THE MAXIMUM STRESS
IN A CYLINDRICAL BAR OF DOUBLE WEDGE
CROSS SECTION UNDER TORSION

Thesis

by

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ABSTRACT

A comparison of an exact calculation of maximum stress in a cylindrical rod under torsion is compared to the usual engineering approximation which takes the maximum stress as that existing on the boundary of the largest circle inscribed in the given cross section. A numerical example for a symmetrical thin double wedge cross section is calculated and essential agreement in stress magnitude found, although there was an eight degree difference in the location of the point of maximum stress.

I. INTRODUCTION

For practical applications, the maximum stress in a solid cylinder of arbitrary cross section under torsion is taken as the maximum shear stress of the largest inscribed circular cross section, which empirical rule has through experience been found sufficiently accurate for engineering purposes. It is the purpose of this paper to investigate this approximation for a solid double wedge, with particular attention to thin symmetrical double wedges that occur in high speed wing cross sections, and to determine the degree of accuracy.

The method of attack which is followed will be to investigate, under St. Venant assumptions, the torsion in a double wedge region R with boundary C by means of a conformal transformation of R onto the interior of a unit circle. The problem then becomes that of Dirichlet, or a boundary value problem of the first kind; specifically, one of solving $\Delta\Psi = 0$ with the boundary condition that $\Psi = \frac{1}{2}\omega(\zeta)\,\bar{\omega}(\bar{\zeta})$ on C if $\omega(\zeta)$ is the mapping function of the region R onto the unit circle. The classical solution, namely a form of Poisson's integral, will yield Ψ from which the desired stresses can be obtained with little difficulty.

II. MATHEMATICAL BACKGROUND

A brief statement of the St. Venant torsion problem is in order, but any detailed account would be superfluous as the literature upon the subject is both extensive and comprehensive.[1] A solid cylindrical bar of homogeneous isotropic material undergoing torsion with no end restraint or body forces may be analyzed by assuming successive cross sections deform or warp identically irrespective of their position along the cylindrical axis. Accordingly the displacements u, v, and w in the x, y, and z directions respectively, where x and y are the coordinates of a point in the cross section and z the direction of the cylindrical axis, become

$$u = -\alpha z y \qquad v = \alpha z x \qquad w = \alpha \Phi(x,y)$$

in which α is the angle of twist per unit length and $\Phi(x,y)$ is as yet an undetermined function which must satisfy the equilibrium, compatibility, and boundary conditions.

An elementary calculation of stresses from the displacements yield

$$\tau_{yz} = \mu\alpha\left(\frac{\partial\Phi}{\partial y} + x\right) \qquad \text{where} \quad \mu = \frac{E}{2(1+\nu)}$$

$$\tau_{xz} = \mu\alpha\left(\frac{\partial\Phi}{\partial x} - y\right) \qquad E = \text{mod. of elasticity}$$

$$\tau_{xy} = \sigma_x = \sigma_y = \sigma_z = 0 \qquad \nu = \text{Poisson's ratio}$$

which one recognizes as the St. Venant conditions. Substitution of these values of the stress into the equations of equilibrium shows that $\Delta\Phi = 0$ is the governing equation for $\Phi(x,y)$, which is usually termed the torsion

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1) See Mathematical Theory of Elasticity, I.S. Sokolnikoff. McGraw Hill Book Co. 1946, Chap. IV, for one readable account.

function. Finally the boundary conditions require that the normal stress on the boundary of the cylinder be zero. In terms of the torsion function this requires

$$\frac{\partial \Phi}{\partial \bar{\nu}} = y\,\bar{\xi}\cdot\bar{\nu} - x\,\bar{\eta}\cdot\bar{\nu}$$

where $\bar{\xi}$, $\bar{\eta}$, and $\bar{\nu}$ are the unit vectors in the x, y, and surface normal directions respectively.

The above derivation presents essentially the problem of Neumann, or the second boundary value problem of potential theory, requiring for its solution an analytic function $\Phi(x,y)$ whose normal derivative on the boundary of the region containing $\Phi(x,y)$ is prescribed. It is convenient, however, to translate the problem of Neumann into one of Dirichlet using the Cauchy relations and the implied existence of a conjugate function, say $\Psi(x,y)$. The problem thus becomes one of solving $\Delta\Psi = 0$ for which the conditions on the boundary of the region can be shown to be $\Psi_b = \frac{1}{2}z\bar{z}$ + constant in which z is the complex coordinate of the cross section. It may also be noted that the constant, actually associated with a rigid body displacement, is non-essential as only derivatives of $\Psi(x,y)$ are involved in the stresses.

The method used in this paper is to determine $\Psi(x,y)$ by the powerful tool of conformal mapping. This artifice is useful in as much as, while an analytic function is invariant under a conformal transformation, the boundary conditions are simplified providing the transformation is properly chosen. If for example, the mapping of an arbitrary cylindrical cross section, R, onto a unit circle is known, say $z = \omega(\zeta)$, ζ being the complex coordinate in the circle plane, we may form

$$\Phi(z) + i\Psi(z) = F(z) = F[\omega(\zeta)] = f(\zeta)$$

where now $f(\zeta)$ is analytic in the region R, with boundary say C. Thus remembering the boundary conditions on $\Psi(x,y) = \frac{1}{2}z\bar{z}$, dropping the non-essential constant, we have $\Psi = \frac{1}{2}\omega(\zeta)\bar{\omega}(\bar{\zeta})$ on the unit circle in addition to $\Delta\Psi(x,y) = 0$. The solution is given by a special form of Poisson's integral, namely

$$f(\zeta) = \Phi + i\Psi = \frac{1}{2\pi}\int_{\gamma}\frac{\omega(\sigma)\,\bar{\omega}(\bar{\sigma})}{\sigma - \zeta}\,d\sigma + \text{constant}$$

where γ is the boundary of the unit circle.

Then providing the integral can be evaluated, $\Phi(x,y)$ and $\Psi(x,y)$ are given by the real and imaginary parts of the integration. Finally the non-zero stresses may be evaluated from the strains using $\Phi(x,y)$ or, as has been shown,[1] directly from the complex torsion function $f(\zeta)$ as

$$\tau_{zx} - i\tau_{zy} = \mu\alpha\left[\frac{df/d\zeta}{d\omega/d\zeta} - i\bar{\omega}(\zeta)\right]$$

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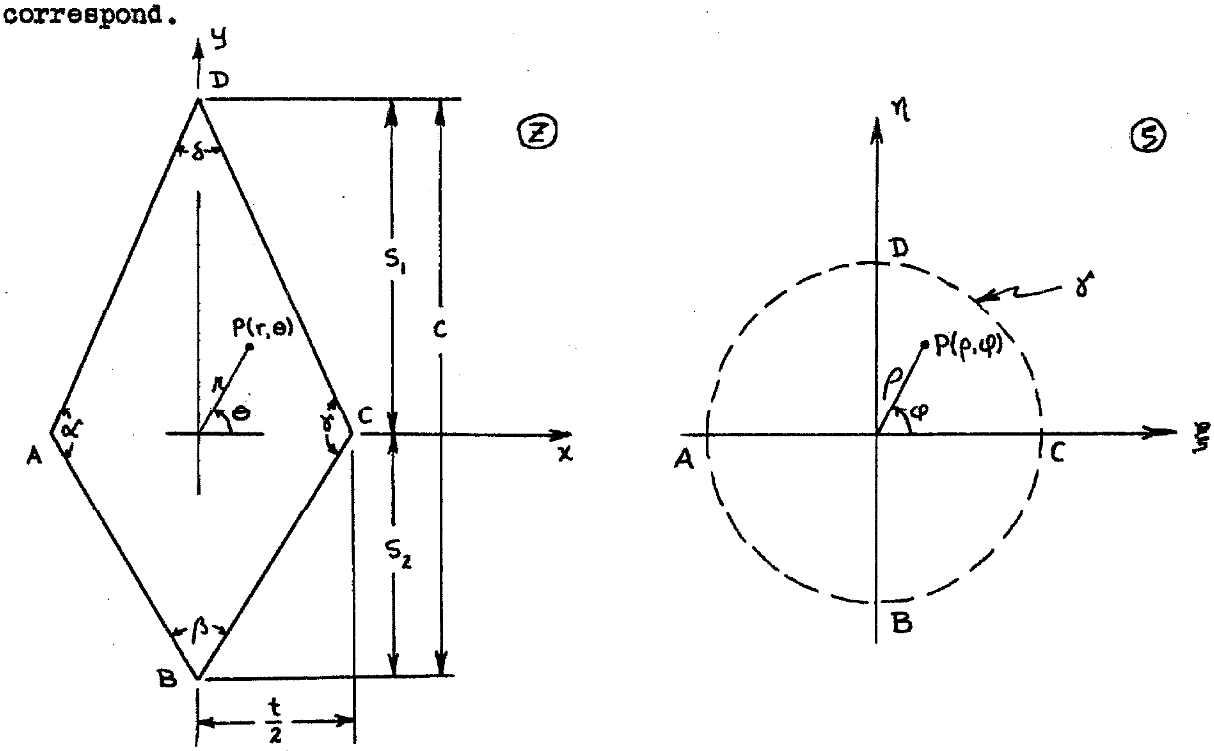
1) Sokolnikoff. op. cit.

III. THE MAPPING FUNCTION

We proceed next to develop the mapping function for a double wedge cross section, and in particular we wish to transform the region inside the double wedge polygon into the interior of the unit circle. This is conveniently done by the well known Schwarz-Christoffel transformation where the geometric relations and points of correspondence are as indicated below. We observe that due to the symmetry about the y axis, four points (instead of the usual three) may be specified, and that in the special case of a diamond shape, or symmetrical double wedge, $\beta = \delta$, the origins correspond.



$$z = x + iy = re^{i\theta} \qquad \zeta = \xi + i\eta = \rho e^{i\varphi}$$

$$\alpha + \beta + \gamma + \delta = 2\pi$$

For $\alpha = \gamma$; $\frac{\delta}{\pi} = 2 - 2\left(\frac{\alpha}{\pi}\right) - \frac{\beta}{\pi}$ $\qquad$ On γ: $\rho = 1$, $\zeta_b = \sigma = e^{i\varphi}$

$$\frac{\delta}{\pi} - 1 = 1 - 2\frac{\alpha}{\pi} - \frac{\beta}{\pi}$$

The Schwarz-Christoffel transformation formula gives for our case

$$\frac{dz}{d\zeta} = K\,(\zeta-a)^{\frac{\alpha}{\pi}-1}(\zeta-b)^{\frac{\beta}{\pi}-1}(\zeta-c)^{\frac{\gamma}{\pi}-1}(\zeta-d)^{\frac{\delta}{\pi}-1} \qquad (1)$$

which for $a = -1$; $b = -i$; $c = +1$; $d = +i$; $\alpha = \gamma$ becomes

$$\frac{dz}{d\zeta} = K\,(\zeta+1)^{\frac{\alpha}{\pi}-1}(\zeta+i)^{\frac{\beta}{\pi}-1}(\zeta-1)^{\frac{\alpha}{\pi}-1}(\zeta-i)^{1-2\frac{\alpha}{\pi}-\frac{\beta}{\pi}}$$

which reduces to

$$\frac{dz}{d\zeta} = K\left[\frac{(\zeta-i)^2}{\zeta^4-1}\left(\frac{\zeta^2-1}{\zeta^2+1}\right)^{\frac{\alpha}{\pi}}\left(\frac{(\zeta+i)^2}{\zeta^2+1}\right)^{\frac{\alpha+\beta}{\pi}}\right] \qquad (2)$$

For the symmetrical case, $\beta = \delta$, to which we shall confine our discussion, the further simplificative results in, $\alpha + \beta = \pi$

$$\frac{dz}{d\zeta} = K\left[\frac{1}{\zeta^2-1}\left(\frac{\zeta^2-1}{\zeta^2+1}\right)^{\frac{\alpha}{\pi}}\right] \qquad (3)$$

As the integration for z in the above involves essentially the incomplete Beta function (see equation 5 following), it can not be easily evaluated in terms of elementary functions. Several approaches were attempted.

a. Successive integration by parts in order to obtain a general term. This resulted in a divergent series; it was hoped that after one or two or so integrations, we could take advantage of the fact that $\frac{\alpha}{\pi} = 1 - \varepsilon$ where ε was a small quantity for normal airfoil sections.

b. Expansion of the integrand in powers of ε. This yielded a logarithmic series which was not particularly useful.

c. Integration of the analytic function along a θ = const. path from $0 \leq r \leq 1$. This gave difficulties similar to those encountered in d following.

d. Integration along the boundary $r = 1$. This procedure was carried out in detail.

Thus employing the last method and using the previously noted correspondence of $z = t/2$ for $\sigma = 1$ we write (3) as

$$\int_{t/2}^{z_b} dz = K\int_1^{\sigma} \frac{1}{\sigma^2-1}\left(\frac{\sigma^2-1}{\sigma^2+1}\right)^{\frac{\alpha}{\pi}} d\sigma \text{ where } z_b = \sigma = e^{i\varphi}$$

$$= K\int_0^{\varphi} \frac{1}{2ie^{i\varphi}\sin\varphi}\left(\frac{i\sin\varphi}{\cos\varphi}\right)^{\frac{\alpha}{\pi}} ie^{i\varphi}\,d\varphi \quad ; \frac{\alpha}{\pi} = 1-\varepsilon$$

$$= K\frac{i^{\frac{\alpha}{\pi}}}{2}\int_0^{\varphi} \sin^{\frac{\alpha}{\pi}-1}\varphi\,\cos^{-\frac{\alpha}{\pi}}\varphi\,d\varphi = \frac{Ki^{\frac{\alpha}{\pi}}}{2}\int_0^{\varphi}\sin^{-\varepsilon}\varphi\,\cos^{\varepsilon-1}\varphi\,d\varphi$$

From this K may be found; as $z_b \to i\frac{c}{2}$, $\varphi \to \frac{\pi}{2}$, then

$$i\frac{c}{2} - \frac{t}{2} = \frac{Ki^{\frac{\alpha}{\pi}}}{2}\int_0^{\frac{\pi}{2}} \cos^{2(\frac{\varepsilon}{2})-1}\vartheta\,\sin^{2(\frac{1-\varepsilon}{2})-1}\vartheta\,d\vartheta$$

$$= \frac{Ki^{\frac{\alpha}{\pi}}}{2}\left[\frac{\Gamma(\frac{\varepsilon}{2})\,\Gamma(\frac{1-\varepsilon}{2})}{2\,\Gamma(\frac{1}{2})}\right]$$

$$\therefore \frac{Ki^{\frac{\alpha}{\pi}}}{2} = -\frac{t-ic}{B(\frac{\varepsilon}{2},\frac{1-\varepsilon}{2})} \tag{4}$$

So in general the transformation becomes

$$z_b = \frac{t}{2} - \frac{t-ic}{B(\frac{\varepsilon}{2},\frac{1-\varepsilon}{2})}\int_0^{\varphi}\sin^{-\varepsilon}\vartheta\,\cos^{\varepsilon-1}\vartheta\,d\vartheta \tag{5}$$

In order to evaluate this integral, consider

$$I_1 = \int_0^{\varphi}\sin^{-\varepsilon}\vartheta\,\cos^{\varepsilon-1}\vartheta\,d\vartheta$$

and let

$$x = \cos\vartheta;\ y = \sin\vartheta;\ d\vartheta = \frac{dx}{-y} = \frac{dy}{x}\ ;\ x = (1-y^2)^{\frac{1}{2}};\ y = (1-x^2)^{\frac{1}{2}}$$

Then $I_1 = \int_0^{\sin\varphi} \frac{(1-y^2)^{\varepsilon-1}}{y^{\varepsilon}}\, dy$

Sucessive integration by parts yields:

$$I_1 = \left[\frac{y^2}{1-y^2}\right]^{1-\frac{\varepsilon}{2}} \frac{1}{(1-\varepsilon)\,y}\left[1 - \frac{2-\varepsilon}{3-\varepsilon}\left(\frac{y^2}{1-y^2}\right) + \frac{2-\varepsilon}{3-\varepsilon}\cdot\frac{4-\varepsilon}{5-\varepsilon}\left(\frac{y^2}{1-y^2}\right)^2 - \cdots\right]_0^{y}$$

And resubstituting ϑ for y, we obtain a tangent series

$$I_1 = \frac{\tan^{2-\varepsilon}\vartheta}{\sin\vartheta}\left[\frac{1}{(1-\varepsilon)} - \frac{1}{(1-\varepsilon)}\cdot\frac{2-\varepsilon}{3-\varepsilon}\tan^2\vartheta + \cdots\right]_0^{\varphi}$$

which converges for $0 \leq \varphi \leq \frac{\pi}{4}$. Writing the general term and summing in an infinite series:

$$I_1 = \frac{\tan^{2-\varepsilon}\varphi}{(1-\varepsilon)\sin\varphi}\left[1 + \sum_{n=1}^{\infty}(-1)^n \frac{B(n, \frac{3-\varepsilon}{2})}{B(n, \frac{2-\varepsilon}{2})}\tan^{2n}\varphi\right] \quad 0 \leq \varphi \leq \frac{\pi}{4} \qquad (6)$$

which gives for the transformation, for $0 \leq \varphi \leq \frac{\pi}{4}$

$$z_b = \frac{t}{2} - \frac{t-ic}{B(\frac{\varepsilon}{2},\frac{1-\varepsilon}{2})}\cdot\frac{\tan^{2-\varepsilon}\varphi}{(1-\varepsilon)\sin\varphi}\left[1 + \sum_{n=1}^{\infty}(-1)^n \frac{B(n,\frac{3-\varepsilon}{2})}{B(n,\frac{2-\varepsilon}{2})}\tan^{2n}\varphi\right] \qquad (7)$$

In the same manner a similar expression can be obtained in the region $\frac{\pi}{4} \leq \varphi \leq \frac{\pi}{2}$ as

$$z_b = i\,\frac{c}{2} + \frac{(t-ic)\cot^{1+\varepsilon}\varphi}{B(\frac{\varepsilon}{2},\frac{1-\varepsilon}{2})\,\varepsilon\cos\varphi}\left[1 + \sum_{n=1}^{\infty}(-1)^n \frac{B(n,\frac{2+\varepsilon}{2})}{B(n,\frac{1+\varepsilon}{2})}\cot^{2n}\varphi\right] \qquad (8)$$

One convenient limit check on the formula exists in the case of a square. As the mapping function as written is exact we may proceed to write $\alpha = \beta = \gamma = \delta = \pi/2$. Then $\alpha/\pi = 1 - \varepsilon = \frac{1}{2}$ implies $\varepsilon = \frac{1}{2}$. Using this value of ε, we may compute

$$B(\tfrac{\varepsilon}{2}, \tfrac{1-\varepsilon}{2}) = B(\tfrac{1}{4}, \tfrac{1}{4}) = 7.420$$

$$B(n, \tfrac{3-\varepsilon}{2}) = B(n, 1.25)$$

$$B(n, \tfrac{2-\varepsilon}{2}) = B(n, 0.75)$$

$$B(n, \tfrac{2+\varepsilon}{2}) = B(n, 1.25)$$

$$B(n, \tfrac{1+\varepsilon}{2}) = B(n, 0.75)$$

Formulas (7) and (8) respectively become

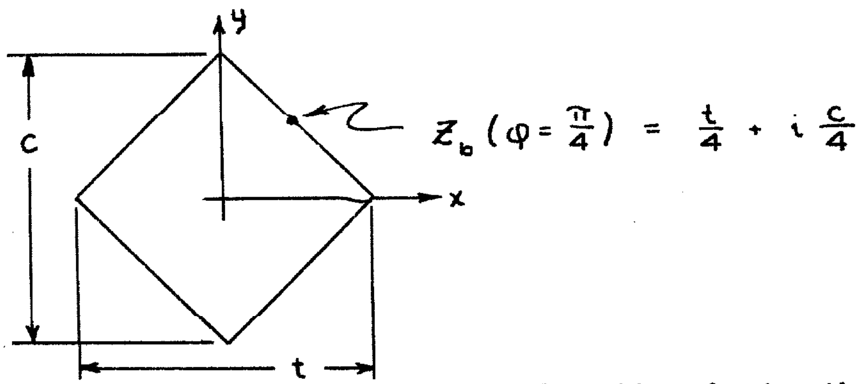
$$z_b = \frac{t}{2} - \frac{t-ic}{7.420} \cdot \frac{\tan^{1.5}\varphi}{0.5\sin\varphi}\left[1 + \sum_{n=1}^{\infty}(-1)^n \frac{B(n,1.25)}{B(n,0.75)}\tan^{2n}\varphi\right]$$

$$z_b = i\frac{c}{2} + \frac{t-ic}{7.420} \cdot \frac{\cot^{1.5}\varphi}{0.5\cos\varphi}\left[1 + \sum_{n=1}^{\infty}(-1)^n \frac{B(n,1.25)}{B(n,0.75)}\cot^{2n}\varphi\right]$$

Now adding the two for $\varphi = \frac{\pi}{4}$, where both expansions are valid, we obtain

$$2z_b = t/2 + i\,c/2 = 2x + i\,2y \quad (\varphi = \pi/4)$$

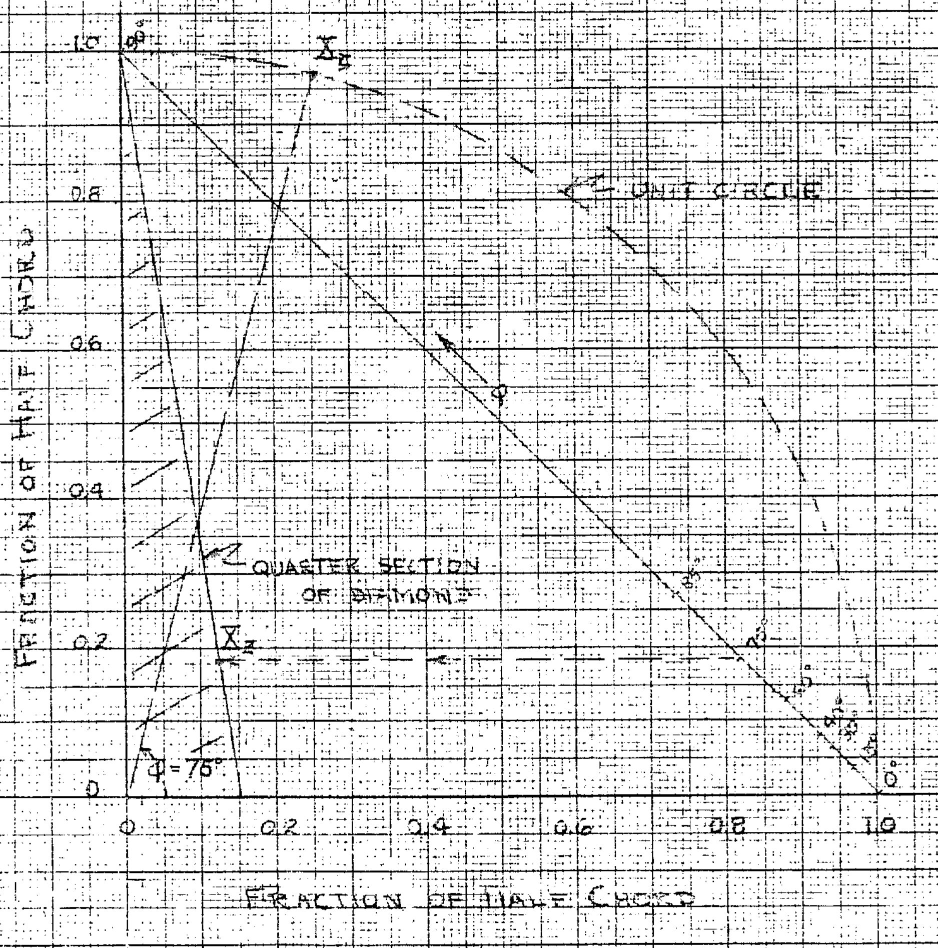
or $x = t/4$ and $y = c/4$ which as we see from the sketch is correct.



As a matter of interest, the transformation showing the correspondence in the two planes has been graphed in Fig. 1. It may be remarked that the summation from $n = 1$ to $n = \infty$ may be dropped from (8) and the formula will be accurate to three figures for the leading (or trailing) three eighths of the chord.

FIGURE 1

TRANSFORMATION CORRESPONDENCE



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IV. THE STRESSES

We shall now solve explicitly the problem of Dirichlet for our case; i.e. given $\Delta\Psi = 0$ and the boundary condition that $\Psi_b = \frac{1}{2} z_b \bar{z}_b = \frac{1}{2}\omega(\sigma)\bar{\omega}(\bar{\sigma})$, find $\Psi(\rho,\varphi)$. Proceed first to calculate Ψ_b from the transformation formula where in particular we are interested the range $0 \le \varphi \le \frac{\pi}{4}$ and more particularly $0 \le \varphi \le \frac{\pi}{12}$ which contains the point of maximum stress.

For convenience write the transformation formula (7) as

$$z_b = \frac{t}{2} - \frac{t-ic}{(1-\varepsilon)B_\varepsilon} F(\varphi) \quad ; \quad \bar{z}_b = \frac{t}{2} - \frac{t+ic}{(1-\varepsilon)B_\varepsilon} F(\varphi) \tag{9}$$

where $$F(\varphi) = \frac{\tan^{2-\varepsilon}\varphi}{\sin\varphi}\left[1 + \sum_{n=1}^{\infty} (\quad)\right]$$

and $$B_\varepsilon = B\left(\frac{\varepsilon}{2}, \frac{1-\varepsilon}{2}\right)$$

Thus

$$\Psi_b = \frac{1}{2} z_b \bar{z}_b = \frac{t^2}{8} - \frac{t^2}{2(1-\varepsilon)B_\varepsilon} F(\varphi) + \frac{c^2+t^2}{2(1-\varepsilon)^2 B_\varepsilon^2} F^2(\varphi)$$

or as

$$\Psi_b = k_1 + k_2 F + k_3 F^2 \tag{10}$$

where we have written

$$k_1 = \frac{t^2}{8} \quad ; \quad k_2 = -\frac{t^2}{2(1-\varepsilon)B_\varepsilon} \quad ; \quad k_3 = \frac{c^2+t^2}{2(1-\varepsilon)^2 B_\varepsilon^2}$$

Next, looking forward to the computation of a typical example, we assume $0 \le \varphi \le \tan^{-1}(1/5)$ and substitute the power series expansion for $\sin\varphi$ and $\tan\varphi$

$$\sin\varphi = \varphi\left(1 - \frac{\varphi^2}{6} + \frac{\varphi^4}{120} - \cdots\right) \quad ; \quad \tan\varphi = \varphi\left(1 + \frac{\varphi^2}{3} + \frac{2\varphi^4}{15} + \cdots\right)$$

and take as the transformation formula that part of the summations up to and including n = 2. These three expansions amount to neglecting terms of the order $\tan^6(1/5)$ when using also that part of the sin φ and tan φ series written above. Then $F(\varphi)$ is of the form

$$F(\varphi) = \varphi^{1-\varepsilon}\left[1 + a_1\varphi^2 + a_2\varphi^4 + a_3\varphi^6 + \cdots\right]$$

which means

$$\Psi_b = k_1 + k_2 F + k_3 F^2$$

becomes

$$\Psi_b = b_1 + b_2\varphi^{1-\varepsilon} + b_3\varphi^{2-2\varepsilon} + b_4\varphi^{3-\varepsilon} + b_5\varphi^{4-2\varepsilon} + b_6\varphi^{5-\varepsilon} + b_7\varphi^{6-2\varepsilon}$$

upon expansion and collection of constants. The relation of the constants is given in the calculation of the example as it is too cumbersome to present here.

Equation (11) must now be written as a Fourier series. Remembering that the constant b_1 is non-essential, we expand Ψ_b as an odd function

$$\Psi_b = \sum_{n=1}^{\infty} \beta_n \sin\frac{n\pi}{2}t \qquad \text{where} \qquad t = \frac{\varphi}{\varphi_0} \tag{12}$$

The coefficients are given by

$$\begin{aligned} \beta_n &= 2\int_0^1 \Psi_b(t)\sin\frac{n\pi}{2}t\,dt \qquad && n = 1,3,5,7,\cdots\cdots \\ \beta_n &= 0 && n = 2,4,6,8,\cdots\cdots \end{aligned} \tag{13}$$

The solution of $\Delta\Psi = 0$ for our case of a circular region with no singularity at the origin is the analytic function

$$\Psi_b(\rho,\varphi) = \sum \rho^{k_0}\Psi_b(\varphi) = \sum_{n=1,3,5,\cdots}^{\infty} \beta_n \rho^{\frac{n\pi}{2\varphi_0}} \sin\frac{n\pi}{2\varphi_0}\varphi \tag{14}$$

Then knowing $\Psi(\rho,\varphi)$, the conjugate function $\Phi(\rho,\varphi)$ can be found as

$$\Phi(\rho,\varphi) = \sum_{n=1,3,\cdots}^{\infty} \beta_n \rho^{k_o} \cos k_o\varphi \qquad \text{where} \qquad k_o = \frac{n\pi}{2\varphi_o} \tag{15}$$

We may now write the complex torsion function

$$f(\zeta) = \Phi + i\Psi = \sum_{n=1,3,\cdots}^{\infty} \beta_n \rho^{k_o} (\cos k_o\varphi + i \sin k_o\varphi)$$

$$= \sum_{n=1,3,\cdots}^{\infty} \beta_n \rho^{k_o} e^{ik_o\varphi} = \sum_{n=1,3,\cdots}^{\infty} \beta_n \zeta^{k_o} \tag{16}$$

In order to compute stresses using the equation in Section II we need

$$(df/d\zeta) = \sum_{n=1,3,\cdots}^{\infty} \beta_n k_o \zeta^{k_o-1} \tag{17}$$

which on the boundary becomes

$$(df/d\zeta)_b = \sum_{n=1,3,\cdots}^{\infty} \beta_n k_o e^{i(k_o-1)\varphi} \tag{18}$$

On the boundary where the maximum will occur, the stresses may be obtained from

$$(\tau_{zx} - i\tau_{zy})_b = \mu\alpha \left[\frac{(df/d\zeta)}{(d\omega/d\zeta)} - i\,\overline{\omega}(\zeta) \right]_b \tag{19}$$

Knowing $\omega'(\zeta)$ is the Schwarz-Christoffel form of the transformation $(dz/d\zeta)$ and $\overline{\omega}(\zeta)$ the mapping function we may substitute in (19) for $\rho = 1$ on the boundary to obtain

$$(\tau_{zx} - i\tau_{zy})_b = \mu\alpha \left\{ \frac{\sum_{n=1,3,\cdots}^{\infty} \beta_n k_o e^{i(k_o-1)\varphi}}{K\left[\frac{1}{\zeta^2-1}\left(\frac{\zeta^2-1}{\zeta^2+1}\right)^{\frac{\alpha}{\pi}}\right]_{\substack{\rho=1\\ \zeta=e^{i\varphi}}}} \right\}$$

$$- i\left[\frac{t}{2} - \frac{t\,F(\varphi)}{(1-\varepsilon)\,B_\varepsilon} - i\,\frac{c\;F(\varphi)}{(1-\varepsilon)\;B_\varepsilon} \right] \tag{20}$$

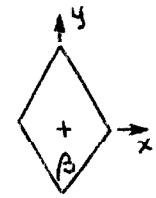
which after some reduction and separation of real and imaginary parts gives finally the stresses on the boundary

$$\tau_{zx_b}(\mu\alpha)^{-1} = \left[\frac{2\sin^{\varepsilon}\varphi\cos^{1-\varepsilon}\varphi}{k}\sum_{n=1,3,5,\cdots}^{\infty}\frac{\beta_n \pi n}{2\varphi_0}\cos\left(\frac{n\pi}{2\varphi_0}\varphi-\vartheta\right)\right]-\frac{c\,F(\varphi)}{(1-\varepsilon)\,B_\varepsilon} \tag{21}$$

$$\tau_{zy_b}(\mu\alpha)^{-1} = \left[\frac{2\sin^{\varepsilon}\varphi\cos^{1-\varepsilon}\varphi}{k}\sum_{n=1,3,5,\cdots}^{\infty}\frac{\beta_n \pi n}{2\varphi_0}\sin\left(\frac{n\pi}{2\varphi_0}\varphi-\vartheta\right)\right]+\frac{t\,F(\varphi)}{(1-\varepsilon)\,B_\varepsilon}-\frac{t}{2} \tag{22}$$

where the complex constant of the transformation formula K has been written

$$K = k e^{i\vartheta} = 2\, i^{-\frac{\alpha}{\pi}}\left[\frac{ic-t}{B_\varepsilon}\right] \quad \text{if} \quad k = \frac{2\sqrt{c^2+t^2}}{B_\varepsilon}, \quad \vartheta = \varepsilon\pi = \beta$$



It remains then to find $\tau = \sqrt{\tau_{zx}^2 + \tau_{zy}^2}$ and set $(d\tau/d\varphi) = 0$ in order to find the angle φ in the circle plane for which the stress is a maximum or minimum. With each value of φ is associated an angle θ in the physical plane through the transpormation formula (7) and thus the position of the maximum stress on the double wedge is determined. The explicit calculation of this position of maximum stress, and its magnitude, is complicated by algebraic difficulties and a trial and error solution has been used.

Finally the result should be compared in both position and magnitude with the conventional approximation. This has been done for one typical airfoil case in the following section.

V. NUMERICAL EXAMPLE

A numerical example has been calculated for the case of a fairly thick but representative diamond shape airfoil cross section with a thickness ratio of .15838. The range of expansion, which is somewhat arbitrary, has been confined to $\varphi_o = 1/5$ which is well within the range of expansion of the transformation, formula (7).

As indicated in equation (11) the relation between the constants is somewhat complicated and those listed here are sufficient for five figure accuracy.

$$a_1 = \frac{1}{6} + \frac{2-\varepsilon}{3} - \frac{2-\varepsilon}{3-\varepsilon} = \frac{15-6\varepsilon}{18} - \frac{2-\varepsilon}{3-\varepsilon}$$

$$a_2 = \frac{1647 - 1152\varepsilon + 180\varepsilon^2}{3240} - \frac{2-\varepsilon}{3-\varepsilon} \cdot \frac{15-6\varepsilon}{18}$$

$$a_3 = \frac{535 - 655\varepsilon + 234\varepsilon^2 - 20\varepsilon^3}{3240} - \frac{2-\varepsilon}{3-\varepsilon} \cdot \frac{1647 - 1152\varepsilon + 180\varepsilon^2}{3240}$$

$$- \frac{2-\varepsilon}{3-\varepsilon}\left(\frac{2}{3} - \frac{4-\varepsilon}{5-\varepsilon}\right)\left(\frac{15-6\varepsilon}{18}\right) - \frac{2-\varepsilon}{3-\varepsilon}\left(\frac{17}{45} - \frac{4}{3}\cdot\frac{4-\varepsilon}{5-\varepsilon} + \frac{4-\varepsilon}{5-\varepsilon} \cdot \frac{6-\varepsilon}{7-\varepsilon}\right)$$

$$b_1 = k_1 = t^2/8 \quad \text{(non-essential)}$$

$$b_2 = k_2 = \frac{-t^2}{2(1-\varepsilon)\,B_\varepsilon}$$

$$b_3 = k_3 = \frac{c^2 + t^2}{2(1-\varepsilon)^2\,B_\varepsilon^2}$$

$$b_4 = k_2 a_1 = k_2\left(\frac{1}{6} + \frac{2-\varepsilon}{3} - \frac{2-\varepsilon}{3-\varepsilon}\right)$$

$$b_5 = 2k_3 a_1 = 2k_3 \left(\frac{1}{6} + \frac{2-\varepsilon}{3} - \frac{2-\varepsilon}{3-\varepsilon}\right)$$

$$b_6 = k_2 a_2 = k_2 \Big[+ \frac{7}{360} + \frac{1}{6} \cdot \frac{2-\varepsilon}{3} + \frac{2(2-\varepsilon)}{15} + \frac{(2-\varepsilon)(1-\varepsilon)}{18} - \frac{1}{6}\cdot\frac{2-\varepsilon}{3-\varepsilon}$$

$$- \frac{2-\varepsilon}{3} \cdot \frac{2-\varepsilon}{3-\varepsilon} - \frac{2}{3}\cdot\frac{2-\varepsilon}{3-\varepsilon} + \frac{2-\varepsilon}{3-\varepsilon} \cdot \frac{4-\varepsilon}{5-\varepsilon}\Big]$$

$$b_7 = k_3(a_1^2 + 2a_2) = k_3 \left[\frac{1}{6} + \frac{2-\varepsilon}{3} - \frac{2-\varepsilon}{3-\varepsilon}\right]^2 + 2k_3 \left[\frac{b_6}{k_2}\right]$$

$$\frac{\alpha}{\pi} = 1 - \varepsilon \qquad \tan\left(\frac{\alpha}{2}\right) = \frac{c}{t} \qquad B_\varepsilon = B\left(\frac{\varepsilon}{2}, \frac{1-\varepsilon}{2}\right)$$

It is convenient to choose a simple value of ε, say 1/10 and find the corresponding thickness ratio. Thus $\varepsilon = 1/10$ implies that $\frac{t}{c} = \tan\left[(\varepsilon\pi)/2\right] = \tan 9^\circ = .15838$. We next write out equations (21) and (22) retaining only those terms necessary for three figure accuracy in the final result.

$$\tau_{zx_b}(\mu\alpha)^{-1} = \frac{\pi}{k\varphi_0} \sin^\varepsilon\varphi \cos^{1-\varepsilon}\varphi \left[\beta_1 \cos\left(\frac{\pi}{2\varphi_0}\varphi - \upsilon\right) + \cdots + 7\beta_7 \cos\left(\frac{7\pi}{2\varphi_0}\varphi - \upsilon\right)\right]$$

$$- \frac{c\,\varphi^{1-\varepsilon}}{(1-\varepsilon)\,B_\varepsilon}\left[1 + a_1\varphi^2 + a_2\varphi^4 + a_3\varphi^6\right]$$

$$\tau_{zy_b}(\mu\alpha)^{-1} = \frac{\pi}{k\varphi_0} \sin^\varepsilon\varphi \cos^{1-\varepsilon}\varphi \left[\beta_1 \sin\left(\frac{\pi}{2\varphi_0}\varphi - \upsilon\right) + \cdots + 7\beta_7 \sin\left(\frac{7\pi}{2\varphi_0}\varphi - \upsilon\right)\right]$$

$$+ \frac{t\,F(\varphi)}{(1-\varepsilon)\,B_\varepsilon} - \frac{t}{2}$$

The computation of constants is straight forward, the values being indicated below

$\varphi_0 = 1/5 \qquad \vartheta = \beta = \varepsilon\pi = .314159 \qquad t/c = .15838 \qquad \varepsilon = 1/10$

$$B_\varepsilon = B\left(\frac{\varepsilon}{2}, \frac{1-\varepsilon}{2}\right) = B(0.05, 0.45) = 21.6193$$

$$k = \frac{2\sqrt{c^2+t^2}}{B_\varepsilon} = \frac{2\sqrt{1.02508}}{21.6193} = .0468846$$

$a_1 = .144828$

$a_2 = .0338781$

$a_3 = -.0948610$

$b_1 = .00313553$ (non-essential)

$b_2 = -.000644594$

$b_3 = .00135382$

$b_4 = -.0000933553$

$b_5 = .000392297$

$b_6 = -.0000218376$

$b_7 = .000120126$

Using the above coefficients and $\varphi = \varphi_0 t = 1/5t$, we find

$$\Psi_b \times 10^3 = 3.13553 - .15143t^{0.9} + .07471t^{1.8} - .00088t^{2.9} + .00086t^{3.8} - .00008t^{4.9} + .00001t^{5.8}$$

from which the first eight Fourier coefficients have been computed as

$\beta_1 = .08080 \qquad = .08080$

$3\beta_3 = 3(.005520) \qquad = .01656$

$5\beta_5 = 5(.001706) \qquad = .008530$

$7\beta_7 = 7(.000654) \qquad = .004578$

$\beta_2 = \beta_4 = \beta_6 = \beta_8 = 0$

Other convenient values to have tabulated are

$$\frac{\pi}{2\varphi_0} = 7.85398 \qquad \frac{5\pi}{2\varphi_0} = 39.2699$$

$$\frac{3\pi}{2\varphi_0} = 23.5619 \qquad \frac{7\pi}{2\varphi_0} = 54.9779$$

$$\frac{c}{(1-\varepsilon)B_z} = .0513940 \qquad \frac{t}{(1-\varepsilon)B_z} = .00813978$$

$$\frac{\pi}{k\varphi_0} = 167.701$$

Finally the stresses τ_{zx}, τ_{yz}, and $\tau = \sqrt{\tau_{zx}^2 + \tau_{zy}^2}$ have been plotted against φ in Fig. 2. The maximum stress was found to be approximately $\tau_{max} = .0797\,\mu\alpha$ at $\varphi = .02$, which corresponds to $\theta = 1.10^\circ$ in the physical plane by equation (9).

The usual approximation is to assume the magnitude of the maximum stress to be that which would occur on the circumference of the largest circle inscribed in the cross section with its center at the center of twist and its location at the point (or points) where the circle and boundary are tangent. In this case such a procedure gives $\tau_{max} = .0782\,\mu\alpha$ at $\theta = 9.0^\circ$.

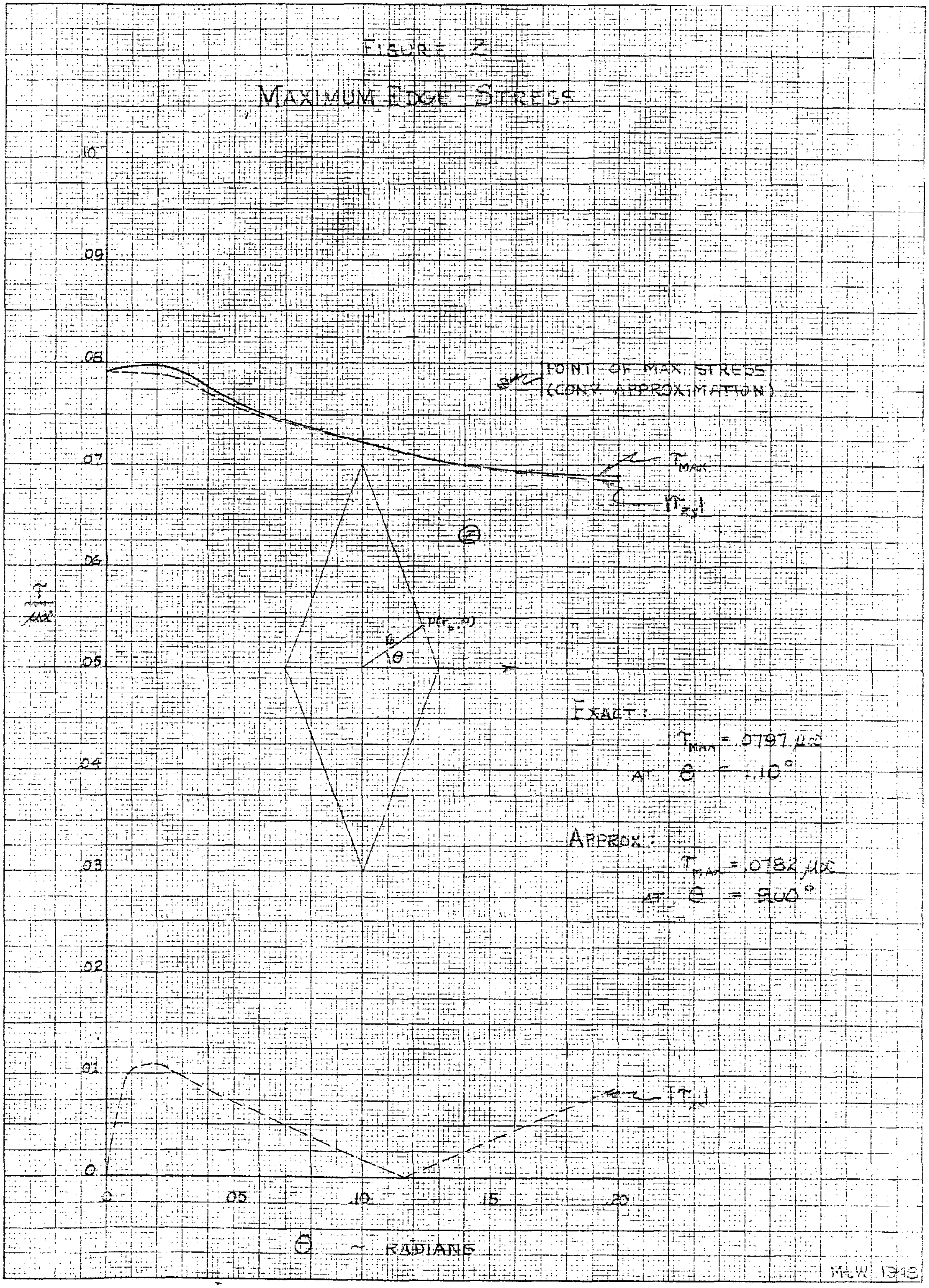


FIGURE 2
MAXIMUM EDGE STRESS
.10
.09
.08
.07
.06
.05
.04
.03
.02
.01
0
.05
.10
.15
.20
θ ~ RADIANS
POINT OF MAX STRESS
(CONV APPROXIMATION)
τMAX
θ
P(rp, θ)
EXACT:
τMAX = .0797 μα
AT θ = 1.10°
APPROX:
τMAX = .0782 μα
AT θ = .900°
MAW 1948

V. CONCLUSIONS

It has been shown that the usual engineering approximation to the maximum stress in a cylindrical body of thin diamond cross section under torsion is accurate enough for practical purposes. While non-conservative by 2% in the example, it is felt that more careful calculation of the Fourier coefficients would perhaps reduce the margin slightly, although as can be seen from Fig. 2, the maximum stress can never be less than τ_{zy} at $\theta = 0$ (i.e. $.07919\,\mu\alpha$). Further as a peak accurs in τ_{zx}, the maximum stress must be greater or at least equal to $.07919\,\mu\alpha$. Regarding the position of the maximum stress, engineering practice seldom permits sharp edges, in missile wings for instance, and lends further justification for the use of the much more easily applied approximate method.