THE EFFECT OF SHEAR INSTABILITY ON THE TRANSVERSE CIRCULATION IN THE ATMOSPHERE

Thesis

by

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NOTATIONS

x, y = Cartesian coordinates on a rotating disc

r, θ = polar " " " " " "

u = velocity in x direction

v = " " y "

V_r = velocity in radial direction

V_θ = velocity perpendicular to radius

ω = angular velocity of disc

g = acceleration due to gravity

$\bar{q}$ = velocity vector (u, v)

$\zeta = \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}$

$X = \frac{2\omega}{\sqrt{g h_0}} x$ (dimensionless coordinates)

$Y = \frac{2\omega}{\sqrt{g h_0}} y$ (" " ")

$\rho = \frac{2\omega}{\sqrt{g h_0}} r$ (" " ")

$U = \frac{u}{\sqrt{g h_0}}$ (dimensionless velocities)

$V = \frac{v}{\sqrt{g h_0}}$ (" " ")

$V_\theta = \frac{v_\theta}{\sqrt{g h_0}}$ (" " ")

$\eta = \frac{h - h_0}{h_0}$ (dimensionless depth)

NOTATIONS (cont'd)

σ = Wave frequency

$\kappa = 2\pi \div$ Wave length

$K_n(x)$ = Modified Bessel function of second type (see Ref. 3)

$I_n(x)$ = " " " " first " (see Ref. 3)

ABSTRACT

In this paper it is shown that the shear fields on either side of the westerlies are dynamically unstable and will roll up to form discrete eddies. A study of the stable vortex systems into which these eddies might collect shows that a "vortex street" is stable for a certain range of the ratio of width to vortex spacing. It is also shown that a formation with vortices placed on the corners of a regular polygon is stable if the number of vortices is less than seven. This still holds if the effect of the shear field north of the westerlies is simulated by a fixed polar cyclone. It further appears that with a strong general circulation, i.e. strong westerlies, only two or three high pressure cells should be found while the numbers, up to six, should be found with weaker circulations. This conclusion is verified by Northern Hemisphere mean pressure charts.

I. INTRODUCTION

The motion of the atmosphere can be considered as a mean flow which is of very large scale and is only slowly changed and, superimposed on this, the low level, smaller scale phenomena usually associated with the polar front. If the mean pressures over a period of about a week are plotted, it is seen that the latter disturbances are averaged out, and only the large scale mean motion is shown. Such a plot of the Northern Hemisphere shows, in addition to the mean westerly flow of air, large scale closed isobaric systems spaced at comparatively regular intervals on the surface of the earth. These include the Aleutian and Icelandic low pressure areas to the north of the westerlies and the Pacific and Bermuda high pressure areas to the south of the westerlies.

As the position and strength of these vortices apparently control the mean path of the low level storms, a knowledge of the properties of these systems would appear to be extremely desirable for any long range weather forecasting. One aspect of this problem has been investigated by Rossby (see Ref. 2) who showed that wave disturbances of about the observed wave length can exist as a result of the variation of the Coriolis acceleration with latitude. In the present paper, an investigation of another aspect of the problem arising from the existence of the shearing motion on either side of the belt of westerlies will be given.

In making these calculations, several approximations have been made. The principal ones and the reasons for their adoption are as follows:

1) As the systems are very deep, i.e. the wind momentum vector for a vertical section is roughly constant, it was felt that the horizontal field of motion was the dominant factor; consequently, vertical velocities were neglected, and horizontal momentum was assumed not to vary with height.

2) With this dynamical setup, the effect of variation of density with altitude would probably be small; consequently the atmosphere was assumed to be a layer of fluid of constant density with its depth being determined by the hydrostatic law.

3) As these systems are continuously being dissipated by friction in the lower layers and fed by the action of friction on the thermodynamically produced westerlies at about the same rate, the effects of friction have been neglected.

4) It has also been assumed that the fluid could be considered as being on a rotating disc rather than on a rotating sphere. As on the earth, the gravitational field normal to the disc was assumed to be deformed by an amount sufficient to cancel out the centrifugal accelerations due to the rotation of the disc.

II. THE FUNDAMENTAL EQUATIONS

Using the assumptions discussed in I, the equations determining the fluid motion on a rotating disc (see Ref. 2, p. 317) are, for a Cartesian coordinate system, the dynamical equations,

$$\frac{Du}{Dt} - 2\omega v = -g\frac{\partial h}{\partial x}$$

and

$$\frac{Dv}{Dt} + 2\omega u = -g\frac{\partial h}{\partial y} \quad , \tag{2.1}$$

and the equation of continuity,

$$\frac{Dh}{Dt} + h\left\{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right\} = 0 \ , \tag{2.2}$$

where

$$\frac{D(\,)}{Dt} = \frac{\partial(\,)}{\partial t} + u\frac{\partial(\,)}{\partial x} + v\frac{\partial(\,)}{\partial y} \quad .$$

In vector form these equations are

$$\frac{D\bar{q}}{Dt} + 2\bar{\omega}\times\bar{q} = -g\nabla h \tag{2.3}$$

and

$$\frac{Dh}{Dt} + h\nabla\cdot\bar{q} = 0 \tag{2.4}$$

These equations may be combined to give a very useful result. If from the curl of Eq. 2.3, one eliminates the term $\nabla\cdot\bar{q}$ by Eq. 2.4, one obtains the equation

$$\frac{D}{Dt}\left\{\frac{\zeta + 2\omega}{h}\right\} = 0 \quad . \tag{2.5}$$

This is the law of conservation of vorticity which could just as well have been taken as one of the fundamental equations. If the motion of the fluid could have been started from rest with a uniform depth, h_o , the last equation can be integrated. This gives the result that

$$\frac{\zeta + 2\omega}{h} = \frac{2\omega}{h_o} . \tag{2.6}$$

As the dynamical equations are non-linear, general solutions to these expressions are not readily found; however the solution for the case of motion in circles can be obtained quite easily. The equilibrium of forces in the radial direction, Fig. 1, requires that

$$\tfrac{1}{2} g r \frac{d}{dr}(h^2) = h V_\theta^2 + 2\omega r h V_\theta . \tag{2.7}$$

From Eq. 2.6,

$$h = h_o \left\{1 + \zeta/2\omega\right\} = h_o \left\{1 + \frac{1}{2\omega r} \frac{d}{dr}(r V_\theta)\right\}. \tag{2.8}$$

The equation to determine the velocity is thus

$$\frac{d^2 V_\theta}{dr^2} + \frac{1}{r}\frac{dV_\theta}{dr} - \left\{\frac{4\omega^2}{g h_o} + \frac{1}{r^2}\right\} V_\theta = \frac{2\omega}{g h_o r} V_\theta^2 . \tag{2.9}$$

This can be put into a somewhat simpler form using the dimensionless variables, $\rho = \frac{2\omega}{\sqrt{g h_o}} r$ and $V_\theta = \frac{V_\theta}{\sqrt{g h_o}}$.

With these variables, Eq. 2.9 becomes

$$\frac{d^2 V_\theta}{d\rho^2} + \frac{1}{\rho}\frac{dV_\theta}{d\rho} - \left\{1 + \frac{1}{\rho^2}\right\} V_\theta = \frac{1}{\rho} V_\theta^2 . \tag{2.10}$$

If V_θ is positive, this is the equation for cyclonic rotation; if V_θ is negative, this is the equation for anticyclonic rotation. The solutions to this equation which vanish for large radii were found numerically at the Massachusetts Institute of Technology by means of the Differential Analyzer. These results are plotted in Fig. 2 together with K_0 (ρ), which they approach asymptotically.

An approximate solution to this same problem may be found by neglecting the quadratic terms of Eq. 2.1, i.e. by neglecting the convective terms in the acceleration of the particles. If these terms are omitted, Eq. 2.1 and Eq. 2.6 may be solved for h, and the following result is obtained:

$$-gh_0\left\{\frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial y^2}\right\}+\frac{\partial^2 h}{\partial t^2}+4\omega^2(h-h_0)=0 \quad . \tag{2.11}$$

In terms of the dimensionless variables X, Y, and η, this equation becomes

$$\frac{\partial^2\eta}{\partial t^2}-\left\{\frac{\partial^2\eta}{\partial X^2}+\frac{\partial^2\eta}{\partial Y^2}\right\}+\eta=0 \ . \tag{2.12}$$

The only steady state solution to this equation which vanishes at infinity and which represents flow in circles (see Ref. 3) is

$$\eta=AK_0(\rho) \tag{2.13}$$

where $\rho=\sqrt{X^2+Y^2}$ is the same as in Eq. 2.10, and A is an arbitrary constant. If A is positive, the motion is anticyclonic; if A is negative, the motion is cyclonic. From Eq. 2.1, we have

$$V_\theta = \frac{v_\theta}{\sqrt{gh_0}} = \frac{\partial \eta}{\partial \rho} = -A K_1(\rho). \qquad (2.14)$$

This plotted for a unit cyclone, $A = -1$, in Fig. 2 with the exact solutions which retained the quadratic terms. From Eq. 2.14 it can be seen that if $\rho \ll 1$, the velocity varies inversely as the radius, just as in an ordinary line vortex in an incompressible non-viscous fluid, and if distances between the vortices of Eq. 2.14 are small, results identical to those found with ordinary vortices will be obtained. However, the linearization of the equations of motion breaks down in this range, so that results with ordinary vortices cannot be applied directly.

If we take h_0 as being the depth of the homogeneous atmosphere, about 8 km., a distance of 2000 km. corresponds to $\rho = 1.04$. Fig. 3 shows that at distances as large as this, the error caused by neglecting the quadratic terms is small, particularly if the constant is adjusted so as to fit near $\rho = 1$ rather than for very large values of ρ. As the spacing of the pressure centers being investigated is of the order of 2000 km., the quadratic terms will be neglected throughout, and the linearized form, Eq. 2.12, will be used. In addition to this, the motions will be considered to be either stationary or changing very slowly so that the local acceleration can be neglected as compared to the Coriolis term. This is equivalent to saying that the fluid motions to be considered can be built up through superposition of vortices of the type given by Eq. 2.13.

With these approximations, Eq. 2.1 and Eq. 2.12 can be written as follows using dimensionless variables:

$$\left.\begin{aligned} V &= \frac{\partial \eta}{\partial X} \\ U &= -\frac{\partial \eta}{\partial Y} \end{aligned}\right\} \qquad (2.15)$$

$$\left\{ \frac{\partial^2 \eta}{\partial X^2} + \frac{\partial^2 \eta}{\partial Y^2} \right\} - \eta = 0 \qquad (2.16)$$

Eq. 2.15 is exactly the usual geostrophic wind equation.

It is of interest to note that Eq. 2.12 (or Eq. 2.16) is the same as the equation for the deflection of a membrane which is elastically supported. The character of the solutions of Eq. 2.12 may often be estimated using this analogy.

III. REVERSE CURRENTS NEAR JETS

In a first approximation the belt of westerlies may be thought of as a uniform rectilinear jet in an otherwise undisturbed atmosphere. In order to estimate the disturbances introduced by the jet, this problem will be considered. If the velocity of the jet is U_0 and its width is $2a$, the boundary conditions for the portion of fluid to the north of the jet are

(a) $\eta = -U_0 a$ for $Y = a$ and (b) $\eta = 0$ for $Y = \infty$.

The differential equation, from Eq. 2.16, is

$$\frac{d^2\eta}{dY^2} - \eta = 0 \ . \tag{3.1}$$

The required solution is thus

$$\eta = -a U_0 e^{-(Y-a)} . \tag{3.2}$$

This corresponds to a velocity field given by

$$U = a U_0 e^{-(Y-a)} . \tag{3.3}$$

This surface deflection is sketched in Fig. 3. The velocity field shows reverse currents existing outside the jet which die out rapidly with distance from the jet.

It was thought that in a two dimensional extension of this problem, such as jet issuing from a gap in a wall as in Fig. 4, the counter-currents might form a series of closed paths. From Eq. 2.15, the boundary conditions are that η be constant along the wall and the edge of the jet. This will be taken as unity for the quadrant where both X and Y are positive. To final the required solution, let

$$\eta = e^{-Y} + \eta' \quad . \tag{3.4}$$

The boundary conditions on η' are

(a) $\eta' = 0$ for $Y = 0$ and (b) $\eta' = 1 - e^{-Y}$ for $X = 0$ (3.5)

and (c) $\eta' = 0$ for X and Y infinite.

The function η' can be built up of basic solutions of the type $\eta_o = F(X) . \sin(\lambda Y)$ by use of the Fourier integral theorum. $F(X)$ is found by substituting η_o in Eq. 2.16. This gives

$$\frac{d^2F}{dX^2} - (1+\lambda^2)F = 0 \, , \tag{3.6}$$

or

$$\eta_o = \sin(\lambda Y)\, e^{-\sqrt{1+\lambda^2}\,X} \, . \tag{3.7}$$

By the Fourier integral theorum,

$$e^{-Y} = \frac{2}{\pi}\int_0^\infty \sin(\lambda Y) \left\{ \int_0^\infty e^{-t} \sin(\lambda t)\, dt \right\} d\lambda \qquad \text{For } Y > 0,$$

or

$$e^{-Y} = \frac{2}{\pi}\int_0^\infty \frac{\lambda}{1+\lambda^2} \sin(\lambda Y)\, d\lambda \qquad \text{For } Y > 0. \tag{3.8}$$

From this

$$1-e^{-Y} = \frac{2}{\pi}\int_0^\infty \sin(\lambda Y)\left\{\frac{1}{\lambda} - \frac{\lambda}{1+\lambda^2}\right\}d\lambda \quad \text{FOR } Y \geqslant 0. \tag{3.9}$$

The integral is an odd function of Y and thus vanishes for $Y=0$. By comparison with this result, the function

$$\eta' = \frac{2}{\pi}\int_0^\infty e^{-\sqrt{1+\lambda^2}X}\sin(\lambda Y)\left\{\frac{1}{\lambda} - \frac{\lambda}{1+\lambda^2}\right\}d\lambda \tag{3.10}$$

satisfies Eq. 2.16 and the boundary conditions, Eq. 3.5 and is thus the required solution. The numerical values of η of Eq. 3.4 can be found by expansion in series in the following manner:

$$\begin{aligned}\frac{\partial^2\eta}{\partial X^2} &= \frac{2}{\pi}\int_0^\infty e^{-\sqrt{1+\lambda^2}X}\sin(\lambda Y)\frac{d\lambda}{\lambda} \\ &= \frac{2}{\pi}\int_0^\infty e^{-X\cosh\alpha}\left\{\sum_{n=0}^\infty (-1)^n\frac{Y^{2n+1}}{(2n+1)!}\sinh^{2n}\alpha\right\}\cosh\alpha\, d\alpha\end{aligned}$$

or

$$\frac{\partial^2\eta}{\partial X^2} = \frac{2}{\pi}\sum_{n=0}^\infty (-1)^n\frac{2n+2}{2n+1}\cdot\frac{Y^{2n+1}}{X^n}\times\frac{K_{n+1}(X)}{2^{n+1}(n+1)!} \quad \text{IF } XY\neq 0. \tag{3.11}$$

A similar expression for $\frac{\partial^2\eta}{\partial Y^2}$ may be obtained by interchanging X and Y. By Eq. 2.16 η is the sum of these two; therefore, if $XY\neq 0$,

$$\eta = \frac{2}{\pi}\sum_{n=0}^\infty \frac{2n+2}{2n+1}\cdot\frac{(-1)^n}{2^{n+1}(n+1)!}\left\{\frac{Y^{2n+1}}{X^n}K_{n+1}(X) + \frac{X^{2n+1}}{Y^n}K_{n+1}(Y)\right\}. \tag{3.12}$$

This is plotted in Fig. 5 for the lines $Y=X$ and for $Y=\infty$. It can be seen that there is no possibility of a maximum or a minimum except on the boundary and that no stationary pressure centers could be produced by this means.

This result might have been anticipated by consideration of the membrane analogy stated in II. From this analogy and the results of the above problem, it appears that no generalization of this basic problem can show closed isobars unless the jet flows in a closed path. If we consider the jet as flowing in a circle about the polar axis, the corresponding solution is simply that

$$\eta = A I_0(\rho) \tag{3.13}$$

where ρ is the dimensionless distance from the pole. This gives a flow purely along the circles of latitude with a pressure maximum at the pole. From this, it must be concluded that, from the point of view of this particular problem, there is no steady motion with closed isobaric fields south of the westerlies and a different mechanism must be found to explain their existence.

IV. STABILITY OF VORTEX SHEETS

The problems discussed in III all show a sharp velocity discontinuity at the edges of the jet; in other words, the jet is bounded by vortex sheets. The vortex sheets are in equilibrium, but if the equilibrium is not stable, it must be assumed that the systems would change until some stable condition is found. That such a vortex sheet is actually unstable will be shown in this section. Similar calculations by Pekeris have yielded the same result. The stability of this vortex sheet will be investigated by assuming that there is a small wave disturbance in the vortex sheet. If the equations of motion, Eq. 2.1, and the equation of continuity, Eq. 2.2, show that the amplitude of this disturbance must increase with time, then the vortex sheet must be considered unstable as any disturbance would thus be amplified.

Suppose we have a mean flow as in Fig. 6 which consists of a stationary body of fluid for $y>0$ and a uniform velocity in the x direction for $y<0$ with the depth being constant, h_0, in both regions in the equilibrium condition and a vortex sheet separating the two regions.

If the disturbance is small, Eq. 2.1 and Eq. 2.2 may be linearised as follows:

$$\begin{aligned}
&\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} - 2\omega v = -g\frac{\partial h}{\partial x} \\
&\frac{\partial v}{\partial t} + U\frac{\partial v}{\partial x} + 2\omega u = -g\frac{\partial h}{\partial y} + 2\omega U \\
&\frac{\partial h}{\partial t} + U\frac{\partial h}{\partial x} + h_o\left\{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right\} = 0
\end{aligned} \tag{4.1}$$

The solutions of these equations which represent a wave in the vortex sheet traveling in the positive direction may be represented as the real parts of the following expressions where the constants A, B, and C may be complex:

$$\begin{aligned}
u &= U + A\,e^{\lambda y + i(\sigma t - \kappa x)} \\
v &= \quad\;\; B\,e^{\lambda y + i(\sigma t - \kappa x)} \\
h &= h_o + C\,e^{\lambda y + i(\sigma t - \kappa x)}
\end{aligned} \tag{4.2}$$

The expressions, Eq. 4.2, will be solutions if

$$\begin{aligned}
&i(\sigma - \kappa U)A - 2\omega B - gi\kappa C = 0 \\
&2\omega A + i(\sigma - \kappa U)B + g\lambda C = 0 \\
&-i\kappa h_o A + h_o\lambda B + i(\sigma - \kappa U)C = 0
\end{aligned} \tag{4.3}$$

In order for Eq. 4.3 to be consistant the following relation must hold:

$$\begin{vmatrix} i(\sigma - \kappa U) & -2\omega & -gi\kappa \\ 2\omega & i(\sigma - \kappa U) & g\lambda \\ -i\kappa h_o & h_o\lambda & i(\sigma - \kappa U) \end{vmatrix} = 0$$

or if $(\sigma - \kappa U) \neq 0$,

$$\lambda^2 = \kappa^2\left\{1 - \frac{(\sigma - \kappa U)^2 - 4\omega^2}{\kappa^2 g h_o}\right\}. \tag{4.4}$$

From this the appropriate form for the moving fluid mass which shows a wave form in the vortex sheet is

$$\lambda = \kappa \sqrt{1 - \frac{(\sigma - \kappa U)^2 - 4\omega^2}{\kappa^2 g h_0}} \, . \tag{4.5}$$

For the stationary fluid mass, designated by $(\ )'$,

$$\lambda' = -\kappa \sqrt{1 - \frac{\sigma^2 - 4\omega^2}{\kappa^2 g h_0}} \, . \tag{4.6}$$

These expressions give disturbances which die away with increasing distance from the vortex sheet. If the displacement of the vortex sheet in the y direction is called η, this displacement must be given by an expression such as

$$\eta = a e^{i(\sigma t - \kappa x)} \, . \tag{4.7}$$

In order to relate the amplitude of the wave with the equations of motion, Eq. 4.2, we have the kinematical condition for small displacements that

$$v = \frac{\partial \eta}{\partial t} + U \frac{\partial \eta}{\partial x} \quad \text{and} \quad v' = \frac{\partial \eta}{\partial t} \tag{4.8}$$

for $y = 0$; this gives

$$\left. \begin{aligned} B &= i a(\sigma - \kappa U) \\ B' &= i a \sigma \end{aligned} \right\} \, . \tag{4.9}$$

We have the additional dynamical boundary condition that the fluid depth must be continuous over the vortex sheet. From this,

$$h_o - \frac{2\omega U}{g}\eta + Ce^{\lambda\eta + i(\sigma t - \kappa x)} = h_o + C'e^{\lambda'\eta + i(\sigma t - \kappa x)}$$

or

$$-\frac{2\omega U}{g}a + C = C' . \tag{4.10}$$

From Eq. 4.3 and 4.9,

$$C = a\frac{\sigma - \kappa U}{g} \times \frac{(\sigma - \kappa U)^2 - 4\omega^2}{2\omega\kappa + \lambda(\sigma - \kappa U)} ,$$

$$\text{and} \qquad C' = a\frac{\sigma}{g} \times \frac{\sigma^2 - 4\omega^2}{2\omega\kappa + \lambda'\sigma} . \tag{4.11}$$

Substituting these expressions in Eq. 4.10 we have

$$-2\omega U + (\sigma - \kappa U)\frac{(\sigma - \kappa U)^2 - 4\omega^2}{2\omega\kappa + \lambda(\sigma - \kappa U)} = \sigma\frac{\sigma^2 - 4\omega^2}{2\omega\kappa + \lambda'\sigma} . \tag{4.12}$$

This expression determines the natural frequencies of the system. It can be seen that $\sigma = 2\omega$ and $\sigma = \kappa U - 2\omega$ are two roots of this equation. This equation can be solved easily if the fluid is very deep so that $\lambda = \kappa$ and $\lambda' = -\kappa$. For this case

$$\sigma = \begin{cases} 2\omega \\ \kappa U - 2\omega \\ \frac{\kappa U}{2}(1 \pm i) \end{cases} \tag{4.13}$$

The third of these frequencies is complex for all wave lengths, and the sheet is thus unstable. This is identical with the result obtained in an infinite fluid mass without Coriolis forces (see Ref. 4, p. 340).

By combining Eq. 4.13 with Eq. 4.5 or Eq. 4.6 it is seen that as long as the wave length of a disturbance in the atmosphere is less than a thousand miles, the magnitude of λ obtained differs from unity by less than one percent, so the approximate complex solution of Eq. 4.13 is sufficiently good to show that a vortex sheet is unstable and will break down into discrete eddies.

V. STABILITY OF ROWS OF VORTICES

When an unstable vortex sheet breaks up into discrete eddies, the vorticity may either diffuse throughout the fluid mass or, if there is a stable vortex formation, the vorticity may collect in a definite pattern. As an example of this type of motion, the vortex sheets shed from a two-dimensional bluff body break down and then form the well-known Kármán "vortex street."

The vortex sheet investigated in the previous section can break up into only one system which is stationary and in equilibrium. This is a row of equal vortices equally spaced as in Fig. 7. In terms of the dimensionless variables introduced in II, the surface deflection for such a system of vortices of unit strength is, by Eq. 2.13,

$$\eta = \sum_{n=-\infty}^{\infty} K_0 [(x-n\ell)^2+y^2]^{1/2}. \tag{5.1}$$

If the n'th vortex is displaced by amounts Δx_n and Δy_n in the x and y directions respectively, the surface deflection in the displaced position is

$$\eta = \sum_{n=-\infty}^{\infty} K_0 [(x-n\ell-\Delta x_n)^2+(y-\Delta y_n)^2]^{1/2} \tag{5.2}$$

The velocity of a vortex is the sum of the velocities induced by all the other vortices. By Eq. 2.15 the velocity of the vortex at the origin is

$$\frac{d}{dt}\Delta x_o = -\frac{\partial}{\partial y}\sum_{n=-\infty}^{\infty} K_o[(x-nl-\Delta x_n)^2+(y-\Delta y_n)^2]^{\frac{1}{2}}\Big|_{\substack{x=\Delta x_o\\ y=\Delta y_o}}$$

$$\frac{d}{dt}\Delta y_n = \frac{\partial}{\partial x}\sum_{n=-\infty}^{\infty} K_o[(x-nl-\Delta x_n)^2+(y-\Delta y_n)^2]^{\frac{1}{2}}\Big|_{\substack{x=\Delta x_o\\ y=\Delta y_o}} \quad . \tag{5.3}$$

If the displacements are small these can be written as

$$\frac{d}{dt}\Delta x_o = \sum_{n=-\infty}^{\infty}\Big|\frac{K_1(nl)}{nl}\Big|\{\Delta y_n-\Delta y_o\} \qquad n\neq 0$$

$$\frac{d}{dt}\Delta y_o = \sum_{n=-\infty}^{\infty}\Big|\frac{K_1(nl)}{nl}+K_o(nl)\Big|\{\Delta x_n-\Delta x_o\}\,, \quad n\neq 0\,. \tag{5.4}$$

Similar expressions for the velocities of the other vortices could be written by symmetry. These form a set of simultaneous differential equations which must be solved in order to determine the motion of the vortices.

As the equations for the displacements are linear, any perturbation can be represented as a sum of terms of the type $\Delta x_n = \Delta x_o e^{in\varphi}$ and $\Delta y_n = \Delta y_o e^{in\varphi}$ where $0 \leq \varphi \leq 2\pi$, and the individual harmonics can be investigated separately. From this,

$$\frac{d}{dt}\Delta x_o = \Delta y_o \sum_{n=1}^{\infty} 2\,\frac{K_1(nl)}{nl}[\cos(n\varphi)-1]\,,$$

and

$$\frac{d}{dt}\Delta y_o = \Delta x_o \sum_{n=1}^{\infty} 2\left\{\frac{K_1(nl)}{nl}+K_o(nl)\right\}[\cos(n\varphi)-1]\,, \tag{5.5}$$

Combining these two equations, we obtain

$$\frac{d^2}{dt^2}(\Delta x_o) - \lambda^2(\Delta x_o) = 0 \quad \text{and} \quad \frac{d^2}{dt^2}(\Delta y_o) - \lambda^2(\Delta y_o) = 0 \tag{5.6}$$

where

$$\lambda^2 = 4\left\{\sum_{n=1}^{\infty} \frac{K_1(n\ell)}{n\ell}[1-\cos(n\varphi)]\right\}\left\{\sum_{n=1}^{\infty}\left[\frac{K_1(n\ell)}{n\ell} + K_o(n\ell)\right][1-\cos n\varphi]\right\} \tag{5.7}$$

From Eq. 5.6 the motion can be stable only if $\lambda^2 < 0$; however from Eq. 5.7, λ^2 can never be negative. A single row of vortices is thus unstable.

As there are two shear regions in the Northern Hemisphere, cyclonic shear to the north of the westerlies and anticyclonic shear to the south, it is possible that a double row of vortices arising from these might be stable. As shown in Fig. 8, there are two possible arrangements of equal and opposite vortices which do not change with time.

We will first investigate the stability of the symmetrical system. If we let $\overline{\Delta x_n}$ and $\overline{\Delta y_n}$ be the displacements of the n'th vortex in the upper row and Δx_n and Δy_n be the displacements of the n'th vortex in the lower row, the surface deflection is given by

$$\eta = \sum_{n=-\infty}^{\infty} K_o[(x-n\ell-\Delta x_n)^2+(y-\Delta y_n)^2]^{\frac{1}{2}} - \sum_{n=-\infty}^{\infty} K_o[(x-n\ell-\overline{\Delta x_n})^2+(y-d-\overline{\Delta y_n})^2]^{\frac{1}{2}}. \tag{5.8}$$

From this expression, the velocities of the vortices can be obtained in the same manner as previously.

The velocity of the zero vortex in the lower row is

$$\frac{d}{dt}(\Delta x_0) = \sum_{n=-\infty}^{\infty}{}' \Big/ \frac{K_1(n\ell)}{n\ell} \Big/ \{\Delta y_n - \Delta y_0\} + \sum_{n=-\infty}^{\infty} \frac{n\ell d}{n^2\ell^2+d^2} \Big\{ 2\frac{K_1[n^2\ell^2+d^2]^{\frac{1}{2}}}{[n^2\ell^2+d^2]^{1/2}} + K_0[n^2\ell^2+d^2]^{\frac{1}{2}} \Big\} \{\overline{\Delta x}_n - \Delta x_0\}$$
$$+ \sum_{n=-\infty}^{\infty} \Big\{ \frac{d^2}{n^2\ell^2+d^2} K_0[n^2\ell^2+d^2]^{1/2} - \frac{n^2\ell^2-d^2}{[n^2\ell^2+d^2]^{3/2}} K_1[n^2\ell^2+d^2]^{1/2} \Big\} \{\overline{\Delta y}_n - \Delta y_0\},$$

(5.9)

and

$$\frac{d}{dt}(\Delta y_0) = \sum_{n=-\infty}^{\infty}{}' \Big/ \frac{K_1(n\ell)}{n\ell} + K_0(n\ell) \Big/ \{\Delta x_n - \Delta x_0\} - \sum_{n=-\infty}^{\infty} \frac{n\ell d}{n^2\ell^2+d^2} \Big\{ 2\frac{K_1[n^2\ell^2+d^2]^{1/2}}{[n^2\ell^2+d^2]^{1/2}} + K_0[n^2\ell^2+d^2]^{1/2} \Big\} \{\overline{\Delta y}_n - \Delta y_0\}$$
$$- \sum_{n=-\infty}^{\infty} \Big\{ \frac{n^2\ell^2-d^2}{[n^2\ell^2+d^2]^{1/2}} K_1[n^2\ell^2+d^2]^{1/2} + \frac{n^2\ell^2}{n^2\ell^2+d^2} K_0[n^2\ell^2+d^2]^{1/2} \Big\} \{\overline{\Delta x}_n - \Delta x_0\} .$$

In these expressions $\sum_{n=-\infty}^{\infty}{}'$ indicates the zero term of the summation is to be left out.

Putting $\Delta x_n = \Delta x_0 e^{in\varphi}$, $\Delta y_n = \Delta y_0 e^{in\varphi}$, $\overline{\Delta x}_n = \overline{\Delta x}_0 e^{in\varphi}$, and $\overline{\Delta y}_n = \overline{\Delta y}_0 e^{in\varphi}$ as before, we can write Eq. 5.9 as

$$\frac{d}{dt}(\Delta x_0) = A\,\Delta y_0 + B\,\overline{\Delta x}_0 + C\,\overline{\Delta y}_0 \qquad (5.10)$$
$$\frac{d}{dt}(\Delta y_0) = D\,\Delta x_0 - B\,\overline{\Delta y}_0 + E\,\overline{\Delta x}_0$$

where

$$A = -2\sum_{n=1}^{\infty} \frac{K_1(n\ell)}{n\ell}[1-\cos(n\varphi)] - 2\sum_{n=1}^{\infty} \Big\{ \frac{d^2}{n^2\ell^2+d^2} K_0[n^2\ell^2+d^2]^{1/2} - (n^2\ell^2-d^2)\frac{K_1[n^2\ell^2+d^2]^{1/2}}{[n^2\ell^2+d^2]^{3/2}} \Big\}$$
$$- K_0(d) - \frac{K_1(d)}{d}$$
$$B = 2i\sum_{n=1}^{\infty} \frac{n\ell d \sin(n\varphi)}{n^2\ell^2+d^2} \Big\{ 2\frac{K_1[n^2\ell^2+d^2]^{1/2}}{[n^2\ell^2+d^2]^{1/2}} + K_0[n^2\ell^2+d^2]^{1/2} \Big\} \qquad (5.11)$$
$$C = 2\sum_{n=1}^{\infty} \Big\{ \frac{d^2}{n^2\ell^2+d^2} K_0[n^2\ell^2+d^2]^{1/2} - (n^2\ell^2-d^2)\frac{K_1[n^2\ell^2+d^2]^{1/2}}{[n^2\ell^2+d^2]^{3/2}} \Big\}\cos(n\varphi)$$
$$+ K_0(d) + \frac{K_1(d)}{d}$$

$$D = A - 2\sum_{n=1}^{\infty} K_0(n\ell)\{1-\cos(n\varphi)\} + \sum_{n=-\infty}^{\infty} K_0[n^2\ell^2+d^2]^{1/2}$$

$$E = C - K_0(d) - 2\sum_{n=1}^{\infty} K_0[n^2\ell^2+d^2]^{1/2}\cos(n\varphi) \quad .$$

From symmetry the equations for the motion of the zero vortex in the upper row can be written as follows:

$$\frac{d}{dt}(\overline{\Delta x_0}) = -A\overline{\Delta y_0} + B\Delta x_0 - C\Delta y_0$$
$$\frac{d}{dt}(\overline{\Delta y_0}) = -D\overline{\Delta x_0} - B\Delta y_0 - E\Delta x_0 \quad . \tag{5.12}$$

We notice the second pair of equations is identical with the first in either of two cases, first if $\Delta x_0 = \overline{\Delta x_0}$ and $\Delta y_0 = -\overline{\Delta y_0}$, and second if $\Delta x_0 = -\overline{\Delta x_0}$ and $\Delta y_0 = \overline{\Delta y_0}$. The first case corresponds to symmetrical perturbations with respect to the center line, and the second case corresponds to antisymmetrical perturbations. In order for the system to be stable, both types of motion must be stable.

For the symmetrical case,

$$\left\{\frac{d}{dt} - B\right\}\Delta x_0 = \{A-C\}\Delta y_0 \quad , \tag{5.13}$$

and
$$\left\{\frac{d}{dt} - B\right\}\Delta y_0 = \{D+E\}\Delta x_0 \quad ,$$

or
$$\left[\left\{\frac{d}{dt} - B\right\}^2 - \{A-C\}\{D+E\}\right]\begin{cases}\Delta x_0 = 0.\\ \Delta y_0 = 0.\end{cases} \tag{5.14}$$

If Δx_0 or $\Delta y_0 \backsim e^{\lambda t}$, the exponent λ is given by Eq. 5.14 as

$$\lambda = B \pm \sqrt{(A-C)(D+E)} \tag{5.15}$$

For the unsymmetrical displacements we obtain the similar result

$$\lambda = -B \pm \sqrt{(A+C)(D-E)} \tag{5.16}$$

As B is pure imaginary, the condition for stability in the two cases is

$$\begin{aligned} &(A-C)(D+E) \leqslant 0 \\ \text{and} \qquad &(A+C)(D-E) \leqslant 0 \;. \end{aligned} \tag{5.17}$$

For the special case where $\varphi = \pi$ it is easily seen that the first of these is violated, for

$$\begin{aligned} D+E = &-4\sum_{n=0}^{\infty}\left\{\frac{K_1[(2n+1)\ell]}{2n+1} - \frac{K_1[(2n+1)^2\ell^2+d^2]^{1/2}}{[(2n+1)^2\ell^2+d^2]^{1/2}}\right\} - 4\sum_{n=0}^{\infty}\left\{K_0[(2n+1)\ell] - K_0[(2n+1)^2\ell^2+d^2]^{1/2}\right\} \\ &-4\sum_{n=0}^{\infty}\left\{2d^2\frac{K_1[(2n+1)^2\ell^2+d^2]^{1/2}}{[(2n+1)^2\ell^2+d^2]^{3/2}} + d^2\,\frac{K_0[(2n+1)^2\ell^2+d^2]^{1/2}}{(2n+1)^2\ell^2+d^2}\right\}. \end{aligned} \tag{5.18}$$

By inspection, $D+E<0$. By using the recurrence formulae for Bessel functions (see Ref. 3, p. 22), A-C can be written as follows:

$$A-C = -2\left\{\frac{K_1(d)}{d} + K_0(d)\right\} - 4\sum_{n=1}^{\infty}\left\{\frac{K_1[(2n-1)\ell]}{(2n-1)\ell} - \frac{K_1[4n^2\ell^2+d^2]^{1/2}}{[4n^2\ell^2+d^2]^{1/2}} + d^2\frac{K_2[4n^2\ell^2+d^2]^{1/2}}{4n^2\ell^2+d^2}\right\}. \tag{5.19}$$

This also is obviously negative. From this, it follows that the symmetrical vortex system of Fig. 8 is unstable.

For the asymmetrical system of Fig. 8, it is found by a similar procedure that Eq. 5.10 and Eq. 5.12 and thus Eq. 5.17 hold if

$$
\begin{aligned}
A &= -2\sum_{n=1}^{\infty}\frac{K_1(n\ell)}{n\ell}[1-\cos(n\varphi)] + 2\sum_{n=0}^{\infty}\left\{\frac{(n+\frac{1}{2})^2\ell^2-d^2}{R_n^3}K_1(R_n)-\frac{d^2}{R_n^2}K_0(R_n)\right\},\\
B &= 2i\sum_{n=0}^{\infty}\frac{\ell d(n+\frac{1}{2})}{R_n^2}\left\{2\,\frac{K_1(R_n)}{R_n}+K_0(R_n)\right\}\sin(n+\tfrac{1}{2})\varphi\,,\\
C &= 2\sum_{n=0}^{\infty}\left\{\frac{d^2}{R_n^2}K_0(R_n)-\frac{(n+\frac{1}{2})^2\ell^2-d^2}{R_n^3}K_1(R_n)\right\}\cos(n+\tfrac{1}{2})\varphi\,,\\
D &= A-2\sum_{n=1}^{\infty}K_0(n\ell)[1-\cos(n\varphi)] + 2\sum_{n=0}^{\infty}K_0(R_n)\,,\\
E &= C-2\sum_{n=0}^{\infty}K_0(R_n)\cos(n+\tfrac{1}{2})\varphi\,,
\end{aligned}
\qquad (5.20)
$$

where $\qquad R_n = [(n+\frac{1}{2})^2\ell^2+d^2]^{1/2}.$

For $\varphi=\pi$, the critical case for stability, $C=E=0$. The stability criterion thus becomes

$$A(\pi)D(\pi)\leqslant 0. \qquad (5.21)$$

From Eq. 5.20 it is apparent that this condition will be satisfied for a given value of ℓ for a range of values of d between the limits which correspond respectively to $A(\pi)=0$ and $D(\pi)=0$.

Stable Range of Values of d/ℓ for Vortex Street

ℓ	Upper limit	Lower limit
0	.281	.281
1	.300	.281
2	.350	.280

These results are shown graphically in Fig. 9. It should be noted that for small values of ℓ, the result, d/ℓ = .281, is exactly that obtained by Kármán (see Ref. 4, p. 345).

This calculation shows that stable configurations do exist. These results would be directly applicable to motions in the earth's atmosphere if the width of the vortex street were small compared to the radius of the earth. Unfortunately, this is not so and the curvature of the shear fields on the edges of the westerlies must be considered. These effects will be considered in the next section.

VI. STABILITY OF RINGS OF VORTICES

If the shear field north of the westerlies is neglected, a ring of equally strong anticyclones evenly spaced around a circle of latitude is the only formation of vortices which will not change its shape with time. The stability of such a system can be approximately investigated by considering the stability of a ring of vortices as given by Eq. 2.13 on a rotating disc as in Fig. 10. If there are N vortices in such a ring of radius a, spaced at equal angles of $\tau = \frac{2\pi}{N}$, the surface deflection for such a system in its equilibrium state is

$$\eta = \sum_{n=1}^{N} K_0 [a^2 + r^2 - 2ar\cos(n\tau - \theta)]^{1/2}. \qquad (6.1)$$

For the calculation of the velocities of the system from the surface deflection, it is useful to know the forms of Eq. 2.15 in polar coordinates. If V_r and V_θ are the dimensionless velocities in the radial and tangential directions respectively, these are

$$V_r = -\frac{1}{r}\frac{\partial \eta}{\partial \theta}$$

and

$$V_\theta = \frac{\partial \eta}{\partial r} . \qquad (6.2)$$

Using the second of these, the system shown in Fig. 10 has an angular velocity, Ω, given by

$$\Omega = -\frac{1}{a}\sum_{n=1}^{N-1} K_1 [2a\sin\tfrac{n\tau}{2}]\sin\tfrac{n\tau}{2} . \qquad (6.3)$$

If the n'th vortex is displaced by Δr_n in the radial direction and by an angle of $\Delta\theta_n$ in the tangential direction, the surface deflection in the displaced condition is

$$\eta = \sum_{n=1}^{N} K_o\left[(a+\Delta r_n)^2 + r^2 - 2(a+\Delta r_n) r \cos(n\tau + \Delta\theta_n - \theta)\right]^{1/2}. \tag{6.4}$$

The velocity of the N'th vortex is

$$\begin{aligned} V_r &= -\frac{1}{r}\frac{\partial}{\partial\theta}\sum_{n=1}^{N-1} K_o\left[(a+\Delta r_n)^2 + r^2 - 2(a+\Delta r_n) r \cos(n\tau + \Delta\theta_n - \theta)\right]^{1/2}, \\ V_\theta &= \frac{\partial}{\partial r}\sum_{n=1}^{N-1} K_o\left[(a+\Delta r_n)^2 + r^2 - 2(a+\Delta r_n) r \cos(n\tau + \Delta\theta_n - \theta)\right]^{1/2} \end{aligned} \tag{6.5}$$

with r placed equal to $a+\Delta r_N$ and θ placed equal to $\Delta\theta_N$ after differentiation.

If the displacements are small, the changes in velocity from the equilibrium value indicated in Eq. 6.3 is, for the N'th vortex,

$$\begin{aligned} \Delta V_r = \frac{d}{dt}(\Delta r_N) &= \sum_{n=1}^{N-1} \frac{\Delta r_n}{2} \sin(n\tau)\, K_o(R_n) \\ &+ \sum_{n=1}^{N-1} a(\Delta\theta_n - \Delta\theta_N)\left\{\frac{K_1(R_n)}{R_n} + \cos^2\left(\frac{n\tau}{2}\right) K_o(R_n)\right\}, \end{aligned} \tag{6.6}$$

and

$$\begin{aligned} \Delta V_\theta = a\frac{d}{dt}(\Delta\theta_N) + \Omega\,\Delta r_N &= \Delta r_N \sum_{n=1}^{N-1}\left\{\sin^2\left(\frac{n\tau}{2}\right) K_o(R_n) - \cos(n\tau)\frac{K_1(R_n)}{R_n}\right\} \\ &+ \sum_{n=1}^{N-1} \Delta r_n\left\{\frac{K_1(R_n)}{R_n} + \sin^2\left(\frac{n\tau}{2}\right) K_o(R_n)\right\} + \sum_{n=1}^{\infty} a\,\Delta\theta_n \cdot \frac{\sin(n\tau)}{2} K_o(R_n) \end{aligned}$$

where

$$R_n = 2a\sin\left(\frac{n\tau}{2}\right).$$

Similar expressions for the velocities of the other vortices could be written from symmetry. These would form a set of simultaneous differential equations for the displacements.

If the N equations in each of the two sets indicated in Eq. 6.6 are added, the following results are obtained:

$$\frac{d}{dt}\left\{\sum_{n=1}^{N}(\Delta r_n)\right\}=0$$
$$\frac{d}{dt}\left\{\sum_{n=1}^{N}(\Delta\theta_n)\right\}=\left\{\sum_{n=1}^{\infty}(\Delta r_n)\right\}\left\{\sum_{n=1}^{N-1}\left[2\sin^2\left(\frac{n\tau}{2}\right)K_0(R_n)+\frac{1}{a}\sin\left(\frac{n\tau}{2}\right)K_1(R_n)\right]\right\}. \qquad (6.7)$$

These can be integrated directly. The result is that

$$\sum_{n=1}^{N}\Delta r_n = C_1 ,$$
$$\sum_{n=1}^{N}\Delta\theta_n = C_1 t\left\{\sum_{n=1}^{N-1}\left[2\sin^2\left(\frac{n\tau}{2}\right)K_0(R_n)+\frac{1}{a}\sin\left(\frac{n\tau}{2}\right)K_1(R_n)\right]\right\}+C_2 . \qquad (6.8)$$

If the system is initially in equilibrium, $C_1=C_2=0$, and

$$\sum_{n=1}^{N}\Delta r_n=\sum_{n=1}^{N}\Delta\theta_n=0 . \qquad (6.9)$$

These results correspond to the similar equations for two dimensional line vortices which state that the impulse of a system having no external forces remains constant (see Ref. 2, p. 220).

The method of harmonic analysis used in V can also be used to advantage in solving Eq. 6.6. Let us assume that

$\Delta r_n = \Delta r_N e^{in\varphi}$ and $\Delta \theta_n = \Delta \theta_N e^{in\varphi}$ where φ is a member of the series, $\frac{2\pi}{N}, \frac{4\pi}{N}, \frac{6\pi}{N}, \cdots, \frac{N-1}{N}2\pi, 2\pi$.

Eq. 6.6 can then be written as follows:

$$\begin{aligned} \frac{d}{dt}\Delta r_N &= A\Delta r_N - B\,a\Delta\theta_N \\ a\frac{d}{dt}\Delta\theta_N &= C\Delta r_N + A\,a\Delta\theta_N \end{aligned} \tag{6.10}$$

where

$$\begin{aligned} A &= \sum_{n=1}^{N-1} \frac{e^{in\varphi}}{2} \sin n\tau\, K_0(R_n)\ , \\ B &= \sum_{n=1}^{N-1} (1-e^{in\varphi})\left\{\frac{K_1(R_n)}{R_n} + \cos^2\left(\tfrac{n\tau}{2}\right)K_0(R_n)\right\}, \\ C &= \sum_{n=1}^{N-1}\left[\left\{\sin^2\left(\tfrac{n\tau}{2}\right)K_0(R_n) - \cos(n\tau)\frac{K_1(R_n)}{R_n}\right\} + e^{in\varphi}\left\{\sin^2\left(\tfrac{n\tau}{2}\right)K_0(R_n) + \frac{K_1(R_n)}{R_n}\right\}\right] \\ &\quad + \sum_{n=1}^{N-1}\frac{K_1(R_n)}{R_n}[1-\cos(n\tau)]\ . \end{aligned} \tag{6.11}$$

If Δr_N and $\Delta\theta_N \backsim e^{\lambda t}$, λ must satisfy

$$(\lambda - A)^2 + BC = 0\ , \tag{6.12}$$

or

$$\lambda = A \pm \sqrt{-BC}\ . \tag{6.13}$$

For any of the N specified values of φ, A may be written as

$$A = i\sum_{n=1}^{\frac{N-1}{2}} \sin(n\tau)\sin(n\varphi)K_0(R_n) \quad \text{for } N \text{ odd} \tag{6.14}$$

or $$A = i\sum_{n=1}^{\frac{N-2}{2}} \sin(n\tau)\sin(n\varphi)K_0(R_n) \quad \text{for } N \text{ even.}$$

As A is pure imaginary the condition for stability is that

$$BC \geqslant 0. \tag{6.15}$$

In the same way,

$$B = 2\sum_{n=1}^{\frac{N-1}{2}} [1-\cos(n\varphi)]\left\{\frac{K_1(R_n)}{R_n} + \cos^2\left(\tfrac{n\tau}{2}\right)K_0(R_n)\right\} \quad \text{for } N \text{ odd,} \tag{6.16}$$

or $$B = 2\sum_{n=1}^{\frac{N-2}{2}} [1-\cos(n\varphi)]\left\{\frac{K_1(R_n)}{R_n} + \cos^2\left(\tfrac{n\tau}{2}\right)K_0(R_n)\right\} + \frac{K_1(2a)}{2a}(1-e^{i\frac{N}{2}\varphi}) \quad \text{for } N \text{ even.}$$

By inspection, B is always positive, and the stability criterion, Eq. 6.15, becomes

$$C \geqslant 0. \tag{6.17}$$

We will now investigate what vortex rings satisfy this condition.

If N = 2,

$$C = \left\{K_0(2a) + \frac{K_1(2a)}{2a}\right\}[1+\cos\varphi] + 2\,\frac{K_1(2a)}{2a}. \tag{6.18}$$

This is positive for all values of φ; so a ring of two vortices is stable.

If N = 3,

$$C = 2\left\{\tfrac{3}{4}K_0(R_1) + \frac{K_1(R_1)}{R_1}\right\}[1+\cos\varphi] + 2\,\frac{K_1(R_1)}{R_1}. \tag{6.19}$$

This, too, is always positive; so a ring of three vortices is stable.

If N = 4,

$$C = 2\left\{\tfrac{1}{2}K_0(R_1) + \frac{K_1(R_1)}{R_1}\right\}[1+\cos\varphi] + \left\{K_0(2a) + \frac{K_1(2a)}{2a}\right\}[1+\cos(2\varphi)] + \frac{K_0(2a)}{a}. \tag{6.20}$$

A system of four vortices is thus stable.

If $N = 5$,

$$C = 2\Big[K_0(R_1)\sin^2(\tfrac{\pi}{5})(1+\cos\varphi) + K_0(R_2)\sin^2(\tfrac{2\pi}{5})(1+\cos 2\varphi) + \frac{K_1(R_1)}{R_1}(1-2\cos\tfrac{2\pi}{5}+\cos\varphi) + \frac{K_1(R_2)}{R_2}(1-2\cos\tfrac{4\pi}{5}+\cos 2\varphi)\Big] . \tag{6.21}$$

For $\varphi = 2\pi$, this is obviously positive. If $\varphi = \frac{2\pi}{5}$ or $\frac{8\pi}{5}$, it is also obviously positive. For $\varphi = \frac{4\pi}{5}$ or $\frac{6\pi}{5}$,

$$C = 2\Big\{.1424\, K_0[1.1756\,a] + 1.0392\, K_0[1.9022\,a] - .2058\,\frac{K_1[1.1756a]}{1.1756a} + 2.4846\,\frac{K_1[1.9022a]}{1.9022a}\Big\} . \tag{6.22}$$

This is an expression with no positive roots, so a ring of five vortices is stable.

If $N = 6$,

$$C = 2\Big\{\tfrac{1}{4}K_0(R_1)(1+\cos\varphi) + \frac{K_1(R_1)}{R_1}\cos\varphi + \tfrac{3}{4}K_0(R_2)(1+\cos 2\varphi) + \frac{K_1(R_2)}{R_2}(2+\cos 2\varphi)\Big\} + K_0(2a)(1+\cos 3\varphi) + \frac{K_1(2a)}{2a}(3+\cos 3\varphi). \tag{6.23}$$

This obviously positive if $\cos\varphi$ is positive. This includes $\varphi = 2\pi$, $\frac{2\pi}{6}$ and $\frac{10\pi}{6}$. For $\varphi = \pi$,

$$C = 2\Big\{-\frac{K_1(R_1)}{R_1} + \tfrac{3}{2}K_0(R_2) + 3\frac{K_1(R_2)}{R_2}\Big\} + 2\,\frac{K_1(2a)}{2a} . \tag{6.24}$$

This is always positive. For $\varphi = \frac{4\pi}{6}$ or $\frac{8\pi}{6}$,

$$C = 2\Big\{\tfrac{1}{8}K_0(R_1) - \tfrac{1}{2}\frac{K_1(R_1)}{R_1} + \tfrac{3}{8}K_0(R_2) + \tfrac{3}{2}\frac{K_1(R_2)}{R_2} + K_0(2a) + 2\frac{K_1(2a)}{2a}\Big\}. \tag{6.25}$$

As this, too, is positive, a ring of six vortices is stable.

If $N = 7$,

$$C = 2\Big\{K_0(R_1)\,\sin^2(\tfrac{\pi}{7})\,(1+\cos\varphi) + \frac{K_1(R_1)}{R_1}(1-2\cos\tfrac{2\pi}{7}+\cos\varphi)$$
$$+K_0(R_2)\,\sin^2(\tfrac{2\pi}{7})(1+\cos 2\varphi) + \frac{K_1(R_2)}{R_2}(1-2\cos\tfrac{4\pi}{7}+\cos 2\varphi) \qquad (6.25)$$
$$+K_0(R_3)\,\sin^2(\tfrac{3\pi}{7})\,(1+\cos 3\varphi) + \frac{K_1(R_3)}{R_3}(1-2\cos\tfrac{6\pi}{7}+\cos 3\varphi)\Big\}.$$

For $\varphi = 2\pi$, $\frac{2\pi}{7}$, or $\frac{12\pi}{7}$, C is evidently positive.

If $\varphi = \frac{4\pi}{7}$ or $\frac{10\pi}{7}$,

$$C = 2\Big\{.1428\,K_0[.8678\,a] + .0609\,K_0[1.5638\,a] + 1.5930\,K_0[1.9506\,a]$$
$$+ .4677\frac{K_1[.8678\,a]}{.8678\,a} + .4509\frac{K_1[1.5638\,a]}{1.5638\,a} + 3.4252\frac{K_1[1.9506\,a]}{1.9506\,a}\Big\} \qquad (6.26)$$

so this is always positive.

If $\varphi = \frac{6\pi}{7}$ or $\frac{8\pi}{7}$,

$$C = 2\Big\{.0187\,K_0[.8678\,a] + .9984\,K_0[1.5638\,a] + .7405\,K_0[1.9506\,a]$$
$$-1.1477\frac{K_1[.8678\,a]}{.8678\,a} + 2.0652\frac{K_1[1.5638\,a]}{1.5638\,a} + .5809\frac{K_1[1.9506\,a]}{1.9506\,a}\Big\}. \qquad (6.27)$$

This is positive if $a > 71$; so that a ring of seven vortices is stable provided $a > 71$.

If $N = 8$ and $\varphi = \pi$,

$$C = 2\Big\{-\sqrt{2}\,\frac{K_1[.7654\,a]}{.7654\,a} + K_0(a\sqrt{2}) + \sqrt{2}\,\frac{K_1(a\sqrt{2})}{a} + \sqrt{2}\,\frac{K_1[1.8478\,a]}{1.8478\,a}\Big\} \qquad (6.28)$$
$$+2\,K_0(2a) + 4\,\frac{K_1(2a)}{2a}.$$

This is always negative so that a ring of eight vortices is not stable. It can be seen that any higher number of vortices in a ring is not stable.

A value of a greater than 7/ corresponds either to disturbances of such a great wave length or to the motion of such a shallow layer of air in the earth's atmosphere that it probably is of no significance. The results of this calculation may then be summarized by the statement that six or less equal vortices placed at the corners of a regular polygon form a stable vortex formation. This result is in agreement with the work of J. J. Thomson who investigated the stability of similar formations of two dimensional line vortices in a frictionless, incompressible fluid (see Ref. 5, p. 94).

As a single ring is stable, the effect of the shear field to the north of the westerlies can roughly be taken into account by placing a fixed vortex at the pole as in Fig. 11. This, of course, breaks down for the cases where the number of anticyclones is large as can be seen from the results of V. The effect of such a vortex is to change the rate of rotation of the system as given by Eq. 6.3 and to add additional terms to Eq. 6.6. If the polar cyclone has a strength A and the anticyclones have a strength of unity, it is found that Eq. 6.10 still holds if

$$C = -A\left[2\frac{K_1(a)}{a} + K_0(a)\right] + \sum_{n=1}^{N-1}\left[\left\{\sin^2\left(\frac{n\tau}{2}\right)K_0(R_n) + \frac{K_1(R_n)}{R_n}\right\} + e^{in\varphi}\left\{\sin^2\left(\frac{n\tau}{2}\right)K_0(R_n) + \frac{K_1(R_n)}{R_n}\right\} - 2\cos(n\tau)\frac{K_1(R_n)}{R_n}\right]. \tag{6.29}$$

The fixed polar cyclone is thus seen to cause a decrease in stability, and if A is large enough, the motion becomes unstable.

If we call the minimum value of A which makes C, as given by Eq. 6.29, vanish $A_{critical}$, then $\frac{1}{N}A_{critical}$ is the ratio of the polar (cyclonic) vorticity to the anticyclonic vorticity in the system. It can be seen from Eq. 6.29 that, except for $N = 2$, as N increases this ratio decreases. From this if the polar vortex is fairly strong only two or three vortices can form a stable system; however as the polar vortex becomes weaker, four, five, or even six anticyclones may form stable configurations.

In order to take into account the northern shear area properly, it would be necessary to consider the stability of double ring systems. This problem is very complex, but the nature of the results can be estimated from the preceding calculations. The "vortex-street" is a limiting case where the spacing is small and shows a stable formation with a very large number of vortices; however the width of the westerlies is too great to permit this system to be reached. The ring of vortices with a polar cyclone is a limiting case where the width of the westerlies is very large and should be much closer to the observed phenomena.

VII. CONCLUSION

As a result of these calculations it is evident that the shear fields next to the westerlies will roll up and form a series of closed isobaric systems with high pressure cells to the south of the westerlies and low pressure cells to the north of the westerlies. It further appears that a maximum of six high pressure cells will be observed although more are possible if the belt of westerlies were to become extremely narrow. Of the two limiting cases, one progressed to the West and the other to the East, so one might expect the intermediate systems to be practically stationary. This is in rough agreement with the idea that the position of such systems ought primarily to be determined by thermodynamic considerations, so that the stable vortex configurations in the atmosphere might be expected to be those that are stationary.

As the polar air mass shows a tendency to rotate as a whole, an increase in the westerly winds, on a percentage basis, increases the shear to the north of the westerlies faster than that to the south. Thus a strong atmospheric circulation is correlated with a large value of the ratio of vorticity to the north of the westerlies to the vorticity south of the westerlies, and a weak atmospheric circulation corresponds to a small value of the ratio. From the results of VI, this would imply that with strong circulation, only two or three high pressure cells would be found whereas with weak circulation, stable formations having up to six cells could be found. This result is well verified by Northern Hemisphere mean pressure charts although no attempt has yet been made to correlate these factors in a quantitative fashion.

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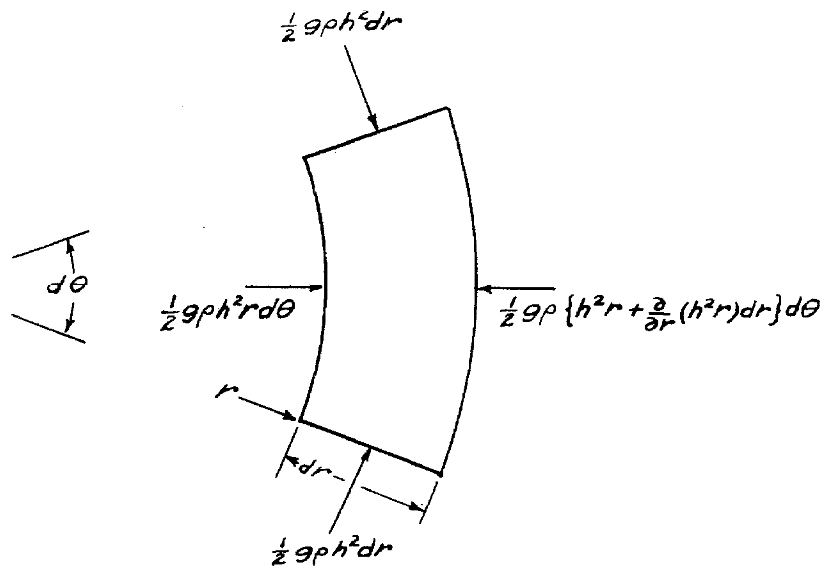

FIGURE 1

DIAGRAM OF FORCES FOR FLOW IN CIRCLES

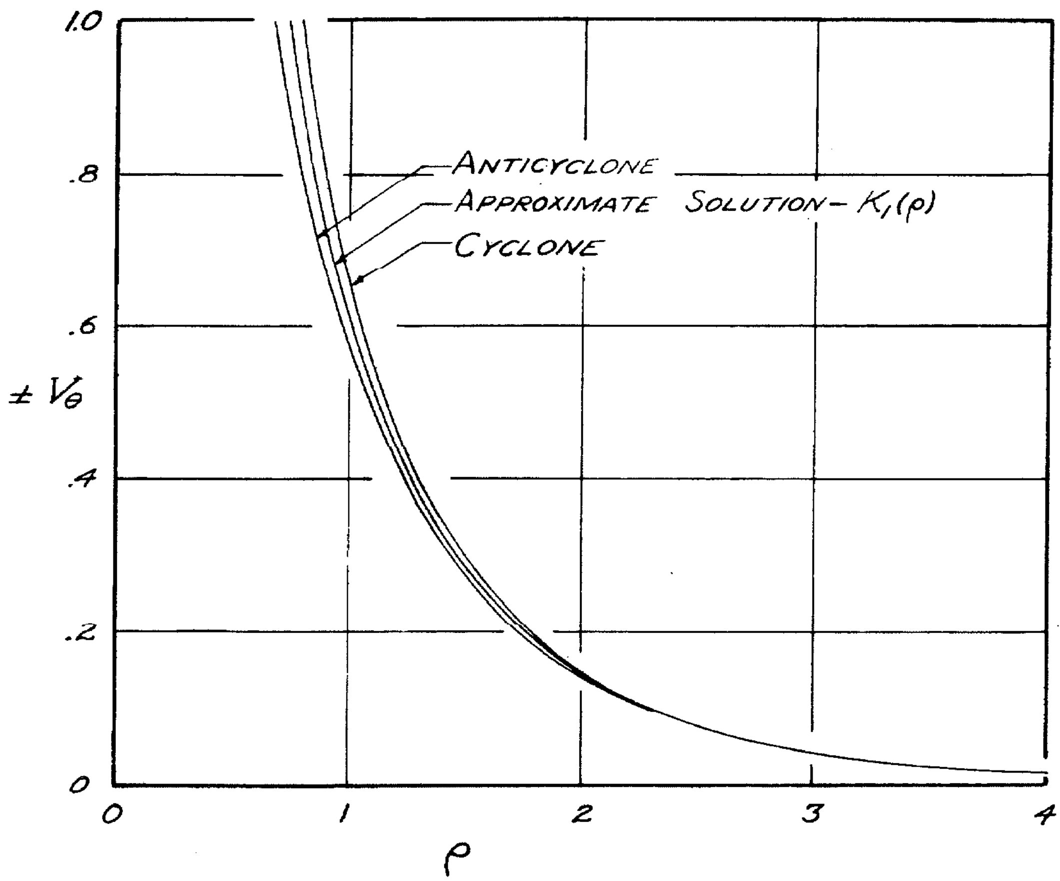

FIGURE 2

VELOCITY PROFILES FOR VORTICES

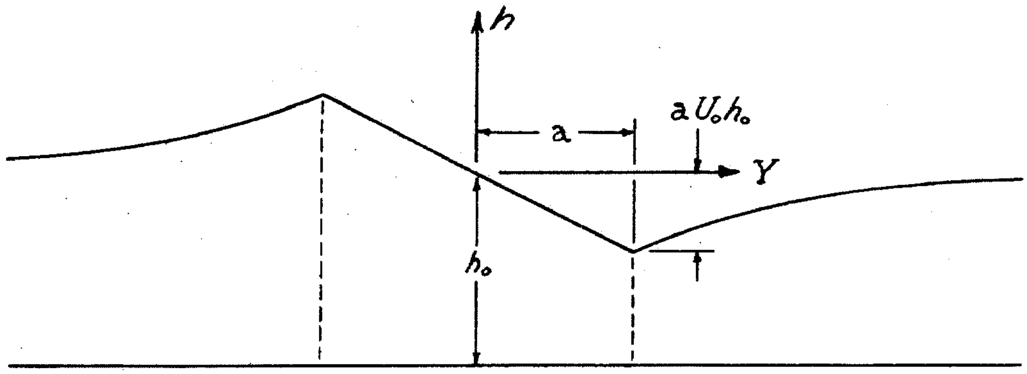

FIGURE 3

SURFACE DEFLECTION FOR A ONE-DIMENSIONAL JET

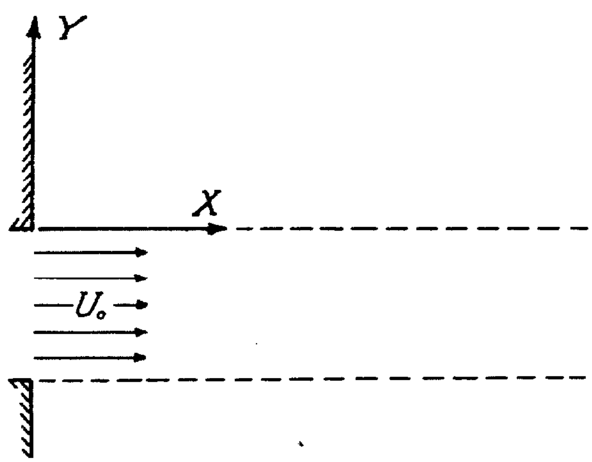

FIGURE 4

JET ISSUING FROM A SLOT IN A WALL

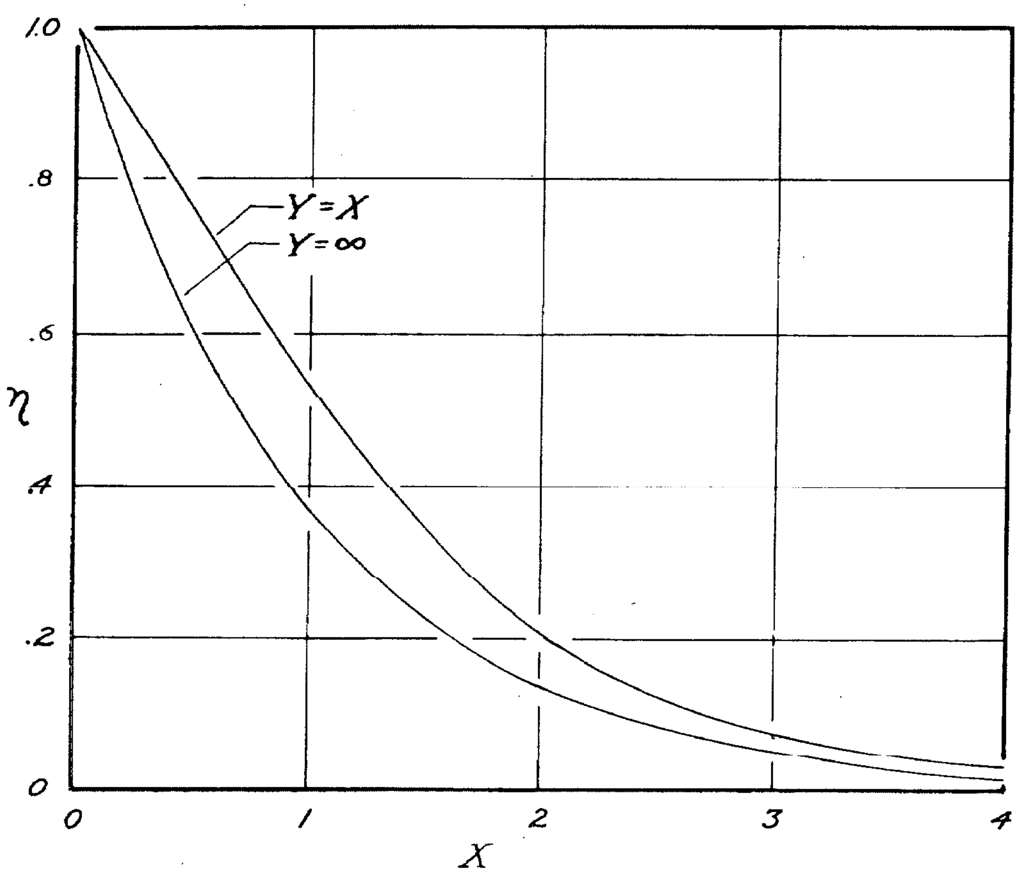

FIGURE 5

SURFACE DEFLECTION FOR A JET ISSUING FROM A SLOT IN A WALL

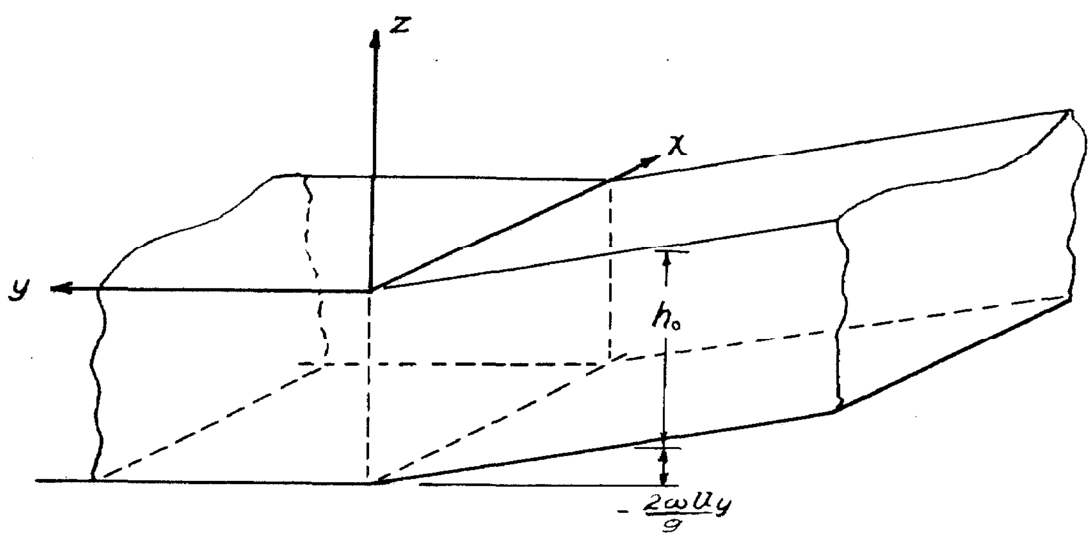

FIGURE 6

VORTEX SHEET BETWEEN STATIONARY

AND MOVING FLUID MASSES

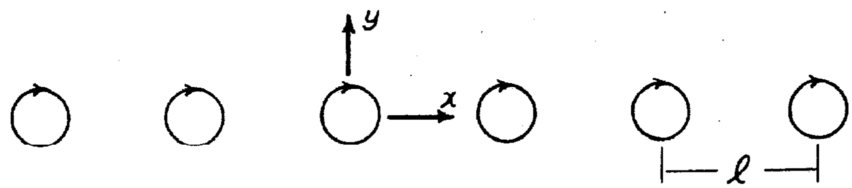

FIGURE 7

SINGLE ROW OF VORTICES

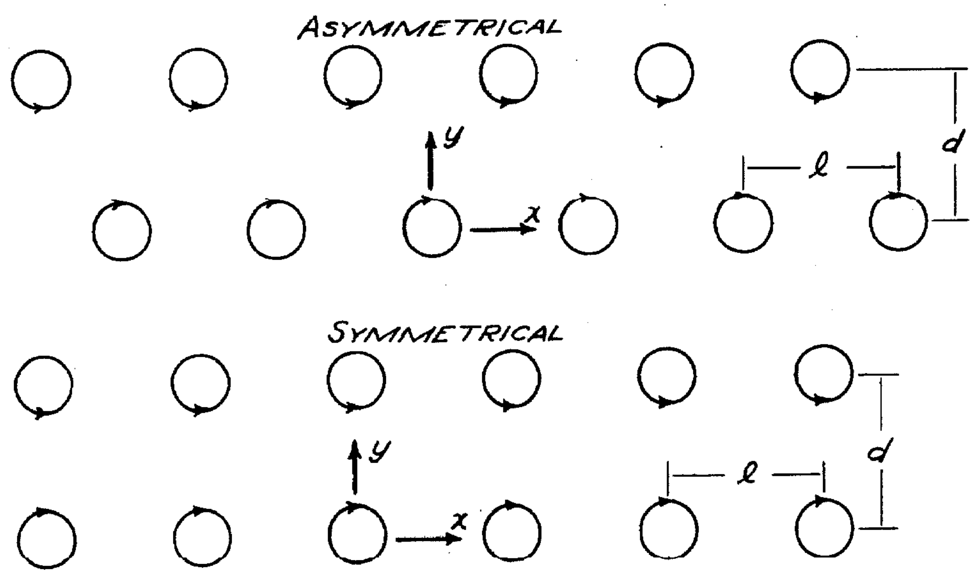

FIGURE 8

DOUBLE ROWS OF VORTICES

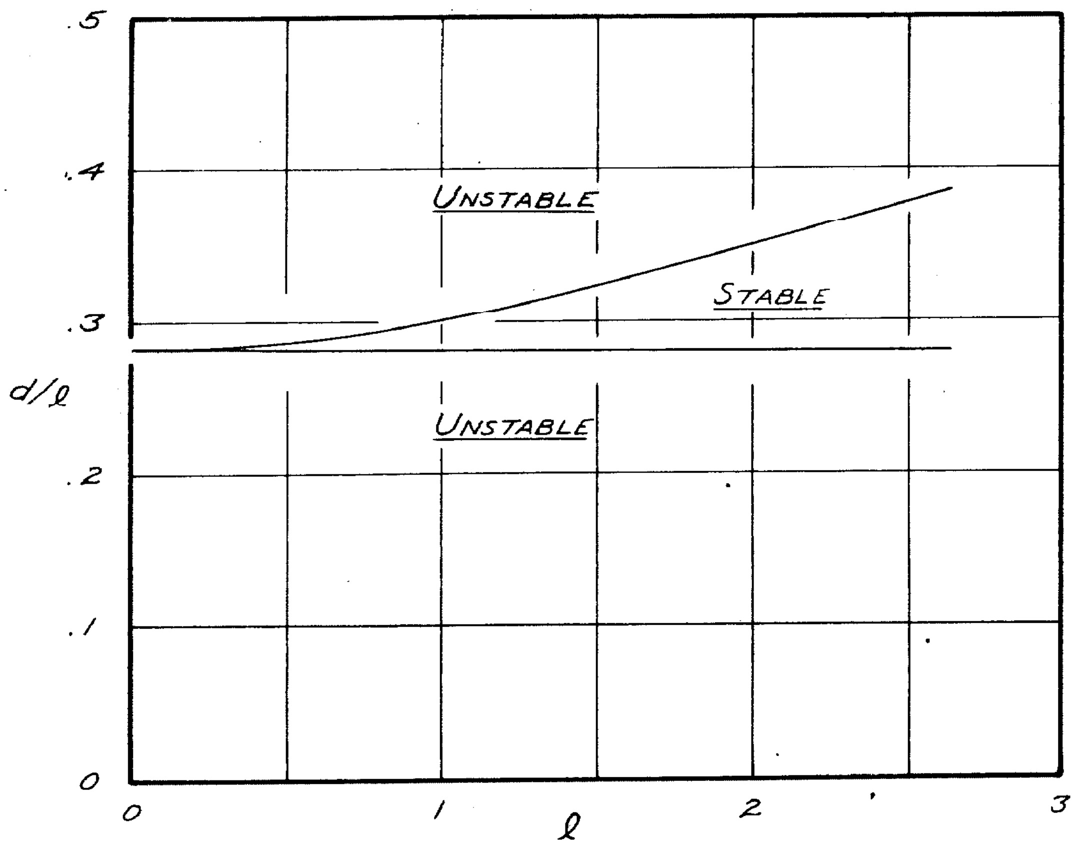

FIGURE 9

STABILITY DIAGRAM FOR DOUBLE VORTEX ROWS

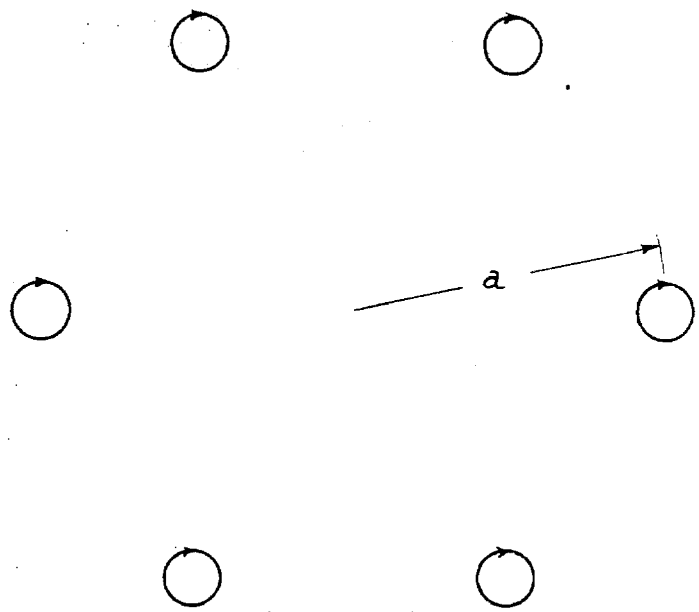

FIGURE 10

RING SYSTEM OF VORTICES

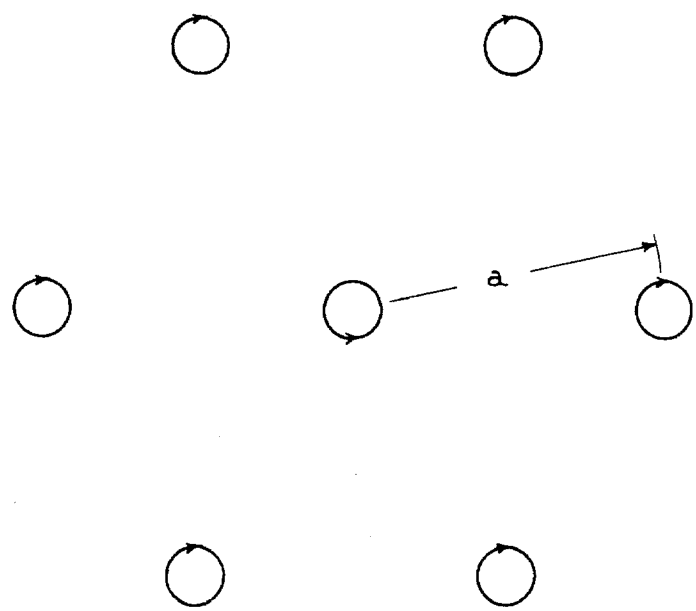

FIGURE 11

RING SYSTEM OF VORTICES WITH POLAR CYCLONE