphi_i(x,y)\}$ form a complete system of functions, then the set of all relations

$$\iint_R D(w_N)\, \varphi_i \, dx\, dy = 0\,, \qquad (i = 1, 2 \cdots N) \tag{4}$$

becomes equivalent to the relation (3).

A generalization due to L. V. Kantorovitch (Ref. 21) consists in replacing the constants c_i by unknown functions of one variable $c_i(x)$, for example, and an application of the minimum principle leads to a system of ordinary differential equations for the functions $c_i(x)$.

The collocation method of Biezeno and Koch (Refs. 22, 23) consists in specifying that $D(w_N)$ be equal to zero at N points of the region, and that of Courant (Ref. 24) in demanding that

$$\delta w(x,y) = \begin{cases} 1 & \text{in } R_i\,, \quad (i = 1, 2 \cdots N) \\ 0 & \text{elsewhere in R}\,, \end{cases} \tag{5}$$

where R_i $(i = 1, 2 \cdots N)$ is a subdivision of the entire region R.

The N equations

$$\iint_{R_i} D(w_N)\,dx\,dy = 0 \qquad (i = 1, 2 \ldots N) \tag{6}$$

are then just enough to determine the constants c_i.

The method of least squares (Refs. 26, 27, and 28), is related to eq. (1.21:10). The constants c_i are to be determined by requiring that the mean square error be as small as possible. That is,

$$\iint_R \left[D(w_N) \right]^2 dx\,dy = \text{Min.} \tag{7}$$

Hence there are N equations

$$\iint_R D(w_N)\,\frac{\partial D(w_N)}{\partial c_i}\,dx\,dy = 0, \qquad (i = 1, 2 \ldots N) \tag{8}$$

to determine the constants c_i.

These methods, although sometimes easier to apply, are narrower in scope than the original Rayleigh-Ritz method inasmuch as all the boundary conditions $B_i(w)$ should be satisfied by w_N; while in the Rayleigh-Ritz method, which starts from the original energy integral, only "rigid" boundary conditions (in contrast to the "natural" boundary conditions) are required to be satisfied by w_N. In cantilever plates, it is the "natural" boundary conditions which cause great difficulty.

All the methods outlined above are characterized by the fact that the potential energy V is approached from <u>above</u> by the approximating functions w_N $\sum_{i=1}^{N} c_i \varphi_i$. In other words, if $V(w_N)$ is the value of V when the true solution w is replaced by w_N, then

$$V(w_N) \geqslant V(w). \tag{9}$$

This follows directly from the general minimum energy theorem. Using Green's formula, it can be verified directly, and one obtains the following result:

$$V(w_N) = V(w) + U(\varepsilon_N) \tag{10}$$

where $U(\varepsilon_N)$ is the integral (1.21:3) when w is substituted by ε_N, and ε_N is the error function

$$\varepsilon_N(x,y) = w(x,y) - w_N(x,y) \quad . \tag{11}$$

$U(\varepsilon_N)$ is the strain energy stored in a plate when it is subjected to a fictitious deflection $w = \varepsilon_N(x,y)$ and is of course a positive quantity.

I.23 METHODS OF RELAXATION OF BOUNDARY CONDITIONS

Equation (1.21:9) suggests immediately another method to obtain an approximate solution for w. As before, assume that w can be represented by an approximating series

$$W_N = \sum_{i=1}^{N} b_i\, g_i(x,y) \qquad (N = 1, 2, 3 \ldots\ldots) \qquad (1)$$

where b_i are unknown constants and $g_i(x,y)$, $(i = 1, 2, \cdots)$ the so-called coordinate functions, satisfying the field equation (1.21:6). Hence W_N is a solution of the field equation and the first part of equation (1.21:8) vanishes. For the plate problem, there remains an equation in the form of a line integral

$$\int_L B_1(w)\, \delta \frac{\partial w}{\partial n}\, ds + \int_L B_2(w)\, \delta w\, ds = 0\,, \qquad (2)$$

where

$$B_1(w) \begin{cases} = D\left[(1-\nu)\left(\cos^2\theta \dfrac{\partial^2 w}{\partial x^2} + \sin 2\theta \dfrac{\partial^2 w}{\partial x \partial y} + \sin^2\theta \dfrac{\partial^2 w}{\partial y^2}\right) + \nu \nabla^2 w\right] + M_n & \text{when the edge is free to rotate} \\ \text{or} \\ = PL \dfrac{\partial w}{\partial n} & \text{when the corresponding edge is clamped} \end{cases} \qquad (3)$$

$$B_2(w) \begin{cases} = D\left[(1-\nu)\dfrac{\partial}{\partial s}\left\{\dfrac{1}{2}\sin 2\theta\left(\dfrac{\partial^2 w}{\partial x^2} - \dfrac{\partial^2 w}{\partial y^2}\right) - \cos 2\theta \dfrac{\partial^2 w}{\partial x \partial y}\right\} \right. \\ \qquad \left. - \cos\theta\, \nabla^2 \dfrac{\partial w}{\partial x} - \sin\theta\, \nabla^2 \dfrac{\partial w}{\partial y}\right] - \left(Q_n - \dfrac{\partial H_{nt}}{\partial s}\right) & \text{when the edge is free to displace} \\ \text{or} \\ = \dfrac{P}{L} w & \text{when the corresponding edge is supported or clamped} \end{cases} \qquad (4)$$

where P and L are characteristic dimensions of force and length respectively.

When (1) is substituted for w into (2), this equation cannot be satisfied unless w is an exact solution. However, the coefficients $\{b_i\}$, $i = 1, 2, 3 \cdots\cdots$ can be determined so that this equation is best approximately satisfied. This method was suggested by Courant (Ref. 15) and Trefftz (Ref. 29) who derived it from the condition of least square error on the boundary, i.e., in specifying particularly

$$\delta W = B_2(w) \quad , \qquad \delta \frac{\partial w}{\partial n} = B_1(w) \tag{5}$$

in equation (2) above. Then the left hand side of (2) is a quadratic function in the constants $\{b_i\}$, $i = 1, 2, \cdots N$. Regarding this as a mean square error, an ordinary maximum-minimum process leads to a set of N linear simultaneous equations:

$$\int_L B_1(w_N)\, B_1(g_i)\, ds + \int_L B_2(w_N)\, B_2(g_i)\, ds = 0 , \quad (i = 1, 2, \cdots) \tag{6}$$

from which the N unknown constants $\{b_i\}$ can be solved.

By assigning to δw and $\delta \frac{\partial w}{\partial n}$ suitable forms, many approximating procedures can be formulated. Every method enumerated in the last part of section (I.22) has a counter part in the present case. It is needless to state them individually in detail. For example, an analogy to Galerkin's method can be obtained by taking

$$\delta W = g_i(x, y) \quad , \qquad \delta \frac{\partial w}{\partial n} = \frac{\partial g_i(x,y)}{\partial n} \quad , \qquad (i = 1, 2, \cdots N); \tag{7}$$

Collocation method by taking δw and $\delta \frac{\partial w}{\partial n}$ to be unity at N points and zero everywhere else; Courant's modified collocation method, by dividing the boundary into N parts, and consider δw and $\delta \frac{\partial w}{\partial n}$ to be unity in one part a time, and zero in others; and so on. The practical value of each depends on the nature of the particular problem.

For Dirichlet's problems, the method of relaxation of boundary conditions always approaches the minimizing integral from <u>below</u>, that is, if the functional $I(\varphi)$ is being minimized, then

$$I(\varphi_{approx.}) \leq I(\varphi_{true}), \qquad \text{for any } N. \tag{8}$$

Hence it is a true counterpart of the Rayleigh-Ritz procedure, which approaches V from <u>above</u>. In the plate problems the situation is different. Let us again define an error function ε_N by

$$\varepsilon_N = w(x,y) - w_N(x,y)\,. \tag{9}$$

Then

$$\begin{aligned} V(w) = V(w_N + \varepsilon_N) &= V(w_N) + U(\varepsilon_N) \\ &+ D\iint_R \Big\{ \nabla^2 w_N \nabla^2 \varepsilon_N + (1-\nu)\Big[\frac{\partial^2 w_N}{\partial x^2}\frac{\partial^2 \varepsilon_N}{\partial y^2} + \frac{\partial^2 w}{\partial y^2}\frac{\partial^2 \varepsilon_N}{\partial x^2} \\ &- 2\frac{\partial^2 w_N}{\partial x \partial y}\frac{\partial^2 \varepsilon_N}{\partial x \partial y}\Big] - \frac{q}{D}\varepsilon_N \Big\}\, dx\,dy - \int_L M_n \frac{\partial \varepsilon_N}{\partial n} ds + \int_L \Big(Q_n - \frac{\partial H_{nt}}{\partial s}\Big)\varepsilon_N ds \end{aligned} \tag{10}$$

where $V(w)$ is given by (1.21:5) and $U(\varepsilon_N)$ is given by (1.21:3).

Transforming the integrals on the right hand side of (10) by Green's formula and integrating by parts, and remembering that

$$\nabla^4 w_N - \frac{q}{D} = 0 \qquad \text{in } R\,,$$

one obtains:

$$V(w) = V(w_N) + U(\varepsilon_N) + \int_L B_1(w_N)\frac{\partial \varepsilon_N}{\partial n} ds + \int_L B_2(w_N)\varepsilon_N ds\,. \tag{11}$$

The last two line integrals are minimized in the process of determining the unknown constants $\{b_i\}$ of w_N, but, in general, they do not vanish. If all the edges are clamped, then the true solution w and $\frac{\partial w}{\partial n}$ will have the boundary values zero, and the last integrals become identical with eq. (6) or corresponding to the analogy to Gaderkin's solution, and they vanish by the process of determining $\{b_i\}$. In this case, the tendency of approaching V from below is certain. However, for plates with free edges, the true values of w and $\frac{\partial w}{\partial n}$ are unknown, and the determination of the sign and magnitude of these integrals are uncertain.

II. AEROELASTIC PROBLEMS OF HIGH SPEED AIRPLANES

§ 2.1 Introduction

In the field of aeroelasticity concerned with thin swept wings of high speed airplanes, the aeronautical engineer is encountering a new phenomenon: the transient nature of aerodynamic coefficients. A supersonic airplane must reach its full speed or recover from its full speed to landing speed in a relatively short time. Thus a very important part of the flight is essentially accelerating or decelerating. Now in the transonic region, the effect of compressibility is particularly pronounced, and the nature of aerodynamic response to an oscillating wing changes in a rapid and subtle manner: from a phase lag in the subsonic case to a phase lead in the supersonic case. The aeroelastic phenomena in this important region can be studied only if the transient nature of the aerodynamic coefficients are taken into account. If the engine power is properly scheduled, so that from performance computations the velocity of flight as a function of time is known, then all the coefficients can be regarded as known functions of time. A particularly important case is that the transient perturbations are small, i.e., that the part of a coefficient which varies with time is small compared to the coefficient itself. In this case the computation is relatively simple.

This report treats the basic mathematical development of the problem. In aeroelasticity the determination of physical constants involved in the problem might be a more difficult task. But granting that these coefficients are known, there is still a need for a systematic method of attack. At present it seems that no work has been published in this field: on problems such as the effect of transient perturbations on flutter, dynamic stability, etc.

This transient perturbation theory includes the ordinary small perturbation theory as a particular case. The perturbation theory extends the meaning of a particular solution which is usually obtained for a simplified picture of the actual case. By the fact that the solution of a boundary value problem depends continuously on the parameters of the differential system , it is natural to look for the possibility of treating problems in which the variation of parameters from the solved case is infinitesimal. For example in the later part of this report it is often necessary to construct a sequence of orthogonal functions with preassigned boundary conditions. This can usually be done by finding the natural modes of free vibrating elastic bodies. If ψ_m is the m^{th} mode of a vibrating string, or bar, or membrane, or plate, with density distribution ρ , then

$$\int \rho\, \psi_m \psi_n \, d\sigma = \begin{cases} \text{Constant} & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases}$$

$\{\psi_m\}$ can usually be determined when ρ is a constant ρ_0 . The perturbation theory then gives $\{\bar{\psi}_m\}$ when ρ is slightly different from ρ_0 . This problem has been treated by Lord Rayleigh (Ref. 5, p. 113), H. Schmidt (Ref. 39, Teil 11, Kap. 3, §6) and W. Meyer zur Capellen (Ref. 40.)

§ 2.2 A General Solution

Assume that the dynamical system can be described with sufficient accuracy by n generalized coordinates $q_1, q_2, \cdots q_n$, n being finite. The equilibrium configuration is taken as $(0, 0, 0, \cdots 0)$. If the system is conservative, the set of displacements $q_1, q_2 \cdots q_n$ will produce forces $Q_1, Q_2, \cdots Q_n$ that are derivatives of the potential energy of the system. In the vicinity of the equilibrium position they can be expressed by the linear expressions

$$Q_i = -\frac{\partial U}{\partial q_i} = -\sum_{j=1}^{n} k_{ij}\, q_j \;,$$

where $i = 1, 2, \cdots n$ and $k_{ij} = k_{ji}$. On the other hand, in a non-conservative system, the forces may still depend on the coordinates, and, in the neighbourhood of the equilibrium position, are given by linear functions of coordinates, but in this case $k_{ij} \neq k_{ji}$. In aeroelasticity, the aerodynamic forces are usually linear functions of the velocity and acceleration components in the neighbourhood of the equilibrium position. The part due to acceleration, the apparent-mass forces, which have their origin in the inertia of the surrounding air, can be included in the inertia forces. The part due to velocity can be written as

$$Q_i = -\sum_{j=1}^{n} \beta_{ij}\, \dot{q}_j$$

the dot indicating differentiation with respect to time. To this may be added a disturbing force γ_i due to gust or control movements. Hence in general

$$Q_i = -\sum_{j=1}^{n} k_{ij}\, q_j - \sum_{j=1}^{n} \beta_{ij}\, \dot{q}_j + \gamma_i \,. \quad (i = 1, 2, \cdots n). \qquad (1)$$

Substituting into Lagrange's equations, one obtains the equations of motion

$$\sum_{j=1}^{n} a_{ij}\, \ddot{q}_j = -\sum_{j=1}^{n} k_{ij}\, q_j - \sum_{j=1}^{n} \beta_{ij}\, \dot{q}_j + \gamma_i \,, \quad (i = 1, 2, \cdots n) \qquad (2)$$

a_{ij}, k_{ij}, β_{ij} and γ_i are functions of airplane geometry and speed of flight, they are either constants or continuous functions of time.

When the coefficients are all constants, the classical approach is to assume $q_j = c_j e^{i\omega t}$, substitute into eqs. (2) and determine ω from the determinantal equation $\Delta(\omega)=0$. The constants c_j are then determined from the initial conditions.

For stability problems, an investigation of the nature of the roots of the determinantal equation will be enough to determine the stability characteristics. For problems of disturbed motion, such as rolling power, controllability, maneuver, divergence, gust loading problems and so on, it is the motion following the disturbance that must be computed. In such cases it is simpler to take care of the initial conditions at the start. This can be done either by the following method or by an application of operational calculus (Ref. 41, 42).

When the coefficients are functions of time, the above method must be modified. It is simpler to reduce eq. (2) into a first-order system. Write

$$x_1 = q_1 , \quad x_2 = \dot{q}_1 , \quad x_3 = q_2 , \quad \cdots , \quad x_{2n} = \dot{q}_n , \tag{3}$$

then eq. (2) reduces into the form

$$\sum_{j=1}^{2n} b_{ij}(t)\,\dot{x}_j = \sum_{j=1}^{2n} c_{ij}(t)\,x_j + \xi_i(t) \tag{4}$$

The set of equations (4) is expressible as the single matrix equation

$$[\, b_{ij} \,] \{ \dot{x}_i \} = [\, c_{ij} \,] \{ x_i \} + \{ \xi_i \} \qquad (4)a$$

or when no ambiguity can result, even more concisely as

$$b\dot{X} = cX + \xi . \tag{4)b$$

If the matrix b is non-singular, $|b| \neq 0$, so that b^{-1} exists, then eq. (4b) can be premultiplied by b^{-1} to give

$$\dot{X} = AX + B. \tag{5)a$$

If the matrix b is singular, of rank r, then (4) will be reducible to a system of r independent equations of the above form and $2n-r$ purely algebraic relations connecting the $2n$ variables. These $2n-r$ linear relations can be used to eliminate $2n-r$ of the variables x_i from the remaining equations of the system.

Therefore it is assumed that the differential system is

$$\begin{cases} \dot{X} = A(t)X + B(t) \\ X(t=0) = X_0 \end{cases} \tag{5}$$

where X is a column matrix of the variables of motion, X_0 the initial conditions, $B(t)$ a column matrix representing the disturbing forces and $A(t)$ a square matrix consists of aerodynamic and inertia and stiffness coefficients.

The classical theory deals with the case when A and B are matrices of constants. The fundamental problem of this section is the case when A and B are functions of time. Special attention is paid to the effect of a small change in the elements of A on the final solution. In other words, if $X(A|t)$ is a solution of equation (5), we look for the solution of the equation

$$\dot{X} = (A+\delta A)X + B, \qquad X(t=0) = X_0. \tag{6}$$

as a perturbation to $\chi(A|t)$.

The general solution of (5) is obtained as follows. Integrating eq. (5) from o to t, one obtains

$$\chi(t) = \int_0^t A(\tau)\chi(\tau)\,d\tau + \int_0^t B(\tau)\,d\tau + \chi_0. \tag{7}$$

Let

$$f(t) \equiv \int_0^t B(\tau)\,d\tau + \chi_0. \tag{8}$$

Then (7) can be written as

$$f(t) = \chi(t) - \int_0^t A(\tau)\chi(\tau)\,d\tau. \tag{9}$$

This is the equation of motion following the disturbance. It is a well-known matrix Volterra integral equation of the second kind. Since $A(\tau)$ is, by nature of the problem, a matric valued continuous function of time, the existence of a unique continuous solution is assured. As a well-known result we can write down the solution explicitly at once:

$$\chi(A|t) = f(t) + \int_0^t A(\tau)f(\tau)\,d\tau + \sum_{i=1}^{\infty}\int_0^t A(\tau)\,d\tau\int_0^{\tau}\cdots\int_0^{\tau_{i-1}} A(\tau_i)f(\tau_i)\,d\tau_i. \tag{10}$$

At this point a slight generalization may be introduced. The kernel $A(\tau)$, $0 \le \tau \le t$, (t is the point of time in question), can always be expressed as a power series in τ. But sometimes it is more convenient to express A as a function of both t and τ, for example, as $e^{-(t-\tau)}$. This case occurs in problems of determining the disturbed motion at the end of an accelerating period of the airplane. We may then write $A(t,\tau)$. But to distinguish from the former case let it be denoted by $K(t,\tau)$, reserving the notation A(t) to functions of τ only.

Equation (9) then corresponds to

$$f(x) = \chi(t) - \int_0^t K(t,\tau)\,\chi(\tau)\,d\tau \tag{11}$$

where $K(t,\tau)$ is a fixed square matrix of continuous functions of two real variables t and τ in the triangle T: $0 \le \tau \le t \le b$; $f(x)$ is a fixed column matrix of continuous functions of t in $0 \le t \le b$, defined by eq. (8); and X is, of course, also matric valued.

The unique continuous solution of (11) is given by the formula (Ref. 43)

$$\chi(t) = f(t) + \int_0^t k(t,\tau)\,f(\tau)\,d\tau \tag{12}$$

where the resolvent kernel $k(t,\tau) \equiv K(t,\tau) + K^{*2}(t,\tau) + K^{*3}(t,\tau) + \cdots$; (13)

is a matric valued continuous function of t and τ in T, with the following notations:

$$\begin{aligned}
K^{*2}(t,\tau) &\equiv \int_\tau^t K(t,\xi)\,K(\xi,\tau)\,d\xi \\
K^{*3}(t,\tau) &\equiv \int_\tau^t K(t,\xi)\,K^{*2}(\xi,\tau)\,d\xi \\
&\cdots\cdots \\
K^{*i+1}(t,\tau) &\equiv \int_\tau^t K(t,\xi)\,K^{*i}(\xi,\tau)\,d\xi \\
&\cdots\cdots
\end{aligned} \tag{14}$$

From the general property of the Volterra integral equation of the second kind, $k(t,\tau)$ is a continuous function of t and τ; the series $K + K^{*2} + K^{*3} + \cdots$ converges uniformly and absolutely for any continuous kernel $K(t,\tau)$.

From the general solutions (10) and (12), it follows that for any dynamical system with a finite degree of freedom, the solution is a continuous function of the initial conditions and of the coefficients of the differential equations, provided that the coefficients are continuous functions of time. This proves the statement quoted in the introduction, § 2.1.

2.3 A Generalized Taylor's Series

The following is a shorter derivation of a series expansion of A.D. Michal (Refs. 31, 44, and 45) in abstract space.

Let us define a particular Banach space E, that is, a complete normed linear space closed under multiplication by real or complex numbers, which forms a normed linear ring. The elements $K_i(t,\tau)$ of E are matric valued continuous functions of two real variables t and τ in the domain T: $a \le \tau \le t \le b$. The addition of elements and multiplication by numbers are defined in the usual way, and the norm of $K_i(t,\tau)$ is defined by any of the equivalent definitions of the norm of a matrix. The ring multiplication is defined by the following:

$$K_1 * K_2 \equiv \int_\tau^t K_1(t,\xi)\, K_2(\xi,\tau)\, d\xi\,, \qquad (K_1, K_2 \in E)\,. \tag{1}$$

In order to simplify the notation for an abstract series, an element I is added to E, which has the properties: $\|I\| = 1$, $K_1 * I = I * K_1 = K_1$, $(K_1 * I) * K_2 = K_1 * (I * K_2)$, $K_1 + I = I + K_1$, $(K_1 + I) + K_2 = K_1 + (I + K_2)$, $(K_1 + I) * K_2 = K_1 * K_2 + I * K_2$, and $K_1 * (I + K_2) = K_1 * I + K_1 * K_2$; for any K_1, K_2 in E. Let this new space be called E$'$.

The composition multiplication defined above is in general not permutable, that is, $K_1 * K_2 \ne K_2 * K_1$. That E$'$ is really a normed linear ring can be easily verified. For example, the associative law of composition multiplication is verified in the next section, § 2.4. As a consequence of the associative law

$$(K_1 * K_2) * K_3 = K_1 * (K_2 * K_3)\,, \tag{2}$$

the continued multiplication of a finite number of elements in E$'$ is uniquely defined. In particular, all the powers of K are uniquely defined. Similarly, terms like $K_1^{*i} * K_2^{*j} * K_3^{*l}$ have unique interpretations.

Now let us consider a power series of K in E$'$. From a general property of the Volterra resolvent kernel, the series $I + K + K^{*2} + \cdots$ converges absolutely in E$'$. Denote the sum by $\Omega(K|t,\tau)$ to signify the functional dependence of $\Omega(t,\tau)$ on the kernel K, and so obtain the formula

$$I + K + K^{*2} + K^{*3} + \cdots \xrightarrow{abs.} \Omega(K|t,\tau) \text{ in } E', \quad \text{(any K in E)}. \qquad (13)$$

In other words, $\Omega(K|t,\tau)$ is an entire functional of K in E$'$.

From this it follows that

$$I + (A+B) + (A+B)^{*2} + (A+B)^{*3} + \cdots \xrightarrow{abs.} \Omega(A+B|t,\tau) \text{ in } E',$$
$$\text{(any A,B in E)}.$$

Evidently the following series, obtained by expanding all the brackets above,

$$I + A + B + A^{*2} + A*B + B*A + B*B + A^{*3} + A^{*2}*B + A*B*A + \cdots\cdot$$

also converges absolutely in E$'$ for any A and B in E.

But an absolutely convergent series in E$'$ can be rearranged without changing its sum. Now rearrange the above series into a double series:

$$I + A + A^{*2} + A^{*3} + \cdots\cdot$$
$$+ (I + A + A^{*2} + \cdots\cdot) * B * (I + A + A^{*2} + A^{*3} + \cdots\cdot)$$
$$+ (I + A + A^{*2} + \cdots\cdot) * B * (I + A + A^{*2} + \cdots) * B * (I + A + A^{*2} + \cdots)$$
$$+ \cdots\cdot,$$

sum by rows and obtain the second formula

$$\Omega\,(A+B\,|\,t,\tau) = \Omega\,(A\,|\,t,\tau) + \Omega\,(A) * B * \Omega\,(A) + \Omega\,(A) * B * \Omega\,(A) * B * \Omega\,(A) + \cdots, \qquad \text{for any } A,\ B \text{ in } E,\ \Omega \text{ in } E' \tag{4}$$

This is the fundamental expansion theorem. It means that the functional $\Omega\,(A+B\,|\,t,\tau)$ with kernel $A\,(t,\tau) + B(t,\tau)$ can be expanded into a generalized Taylor's series in "powers" of $B\,(t,\tau)$ with "coefficients" the functionals $\Omega\,(A\,|\,t,\tau)$.

Apply this result to the matric integral equation

$$f(t) = \chi(t) - \int_0^t K\,(t,\tau)\,\chi\,(\tau)\,d\tau \qquad (2.2{:}11)_{\text{bis}}$$

which has the unique continuous solution

$$\chi\,(K\,|\,t) = f(t) + \int_0^t k\,(t,\tau)\,f(\tau)\,d\tau \qquad (2.2{:}12)_{\text{bis}}$$

where $\quad k\,(t,\tau) = K\,(t,\tau) + K^{*2}(t,\tau) + K^{*3}\,(t,\tau) + \cdots .$ $\qquad (2.2{:}13)_{\text{bis}}$

Recall that $\Omega\,(K\,|\,t,\tau)$ is I plus a continuous function of t, τ in T, which is the sum of an absolutely convergent series with the general term a continuous function of t, τ in T, so it can be postmultiplied by a column matrix $f\,(\tau)$ and integrated from o to t term-by-term. Thus the solution (2.2:12) of the integral eq. (2.2:11) can be written as

$$\chi\,(K\,/\,t) = \Omega\,(K) * f\,,$$

and therefore, following (4),

$$\begin{aligned}
\chi\,(K + \delta K\,|\,t) &= \Omega\,(K + \delta K) * f \\
&= [\,\Omega\,(K) + \Omega(K) * \delta K * \Omega(K) + \cdots\,] * f \\
&= \chi\,(K) + \Omega\,(K) * \delta K * \chi(t) + \sum \Omega\,(K) * \delta K * \cdots * \Omega(K) * \delta K * \chi(K).
\end{aligned}$$

This expansion is valid for all continuous $\delta K(t,\tau)$ in T: $0 \le \tau \le t \le b$. Further explanation is given in the next Section, § 2.4.

2.4 Elementary Derivation of the Series Expansion of Solutions

The results of the preceding section will be derived again in elementary way without using the notion of abstract space.

In section 2.2, the composition powers of $K(t,\tau)$ are defined by equations (2.2:14). A generalization of this definition to products of two functions $K_1(t,\tau)$ and $K_2(t,\tau)$ is as follows:

$$K_1 * K_2(t,\tau) = \int_\tau^t K_1(t,\xi)\, K_2(\xi,\tau)\, d\xi. \tag{1}$$

The star product is a convenient shorthand for writing integrals. $K_1(t,\tau)$ and $K_2(t,\tau)$ are square matrices of continuous functions of two variables t and τ in the domain T: $a \le \tau \le t \le b$, each matrix in particular may consist of a single function. The formal work in the following is the same whether one thinks of K as a single continuous function or a square matrix of continuous functions.

Now for three functions $K_1(t,\tau)$, $K_2(t,\tau)$ and $K_3(t,\tau)$, it can be verified that

$$K_1 * (K_2 * K_3) = (K_1 * K_2) * K_3. \tag{2}$$

For let $L(t,\tau) = K_1 * K_2$, $M(t,\tau) = K_2 * K_3$;

Then

$$(K_1 * K_2) * K_3 \overset{\text{def.}}{=} \int_\tau^t L(t,\xi)\, K_3(\xi,\tau)\, d\xi$$

$$= \int_\tau^t d\xi \int_\xi^t K_1(x,\eta)\, K_2(\eta,\xi)\, K_3(\xi,\tau)\, d\eta$$

which, by Dirichlet's lemma on the change of order of integration.

$$= \int_\tau^t d\eta \int_\tau^\eta K_1(t,\eta) K_2(\eta,\xi)\, K_3(\xi,\tau)\, d\xi$$

$$= \int_\tau^t K_1(t,\eta)\, M(\eta,\tau)\, d\eta = K_1 * (K_2 * K_3).$$

As a consequence there is no ambiguity in writing the star product of three continuous functions K_1, K_2, K_3 as $K_1 * K_2 * K_3$, which can be interpreted either as $K_1 * (K_2 * K_3)$ or $(K_1 * K_2) * K_3$. In particular, if $K_1 = K_2 = K_3$, then K_1^{*3} is uniquely defined as $K_1 * K_1 * K_1$. Similar definitions apply to higher powers of K_1, and to terms like $K_1^{*i} * K_2^{*j} * K_3^{*l}$. The star product is, in general, non-commutative, i.e. $K_1 * K_2 \neq K_2 * K_1$.

For convenience, let us define a function I, which has the value unity regarded as a function of (t, τ), but its star multiplication with other functions is defined by the special rule

$$I * K = K = K * I \tag{3}$$

instead of an integration. Obviously the associative law holds for continued multiplication of I and K's.

Now consider the series

$$I + K(t,\tau) + K^{*2}(t,\tau) + \cdots + K^{*n}(t,\tau) + \cdots, \tag{4}$$

or

$$I + K(t,\tau) + \int_\tau^t K(t,\xi)K(\xi,\tau)\,d\xi + \int_\tau^t K(t,\xi)\,d\xi \int_\tau^\xi K(\xi,\eta)K(\eta,\tau)\,d\eta + \cdots .$$

It will be shown that this series converges uniformly and absolutely for t, τ in T: $0 \le \tau \le t \le b$. For this purpose, let us define the norm of a matrix $[K_{ij}(t,\tau)]$ as $\| K_{ij}(t,\tau)\| = \underset{0\le\tau\le t\le b}{\mathrm{Max}} \left| \sqrt{\sum_{i,j} K_{ij}(t,\tau)^2} \right|$.

Let

$$M = \| K(t,\tau) \|,$$

then

$$\| K^{*2}(t,\tau) \| = \| \int_\tau^t K(t,\xi)\,K(\xi,\tau)\,d\xi \| \le \| \int_\tau^t M^2\,d\xi \|$$

$$\le M^2 (t-\tau).$$

By induction $\| K^{*n}(t,\tau) \| \leq \frac{M^n}{(n-1)!}(t-\tau)^{n-1}$.

Therefore the series (4) is dominated term-by-term by the series

$$1 + \sum_{n=1}^{\infty} \frac{M^n}{(n-1)!}(b-a)^{n-1}$$

which is convergent for all finite values of b and M. Hence the series (4) converges uniformly and absolutely in T: $0 \leq \tau \leq t \leq b$.

Let the sum of the series (4) be denoted by $\Omega(K|t,\tau)$, to signify the functional dependence of $\Omega(t,\tau)$ on the kernel K. Then

$$\Omega(K|t,\tau) \equiv I + K(t,\tau) + K^{*2}(t,\tau) + K^{*3}(t,\tau) + \cdots \tag{5}$$

Since (5) is true for any continuous function $K(t,\tau)$, so it is true when the kernel is a sum of two continuous functions $K(t,\tau)$ and $\delta K(t,\tau)$.

Hence

$$\Omega(K+\delta K|t,\tau) = I + (K+\delta K) + (K+\delta K)^{*2} + (K+\delta K)^{*3} + \cdots \tag{6}$$

But

$$(K+\delta K)^{*2}(t,\tau) = \int_\tau^t [K(t,\xi)+\delta K(t,\xi)][K(\xi,\tau)+\delta K(\xi,\tau)]\,d\xi$$

$$= \int_\tau^t [K(t,\xi)K(\xi,\tau) + \delta K(t,\xi)K(\xi,\tau) + K(t,\xi)\delta K(\xi,\tau) + \delta K(t,\xi)\delta K(\xi,\tau)]\,d\xi$$

$$= K^{*2} + \delta K * K + K * \delta K + \delta K^{*2}.$$

A similar relation holds for higher powers of $K+\delta K$. The expansion is formally the same as in ordinary non-commutative algebra. Now expand all the terms in the right hand side of (6) as above:

$$\Omega(K+\delta K) = I + K + \delta K + K^{*2} + K*\delta K + \delta K * K + \delta K^{*2} + K^{*3}$$
$$+ K^{*2}*\delta K + K*\delta K*K + \delta K * K^{*2} + K*\delta K^{2} + \delta K*K*\delta K$$
$$+ \delta K^{*2}*K + \delta K^{*3} + K^{*4} + \cdots .$$

The series on the right converges absolutely, and hence can be rearranged in any manner whatsoever without changing its value. Rearrange into a double series

$$I + K + K^{*2} + K^{*3} + \cdots$$
$$+ (I + K + K^{*2} + \cdots) * \delta K * (I + K + K^{*2} + \cdots)$$
$$+ (I + K + K^{*2} + \cdots) * \delta K * (I + K + K^{*2} + \cdots) * \delta K * (I + K + K^{*2} + \cdots)$$
$$+ \cdots ,$$

and sum by rows, to obtain

$$\Omega(K+\delta K \mid t,\tau) = \Omega(K) + \Omega(K) * \delta K * \Omega(K) + \Omega(K)*\delta K*\Omega(K)*\delta K*\Omega(K) + \cdots . \quad (7)$$

This is the fundamental expansion theorem. It is analogous to the Taylor's series. The functional $\Omega(K+\delta K)$ can be expanded into a series in "powers" of δK with "coefficients" as functions $\Omega(K)$.

Apply this result to the matric integral equation (2.2:11) which has the unique continuous solution

$$\chi(K \mid t) = f(t) + \int_0^t k(t,\tau) f(\tau)\, d\tau \quad (2.2{:}12)_{\text{bis}}$$

where $\quad k(t,\tau) = K(t,\tau) + K^{*2}(t,\tau) + K^{*3}(t,\tau) + \cdots .$

Notice that

$$\Omega(K \mid t,\tau) = I + k(t,\tau), \quad (8)$$

hence by defining the star product of k or I to a column matrix $\{f_i(t)\}$ as

$$I * f = f, \qquad k * f = \int_0^t \sum_j k_{ij}(t,\tau) f_j(\tau) d\tau, \tag{9}$$

then the solution (2.2:12) can be written as

$$\chi(K|t) = \Omega(K) * f \tag{10}$$

and therefore from (7):

$$\begin{aligned} \chi(K+\delta K|t) &= \Omega(K+\delta K) * f \\ &= \chi(K) + \Omega(K) * \delta K * \chi(K) + \sum \Omega(K) * \delta K * \cdots * \Omega(K) * \delta K * \chi(K). \end{aligned} \tag{11}$$

This expansion is valid for all continuous $\delta K(t,\tau)$ in $T: a \leq \tau \leq t \leq b$.

The result can be stated in the following theorem:

Let $\Omega(K|t,\tau)$ be a functional of $K(t,\tau)$ defined by the series (5), and $\chi(K|t)$ be the unique continuous solution of the Volterra integral equation of the second kind (2.2:11) with continuous kernel $K(t,\tau)$; then the unique continuous solution of the equation (2.2:11) when the kernel becomes $K(t,\tau) + \delta K(t,\tau)$ can be written as

$$\chi(K+\delta K|t) = \chi(K) + \Omega(K) * \delta K * \chi(K) + \sum \Omega(K) * \delta K * \cdots * \delta K * \chi(K). \tag*{(11)$_{\text{bis}}$}$$

The star products in this formula are defined by eqs. (1), (3), and (9).

More explicitly

$$\begin{aligned} \chi(K+\delta K|t) = \chi(K) + \sum_{n=3,5,7\cdots}^{\infty} \int_0^t \Omega(x,\eta_1)\, d\eta_1 \int_0^{\eta_1} \delta K(\eta_1,\eta_2)\, d\eta_2 \int_0^{\eta_2} \Omega(\eta_2,\eta_3)\, d\eta_3 \int_0^{\eta_3} \cdots \\ \cdots \int_0^{\eta_{n-1}} \delta K(\eta_{n-1},\eta_n)\, \chi(K|\eta_n)\, d\eta_n . \end{aligned} \tag{12}$$

where for shortness $\Omega(t,\tau)$ has been written in place of $\Omega(K|t,\tau)$.

It should be noted, however, that the expression (11) is in fact easier to interpret than (12). The latter is a particular way of expressing (11), and may not be the simplest. For example, it may be desirable to interpret the general term of (11) as (Cf. eq. 17 below)

$$\{\Omega(K)*\delta K\}*\{\Omega(K)*\delta K\}\cdots\{\Omega(K)*\delta K\}*X(K).$$

Returning now to the original form of the dynamical equation (2.2:6) or (2.2:9) of which the solution is given by (2.2:10), it is useful to have the explicit formulas:

When $B(t)=0$, i.e. when no gust or control forces and moments are acting, the matric equation of motion becomes

$$\dot{X}_i = \sum_{j=1}^{2n} A_{ij}(t)\, X_j(t), \quad X_i(t=0) = X_{0i}, \quad (i=1,2,\cdots 2n). \tag{13}$$

The solution is given by

$$X_i(A|t) = X_{0i} + \left\{\int_0^t A(\tau)\,d\tau + \int_0^t A(\tau)\,d\tau\int_0^\tau A(s)\,ds + \cdots\right\} X_{0i}. \tag{14}$$

The so-called "matrizant functional" of A(t) is defined as follows:

$$\begin{aligned} R_h^t(A) = I &+ \int_h^t A(s)\,ds + \int_h^t A(s)\,ds\int_h^s A(s_1)\,ds_1 + \\ &+ \sum_{i=2}^{\infty} \int_h^t A(s)\,ds\int_h^s A(s_1)\,ds_1\int_h^{s_1}\cdots\int_h^{s_{i-1}} A(s_i)\,ds_i, \end{aligned} \tag{15}$$

where I is the idemfactor.

With this notation, (14) can be written as

$$X(A|t) = R_0^t(A)\,X_0. \tag{16}$$

The generalized Taylor's expansion then leads to the following relation:

$$\chi(A+\delta A|t) = \chi(A|t) + \int_0^t R_{s_1}^{t}(A)\,\delta A(s_1)\,\chi(A|s_1)\,ds_1$$
$$+ \sum_{i=3,5\cdots}^{\infty} \int_0^t R_{s_1}^{t}(A)\,\delta A(s_1)\,ds_1 \int_0^{s_1} R_{s_2}^{s_1}(A)\,\delta A(s_2)\,ds_2 \int_0^{s_2} \cdots$$
$$\cdots \int_0^{s_{i-1}} R_{s_i}^{s_{i-1}}(A)\,\delta A(s_i)\,ds_i\,\chi(A|s_i). \tag{17}$$

When $B(t) \neq 0$, i.e. the gust or control forces or other exciting forces exist, the differential system

$$\dot{\chi} = A(t)\chi(t) + B(t), \qquad \chi(t=0) = \chi_0 \tag*{(2.2:5)$_{\text{bis}}$}$$

where $A(t)$ is a square matrix of continuous functions of t, and $B(t)$ a column matrix of functions of t, (integrable, but not necessarily continuous) has the unique continuous solution (2.2:10) which can be written as

$$\chi(A|t) = f(t) + A * R_{\tau}^{t}(A) * f(\tau), \tag{18}$$

where
$$f(t) = \int_0^t B(t)\,dt + \chi_0 .$$

The generalized Taylor's series of $\chi(A+\delta A|t)$, where $\delta A(t)$ is another square matrix of continuous function of t, now takes exactly the same form as (17).

§ 2.5 The Solution in Canonical Form

The nature of the solution can be best recognized by reducing the result into canonical form. The dynamical system is represented by the differential equation:

$$\dot{\chi} = A(t)\chi(t) + B(t), \qquad \chi(t=0) = \chi_0, \tag*{(2.2:5)$_{\text{bis}}$}$$

where $A(t)$ is a square matrix of continuous functions of t, $B(t)$ a column matrix of integrable functions of t, and $\chi(t)$ a column matrix of the variables in question. Now let the variables X be subjected to a linear transformation P, so that a column matrix of new variables W is defined by

$$PX = W, \tag{1}$$

where P is a non-singular square matrix of constants. The differential system (2.2:5) will be transformed into:

$$\dot{W} = \Lambda W + \mathrm{B}, \qquad W_0 = PX_0, \tag{2}$$

where

$$\Lambda = PAP^{-1}, \qquad \mathrm{B} = PB. \tag{3}$$

A collineatory transformation P which renders the new kernel Λ to the simplest form is to be found.

(I) Consider first the case when <u>A is a square matrix of constants</u>.

From the general properties of λ-matrices and elementary divisors, it follows that: (Ref. 46, Chap. 20)

If a_1, and a_2 are two matrices independent of λ, then the necessary and sufficient condition for the existence of a non-singular matrix P, such that

$$a_2 = P a_1 P^{-1}$$

is that the characteristic matrices of a_1, and a_2 have the same invariant factors, or if one prefers, the same elementary divisors.

Now the elementary divisors of the characteristic matrix $A - \lambda I$ of A depend on the nature of the latent roots of A, (i.e., the roots of the determinantal equation $|A - \lambda I| = 0$, which are also called characteristic roots). Naturally one looks for the condition of existence of a collineatory transformation which renders A into a diagonal matrix. It is easily verified by the above general theorem, that this collineatory transformation is always possible when the latent roots of A are all distinct. However, it is not always possible to reduce A into a diagonal matrix by collineatory transformation when there are repeated latent roots. It is found (Chap. VI, Ref. 47, and Chap. 3, Ref. 48), that the general canonical form is a diagonal matrix with a certain number of unit elements added in the superdiagonal (i.e., the elements immediately to the right of the principal diagonal). The rule is that unit element is certainly absent if the place in the superdiagonal is adjacent horizontally and vertically to distinct elements in the principal diagonal, but that it may be present when the adjacent elements in the principal diagonal are the same. Thus the typical canonical form is

$$\left(\begin{array}{ccc|cc|ccc|c}
\lambda_1 & 1 & & & & & & & \\
 & \lambda_1 & 1 & & & & & & \\
 & & \lambda_1 & & & & & & \\
\hline
 & & & \lambda_1 & 1 & & & & \\
 & & & & \lambda_1 & & & & \\
\hline
 & & & & & \lambda_2 & 1 & & \\
 & & & & & & \lambda_2 & 1 & \\
 & & & & & & & \lambda_2 & \\
\hline
 & & & & & & & & \\
\end{array}\right) \qquad (4)$$

where the elements in the principal diagonal are, of course, the latent roots. It is clear that the canonical matrix can be partitioned into a diagonal matrix of square submatrices whose rows and columns do not overlap, and which are all of the <u>simple classical type</u>:

$$\begin{bmatrix} \lambda_1 & 1 & & \\ & \lambda_1 & 1 & \\ & & \lambda_1 & 1 \\ & & & \lambda_1 \end{bmatrix} \qquad \text{(all the rest of places are zeros)} \qquad (5)$$

The order of the simple classical submatrices can range from 1 to n, depending on the degree of multiplicity of the particular latent root λ_i and the rank of the λ-matrix $A - \lambda_i I$.

This being established, several particular cases follow:

Case Ia. All latent roots distinct.

Let $\{\lambda_i\}$ be the latent roots of A, then there exists a non-singular matrix P such that $PAP^{-1}=\Lambda$ where Λ is a diagonal matrix:

$$\Lambda = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_{2n} \end{pmatrix} \qquad (6)$$

the order of A being 2n.

Write (6) as a sum:

$$\Lambda = \lambda_i Z_i \qquad (7)$$

where Z_i is a diagonal matrix with all elements zero except the one occupying the i th column which is unity; the summation convention for the subscripts is used in (7).

Since

$$Z_i Z_j = \begin{cases} 0 & \text{if } i \neq j, \\ Z_i & \text{if } i = j, \end{cases}$$

it follows that $\Lambda^n = \lambda_i^n Z_i$.

Therefore substituting Λ as kernel into (2.4:15), one obtains the matrizant functional

$$R_r^t(\Lambda) = e^{\Lambda(t-r)} = e^{\lambda_i(t-r)} Z_i . \qquad (8)$$

Hence from (2) and(2.4:16), it follows that when external forces (gust or control) are absent, the solution is

$$W(\Lambda|t) = \left(\sum_{i=1}^{2n} e^{\lambda_i t} Z_i \right) W_o .$$

Hence by (1),

$$X(A|t) = P^{-1} \left(\sum_{i=1}^{2n} e^{\lambda_i t} Z_i \right) PX_o \qquad (9)$$

When external forces do exist, one must use (2.2:10) or its equivalent (2.4:18), the solution is then

$$\chi(A|t) = \int_0^t B(t)\,dt + \chi_0 + P^{-1}\left(\sum_{i=1}^{2n} \lambda_i Z_i \int_0^t e^{\lambda_i(t-r)} f(r)\,dr\right). \tag{10}$$

where $\quad f(t) = P\int_0^t B(t)\,dt + P\chi_0.$

It is at once recognised that this is identical with the classical solution. However, the advantage of the Heaviside Operational methods is preserved in the way the initial conditions are taken care of. In fact, the amount of work necessary in evaluating the transformation P is just the same as that needed by the classical or Heaviside solution. The advantage of the present method lies in the fact that in many cases one is not interested in reducing the result into its canonical form; and then it is simpler to work directly with the method developed in Section 2.4.

Case Ib. With repeated latent roots.

Consider a typical simple classical submatrix.

$$\Lambda_i = \begin{bmatrix} \lambda_i & 1 & \\ & \lambda_i & 1 \\ & & \lambda_i \end{bmatrix}. \tag{11}$$

The series (2.4:15) gives

$$R_r^t(\Lambda_i) = I + \Lambda_i(t-r) + \Lambda_i^2\frac{(t-r)^2}{2!} + \cdots + \Lambda_i^n\frac{(t-r)^n}{n!} + \cdots$$

$$= \begin{pmatrix} e^{\lambda_i(t-r)} & (t-r)e^{\lambda_i(t-r)} & \frac{(t-r)^2}{2!}e^{\lambda_i(t-r)} \\ 0 & e^{\lambda_i(t-r)} & (t-r)e^{\lambda_i(t-r)} \\ 0 & 0 & e^{\lambda_i(t-r)} \end{pmatrix} \tag{12}$$

This shows that the solution includes terms like $t\,e^{\lambda_i t}$ and $t^2 e^{\lambda_i t}$ which is to be expected from considerations on the familiar solution of a system of linear ordinary differential equations with constant coefficients when the auxiliary equation has multiple roots.

The above result is typical, and generalization to any order of the classical type is obvious. Recall that the general form of the canonical matrix (4) can be partitioned into a diagonal matrix of simple classical submatrices Λ_i, so write

$$\Lambda = \begin{pmatrix} \Lambda_1 & & & \\ & \Lambda_2 & & \\ & & \ddots & \\ & & & \Lambda_k \end{pmatrix};$$

then

$$R_n^t(\Lambda) = \begin{pmatrix} R_n^t(\Lambda_1) & & & \\ & R_n^t(\Lambda_2) & & \\ & & \ddots & \\ & & & R_n^t(\Lambda_k) \end{pmatrix} \tag{13}$$

and as before

$$\chi(A|t) = f(t) + P^{-1}\Lambda \int_0^t R_\tau^t(\Lambda) P f(\tau)\, d\tau , \tag{14}$$

where $\quad f(t) = \int_0^t B(t)\, dt + X_0 .$

(II). <u>When A is a square matrix of functions of time.</u>

Although the solution of the dynamical equation given in Section 2.4 is an entire analytic functional of the kernel A(t), the behaviour of the solution is in general complicated. However, in aeroelastic problems, the variation with time of the elements of the matrix A(t) is in general small; that is, if A(t) is written as

$$A(t) = A_0 + \delta A(t) , \tag{15}$$

where A_0 is a matrix of constants, then in general

$$\| \delta A_{ij}(t) \| \ll \| A_{0ij} \| . \tag{16}$$

Such small perturbation problems are discussed in section 2.6.

§ 2.6 Small Perturbations

Although the results derived in section 2.4 are perfectly general so that there is no limitation to the magnitude of the norm of $\delta K(t,\tau)$ (or $\delta A(t)$) in the series expansions (2.4:11) or (2.4:17), these expansions are of practical interest only when $\|\delta K\|$ is small.

An upper bound to the remainder after n terms of the expansion (2.4:17) can be obtained as follows: the norm of the nth term (the term with $\delta A(t)$ occuring n times) is bounded by $\| X(A(t))\| \frac{M^n}{n!}$ where $M = T \| R(K)\| \|\delta K\|$, T being the time interval in question. Recall that the ordinary Taylor's expansion of e^u with remainder is:

$$e^u = 1 + u + \frac{u^2}{2!} + \cdots + \frac{u^{n-1}}{(n-1)!} + R_n ,$$

where

$$R_n = \frac{u^n}{(n-1)!} \int_0^1 (1-z)^{n-1} e^{uz} dz ,$$

So similarly we have,

$$\| X(K+\delta K) - X(K) - \sum_{i=1}^{2n-1} \int \cdots \int \| \leq \| X(K)\| \frac{M^{n+1}}{n!} \int_0^1 (1-z)^n e^{Mz} dz$$

$$\leq \| X(K)\| \frac{M^{n+1}}{(n+1)!} e^M . \tag{1}$$

Hence the expansion converges at least as fast as an exponential series.

According to the assumption of small perturbation, if

$$A(t) = A_0 + \delta A(t) \tag{2}$$

where A_0 is a square matrix of constants, then

$$\| \delta A_{ij}(t) \| \ll \| A_0 \| \tag{3}$$

for t in the time interval in question. The solution with the kernel A_0 may be regarded as the initial solution, and that with the kernel $A_0 + \delta A$ the perturbed solution, where δA is the perturbation kernel.

From the general expansion theorem, if only the first few terms are taken, the perturbed solution is

$$\chi(A_0 + \delta A \mid t) = \chi(A_0 \mid t) + \int_0^t R_s^t(A_0)\,\delta A(s)\,\chi(A_0 \mid s)\,ds + \text{remainder}, \tag{4}$$

where $R_s^t(A_0)$ is the matrizant functional with kernel A_0, given by eq. (2.4:15).

The effect of perturbation can be appreciated in a few simple cases as follows:

Let P be a collineatory transformation which transforms A_0 into the canonical form Λ_0. Assuming that all the roots of the characteristic equation of A_0 are distinct, then Λ_0 is a diagonal matrix:

$$P A_0 P^{-1} = \Lambda_0 = \sum_i \lambda_i Z_i \tag{5}$$

where $\{\lambda_i\}$ are the roots of the characteristic equation $| A_0 - \lambda I | = 0$ and Z_i the unitary matrix defined by (2.5:7). Then, according to (2.5:8),

$$R_r^t(\Lambda_0) = \sum_i e^{\lambda_i (t - r)} Z_i \,. \tag{6}$$

Write also

$$\delta \Lambda = P\, \delta A\, P^{-1} \tag{7}$$

and let the elements of $\delta\Lambda$ be denoted by δa_{ij}, $i = 1, 2, \cdots 2n$. The variables X are reduced to W by PX = W as in the last section.

Consider the simplest case when the differential system consists of

$$\dot{W} = (\Lambda_0 + \delta\Lambda)\, W \quad ; \quad W(t=0) = W^{(0)}, \tag{8}$$

and $\delta\Lambda$ is a <u>matrix of constants</u>. Then

$$\int_0^t R_s^t(\Lambda_0)\,\delta\Lambda\, R_0^s(\Lambda_0)\,ds = \left(\int_0^t e^{\lambda_i (t-s) + \lambda_j s} \delta a_{ij}\, ds \right) \quad = \text{a matrix with}$$

elements:

$$\begin{cases} \dfrac{e^{\lambda_j t} - e^{-\lambda_i t}}{\lambda_j - \lambda_i}\, \delta a_{ij} & \text{if } i \neq j, \\[2ex] t\; e^{\lambda_i t}\, \delta a_{ij} & \text{if } i = j, \end{cases}$$

and the solution of disturbed motion is approximately:

$$\begin{aligned} W(\Lambda_0 + \delta\Lambda) = & \sum_{i=1}^{2n} e^{\lambda_i t}\, W_i^{(0)} + \sum_{i=1}^{2n} t\, e^{\lambda_i t}\, \delta a_{ii}\, W_i^{(0)} \\ & + \sum_{j \neq i}^{2n} \frac{e^{\lambda_j t} - e^{\lambda_i t}}{\lambda_j - \lambda_i}\, \delta a_{ij}\, W_j^{(0)}. \end{aligned} \tag{9}$$

In general $\{\lambda_i\}$ are imaginary numbers or complex numbers. The formula above of course holds only for small values of t.

The case discussed above has an important engineering application. It tells the effect of a slight change in design on the final behaviour of an airplane. As particular examples, problems in dynamic stability, controllability, and maneuverability, etc., may be mentioned.

If $\delta\Lambda(t)$ is proportional to $e^{-\alpha t}$, with elements $\delta a_{ij}\, e^{-\alpha t}$, the corresponding result will be

$$W(\Lambda_0 + \delta\Lambda) = \sum_{i=1}^{2n} e^{\lambda_i t}\, W_i^{(0)} + \sum_{i,j}{}' \frac{e^{(\lambda_j - \alpha)t} - e^{-\lambda_i t}}{\lambda_j - \lambda_i - \alpha}\, \delta a_{ij}\, W_j^{(0)}. \tag{10}$$

the summation over i,j excludes those terms for which $\lambda_j - \lambda_i - \alpha = 0$, for such terms the coefficients before δa_{ij} should be replaced by $te^{\lambda_i t}$. In the reduced form $W_i(\Lambda_0)$ are normal coordinates: $W_i^{(0)}$ excites the motion of W_i alone. For the perturbed case, however, $W_i^{(0)}$ excites all the coordinates.

III. DEFLECTION OF CANTILEVER PLATES

§3.1. PRELIMINARY DISCUSSIONS

In second order (first) approximation the problem of bending of flat plates consists in integrating the equation

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{1}{D} q(x,y) \tag{1}$$

where $w(x,y)$ satisfies along the boundary L of a region R certain boundary conditions defined by the problem, and w is continuous $(D,4)$ in R and on L, and q satisfies desirable continuity properties.

There are four kinds of problems, classified according to the nature of prescribed boundary conditions:

(A) Built-in edge problems

$$w = \frac{\partial w}{\partial n} = 0 \quad \text{on L, where n is the outer normal to L.} \tag{2}$$

(B) Supported edge problems

$$w = 0, \qquad \frac{\partial^2 w}{\partial n^2} + \nu \frac{\partial^2 w}{\partial t^2} = f(x,y) \quad \text{on L}, \tag{3}$$

where n and t are outer normal and tangent to L respectively, the sense of n is to t as x is to y-axis.

Subcase (B_1). Simply-supported edge problems, $f \equiv 0$.

(C) Unsupported edge problems

$$\left.\begin{aligned} &\frac{\partial^2 w}{\partial n^2} + \nu \frac{\partial^2 w}{\partial t^2} = g(x,y), \\ &\frac{\partial^3 w}{\partial n^3} + (2-\nu)\frac{\partial^3 w}{\partial n\, \partial t^2} = h(x,y) \quad \text{on L}. \end{aligned}\right\} \tag{4}$$

Subcase (C_1). Free-edge problems: $g(x,y) = h(x,y) \equiv 0$.

(D) Mixed edge-condition problems.

This can be further classified into

(D_1) BS. When L is partly simply-supported and partly built-in.

(D_2) BU. When L is partly built-in and partly unsupported.

(D_3) SU. When L is partly supported and partly unsupported.

(D_4) SBU. When L is partly built-in, partly supported, and partly unsupported.

Problem (A) is often made a subject of mathematical investigation. Existence theorems and solutions in fairly general forms have been obtained by Almansi, Lauricella, J. Hadamard, A. Korn, and others. (Refs. 32, 33, 34, 35). Approximate solutions for rectangular plates have been given by W. Ritz, Timoshenko and others (Refs. 17, 10).

Problem (B) is entensively studied by Timoshenko and others. (Ref. 10). For rectangular plates an application of Fourier series makes the problem particularly simple. Both problems (A) and (B) can be reduced to a set of first boundary value problems of the potential theory. (Ref. 36, Chapter 19).

Problem (C) is much more difficult than the first two and few existing results can be quoted. The problem of vibration of a rectangular plate with free edges which is akin to (C_1) was studied by Ritz in his 1909 paper in which he illustrated a spectacular success of his energy method.

Of problem (D), a few cases of (D_1), (D_3), and (D_4) have been studied. (See Ref. 10 for references to original works). The solved cases are:

(1) Rectangular plates with two opposite edges simply supported and the other two edges built-in.

(2) Rectangular plates with three edges simply supported and one edge built-in.

(3) Rectangular plates with two opposite edges simply supported, the third edge free and the fourth edge built-in or simply supported.

(4) Rectangular plates with two opposite edges simply supported and the other two edges supported elastically.

(5) Plate of the form of a parallelogram with two opposite edges simply supported. (Ref. 54).

In all the above cases, the solution can be put in the form of an infinite series.

$$W = \sum_{m=1,2,\ldots} Y_m(y) \sin \frac{m\pi x}{a}$$

where $\{Y_m(y)\}$ are functions of y alone, provided that the edges x equal to constants are simply supported.

(6) Circular plates supported at several points along the boundary.

(7) Circular plates in the form of a sector, the straight edges of which are simply supported.

For other problems, there is little available information.

The swept wing problem is a problem (D_2) with BU boundary. Exact solutions are in general hard to get from the standpoint of biharmonic analysis but approximate solutions of engineering significance can be found by the various direct methods of solution.

Solution of the Differential Equation

In the application of the method of relaxation of boundary conditions, it is usually convenient to write the solution of eq. (3.1:1) in the form of a sum of a particular integral and a biharmonic function:

$$w(x,y) = w_p(x,y) + w_c(x,y) \qquad (1)$$

so that

$$\frac{\partial^4 w_p}{\partial x^4} + 2\frac{\partial^4 w_p}{\partial x^2 \partial y^2} + \frac{\partial^4 w_p}{\partial y^4} = \frac{1}{D} q(x,y),$$

and

$$\frac{\partial^4 w_c}{\partial x^4} + 2\frac{\partial^4 w_c}{\partial x^2 \partial y^2} + \frac{\partial^4 w_c}{\partial y^4} = 0. \qquad (2)$$

The particular integral w_p can be easily obtained if $q(x,y)$ is simple, say, being a constant, a function of single variable x or y, or a polynomial in x,y. For other cases it was shown by Mathieu (Ref. 37), that a general expression for w_p can be obtained by an extension of the potential theory. It can be verified that if

$$w_p(x,y) = -\frac{1}{8\pi D}\iint_R \left(r^2 \log\frac{1}{r} + \frac{r^2}{2}\right) q(a,b)\,da\,db, \qquad (3)$$

where

$$r^2 = (x-a)^2 + (y-b)^2,$$

then

$$\nabla^4 w_p(x,y) = \begin{cases} 0 & \text{if the point } (x,y) \text{ is outside of the region } R, \\ \dfrac{q(x,y)}{D} & \text{if the point } (x,y) \text{ is inside of the region } R. \end{cases}$$

The biharmonic function $w_c(x,y)$ can be constructed conveniently from harmonic functions, for which there is a very extensive store of information. The following results are well-known:

If w_1 and w_2 are two harmonic functions in a region R, then

$$w = xw_1 + w_2 \tag{4}$$

is a biharmonic function in the same region. Conversely, if the boundary L of R intersects every line parallel to the x-axis in at most two points, then for every biharmonic function w in R, there exist two harmonic functions w_1 and w_2, so that w can be represented by the formula (4).

Similar result is true when x in the above theorem is replaced by y. And furthermore, if the origin is enclosed in R and every radius vector intersects L in only one point, then every biharmonic function w in R can be represented by functions of the form

$$w = (r^2 - r_0^2)\, w_1 + w_2 , \tag{5}$$

where

$$r^2 = x^2 + y^2 ,$$

w_1 and w_2 are two harmonic functions in R, and r_0 is any preassigned constant. Conversely, every function w in this form is a biharmonic function in R.

In the application of the method of relaxation of boundary conditions to swept plate problems, a suitable form of particular solution w_p is first obtained. This w_p, while balancing the distributed normal loads over the faces of the plate, will induce bending moments and shear reactions along the edges of the plate. The next step is then to find a solution for a plate loaded along the edges, and to this, the Trefftz method is applied by taking the biharmonic functions as elementary solutions.

A remark on the places where the maximum mean stresses occur may be useful. When the function $q(x,y)$ is zero over a subregion R_1 of R, the differential equation (3.1:1) becomes

$$\nabla^2 (\nabla^2 w) = 0 \text{ in } R_1. \tag{1}$$

Hence $\nabla^2 w$ is a harmonic function over R_1. But a harmonic function in R_1 cannot have a maximum or minimum at a point in the interior of the region R_1. (§2.26, Ref. 13, or §37, Ref. 38). Its extremum is reached only on the boundary of the region. Now $\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}$ is the mean curvature of the bent middle surface of the plate. Hence the mean curvature of a bent plate reaches its maximum or minimum either on the edges of the plate or under the points of loading. Thus when the external load is applied along the edges of the plate only, the mean curvature is assured to reach its maximum only on the edges of the plate. The mean curvature is proportional to the mean maximum bending stresses (e.g. $\sigma_x + \sigma_y$) in the plate. Hence the mean maximum bending stresses occur either under the load or on the boundary of the plate.

Similar reasoning leads to corresponding results regarding to shearing stresses. If, in a certain subregion R_1, q is a function of y alone, then

$$\frac{\partial}{\partial x} \nabla^2 (\nabla^2 w) = \nabla^2 \left(\nabla^2 \frac{\partial w}{\partial x} \right) = \frac{1}{D} \frac{\partial q(x,y)}{\partial x} = 0 .$$

But

$$-D \nabla^2 \frac{\partial w}{\partial x} = Q_x , \qquad \text{the vertical shearing force on } x = \text{const. sections}$$

Hence

$$\nabla^2 Q_x = 0 \qquad \text{where } q = q(y)$$

In the same way,

$$\nabla^2 Q_y = 0 \qquad \text{where } q = q\ (x).$$

Q_x and Q_y are then harmonic functions in the respective cases. Hence Q_x (or Q_y) reaches its maximum or minimum either on the edges of the plate or at points where $\frac{\partial q}{\partial x}$ (or $\frac{\partial q}{\partial y}$) is different from zero.

In the application of the method of relaxation of boundary conditions, an error function is defined by the relation

$$\varepsilon_N = w - w_N$$

where w_N is an approximating function to the true solution w. Since w and w_N both satisfy the differential equation

$$\nabla^4 w = \frac{q}{D},$$

so

$$\nabla^4 \varepsilon_N = 0.$$

over the whole plate. Therefore the mean curvature of the error function reaches the maximum only on the boundary and hence the maximum mean error in stresses occurs only on the boundary of the plate.

§ 3.2 Rectangular Cantilever Plates

The deflection of a rectangular cantilever plate can be obtained by the Rayleigh-Ritz method and the method of relaxation of boundary conditions. In this section both procedures are carried out in detail. It is shown that the homogeneous part of the system of linear equations for the determination of the unknown coefficients is independent of the external loading, and can be formulated once and for all. The loading condition effects the constant terms of these equations only. Hence the sample computation is carried out only for the case of distributed shear load at the tip section.

For simplicity the coordinates (x,y) are expressed in dimensionless form. The length is measured in terms of $\frac{c}{2}$, the half-chord of the plate (Fig. 3.2:1). Without modifying the dimensions of w and q, the flexural rigidity is here defined as

$$D = \frac{4Et^3}{3(1-\nu^2)c^4}$$

where t is the thickness of the plate. E and ν are the Young's modulus and Poisson's ratio of the material of the plate. With this modification, the plate dimension is shown in Fig. 3.2:2.

§ 3.21 Solution by Rayleigh-Ritz Method

The Rayleigh-Ritz method can be applied with success to find the deflection surface of a cantilever plate. According to the energy principle, the deflection $w(x,y)$ of the plate is one of the "admissible" functions which minimizes the potential energy of the plate:

$$V = \frac{1}{2} \iint D\left\{ \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)^2 - 2(1-\nu)\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right]\right\} dx\,dy$$

$$- \iint q\,w\,dx\,dy + \int M_n \frac{\partial w}{\partial n} ds - \int \left(Q_n - \frac{\partial H_{nt}}{\partial s}\right) w\,ds, \qquad (1)$$

where the line integral is extended over the free edges of the plate. A function w is called admissible if and only if it satisfies the "rigid" support conditions on the boundary, for example, $w = 0$ on a simply supported edge and $w = \frac{\partial w}{\partial n} = 0$ on a built-in edge.

It is assumed that w can be represented by a series of admissible functions in the form

$$w = w_0(x,y) + \sum_{m,n} a_{mn} w_{mn}(x,y). \qquad (2)$$

where the coefficients $\{a_{mn}\}$ are determined by the minimization of the energy integral (1).

The simplest procedure is to choose $\{w_{mn}(x,y)\}$ which possess the best orthogonality properties, such as the normal modes of free vibration of a plate of the same form and same supporting conditions as the given plate. Since, however, the vibration modes of a cantilever plate are not conveniently expressible in elementary functions, their use is impractical.

In the following, $w_{mn}(x,y)$ is taken in the form

$$W_{mn}(x,y) = f_m(x) \cdot g_n(y) \qquad (3)$$

Where f_m (x) is the $m^{\underline{th}}$ free vibration mode of a thin elastic bar of length L, clamped at the end x = o and free at the end x = L; and g_n (y) is the $n^{\underline{th}}$ mode of a bar of length 2, with both ends y = $\pm$ 1 free. The f_m and g_n are normalized so that

$$\int_0^L f_m(x)\, f_n(x)\, dx = \delta_{mn}, \qquad \int_{-1}^{1} g_m(y)\, g_n(y)\, dy = \delta_{mn} \qquad (4)$$

Where $\delta_{mn} = 1$ if $m = n$, and $\delta_{mn} = o$ if $m \neq n$.

Substituting (2), (3) into (1), one obtains a quadratic function of the constants $\{a_{mn}\}$. When the $\{a_{mn}\}$ is so chosen as to minimize V, the conditions

$$\frac{\partial V}{\partial a_{mn}} = 0, \qquad \text{(any m, n)} \qquad (5)$$

lead to a system of simultaneous linear equations from which $\{a_{mn}\}$ can be determined.

To obtain explicit formulas, let us introduce the following notations:

$$\alpha_{mn} = \int_0^L f_m''(x)\, f_n''(x)\, dx, \qquad \kappa_{mn} = \int_{-1}^{1} g_m''(y)\, g_n''(y)\, dy,$$

$$\beta_{mn} = \int_0^L f_m'(x)\, f_n'(x)\, dx = \beta_{nm}, \qquad \lambda_{mn} = \int_{-1}^{1} g_m'(y)\, g_n'(y)\, dy = \lambda_{nm},$$

$$\gamma_{mn} = \int_0^L f_m''(x)\, f_n(x)\, dx, \qquad \mu_{mn} = \int_{-1}^{1} g_m''(y)\, g_n(y)\, dy. \qquad (6)$$

and

$$\frac{\partial V_o}{\partial a_{mn}} = \int_{-1}^{1}\int_0^L D\left\{ \frac{\partial^2 w_o}{\partial x^2}\left(f_m'' g_n + \nu f_m g_n''\right) + \frac{\partial^2 w_o}{\partial y^2}\left(f_m g_n'' + \nu f_m'' g_n\right) + 2(1-\nu)\frac{\partial^2 w_o}{\partial x\,\partial y} f_m' g_n' \right\} dx\, dy$$
$$- \int_{-1}^{1}\int_0^L q\, f_m g_n\, dx\, dy + \int M_n \frac{\partial (f_m g_n)}{\partial n}\, ds + \int \left(Q_n - \frac{\partial H_{nt}}{\partial s}\right) f_m g_n\, ds \qquad (7)$$

Where the primes denote differentiation with respect to the independent variables, and in (7), the line integrals are extended over that part of the free edges on which external shear and moments are applied.

The aforesaid system of linear equations (5) now takes the form:

$$2a_{mn}\left[\alpha_{mm} + \kappa_{nn} + 2\nu\gamma_{mm}\mu_{nn} + 2(1-\nu)\beta_{mm}\lambda_{nn}\right]$$

$$+ \sum_{ij\neq mn} a_{ij}\left[2\nu\gamma_{mi}\mu_{jn} + 2(1-\nu)\beta_{mi}\lambda_{jn}\right] = -\frac{2}{D}\frac{\partial V_0}{\partial a_{mn}},$$

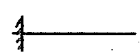
$$(\text{any } m,\ n). \qquad (8)$$

The left hand side of the equations (8) are definite linear forms in $\{a_{mn}\}$, and can be determined once and for all for a given aspect ratio of the plate. For different loading conditions, only the terms $\left\{\frac{\partial V_0}{\partial a_{mn}}\right\}$ are different.

The functions $\{f_m(x)\}$ and $\{g_n(y)\}$ are solutions of the differential equation

$$\frac{d^4u}{dx^4} = p^4u$$

with the boundary conditions $u = \frac{\partial u}{\partial x} = 0$ for a clamped end and $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^3 u}{\partial x^3} = 0$ for free ends. They are found to be as follows: (Ref. 5)

(I) $f_n(x)$. Normal mode of a clamped-free bar of length L, clamped at $x = 0$, free at $x = L$: 

$$f_n(x) = \frac{1}{\sqrt{L}}\left(-\cos\frac{p_n x}{L} + \cosh\frac{p_n x}{L}\right) - \frac{\cos p_n + \cosh p_n}{\sqrt{L}\,(\sin p_n + \sinh p_n)}\left(-\sin\frac{p_n x}{L} + \sinh\frac{p_n x}{L}\right),$$

Where $\{p_n\}$ are the roots of the equation (9)

$$\cos p_n \cosh p_n + 1 = 0.$$

$p_1 = 1.875104$, $p_2 = 4.694091$, $p_3 = 7.854757$

$p_4 = 10.995541$, $p_5 = 14.137168$.

$n > 5$, $\quad p_n = \frac{\pi}{2}(2n-1)$ to 6 decimal places.

For $n > 2$,

$$\sqrt{L}\, f_n(x) = -\cos\frac{p_n x}{L} + \left[1-(-1)^{n+1}\cdot 2e^{-p_n}\right]\sin\frac{p_n x}{L} + (-)^{n+1}\cdot e^{-p_n(1-\frac{x}{L})}$$

$$+\left[1-(-1)^{n+1}\cdot e^{-p_n}\right] e^{-p_n\frac{x}{L}},$$

within 5 significant figures.

(II). $g_n(y)$, normal mode of a free-free lear of length 2, with origin at the middle-point. 

If n is even: $\quad g_n(y) = \dfrac{\cosh q_n \cos q_n y + \cos q_n \cosh q_n y}{\sqrt{\cosh^2 q_n + \cos^2 q_n}}$ (10)a

Where $\quad \tan q_n + \tanh q_n = 0.$

If n is odd: $\quad g_n(y) = \dfrac{\sinh q_n \sin q_n y + \sin q_n \sinh q_n y}{\sqrt{\sinh^2 q_n - \sin^2 q_n}}$ (10)b

Where $\quad \tan q_n - \tanh q_n = 0.$

The roots q_0, and $q_1 = 0$ correspond to $g_0(y) = \frac{1}{\sqrt{2}}$, and $g_1(y) = \sqrt{\frac{3}{2}}\, y$. The other roots are

$$q_2 = 2.365020, \quad q_3 = 3.926602, \quad q_4 = 5.497804,$$

$$n \geqslant 5, \quad q_n = \left(n-\tfrac{1}{2}\right)\frac{\pi}{2} \quad \text{to 6 decimal places.}$$

In general, for $n \geqslant 2$, up to four decimal places,

$$g_n(y) = \cos\left(\frac{n}{2}-\frac{1}{4}\right)\pi y + \frac{(-)^{\frac{n}{2}}\cosh\left(\frac{n}{2}-\frac{1}{4}\right)\pi y}{\sqrt{2}\cosh\left(\frac{n}{2}-\frac{1}{4}\right)\pi} \text{ for even } n. \quad \text{(10)c}$$

$$g_n(y) = \sin\left(\frac{n}{2}-\frac{1}{4}\right)\pi y + \frac{(-)^{\frac{n}{2}}\sinh\left(\frac{n}{2}-\frac{1}{4}\right)\pi y}{\sqrt{2}\sinh\left(\frac{n}{2}-\frac{1}{4}\right)\pi} \text{ for odd } n. \quad \text{(10)d}$$

For small values of y and large n, the hyperbolic part of these expressions becomes negligible, and $g_n(y)$ behaves like trigonometric functions.

The integrals α_{mn}, β_{mn} etc., are found as follows:

$$\text{(I).}\quad \alpha_{mn} = \begin{cases} \dfrac{p_m^4}{L^4} & \text{if } m = n. \\ 0 & \text{if } m \neq n. \end{cases} \tag{11}$$

$$\begin{aligned}
\text{(II).}\quad \underset{(m\neq n)}{\beta_{mn}} = \frac{p_m p_n}{L^2} \Bigg\{ & \frac{(1+\sigma_m\sigma_n)}{2}\left[\frac{\sin(p_m-p_n)}{p_m-p_n} + \frac{\sinh(p_m+p_n)}{p_m+p_n}\right] - \\
& -\frac{(1-\sigma_m\sigma_n)}{2}\left[\frac{\sin(p_m+p_n)}{p_m+p_n} + \frac{\sinh(p_m-p_n)}{p_m-p_n}\right] \\
& + \frac{(\sigma_m-\sigma_n)}{2}\,\frac{\cos(p_m-p_n)+\cosh(p_m-p_n)}{p_m-p_n} - \frac{\sigma_m+\sigma_n}{2}\,\frac{\cos(p_m+p_n)+\cosh(p_m+p_n)}{p_m+p_n} \\
& + 2\,\frac{\sigma_m p_n+\sigma_n p_m}{p_m^2+p_n^2} + 2\,\frac{\sigma_n p_m-\sigma_m p_n}{p_m^2-p_n^2} \\
& + \frac{1}{p_m^2+p_n^2}\Big[p_m(\cos p_m \sinh p_n - \cosh p_m \sin p_n) + p_n(\cos p_n \sinh p_m - \cosh p_n \sin p_m)\Big] \\
& + \frac{\sigma_m}{p_m^2+p_n^2}\Big[p_m(\sin p_n \sinh p_m - \sin p_m \sinh p_n) - p_n(\cos p_m \cosh p_n + \cos p_n \cosh p_m)\Big] \\
& + \frac{\sigma_n}{p_m^2+p_n^2}\Big[p_n(\sin p_m \sinh p_n - \sin p_n \sinh p_m) - p_m(\cos p_n \cosh p_m + \cos p_m \cosh p_n)\Big] \\
& + \frac{\sigma_m\sigma_n}{p_m^2+p_n^2}\Big[p_m(\cos p_n \sinh p_m + \cosh p_n \sin p_m) + p_n(\cos p_m \sinh p_n + \cosh p_m \sin p_n)\Big]\Bigg\},
\end{aligned} \tag{12a}$$

$$\begin{aligned}
\beta_{mm} = \frac{p_m}{L^2}\Bigg\{ & (1+\sigma_m^2)\,\frac{\sinh 2p_m}{4} - \frac{(1-\sigma_m^2)}{4}\sin 2p_m + \sigma_m^2 p_m \\
& - (1-\sigma_m^2)\sin p_m \cosh p_m + (1+\sigma_m^2)\sinh p_m \cos p_m \\
& + \sigma_m(4 - \sinh^2 p_m + \sin^2 p_m)\Bigg\};
\end{aligned} \tag{12b}$$

where

$$\sigma_m = \frac{\cos p_m + \cosh p_m}{\sin p_m + \sinh p_m}. \tag{12c}$$

These exact formulae are needed for small values of m and n. For moderate values of m, n; the following approximate formulae are valid to four significant figures:

For m, n>2: m≠n

$$\frac{L^2}{p_m p_n}\beta_{mn} = \frac{\sin(p_m - p_n)}{p_m - p_n} + 2\frac{(-1)^{m+n}+1}{p_m + p_n} + \left[(-1)^{m+1} + (-1)^{n+1}\right]\frac{e^{-p_m} - e^{-p_n}}{p_m - p_n}$$

$$+ \frac{1}{p_m^2 + p_n^2}\Big\{\left[1 + (-1)^{m+n+1}\right](p_m + p_n) + \left[1 + (-1)^n\right]p_m + \left[1 + (-1)^m\right]p_n$$

$$+ \left[(-1)^m + (-1)^n\right]\left[e^{-p_n}p_m + e^{-p_m}p_n\right]$$

$$+ \left[(-1)^m - (-1)^n\right]\left[e^{-p_n}p_n - e^{-p_m}p_m\right]\Big\}. \qquad (12)d$$

$$m, n > 4, \quad \frac{L^2}{p_m}\beta_{mm} = 2 + p_m + (-)^m\left[4p_m + 11\right]p_m \; e^{-p_m}. \qquad (12)e$$

$$\text{(III).} \quad \gamma_{mn} = \frac{4\sigma_m p_m}{L^2} - \beta_{mn} \qquad (13)$$

$$\text{(IV).} \quad \kappa_{mn} = \int_{-1}^{1} g_m''(y)\, g_n''(y)\, dy = \begin{cases} q_m^4 & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases} \qquad (14)$$

$$\text{(V).} \quad \lambda_{mi} = \lambda_{im} = \int_{-1}^{1} g_m'(y)\, g_i'(y)\, dy \qquad (15)a$$

When m and i are both even:

$$\lambda_{mi} \underset{\substack{\text{even-even,} \\ m \neq i}}{=} \frac{q_m q_i}{\left(1 + \frac{\cos^2 q_m}{\cosh^2 q_i}\right)\left(1 + \frac{\cos^2 q_i}{\cosh^2 q_i}\right)}\left\{\frac{\sin(q_m - q_i)}{q_m - q_i} - \frac{\sin(q_m + q_i)}{q_m + q_i}\right.$$

$$+ \cos q_m \cos q_i\left[\frac{\tanh q_m + \tanh q_i}{q_m + q_i} - \frac{\tanh q_m - \tanh q_i}{q_m - q_i}\right] -$$

$$- \frac{2\cos q_i}{q_m^2 + q_i^2}\left[- q_m \cos q_m \tanh q_i + q_i \sin q_m\right] - \frac{2\cos q_m}{q_m^2+q_i^2}\left[- q_i \cos q_i \tanh q_m + q_m \sin q_i\right]\Big\}. \qquad (15)_b$$

When m and i are both odd: (except for index 1, which can be easily integrated)

$$\lambda_{mi} \underset{\substack{\text{odd-odd} \\ m \neq i}}{=} \frac{q_m \; q_i}{\sqrt{\left(1 - \frac{\sin^2 q_m}{\sinh^2 q_m}\right)\left(1 - \frac{\sin^2 q_i}{\sinh^2 q_i}\right)}} \left\{ \frac{\sin (q_m - q_i)}{q_m - q_i} + \frac{\sin (q_m + q_i)}{q_m + q_i} \right.$$

$$+ \sin q_m \sin q_i \left[\frac{\coth q_m + \coth q_i}{q_m + q_i} - \frac{\coth q_m - \coth q_i}{q_m - q_i}\right]$$

$$+ \frac{2 \sin q_i}{q_m^2 + q_i^2}\left[q_m \sin q_m \coth q_i + q_i \cos q_m \right]$$

$$\left. + \frac{2 \sin q_m}{q_m^2 + q_i^2}\left[q_i \sin q_i \coth q_m + q_m \cos q_i \right]\right\}. \qquad (15)_c$$

$$\lambda_{mm} \underset{m,\ \text{even}}{=} \frac{q_m^2}{1 + \frac{\cos^2 q_m}{\cosh^2 q_m}} \left\{ 1 - \frac{\sin 2 q_m}{2 q_m} + \cos^2 q_m \left[\frac{\tanh q_m}{q_m} - \frac{1}{\cosh^2 q_m}\right]\right.$$

$$\left. + \frac{2 \cos q_m}{q_m}\left[\cos q_m \tanh q_m - \sin q_m \right]\right\}. \qquad (15)d$$

$$\lambda_{mm} \underset{m,\ \text{odd}}{=} \frac{q_m^2}{1 - \frac{\sin^2 q_m}{\sinh^2 q_m}} \left\{ 1 + \frac{\sin 2 q_m}{2 q_m} + \sin^2 q_m \left[\frac{\coth q_m}{q_m} - \frac{1}{\sinh^2 q_m}\right]\right.$$

$$\left. + \frac{2 \sin q_m}{q_m}\left[\sin q_m \coth q_m + \cos q_m \right]\right\}. \qquad (15)e$$

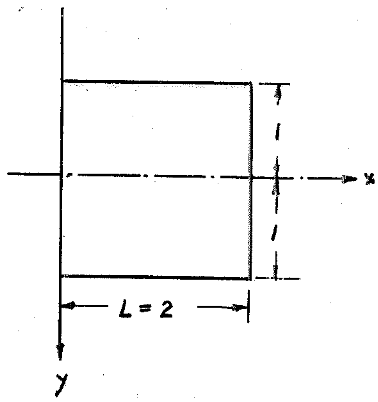
When one of m and n is even and the other is odd, $\lambda_{mn} = 0$. $(15)_f$

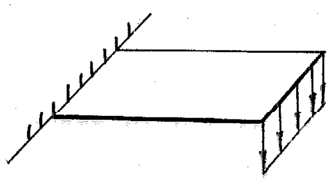
(VI) $\mu_{mi} = \int_{-1}^{1} g_m''(y)\, g_i(y)\,dy = 2\, g_m'(1)\, g_i(1) - \lambda_{im}$. (16)

The values of the constants α_{mn}, β_{mn} etc., are computed according to formulas (11) through (16) and listed in Table 3.21:1 through Table 3.21:6, for a range of m and n from o to 4. Coefficients with small subscripts are the more difficult ones to compute; for m, n > 4, approximate calculations of these coefficients can be carried out relatively easily.

Values of fm (L), fm (L), gn (l) and gn (l) together with σ_m and p_m, q_m, are given in Tables 3.21:7 and 3.21:8. σ_m is defined by equation (12)c.

Example 1. Cantilever square plate loaded by uniform shear along the tip section.





As an example consider the deflection of a square cantilever plate loaded by a uniform shear load along the tip section. It is well-known that the classical solution of St. Venant's problem of flexure of a beam of rectangular cross-section holds only for a particular distribution of the shear load along the tip, and that the ordinary engineering beam theory holds only for comparatively long beams with the width and depth of its cross-section of the same order of magnitude. The present method solves the case when the thickness of the beam is much smaller than the width under various loading conditions, which is taken as uniform shear in this example.

By symmetry, only even indices of $g_n(y)$ will appear in the deflection function. Let us take six indeterminate coefficients:

$$a_{10},\quad a_{12},\quad a_{14},$$
$$a_{20},\quad a_{22},\quad a_{24}\ ,$$

so that the deflection is expressed as:

$$w = w_0(x,y) + a_{10}\, f_1(x)g_0(y) + a_{12}\, f_1(x)g_2(y) + \cdot\ \cdot\ \cdot + a_{24}f_2(x)g_4(y) \tag{17}$$

The function $w_0(x,y)$ is taken as the deflection function given by the ordinary beam theory, so that

$$w_0(x,y) = \frac{q}{(1-\nu^2)D}\left\{ L\,\frac{x^2}{2} - \frac{x^3}{6} \right\}. \tag{18}$$

where q is the load per unit length applied along the edge of the plate.

Substituting (18) into (7) and integrating, one obtains

$$\frac{\partial V_0}{\partial a_{mn}} = \frac{q}{(1-\nu^2)}\int_0^L\int_{-1}^{1}(L-x)\left[f_m''(x)g_n(y)+\nu f_m(x)g_n''(y)\right]dx\,dy - q\,f_m(L)\int_{-1}^{1}g_n(y)\,dy$$

$$= \begin{cases} \sqrt{2}\,\dfrac{\nu^2}{1-\nu^2}\,f_m(L)\,q & \text{if } n=0, \\[2ex] 2\,\dfrac{\nu}{1-\nu^2}\,g_n'(1)\left\{2L\sqrt{L}\,\sigma_m - L^2 f_m(L)\right\}\dfrac{1}{p_m}\,q\,, & \text{if } n\neq 0. \end{cases} \qquad (19)$$

where p_m is the root corresponding to $f_m(x)$. Values of $g_n'(1)$, $f_m(L)$, and σ_m, p_m are listed in Table 3.21:7 and 8. For the particular case of L = 2, and $\nu = .3$,

$$\frac{\partial V_0}{\partial a_{10}} = .19583\ q\,, \qquad \frac{\partial V_0}{\partial a_{20}} = -.19701\ q\,,$$

$$\frac{\partial V_0}{\partial a_{12}} = 1.6153\ q\,, \qquad \frac{\partial V_0}{\partial a_{22}} = -5.2599\ q\,,$$

$$\frac{\partial V_0}{\partial a_{14}} = -3.8217\ q\,, \qquad \frac{\partial V_0}{\partial a_{24}} = 12.445\ q\,.$$

Substituting these into equations (8), with m ranges from 1 and 2, and n from 0, 2, 4, one obtains the system of linear simultaneous equations as follows, where ν is taken to be 0.3:

$$\begin{aligned}
+1.5453\,a_{10} + 2.7528\,a_{12} - 6.5130\,a_{14} - 8.1768\,a_{22} + 19.346\,a_{24} &= -.19583\ \tfrac{2q}{D}, \\
+158.469\,a_{12} - 32.421\,a_{14} + 14.327\,a_{22} - 15.202\,a_{24} &= -1.6153\ \tfrac{2q}{D}, \\
-33.674\,a_{12} + 2188.97\,a_{14} - 11.4798\,a_{22} + 48.341\,a_{24} &= 3.8217\ \tfrac{2q}{D}, \\
+60.690\,a_{20} + 1.2680\,a_{12} - 3.00006\,a_{14} + 13.705\,a_{22} + 121.56\,a_{24} &= +.19701\ \tfrac{2q}{D}, \\
+34.745\,a_{12} - 24.329\,a_{14} + 518.313\,a_{22} - 134.45\,a_{24} &= +5.2599\ \tfrac{2q}{D}, \\
-24.907\,a_{12} + 132.12\,a_{14} - 140.69\,a_{22} + 3399.2\,a_{24} &= -12.445\ \tfrac{2q}{D}.
\end{aligned}$$

This can be solved by iteration or otherwise. The solution is:

$a_{10} = -.003959\frac{2q}{D}$, $a_{12} = -.01108\frac{2q}{D}$, $a_{14} = -.001703\frac{2q}{D}$,

$a_{20} = .008075\frac{2q}{D}$, $a_{22} = .01009\frac{2q}{D}$, $a_{24} = -.003391\frac{2q}{D}$

Hence the deflection surface is given by:

$$w(x,y) = \frac{q}{D}\left\{\frac{1}{1-\nu^2}\left(\frac{Lx}{2}-\frac{x^3}{6}\right) - .007918\ f_1(x) - .02216\ f_1(x)\ g_2(y) + .003406\ f_1(x)\ g_4(y) + .01615\ f_2(x) + .02018\ f_2(x)\ g_2(y) - .006782\ f_2(x)\ g_4(y)\right\}.$$

where f(x) and g(y) are functions given by (9) and (10).

3.22. Solution by Relaxation of Boundary Conditions

The problem of the deflection of rectangular cantilever plates can be solved by the method of relaxation of boundary conditions. This is essentially a modification of biharmonic analysis to obtain approximate solutions. The deflection $w(x,y)$ is represented by a sum of a particular integral and a complementary function. The latter is then taken as a series of biharmonic functions, the coefficients of this series are determined by minimizing a weighted mean error on the boundary of the plate. The principle of this method is discussed in Section 1.23.

The procedure is simplified if one can construct a convenient sequence of biharmonic functions which possess orthogonality properties along the boundary curve of the plate. However, the classical Schmidt process of normalization (p. 131, Ref. 25) is rather tedious for the present case. Hence a more direct, though less systematic procedure is adopted in the following.

Biharmonic Functions

Since the functions

$$\begin{matrix}\sinh\\ \cosh\end{matrix}\lambda x \begin{matrix}\sin\\ \cos\end{matrix}\lambda y \qquad \text{and} \qquad \begin{matrix}\sin\\ \cos\end{matrix}\lambda x \quad \begin{matrix}\sinh\\ \cosh\end{matrix}\lambda y$$

are harmonic functions, where λ is any real constant, it follows that (Cf. section 3.12):

$$(A_1 - A_2 x - A_3 y)\left\{ \quad B \begin{matrix}\sin\\ \cos\end{matrix}\lambda x \quad \begin{matrix}\sinh\\ \cosh\end{matrix}\lambda y - C \begin{matrix}\sinh\\ \cosh\end{matrix}\lambda x \quad \begin{matrix}\sin\\ \cos\end{matrix}\lambda y\right)$$

are biharmonic functions, where A, B, C are arbitrary constants. The notation on the possible combinations of trigonometric and hyperbolic functions seems to be obvious.

The following functions are also biharmonic:

$$(a+bx)\ p^{3}(y), \qquad (a+by)\ p^{3}(x)$$

where $p^{3}(y)$ and $p^{3}(x)$ are polynomials of degree not higher than third in y and x respectively. It can be verified that the most general form of biharmonic functions when the variables are separable is given by the above two types.

Approximating Sequence

Let $w_0(x,y)$ be a particular integral of the differential equation

$$\nabla^4 w_0 = \frac{q}{D}$$

where q is the distributed load over the plate. This function in general does not satisfy the boundary conditions along the edges of the plate (otherwise w_0 is an exact solution and the problem is solved). For convenience, however, assume that w_0 satisfies the boundary conditions along the built-in edge of the plate.

The following examples are carried out for the case when the loading and hence the deflection of the plate is symmetric about the x-axis. The anti-symmetric case can be similarly worked out.

The deflection surface is assumed to be expressible in series of the form:

$$w = w_0 + \sum_{\lambda} A_\lambda\, x \sinh \lambda x \cos \lambda y + \sum_{\mu} B_\mu\, x \sin \mu x \cosh \mu y, \quad (1)$$

where λ, μ are constants. In this form the boundary conditions along the edge $x = o$,

$$w = \frac{\partial w}{\partial x} = o,$$

are satisfied. Since this is the edge where the maximum stress occurs, it seems reasonable to have boundary conditions on this edge satisfied exactly to start with.

Boundary Functions.

The boundary functions $B_1(w)$ and $B_2(w)$ (Cf. section 1.23) are defined as functions along the boundary:

$$B_1(w) = \begin{cases} D\left[\frac{\partial^2 w}{\partial n^2} + \nu \frac{\partial^2 w}{\partial t^2} \right] + M_n & \text{on unsupported edges,} \\ PL \frac{\partial w}{\partial n} & \text{on built-in edges} \end{cases}$$

$$B_2(w) = \begin{cases} -D\left[\frac{\partial^3 w}{\partial n^3} + (2-\nu) \frac{\partial^3 w}{\partial n \partial t^2} \right] - \left(Q_n + \frac{\partial H_{nt}}{\partial t} \right) & \text{on free edges ,} \\ Pw & \text{on supported edges .} \end{cases} \tag{2}$$

where n and t denote the normal and tangential directions on the boundary and M_n, Q_n, H_{nt} the external moment, shear, and twisting couple acting on the boundary respectively. Specifically,

$$B_1^{\perp x}(w) \equiv B_1(w) \quad \text{on free edge perpendicular to x-axis}$$
$$= D\left[\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right] - M_x \;,$$
$$B_1^{\perp y}(w) \equiv B_1(w) \text{ on free edges perpendicular to y axis}$$
$$= D\left[\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right] - M_y, \text{ and so on.}$$

Similarly let the following notations be used:

$$B_1^{\perp tip}(w) = \left\{ B_1^{\perp x}(w) \right\}_{x=L} , \qquad B_2^{\perp tip}(w) = \left\{ B_2^{\perp x}(w) \right\}_{x=L} ,$$
$$B_1^{\perp side}(w) = \left\{ B_1^{\perp y}(w) \right\}_{y=\pm 1} , \qquad B_2^{\perp side}(w) = \left\{ B_2^{\perp y}(w) \right\}_{y=\pm 1} . \tag{3}$$

By symmetry $\left\{ B_1^{\perp y}(w) \right\}_{y=+1} = \left\{ B_1^{\perp y}(w) \right\}_{y=-1}$, so $B_1^{\perp side}(w)$ means $B_1(w)$ on either side of the plate: $y = \pm 1$.

Of course, all the boundary functions vanish for exact solutions. When w is not an exact solution, the boundary functions represent the error on the boundary.

According to (1) and (2), by substitution, one obtains

$$
\begin{aligned}
B_1^{\text{tip}}(w) &= B_1^{\text{tip}}(w_0) + D\sum_{\lambda} A_\lambda\left[(1-\nu)L\lambda^2 \sinh\lambda L - 2\lambda\cosh\lambda L)\right]\cos\lambda y \\
&\quad + D\sum_{\mu} B_\mu\left[(\nu-1)\mu^2 L\sin\mu L - 2\mu\cos\mu L\right]\cosh\mu y\,, \\
B_2^{\text{tip}}(w) &= B_2^{\text{tip}}(w_0) - D\sum_{\lambda} A_\lambda\left[(\nu-1)\lambda^3 L\cosh\lambda L + (1+\nu)\lambda^2\sinh\lambda L\right]\cos\lambda y \\
&\quad - D\sum_{\mu} B_\mu\left[(1-\nu)\mu^3 L\cos\mu L - (1+\nu)\mu^2\sin\mu L\right]\cosh\mu y\,, \\
B_1^{\text{side}}(w) &= B_1^{\text{side}}(w_0) + D\sum_{\lambda} A_\lambda\left[(\nu-1)\lambda^2(\cos\lambda)\,x\sinh\lambda x + 2\nu\cos\lambda\cosh\lambda x\right] \\
&\quad - D\sum_{\mu} B_\mu\left[(1-\nu)(\cosh\mu)\,x\sin\mu x + 2\nu\mu\cosh\mu\cos\mu x\right], \\
B_2^{\text{side}}(w) &= B_2^{\text{side}}(w_0) - D\sum_{\lambda} A_\lambda\left[(\nu-1)\lambda^3(\sin\lambda)\,x\sin\lambda x - 2(2-\nu)\lambda^2\sin\lambda\cosh\lambda x\right] \\
&\quad - D\sum_{\mu} B_\mu\left[(\nu-1)\mu^3(\sinh\mu)\,x\sin\mu x + 2(2-\nu)\mu\sinh\mu\cos\mu x\right].
\end{aligned}
\tag{4}
$$

<u>Equations for the Determination of the Unknown Coefficients</u>

The equations for the determination of the unknown coefficients $\{A_\lambda\}$ and $\{B_\mu\}$ can be formed in various ways according to section 1.23. The following is one of the simplest ways for the present problem:

$$
\begin{aligned}
&\int_0^1 B_1^{\text{tip}}(w)\,L\sinh\lambda L\cdot\cos\lambda y\,dy + \int_0^L B_1^{\text{side}}(w)\,x\sinh\lambda x(\cos\lambda)\,dx = 0 \\
&\int_0^1 B_2^{\text{tip}}(w)\,L\sinh\lambda L\cdot\cos\lambda y\,dy + \int_0^L B_2^{\text{side}}(w)\,x\sinh\lambda x\cdot\cos\lambda\cdot dx = 0 \\
&\int_0^1 B_1^{\text{tip}}(w)\,L\sin\mu L\cdot\cosh\mu y\,dy + \int_0^L B_1^{\text{side}}(w)\,x\sin\mu x\cdot\cosh\mu\cdot dx = 0. \\
&\int_0^1 B_2^{\text{tip}}(w)\,L\sin\mu L\cdot\cosh\mu y\,dy + \int_0^L B_2^{\text{side}}(w)\,x\sin\mu x\cdot\cosh\mu\cdot dx = 0.
\end{aligned}
\tag{5}
$$

This gives a system of linear simultaneous equations in $\{A_\lambda\}$ and $\{B_\mu\}$ from which these constants can be determined.

<u>Example 1.</u> <u>Rectangular Plates with Uniform Shear Load at the Tip Section</u>

For a cantilever plate loaded at the tip section (See Fig. 3.3:1) with a uniform load of q per unit length, the particular solution $w_0(x,y)$ may be obtained by considering the plate as a cantilever beam:

$$
w_0(x,y) = \frac{q}{6D}\left\{(L-x)^3 - L^2(L-3x)\right\}, \tag{6}
$$

This satisfies all the boundary conditions except the bending moment along the edges $y = \pm 1$; that is, $B_1^{tip}(w_0) = B_2^{tip}(w_0) = B_1^{side}(w_0) = B_2^{side}(w_0) = 0$, but $B_1^{side}(w_0) = \frac{\nu q (L-x)}{D}$.

Let four indeterminate constants be taken for the first approximation. Let $\{\lambda\}$ take the values $\{n\pi\}$ and $\{\mu\}$ the values $\{\frac{n\pi}{L}\}$, where $\{n\}$ are positive integers. Then the approximating deflection function is

$$w = \frac{q}{6D}\left\{(L-x)^3 - L^2(L-3x)\right\} + A_1 x \sinh \pi x \cos \pi y + A_2 x \sinh 2\pi x \cos 2\pi y + B_1 x \sin\frac{\pi x}{L}\cosh\frac{\pi y}{2L} + B_2 x \sin\frac{2\pi x}{L}\cosh\frac{\pi y}{L}. \tag{7}$$

A_1, A_2, B_1, B_2, are to be determined from the boundary conditions. The condition that there is no resultant force at the free corners demands that

$$\frac{\partial^2 w}{\partial x \partial y} = 0 \qquad \text{for} \quad x = L, \quad y = \pm 1.$$

This condition imposes the restriction that

$$B_2 = \frac{\sinh \pi/L}{4 \sinh 2\pi/L} B_1. \tag{8}$$

Hence in fact, only three unknown constants are to be determined.

The linear equations for the determination of the constants $\{A_n\}$, $\{B_n\}$ are formed according to equations (5). In the present example they are the following:

$$\begin{aligned} L \sinh \pi L \int_0^1 B_1^{tip}(w) \cos \pi y \cdot dy + \int_0^L B_1^{side}(w)\, x \sinh \pi x \cdot dx &= 0, \\ \int_0^L B_1^{side}(w)\, x \cdot \sin\frac{\pi x}{L}\, dx &= 0, \\ L \sinh \pi L \int_0^1 B_2^{tip}(w) \cos \pi y \cdot dy + \int_0^L B_2^{side}(w)\, x \cdot \sinh \pi x \cdot dx &= 0. \end{aligned} \tag{9}$$

Let us write

$$\frac{1}{D}\, B_1^{\text{tip}}(w) = C_{A_1} A_1 \cos y + C_{A_2} A_2 \cos 2\pi y + C_{B_1} B_1 \cosh\frac{\pi y}{L} + C_{B_2} k B_1 \cosh\frac{2\pi y}{L},$$

$$\frac{1}{D}\, B_2^{\text{tip}}(w) = C_{A_1}' A_1 \cos\pi y + C_{A_2}' A_2 \cos 2\pi y + C_{B_1}' B_1 \cosh\frac{\pi y}{L} + C_{B_2}' k B_1 \cosh\frac{2\pi y}{L},$$

$$\frac{1}{D}\, B_1^{\text{side}}(w) = \frac{\nu q}{D}(L-x) + k_{A_1} A_1\, x \sinh\pi x - 2\nu A_1 \cosh\pi x + k_{A_2} A_2 x \sinh 2\pi x$$
$$+ 2\nu A_2 \cosh 2\pi x + k_{B_1} B_1 x \sin\frac{\pi x}{L} + L_{B_1} B_1 \cos\frac{\pi x}{L} + k_{B_2} k B_1 x \sin\frac{2\pi x}{L} + l_{B_2} k B_1 \cos\frac{2\pi x}{L},$$

$$\frac{1}{D}\, B_2^{\text{side}}(w) = k_{B_1}' B_1 x \sin\frac{\pi x}{L} + l_{B_1}' B_1 \cos\frac{\pi x}{L} + k_{B_2}' k B_1 x \sin\frac{2\pi x}{L} + l_{B_2}' k B_1 \cos\frac{2\pi x}{L}. \tag{10}$$

where

$$c_{A_1} = (1-\nu) L \pi^2 \sinh\pi L + 2\pi\cosh\pi L,$$

$$c_{A_2} = (1-\nu)\, 4\pi^2 L \sinh 2\pi L + 4\pi\cosh 2\pi L,$$

$$c_{A_1}' = (1-\nu)\pi^3 L \cosh\pi L - (1+\nu)\pi^2 \sinh\pi L,$$

$$c_{A_2}' = (1-\nu)\, 8\pi^3 L \cosh 2\pi L - (1+\nu)\, 4\pi^2 \sinh 2\pi L,$$

$$c_{B_1} = -2\frac{\pi}{L}, \quad c_{B_2} = 4\frac{\pi}{L}, \quad c_{B_1}' = (1-\nu)\frac{\pi^3}{L^3}, \quad c_{B_2}' = -(1-\nu)\frac{8\pi^3}{L^2},$$

$$k_{A_1} = (1-\nu)\pi^2, \quad k_{A_2} = -4(1-\nu)\pi^2, \quad k_{B_1} = (1-\nu)\cosh\frac{\pi}{L}, \quad k_{B_2} = (1-\nu)\cosh\frac{2\pi}{L},$$

$$k_{B_1}' = (1-\nu)\frac{\pi^3}{L^3}\sinh\frac{\pi}{L}, \quad k_{B_2}' = (1-\nu)\frac{8\pi^3}{L^3}\sinh\frac{2\pi}{L}, \tag{11}$$

$$l_{B_1} = 2\nu\frac{\pi}{L}\cosh\frac{\pi}{L}, \quad l_{B_2} = 4\nu\frac{\pi}{L}\cosh\frac{2\pi}{L}, \quad l_{B_1}' = -2(2-\nu)\frac{\pi^2}{L^2}\sinh\frac{\pi}{L},$$

$$l_{B_2}' = -2(2-\nu)\frac{4\pi^2}{L^2}\sinh\frac{2\pi}{L}. \quad K = \sinh\frac{\pi}{L}\Big/ 4\sinh\frac{2\pi}{L}.$$

Then the explicit expression of the system of equations (9) becomes:

$$\frac{\nu q}{D}(\mathrm{I}) + \left[L\sinh(\pi L)\frac{c_{A_1}}{2} + k_{A_1}(\mathrm{III}) - 2\nu(\mathrm{IV})\right]A_1 + \left[k_{A_2}(\mathrm{V}) + 2\nu(\mathrm{VI})\right]A_2$$
$$+ \Big[L\sinh(\pi L)\{c_{B_1}(\alpha) + c_{B_2}K(\beta)\} + k_{B_1}(\mathrm{VII}) + l_{B_1}(\mathrm{VIII})$$
$$+ k_{B_2}K(\mathrm{IX}) + l_{B_2}K(\mathrm{X})\Big]B_1 = 0,$$

$$\frac{\nu q}{D}(\mathrm{II}) + \left[k_{A_1}(\mathrm{VII}) - 2\nu(\mathrm{XI})\right]A_1 + \left[k_{A_2}(\mathrm{XII}) + 2\nu(\mathrm{XIII})\right]A_2$$
$$+ \left[k_{B_1}(\mathrm{XIV}) + l_{B_1}(\mathrm{XV}) + k_{B_2}k(\mathrm{XVI}) + l_{B_2}k(\mathrm{XVII})\right]B_1 = 0,$$

$$L \sinh \pi L \, \frac{c'_{A1}}{2} A_1 + \Big[L \sinh \pi L \{ c'_{B1} (\alpha) + c'_{B2} \, k (\beta) \} + k'_{B1} \, (\mathrm{VII})$$

$$+ \, \ell'_{B1} \, (\mathrm{VIII}) + k'_{B2} \, k \, (\mathrm{IX}) + \ell'_{B2} \, k \, (\mathrm{X}) \Big] B_1 = 0 \, , \qquad (12)$$

where (α), (β), (I), (II), (III) etc. are integrals listed in Table 3.22:I.

<u>Sample Computation of a Square Cantilever Plate.</u>

As a particular case consider a square cantilever plate, L = 2. When the Poisson's ratio ν is taken as 0.3, the value of the constants listed in equations (11) are then as follows:

$k = .048395$,

$C_{A1} = 10.0503 e^{2\pi}$, $\quad C_{A2} = 83.9181 e^{4\pi}$, $\quad C'_{A1} = 15.2891 e^{2\pi}$, $\quad C'_{A2} = 147.9742 e^{4\pi}$

$C_{B1} = -3.14159$, $\quad C_{B2} = 6.28318$, $\quad C'_{B1} = 5.42609$, $\quad C'_{B2} = -43.4087$,

$k_{A1} = 6.90872$, $\quad k_{A2} = -27.6349$, $\quad k_{B1} = 1.75643$, $\quad k_{B2} = 8.11443$,

$k'_{B1} = 6.24354$, $\quad k'_{B2} = 250.6602$,

$\ell_{B1} = 2.36485$, $\quad \ell_{B2} = 21.8505$, $\quad \ell'_{B1} = -19.3060$, $\quad \ell'_{B2} = -387.5397$.

Then equations (12) become, when the numerical values are substituted from table 3.22:1,

$$5.919684 e^{2\pi} A_1 - 2.607414 e^{4\pi} A_2 + .116980 \, B_1 = -.0690695 \frac{\nu q}{D} ,$$

$$.963228 e^{2\pi} A_1 - 1.402525 e^{4\pi} A_2 + .501483 \, B_1 = -1.03205 \frac{\nu q}{D} ,$$

$$7.64455 e^{2\pi} A_1 + 0 \qquad\qquad + .731412 \, B_1 = 0 .$$

The solution is easily found to be

$$e^{2\pi} A_1 = .0044641 \frac{q}{D} ,$$

$$e^{4\pi} A_2 = .207136 \frac{q}{D}$$

$$B_1 = -.0466575 \frac{q}{D}$$

$$B_2 = .0022580 \frac{q}{D}$$

Hence the deflection surface is approximately:

$$w = \frac{q}{D}\left\{ L\frac{x^2}{2} - \frac{x^3}{6} \qquad + .0044641\, e^{-2\pi}\, x \sinh \pi x \cos \pi y \right.$$

$$+ .207136 \quad e^{-4\pi}\, x \sin 2\pi x \cos 2\pi y$$

$$-.0466575 \cdot \quad x \sin\frac{\pi x}{L}\cos\frac{\pi y}{2L} - .0022580 \cdot \quad x \sin\frac{2\pi x}{L}\cos\frac{\pi y}{L}.$$

At the tip section, $x = L$:

$$w = \frac{q}{D}\left\{ \frac{L^3}{3} + .0044641 \cos\pi y + .207136 \cos 2\pi y \right\}.$$

This is perhaps less accurate than the previous result obtained by the Rayleigh-Ritz method, since only three independent coefficients have been taken. As a conclusion we may say that in solving the plate problem the method of relaxation of boundary conditions is less accurate and more tedious than the procedure described in Section 3.21. Accordingly the Rayleigh-Ritz method is used in the next section to solve the probelms of deflection of swept plates.

3.3 Swept Cantilever Plates

With the coordinate system and notations used in Section (1.13), the potential energy integral (3.21:1) becomes

$$V = \frac{1}{2}\frac{D}{\cos^3\theta}\iint_R\left\{\left(\frac{\partial^2 w}{\partial\xi^2} - 2\sin\theta\frac{\partial^2 w}{\partial\xi\partial\eta} + \frac{\partial^2 w}{\partial\eta^2}\right)^2 - 2(1-\nu)\cos^2\theta\left[\frac{\partial^2 w}{\partial\xi^2}\frac{\partial^2 w}{\partial\eta^2} - \left(\frac{\partial^2 w}{\partial\xi\partial\eta}\right)^2\right]\right\}d\xi\, d\eta$$

$$- \iint_R q\, w\cos\theta\, d\xi\, d\eta + \int_L M_n\frac{\partial w}{\partial n}ds - \int_L\left(Q_n - \frac{\partial H_{nt}}{\partial s}\right)w\, ds \qquad (1)$$

where R is extended over the whole plate and L over the free edges; M_n, H_{nt}, and Q_n being the externally applied couples and shear on the edges. The sweep angle is θ.

The same method used in Section (s.21) can be applied here. The boundary conditions on displacements at the restrained edge are

$$w = 0 \quad \text{and} \quad \frac{\partial w}{\partial\xi} = 0 \qquad \text{when} \quad \xi = 0 \qquad (2)$$

Any function w continuous (D,4) and which satisfies (2) is called "admissible". The solution of a loaded swept cantilever plate is given by one of the admissible functions which minimizes the potential energy.

As before, write

$$w(\xi,\eta) = w_0(\xi,\eta) + \sum_{m,n} a_{mn} w_{mn}(\xi,\eta) \qquad (3)$$

where $w_0(\xi,\eta)$ is an approximate solution. Of course all w_0 and $\{w_{mn}\}$ must be admissible functions. Furthermore, write

$$w_{mn}(\xi,\eta) = f_m(\xi)\, g_n(\eta) \qquad (4)$$

where $\{f_m(\xi)\}$ and $\{g_n(\eta)\}$ are the normal modes of a vibrating bar as defined in Section (3.21) and given by equations (3.21:4) and 3.21:10). Obviously w_{mn} in this form satisfies the supporting conditions (3.3:2). Using the definitions adopted in Section (3.21) for the sysbols α_{mi}, β_{mi},

γ_{mi}, κ_{mi}, λ_{mi}, μ_{mi}, and in addition, let

$$\psi_{mi} = \int_0^L f_m'' f_i' \, dx , \qquad \pi_{mi} = \int_{-1}^{1} g_m'' g_i' \, dy ,$$
$$\tau_{mi} = \int_0^L f_m' f_i \, dx , \qquad \zeta_{mi} = \int_{-1}^{1} g_m' g_i \, dy . \tag{5}$$

Substituting (3.3:3) into (3.3:1) and equating its partial derivatives with respect to $\{a_{mn}\}$ to zero, one obtains a system of linear simultaneous equations for the determination of $\{a_{mn}\}$ as follows:

$$\begin{aligned} 2\, a_{mn} \Big[& \alpha_{mm} + \kappa_{nn} + 2(1+\sin^2\theta - \nu\cos^2\theta)\, \beta_{mm}\lambda_{nn} + 2(\sin^2\theta + \nu\cos^2\theta)\, \gamma_{mm}\mu_{nn} \\ & - 4\sin\theta\, (\psi_{mm}\zeta_{nn} + \tau_{mm}\pi_{nn}) \Big] \\ + \sum_{ij \neq mn} a_{ij} \Big[& 2(1+\sin^2\theta - \nu\cos^2\theta)\beta_{mi}\lambda_{jn} + 2(\sin^2\theta + \nu\cos^2\theta)\, \gamma_{mi}\mu_{jn} \\ & - 4\sin\theta\, (\psi_{mi}\zeta_{jn} + \tau_{mi}\pi_{jn}) \Big] = - \frac{2\cos^3\theta}{D} \frac{\partial V_0}{\partial a_{mn}} . \end{aligned} \tag{6}$$

where

$$\begin{aligned} \frac{\partial V_0}{\partial a_{mn}} = \frac{D}{\cos^3\theta} \int_{-1}^{1}\int_0^{L} \Big\{ & f_m'' g_n \Big[\frac{\partial^2 w_0}{\partial \xi^2} - 2\sin\theta \frac{\partial^2 w_0}{\partial \xi \partial \eta} + (\sin^2\theta + \nu\cos^2\theta) \frac{\partial^2 w_0}{\partial \eta^2} \Big] \\ & + 2 f_m' g_n' \Big[-\sin\theta \Big(\frac{\partial^2 w_0}{\partial \xi^2} + \frac{\partial^2 w_0}{\partial \eta^2} \Big) + (1+\sin^2\theta - \nu\cos^2\theta) \frac{\partial^2 w_0}{\partial \xi \partial \eta} \Big] \\ & + f_m g_n'' \Big[(\sin^2\theta + \nu\cos^2\theta) \frac{\partial^2 w_0}{\partial \xi^2} - 2\sin\theta \frac{\partial^2 w_0}{\partial \xi \partial \eta} + \frac{\partial^2 w_0}{\partial \eta^2} \Big] \Big\} \, d\xi \, d\eta \\ & - \cos\theta \int_{-1}^{1}\int_0^{L} q\, f_m g_n \, d\xi\, d\eta + \int M_n \frac{\partial (f_m g_n)}{\partial n} ds - \int \Big(Q_n - \frac{\partial H_{nt}}{\partial s} \Big) f_m g_n \, ds \end{aligned} \tag{7}$$

These equations reduce correctly to (3.21:7) and (3.21:8) when $\theta = 0$.

The expressions for α_{mm}, β_{mi}, γ_{mi}, κ_{mm}, λ_{mi}, μ_{mi} are given before in Section 3.21. Those for ψ_{mi}, τ_{mi}, π_{mi}, ζ_{mi} are given below.

$$\text{(I)} \quad \psi_{mi} = \int_0^L f_m'' f_i' \, dx = f_m'(L) f_i'(L) - \psi_{im} \tag{8a}$$

$$\begin{aligned} \frac{L^3}{p_i p_m^2} \psi_{mi} = & \frac{1+\sigma_i\sigma_m}{2} \Big[-\frac{\cos(p_i - p_m)}{p_i - p_m} + \frac{\cosh(p_i + p_m)}{p_i + p_m} \Big] + \frac{1-\sigma_i\sigma_m}{2} \Big[-\frac{\cos(p_i+p_m)}{p_i+p_m} + \frac{\cosh(p_i - p_m)}{p_i - p_m} \Big] \\ & + \frac{2\sigma_i\sigma_m p_m}{p_i^2 - p_m^2} + \frac{\sigma_i - \sigma_m}{2} \Big[\frac{\sin(p_i - p_m)}{p_i - p_m} - \frac{\sinh(p_i - p_m)}{p_i - p_m} \Big] \\ & + \frac{\sigma_i + \sigma_m}{2} \Big[\frac{\sin(p_i + p_m)}{p_i + p_m} - \frac{\sinh(p_i + p_m)}{p_i + p_m} \Big] + \end{aligned}$$

$$+\frac{1}{p_m^2+p_i^2}\Big\{ p_m(\sin p_i \sinh p_m + \sin p_m \sinh p_i) + p_i(-\cosh p_m \cos p_i + \cos p_m \cosh p_i)$$
$$+\sigma_m\left[p_m(\sinh p_i \cos p_m - \sin p_i \cosh p_m) + p_i(\sinh p_m \cos p_i - \sin p_m \cosh p_i)\right]$$
$$+\sigma_i\left[p_m(\cos p_i \sinh p_m - \cosh p_i \sin p_m) + p_i(\cosh p_m \sin p_i - \cos p_m \sinh p_i)\right]$$
$$+\sigma_i\sigma_m\left[p_m(2-\cosh p_i \cos p_m - \cos p_i \cosh p_m) + p_i(\sin p_m \sinh p_i - \sin p_i \sinh p_m)\right]\Big\}$$
$$(\text{for } m \neq i). \qquad (8)_b$$

$$\frac{L^3}{p_m^2}\psi_{mm} = \frac{1+\sigma_m^2}{2}\frac{\cosh 2p_m}{2} - \frac{1-\sigma_m^2}{2}\frac{\cos 2p_m}{2} + \frac{3\sigma_m^2}{2} + \frac{\sigma_m}{2}\left[\sin 2p_m - \sinh 2p_m\right]$$
$$+\sin p_m \sinh p_m + \sigma_m(\sinh p_m \cos p_m - \sin p_m \cosh p_m) \qquad (8)_c$$

For $m > 4$, to four significant figures: $\frac{L^3}{p_m^2}\psi_{mm} = 2 + (-)^m e^{-p_m}$.

$$(\text{II}) \quad \tau_{mi} = \int_0^L f_m' f_i \, dx = f_m(L) f_i(L) - \tau_{im} \qquad (9)$$

$$\frac{L}{p_i}\tau_{im}\,_{(i\neq m)} = \frac{1+\sigma_i\sigma_m}{2}\left[\frac{\cos(p_i-p_m)}{p_i-p_m} + \frac{\cosh(p_i+p_m)}{p_i+p_m}\right] + \frac{1-\sigma_i\sigma_m}{2}\left[\frac{\cos(p_i+p_m)}{p_i+p_m} + \frac{\cosh(p_i-p_m)}{p_i-p_m}\right]$$
$$-\frac{2p_i}{p_i^2-p_m^2} - \frac{\sigma_i-\sigma_m}{2}\left[\frac{\sin(p_i-p_m)}{p_i-p_m} + \frac{\sinh(p_i-p_m)}{p_i-p_m}\right] - \frac{\sigma_i+\sigma_m}{2}\left[\frac{\sin(p_i+p_m)}{p_i+p_m} + \frac{\sinh(p_i+p_m)}{p_i+p_m}\right]$$

$$+\frac{1}{p_m^2+p_i^2}\Big\{ p_m(\sin p_i \sinh p_m - \sin p_m \sinh p_i) + p_i(-\cos p_i \cosh p_m - \cos p_m \cosh p_i + 2)$$
$$-\sigma_m\left[p_m(\sin p_i \cosh p_m + \sinh p_i \cos p_m) - p_i(\sinh p_m \cos p_i + \sin p_m \cosh p_i)\right]$$
$$+\sigma_i\left[p_m(\cos p_i \sinh p_m + \cosh p_i \sin p_m) + p_i(\cosh p_m \sin p_i + \cos p_m \sinh p_i)\right]$$
$$+\sigma_i\sigma_m\left[p_m(\cosh p_i \cos p_m - \cos p_i \cosh p_m) - p_i(\sin p_m \sinh p_i + \sin p_i \sinh p_m)\right]\Big\}.$$

$$L\,\tau_{mm} = \frac{1+\sigma_m^2}{4}\cosh 2p_m + \frac{1-\sigma_m^2}{4}\cos 2p_m + \frac{3}{2} - \frac{\sigma_m}{2}\left[\sin 2p_m + \sinh 2p_m\right]$$
$$+\sigma_m(\cos p_m \sinh p_m + \cosh p_m \sin p_m) - \sigma_m^2 \sin p_m \sinh p_m. \qquad m>4, \ \tau_{mm} = \frac{2}{L}.$$

$$(\text{III}). \quad \pi_{mi} = \int_{-1}^{1} g_m'' g_i' \, dy = 2 g_m'(1) g_i'(1) - \pi_{im}. \qquad (10)_a$$

When m and i are both even numbers or both odd numbers, $\pi_{mi} = 0$.

When m is even and i is odd,

$$\frac{1}{q_i q_m^2}\pi_{mi} = \frac{1}{\sqrt{\left(1+\frac{\cos^2 q_m}{\cosh^2 q_m}\right)\left(1-\frac{\sin^2 q_i}{\sinh^2 q_i}\right)}}\Big\{-\frac{\sin(q_m-q_i)}{q_m-q_i} - \frac{\sin(q_m+q_i)}{q_m+q_i}$$
$$+\cos q_m \sin q_i\left[\frac{\tanh q_m \coth q_i + 1}{q_m+q_i} + \frac{\tanh q_m \coth q_i - 1}{q_m-q_i}\right] +$$

$$+ \frac{2q_m}{q_m^2+q_i^2}\left(\cos q_i \cos q_m \tanh q_m - \sin q_i \sin q_m \coth q_i\right)\Big\}. \qquad (10)_b$$

$$(\text{IV}).\quad \zeta_{mi} = \int_{-1}^{1} g_m' g_i \, dy = 2 g_m(1) g_i(1) - \zeta_{im}. \qquad (11)_a$$

Similar to Π_{mi}, ζ_{mi} vanishes when m, i are both even or both odd. When m is even and i is odd,

$$\frac{1}{q_i}\zeta_{im} = \frac{1}{\sqrt{1+\cos^2 q_m/\cosh^2 q_m}\,\sqrt{1-\sin^2 q_i/\sinh^2 q_i}}\Big\{\frac{\sin(q_m-q_i)}{q_m-q_i} + \frac{\sin(q_m+q_i)}{q_m+q_i}$$

$$+ \frac{2}{p_m^2+p_i^2}\left[2q_i \sin q_i \cos q_m + q_m(\cos q_m \cos q_i \tanh q_m + \sin q_m \sin q_i \coth q_i)\right]$$

$$+ \cos q_m \sin q_i \left[\frac{\tanh q_m \coth q_i + 1}{q_m+q_i} + \frac{\tanh q_m \coth q_i - 1}{q_m - q_i}\right]\Big\}. \qquad (11)_b$$

The values of the constants ψ_{mi}, τ_{mi} etc. are computed according to the above formulas and listed in Table 3.3:1 through Table 3.3:4, for a range of m and i from 0 to 4. Coefficients with larger subscripts can be computed relatively easily.

Empirical Results

The choice of the first approximation $w_o(x,y)$ is guided by two principles: It should be simple and general enough so that w_o can be easily constructed for various loading conditions, and it should be powerful enough to take care of a major part of the deflection. For this purpose any empirical information would be invaluable.

From the test results obtained at GALCIT Structural Laboratory on the deflection of thin elastic swept plates (Ref. 52), it has been pointed out that the deflection of a swept cantilever plate at a distance fairly far away from the built-in edge can be approximated by assuming the plate to be built-in along a fictitious line A-A perpendicular to the longitudinal axis of the plate (Fig. 3.3:1). The point of intersection (point D in Fig. 3.3:1) of this line A-A with the actual built-in edge BC varies with the sweep angle and loading conditions. This variation, however, is not conveniently predictable.

The following approximate theory, together with an experimentally determined empirical factor, seems to give desirable results.

Consider the swept cantilever plate as being formed by a series of elementary symmetrical rectangular beams of length <u>dx</u> each, set one against the other so that in the limit they approach the form of the given plate (Fig. 3.4:2). Assume that, for each of these elementary beams, the ordinary beam formulae for bending and torsion hold. In the swept beam, however, it is evident that the stress distribution is no longer uniform across the width of the beam. To take care of the effect of this non-uniform stress distribution, an <u>effectiveness factor k</u> is assumed, so that when the actual width c of the elementary beam is reduced to kc, the result will give a correct estimation of deflection.

With the notations and sign conventions illustrated in Fig. 3.3:2, and according to the assumption on torsion formula above, the torsional deflection of any elementary beam is given by

$$d\varphi = \frac{T}{GJ'}\, dx$$

where GJ' is the modified torsional rigidity of the beam. The deflection at (x,y) due to the twist of the element at ξ is

$$dw = (y-\eta)\, d\varphi = (y-\eta)\frac{T}{GJ'}\, d\xi .$$

The total deflection at (x,y) due to the torsion of the whole plate is then

$$w = \int_0^x (y - \eta) \frac{T}{GJ'} d\xi$$

The deflection due to bending moment is given by

$$\frac{\partial^2 w}{\partial x^2} = - \frac{M}{EI'},$$

where EI' is the modified rigidity of the beam and M the bending moment at the section. Since the slope and deflection of the beam at the built-in edge x = 0 are both zero, so a repeated integration gives

$$w = - \int_0^x ds \int_0^s \frac{M(\xi')}{EI'(\xi')} d\xi'. \tag{13}$$

Therefore, the deflection of the plate is given by superposing (12) and (13):

$$w(x,y) = -\int_0^x ds \int_0^s \frac{M(\xi')}{EI'(\xi')} d\xi' + \int_0^x [y - \eta(\xi)] \frac{T(\xi)}{GJ'(\xi)} d\xi. \tag{14}$$

The function w(x,y) obtained above is linear in y. The second term on the right hand side of (14) in general does not satisfy the boundary condition $\left(\frac{\partial w}{\partial x}\right)_{\xi=0} = 0$ at the built-in edge. This is due to the fact that the free torsion formula is used in computing the torsional deflection. Actually the built-in end is restrained from warping. However, from the results of Timoshenko (Ref. 49), Senft (Ref. 50), and Föpple (Ref. 51) it appears that the effect of end-restraint dies out exponentially as the distance from that end increases. The effect of end-restraint amounts approximately to multiplying the second term in (14) by a factor

$$(1 - e^{-\lambda x}),$$

where λ is a constant, and then w(x,y) given by equation (14) satisfies all the boundary conditions at the built-in edge. The correction due to the factor $e^{-\lambda x}$ is, however, quite negligible at finite distances from the built-in edge. Hence, with this understanding, the function $w(x,y)$ given by equation (14) may be taken as the first approximation $w_0(x,y)$ in the Rayleigh-Ritz procedure.

Returning to the effectiveness factor, the modified rigidities for narrow rectangular cross-sections are given by

$$EI' = E\frac{kct^3}{12} \quad , \text{ and} \qquad (15)$$
$$GJ' = G\frac{kct^3}{3} .$$

By comparison with experimental results an approximate value

$$k = \cos^2\theta \qquad (16)$$

is obtained, where θ represents the sweep angle. That this formula gives fairly good results is illustrated by Fig. (3.3:3).

Several particular cases for cantilever plates of uniform thickness are listed below:

(I) Pure bending,
$$w = -\int_0^x ds \int_0^s \frac{M(\xi')}{EI'(\xi')} d\xi' . \qquad (17)$$

If $M =$ const. then $w = -\frac{Mx^2}{2EI'}$.

(II) Pure torsion
$$w = \int_0^x [\, y - \eta(\xi)] \; \frac{T(\xi)}{GJ'} d\xi . \qquad (18)$$

If $T =$ const.,
$$w = \frac{T}{GJ'} [\; yx - \frac{\tan\theta \; x^2}{2} \;].$$

(III) Concentrated load at the tip section:

If the coordinates of the point of application of the concentrated load P is (X,Y), then

$$w(x,y) = \frac{P}{EI'} \left\{ X\frac{x^2}{2} - \frac{x^3}{6} \right\} + \frac{P}{GJ'} \left\{ \tan^2\theta \; \frac{x^3}{3} - (Y + y)\frac{\tan\theta \; x^2}{2} + Yxy \right\} \qquad (19)$$

(IV) Uniformly loaded

If q = load per unit area, and b and c the span and chord respectively, then

$$w(x,y) = \frac{qcx^4}{24EI'} + \frac{qc\tan\theta}{2GJ'}\left[-\frac{\tan\theta}{4}\left(x^4 - \frac{8}{3}bx^3 + 2b^2x^2\right) + \left(\frac{x^3}{3} - bx^2 + b^2x\right)y\right]. \tag{20}$$

In actually carrying out the computation, it is often advantageous to expand $w_0(x,y)$ and its derivatives into series in terms of functions $\{f_m(x)\}$, $\{g_n(y)\}$ as defined in connection with equation (4).

Example. <u>Cantilever Swept Plate Under Uniform Shear Load Along the Tip Section.</u>

As an application of this method, the deflection of a cantilever swept plate under uniform shear load along the tip section is carried out. The angle of sweep is denoted by θ, and the load is applied along the edge EF (Fig. 3.3:1). The length of EF is taken as 2 and that of the edge CF is L. The approximate solution $w_0(x,y)$ is constructed according to the elementary theory given above, taking $k = \cos^2\theta$, and according to equations (15) and (19):

$$w_0(x,y) = \frac{q}{(1-\nu^2)D\cos^2\theta}\left\{\left[X\frac{x^2}{2} - \frac{x^3}{6}\right] + \frac{1+\nu}{2}\left[\tan^2\theta\,\frac{x^3}{3} - (Y+y)\frac{\tan\theta\, x^2}{2} + Yxy\right]\right\}. \tag{21}$$

where 2q is the total shear load acting on the tip section. The coordinates X and Y should now be taken as $L\cos\theta$ and $L\sin\theta$ respectively.

Transformed into the oblique coordinates (ξ, η) by the relation

$$\begin{aligned} x &= \xi\cos\theta \\ y &= \eta + \xi\sin\theta \end{aligned} \tag{22}$$

equation (21) becomes

$$w_0(\xi,\eta) = \frac{q}{(1-\nu^2)\cos\theta\, D}\left\{\cos^2\theta\left[\frac{L\xi^2}{2} - \frac{\xi^3}{6}\right] + \frac{1+\nu}{2}\sin\theta\left[-\sin\theta\,\frac{\xi^3}{6} + L\sin\theta\,\frac{\xi^2}{2} - \frac{\eta\xi^2}{2} + L\xi\eta\right]\right\}. \tag{23}$$

Substituting (23) into equation (7), one obtains

$$\frac{\partial V_o}{\partial a_{mn}} = \frac{q}{(1-\nu^2)\cos^4\theta}\int_{-1}^{1}\int_{0}^{L}\left\{ f_m'' g_n (A+B\xi+C\eta) + f_m' g_n' (D+E\xi+F\eta) + f_m g_n'' (G+H\xi+K\eta)\right\} d\xi\, d\eta - q\, f_m(L)\int_{-1}^{1} g_n(\eta)\, d\eta. \tag{24}$$

where $A = L\left[1-\frac{3+\nu}{2}\sin^2\theta\right]$, $B = -A/L$,

$C = -\frac{1+\nu}{2}\sin\theta$, $D = -EL$,

$E = 2\sin\theta\left[-\frac{1+\nu}{2}+\left(1+\frac{\nu(1+\nu)}{2}\right)\cos^2\theta\right]$, $F = (1+\nu)\sin^2\theta$.

$G = L\left[(\sin^2\theta+\nu\cos^2\theta)(\cos^2\theta+\frac{1+\nu}{2}\sin^2\theta)-(1+\nu)\sin^2\theta\right]$, $H = -G/L$

$$K = -\frac{1+\nu}{2}\sin\theta\,(\sin^2\theta+\nu\cos^2\theta). \tag{25}$$

The integrals involved in (24) can be worked out without difficulty. They are listed in the following:

$(I)_{mn} = \int_{-1}^{1}\int_{0}^{L} f_m'' g_n \, d\xi\, d\eta = \sqrt{2}\ f_m'(L)$ if $n = 0$; $= 0$ if $n \neq 0$.

$(II)_{mn} = \int_{-1}^{1}\int_{0}^{L} f_m'' g_n \xi \, d\xi\, d\eta = \sqrt{2}\left[L f_m'(L) - f_m(L)\right]$ if $n = 0$; $= 0$ if $n \neq 0$.

$(III)_{mn} = \int_{-1}^{1}\int_{0}^{L} f_m'' g_n \eta \, d\xi\, d\eta = \sqrt{\frac{2}{3}}\, f_m'(L)$ if $n = 1$; $= 0$ if $n \neq 1$.

$(IV)_{mn} = \int_{-1}^{1}\int_{0}^{L} f_m' g_n' \, d\xi\, d\eta = 2\, f_m(L)\, g_n(1)$ if n is odd; $= 0$ if n is even.

$(V)_{mn} = \int_{-1}^{1}\int_{0}^{L} f_m' g_n' \xi \, d\xi\, d\eta = 2\, g_n(1)\left[L\, f_m(L) - \frac{L^2}{p_m^2} f_m'(L)\right]$ if n is odd; $= 0$ if n is even.

$(VI)_{mn} = \int_{-1}^{1}\int_{0}^{L} f_m' g_n' \eta \, d\xi\, d\eta = \sqrt{\frac{2}{3}}\, f_m(L)\, \zeta_{n1}$.

$(VII)_{mn} = \int_{-1}^{1}\int_{0}^{L} f_m g_n'' \, d\xi\, d\eta = 2\,\frac{L^2}{p_m^2} f_m'(L)\, g_n'(1)$ if n is even; $= 0$ if n is odd.

$(VIII)_{mn} = \int_{-1}^{1}\int_{0}^{L} f_m g_n'' \xi \, d\xi\, d\eta = 2\,\frac{L^2}{p_m^2} f_m(L) g_n'(1)$ if n is even; $= 0$ if n is odd

$$(IX)_{mn} = \int_{-1}^{1}\int_{0}^{L} f_m g_n'' \eta \, d\xi\, d\eta = 2\,\frac{L^2}{p_m^2} f_m'(L)\ \mu_{n1}. \tag{26}$$

All the quantities involved in the right-hand-side of these equations are tabulated in Tables 3.21:6, 7, 8 and 3.3:4. For the particular cases of $\theta = 30^\circ$, 45°, and 60°; and Poisson's ratio $\nu = 0.3$, the numerical values of the constants A, B, C, etc. and the integrals $(I)_{mn}$, $(II)_{mn}$, $(III)_{mn}$, etc. are listed in Tables 3.3:5 and 3.3:6. Equation (24)can now be written as

$$\frac{\partial V_0}{\partial a_{mn}} = \frac{q}{(1-\nu^2)\cos^4\theta} \sum_{mn} - q f_m(L) \int_{-1}^{1} g_n(\eta)\, d\eta, \tag{27}$$

where $\sum_{mn} \equiv A(I)_{mn} + B\,(II)_{mn} + C(III)_{mn} + D(IV)_{mn} + E(V)_{mn} + F(VI)_{mn} + G(VII)_{mn} + H\,(VIII)_{mn} + K(IX)_{mn}$,

and $\int_{-1}^{1} g_n(\eta)\,d\eta = \sqrt{2}$ if $n = 0$; and $= 0$ if $n \neq 0$.

Sample Computation. Swept Plate, $L = 4$, $\theta = 45°$

Let six indeterminate coefficients be taken in this case, so that the deflection function is represented by

$$\begin{aligned} w(\xi,\eta) = w_0(\xi,\eta) &+ a_{10} f_1(\xi) g_0(\eta) + a_{11} f_1(\xi) g_1(\eta) + a_{12} f_1(\xi) g_2(\eta) \\ &+ a_{20} f_2(\xi) g_0(\eta) + a_{21} f_2(\xi) g_1(\eta) + a_{22} f_2(\xi) g_2(\eta). \end{aligned} \tag{28}$$

Now the equations for the determination of the coefficients a_{11}, a_{12} etc. are, according to equation (6), and taking $\nu = 0.3$, $\theta = 45°$, $L = 4$:

$$\begin{aligned} &2a_{mn}\left[\alpha_{mn} + \kappa_{nn} + 2.7\,\beta_{mm}\lambda_{nn} + 1.3\,\gamma_{mm}\mu_{nn} - 2.82843(\psi_{mm}\zeta_{nn} + \tau_{mm}\pi_{nn})\right] \\ &+ \sum_{ij \neq mn} a_{ij}\left[2.7\,\beta_{mi}\lambda_{jn} + 1.3\,\gamma_{mi}\mu_{jn} - 2.82843\,(\psi_{mi}\zeta_{jn} + \tau_{mi}\pi_{jn})\right] = -\frac{2\cos^3\theta}{D}\frac{\partial V_0}{\partial a_{mn}}. \end{aligned} \tag{29}$$

Substituting the numerical values in, one obtains:

$$\begin{aligned} .09658a_{10} - .3032a_{11} + 1.491a_{12} + 1.292a_{21} - 4.429a_{22} &= -.29091 \\ 33.10a_{11} + .6064a_{12} - 2.584a_{21} - 2.584a_{22} &= +.566476 \\ -45.10a_{12} \qquad\qquad - 285.4a_{22} &= -2.4454 \\ 3.794a_{20} + .6873a_{11} + .6869a_{12} - 3.516a_{21} + 7.425a_{22} &= +.29264 \\ -23.12a_{11} - 1.375a_{12} + 65.74a_{21} + 7.032a_{22} &= +1.19170 \\ +193.7a_{12} \qquad\qquad +111.1a_{22} &= -.91876 \end{aligned} \tag{30}$$

where the right-hand-side of these equations are written in terms of $-\frac{2\cos^3\theta\, q}{(1-\nu^2)D}$.

The solution is

$a_{10} = 1.863$, $\quad a_{11} = -.01412$, $\quad a_{12} = .007508$,

$a_{20} = -.05989$, $\quad a_{21} = -.01685$, $\quad a_{22} = -.007245$,

expressed in terms of $\frac{q}{(1-\nu^2)D}$, i.e., $a_{10} = 1.863\frac{q}{(1-\nu^2)D}$, etc. Hence the correction to $w_0(x,y)$ is quite small. The deflection surface is given by equation (28).

IV. Stresses in Rectangular Cantilever Plates

§ 4.1 Introduction

The stresses in a cantilever plate can be obtained from the deflection function according to equations (1.12:33) through (1.12:35) of section (1.12) for rectangular plate and equations (1.13:5) through (1.13:10) of section (1.13) for swept plate.

Hence the strength computation can be carried out easily if the deflection problem is solved.

However, an exact solution for deflection of a cantilever plate can be obtained only in a very limited number of particular cases. In general, direct procedures such as those developed in the last chapter are often used to obtain practical approximate solutions within desired accuracy. These procedures are satisfactory for the determination of the deflection w as far as convergence is concerned. But in order to obtain stresses from the approximate expression of w, one has to differentiate w twice or thrice. Unfortunately, in many cases this differentiation is not permissible. In differentiating the approximating series for w, either the convergence of the resulting series is no longer guaranteed or the degree of convergence becomes so poor that the result is of no practical use. The approximating function always oscillates about the true solution. Indeed, the very proof of the possibility of an approximating procedure (Ref, 18) lies in the fact that the function $(w_{true} - w_{approx.})$ possesses a large number of zeroes somewhat uniformly distributed over the domain in question, and that the number of zeroes increases without limit as the number of terms in the approximating series $w_{approx.}$ tends to infinity. If w and $w_{approx.}$ are continuous functions, this guarantees the convergence of

$w_{approx.}$ to w. But apparently the derivatives of $w_{approx.}$ may not converge to the derivatives of w. As an example one may take Ritz' solution on the normal mode of a vibrating square plate: here the $w_{approx.}$ series behaves like the series $\sum_m \sum_n \frac{1}{m^3 n^3}$, and obviously would not permit differentiation higher than the second order. Hence *a theory of direct stress approximation without using intermediate* deflection functions is desirable. For rectangular plates this can be carried out by a procedure somewhat similar in nature to the Rayleigh-Ritz method. A sequence of solutions is first constructed which satisfy the equations of equilibrium and boundary condition (on stresses), but not necessarily the equation of compatibility. Then it is assumed that the state of stresses can be represented by a linear combination of elements of this sequence of functions. To choose from the infinitude of possible linear combinations the one that satisfies (in the limit) the equation of compatibility, the basic minimum energy principle is used from which the coefficients in the approximating series can be determined.

§ 4.2 Method of Solution

The method of solution is as follows.

Let Mx and My be the bending couples and H_{xy} the twisting couple per unit length in the plate, Qx and Qy the transverse shear-stress resultants per unit length, and p the surface load per unit area acting on the plate. (See Fig. 4.1:1) The equations of equilibrium are

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + p = 0, \tag{1}$$

$$\frac{\partial M_x}{\partial x} + \frac{\partial H_{xy}}{\partial y} - Q_x = 0,$$

$$\frac{\partial H_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y = 0.$$

These equations are independent of the way in which the stresses are distributed over the thickness of the plate. In terms of the stresses,

$$M_x = \int_{-\frac{t}{2}}^{\frac{t}{2}} z\,\sigma_x\,dz, \qquad M_y = \int_{-\frac{t}{2}}^{\frac{t}{2}} z\,\sigma_y\,dz, \qquad H_{xy} = \int_{-\frac{t}{2}}^{\frac{t}{2}} z\,\tau_{xy}\,dz,$$
$$Q_x = \int_{-\frac{t}{2}}^{\frac{t}{2}} \tau_{xz}\,dz, \qquad Q_y = \int_{-\frac{t}{2}}^{\frac{t}{2}} \tau_{yz}\,dz. \tag{2}$$

Equations (1) consist of three equations in five unknowns. Solutions can be obtained by superposing a particular integral and a complementary function. The complementary function satisfies the homogeneous equations

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = 0, \qquad \frac{\partial M_x}{\partial x} + \frac{\partial H_{xy}}{\partial y} - Q_x = 0, \qquad \frac{\partial H_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y = 0. \tag{3}$$

Let a stress function χ be defined by the equations

$$Q_x = \frac{\partial \chi}{\partial y}, \qquad Q_y = -\frac{\partial \chi}{\partial x} \tag{4}$$

Then the first equation of (3) is satisfied by an arbitrary function χ. The second and third equations of (3) now become

$$\frac{\partial M_x}{\partial x} + \frac{\partial (H_{xy} - \chi)}{\partial y} = 0,$$
$$\frac{\partial M_y}{\partial y} + \frac{\partial (H_{xy} + \chi)}{\partial x} = 0. \tag{5}$$

Define two more stress functions ϕ and ψ as follows:

$$M_x = \frac{\partial \phi}{\partial y}, \qquad M_y = -\frac{\partial \psi}{\partial x},$$
$$H_{xy} + \chi = \frac{\partial \psi}{\partial y}, \qquad H_{xy} - \chi = -\frac{\partial \phi}{\partial x}; \tag{6}$$

and hence
$$H_{xy} = \frac{1}{2} \left(\frac{\partial \psi}{\partial y} - \frac{\partial \phi}{\partial x} \right),$$
$$\chi = \frac{1}{2} \left(\frac{\partial \psi}{\partial y} + \frac{\partial \phi}{\partial x} \right).$$

Then the system of equations (5) are satisfied by arbitrary continuous (D,1) functions ϕ and ψ. The boundary conditions on the free edges are obviously

$$\chi = \text{const.} \quad \text{and} \quad H_{nt} = M_n = 0 , \tag{7}$$

where n denotes direction of outer normal to the edge in question. The system of moments and shears derived from the two arbitrary functions ϕ and ψ according to equations (6) may be called a "self-equilibrating system".

To determine the particular self-equilibrating system which in conjunction with the particular integral solves the problem, use has to be made of the basic energy principle for the stresses (Castigliano's theorem of least work), according to which "the true state of stress is distinguished from all statically correct states of stress by the condition that the complementary energy be a minimum".

For materials obeying Hooke's law, and for given surface stresses or displacements, the complementary energy is the difference of the strain-energy U and the work W_c which the surface stresses do over that portion of the surface where the displacements are prescribed.

In the particular case of a cantilever plate, that portion of the surface where the displacements are prescribed is the built-in edge, where the deflection and rotations are zero. Hence the function W_c vanishes in all cases and the complementary energy is identical with the strain-energy in the plate.

§ 4.3 Applications

For plates of uniform thickness and isotropic material, the stress-strain relations are

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right],$$

$$\gamma_{xy} = \frac{2(1+\nu)}{E}\tau_{xy}$$

etc., by cyclic substitution of indices. The strain energy stored in the plate is then

$$U = \frac{1}{2E}\iiint_{-\frac{t}{2}}^{\frac{t}{2}}\left\{\sigma_x^2 + \sigma_y^2 + \sigma_z^2 - 2\nu(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x) + 2(1+\nu)(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)\right\} dz\,dx\,dy.$$

Assuming that the normal stresses are distributed linearly over the thickness of the plate and that the contribution to the total energy due to σ_z and τ_{xz}, τ_{yz} are negligible (being of order t^2:1 compared to other terms), so that

$$\sigma_x = \frac{M_x}{t^2/6}\frac{z}{t/2}, \qquad \sigma_y = \frac{M_y}{t^2/6}\frac{z}{t/2}, \qquad \tau_{xy} = \frac{H_{xy}}{t^2/6}\frac{z}{t/2}, \tag{1}$$

and hence

$$U = \frac{1}{2}\iint\frac{1}{(1-\nu^2)D}\left\{M_x^2 + M_y^2 - 2\nu M_x M_y + 2(1+\nu)H_{xy}^2\right\} dx\,dy. \tag{2}$$

Let a particular solution of the system of equations of equilibrium for the given load distribution be denoted by M_{xo}, M_{yo}, H_{xyo}, Q_{xo}, and Q_{yo}, and let a system of self-equilibrating stresses derived from the stress-functions ϕ and ψ according to eq (4.1:6) be added to them so that

$$\begin{gathered} M_x = M_{xo} + \frac{\partial\phi}{\partial y}, \qquad M_y = M_{yo} - \frac{\partial\psi}{\partial x}, \\ H_{xy} = H_{xyo} + \frac{1}{2}\left(\frac{\partial\psi}{\partial y} - \frac{\partial\phi}{\partial x}\right). \end{gathered} \tag{3}$$

Then the strain energy U in the plate can be written as

$$U = U_0 + U_1 + U_2 \tag{4}$$

where

$$U_0 = \frac{1}{2(1-\nu^2)} \iint \frac{1}{D} \left\{ M_{x_0}^2 + M_{y_0}^2 + 2\nu M_{x_0} M_{y_0} + 2(1+\nu) H_{xy_0}^2 \right\} dx\,dy, \tag{4a}$$

$$U_1 = \frac{1}{2(1-\nu^2)} \iint \frac{1}{D} \left\{ 2 M_{x_0} \frac{\partial\phi}{\partial y} + 2 M_{y_0} \frac{\partial\psi}{\partial x} + 2\nu \left(M_{x_0} \frac{\partial\psi}{\partial x} + M_{y_0} \frac{\partial\phi}{\partial y} \right) + 2(1+\nu) H_{xy_0} \left(\frac{\partial\psi}{\partial y} - \frac{\partial\phi}{\partial x} \right) \right\} dx\,dy, \tag{4b}$$

$$U_2 = \frac{1}{2(1-\nu^2)} \iint \frac{1}{D} \left\{ \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\psi}{\partial x}\right)^2 + 2\nu \frac{\partial\phi}{\partial y} \frac{\partial\psi}{\partial x} + \frac{1+\nu}{2} \left(\frac{\partial\psi}{\partial y} - \frac{\partial\phi}{\partial x} \right)^2 \right\} dx\,dy. \tag{4c}$$

According to the eqs. (4.2:7), the boundary conditions to be imposed on ϕ and ψ are

$$x = L: \quad \frac{\partial\phi}{\partial y} = 0, \quad \frac{\partial\psi}{\partial y} = 0, \quad \frac{\partial\phi}{\partial x} = 0. \tag{5a}$$

$$y = \pm 1: \quad \frac{\partial\psi}{\partial x} = 0, \quad \frac{\partial\psi}{\partial y} = 0, \quad \frac{\partial\phi}{\partial x} = 0.$$

This can be replaced by

$$x = L: \quad \phi = \psi = 0, \quad \frac{\partial\phi}{\partial x} = 0, \quad \text{and} \tag{5}$$

$$y = \pm 1: \quad \phi = \psi = 0, \quad \frac{\partial\psi}{\partial y} = 0.$$

ϕ and ψ are otherwise arbitrary.

To construct the functions ϕ and ψ, use is made of the vibration modes of thin elastic rods. Let $u_n(y)$ be the amplitude of the n th mode of a free vibrating thin elastic bar of length 2 clamped at both ends; and let $v_n(x)$ be the corresponding function for a bar of length L which is clamped at the end $x = L$ and free at the end $x = 0$. Let $w_n(x)$ be that for a bar of length L and simply supported at its ends $x = 0$ and $x = L$, and $r_n(y)$ be that for a bar of length 2 and simply supported at both ends $y = \pm 1$. Then the sets of functions $\{u_n(x)\}$ etc., are orthogonal sequences.

Let them be normalized so that relations like

$$\int u_m(x)\, u_n(x)\, dx = \delta_n^m = \begin{cases} 0 & \text{when } m \neq n \\ 1 & \text{when } m = n. \end{cases} \tag{6}$$

hold, δ_n^m being the Kronecker delta. The derived sequences $\{u_n'(x)\}$ and $\{v_n'(x)\}$, where primes indicate differentiation with respect to x, are not orthogonal, since although $u_n'(x)$, $v_n'(x)$ satisfy the same differential equation as $u_n(x)$ and $v_n(x)$ respectively, their corresponding boundary conditions are not admissible. On the other hand, the sequences $\{r_n'(x)\}$ and $\{w_n'(x)\}$ are orthogonal, since the derived boundary conditions are admissible. Let the following notations be used:

$$\begin{aligned}
\omega_{mn} = \omega_{nm} &= \int_{-1}^{1} u_m'(y)\, u_n'(y)\, dy, & \beta_{mn} = \beta_{nm} &= \int_0^L v_m'(x)\, v_n'(x)\, dx, \\
\theta_{mm}\delta_n^m &= \int_{-1}^{1} r_m'(y)\, r_n'(y)\, dy, & \epsilon_{mm}\delta_n^m &= \int_0^L w_m'(x)\, w_n'(x)\, dx, \\
\rho_{mn} &= \int_{-1}^{1} u_m(y)\, r_n'(y)\, dy, & \varphi_{mn} &= \int_0^L v_m(x)\, w_n'(x)\, dx, \\
\eta_{mn} &= \int_{-1}^{1} u_m'(y)\, r_n(y)\, dy, & \xi_{mn} &= \int_0^L v_m'(x)\, w_n(x)\, dx.
\end{aligned} \tag{7}$$

Now a set of stress functions ϕ and ψ, which satisfy the boundary conditions (5) can be written as

$$\begin{aligned}
\phi &= \sum_{m,n} a_{mn}\, v_m(x)\, r_n(y), \\
\psi &= \sum_{m,n} b_{mn}\, w_m(x)\, u_n(y).
\end{aligned} \tag{8}$$

Substituting (8) into (4), one obtains an expression for U as a quadratic function of the constants $\{a_{mn}\}$ and $\{b_{mn}\}$. The constants $\{a_{mn}\}$, $\{b_{mn}\}$ must be so chosen as to make U a minimum.

By ordinary maximum minimum process this requires that

$$\frac{\partial U_1}{\partial a_{mn}} + \frac{\partial U_2}{\partial a_{mn}} = 0, \qquad \frac{\partial U_1}{\partial b_{mn}} + \frac{\partial U_2}{\partial b_{mn}} = 0, \qquad \text{for any } m, n, \tag{9}$$

U_0 being independent of $\{a_{mn}\}$ and $\{b_{mn}\}$. Now

$$\frac{\partial U_1}{\partial a_{mn}} = \frac{1}{(1-\nu^2)D} \iint \left\{ (M_{x_0} + \nu M_{y_0}) v_m(x)\, r_n'(y) - (1+\nu) H_{xy_0} v_m'(x)\, r_n(y) \right\} dx\, dy. \tag{10)a}$$

$$\frac{\partial U_1}{\partial b_{mn}} = \frac{1}{(1-\nu^2)D} \iint \left\{ (M_{y_0} + \nu M_{x_0}) w_m'(x)\, u_n(y) + (1+\nu) H_{xy_0} w_m(x)\, u_n'(y) \right\} dx\, dy. \tag{10)b}$$

$$\frac{\partial U_2}{\partial a_{mn}} = \left[2\,\theta_{nn} + (1+\nu)\beta_{mm} \right] a_{mn} + \frac{1+\nu}{2} \sum_{j \neq m} \beta_{mj}\, a_{jn}$$
$$+ \sum_{j,k} \left[2\nu\, \varphi_{mj}\, \rho_{kn} - (1+\nu)\, \xi_{mj}\, \eta_{kn} \right] b_{jk}. \tag{10)c}$$

$$\frac{\partial U_2}{\partial b_{mn}} = \left[2\,\epsilon_{mm} + (1+\nu)\,\omega_{nn} \right] b_{mn} + \frac{1+\nu}{2} \sum_{k \neq n} \omega_{nk}\, b_{mk}$$
$$+ \sum_{j,k} \left[2\nu\, \varphi_{jm}\, \rho_{nk} - (1+\nu)\, \xi_{jm}\, \eta_{nk} \right] a_{jk}. \tag{10)d}$$

It should be noted that the right hand side of (10)a and (10)b are known quantities and that of (10)c and (10)d are linear in $\{a_{mn}\}$ and $\{b_{mn}\}$. Hence equations (9) lead to a system of linear equations from which $\{a_{mn}\}$, $\{b_{mn}\}$ can be determined.

The linear forms (10)c and (10)d are functions of plate geometry and independent of the external loading conditions, hence once tabulated can be used for all loading conditions.

Convergence can be established from the fact that the sequences

$\{u_n(x)\}$ etc., are complete systems. Physically, good convergence can be expected since good estimations on M_{xo}, M_{yo}, etc., from the point of view of statics can usually be made, and hence those due to ϕ and ψ are only small correction terms.

It remains to write out the explicit formulae for $u_n(y)$, $v_n(x)$, α_{mn}, β_{mn} etc. The several functions $u_n(x)$, $v_n(x)$, $r_n(x)$, $w_n(x)$ all satisfy the differential equation

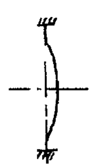
$$\frac{d^4 u}{dx^4} = p^4 u, \tag{11}$$

and the boundary conditions $u = \frac{\partial u}{\partial x} = 0$ for clamped ends, $u = \frac{\partial^2 u}{\partial x^2} = 0$ for supported ends and $\frac{\partial^3 u}{\partial x^3} = \frac{\partial^2 u}{\partial x^2} = 0$ for free ends. A fourth set of admissible boundary conditions are $\frac{\partial u}{\partial x} = \frac{\partial^3 u}{\partial x^3} = 0$, which are the conditions satisfied by $\{r_n(x)\}$ and $\{w_n(x)\}$.

The functions u, v, w etc., are found to be as follows:

(I) $u_n(y)$, normal mode of a clamped-clamped bar of length 2, with origin at the middle point. This is similar to the function $g_n(y)$ of section 3.21.

If n is even:



$$u_n(y) = \frac{-\cosh p_n \cos p_n y + \cos p_n \cosh p_n y}{\sqrt{\cosh^2 p_n + \cos^2 p_n}},$$

$$\text{where} \quad \tan p_n + \tanh p_n = 0. \tag{12}$$

If n is odd:

$$u_n(y) = \frac{-\sinh p_n \sin p_n y + \sin p_n \sinh p_n y}{\sqrt{\sinh^2 p_n - \sin^2 p_n}},$$

$$\text{where} \quad \tan p_n - \tanh p_n = 0.$$

The roots p_n are approximately equal to $(n - \frac{1}{2})\frac{\pi}{2}$. Their exact values are given in Section 3.21.

(II). $v_n(x)$: Normal mode of a free-clamped bar of length L, free at $x=0$, clamped at $x=L$.

$$v_n(x) = \frac{1}{\sqrt{L}}\left\{\cos\frac{p_n x}{L} + \cosh\frac{p_n x}{L} - \frac{\cos p_n + \cosh p_n}{\sin p_n + \sinh p_n}\left(\sin\frac{p_n x}{L} + \sinh\frac{p_n x}{L}\right)\right. , \tag{13}$$

Where p_n is the root of $\cos p_n \cosh p_n + 1 = 0$.

This is equal to the function $f_n(x)$ of Section 3.21 with the origin changed to $x = L$. The roots p_n are given in Section 3.21.

(III). $w_n(x)$: Normal mode of a supported-supported bar of length L, with ends at $x = 0$ and $x = L$.

$$w_n(x) = \sqrt{\frac{2}{L}}\,\sin\frac{n\pi x}{L} \qquad \text{n, any integer.} \tag{14}$$

(IV). $r_n(y)$: Normal mode of a supported-supported bar of length 2 with ends at $y = \pm 1$.

$$r_n(y) = (-1)^{\frac{n}{2}} \sin\frac{n\pi y}{2} \qquad \text{if n is even,}$$

$$r_n(y) = (-1)^{\frac{n+1}{2}} \cos\frac{n\pi y}{2} \qquad \text{if n is odd.} \tag{15}$$

The integrals ω_{mn}, β_{mn} etc., are then found to be as follows:

$$\text{(I).}\quad \omega_{ml} = \omega_{lm} = \int_{-1}^{1} u_m'(y)\, u_l'(y)\, dy . \tag{16}$$

When $m \neq l$, m and l both even:

$$\omega_{ml} = \frac{p_m p_l}{\sqrt{\left(1+\frac{\cos^2 p_m}{\cosh^2 p_m}\right)\left(1+\frac{\cos^2 p_l}{\cosh^2 p_l}\right)}}\left\{\frac{\sin(p_m - p_l)}{p_m - p_l} - \frac{\sin(p_m + p_l)}{p_m + p_l}\right.$$

$$+ \cos p_m \cos p_\ell \left[\frac{\tanh p_m + \tanh p_\ell}{p_m + p_\ell} - \frac{\tanh p_m - \tanh p_\ell}{p_m - p_\ell} \right]$$

$$+ \frac{2}{p_m^2 + p_\ell^2} \left[p_m (\sin p_\ell \cos p_m - \cos p_\ell \cos p_m \tanh p_\ell) + p(\sin p_m \cos p_\ell - \cos p_\ell \cos p_m \tanh p_m) \right] \Bigg\} \qquad (16)a$$

When $m \neq \ell$, and m and ℓ are both odd:

$$\omega_{m\ell} = \frac{p_m p_\ell}{\sqrt{\left(1 - \frac{\sin^2 p_m}{\sinh^2 p_m}\right)\left(1 - \frac{\sin^2 p_\ell}{\sinh^2 p_\ell}\right)}} \Bigg\{ \frac{\sin(p_m - p_\ell)}{p_m - p_\ell} + \frac{\sin(p_m + p_\ell)}{p_m + p_\ell}$$

$$+ \sin p_m \sin p_\ell \left[\frac{\coth p_m + \coth p_\ell}{p_m + p_\ell} - \frac{\coth p_m - \coth p_\ell}{p_m - p_\ell} \right]$$

$$- \frac{2 \sin p_\ell}{p_m^2 + p_\ell^2} \left[p_m \sin p_m \coth p_\ell + p_\ell \cos p_m \right] - \frac{2 \sin p_m}{p_m^2 + p_\ell^2} \left[p_\ell \sin p_\ell \coth p_m + p_m \cos p_\ell \right] \Bigg\} \qquad (16)b$$

When $m = \ell$, and m is an even number:

$$\omega_{mm} = \frac{p_m^2}{1 + \frac{\cos^2 p_m}{\cosh^2 p_m}} \Bigg\{ 1 - \frac{\sin 2p_m}{2 p_m} + \cos^2 p_m \left[\frac{\tanh p_m}{p_m} - \frac{1}{\cosh^2 p_m} \right]$$

$$+ \frac{2 \cos p_m}{p_m} \left[- \cos p_m \tanh p_m + \sin p_m \right] \Bigg\}. \qquad (16)c$$

When $m = \ell$, and m is an odd number:

$$\omega_{mm} = \frac{p_m^2}{1 - \frac{\sin^2 p_m}{\sinh^2 p_m}} \Bigg\{ 1 + \frac{\sin 2p_m}{2 p_m} + \sin^2 p_m \left[\frac{\coth p_m}{p_m} - \frac{1}{\sinh^2 p_m} \right]$$

$$- \frac{2 \sin p_m}{p_m} \left[\sin p_m \coth p_m + \cos p_m \right] \Bigg\}. \qquad (16)d$$

When one of m and n is even and the other is odd, $\omega_{mn} = 0$. (16)e

The p's in the above formulas are the roots corresponding to $u_n(y)$.

(II). $\beta_{mn} = \beta_{nm}$ is given in Section 3.21.

(III). $\theta_{mm} = \left(\frac{m\pi}{2}\right)^2$ (17)

(IV). $\epsilon_{mm} = \left(\frac{m\pi}{L}\right)^2.$ (18)

(V). $$\rho_{mn} = \frac{4n\pi p_m}{\sqrt{1+\frac{\cos^2 p_m}{\cosh^2 p_m}}}\left[-\frac{\sin p_m}{4p_m^2-n^2\pi^2}-\frac{\cos p_m \tanh p_m}{4p_m^2+n^2\pi^2}\right],$$ if m, n are both even. (19)a

$$\rho_{mn} = \frac{4n\pi p_m}{\sqrt{1-\frac{\sin^2 p_m}{\sinh^2 p_m}}}\left[\frac{\cos p_m}{4p_m^2-n^2\pi^2}+\frac{\sin p_m \coth p_m}{4p_m^2+n^2\pi^2}\right],$$ if m, n are both odd. (19)b

$\rho_{mn}=0$, if one of the m, n is even and the other is odd. (19)c

The p_m in these formulas are the roots belonging to $u_n(y)$.

(VI). $$\varphi_{mn} = \frac{(-1)^n}{L}\sqrt{2}\,n\pi p_m\left[\frac{\sin p_m+\sigma_m\cos p_m}{p_m^2-n^2\pi^2}+\frac{(\tanh p_m-\sigma_m)\cosh p_m}{p_m^2+n^2\pi^2}+(-1)^n\frac{2p_m^2\sigma_m}{n^4\pi^4-p_m^4}\right], \tag{20}$$

where $\sigma_m = \dfrac{\cos p_m + \cosh p_m}{\sin p_m + \sinh p_m}$, and p_m are the roots belonging to $v_m(x)$. For $m > 2$, $\sigma_m = 1 - 2e^{-p_m}$ to five decimal places, and

$$\varphi_{mn} = \frac{(-1)^n}{L}\sqrt{2}\,n\pi p_m\left[\frac{\sin p_m+\sigma_m\cos p_m}{p_m^2-n^2\pi^2}+\frac{1}{p_m^2+n^2\pi^2}+(-1)^n\frac{2\sigma_m p_m^2}{n^4\pi^4-p_m^4}\right].$$

(VII). η_{mn} is the same as ρ_{mn} except that the term $\sin p_m$ should be changed into $-\sin p_m$.

(VIII). ξ_{mn} is the same as $-\varphi_{mn}$ except that the last term in the bracket should be changed into $(-1)^n\dfrac{2\sigma_m(n\pi)^2}{(n\pi)^4-p_m^4}$.

The numerical values of these functions are given in Tables 4.3:1 through 4.3:7. No practical example is given since the application of this method is quite similar to the Rayleigh-Ritz method of sections 3.21 and 3.3. **

** At the time of writing this thesis, an elaborate experimental program is going on at the GALCIT structural laboratory. When such experimental data become available, then it will be interesting to compare the numerical solutions of particular cases.

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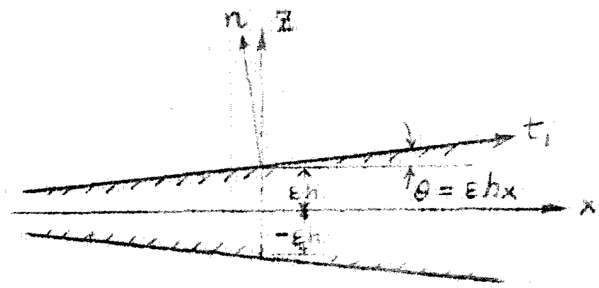
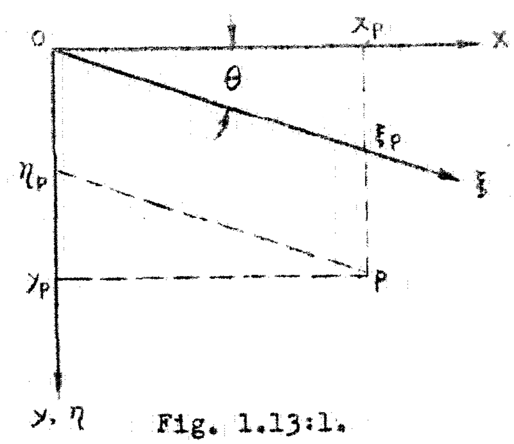


Fig. 1.12:1.



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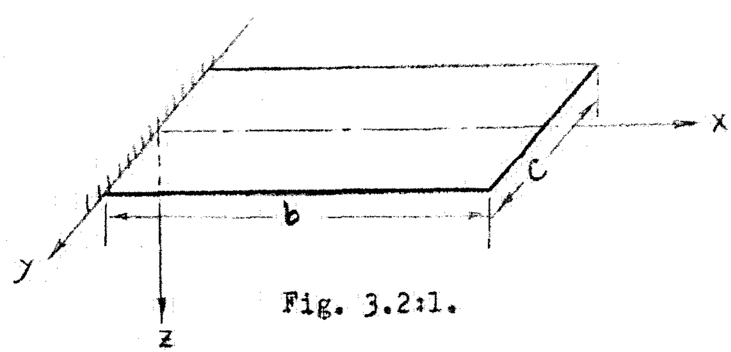

Fig. 3.2:1.

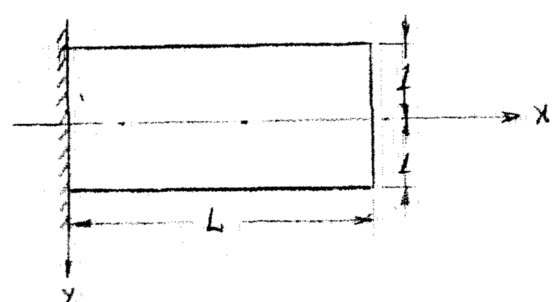

Fig. 3.2:2.

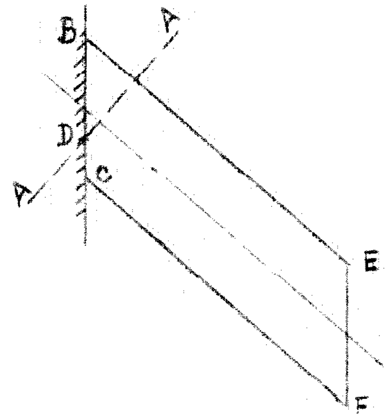

Fig. 3.3:1.

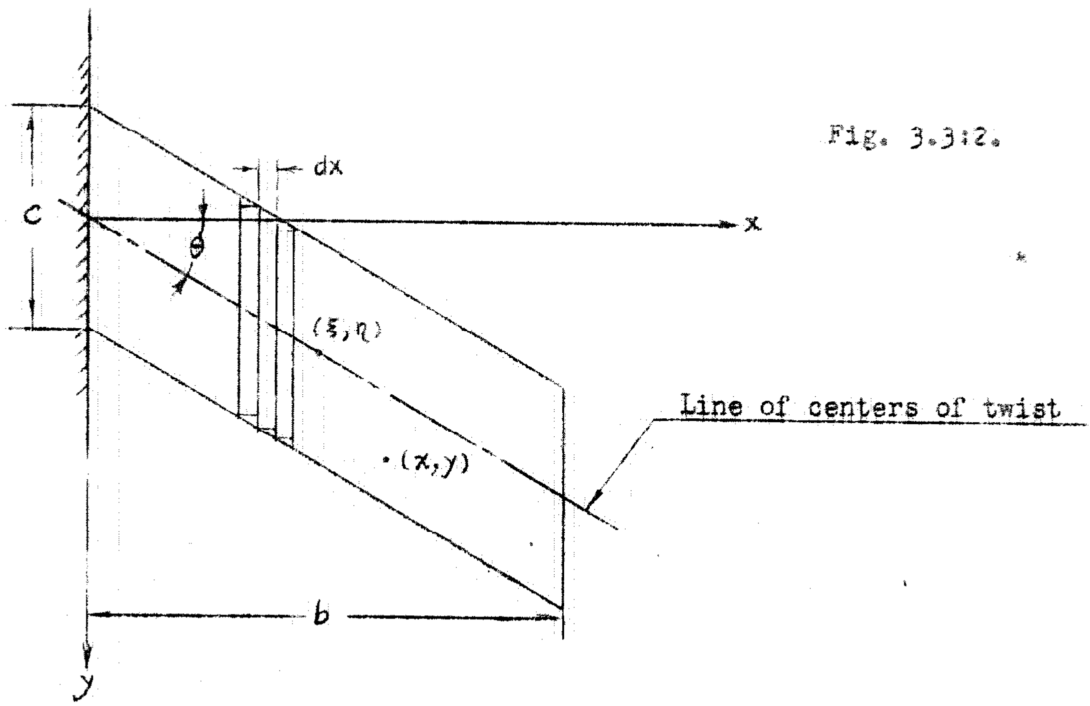

Fig. 3.3:2.

(x,y,z) -- right-handed coordinate system.

(x,y) --- coordinates of field point.

(ξ,η) --- running coordinates of the points along the line of the centers of twist.

The positive sense of torsional couple coincides with the x-axis, and the twisting angle φ is measured in the same sense as the torsional couple.

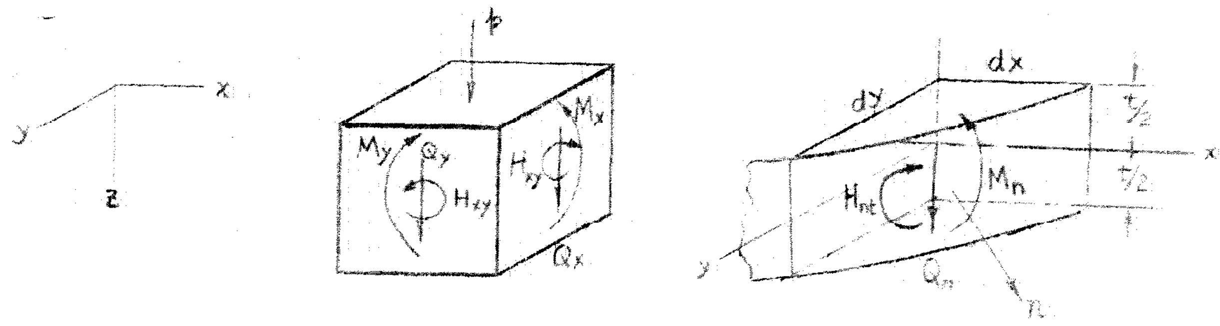

Fig. 4.1:1.

FIG. 3.3:3

THE DEFLECTION OF A 40 DEGREE SWEPT PLATE

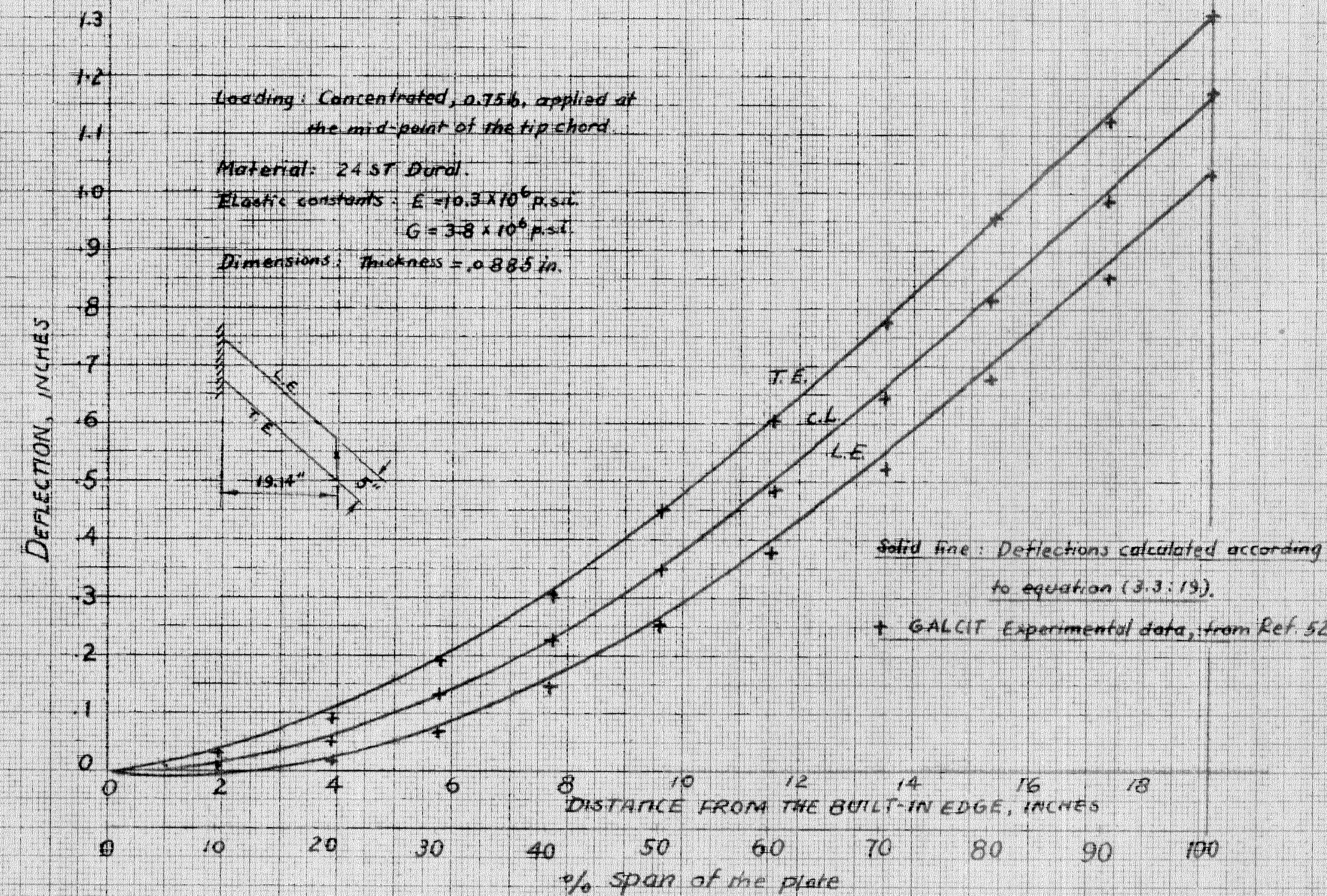

Table 3.21 : 1 $\alpha_{mn} \times L^4$

m \ n	1	2	3	4
1	12.36236	0	0	0
2	0	485.5187	0	0
3	0	0	3806.545	0
4	0	0	0	14617.274

Table 3.21 : 2 $\beta_{mn} \times L^2$

m \ n	1	2	3	4
1	9.522	7.392	3.828	7.126
2	7.392	38.78	17.39	24.56
3	3.828	17.39	76.39	25.29
4	7.126	24.56	25.29	143.0

Table 3.21 : 3 $\gamma_{mn} \times L^2$

m \ n	1	2	3	4
1	-3.949	11.73	27.56	36.17
2	-1.819	-19.66	14.00	18.74
3	+1.745	+1.727	-44.99	18.69
4	-1.553	-5.439	+6.105	99.02

Table 3.21 : 4 χ_{mn}

m \ n	0	1	2	3	4
0	0	0	0	0	0
1	0	0	0	0	0
2	0	0	31.28522	0	0
3	0	0	0	237.7210	0
4	0	0	0	0	913.6019

Table 3.21 : 5 λ_{mn}

m \ n	0	1	2	3	4
0	0	0	0	0	0
1	0	3.00000	0	3.46410	0
2	0	0	12.3701	0	-8.93007
3	0	3.46410	0	27.2023	0
4	0	0	-8.93007	0	46.7174

Table 3.21 : 6 μ_{mn}

m \ n	0	1	2	3	4
0	0	0	0	0	0
1	0	0	0	0	0
2	-4.64724	0	-10.0465	0	6.60643
3	0	-10.1486	0	-11.4837	0
4	+10.9953	0	4.49092	0	-41.2197

Table 3.21 : 7 $f_m(x)$

m	1	2	3	4
$\sqrt{L}\, f_m(L)$	1.980194	-1.992014	1.998525	-2.000033
$L^{3/2} f_m'(L)$	2.692451	-9.600801	15.691245	-21.992182
σ_m	.742976	1.018474	.999224	1.000032
p_m	1.875104	4.694091	7.854757	10.995541

Table 3.21 : 8 $g_m(y)$

m	0	1	2	3	4
$g_m(1)$	.707107	1.22474	-1.41420	-1.41421	1.41422
$g_m'(1)$	0	1.22474	-3.28610	-5.55737	7.77483
p_m	0	0	2.365020		5.497804

Table 3.3 : 1 $\Psi_{mn} \cdot L^3$

	1	2	3	4
1	3.9604	-16.874	11.505	1.1360
2	-8.9761	45.935	67.291	143.90
3	30.743	-217.94	123.37	-1132.1
4	-58.077	67.244	786.96	241.80

Table 3.3 : 2 $\tau_{mn} \cdot L$

	1	2	3	4
1	2.0557	.77618	-.20337	.33610
2	-4.7208	1.9933	2.4215	-.30113
3	4.1608	-6.4026	2.0000	3.4945
4	-4.2966	4.2852	-7.4916	2.0000

Table 3.3 : 3 π_{mn}

	0	1	2	3	4
0	0	0	0	0	0
1	0	-8.04927	0	-16.4493	0
2	0	0	52.9734	0	32.4924
3	0	19.0444	0	-118.908	0

Table 3.3 : 4 ζ_{mn}

m \ n	0	1	2	3	4
0	0	0	0	0	0
1	1.73206	0	0	0	0
2	0	-3.46407	0	-.624436	0
3	-2.00000	0	4.62440	0	1.464124
4	0	3.46411	0	-5.46413	0

Table 3.3 : 5

	30^0	45^0	60^0
A	.58778L	.17555L	-.23666
B	-.58778	-.17555	.23666
C	-.32500	-.45962	-.56290
D	-.24925L	.071418L	.51918L
E	.24925	-.071418	-.51918
F	.32500	.65000	.97500
G	.10844L	-.11350	-.36656
H	-.10844	.11350	.36656
K	-.154375	-.29875	-.4644

Table 3.3 : 6

Subscripts m n	(1,0)	(1,1)	(1,2)	(2,0)	(2,1)	(2,2)
$(I)_{mn}$	$(3.80770)\frac{1}{L\sqrt{L}}$	0	0	$(-13.57758)\frac{1}{L\sqrt{L}}$	0	0
$(II)_{mn}$	$(1.00728)\frac{1}{\sqrt{L}}$	0	0	$(-10.76045)\frac{1}{\sqrt{L}}$	0	0
$(III)_{mn}$	0	$(2.19838)\frac{1}{L\sqrt{L}}$	0	0	$(-7.83902)\frac{1}{L\sqrt{L}}$	0
$(IV)_{mn}$	0	$(4.850445)\frac{1}{\sqrt{L}}$	0	0	$(-4.87939)\frac{1}{\sqrt{L}}$	0
$(V)_{mn}$	0	$(2.97471)\sqrt{L}$	0	0	$(-3.812118)\sqrt{L}$	0
$(VI)_{mn}$	0	0	$(-5.600784)\frac{1}{\sqrt{L}}$	0	0	$(5.63421)\frac{1}{\sqrt{L}}$
$(VII)_{mn}$	0	0	$(-5.03278)\sqrt{L}$	0	0	$(2.86362)\sqrt{L}$
$(VIII)_{mn}$	0	0	$(-3.70141)L\sqrt{L}$	0	0	$(.594156)L\sqrt{L}$
$(IX)_{mn}$	0	0	0	0	0	0

Table 4.2:1 ω_{mn}

m \ n	2	3	4	5
2	4.14044	0	2.48531	0
3	0	11.48810	0	22.51042
4	2.48531	0	24.72654	0
5	0	22.51042	0	42.89623

Table 4.2:2 θ_{mn}

m \ n	2	3	4	5
2	9.86960	0	0	0
3	0	22.2066	0	0
4	0	0	39.4784	0
5	0	0	0	61.6850

Table 4.2:3 ρ_{mn}

m \ n	2	3	4	5
2	3.0818	0	1.0674	0
3	0	3.1711	0	.38478
4	.59069	0	-5.9807	0
5	0	2.3495	0	-5.9958

Table 4.2:4 η_{mn}

m \ n	2	3	4	5
2	-1.7465	0	-.15131	0
3	0	4.5445	0	1.5018
4	-1.8091	0	4.5791	0
5	0	1.0443	0	-7.4006

Table 4.2:5 $\epsilon_{mn} \times L^2$

m \ n	1	2	3	4
1	.986959	0	0	0
2	0	3.94784	0	0
3	0	0	8.88263	0
4	0	0	0	15.7912

Table 4.2:6 $\varphi_{mn} \times L$

m \ n	1	2	3	4
1	1.03219	-.056033	.032755	-.013299
2	.08905	3.48312	.004767	.16434
3	1.2812	.002338	6.3241	.01117
4	0	-3.6221	0	9.1579

Table 4.2:7 ξ_{mn}

m \ n	1	2	3	4
1	-1.9560	-.56705	-.44184	-.30173
2	-1.3372	-4.8638	-1.1543	-1.2465
3	1.3462	-1.3773	7.7149	-1.2723
4	-.74714	2.4021	-1.3977	-10.559